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LARGE DEVIATION PRINCIPLE FOR EMPIRICAL FIELDS OF LOG AND RIESZ GASES

THOMAS LEBLÉ AND SYLVIA SERFATY

Abstract. We study the Gibbs measure associated to a system of N particles with logarithmic, Coulomb or Riesz pair interactions under a fairly general confining potential, in the limit $N \to \infty$. After rescaling we examine a microscopic quantity, the associated empirical point process, for which we prove a large deviation principle whose rate function is the sum of a specific relative entropy weighted by the temperature and of a "renormalized energy" which measures the disorder of a configuration. This indicates that the configurations should cristallize as the temperature vanishes and behave microscopically like Poisson point processes as the temperature tends to infinity.

We deduce a variational characterization of the sine-beta and Ginibre point processes which arise in random matrix theory. We also give a next-to-leading order expansion of the free energy of the system, thus proving the existence of a thermodynamic limit.

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1. Introduction

1.1. **General setting and main results.** We consider the Hamiltonian of a system of N points in the Euclidean space $\mathbb{R}^d$ ($d \geq 1$) interacting via logarithmic, Coulomb or Riesz pairwise interactions, in a potential V:

$$\mathcal{H}_N(x_1, \dots, x_N) := \sum_{1 \leq i \neq j \leq N} g(x_i - x_j) + N \sum_{i=1}^N V(x_i), \quad x_i \in \mathbb{R}^d, \tag{1.1}$$

where the interaction kernel is given by either

$$g(x) = -\log |x|, \quad \text{in dimension } d = 1, \tag{1.2}$$

or

$$g(x) = -\log |x|, \quad \text{in dimension } d = 2, \tag{1.3}$$

or in general dimension

$$g(x) = \frac{1}{|x|^s}, \quad \max(0, d-2) \leq s < d. \tag{1.4}$$

Whenever the parameter s appears, it will be with the convention that s is taken to mean 0 if we are in the cases (1.2) or (1.3). The potential V is a confining potential, growing fast enough at infinity, on which we shall make assumptions later.

We are interested in proving a Large Deviation Principle (LDP) for the Gibbs measure associated to this Hamiltonian

$$d\mathbb{P}_{N,\beta}(x_1, \dots, x_N) = \frac{1}{Z_{N,\beta}} e^{-\frac{\beta}{2} N^{-\frac{s}{d}} \mathcal{H}_N(x_1, \dots, x_N)} dx_1 \dots dx_N, \tag{1.5}$$

where $\beta > 0$ is a constant that represents an inverse temperature, and the temperature scaling $\beta N^{-s/d}$ (understood with the convention $s = 0$ in cases (1.2)–(1.3)) is chosen to obtain non-trivial results.

In the case (1.2), this Gibbs measure corresponds to a "1D log-gas" system, also called a "β-ensemble". As is well known, particular instances of these occur in random matrix theory, for example when $\beta = 1, 2, 4$ with a quadratic potential V (with the GOE, GUE, GSE ensembles) and they have been intensively studied. In the case (1.3) it corresponds to a two-dimensional log-gas or Coulomb gas or "one-component plasma", a particular instance being the Ginibre ensemble of random matrices obtained with the choice $\beta = 2$ and V quadratic. For a general presentation of these we refer to the textbooks [Meh04, For10, AGZ10] and the foundational papers [Wig55, Dys62] where the connection between the law of the eigenvalues of random matrices and Coulomb gases was first noticed. A related version, with particles of opposite signs, also called classical Coulomb gas is also a fundamental model of statistical mechanics, cf. the review [Spe97] and references therein.

The case $d \geq 3$ and $s = d - 2$ corresponds to a higher dimensional Coulomb gas, which can be seen as a toy (classical) model for matter. The study of these was pioneered e.g. in [PS72, JLM93, LL69, LN75].

Finally, the case (1.4) can be seen as a generalization of the Coulomb case with more general Riesz interactions. By extension, we may call such a system a Riesz gas. Motivations for studying Riesz gases are numerous in the physics literature, see for instance [Maz11, BBDR05]: they can also correspond to physically meaningful particle systems, such as systems with Coulomb interaction constrained to a lower-dimensional subspace. Another important motivation for studying such systems is the topic of approximation theory. We refer to the forthcoming monograph of Borodachev-Hardin-Saff [BHS], the review papers [SK97, BHS12] and references therein. In that context such systems are mostly studied on the d-dimensional sphere or torus.

In all cases of interactions, the ensembles governed by the law (1.5) are considered as difficult systems in statistical mechanics because the interactions they contain are truly long-range, and the points are not constrained to a lattice. As always in statistical mechanics, one would like to understand if there are phase-transitions for particular values of the (inverse) temperature β. For such systems, one may expect what physicists call a liquid for small β, and a crystal for large β, cf. for instance [HM13]. In the case of the two-dimensional Coulomb gas (or one-component plasma) there are in fact important controversies in the physic communities (see for instance [Sti98]) as to whether there is a finite β for which the system cristallizes, and what its value is. This crystallization phenomenon has only been justified numerically, the first instance seems to be [BST66]. The exact definition of crystallization matters a lot of course, the one taken by physicists is that of non-decay of the two-point correlation function, a rather weak criterion. One consequence of the results we prove here will be that there is no finite temperature of cristallization with the strict definition of the configuration being a crystal. In other words crystallization in that sense can happen only in the limit $\beta \to \infty$. In one dimension, the result is complete thanks to the result of [Leb14, Leb]: we will see that the crystallization happens if and only if β is infinite.

Such systems naturally exhibit two lengthscales: a mesoscopic (or macroscopic) scale corresponding to the scale of confinement of the potential V – here 1 – at which one can study the average (or mean-field) distributions of the points, and a microscopic scale corresponding to the interparticle distance – here $N^{-1/d}$ – at which one can study the "local laws" for the distributions of points. Of course, crystallization is a phenomenon that happens at the microscopic or local scale.

Our approach in this paper is in line with the approaches of [SS15] for the case (1.2), [SS12a] for the case (1.3), [RS13] for the Coulomb cases, and [PS14] for the general Riesz case. As in those previous papers, it allows to treat the case of arbitrary β and quite general V. As in [PS14], it also allows to treat all cases (1.2)–(1.3)–(1.4) in one unified approach.

Prior to these works, the case (1.2) is certainly the one that has been most intensively studied and for general values of β and general V's. This culminated with very detailed results in the most recent papers which obtain on the one hand very precise asymptotic expansions of the partition function [BG13b, BG13a, Shc13, BFG13] and on the other hand complete characterizations of the point processes at the microscopic level, including spacing between the points [VV09, BEY14, BEY12]. The case (1.3) has been studied for general V in the particular case $\beta = 2$, which allows to use determinantal representations, and characterize the limiting processes at the microscopic level [Gin65, BS09]. Central Limit Theorems for fluctuations were also obtained [Joh98, RV07, AHM11, AHM]. The case (1.3) without temperature (formally $\beta = \infty$) is also well understood with rigidity results on the number of points in microscopic boxes [AOC12,NS14]. There was however little on the case of general β (away from the determinantal case) for (1.3) or for any β with the Riesz interaction kernel.

It is well-known since [Cho58] (see [ST97] for the logarithmic case, or [Ser15, Chap. 2] for a simple proof in the general case) that to leading order, under suitable assumptions on V, and if $s < d$ in (1.4), we have

$$\min \mathcal{H}_N = N^2 \mathcal{I}(\mu_V) + o(N^2) \tag{1.6}$$

in the limit $N \to \infty$, where

$$\mathcal{I}(\mu) = \iint_{\mathbb{R}^d \times \mathbb{R}^d} g(x-y)\, d\mu(x)\, d\mu(y) + \int_{\mathbb{R}^d} V(x)\, d\mu(x) \tag{1.7}$$

is the mean-field energy functional defined for Radon measures μ, and the so-called *equilibrium measure* μ_V is the minimizer of $\mathcal{I}$ in the space of probability measures on $\mathbb{R}^d$, denoted $\mathcal{P}(\mathbb{R}^d)$. This is true only for $s < d$, which is the condition for (1.7) to make sense and to have a minimizer. We will always assume that μ_V is a measure with a Hölder continuous density on its support, we abuse notation by denoting its density $\mu_V(x)$ and we also assume that its support Σ is a compact set with a nice boundary. We allow for several connected components of Σ (also called the multi-cut regime in the logarithmic case of dimension 1). The detailed assumptions are listed in Section 2.1.

An LDP for the law of the "empirical measure" $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ under the Gibbs measure

$$\frac{1}{Z_{N,\beta}} e^{-\frac{\beta}{2}\mathcal{H}_N(x_1,\dots,x_N)} dx_1 \dots dx_N$$

(i.e. (1.5) but with a different temperature scaling) also holds: there exists an LDP at speed N^2 with rate function $\beta/2(\mathcal{I} - \mathcal{I}(\mu_V))$. This was shown in [HP00, AG97] (for the case (1.2)), [AZ98,BG99] (for the case (1.3) for $\beta = 2$), [CGaZ] for a more general setting including the Riesz one, see also [Ser15, Chap. 2] for a simple presentation.

This settles in some sense the understanding of the leading order macroscopic behavior of these systems: at finite temperature, all empirical measures $\frac{1}{N}\sum_{i=1}^N \delta_{x_i}$ ressemble the equilibrium measure μ_V, except with exponentially small probability.

On the other hand, the behavior of the Hamiltonian $\mathcal{H}_N$ and of its minimizers has been understood at the next order and at the microscopic scale where the points become well-separated, i.e. $N^{-1/d}$. First, it was remarked in [SS12a] (for the case (1.3)), [SS15] (for the case (1.2)), [RS13] (for all the Coulomb cases), and [PS14] (for the general situation) that $\mathcal{H}_N$ can be exactly split into the sum of a constant leading order term and a typically next order term, as

$$\mathcal{H}_N(x_1,\dots,x_N) = N^2\mathcal{I}(\mu_V) + 2N\sum_{i=1}^N \zeta(x_i) + N^{1+s/d} w_N(x_1,\dots,x_N) \tag{1.8}$$

in the case (1.4) and respectively

$$\mathcal{H}_N(x_1,\dots,x_N) = N^2\mathcal{I}(\mu_V) - \frac{N}{d}\log N + 2N\sum_{i=1}^N \zeta(x_i) + N w_N(x_1,\dots,x_N) \tag{1.9}$$

in the cases (1.2)–(1.3), where w_N will be defined in (2.30), and ζ is a function depending only on V, which is nonnegative and vanishes exactly in a set that we denote ω and which contains Σ (precise definitions will be given in (2.4) in Section 2.1). It was shown in [SS12a,SS15,RS13, PS14] that the object w_N has a limit $\mathcal{W}$ as $N \to \infty$ called the "renormalized energy", which is expressed in terms of the potential generated by the limits of configurations blown-up at the scale $N^{1/d}$. Its precise definition is given in Section 2.3. As a consequence, minimizers of $\mathcal{H}_N$ converge, after blow-up, to minimizers of $\mathcal{W}$. It is expected (but this remains at the level of a conjecture except in dimension 1), that at least in low dimensions, the minimum of $\mathcal{W}$ is achieved at simple (Bravais) lattice configurations, i.e. minimizers of $\mathcal{W}$ are expected to be crystalline and ressemble perfect lattices. This settled in [SS12a, SS15, RS13, PS14] the analysis of the microscopic behavior of minimizers in the formal case $\beta = \infty$ by connecting $\mathcal{H}_N$ to $\mathcal{W}$ and minimizers of $\mathcal{H}_N$ to the cristallization question of minimizing $\mathcal{W}$. The information obtained this way on $\mathcal{H}_N$ also allowed to deduce information on the case with temperature i.e. on $\mathbb{P}_{N,\beta}$: an asymptotic expansion of the logarithm of the partition function $Z_{N,\beta}$ and a qualitative description of the limit of $\mathbb{P}_{N,\beta}$, which become sharp only as $\beta \to \infty$, and hints at a crystallization phenomenon. In dimension 1, the crystallization was rigorously established in [SS15], using the result of [Leb14].

