COMPLEMENTS ON DISCONNECTED REDUCTIVE GROUPS

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F. DIGNE AND J. MICHEL

Dedicated to the memory of Robert Steinberg

Abstract. We present various results on disconnected reductive groups, in particular about the characteristic 0 representation theory of such groups over finite fields.

1. Introduction

Let $\mathbf{G}$ be a (possibly disconnected) linear algebraic group over an algebraically closed field. We assume that the connected component $\mathbf{G}^0$ is reductive, and then call $\mathbf{G}$ a (possibly disconnected) reductive group. This situation was studied by Steinberg in [St] where he introduced the notion of quasi-semi-simple elements.

Assume now that $\mathbf{G}$ is over an algebraic closure $\overline{\mathbb{F}}_q$ of the finite field $\mathbb{F}_q$, defined over $\mathbb{F}_q$ with corresponding Frobenius endomorphism F. Let $\mathbf{G}^1$ be an F-stable connected component of $\mathbf{G}$. We want to study $(\mathbf{G}^0)^F$-class functions on $(\mathbf{G}^1)^F$; if $\mathbf{G}^1$ generates $\mathbf{G}$, they coincide with $\mathbf{G}^F$-class functions on $(\mathbf{G}^1)^F$.

This setting we adopt here is also taken up by Lusztig in his series of papers on disconnected groups [Lu] and is slightly more general than the setting of [DM94], where we assumed that $\mathbf{G}^1$ contains an F-stable quasi-central element. A detailed comparison of both situations is done in the next section.

As the title says, this paper is a series of complements to our original paper [DM94] which are mostly straightforward developments that various people asked us about and that, as far as we know, have not appeared in the literature; we thank in particular Olivier Brunat, Gerhard Hiss, Cheryl Praeger and Karine Sorlin for asking these questions.

In section 2 we show how quite a few results of [DM94] are still valid in our more general setting.

In section 3 we take a "global" viewpoint to give a formula for the scalar product of two Deligne-Lusztig characters on the whole of $\mathbf{G}^F$.

In section 4 we show how to extend to disconnected groups the formula of Steinberg [St, 15.1] counting unipotent elements.

In section 5 we extend the theorem that tensoring Lusztig induction with the Steinberg character gives ordinary induction.

In section 6 we give a formula for the characteristic function of a quasi-semi-simple class, extending the case of a quasi-central class which was treated in [DM94].

In section 7 we show how to classify quasi-semi-simple conjugacy classes, first for a (possibly disconnected) reductive group over an arbitrary algebraically closed field, and then over $\mathbb{F}_q$.

Finally, in section 8 we extend to our setting previous results on Shintani descent.

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2. Preliminaries

In this paper, we consider a (possibly disconnected) algebraic group $\mathbf{G}$ over $\overline{\mathbb{F}}_q$ (excepted at the beginning of section 7 where we accept an arbitrary algebraically closed field), defined over $\mathbb{F}_q$ with corresponding Frobenius automorphism F. If $\mathbf{G}^1$ is an F-stable component of $\mathbf{G}$, we call class functions on $(\mathbf{G}^1)^F$ the complex-valued functions invariant under $(\mathbf{G}^0)^F$-conjugacy (or equivalently under $(\mathbf{G}^1)^F$-conjugacy). Note that if $\mathbf{G}^1$ does not generate $\mathbf{G}$, there may be less functions invariant by $\mathbf{G}^F$-conjugacy than by $(\mathbf{G}^1)^F$-conjugacy; but the propositions we prove will apply *in particular* to the $\mathbf{G}^F$-invariant functions so we do not lose any generality. The class functions on $(\mathbf{G}^1)^F$ are provided with the scalar product $\langle f,g\rangle_{(\mathbf{G}^1)^F} = |(\mathbf{G}^1)^F|^{-1}\sum_{h\in(\mathbf{G}^1)^F} f(h)\overline{g(h)}$. We call $\mathbf{G}$ reductive when $\mathbf{G}^0$ is reductive.

When $\mathbf{G}$ is reductive, following [St] we call quasi-semi-simple an element which normalizes a pair $\mathbf{T}^0\subset\mathbf{B}^0$ of a maximal torus of $\mathbf{G}^0$ and a Borel subgroup of $\mathbf{G}^0$. Following [DM94, 1.15], we call quasi-central a quasi-semi-simple element σ which has maximal dimension of centralizer $C_{\mathbf{G}^0}(\sigma)$ (that we will also denote by $\mathbf{G}^{0\sigma}$) amongst all quasi-semi-simple elements of $\mathbf{G}^0\cdot\sigma$.

In the sequel, we fix a reductive group $\mathbf{G}$ and (excepted in the next section where we take a "global" viewpoint) an F-stable connected component $\mathbf{G}^1$ of $\mathbf{G}$. In most of [DM94] we assumed that $(\mathbf{G}^1)^F$ contained a quasi-central element. Here we do not assume this. Note however that by [DM94, 1.34] $\mathbf{G}^1$ contains an element σ which induces an F-stable quasi-central automorphism of $\mathbf{G}^0$. Such an element will be enough for our purpose, and we fix one from now on.

By [DM94, 1.35] when $H^1(F, Z\mathbf{G}^0)=1$ then $(\mathbf{G}^1)^F$ contains quasi-central elements. Here is an example where $(\mathbf{G}^1)^F$ does not contain quasi-central elements.

Example 2.1. Take $s=\begin{pmatrix}\xi & 0\\ 0 & 1\end{pmatrix}$ where $\xi\in\mathbb{F}_q-\mathbb{F}_q^2$, take $\mathbf{G}^0=\mathrm{SL}_2$ and let $\mathbf{G}=<\mathbf{G}^0,s>\subset\mathrm{GL}_2$ endowed with the standard Frobenius endomorphism on GL_2, so that s is F-stable and $\mathbf{G}^F=\mathrm{GL}_2(\mathbb{F}_q)$. We take $\mathbf{G}^1=\mathbf{G}^0\cdot s$. Here quasi-central elements are central and coincide with $\mathbf{G}^0\cdot s\cap Z\mathbf{G}$ which is nonempty since if $\eta\in\mathbb{F}_{q^2}$ is a square root of ξ then $\begin{pmatrix}\eta & 0\\ 0 & \eta\end{pmatrix}\in\mathbf{G}^0\cdot s\cap Z\mathbf{G}$; but $\mathbf{G}^0\cdot s$ does not meet $(Z\mathbf{G})^F$. □

In the above example $\mathbf{G}^1/\mathbf{G}^0$ is semisimple. No such example exists when $\mathbf{G}^1/\mathbf{G}^0$ is unipotent:

Lemma 2.2. *Let $\mathbf{G}^1$ be an F-stable connected component of $\mathbf{G}$ such that $\mathbf{G}^1/\mathbf{G}^0$ is unipotent. Then $(\mathbf{G}^1)^F$ contains unipotent quasi-central elements.*

Proof. Let $\mathbf{T}^0\subset\mathbf{B}^0$ be a pair of an F-stable maximal torus of $\mathbf{G}^0$ and an F-stable Borel subgroup of $\mathbf{G}^0$. Then $N_{\mathbf{G}^F}(\mathbf{T}^0\subset\mathbf{B}^0)$ meets $(\mathbf{G}^1)^F$, since any two F-stable pairs $\mathbf{T}^0\subset\mathbf{B}^0$ are $(\mathbf{G}^0)^F$-conjugate. Let su be the Jordan decomposition of an element of $N_{(\mathbf{G}^1)^F}(\mathbf{T}^0\subset\mathbf{B}^0)$. Then $s\in\mathbf{G}^0$ since $\mathbf{G}^1/\mathbf{G}^0$ is unipotent, and u is F-stable, unipotent and still in $N_{(\mathbf{G}^1)^F}(\mathbf{T}^0\subset\mathbf{B}^0)$ thus quasi-semi-simple, so is quasi-central by [DM94, 1.33]. □

Note, however, that there may exist a unipotent quasi-central element σ which is rational as an automorphism but such that there is no rational element inducing the same automorphism.

Example 2.3. We give an example in $\mathbf{G} = \mathrm{SL}_5 \rtimes <\sigma'>$ where $\mathbf{G}^0 = \mathrm{SL}_5$ has the standard rational structure over a finite field $\mathbb{F}_q$ of characteristic 2 with $q \equiv 1 \mod 5$ and σ' is the automorphism of $\mathbf{G}^0$ given by $g \mapsto J{}^t g^{-1} J$ where J is the antidiagonal matrix with all non-zero entries equal to 1, so that σ' stabilizes the pair $\mathbf{T}^0 \subset \mathbf{B}^0$ where $\mathbf{T}^0$ is the maximal torus of diagonal matrices and $\mathbf{B}^0$ the Borel subgroup of upper triangular matrices, hence σ' is quasi-semi-simple. Let t be the diagonal matrix with entries (a, a, a^{-4}, a, a) where a^{q-1} is a non trivial 5-th root of unity $\zeta \in \mathbb{F}_q$. We claim that $\sigma = t\sigma'$ is as announced: it is still quasi-semi-simple; we have $\sigma^2 = t\sigma'(t) = tt^{-1} = 1$ so that σ is unipotent; we have ${}^F\sigma = {}^F t t^{-1}\sigma = \zeta\sigma$, so that σ is rational as an automorphism but not rational. Moreover a rational element inducing the same automorphism must be of the form $z\sigma$ with z central in $\mathbf{G}^0$ and $z \cdot {}^F z^{-1} = \zeta \,\mathrm{Id}$; but the center $Z\mathbf{G}^0$ is generated by $\zeta \,\mathrm{Id}$ and for any $z = \zeta^k \,\mathrm{Id} \in Z\mathbf{G}^0$ we have $z \cdot {}^F z^{-1} = \zeta^{k(q-1)} \,\mathrm{Id} = \mathrm{Id} \neq \zeta \,\mathrm{Id}$. □

As in [DM94] we call "Levi" of $\mathbf{G}$ a subgroup $\mathbf{L}$ of the form $N_{\mathbf{G}}(\mathbf{L}^0 \subset \mathbf{P}^0)$ where $\mathbf{L}^0$ is a Levi subgroup of the parabolic subgroup $\mathbf{P}^0$ of $\mathbf{G}^0$. A particular case is a "torus" $N_{\mathbf{G}}(\mathbf{T}^0, \mathbf{B}^0)$ where $\mathbf{T}^0 \subset \mathbf{B}^0$ is a pair of a maximal torus of $\mathbf{G}^0$ and a Borel subgroup of $\mathbf{G}^0$; note that a "torus" meets all connected components of $\mathbf{G}$, while (contrary to what is stated erroneously after [DM94, 1.4]) this may not be the case for a "Levi".

We call "Levi" of $\mathbf{G}^1$ a set of the form $\mathbf{L}^1 = \mathbf{L} \cap \mathbf{G}^1$ where $\mathbf{L}$ is a "Levi" of $\mathbf{G}$ and the intersection is nonempty; note that if $\mathbf{G}^1$ does not generate $\mathbf{G}$, there may exist several "Levis" of $\mathbf{G}$ which have same intersection with $\mathbf{G}^1$. Nevertheless $\mathbf{L}^1$ determines $\mathbf{L}^0$ as the identity component of $<\mathbf{L}^1>$.

We assume now that $\mathbf{L}^1$ is an F-stable "Levi" of $\mathbf{G}^1$ of the form $N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$. If $\mathbf{U}$ is the unipotent radical of $\mathbf{P}^0$, we define $\mathbf{Y}^0_{\mathbf{U}} = \{x \in \mathbf{G}^0 \mid x^{-1} \cdot {}^F x \in \mathbf{U}\}$ on which $(g, l) \in \mathbf{G}^F \times \mathbf{L}^F$ such that $gl \in \mathbf{G}^0$ acts by $x \mapsto gxl$, where $\mathbf{L} = N_{\mathbf{G}}(\mathbf{L}^0, \mathbf{P}^0)$. Along the same lines as [DM94, 2.10] we define

Definition 2.4. *Let $\mathbf{L}^1$ be an F-stable "Levi" of $\mathbf{G}^1$ of the form $N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$ and let $\mathbf{U}$ be the unipotent radical of $\mathbf{P}^0$. For λ a class function on $(\mathbf{L}^1)^F$ and $g \in (\mathbf{G}^1)^F$ we set*

$$R^{\mathbf{G}^1}_{\mathbf{L}^1}(\lambda)(g) = |(\mathbf{L}^1)^F|^{-1} \sum_{l \in (\mathbf{L}^1)^F} \lambda(l) \,\mathrm{Trace}((g, l^{-1}) \mid H^*_c(\mathbf{Y}^0_{\mathbf{U}}))$$

and for γ a class function on $(\mathbf{G}^1)^F$ and $l \in (\mathbf{L}^1)^F$ we set

$${}^*R^{\mathbf{G}^1}_{\mathbf{L}^1}(\gamma)(l) = |(\mathbf{G}^1)^F|^{-1} \sum_{g \in (\mathbf{G}^1)^F} \gamma(g) \,\mathrm{Trace}((g^{-1}, l) \mid H^*_c(\mathbf{Y}^0_{\mathbf{U}})).$$

In the above H^*_c denotes the ℓ-adic cohomology with compact support, where we have chosen once and for all a prime number $\ell \neq p$. In order to consider the virtual character $\mathrm{Trace}(x \mid H^*_c(\mathbf{X})) = \sum_i (-1)^i \,\mathrm{Trace}(x \mid H^i_c(\mathbf{X}, \overline{\mathbb{Q}}_\ell))$ as a complex character we chose once and for all an embedding $\overline{\mathbb{Q}}_\ell \hookrightarrow \mathbb{C}$.

Writing $R^{\mathbf{G}^1}_{\mathbf{L}^1}$ and ${}^*R^{\mathbf{G}^1}_{\mathbf{L}^1}$ is an abuse of notation: the definition needs the choice of a $\mathbf{P}^0$ such that $\mathbf{L}^1 = N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$. Our subsequent statements will use an implicit choice. Under certain assumptions we will prove a Mackey formula (Theorem 2.6) which when true implies that $R^{\mathbf{G}^1}_{\mathbf{L}^1}$ and ${}^*R^{\mathbf{G}^1}_{\mathbf{L}^1}$ are independent of the choice of $\mathbf{P}^0$.

By the same arguments as for [DM94, 2.10] (using that $(\mathbf{L}^1)^F$ is nonempty and [DM94, 2.3]) definition 2.4 agrees with the restriction to $(\mathbf{G}^1)^F$ and $(\mathbf{L}^1)^F$ of [DM94, 2.2].

