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On the algebraic structure of rational discrete dynamical systems

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Abstract

We show how singularities shape the evolution of rational discrete dynamical systems. The stabilisation of the form of the iterates suggests a description providing among other things generalised Hirota form, exact evaluation of the algebraic entropy as well as remarkable polynomial factorisation properties. We illustrate the phenomenon explicitly with examples covering a wide range of models.

1 Introduction

Many of the algebraic aspects of rational discrete systems have already been investigated, especially in view of their integrability. A number of these aspects are extensions to the discrete case of features of continuous systems, but the rationality of the evolution made the algebro-geometric approach inescapable. It is indeed at the basis of any classification attempt, symmetry or multidimensional consistency analysis, as well as complexity measure via algebraic entropy. See for example the numerous results exposed in the series of SIDE meetings [1].

The nature of the evolution is also responsible for one recurrent fact: looking at a finite number of steps of the discrete evolution yields informations which are in essence of asymptotic nature, like integrability, hierarchies, or value of the entropy.

Motivated by the original works [2, 3] and comforted by the more modern approaches [4, 5] on continuous systems, the importance of the singularity structure was recognised very early [6]. The use of the apparatus available in two dimensions, and notably the theory of intersection of curves on algebraic surfaces, then lead to powerful theorems, in particular on discrete Painlevé equations [7] and “QRT” maps [8–10].

Direct computations of discrete evolutions have also been performed, especially to detect integrability, endeavouring to reduce the size of the calculations. For instance, looking at the images of a straight line in the space of initial conditions, inspired by the geometrical idea of [11], allows to produce an exact sequence of degrees of the iterates, and in turn to

evaluate exactly the algebraic entropy [12, 13]. Restricting the evolution to integers lightens even more the calculations. Looking then at the growth of their height [14, 15], gives an approximate but efficient way to evaluate the algebraic entropy. Going even further one may perform the calculations on finite fields, analyse various statistical properties [16, 17], and eventually detect integrability.

We take here an opposite attitude, and choose an ingenious option: we do evaluate exactly the first steps of the evolutions, and analyse their structure, and especially their factorisation properties.

The main outcome is that the form of the iterates, shaped by the singularities, suggests changes of description which automatically provide:

- Exact calculations of the algebraic entropy, and proofs of its algebraicity (see [18–20] for a general point of view on this question)
- Generalisations of the discrete Hirota-Sato form and τ functions [21–27], reducing to the standard quadratic form in specific integrable cases
- Various polynomial and integer factorisation properties similar the Laurent property [28–33]

The plan of the paper is the following:

In section 2 we briefly recall basic algebro-geometric notions which will be at the core of the phenomenon we exhibit.

In section 3 we describe explicitly a number of models in various dimensions:

- 1 - An algebraically integrable two dimensional map in the QRT family: McMillan
- 2 - A discrete Painlevé equation dP_{II} , i.e. a non autonomous extension of the previous
- 3 - Another discrete Painlevé equation: qP_{VI}
- 4 - A non integrable non confining map in two dimensions: Jaeger
- 5 - A confining non integrable map in two dimensions: JNH-CMV
- 6 - Another confining non integrable map in two dimensions
- 7 - A three dimensional algebraically integrable map: N=3 periodic Volterra
- 8 - A linearisable map
- 9 - An unruly model in three dimensions
- 10 - A recurrence of order 2 on functional space, Delay Differential equation
- 11 - An integrable lattice map: Q_V

We conclude with suggestions for further explorations.

2 The importance of being singular

We use complex projective spaces as spaces of initial conditions \mathcal{I} . Suppose for simplicity that \mathcal{I} is of dimension N , with $N + 1$ homogeneous coordinates and call φ is the forward map, and ψ the backward map. Then

$$\begin{aligned}\varphi &: [x_0, x_1, \dots, x_N] \rightarrow [x'_0, x'_1, \dots, x'_N] \\ \psi &: [y_0, y_1, \dots, y_N] \rightarrow [y'_0, y'_1, \dots, y'_N]\end{aligned}$$

The evolution step will always be given by a birational map, so that both the forward and the backward evolution are described by polynomial maps, i.e. $x'_j, j = 0 \dots n$ and $y'_j, j = 0 \dots n$ are polynomials of degree d_φ and d_ψ respectively (usually $d_\varphi = d_\psi > 1$).

The singular points of φ (resp. ψ) are the ones for which $x'_j = 0, j = 0 \dots n$, (resp. $y'_j = 0, j = 0 \dots n$). We know that the sets of singular points are algebraic varieties of dimension $\leq N - 2$.

Since ' ψ is the inverse of φ ' means that the composition $\varphi \cdot \psi$ appears as a multiplication of all coordinates by a common factor, we have the two basic relations

$$\psi \cdot \varphi \simeq \kappa_\varphi \cdot id, \quad \varphi \cdot \psi \simeq \kappa_\psi \cdot id \tag{1}$$

The two polynomials κ_φ and κ_ψ , both of degree $d_\varphi d_\psi - 1$, may be decomposable.

$$\kappa_\varphi = \prod_{j=1}^p (K_j^+)^{l_j}, \quad \kappa_\psi = \prod_{j=1}^q (K_j^-)^{m_j} \tag{2}$$

Each factor K_j^\pm defines an algebraic variety of codimension 1 playing an important rôle in the sequel.

There is a simple relation between the varieties κ_φ and κ_ψ and the singular locus of φ and ψ : the varieties of equation $K_j^+ = 0$ are blown down by φ and their images are entirely made of singular points of ψ . This reflects the fact that one cannot take one step forward and then a step backward when starting from a point on $\kappa_\varphi = 0$. The same applies to ψ *mutatis mutandis*.

Suppose Σ is an indecomposable variety of codimension 1 of equation $E_\Sigma = 0$. The pullback by φ of the equation of Σ gives the equation $E_{\Sigma'}$ of the image Σ' of Σ by ψ . The important point is that this pullback may contain additional factors.

$$\varphi^*(E_\Sigma) = E_{\Sigma'} (K_1^+)^{n_1} (K_2^+)^{n_2} \dots (K_p^+)^{n_p}$$

Such factors are necessarily built from components of κ_φ . Their presence reflects the difference between total and proper transform: the non-singular points of Σ go into the proper transform. The singular subvarieties contained in Σ are blown up to the other components. In a way the proper transform is the true image, disregarding the singularities.

Singularity confinement [6] in this context is just that some iterate of φ sends components of κ_φ into components of κ_ψ . The regularisation comes, when described with homogeneous coordinates, from the removal of factor common to all coordinates. This is also the origin of the possible drop of degree of the iterates of φ .

Remark 1: A consequence of the existence of the variety $\kappa_\varphi = 0$ is that the image by φ of a generic line always hits some singular point of ψ . The reason is that any generic line

crosses κ_φ , since we are working with complex projective space. In other words, while the image of a generic point is always a generic point, the image of a generic line is never a generic line.

Remark 2: A variety could be its own transform. It is then covariant by φ and ψ . Covariant objects play a fundamental rôle in the description of the algebraic invariants [34].

By abuse of language we will say that S' is the proper transform of S , if S and S' are the equations of varieties which are the proper transforms of each other by φ or ψ .