Our goal here is to obtain a complete LDP that lies at this next order and is valid for all β. It describes the configurations after blow-up at the microscopic scale around points in the support Σ of the equilibrium measure μ_V and gives a rate function on the random point processes obtained via the blown-up limits. Equivalently it is an LDP "at the process level" also called "type-III LDP", cf. for example [RAS09]. For general reference on large deviations one may see e.g. [DZ10]. The idea of using large deviations methods for such systems already appeared in [BBDR05] where results of the same flavor but at a more formal level are presented.

1.1.1. *Preliminary notation.* Before giving a statement, let us introduce some notation. We denote by $\mathcal{X}$ the set of locally finite (not necessarily simple) point configurations in $\mathbb{R}^d$, or equivalently the set of purely atomic Radon measures giving an integer mass to singletons, cf. [DVJ88]. The topology of vague convergence induces a topology on $\mathcal{X}$. A point process is then defined to be a probability measure on $\mathcal{X}$, i.e. an element of $\mathcal{P}(\mathcal{X})$, cf. [DVJ88]. We can then see configurations $(x_1,\dots,x_N)$ as elements of the space $\mathcal{X}$ of discrete (finite or infinite) point configurations in $\mathbb{R}^d$. When starting from an N-uple of points $(x_1,\dots,x_N)$, we first rescale the associated finite configuration $\sum_{i=1}^N \delta_{x_i}$ by a factor $N^{1/d}$ and then define the map

$$\begin{aligned} i_N : (\mathbb{R}^d)^N &\to \mathcal{P}(\Sigma \times \mathcal{X}) \\ (x_1, \dots, x_N) &\mapsto \fint_{\Sigma} \delta_{(x, \theta_{N^{1/d}x} \cdot (\sum_{i=1}^N \delta_{N^{1/d}x_i}))} dx \end{aligned} \tag{1.10}$$

where θ_λ denotes the action of translation by λ and δ is the Dirac mass.

The space $\mathcal{P}(\Sigma \times \mathcal{X})$ is defined as the space of "tagged point processes", where we keep as a tag the point $x \in \Sigma$ around which the configuration was blown up. It is equipped with the topology of weak convergence of measures on $\Sigma \times \mathcal{X}$ (the topology is discussed further in Section 2.4). If $\bar{P}$ is a tagged point process we will always assume that the first marginal of $\bar{P}$ is the normalized Lebesgue measure on Σ. We will generally denote with bars the quantities that correspond to tagged processes, and without bars the quantities that correspond to non-tagged processes. We denote by $\mathcal{P}_s(\mathcal{X})$ the set of translation-invariant, or stationary point processes. We also call stationary a tagged point process $\bar{P}$ such that the disintegration measure $\bar{P}^x$ (cf. [AGS05, Section 5.3] for a definition) is stationary for (Lebesgue-)a.e. $x \in \Sigma$ and denote by $\mathcal{P}_s(\Sigma \times \mathcal{X})$ the set of stationary tagged point processes.

In [PS14] and previous articles, a renormalized energy $\mathcal{W}$ was defined at the level of the potentials generated by a point configuration. In the particular case (1.2), it can be interpreted as the L^2 norm of the Stieltjes transform, properly normalized (cf. [SS15]). This energy may be "projected down" to a renormalized energy $\mathbb{W}$ defined on point configurations themselves (all definitions will be recast more completely below in Section 2.3). One can then extend it as an energy on point processes $P \in \mathcal{P}(\mathcal{X})$ by

$$\widetilde{\mathbb{W}}_m(P) := \int \mathbb{W}_m(\mathcal{C}) \, dP(\mathcal{C}) \tag{1.11}$$

where we keep as index m the intensity of the point process, or equivalently the background density. We then define the renormalized energy of a tagged point process as

$$\overline{\mathbb{W}}_{\mu_V}(\bar{P}) := \frac{1}{c_{d,s}} \int_{\Sigma} \widetilde{\mathbb{W}}_{\mu_V(x)}(\bar{P}^x) dx \tag{1.12}$$

where $c_{d,s}$ is a constant depending only on d, s.

Next, we define a specific relative entropy as the infinite-volume limit of the usual relative entropy with respect to some reference measure. Below C_N denotes the hypercube of sidelength N, $[-N/2, N/2]^d$ and $|U|$ denote the Lebesgue measure (or volume) of a set U.

Definition 1.1. *Let P be a stationary point process on $\mathbb{R}^d$. The relative specific entropy $\mathsf{ent}[P|\mathbf{\Pi}^1]$ of P with respect to $\mathbf{\Pi}^1$, the Poisson point process of uniform intensity 1, is given by*

$$\mathsf{ent}[P|\mathbf{\Pi}^1] := \lim_{N\to\infty} \frac{1}{|C_N|} \mathrm{Ent}\left(P_{|C_N} | \mathbf{\Pi}^1_{|C_N}\right) \tag{1.13}$$

where $P_{|C_N}$ denotes the process induced on (the point configurations in) C_N, and $\mathrm{Ent}(\cdot|\cdot)$ denotes the usual relative entropy (or Kullbak-Leibler divergence) of two probability measures defined on the same probability space.

We take the appropriate sign convention for the entropy so that $\mathsf{ent} \geq 0$ i.e. if μ, ν are two probability measures defined on the same space we let

$$\mathrm{Ent}\,(\mu|\nu) := \int \log \frac{d\mu}{d\nu} d\mu$$

if μ is absolutely continuous with respect to ν and $+\infty$ otherwise. It is known (see e.g. [RAS09]) that the limit (1.13) exists for all stationary processes, hence the relative specific entropy is well-defined, and also that the functional $P \mapsto \mathsf{ent}[P|\mathbf{\Pi}^1]$ is affine lower semi-continous and that its sub-level sets are compact.

We end this section by recalling the definition of LDP.

Definition 1.2. *A sequence $\{\mu_N\}_N$ of probability measures on a metric space X is said to satisfy a Large Deviation Principle (LDP) at speed r_N with rate function $I : X \to [0, +\infty]$ if the following holds for any $A \subset X$*

$$-\inf_{\mathring{A}} I \le \liminf_{N\to\infty} \frac{1}{r_N} \log \mu_N(A) \le \limsup_{N\to\infty} \frac{1}{r_N} \log \mu_N(A) \le -\inf_{\overline{A}} I,$$

where $\mathring{A}$ (resp. $\overline{A}$) denotes the interior (resp. the closure) of A. The functional I is said to be a "good rate function" if it is lower semi-continuous and has compact sub-level sets.

We refer to [DZ10] for a detailed treatment of the theory of large deviations and to [RAS09] for an introduction to the applications of LDP's in the statistical physics setting.

1.1.2. *Main result and consequences.* We may now state our main LDP result.

Theorem 1 (Large Deviation Principle)**.** *Assume V satisfies the assumptions of Section 2.1. Let $\overline{\mathfrak{P}}_{N,\beta}$ be the random tagged empirical field associated to the Gibbs measure by pushing forward $\mathbb{P}_{N,\beta}$ by the map (1.10). Then for any $\beta > 0$, the sequence $\{\overline{\mathfrak{P}}_{N,\beta}\}_N$ satisfies a large deviation principle at speed N with good rate function $\beta(\overline{\mathcal{F}}_\beta - \inf \overline{\mathcal{F}}_\beta)$ where*

$$\overline{\mathcal{F}}_\beta(\bar{P}) := \frac{1}{2}\overline{\mathbb{W}}_{\mu_V}(\bar{P}) + \frac{1}{\beta}\left(\int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]\,dx + 1 - |\Sigma|\right). \tag{1.14}$$

A first consequence of the LDP is that in the limit $N \to \infty$, the Gibbs measure (more precisely the limit of $\overline{\mathfrak{P}}_{N,\beta}$) concentrates on minimizers of $\overline{\mathcal{F}}_\beta$. Also, it is easy to see that $\bar{P}$ minimizes $\overline{\mathcal{F}}_\beta$ if and only if its disintegration measures $\bar{P}^x$ minimize for a.e. $x \in \Sigma$ the non averaged rate function

$$\mathcal{F}_\beta(P) := \frac{1}{2c_{d,s}}\widetilde{\mathbb{W}}_{\mu_V(x)}(P) + \frac{1}{\beta}\mathsf{ent}[P|\mathbf{\Pi}^1]. \tag{1.15}$$

Identifying the minimizers of either $\mathcal{F}_\beta$ or $\overline{\mathcal{F}}_\beta$ is a hard question in general, even if one knew what the minimizers of $\mathbb{W}$ are. However, one readily sees the effect of the temperature: in the minimization there is a competition between the term $\overline{\mathbb{W}}_{\mu_V}$ or $\widetilde{\mathbb{W}}_{\mu_V(x)}$ based on the renormalized energy, and which is thus expected to favor crystalline (hence very ordered) configurations, and the entropy term which to the contrary favors disorder. The temperature determines the relative weight of these two competing effects: as $\beta \to 0$ (i.e. temperature gets large) the entropy term dominates and configurations can be expected to behave like a Poisson point process, while as $\beta \to \infty$ (i.e. temperature gets very small), the renormalized energy dominates, and configurations can be expected to crystallize. In particular, as we will observe later, our result implies that crystallization, in the strict sense of configurations being cristalline, should not be expected to happen at a finite (fixed) β because cristalline configurations give rise to an infinite entropy. Thus the cristallization expected in the physics literature can only be a weaker form of crystallization such as a transition to slower decaying or non-decaying correlation functions.

Our result naturally raises two questions: the first is to understand better $\overline{\mathbb{W}}$ and its minimizers and the second is to better understand the specific relative entropy, about which not much seems to be known in general.

In the particular case of (1.2) with a quadratic potential $V(x) = x^2$, the equilibrium measure is known to be the semi-circular law whose density is given by

$$x \mapsto \frac{1}{2\pi}\mathbf{1}_{[-2,2]}\sqrt{4-x^2},$$

and the limiting point process at the microscopic level around a point $x \in [-2,2]$ (let us emphasize that in this case there is *no* averaging over translations) is identified for any $\beta > 0$ in [VV09] to be the "sine-β" point process, which we will denote by $\mathrm{Sine}_\beta(x)$ (so that $\mathrm{Sine}_\beta(x)$ has intensity $\frac{1}{2\pi}\sqrt{4-x^2}$). They are all equal in law up to rescaling and we denote by Sine_β the corresponding process with intensity 1. It is also proven to be the limit of the β-circular ensemble [Nak14]. A corollary of our result is then a new characterization of these processes:

Corollary 1.3 (Sine-beta process). *For any $\beta > 0$, the tagged point process*

$$\overline{\mathrm{Sine}_\beta} := \int_{[-2,2]} \delta_{(x, \mathrm{Sine}_\beta(x))}$$

minimizes the rate function $\overline{\mathcal{F}}_\beta$ among tagged point processes in $\mathcal{P}(\Sigma \times \mathcal{X})$. The point process Sine_β *minimizes*

$$\mathcal{F}_\beta(P) = \frac{1}{4\pi}\widetilde{\mathbb{W}}_1(P) + \frac{1}{\beta}\mathsf{ent}[P|\mathbf{\Pi}^1]$$

among stationary point processes of intensity 1 *in* $\mathbb{R}$.

