Singular solutions for the plasma at the resonance

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Bruno Després[†], Lise-Marie Imbert-Gérard[‡] and Olivier Lafitte[§]

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Abstract

Little is known on the mathematical theory of hybrid and cyclotron solutions of the Maxwell equations with the cold plasma dielectric tensor. Such equations arise in magnetic plasmas such the ones needed for the modeling the an electromagnetic wave in Tokamaks. The behavior of solutions can be extremely different to those in vacuum. This work intends to contribute to the local theory by means of original representation formulas based on special functions and a certain eikonal equation, and with a careful treatment of the singularity for the hybrid resonance.

1 Introduction

The goal of this work is to study with an original mathematical method some singular solutions of the cold plasma model at different resonances, singular meaning here that at a point x_* a function u satisfies $|(x-x_*)u(x)| \geq c$ for some $c > 0$. In a more general sense, the function u can also be a Dirac mass or a principal value. The problem comes from the modeling of electromagnetic waves in strongly magnetized plasmas and from the so-called resonant heating phenomenon. We more specifically consider the one species cold plasma model which represents a collection of zero-temperature electrons immersed in a uniform static magnetic field. Resonances correspond to limits of this model, and will be studied in this work thanks to the introduction of the friction of electrons on the background ions.
In slab geometry in dimension two, $(x, y) \in \mathbb{R}^2$, and time harmonic regime, there is no resonance in the equation for the transverse magnetic polarization (also denoted as O-mode equations in plasma physics), while the equations for the transverse electric polarization (also denoted as X-mode) have two resonances. This work focuses on the latter equations, that can be reduced to

$$\left\{ \begin{array}{lllll} -\partial_y^2 E_1 & +\partial_{xy}^2 E_2 & -\epsilon_{11} E_1 & -\epsilon_{12} E_2 & = 0, \\ \partial_{xy}^2 E_1 & -\partial_x^2 E_2 & -\epsilon_{21} E_1 & -\epsilon_{22} E_2 & = 0, \end{array} \right. \tag{1}$$

where the electric field (E_1, E_2) is transverse to the bulk magnetic field. Denote by ω the pulsation of the wave. A first order approximation as in [17, 18, 6, 11] through the cold plasma approximation yields the following dielectric tensor

$$\epsilon = \begin{pmatrix} 1 - \frac{\omega_p^2}{(\omega^2-\omega_c^2)} & i\frac{\omega_c \omega_p^2}{\omega(\omega^2-\omega_c^2)} \\ -i\frac{\omega_c \omega_p^2}{\omega(\omega^2-\omega_c^2)} & 1 - \frac{\omega_p^2}{(\omega^2-\omega_c^2)} \end{pmatrix}, \tag{2}$$

where the cyclotron frequency is defined by $\omega_c = \frac{e|\mathbf{B}_0|}{m_e}$ while the plasma frequency plasma frequency is defined by $\omega_p = \sqrt{\frac{e^2 N_e}{\epsilon_0 m_e}}$. Assume that the magnetic field and the electronic density depend on the horizontal variable x only: that is $\mathbf{B}_0 = \mathbf{B}_0(x)$ and $N_e = N_e(x)$. It corresponds to a plasma facing a wall as depicted in figure 1. This is typically what occurs at a wall in a Tokamak where an antenna sends an electromagnetic wave toward the plasma in order to probe or to heat the plasma. Heating of a magnetic fusion plasma with such devices

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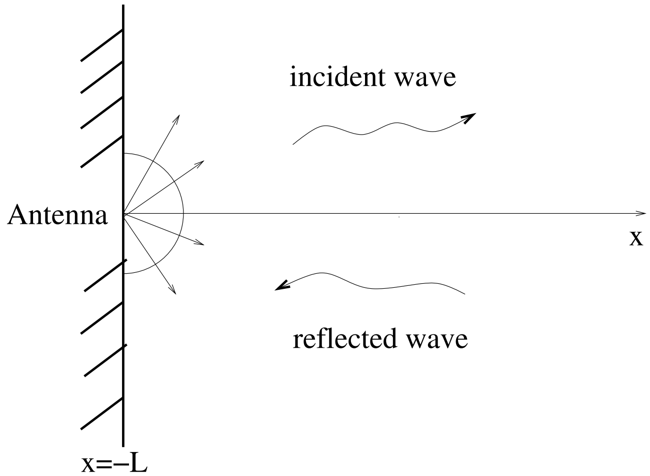

Figure 1: X-mode in slab geometry: the domain. In a real physical device an antenna is on the wall on the left and sends an incident electromagnetic wave through a medium which is assumed to be infinite for simplicity. The incident wave generates a reflected wave which can be used to identify the properties of the plasma. The medium is filled with a plasma with dielectric tensor given by (3).

is important issue for the ITER project [12]. The physical coefficients at the wall of a Tokamak are described in figure 2. More realistic plasma parameters may be much more oscillating as shown in the numerical studies [8, 9].

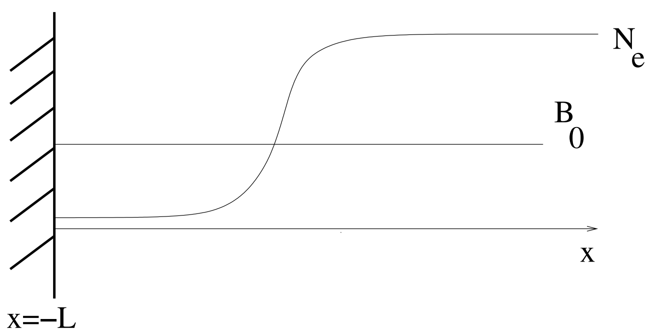

Figure 2: The physical parameters for the X-mode equations in slab geometry. The electronic density $x \mapsto N_e(x)$ is low at the boundary, and increases towards a plateau. The background magnetic field B_0 is here taken as constant for simplicity.

Resonances correspond to a limit of the geometric optics, as the local wave number approaches infinity. They require a careful examination since they are related to singular behavior and absorption. The interested reader can refer to the physics textbooks [17, 18], or to [6] for a historical physical paper on the topic. A recent physical reference is [10] and therein. See also for singular solutions of wave equations for the theory of cloaking [20] and for metamaterials [5, 2]. As announced earlier, it is then natural to model the effect of collisions with a bath of static ions by an additional friction term, since the energy is absorbed by the ions. Describing the behavior of the solutions as the friction parameter $\nu > 0$ approaches zero will evidence the singularity of the solutions to the limit problem for $\nu = 0$. The singular solutions will be built by means of new explicit representation formulas which allow a careful extraction of the singularities.
The singular behavior of the limit solutions is strongly related to the structure of the regularized dielectric tensor. The complex pulsation $\widetilde{\omega} = \omega + i\nu$ shifts the exact pulsation in the upper half-plane by the factor $\nu > 0$.

So the regularized dielectric tensor reads

$$\epsilon^\nu = \begin{pmatrix} 1 - \frac{\widetilde{\omega}\omega_p^2}{\omega(\widetilde{\omega}^2-\omega_c^2)} & i\frac{\omega_c\omega_p^2}{\omega(\widetilde{\omega}^2-\omega_c^2)} \\ -i\frac{\omega_c\omega_p^2}{\omega(\widetilde{\omega}^2-\omega_c^2)} & 1 - \frac{\widetilde{\omega}\omega_p^2}{\omega(\widetilde{\omega}^2-\omega_c^2)} \end{pmatrix}, \quad (3)$$

where ω_p and ω_c respectively are the plasma and cyclotron frequencies, and the regularized dielectric tensor is translation invariant in the y-direction. The equations are

$$\begin{cases} -\partial_y^2 E_1^\nu & +\partial_{xy}^2 E_2^\nu & -\epsilon_{11}^\nu E_1^\nu & -\epsilon_{12}^\nu E_2^\nu & = 0, \\ \partial_{xy}^2 E_1^\nu & -\partial_x^2 E_2^\nu & -\epsilon_{21}^\nu E_1^\nu & -\epsilon_{22}^\nu E_2^\nu & = 0. \end{cases} \quad (4)$$

For a recent mathematical treatment by means of singular integral equations refer to [11]. In fusion plasmas, the value of the friction parameter can be extremely small. For example relative values of $\frac{\nu}{\omega} \approx 10^{-7}$ are common in fusion plasmas. This extremely small value shows that the frictionless limit regime $\nu \to 0$ is relevant for some fusion applications. In such regimes the formal first order approximation of the dielectric tensor reads

$$\epsilon^\nu = \epsilon^0 + i\nu\mathbb{D} + O(\nu^2), \quad (5)$$

with

$$\epsilon^0 = \begin{pmatrix} 1 - \frac{\omega_p^2}{\omega^2-\omega_c^2} & i\frac{\omega_c\omega_p^2}{\omega(\omega^2-\omega_c^2)} \\ -i\frac{\omega_c\omega_p^2}{\omega(\omega^2-\omega_c^2)} & 1 - \frac{\omega_p^2}{\omega^2-\omega_c^2} \end{pmatrix} \text{ and } \mathbb{D} = \begin{pmatrix} \lambda_1 & -i\lambda_2 \\ i\lambda_2 & \lambda_1 \end{pmatrix}. \quad (6)$$

The dissipation tensor $\mathbb{D}$ accounts for the underlying physical dissipation. Its coefficients are $\lambda_1 = \frac{\omega_p^2(\omega^2+\omega_c^2)}{\omega(\omega^2-\omega_c^2)^2} > 0$ and $\lambda_2 = \frac{2\omega_c\omega_p^2}{(\omega^2-\omega_c^2)^2} \in \mathbb{R}$. Since $\lambda_1 - \lambda_2 = \frac{\omega_p^2}{\omega(\omega+\omega_c)^2} > 0$, one has that $\mathbb{D} = \mathbb{D}^* > 0$ is indeed a positive matrix. As previously mentioned, and stressed in figure 2, the coefficients of these tensors are a priori non constant functions of the horizontal variable x.
The structure of the limit tensor (2) shows two different resonances in the limit model $\nu = 0$, defined thanks to the dispersion relation, see [17]. The first one is related to the **cyclotron resonance** and corresponds to a vanishing denominator: it writes

$$\omega_c(x) = \omega$$

where ω_c is a function of the x variable, a priori non constant. We will show that mathematical solutions of Maxwell's equations (4) actually have no singular behavior: they are bounded in this regime, despite the fact that the dielectric coefficients are not. The other interesting regime corresponds to the vanishing of the diagonal part of ϵ^0, that is to say

$$\omega_p^2(x) + \omega_c^2(x) = \omega^2.$$

This is the **hybrid resonance**. In this regime the dielectric tensor is non singular since the denominator is non zero, i.e. the coefficients of the dielectric tensor are bounded. However the solutions are known to be highly singular: they are not bounded near the resonance. As stressed in [11], a useful quantity that can be used to characterize the singular behavior of the solutions is the **resonant heating** of the plasma. The resonant heating [3, 11] is defined by

$$\mathcal{Q} = \lim_{\nu \to 0^+} \left(\Im \int (\epsilon^\nu \mathbf{E}^\nu \cdot \mathbf{E}^\nu)\, dx \right) \quad (7)$$

where the total electric field is $\mathbf{E}^\nu = (E_1^\nu, E_2^\nu)$ and $(\mathbf{A} \cdot \mathbf{B}) = \mathbf{A} \cdot \overline{\mathbf{B}}$ denotes the hermitian product of the complex vectors $\mathbf{A}$ and $\mathbf{B}$. This formula has been justified in [11] where it is shown that $\mathcal{Q} > 0$ in case of resonant heating using the simplification $\epsilon^\nu = \epsilon + i\nu\mathbf{I}$. It means that the dissipation tensor introduced in the ν expansion of ϵ^ν (5) is diagonal and $\mathbb{D} = \mathbb{I}$. This simplification is mathematically justified since it corresponds to the **limiting absorption principle**. In this work we will show that the resonant heating is well defined and takes the same value for a large class of dissipation tensors that includes physically based tensors.
The method of analysis uses $W^\nu = \partial_x E_2^\nu - \partial_y E_1^\nu$ which is, after linearization around the bulk magnetic field, the parallel component of the magnetic field. The system (4) is then equivalent to the following first order system

$$\begin{cases} \partial_y W^\nu & -\epsilon_{11}^\nu E_1^\nu & -\epsilon_{12}^\nu E_2^\nu & = 0, \\ -\partial_x W^\nu & -\epsilon_{21}^\nu E_1^\nu & -\epsilon_{22}^\nu E_2^\nu & = 0, \\ W^\nu & +\partial_y E_1^\nu & -\partial_x E_2^\nu & = 0. \end{cases}$$

Thanks to the fact that the coefficients ϵ^ν_{ij} depend only on x, using the Fourier transform in y, denoting by $i\theta$ the Fourier coefficient, it yields

$$\left\{\begin{array}{cccc} i\theta W^\nu & -\epsilon^\nu_{11}E^\nu_1 & -\epsilon^\nu_{12}E^\nu_2 & =0, \\ -\frac{d}{dx}W^\nu & -\epsilon^\nu_{21}E^\nu_1 & -\epsilon^\nu_{22}E^\nu_2 & =0, \\ W^\nu & +i\theta E^\nu_1 & -\frac{d}{dx}E^\nu_2 & =0. \end{array}\right. \tag{8}$$

The Fourier coefficient θ can be considered as a frozen parameter from now on, and ∂_x will be replaced by $\frac{d}{dx}$ for the derivative with respect to x. In this system, the derivative with respect to x appears only on W^ν and E^ν_2. The energy integral (7) can be expressed differently using the identity

$$\int_a^b (\epsilon^\nu \mathbf{E}^\nu \cdot \mathbf{E}^\nu)\, dx = \int_a^b \left(i\theta W^\nu \overline{E^\nu_1} - \frac{d}{dx} W^\nu \overline{E^\nu_2} \right) dx$$

$$= \int_a^b \left(i\theta W^\nu \overline{E^\nu_1} + W^\nu \frac{d}{dx}\overline{E^\nu_2} \right) dx - W^\nu(b)\overline{E^\nu_2}(b) + W^\nu(a)\overline{E^\nu_2}(a) = -\int_a^b |W^\nu|^2\, dx - W^\nu(b)\overline{E^\nu_2}(b) + W^\nu(a)\overline{E^\nu_2}(a).$$

One obtains another expression

$$\Im \int_a^b (\epsilon^\nu \mathbf{E}^\nu \cdot \mathbf{E}^\nu)\, dx = -\Im\left(W^\nu(b)\overline{E^\nu_2}(b)\right) + \Im\left(W^\nu(a)\overline{E^\nu_2}(a)\right) \tag{9}$$

which expresses a balance of energy.
It is convenient to consider a differential system with only two unknowns, namely (E^ν_2, W^ν), while E^ν_1 can be computed thanks to the first equation of (8) by

$$E^\nu_1 = \frac{i\theta W^\nu - \epsilon^\nu_{12}E^\nu_2}{\epsilon^\nu_{11}} \tag{10}$$

The reduced system then reads

$$\frac{d}{dx}\left(\begin{array}{c} E^\nu_2(x) \\ W^\nu(x) \end{array}\right) = \mathbf{M}_\nu(x,\theta)\left(\begin{array}{c} E^\nu_2(x) \\ W^\nu(x) \end{array}\right), \qquad x\in\mathbb{R}, \tag{11}$$

where the matrix is defined as follows.

Definition 1. *The matrix $\mathbf{M}_\nu(x,\theta)$ is constructed from the dielectric tensor by*

$$\mathbf{M}_\nu(\cdot,\theta) = \left(\begin{array}{cc} -\dfrac{i\theta\epsilon^\nu_{12}}{\epsilon^\nu_{11}} & 1-\dfrac{\theta^2}{\epsilon^\nu_{11}} \\ -\dfrac{d^\nu}{\epsilon^\nu_{11}} & -\dfrac{i\theta\epsilon^\nu_{21}}{\epsilon^\nu_{11}} \end{array}\right) \tag{12}$$

where the coefficient d^ν is the determinant of the dielectric tensor ϵ^ν

$$d^\nu = \epsilon^\nu_{11}\epsilon^\nu_{22} - \epsilon^\nu_{12}\epsilon^\nu_{21}. \tag{13}$$

The determinant of $\mathbf{M}_\nu(x,\theta)$ will be denoted $D_\nu(\cdot,\theta) = det\, \mathbf{M}_\nu(\cdot,\theta)$.