Given an evolution map φ , denote by p_k the successive images¹ of $p_0 = [x_0, x_1, \dots, x_N]$. The components of p_k will factor into indecomposable *blocks*. These blocks are necessarily either the factors K_i^+ of the multiplier κ_φ and their transforms, either the transforms of the coordinate planes. *These blocks verify remarkable algebraic recurrence relations, and this is the subject of this paper.*

Remark 3: Birational changes of coordinates, which define the natural equivalence relation between different descriptions of the same model, affect the singularity structure, the value of κ_φ and κ_ψ , and the form of the equations relating the various blocks. They however will not spoil the general features of the recurrences between blocks.

3 Eleven models

The simplest possible type of discrete systems is given by recurrences of finite order. A recurrence of order k may be looked at as a map in its k dimensional space of initial data. There are two natural generalisations, leading to infinite dimensional space of initial conditions: recurrences defined over functional space, and recurrences with multi-indices (lattice maps). Both will be considered, again supposing rational invertibility of the evolutions.

This section contains the explicit description of the aforementioned blocks and recurrence relations for eleven different models, integrable as well as not integrable, finite dimensional as well as infinite dimensional, to offer a panoramic view on the property we describe, including a limiting case (section 3.9).

3.1 McMillan

The model is a prototype of algebraically integrable map in two dimension, belonging to the Quispel-Roberts-Thompson family [8–10, 35].

The map φ associated to the model reads

$$\varphi : [x, y, z] \longrightarrow [-y(x^2 - z^2) + 2axz^2, x(x^2 - z^2), z(x^2 - z^2)] \quad (3)$$

Its inverse ψ is

$$\psi : [x, y, z] \longrightarrow [y(y - z)(y + z), xz^2 - y^2x + 2yaz^2, z(y - z)(y + z)] \quad (4)$$

and

$$\psi \cdot \varphi \simeq \kappa_\varphi = (x - z)^4 (z + x)^4 = B_1^4 C_1^4, \quad \varphi \cdot \psi \simeq \kappa_\psi = (y - z)^4 (y + z)^4 \quad (5)$$

¹The point p_{k+1} can be obtained from p_k either by the action of φ on p_k , either by pulling back the coordinates of p_k . The homogeneous coordinates obtained in these two ways may differ, but they represent the same point projectively.

The form of the first iterates is²:

$$\begin{aligned} p_0 &= [x, y, z] \\ p_1 &= [A_1, x B_1 C_1, z B_1 C_1] \\ p_2 &= [A_2 B_1 C_1, A_1 B_2 C_2, z B_1 C_1 B_2 C_2] \\ &\dots \end{aligned} \tag{6}$$

$$p_k = [A_k B_{k-1} C_{k-1}, A_{k-1} B_k C_k, z B_{k-1} C_{k-1} B_k C_k] \tag{7}$$

Expressing that $p_{k+1} = \varphi(p_k)$ gives only one condition:

$$\begin{aligned} (A_k - z B_k C_k) (A_k + z B_k C_k) (A_{k-1} B_{k+1} C_{k+1} + A_{k+1} B_{k-1} C_{k-1}) \\ - 2 a z^2 A_k B_{k-1} C_{k-1} B_k C_k B_{k+1} C_{k+1} = 0. \end{aligned} \tag{8}$$

This condition does not suffice to determine $\{A_{k+1}, B_{k+1}, C_{k+1}\}$.

Claim: There exist algebraic relations between the blocks A, B, C . These relations allow to calculate $\{A_{k+1}, B_{k+1}, C_{k+1}\}$ in terms of the previous A, B, C 's. Moreover $A_{k+1}, B_{k+1}, C_{k+1}$ are the proper transforms of A_k, B_k, C_k .

Proof. We have, for $k \geq 3$:

$$\boxed{\begin{aligned} A_k - z B_k C_k + \mathbf{B}_{k+1} C_{k-1} &= 0, & A_k + z B_k C_k - B_{k-1} \mathbf{C}_{k+1} &= 0 \\ A_{k-1} \mathbf{B}_{k+1} \mathbf{C}_{k+1} + \mathbf{A}_{k+1} B_{k-1} C_{k-1} + 2 a z^2 A_k B_k C_k &= 0 \end{aligned}} \tag{9}$$

Equations (9) imply (8), and can be verified directly for $k=3$ and $k=4$. The validity for general k is obtained by recursion. Proving that the form of (7) and of relations (9) is stable is just a matter of counting factors B 's and C 's. We know from section (2) that

$$\begin{cases} \varphi^*(A_k) = B_1^{\alpha_B(k)} C_1^{\alpha_C(k)} A_{k+1} \\ \varphi^*(B_k) = B_1^{\beta_B(k)} C_1^{\beta_C(k)} B_{k+1} \\ \varphi^*(C_k) = B_1^{\gamma_B(k)} C_1^{\gamma_C(k)} C_{k+1} \end{cases}$$

for some exponents α, β, γ . Using (9) we get A_{k+1} from the previous A, B, C 's. We may thus evaluate all the exponents $\alpha(k+1), \beta(k+1), \gamma(k+1)$. The outcome is that $A_{k+1}, B_{k+1}, C_{k+1}$ are the proper transforms of A_k, B_k, C_k and do not factorise, as no new factors B_1 or C_1 are left over in the components of p_k after $k = 3$. QED.

We will present similar properties in the subsequent sections. Their proof goes along the same lines and will not be detailed.

Relations (9) define completely the evolution of $\{A_k, B_k, C_k\}$. They extend over a string of points of length 3. Although their solution is written as fractions, the result is automatically a polynomial in terms of the initial conditions $[x, y, z]$. They moreover enjoy a Laurent property [28–31, 33].

Define a map $f_\varphi : [P, Q, R, U, V, W] \longrightarrow [U, V, W, U', V', W']$ with

$$U' = \frac{-P V' W' - 2 a z^2 U V W}{Q R}, \quad V' = -\frac{U - z V W}{R}, \quad W' = \frac{U + z V W}{Q}$$

²There always is a rescaling possibility of the various factors, in particular an ambiguity in the signs, together with a possibility of exchanging B and C .

which implements the solution of (9) as a map. We may consider iterations of f_φ starting from arbitrary initial data $[p, q, r, u, v, w]$. The images are Laurent polynomials in $[p, q, r, u, v, w]$. If in addition the triplets $[p, q, r]$ and $[u, v, w]$ happen to be of the form $[A_{k-1}, B_{k-1}, C_{k-1}]$ and $[A_k, B_k, C_k]$, then the iterates are polynomials.

Setting $\Gamma_k = B_k C_k$, one may rewrite the 'raw' form (9) as

$$\begin{cases} A_k^2 - z^2 \Gamma_k^2 + \Gamma_{k-1} \Gamma_{k+1} = 0 \\ A_{k-1} \Gamma_{k+1} + A_{k+1} \Gamma_{k-1} + 2a z^2 A_k \Gamma_k = 0 \end{cases}$$

which is nothing but the bilinear quadratic discrete Hirota form, the inhomogeneous coordinates of the k th iterate being just $[A_k/(z\Gamma_k), A_{k-1}/(z\Gamma_{k-1})]$, as suggested in [36–38], following [3, 5].