The one other case in which the limiting Gibbsian point process is identified is the case (1.3) with V quadratic, which gives rise to the so-called "Ginibre point process" [Gin65, BS09]. In this case we also obtain the new characterization:

Corollary 1.4 (Ginibre process). *The Ginibre point process minimizes*

$$\mathcal{F}_2(P) = \frac{1}{4\pi}\widetilde{\mathbb{W}}_1(P) + \frac{1}{2}\mathsf{ent}[P|\mathbf{\Pi}^1]$$

among stationary point processes of intensity 1 *in* $\mathbb{R}^2$.

Corollaries 1.3 and 1.4 are proven in Section 4.3.

As mentioned above, the infimum $\inf \overline{\mathcal{F}}_\beta$ is unknown in general and its determination seems to be a difficult problem. However we know exactly how $\overline{\mathcal{F}}_\beta$ depends on μ_V hence on V, because we know how the $\overline{\mathbb{W}}$ and entropy terms scale in terms of the equilibrium measure density (which is the same as the point process intensity). For any $m > 0$, we let σ_m be the map which rescales a point configuration by the factor $m^{1/d}$, i.e. turns a configuration of density m into one of intensity 1. Then we may consider $\bar{P}'$ the push-forward of $\bar{P}$ by the map on $\Sigma \times \mathcal{X}$

$$(x, \mathcal{C}) \mapsto (x, \sigma_{\mu_V(x)}\mathcal{C}).$$

In the case (1.4), the rescaling yields

$$\overline{\mathcal{F}}_\beta(\bar{P}) = \frac{1}{2c_{d,s}} \int_\Sigma \widetilde{\mathbb{W}}_1(\bar{P}'^x)\mu_V(x)^{1+s/d}\,dx + \frac{1}{\beta}\left(\int_\Sigma \mathsf{ent}[\bar{P}'^x|\mathbf{\Pi}^1]\mu_V(x)\,dx + \int_\Sigma \mu_V(x)\log\mu_V(x)\,dx\right). \tag{1.16}$$

In the cases (1.2)–(1.3), the rescaling yields

$$\overline{\mathcal{F}}_\beta(\bar{P}) = \frac{1}{2}\left(\frac{1}{c_{d,s}}\int_\Sigma \widetilde{\mathbb{W}}_1(\bar{P}'^x)\mu_V(x)\,dx - \frac{1}{d}\int_\Sigma \mu_V(x)\log\mu_V(x)\,dx\right) + \frac{1}{\beta}\left(\int_\Sigma \mathsf{ent}[\bar{P}'^x|\mathbf{\Pi}^1]\,\mu_V(x)\,dx + \int_\Sigma \mu_V(x)\log\mu_V(x)\,dx\right)$$

and in these particular cases, the terms recombine into

$$\overline{\mathcal{F}}_\beta(\bar{P}) = \int_\Sigma \left(\frac{1}{2c_{d,s}}\widetilde{\mathbb{W}}_1(\bar{P}'^x) + \frac{1}{\beta}\mathsf{ent}[\bar{P}'^x|\mathbf{\Pi}^1]\right)\mu_V(x)\,dx + \left(\frac{1}{\beta} - \frac{1}{2d}\right)\int_\Sigma \mu_V(x)\log\mu_V(x)\,dx. \tag{1.17}$$

There has been a lot of interest recently in proving "universality results" for such systems, i.e. proving that their microscopic behavior is independent of V, hence of μ_V. Such results have been obtained in the cases (1.2) in [BEY14,BEY12,BFG13], etc. In the above formulae, the terms not involving μ_V are independent of μ_V and V, hence universal. In the cases (1.2)–(1.3), since μ_V is a probability measure, one can deduce from (1.17) that

$$\boxed{\min \overline{\mathcal{F}}_\beta = \min\left(\frac{1}{2c_{d,s}}\widetilde{\mathbb{W}}_1 + \frac{1}{\beta}\mathsf{ent}[\cdot|\mathbf{\Pi}^1]\right) + \left(\frac{1}{\beta} - \frac{1}{2d}\right)\int_\Sigma \mu_V(x)\log\mu_V(x)\,dx.}$$

Hence the dependence of $\min \overline{\mathcal{F}}_\beta$ in μ_V is just an additive constant which happens to vanish when $\beta = 2$ in dimension 1 and $\beta = 4$ in dimension 2. This is in agreement with the universality known in these cases: minimizers of $\overline{\mathcal{F}}_\beta$ are independent of V hence universal.

In contrast, in (1.16) μ_V comes as a multiplicative weight and in the minimization of $\overline{\mathcal{F}}_\beta$ the relative weights of the energy $\widetilde{\mathbb{W}}_1$ and of the entropy depend on μ_V : this can be seen as creating an effective temperature $\beta\mu_V^{s/d}$. Hence the minimizers of $\overline{\mathcal{F}}_\beta$ will not be universal, and this indicates that universality, in the sense previously used, fails in higher dimensional Coulomb cases or in Riesz cases. In other words, universality seems to be directly tied with the logarithmic nature of the interaction. We note that no positive or negative prediction in that direction seemed to have been proposed.

A byproduct of the LDP is naturally the existence of a thermodynamic limit for these systems, i.e. an asymptotic term in N in the expansion of $\log Z_{N,\beta}$, which in view of the above discussion is given by :

Corollary 1.5 (Thermodynamic limit)**.** *Under the same assumptions, we have, as $N \to \infty$,*

$$\log Z_{N,\beta} = -\frac{\beta N^{2-\frac{s}{d}}}{2}\mathcal{I}(\mu_V) - N\beta\min\overline{\mathcal{F}}_\beta + o((\beta+1)N) \tag{1.18}$$

in the cases (1.4)*; and in the cases* (1.2)–(1.3)

$$\log Z_{N,\beta} = -\frac{\beta N^2}{2}\mathcal{I}(\mu_V) + \frac{\beta N}{2d}\log N - N\beta \min \overline{\mathcal{F}}_\beta + o((\beta+1)N)$$

or more explicitly

$$\log Z_{N,\beta} = -\frac{\beta N^2}{2}\mathcal{I}(\mu_V) + \frac{\beta N}{2d}\log N - N\beta \min\left(\frac{1}{2c_{d,s}}\widetilde{\mathbb{W}}_1 + \frac{1}{\beta}\mathsf{ent}[\cdot|\mathbf{\Pi}^1]\right) - N\beta\left(\frac{1}{\beta} - \frac{1}{2d}\right)\int_\Sigma \mu_V(x)\log\mu_V(x)\,dx + o((\beta+1)N). \tag{1.19}$$

Here the $o(1)$ *tend to zero as* $N \to \infty$ *independently of* β.

This provides an asymptotic expansion of the free energy (i.e. $-\frac{1}{\beta}\log Z_{N,\beta}$) up to order N, where in view of (1.14), the order N term itself has the structure of a free energy.

The existence of such a thermodynamic limit had been known for a long time for the two and three dimensional Coulomb cases [LN75, SM76, PS72]. Our formulae are to be compared with the recent results of [Shc13, BG13b, BG13a, BFG13] in the dimension 1 logarithmic case. These authors obtain asymptotic expansions of $\log Z_{N,\beta}$ to much lower orders than this, however they make quite strong assumptions on the regularity of the potential V, and sometimes the coefficients are not easy to explicitly compute. Since in this setting (1.2), $\log Z_{N,\beta}$ is known explicitly for $V(x) = x^2$ via Selberg integrals, by comparing to (1.19) this allows to identify the value of $\mathcal{F}_\beta(\mathrm{Sine}_\beta) = \min \mathcal{F}_\beta$ (where $\mathcal{F}_\beta$ is defined in (1.15)), and then to immediately deduce the explicit coefficients in the expansion of $\log Z_{N,\beta}$ up to order N for general V (and the difference in the order N coefficient only involves the difference in $\int \mu_V \log \mu_V$). In the case (1.3) our result can also be compared to the formal result of [ZW06].

In both logarithmic cases, we recover in (1.19) the cancellation of the order N term when $\beta = 4$ in dimension 2 and $\beta = 2$ in dimension 1 that was first observed in [Dys62, Part II, section II] and [ZW06], and when this happens then, V_1 and V_2 being two potentials satisfying our assumptions, we obtain

$$\log Z_{N,\beta}(V_2) - \log Z_{N,\beta}(V_1) = -\frac{\beta N^2}{2}(\mathcal{I}(\mu_{V_2}) - \mathcal{I}(\mu_{V_1})) + o((\beta+1)N)$$

in agreement with the well-known fact [BIZ80, EM03] that expansions of $\log Z_{N,\beta}$ corresponding to different potentials V then differ by an expansion in even powers of N only.

Finally, in the general case of Riesz gases (1.4), our result (1.18) seems to be the first rigorous one of its kind.

1.2. **Proof outline.** Using the splitting formula (1.8)–(1.9), one can factor out the constant terms from the Hamiltonian and the partition function, and reduce to studying only

$$d\mathbb{P}_{N,\beta}(x_1,\dots,x_N) = \frac{1}{K_{N,\beta}} e^{-\frac{\beta N}{2} w_N(x_1,\dots,x_N)} e^{-N\beta\sum_{i=1}^N \zeta(x_i)}\,dx_1\dots dx_N \tag{1.20}$$

where

$$K_{N,\beta} = Z_{N,\beta} e^{\frac{1}{2}\beta N^{2-s/d}\mathcal{I}(\mu_V)} e^{-\frac{1}{2}\beta\frac{N}{d}\log N}$$

(here the second multiplicative term exists only in the cases (1.2)–(1.3)). It is already proven in [PS14, Theorem 6.] (note the different normalization of temperature there) that

$$|\log K_{N,\beta}| \le C_\beta \beta N \tag{1.21}$$

with C_β bounded on any interval $[\beta_0, +\infty)$ with $\beta_0 > 0$. We next define the reference measure $\mathbb{Q}_{N,\beta}$ as the probability measure on $(\mathbb{R}^d)^N$ with density

$$d\mathbb{Q}_{N,\beta}(x_1, \dots, x_N) := \frac{e^{-N\beta \sum_{i=1}^N \zeta(x_i)}}{\left(\int_{\mathbb{R}^d} e^{-N\beta\zeta(x)}\, dx\right)^N} dx_1 \dots dx_N. \tag{1.22}$$

The effect of ζ is that of confining the points to the set ω containing the support Σ of the equilibrium measure. Thus, one can think of $\mathbb{Q}_{N,\beta}$ as being essentially the N times tensor product of the normalized Lebesgue measure on ω, and of (1.20) as being formally

$$\frac{1}{K_{N,\beta}} e^{-\frac{\beta N}{2} w_N(x_1,\dots,x_N)} \prod_{i=1}^N \mathbf{1}_\omega(x_i) dx_i.$$

To prove an LDP, the standard method consists in evaluating the logarithm of the probability $\mathbb{P}_{N,\beta}(B(\bar{P}, \varepsilon))$, where $\bar{P}$ is a given tagged point process, element of $\mathcal{P}(\Sigma \times \mathcal{X})$ (recall $(\mathbb{R}^d)^N$ embeds into this space via (1.10)) and $B(\bar{P}, \varepsilon)$ is a ball of small radius ε around it, for a distance that metrizes the weak topology we are working with.
Since

$$\mathbb{P}_{N,\beta}(B(\bar{P}, \varepsilon)) \simeq \frac{1}{K_{N,\beta}} \int_{i_N(x_1,\dots,x_N) \in B(\bar{P},\varepsilon)} e^{-\frac{\beta N}{2} w_N(x_1,\dots,x_N)} \prod_{i=1}^N \mathbf{1}_\omega(x_i) dx_i$$

we may formally write

$$\begin{aligned} \lim_{\varepsilon \to 0} \log \mathbb{P}_{N,\beta}(B(\bar{P}, \varepsilon)) = -\log K_{N,\beta} - \frac{\beta N}{2} w_N(\bar{P}) \\ + \lim_{\varepsilon \to 0} \log \left(|\{(x_1, \dots, x_N) \in \omega^N, i_N(x_1, \dots, x_N) \in B(\bar{P}, \varepsilon)\}| \right). \end{aligned} \tag{1.23}$$

Extracting this way the exponential of a function is the idea of the Varadhan integral lemma (cf. [DZ10, Theorem 4.3.1]), and works when the function is continuous. In similar contexts to ours, this is used e.g. in [Geo93, GZ93].