The determinant of the matrix can be simplified as follows

$$D_\nu(\cdot,\theta) = -\left(\frac{i\theta\epsilon^\nu_{21}}{\epsilon^\nu_{11}}\right)^2 + \frac{d^\nu}{\epsilon^\nu_{11}}\left(1-\frac{\theta^2}{\epsilon^\nu_{11}}\right) = \frac{d^\nu}{\epsilon^\nu_{11}} - \theta^2\frac{\epsilon^\nu_{22}}{\epsilon^\nu_{11}} = \frac{d^\nu}{\epsilon^\nu_{11}} - \theta^2. \tag{14}$$

It is already foreseeable that the roots of ϵ^ν_{11} will play a crucial part in some cases: the analysis of singular solutions relies on the analyticity of the coefficients of the dielectric tensor, at least locally around $x = x_*$, where x_* is a hybrid resonance. This assumption is very useful to study the problem with a convenient shift in the complex plane. The main results of this paper stem from a precise local study of the solution of the reduced system when the entries of M_ν have a simple pole at x_*, and the determinant of the matrix has a simple pole as well. The mathematical method of singularity extraction is completely new to our knowledge.
Our main results can be summarized as follows:

Lemma 2. In our geometry, the cyclotron singularity of the dielectric tensor ($\omega = \omega_c$) is only an apparent singularity in the sense that the electromagnetic field is bounded and the limit heating is zero $\mathcal{Q}_{cyc} = 0$.

Theorems 1 and 2. The hybrid resonance ($\omega^2 = \omega_c^2 + \omega_p^2$) yields a singular horizontal component of the electric field, i.e. $|(x - x_*)E_1^{\nu=0}| \geq c > 0$, and a generic positive resonant heating $\mathcal{Q}_{hyb} > 0$. The limit of the horizontal component of the electric field can be expressed as the sum of a Dirac mass, of a Principal Value and of a bounded term, as in [11].

Corollary 1. The value of the hybrid resonant heating is independent of the local dissipation tensor, and moreover the local dissipation tensor can be non physical as well. The limit value for $\nu = 0^+$ of the electromagnetic unknowns is independent of the local dissipation tensor. It establishes the uniqueness of the limit solution with respect to the local dissipation tensor. This is a new result that was not possible with the techniques developed in [11] where the limit was established only for $\mathbb{D} = \mathbb{I}$.

Numerical section We provide numerical evidence of our claims with the help of a numerical method that can be used to treat other problems with same structure.

Extension to multi-species We finally provide some formulas which shows the method of singularity extraction can be used for multi-species, such as ions and electrons. These formulae are easier to handle than the sum of dielectric tensors generally used both for the cyclotron resonances and the hybrid resonances.

The work is organized as follows. The cyclotron singularity of the dielectric tensor is studied in section 2, where the electromagnetic field is shown to stay bounded in the neighborhood of this singularity. The hybrid resonance is studied in detail in section 3 where the singularity is carefully extracted using convenient representation formulas with Bessel functions. We prove both the generic resonant heating and the fact that its value is independent of the local dissipation tensor. Numerical results are provided in section 4. Some alternative formulas are collected in the appendix.

2 The cyclotron resonance

The cyclotron singularity corresponds to a formal singularity in the denominator of (3). We consider the situation where there exists at least one x_c in the domain such that $\omega_c(x_c) = \omega$. For the simplicity of the analysis, we will assume that $\frac{d}{dx}\omega_c(x_c) \neq 0$ so that x_c is isolated. Clearly the dielectric tensor (3) is singular at x_c at the limit $\nu = 0^+$, while it is not singular for the regularized system, i.e. $\nu \neq 0$. In order to focus on the cyclotron resonance, we want it to be away from the hybrid resonance, that is $\omega_p \neq 0$. This is a generic case for a plasma.

Definition 2. *A point x_c is referred to as an isolated cyclotron singularity if there exists $r > 0$ such that $\omega_c(x_c) = \omega$ and $\frac{d}{dx}\omega_c(x_c) \neq 0$.*

Note that such a singularity does not depend on ν, since ω_c does not depend on ν. Let us now consider the reduced system (11), where the system's matrix $\mathbf{M}_\nu(\cdot, \theta)$ is defined with the physical dielectric tensor (3). We assume there is an isolated singularity x_c. Then the system is well defined for all $\nu \in \mathbb{R}$, and one can pass to the limit in the following sense.

Lemma 1. *Assume that the magnetic field B_0 and the density N_e are smooth. Assume an isolated cyclotron singularity at x_c, and assume there is no hybrid resonance on $[x_c - r, x_c + r]$. Assume (E_2^ν, W^ν) solves the well posed system (11) on $[x_c - r, x_c + r]$ for all $\nu \in \mathbb{R}$, with a given Cauchy data which admits a finite limit when ν goes to 0 written as $(E_2^\nu, W^\nu)(x_c - r) = (a, b)$.*
Then (E_2^ν, W^ν) tends to a bounded limit (E_2, W) as ν tends to zero, and (E_2, W) solves the limit system (11) on $[x_c - r, x_c + r]$ with $\nu = 0$ and with the same Cauchy data $(E_2, W)(x_c - r) = (a, b)$

Proof. Definition 1 with the physical tensor (3) yields after some careful calculations

$$\mathbf{M}_\nu(\cdot, \theta) = \frac{1}{\alpha_\nu(\cdot, \theta)} \begin{pmatrix} \theta\omega_c\omega_p^2 & (1-\theta^2)\omega(\tilde{\omega}^2 - \omega_c^2) - \tilde{\omega}\omega_p^2 \\ -\omega(\tilde{\omega}^2 - \omega_c^2) + 2\tilde{\omega}\omega_p^2 - \dfrac{\omega_p^4}{\omega} & -\theta\omega_c\omega_p^2 \end{pmatrix} \tag{15}$$

where $\alpha_\nu(\cdot,\theta) = \omega(\tilde\omega^2 - \omega_c^2) - \tilde\omega\omega_p^2$. The determinant (14) of this matrix can be recast as $D_\nu(\cdot,\theta) = \frac{d^\nu}{\epsilon_{11}^\nu} - \theta^2 = \frac{\omega(\tilde\omega^2-\omega_c^2)-2\tilde\omega\omega_p^2+\frac{\omega_p^4}{\omega}}{\alpha_\nu(\cdot,\theta)} - \theta^2$. The limit value of α as ν approaches 0^+ is $\alpha_0(\cdot,\theta) = \omega\left(\omega^2 - \omega_c^2 - \omega_p^2\right)$. Since the cyclotron resonance is not also an hybrid resonance, $\omega_p > 0$, one has that

$$\alpha_0(x_c,\theta) = -\omega\omega_p^2 \neq 0 \text{ since } \omega = \omega_c(x_c) \neq 0. \tag{16}$$

In this situation the matrix $\mathbf{M}_\nu(\cdot,\theta)$ is locally bounded and smooth. So the solutions of the reduced system (11) are smooth locally around x_c and one can pass to the limit in this system. □

Note that there is still a particular behavior in the neighborhood of this cyclotron resonance, known as 'turning point behavior' [7, 13], but which does not induce any singular regime, except in the limit $\omega \to +\infty$, which is not considered here. The consequences of the representation (15) on physical quantities are characterized in the following result.

Lemma 2. *Under the assumptions of Lemma 1, then E_1^ν also tends to a bounded limit E_1 and the value of the limit heating is zero*

$$\lim_{\nu\to 0^+}\left(\operatorname{Im}\int_{x_c-r}^{x_c+r}\left(\epsilon^\nu \mathbf{E}^\nu \cdot \mathbf{E}^\nu\right)dx\right) = 0. \tag{17}$$

Proof. The horizontal component of the electric field can be evaluated using (10): after some computations the coefficients in (10) can be expressed as

$$\frac{i\theta}{\epsilon_{11}^\nu} = \frac{i\theta\omega\left(\tilde\omega^2 - \omega_c(\cdot)^2\right)}{\alpha_\nu(\cdot,\theta)} \text{ and } \frac{-\epsilon_{12}^\nu}{\epsilon_{11}^\nu} = \frac{-i\omega_c(\cdot)\omega_p^2}{\alpha_\nu(\cdot,\theta)}.$$

Using (16), the limit $\nu = 0$ value of these coefficients at x_c is

$$\left(\frac{i\theta}{\epsilon_{11}^0}\right)(x_c) = 0 \text{ and } \left(\frac{-\epsilon_{12}^0}{\epsilon_{11}^0}\right)(x_c) = i. \tag{18}$$

So one can pass to the limit in (10) since the coefficients are bounded in the interval $[x_c - r, x_c + r]$. Therefore the horizontal part of the electric field E_1^ν admits a bounded and smooth limit E_1.
Consider now the heating term. On the first hand we notice that the electric field can be split into two parts

$$\mathbf{E}^\nu = E_2^\nu\begin{pmatrix} i \\ 1\end{pmatrix} + \begin{pmatrix} E_1^\nu - iE_2^\nu \\ 0\end{pmatrix},$$

the first vector being an eigenvector of the dielectric tensor : $\epsilon^\nu\begin{pmatrix} i \\ 1\end{pmatrix} = \lambda\begin{pmatrix} i \\ 1\end{pmatrix}$ where the eigenvalue $\lambda = 1 - \frac{\omega_p^2}{\omega(\tilde\omega+\omega_c)}$ is bounded so non singular at x_c. The second vector is $O\left(\nu + |x - x_c|\right)$ due to (10) and (18), which means that it counterbalances the singularity of the dielectric tensor. It shows that $\epsilon^\nu\mathbf{E}^\nu$ is bounded uniformly on $[x_c - r, x_c + r]$ with respect to ν. Since $\mathbf{E}^\nu$ is also uniformly bounded, it yields that $(\epsilon^\nu\mathbf{E}^\nu, \mathbf{E}^\nu)$ is bounded uniformly on $[x_c - r, x_c + r]$ with respect to ν.
On the other hand the tensor ϵ^ν tends almost everywhere (that is for $x \neq x_c$) to an hermitian tensor, so that $\Im\left(\epsilon^\nu\mathbf{E}^\nu, \mathbf{E}^\nu\right)$ tends almost everywhere to zero. The Lebesgue dominated convergence theorem states that

$$\lim_{\nu\to 0^+}\left(\Im\int_{x_c-r}^{x_c+r}\left(\epsilon^\nu \mathbf{E}^\nu \cdot \mathbf{E}^\nu\right)dx\right) = 0.$$

The proof is ended. □

It evidences the fact that there is no resonant cyclotron heating in our model. This result is compatible with the literature [17, 18, 6] where a resonant cyclotron heating is possible, but only in another configuration with wave number parallel to the bulk magnetic field. In our case it yields $\partial_x = 0$ and so is excluded from our discussion.

3 The hybrid resonance

This section gathers the main theoretical contributions of this work. The required hypothesis on the general dielectric tensor

$$\epsilon^\nu(x,y) = \begin{pmatrix} \epsilon^\nu_{11}(x) & \epsilon^\nu_{12}(x) \\ \epsilon^\nu_{21}(x) & \epsilon^\nu_{22}(x) \end{pmatrix} \in \mathbb{C}^{2\times 2}, \forall \nu \in \mathbb{R}, \tag{19}$$

for the following work to hold will be emphasized in Assumptions 1, 2 and 3. Note that each one of these assumptions is satisfied by the physical tensor (3), but that all the following results hold for the wider range of tensors satisfying these three assumptions.

The hybrid resonance corresponds to a singularity in the limit problem $\nu = 0$, it more precisely corresponds to the vanishing of ϵ^0_{11}, which occurs in the expression of the x component of the electric field E^0_1, see (10), as well as in the matrix of the (E^0_2, W^0) system, see (12) with $\nu = 0$.

Definition 3. *A point x_h is referred to as a local hybrid resonance for the limit system* (4)-(19)*, i.e. with $\nu = 0$, if the limit diagonal coefficient of $\epsilon^\nu_{11}(x)$, is such that*

$$\epsilon^0_{11}(x_h) = 0 \text{ and } \partial_x \epsilon^0_{11}(x_h) \in \mathbb{R}^* \tag{20}$$

and if the extra-diagonal part is locally non zero

$$\epsilon^0_{12}(x_h) \neq 0. \tag{21}$$

We will moreover assume that

$$i\partial_\nu \epsilon^0_{11}(x_h) \in \mathbb{R}^* \tag{22}$$

which is compatible with the physical context as visible in equation (6).

Going back to the physical dielectric tensor (3), one sees that $\epsilon^0_{11}(x_h) = 0$ corresponds to

$$\omega^2 = \omega_c^2(x_h) + \omega_p^2(x_h).$$

This case is referred to as hybrid regime in the literature, see [17, 18, 6]. Unlike in the cyclotron resonance case, there exist singular solutions at the hybrid resonance. This is known since the seminal work of Budden [4], for an extremely particular case where $\epsilon^0_{11} = 1 + \frac{x}{x_h}$. A recent mathematical analysis, performed in [11] with a singular integral equation, has evidenced the role of the limit resonant heating. Our main objective of the present work is to provide additional understanding of the resonant heating by means of a local extraction of the singularity.

Stemming directly from the definition, a first property of the 2×2 differential system (11) is the following:

Lemma 3. *Assume x_h is local hybrid resonance, then for $\nu = 0$ the determinant (13) of the dielectric tensor is such*

$$d^0(x_h) = -\left|\epsilon^0_{12}(x_h)\right|^2 < 0 \tag{23}$$

Hereafter follows the list of some assumptions on the general regularized tensor (19). This is required for the upcoming analysis.

Assumption 1 (Uniqueness and Analyticity of the coefficients)**.** *Assume x_h is a local hybrid resonance. There exists $\Lambda_1 > 0$ such that*

- *the dielectric tensor ϵ^ν is analytic in the ball $x \in B(x_h, \Lambda_1)$ for all $\nu \in [-\Lambda_1, \Lambda_1]$,*
- *x_h is the unique root of ϵ^0_{11} in $B(x_h, \Lambda_1)$.*

Note that, for F satisfying this assumption of analyticity, $F(z) = \sum_{n\geq 0} a_n z^n$, with $\sum |a_n| R^n < +\infty$ for $R < \Lambda_1$. One shall denote the derivative as usual by $F'(z) = \sum_{n\geq 0} n a_n z^{n-1}$. One has the chain rule $\frac{d}{dx}(F(\phi(x))) = F'(\phi(x))\frac{d\phi}{dx}(x)$ for ϕ a function from an interval I of $\mathbb{R}$ to $\mathbb{C}$.

Assumption 2 (Antisymmetry of the dielectric tensor)**.** *The dielectric tensor ϵ^ν is such that*

$$\epsilon^\nu_{12} = -\epsilon^\nu_{21}, \forall \nu \in [-\Lambda_1, \Lambda_1]. \tag{24}$$

As a result, the trace of the matrix $\mathbf{M}_\nu(x,\theta)$ of the 2×2 system is identically zero on its domain.

The last assumption is satisfied by the physical tensor (2) and by its first order approximation (5).
Also note that as a consequence of this assumption, the determinant of $\mathbf{M}_\nu$ is less singular than expected: since all the entries of the matrix are $O((\epsilon_{11}^\nu)^{-1})$, the determinant would be expected to be $O((\epsilon_{11}^\nu)^{-2})$, but from (24) it actually is only $O((\epsilon_{11}^\nu)^{-1})$. For the sake of clarity, consider the following notation:

Definition 4. *Define for all* $\nu \in [-\Lambda_1, \Lambda_1]$

$$a_\nu = -\frac{i\theta\epsilon_{12}^\nu}{\epsilon_{11}^\nu}, \quad b_\nu = 1 - \frac{\theta^2}{\epsilon_{11}^\nu}, \quad c_\nu = -\frac{d^\nu}{\epsilon_{11}^\nu}, \tag{25}$$

so that thanks to the symmetry property (24)*, the matrix* $\mathbf{M}_\nu$ *introduced in Definition 1 reads*

$$\mathbf{M}_\nu(x,\theta) = \begin{pmatrix} a_\nu(x,\theta) & b_\nu(x,\theta) \\ c_\nu(x) & -a_\nu(x,\theta) \end{pmatrix}. \tag{26}$$

These quantities are well defined except possibly at the roots of ϵ_{11}^ν*.*

In the model case $\epsilon^\nu = \epsilon^0 + i\nu\mathbb{D}$ with a diagonal dissipation tensor $\mathbb{D} = \mathbb{I}$ studied in [11], one has $\epsilon_{11}^\nu = \alpha(x) + i\nu$ and $\epsilon_{12}^\nu = i\delta(x)$. In this case the coefficients are exactly $a_\nu(x,\theta) = \frac{\theta\delta(x)}{\alpha(x)+i\nu}$, $b_\nu(x,\theta) = 1 - \frac{\theta^2}{\alpha(x)+i\nu}$ and $c_\nu(x) = \frac{\delta^2(x)}{\alpha(x)+i\nu} - (\alpha(x)+i\nu)$, and so c_ν does not depend on θ.

