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# Numerical methods for one-dimensional aggregation equations

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## Abstract

We focus in this work in the numerical discretization of the one dimensional aggregation equation  $\partial_t \rho + \partial_x(v\rho) = 0$ ,  $v = a(W' * \rho)$ , in the attractive case. Finite time blow up of smooth initial data occurs for potential  $W$  having a Lipschitz singularity at the origin. A numerical discretization is proposed for which the convergence towards duality solutions of the aggregation equation is proved. It relies on a careful choice of the discretized macroscopic velocity  $v$  in order to give a sense to the product  $v\rho$ . Moreover, using the same idea, we propose an asymptotic preserving scheme for a kinetic system in hyperbolic scaling converging towards the aggregation equation in hydrodynamical limit. Finally numerical simulations are provided to illustrate the results.

**Keywords.** aggregation equation, duality solutions, finite volume schemes, asymptotic preserving schemes, weak measure solutions, hydrodynamical limit.

**2010 AMS classifications.** 35B40, 35D30, 35L60, 35Q92, 65M08.

## 1 Introduction

This paper is devoted to the numerical approximation of the so-called aggregation equation which writes in one space dimension

$$\partial_t \rho + \partial_x(a(W' * \rho)\rho) = 0, \quad (1.1)$$

complemented with some initial data  $\rho(0, x) = \rho^{ini}(x)$ . This nonlocal and nonlinear conservation equation is involved in many applications in physics and biology, where it describes the behaviour of a population of particles (in physical applications) or cells (in biological applications) interacting under a continuous interaction potential  $W$ . The quantity  $\rho$  denotes the density of these particles or cells. The function  $a$  is often linear ( $a(u) = \pm u$ ), see e.g. [3, 11, 34, 35, 37, 33], but in several applications, such as pedestrian motion [14, 15] or chemotaxis (see [24] and Section 4.2 below) a specific nonlinearity has to be considered. Depending on the choice of the potential  $W$  and the function  $a$ , one can be in the repulsive or in the attractive case, the latter leading to aggregation phenomena.

In this work we focus on the case involving attractive forces. Individuals attract one another under the action of the potential  $W$ , assumed to be smooth away from 0 and bounded from below. More precisely,  $W$  satisfies the following properties:

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**Assumption 1.1** We assume that  $W \in C^1(\mathbb{R} \setminus \{0\})$  is a pointy attractive potential, i.e.

$$W'' = -\delta_0 + w, \quad w \in C_0(\mathbb{R}), \quad \text{with } \|w\|_{L^1(\mathbb{R})} = w_0 < \infty, \quad (1.2)$$

in the distributional sense, where  $\delta_0$  is the Dirac measure at 0.

Under Assumption 1.1, attractivity in the nonlinear case is ensured provided the function  $a$  satisfies

$$a \in C^1(\mathbb{R}), \quad 0 \leq a'(x) \leq \alpha, \quad \forall x \in \mathbb{R}. \quad (1.3)$$

This case has been extensively studied in the linear case  $a = \text{id}$  [4, 5, 6] and it is known that if the potential  $W$  has a Lipschitz singularity then weak solutions blow up in finite time (see e.g. [4, 24]), so that measure valued solutions arise. At the theoretical level, global in time existence has been obtained in the linear case  $a = \text{id}$  and in any space dimension by Carrillo et al. [13], in the space  $\mathcal{P}_2$  of probability measures with finite second moment, using the geometrical approach of gradient flows. In the nonlinear case, but in one space dimension, global existence of measure solutions has been obtained by completely different means in [27], namely thanks to the notion of duality solutions. It has also been proved in [27] that in the linear case  $a = \text{id}$ , both notions coincide. In the same respect, we refer to [7], where gradient flow solutions are proved to be equivalent to entropy solutions of the Burgers equation, with a particular focus on the repulsive case. The key point leading to both uniqueness of solutions and equivalence between the two notions is the definition of the macroscopic velocity. In the framework of gradient flows, it is defined as the unique element with minimal norm in the subdifferential of the interaction energy associated to  $W$ , where  $\mathcal{P}_2$  is endowed with the Wasserstein distance (see [2, 13] for more details). In [27], the macroscopic velocity is defined using the chain rule for BV functions, and this is the viewpoint adopted for numerical analysis.

In some applications, the aggregation equation can be obtained as the hydrodynamic limit of some kinetic system. For instance, we consider here the following two velocities kinetic model in hyperbolic scaling which is used to model the so-called run-and-tumble process in bacterial chemotaxis (see [16, 17, 24]). The dynamics of the distribution function of cells  $f_\varepsilon^\pm$  at time  $t$ , position  $x$  and with velocity  $\pm 1$  is governed by

$$\partial_t f_\varepsilon^\pm \pm \partial_x f_\varepsilon^\pm = \pm \frac{1}{\varepsilon} (\phi(-W' * \rho_\varepsilon) f_\varepsilon^- - \phi(W' * \rho_\varepsilon) f_\varepsilon^+). \quad (1.4)$$

The left hand side corresponds to the free transport (run phase) whereas the right hand side models the velocity reorientation, called tumble process. The function  $\phi$  is given by  $\phi(x) = \frac{1}{2}(1 - a(x))$  where the function  $a$  is assumed to be odd, in addition to (1.3). Existence of global in time  $L^\infty$  weak solutions for such a kinetic equation with fixed  $\varepsilon > 0$  is well-known (see e.g. [10, 38]). However the bound is not uniform in  $\varepsilon$  and it actually turns out that limit when  $\varepsilon$  goes to zero of the total density  $\rho = \lim_{\varepsilon \rightarrow 0} (f_\varepsilon^+ + f_\varepsilon^-)$  solves the aggregation equation (1.1), see Section 3 below for details.

Therefore an interesting issue consists in providing a numerical scheme for the kinetic system (1.4) which allows to recover the asymptotic limit when  $\varepsilon \rightarrow 0$ . Such schemes are usually called *asymptotic preserving (AP)* [29]. They are of great interest for kinetic equations since letting  $\varepsilon \rightarrow 0$  with the mesh size and time step fixed, the scheme becomes a scheme for the macroscopic limit (see e.g. [18, 30, 32]). In other words, AP schemes allow a numerical discretization whose time step is not constrained by some constant depending on  $\varepsilon$ . We refer to [28] for a review on AP schemes.

The aim of this work is precisely to design numerical methods for (1.1) and (1.4) that are able to capture the measure solutions after blow-up. The main difficulty is that after blow up the velocity  $a(W' * \rho)$  is discontinuous, so that the definition of the flux has to be considered with great care. Following the principle that holds at the continuous level, the numerical velocity is obtained thanks a careful discretization of the Vol'pert calculus for BV functions. We emphasize that the numerical solution may depend upon the way of discretizing the velocity. For equation (1.1) we work directly on the definition of  $a(W' * \rho)$ , for the kinetic model, the discretization is defined through the right-hand side of equation (1.4). The final scheme is obtained by a splitting technique, as in for instance [30, 31], which is in this particular case very easy to implement. As we shall see, this is not completely satisfactory in the sense that the fluxes at equilibrium are not correctly computed. A more sophisticated technique consists in using well-balanced schemes [23, 21] as it has been successfully used for chemotaxis models in [22, 19, 20] for smooth solutions. However, it is not clear that such schemes allow to recover the solutions after blow up.

The outline of the paper is as follows. Section 2 is devoted to the aggregation equation (1.1). After recalling existence and uniqueness result for this system, we provide a numerical scheme and prove its convergence. In Section 3, we consider the kinetic equation (1.4). We first establish the rigorous derivation of the aggregation equation thanks to a hyperbolic limit  $\varepsilon \rightarrow 0$ . Then we propose an asymptotic preserving scheme and prove its convergence. Finally, Section 4 is devoted to some numerical simulations.

Part of these results were announced in [26].

## 2 Aggregation equation

### 2.1 Existence of duality solutions

For  $Y$  and  $Z$  two metric spaces we denote  $C(Y, Z)$  the set of continuous functions from  $Y$  to  $Z$ . We denote  $C_0(\mathbb{R})$  the set of continuous functions that vanish at infinity and  $C_c(\mathbb{R})$  the set of those with compact support. Let  $\mathcal{M}_b(\mathbb{R})$  be the set of bounded Radon measures and by  $\mathcal{P}_1(\mathbb{R})$  the set of positive measure in  $\mathcal{M}_b(\mathbb{R})$  such that  $\int_{\mathbb{R}} |x| d\mu(x) < \infty$ . From now on, the space  $\mathcal{M}_b(\mathbb{R}^N)$  is always endowed with the weak topology  $\sigma(\mathcal{M}_b, C_0)$ . We denote  $\mathcal{S}_{\mathcal{M}} := C([0, T]; \mathcal{M}_b(\mathbb{R}^N) - \sigma(\mathcal{M}_b, C_0))$ .

Duality solutions have been introduced in [8] to solve scalar conservation laws with discontinuous coefficients. More precisely, it gives sense to measure valued solutions of the scalar conservation law

$$\partial_t \rho(t, x) + \partial_x (b(t, x) \rho(t, x)) = 0,$$

where  $b \in L^\infty((0, T) \times \mathbb{R})$  satisfies the so-called one-sided Lipschitz condition

$$\partial_x b(t, \cdot) \leq \beta(t) \quad \text{for } \beta \in L^1(0, T), \text{ in the distributional sense.} \quad (2.1)$$

This key point suggests that the velocity field should be compressive. We refer to [8] for the precise definition and general properties of these solutions.

Let us first define a notion of duality solution for the aggregation equation (1.1) in the spirit of [9, 25]:

**Definition 2.1** *We say that  $\rho \in C([0, T]; \mathcal{M}_b(\mathbb{R}))$  is a duality solution to (1.1) if there exists  $\hat{a}_\rho \in L^\infty((0, T) \times \mathbb{R})$  and  $\alpha \in L^1_{loc}(0, T)$  satisfying  $\partial_x \hat{a}_\rho \leq \alpha$  in  $\mathcal{D}'((0, T) \times \mathbb{R})$ , such that for all*

$0 < t_1 < t_2 < T$ ,

$$\partial_t \rho + \partial_x(\widehat{a}_\rho \rho) = 0 \quad \text{in the sense of duality on } (t_1, t_2),$$

and  $\widehat{a}_\rho = a(W' * \rho)$  a.e.