The recurrence (9) also gives the constraints obeyed by the sequence of degrees $\delta(A_n), \dots$ of the successive A, B, C 's,

$$\begin{cases} \delta(A_{k+1}) + \delta(B_{k-1} C_{k-1}) = \delta(A_{k-1}) + \delta(B_{k+1} C_{k+1}) = \delta(A_k) + \delta(B_k C_k) + 2 \\ \delta(B_{k+1}) + \delta(C_{k-1}) = \delta(C_{k+1}) + \delta(B_{k-1}) = \delta(A_k) = \delta(B_k C_k) + 1 \end{cases}$$

and consequently the one verified by the degree $d_n = \delta(A_n) + \delta(B_{n-1} C_{n-1})$ of p_n

$$d_n - 3d_{n-1} + 3d_{n-2} - d_{n-3} = 0,$$

which proves quadratic growth of $\{d_n\}$ and vanishing of the algebraic entropy.

Finally the invariant of the model may be rewritten

$$I = \left(\frac{A_k}{z\Gamma_k} \right)^2 + \left(\frac{A_{k+1}}{z\Gamma_{k+1}} \right)^2 - \left(\frac{A_k A_{k+1}}{z^2 \Gamma_k \Gamma_{k+1}} - a \right)^2.$$

3.2 dP_{II}

This an integrable non autonomous extension of the previous model, and has the Painlevé II equation as a continuous limit [39]. Its non autonomous nature invites us to write the map in three dimensions, the added variable having a linear evolution.

$$\varphi : [x, y, z, t] \longrightarrow [-y(t^2 - x^2) + (cz + bt)xt, x(t^2 - x^2), (z + t)(t^2 - x^2), t(t^2 - x^2)]$$

and

$$\psi : [x, y, z, t] \longrightarrow [y(t^2 - y^2), -x(t^2 - y^2) + t(bt + c(z - t))y, (t - z)(t^2 - y^2), t(t^2 - y^2)]$$

$$\kappa_\varphi = (t + x)^4 (t - x)^4 = B_1^4 C_1^4, \quad \kappa_\psi = (t + y)^4 (t - y)^4$$

The sequence of point we get from $p_0 = [x, y, z, t]$ is

$$\begin{aligned} p_1 &= [A_1, x B_1 C_1, (z + t) B_1 C_1, t B_1 C_1] \\ p_2 &= [A_2 B_1 C_1, A_1 B_2 C_2, (z + 2t) B_1 C_1 B_2 C_2, t B_1 C_1 B_2 C_2] \\ &\dots \\ p_k &= [A_k B_{k-1} C_{k-1}, A_{k-1} B_k C_k, (z + kt) B_{k-1} C_{k-1} B_k C_k, t B_{k-1} C_{k-1} B_k C_k] \end{aligned} \quad (10)$$

The recurrence on $A_k B_k C_k$ is given by the following constraints, generalising straightforwardly (9):

$$\boxed{\begin{aligned} A_k - t B_k C_k + \mathbf{B}_{k+1} C_{k-1} &= 0, & A_k + t B_k C_k - B_{k-1} \mathbf{C}_{k+1} &= 0 \\ A_{k-1} \mathbf{B}_{k+1} \mathbf{C}_{k+1} + \mathbf{A}_{k+1} B_{k-1} C_{k-1} - t(b t + c(z + k t)) A_k B_k C_k &= 0 \end{aligned}} \quad (11)$$

The proof is similar to the one given in the previous section. Setting here again $\Gamma_k = B_k C_k$ we get the ‘Hirota form’ found in [37], extending over a string of length 3.

We can use these relations to prove vanishing of the entropy, and check that the various A_k, B_k, C_k are the proper transforms of $A_{k-1}, B_{k-1}, C_{k-1}$. The algebraic invariant has disappeared, but the overall algebraic structure is essentially unchanged, compared to the previous model, apart from one coefficient which became non-constant.

3.3 qP_{VI}

The map φ , which is a discrete version of the Painlevé equation P_{VI} , may be written as the composition three maps, taken from equations (19,20,21) of [40].

$$\begin{aligned} \varphi(p_0) &= \varphi([x, y, z, t]) = \varphi_3 \cdot \varphi_2 \cdot \varphi_1([x, y, z, t]) \quad \text{with} \\ \varphi_1(p_0) &= [xy(x-ct)(x-dt), sh(x-az)(x-bz)t^2, zy(x-ct)(x-dt), ty(x-ct)(x-dt)] \\ \varphi_2(p_0) &= [cd(y-pz)(y-zr)t^2, yx(-st+y)(y-ht), zx(-st+y)(y-ht), tx(-st+y)(y-ht)] \\ \varphi_3(p_0) &= [x, y, qz, t], \quad q = (cdpr)/(absh) \end{aligned}$$

The first iterates yield the following sequence of points, setting $\alpha = ab$, $\beta = cd$, $\sigma = sh$, and $\rho = pr$:

$$\begin{aligned} p_1 &= [\alpha\beta y A_1 B_1 E_1 F_1, \alpha\sigma^2 xt^2 C_1 D_1 G_1 H_1, \beta\rho xyz C_1 D_1 E_1 F_1, \alpha\sigma xyt C_1 D_1 E_1 F_1] \\ p_2 &= [(\alpha\beta)^2\sigma xt^2 A_2 B_2 E_2 F_2, \alpha\beta\sigma^2 y C_2 D_2 G_2 H_2, (\beta\rho)^2 z C_2 D_2 E_2 F_2, (\alpha\sigma)^2 t C_2 D_2 E_2 F_2] \\ p_3 &= [(\alpha\beta)^3\sigma^2 y A_3 B_3 E_3 F_3, \alpha\beta^2\sigma^2 x C_3 D_3 G_3 H_3, (\beta\rho)^3 xyz C_3 D_3 E_3 F_3, (\alpha\sigma)^3 xyt C_3 D_3 E_3 F_3] \\ p_4 &= [(\alpha\beta)^4\sigma^3 x A_4 B_4 E_4 F_4, \alpha\beta^3\sigma^2 yt^2 C_4 D_4 G_4 H_4, (\beta\rho)^4 z C_4 D_4 E_4 F_4, (\alpha\sigma)^4 t C_4 D_4 E_4 F_4] \\ p_5 &= [(\alpha\beta)^5\sigma^4 yt^2 A_5 B_5 E_5 F_5, \alpha\beta^4\sigma^2 x C_5 D_5 G_5 H_5, (\beta\rho)^5 xyz C_5 D_5 E_5 F_5, (\alpha\sigma)^5 xyt C_5 D_5 E_5 F_5] \\ p_6 &= [(\alpha\beta)^6\sigma^5 x A_6 B_6 E_6 F_6, \alpha\beta^5\sigma^2 y C_6 D_6 G_6 H_6, (\beta\rho)^6 z C_6 D_6 E_6 F_6, (\alpha\sigma)^6 t C_6 D_6 E_6 F_6] \\ &\dots \end{aligned}$$

The form of p_k is thus

$$p_k = [(\alpha\beta)^k \sigma^{k-1} f_{1,k} A_k B_k E_k F_k, \alpha \beta^{k-1} \sigma^2 f_{2,k} C_k D_k G_k H_k, (\beta\rho)^k f_{3,k} C_k D_k E_k F_k, (\alpha\sigma)^k f_{4,k} C_k D_k E_k F_k] \quad (12)$$

where the $f_{i,k}$ depend on the initial conditions, are such that $f_{i,k} = f_{i,k+6}$, and can be read from the iterates given above. They are summarised in the table

k mod 6	$i = 1$	$i = 2$	$i = 3$	$i = 4$
0	x	y	z	t
1	y	xt^2	xyz	xyt
2	xt^2	y	z	t
3	y	x	xyz	xyt
4	x	yt^2	z	t
5	yt^2	x	xyz	xyt