In (1.23) the term in the second line is the logarithm of the volume of point configurations whose associated "empirical field" is close to $\bar{P}$. By classical large deviations theorems, such a quantity is expected to be the entropy of $\bar{P}$. More precisely since we are dealing with blown-up configurations, or empirical fields, we need to use a relative specific entropy as defined above (cf. also [RAS09, Geo93]) as opposed to a usual entropy.

The most problematic term in (1.23) is the second one in the right-hand side, $w_N(\bar{P})$, which really makes no sense. The idea is that it should be close to $\overline{\mathbb{W}}(\bar{P})$ which is the well-defined quantity appearing in the rate function (1.14). If we were dealing with a continuous function of $\bar{P}$ then the replacement of $w_N(\bar{P})$ by $\overline{\mathbb{W}}(\bar{P})$ would be fine. However there are three difficulties here:

1. w_N depends on N and we need to take the limit $N \to \infty$,
2. this limit cannot be uniform because the quantities that define w_N becomes infinite when two points approach each other,
3. w_N is not adapted to the topology that we are working with, which is a weak topology which retains only local information on the point configurations, while w_N contains long-range interactions and does not depend only on local data of the points but on the whole configuration.

Thus, the approach outlined in (1.23) cannot work directly. Instead, we have to look again at the whole ball $B(\bar{P}, \varepsilon)$ and to show that in that ball there is a logarithmically large enough volume of configurations for which we can replace w_N by $\overline{\mathbb{W}}(\bar{P})$. This will give a lower bound on $\log \mathbb{P}_{N,\beta}(B(\bar{P},\varepsilon))$ and the upper bound is in fact much easier to deduce from the previously known results of [PS14]. The second obstacle above, related to the discontinuity of the Hamiltonian near the diagonals of $(\mathbb{R}^d)^N$, is similar to the difficulty encountered in [BG99]. It is handled differently though, by controlling the difference between w_N and a version of it where the singularities are truncated at some small level η. This works out precisely because the renormalized energies are defined as limits as $\eta \to 0$ of quantities truncated at the level η. By controlling this difference thanks to the tools of [PS14], we are able to show that it is small often enough, i.e. the volume of the configurations where it is small is logarithmically large enough.

The third point above, the fact that the total energy is nonlocal in the data of the configuration, creates the most delicate difficulty. The way we circumvent it is via the "screening procedure" developed in [SS12b, SS12a, SS15, RS13, PS14]. Each configuration generates a potential, denoted H, and an "electric field" $E = \nabla H$, and the energy really corresponds to the (renormalized) integral of $|E|^2$. We show that thanks to the screening, we can always modify a bit each configuration so as to make the energy that it generates additive (hence local) in space, while not moving the configuration too far from $\bar{P}$ and not losing too much logarithmic volume in phase-space. This will be detailed in Sections 5 and 6.

In the end our result is a consequence of two intermediate results.

The first one is a large deviation result for the "reference" empirical field i.e. for the measure $\bar{\mathfrak{Q}}_{N,\beta}$, defined as the push-forward of $\mathbb{Q}_{N,\beta}$ by i_N, cf. (1.10) and (1.22). We let

$$c_{\omega,\Sigma} := \log|\omega| - |\Sigma| + 1, \tag{1.24}$$

where ω is the zero-set of ζ, as mentioned above.

Proposition 1.6. *For any $A \subset \mathcal{P}_s(\Sigma \times \mathcal{X})$, we have*

$$\begin{aligned}(1.25)\quad -\inf_{\mathring{A}\cap\mathcal{P}_{s,1}} \int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx - c_{\omega,\Sigma} &\le \liminf_{N\to\infty} \frac{1}{N}\log \bar{\mathfrak{Q}}_{N,\beta}(A) \\ &\le \limsup_{N\to\infty} \frac{1}{N}\log \bar{\mathfrak{Q}}_{N,\beta}(A) \le -\inf_{\bar{P}\in\bar{A}} \int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx - c_{\omega,\Sigma}.\end{aligned}$$

In (1.25) and in the rest of the paper, if A is a set of (tagged) configurations, $\mathring{A}$ denotes the interior of A and $\bar{A}$ denotes the closure of A. The meaning of the (technical) restriction $\mathring{A} \cap \mathcal{P}_{s,1}$ will be precised later, let us say that $\mathcal{P}_{s,1}(\Sigma \times \mathcal{X})$ denotes the set of tagged point process of total intensity 1 (i.e. there is an average number of 1 point by unit volume, see Section 2.4).

Quantities obtained by averaging a given configuration over translations as in (1.10), are called "empirical fields". The first large deviations principles for empirical fields seem to be stated in [Var88], [Föl88] and the relative specific entropy was then formalized by Föllmer and Orey [FO88] in the non-interacting discrete case (see also [RAS09] for another approach), by Georgii [Geo93] in the interacting discrete case and Georgii and Zessin [GZ93] in the interacting continuous case. In that light, the result of this proposition is not too surprising, however our setting differs from the one of [GZ93] in that the reference measure $\mathbb{Q}_{N,\beta}$ is not the restriction of a Poisson point process to a hypercube but somehow only approximates a Bernoulli point process on some domain ω - which is not a hypercube - with the possibility of some points falling outside ω. Moreover we want to study large deviations for tagged point processes (let us emphasize that our use of "tags" is not the same as the "marks" in [GZ93]) which requires an additional approximation argument. The proof of these successive adaptations to our context occupies Section 7.

Let us say a word about the choice of topology on $\mathcal{X}$. It is well known that large deviation principles hold for empirical fields after endowing $\mathcal{X}$ with a strong topology, namely the τ-topology (the initial topology on $\mathcal{X}$ associated to the maps $\mathcal{C} \mapsto f(\mathcal{C})$ for any bounded measurable function f which is local in the sense of (2.40)), see e.g. [Geo93], [RAS09]. Altough we expect both Proposition 1.6 and Theorem 1 to hold with this stronger topology, with essentially the same proof, we do not pursue this generality here. Let us here emphasize that even when restating Proposition 1.6 in the τ-topology our main theorem does not follow from an application of Varadhan's integral lemma, because w_N is neither bounded nor local.

Proposition 1.6 is then complemented by the following result, which essentially yields the main theorem:

Proposition 1.7. *Let $\bar{P} \in \mathcal{P}_{s,1}(\Sigma \times \mathcal{X})$. For all $\delta_1, \delta_2 > 0$ we have*

$$\lim_{N\to\infty} \frac{1}{N} \log \bar{\mathfrak{Q}}_{N,\beta} \left(B(\bar{P}, \delta_1) \cap T_{N,\delta_2}(\bar{P}) \right) \geq - \int_{\Sigma} \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx - c_{\omega,\Sigma}, \tag{1.26}$$

where $T_{N,\delta_2}(\bar{P})$ denotes a set of point processes obtained from N-uples of points $(x_1, \dots, x_N)$ by $i_N(x_1, \dots, x_N)$ (where i_N is defined in (1.10)) which satisfy

$$w_N(x_1, \dots, x_N) \leq \overline{\mathbb{W}}_{\mu_V}(\bar{P}) + \delta_2.$$

This proposition is the hard part of the proof. To obtain it we need to show that the set $T_{N,\delta_2}(\bar{P})$ has enough volume in phase-space. This relies on taking arbitrary configurations in $B(\bar{P}, \delta_1)$ and showing that a large enough fraction of them (in the sense of volume) can be screened and modified to generate a small energy truncation error, as alluded to above.

1.3. Open questions and further study.

1.3.1. *Crystallization and phase transitions.* Let us observe the following :

Lemma 1.8. *Let Γ be a point configuration periodic with respect to some lattice $\Lambda \in \mathbb{R}^d$ and let P_Γ be the associated stationary point process defined by*

$$P_\Gamma := \fint_{\Lambda} \delta_{\theta_x \cdot \Gamma} dx.$$

Then the specific relative entropy $\mathsf{ent}[P_\Gamma | \mathbf{\Pi}^1]$ is equal to $+\infty$.

Proof. It is in fact easy to see that for any integer N the point process induced by P_Γ in the hypercube C_N is absolutely singular with respect to the Poisson point process $\mathbf{\Pi}^1_{|C_N}$ hence the usual relative entropy $\mathrm{Ent}[P_{\Gamma|C_N} | \mathbf{\Pi}^1_{|C_N}]$ is infinite. Thus by definition we also have $\mathsf{ent}[P_\Gamma | \mathbf{\Pi}^1] = +\infty$. □

It follows from Lemma 1.8 that if β is finite, the minimizer of $\mathcal{F}_\beta$ cannot be a periodic point process and in particular it cannot be the point process associated to some lattice (or crystal). Hence there is no crystallization in the strong sense i.e. the particles cannot concentrate on an exact lattice. However some weaker crystallization could occur at finite β e.g. if the connected two-point correlation function $\rho_2 - 1$ of minimizers of $\mathcal{F}_\beta$ decays more slowly to 0 as β gets larger or ceases to be in L^1 for β greater than some β_c. Hints towards such a transition in the behavior of $\rho_2 - 1$ for the one-dimensional log-gas may be found in [For93] where an explicit formula for the two-point correlation function is computed for the limiting point processes associated to the β-Circular Ensemble (which according to [Nak14] turn out to also be Sine_β).

Such a change in the long-distance behavior of ρ_2 would not necessarily imply a first-order phase transition i.e. a singularity in the first derivative of $\beta \mapsto \min \mathcal{F}_\beta$, as would be implied e.g. by the existence of two minimizers of $\mathcal{F}_\beta$ with different energies. The existence of a first-order phase transition in the two-dimensional logarithmic case (also called the two-dimensional one-component plasma) is discussed in the physics literature, see e.g. [Sti98]. On the other hand, it might be that for some β there exists several minimizers of $\mathcal{F}_\beta$ with the same energy and the physical implications of such a situation is unclear to us. Let us note that uniqueness of the minimizers (or at least the fact that they all have the same energy, hence the same entropy) would for example allow to retrieve as a straightforward corollary of our LDP the equiparitition property shown in [BMSS13] for β-models.

In the following paragraph we collect some open questions, stemming from the ones discussed above.