Assumption 3. *The regularized dielectric tensor coefficient* $\epsilon_{11}^\nu(z)$ *is locally analytic with respect to both variables* z *and* ν*.*

The techniques developed in this work are direct consequences of these properties. The study that follows is performed in two steps: the regularized problem, with $\nu \neq 0$ is first analyzed, then the limit as ν approaches zero is studied, to go back to the original problem. To this purpose it is crucial to develop uniform tools with respect to the regularization parameter ν, so that their properties still hold as ν goes to zero.

3.1 A convenient second order equation

System (12) with matrix (26) can provide two different second order equations, by elimination of either of the two unknowns. In order to choose which of these two equations to study, consider the following consequence of Definition 3.

Lemma 4. *Assume that* x_h *is a local hybrid resonance. Then the dielectric tensor* ϵ^ν *(3) is such that:*

- *There exists* Λ_2 *satisfying* $0 < \Lambda_2 \leq \Lambda_1$ *and* $C > 0$ *independent of the angle of incidence* θ *such that*

$$\left|\frac{a_\nu(x)}{c_\nu(x)}\right| \leq C|\theta| \text{ for } x \in B(x_h, \Lambda_2) \text{ and for } |\nu| \leq \Lambda_2. \tag{27}$$

- *The quantity* $\left|\frac{a_\nu(x_h)}{b_\nu(x_h)}\right| = \left|\frac{\epsilon_{12}^\nu(x_h)}{\theta}\right|$ *is unbounded for* $\theta \to 0$*.*

The fact that the upper extra diagonal coefficient b_ν does not dominate the diagonal part a_ν for vanishing θ results in a singularity in this regime which makes the analysis trickier than with for the equation on E_2^ν, therefore the formulation considered is the one on W^ν:

$$\frac{d}{dx}\left(\frac{1}{c_\nu}\frac{d}{dx}W^\nu\right) = \left(\frac{a_\nu^2}{c_\nu} + b_\nu - \left(\frac{a_\nu}{c_\nu}\right)'\right)W_\nu. \tag{28}$$

Remark 1. *At the resonance, the limit equation* (28) *has a 'regular singular point' [7, 13]. Indeed, thanks to Definition 3 and Lemma 4, the* $\frac{1}{c_\nu}$ *term is bounded while the* $\frac{a_\nu^2}{c_\nu}$*,* b_ν *and* $\left(\frac{a_\nu}{c_\nu}\right)'$ *term all behave as* $O((\epsilon^\nu)^{-1})$ *at the resonance.*

An adequate scaling of the unknown W^ν then provides an equation with no first order term. The rescaled unknown $y(x) = \frac{1}{\sqrt{-c_\nu(x)}}W^\nu(x)$ indeed satisfies the equation

$$\frac{d^2y}{dx^2} = \left(a_\nu^2 + b_\nu c_\nu - c_\nu\left(\frac{a_\nu}{c_\nu}\right)' + \sqrt{-c_\nu}\left(\frac{1}{\sqrt{-c_\nu}}\right)''\right)y, \qquad x \in \mathbb{R}. \tag{29}$$

Remark 2. *In this whole work, the square root has to be understood as the principal square root on complex numbers.*

A consequence of Assumption 3 provides a unique complex root of the coefficient ϵ_{11}^{ν} in a neighborhood of the root of the limit coefficient ϵ_{11}^{0}, thanks to the open mapping theorem:

Lemma 5. *Suppose x_h is a local hybrid resonance. Under Assumptions 1, 2 and 3, there exists Λ_3 satisfying $0 < \Lambda_3 \leq \Lambda_2$ such that if $|\nu| \leq \Lambda_3$ there exists a unique point $X_\nu \in B(x_h, \Lambda_3) \subset \mathbb{C}$ in a neighborhood of x_h solution of*

$$\epsilon_{11}^{\nu}(X_\nu) = 0. \tag{30}$$

A first order expansion is

$$X_\nu = x_h - \frac{\partial_\nu \epsilon_{11}^{0}(x_h)}{\partial_x \epsilon_{11}^{0}(x_h)} \nu + O(\nu^2) \tag{31}$$

which shows the limit of X_ν as ν approaches zero is x_h and X_ν is non real for small $\nu \neq 0$ in view of (22).

Proof. From Theorem 10.30 of [16] and Definition 3, there exists neighborhoods $\mathcal{V}^0$ and $\mathcal{V}^\nu$ of x_h on which respectively ϵ_{11}^{0} and $\epsilon_{11}^{\nu} - \left(\epsilon_{11}^{\nu} - \epsilon_{11}^{0}\right)(x_h)$ are one-to-one. Assumption 3 then implies that there exists a $\Lambda_4 > 0$ such that for all ν satisfying $|\nu| < \Lambda_4$, $-\left(\epsilon_{11}^{\nu} - \epsilon_{11}^{0}\right)(x_h) \in \mathcal{V}^\nu$, and that there exists $\Lambda_5 > 0$ such that for all ν satisfying $|\nu| < \Lambda_5$, $(\epsilon_{11}^{\nu})'(x_h) \neq 0$. Set $\Lambda_3 = \min(\Lambda_2, \Lambda_4, \Lambda_5)$. Then for all ν satisfying $|\nu| < \Lambda_3$ there is a unique solution of the equation

$$\epsilon_{11}^{\nu}(x) - \left(\epsilon_{11}^{\nu} - \epsilon_{11}^{0}\right)(x_h) = -\left(\epsilon_{11}^{\nu} - \epsilon_{11}^{0}\right)(x_h), \quad \forall x \in \mathcal{V}^\nu,$$

that is to say a unique $X_\nu \in \mathcal{V}^\nu$ satisfying Equation (30). The formula (31) is a consequence of the local expansion

$$0 = \epsilon_{11}^{\nu}(X_\nu) = \partial_x \epsilon_{11}^{0}(x_h)(X_\nu - x_h) + \partial_x \epsilon_{11}^{0}(x_h)\nu + O(\nu^2).$$

□

Definition 5. *The unique point $X_\nu \in \mathbb{C}$ solution of (30) is called the translated hybrid resonance.*

The case $\epsilon_{11}^{\nu} = \alpha(x) + i\nu$ mentioned above satisfies these hypotheses, with $\alpha(0) = 0$, $\alpha'(0) < 0$, and for this case one has, at first order, $X_\nu = -i\frac{\nu}{\alpha'(0)} + O(\nu^2)$, which means that the dominant part of the translated resonance is pure imaginary with positive imaginary part.
In order to solve Equation (29), it is then crucial to isolate the singularity of the solution. We propose here to first understand the singular structure of the equation's coefficient.
In the case of a local hybrid resonance as in Definition 3, we notice that the most singular term of the coefficient is

$$\sqrt{-c_\nu}\left(\frac{1}{\sqrt{-c_\nu}}\right)''.$$

We then consider the following quantity:

Definition 6. *Suppose x_h is a local hybrid resonance. Under Assumptions 1, 2 and 3, we define the 'coefficient function' $R_\nu(\cdot)$ in $B(x_h, \Lambda_3)$ as:*

$$R_\nu(.) = (\cdot - X_\nu)\left(a_\nu^2 + b_\nu c_\nu - c_\nu\left(\frac{a_\nu}{c_\nu}\right)' + \sqrt{-c_\nu}\left(\frac{1}{\sqrt{-c_\nu}}\right)'' + \frac{1}{4(\cdot - X_\nu)^2}\right).$$

With this notation, equation (29) reads

$$\frac{d^2 y(x)}{dx^2} = \left(-\frac{1}{4(x - X_\nu)^2} + \frac{R_\nu(x)}{x - X_\nu}\right) y(x), \qquad x \in \mathbb{R}. \tag{32}$$

Since the function R_ν can also be defined by continuity at X_μ, as stated in the following result, this form of the equation evidences the singularity of the coefficient.
Note that the method described below extends in a more general case than the one of Definition 3, namely when $\epsilon_{11}^{0}(x_h) = \partial_x \epsilon_{11}^{0}(x_h) = 0$ and $\partial_{x^2}^2 \epsilon_{11}^{0}(x_h) \in \mathbb{R}^*$. In this case one considers k_ν the limit, for $x \to X_\nu$, of

$$(x - X_\nu)^2\left(a_\nu^2 + b_\nu c_\nu - c_\nu\left(\frac{a_\nu}{c_\nu}\right)' + \sqrt{-c_\nu}\left(\frac{1}{\sqrt{-c_\nu}}\right)''\right)$$

which is finite. The value of k_ν will generate the family of approximate solutions needed for the subsequent study.
Returning to the situation of Definition 3, one has

Lemma 6. *Suppose x_h is a local hybrid resonance. Under Assumptions 1, 4 and 3, there exists Λ_6 satisfying $0 < \Lambda_6 \le \Lambda_3$ such that the coefficient function $z \mapsto R_\nu(z)$ is analytic on a ball $z \in B(x_h, \Lambda_6)$ uniformly for $|\nu| < \Lambda_6$. Moreover*

$$R_\nu(X_\nu) = \left[\frac{i\theta(\epsilon_{12}^\nu)' - (\epsilon_{12}^\nu)^2}{(\epsilon_{11}^\nu)'} - \frac{(\epsilon_{12}^\nu)'}{\epsilon_{12}^\nu} + \frac{1}{4}\frac{(\epsilon_{11}^\nu)''}{(\epsilon_{11}^\nu)'}\right](X_\nu)$$

where X_ν is the translated resonance. This makes sense by continuity for ν small since $(\epsilon_{11}^\nu)'(X_\nu) = \partial_x \epsilon_{11}^\nu(X_\nu) \neq 0$ using (20).

The proof of this Lemma is given in Section A.
As announced, since $R_\nu(X_\nu)$ is bounded, the singularity is explicit in Equation (32): there is a $(x - X_\nu)^{-2}$ singularity, and only if $R_\nu(X_\nu) \neq 0$ then there is also a $(x - X_\nu)^{-1}$ singularity.
Since R_ν is bounded and analytic, one can formally shift the equation and the unknown in the complex plane. That is we formally consider the function

$$y(\cdot) = \frac{1}{\sqrt{-c_\nu(\cdot + X_\nu)}} W^\nu(\cdot + X_\nu)$$

and write the equation for this new function. It can be justified that W^ν solution of (28) admits a convenient extension in a complex neighborhood of the hybrid resonance x_h. It provides a new form of equation (29). However this is only formal at this level.

Definition 7 (Shifted equation)**.** *The shifted equation writes*

$$\frac{d^2y(z)}{dz^2} = \left(-\frac{1}{4z^2} + \frac{R_\nu(z + X_\nu)}{z}\right) y(z), \qquad z \in \mathbb{C}^*. \tag{33}$$

Our strategy if first to construct an explicit quasi-solution of the shifted equation, and second to shift back all quantities. It will construct an explicit quasi-solution of the original equation (32). The modifications to obtained exact solutions of the original equation (32) will be performed in a third step.

3.2 Freezing and defreezing the coefficient-function

This section focuses on equation (33) for all $\nu \neq 0$. It is important to see that the only hypothesis that is required to that purpose is the fact that the function $x \mapsto R_\nu(x + X_\nu)$ is bounded. As a consequence we choose to focus on the more general equation

$$\frac{d^2y(z)}{dz^2} = \left(-\frac{1}{4z^2} + \frac{\mathcal{R}(z)}{z}\right) y(z), \qquad z \in \mathbb{C}^*, \tag{34}$$

with the only assumption that the function $\mathcal{R}$ is continuous in a neighborhood of the origin. We will try to distinguish between the results that only depend on this feature of $\mathcal{R}$ and those that are specific to the case $\mathcal{R} = R_\nu(\cdot + X_\nu)$.
It turns out that equation (34) is related to the Bessel equation: to see that, it is sufficient to replace the coefficient function $\mathcal{R}$ by its value at the singularity. Indeed, consider the resulting new equation

$$\frac{d^2y(z)}{dz^2} = \left(-\frac{1}{4z^2} + \frac{\mathcal{R}(0)}{z}\right) y(z), \qquad z \in \mathbb{C}^*. \tag{35}$$

This equation is presented in the chapter on Bessel functions in the classical textbook [1], see equation (9.1.50). According to this reference, solutions are then expressed in terms of the Bessel functions J_0 and Y_0. Even if the equation is singular at the origin, its solutions are locally bounded as stressed below.

Lemma 7. *If $\mathcal{R}(0) \neq 0$, a pair of independent solutions of (35) can be expressed with Bessel functions under the form*

$$\begin{cases} z \mapsto \sqrt{z} J_0\left(\lambda\sqrt{z}\right), & z \in \mathbb{C}, \\ z \mapsto \sqrt{z} Y_0\left(\lambda\sqrt{z}\right), & z \in \mathbb{C}, \end{cases}$$

where $\lambda = 2\sqrt{-\mathcal{R}(0)}$.
If $\mathcal{R}(0) = 0$, *a pair of independent solutions of (35) can be expressed with the simpler expression*

$$\left\{ \begin{array}{l} z \mapsto \sqrt{z} \\ z \mapsto \sqrt{z} \log(x) \, . \end{array} \right.$$

Remark 3. *The function* $J_0(z)$ *admits a infinite converging expansion in powers of* z^2

$$J_0(z) = \sum_{k=0}^{\infty} \frac{(-\frac{1}{4}z^2)^k}{(k!)^2}, \tag{36}$$

and the function $Y_0(z) - \frac{2}{\pi} \log\left(\frac{z}{2}\right) J_0(z)$ *admits a infinite converging expansion in powers of* z^2

$$T_0(z) := Y_0(z) - \frac{2}{\pi} \log\left(\frac{z}{2}\right) J_0(z) = -\frac{2}{\pi} \sum_{k=0}^{\infty} \frac{\psi(k+1)}{(k!)^2} (-\frac{z^2}{4})^k, \tag{37}$$

where $\psi(k) = \sum_{l=1}^{k} 1/l$ *for all* $k \in \mathbb{N}^*$.
Therefore a more convenient pair of independent solutions of (35) also valid for $\lambda = 0$ *reads*

$$\left\{ \begin{array}{ll} z \mapsto \sqrt{z} J_0\left(\lambda\sqrt{z}\right), & z \in \mathbb{C}, \\ z \mapsto \sqrt{z} \left[Y_0\left(\lambda\sqrt{z}\right) - \frac{2}{\pi} \log\left(\frac{\lambda}{2}\right) J_0\left(\lambda\sqrt{z}\right) \right], & z \in \mathbb{C}. \end{array} \right. \tag{38}$$

The reason for using the pair (38) of independent solutions instead of the one proposed in Lemma 7 is that it is uniform with respect to λ, so when applied to the case

$$\mathcal{R} = R_\nu(\cdot + X_\nu) \tag{39}$$

it will be uniform with respect to ν. Indeed in this case, the parameter λ will depend on ν since then $\lambda = \lambda_\nu = 2\sqrt{-R_\nu(X_\nu)}$. So the second function from Lemma 7 has a logarithmic divergence when $\lambda_\nu\sqrt{z} \approx 0$, which means that it diverges if either $\lambda_\nu \approx 0$ or $\sqrt{z} \approx 0$. On the other hand the logarithmic divergence of the second function of (38) is by construction only for $\sqrt{z} \approx 0$. As a result the functions (38) are well defined for $z \neq 0$ as ν goes to zero, as well as for $\nu = 0$.
The Bessel function (37) recasts as $Y_0(z) = T_0(z) + \frac{2}{\pi} \log\left(\frac{z}{2}\right) J_0(z)$ and the second function in (38) is also

$$z \mapsto \sqrt{z} \left[T_0\left(\lambda\sqrt{z}\right) + \frac{1}{\pi} \log(z) J_0\left(\lambda\sqrt{z}\right) \right]. \tag{40}$$

The Bessel based functions, solutions of the equation with frozen coefficient (35), provide the formal dominant behavior of the solutions of equation (34). Nevertheless the rigorous justification of this statement is not evident, mainly because the frozen coefficient $\mathcal{R}(0)$ is only a first order local approximation of the coefficient function $\mathcal{R}$. Indeed the singularity of the $\frac{1}{4z^2}$ term is of second order, so any naive iteration technique that would aim at controlling the error of approximation between both may diverge as well like $O\left(z^{-1}\right)$. Overcoming this difficulty is the purpose of the next paragraph, which presents a technical process to control these errors.
The next idea is to explicit the link between Equation (34) and the frozen coefficient equation (35). Yet the singularity of the equation requires specific attention, and following process is two folded: an intermediate equation is introduced to focus in a first step on the singularity, before finally going back to the desired equation, namely (34).