The various factors A, B, C, D, E, F, G, H verify simple recurrence relations, which can be checked for the first few ones, and then proved by recursion, as in the previous sections. They read

$$\begin{aligned}
p^{k-1}r^k \nu_{1,k} E_k F_k - sh \omega_{1,k} G_k H_k &= \mathbf{A}_k B_{k-1} \\
p^k r^{k-1} \nu_{1,k} E_k F_k - sh \omega_{1,k} G_k H_k &= A_{k-1} \mathbf{B}_k \\
(ab)^{k-1} s^{k-2} h^{k-1} \nu_{2,k} E_k F_k - (cd)^{k-1} \omega_{2,k} G_k H_k &= \mathbf{C}_k D_k \\
(ab)^{k-1} s^{k-1} h^{k-2} \nu_{2,k} E_k F_k - (cd)^{k-1} \omega_{2,k} G_k H_k &= C_{k-1} \mathbf{D}_k
\end{aligned} \tag{13}$$

giving A_k, B_k, C_k, D_k in terms of $A_{k-1}, B_{k-1}, E_k, F_k, G_k, H_k$, and

$$\begin{aligned}
c^k d^{k-1} \lambda_{1,k} A_k B_k - sh \mu_{1,k} C_k D_k &= \mathbf{E}_{k+1} F_k \\
c^{k-1} d^k \lambda_{1,k} A_k B_k - sh \mu_{1,k} C_k D_k &= E_k \mathbf{F}_{k+1} \\
a^k b^{k-1} (sh)^{k-1} \lambda_{2,k} A_k B_k - (pr)^k \mu_{2,k} C_k D_k &= \mathbf{G}_{k+1} H_k \\
a^{k-1} b^k (sh)^{k-1} \lambda_{2,k} A_k B_k - (pr)^k \mu_{2,k} C_k D_k &= G_k \mathbf{H}_{k+1}
\end{aligned} \tag{14}$$

giving $E_{k+1}, F_{k+1}, G_{k+1}, H_{k+1}$ in terms of A_k, B_k, C_k, D_k . The coefficients $\lambda, \mu, \nu, \omega$ appearing in the previous relations depend on the initial conditions and are given by the following table:

k mod 6	$\nu_{1,k}$	$\nu_{2,k}$	$\omega_{1,k}$	$\omega_{2,k}$	$\lambda_{1,k}$	$\lambda_{2,k}$	$\mu_{1,k}$	$\mu_{2,k}$
0	z	t	y	y	x	x	t	z
1	yz	y	t^2	t	1	1	t	xz
2	z	t	y	y	xt	xt^2	1	z
3	y	yt	1	1	1	1	xt	xz
4	z	1	yt^2	yt	x	x	t	z
5	y	yt	1	1	t	t^2	x	xz

As in the previous case, equations (14,13) can be put in a quadratic form by setting

$$U_k = A_k B_k, V_k = C_k D_k, W_k = E_k F_k, T_k = G_k H_k,$$

with coefficients depending on the initial conditions and on k . They differ from the ones found in [37].

Relations (13) and (14) define an iteration, starting from $[A_1, B_1, C_1, D_1, E_1, F_1, G_1, H_1]$. The explicit calculation of the first few iterates indicates that they are Laurent polynomials in the initial conditions. If in addition the initial conditions were produced by the action of the map φ on some $p_0 = [x, y, z, t]$, then all resulting quantities are *polynomials* in x, y, z, t . Relations (13) and (14), as well as (12) may also be used to prove vanishing of the entropy, because they provide an exact evaluation of the successive degrees of the iterates of φ .

3.4 A non integrable non confining map in P_2

The model was proposed by F. Jaeger in relation to studies of Bose-Meisner algebras (see for example [41]). It is the product of two birational involutions of P_2 , constructed from a

projective linear map L defined by the matrix

$$\begin{bmatrix} 1 & 2 & 2 \\ 1 & 3 & -4 \\ 1 & 4 & -5 \end{bmatrix}$$

and the fundamental involution

$$j : [x, y, z] \longrightarrow [y z, z z, x y].$$

Define

$$i = L^{-1} \cdot j \cdot L, \quad \varphi = i \cdot j, \quad \psi = j \cdot i, \quad \kappa_\varphi = \kappa_\psi = 25 x y z. \quad (15)$$

It will appear that the pattern leads naturally to decompose φ as the product

$$\varphi = (L^{-1} j) \cdot (L j)$$

The sequence of iterates $p_0, q_1 = L^{-1}j(p_0), p_1 = Lj(q_1), \dots$ we obtain is

$$\begin{aligned} p_0 &= [x, y, z] \\ q_1 &= [U_1, V_1, W_1] \\ p_1 &= [A_1, x B_1, x C_1] \\ q_2 &= [U_2, U_1 V_2, U_1 W_2] \\ p_2 &= [A_2, A_1 B_2, A_1 C_2] \\ &\dots \\ q_k &= [U_k, U_{k-1} V_k, U_{k-1} W_k] \\ p_k &= [A_k, A_{k-1} B_k, A_{k-1} C_k] \end{aligned}$$

The recurrence on the blocks A, B, C, U, V, W take now the form

$$\boxed{\begin{aligned} A_{k-1} B_k C_k + 2 C_k A_k + 2 A_k B_k &= 5 \mathbf{U}_{k+1} \\ A_{k-1} B_k C_k + 3 C_k A_k - 4 A_k B_k &= 5 U_k \mathbf{V}_{k+1} \\ A_{k-1} B_k C_k + 4 C_k A_k - 5 A_k B_k &= 5 U_k \mathbf{W}_{k+1} \end{aligned}} \quad (16)$$

$$\boxed{\begin{aligned} U_k V_{k+1} W_{k+1} + 18 W_{k+1} U_{k+1} - 14 U_{k+1} V_{k+1} &= \mathbf{A}_{k+1}, \\ U_k V_{k+1} W_{k+1} - 7 W_{k+1} U_{k+1} + 6 U_{k+1} V_{k+1} &= A_k \mathbf{B}_{k+1}, \\ U_k V_{k+1} W_{k+1} - 2 W_{k+1} U_{k+1} + U_{k+1} V_{k+1} &= A_k \mathbf{C}_{k+1} \end{aligned}} \quad (17)$$

The recurrence relations we get here are not quadratic. The main point is that in (16) and (17), U_{k+1} and A_{k+1} have polynomial expressions. This reflects the singularity structure of the map: in the iteration process, the lines $\{y = 0\}$ and $\{z = 0\}$ are blown down to points which whose images never meets any singularity, while the line $\{x = 0\}$ goes to $[1, 1, 1]$, and then $[1, 0, 0]$ which is singular. Relations (16) and (17) allow us to calculate exactly the entropy $\epsilon = \log((3 + \sqrt{5})/2)$ of φ .