From now on, we denote by  $A$  the antiderivative of  $a$  such that  $A(0) = 0$ . Using the chain rule, a natural definition of the flux is

$$J := -\partial_x A(W' * \rho) + a(W' * \rho)(w * \rho). \quad (2.2)$$

In fact, a formal computation shows that

$$-\partial_x A(W' * \rho) = -a(W' * \rho)(W'' * \rho) = a(W' * \rho)(\rho - w * \rho),$$

where we use (1.2) for the last identity.

Then we are in position to state the existence and uniqueness result of [27].

**Theorem 2.2 ([27], Theorem 3.9)** *Let us assume that  $\rho^{ini}$  is given in  $\mathcal{P}_1(\mathbb{R})$ . Under Assumption 1.1 on the potential  $W$  and (1.3) for the nonlinear function  $a$ , for all  $T > 0$  there exists a unique duality solution  $\rho$  of (1.1) in the sense of Definition 2.1 with  $\rho \geq 0$ ,  $\rho(t) \in \mathcal{P}_1(\mathbb{R})$  for  $t \in (0, T)$  and which satisfies in the distributional sense:*

$$\partial_t \rho + \partial_x J = 0, \quad (2.3)$$

where  $J$  is defined in (2.2). Moreover, there exists  $\widehat{a}$ , called universal representative, such that  $\widehat{a} = a(W' * \rho)$  a.e. Then  $\rho = X_{\#} \rho^{ini}$ , where  $X$  is the Filippov flow associated to the velocity  $\widehat{a}$ .

## 2.2 Numerical discretization

Let us consider a uniform space discretization with step  $\delta x$  and denote by  $\delta t$  the time step; then  $t_n = n\delta t$  and  $x_i = x_0 + i\delta x$ ,  $i \in \mathbb{Z}$ . We assume that the initial datum  $\rho^{ini}$  is given in  $\mathcal{P}_1(\mathbb{R})$ .

For  $n \in \mathbb{N}$ , we assume to have computed an approximation  $(\rho_i^n)_{i \in \mathbb{Z}}$  of  $(\rho(t_n, x_i))_{i \in \mathbb{Z}}$ , we denote by  $(S_i^n)_{i \in \mathbb{Z}}$  an approximation of  $(W * \rho(t_n, x_i))_{i \in \mathbb{Z}}$  and by  $(\nu_i^n)_{i \in \mathbb{Z}}$  an approximation of  $(w * \rho(t_n, x_i))_{i \in \mathbb{Z}}$ . Let us denote  $\lambda = \delta t / \delta x$  and  $M$  the total mass of the system,  $M = |\rho^{ini}|(\mathbb{R})$ . We obtain an approximation of  $\rho(t_{n+1}, x_i)$  denoted  $\rho_i^{n+1}$  by using the following Lax-Friedrichs discretization of equation (2.3)–(2.2):

$$\rho_i^{n+1} = \rho_i^n - \lambda(J_{i+1/2}^n - J_{i-1/2}^n) + \frac{\lambda}{2} c (\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n) \quad (2.4)$$

$$J_{i+1/2}^n = -\frac{A(\partial_x S_{i+1}^n) - A(\partial_x S_i^n)}{\delta x} + a_{i+1/2}^n \frac{\nu_{i+1}^n + \nu_i^n}{2}, \quad (2.5)$$

where we have defined

$$c := \max_{x \in [-M(1+w_0), M(1+w_0)]} |a(x)|. \quad (2.6)$$

In this scheme, we use the discretization

$$\partial_x S_{i+1}^n = \frac{S_{i+2}^n - S_i^n}{2\delta x}, \quad (2.7)$$

and the approximation

$$a_{i+1/2}^n = \begin{cases} 0 & \text{if } \partial_x S_{i+1}^n = \partial_x S_i^n, \\ \frac{A(\partial_x S_{i+1}^n) - A(\partial_x S_i^n)}{\partial_x S_{i+1}^n - \partial_x S_i^n} & \text{otherwise.} \end{cases} \quad (2.8)$$

We need now a scheme for  $S_i^n$ . From Assumption 1.1, we deduce by taking the convolution of (1.2) with  $\rho$  that  $-W'' * \rho + w * \rho = \rho$ . This equation is discretized by using a standard finite difference scheme:

$$-\frac{S_{i+1}^n - 2S_i^n + S_{i-1}^n}{\delta x^2} + \nu_i^n = \rho_i^n. \quad (2.9)$$

For the computation of  $(\nu_i^n)_i$ , there are multiple ways; here we propose to use a piecewise constant approximation for  $\rho$  on each interval  $[x_i, x_{i+1})$ , so that

$$\nu_i^n = \sum_{k \in \mathbb{Z}} \int_{x_k}^{x_{k+1}} \rho_i^n w(x_i - y) dy,$$

which can be rewritten

$$\nu_i^n = \sum_{k \in \mathbb{Z}} \rho_k^n w_{ki}, \quad w_{ki} = \int_{x_i - x_{k+1}}^{x_i - x_k} w(z) dz = \int_{(i-1-k)\delta x}^{(i-k)\delta x} w(z) dz. \quad (2.10)$$

We turn back now to the discrete macroscopic flux (2.5), and notice that using (2.8) we have

$$J_{i+1/2}^n = a_{i+1/2}^n \left( -\frac{\partial_x S_{i+1}^n - \partial_x S_i^n}{\delta x} + \frac{\nu_{i+1}^n + \nu_i^n}{2} \right).$$

Using then (2.7) and (2.9) we obtain the following result.

**Lemma 2.3** *With the choice (2.8) for the discretization of the macroscopic velocity  $a_{i+1/2}^n$ , the macroscopic flux (2.5) rewrites*

$$J_{i+1/2}^n = a_{i+1/2}^n \frac{\rho_{i+1}^n + \rho_i^n}{2}. \quad (2.11)$$

This lemma is actually the discretization of the universal representative  $\hat{a}$  of Theorem 2.2. It emphasizes the importance of the choice of the discretization of the macroscopic velocity  $a_{i+1/2}^n$  in (2.8). Numerical examples showing wrong dynamics with a different discretization choice will be provided in Section 4.2.

Finally, injecting the expression (2.11) of the flux into (2.4), we obtain

$$\rho_i^{n+1} = \rho_i^n \left( 1 - \lambda c + \frac{\lambda}{4} (a_{i-1/2}^n - a_{i+1/2}^n) \right) + \frac{\lambda}{2} \left( c + \frac{a_{i-1/2}^n}{2} \right) \rho_{i-1}^n + \frac{\lambda}{2} \left( c - \frac{a_{i+1/2}^n}{2} \right) \rho_{i+1}^n. \quad (2.12)$$

**Remark 2.4** *The choice of the discretization (2.8) for the macroscopic velocity can be seen as a consequence of the chain rule (or Vol'pert calculus) for BV functions [39] (see also remark 3.98 of [1]): for a BV function  $u$ , the fonction  $\hat{a}_V$  defining the chain rule  $\partial_x A(u) = \hat{a}_V \partial_x u$  is constructed by*

$$\hat{a}_V(x) = \int_0^1 a(tu_1(x) + (1-t)u_2(x)) dt,$$

where

$$(u_1, u_2) = \begin{cases} (u, u) & \text{if } x \in \mathbb{R} \setminus S_u, \\ (u^+, u^-) & \text{if } x \in J_u, \\ \text{arbitrary} & \text{elsewhere.} \end{cases}$$

We have denoted by  $S_u$  the set of  $x \in \mathbb{R}$  where  $u$  does not admit an approximate limit and by  $J_u \subset S_u$  the set of jump points. Applying that to  $u = \partial_x S$ , we obtain,

$$\widehat{a}_V(x) = \begin{cases} a(\partial_x S(x)) & \text{if } x \in \mathbb{R} \setminus S_u, \\ \frac{A(\partial_x S(x^+)) - A(\partial_x S(x^-))}{\partial_x S(x^+) - \partial_x S(x^-)} & \text{if } x \in J_u, \\ \text{arbitrary} & \text{elsewhere.} \end{cases}$$

### 2.3 Numerical analysis

In this subsection, we prove the convergence of the numerical scheme defined in (2.4)–(2.8) towards the unique duality solution of Theorem 2.2. We first state a Lemma which proves a CFL-like condition for the scheme:

**Lemma 2.5** *Let us assume that (1.3) holds and that the condition*

$$\lambda := \frac{\delta t}{\delta x} \leq \frac{1}{2c}, \quad (2.13)$$

*is satisfied with  $c$  defined in (2.6). Let us assume that  $\rho^{ini} \in \mathcal{P}_1(\mathbb{R})$  is given, compactly supported and nonnegative, and we define  $\rho_i^0 = \frac{1}{\delta x} \int_{x_i}^{x_{i+1}} \rho^{ini}(dx) \geq 0$ .*

*Then for all  $i$  and  $n \in \mathbb{N}$ , the sequences computed thanks to the scheme defined in (2.4)–(2.9) satisfy*

$$\rho_i^n \geq 0, \quad |a_{i+1/2}^n| \leq c.$$

**Proof.**

• *Preliminaries.* Let us define  $M_i^n = \delta x \sum_{j \leq i} \rho_j^n$ . Since the scheme (2.4) is conservative, the total mass of the system is conserved, then  $M_\infty^n = M_\infty^0 = M$ . Clearly,  $\rho_i^n = (M_i^n - M_{i-1}^n)/\delta x$  and from (2.11) we have  $J_{i+1/2}^n = a_{i+1/2}^n (M_{i+1}^n - M_{i-1}^n)/(2\delta x)$ . Then we deduce from (2.4) that

$$M_i^{n+1} = (1 - \lambda c)M_i^n + \frac{\lambda}{2} (c - a_{i+1/2}^n) M_{i-1}^n + \frac{\lambda}{2} (c + a_{i+1/2}^n) M_{i+1}^n. \quad (2.14)$$

Thus  $M_i^{n+1}$  can be written as a convex combination of  $M_{i-1}^n$ ,  $M_i^n$  and  $M_{i+1}^n$  provided the CFL condition is satisfied and  $|a_{i+1/2}^n| \leq c$ , which we prove below by induction.