3.2.1 Eikonal equation and stretching function

Consider first another intermediate equation by reintroducing a varying coefficient in Equation (35):

Definition 8 (Streching procedure)**.** *Let* ρ *be an arbitrary smooth function such that* $\rho(0) = 0$ *and* $\rho'(0) \neq 0$. *We will refer to it as the stretching function. This name comes, in particular, from Mc Kelvey [15].*
Let u *be a given function. The stretching of* u *is the function* $\tilde{u}$ *defined by*

$$\tilde{u}(z) = (\rho'(z))^{-\frac{1}{2}} u(\rho(z)).$$

The equation for the stretched function is the following.

Lemma 8. *Assume ρ is a stretching function and that u solves the frozen coefficient equation* (35). *Set* $s = \frac{\left((\rho')^{-\frac{1}{2}}\right)''}{(\rho')^{-\frac{1}{2}}}$. *Then the stretched unknown $y = \tilde{u}$ satisfies*

$$\frac{d^2 y(z)}{dz^2} = \left(-\frac{\rho'(z)^2}{4\rho(z)^2} + \frac{\mathcal{R}(0)\rho'(z)^2}{\rho(z)} + s(z) \right) y(z), \qquad z \in \mathbb{C}^*. \tag{41}$$

Proof. The proof relies on the scaling of the stretched unknown: the coefficient of the first order derivative term in the computation of $\tilde{u}''$ is $((\rho')^{-1/2})'\rho' + ((\rho')^{1/2})'$, which happens to be zero. □

Note that the function s is a priori bounded around 0 since $\rho'(0) \neq 0$, so that this form of equation evidences again, as Equation (34), the singularities of its coefficient.
At this point the stretching function is a tool that will be fit toward the approximation of Equation (34). Indeed, although Equation (41) is different from the initial equation, namely (32), an adequate choice of stretching function leads the way back to it. We use a stretching function as the solution of the following eikonal equation

$$\rho'(z)^2 \left(-\frac{1}{4\rho(z)^2} + \frac{\mathcal{R}(0)}{\rho(z)} \right) = -\frac{1}{4z^2} + \frac{\mathcal{R}(z)}{z}, \qquad z \in \mathbb{C}^*. \tag{42}$$

Note that we call this equation the 'eikonal' equation because it plays the same role as the equation for ϕ if one seeks a solution of the wave equation $(\Delta - c^{-2}(x)\partial_{t^2}^2)u = 0$ under the form $u(x,t) = a(x,k)e^{ik(\phi(x)-t)}$.
Note that (42) implies $\frac{\rho'(z)}{\rho(z)} = \pm\frac{1}{z}\sqrt{\frac{1-4zR(z)}{1-4\rho(z)R(0)}}$ for $z, \rho(z)$ non zero. As we choose ρ bounded in the neighborhood of 0, only the + sign is relevant. The equation rewrites

$$\left(\frac{\rho(z)}{z}\right)'(z) = \frac{\rho(z)}{z} F\left(z, \frac{\rho(z)}{z}\right), \quad z \in \mathbb{C}^*, \tag{43}$$

where F is defined by

$$F(z,a) = \frac{4\left[\mathcal{R}(0)a - \mathcal{R}(z)\right]}{\sqrt{1-4\mathcal{R}(z)z}\sqrt{1-4\mathcal{R}(0)az} + (1-4\mathcal{R}(0)az)}.$$

We notice that the equation (42) is be defined also at $z = 0$ since $F(0,a)$ makes sense, at least for small z and for bounded $\mathcal{R}$ since it guarantees that the square roots are defined without ambiguity.
We now reintroduce $\mathcal{R} = R_\nu(\cdot + X_\nu)$ and consider the more general equation in the complex plane

$$\sigma_\nu'(z) = \sigma_\nu(z) F_\nu\left(z, \sigma_\nu(z)\right), \quad \rho_\nu(z) = z\sigma_\nu(z), \qquad z \in \mathbb{C}, \tag{44}$$

where F_ν is defined by

$$F_\nu(z,a) = \frac{4\left[R_\nu(X_\nu)a - R_\nu(z+X_\nu)\right]}{\sqrt{1-4R_\nu(z+X_\nu)z}\sqrt{1-4R_\nu(X_\nu)za} + (1-4R_\nu(X_\nu)za)}.$$

Lemma 9 (Solution of the eikonal equation)**.** *Suppose x_h is a local hybrid resonance. Under Assumptions 1, 2 and 3, consider Equation (44).*
Then there exists a constant Λ_7 satisfying $0 < \Lambda_7 \leq \Lambda_6$ such that:

- *For all $|\nu| \leq \Lambda_7$ there exists a solution to Equation (42) on a maximal interval $I_\nu =]a_\nu, b_\nu[\subset] - \Lambda_3, \Lambda_3[$ which contains the resonance x_h.*
- *There exists $a < 0 < b$ such that, for all $|\nu| < \Lambda_7$, $x_h \in I =]a, b[\subset I_\nu$.*
- *A solution of the eikonal equation (44) is $\rho_\nu(z)$ for $z \in B(0, \Lambda_7)$ defined*

$$\rho_\nu(z) = z \exp\left(z \int_0^1 F_\nu(zt, \sigma_\nu(zt))dt \right). \tag{45}$$

  Note that F is evaluated on the straight line from 0 to z

- *This function is analytic in the ball $z \in B(0, \Lambda_7)$.*

Proof. The function $\sigma_\nu = \frac{\rho_\nu}{x}$ is solution of the equation

$$\sigma_\nu'(z) = \sigma_\nu(z) F_\nu(z, \sigma_\nu(z)), \qquad z \in \mathbb{C}. \tag{46}$$

The value of $F_\nu(0, a) = 2R_\nu(X_\nu)\,(a-1)$ is a well defined (i.e. non singular, or bounded) real number. Equation (46) can be solved in the complex plane using a Theorem from Coddington-Levinson [7][theorem 8.1, page 34]. For simplicity consider the initial data $\sigma_\nu(0) = 1$ which yields $\sigma_\nu'(0) = F_\nu(0,1) = 0$. This choice is arbitrary. Since $\sigma_\nu'(0) = 0$, one has the Taylor expansion $\rho_\nu(z) = z + O_\nu(z^3)$ where the constant in $\left|O_\nu(z^3)\right| \le C|z|^3$ is uniform with respect to ν. All other coefficients of the Taylor expansion of the stretching function ρ_ν can be easily computed from the **integral representation in the complex plane** (45) where the path of integration is the straight line. The analyticity is provided by the Coddington-Levinson theorem. □

Thanks to the stretching function described in Lemma 9, it is now possible to explicit the solutions of Equation (41) where, in view of lemma 7 and equation (39), one needs to set

$$\lambda_\nu = 2\sqrt{\mathcal{R}_\nu(X_\nu)}.$$

Definition 9. *Suppose x_h is a local hybrid resonance. Under Assumptions 1, 2 and 3, consider for all $|\nu| \le \Lambda_7$ the stretching function ρ_ν described in Lemma 9. A pair (U_ν, V_ν) of independent solutions of Equation (41) is defined for all $(\nu, z) \in (-\Lambda_7, \Lambda_7) \times B(0, \Lambda_7)$ by*

$$\begin{cases} U_\nu(z) &= \sqrt{\dfrac{\rho_\nu(z)}{\rho_\nu'(z)}} J_0\left(\lambda_\nu \sqrt{\rho_\nu(z)}\right), \\ V_\nu(z) &= \sqrt{\dfrac{\rho_\nu(z)}{\rho_\nu'(z)}} \left[Y_0\left(\lambda_\nu \sqrt{\rho_\nu(z)}\right) \qquad -\frac{2}{\pi}\log\left(\frac{\lambda_\nu}{2}\right) J_0\left(\lambda_\nu\sqrt{\rho_\nu(z)}\right)\right] \\ &= \sqrt{\dfrac{\rho_\nu(z)}{\rho_\nu'(z)}} \left[T_0\left(\lambda_\nu \sqrt{\rho_\nu(z)}\right) \quad +\frac{1}{\pi}\log\left(\rho_\nu(z)\right) J_0\left(\lambda_\nu\sqrt{\rho_\nu(z)}\right)\right] \end{cases}$$

These functions are uniformly bounded with respect to ν.

Lemma 10. *Suppose x_h is a local hybrid resonance. Under Assumptions 1, 2 and 3, consider for all $|\nu| \le \Lambda_7$ the stretching function ρ_ν (45).*
Then the functions U_ν and V_ν are bounded in a complex neighborhood of the resonance $x_h \in \mathbb{C}$, uniformly for small ν: there exists $\Lambda_8 > 0$ such that for all $(\nu, z) \in (-\Lambda_8, \Lambda_8) \times B(0, \Lambda_8)$

$$|U_\nu(z)| + |V_\nu(z)| \le C.$$

Moreover Λ_8 can be determined such that for all $\nu \in B(0, \Lambda_8)$

$$|\rho_\nu'(z)| \ge \delta > 0$$

in the same neighborhood.

Proof. The property on ρ_ν is immediate since $\sigma_\nu(0) = 1$. The uniform boundedness of U_ν stems from the boundedness of J_0 (36). Concerning V_ν, the $\log \lambda_\nu$ term has been carefully removed, see (37), so that the boundedness is achieved even for vanishing λ_ν: the remaining $\log z$ is controlled by the $\sqrt{z}$ contribution that comes from $\sqrt{\rho_\nu(z)}$. □

It is now possible to apply the reverse shift.

Lemma 11. *The functions $x \mapsto U_\nu(x - X_\nu)$ and $x \mapsto V_\nu(x - X_\nu)$ are solutions of the equation on the real line*

$$\frac{d^2}{dx^2} y(x) = \left[-\frac{1}{4(x-X_\nu)^2} + \frac{R_\nu(x)}{x - X_\nu} + s_\nu(x - X_\nu)\right] y(x), \quad x \in B(x_h, \Lambda_8) \cap \mathbb{R}, \tag{47}$$

where $s_\nu = \dfrac{\left((\rho_\nu')^{-\frac{1}{2}}\right)''}{(\rho_\nu')^{-\frac{1}{2}}}$.

Proof. By definition, all these fonctions are analytic in the ball $B(x_h, \Lambda_8) - \{X_\nu\}$ (that is except at X_ν) which is uniform with respect to ν. But, when ρ_ν satisfies the eikonal relation (42) with $\mathcal{R}(z) = R_\nu(z + X_\nu)$, the function $z \mapsto U_\nu(z)$ (without the shift) is solution of (41). Therefore the identity (47) between analytic functions is true on $B(x_h, \Lambda_8) \cap \mathbb{R}$, except at X_ν for $X_\nu \in \mathbb{R}$. So this relation is true everywhere by analytic continuation. The proof is ended. □

Since X_ν is non real for small non zero ν, this equation is valid also for small non zero ν.

3.2.2 General solution with a Duhamel's principle and limit process

In order to go back to the equation with a varying coefficient, let us now consider the original problem (32). One can rewrite this equation as: for all $x \in \mathbb{R}$

$$\frac{d^2}{dx^2}y(x) = \left[-\frac{1}{4(x-X_\nu)^2} + \frac{R_\nu(x)}{x-X_\nu} + s_\nu(x-X_\nu)\right] y(x) - s_\nu(x-X_\nu)y(x). \tag{48}$$

The last term is a perturbation with respect to Equation (47). As usual for such problems, this term is treated by means of the Duhamel principle where we will make major profit of the fact that the fundamental solutions of the singular equation (47) are bounded.

Lemma 12. *Suppose $x_h \in \mathbb{R}$ is an isolated hybrid resonance, and assumptions 1, 2 and 3 hold. Consider for all ν a given reference point $x^* \in \mathbb{R}$ close to x_h in the ball of analyticity $B(x_h, \Lambda_8)$. Then the function*

$$y(x) = A_\nu(x)U_\nu(x-X_\nu) + B_\nu(x)V_\nu(x-X_\nu) \tag{49}$$

is solution of (48) for all $(\nu, x) \in (-\Lambda_8, \Lambda_8) \times B(x_h, \Lambda_8) \cap \mathbb{R}$ if and only if $\mathbf{C}_\nu$, defined by

$$\mathbf{C}_\nu(x) = \begin{pmatrix} A_\nu(x) \\ B_\nu(x) \end{pmatrix}, \quad x \in \mathbb{B}(x_h, \Lambda_8) \cap \mathbb{R}, \tag{50}$$

is the solution of the integral equation

$$\mathbf{C}_\nu(x) = \mathbf{C}_\nu(x^*) + \pi \int_{x^*}^{x} \mathcal{M}_\nu(t)\mathbf{C}_\nu(t)dt \tag{51}$$

where the matrix

$$\mathcal{M}_\nu(t) = s_\nu(t-X_\nu)\begin{pmatrix} -U_\nu(t-X_\nu)V_\nu(t-X_\nu) & -U_\nu(t-X_\nu)^2 \\ V_\nu(t-X_\nu)^2 & -U_\nu(t-X_\nu)V_\nu(t-X_\nu) \end{pmatrix}, \tag{52}$$

is uniformly bounded in L^∞ for $(\nu, t) \in (-\Lambda_8, \Lambda_8) \times B(x_h, \Lambda_8) \cap \mathbb{R}$.

As usual for such Volterra integral equations, the initial point $x^* \in \mathbb{R}$ can be chosen arbitrarily.