3.5 A confining non integrable map in P_2

We briefly mention here what has become a prototype of confining but chaotic map in two dimension, described in [42].

$$\varphi : [x, y, z] \longrightarrow [x^3 + az^3 - yx^2, x^3, x^2z]$$

coming from the simple order 2 recurrence

$$u_{n+1} + u_{n-1} = u_n + \frac{a}{u_n^2}$$

Here

$$\kappa_\varphi = x^3, \quad \kappa_\psi = y^3$$

This map has been shown to have positive entropy by various methods, among which the construction of a rational surface over P_2 where the singularities are resolved [43]. The lift of the map to the Picard group of this variety is a linear map whose maximal eigenvalues gives the entropy.

The generic iterate has the form

$$p_k = [A_{k-3}^3 A_k, A_{k-4} A_{k-1}^3, z A_{k-3}^2 A_{k-2}^2 A_{k-1}^2] \quad (18)$$

and the recurrence relation between the blocks A becomes

$$\boxed{A_k^3 A_{k-3}^3 + a z^3 A_{k-1}^6 A_{k-2}^6 - A_{k-1}^3 A_{k-4} A_k^2 = A_{k-3}^2 A_{k-2}^3 \mathbf{A}_{k+1}} \quad (19)$$

which is equ (4.6) of [30], extending over a string of length 6. This relation is not quadratic nor multilinear, but it allows to prove (18) with the same type of argument as in the previous sections, providing the recurrence condition on the degrees of the iterates of φ , and the value of the entropy $\epsilon = (3 + \sqrt{5})/2$.

3.6 Another confining non integrable map in P_2

Another interesting confining non integrable map described in [13], eq 29, was also examined for $a = 0$ in [30]. The map comes from the order 2 recurrence

$$u_{n+1} \cdot u_{n-1} = u_n + \frac{1}{u_n} + a$$

$$\begin{aligned} \varphi : [x, y, z] &\rightarrow [z(x^2 + z^2 + axz), yx^2, xyz], \\ \psi : [x, y, z] &\rightarrow [y^2x, z(y^2 + z^2 + azy), xyz]. \\ \kappa_\varphi &= x^3 y^2 z (x + z(a + \alpha)/2) (x + z(a - \alpha)/2) = x^3 y^2 z B_1 C_1, \\ \kappa_\psi &= x^2 y^3 z (y + z(a + \alpha)/2) (y + z(a - \alpha)/2) \end{aligned}$$

with $\alpha = \sqrt{a^2 - 4}$.

By blowing up 18 points, we may define a rational surface over P_2 where the lift of the map becomes a diffeomorphism, and the entropy is given as the logarithm of the inverse of the root of $s^4 - s^3 - 2s^2 - s + 1 = 0$ of smallest modulus ($\simeq \log 2.08102$).

The iterates of φ are³

$$\begin{aligned}
p_0 &= [x, y, z] \\
p_1 &= [z A_1, x^2 y, x y z] \\
p_2 &= [z A_2, x A_1^2, x^2 y A_1] \\
p_3 &= [x y A_3, z^2 A_1 A_2^2, x^2 y z A_1^2 A_2] \\
p_4 &= [x^2 y A_1 A_4, z A_2 A_3^2, x z^2 A_1^2 A_2^2 A_3] \\
p_5 &= [z A_1^2 A_2 A_5, x y^2 A_3 A_4^2, y z^2 A_1 A_2^2 A_3^2 A_4] \\
p_6 &= [z^2 A_2^2 A_3 A_6, x y A_1 A_4 A_5^2, x y^2 z A_2 A_3^2 A_4^2 A_5] \\
&\dots
\end{aligned}$$

The form stabilises into

$$p_k = [g_{1,k} A_{k-4}^2 A_{k-3} A_k, g_{2,k} A_{k-5} A_{k-2} A_{k-1}^2, g_{3,k} A_{k-4} A_{k-3}^2 A_{k-2}^2 A_{k-1}]. \quad (20)$$

The coefficients $g_{i,k}$ depend on the initial condition $\{x, y, z\}$, and verify $g_{i,k+9} = g_{i,k}$. They are given by:

k mod 9	$i = 1$	$i = 2$	$i = 3$
0	x	y	z
1	z	$x^2 y$	$x y z$
2	z	x	$x^2 y$
3	$x y$	z^2	$x^2 y z$
4	$x^2 y$	z	$x z^2$
5	z	$x y^2$	$y z^2$
6	z^2	$x y$	$x y^2 z$
7	y	z	$x y^2$
8	$x y^2$	z	$x y z$

Notice that the presence of the factors $g_{i,k}$ shows that the coordinate planes appear periodically in the iterates.

The recurrence relations between the A 's also have coefficients depending on the initial conditions, and on k in a periodic way (period 9).

$$\boxed{\lambda_k^2 A_{k-4}^2 A_k^2 + \mu_k^2 A_{k-3}^2 A_{k-2}^4 A_{k-1}^2 + a \lambda_k \mu_k A_{k-4} A_{k-3} A_{k-2}^2 A_{k-1} A_k = A_{k-5} \mathbf{A}_{k+1}} \quad (21)$$

The coefficients are given by the following table

k mod 9	λ	μ
0	x	z
1	1	$x y$
2	z	$x^2 y$
3	1	$x z$
4	$x y$	z^2
5	1	$y z$
6	z	$x y^2$
7	1	$x y$
8	y	z

³From $A_1 = B_1 C_1$, the blocks A_k are products of two blocks, which we do not write for simplicity

The recurrence relation (21) differs from the one given for $a = 0$ in [30], since it depends on the order k . The explicit calculation of the first iterations indicates that it verifies the Laurent property for arbitrary x, y, z and a .

3.7 An integrable map in P_3 : N=3 Periodic Volterra

$$\varphi := [x, y, z, t] \longrightarrow [x', y', z', t'] \quad \text{with}$$

$$x' = -x (t^2 + 2et(y-z) - e^2x^2 + e^2(y+z)^2) \quad (22)$$

$$t' = -t (t^2 + e^2(x^2 + y^2 + z^2 - 2xy - 2yz - 2xz)) \quad (23)$$

y' and z' being obtained from (22) by circular permutations of x, y, z .

The map comes from a discretisation of a continuous integrable system, and is known to have two algebraic invariants [44].

Starting from $p_0 = [A_0, B_0, C_0, D_0]$ we get a sequence of points of the form:

$$\begin{aligned} p1 &= [A_0 A_1, B_0 B_1, C_0 C_1, D_0 D_1] \\ p2 &= [A_0 A_1 A_2, B_0 B_1 B_2, C_0 C_1 C_2, D_0 A_1 B_1 C_1] \\ p3 &= [A_0 A_1 A_2 A_3, B_0 B_1 B_2 B_3, C_0 C_1 C_2 C_3, D_0 D_1 B_2 A_2 C_2] \\ p4 &= [A_0 A_2 A_3 A_4, B_0 B_2 B_3 B_4, C_0 C_2 C_3 C_4, D_0 B_3 A_3 C_3] \\ p5 &= [A_0 A_3 A_4 A_5, B_0 B_3 B_4 B_5, C_0 C_3 C_4 C_5, D_0 D_1 B_4 A_4 C_4] \\ &\dots \end{aligned}$$

The form of the iterates stabilises after three steps, with as slight difference in the structure of the last component between odd and even order.