We first establish some estimates. By definition of  $\nu_i^n$  in (2.10), we deduce from (1.2) that for all  $i \in \mathbb{Z}$ ,

$$\delta x \sum_{j \leq i} |\nu_j^n| \leq \delta x \sum_{k \in \mathbb{Z}} \rho_k^n w_0 = M w_0. \quad (2.15)$$

Moreover, from the definition of  $\partial_x S_i^n$  in (2.7) and using equation (2.9), we deduce

$$-\frac{\partial_x S_{i+1}^n - \partial_x S_i^n}{\delta x} + \frac{\nu_{i+1}^n + \nu_i^n}{2} = \frac{\rho_{i+1}^n + \rho_i^n}{2}.$$

Summing this latter equation over  $i$ , we obtain

$$\partial_x S_{i+1}^n = -\frac{1}{2} \left( M_{i+1}^n + M_i^n - \delta x \left( 2 \sum_{j \leq i} \nu_j^n + \nu_{i+1}^n - \nu_0^n \right) \right). \quad (2.16)$$

• *Induction on  $n$ .* We are now in position to prove the lemma by an induction on  $n$ . For  $n = 0$ , by construction of the initial data, we have  $\rho_i^0 \geq 0$ . Then for all  $i \in \mathbb{Z}$ , we have  $0 \leq M_i^0 \leq M$  and with (2.16) and (2.15) we deduce that

$$|\partial_x S_{i+1}^0| \leq M(1 + w_0), \quad \text{for all } i \in \mathbb{Z}.$$

Futhermore, since we have

$$\frac{A(\partial_x S_{i+1}^0) - A(\partial_x S_i^0)}{\partial_x S_{i+1}^0 - \partial_x S_i^0} = a(\theta_i^0), \quad \theta_i^0 \in (\partial_x S_i^0, \partial_x S_{i+1}^0) \subset (-M(1 + w_0), M(1 + w_0)),$$

we deduce with (2.8) that  $|a_{i+1/2}^0| \leq c$ , which proves the result for  $n = 0$ .

Let us assume that  $\rho_i^n \geq 0$  and  $|a_{i+1/2}^n| \leq c$ , for some  $n \in \mathbb{N}$ . From condition (2.13) and the induction assumption  $|a_{i+1/2}^n| \leq c$ , we deduce that in the scheme (2.12), all the coefficients in front of  $\rho_{i-1}^n$ ,  $\rho_i^n$  and  $\rho_{i+1}^n$  are nonnegative. Thus  $\rho_i^{n+1} \geq 0$  for all  $i$ . Moreover, we have clearly by definition that  $0 \leq M_i^n \leq M$ . Then, from the condition (2.13) and induction assumption  $|a_{i+1/2}^n| \leq c$ , we deduce with (2.14) that  $M_i^{n+1}$  is a convex combination of  $M_{i+1}^n$ ,  $M_i^n$  and  $M_{i-1}^n$ . Then  $0 \leq M_i^{n+1} \leq M$ . Thus, as above, using (2.16) with  $n + 1$  instead of  $n$ , we have  $|\partial_x S_{i+1}^{n+1}| \leq M(1 + w_0)$ , which implies  $|a_{i+1/2}^{n+1}| \leq c$ .  $\square$

For any given sequence  $(u_i^n)_i$ , we define the corresponding piecewise constant reconstruction

$$u_\delta(t, x) = \sum_{n \in \mathbb{N}} \sum_{i \in \mathbb{Z}} u_i^n \mathbf{1}_{[n\delta t, (n+1)\delta t) \times [x_i, x_{i+1})}(t, x). \quad (2.17)$$

Then we have the following convergence result:

**Theorem 2.6** *Let us assume that  $\rho^{ini} \in \mathcal{P}_1(\mathbb{R})$  is given, compactly supported and nonnegative and define  $\rho_i^0 = \frac{1}{\delta x} \int_{x_i}^{x_{i+1}} \rho^{ini}(dx) \geq 0$ . Under assumption (1.3), if (2.13) is satisfied, then the discretization  $\rho_\delta$  converges in  $\mathcal{S}_M$  towards the unique duality solution  $\rho$  of Theorem 2.2 as  $\delta t$  and  $\delta x$  go to 0.*

**Proof.** The proof is divided into several steps.

(i) *Estimates.* Recall that from the proof of Lemma 2.5, we have  $0 \leq M_i^n \leq M$ . Therefore, since  $0 \leq \rho_i^n = (M_i^n - M_{i-1}^n)/\delta x$ , equation (2.12) implies a  $BV(\mathbb{R})$  estimate on  $(M_i^n)_i$ , provided (2.13) is satisfied. More precisely the scheme is TVD for the sequence  $(M_i^n)_i$ .

(ii) *Convergence.* Standard techniques imply a  $L^\infty \cap BV((0, T) \times \mathbb{R})$  estimate on the function  $M_\delta$  defined from  $(M_i^n)$  by (2.17). It implies the convergence, up to a subsequence, of  $M_\delta$  in  $L_{loc}^1(\mathbb{R}^+ \times \mathbb{R})$  towards a function  $\widetilde{M} \in L^\infty \cap BV((0, T) \times \mathbb{R})$  when  $\delta t$  and  $\delta x$  go to 0 and satisfy (2.13).

Let us define  $\rho = \partial_x \widetilde{M} \in L^\infty((0, T); \mathcal{M}_b(\mathbb{R}))$ . Obviously, noting that  $\rho_i^n = (M_i^n - M_{i-1}^n)/\delta x$ , we deduce that  $\rho$  is the limit in  $\mathcal{S}_M$  of  $\rho_\delta$ . By definition (2.10), we have that  $\nu_\delta = w * \rho_\delta$ . Therefore, the sequence  $(\nu_\delta)_\delta$  converges, up to a subsequence, towards  $\nu := w * \rho$  for a.e.  $t > 0$  and  $x \in \mathbb{R}$ .



From (2.16), we deduce that we have the same bound on the sequence  $(\partial_x S_i^n)_{i,n}$  as on  $(M_i^n)_{i,n}$ . We conclude that the sequence  $(\partial_x S_i^n)_{i,n}$  is bounded in  $L^\infty \cap BV((0, T) \times \mathbb{R})$ . As above, we get the convergence, up to a subsequence, in  $L^1_{loc}(\mathbb{R}^+ \times \mathbb{R})$  of  $\partial_x S_\delta$  towards a function  $\partial_x S$  belonging to  $L^\infty \cap BV((0, T) \times \mathbb{R})$  as  $\delta t$  and  $\delta x$  go to 0 and satisfy (2.13). By definition of  $\partial_x S_\delta$ , we have the strong convergence up to a subsequence in  $L^1_{loc}(\mathbb{R}^+, W^{1,1}_{loc}(\mathbb{R}))$  of  $S_\delta$  towards  $S$ .

(iii) *Passing to the limit.* Passing to the limit in the equation (2.9) we deduce that  $S$  and  $w$  satisfy in the weak sense the equation

$$-\partial_{xx}S + w = \rho.$$

Moreover, from Lemma 2.5, we deduce that the sequence  $(a_\delta)_\delta$  is bounded in  $L^\infty$ , thus we can extract a subsequence converging in  $L^\infty - weak^*$  towards  $\tilde{a}$ . From the  $L^1_{loc}$  convergence of  $(\partial_x S_\delta)_\delta$ , we deduce that  $\tilde{a} = a(\partial_x S)$  a.e. Then, from (2.5), we have the convergence in the sense of distribution of  $J_\delta$  towards  $J = -\partial_x(A(\partial_x S)) + \tilde{a}w$  a.e. Finally, taking the limit in the distributional sense of equation (2.4) we deduce that  $\rho$  is a solution in the sense of distribution of (2.3)–(2.2). By uniqueness of this solution, we deduce that  $\rho$  is the unique duality solution of Theorem 2.2.  $\square$

Finally, we notice that, as in the continuous case (see [25]), the nonnegativity of the density  $\rho$  allows to ensure an one-sided Lipschitz condition on the discretized macroscopic velocity.

**Proposition 2.7** *With the notations and assumptions of Theorem 2.6, the discrete macroscopic velocity in (2.8) satisfies the discrete one-sided Lipschitz condition:*

$$\frac{1}{\delta x} (a_{i+1/2}^n - a_{i-1/2}^n) \leq 2M\alpha \|w\|_\infty,$$

where  $M$  is the total mass of the system and  $\alpha$  is defined in (1.3).