The two maps $R_{\mathbf{L}^1}^{\mathbf{G}^1}$ and ${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}$ are adjoint with respect to the scalar products on $(\mathbf{G}^1)^F$ and $(\mathbf{L}^1)^F$.

We note the following variation on [DM94, 2.6].

Proposition 2.5. *Let su be the Jordan decomposition of an element of $(\mathbf{G}^1)^F$ and λ a class function on $(\mathbf{L}^1)^F$;*

(i) *if s is central, if we set*

$$Q_{\mathbf{L}^0}^{\mathbf{G}^0}(u,v)=\begin{cases}\operatorname{Trace}((u,v)\mid H_c^*(\mathbf{Y}_{\mathbf{U}}^0)) & \text{if } uv\in\mathbf{G}^0\\ 0 & \text{otherwise}\end{cases},$$

we have

$$(R_{\mathbf{L}^1}^{\mathbf{G}^1}\lambda)(su)=|(\mathbf{L}^0)^F|^{-1}\sum_{v\in(\mathbf{L}^0\cdot u)^F_{unip}}Q_{\mathbf{L}^0}^{\mathbf{G}^0}(u,v^{-1})\lambda(sv)$$

(ii) *in general*

$$(R_{\mathbf{L}^1}^{\mathbf{G}^1}\lambda)(su)=\sum_{\{h\in(\mathbf{G}^0)^F\mid {}^h\mathbf{L}\ni s\}}\frac{|{}^h\mathbf{L}^0\cap C_{\mathbf{G}}(s)^{0F}|}{|(\mathbf{L}^0)^F||C_{\mathbf{G}}(s)^{0F}|}R_{{}^h\mathbf{L}^1\cap C_{\mathbf{G}}(s)^0\cdot su}^{C_{\mathbf{G}}(s)^0\cdot su}({}^h\lambda)(su).$$

(iii) *if tv is the Jordan decomposition of an element of $(\mathbf{L}^1)^F$ and γ a class function on $(\mathbf{G}^1)^F$; we have*

$$({}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}\gamma)(tv)=|(\mathbf{G}^{t0})^F|^{-1}\sum_{u\in(\mathbf{G}^{t0}\cdot v)^F_{unip}}Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(u,v^{-1})\gamma(tu).$$

In the above we abused notation to write ${}^h\mathbf{L}\ni s$ for $<\mathbf{L}^1>\ni {}^{h^{-1}}s$.

Proof. (i) results from [DM94, 2.6(i)] using the same arguments as the proof of [DM94, 2.10]; we then get (ii) by plugging back (i) in [DM94, 2.6(i)]. □

In our setting the Mackey formula [DM94, 3.1] is still valid in the cases where we proved it [DM94, Théorème 3.2] and [DM94, Théorème 4.5]. Before stating it notice that [DM94, 1.40] remains true without assuming that $(\mathbf{G}^1)^F$ contains quasi-central elements, replacing in the proof $(\mathbf{G}^0)^F.\sigma$ with $(\mathbf{G}^1)^F$, which shows that any F-stable "Levi" of $\mathbf{G}^1$ is $(\mathbf{G}^0)^F$-conjugate to a "Levi" containing σ. This explains why we only state the Mackey formula in the case of "Levis" containing σ.

Theorem 2.6. *If $\mathbf{L}^1$ and $\mathbf{M}^1$ are two F-stable "Levis" of $\mathbf{G}^1$ containing σ then under one of the following assumptions:*

- *$\mathbf{L}^0$ (resp. $\mathbf{M}^0$) is a Levi subgroup of an F-stable parabolic subgroup normalized by $\mathbf{L}^1$ (resp. $\mathbf{M}^1$).*
- *one of $\mathbf{L}^1$ and $\mathbf{M}^1$ is a "torus"*

we have

$${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}R_{\mathbf{M}^1}^{\mathbf{G}^1}=\sum_{x\in[\mathbf{L}^{\sigma0F}\backslash\mathcal{S}_{\mathbf{G}^{\sigma0}}(\mathbf{L}^{\sigma0},\mathbf{M}^{\sigma0})^F/\mathbf{M}^{\sigma0F}]}R_{(\mathbf{L}^1\cap{}^x\mathbf{M}^1)}^{\mathbf{L}^1}\,{}^*R_{(\mathbf{L}^1\cap{}^x\mathbf{M}^1)}^{{}^x\mathbf{M}^1}\operatorname{ad}x$$

where $\mathcal{S}_{\mathbf{G}^{\sigma0}}(\mathbf{L}^{\sigma0},\mathbf{M}^{\sigma0})$ is the set of elements $x\in\mathbf{G}^{\sigma0}$ such that $\mathbf{L}^{\sigma0}\cap{}^x\mathbf{M}^{\sigma0}$ contains a maximal torus of $\mathbf{G}^{\sigma0}$.

Proof. We first prove the theorem in the case of F-stable parabolic subgroups $\mathbf{P}^0 = \mathbf{L}^0 \ltimes \mathbf{U}$ and $\mathbf{Q}^0 = \mathbf{M}^0 \ltimes \mathbf{V}$ following the proof of [DM94, 3.2]. The difference is that the variety we consider here is the intersection with $\mathbf{G}^0$ of the variety considered in *loc. cit.*. Here, the left-hand side of the Mackey formula is given by $\overline{\mathbb{Q}}_\ell[(\mathbf{U}^F\backslash(\mathbf{G}^0)^F/\mathbf{V}^F)^\sigma]$ instead of $\overline{\mathbb{Q}}_\ell[(\mathbf{U}^F\backslash(\mathbf{G}^0)^F.{<}\sigma{>}/\mathbf{V}^F)^\sigma]$. Nevertheless we can use [DM94, Lemma 3.3] which remains valid with the same proof. As for [DM94, Lemma 3.5], we have to replace it with

Lemma 2.7. *For any $x \in \mathcal{S}_{\mathbf{G}^{\sigma 0}}(\mathbf{L}^{\sigma 0}, \mathbf{M}^{\sigma 0})^F$ the map $(l(\mathbf{L}^0 \cap {}^x\mathbf{V}^F), ({}^x\mathbf{M}^0 \cap \mathbf{U}^F)\cdot {}^xm) \mapsto \mathbf{U}^F lxm\mathbf{V}^F$ is an isomorphism from $(\mathbf{L}^0)^F/(\mathbf{L}^0 \cap {}^x\mathbf{V}^F) \times_{(\mathbf{L}^0\cap {}^x\mathbf{M}^0)^F} ({}^x\mathbf{M}^0 \cap \mathbf{U}^F)\backslash {}^x(\mathbf{M}^0)^F$ to $\mathbf{U}^F\backslash(\mathbf{P}^0)^F x(\mathbf{Q}^0)^F/\mathbf{V}^F$ which is compatible with the action of $(\mathbf{L}^1)^F \times ((\mathbf{M}^1)^F)^{-1}$ where the action of $(\lambda, \mu^{-1}) \in (\mathbf{L}^1)^F \times ((\mathbf{M}^1)^F)^{-1}$ maps $(l(\mathbf{L}^0 \cap {}^x\mathbf{V}^F), ({}^x\mathbf{M}^0 \cap \mathbf{U}^F)\cdot {}^xm)$ to the class of $(\lambda l\nu^{-1}(\mathbf{L}^0 \cap {}^x\mathbf{V}^F), ({}^x\mathbf{M}^0 \cap \mathbf{U}^F)\cdot \nu {}^xm\mu^{-1})$ with $\nu \in (\mathbf{L}^1)^F \cap {}^x(\mathbf{M}^1)^F$ (independent of ν).*

Proof. The isomorphism of the lemma involves only connected groups and is a known result (see *e.g.* [DM91, 5.7]). The compatibility with the actions is straightforward. □

This allows to complete the proof in the first case.

We now prove the second case following section 4 of [DM94]. We first notice that the statement and proof of Lemma 4.1 in [DM94] don't use the element σ but only its action. In Lemma 4.2, 4.3 and 4.4 there is no σ involved but only the action of the groups $\mathbf{L}^F$ and $\mathbf{M}^F$ on the pieces of a variety depending only on $\mathbf{L}$, $\mathbf{M}$ and the associated parabolics. This gives the second case. □

We now rephrase [DM94, 4.8] and [DM94, 4.11] in our setting, specializing the Mackey formula to the case of two "tori". Let $\mathcal{T}_1$ be the set of "tori" of $\mathbf{G}^1$; if $\mathbf{T}^1 = N_{\mathbf{G}^1}(\mathbf{T}^0, \mathbf{B}^0) \in \mathcal{T}_1^F$ then $\mathbf{T}^0$ is F-stable. We define $\mathrm{Irr}((\mathbf{T}^1)^F)$ as the set of restrictions to $(\mathbf{T}^1)^F$ of extensions to ${<}(\mathbf{T}^1)^F{>}$ of elements of $\mathrm{Irr}((\mathbf{T}^0)^F)$.

Proposition 2.8. *If $\mathbf{T}^1, \mathbf{T}'^1 \in \mathcal{T}_1^F$ and $\theta \in \mathrm{Irr}((\mathbf{T}^1)^F), \theta' \in \mathrm{Irr}((\mathbf{T}'^1)^F)$ then*

$$\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}'^1}^{\mathbf{G}^1}(\theta')\rangle_{(\mathbf{G}^1)^F} = 0 \textit{ unless } (\mathbf{T}^1, \theta) \textit{ and } (\mathbf{T}'^1, \theta') \textit{ are } (\mathbf{G}^0)^F\textit{-conjugate.}$$

And

(i) *If for some $n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)$ and $\zeta \neq 1$ we have ${}^n\theta = \zeta\theta$ then $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta) = 0$.*

(ii) *Otherwise $\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F} = |\{n \in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1) \mid {}^n\theta = \theta\}|/|(\mathbf{T}^1)^F|$.*

If $\mathbf{T}^1 = \mathbf{T}'^1$ the above can be written

$$\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta')\rangle_{(\mathbf{G}^1)^F} = \langle \mathrm{Ind}_{(\mathbf{T}^1)^F}^{N_{\mathbf{G}^1}(\mathbf{T}^0)^F}\theta, \mathrm{Ind}_{(\mathbf{T}^1)^F}^{N_{\mathbf{G}^1}(\mathbf{T}^0)^F}\theta'\rangle_{N_{\mathbf{G}^1}(\mathbf{T}^0)^F}$$

where when $A^1 \subset B^1$ are cosets of finite groups $A^0 \subset B^0$ and χ is a A^0-class function on A^1 we set $\mathrm{Ind}_{A^1}^{B^1}\chi(x) = |A^0|^{-1}\sum_{\{y \in B^0 \mid {}^yx \in A^1\}}\chi({}^yx)$.

Proof. As noticed above Theorem 2.6 we may assume that $\mathbf{T}^1$ and $\mathbf{T}'^1$ contain σ. By [DM94, 1.39], if $\mathbf{T}^1$ and $\mathbf{T}'^1$ contain σ, they are $(\mathbf{G}^0)^F$ conjugate if and only if they are conjugate under ${\mathbf{G}^{\sigma 0}}^F$. The Mackey formula shows then that the scalar product vanishes when $\mathbf{T}^1$ and $\mathbf{T}'^1$ are not $(\mathbf{G}^0)^F$-conjugate.

Otherwise we may assume $\mathbf{T}^1 = \mathbf{T}'^1$ and the Mackey formula gives

$$\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta), R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F} = |(\mathbf{T}^{\sigma 0})^F|^{-1}\sum_{n \in N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F}\langle \theta, {}^n\theta\rangle_{(\mathbf{T}^1)^F}.$$

The term $\langle\theta,{}^n\theta\rangle_{(\mathbf{T}^1)^F}$ is 0 unless ${}^n\theta=\zeta_n\theta$ for some constant ζ_n and in this last case $\langle\theta,{}^n\theta\rangle_{(\mathbf{T}^1)^F}=\overline{\zeta}_n$. If ${}^{n'}\theta=\zeta_{n'}\theta$ then ${}^{nn'}\theta=\zeta_{n'}{}^n\theta=\zeta_{n'}\zeta_n\theta$ thus the ζ_n form a group; if this group is not trivial, that is some ζ_n is not equal to 1, we have $\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta),R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F}=0$ which implies that in this case $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)=0$. This gives (i) since by [DM94, 1.39], if $\mathbf{T}^1\ni\sigma$ then $N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)=N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F\cdot(\mathbf{T}^0)^F$, so that if there exists n as in (i) there exists an $n\in N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F$ with same action on θ since $(\mathbf{T}^0)^F$ has trivial action on θ.

In case (ii), for each non-zero term we have ${}^n\theta=\theta$ and we have to check that the value $|((\mathbf{T}^\sigma)^0)^F|^{-1}\{n\in N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F\mid {}^n\theta=\theta\}|$ given by the Mackey formula is equal to the stated value. This results again from [DM94, 1.39] written $N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)=N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^1)^F\cdot(\mathbf{T}^0)^F$, and from $N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^1)^F\cap(\mathbf{T}^0)^F=((\mathbf{T}^\sigma)^0)^F$.