Denoting the k th iterate p_k as $[X_k, Y_k, Z_k, T_k]$ with

$$\begin{cases} X_k = A_0 A_{k-2} A_{k-1} A_k, & Y_k = B_0 B_{k-2} B_{k-1} B_k, & Z_k = C_0 C_{k-2} C_{k-1} C_k \\ T_{2m+1} = D_0 D_1 A_{2m} B_{2m} C_{2m}, & T_{2m+2} = D_0 A_{2m+1} B_{2m+1} C_{2m+1}, \end{cases} \quad (24)$$

the various A, B, C, D 's verify sets of constraints of the form

$$\begin{cases} D_0(X_k + Y_k + Z_k) - (A_0 + B_0 + C_0) T_k = 0 \\ T_k + e(\pm X_k \pm Y_k \pm Z_k) - (D_0 + e(\pm A_0 \pm B_0 \pm C_0)) \alpha_{k-1} \beta_k \gamma_{k+1} = 0 \end{cases} \quad (25)$$

where $\{\alpha, \beta, \gamma\}$ is some permutation of $\{A, B, C\}$.

These constraints, extending over strings of successive points of length 3, are responsible for the factorisation properties: they define various ideals, and the factorisations take place in the the algebra generated by the A, B, C, \dots *quotiented by these ideals*. Moreover the constraints are conserved by the evolution. The various \pm signs in (25) depend on k in a periodic way (period 3).

A typical example of these relations is (for $k = 3$):

$$\left\{ \begin{array}{l} D_0 (X_3 + Y_3 + Z_3) - (A_0 + B_0 + C_0) T_3 = 0 \\ T_3 + e(-X_3 + Y_3 + Z_3) - (D_0 - eA_0 + eB_0 + eC_0) A_2 B_3 C_1 = 0 \\ T_3 + e(X_3 - Y_3 + Z_3) - (D_0 + eA_0 - eB_0 + eC_0) A_1 B_2 C_3 = 0 \\ T_3 + e(X_3 + Y_3 - Z_3) - (D_0 + eA_0 + eB_0 - eC_0) A_3 B_1 C_2 = 0 \\ T_3 + e(-X_3 - Y_3 + Z_3) - (D_0 - eA_0 - eB_0 + eC_0) A_1 B_3 C_2 = 0 \\ T_3 + e(-X_3 + Y_3 - Z_3) - (D_0 - eA_0 + eB_0 - eC_0) A_3 B_2 C_1 = 0 \\ T_3 + e(X_3 - Y_3 - Z_3) - (D_0 + eA_0 - eB_0 - eC_0) A_2 B_1 C_3 = 0 \end{array} \right.$$

The set of constraints is invariant by circular permutation $A \rightarrow B \rightarrow C \rightarrow A$. It cannot be written solely in term of the components X, Y, Z, T . It cannot be solved straightforwardly for any set A_k, B_k, C_k 's, because it is then over-determined.

On the other hand, writing that $[X_{k+1}, Y_{k+1}, Z_{k+1}, T_{k+1}]$ is the image of $[X_k, Y_k, Z_k, T_k]$ by φ yields algebraic equations for $\{A_{k+1}, B_{k+1}, C_{k+1}\}$, which one can solve rationally.

The effect of the set of constraints (25) is that these expressions can be simplified. The factors of $\{A_{k+1}, B_{k+1}, C_{k+1}\}$, reduce to monomials in $A_{k-1}, B_{k-1}, C_{k-1}, A_{k-2}, B_{k-2}, C_{k-2}$ and possibly D_1 for odd k .

The simplified relations defining the iteration read:

$$\boxed{\begin{array}{l} T_k^2 + 2eT_k(Y_k - Z_k) + e^2(Y_k + Z_k)^2 - e^2X_k^2 + \Delta_k \mathbf{A}_{k+1} B_{k-2} C_{k-2} A_{k-1} B_{k-1} C_{k-1} = 0 \\ T_k^2 + 2eT_k(Z_k - X_k) + e^2(Z_k + X_k)^2 - e^2Y_k^2 + \Delta_k A_{k-2} \mathbf{B}_{k+1} C_{k-2} A_{k-1} B_{k-1} C_{k-1} = 0 \\ T_k^2 + 2eT_k(X_k - Y_k) + e^2(X_k + Y_k)^2 - e^2Z_k^2 + \Delta_k A_{k-2} B_{k-2} \mathbf{C}_{k+1} A_{k-1} B_{k-1} C_{k-1} = 0 \end{array}} \quad (26)$$

with $\Delta_k = D_1$ for odd k and $\Delta_k = 1$ for even k .

These equations are linear in $\{A_{k+1}, B_{k+1}, C_{k+1}\}$. They tell us that the factor

$$f_k = \Delta_k A_{k-2} B_{k-2} C_{k-2} A_{k-1} B_{k-1} C_{k-1}$$

goes away from the homogeneous coordinates when calculating p_{k+1} as $\varphi(p_k)$. They extend over a *string of successive points of length 4*. They are the generalisation of the Hirota bilinear formalism for the map under consideration, but they are not quadratic anymore. They do not have the Laurent property.

Thanks to the relations (25), their solution in $\{A_{k+1}, B_{k+1}, C_{k+1}\}$ is polynomial in terms of the initial conditions $\{A_0, B_0, C_0, D_0\}$, and it is possible to show that $A_{k+1}, B_{k+1}, C_{k+1}$ are the proper transforms of A_k, B_k, C_k . In other words, the A, B, C 's do not factorise.

3.8 A linearisable map

Linearisable recurrence are known to have special singularity structure. As an example we can take the one studied in [45], where it was shown to be non-confining, but integrable.

The recurrence is

$$u_{n+1} = u_n + \frac{u_n - u_{n-1}}{1 + u_n - u_{n-1}}$$

so that

$$\begin{aligned}\varphi : [x, y, z] &\longrightarrow [2xz + x^2 - xy - yz, x(x - y + z), (x - y + z)z] \\ \psi : [x, y, z] &\longrightarrow [y(x - y - z), xz + xy - 2yz - y^2, z(x - y - z)] \\ \kappa_\varphi &= z^2(x - y + z) = z^2 B_1, \quad \kappa_\psi = z^2(x - y - z)\end{aligned}$$

The iterates p_k take the form

$$p_k = [A_k, A_{k-1} B_k, z B_1 B_2 B_3 \dots B_k]$$

with

$$\varphi^*(B_k) = z B_{k+1}$$

There is regularity of the pattern, but the number of factors increases with k . This is related to the fact that at each step, one more factor B_1 appears, and is at the origin of the low (linear) growth of the degrees of the iterates.

The recurrence relations between blocks read

$$\boxed{\begin{aligned}z B_1 B_2 \dots B_{k-1} (2 A_k B_k - A_{k-1} B_k^2 - \mathbf{A}_{k+1}) + A_k (A_k - A_{k-1} B_k) &= 0 \\ z B_1 B_2 \dots B_{k-1} (B_k - \mathbf{B}_{k+1}) + A_k - A_{k-1} B_k &= 0\end{aligned}} \quad (27)$$

The linear growth of the degrees can be read from the previous relations.

3.9 An unruly model

We know of maps for which the sequence of degrees does not verify any finite recurrence relation. Although this does not prevent their entropy from being the logarithm of an algebraic integer, it will prevent the existence of the Hirota like forms we have seen in the previous cases.

A simple example was found in [29]. It is a monomial map in three dimensions:

$$\begin{aligned}\varphi : [x, y, z, t] &\longrightarrow [ty, tz, x^2, tx], \quad \psi : [x, y, z, t] \longrightarrow [tz, xz, yz, t^2] \\ \kappa_\varphi &= x^2 t, \quad \kappa_\psi = z t^2\end{aligned}$$

This map has the peculiarity that the entropies of φ and of ψ differ [29].