**Proof.** From definition (2.8) we have, applying the mean value Theorem:

$$\frac{1}{\delta x} (a_{i+1/2}^n - a_{i-1/2}^n) = \frac{1}{\delta x} (a(\theta_{i+1/2}^n) - a(\theta_{i-1/2}^n)) = \frac{a'(\gamma_i^n)}{\delta x} (\theta_{i+1/2}^n - \theta_{i-1/2}^n),$$

where  $\theta_{i+1/2}^n \in (\partial_x S_i^n, \partial_x S_{i+1}^n)$  and  $\gamma_i^n \in (\theta_{i-1/2}^n, \theta_{i+1/2}^n)$  (where the notation  $(\mu, \nu)$  stands for the interval  $(\nu, \mu)$  when  $\nu < \mu$ ). Then, using assumption (1.3),

$$\frac{1}{\delta x} (a_{i+1/2}^n - a_{i-1/2}^n) \leq \frac{a'(\gamma_i^n)}{\delta x} \max \{0, \partial_x S_{i+1}^n - \partial_x S_i^n, \partial_x S_i^n - \partial_x S_{i-1}^n, \partial_x S_{i+1}^n - \partial_x S_{i-1}^n\}.$$

From the definition (2.7) and with (2.9), we have

$$\frac{\partial_x S_{i+1}^n - \partial_x S_i^n}{\delta x} = \frac{\nu_{i+2}^n + \nu_i^n}{2} - \frac{\rho_{i+1}^n + \rho_i^n}{2} \leq \frac{\nu_{i+2}^n + \nu_i^n}{2},$$

and

$$\frac{\partial_x S_{i+1}^n - \partial_x S_{i-1}^n}{\delta x} = \frac{\nu_{i+2}^n + 2\nu_i^n + \nu_{i-1}^n}{2} - \frac{\rho_{i+1}^n + 2\rho_i^n + \rho_{i-1}^n}{2} \leq \frac{\nu_{i+2}^n + 2\nu_i^n + \nu_{i-1}^n}{2},$$

where we use the nonnegativity on the sequence  $(\rho_i^n)_{i,n}$ . By definition (2.10), the sequence  $(\nu_i^n)_{i,n}$  is uniformly bounded in  $L^\infty$  by  $\|w\|_\infty M$ . We deduce that  $(\partial_x S_{i+1}^n - \partial_x S_i^n)/\delta x$  and  $(\partial_x S_{i+1}^n - \partial_x S_{i-1}^n)/\delta x$  are uniformly bounded from above by  $\|w\|_\infty M$ . Using moreover assumption (1.3) allows to conclude the proof.  $\square$

**Remark 2.8** *In some applications, we have  $w = W$ . In this case, we prefer to set  $v_i^n = S_i^n$  instead of (2.10). The sequence  $(S_i^n)_{i,n}$  is then entirely determined by solving (2.9). The convergence result in Theorem 2.6 still holds true, with a straightforward adaptation of the proof.*

### 3 Asymptotic preserving scheme

In this section, we consider an asymptotic preserving scheme allowing to recover the numerical discretization (2.4)–(2.5) from a kinetic model (1.4).

Asymptotic preserving (AP) schemes have been widely developed since the 90s for a wide range of time-dependent kinetic and hyperbolic equations. The basic idea is to develop a numerical discretization that preserves the asymptotic limits from the microscopic to the macroscopic models [28]. Moreover, in the definition of [29], an AP scheme should be implemented explicitly (or at least more efficiently than using a Newton type solvers for nonlinear algebraic systems).

#### 3.1 Hydrodynamic limit

As already mentioned, aggregation equation (1.1) can be derived as the hydrodynamic limit of the kinetic system (1.4). We recall that the function  $\phi$  satisfies

$$\phi(x) = \frac{1}{2}(1 - a(x)) \text{ with } a \text{ an odd function satisfying (1.3) and } \|a\|_\infty \leq 1. \quad (3.1)$$

We introduce the potential  $S_\varepsilon = W * \rho_\varepsilon$ , which, due to (1.2) is a weak solution to

$$-\partial_{xx}S_\varepsilon + w * \rho_\varepsilon = \rho_\varepsilon. \quad (3.2)$$

Using the fact that  $\phi(x) + \phi(-x) = 1$ , we can rewrite these equations as

$$\partial_t f_\varepsilon^\pm \pm \partial_x f_\varepsilon^\pm = \frac{1}{\varepsilon}(\phi(\mp \partial_x S_\varepsilon)\rho_\varepsilon - f_\varepsilon^\pm), \quad (3.3)$$

**Remark 3.1** *For the sake of clarity, and since our examples are limited to this case, we consider only the two-velocities model (3.3). However the results can be adapted to more general models with continuous and bounded velocities.*

As above, we denote by  $A$  an antiderivative of  $a$  and by  $\Phi$  an antiderivative of  $\phi$ , so that, from (3.2),

$$\Pi_\varepsilon^\pm := \phi(\pm \partial_x S_\varepsilon)\rho_\varepsilon = \mp \partial_x(\Phi(\pm \partial_x S_\varepsilon)) + (w * \rho_\varepsilon)\phi(\pm \partial_x S_\varepsilon). \quad (3.4)$$

Now we recall briefly the derivation of the aggregation equation (1.1) from the two velocities kinetic system (3.3). We first notice that the momentum equations obtained by adding and subtracting the two equations in (3.3) are given by

$$\partial_t \rho_\varepsilon + \partial_x J_\varepsilon = 0, \quad (3.5)$$

$$\partial_t J_\varepsilon + \partial_x \rho_\varepsilon = \frac{1}{\varepsilon}(\Pi_\varepsilon^- - \Pi_\varepsilon^+ - J_\varepsilon), \quad (3.6)$$

where  $\rho_\varepsilon = f_\varepsilon^+ + f_\varepsilon^-$  and  $J_\varepsilon = f_\varepsilon^+ - f_\varepsilon^-$ . From (3.1) and (3.4), we deduce

$$\Pi_\varepsilon^- - \Pi_\varepsilon^+ = a(\partial_x S_\varepsilon)\rho_\varepsilon = -\partial_x A(\partial_x S_\varepsilon) + (w * \rho_\varepsilon)a(\partial_x S_\varepsilon). \quad (3.7)$$

Letting formally  $\varepsilon \rightarrow 0$  in (3.6), we get

$$J_\varepsilon \rightarrow J_0 := \Pi_0^- - \Pi_0^+ = -\partial_x A(\partial_x S_0) + (w * \rho_0)a(\partial_x S_0).$$

Inserting in (3.5), we recover the aggregation equation

$$\partial_t \rho_0 + \partial_x J_0 = 0, \quad J_0 = -\partial_x A(\partial_x S_0) + (w * \rho_0)a(\partial_x S_0), \quad (3.8)$$

where  $S_0 = W * \rho_0$ .

The preceding formal computations can be made rigorous. Following the lines of proof of Theorem 3.10 of [25], we can indeed state the following theorem.

**Theorem 3.2** *Assume  $\phi$  is given by (3.1) and  $W$  as in (1.2). Let  $f^{ini} \geq 0$  be given such that  $\rho^{ini} := f^{ini,+} + f^{ini,-}$  belongs to  $\mathcal{P}_1(\mathbb{R})$ . Let  $f_\varepsilon^\pm$  be solutions to (3.3)–(3.2). Then, as  $\varepsilon \rightarrow 0$ ,  $f_\varepsilon^\pm$  converge in the following sense:*

$$\rho_\varepsilon := f_\varepsilon^+ + f_\varepsilon^- \rightharpoonup \rho \quad \text{in} \quad \mathcal{S}_M := C([0, T]; \mathcal{M}_b(\mathbb{R}) - \sigma(\mathcal{M}_b, C_0)),$$

where  $\rho$  is the unique duality solution of Theorem 2.2.

### 3.2 An AP numerical scheme

As above, we consider a time discretization of step  $\delta t$  and a uniform space discretization of step  $\delta x$ . We denote  $f_{h,\varepsilon}^{n,\pm}$  an approximation of  $f_\varepsilon^\pm$  at time  $t_n = n\delta t$ , for  $n \in \mathbb{N}$ . Before giving the numerical scheme, we explain formally the main idea which is based on the following time splitting argument:

- Assuming that approximations  $f_{h,\varepsilon}^{n,\pm}$  and  $S_{h,\varepsilon}^n$  of  $f_\varepsilon^\pm$  and  $S_\varepsilon$  are known at time  $t_n$ , we set  $\rho_{h,\varepsilon}^n = f_{h,\varepsilon}^{n,+} + f_{h,\varepsilon}^{n,-}$ . We have now everything at hand to compute  $\Pi_{h,\varepsilon}^{n,\pm}$  from (3.4):

$$\Pi_{h,\varepsilon}^{n,\pm} := \mp \partial_x \Phi(\pm \partial_x S_{h,\varepsilon}^n) + (w * \rho_{h,\varepsilon}^n)\phi(\pm \partial_x S_{h,\varepsilon}^n). \quad (3.9)$$

We solve in this first step, during a time step  $\delta t$ , the relaxation equation

$$\partial_t f_{h,\varepsilon}^\pm = \frac{1}{\varepsilon} (\Pi_{h,\varepsilon}^{\mp} - f_{h,\varepsilon}^\pm), \quad (3.10)$$

which allows to compute  $f_{h,\varepsilon}^{n+1/2,\pm}$ . Summing the equations for  $f_{h,\varepsilon}^+$  and  $f_{h,\varepsilon}^-$  in (3.10), we deduce that  $\partial_t \rho_{h,\varepsilon} = 0$ . Then  $\rho_{h,\varepsilon}^{n+1/2} = \rho_{h,\varepsilon}^n$  and since (3.2) depends only on  $\rho_{h,\varepsilon}$ , we have  $S_{h,\varepsilon}^{n+1/2} = S_{h,\varepsilon}^n$ . Therefore  $\Pi_{h,\varepsilon}^{n,\pm}$  is constant during this time step:  $\Pi_{h,\varepsilon}^{n+1/2,\pm} = \Pi_{h,\varepsilon}^{n,\pm}$ , so that we can solve exactly equation (3.10) by

$$f_{h,\varepsilon}^{n+1/2,\pm} = e^{-\delta t/\varepsilon} (f_{h,\varepsilon}^{n,\pm} - \Pi_{h,\varepsilon}^{n,\mp}) + \Pi_{h,\varepsilon}^{n,\mp}. \quad (3.11)$$

- In a second step, we discretize during a time step  $\delta t$  the free transport equation:

$$\partial_t f_{h,\varepsilon}^{n,\pm} \pm \partial_x f_{h,\varepsilon}^{n,\pm} = 0.$$

Denoting by  $D_x$  some discrete derivative with respect to  $x$ , we obtain

$$f_{h,\varepsilon}^{n+1,\pm} = f_{h,\varepsilon}^{n+1/2,\pm} \mp \delta t D_x f_{h,\varepsilon}^{n+1/2,\pm}. \quad (3.12)$$

Then we compute  $\rho_{h,\varepsilon}^{n+1} = f_{h,\varepsilon}^{n+1,+} + f_{h,\varepsilon}^{n+1,-}$  and solve (3.2) to obtain  $S_{h,\varepsilon}^{n+1}$ . We can start then a new iteration.