We now prove the final remark. By definition we have

$$\langle \operatorname{Ind}_{(\mathbf{T}^1)^F}^{N_{\mathbf{G}^1}(\mathbf{T}^0)^F}\theta,\operatorname{Ind}_{(\mathbf{T}^1)^F}^{N_{\mathbf{G}^1}(\mathbf{T}^0)^F}\theta'\rangle_{N_{\mathbf{G}^1}(\mathbf{T}^0)^F}= \\ |N_{\mathbf{G}^1}(\mathbf{T}^0)^F|^{-1}|(\mathbf{T}^1)^F|^{-2}\sum_{x\in N_{\mathbf{G}^1}(\mathbf{T}^0)^F}\sum_{\{n,n'\in N_{\mathbf{G}^1}(\mathbf{T}^0)^F\mid {}^nx,{}^{n'}x\in\mathbf{T}^1\}}\theta({}^nx)\overline{\theta({}^{n'}x)}.$$

Doing the summation over $t={}^nx$ and $n''=n'n^{-1}\in N_{\mathbf{G}^0}(\mathbf{T}^0)^F$ we get

$$|N_{\mathbf{G}^1}(\mathbf{T}^0)^F|^{-1}|(\mathbf{T}^1)^F|^{-2}\sum_{t\in(\mathbf{T}^1)^F}\sum_{n\in N_{\mathbf{G}^1}(\mathbf{T}^0)^F}\sum_{\{n''\in N_{\mathbf{G}^0}(\mathbf{T}^0)^F\mid {}^{n''}t\in\mathbf{T}^1\}}\theta(t)\overline{\theta({}^{n''}t)}.$$

The conditions $n''\in N_{\mathbf{G}^0}(\mathbf{T}^0)^F$ together with ${}^{n''}t\in\mathbf{T}^1$ are equivalent to $n''\in N_{\mathbf{G}^0}(\mathbf{T}^1)^F$, so that we get $|(\mathbf{T}^1)^F|^{-1}\sum_{n''\in N_{\mathbf{G}^0}(\mathbf{T}^1)^F}\langle\theta,{}^{n''}\theta\rangle_{(\mathbf{T}^1)^F}$. As explained in the first part of the proof, the scalar product $\langle\theta,{}^{n''}\theta\rangle_{(\mathbf{T}^1)^F}$ is zero unless ${}^{n''}\theta=\zeta_{n''}\theta$ for some root of unity $\zeta_{n''}$ and arguing as in the first part of the proof we find that the above sum is zero if there exists n'' such that $\zeta_{n''}\neq 1$ and is equal to $|(\mathbf{T}^1)^F|^{-1}|\{n\in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)\mid {}^n\theta=\theta\}|$ otherwise. □

Remark 2.9. In the context of Proposition 2.8, if σ is F-stable then we may apply θ to it and for any $n\in N_{\mathbf{G}^{\sigma 0}}(\mathbf{T}^{\sigma 0})^F$ we have $\theta({}^n\sigma)=\theta(\sigma)$ so for any $n\in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)$ and ζ such that ${}^n\theta=\zeta\theta$ we have $\zeta=1$. When $H^1(F,Z\mathbf{G}^0)=1$ we may choose σ to be F-stable, so that $\zeta\neq 1$ never happens.

Here is an example where $\zeta_n=-1$, thus $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)=0$: we take again the context of Example 2.1 and take $\mathbf{T}^0=\begin{pmatrix}a&0\\0&a^{-1}\end{pmatrix}$ and let $\mathbf{T}^1=\mathbf{T}^0\cdot s$; let us define θ on $ts\in(\mathbf{T}^1)^F$ by $\theta(ts)=-\lambda(t)$ where λ is the non-trivial order 2 character of $(\mathbf{T}^0)^F$ (Legendre symbol); then for any $n\in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)\backslash\mathbf{T}^0$ we have ${}^n\theta=-\theta$. □

Proposition [DM94, 4.11] extends as follows to our context:

Corollary 2.10 (of 2.8)**.** *Let $p^{\mathbf{G}^1}$ be the projector to uniform functions on $(\mathbf{G}^1)^F$. We have*

$$p^{\mathbf{G}^1}=|(\mathbf{G}^1)^F|^{-1}\sum_{\mathbf{T}^1\in\mathcal{T}_1^F}|(\mathbf{T}^1)^F|R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ {}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}.$$

Proof. We have only to check that for any $\theta\in\operatorname{Irr}((\mathbf{T}^1)^F)$ such that $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\neq 0$ and any class function χ on $(\mathbf{G}^1)^F$ we have $\langle p^{\mathbf{G}^1}\chi,R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F}=\langle\chi,R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F}$.

By Proposition 2.8, to evaluate the left-hand side we may restrict the sum to tori conjugate to $\mathbf{T}^1$, so we get

$$\begin{aligned}\langle p^{\mathbf{G}^1}\chi, R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F} &= |N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)|^{-1}|(\mathbf{T}^1)^F|\langle R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ {}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}\chi, R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F}\\ &= |N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)|^{-1}|(\mathbf{T}^1)^F|\langle \chi, R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ {}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)\rangle_{(\mathbf{G}^1)^F}.\end{aligned}$$

The equality to prove is true if $R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)=0$; otherwise by Proposition 2.8 we have ${}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)=|(\mathbf{T}^1)^F|^{-1}\sum_{n\in N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)} {}^n\theta$, whence in that case

$$R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ {}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)=|(\mathbf{T}^1)^F|^{-1}|N_{(\mathbf{G}^0)^F}(\mathbf{T}^1)|R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta),$$

since $R_{\mathbf{T}^1}^{\mathbf{G}^1}({}^n\theta)=R_{\mathbf{T}^1}^{\mathbf{G}^1}(\theta)$, whence the result. □

We now adapt the definition of duality to our setting.

Definition 2.11.
- *For a connected reductive group* $\mathbf{G}$*, we define the* $\mathbb{F}_q$*-rank as the maximal dimension of a split torus, and define* $\varepsilon_{\mathbf{G}}=(-1)^{\mathbb{F}_q\text{-rank of }\mathbf{G}}$ *and* $\eta_{\mathbf{G}}=\varepsilon_{\mathbf{G}/\operatorname{rad}\mathbf{G}}$*.*
- *For an* F*-stable connected component* $\mathbf{G}^1$ *of a (possibly disconnected) reductive group we define* $\varepsilon_{\mathbf{G}^1}=\varepsilon_{\mathbf{G}^{\sigma 0}}$ *and* $\eta_{\mathbf{G}^1}=\eta_{\mathbf{G}^{\sigma 0}}$ *where* σ *is a quasi-central element of* $\mathbf{G}^1$ *which induces an* F*-stable automorphism of* $\mathbf{G}^0$*.*

Let us see that these definitions agree with [DM94]: in [DM94, 3.6(i)], we define $\varepsilon_{\mathbf{G}^1}$ to be $\varepsilon_{\mathbf{G}^{0\tau}}$ where τ is any quasi-semi-simple element of $\mathbf{G}^1$ which induces an F-stable automorphism of $\mathbf{G}^0$ and lies in a "torus" of the form $N_{\mathbf{G}^1}(\mathbf{T}_0\subset\mathbf{B}_0)$ where both $\mathbf{T}^0$ and $\mathbf{B}^0$ are F-stable; by [DM94, 1.36(ii)] a σ as above is such a τ.

We fix an F-stable pair $(\mathbf{T}_0\subset\mathbf{B}_0)$ and define duality on $\operatorname{Irr}((\mathbf{G}^1)^F)$ by

$$D_{\mathbf{G}^1}=\sum_{\mathbf{P}^0\supset\mathbf{B}^0}\eta_{\mathbf{L}^1}R_{\mathbf{L}^1}^{\mathbf{G}^1}\circ {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} \tag{2.12}$$

where in the sum $\mathbf{P}^0$ runs over F-stable parabolic subgroups containing $\mathbf{B}^0$ such that $N_{\mathbf{G}^1}(\mathbf{P}^0)$ is non empty, and $\mathbf{L}^1$ denotes $N_{\mathbf{G}^1}(\mathbf{L}^0\subset\mathbf{P}^0)$ where $\mathbf{L}^0$ is the Levi subgroup of $\mathbf{P}^0$ containing $\mathbf{T}^0$. The duality thus defined coincides with the duality defined in [DM94, 3.10] when σ is in $(\mathbf{G}^1)^F$.

In our context we can define $\operatorname{St}_{\mathbf{G}^1}$ similarly to [DM94, 3.16], as $D_{\mathbf{G}^1}(\operatorname{Id}_{\mathbf{G}^1})$, and [DM94, 3.18] remains true:

Proposition 2.13. $\operatorname{St}_{\mathbf{G}^1}$ *vanishes outside quasi-semi-simple elements, and if* $x\in(\mathbf{G}^1)^F$ *is quasi-semi-simple we have*

$$\operatorname{St}_{\mathbf{G}^1}(x)=\varepsilon_{\mathbf{G}^1}\varepsilon_{(\mathbf{G}^x)^0}|(\mathbf{G}^x)^0|_p.$$

3. A global formula for the scalar product of Deligne-Lusztig characters.

In this section we give a result of a different flavor, where we do not restrict our attention to a connected component $\mathbf{G}^1$.

Definition 3.1. *For any character* θ *of* $\mathbf{T}^F$*, we define* $R_{\mathbf{T}}^{\mathbf{G}}$ *as in* [DM94, 2.2]*. If for a "torus"* $\mathbf{T}$ *and* $\alpha=g\mathbf{G}^0\in\mathbf{G}/\mathbf{G}^0$ *we denote by* $\mathbf{T}^{[\alpha]}$ *or* $\mathbf{T}^{[g]}$ *the unique connected component of* $\mathbf{T}$ *which meets* $g\mathbf{G}^0$*, this is equivalent for* $g\in\mathbf{G}^F$ *to*

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g)=|(\mathbf{T}^0)^F|/|\mathbf{T}^F|\sum_{\{a\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\mid {}^ag\in\mathbf{T}^F(\mathbf{G}^0)^F\}} R_{\mathbf{T}^{[{}^ag]}}^{\mathbf{G}^{[{}^ag]}}(\theta)({}^ag)$$

where the right-hand side is defined by 2.4 (see [DM94, 2.3]*).*

We deduce from Proposition 2.8 the following formula for the whole group $\mathbf{G}$:

Proposition 3.2. *Let* $\mathbf{T}$, $\mathbf{T}'$ *be two "tori" of* $\mathbf{G}$ *and let* $\theta\in\operatorname{Irr}(\mathbf{T}^F)$, $\theta'\in\operatorname{Irr}(\mathbf{T}'^F)$. *Then* $\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle_{\mathbf{G}^F}=0$ *if* $\mathbf{T}^0$ *and* $\mathbf{T}'^0$ *are not* $\mathbf{G}^F$*-conjugate, and if* $\mathbf{T}^0=\mathbf{T}'^0$ *we have*

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle_{\mathbf{G}^F}=\langle \operatorname{Ind}_{\mathbf{T}^F}^{N_{\mathbf{G}}(\mathbf{T}^0)^F}(\theta), \operatorname{Ind}_{\mathbf{T}'^F}^{N_{\mathbf{G}}(\mathbf{T}^0)^F}(\theta')\rangle_{N_{\mathbf{G}}(\mathbf{T}^0)^F}.$$

Proof. Definition 3.1 can be written

$$R_{\mathbf{T}}^{\mathbf{G}}(\theta)(g)=|(\mathbf{T}^0)^F|/|\mathbf{T}^F|\sum_{\{a\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\mid {}^a g\in\mathbf{T}^F(\mathbf{G}^0)^F\}} R^{\mathbf{G}^{[g]}}_{({}^{a^{-1}}\mathbf{T})^{[g]}}({}^{a^{-1}}\theta)(g).$$

So the scalar product we want to compute is equal to

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle_{\mathbf{G}^F}=\frac{1}{|\mathbf{G}^F|}\frac{|(\mathbf{T}^0)^F|}{|\mathbf{T}^F|}\frac{|\mathbf{T}'^{0F}|}{|\mathbf{T}'^F|}$$
$$\sum_{\substack{\alpha\in\mathbf{G}^F/\mathbf{G}^{0F}\\ g\in(\mathbf{G}^0)^F.\alpha}}\ \sum_{\substack{\{a\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\mid {}^a\alpha\in\mathbf{T}^F(\mathbf{G}^0)^F\}\\ \{a'\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\mid {}^{a'}\alpha\in\mathbf{T}'^F(\mathbf{G}^0)^F\}}} R^{\mathbf{G}.\alpha}_{({}^{a^{-1}}\mathbf{T})^{[\alpha]}}({}^{a^{-1}}\theta)(g)\overline{R^{\mathbf{G}.\alpha}_{({}^{a'^{-1}}\mathbf{T}')^{[\alpha]}}({}^{a'^{-1}}\theta')(g)},$$

which can be written

$$\langle R_{\mathbf{T}}^{\mathbf{G}}(\theta), R_{\mathbf{T}'}^{\mathbf{G}}(\theta')\rangle_{\mathbf{G}^F}=\frac{|(\mathbf{G}^0)^F|}{|\mathbf{G}^F|}\frac{|(\mathbf{T}^0)^F|}{|\mathbf{T}^F|}\frac{|\mathbf{T}'^{0F}|}{|\mathbf{T}'^F|}$$
$$\sum_{\alpha\in\mathbf{G}^F/\mathbf{G}^{0F}}\ \sum_{\substack{\{a\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\mid {}^a\alpha\in\mathbf{T}^F(\mathbf{G}^0)^F\}\\ \{a'\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\mid {}^{a'}\alpha\in\mathbf{T}'^F(\mathbf{G}^0)^F\}}} \langle R^{\mathbf{G}.\alpha}_{({}^{a^{-1}}\mathbf{T})^{[\alpha]}}({}^{a^{-1}}\theta), R^{\mathbf{G}.\alpha}_{({}^{a'^{-1}}\mathbf{T}')^{[\alpha]}}({}^{a'^{-1}}\theta')\rangle_{(\mathbf{G}^0)^F.\alpha}.$$

By Proposition 2.8 the scalar product on the right-hand side is zero unless $({}^{a^{-1}}\mathbf{T})^{[\alpha]}$ and $({}^{a'^{-1}}\mathbf{T}')^{[\alpha]}$ are $(\mathbf{G}^0)^F$-conjugate, which implies that $\mathbf{T}^0$ and $\mathbf{T}'^0$ are $(\mathbf{G}^0)^F$-conjugate. So we can assume that $\mathbf{T}^0=\mathbf{T}'^0$. Moreover for each a' indexing a non-zero summand, there is a representative $y\in a'^{-1}(\mathbf{G}^0)^F$ such that $({}^y\mathbf{T}')^{[\alpha]}=({}^{a^{-1}}\mathbf{T})^{[\alpha]}$. This last equality and the condition on a imply the condition ${}^{a'}\alpha\in\mathbf{T}'^F(\mathbf{G}^0)^F$ since this condition can be written $({}^y\mathbf{T}')^{[\alpha]}\neq\emptyset$. Thus we can do the summation over all such $y\in\mathbf{G}^F$, provided we divide by $|N_{(\mathbf{G}^0)^F}(({}^{a^{-1}}\mathbf{T})^{[\alpha]})|$. So we get, applying Proposition 2.8 that the above expression is equal to

$$\frac{|(\mathbf{G}^0)^F|}{|\mathbf{G}^F|}\frac{|(\mathbf{T}^0)^F|^2}{|\mathbf{T}^F||\mathbf{T}'^F|}\sum_{\alpha\in\mathbf{G}^F/\mathbf{G}^{0F}}\ \sum_{\{a\in[\mathbf{G}^F/(\mathbf{G}^0)^F]\mid {}^a\alpha\in\mathbf{T}^F(\mathbf{G}^0)^F\}}|N_{(\mathbf{G}^0)^F}(({}^{a^{-1}}\mathbf{T})^{[\alpha]})|^{-1}$$
$$\sum_{\{y\in\mathbf{G}^F\mid({}^y\mathbf{T}')^{[\alpha]}=({}^{a^{-1}}\mathbf{T})^{[\alpha]}\}}\langle\operatorname{Ind}_{({}^{a^{-1}}\mathbf{T})^{[\alpha]}}^{N_{\mathbf{G}^0.\alpha}({}^{a^{-1}}\mathbf{T}^0)^F}{}^{a^{-1}}\theta, \operatorname{Ind}_{({}^{a^{-1}}\mathbf{T})^{[\alpha]}}^{N_{\mathbf{G}^0.\alpha}({}^{a^{-1}}\mathbf{T}^0)^F}{}^{y}\theta'\rangle_{N_{\mathbf{G}^0.\alpha}(\mathbf{T}^0)^F}.$$