The structure of the iterates is simple, since they are all written in term of the coordinate planes

$$p_k = [x^{\delta_1^x} y^{\delta_1^y} z^{\delta_1^z} t^{\delta_1^t}, x^{\delta_2^x} y^{\delta_2^y} z^{\delta_2^z} t^{\delta_2^t}, x^{\delta_3^x} y^{\delta_3^y} z^{\delta_3^z} t^{\delta_3^t}, x^{\delta_4^x} y^{\delta_4^y} z^{\delta_4^z} t^{\delta_4^t}]$$

for some powers δ_*^* . For ψ , the form of the iterates stabilises and sequence of degrees verifies a finite recurrence relation, but this is not the case for φ . The peculiarity of the model is that the singularity structure is such that the sequences of proper transforms which are at the basis of the observation we made for all other examples do not appear here. This model is a limiting case to keep in mind for further developments.

3.10 A delay-differential / differential difference equation

Consider the following equation:

$$a u(t) - b \partial_t u(t) = u(t) (u(t+1) - u(t-1)) \quad (28)$$

where ∂_t means time derivative.

Equation (28) was obtained in [46] by a non trivial reduction of a semi-discrete equation [47]. This equation is a delay difference equation of which the entropy has been evaluated in [48], and found to be vanishing. One may equivalently consider the differential difference equation, or recurrence of order two defined on functional space:

$$a u_n(t) - b \partial_t u_n(t) = u_n(t) (u_{n+1}(t) - u_{n-1}(t)) \quad (29)$$

The maps φ and ψ associated to these equations are:

$$\varphi : [x, y, z] \longrightarrow [axz - b(x'z - xz') + xy, x^2, xz] \quad (30)$$

$$\psi : [x, y, z] \longrightarrow [y^2, -ayz + b(y'z - yz') + xy, yz] \quad (31)$$

where prime (') means derivative. Here x, y, z should be considered as a container for the infinite sequences $[x(t), x'(t), x''(t), \dots]$, $[y(t), y'(t), y''(t), \dots]$, and $[z(t), z'(t), z''(t), \dots]$.

For this map

$$\kappa_\varphi([x, y, z]) = x^3, \quad \kappa_\psi([x, y, z]) = y^3.$$

and we get the following form for the first iterates starting from p_0

$$\begin{aligned} p_0 &= [A_0, B_0, C_0] \\ p_1 &= [A_1, A_0^2, A_0 C_0] \\ p_2 &= [A_2, A_1^2, A_0 A_1 C_0] \\ p_3 &= [A_0^2 A_3, A_2^2, A_0 A_1 A_2 C_0] \\ p_4 &= [A_1^2 A_4, A_0 A_3^2, A_1 A_2 A_3 C_0] \\ &\dots \\ p_k &= [A_{k-3}^2 A_k, A_{k-4} A_{k-1}^2, A_{k-3} A_{k-2} A_{k-1} C_0] \end{aligned} \quad (32)$$

The recurrence for A_k reads

$$\boxed{a C_0 A_k A_{k+1} A_{k+2} A_{k+3} + A_0 A_{k+3} A_{k+2}^2 + c A_k^2 A_{k+3}^2 \partial_t \left(\frac{C_0 A_{k+1} A_{k+2}}{A_k A_{k+3}} \right) = A_k A_{k+1}^2 \mathbf{A}_{k+4}} \quad (33)$$

This relation extends over a string of length 5. Again, although given as a fraction, A_{k+4} is a differential polynomial in the initial conditions.

The proof of relation (33) is done by recursion. It is verified for $k = 0$ and $k = 1$. We moreover know that the pullback of any A_k by φ is of the form $A_0^n A_{k+1}$, it is easy to show the validity of (33) for $k + 1$. In particular, one finds that A_{k+1} is the proper transform of A_k and moreover

$$\varphi^* \left(\frac{C_0 A_{k+1} A_{k+2}}{A_k A_{k+3}} \right) = \frac{C_0 A_{k+2} A_{k+3}}{A_{k+1} A_{k+4}},$$

so that the pullback of the derivative term appearing in (33) does not contain factors A_0 . We also get relations on the various degrees $\delta(A_k)$:

$$\begin{aligned} 1 + \delta(A_k) + \delta(A_{k+1}) + \delta(A_{k+2}) + \delta(A_{k+3}) &= 1 + 2\delta(A_{k+2}) + \delta(A_{k+3}) \\ &= \delta(A_k) + 2\delta(A_{k+1}) + \delta(A_{k+4}). \end{aligned}$$

One then easily proves the result on the sequence of degrees of p_n given in [48]

$$\delta(p_n) = \frac{1}{8} \left(6n^2 + 9 - (-1)^n \right)$$

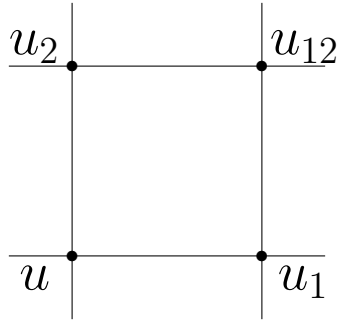
ensuring the vanishing of the entropy.

One could always question the notion of integrability for differential-difference equations, and even more for delay-difference equations, but the vanishing of the algebraic entropy is a very strong structural constraint on the equation.

3.11 An integrable lattice map: Q_V

The model, introduced in [49,50] is defined on a plane square lattice, by a multilinear relations between the values of an unknown function $u_{n,m}$, $n \in \mathbb{Z}, m \in \mathbb{Z}$. It interpolates between the various models of the Adler-Bobenko-Suris list [51–53], and has seven free parameters. The integrability of the model was originally based on the evaluation of its algebraic entropy, which vanishes. Subsequently this model was shown to have an infinite set of symmetries, implemented by two recursion operators related by a elliptic condition [54,55].

The elementary cell of the lattice, written with the usual convention $u_{n,m} = u$, $u_{n+1,m} = u_1$, $u_{n,m+1} = u_2$, $u_{n+1,m+1} = u_{12}$ looks like



and the local relation defining the model reads:

$$\begin{aligned} a_1 u u_1 u_2 u_{12} + a_2 (u u_1 u_2 + u u_2 u_{12} + u u_1 u_{12} + u_1 u_2 u_{12}) + a_3 (u u_1 + u_2 u_{12}) \\ + a_4 (u_1 u_2 + u u_{12}) + a_5 (u u_2 + u_1 u_{12}) + a_6 (u + u_1 + u_2 + u_{12}) + a_7 = 0 \end{aligned} \quad (34)$$

The previous relation being multilinear, it is possible to calculate any of the corner variables in term of the other three. On each cell set $\varphi_{n,m} : u \rightarrow u_{12}$ and $\psi_{n,m} : u_{12} \rightarrow u$.