The main advantage of this method is that the small parameter  $\varepsilon$  is taken into account only in the first step. Letting  $\varepsilon \rightarrow 0$ , we deduce easily from (3.11) that at the limit  $\varepsilon \rightarrow 0$ , we have

$$f_{h,\varepsilon}^{n+1/2,\pm} \rightarrow f_{h,0}^{n+1/2,\pm} = \Pi_0^{n+1/2,\pm} = \Pi_0^{n,\pm},$$

since as explained above  $\rho_{h,\varepsilon}^{n+1/2} = \rho_{h,\varepsilon}^n$  and  $S_{h,\varepsilon}^{n+1/2} = S_{h,\varepsilon}^n$ . Moreover, with (3.9) we obtain

$$\Pi_{h,0}^{n,\pm} := -\partial_x \Phi(\pm \partial_x S_{h,0}^n) + (w * \rho_{h,0}^n) \phi(\pm \partial_x S_{h,0}^n).$$

Then by applying the first step (3.12), we have

$$f_{h,0}^{n+1,\pm} = \Pi_{h,0}^{n+1/2,\pm} \mp \delta t D_x \Pi_0^{n+1/2,\pm}.$$

Summing, we deduce, using the notations  $\rho_{h,0} = f_{h,0}^+ + f_{h,0}^-$  and  $J_{h,0} = f_{h,0}^+ - f_{h,0}^-$  that

$$\rho_{h,0}^{n+1} = \rho_{h,0}^{n+1/2} - \delta t D_x J_{h,0}^{n+1/2}.$$

Moreover, we have  $\rho_0^{n+1/2} = \rho_0^n$  and  $J_0^{n+1/2} = J_0^n$ . Then,

$$\rho_{h,0}^{n+1} = \rho_{h,0}^n - \delta t D_x J_{h,0}^n,$$

which is an explicit in time discretization of the conservation equation (3.8).

**Numerical discretization.** We recall the time and space discretization  $t_n = n\delta t$ ,  $n \in \mathbb{N}$  and  $x_i = i\delta x$ ,  $i \in \mathbb{Z}$ . As above,  $(\rho_i^n)_{i,n}$ ,  $(S_i^n)_{i,n}$  and  $(\nu_i^n)_{i,n}$  are approximations of resp.  $(\rho(t_n, x_i))_{i,n}$ ,  $(S(t_n, x_i))_{i,n}$  and  $(W * \rho(t_n, x_i))_{i,n}$ . Moreover, we denote by  $(f_i^{n,\pm})_{i,n}$  an approximation of  $(f_\varepsilon^\pm(t_n, x_i))_{i,j,n}$  and by  $(\Pi_i^{n,\pm})_{i,n}$  an approximation of  $(\Pi_\varepsilon^\pm(t_n, x_i))_{i,n}$  defined in (3.4).

Assuming  $(f_i^{n,\pm})_{i,n}$  are known for some  $n \in \mathbb{N}$ , we compute the approximated density

$$\rho_i^n = f_i^{n,+} + f_i^{n,-}, \quad J_i^n = f_i^{n,+} - f_i^{n,-}.$$

Then we compute  $(\nu_i^n)_i$  as in previous Section thanks to (2.10) and the macroscopic potential  $(S_i^n)_i$  by solving

$$-\frac{S_{i+1}^n - 2S_i^n + S_{i-1}^n}{\delta x^2} + \nu_i^n = \rho_i^n. \quad (3.13)$$

We have seen in the previous Section that the flux and therefore the corresponding macroscopic velocity should be defined with care. Then we use the following discretization of the macroscopic velocity

$$a_i^n = \begin{cases} 0, & \text{if } \partial_x S_{i+1/2}^n = \partial_x S_{i-1/2}^n, \\ \frac{A(\partial_x S_{i+1/2}^n) - A(\partial_x S_{i-1/2}^n)}{\partial_x S_{i+1/2}^n - \partial_x S_{i-1/2}^n}, & \text{otherwise.} \end{cases} \quad (3.14)$$

In this expression, we use the notation  $\partial_x S_{i+1/2}^n = (S_{i+1}^n - S_i^n)/\delta x$ . We define then the approximations  $\phi_i^{n,\pm}$  of  $\phi(\pm \partial_x S_\varepsilon(t_n, x_i))$  by

$$\phi_i^{n,\pm} = \frac{1}{2}(1 \pm a_i^n). \quad (3.15)$$

Then the quantity  $\Pi$  defined in (3.4) is approximated by

$$\Pi_i^{n,\pm} = \phi_i^{n,\pm} \rho_i^n. \quad (3.16)$$

Using (3.14) and (3.13), we also have

$$\Pi_i^{n,\pm} = \frac{1}{2}(\rho_i^n \pm a_i^n \nu_i^n) \mp \frac{1}{2} \frac{A(\partial_x S_{i+1/2}^n) - A(\partial_x S_{i-1/2}^n)}{\delta x}. \quad (3.17)$$

Then we compute

$$f_i^{n+1/2,\pm} = e^{-\delta t/\varepsilon} f_i^{n,\pm} + (1 - e^{-\delta t/\varepsilon}) \Pi_i^{n,\pm}. \quad (3.18)$$

Summing the two equations in (3.18), we deduce using the definition (3.15)-(3.16) that  $\rho_i^{n+1/2} = \rho_i^n$ . In fact, from (3.15) we have clearly that  $\phi_i^{n,+} + \phi_i^{n,-} = 1$ . Then, we obtain  $(f_{ij}^{n+1})_{i,j}$  by, for instance, applying an upwind discretization for the step 2:

$$\begin{cases} f_i^{n+1,+} = f_i^{n+1/2,+} - \lambda(f_i^{n+1/2,+} - f_{i-1}^{n+1/2,+}), \\ f_i^{n+1,-} = f_i^{n+1/2,-} + \lambda(f_{i+1}^{n+1/2,-} - f_i^{n+1/2,-}), \end{cases} \quad (3.19)$$

where we recall that  $\lambda = \delta t/\delta x$ .

The following theorem states the AP property of the scheme defined above.

**Theorem 3.3** *Let us assume that the initial distributions  $f^{ini,\pm}$  are nonnegative and such that  $\rho^{ini} := f^{ini,+} + f^{ini,-}$  belongs to  $\mathcal{P}_1(\mathbb{R})$ . Let us assume that  $\phi$  satisfies (3.1). Let us consider the sequence  $(f_i^{n,\pm})_{i,n}$  computed thanks to (3.13)–(3.19). Then, under the CFL condition,*

$$\lambda := \frac{\delta t}{\delta x} \leq 1, \quad (3.20)$$

as  $\varepsilon \rightarrow 0$ , the sequence  $(\rho_i^n)_{i,n} := (f_i^{n,+} + f_i^{n,-})_{i,n}$  converges weakly, up to a subsequence, towards the sequence  $(\tilde{\rho}_i^n)_{i,n}$ , computed by a Lax-Friedrichs discretization as in (2.4)–(2.5) of the equation

$$\partial_t \tilde{\rho} + \partial_x \tilde{J} = 0, \quad \tilde{J} = -\partial_x A(\partial_x \tilde{S}) + a(\partial_x \tilde{S}) \tilde{S}.$$

**Proof.** The proof is divided into several steps.

(i) *Nonnegativity.* Assume  $f_i^{n,\pm} \geq 0$  for all  $i \in \mathbb{Z}$ . By the assumption  $\|a\|_\infty \leq 1$ , we have that the function  $\phi$  is nonnegative and therefore  $\phi_i^{n,\pm}$  defined in (3.15) is nonnegative. With (3.16), we deduce  $\Pi_{ij}^{n,\pm} \geq 0$ . Then, from (3.18), we conclude that  $f_{ij}^{n+1/2}$  is nonnegative. Next, from (3.19) and provided the CFL condition (3.20) is satisfied, we deduce that  $f_i^{n+1,\pm}$  is a convex combination of  $f_i^{n+1/2,\pm}$ ,  $f_{i-1}^{n+1/2,\pm}$  and  $f_{i+1}^{n+1/2,\pm}$ . Then  $f_i^{n+1,\pm} \geq 0$ .

(ii) *Mass conservation.* We recall that we have from (3.18) that  $\rho_i^{n+1/2} = \rho_i^n$ . Summing (3.19) over  $i \in \mathbb{Z}$ , we deduce that

$$\delta x \sum_{i \in \mathbb{Z}} \rho_i^{n+1} = \delta x \sum_{i \in \mathbb{Z}} \rho_i^{n+1/2}.$$

Then the scheme is conservative:

$$\delta x \sum_{i \in \mathbb{Z}} \rho_i^{n+1} = \delta x \sum_{i \in \mathbb{Z}} \rho_i^n = M.$$

(iii) *Estimates.* Since  $\rho_i^{n+1/2} = \rho_i^n$ , we have  $S_i^{n+1/2} = S_i^n$  and  $\nu_i^{n+1/2} = \nu_i^n$ . Due to the mass conservation, we still have the bound in (2.15), i.e. for all  $i$  and  $n$ ,

$$\delta x \sum_{j \leq i} |\nu_i^n| \leq M w_0. \quad (3.21)$$

Adding the equations (3.19), we deduce,

$$\rho_i^{n+1} = \rho_i^{n+1/2} + \frac{\lambda}{2}(\rho_{i+1}^{n+1/2} - 2\rho_i^{n+1/2} + \rho_{i-1}^{n+1/2}) - \frac{\lambda}{2}(J_{i+1}^{n+1/2} - J_{i-1}^{n+1/2}), \quad (3.22)$$

where  $J_i^{n+1/2} = f_i^{n+1/2,+} - f_i^{n+1/2,-}$ . We introduce the quantity

$$\gamma_{i+1/2}^{n+1/2} = \begin{cases} \frac{J_{i+1}^{n+1/2} + J_i^{n+1/2}}{\rho_{i+1}^{n+1/2} + \rho_i^{n+1/2}}, & \text{if } \rho_{i+1}^{n+1/2} + \rho_i^{n+1/2} \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