We now conjugate everything by a, take ay as new variable y and set $b={}^a\alpha$. We get

$$\frac{|(\mathbf{T}^0)^F|^2}{|\mathbf{T}^F||\mathbf{T}'^F|}\sum_{b\in\mathbf{T}^F/(\mathbf{T}^0)^F}|N_{(\mathbf{G}^0)^F}(\mathbf{T}^{[b]})|^{-1}$$
$$\sum_{\{y\in\mathbf{G}^F\mid({}^y\mathbf{T}')^{[b]}=\mathbf{T}^{[b]}\}}\langle\operatorname{Ind}_{\mathbf{T}^{[b]F}}^{N_{\mathbf{G}^0.b}(\mathbf{T}^0)^F}\theta, \operatorname{Ind}_{\mathbf{T}^{[b]F}}^{N_{\mathbf{G}^0.b}(\mathbf{T}^0)^F}{}^y\theta'\rangle_{N_{\mathbf{G}^0.b}(\mathbf{T}^0)^F}, \tag{3.3}$$

since for $b\in \mathbf{T}^F/(\mathbf{T}^0)^F$ any choice of $a\in \mathbf{G}^F/(\mathbf{G}^0)^F$ gives an $\alpha={}^{a^{-1}}b$ which satisfies the condition ${}^a\alpha\in\mathbf{T}^F(\mathbf{G}^0)^F$.

Let us now transform the right-hand side of 3.2. Using the definition we have

$$\langle \operatorname{Ind}_{\mathbf{T}^F}^{N_{\mathbf{G}}(\mathbf{T}^0)^F}(\theta),\operatorname{Ind}_{\mathbf{T}'^F}^{N_{\mathbf{G}}(\mathbf{T}^0)^F}(\theta)\rangle_{N_{\mathbf{G}}(\mathbf{T}^0)^F}=$$
$$|\mathbf{T}^F|^{-1}|\mathbf{T}'^F|^{-1}|N_{\mathbf{G}}(\mathbf{T}^0)^F|^{-1}\sum_{\{n,x,x'\in N_{\mathbf{G}}(\mathbf{T}^0)^F\mid {}^xn\in\mathbf{T},{}^{x'}n\in\mathbf{T}'\}}\theta({}^xn)\overline{\theta'({}^{x'}n)}=$$
$$|\mathbf{T}^F|^{-1}|\mathbf{T}'^F|^{-1}|N_{\mathbf{G}}(\mathbf{T}^0)^F|^{-1}$$
$$\sum_{b,a,a'\in[N_{\mathbf{G}}(\mathbf{T}^0)^F/N_{\mathbf{G}^0}(\mathbf{T}^0)^F]}\ \sum_{\left\{\begin{smallmatrix} n\in N_{\mathbf{G}^0}(\mathbf{T}^0)^Fb\\ x_0,x_0'\in N_{\mathbf{G}^0}(\mathbf{T}^0)^F\end{smallmatrix}\,\middle|\,\begin{smallmatrix}{}^{x_0}n\in({}^{a^{-1}}\mathbf{T})^{[b]}\\ {}^{x_0'}n\in({}^{a'^{-1}}\mathbf{T}')^{[b]}\end{smallmatrix}\right\}}{}^{a^{-1}}\theta({}^{x_0}n)\overline{{}^{a'^{-1}}\theta'({}^{x_0'}n)}=$$
$$\frac{|(\mathbf{T}^0)^F|}{|\mathbf{T}^F|}\frac{|\mathbf{T}'^{0F}|}{|\mathbf{T}'^F|}\frac{|N_{\mathbf{G}^0}(\mathbf{T}^0)^F|}{|N_{\mathbf{G}}(\mathbf{T}^0)^F|}$$
$$\sum_{b,a,a'\in[N_{\mathbf{G}}(\mathbf{T}^0)^F/N_{\mathbf{G}^0}(\mathbf{T}^0)^F]}\langle \operatorname{Ind}_{({}^{a^{-1}}\mathbf{T})^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F\cdot b}\,{}^{a^{-1}}\theta,\operatorname{Ind}_{({}^{a'^{-1}}\mathbf{T}')^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F\cdot b}\,{}^{a'^{-1}}\theta'\rangle_{N_{\mathbf{G}^0}(\mathbf{T}^0)^Fb}.$$

We may simplify the sum by conjugating by a the terms in the scalar product to get

$$\langle \operatorname{Ind}_{\mathbf{T}^{[{}^ab]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F\cdot{}^ab}\,\theta,\operatorname{Ind}_{({}^{aa'^{-1}}\mathbf{T}')^{[{}^ab]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F\cdot{}^ab}\,{}^{aa'^{-1}}\theta'\rangle_{N_{\mathbf{G}^0}(\mathbf{T}^0)^F\,{}^ab}$$

then we may take, given a, the conjugate ${}^ab$ as new variable b, and aa'^{-1} as the new variable a' to get

$$\frac{|(\mathbf{T}^0)^F|}{|\mathbf{T}^F|}\frac{|\mathbf{T}'^{0F}|}{|\mathbf{T}'^F|}\sum_{b,a'\in[\frac{N_{\mathbf{G}}(\mathbf{T}^0)^F}{N_{\mathbf{G}^0}(\mathbf{T}^0)^F}]}\langle \operatorname{Ind}_{\mathbf{T}^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F\cdot b}\,\theta,\operatorname{Ind}_{({}^{a'}\mathbf{T}')^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F\cdot b}\,{}^{a'}\theta'\rangle_{N_{\mathbf{G}^0}(\mathbf{T}^0)^Fb}.$$

Now, by Frobenius reciprocity, for the inner scalar product not to vanish, there must be some element $x\in N_{\mathbf{G}^0}(\mathbf{T}^0)^F$ such that ${}^x({}^{a'}\mathbf{T}')^{[b]F}$ meets $\mathbf{T}^{[b]F}$ which, considering the definitions, implies that $({}^{xa'}\mathbf{T}')^{[b]}=\mathbf{T}^{[b]}$. We may then conjugate the term $\operatorname{Ind}_{({}^{a'}\mathbf{T}')^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F\cdot b}\,{}^{a'}\theta'$ by such an x to get $\operatorname{Ind}_{\mathbf{T}^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F\cdot b}\,{}^{xa'}\theta'$ and take $y=xa'$ as a new variable, provided we count the number of x for a given a', which is $|N_{\mathbf{G}^0}(\mathbf{T}^{[b]})^F|$. We get

$$\text{(3.4)}\quad \frac{|(\mathbf{T}^0)^F|}{|\mathbf{T}^F|}\frac{|\mathbf{T}'^{0F}|}{|\mathbf{T}'^F|}\sum_{b\in[N_{\mathbf{G}}(\mathbf{T}^0)^F/N_{\mathbf{G}^0}(\mathbf{T}^0)^F]}|N_{\mathbf{G}^0}(\mathbf{T}^{[b]})^F|^{-1}$$
$$\sum_{\{y\in N_{\mathbf{G}}(\mathbf{T}^0)^F\mid({}^y\mathbf{T}')^{[b]}=\mathbf{T}^{[b]}\}}\langle \operatorname{Ind}_{\mathbf{T}^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F\cdot b}\,\theta,\operatorname{Ind}_{\mathbf{T}^{[b]F}}^{N_{\mathbf{G}^0}(\mathbf{T}^0)^F\cdot b}\,{}^y\theta'\rangle_{N_{\mathbf{G}^0}(\mathbf{T}^0)^Fb}.$$

Since any $b\in[N_{\mathbf{G}}(\mathbf{T}^0)^F/N_{\mathbf{G}^0}(\mathbf{T}^0)^F]$ such that $\mathbf{T}^{[b]F}$ is not empty has a representative in $\mathbf{T}^F$ we can do the first summation over $b\in[\mathbf{T}^F/(\mathbf{T}^0)^F]$ so that 3.3 is equal to 3.4. □

4. Counting unipotent elements in disconnected groups

Proposition 4.1. *Assume $\mathbf{G}^1/\mathbf{G}^0$ unipotent and take $\sigma\in\mathbf{G}^1$ unipotent F-stable and quasi-central (see 2.2). Then the number of unipotent elements of $(\mathbf{G}^1)^F$ is given by $|(\mathbf{G}^{\sigma 0})^F|_p^2|\mathbf{G}^{0F}|/|(\mathbf{G}^{\sigma 0})^F|$.*

Proof. Let $\chi_\mathcal{U}$ be the characteristic function of the set of unipotent elements of $(\mathbf{G}^1)^F$. Then $|(\mathbf{G}^1)^F_{\text{unip}}|=|(\mathbf{G}^1)^F|\langle\chi_\mathcal{U},\mathrm{Id}\rangle_{(\mathbf{G}^1)^F}$ and

$$\langle\chi_\mathcal{U},\mathrm{Id}\rangle_{(\mathbf{G}^1)^F}=\langle \mathrm{D}_{\mathbf{G}^1}(\chi_\mathcal{U}),\mathrm{D}_{\mathbf{G}^1}(\mathrm{Id})\rangle_{(\mathbf{G}^1)^F}=\langle \mathrm{D}_{\mathbf{G}^1}(\chi_\mathcal{U}),\mathrm{St}_{\mathbf{G}^1}\rangle_{(\mathbf{G}^1)^F},$$

the first equality since $\mathrm{D}_{\mathbf{G}^1}$ is an isometry by [DM94, 3.12]. According to [DM94, 2.11], for any σ-stable and F-stable Levi subgroup $\mathbf{L}^0$ of a σ-stable parabolic subgroup of $\mathbf{G}^0$, setting $\mathbf{L}^1=\mathbf{L}^0.\sigma$, we have $R_{\mathbf{L}^1}^{\mathbf{G}^1}(\pi.\chi_\mathcal{U}|_{(\mathbf{L}^1)^F})=R_{\mathbf{L}^1}^{\mathbf{G}^1}(\pi).\chi_\mathcal{U}$ and ${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\varphi).\chi_\mathcal{U}|_{(\mathbf{L}^1)^F}={}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\varphi.\chi_\mathcal{U})$, thus, by 2.12, $\mathrm{D}_{\mathbf{G}^1}(\pi.\chi_\mathcal{U})=\mathrm{D}_{\mathbf{G}^1}(\pi).\chi_\mathcal{U}$; in particular $\mathrm{D}_{\mathbf{G}^1}(\chi_\mathcal{U})=\mathrm{D}_{\mathbf{G}^1}(\mathrm{Id}).\chi_\mathcal{U}=\mathrm{St}_{\mathbf{G}^1}.\chi_\mathcal{U}$. Now, by Proposition 2.13, the only unipotent elements on which $\mathrm{St}_{\mathbf{G}^1}$ does not vanish are the quasi-semi-simple (thus quasi-central) ones; by [DM94, 1.37] all such are in the $\mathbf{G}^{0F}$-class of σ and, again by 2.13 we have $\mathrm{St}_{\mathbf{G}^1}(\sigma)=|(\mathbf{G}^{\sigma 0})^F|_p$. We get

$$\begin{aligned}|(\mathbf{G}^1)^F|\langle \mathrm{D}_{\mathbf{G}^1}\chi_\mathcal{U},\mathrm{St}_{\mathbf{G}^1}\rangle_{(\mathbf{G}^1)^F}&=|(\mathbf{G}^1)^F|\langle \mathrm{St}_{\mathbf{G}^1}.\chi_\mathcal{U},\mathrm{St}_{\mathbf{G}^1}\rangle_{(\mathbf{G}^1)^F}\\&=|\{\mathbf{G}^{0F}\text{-class of }\sigma\}||(\mathbf{G}^{\sigma 0})^F|_p^2\end{aligned}$$

whence the proposition. □

Example 4.2. The formula of Proposition 4.1 applies in the following cases where σ induces a diagram automorphism of order 2 and q is a power of 2:

- $\mathbf{G}^0=\mathrm{SO}_{2n}$, $(\mathbf{G}^{\sigma 0})^F=\mathrm{SO}_{2n-1}(\mathbb{F}_q)$;
- $\mathbf{G}^0=\mathrm{GL}_{2n}$, $(\mathbf{G}^{\sigma 0})^F=\mathrm{Sp}_{2n}(\mathbb{F}_q)$;
- $\mathbf{G}^0=\mathrm{GL}_{2n+1}$, $(\mathbf{G}^{\sigma 0})^F=\mathrm{SO}_{2n+1}(\mathbb{F}_q)\simeq\mathrm{Sp}_{2n}(\mathbb{F}_q)$;
- $\mathbf{G}^0=E_6$, $(\mathbf{G}^{\sigma 0})^F=F_4(\mathbb{F}_q)$;

And it applies to the case where $\mathbf{G}^0=\mathrm{SO}_8$ where σ induces a diagram automorphism of order 3 and q is a power of 3, in which case $(\mathbf{G}^{\sigma 0})^F=G_2(\mathbb{F}_q)$.