1.3.2. *Open questions.*

- Is the minimum of $\mathcal{F}_\beta$ unique? Let us observe that the specific relative entropy $\mathsf{ent}[\cdot|\mathbf{\Pi}^1]$ is affine, hence so is $\mathcal{F}_\beta$ and no easy "strict convexity" argument seems to hold. Do at least all the minimizers of $\mathcal{F}_\beta$ share the same energy and entropy?
- Can the variational characterization of the Sine_β, Ginibre and other limiting point processes be used to provide more information on these processes?
- Does crystallization hold in a weak sense, e.g. at the level of a change in the large-distance behaviour of the two-point correlation function of minimizers of $\mathcal{F}_\beta$ when β crosses some critical value?
- Can we characterize the minima (minimum?) of $\widetilde{\mathbb{W}}_1$ for $d \geq 2$? Can we at least prove that any minimizer of $\widetilde{\mathbb{W}}_1$ has infinite specific relative entropy, which would be a first hint towards their conjectural "ordered" nature?
- Is there a limit to the Gibbsian point process $\mathbf{P}_{N,\beta}$, defined as the push-forward of $\mathbb{P}_{N,\beta}$ by $(x_1, \dots, x_N) \mapsto \sum_{i=1}^N \delta_{N^{1/d}x_i}$ and of their translates $\theta_x \cdot \mathbf{P}_{N,\beta}$ for x in the "bulk" (the interior of Σ) ? In the cases where the existence of a limit is known, can we find a purely "energy based" proof? Can we at least prove that any limit point of $\mathbf{P}_{N,\beta}$ is translation invariant? Can we at least prove the mere existence of limit points for a general class of V, s, d?
- Can one somehow use the next-order information on $Z_{N,\beta}$ of Corollary 1.5 to prove a central limit theorem for the fluctuations?

1.3.3. *Further study.* In order to deduce further consequences of the LDP, it is more convenient to express the renormalized energy of a point process in terms of its two-point correlation functions. This is inspired by [BS13] and is the object of [Leb]. This approach allows one to obtain further qualitative information, as the convergence of minimizers of $\mathcal{F}_\beta$ to a Poisson point process in the limit $\beta \to 0$, thus retrieving results of [AD14] in the special case of sine-beta processes.

The rest of the paper is organized as follows: Section 2 contains our assumptions, the definitions of the renormalized energy and of the specific relative entropy, as well as some important notation. In Section 3 we present some preliminary results on the renormalized energy. Section 4 contains the proofs of the main results and corollaries, assuming the results of

Propositions 1.6 and 1.7. In Section 5 we recall the screening result and describe the procedure to screen random point configurations. We also describe the regularization procedure. In Section 6 we complete the proof of Proposition 1.7 by showing that given a random point configuration we can often enough screen it and regularize it to have the right energy. In Section 7 we prove Proposition 1.6. In Section 8, we collect miscellaneous additional proofs.

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2. Assumptions and main definitions

2.1. **Our assumptions.** We now start to describe more precisely the setting in which we work, which is identical to that of [PS14].

We first place assumptions on V that ensure the existence of the equilibrium μ_V from standard potential theory:

$$V \text{ is lower semi-continuous (l.s.c.) and bounded below} \tag{2.1}$$

$$\{x : V(x) < \infty\} \text{ has positive } g\text{-capacity} \tag{2.2}$$

$$\lim_{|x|\to\infty} V(x) = +\infty, \quad \text{resp. } \lim_{|x|\to\infty} \frac{V(x)}{2} - \log|x| = +\infty \text{ in cases } (1.2) - (1.3) \tag{2.3}$$

The following theorem, due to Frostman [Fro35] (cf. also [ST97]) then gives the existence and characterization of the equilibrium measure (let us recall that the energy functional was defined in (1.7)):

Theorem 2 (Frostman)**.** *Assume that V satisfies* (2.1)–(2.2)*, then there exists a unique minimizer $\mu_V \in \mathcal{P}(\mathbb{R}^d)$ of $\mathcal{I}$ and $\mathcal{I}(\mu_V)$ is finite. Moreover the following properties hold:*
- *$\Sigma := \mathrm{supp}(\mu_V)$ is bounded and has positive g-capacity,*
- *for $c := \mathcal{I}(\mu_V) - \int \frac{V}{2} d\mu_V$ and $H^{\mu_V}(x) := \int g(x-y) d\mu_V(y)$ there holds*

$$\begin{cases} H^{\mu_V} + \frac{V}{2} \geq c & \text{quasi everywhere (q.e.)}, \\ H^{\mu_V} + \frac{V}{2} = c & \text{q.e. on } \Sigma. \end{cases}$$

We now define the function ζ that appeared above:

$$\zeta := H^{\mu_V} + \frac{V}{2} - c \geq 0. \tag{2.4}$$

We let ω be the zero set of ζ and by Theorem 2 we have

$$\Sigma \subset \omega := \{\zeta = 0\}. \tag{2.5}$$

The function H^{μ_V} is the solution to a classical obstacle problem in the Coulomb case, respectively a fractional obstacle problem in the other cases (cf. [CSS08]). The set ω then corresponds to the *contact set* or *coincidence set* of the obstacle problem, and Σ is the set where the obstacle is "active", sometimes called the *droplet.*

We will assume that μ_V is really a d-dimensional measure (i.e. Σ is a nice d-dimensional set), with a density, and we need to assume that this density (that we still denote μ_V) is bounded and sufficiently regular on its support. More precisely, we make the following assumptions (which are technical and could certainly be somewhat relaxed):

$$\partial\Sigma \text{ is } C^1 \tag{2.6}$$

$$\mu_V \text{ has a density which is } C^{0,\kappa} \text{ in } \Sigma, \tag{2.7}$$

$$\exists c_1, c_2, \overline{m} > 0 \text{ s.t. } c_1 \mathrm{dist}(x, \partial\Sigma)^\alpha \leq \mu_V(x) \leq \min(c_2 \mathrm{dist}(x,\partial\Sigma)^\alpha, \overline{m}) < \infty \text{ in } \Sigma, \tag{2.8}$$

with the conditions

$$0 < \kappa \le 1, \qquad 0 \le \alpha \le \frac{2\kappa d}{2d - s}. \tag{2.9}$$

Of course if $\alpha < 1$ one should take $\kappa = \alpha$, and if $\alpha \ge 1$, one should take $\kappa = 1$ and $\alpha \le \frac{2d}{d-s}$. These assumptions are meant to include the case of the semi-circle law $\frac{1}{2\pi}\sqrt{4 - x^2}\mathbf{1}_{[-2,2]}(x)$ arising for a quadratic potential in the setting (1.2). We also know that in the Coulomb cases, a quadratic potential gives rise to an equilibrium measure which is a multiple of a characteristic function of a ball, also covered by our assumptions with $\alpha = 0$. Finally, in the Riesz case, it was noticed in [CGaZ, Corollary 1.4] that any compactly supported radial profile can be obtained as the equilibrium measure associated to some potential. Our assumptions are thus never empty.

The last assumption is that there exists $\beta_1 > 0$ such that

$$\begin{cases} \int e^{-\beta_1 V(x)/2}\, dx < \infty & \text{in the case (1.4)} \\ \int e^{-\beta_1 (\frac{V(x)}{2} - \log|x|)}\, dx < \infty & \text{in the cases (1.2)–(1.3),} \end{cases} \tag{2.10}$$

It is a standard assumption ensuring the existence of the partition function.

2.2. **The extension representation for the fractional Laplacian.** In the next two sections, we recall elements from [PS14]. Our method of proof relies on expressing the interaction part of the Hamiltonian as a quadratic integral of the potential generated by the point configuration via

$$g * \sum_i \delta_{x_i}$$

(where $*$ denotes the convolution product) and expanding this integral interaction to next order in N. Outside of the Coulomb case, the Riesz kernel g is not the convolution kernel of a local operator, but rather of a fractional Laplacian. However, according to Caffarelli and Silvestre [CS07], when $d - 2 < s < d$, this fractional Laplacian nonlocal operator can be transformed into a local but inhomogeneous operator of the form $\operatorname{div}(|y|^\gamma \nabla \cdot)$ by adding one space variable $y \in \mathbb{R}$ to the space $\mathbb{R}^d$. We refer to [PS14] for more details. In what follows, k will denote the dimension extension. We will take $k = 0$ in all the Coulomb cases, i.e. $s = d - 2$ and $d \ge 3$ or (1.3). In all other cases, we will need to take $k = 1$. Points in the space $\mathbb{R}^d$ will be denoted by x, and points in the extended space $\mathbb{R}^{d+k}$ by X, with $X = (x, y)$, $x \in \mathbb{R}^d$, $y \in \mathbb{R}^k$. We will often identify $\mathbb{R}^d \times \{0\}$ and $\mathbb{R}^d$.

If γ is chosen such that

$$d - 2 + k + \gamma = s\,, \tag{2.11}$$

then, given a measure μ on $\mathbb{R}^d$, the potential $H^\mu(x)$ generated by μ defined in $\mathbb{R}^d$ by

$$H^\mu(x) = g * \mu(x) = \int_{\mathbb{R}^d} \frac{1}{|x - x'|^s}\, d\mu(x') \tag{2.12}$$

can be extended to a function $H^\mu(X)$ on $\mathbb{R}^{d+k}$ defined by

$$H^\mu(X) = \int_{\mathbb{R}^d} \frac{1}{|X - (x', 0)|^s}\, d\mu(x') \tag{2.13}$$

and this function satisfies

$$-\operatorname{div}(|y|^\gamma \nabla H^\mu) = c_{d,s}\, \mu \delta_{\mathbb{R}^d} \tag{2.14}$$

where by $\delta_{\mathbb{R}^d}$ we mean the uniform measure on $\mathbb{R}^d \times \{0\}$ i.e. $\mu\delta_{\mathbb{R}^d}$ acts on test functions φ by

$$\int_{\mathbb{R}^{d+k}} \varphi(X) d(\mu\delta_{\mathbb{R}^d})(X) = \int_{\mathbb{R}^d} \varphi(x,0)\, d\mu(x),$$

and

$$c_{d,s} = \begin{cases} 2s\, \frac{2\pi^{\frac{d}{2}}\Gamma\left(\frac{s+2-d}{2}\right)}{\Gamma\left(\frac{s+2}{2}\right)} & \text{for } s > \max(0, d-2)\ , \\ (d-2)\frac{2\pi^{\frac{d}{2}}}{\Gamma(d/2)} & \text{for } s = d-2 > 0\ , \\ 2\pi & \text{in cases (1.2), (1.3)}\ . \end{cases} \tag{2.15}$$

In particular $g(X) = |X|^{-s}$ seen as a function of $\mathbb{R}^{d+k}$ satisfies

$$-\operatorname{div}(|y|^\gamma \nabla g) = c_{d,s}\delta_0. \tag{2.16}$$

In order to recover the Coulomb cases, it suffices to take $k = \gamma = 0$, in which case we retrieve the fact that g is a multiple of the fundamental solution of the Laplacian. If $s > d-2$ we take $k = 1$ and γ satisfying (2.11). In the case (1.2), we note that $g(x) = -\log|x|$ appears as the $y = 0$ restriction of $-\log|X|$, which is (up to a factor 2π) the fundamental solution to the Laplacian operator in dimension $d + k = 2$. In this case, we may thus choose $k = 1$ and $\gamma = 0$, $c_{d,s} = c_{1,0} = 2\pi$, and the potential $H^\mu = g * \mu$ still satisfies (2.14), while g still satisfies (2.16).

To summarize, we will take

$$\text{in the case } \max(0, d-2) < s < d, \quad \text{then } \ k = 1,\ \gamma = s - d + 2 - k\,, \tag{2.17}$$

$$\text{in the case (1.2)}, \quad \text{then } \ k = 1,\ \gamma = 0\,, \tag{2.18}$$

$$\text{in the case (1.3) or } d \geq 3, s = d-2, \quad \text{then } \ k = 0,\ \gamma = 0\,. \tag{2.19}$$

We note that the formula (2.11) always remains formally true when taking the convention that $s = 0$ in the case $g(x) = -\log|x|$, and we also note that the assumption $d - 2 \leq s < d$ implies that in all cases $\gamma \in (-1, 1)$.