Proof. This is the standard procedure of variation of parameters. Consider the following equation:

$$y'' - \left[-\frac{1}{4(x-X_\nu)^2} + \frac{R_\nu(x)}{x-X_\nu} + s_\nu(x-X_\nu)\right] y = f \tag{53}$$

with $f(x) = -s_\nu(x-X_\nu)y$. The solution y can be expressed as a combination of $U_\nu(x-X_\nu)$ and $V_\nu(x-X_\nu)$, with appropriate coefficients $A_\nu(x)$ and $B_\nu(x)$

$$y(x) = A_\nu(x)U_\nu(x-X_\nu) + B_\nu(x)V_\nu(x-X_\nu).$$

To construct the coefficients $A_\nu(x)$ and $B_\nu(x)$, we first assume that

$$A'_\nu(x)U_\nu(x-X_\nu) + B'_\nu(x)V_\nu(x-X_\nu) = 0. \tag{54}$$

So the first derivative of y is $y'(x) = A_\nu(x)U'_\nu(x-X_\nu) + B_\nu(x)V'_\nu(x-X_\nu)$. The second derivative reads

$$y''(x) = A_\nu(x)U''_\nu(x-X_\nu) + B_\nu(x)V''_\nu(x-X_\nu) + A'_\nu(x)U'_\nu(x-X_\nu) + B'_\nu(x)V'_\nu(x-X_\nu).$$

Using (47), one gets

$$y''(x) = \left[-\frac{1}{4(x-X_\nu)^2} + \frac{R_\nu(x)}{x-X_\nu} + s_\nu(x-X_\nu)\right] y(x) + A'_\nu(x)U'_\nu(x-X_\nu) + B'_\nu(x)V'_\nu(x-X_\nu).$$

Since y is solution of (53), one gets

$$A'_\nu(x)U'_\nu(x-X_\nu) + B'_\nu(x)V'_\nu(x-X_\nu) = f \tag{55}$$

with $f(x) = -s_\nu(x-X_\nu)\left(A_\nu(x)U_\nu(x-X_\nu) + B_\nu(x)V_\nu(x-X_\nu)\right)$. One can solve now the linear system made of (54) and (55).
To compute the determinant of the linear system (54-55), consider the Wronskian of U_ν and V_ν. Since $\mathcal{W}_{(J_0,Y_0)}(x) = \frac{2}{\pi x}$, elementary calculations yield

$$\mathcal{W}_{(U_\nu,V_\nu)}(x) = \lambda_\nu \left(\sqrt{\frac{\rho_\nu(x-X_\nu)}{\rho_\nu'(x-X_\nu)}}\right)^2 \frac{\rho_\nu'(x-X_\nu)}{2\sqrt{\rho_\nu(x-X_\nu)}} \mathcal{W}_{(J_0,Y_0)}\left(\lambda_\nu\sqrt{\rho_\nu(x-X_\nu)}\right),$$

so that $\mathcal{W}_{(U_\nu,V_\nu)}(x) = \frac{1}{\pi}$. By analytic continuation the determinant of the linear system (54-55) is also equal to the same value.
Therefore the solution of the linear system (54-55) reads

$$\begin{cases} A_\nu'(x) &= -\pi s_\nu(x-X_\nu)\left[A_\nu(x)U_\nu(x-X_\nu) + B_\nu(x)V_\nu(x-X_\nu)\right]V_\nu(x-X_\nu), \\ B_\nu'(x) &= \pi s_\nu(x-X_\nu)\left[A_\nu(x)U_\nu(x-X_\nu) + B_\nu(x)V_\nu(x-X_\nu)\right]U_\nu(x-X_\nu). \end{cases}$$

After integration, it yields the representation formula (51). The boundedness of the kernel comes from the properties of s_ν, U_ν and V_ν in Lemma 10. It completes the proof. $\square$

Under the same conditions, one can define the integral operator $\mathcal{K}_\nu$

$$\mathcal{K}_\nu(f)(x) = \pi \int_{x^*}^{x} \mathcal{M}_\nu(t) f(t)dt,$$

so that the integral equation (51) reads $\mathbf{C}_\nu - \mathcal{K}_\nu\mathbf{C}_\nu = \mathbf{C}_\nu(x^*)$, with a constant right hand side. Classically for this Volterra second type integral equation (see [19]), the solution is expressed with the resolvent kernel $\mathfrak{Q}_\nu[f] = \sum_{n\geq 1}\mathcal{K}_\nu^{(n)}(f)$ under the form

$$\begin{aligned} \mathbf{C}_\nu(x) &= \mathbf{C}_\nu(x^*) + \mathfrak{Q}_\nu[\mathbf{C}_\nu(x^*)](x), \\ &= (Id + \mathfrak{Q}_\nu[1])(x)\mathbf{C}_\nu(x^*), \end{aligned} \quad \forall x \in \mathbb{R}, \tag{56}$$

where $\mathbf{C}_\nu(x^*)$ is the initial condition.
Foreseeing the limit process, the following results state the limit properties of ρ_ν and $\mathcal{M}_\nu$, and of additional quantities, as ν approaches zero.

Lemma 13. *Suppose x_h is a local hybrid resonance. Under Assumptions 1, 2 and 3, consider for all $|\nu| \leq \Lambda_8$ the stretching function ρ_ν described in Lemma 9. One has that $\lim_{\nu\to 0^\pm}\rho_\nu(z) = z\sigma_0(z)$ on $B(x_h,\Lambda_8)$ where σ_0 is the solution of the integral equation (46) such that $\sigma_0(0) = 1$. This limit function will naturally be called ρ_0. Then the* log *and square root terms have limits which depend on the sign of ν: we set*

$$\log\rho_0(x-x_h)^\pm = \lim_{\nu\to 0^\pm}\log\rho_\nu(x-X_\nu) = \begin{cases} \log\left(\rho_0(x-x_h)\right) & \text{for } x_h < x, \\ \log\left(-\rho_0(x-x_h)\right) \mp i\pi\mathfrak{s} & \text{for } x < x_h, \end{cases} \tag{57}$$

where $\mathfrak{s} = sign\left(\frac{i\partial_\nu\epsilon_{11}^\nu(x)}{\partial_x\epsilon_{11}^\nu(x)}\right)_{|x=x_h,\nu=0}$, and

$$\sqrt{\rho_0(x-x_h)}^\pm = \lim_{\nu\to 0^\pm}\sqrt{\rho_\nu(x-X_\nu)} = \begin{cases} \sqrt{\rho_0(x-x_h)} & \text{for } x_h < x, \\ \mp i\mathfrak{s}\sqrt{-\rho_0(x-x_h)} & \text{for } x < x_h, \end{cases} \tag{58}$$

Proof. The claim on ρ_0 is evident. The limit of the log term comes from the principal value of the logarithm: $\log z = a(z) + ib(z)$ where $a(z) = \log|z|$ and $b(z) \in]-\pi,\pi]$. And the limit of the square root term comes from the principal value of the complex square root: $\sqrt{z} = \sqrt{|z|}e^{i\frac{\theta(z)}{2}}$ where $\theta(z) \in]-\pi,\pi]$ is the argument of z. $\square$

Lemma 14. *Suppose x_h is a local hybrid resonance. Under Assumptions 1, 2 and 3, the matrix $\mathcal{M}_\nu$ defined in* (52) *has a limit in the following sense.*

- $\mathcal{M}_\nu(x) \underset{\nu\to 0^+}{\overset{C^0(x_h-\Lambda_8,x_h+\Lambda_8)}{\longrightarrow}} \mathcal{M}^+(x)$
- $\mathcal{M}_\nu(x) \underset{\nu\to 0^-}{\overset{C^0(x_h-\Lambda_8,x_h+\Lambda_8)}{\longrightarrow}} \mathcal{M}^-(x)$

- *The continuous kernels $\mathcal{M}^+$ and $\mathcal{M}^-$ are such that: $\mathcal{M}^+ - \mathcal{M}^- = 0$ vanishes on $[x_h, x_h + \Lambda_8)$, while $\mathcal{M}^+ - \mathcal{M}^- \neq 0$ is non identically zero on $(x_h - \Lambda_8, x_h)$.*

Consider (for simplicity) a Cauchy data $C_\nu(x^*)$ which admits a finite (given) limit in ν in the representation (56) of the solution of integral equation. Then the functions A_ν and B_ν also converge in $C^0(x_h - \Lambda_8, x_h + \Lambda_8)$ to smooth limit functions $A_0^\pm$ and $B_0^\pm$.

Proof. The continuous limits of the kernel $\mathcal{M}_\nu$ are given by application of Lemmas 13-14 since the matrix entries only depend on the functions U_ν and V_ν. The limit of functions A_ν and B_ν then directly stems from the limit of (56) where the kernel has a limit which is bounded. It ends the proof. ☐

Definition 10. *We define three additional functions as follows:*

$$\begin{cases} W_1^\nu(x) &= \sqrt{-c_\nu(x)}\sqrt{\frac{\rho_\nu(x-X_\nu)}{\rho_\nu'(x-X_\nu)}}J_0\left(\lambda_\nu\sqrt{\rho_\nu(x-X_\nu)}\right) = \sqrt{-c_\nu(x)}U_\nu\left(x-X_\nu\right) \\ W_2^\nu(x) &= \sqrt{-c_\nu(x)}\sqrt{\frac{\rho_\nu(x-X_\nu)}{\rho_\nu'(x-X_\nu)}}\left(T_0\left(\lambda_\nu\sqrt{\rho_\nu(x-X_\nu)}\right) + \frac{1}{\pi}\log(\rho_\nu(x-X_\nu))J_0\left(\lambda_\nu\sqrt{\rho_\nu(x-X_\nu)}\right)\right) \\ &= \sqrt{-c_\nu(x)}V_\nu\left(x-X_\nu\right), \\ W_3^\nu(x) &= \sqrt{-c_\nu(x)}\sqrt{\frac{\rho_\nu(x-X_\nu)}{\rho_\nu'(x-X_\nu)}}T_0\left(\lambda_\nu\sqrt{\rho_\nu(x-X_\nu)}\right). \end{cases}$$

Since $c_\nu(x)\rho_\nu(x-X_\nu)$ is bounded, the functions W_1^ν and W_3^ν are uniformly bounded.
On the other hand the function

$$W_2^\nu(x) = W_3^\nu(x) + \frac{1}{\pi}\log(\rho_\nu(x-X_\nu))W_1^\nu(x)$$

is not bounded due the log *term.*

Lemma 15. *Suppose x_h is a local hybrid resonance. Under Assumptions 1, 2 and 3, consider for all $|\nu| \leq \Lambda_8$ the stretching function ρ_ν described in Lemma 9. One has*

$$\lim_{\nu\to 0^\pm} W_1^\nu(x) = W_1(x) := \sqrt{\frac{d^0(x)\rho_0(x-x_h)}{\epsilon_{11}^0(x)\rho_0'(x-x_h)}}J_0(\lambda_0\sqrt{\rho_0(x-x_h)}), \quad \forall x \in B(x_h, \Lambda_8) \cap \mathbb{R},$$

where the limit function W_1 is analytic on $B(x_h, \Lambda_8) \cap \mathbb{R}$. Similarly

$$\lim_{\nu\to 0^\pm} W_3^\nu(x) = W_3(x) := \sqrt{\frac{d^0(x)\rho_0(x-x_h)}{\epsilon_{11}^0(x)\rho_0'(x-x_h)}}T_0(\lambda_0\sqrt{\rho_0(x-x_h)}), \quad \forall x \in B(x_h, \Lambda_8) \cap \mathbb{R},$$

and W_3 analytic on $B(x_h, \Lambda_8) \cap \mathbb{R}$.

Proof. The point is to take the limit in the series (36) that defines J_0. This is straightforward. Only even terms show up this series, so $J_0(\lambda_0\sqrt{z\sigma_0(z)})$ is analytic. For small $x - x_h$ the weight is the square root of $\frac{d^0(x)\rho_0(x-x_h)}{\epsilon_{11}^0(x)\rho_0'(x-x_h)}$ which is well defined since $\frac{d^0(x)\rho_0(x-x_h)}{\epsilon_{11}^0(x)\rho_0'(x-x_h)} \approx \frac{d^0(x_h)}{(\epsilon_{11}^0)'(x_h)} > 0$ due to our normalization. So W_1 is analytic in the neighborhood of x_h. Mutatis mutandis the proof is the same for the function W_3^ν. ☐

Considering the logarithmic part in Definition (37) of Y_0, the situation is a little more involved concerning W_2^ν since the limits are dependent of the sign of ν.

Lemma 16. *Suppose x_h is a local hybrid resonance. Under Assumptions 1, 2 and 3, consider for all $|\nu| \leq \Lambda_8$ the stretching function ρ_ν described in Lemma 9. One has*

$$\lim_{\nu\to 0^\pm} W_2^\nu(x) = W_2^\pm(x), \quad \forall x \in B(x_h, \Lambda_8) \cap \mathbb{R},$$

where the limit functions are

$$W_2^\pm = \sqrt{\frac{d^0(x)\rho_0(x-x_h)}{\epsilon_{11}^0(x)\rho_0'(x-x_h)}}\left(T_0(\lambda_0\sqrt{\rho_0(x-x_h)}) + \frac{1}{\pi}\log\left(\rho_0(x-x_h)\right)^\pm J_0\left(\lambda_0\sqrt{\rho_0(x-x_h)}\right)\right) \tag{59}$$

The limit functions W_2^+ and W_2^- are equal on the interval $(x_h, x_h + \Lambda_8)$, but are different on the interval $(x_h - \Lambda_8, x_h)$, where they satisfy the jump relation

$$W_2^+ - W_2^- = -2iW_1. \tag{60}$$

Proof. Formula (59) is an immediate consequence of the definition of W_3^ν. The second relation comes from (57): we note that a similar relation has been proved in [11]. ☐

3.3 Representation of the physical unknowns

One can now obtain meaningful representation formulas for the physical quantities we are interested in, being particularly careful about the various singularities encountered.

Proposition 1. *Suppose $x_h \in \mathbb{R}$ is an isolated hybrid resonance, and assumptions 1, 2 and 3 hold. Consider for all $|\nu| \le \Lambda_8$ the stretching function ρ_ν described in Lemma 9. Consider for all ν a given reference point $x^* \in \mathbb{R}$ close to x_h in the ball of analyticity $B(x_h, \Lambda_8)$, as well as a Cauchy data $C_\nu(x^*)$ which goes to a finite limit when ν goes to 0.*
Then the unique solution $(E_1^\nu, E_2^\nu, W^\nu)$ of the regularized system (4)-(19) on $B(x_h, \Lambda_8) \cap \mathbb{R}$ is given by

$$\left\{\begin{array}{l} W^\nu = A_\nu W_1^\nu + B_\nu W_3^\nu + B_\nu \frac{1}{\pi} \log \rho_\nu(\cdot - X_\nu) W_1^\nu, \\ E_2^\nu = A_\nu \left[\frac{i\theta\epsilon_{12}^\nu}{d^\nu} W_1^\nu - \frac{\epsilon_{11}^\nu}{d^\nu}\frac{d}{dx} W_1^\nu\right] + B_\nu \left[\frac{i\theta\epsilon_{12}^\nu}{d^\nu} W_3^\nu - \frac{\epsilon_{11}^\nu}{d^\nu}\frac{d}{dx} W_3^\nu\right] \\ \quad + B_\nu \frac{1}{\pi} \log \rho_\nu(\cdot - X_\nu) \left[\frac{i\theta\epsilon_{12}^\nu}{d^\nu} W_1^\nu - \frac{\epsilon_{11}^\nu}{d^\nu}\frac{d}{dx} W_1^\nu\right] - B_\nu \frac{1}{\pi} \frac{\rho_\nu'(\cdot - X_\nu)}{\rho_\nu(\cdot - X_\nu)} \frac{\epsilon_{11}^\nu}{d^\nu} W_1^\nu, \\ E_1^\nu = A_\nu \left[\frac{i\theta\epsilon_{11}^\nu}{d^\nu} W_1^\nu + \frac{\epsilon_{12}^\nu}{d^\nu}\frac{d}{dx} W_1^\nu\right] + B_\nu \left[\frac{i\theta\epsilon_{11}^\nu}{d^\nu} W_3^\nu + \frac{\epsilon_{12}^\nu}{d^\nu}\frac{d}{dx} W_3^\nu\right] \\ \quad + B_\nu \frac{1}{\pi} \log \rho_\nu(\cdot - X_\nu) \left[\frac{i\theta\epsilon_{11}^\nu}{d^\nu} W_1^\nu + \frac{\epsilon_{12}^\nu}{d^\nu}\frac{d}{dx} W_1^\nu\right] + B_\nu \frac{1}{\pi} \frac{\rho_\nu'(\cdot - X_\nu)}{\rho_\nu(\cdot - X_\nu)} \frac{\epsilon_{12}^\nu(x)}{d^\nu} W_1^\nu. \end{array}\right. \tag{61}$$

Proof. The expression of $W^\nu = \sqrt{-c_\nu}\,(A_\nu U_\nu(\cdot - X_\nu) + B_\nu V_\nu(\cdot - X_\nu))$ is immediate from Lemma 12, since it gives the general solution (49) of the corresponding second order equation. It yields the first identity.
Concerning the second identity, one can start from $E_2^\nu = \frac{a_\nu}{c_\nu} W^\nu + \frac{1}{c_\nu}\frac{d}{dx} W^\nu$ by means of (11) and (26). Using (25) one has $\frac{1}{c_\nu} = -\frac{\epsilon_{11}^\nu}{d^\nu}$ and $\frac{a_\nu}{c_\nu} = \frac{i\theta\epsilon_{12}^\nu}{d^\nu}$. So one can write

$$E_2^\nu = \frac{i\theta\epsilon_{12}^\nu}{d^\nu} W^\nu - \frac{\epsilon_{11}^\nu}{d^\nu}\frac{d}{dx} W^\nu. \tag{62}$$