5. Tensoring by the Steinberg character

Proposition 5.1. *Let $\mathbf{L}^1$ be an F-stable "Levi" of $\mathbf{G}^1$. Then, for any class function γ on $(\mathbf{G}^1)^F$ we have:*

$${}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\gamma\cdot\varepsilon_{\mathbf{G}^1}\mathrm{St}_{\mathbf{G}^1})=\varepsilon_{\mathbf{L}^1}\mathrm{St}_{\mathbf{L}^1}\mathrm{Res}_{(\mathbf{L}^1)^F}^{(\mathbf{G}^1)^F}\gamma.$$

Proof. Let su be the Jordan decomposition of a quasi-semi-simple element of $\mathbf{G}^1$ with s semisimple. We claim that u is quasi-central in $\mathbf{G}^s$. Indeed su, being quasi-semi-simple, is in a "torus" $\mathbf{T}$, thus s and u also are in $\mathbf{T}$. By [DM94, 1.8(iii)] the intersection of $\mathbf{T}\cap\mathbf{G}^s$ is a "torus" of $\mathbf{G}^s$, thus u is quasi-semi-simple in $\mathbf{G}^s$, hence quasi-central since unipotent.

Let tv be the Jordan decomposition of an element $l\in(\mathbf{L}^1)^F$ where t is semisimple. Since $\mathrm{St}_{\mathbf{L}^1}$ vanishes outside quasi-semi-simple elements the right-hand side of the proposition vanishes on l unless it is quasi-semi-simple which by our claim means that v is quasi-central in $\mathbf{L}^t$. By the character formula 2.5 the left-hand side of the proposition evaluates at l to

$$^*R_{\mathbf{L}^1}^{\mathbf{G}^1}(\gamma\cdot\varepsilon_{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1})(l)=|(\mathbf{G}^{t0})^F|^{-1}\sum_{u\in(\mathbf{G}^{t0}\cdot v)^F_{\text{unip}}}Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(u,v^{-1})\gamma(tu)\varepsilon_{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1}(tu).$$

By the same argument as above, applied to $\operatorname{St}_{\mathbf{G}^1}$, the only non zero terms in the above sum are for u quasi-central in $\mathbf{G}^t$. For such u, by [DM94, 4.16], $Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(u,v^{-1})$ vanishes unless u and v are $(\mathbf{G}^{t0})^F$-conjugate. Hence both sides of the equality to prove vanish unless u and v are quasi-central and $(\mathbf{G}^{t0})^F$-conjugate. In that case by [DM94, 4.16] and [DM91, (**) page 98] we have $Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}(u,v^{-1})=Q_{\mathbf{L}^{l0}}^{\mathbf{G}^{l0}}(1,1)=\varepsilon_{\mathbf{G}^{l0}}\varepsilon_{\mathbf{L}^{l0}}|(\mathbf{G}^{l0})^F|_{p'}|(\mathbf{L}^{l0})^F|_p$. Taking into account that the $(\mathbf{G}^{t0})^F$-class of v has cardinality $|(\mathbf{G}^{t0})^F|/|(\mathbf{G}^{l0})^F|$ and that by 2.13 we have $\operatorname{St}_{\mathbf{G}^1}(l)=\varepsilon_{\mathbf{G}^{\sigma 0}}\varepsilon_{\mathbf{G}^{l0}}|(\mathbf{G}^{l0})^F|_p$, the left-hand side of the proposition reduces to $\gamma(l)\varepsilon_{\mathbf{L}^{l0}}|(\mathbf{L}^{l0})^F|_p$, which is also the value of the right-hand side by applying 2.13 in $\mathbf{L}^1$. □

By adjunction, we get

Corollary 5.2. *For any class function λ on $(\mathbf{L}^1)^F$ we have:*

$$R_{\mathbf{L}^1}^{\mathbf{G}^1}(\lambda)\varepsilon_{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1}=\operatorname{Ind}_{(\mathbf{L}^1)^F}^{(\mathbf{G}^1)^F}(\varepsilon_{\mathbf{L}^1}\operatorname{St}_{\mathbf{L}^1}\lambda)$$

6. Characteristic functions of quasi-semi-simple classes

One of the goals of this section is Proposition 6.4 where we give a formula for the characteristic function of a quasi-semi-simple class which shows in particular that it is uniform; this generalizes the case of quasi-central elements given in [DM94, 4.14].

If $x\in(\mathbf{G}^1)^F$ has Jordan decomposition $x=su$ we will denote by d_x the map from class functions on $(\mathbf{G}^1)^F$ to class functions on $(C_{\mathbf{G}}(s)^0\cdot u)^F$ given by

$$(d_xf)(v)=\begin{cases}f(sv) & \text{if } v\in(C_{\mathbf{G}}(s)^0\cdot u)^F \text{ is unipotent}\\ 0 & \text{otherwise}\end{cases}$$

Lemma 6.1. *Let $\mathbf{L}^1$ be an F-stable "Levi" of $\mathbf{G}^1$. If $x=su$ is the Jordan decomposition of an element of $(\mathbf{L}^1)^F$ we have $d_x\circ{}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}={}^*R_{C_{\mathbf{L}}(s)^0\cdot u}^{C_{\mathbf{G}}(s)^0\cdot u}\circ d_x$.*

Proof. For v unipotent in $(C_{\mathbf{G}}(s)^0\cdot u)^F$ we have

$$(d_x{}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}f)(v)=({}^*R_{\mathbf{L}^1}^{\mathbf{G}^1}f)(sv)=({}^*R_{C_{\mathbf{L}}(s)^0\cdot su}^{C_{\mathbf{G}}(s)^0\cdot su}f)(sv)=({}^*R_{C_{\mathbf{L}}(s)^0\cdot u}^{C_{\mathbf{G}}(s)^0\cdot u}d_xf)(v)$$

where the second equality is [DM94, 2.9] and the last is by the character formula 2.5(iii). □

Proposition 6.2. *If $x=su$ is the Jordan decomposition of an element of $(\mathbf{G}^1)^F$, we have $d_x\circ p^{\mathbf{G}^1}=p^{C_{\mathbf{G}}(s)^0\cdot u}\circ d_x$.*

Proof. Let f be a class function on $(\mathbf{G}^1)^F$. For $v\in(C_{\mathbf{G}}(s)^0\cdot u)^F$ unipotent, we have, where the last equality is by 2.10:

$$(d_xp^{\mathbf{G}^1}f)(v)=p^{\mathbf{G}^1}f(sv)=|(\mathbf{G}^1)^F|^{-1}\sum_{\mathbf{T}^1\in\mathcal{T}_1^F}|(\mathbf{T}^1)^F|(R_{\mathbf{T}^1}^{\mathbf{G}^1}\circ{}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}f)(sv)$$

which by Proposition 2.5(ii) is:

$$\sum_{\mathbf{T}^1\in\mathcal{T}_1^F}\ \sum_{\{h\in(\mathbf{G}^0)^F\mid {}^h\mathbf{T}\ni s\}} \frac{|{}^h\mathbf{T}^0\cap C_{\mathbf{G}}(s)^{0F}|}{|(\mathbf{G}^0)^F||C_{\mathbf{G}}(s)^{0F}|}(R^{C_{\mathbf{G}}(s)^0\cdot su}_{{}^h\mathbf{T}\cap C_{\mathbf{G}}(s)^0\cdot su}\circ {}^{h*}R^{\mathbf{G}^1}_{\mathbf{T}^1}f)(sv).$$

Using that ${}^{h*}R^{\mathbf{G}^1}_{\mathbf{T}^1}f={}^*R^{\mathbf{G}^1}_{{}^h\mathbf{T}^1}f$ and summing over the ${}^h\mathbf{T}^1$, this becomes

$$\sum_{\{\mathbf{T}^1\in\mathcal{T}_1^F\mid \mathbf{T}\ni s\}} \frac{|\mathbf{T}^0\cap C_{\mathbf{G}}(s)^{0F}|}{|C_{\mathbf{G}}(s)^{0F}|}(R^{C_{\mathbf{G}}(s)^0\cdot su}_{\mathbf{T}^1\cap C_{\mathbf{G}}(s)^0\cdot su}\circ {}^*R^{\mathbf{G}^1}_{\mathbf{T}^1}f)(sv).$$

Using that by Proposition 2.5(i) for any class function χ on $\mathbf{T}^1\cap C_{\mathbf{G}}(s)^0\cdot su^F$

$$\begin{aligned}(R^{C_{\mathbf{G}}(s)^0\cdot su}_{\mathbf{T}^1\cap C_{\mathbf{G}}(s)^0\cdot su}\chi)(sv)&=|\mathbf{T}^0\cap C_{\mathbf{G}}(s)^{0F}|^{-1}\sum_{v'\in(\mathbf{T}\cap C_{\mathbf{G}}(s)^0\cdot u)^F_{\text{unip}}} Q^{(\mathbf{G}^s)^0}_{(\mathbf{T}^s)^0}(v,v'^{-1})\chi(sv')\\ &=R^{C_{\mathbf{G}}(s)^0\cdot u}_{\mathbf{T}\cap C_{\mathbf{G}}(s)^0\cdot u}(d_x\chi)(v),\end{aligned}$$

and using Lemma 6.1, we get

$$|C_{\mathbf{G}}(s)^0\cdot su^F|^{-1}\sum_{\{\mathbf{T}^1\in\mathcal{T}_1^F\mid\mathbf{T}\ni s\}}|(\mathbf{T}^s)^{0F}|(R^{C_{\mathbf{G}}(s)^0\cdot u}_{\mathbf{T}\cap C_{\mathbf{G}}(s)^0\cdot u}\circ {}^*R^{C_{\mathbf{G}}(s)^0\cdot u}_{\mathbf{T}\cap C_{\mathbf{G}}(s)^0\cdot u}d_xf)(v)$$

which is the desired result if we apply Corollary 2.10 in $C_{\mathbf{G}}(s)^0\cdot u$ and remark that by [DM94, 1.8 (iv)] the map $\mathbf{T}^1\mapsto\mathbf{T}\cap C_{\mathbf{G}}(s)^0\cdot u$ induces a bijection between $\{\mathbf{T}^1\in\mathcal{T}_1^F\mid\mathbf{T}\ni s\}$ and F-stable "tori" of $C_{\mathbf{G}}(s)^0\cdot u$. □

Corollary 6.3. *A class function f on $(\mathbf{G}^1)^F$ is uniform if and only if for every $x\in(\mathbf{G}^1)^F$ the function d_xf is uniform.*

Proof. Indeed, $f=p^{\mathbf{G}^1}f$ if and only if for any $x\in(\mathbf{G}^1)^F$ we have $d_xf=d_xp^{\mathbf{G}^1}f=p^{C_{\mathbf{G}}(s)^0\cdot u}d_xf$, the last equality by Proposition 6.2. □

For $x\in(\mathbf{G}^1)^F$ we consider the class function $\pi_x^{\mathbf{G}^1}$ on $(\mathbf{G}^1)^F$ defined by

$$\pi_x^{\mathbf{G}^1}(y)=\begin{cases}0 & \text{if } y \text{ is not conjugate to } x\\ |C_{\mathbf{G}^0}(x)^F| & \text{if } y=x\end{cases}$$

Proposition 6.4. *For $x\in(\mathbf{G}^1)^F$ quasi-semi-simple the function $\pi_x^{\mathbf{G}^1}$ is uniform, given by*

$$\begin{aligned}\pi_x^{\mathbf{G}^1}&=\varepsilon_{\mathbf{G}^{x0}}|C_{\mathbf{G}}(x)^0|_p^{-1}\sum_{\{\mathbf{T}^1\in\mathcal{T}_1^F\mid\mathbf{T}^1\ni x\}}\varepsilon_{\mathbf{T}^1}R^{\mathbf{G}^1}_{\mathbf{T}^1}(\pi_x^{\mathbf{T}^1})\\ &=|W^0(x)|^{-1}\sum_{w\in W^0(x)}\dim R^{C_{\mathbf{G}}(x)^0}_{\mathbf{T}_w}(\mathrm{Id})R^{\mathbf{G}^1}_{C_{\mathbf{G}^1}(\mathbf{T}_w)}(\pi_x^{C_{\mathbf{G}^1}(\mathbf{T}_w)})\end{aligned}$$

where in the second equality $W^0(x)$ denotes the Weyl group of $C_{\mathbf{G}}(x)^0$ and $\mathbf{T}_w$ denotes an F-stable torus of type w of this last group.

Proof. First, using Corollary 6.3 we prove that $\pi_x^{\mathbf{G}^1}$ is uniform. Let su be the Jordan decomposition of x. For $y\in(\mathbf{G}^1)^F$ the function $d_y\pi_x^{\mathbf{G}^1}$ is zero unless the semi-simple part of y is conjugate to s. Hence it is sufficient to evaluate $d_y\pi_x^{\mathbf{G}^1}(v)$ for elements y whose semisimple part is equal to s. For such elements $d_y\pi_x^{\mathbf{G}^1}(v)$ is up to a coefficient equal to $\pi_u^{C_{\mathbf{G}}(s)^0\cdot u}$. This function is uniform by [DM94, 4.14], since u being the unipotent part of a quasi-semi-simple element is quasi-central in $C_{\mathbf{G}}(s)$ (see beginning of the proof of Proposition 5.1).