2.3. **The renormalized energy for electric fields.** In this section, we recall the definition from [PS14]. First, let us define the truncated Riesz (or logarithmic) kernel as follows: for $1 > \eta > 0$ and $X \in \mathbb{R}^{d+k}$, let

$$f_\eta(X) = (g(X) - g(\eta))_+\,. \tag{2.20}$$

We note that the function f_η vanishes outside of $B(0,\eta) \subset \mathbb{R}^{d+k}$ and satisfies that

$$\delta_0^{(\eta)} := \frac{1}{c_{d,s}} \operatorname{div}(|y|^\gamma \nabla f_\eta) + \delta_0 \tag{2.21}$$

is a positive measure supported on $\partial B(0,\eta)$, and which is such that for any test-function φ,

$$\int \varphi \delta_0^{(\eta)} = \frac{1}{c_{d,s}} \int_{\partial B(0,\eta)} \varphi(X)|y|^\gamma g'(\eta).$$

One can thus check that $\delta_0^{(\eta)}$ is a positive measure of mass 1, and we may write

$$-\operatorname{div}(|y|^\gamma \nabla f_\eta) = c_{d,s}(\delta_0 - \delta_0^{(\eta)}) \quad \text{in } \mathbb{R}^{d+k}. \tag{2.22}$$

We will also denote by $\delta_p^{(\eta)}$ the measure $\delta_0^{(\eta)}(X-p)$, for $p \in \mathbb{R}^d \times \{0\}$. Again, we note that this includes the cases (2.18)–(2.19). In the Coulomb cases, i.e. when $k=0$, then $\delta_0^{(\eta)}$ is simply the normalized surface measure on $\partial B(0,\eta)$.

We are now in a position to define the renormalized energy for a finite configuration of points, i.e. the quantity w_N appearing in (1.8)–(1.9). It is defined via the gradient of the potential generated by the point configuration, embedded into the extended space $\mathbb{R}^{d+k}$. More precisely, for a configuration of points $(x_1,\dots,x_N)$, we introduce the potential H_N generated by the points and the "background charge" $N\mu_V$:

$$H_N = g * \left(\sum_{i=1}^N \delta_{x_i} - N\mu_V \delta_{\mathbb{R}^d}\right). \tag{2.23}$$

We also introduce the blown-up configuration $(x_1',\dots,x_N') = (N^{1/d}x_1,\dots,N^{1/d}x_N)$, the blown-up equilibrium measure of density $\mu_V'(x') = \mu_V(N^{-1/d}x')$, and the blown-up potential

$$H_N' = g * \left(\sum_{i=1}^N \delta_{x_i'} - \mu_V' \delta_{\mathbb{R}^d}\right). \tag{2.24}$$

In view of the discussion of Section 2.2, H_N and H_N' can be viewed as functions on $\mathbb{R}^{d+k}$ satisfying the relations
(2.25)

$$-\operatorname{div}(|y|^\gamma \nabla H_N) = c_{d,s}\left(\sum_{i=1}^N \delta_{x_i} - N\mu_V \delta_{\mathbb{R}^d}\right) \qquad -\operatorname{div}(|y|^\gamma \nabla H_N') = c_{d,s}\left(\sum_{i=1}^N \delta_{x_i'} - \mu_V' \delta_{\mathbb{R}^d}\right)$$

The gradients of these potentials are called "electric fields" and denoted E, by analogy with the Coulomb case in which they correspond to the physical electric field generated by the points viewed as singular charges.

In the sequel, we will always identify a point configuration with the sum (possibly with multiplicity) of Dirac masses that it generates, i.e. $\mathcal{C}$ will mean a sum of Dirac masses with integer weights, as well as a point configuration.

For any electric field E solving a relation of the form

$$-\operatorname{div}(|y|^\gamma E) = c_{d,s}\Big(\mathcal{C} - m(x)\delta_{\mathbb{R}^d}\Big) \quad \text{in } \mathbb{R}^{d+k}, \tag{2.26}$$

where $\mathcal{C}$ is some locally finite point configuration in $\mathbb{R}^d \times \{0\}$ (identified with $\mathbb{R}^d$) we define

$$E_\eta := E - \nabla f_\eta * \mathcal{C} \tag{2.27}$$

where $*$ denotes again the convolution product i.e. $(\nabla f_\eta * \mathcal{C})(x) = \sum_{p\in\mathcal{C}} \nabla f_\eta(x-p)$. Let us note that this convolution product is well-defined because f_η is supported on $B(0,\eta)$ and $\mathcal{C}$ is locally finite. If E happens to be the gradient of a function H, then we will also denote

$$H_\eta := H - f_\eta * \mathcal{C}. \tag{2.28}$$

This corresponds to "truncating off" the infinite peak in the potential around each point of the configuration:

Remark 2.1. *If $H = g * \left(\sum_{i=1}^N \delta_{x_i} - m(x)\delta_{\mathbb{R}^d}\right)$ then the transformation from H to H_η amounts to truncating the kernel g, but only for the Dirac part of the r.h.s. Indeed, letting*

$g_\eta(x) = \min(g(x), g(\eta))$ *be the truncated kernel, we have*

$$H_\eta = g_\eta * \left(\sum_{i=1}^N \delta_{x_i}\right) - g * (m\delta_{\mathbb{R}^d}).$$

We may then define the truncated versions of H_N and H'_N as in (2.28), in particular

$$H'_{N,\eta} = H_{N,\eta} - \sum_{i=1}^N f_\eta(x - x'_i). \tag{2.29}$$

With this notation, we let

$$w_N(x_1, \ldots, x_N) := \frac{1}{N c_{d,s}} \lim_{\eta \to 0} \left(\int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla H'_{N,\eta}|^2 - N c_{d,s} g(\eta) \right). \tag{2.30}$$

It is proven in [PS14] that this limit exists and that with this definition, the exact relations (1.8)–(1.9) hold. With the presence of the factor $\frac{1}{N}$ the quantity w_N is expected to be typically of order 1.

The renormalized energy of an infinite configuration of points (already at the blown-up scale) is defined in a similar way, via an electric field which is the gradient of a potential associated to the configuration. Note that while for a finite configuration of N points, we may find a unique potential generated by it via (2.23), for an infinite configuration there is no canonical choice of such a potential (one may always add the gradient of a function satisfying $-\mathrm{div}\,(|y|^\gamma \nabla H) = 0$. This explains the need for a definition based on the electric field, and a definition down at the level of points.

Definition 2.2 (Admissible vector fields)**.** *Given a number $m \geq 0$, we define the class $\mathcal{A}_m$ to be the class of gradient vector fields $E = \nabla H$ that satisfy*

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\Big(\mathcal{C} - m\delta_{\mathbb{R}^d}\Big) \quad \text{in } \mathbb{R}^{d+k} \tag{2.31}$$

where $\mathcal{C}$ is a point configuration in $\mathbb{R}^d \times \{0\}$.

This class corresponds to vector fields that will be limits of those generated by the original configuration $(x_1, \ldots, x_N)$ after blow-up at the scale $N^{1/d}$ near the point x, where $m = \mu_V(x)$ can be understood as the local density of points.

We are now in a position to define the renormalized energy. In the definition, C_R denotes as before the hypercube $[-R/2, R/2]^d$.

Definition 2.3 (Renormalized energy)**.** *For any $m > 0$, $\nabla H \in \mathcal{A}_m$ and $0 < \eta < 1$, we define*

$$\mathcal{W}_\eta(\nabla H) = \limsup_{R \to \infty} \left(\frac{1}{R^d} \int_{C_R \times \mathbb{R}^k} |y|^\gamma |\nabla H_\eta|^2 - m c_{d,s} g(\eta) \right) \tag{2.32}$$

where H_η is as in (2.28)*, and*

$$\mathcal{W}(\nabla H) = \lim_{\eta \to 0} \mathcal{W}_\eta(\nabla H). \tag{2.33}$$

Let us observe that the value of the parameter m is implicit in the notation $\mathcal{W}(\nabla H)$. In fact for any given $E = \nabla H$, there exists at most one $m > 0$ such that E is in $\mathcal{A}_m$ hence there is in fact no ambiguity.

Definition 2.4 (Scaling on E)**.** *We define the following "scaling" map allowing us to pass bijectively from an electric field in $\mathcal{A}_m$ to an electric field in $\mathcal{A}_1$. We let*

$$\sigma_m E := m^{-\frac{s+1}{d}} E(\cdot\, m^{-1/d}) \tag{2.34}$$

By scaling, we may then always reduce to studying the class $\mathcal{A}_1$, indeed, if $E \in \mathcal{A}_m$, then $\sigma_m E \in \mathcal{A}_1$ and

$$\mathcal{W}_\eta(E) = m^{1+s/d}\mathcal{W}_{\eta m^{1/d}}(\sigma_m E) \qquad \mathcal{W}(E) = m^{1+s/d}\mathcal{W}(\sigma_m E) \tag{2.35}$$

in the case (1.4), and respectively

$$\mathcal{W}_\eta(E) = m\left(\mathcal{W}_{m\eta}(\sigma_m E) - \frac{2\pi}{d}\log m\right) \qquad \mathcal{W}(E) = m\left(\mathcal{W}(\sigma_m E) - \frac{2\pi}{d}\log m\right) \tag{2.36}$$

in the cases (1.2)–(1.3).

The name renormalized energy (originating in Bethuel-Brezis-Hélein [BBH94] in the context of two-dimensional Ginzburg-Landau vortices) reflects the fact that $\int |y|^\gamma |\nabla H|^2$ which is infinite, is computed in a renormalized way by first changing H into H_η and then removing the appropriate divergent part $c_{d,s} g(\eta)$ per point.

It is proven in [PS14] that the limit in (2.33) exists, $\{\mathcal{W}_\eta\}_{\eta<1}$ are uniformly bounded below on $\mathcal{A}_1$ by a finite constant depending only on s and d, and $\mathcal{W}$ and $\mathcal{W}_\eta$ have a minimizer over the class $\mathcal{A}_1$. We can also note that $\mathcal{W}$ does not feel compact perturbations of the points in $\mathcal{C}$. As already mentioned the questions of identifying $\min_{\mathcal{A}_1} \mathcal{W}$ is open, and we expect some (Bravais) lattice configuration to achieve the minimum, at least in low dimension. In [SS12b] it is proven that in the case (1.3), $\mathcal{W}$ achieves its minimum over lattice configurations of volume 1 at the triangular lattice. The same result is extended to the general case (1.4) in dimension 2 in [PS14]. In dimension 1, the minimum of $\mathcal{W}$ is known in the case (1.2): it is the value obtained at the lattice $\mathbb{Z}$ [SS15], cf. [Leb14] for uniqueness.

We let $\mathcal{A} = \cup_{m>0}\mathcal{A}_m$ be the class of all m-admissible gradient vector fields for any $m > 0$ (let us note that in the definition of $\mathcal{A}$ the union over $m > 0$ is in fact disjoint). It is observed in [PS14] that $\mathcal{A}$ is contained in $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ for any $p < p_{\max} := \min(2, \frac{2}{\gamma+1}, \frac{d+k}{s+1})$ (note that they blow up exactly like $1/|X|^{s+1}$ near each point of $\mathcal{C}$). These spaces $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ are endowed with the strong local topology. We note that $\mathcal{A}$ is a Borel subset of $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ for $p < p_{\max}$ and that $\mathcal{W} : \mathcal{A} \to \mathbb{R} \cup \{+\infty\}$ is measurable.