To compute $\frac{d}{dx} W^\nu$ we directly differentiate all terms in the expression already obtained for W^ν (first line of (61)). We observe that the derivatives of A_ν and B_ν vanish since $A_\nu' U_\nu(\cdot - x_h) + B_\nu' V_\nu(\cdot - x_h) = 0$, see (54). It yields

$$\frac{d}{dx} W^\nu = A_\nu \frac{d}{dx} W_1^\nu + B_\nu \left(\frac{d}{dx} W_3^\nu + \frac{1}{\pi} \log \rho_\nu(\cdot - X_\nu) \frac{d}{dx} W_1^\nu\right) + B_\nu \frac{1}{\pi} \frac{\rho_\nu'(\cdot - X_\nu)}{\rho_\nu(\cdot - X_\nu)} W_1^\nu. \tag{63}$$

It is then sufficient to plug this expression in (62) and to reorganize the sum to get the the representation formula for E_2^ν in the second line of (61).
From (8), multiplying the first equality by ϵ_{11}^ν and the second equality by $-\epsilon_{12}^\nu$, adding, and using the symmetries of the dielectric tensor, one gets

$$E_1^\nu = \frac{i\theta\epsilon_{11}^\nu}{d^\nu} W^\nu + \frac{\epsilon_{12}^\nu}{d^\nu}\frac{d}{dx} W^\nu. \tag{64}$$

Even if very simple, this algebra seems to be important since various cancellations of potential singular terms have been performed. It is then sufficient to plug the representation formulas for W^ν and $\frac{d}{dx} W^\nu$ in (64) to obtain the third line of the claim. □

At inspection of the representation formulas (61), it is clear that the convergence with respect to ν is not the same for E_2^ν and W_ν on the one hand, and for E_1^ν on the other hand. Indeed the most singular term in E_2^ν and W_ν is the logarithm $\log \rho_\nu(\cdot - X_\nu)$: the last term in E_2^ν is non singular since $\frac{\epsilon_{11}^\nu}{\rho_\nu(\cdot - X_\nu)}$ is the ratio of two singular terms; in terms of singular behavior these to terms cancel each other.
So these terms all terms are bounded in L^p_{loc} for $p < \infty$ and pass to the limit point wise except at the singularity $x_h = \lim_{\nu \to 0} X_\nu$. This result was already obtained for the slab geometry with a completely different technique in [11][proposition 5.14] and is generalized here. So we state without detail the representation formulas for the limits.

Proposition 2. *Suppose x_h is a local hybrid resonance, and assumptions 1, 4 and 3 hold. Consider for all $|\nu| \le \Lambda_8$ the stretching function ρ_ν described in Lemma 9, and $C_\nu(x^*)$ which goes to a finite limit when ν goes to 0. Consider the unique solution $(E_1^\nu, E_2^\nu, W^\nu)$ of the regularized system (4) on $B(x_h, \Lambda_8)$.*
Then for all $1 \le p < \infty$ the L^p limit as $\nu \to 0$ of W^ν and E_2^ν are

$$W^\pm = A_0^\pm W_1 + B_0^\pm(x) W_3 + B_0^\pm \frac{1}{\pi} \log \rho_0(\cdot - x_h)^\pm W_1 \tag{65}$$

and

$$E_2^\pm = A_0^\pm \left[\frac{i\theta\epsilon_{12}^0}{d^0} W_1 - \frac{\epsilon_{11}^0}{d^0}\frac{d}{dx}W_1\right] + B_0^\pm \left[\frac{i\theta\epsilon_{12}^0}{d^0} W_3 - \frac{\epsilon_{11}^0}{d^0}\frac{d}{dx}W_3\right]$$
$$+ B_0^\pm \frac{1}{\pi}\log\rho_0(\cdot - x_h)^\pm \left[\frac{i\theta\epsilon_{12}^0}{d^0} W_1 - \frac{\epsilon_{11}^0}{d0}\frac{d}{dx}W_1\right] - B_0^\pm \frac{1}{\pi}\frac{\rho_0'(\cdot - x_h)}{\rho_0(\cdot - x_h)}\frac{\epsilon_{11}^0}{d^0}W_1 \tag{66}$$

where $\log\rho_0(\cdot - x_h)^\pm$ *is defined in (57). The limits hold also on pointwise in* $\mathbb{R} - \{x_h\} \cap B(x_h, \Lambda_8)$.

The situation is different for the E_1^ν component (61) since the division by $\rho_\nu(\cdot - X_\nu)$ in the last term yields to a singularity as ν approaches zero. This behavior is the same that already demonstrated in [11] using an singular integral equation technique. The difference is that we have now an explicit representation of this singular behavior for small ν. To express the limit of the singular term, the more efficient way is to use principal value and Dirac mass. Since the result is essentially the same as in [11] and all calculations are now evident starting from the representation formula (61) we state the result without details of the proof.

Theorem 1. *Suppose* x_h *is a local hybrid resonance, and assumptions 1, 4 and 3 hold. Consider for all* $|\nu| \le \Lambda_8$ *the stretching function* ρ_ν *described in Lemma 9, and* $C_\nu(x^*)$ *which goes to a finite limit when* $\nu \to 0$. *Consider the unique solution* $(E_1^\nu, E_2^\nu, W^\nu)$ *of the regularized system* (4) *on* $B(x_h, \Lambda_8)$. *Then the limit of* E_ν^1 *is on* $B(x_h, \Lambda_8)$

$$E_1^\pm = A_0^\pm \left[\tfrac{i\theta\epsilon_{11}^0}{d^0} W_1 + \tfrac{\epsilon_{12}^0}{d^0}\tfrac{d}{dx}W_1\right] + B_0^\pm \left[\tfrac{i\theta\epsilon_{11}^0}{d^0} W_3 + \tfrac{\epsilon_{12}^0}{d^0}\tfrac{d}{dx}W_3\right] + B_0^\pm \tfrac{1}{\pi}\log\rho_0(\cdot - x_h)\left[\tfrac{i\theta\epsilon_{11}^0}{d^0} W_1 + \tfrac{\epsilon_{12}^0}{d^0}\tfrac{d}{dx}W_1\right]$$
$$+iB_0^\pm(0)\tfrac{\epsilon_{12}^0(0)}{d^0} D(\cdot - x_h) + P.V.\left(B_0^\pm \tfrac{1}{\pi}\tfrac{\rho_0'(\cdot - x_h)}{\rho_0(\cdot - x_h)}\tfrac{\epsilon_{12}^0(x)}{d^0}W_1\right),$$

where D *is the Dirac mass and the principal value is defined by* $< P.V.(\alpha), \phi >= \lim_{\tau\to 0^+}\int_{x\notin[x_h-\tau, x_h+\tau]}\alpha(x)\phi(x)dx$.

Of course the right hand side terms on the first line converge also in L^p_{loc} as in the previous proposition, and the point wise limit holds away from the hybrid singularity x_h.

3.4 Limit heating term

The heating was defined in the formula (9). For $a < b$ in $B(x_h, \Lambda_8) \cap \mathbb{R}^*$ and $\nu \in (-\Lambda_8, \Lambda_8)$ we note

$$Q^\nu(a,b) = \Im \int_a^b (\epsilon^\nu(x)\mathbf{E}^\nu(x)) \cdot \overline{\mathbf{E}^\nu(x)}dx = -\Im\left(W^\nu(b)\overline{E_2^\nu(b)}\right) + \Im\left(W^\nu(a)\overline{E_2^\nu(a)}\right) \tag{67}$$

where $(E_1^\nu, E_2^\nu, W^\nu)$ is the regularized solution system (4) on $B(x_h, \Lambda_8)$. The limit value for vanishing ν is the resonant heating and can be characterized.

Theorem 2 (Resonant heating)**.** *Suppose* x_h *is a local hybrid resonance, and assumptions 1, 4 and 3 hold. Consider for all* $|\nu| \le \Lambda_8$ *the stretching function* ρ_ν *described in Lemma 9, and* $C_\nu(x^*)$ *which goes to a finite limit when* ν *goes to 0.*
One can pass to the limit for $(a,b) \in B(x_h, \Lambda_8)^2 \cap \mathbb{R}^*$ *as follows.*

- *i) For* $(a - x_h)(b - x_h) > 0$ *then* $\lim_{\nu\to 0} Q^\nu(a,b) = 0$.
- *ii) If* $a < x_h < b$, *then*

$$\lim_{\nu\to 0^+} Q^\nu(a,b) = \frac{1}{\pi}|B_0(x_h)|^2 \left|\epsilon_{12}^0(0)\right|^2 sign\left(\partial_\nu\epsilon_{11}^0\right). \tag{68}$$

Proof. There are many possibilities to compute this limit value from the previous representation formulas.

First case) This is a direct consequence on the fact that a and b are on the same side of x_h, that the solution is smooth away from x_h and that $\Im\left[\left(\epsilon^0(x)\mathbf{X}\right)\cdot\overline{\mathbf{X}}\right] = 0$ for all complex vector $\mathbf{X}$ since $\epsilon^0(x)$ is hermitian.

Second case) It is sufficient to decompose the electric field in two parts, a regular part $\mathbf{R}^\nu$ and a singular part $\mathbf{S}^\nu$, such that $\mathbf{R}^\nu$ is regular enough, for example bounded in $L^p(-\Lambda_8, \Lambda_8)$ uniformly with respect to ν and with a point wise limit for $x \ne x_h$ and the singular part $\mathbf{S}^\nu$ is such that $(x - X_\nu)\mathbf{S}^\nu$ is bounded in $L^\infty(-\Lambda_8, \Lambda_8)$. It is then clear that one can choose

$$\mathbf{S}^\nu(x) = \left(B_\nu \frac{1}{\pi}\frac{\rho_\nu'(\cdot - X_\nu)}{\rho_\nu(\cdot - X_\nu)}\frac{\epsilon_{12}^\nu}{d^\nu}W_1^\nu, 0\right).$$

One has thus

$$Q(a,b)=\Im\int_a^b\Big[(\epsilon^\nu(x)\mathbf{R}^\nu(x))\cdot\overline{\mathbf{R}^\nu(x)}dx+(\epsilon^\nu(x)\mathbf{S}^\nu(x))\cdot\overline{\mathbf{R}^\nu(x)}dx+$$

$$(\epsilon^\nu(x)\mathbf{R}^\nu(x))\cdot\overline{\mathbf{S}^\nu(x)}dx+(\epsilon^\nu(x)\mathbf{S}^\nu(x))\cdot\overline{\mathbf{S}^\nu(x)}\Big]\,dx$$

One checks that $\Im\int_a^b(\epsilon^\nu(x)\mathbf{S}^\nu(x))\cdot\overline{\mathbf{R}^\nu(x)}dx=\Im\int_a^b\epsilon^\nu(x)_{11}\mathbf{S}^\nu_1(x)\overline{\mathbf{R}^\nu_1(x)}dx$ and that

$$\Im\int_a^b(\epsilon^\nu(x)\mathbf{R}^\nu(x))\cdot\overline{\mathbf{S}^\nu(x)}dx=\Im\int_a^b(\mathbf{R}^\nu(x))\cdot\overline{\epsilon^\nu(x)\mathbf{S}^\nu(x)}dx=\Im\int_a^b(\mathbf{R}^\nu_{11}(x))\cdot\overline{\epsilon^\nu_{11}(x)\mathbf{S}^\nu_1(x)}dx.$$

These two terms converge to 0 using the dominated convergence theorem as $\nu\to0$ thanks to the regularity of $(x-X_\nu)S^\nu_1(x)$ and of $(x-\overline{X}_\nu)S^\nu_1(x)$ in the neighborhood of x_h. Far better is, of course, the term $\Im\int_a^b(\epsilon^\nu(x)\mathbf{R}^\nu(x))\cdot\overline{\mathbf{R}^\nu(x)}dx$, which limit is also 0.

Let us study the remaining term which is

$$J=\int_a^b(\epsilon^\nu(x)\mathbf{S}^\nu(x))\cdot\overline{\mathbf{S}^\nu(x)}=\int_a^b\epsilon^\nu_{11}(x)\left|B_\nu(x)\frac{1}{\pi}\frac{\rho'_\nu(x-X_\nu)}{\rho_\nu(x-X_\nu)}\frac{\epsilon^\nu_{12}(x)}{d^\nu(x)}W^\nu_1(x)\right|^2dx.$$

One denotes by $z_\nu(x)=\frac{\epsilon^\nu_{11}(x)}{x-X_\nu}$ and by $K_\nu(x)=(x-X_\nu)B_\nu(x)\frac{1}{\pi}\frac{\rho'_\nu(x-X_\nu)}{\rho_\nu(x-X_\nu)}\frac{\epsilon^\nu_{12}(x)}{d^\nu(x)}W^\nu_1(x)$. These two terms are regular, hence

$$J=\int_a^b z_\nu(x)|K_\nu(x)|^2\frac{x-X_\nu}{|x-X_\nu|^2}dx.$$

One introduces $J_0=\int_a^b z_\nu(X_\nu)|K_\nu(X_\nu)|^2\frac{x-X_\nu}{|x-X_\nu|^2}dx$. One checks that there exists a regular function $H_\nu(x)$ such that $z_\nu(x)|K_\nu(x)|^2-z_\nu(X_\nu)|K_\nu(X_\nu)|^2=(x-X_\nu)H_\nu(x)$. We thus deduce that

$$J-J_0=\int_a^b H_\nu(x)\frac{(x-X_\nu)^2}{|x-X_\nu|^2}dx.$$

As the limit exists in L^1, we use the dominated convergence theorem to deduce that the limit of $J-J_0$ is equal to $\int_a^b H_0(x)dx$. It is real hence its imaginary part is zero.

Consider now J_0. Use $\frac{x-X_\nu}{|x-X_\nu|^2}=\frac{x-\Re X_\nu-i\Im X_\nu}{(x-\Re X_\nu)^2+(\Im X_\nu)^2}$. The change of variable $t=\frac{x-\Re X_\nu}{\Im X_\nu}$ yields

$$J_0=z_\nu(X_\nu)|K_\nu(X_\nu)|^2\int_{\frac{a-\Re X_\nu}{\Im X_\nu}}^{\frac{b-\Re X_\nu}{\Im X_\nu}}\frac{t-i}{t^2+1}dt.$$

We thus notice that

$$J_0=z_\nu(X_\nu)|K_\nu(X_\nu)|^2[A^\nu-iB^\nu]$$

where A^ν is given thanks to $\frac{1}{2}\ln(1+t^2)$ and $B^\nu=\tan^{-1}(\frac{b-\Re X_\nu}{\Im X_\nu})-\tan^{-1}(\frac{a-\Re X_\nu}{\Im X_\nu})$. One observes that the limit of $z_\nu(X_\nu))$ when $\nu\to0$ is real, equal to $\partial_x\epsilon^0_{11}(x_h)$ (because the limit of z_ν is equal to $\frac{\epsilon^0_{11}(x)}{x-x_h}$), and the limit of $\frac{\epsilon^0_{11}(x)}{x-x_h}$ when $x\to x_h$ is $\partial_x\epsilon^0_{11}(x_h)$, real. One has

$$A^\nu=\frac{1}{2}\ln\frac{(b-\Re X_\nu)^2+(\Im X_\nu)^2}{(a-\Re X_\nu)^2+(\Im X_\nu)^2}\to\frac{1}{2}\ln\frac{(b-x_h)^2}{(a-x_h)^2}\in\mathbb{R}.$$

When $a<x_h<b$, the limit of $\frac{a-\Re X_\nu}{\Im X_\nu}$ is $-\infty$ when $\Im X_\nu$ goes to 0 by positive values, and the limit of $\frac{a-\Re X_\nu}{\Im X_\nu}$ is $+\infty$ when $\Im X_\nu$ goes to 0 by negative values. We thus deduce

$$B^\nu\longrightarrow\pi\mathrm{sign}(\Im X_\nu).$$

Hence one deduces that the limit of Im J_0 is $-\pi\partial_x\epsilon^0_{11}(x_h)|B^\pm_0(0)W_1(0)\frac{\epsilon^0_{12}(x_h)}{\pi d^0(x_h)}|^2$. Using $d^0(x_h)=(\epsilon^0_{12}(x_h))^2$, one obtains that

$$\Im J\longrightarrow-\mathrm{sign}\left(\Im X_\nu\frac{\partial_x\epsilon^0_{11}(x_h)}{\pi|\epsilon^0_{12}(x_h)|^2}|B^\pm_0(0)W_1(0)|^2\right).$$

Using finally that $|W_1(0)|^2=\frac{\left|\epsilon^0_{12}(x_h)\right|^2}{\left|\partial_x\epsilon^0_{11}(x_h)\right|}$, one gets $\Im J\longrightarrow-\mathrm{sign}\left(\Im X_\nu|B^\pm_0(0)|^2\right)$ as ν tends to zero.