We have thus $\pi_x^{\mathbf{G}^1}=p^{\mathbf{G}^1}\pi_x^{\mathbf{G}^1}$. We use this to get the formula of the proposition. We start by using Proposition 2.13 to write $\pi_x^{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1}=\varepsilon_{\mathbf{G}^1}\varepsilon_{\mathbf{G}^{x0}}|(\mathbf{G}^{x0})^F|_p\pi_x^{\mathbf{G}^1}$, or equivalently $\pi_x^{\mathbf{G}^1}=\varepsilon_{\mathbf{G}^1}\varepsilon_{\mathbf{G}^{x0}}|(\mathbf{G}^{x0})^F|_p^{-1}p^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1})$. Using Corollary 2.10 and that by Proposition 5.1 we have ${}^*R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1})=\varepsilon_{\mathbf{G}^1}\varepsilon_{\mathbf{T}^1}\operatorname{St}_{\mathbf{T}^1}\operatorname{Res}_{(\mathbf{T}^1)^F}^{(\mathbf{G}^1)^F}(\pi_x^{\mathbf{G}^1})$, we get

$$p^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1})=\varepsilon_{\mathbf{G}^1}|(\mathbf{G}^1)^F|^{-1}\sum_{\mathbf{T}^1\in\mathcal{T}_1^F}|(\mathbf{T}^1)^F|\varepsilon_{\mathbf{T}^1}R_{\mathbf{T}^1}^{\mathbf{G}^1}(\operatorname{St}_{\mathbf{T}^1}\operatorname{Res}_{(\mathbf{T}^1)^F}^{(\mathbf{G}^1)^F}(\pi_x^{\mathbf{G}^1})).$$

The function $\operatorname{St}_{\mathbf{T}^1}$ is constant equal to 1. Now we have

$$\operatorname{Res}_{(\mathbf{T}^1)^F}^{(\mathbf{G}^1)^F}\pi_x^{\mathbf{G}^1}=|(\mathbf{T}^0)^F|^{-1}\sum_{\{g\in(\mathbf{G}^0)^F\mid {}^gx\in\mathbf{T}^1\}}\pi_{{}^gx}^{\mathbf{T}^1}.$$

To see this, do the scalar product with a class function f on $(\mathbf{T}^1)^F$:

$$\langle\operatorname{Res}_{(\mathbf{T}^1)^F}^{(\mathbf{G}^1)^F}\pi_x^{\mathbf{G}^1},f\rangle_{(\mathbf{T}^1)^F}=\langle\pi_x^{\mathbf{G}^1},\operatorname{Ind}_{\mathbf{T}^1}^{\mathbf{G}^1}f\rangle_{(\mathbf{G}^1)^F}=|(\mathbf{T}^0)^F|^{-1}\sum_{\{g\in(\mathbf{G}^0)^F\mid {}^gx\in\mathbf{T}^1\}}f({}^gx).$$

We then get using that $|(\mathbf{T}^0)^F|=|(\mathbf{T}^1)^F|$

$$p^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1})=\varepsilon_{\mathbf{G}^1}|(\mathbf{G}^1)^F|^{-1}\sum_{\mathbf{T}^1\in\mathcal{T}_1^F}\sum_{\{g\in(\mathbf{G}^0)^F\mid {}^gx\in\mathbf{T}^1\}}\varepsilon_{\mathbf{T}^1}R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_{{}^gx}^{\mathbf{T}^1}).$$

Taking ${}^{g^{-1}}\mathbf{T}^1$ as summation index we get

$$p^{\mathbf{G}^1}(\pi_x^{\mathbf{G}^1}\operatorname{St}_{\mathbf{G}^1})=\varepsilon_{\mathbf{G}^1}\sum_{\{\mathbf{T}^1\in\mathcal{T}_1^F\mid\mathbf{T}^1\ni x\}}\varepsilon_{\mathbf{T}^1}R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_x^{\mathbf{T}^1}),$$

hence

$$\pi_x^{\mathbf{G}^1}=\varepsilon_{\mathbf{G}^{x0}}|(\mathbf{G}^{x0})^F|_p^{-1}\sum_{\{\mathbf{T}^1\in\mathcal{T}_1^F\mid\mathbf{T}^1\ni x\}}\varepsilon_{\mathbf{T}^1}R_{\mathbf{T}^1}^{\mathbf{G}^1}(\pi_x^{\mathbf{T}^1}),$$

which is the first equality of the proposition.

For the second equality of the proposition, we first use [DM94, 1.8 (iii) and (iv)] to sum over tori of $C_{\mathbf{G}}(x)^0$: the $\mathbf{T}^1\in\mathcal{T}_1^F$ containing x are in bijection with the maximal tori of $C_{\mathbf{G}}(x)^0$ by $\mathbf{T}^1\mapsto(\mathbf{T}^{1x})^0$ and conversely $\mathbf{S}\mapsto C_{\mathbf{G}^1}(\mathbf{S})$. This bijection satisfies $\varepsilon_{\mathbf{T}^1}=\varepsilon_{\mathbf{S}}$ by definition of ε.

We then sum over $(C_{\mathbf{G}}(x)^0)^F$-conjugacy classes of maximal tori, which are parameterized by F-conjugacy classes of $W^0(x)$. We then have to multiply by $|(C_{\mathbf{G}}(x)^0)^F|/|N_{(C_{\mathbf{G}}(x)^0)}(\mathbf{S})^F|$ the term indexed by the class of $\mathbf{S}$. Then we sum over the elements of $W^0(x)$. We then have to multiply the term indexed by w by $|C_{W^0(x)}(wF)|/|W^0(x)|$. Using $|N_{(C_{\mathbf{G}}(x)^0)}(\mathbf{S})^F|=|\mathbf{S}^F||C_{W^0(x)}(wF)|$, and the formula for $\dim R_{\mathbf{T}_w}^{C_{\mathbf{G}}(x)^0}(\mathrm{Id})$ we get the result. □

7. Classification of quasi-semi-simple classes

The first items of this section, before 7.7, apply for algebraic groups over an arbitrary algebraic closed field k.

We denote by $\mathcal{C}(\mathbf{G}^1)$ the set of conjugacy classes of $\mathbf{G}^1$, that is the orbits under $\mathbf{G}^0$-conjugacy, and denote by $\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}}$ the set of quasi-semi-simple classes.

Proposition 7.1. *For $\mathbf{T}^1 \in \mathcal{T}_1$ write $\mathbf{T}^1 = \mathbf{T}^0 \cdot \sigma$ where σ is quasi-central. Then $\mathcal{C}(\mathbf{G}^1)_{qss}$ is in bijection with the set of $N_{\mathbf{G}^0}(\mathbf{T}^1)$-orbits in $\mathbf{T}^1$, which itself is in bijection with the set of W^σ-orbits in $\mathcal{C}(\mathbf{T}^1)$, where $W = N_{\mathbf{G}^0}(\mathbf{T}^0)/\mathbf{T}^0$. We have $\mathcal{C}(\mathbf{T}^1) \simeq \mathbf{T}^1/\mathcal{L}_\sigma(\mathbf{T}^0)$ where $\mathcal{L}_\sigma$ is the map $t \mapsto t^{-1}.{}^\sigma t$.*

Proof. By definition every quasi-semi-simple element of $\mathbf{G}^1$ is in some $\mathbf{T}^1 \in \mathcal{T}_1$ and $\mathcal{T}_1$ is a single orbit under $\mathbf{G}^0$-conjugacy. It is thus sufficient to find how classes of $\mathbf{G}^1$ intersect $\mathbf{T}^1$. By [DM94, 1.13] two elements of $\mathbf{T}^1$ are $\mathbf{G}^0$-conjugate if and only if they are conjugate under $N_{\mathbf{G}^0}(\mathbf{T}^0)$. We can replace $N_{\mathbf{G}^0}(\mathbf{T}^0)$ by $N_{\mathbf{G}^0}(\mathbf{T}^1)$ since if ${}^g(\sigma t) = \sigma t'$ where $g \in N_{\mathbf{G}^0}(\mathbf{T}^0)$ then the image of g in W lies in W^σ. By [DM94, 1.15(iii)] elements of W^σ have representatives in $\mathbf{G}^{\sigma 0}$. Write $g = s\dot{w}$ where $\dot{w}$ is such a representative and $s \in \mathbf{T}^0$. Then ${}^{s\dot{w}}(t\sigma) = \mathcal{L}_\sigma(s^{-1})^w t\sigma$ whence the proposition. □

Lemma 7.2. $\mathbf{T}^0 = \mathbf{T}^{\sigma 0}.\mathcal{L}_\sigma(\mathbf{T}^0)$.

Proof. This is proved in [DM94, 1.33] when σ is unipotent (and then the product is direct). We proceed similarly to that proof: $\mathbf{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\mathbf{T}^0)$ is finite, since its exponent divides the order of σ (if $\sigma(t^{-1}\,{}^\sigma t) = t^{-1}\,{}^\sigma t$ then $(t^{-1}\,{}^\sigma t)^n = t^{-1}\,{}^{\sigma^n} t$), and $\dim(\mathbf{T}^{\sigma 0}) + \dim(\mathcal{L}_\sigma(\mathbf{T}^0)) = \dim(\mathbf{T}^0)$ as the exact sequence $1 \to \mathbf{T}^{0^\sigma} \to \mathbf{T}^0 \to \mathcal{L}_\sigma(\mathbf{T}^0) \to 1$ shows. □

It follows that $\mathbf{T}^0/\mathcal{L}_\sigma(\mathbf{T}^0) \simeq \mathbf{T}^{\sigma 0}/(\mathbf{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\mathbf{T}^0))$; since the set $\mathcal{C}(\mathbf{G}^{\sigma 0})_{\mathrm{ss}}$ of semi-simple classes of $\mathbf{G}^{\sigma 0}$ identifies with the set of W^σ-orbits on $\mathbf{T}^{\sigma 0}$ this induces a surjective map $\mathcal{C}(\mathbf{G}^{\sigma 0})_{\mathrm{ss}} \to \mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}}$.

Example 7.3. We will describe the quasi-semi-simple classes of $\mathbf{G}^0 \cdot \sigma$, where $\mathbf{G}^0 = \mathrm{GL}_n(k)$ and σ is the quasi-central automorphism given by $\sigma(g) = J\,{}^t g^{-1} J^{-1}$, where, if n is even J is the matrix $\begin{pmatrix} 0 & -J_0 \\ J_0 & 0 \end{pmatrix}$ with $J_0 = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$ and if n is odd J is the antidiagonal matrix $\begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}$ (any outer algebraic automorphism of GL_n is equal to σ up to an inner automorphism).

The automorphism σ normalizes the pair $\mathbf{T}^0 \subset \mathbf{B}$ where $\mathbf{T}^0$ is the diagonal torus and $\mathbf{B}$ the group of upper triangular matrices. Then $\mathbf{T}^1 = \mathbf{T}^0 \cdot \sigma \in \mathcal{T}_1$. For $\mathrm{diag}(x_1, \ldots, x_n) \in \mathbf{T}^0$, where $x_i \in k^\times$, we have $\sigma(\mathrm{diag}(x_1, \ldots, x_n)) = \mathrm{diag}(x_n^{-1}, \ldots, x_1^{-1})$. It follows that $\mathcal{L}_\sigma(\mathbf{T}^0) = \{\mathrm{diag}(x_1, x_2, \ldots, x_2, x_1)\}$ — here x_{m+1} is a square when $n = 2m+1$ but this is not a condition since k is algebraically closed. As suggested above, we could take as representatives of $\mathbf{T}^0/\mathcal{L}_\sigma(\mathbf{T}^0)$ the set $\mathbf{T}^{\sigma 0}/(\mathbf{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\mathbf{T}^0))$, but since $\mathbf{T}^{\sigma 0} \cap \mathcal{L}_\sigma(\mathbf{T}^0)$ is not trivial (it consists of the diagonal matrices with entries ± 1 placed symmetrically), it is more convenient to take for representatives of the quasi-semi-simple classes the set $\{\mathrm{diag}(x_1, x_2, \ldots, x_{\lfloor \frac{n}{2} \rfloor}, 1, \ldots, 1)\}\sigma$. In this model the action of W^σ is generated by the permutations of the $\lfloor \frac{n}{2} \rfloor$ first entries, and by the maps $x_i \mapsto x_i^{-1}$, so the quasi-semi-simple classes of $\mathbf{G}^0 \cdot \sigma$ are parameterized by the quasi-semi-simple classes of $\mathbf{G}^{\sigma 0}$.

Proposition 7.4. *Let $s\sigma = \mathrm{diag}(x_1, x_2, \ldots, x_{\lfloor \frac{n}{2} \rfloor}, 1, \ldots, 1)\sigma$ be a quasi-semi-simple element as above. If char $k = 2$ then $C_{\mathbf{G}^0}(s\sigma)$ is connected. Otherwise, if n is odd, $A(s\sigma) := C_{\mathbf{G}^0}(s\sigma)/C_{\mathbf{G}^0}(s\sigma)^0$ is of order two, generated by $-1 \in Z\mathbf{G}^0 = Z\,\mathrm{GL}_n(k)$.*

If n is even, $A(s\sigma)\neq 1$ if and only if for some i we have $x_i=-1$; then $x_i\mapsto x_i^{-1}$ is an element of W^σ which has a representative in $C_{\mathbf{G}^0}(s\sigma)$ generating $A(s\sigma)$, which is of order 2.

Proof. We will use that for a group G and an automorphism σ of G we have an exact sequence (see for example [St, 4.5])

$$1\to(ZG)^\sigma\to G^\sigma\to(G/ZG)^\sigma\to(\mathcal{L}_\sigma G\cap ZG)/\mathcal{L}_\sigma ZG\to 1 \tag{7.5}$$

If we take $G=\mathbf{G}^0=\mathrm{GL}_n(k)$ in 7.5 and $s\sigma$ for σ, since on $Z\mathbf{G}^0$ the map $\mathcal{L}_\sigma=\mathcal{L}_{s\sigma}$ is $z\mapsto z^2$, hence surjective, we get that $\mathbf{G}^{0s\sigma}\to\mathrm{PGL}_n^{s\sigma}$ is surjective and has kernel $(Z\mathbf{G}^0)^\sigma=\{\pm1\}$.

Assume n odd and take $G=\mathrm{SL}_n(k)$ in 7.5. We have $Z\,\mathrm{SL}_n^\sigma=\{1\}$ so that we get the following diagram with exact rows:

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \{\pm1\} & \longrightarrow & \mathrm{GL}_n^{s\sigma} & \longrightarrow & \mathrm{PGL}_n^{s\sigma} & \longrightarrow & 1\\
 & & & & \uparrow & & \| & & \\
 & & 1 & \longrightarrow & \mathrm{SL}_n^{s\sigma} & \longrightarrow & \mathrm{PGL}_n^{s\sigma} & \longrightarrow & 1
\end{array}$$

This shows that $\mathrm{GL}_n^{s\sigma}/\mathrm{SL}_n^{s\sigma}\simeq\{\pm1\}$; by [St, 8.1] $\mathrm{SL}_n^{s\sigma}$ is connected, hence $\mathrm{PGL}_n^{s\sigma}$ is connected thus $\mathrm{GL}_n^{s\sigma}=(\mathrm{GL}_n^{s\sigma})^0\times\{\pm1\}$ is connected if and only if char $k=2$.