2.4. **Point configurations and point processes.**

In this section we introduce or recall the notation that we will use throughout the paper.

2.4.1. *General.* If (X, d_X) is a metric space we endow the space $\mathcal{P}(X)$ of Borel probability measures on X with the Dudley distance:

$$d_{\mathcal{P}(X)}(P_1, P_2) = \sup\left\{\int F(dP_1 - dP_2)|\ F \in \mathrm{Lip}_1(X)\right\} \tag{2.37}$$

where $\mathrm{Lip}_1(X)$ denotes the set of functions $F : X \to \mathbb{C}$ that are 1-Lipschitz with respect to d_X and such that $\|F\|_\infty \le 1$. It is well-known that the distance $d_{\mathcal{P}(X)}$ metrizes the topology of weak convergence on $\mathcal{P}(X)$. If $P \in \mathcal{P}(X)$ is a probability measure and $f : X \to \mathbb{R}^d$ a measurable function, we denote by $\mathbf{E}_P[f]$ the expectation of f under P.

2.4.2. *Point configurations.* If A is a Borel set of $\mathbb{R}^d$ we denote by $\mathcal{X}(A)$ the set of locally finite point configurations in A or equivalently the set of non-negative, purely atomic Radon measures on A giving an integer mass to singletons (see [DVJ88]). As mentioned before, we will write $\mathcal{C}$ for $\sum_{p\in\mathcal{C}} \delta_p$.

We endow the set $\mathcal{X} := \mathcal{X}(\mathbb{R}^d)$ (and the sets $\mathcal{X}(A)$ for A Borel) with the topology induced by the topology of weak convergence of Radon measure (also known as vague convergence or convergence against compactly supported continuous functions). If B is a compact subset of $\mathbb{R}^d$ we endow $\mathcal{X}(B)$ with the following distance:

$$d_{\mathcal{X}(B)}(\mathcal{C}, \mathcal{C}') := \sup \left\{ \int F(d\mathcal{C} - d\mathcal{C}') |\ F \in \mathrm{Lip}_1(B) \right\}. \tag{2.38}$$

The total mass $\mathcal{C}(B)$ of $\mathcal{C}$ on B corresponds to the number of points of the point configuration in B and when $\mathcal{C}(B) = \mathcal{C}'(B)$ the distance $d_{\mathcal{X}(B)}$ coincides with the "minimal connection" distance.

We endow $\mathcal{X} := \mathcal{X}(\mathbb{R}^d)$ with the following distance:

$$d_{\mathcal{X}}(\mathcal{C}, \mathcal{C}') := \sum_{k\geq 1} \frac{1}{2^k} \left(\frac{d_{\mathcal{X}(C_k)}(\mathcal{C}, \mathcal{C}')}{(\mathcal{C}(C_k) + \mathcal{C}'(C_k)) \vee 1} \wedge 1 \right). \tag{2.39}$$

We denote by $\mathrm{Lip}_1(\mathcal{X})$ the set of all functions $F : \mathcal{X} \to \mathbb{C}$ that are 1-Lipschitz with respect to $d_{\mathcal{X}}$ and such that $\|F\|_\infty \leq 1$. We say that a measurable function $f : \mathcal{X} \to \mathbb{C}$ is local when

$$f(\mathcal{C}) = f(\mathcal{C} \cap C_k) \text{ for all } \mathcal{C} \in \mathcal{X} \tag{2.40}$$

for some integer k. We denote by $\mathrm{Loc}_k(\mathcal{X})$ the set of functions that satisfies (2.40) for a fixed integer k and we let $\mathrm{Loc}(\mathcal{X}) := \cup_{k\geq 1} \mathrm{Loc}_k(\mathcal{X})$.

Lemma 2.5. *The following properties hold:*

(1) *The topological space $\mathcal{X}$ is Polish.*
(2) *The distances $d_{\mathcal{X}(B)}$ and $d_{\mathcal{X}}$ are actual distances compatible with the respective topologies on $\mathcal{X}(B)$ and $\mathcal{X}$.*
(3) *Any bounded continuous function $F : \mathcal{X} \to \mathbb{C}$ can be approximated by a sequence of bounded local functions. Moreover the approximation is uniform on the set of* 1*-Lipschitz functions in that for any $\delta > 0$ there exists an integer k such any function $F \in \mathrm{Lip}_1(\mathcal{X})$ is δ-close (in sup-norm) to some local function $f \in \mathrm{Loc}_k(\mathcal{X})$.*

Lemma 2.5 is proven in Section 8.

2.4.3. *Translations, volume, compactness.* The additive group $\mathbb{R}^d$ acts on $\mathcal{X}$ by translations $\{\theta_t\}_{t\in\mathbb{R}^d}$: if $\mathcal{C} = \{x_i, i \in I\} \in \mathcal{X}$ we let

$$\theta_t \cdot \mathcal{C} := \{x_i - t, i \in I\}. \tag{2.41}$$

We will use the same notation for the action of $\mathbb{R}^d$ on Borel sets of $\mathbb{R}^d$: if A is Borel and $t \in \mathbb{R}^d$, we denote by $\theta_t \cdot A$ the translation of A by the vector $-t$.

For any integer N we identify a configuration $\mathcal{C}$ that has N points with all the N-uples of points in $\mathbb{R}^d$ which correspond to $\mathcal{C}$ and if A is a set of configurations with N points we denote by $\mathbf{Leb}^{\otimes N}(A)$ the Lebesgue measure of the corresponding subset of $(\mathbb{R}^d)^N$.

If $\mathcal{C}$ is a point configuration, $x \in \mathbb{R}^d$ and $R > 0$ we denote by $\mathcal{N}(x, R)(\mathcal{C})$ the number of point of $\mathcal{C}$ in C_R. The following lemma is elementary:

Lemma 2.6 (Compactness in $\mathcal{X}$). *Let $C : \mathbb{R} \to \mathbb{R}^+$ be an arbitrary function, then the following set is compact in $\mathcal{X}$:*

$$\{\mathcal{C} \in \mathcal{X} \mid \mathcal{N}(x, R)(\mathcal{C}) \leq C(R) \text{ for all } R > 0\}.$$

Proof. It follows from the compactness of the hypercubes C_R for all $R > 0$ (hence of their powers C_R^n) and from the definition (2.39) of the distance on $\mathcal{X}$, together with a diagonal extraction procedure in order to extract a subsequence converging on each C_k for $k \geq 1$. □

2.4.4. *Point processes.* A point process is a probability measure on $\mathcal{X}$, a tagged point process is a probability measure on $\Lambda \times \mathcal{X}$ where Λ is some Borel set of $\mathbb{R}^d$ with non-empty interior. Usually Λ will be Σ (the support of the equilibrium measure μ).

When Λ is fixed, we shall always assume that the first marginal of a tagged point process $\bar{P}$ is the normalized Lebesgue measure on Λ hence we may consider the disintegration measures $\{\bar{P}^x\}_{x\in\Lambda}$ of $\bar{P}$ (for a definition see [AGS05, Section 5.3]), such that for any measurable function F on $\Lambda \times \mathcal{X}$ we have

$$\mathbf{E}_{\bar{P}}[F] = \fint_{\Lambda} \mathbf{E}_{\bar{P}^x}[F(x, \cdot)]dx.$$

We denote by $\mathcal{P}_s(\mathcal{X})$ the set of translation-invariant (or stationary) point processes. We also call stationary a tagged point process such that the disintegration measure $\bar{P}^x$ is stationary for (Lebesgue-)a.e. $x \in \Lambda$ and denote by $\mathcal{P}_s(\Lambda \times \mathcal{X})$ the set of stationary tagged point processes.

Let P be a point process. If there exists a measurable function $\rho_{1,P}$ such that for any function $\varphi \in C_c^0(\mathbb{R}^d)$ we have

$$\mathbf{E}_P\left[\sum_{x\in\mathcal{C}} \varphi(x)\right] = \int_{\mathbb{R}^d} \rho_{1,P}(x)\varphi(x)dx, \tag{2.42}$$

then we say that $\rho_{1,P}$ is the one-point correlation function (or "intensity") of the point process P. For $m \geq 0$ we say that a point process P is of intensity m when the function $\rho_{1,P}$ of (2.42) exists and satisfies $\rho_{1,P} \equiv m$.

We will denote by $\mathcal{P}_{s,1}(\mathcal{X})$ the set of stationary point processes of intensity 1 and by $\mathcal{P}_{s,1}(\Lambda \times \mathcal{X})$ the set of stationary tagged point processes (with space coordinate taken in Λ) such that the integral on $x \in \Lambda$ of the intensity of the disintegration measure $\bar{P}^x$ (which is by assumption a stationary point process) is 1.

We define the following "scaling" map allowing us to pass bijectively from a point process of intensity m to a point process of intensity 1.

$$\sigma_m P := \text{ the push-forward of } P \text{ by } \mathcal{C} \mapsto m^{1/d}\mathcal{C}. \tag{2.43}$$

Let us conclude this paragraph with a remark on the notion of convergence of point processes used here.

Remark 2.7. *We endow $\mathcal{P}(\mathcal{X})$ with the usual topology of weak convergence of probability measures (for the Borel σ-algebra on $\mathcal{X}$). This induces by definition a notion of convergence that corresponds to the weak convergence of probability distributions on $\mathcal{X}$. Another natural topology on $\mathcal{P}(\mathcal{X})$ is "convergence of the finite distributions" [DVJ08, Section 11.1] - sometimes also called the "convergence with respect to vague topology for the counting measure of the point process". The latter might seem weaker than the former, however the two notions of convergence coincide as stated in [DVJ08, Theorem 11.1.VII].*

2.4.5. *Electric field processes.* An electric field process is an element of $\mathcal{P}(L^p_{\text{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k}))$ where $p < p_{\max}$, concentrated on $\mathcal{A}$. It will usually be denoted by P^{elec}. We say that P^{elec} is stationary when it is law-invariant under the (push-forward by) translations $\theta_x \cdot E = E(\cdot - x)$ for any $x \in \mathbb{R}^d \subset \mathbb{R}^d \times \{0\}^k$. A tagged electric field process is an element of $\mathcal{P}(\Sigma \times L^p_{\text{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k}))$ whose first marginal is the normalized Lebesgue measure on Σ and whose disintegration slices are electric field processes. It will be denoted by $\bar{P}^{\text{elec}}$. We say that a tagged electric field process $\bar{P}^{\text{elec}}$ is stationary if for a.e. $x \in \Sigma$, the disintegration measure $\bar{P}^{\text{elec},x}$ is stationary.

2.4.6. *Application of the stationarity.* We end this section with an elementary lemma exposing a consequence of the stationarity assumptions which we will make a constant use of.