We clearly have that  $|\gamma_{i+1/2}^{n+1/2}| \leq 1$ . Then, equation (3.22) rewrites

$$\rho_i^{n+1} = \rho_i^n + \frac{\lambda}{2}(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n) - \frac{\lambda}{2}(\gamma_{i+1/2}^{n+1/2}(\rho_{i+1}^n + \rho_i^n) - \gamma_{i-1/2}^{n+1/2}(\rho_i^n + \rho_{i-1}^n)). \quad (3.23)$$

As in the proof of Theorem 2.6, we introduce the quantity  $M_i^n = \delta x \sum_{k \leq i} \rho_k^n$ . By definition of  $M_i^n$ , we have  $\delta x(\rho_{i+1}^n + \rho_i^n) = M_{i+1}^n - M_{i-1}^n$ . Therefore, summing (3.23), we deduce

$$M_i^{n+1} = M_i^n(1 - \lambda) + \frac{\lambda}{2}M_{i+1}^n(1 - \gamma_{i+1/2}^{n+1/2}) + \frac{\lambda}{2}M_{i-1}^n(1 + \gamma_{i+1/2}^{n+1/2}). \quad (3.24)$$

Thus  $M_i^{n+1}$  is a convex combination of  $M_{i-1}^n$ ,  $M_i^n$  and  $M_{i+1}^n$ , provided the CFL condition (3.20) is satisfied. It is then obvious by an induction on  $n$  to deduce that for all  $i$  and  $n$ ,  $0 \leq M_i^n \leq M$  and that we have a *BV*-estimate on the sequence  $(M_i^n)$ .

Using (2.10), we deduce, by using  $\rho_k^n = (M_k^n - M_{k-1}^n)/\delta x$ , that

$$\nu_i^n = \sum_{k \in \mathbb{Z}} M_k^n \int_{(i-1-k)\delta x}^{(i-k)\delta x} \frac{w(z + \delta x) - w(z)}{\delta x} dz.$$

Since the function  $w$  is bounded, we deduce that the integral in the right hand side is bounded. Therefore, from the bound on  $(M_i^n)_{i,n}$ , we deduce a *BV*-bound on the sequence  $(\nu_i^n)_{i,n}$ .

From (3.13), we have

$$-\frac{\partial_x S_{i+1/2}^n - \partial_x S_{i-1/2}^n}{\delta x} + \nu_i^n = \rho_i^n. \quad (3.25)$$

Summing this latter equation on  $i$ , we deduce,

$$\partial_x S_{i+1/2}^n = \sum_{k \leq i} \delta x \nu_k^n - M_i^n. \quad (3.26)$$

Using moreover (3.21), we have the bound for all  $i, n$

$$|\partial_x S_{i+1/2}^n| \leq M(1 + w_0).$$

(iv) *Passing to the limit  $\varepsilon \rightarrow 0$ .* We will denote with a tilde  $\sim$  all the limits of considered quantities when they exist.

From the  $L^\infty \cap BV$  bound independant of  $\varepsilon$  on the sequences  $(M_i^n)_{i,n}$  and  $(\nu_i^n)_{i,n}$ , we deduce that we can extract a subsequence that converges strongly in  $L^1_{loc}$  as  $\varepsilon \rightarrow 0$  to  $(\widetilde{M}_i^n)_{i,n}$  and  $(\widetilde{\nu}_i^n)_{i,n}$  respectively. We deduce using moreover (3.26) that, up to a subsequence,  $(\partial_x S_{i+1/2}^n)_{i,n}$  converges strongly in  $L^1_{loc}$  as  $\varepsilon \rightarrow 0$ . Moreover, since the sequence  $(\gamma_{i+1/2}^{n+1/2})_{i,n}$  is bounded in  $L^\infty$  independantly of  $\varepsilon$ , we can extract a subsequence converging in  $L^\infty - weak^*$  as  $\varepsilon \rightarrow 0$  to  $(\widetilde{\gamma}_{i+1/2}^{n+1/2})_{i,n}$ .

Taking the limit  $\varepsilon \rightarrow 0$  in (3.24), we deduce that the limit sequences  $(\widetilde{M}_i^n)_{i,n}$  and  $(\widetilde{\gamma}_{i+1/2}^{n+1/2})_{i,n}$  satisfy the same relation (3.24). Defining  $\widetilde{\rho}_i^n = (\widetilde{M}_i^n - \widetilde{M}_{i-1}^n)/\delta x$ , this sequence satisfies equation (3.23). Moreover, we have the weak convergence in  $\mathcal{S}\mathcal{M}$  of  $(\rho_i^n)_{i,n}$  towards  $(\widetilde{\rho}_i^n)$  as  $\varepsilon \rightarrow 0$ . Letting  $\varepsilon \rightarrow 0$  the relation (3.25) is still available for the limit quantities.

From the strong convergence, we can pass to the limit in (3.14) to deduce the  $L^\infty - weak^*$  convergence of  $(a_i^n)_{i,n}$ . Passing to the limit in (3.17), we deduce that the sequence  $(\Pi_i^{n,\pm})_{i,n}$  converges to a limit still satisfying (3.17) with a tilde on all quantities. This equation can also be rewritten

$$\widetilde{\Pi}_i^{n,\pm} = \widetilde{\phi}_i^{n,\pm} \widetilde{\rho}_i^n = \frac{1}{2}(1 \pm \widetilde{a}_i^n) \widetilde{\rho}_i^n.$$

Finally, from (3.18), we have

$$J_i^{n+1/2} = e^{-\delta t/\varepsilon} J_i^n + (1 - e^{-\delta t/\varepsilon})(\Pi_i^{n,+} - \Pi_i^{n,-}).$$

Since  $|J_i^n| \leq M$ , we can take the weak limit as  $\varepsilon \rightarrow 0$  and deduce that

$$J_i^{n+1/2} \rightharpoonup \widetilde{\Pi}_i^{n,+} - \widetilde{\Pi}_i^{n,-} = \widetilde{a}_i^n \widetilde{\rho}_i^n.$$

We conclude the proof by passing to the limit  $\varepsilon \rightarrow 0$  in (3.22). □

## 4 Numerical simulations

We present in this Section some numerical examples to illustrate our results. In particular, we present two examples with applications in biology or plasma physics, where  $w = 0$  or  $w = W$ .

The analytical results presented above are setted on the whole real line  $\mathbb{R}$ . For the numerical simulations, we have to restrict our domain of computation to a finite domain of  $\mathbb{R}$ . Then boundary conditions should be fixed. This point is really challenging due to the convolution products which requires to know the solution on the whole real line. Since we are focusing on aggregation equation, it is known (see e.g. [25, 13]) that initial data with compact support are compactly supported and collapse in finite time into a single Dirac delta. Therefore analytical solutions never reach the boundaries. Then, since we expect the density to be 0 outside of the domain, we impose homogeneous Dirichlet boundary conditions on the density (and on the distribution function for the kinetic equation) and homogeneous Neumann boundary conditions for the potential  $S$ .

### 4.1 Simulation of an aggregation equation

In this subsection we consider the case  $W = -\frac{1}{2}|x|$  and  $a = \text{id}$ . Then the equation writes

$$\partial_t \rho + \partial_x((W' * \rho)\rho) = 0.$$

This equation appears in several applications in biology or physics, see for instance [36] where this system is the high field limit of a Vlasov-Poisson-Fokker-Planck system, the quantity  $S = W * \rho$  being the solution of the Poisson equation. In biology, it can be seen as a Patlack-Keller-Segel model without diffusion, the quantity  $S$  being the chemoattractant concentration.

Numerical scheme (2.4)–(2.5) is implemented. We notice that in the case  $a(x) = x$ , (2.8) rewrites  $a_{i+1/2}^n = \frac{1}{2}(\partial_x S_{i+1}^n + \partial_x S_i^n)$ . Numerical results are display in Figure 1 for two different initial data. In Figure 1 left, we take  $\rho^{ini}(x) = e^{-10x^2}$ . We observe that the initial bump stiffens and the solutions blows up in finite time to form one stationary single Dirac. In Figure 1 right we take  $\rho^{ini}(x) = e^{-10(x-1.25)^2} + 0.8e^{-20x^2} + e^{-10(x+1)^2}$ . As for the previous initial data, the initial bumps blow up and collapse in one single Dirac mass in finite time.

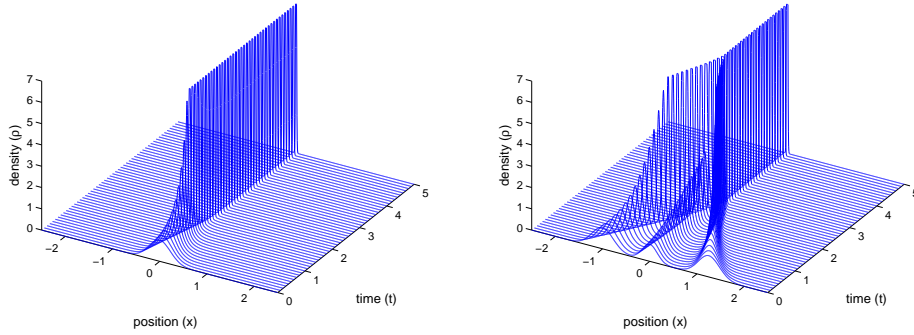


Figure 1: Dynamics of the density  $\rho$  with two different initial data in the case  $W = -\frac{1}{2}|x|$

In this case  $W'' = -\delta_0$ . Then, setting  $S = W * \rho$ , we have that  $-\partial_{xx}S = \rho$  and  $-\partial_x S$  is an antiderivative of  $\rho$ . Then, integrating the aggregation equation, we can rewrite it as

$$\partial_t \partial_x S + \frac{1}{2} \partial_x (\partial_x S)^2 = 0.$$

We recognize the Burgers equation for  $\partial_x S$ . Moreover, in this particular case where  $\nu_i^n = 0$ , we can deduce from (2.4)–(2.9) a scheme on  $(\partial_x S_i^n)_{i,n}$ . First, (2.5) rewrites

$$J_{i+1/2}^n = -\frac{1}{\delta x} (A(\partial_x S_{i+1}^n) - A(\partial_x S_i^n)).$$