Assume now n even; then $(\mathbf{T}^0)^\sigma$ is connected hence $-1\in(\mathrm{GL}_n^{s\sigma})^0$ for all $s\in\mathbf{T}_0$. Using this, the exact sequence $1\to\{\pm1\}\to\mathrm{GL}_n^{s\sigma}\to\mathrm{PGL}_n^{s\sigma}\to1$ implies $A(s\sigma)=\mathbf{G}^{s\sigma}/\mathbf{G}^{0s\sigma}=\mathrm{GL}_n^{s\sigma}/(\mathrm{GL}_n^{s\sigma})^0\simeq\mathrm{PGL}_n^{s\sigma}/(\mathrm{PGL}_n^{s\sigma})^0$. To compute this group we use 7.5 with $\mathrm{SL}_n(k)$ for G and $s\sigma$ for σ:

$$1\to\{\pm1\}\to\mathrm{SL}_n^{s\sigma}\to\mathrm{PGL}_n^{s\sigma}\to(\mathcal{L}_{s\sigma}\mathrm{SL}_n\cap Z\,\mathrm{SL}_n)/\mathcal{L}_\sigma Z\,\mathrm{SL}_n\to1$$

which, since $\mathrm{SL}_n^{s\sigma}$ is connected, implies that $A(s\sigma)=(\mathcal{L}_{s\sigma}\mathrm{SL}_n\cap Z\,\mathrm{SL}_n)/\mathcal{L}_\sigma Z\,\mathrm{SL}_n$ thus is non trivial (of order 2) if and only if $\mathcal{L}_{s\sigma}\mathrm{SL}_n\cap Z\,\mathrm{SL}_n$ contains an element which is not a square in $Z\,\mathrm{SL}_n$; thus $A(s\sigma)$ is trivial if char $k=2$. We assume now char $k\neq2$. Then a non-square is of the form $\mathrm{diag}(z,\dots,z)$ with $z^m=-1$ if we set $m=n/2$.

The following lemma is a transcription of [St, 9.5].

Lemma 7.6. *Let σ be a quasi-central automorphism of the connected reductive group $\mathbf{G}$ which stabilizes the pair $\mathbf{T}\subset\mathbf{B}$ of a maximal torus and a Borel subgroup; let W be the Weyl group of $\mathbf{T}$ and let $s\in\mathbf{T}$. Then $\mathbf{T}\cap\mathcal{L}_{s\sigma}(\mathbf{G})=\{\mathcal{L}_w(s^{-1})\mid w\in W^\sigma\}\cdot\mathcal{L}_\sigma(\mathbf{T})$.*

Proof. Assume $t=\mathcal{L}_{s\sigma}(x)$ for $t\in\mathbf{T}$, or equivalently $xt={}^{s\sigma}x$. Then if x is in the Bruhat cell $\mathbf{B}w\mathbf{B}$, we must have $w\in W^\sigma$. Taking for w a σ-stable representative $\dot w$ and writing the unique Bruhat decomposition $x=u_1\dot wt_1u_2$ where $u_2\in\mathbf{U},t_1\in\mathbf{T}$ and $u_1\in\mathbf{U}\cap{}^w\mathbf{U}^-$ where $\mathbf{U}$ is the unipotent radical of $\mathbf{B}$ and $\mathbf{U}^-$ the unipotent radical of the opposite Borel, the equality $xt={}^{s\sigma}x$ implies that $\dot wt_1t={}^{s\sigma}(\dot wt_1)$ or equivalently $t=\mathcal{L}_{w^{-1}}(s^{-1})\mathcal{L}_\sigma(t_1)$, whence the lemma. □

From this lemma we get $\mathcal{L}_{s\sigma}(\mathrm{SL}_n)\cap Z\,\mathrm{SL}_n=\{\mathcal{L}_w(s^{-1})\mid w\in W^\sigma\}\cdot\mathcal{L}_\sigma(\mathbf{T})\cap Z\,\mathrm{SL}_n$, where $\mathbf{T}$ is the maximal torus of SL_n. The element $\mathrm{diag}(x_1,x_2,\dots,x_m,1,\dots,1)\sigma$ is conjugate to $s\sigma=\mathrm{diag}(y_1,y_2,\dots,y_m,y_m^{-1},\dots,y_1^{-1})\sigma\in\mathbf{T}^\sigma\cdot\sigma$ where $y_i^2=x_i$. It will have a non connected centralizer if and only if for some $w\in W^\sigma$ and some $t\in\mathbf{T}$ we have $\mathcal{L}_w(s^{-1})\cdot\mathcal{L}_\sigma(t)=\mathrm{diag}(z,\dots,z)$ with $z^m=-1$ and then an appropriate representative of w (multiplying if needed by an element of $Z\,\mathrm{GL}_n$) will be in $C_{\mathbf{G}^0}(s\sigma)$ and have a non-trivial image in $A(s\sigma)$. Since s and w are σ-fixed, we have $\mathcal{L}_w(s)\in\mathbf{T}^\sigma$, thus is of the form $\mathrm{diag}(a_1,\ldots,a_m,a_m^{-1},\ldots a_1^{-1})$. Since $\mathcal{L}_\sigma(\mathbf{T})=\{\mathrm{diag}(t_1,\ldots,t_m,t_m\ldots,t_1)\mid t_1t_2\ldots t_m=1\}$, we get $z=a_1t_1=a_2t_2=\ldots=a_mt_m=a_m^{-1}t_m=\ldots=a_1^{-1}t_1$; in particular $a_i=\pm1$ for all i and $a_1a_2\ldots a_m=-1$. We can take w up to conjugacy in W^σ since $\mathcal{L}_{vwv^{-1}}(s^{-1})={}^v\mathcal{L}_w({}^{v^{-1}}s^{-1})$ and $\mathcal{L}_\sigma(\mathbf{T})$ is invariant under W^σ-conjugacy. We see W^σ as the group of permutations of $\{1,2,\ldots,m,-m,\ldots,-1\}$ which preserves the pairs $\{i,-i\}$. A non-trivial cycle of w has, up to conjugacy, the form either $(1,-1)$ or $(1,-2,3,\ldots,(-1)^{i-1}i,i+1,i+2,\ldots,k)$ with $0\le i\le k\le n$ and i even, or $(1,-2,3,\ldots,(-1)^{i-1}i,-(i+1),-(i+2),\ldots,-k,-1,2,-3,\ldots,k)$ and i odd. The contribution to $a_1\ldots a_m$ of the orbit $(1,-1)$ is $a_1=y_1^2$ hence is 1 except if $y_1^2=x_1=-1$. Let us consider an orbit of the second form. The k first coordinates of $\mathcal{L}_w(s^{-1})$ are $(y_1y_2,\ldots,y_iy_{i+1},y_{i+1}/y_{i+2},\ldots,y_k/y_1)$. Hence there must exist signs ε_j such that $y_2=\varepsilon_1/y_1$, $y_3=\varepsilon_2/y_2,\ldots$, $y_{i+1}=\varepsilon_i/y_i$ and $y_{i+2}=\varepsilon_{i+1}y_{i+1},\ldots$, $y_k=\varepsilon_{k-1}y_{k-1}$, $y_1=\varepsilon_ky_k$. This gives $y_1=\begin{cases}\varepsilon_1\ldots\varepsilon_ky_1 & \text{if } i \text{ is even}\\ \varepsilon_1\ldots\varepsilon_k/y_1 & \text{if } i \text{ is odd}\end{cases}$. The contribution of the orbit to $a_1\ldots a_m$ is $\varepsilon_1\ldots\varepsilon_k$ thus is 1 if i is even and $x_1=y_1^2$ if i is odd. Again, we see that one of the x_i must equal -1 to get $a_1\ldots a_m=-1$. Conversely if $x_1=-1$, for any z such that $z^m=-1$, choosing t such that $\mathcal{L}_\sigma(t)=\mathrm{diag}(-z,z,z,\ldots,z,-z)$ and taking $w=(1,-1)$ we get $\mathcal{L}_w(s^{-1})\mathcal{L}_\sigma(t)=\mathrm{diag}(z,\ldots,z)$ as desired. □

We now go back to the case where $k=\overline{\mathbb{F}}_q$, and in the context of Proposition 7.1, we now assume that $\mathbf{T}^1$ is F-stable and that σ induces an F-stable automorphism of $\mathbf{G}^0$.

Proposition 7.7. *Let $\mathbf{T}^{1rat}=\{s\in\mathbf{T}^1\mid \exists n\in N_{\mathbf{G}^0}(\mathbf{T}^1), {}^{nF}s=s\}$; then $\mathbf{T}^{1rat}$ is stable by $\mathbf{T}^0$-conjugacy, which gives a meaning to $\mathcal{C}(\mathbf{T}^{1rat})$. Then $c\mapsto c\cap\mathbf{T}^1$ induces a bijection between $(\mathcal{C}(\mathbf{G}^1)_{qss})^F$ and the W^σ-orbits on $\mathcal{C}(\mathbf{T}^{1rat})$.*

Proof. A class $c\in\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}}$ is F-stable if and only if given $s\in c$ we have ${}^Fs\in c$. If we take $s\in c\cap\mathbf{T}^1$ then ${}^Fs\in c\cap\mathbf{T}^1$ which as observed in the proof of 7.1 implies that ${}^Fs$ is conjugate to s under $N_{\mathbf{G}^0}(\mathbf{T}^1)$, that is $s\in\mathbf{T}^{1\mathrm{rat}}$. Thus c is F-stable if and only if $c\cap\mathbf{T}^1=c\cap\mathbf{T}^{1\mathrm{rat}}$. The proposition then results from Proposition 7.1 observing that $\mathbf{T}^{1\mathrm{rat}}$ is stable under $N_{\mathbf{G}^0}(\mathbf{T}^1)$-conjugacy and that the corresponding orbits are the W^σ-orbits on $\mathcal{C}(\mathbf{T}^{1\mathrm{rat}})$. □

Example 7.8. When $\mathbf{G}^1=\mathrm{GL}_n(\overline{\mathbb{F}}_q)\cdot\sigma$ with σ as in Example 7.3, the map

$$\mathrm{diag}(x_1,x_2,\ldots,x_{\lfloor\frac n2\rfloor},1,\ldots,1)\mapsto\mathrm{diag}(x_1,x_2,\ldots,x_{\lfloor\frac n2\rfloor},\dagger,x_{\lfloor\frac n2\rfloor}^{-1},\ldots,x_2^{-1},x_1^{-1})$$

where † represents 1 if n is odd and an omitted entry otherwise, is compatible with the action of W^σ as described in 7.3 on the left-hand side and the natural action on the right-hand side. This map induces a bijection from $\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}}$ to the semi-simple classes of $(\mathrm{GL}_n^\sigma)^0$ which restricts to a bijection from $(\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}})^F$ to the F-stable semi-simple classes of $(\mathrm{GL}_n^\sigma)^0$.

We will now compute the cardinality of $(\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}})^F$.

Proposition 7.9. *Let f be a function on $(\mathcal{C}(\mathbf{G}^1)_{qss})^F$. Then*

$$\sum_{c\in(\mathcal{C}(\mathbf{G}^1)_{qss})^F} f(c)=|W^\sigma|^{-1}\sum_{w\in W^\sigma}\tilde f(w)$$

where $\tilde f(w):=\sum_s f(s)$, *where* s *runs over representatives in* $\mathbf{T}^{1wF}$ *of* $\mathbf{T}^{1wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}$.

Proof. We have $\mathcal{C}(\mathbf{T}^{1\mathrm{rat}})=\bigcup_{w\in W^\sigma}\{s\mathcal{L}_\sigma(\mathbf{T}^0)\in\mathbf{T}^1/\mathcal{L}_\sigma(\mathbf{T}^0)\mid s\mathcal{L}_\sigma(\mathbf{T}^0)\text{ is }wF\text{-stable}\}$. The conjugation by $v\in W^\sigma$ sends a wF-stable coset $s\mathcal{L}_\sigma(\mathbf{T}^0)$ to a $vwFv^{-1}$-stable coset; and the number of w such that $s\mathcal{L}_\sigma(\mathbf{T}^0)$ is wF-stable is equal to $N_{W^\sigma}(s\mathcal{L}_\sigma(\mathbf{T}^0))$. It follows that

$$\sum_{c\in(\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}})^F} f(c)=|W^\sigma|^{-1}\sum_{w\in W^\sigma}\sum_{s\mathcal{L}_\sigma(\mathbf{T}^0)\in(\mathbf{T}^1/\mathcal{L}_\sigma(\mathbf{T}^0))^{wF}} f(s\mathcal{L}_\sigma(\mathbf{T}^0)).$$

The proposition follows since, $\mathcal{L}_\sigma(\mathbf{T}^0)$ being connected, we have $(\mathbf{T}^1/\mathcal{L}_\sigma(\mathbf{T}^0))^{wF}=\mathbf{T}^{1wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}$. □

Corollary 7.10. *We have* $|(\mathcal{C}(\mathbf{G}^1)_{qss})^F|=|(\mathcal{C}(\mathbf{G}^{\sigma0})_{ss})^F|$.

Proof. Let us take $f=1$ in 7.9. We need to sum over $w\in W^\sigma$ the value $|\mathbf{T}^{1wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}|$. First note that $|\mathbf{T}^{1wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}|=|\mathbf{T}^{0wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}|$. By Lemma 7.2 we have the exact sequence

$$1\to\mathbf{T}^{\sigma0}\cap\mathcal{L}_\sigma(\mathbf{T}^0)\to\mathbf{T}^{\sigma0}\times\mathcal{L}_\sigma(\mathbf{T}^0)\to\mathbf{T}^0\to1$$

whence the Galois cohomology exact sequence:

$$1\to(\mathbf{T}^{\sigma0}\cap\mathcal{L}_\sigma(\mathbf{T}^0))^{wF}\to\mathbf{T}^{\sigma0wF}\times(\mathcal{L}_\sigma(\mathbf{T}^0))^{wF}\to \mathbf{T}^{0wF}\to H^1(wF,(\mathbf{T}^{\sigma0}\cap\mathcal{L}_\sigma(\mathbf{T}^0)))\to1.$$

Using that for any automorphism τ of a finite group G we have $|G^\tau|=|H^1(\tau,G)|$, we have $|(\mathbf{T}^{\sigma0}\cap\mathcal{L}_\sigma(\mathbf{T}^0))^{wF}|=|H^1(wF,(\mathbf{T}^{\sigma0}\cap\mathcal{L}_\sigma(\mathbf{T}^0)))|$. Together with the above exact sequence it implies that $|\mathbf{T}^{0wF}/\mathcal{L}_\sigma(\mathbf{T}^0)^{wF}|=|\mathbf{T}^{\sigma0wF}|$ whence

$$|(\mathcal{C}(\mathbf{G}^1)_{\mathrm{qss}})^F|=|W^\sigma|^{-1}\sum_{w\in W^\sigma}|\mathbf{T}^{\sigma0wF}|.$$

The corollary follows by either applying the same formula for the connected group $\mathbf{G}^{\sigma0}$, or referring to [Le, Proposition 2.1]. □

8. Shintani descent.

We now look at Shintani descent in our context; we are able to obtain a result when $\mathbf{G}^1/\mathbf{G}^0$ is semi-simple. We should mention previous work on this subject: Eftekhari ([E96, II. 3.4]) has the same result in the case of a torus; he does not need to assume p good but needs q to be large enough to apply results of Lusztig identifying Deligne-Lusztig induction with induction of character sheaves; Digne ([D99, 1.1]) has the result in the same generality excepted that he needs the assumption that $\mathbf{G}^1$ contains an F-stable quasi-central element; however a defect of his proof is the use without proof of the property given in Lemma 8.4 below.