The proof is ended.

□

An interesting and somewhat counter intuitive corollary is the following.

Corollary 1. *Consider any dielectric tensor satisfying (4-24) but not necessarily with the exact form (3). Assume ϵ^0 is a smooth hermitian matrix, and assume $x_h = 0$ is an hybrid resonance. Define the local dissipation tensor*

$$\mathbb{D} = -i\,(\partial_\nu \epsilon)^0\,(0) = \begin{pmatrix} d_1 & d_2 \\ -d_2 & d_1 \end{pmatrix}$$

provided the first coefficient is positive $d_1 > 0$.
Then the value of the resonant heating is independent of $\mathbb{D}$. The same for the pointwise limits of the electric, of magnetic fields and of the numerical vale of the resonant heating.

Proof. It is sufficient to realize that the major assumption that was made, that is (20-13) is independent of the exact value of $d_1 > 0$, and that the other coefficients in $\mathbb{D}$ do not show up in the proof of the previous theorem. This is also clear in view of the value of the resonant heating (68). □

An interesting consequence is that can replace the initial dielectric tensor (3) with the linear approximation $\epsilon^0 + i\nu\mathbb{D}$ with $\mathbb{D}$ defined in (6). If one is interested only in the limit value, which is a reasonable assumption for fusion plasmas where $\nu \approx 10^{-7}$ may be encountered, this is a valid assumption.

4 Numerical illustrations and analytic solutions

4.1 Numerics

We show numerical results which illustrate some of the theoretical results. These results have been computed with the Matlab solver developed by Lise-Marie Imbert-Gérard during her PhD [14]. The code solves the system under the form (11) with the ode23 subroutine of Matlab. This routine is adapted to stiff problems.
We use three different dielectric tensors that all have the same limit for $\nu = 0$. The three dielectric tensors are denoted by three different "dissipation" tensors. The first one denoted as $\mathbb{D}_0$ means that we use the exact formula (3). The second one $\mathbb{D}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ means that we use the linear approximation $\epsilon^0 + i\nu\mathbb{D}_1$. One can notice that $\mathbb{D}_1 = \mathbb{I}$ corresponds to the limiting absorption principle that was studied in [11]. The last $\mathbb{D}_2 = \mathbb{D}_2^* = \begin{pmatrix} 1 & -2i \\ 2i & 5 \end{pmatrix}$ is a hermitian and positive matrix as is required on the physical basis (see (6) for example). These three methods satisfy the main hypothesis of corollary 1 that is $(\mathbb{D}_i)_{11} > 0$ for $i = 0, 1, 2$. Once the dissipation tensor is set, the matrix $\mathbf{M}_\nu(x, \theta)$ is constructed, and the problem solved numerically in the interval $[-5, 5]$ with Cauchy data $E_2(5) = 1$ and $W(5) = 0$. The other parameters are chosen so that $x_h = 0$ is an isolated hybrid resonance, that $x < 0$ is a propagating and that < 0 is non propagative. This is clearly visible in table 1. In figure 3 we display the real part of E_x^ν, E_y^ν, W^ν for four different values of ν, and for the linear models with $\mathbb{D}_1$ and $\mathbb{D}_2$. We observe that even if the results for $\nu = 0.5$ are very different, the functions converge numerically to the same limit for $\nu = 0.5 \times 10^{-3}$. The next figure 4 confirms this analysis for the imaginary of the electromagnetic field. The final figure 5 shows the same results but computed with the "real" dissipation tensor. One clearly observes the same limit as in figures 3 and 4, perhaps with smaller oscillations for large value ν.

ν	.5	$.5 \times 10^{-1}$	$.5 \times 10^{-2}$	$.5 \times 10^{-3}$	$.5 \times 10^{-4}$	$.5 \times 10^{-5}$
$\mathbb{D}_0$	9898	19484	18724	18623	18604	18590
$\mathbb{D}_1$	1534268	36976	20272	18771	18613	18588
$\mathbb{D}_2$	109836517	89603	22761	19001	18637	18591

Table 1: Values of the heating calculated with formula (67) with different ν and different dissipation tensors. As predicted by the theory, the limit resonant heating in the last column is independent of the dissipation tensor.

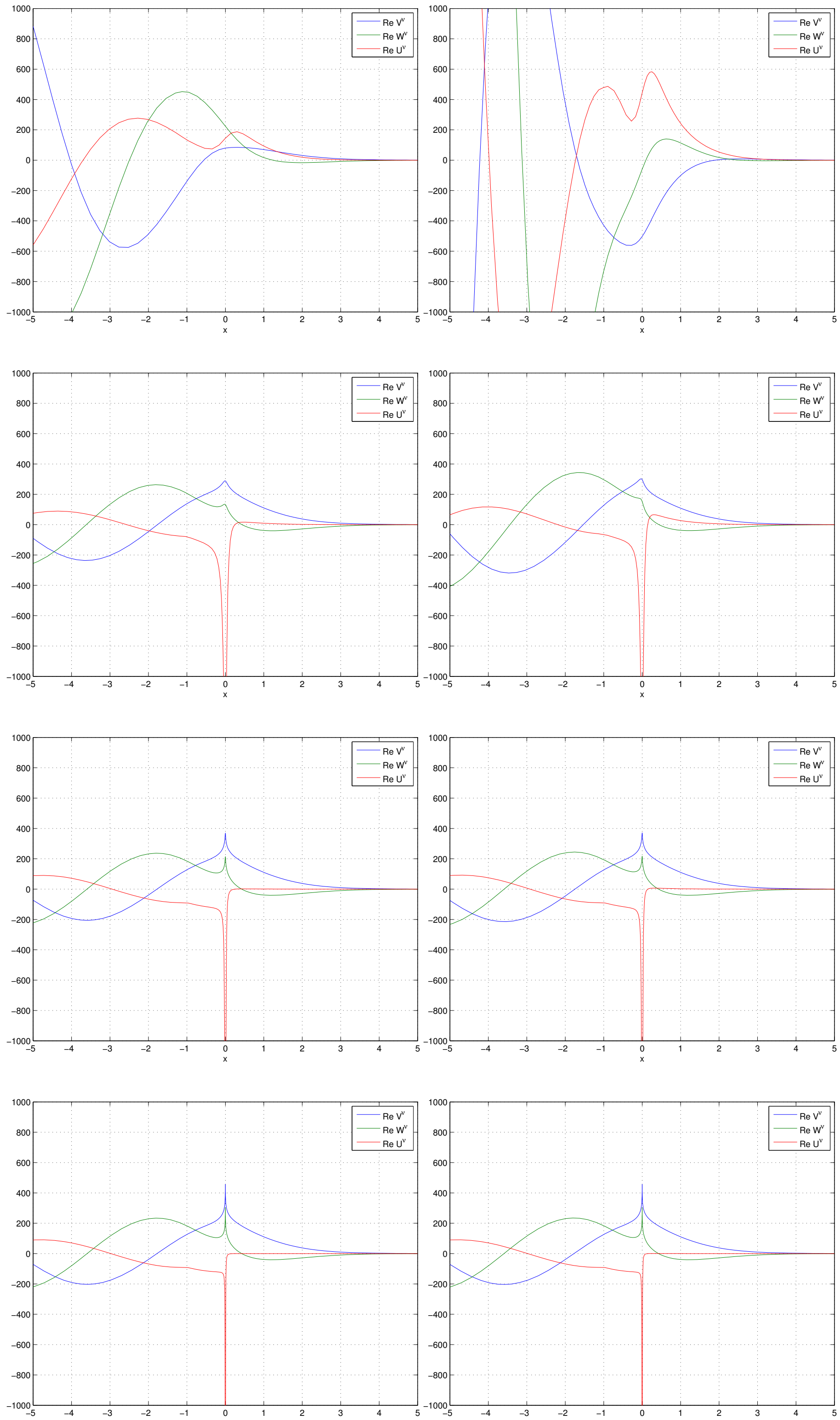

Figure 3: Real part, $\mathbb{D}_1$ on the left, $\mathbb{D}_2$ on the right, from top to bottom: $\nu = .5, .5 \cdot 10^{-1}, .5 \cdot 10^{-2}, .5 \cdot 10^{-3}$. One observes the convergence to the same limit.

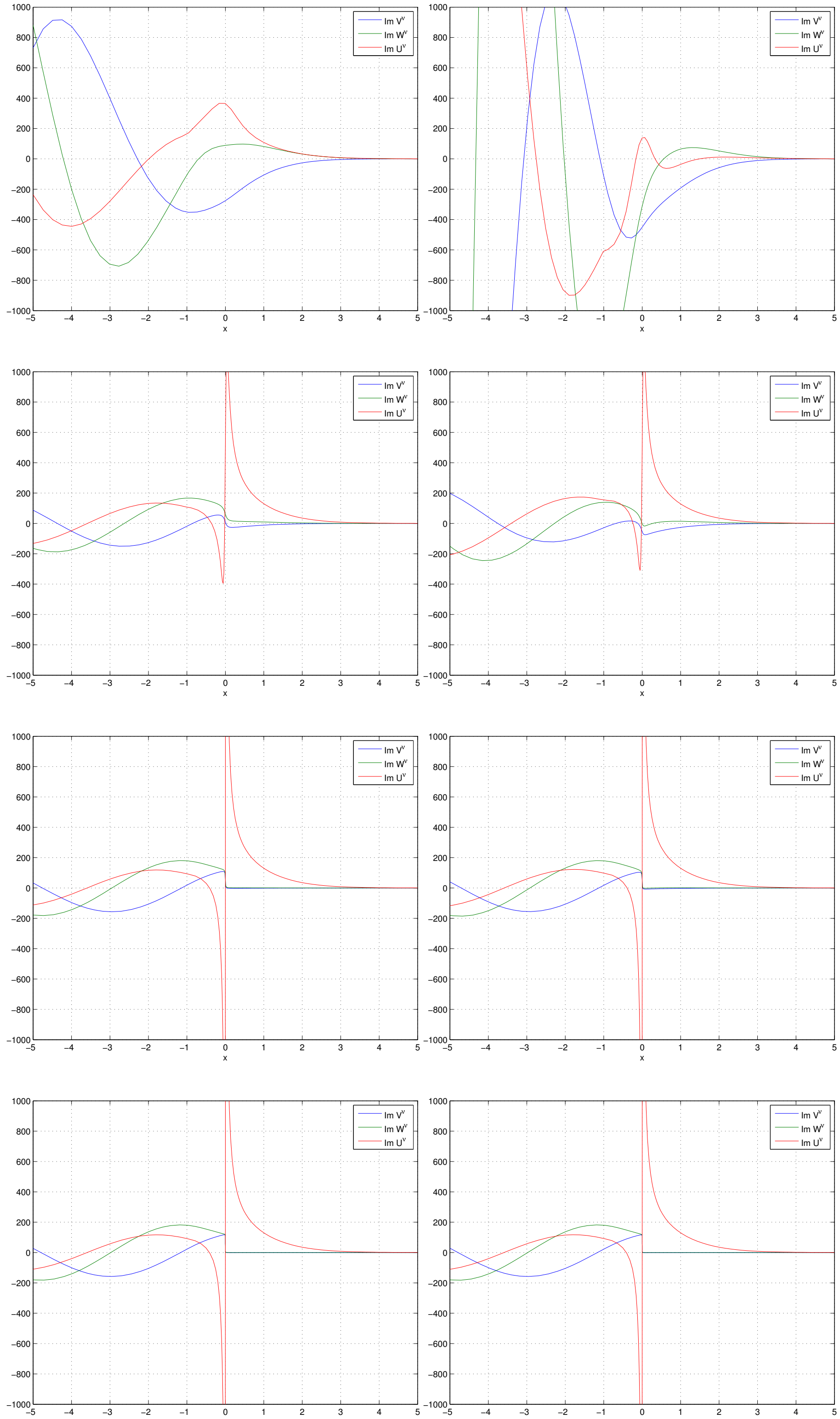

Figure 4: Imaginary part, $\mathbb{D}_1$ on the left, $\mathbb{D}_2$ on the right, from top to bottom: $\nu = .5, .5 \cdot 10^{-1}, .5 \cdot 10^{-2}, .5 \cdot 10^{-3}$. One observes the convergence to the same limit.

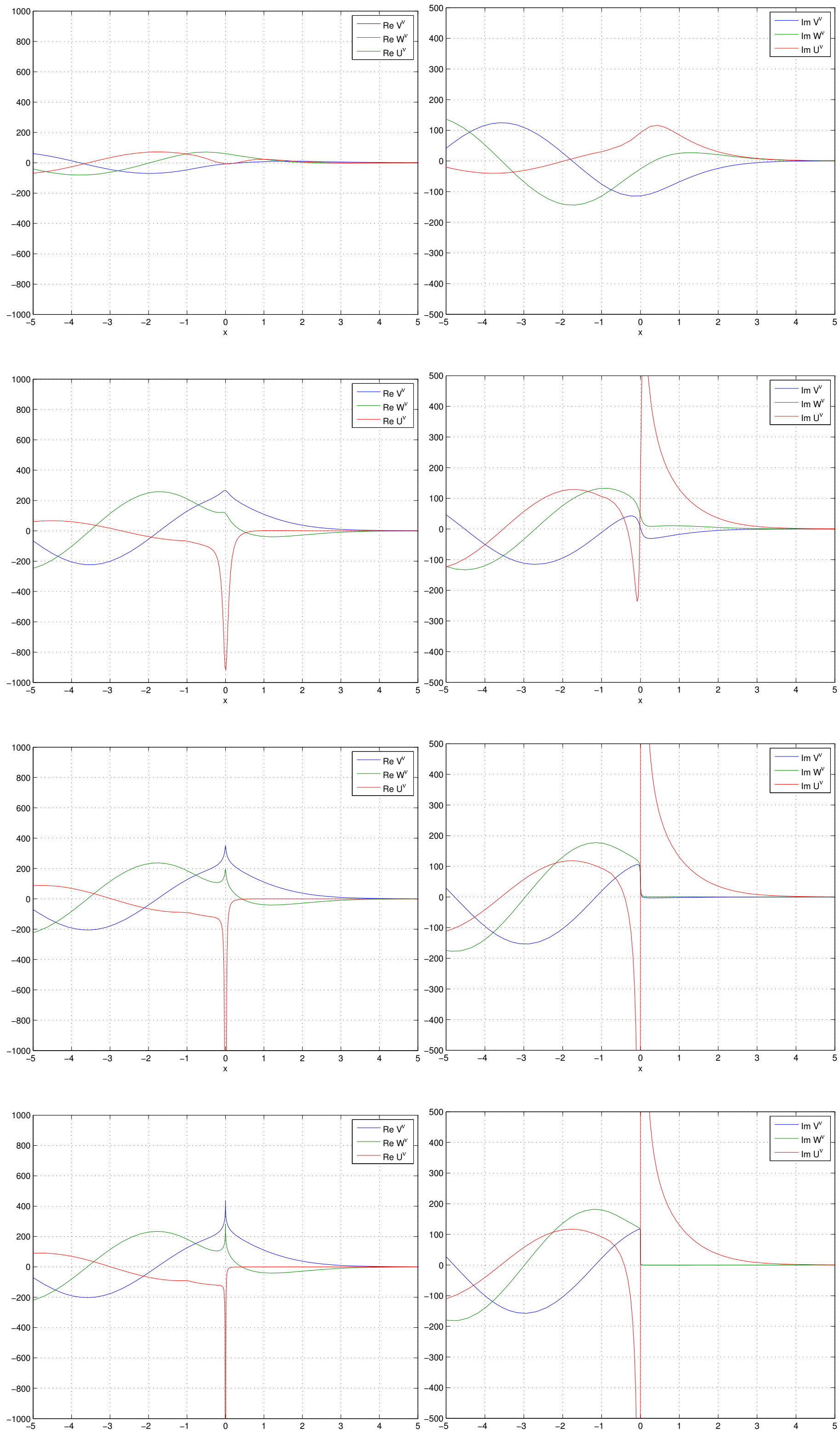

Figure 5: $\mathbb{D}_0$, real part on the left, imaginary part on the right, from top to bottom: $\nu = .5, .5 \cdot 10^{-1}, .5 \cdot 10^{-2}, .5 \cdot 10^{-3}$. For the smallest value of ν, the results are almost the same as in figures 3 and 4 computed with the linear models $\mathbb{D}_1$ and $\mathbb{D}_2$.