Moreover, denoting  $\rho_{i+1/2}^n = \frac{1}{2}(\rho_i^n + \rho_{i+1}^n)$ , we deduce from (2.9) and (2.7)

$$\rho_{i+1/2}^n = -\frac{1}{\delta x} (\partial_x S_{i+1}^n - \partial_x S_i^n).$$

We deduce

$$\sum_{k \leq i} \rho_{k+1/2}^n = -\frac{1}{\delta x} \partial_x S_{i+1}^n.$$

Equation (2.4) implies straightforwardly

$$\rho_{i+1/2}^{n+1} = \rho_{i+1/2}^n (1 - \lambda c) + \frac{\lambda}{2} c (\rho_{i-1/2}^n + \rho_{i+3/2}^n) + \frac{\lambda}{2} (J_{i-1/2}^n - J_{i+3/2}^n).$$



Summing this latter equation, we deduce

$$\partial_x S_i^{n+1} = \partial_x S_i^n (1 - \lambda c) + \frac{\lambda}{2} c (\partial_x S_{i-1}^n + \partial_x S_{i+1}^n) - \frac{\lambda}{2} (A(\partial_x S_{i+1}^n) - A(\partial_x S_{i-1}^n)).$$

In the case at hand where  $a = id$ , we have  $A(x) = x^2/2$ , we recognize the well-known Lax-Friedrichs discretization for the Burgers equation. Here we have  $c = M$  where  $M$  is the total mass of the system. Then the numerical results of Figure 1 were expected; we recover the convergence in finite time in a single Dirac mass as established for instance in [13, Section 4].

## 4.2 A kinetic model for chemotaxis

Let us consider the so-called Othmer-Dunbar-Alt model, describing the motion of cells by chemotaxis, in one dimension. This model has been used since the 80's when it has been observed that the motion of bacteria is due to the alternance of straight swim in a given direction, called *run phase*, with cells reorientation to choose a new direction, called *tumble phase*. This system governs the dynamics of the distribution function  $f_\varepsilon$ . In the hyperbolic scaling, it reads:

$$\partial_t f_\varepsilon + v \partial_x f_\varepsilon = \frac{1}{\varepsilon} \int_V (T[v' \rightarrow v] f_\varepsilon(v') - T[v \rightarrow v'] f_\varepsilon(v)) dv'.$$

In this equation  $T[v' \rightarrow v]$  is the turning rate, corresponding to the probability of cells to change their velocities from  $v'$  to  $v$  during a tumble phase. In [16] the following expression has been taken for the turning rate

$$T[v' \rightarrow v] = \phi(v' \partial_x S_\varepsilon).$$

In this equation, the quantity  $S_\varepsilon$  corresponds to the chemoattractant concentration which solves the elliptic equation

$$-\partial_{xx} S_\varepsilon + S_\varepsilon = \rho_\varepsilon, \quad (4.1)$$

where  $\rho_\varepsilon = \int_V f_\varepsilon(v) dv$  is the density of cells. This latter equation can be rewritten  $S_\varepsilon = W * \rho_\varepsilon$  for  $W = \frac{1}{2} e^{-|x|}$ ; therefore we have  $W = w$  in (1.2).

The velocity  $v$  is assumed to have a constant modulus which is fixed to 1 by normalization. Then, in one dimension, the kinetic equation reduces to:

$$\partial_t f_\varepsilon^\pm \pm \partial_x f_\varepsilon^\pm = \frac{1}{\varepsilon} (\phi(\mp \partial_x S_\varepsilon) f_\varepsilon^\pm - \phi(\pm \partial_x S_\varepsilon) f_\varepsilon^\pm). \quad (4.2)$$

The density of cells is defined by  $\rho_\varepsilon := f_\varepsilon^+ + f_\varepsilon^-$ . We assume that the turning rate satisfies (3.1). Then, we can rewrite (4.2) as (3.3). Applying Theorem 3.2, the limiting model when  $\varepsilon \rightarrow 0$  is given by

$$\partial_t \rho + \partial_x J = 0, \quad J = -\partial_x A(\partial_x S) + a(\partial_x S) S. \quad (4.3)$$

This equation has been studied in [25].

**Remark 4.1** *As in the first example of this Section, we can recover an equation for the potential  $S = W * \rho$ , which turns out here to be nonlocal. Indeed, taking the convolution with  $W = \frac{1}{2} e^{-|x|}$  of (4.3), we obtain*

$$\partial_t S + A(\partial_x S) - W * A(\partial_x S) + \partial_x W * (a(\partial_x S) S) = 0.$$

Then by recombining (4.1) and (4.3), this latter equation can rewrite

$$\partial_t S - \partial_{txx} S + \partial_x [-\partial_x A(\partial_x S) + a(\partial_x S) S] = 0.$$

It bears some resemblance with the well-known Camassa-Holm equation [12], and exhibits the same peakon-like solutions. However the underlying dynamics is completely different and in the present case peakons collapse. Notice also that there are no anti-peakons because of the positivity of  $\rho$ .

The computational domain is assumed to be  $[-2.5, 2.5]$  and the velocity  $v$  is normalized to 1. We consider the function  $a(x) = 2/\pi \operatorname{Arctan}(10x)$ , which clearly satisfies (1.3), and recall that  $W = \frac{1}{2}e^{-|x|}$ .

#### 4.2.1 Macroscopic model

First we consider (4.3) and discretize the system thanks to (2.4)–(2.9), with  $\nu_i^n$  replaced by  $S_i^n$  in (2.9). Figure 2 displays the numerical results for the following two initial data:

$$\rho^{ini}(x) = e^{-10(x-0.7)^2} + e^{-10(x+0.7)^2}, \quad (4.4)$$

$$\rho^{ini}(x) = e^{-10(x-1.25)^2} + 0.8e^{-20x^2} + e^{-10(x+1)^2}. \quad (4.5)$$

As expected, we have a fast blow up of the regular solution and a finite-time collapse in a single Dirac mass, at a rate much higher than in the case  $W = -\frac{1}{2}|x|$ , compared with Figure 1-right.

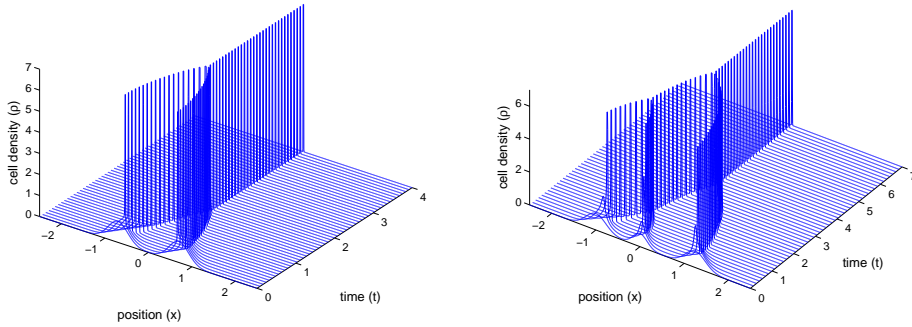


Figure 2: Macroscopic model (4.3): cell density  $\rho$  for initial data (4.4) (left) and (4.5) (right)

The behaviour of such Dirac solutions can be recovered by studying solutions in the form  $\rho(t, x) = \sum_{i=1}^n m_i \delta(x - x_i(t))$ . Then we have  $S(t, x) = W * \rho(t, x) = \frac{1}{2} \sum_{i=1}^n m_i e^{-|x - x_i(t)|}$ . After straightforward computations, we deduce from the expression in (4.3) that

$$J = - \sum_{i=1}^n [A(\partial_x S)]_{x_i} \delta(x - x_i(t)),$$

where the notation  $[f]_{x_i}$  denotes the jump of the function  $f$  at the point  $x_i$ . In particular, we have that  $\rho$  satisfies system (4.3) provided,

$$m_i x_i'(t) = -[A(\partial_x S)]_{x_i}.$$

Moreover, the function  $a$  being increasing and odd, the function  $A$  is strictly convex and can be chosen even. Then, equilibrium states satisfy

$$-[A(\partial_x S)]_{x_i} = A\left(\frac{1}{2}\left(m_i + \sum_{j \neq i} m_j e^{-|x_j - x_i|}\right)\right) - A\left(\frac{1}{2}\left(-m_i + \sum_{j \neq i} m_j e^{-|x_j - x_i|}\right)\right) = 0.$$

This equality is true only if  $\sum_{j \neq i} m_j e^{-|x_j - x_i|} = 0$ , which implies  $n = 1$ . Therefore stationary states are given by a single stationary Dirac mass. Convergence towards this equilibrium is proved in [25, 13].

#### 4.2.2 Kinetic framework

We turn now to the kinetic framework and implement the scheme described in Section 3. In Figure 3 and 4 we display the dynamics of the cell density  $\rho$  for the two regular bumps initial data (4.4) and an initial distribution function given by  $f^{ini, \pm}(x) = \frac{1}{2} \rho^{ini}(x)$ . We plot in the left part of the figures the numerical results corresponding to the macroscopic model (4.3), whereas the right part corresponds to numerical solution of (4.2) with  $\varepsilon = 0.1$ . The macroscopic behaviour is the same as in Fig. 2-left, while in the kinetic case (right), the solution does not blow up, as it is expected. However, the behaviour is similar to the one for the macroscopic model: we observe the formation of two interacting aggregates that attract one another to collapse in a single aggregate in finite time.