In order to define an evolution we need to specify initial conditions. We choose to give initial conditions on two adjacent diagonals, labelled -1 and 0 , and use the local condition to fill the entire lattice. Points on the diagonal k have coordinates $\{n, m\}$ with $n + m = k$. We may then define a map φ from diagonal k to $k + 2$, and ψ from diagonal $k + 2$ to k

(straight arrows in Figure 1), for which the values on diagonal $k + 1$ enter as parameters. Although the space of initial conditions is infinite, a point $\{n, m\}$, 'sees' only a finite number of initial points on the diagonals -1 and 0 . We projectivise the system by turning the space of values at each point in to a projective line P_1 , with homogeneous coordinates $[X_{n,m}, Y_{n,m}]$ so that $u_{n,m} = X_{n,m}/Y_{n,m}$, and writing only polynomial expressions, keeping in mind that any common factor to X and Y ought to be removed.

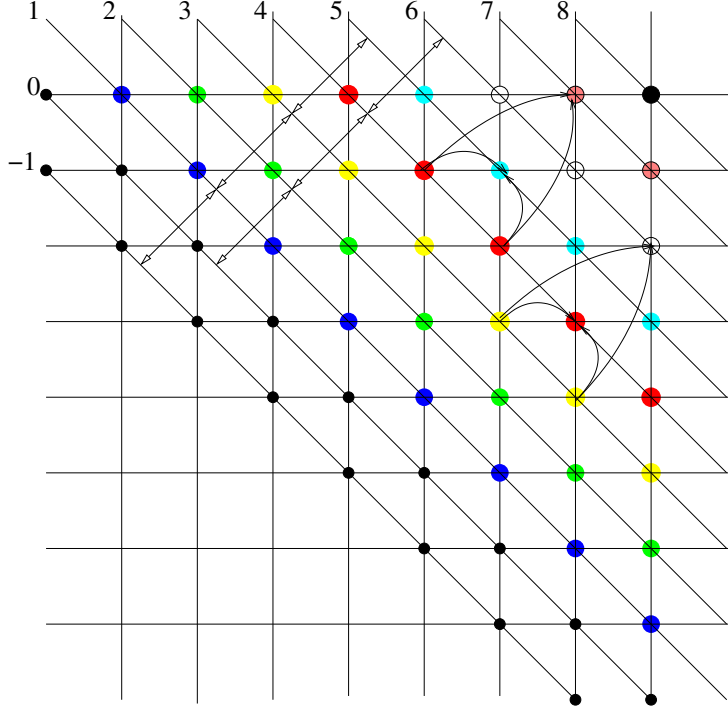


Figure 1: Initial conditions and (north-east) evolution

We know from [50] that the drop of the degrees of the successive iterates is intimately related to one of the biquadratics given in [51]. This biquadratic is nothing but the multiplier $\kappa_{n,m} \simeq \psi_{n,m} \cdot \varphi_{n,m}$ calculated on one cell: to any pair of adjacent points $\{n, m+1\}$, $\{n+1, m\}$ on a diagonal we associate the polynomial

$$\left\{ \begin{array}{l} \kappa_{n,m} = (a_1 a_7 - a_3^2 + a_4^2 - a_5^2) X_{n,m+1} Y_{n,m+1} X_{n+1,m} Y_{n+1,m} \\ + (a_1 a_6 + a_2 a_4 - a_2 a_3 - a_2 a_5) (X_{n,m+1}^2 Y_{n+1,m} X_{n+1,m} + X_{n+1,m}^2 X_{n,m+1} Y_{n,m+1}) \\ + (a_2 a_7 + a_4 a_6 - a_3 a_6 - a_5 a_6) (Y_{n,m+1}^2 Y_{n+1,m} X_{n+1,m} + Y_{n+1,m}^2 X_{n,m+1} Y_{n,m+1}) \\ + (a_2 a_6 - a_3 a_5) (X_{n,m+1}^2 Y_{n+1,m}^2 + Y_{n,m+1}^2 X_{n+1,m}^2) \\ + (a_1 a_4 - a_2^2) X_{n,m+1}^2 X_{n+1,m}^2 + (a_4 a_7 - a_6^2) Y_{n,m+1}^2 Y_{n+1,m}^2 \end{array} \right. \quad (35)$$

The key fact is that these polynomials split into two factors as soon as $n + m \geq 2$. One of these two factors is common to $X_{n+1,m+1}$ and $Y_{n+1,m+1}$ the second one is common to

$X_{n+2,m+2}$ and $Y_{n+2,m+2}$. The evolution equations may be rewritten as the recurrence

$$\begin{aligned}
\Omega_{n,m} \Omega_{n+1,m+1} &= \kappa_{n-1,m-1} \\
\mathbf{X}_{n+1,m+1} \Omega_{n+1,m+1} &= -a_2 X_{n,m+1} X_{n,m} X_{n+1,m} - a_3 X_{n,m} X_{n+1,m} Y_{n,m+1} \\
&\quad - a_4 X_{n,m+1} X_{n+1,m} Y_{n,m} - a_5 X_{n,m+1} X_{n,m} Y_{n+1,m} - a_7 Y_{n,m+1} Y_{n,m} Y_{n+1,m} \\
&\quad - a_6 (X_{n,m+1} Y_{n,m} Y_{n+1,m} + X_{n,m} Y_{n,m+1} Y_{n+1,m} + X_{n+1,m} Y_{n,m+1} Y_{n,m}) \\
\mathbf{Y}_{n+1,m+1} \Omega_{n+1,m+1} &= a_1 X_{n,m+1} X_{n,m} X_{n+1,m} + a_3 X_{n,m+1} Y_{n,m} Y_{n+1,m} \\
&\quad + a_4 X_{n,m} Y_{n,m+1} Y_{n+1,m} + a_5 X_{n+1,m} Y_{n,m+1} Y_{n,m} + a_6 Y_{n,m+1} Y_{n,m} Y_{n+1,m} \\
&\quad + a_2 (X_{n,m+1} X_{n,m} Y_{n+1,m} + X_{n,m} X_{n+1,m} Y_{n,m+1} + X_{n,m+1} X_{n+1,m} Y_{n,m}),
\end{aligned} \tag{36}$$

with the added initial condition that $\Omega_{k,l} = 1$ for $k+l=1$. The various points entering the defining relation of the Ω 's are pictured in Figure 1 with the curved arrows. The Ω 's and the X, Y 's given by (36) are *polynomials in the initial conditions, and there is no additional common factor to $X_{n+1,m+1}$ and $Y_{n+1,m+1}$ for generic values of the parameters a* . Relations (36) yield the sequence of degrees of the iterates found in [49, 50], quadratic growth and vanishing entropy. More details will be given elsewhere.

4 Conclusion and perspectives

We have shown that, for systems undergoing a rational discrete evolution, a self-organisation takes place after a finite number of steps: the structure of the iterates stabilises - this is to be compared with the results of [56] - and the τ functions pop out spontaneously as pieces of the components of the iterates, providing a new set of variables to describe the evolution.

This change of description of the models is not to be confused with a usual (birational) change of coordinates: we barter the original coordinates for pieces of the components of strings of successive iterates and transforms of the factors of the multipliers κ_φ .

The recurrence relations obeyed by the new variables provide us with an exact evaluation of the algebraic entropy and of its avatars obtained by reductions to integers and finite fields, without restriction to integrability. They support, by the form they take, the fundamental conjecture presented in [12] that the entropy is always the logarithm of an algebraic integer. They also invite us to make contact with the results of [57, 58] on orthogonal polynomials.

In the integrable cases our approach may finally yield, in addition to the known applications (Lax pairs, special solutions, soliton solutions, grassmanian description) a new classification tool.

All of this is matter for future work.

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