4.2 Two analytic solutions

The first case of interest (for which $s = 0$) is $\rho_\nu(x) = \frac{x}{1+l_0^{-1}x}$, l_0 being an arbitrary length. Choose arbitrarily X_ν, (such as $X_\nu = -i\nu$). Consider:

$$c_\nu(x) = -\frac{\rho'_\nu}{\rho_\nu}(x - X_\nu) = -\frac{1}{(x - X_\nu)(1 + l_0^{-1}(x - X_\nu))} a_\nu(x) = -c_\nu(x)$$

$$b_\nu(x) = -c_\nu(x) + \frac{1}{4}\frac{\lambda^2}{(1 + l_0^{-1}(x - X_\nu))^2}$$

where λ, l_0 are arbitrary constants.

Lemma 17. *The family of solutions W^ν and E_2^ν of (11), with the above coefficients of M_ν defined in (12) is given by*

$$\begin{array}{ll} W^\nu(x) & = (-c_\nu(x))^{\frac{1}{2}}(AU_\nu(x - X_\nu) + BV_\nu(x - X_\nu)), \\ E_2^\nu(x) & = A[c_\nu(x)^{-1}(((-c_\nu(x))^{\frac{1}{2}}U_\nu(x - X_\nu))' + a_\nu(x)(-c_\nu(x))^{\frac{1}{2}}V_\nu(x - X_\nu))] \\ & +B[c_\nu(x)^{-1}(((-c_\nu(x))^{\frac{1}{2}}V_\nu(x - X_\nu))' + a_\nu(x)(-c_\nu(x))^{\frac{1}{2}}V_\nu(x - X_\nu))] \end{array}$$

Proof. One checks that $(\rho_\nu)'(x) = \frac{K}{(1+l_0^{-1}(x-X_\nu))^2}$, hence $(\rho'_\nu(x))^{-\frac{1}{2}}$ is a polynomial of degree 1, hence $s = 0$. We construct coefficients a_ν, b_ν, c_ν according to the form of the solution. Introduce $f_\nu = (\frac{-c_\nu}{\rho'_\nu})^{\frac{1}{2}}$.One checks that the equality $(E_2^\nu)' = a_\nu E_2^\nu + b_\nu W^\nu$ yields

$$(\frac{f'_\nu}{c_\nu})' + (\frac{a_\nu}{c_\nu})' f_\nu + \frac{\rho'_\nu}{4\rho_\nu^2 f_\nu}(1 + \lambda^2\rho_\nu) = (\frac{a_\nu^2}{c_\nu} + b_\nu)f_\nu,$$

and the choice of a_ν, b_ν, c_ν satisfy this equality. Note in particular that the relation $c_\nu = -\frac{\rho'_\nu}{\rho_\nu}$ yields the identity $(\frac{f'}{c_\nu})' + \frac{\rho'_\nu}{4\rho_\nu^2 f} = 0$. □

The second case is the case where the solution is singular, namely

$$W^\nu = AJ_0(\rho_\nu(x - X_\nu)) + BY_0(\rho_\nu(x - X_\nu))$$

(we consider the solution as a combination of Bessel functions),
and $\rho_\nu(x) = x(1 + l_0^{-2}x^2)$. One may choose in this case $c_\nu(x) = K\frac{\rho'_\nu(x+X_\nu)}{\rho_\nu(x+X_\nu)}$, and $b_\nu(x) = c_\nu(x)(\frac{a_\nu(x)}{c_\nu(x)})^2 - (\frac{a_\nu(x)}{c_\nu(x)})' - \frac{1}{K}\rho_\nu\rho'_\nu(x + X_\nu)$, that is

$$c_\nu(x) = K[\frac{1}{x - X_\nu} + \frac{2(x - X_\nu)}{l_0^2 + (x - X_\nu)^2}], a_\nu(x) = C_0 c_\nu(x),$$

and

$$b_\nu(x) = C_0^2 c_\nu(x) - \frac{1}{K}\rho'_\nu(x - X_\nu)\rho_\nu(x - X_\nu) = C_0^2 c_\nu(x) - \frac{1}{K}(x - X_\nu)(1 + l_0^{-2}(x - X_\nu)^2)(1 + 3l_0^{-2}(x - X_\nu)^2).$$

With a choice of the arbitrary constants K, C_0 and l_0, and with a convenient choice of X_ν, one has

Lemma 18. *All solutions of*

$$\begin{pmatrix} E_2^\nu \\ W^\nu \end{pmatrix}' = \begin{pmatrix} C_0 c_\nu & C_0^2 c_\nu - \frac{1}{K}\rho_\nu\rho'_\nu(x - X_\nu) \\ c_\nu(x) & -C_0 c_\nu(x) \end{pmatrix} \begin{pmatrix} E_2^\nu \\ W^\nu \end{pmatrix}$$

with $c_\nu(x) = K\frac{\rho'_\nu}{\rho_\nu}(x - X_\nu)$, $\rho_\nu(z) = z(1 + l_0^{-2}z^2)$ are

$$\begin{array}{ll} W^\nu(x) & = AJ_0(\rho_\nu(x)) + BY_0(\rho_\nu(x)), \\ E_2^\nu(x) & = K^{-1}\rho_\nu(x - X_\nu)(AJ'_0(\rho_\nu((x - X_\nu)) + BY'_0(\rho_\nu((x - X_\nu))) + C_0(AJ_0(\rho_\nu(x)) + BY_0(\rho_\nu(x))). \end{array}$$

4.3 A short conclusion

All our numerical results confirm the singular solution predicted by the theoretical resuls. It allows us now to have a comprehensive understanding of the local structure of the hybrid singular solutions. With a careful expression of $C_\nu(x_*)$ through $E_2^\nu(x_*)$ and $W^\nu(x_*)$ and using the methods developed in [13], one can, as in [11], connect this local structure near the singularity to the condition at infinity.

A Proof of Lemma 6

Proof. This function is a Laurent series by definition. Hence we will show that there exists $\Lambda_6 > 0$ such that $R_\nu(x)$ is uniformly bounded, i.e. bounded for $x \in B(x_h, \Lambda_6)$ uniformly for $\nu \in B(0, \Lambda_6)$. And such a Λ_6 can be replaced by Λ_3 in case this one was smaller. This will prove that it is an analytic function with a uniform radius of convergence.
The function R_ν is the product of $x - X_\nu$ times a contribution which is the sum of four terms. The first term is $a_\nu^2 + b_\nu c_\nu = -D_\nu = c_\nu + \theta^2$, which is at most $O\left((\epsilon_{11}^\nu)^{-1}\right)$. The product by $x - X_\nu$ is bounded and admits a finite limit for $x \to X_\nu$. The second term depends on $\frac{a_\nu}{c_\nu}$ which is locally bounded, so does not yield any difficulty either: its contribution to R_ν is bounded. The third term can be written as

$$\sqrt{-c_\nu}\left(\frac{1}{\sqrt{-c_\nu}}\right)'' = -\frac{1}{2}\left(\frac{c_\nu'}{c_\nu}\right)' + \frac{1}{4}\left(\frac{c_\nu'}{c_\nu}\right)^2.$$

Considering the bounded function $k_\nu(x) = c_\nu(x)(x - X_\nu)$, one has

$$\begin{aligned}\sqrt{-c_\nu}\left(\frac{1}{\sqrt{-c_\nu}}\right)'' &= -\frac{1}{2}\left(\frac{k_\nu'}{k_\nu}\right)' - \frac{1}{2}\frac{1}{(\cdot - X_\nu)^2} + \frac{1}{4}\left(\frac{k_\nu'}{k_\nu} - \frac{1}{\cdot - X_\nu}\right)^2, \\ &= -\frac{1}{4}\frac{1}{(\cdot - X_\nu)^2} - \frac{1}{2}\frac{k_\nu'}{k_\nu}\frac{1}{\cdot - X_\nu} - \frac{1}{2}\left(\frac{k_\nu'}{k_\nu}\right)' + \frac{1}{4}\left(\frac{k_\nu'}{k_\nu}\right)^2.\end{aligned}$$

It yields that $\sqrt{-c_\nu}\left(\frac{1}{\sqrt{-c_\nu}}\right)'' = -\frac{1}{4(\cdot - X_\nu)^2} + \frac{\tau_\nu(x)}{\cdot - X_\nu}$ where the function τ_ν is uniformly bounded in the sense that $|\tau_\nu(x)| \le C$, i.e. bounded for $x \in B(x_h, \Lambda_6)$ uniformly for $\nu \in B(0, \Lambda_6)$. It shows that the most singular part of the two last terms cancel in the definition of R_ν. The exact value of $R_\nu(X_\nu)$ may be computed as follows using d^ν defined in (13). From $\frac{a_\nu}{c_\nu} = i\theta\frac{\epsilon_{12}^\nu}{d^\nu}$, one has

$$\left(\frac{a_\nu}{c_\nu}\right)'(X_\nu) = -i\theta\frac{(\epsilon_{12}^\nu)'}{(\epsilon_{12}^\nu)^2}(X_\nu).$$

Moreover, noticing that $k_\nu = -d^\nu\frac{x - X_\nu}{\epsilon_{11}^\nu}$, and using the l'Hopital rule with $\epsilon_{11}^\nu(x) = (\epsilon_{11}^\nu)'(X_\nu)(x - X_\nu) + \frac{1}{2}((\epsilon_{11}^\nu)''(X_\nu) + o(1))(x - X_\nu)^2$,

$$k_\nu(X_\nu) = -\frac{(\epsilon_{12}^\nu)^2}{(\epsilon_{11}^\nu)'}(X_\nu) \text{ and } \frac{k_\nu'}{k_\nu}(X_\nu) = 2\frac{(\epsilon_{12}^\nu)'}{\epsilon_{12}^\nu}(X_\nu) - \frac{1}{2}\frac{(\epsilon_{11}^\nu)''}{(\epsilon_{11}^\nu)'}(X_\nu).$$

It completes the proof since

$$R_\nu(X_\nu) = k_\nu(X_\nu)(1 - (\frac{a_\nu}{c_\nu})')(X_\nu) - \frac{1}{2}\frac{k_\nu'}{k_\nu}(X_\nu) = \left[\frac{i\theta(\epsilon_{12}^\nu)' - (\epsilon_{12}^\nu)^2}{(\epsilon_{11}^\nu)'} - \frac{(\epsilon_{12}^\nu)'}{\epsilon_{12}^\nu} + \frac{1}{4}\frac{(\epsilon_{11}^\nu)''}{(\epsilon_{11}^\nu)'}\right](X_\nu).$$

□

B Multi species

Our aim here is to show that our method can be used to study the resonances in the case of multi-species.
For simplicity, let us consider the case of the electron-ion system. After elementary calculations based on the fundamental equations that one may find in [17, 18], one obtains for example the generalization of (26)

$$\mathbf{M}_\nu(x,\theta) = \frac{1}{D_\nu(x)}\begin{pmatrix} A_\nu(x,\theta) & B_\nu(x,\theta) \\ C_\nu(x) & -A_\nu(x,\theta)\end{pmatrix}. \tag{69}$$

Here D_ν satisfies

- $D_0(x)$ vanishes at points x_{hi} such that $A_0(x_{hi}), B_0(x_{hi}), C_0(x_{h_i})$ do not vanish and $(A_0^2 + B_0C_0)(x_{hi}) = 0$
- $i\partial_\nu D_\nu(x_{hi})|_{\nu=0} \in \mathbb{R} \neq 0$

which are the unique conditions needed for the theoretical set-up of our main result. This coefficient D_ν is analogous to ϵ_{11}^ν.
For simplicity we describe a two species case, one is electron and one is ions and just give the main ideas. To obtain (69), one can start from the Maxwell equations

$$\nabla \wedge E = i\omega B, \quad c^2\nabla \wedge B = -i\omega E + 4\pi j$$

with the so-called fluid equations for the current j (the external magnetic field B_0 is $B_0(x), b$, b unit vector) :

$$(-i\omega + \nu)j = \frac{eB_0(x)}{m_e} j \wedge b - \frac{\omega_p^2}{4\pi}(E + B_0(x)v \wedge b)$$

where the ion velocity v is given by $(-i\omega + \nu)v = \frac{Ze}{m_i}(E + B_0(x)v \wedge b)$. The physical relevant parameters are $\omega_c^i = \frac{ZeB_0}{m_i}, \omega_c^e = \frac{eB_0}{m_e}$, $p = \frac{\omega_c^i}{\omega_c^e} = \frac{Zm_e}{m_i}$.
The traditional approach is to calculate v in terms of E (which introduces a singularity at x such that $\omega_c^i(x) = \omega$), then to deduce j in terms of E (which introduces a singularity at x such that $\omega_c^e(x) = \omega$) and finally to replace j in Maxwell equations. This structure is not the natural one in the sense that the dielectric tensor that one obtains has singularities when ν goes to 0. Instead of using this approach, we seek the TE field $(E_2(x), B_3(x) := W(x))e^{i\theta y}$, solution of

$$\begin{cases} c^2W' = i\omega E_2 - 4\pi j_2 \\ E_2' = i\theta E_1 + i\omega W \end{cases}$$

We have thus to obtain (j_2, E_1). It can be checked they are solution of

$$\begin{cases} (-i\omega + \nu)j_2 + \omega_c^e j_1 - \frac{\omega_p^2}{4\pi}(1+p)B_0v_1 & = -\frac{\omega_p^2}{4\pi}(1+p)E_2 \\ -\omega_c^e j_2 + (-i\omega + \nu)j_1 + \frac{\omega_p^2}{4\pi}(1+p)B_0v_2 + \frac{\omega_p^2}{4\pi}(1+p)E_1 & = 0 \\ (-i\omega + \nu)v_1 - \omega_c^i v_2 - \frac{Ze}{m_i}E_1 & = 0 \\ \omega_c^i v_2 + (-i\omega + \nu)v_1 & = \frac{Ze}{m_i}E_2 \\ 4\pi j_1 - i\omega E_1 & = c^2 i\theta W \end{cases}$$

Treating this system globally leads to a determinant D_ν:

$$D_\nu(x) = i\omega((\omega_c^i)^2 + (i\omega - \nu)^2)((\omega_c^e)^2 - (i\omega - \nu)^2) + (i\omega - \nu)((i\omega - \nu)^2 + \omega_c^i\omega_c^e)\omega_p^2(1+p).$$

One has $\frac{D_0(x)}{i\omega} = (p^2(\omega_c^e)^2 - \omega^2)((\omega_c^e)^2 + \omega^2) + (1+p)\omega_p^2(p(\omega_c^e)^2 - \omega^2)$. In this case, for all x, there exist a unique value of ω for which there is an hybrid resonance such that $D_0(x) = 0$. If one denotes $X = (\frac{\omega}{\omega_c^e})^2$, X is solution of $-X^2 + (p+1)(p - 1 - \frac{\omega_p^2}{\omega_c^2})X + p(p + (1+p)\frac{\omega_p^2}{\omega_c^2} = 0$ hence $X = X_*(x)$, equivalent to $\omega = \omega_c^e(x)(X_*(x))^{\frac{1}{2}}$. One is thus left with finding x_{hi} such that $\omega = \omega_c^e(x_{hi})(X_*(x_{hi}))^{\frac{1}{2}}$. Other formulas can be sought for.

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