As above $\mathbf{G}^1$ denotes an F-stable connected component of $\mathbf{G}$ of the form $\mathbf{G}^0\cdot\sigma$ where σ induces a quasi-central automorphism of $\mathbf{G}^0$ commuting with F.

Applying Lang's theorem, one can write any element of $\mathbf{G}^1$ as $x\cdot{}^{\sigma F}x^{-1}\sigma$ for some $x\in\mathbf{G}^0$, or as $\sigma{}^F x^{-1}\cdot x$ for some $x\in\mathbf{G}^0$. Using that σ, as automorphism, commutes with F, it is easy to check that the correspondence $x\cdot{}^{\sigma F}x^{-1}\sigma\mapsto\sigma{}^F x^{-1}\cdot x$ induces a bijection $n_{F/\sigma F}$ from the $(\mathbf{G}^0)^F$-conjugacy classes of $(\mathbf{G}^1)^F$ to the $\mathbf{G}^{0\sigma F}$-conjugacy classes of $(\mathbf{G}^1)^{\sigma F}$ and that $|\mathbf{G}^{0\sigma F}||c|=|(\mathbf{G}^0)^F||n_{F/\sigma F}(c)|$ for any $(\mathbf{G}^0)^F$-class c in $(\mathbf{G}^1)^F$. It follows that the operator $\mathrm{sh}_{F/\sigma F}$ from $(\mathbf{G}^0)^F$-class functions on $(\mathbf{G}^1)^F$

to $\mathbf{G}^{0\sigma F}$-class functions on $(\mathbf{G}^1)^{\sigma F}$ defined by $\mathrm{sh}_{F/\sigma F}(\chi)(n_{F/\sigma F}x) = \chi(x)$ is an isometry.

The end of this section is devoted to the proof of the following

Proposition 8.1. *Let $\mathbf{L}^1 = N_{\mathbf{G}^1}(\mathbf{L}^0 \subset \mathbf{P}^0)$ be a "Levi" of $\mathbf{G}^1$ containing σ, where $\mathbf{L}^0$ is F-stable; we have $\mathbf{L}^1 = \mathbf{L}^0 \cdot \sigma$. Assume that σ is semi-simple and that the characteristic is good for $\mathbf{G}^{\sigma 0}$. Then*

$$\mathrm{sh}_{F/\sigma F} \circ {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} = {}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} \circ \mathrm{sh}_{F/\sigma F} \quad \textit{and} \quad \mathrm{sh}_{F/\sigma F} \circ R_{\mathbf{L}^1}^{\mathbf{G}^1} = R_{\mathbf{L}^1}^{\mathbf{G}^1} \circ \mathrm{sh}_{F/\sigma F}.$$

Proof. The second equality follows from the first by adjunction, using that the adjoint of $\mathrm{sh}_{F/\sigma F}$ is $\mathrm{sh}_{F/\sigma F}^{-1}$. Let us prove the first equality.

Let χ be a $(\mathbf{G}^0)^F$-class function on $\mathbf{G}^1$ and let $\sigma l u = u \sigma l$ be the Jordan decomposition of an element of $(\mathbf{L}^1)^{\sigma F}$ with u unipotent and σl semi-simple. By the character formula 2.5(iii) and the definition of $Q_{\mathbf{L}^{t0}}^{\mathbf{G}^{t0}}$ we have

$$({}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} \mathrm{sh}_{F/\sigma F}(\chi))(\sigma l u) = \\ |(\mathbf{G}^{\sigma l})^{0\sigma F}|^{-1} \sum_{v \in (\mathbf{G}^{\sigma l})^{0\sigma F}_{\mathrm{unip}}} \mathrm{sh}_{F/\sigma F}(\chi)(\sigma l v) \,\mathrm{Trace}((v, u^{-1}) | H_c^*(Y_{\mathbf{U},\sigma F})),$$

where v (resp. u) acts by left- (resp. right-) translation on $Y_{\mathbf{U},\sigma F} = \{x \in (\mathbf{G}^{\sigma l})^0 \mid x^{-1} \cdot {}^{\sigma F}x \in \mathbf{U}\}$ where $\mathbf{U}$ denotes the unipotent radical of $\mathbf{P}^0$; in the summation v is in the identity component of $\mathbf{G}^{\sigma l}$ since, σ being semi-simple, u is in $\mathbf{G}^0$ hence in $(\mathbf{G}^{\sigma l})^0$ by [DM94, 1.8 (i)] since σl is semisimple.

Let us write $l = {}^F\lambda^{-1} \cdot \lambda$ with $\lambda \in \mathbf{L}^0$, so that $\sigma l = n_{F/\sigma F}(l'\sigma)$ where $l' = \lambda \cdot {}^{\sigma F}\lambda^{-1}$.

Lemma 8.2. *For $v \in (\mathbf{G}^{\sigma l})^{0\sigma F}_{unip}$ we have $\sigma l v = n_{F/\sigma F}((\sigma l \cdot v')^{\sigma^F \lambda^{-1}})$ where $v' = n_{\sigma F/\sigma F} v \in (\mathbf{G}^{\sigma l})^{0\sigma F}$ is defined by writing $v = {}^{\sigma F}\eta \cdot \eta^{-1}$ where $\eta \in (\mathbf{G}^{\sigma t})^0$ and setting $v' = \eta^{-1} \cdot {}^{\sigma F}\eta$.*

Proof. We have $\sigma l v = \sigma l {}^{\sigma F}\eta \cdot \eta^{-1} = {}^{\sigma F}\eta \sigma l \eta^{-1} = \sigma^F(\eta\lambda^{-1})\lambda\eta^{-1}$, thus $\sigma l v = n_{F/\sigma F}((\lambda\eta^{-1}) \cdot \sigma^F(\eta\lambda^{-1})\sigma)$. And we have $(\lambda\eta^{-1}) \cdot \sigma^F(\eta\lambda^{-1})\sigma = \lambda v' {}^{\sigma F}\lambda^{-1}\sigma = {}^F\lambda l v' \sigma^F \lambda^{-1} = (\sigma l v')^{\sigma^F\lambda^{-1}}$, thus $\mathrm{sh}_{F/\sigma F}(\chi)(\sigma l v) = \chi((\sigma l v')^{\sigma^F \lambda^{-1}})$. □

Lemma 8.3. (i) *We have $(\sigma l)^{\sigma^F\lambda^{-1}} = l'\sigma$.*

(ii) *The conjugation $x \mapsto x^{\sigma^F\lambda^{-1}}$ maps $\mathbf{G}^{\sigma l}$ and the action of σF on it, to $\mathbf{G}^{l'\sigma}$ with the action of F on it; in particular it induces bijections $(\mathbf{G}^{\sigma l})^{0\sigma F} \xrightarrow{\sim} (\mathbf{G}^{l'\sigma})^{0^F}$ and $Y_{\mathbf{U},\sigma F} \xrightarrow{\sim} Y_{\mathbf{U},F}$, where $Y_{\mathbf{U},F} = \{x \in (\mathbf{G}^{l'\sigma})^0 \mid x^{-1\,F}x \in \mathbf{U}\}$.*

Proof. (i) is an obvious computation and shows that if $x \in \mathbf{G}^{\sigma l}$ then $x^{\sigma^F\lambda^{-1}} \in \mathbf{G}^{l'\sigma}$. To prove (ii), it remains to show that if $x \in \mathbf{G}^{\sigma l}$ then ${}^F(x^{\sigma^F\lambda^{-1}}) = ({}^{\sigma F}x)^{\sigma^F\lambda^{-1}}$. From $x^\sigma = x^{l^{-1}} = x^{\lambda^{-1}\cdot{}^F\lambda}$, we get $x^{\sigma^F\lambda^{-1}} = x^{\lambda^{-1}}$, whence ${}^F(x^{\sigma^F\lambda^{-1}}) = ({}^Fx)^{{}^F\lambda^{-1}} = (({}^{\sigma F}x)^\sigma)^{{}^F\lambda^{-1}} = ({}^{\sigma F}x)^{\sigma^F\lambda^{-1}}$. □

Applying lemmas 8.2 and 8.3 we get

$$({}^*R_{\mathbf{L}^1}^{\mathbf{G}^1} \mathrm{sh}_{F/\sigma F}(\chi))(\sigma l u) = \\ |(\mathbf{G}^{\sigma l})^{0\sigma F}|^{-1} \sum_{v \in (\mathbf{G}^{\sigma l})^{0\sigma F}_{\mathrm{unip}}} \chi((\sigma l v')^{\sigma^F\lambda^{-1}}) \,\mathrm{Trace}((v^{\sigma^F\lambda^{-1}}, (u^{\sigma^F\lambda^{-1}})^{-1}) \mid H_c^*(Y_{\mathbf{U},F})).$$

Lemma 8.4. *Assume that the characteristic is good for $\mathbf{G}^{\sigma 0}$, where σ is a quasi-central element of $\mathbf{G}$. Then it is also good for $(\mathbf{G}^s)^0$ where s is any quasi-semi-simple element of $\mathbf{G}^0\cdot\sigma$.*

Proof. Let Σ_σ (resp. Σ_s) be the root system of $\mathbf{G}^{\sigma 0}$ (resp. $(\mathbf{G}^s)^0$). By definition, a characteristic p is good for a reductive group if for no closed subsystem of its root system the quotient of the generated lattices has p-torsion. The system Σ_s is not a closed subsystem of Σ_σ in general, but the relationship is expounded in [DM02]: let Σ be the root system of $\mathbf{G}^0$ with respect to a σ-stable pair $\mathbf{T}\subset\mathbf{B}$ of a maximal torus and a Borel subgroup of $\mathbf{G}^0$. Up to conjugacy, we may assume that s also stabilizes that pair. Let $\overline{\Sigma}$ the set of sums of the σ-orbits in Σ, and Σ' the set of averages of the same orbits. Then Σ' is a non-necessarily reduced root system, but Σ_σ and Σ_s are subsystems of Σ' and are reduced. The system $\overline{\Sigma}$ is reduced, and the set of sums of orbits whose average is in Σ_σ (resp. Σ_s) is a closed subsystem that we denote by $\overline{\Sigma}_\sigma$ (resp. $\overline{\Sigma}_s$).

We need now the following generalization of [Bou, chap VI, §1.1, lemme1]

Lemma 8.5. *Let $\mathcal{L}$ be a finite set of lines generating a vector space V over a field of characteristic 0; then two reflections of V which stabilize $\mathcal{L}$ and have a common eigenvalue $\zeta\neq 1$ with ζ-eigenspace the same line of $\mathcal{L}$ are equal.*

Proof. Here we mean by reflection an element $s\in\mathrm{GL}(V)$ such that $\ker(s-1)$ is a hyperplane. Let s and s' be reflections as in the statement. The product $s^{-1}s'$ stabilizes $\mathcal{L}$, so has a power which fixes $\mathcal{L}$, thus is semisimple. On the other hand $s^{-1}s'$ by assumption fixes one line $L\in\mathcal{L}$ and induces the identity on V/L, thus is unipotent. Being semi-simple and unipotent it has to be the identity. □

It follows from 8.5 that two root systems with proportional roots have same Weyl group, thus same good primes; thus:

- Σ_s and $\overline{\Sigma}_s$ have same good primes, as well as Σ_σ and $\overline{\Sigma}_\sigma$.
- The bad primes for $\overline{\Sigma}_s$ are a subset of those for $\overline{\Sigma}$, since it is a closed subsystem.

It only remains to show that the good primes for $\overline{\Sigma}$ are the same as for $\overline{\Sigma}_\sigma$, which can be checked case by case: we can reduce to the case where Σ is irreducible, where these systems coincide excepted when Σ is of type A_{2n}; but in this case $\overline{\Sigma}$ is of type B_n and Σ_σ is of type B_n or C_n, which have the same set $\{2\}$ of bad primes. □

Since the characteristic is good for $\mathbf{G}^{\sigma 0}$, hence also for $(\mathbf{G}^{\sigma l})^0$ by lemma 8.4, the elements v' et v are conjugate in $(\mathbf{G}^{\sigma l})^{0\,\sigma F}$ (see [DM85, IV Corollaire 1.2]). By Lemma 8.3(ii), the element $v^{\sigma^F\lambda^{-1}}$ runs over the unipotent elements of $(\mathbf{G}^{l'\sigma})^{0\,F}$ when v runs over $(\mathbf{G}^{\sigma l})^{0\,\sigma F}_{\mathrm{unip}}$. Using moreover the equality $|(\mathbf{G}^{\sigma l})^{0\,\sigma F}|=|(\mathbf{G}^{l'\sigma})^{0\,F}|$ we get

$$(*)\quad ({}^*R^{\mathbf{G}^1}_{\mathbf{L}^1}\,\mathrm{sh}_{F/\sigma F}(\chi))(\sigma l u)=\frac{1}{|(\mathbf{G}^{l'\sigma})^{0\,F}|}\sum_{u_1\in(\mathbf{G}^{l'\sigma})^{0\,F}_{\mathrm{unip}}}\chi(u_1 l'\sigma)\qquad \mathrm{Trace}((u_1,(u^{\sigma^F\lambda^{-1}})^{-1})|H^*_c(Y_{\mathbf{U},F}))$$

On the other hand by Lemma 8.2 applied with $v=u$, we have

$$(\mathrm{sh}_{F/\sigma F}\,{}^*R^{\mathbf{G}^1}_{\mathbf{L}^1}(\chi))(\sigma l u)={}^*R^{\mathbf{G}^1}_{\mathbf{L}^1}(\chi)((\sigma l u)^{\sigma^F\lambda^{-1}})={}^*R^{\mathbf{G}^1}_{\mathbf{L}^1}(\chi)(l'\sigma\cdot u^{\sigma^F\lambda^{-1}}),$$

the second equality by Lemma 8.3(i). By the character formula this is equal to the right-hand side of formula (*). □

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