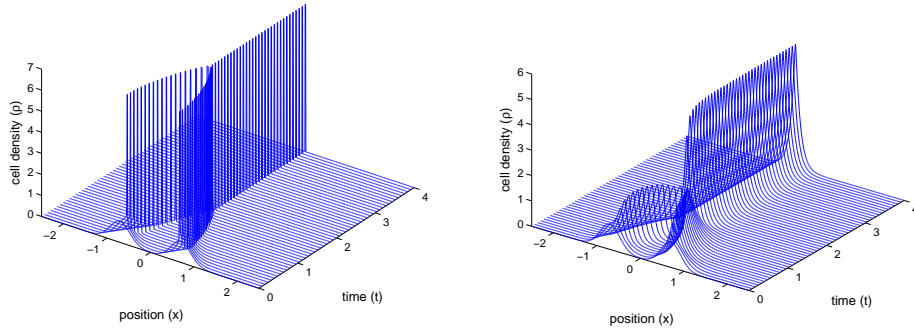


Figure 3: Dynamics of the cell density  $\rho$  for an initial data given by a two regular bumps: comparison between the macroscopic model (left) and the kinetic model for  $\varepsilon = 0.1$  (right).

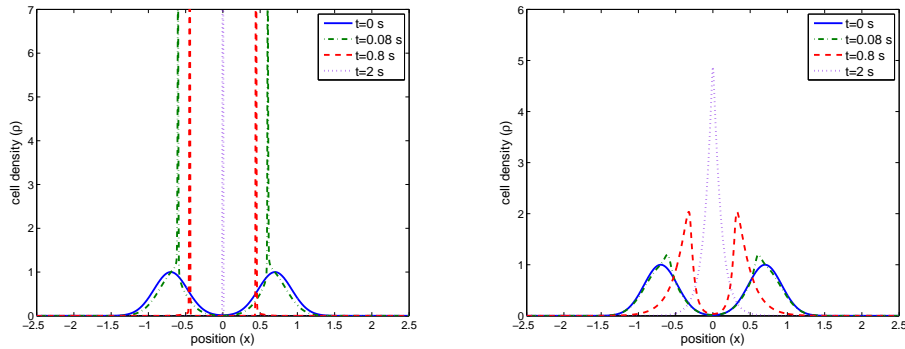


Figure 4: Dynamics of the cell density  $\rho$  for an initial data given by a two regular bumps: comparison between the macroscopic model (left) and the kinetic model for  $\varepsilon = 0.1$  (right).

This behaviour can be interpreted formally, since in the kinetic case, stationary states for

(4.2) are given by

$$\pm \partial_x f_\varepsilon^\pm = \frac{1}{\varepsilon} (\phi(\mp \partial_x S_\varepsilon) \rho_\varepsilon - f_\varepsilon^\pm).$$

Summing these two equations, we easily deduce that  $\partial_x(f_\varepsilon^+ - f_\varepsilon^-) = \partial_x J_\varepsilon = 0$ . Since we expect  $J_\varepsilon = 0$  at infinity, we can take  $f_\varepsilon^+ = f_\varepsilon^- = f_\varepsilon$ . Then, using the expression of  $\phi$  in (3.1), the previous equation rewrites  $\partial_x f_\varepsilon = \frac{1}{\varepsilon} a(\partial_x S_\varepsilon) f_\varepsilon$ . Formally, if we consider that the function  $a$  is an approximation of the sign function, then the latter equation is a linear ODE which can be solved easily and implies that  $f$  is given by a sum of exponential functions with tail  $\pm \frac{1}{\varepsilon}$ . This behaviour corresponds to what is observed in Figure 4 right.

We illustrate now the asymptotic preserving property of the scheme: in Figure 5 are displayed numerical results for different values of  $\varepsilon$  and for the three regular bumps initial data (4.5). As above, we notice the formation of 3 aggregates that merge into one single aggregate. When  $\varepsilon \rightarrow 0$ , we observe that the numerical solutions converges to the one computed in the macroscopic case, which is an illustration of the result of Theorem 3.3, compare with Figure 2-right.

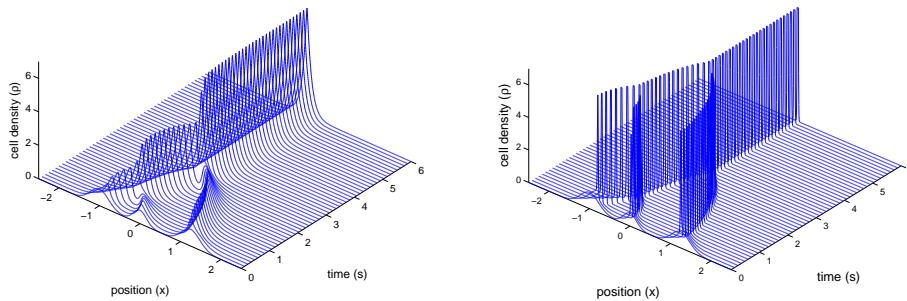


Figure 5: Dynamics of the cell density for an initial data given by a sum of 3 regular bumps: simulation of the kinetic model with  $\varepsilon = 0.1$  (left) and  $\varepsilon = 10^{-3}$  (right).

### 4.2.3 Specific issues

We conclude this section with some important remarks concerning the choice of the schemes above. First, we emphasize the importance of the choice of the discretized macroscopic velocity. For instance, in the aggregation equation (1.1), if instead of defining the discretization (2.8) we take  $a_{i+1/2}^n = a((S_{i+1}^n - S_i^n)/\delta x)$ , we obtain Figure 6, to be compared with Figure 3. Concerning the kinetic model, we display in Figure 7 the results obtained when the discretization of  $a_i^n$  in (3.14) is replaced by  $a_i^n = a(\partial_x S_i^n)$ , with  $\partial_x S_i^n = (S_{i+1}^n - S_{i-1}^n)/2\delta x$ . We notice in Figure 7 (left) that for  $\varepsilon = 10^{-3}$  the behaviour of the density remains comparable with the macroscopic model (see Figure 2). When  $\varepsilon$  goes to zero, namely here  $\varepsilon = 10^{-5}$ , we observe the same kind of result as for the macroscopic case, compare Figures 7 (right) and 6. This emphasizes that in some sense the asymptotic preserving property has to be complemented with a careful definition of the nonconservative product when Dirac masses appear in the solution. Notice also that for weak relaxation this definition is not so crucial.

Next we recall that the time-splitting strategy we have chosen here for its simplicity, which allows us to prove rigorous convergence results, is known to have some drawbacks concerning

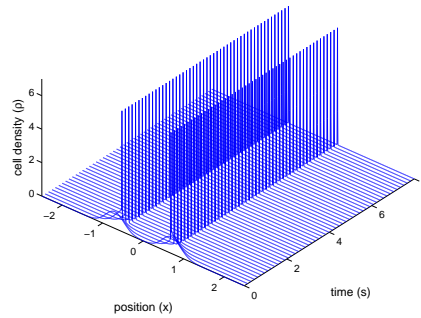


Figure 6: Wrong velocity discretization for (1.1)

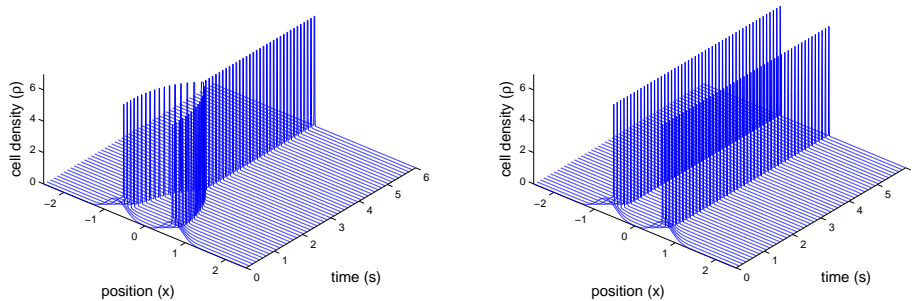


Figure 7: Wrong velocity discretization for kinetic model:  $\varepsilon = 10^{-3}$  (left) –  $\varepsilon = 10^{-5}$  (right)

the macroscopic fluxes. Indeed these are expected to be flat for stationary solutions: this is the so-called well-balanced property. This is not the case here, as evidenced in Figure 8, where are displayed the numerical fluxes ( $J_\varepsilon = f_\varepsilon^+ - f_\varepsilon^-$ ) at the final time corresponding to the numerical simulations presented in Figure 5. This illustrates the fact that the flux converges to zero merely in a weak sense. To obtain better results, well-balanced schemes have to be considered, as in e.g. [21, 22, 19]. However these results are obtained mostly for smooth solutions, hence likely can be applied for weak relaxation (that is “large” values of  $\epsilon$ ), but their behaviour when  $\epsilon$  goes to zero is far from clear. The problem is actually twofold: first developing an asymptotic preserving scheme with the well-balanced property for smooth solutions, next making such a scheme compatible with the nonconservative product when measure-valued solutions appear.

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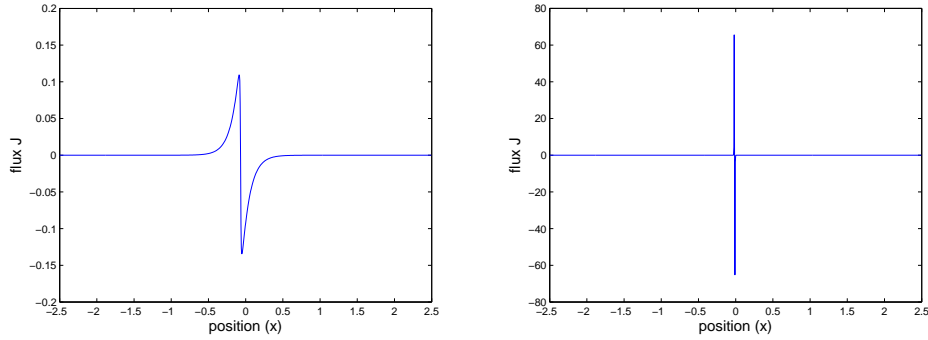


Figure 8: Flux at the final time  $T = 6$  corresponding to the simulations in the Figure 5: kinetic model with  $\varepsilon = 0.1$  (left) and  $\varepsilon = 10^{-3}$  (right).

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