Lemma 2.8. *For any P stationary (point or electric) process, resp. $\bar{P}$ stationary (point or electric) tagged process, for every $T, R > 0$, for any Φ scalar nonnegative function of the point configuration or electric field X, we have*

$$\mathbf{E}_P \left[\fint_{C_T \times \mathbb{R}^k} \Phi(X(x))\, dx \right] = \mathbf{E}_P \left[\fint_{C_R \times \mathbb{R}^k} \Phi(X(x)) \right].$$

Moreover $\mathbf{E}_P \left[\lim_{R\to\infty} \fint_{C_R \times \mathbb{R}^k} \Phi(X(x)) \right]$ *exists and coincides with* $\mathbf{E}_P \left[\fint_{C_T \times \mathbb{R}^k} \Phi(X(x))\, dx \right]$ *for any $T > 0$.*

Proof. The multiparameter ergodic theorem (cf. [Bec81]) ensures that for any $T > 0$

$$\mathbf{E}_P \left[\fint_{C_T \times \mathbb{R}^k} \Phi(X(x))\, dx \right] = \mathbf{E}_P \left[\lim_{R\to\infty} \frac{1}{R^d} \int_{C_R} \fint_{C_T \times \mathbb{R}^k} \Phi(X(\lambda + x))\, dx\, d\lambda \right]$$
$$= \mathbf{E}_P \left[\lim_{R\to\infty} \frac{1}{R^d} \fint_{C_R} \Phi(X(x)) \right]$$

where we used Fubini's theorem and the fact that $\mathbf{1}_{C_{R-T}} \le \mathbf{1}_{C_R} * \mathbf{1}_{C_T} \le \mathbf{1}_{C_{R+T}}$ and Φ nonnegative. The result follows. □

2.5. The renormalized energy for point configurations and processes.

2.5.1. *The case of electric field processes.* We may now define the renormalized energy for random electric fields (in all the rest of the paper we take $p < p_{\max}$).

Definition 2.9. *If P^{elec} is an electric field process, we let*

$$\widetilde{\mathcal{W}}_\eta(P^{\text{elec}}) := \int \mathcal{W}_\eta(E) dP^{\text{elec}}(E) \qquad \widetilde{\mathcal{W}}(P^{\text{elec}}) := \int \mathcal{W}(E) dP^{\text{elec}}(E) \tag{2.44}$$

whenever the expressions in the right-hand side make sense.
We also define

$$\overline{\mathcal{W}}_{\mu_V(x)}(\bar{P}^{\text{elec}}) := \fint_{\Sigma} \widetilde{\mathcal{W}}(\bar{P}^{\text{elec},x}) dx, \tag{2.45}$$

whenever $\bar{P}^{\text{elec}}$ is a tagged electric field process such that for a.e. $x \in \Sigma$, the disintegration measure $\bar{P}^{\text{elec},x}$ is concentrated on $\mathcal{A}_{\mu_V(x)}$ (otherwise we set $\overline{\mathcal{W}}_{\mu_V(x)}(\bar{P}^{\text{elec}}) = +\infty$).

2.5.2. *The case of point configurations and point processes.* For any $m > 0$ and for any admissible gradient vector field $E \in \mathcal{A}_m$ we let

$$\mathrm{Conf}_m(E) := \frac{-\mathrm{div}\,(|y|^\gamma E)}{c_{d,s}} + m\delta_{\mathbb{R}^d} \tag{2.46}$$

be the underlying point configuration. For any $E \in \mathcal{A}$ there is exactly one value of $m > 0$ such that $E \in \mathcal{A}_m$ and we let $\mathrm{Conf}(E) := \mathrm{Conf}_m(E)$ for the suitable value of m, this defines a map $\mathcal{A} \to \mathcal{X}$ and we denote by $\mathcal{X}^o \subset \mathcal{X}$ its image i.e. the set of point configurations $\mathcal{C}$ for which there exists at least one admissible gradient vector field E such that $\mathrm{Conf}(E) = \mathcal{C}$. It is clear that the maps $\mathrm{Conf}_m : \mathcal{A}_m \to \mathcal{X}$ and $\mathrm{Conf} : \mathcal{A} \to \mathcal{X}$ are measurable. Let us note that the fiber of Conf at any $\mathcal{C} \in \mathcal{X}^o$ is always infinite, if E is in the fiber of $\mathcal{C}$ we can simply add to E the gradient of any function satisfying $\mathrm{div}\,(|y|^\gamma \nabla H) = 0$ on $\mathbb{R}^{d+k}$ and by doing so we recover exactly the fiber of $\mathcal{C}$.

We may then define the renormalized energy of a point configuration/process by means of the renormalized energy of electric field/processes in the fiber of Conf.

Definition 2.10. *If $\mathcal{C}$ is a point configuration and $m > 0$ we let*

$$\mathbb{W}_m(\mathcal{C}) := \inf\{\mathcal{W}(E) \mid E \in \mathcal{A}_m, \mathrm{Conf}_m(E) = \mathcal{C}\}$$

with the convention $\inf(\varnothing) = +\infty$ *(hence $\mathbb{W}(\mathcal{C}) = +\infty$ when $\mathcal{C} \in \mathcal{X}\backslash\mathcal{X}^o$).*
If P is a point process and $m > 0$ we let as in (1.11)

$$\widetilde{\mathbb{W}}_m(P) := \int \mathbb{W}_m(\mathcal{C})dP(\mathcal{C}).$$

If $\bar{P} \in \mathcal{P}(\Sigma \times \mathcal{X})$ is a tagged point process we let as in (1.12)

$$\overline{\mathbb{W}}_{\mu_V}(\bar{P}) := \frac{1}{c_{d,s}} \int_\Sigma \widetilde{\mathbb{W}}_{\mu_V(x)}(\bar{P}^x)dx.$$

In Section 8, we will prove the following :

Lemma 2.11. *If $k = 0$, two E's in $\mathcal{A}_m$ such that $\mathrm{Conf}_m(E) = \mathcal{C}$ and $\mathcal{W}(E)$ is finite differ by a constant vector field, and if $k = 1$, an $E \in \mathcal{A}_m$ such that $\mathrm{Conf}_m(E) = \mathcal{C}$ and $\mathcal{W}(E)$ is finite is unique. In all cases, the* inf *in the definition of $\mathbb{W}_m$ is a uniquely achieved minimum.*

The following lemma, proven in Section 8, is stated for point processes of intensity 1 and probability P^{elec} concentrated on $\mathcal{A}_1$ and is easily extended to any intensity $m > 0$ and class $\mathcal{A}_m$ by the scaling map (2.34) and the scaling relations (2.35), (2.36).

Lemma 2.12. *Let P be a stationary point process such that $\widetilde{\mathbb{W}}_1(P)$ is finite. Then there exists at least one stationary probability measure P^{elec} concentrated on $\mathcal{A}_1$ such that the push-forward of P^{elec} by Conf_1 is P and $\widetilde{\mathcal{W}}(P^{\mathrm{elec}}) < +\infty$. Moreover we have*

$$\widetilde{\mathbb{W}}_1(P) = \min\{\widetilde{\mathcal{W}}(P^{\mathrm{elec}}) \mid P^{\mathrm{elec}} \text{ stationary and } \mathrm{Conf}_1\#P^{\mathrm{elec}} = P\}, \tag{2.47}$$

where $\mathrm{Conf}_1\#P^{\mathrm{elec}}$ denotes the push-forward of P^{elec} by Conf_1.

The identity (2.47) extends readily not only to point processes P such that $\widetilde{\mathbb{W}}_m(P)$ is finite for some $m > 0$ (with of course Conf_m instead of Conf_1) but also to the context of tagged point processes, by applying the result of Lemma 2.12 to each disintegration $\bar{P}^x$ $(x \in \Sigma)$ of a tagged point process $\bar{P}$.

3. Preliminaries on the energy

In this section we recall a few facts about the renormalized energies from [PS14, SS12a, SS15, RS13] and we deduce a few new properties which will be crucial for us.

3.1. **Splitting and lower bound estimates.** Here we recall how $\overline{\mathbb{W}}$ is related to the Hamiltonian $\mathcal{H}_N$. The connection originates in the exact "splitting formula" mentioned in the introduction in (1.8)–(1.9), where ζ is as in (2.4) and w_N is as in (2.30). For a proof of this formula in our situation, see [PS14]. Once this is established, one needs to analyze the limit as $N \to \infty$ of w_N. This was done in the previous works, and we will make repeated use of the following lower bound, which is an immediate consequence of [PS14, Proposition 5.2] (in [PS14] it was given in terms of the electric field process, but it can be "projected down" at the level of the point processes via the map (2.46) and Definition 2.10):

Lemma 3.1. *Assume V satisfies* (2.1), (2.2), (2.3), *and that μ_V is a measure with a density which is bounded and almost everywhere (a.e.) continuous. For any N, let $x_1, \dots, x_N \in \mathbb{R}^d$ and define $\bar{P}_{\nu_N} = i_N(\{x_1, \dots, x_N\})$ as in* (1.10). *Assume that $w_N(x_1, \dots, x_N) \le C$ for some C independent of N. Then up to extraction of a subsequence, $\bar{P}_{\nu_N}$ converges weakly in the sense of probability measures to a measure $\bar{P} \in \mathcal{P}(\Sigma \times \mathcal{X})$ which is stationary and*

$$\liminf_{N\to\infty} N^{-1-\frac{s}{d}} \left(\mathcal{H}_N(x_1, \dots, x_N) - N^2 \mathcal{I}(\mu_V)\right) \ge \overline{\mathbb{W}}_{\mu_V}(\bar{P}) \text{ in case (1.4)} \tag{3.1}$$

respectively

$$\liminf_{N\to\infty} \frac{1}{N}\left(\mathcal{H}_N(x_1, \dots, x_N) - N^2 \mathcal{I}(\mu_V) + \frac{N}{d}\log N\right) \ge \overline{\mathbb{W}}_{\mu_V}(\bar{P}) \text{ in cases (1.2)} - -\text{(1.3)}, \tag{3.2}$$

or equivalently

$$\liminf_{N\to\infty} w_N(x_1, \dots, x_N) \ge \overline{\mathbb{W}}_{\mu_V}(\bar{P}). \tag{3.3}$$

This of course gives a formal lower bound for the energy part in the rate function (or an LDP upper bound), the difficulty will be in obtaining the corresponding upper bound (respectively an LDP lower bound).

Remark 3.2. *The relation* (3.3) *constitutes the* Γ-lim inf *(or lower bound) part of* Γ*-convergence. The result of Proposition 1.7 implies in particular that given any $\bar{P} \in \mathcal{P}_{s,1}(\Sigma \times \mathcal{X})$ which has a finite entropy (average of the relative specific entropy), and $\delta_1, \delta_2 > 0$, there exists (for any N large enough) $(x_1, \dots, x_N)$ such that $i_N(x_1, \dots, x_N) \in B(\bar{P}, \delta_1)$ and $w_N(x_1, \dots, x_N) \le \overline{\mathbb{W}}_{\mu_V}(\bar{P}) + \delta_2$. Taking δ_1, δ_2 tending to 0 as $N \to \infty$, we obtain the upper bound (or* Γ*-limsup) part of* Γ*-convergence, for stationary $\bar{P}$ with finite entropy. In fact a careful inspection of the proof of Proposition 1.7 shows that we may remove the assumption on the entropy. Indeed for any $\bar{P}$ stationary such that $\overline{\mathbb{W}}_{\mu_V}(\bar{P})$ is finite (otherwise there is nothing to prove), starting with a sequence of N-tuples $\{X_N\}_N = \{(x_1, \dots, x_N)_N\}_N$ such that $i_N(X_N)$ is in $B(\bar{P}, \delta_1)$ for N large enough (which can always be constructed by sampling $\bar{P}$ on large hypercubes) we may, by first regularizing X_N and then applying the screening procedure, get another sequence $\{X'_N\}_N$ such that $i_N(X'_N)$ is eventually in $B(\bar{P}, 2\delta_1)$ and $\limsup_{N\to\infty} w_N(X'_N) \le \overline{\mathbb{W}}_{\mu_V}(\bar{P}) + \delta_2$. Together with a standard diagonal argument this implies the full* Γ*-convergence of w_N to $\overline{\mathbb{W}}_{\mu_V}$.*

On the other hand, it was proven in [SS12a, SS15] that in the logarithmic cases, the Γ*-limit of w_N is the analogue of $\overline{\mathbb{W}}$ but defined from a different variant of the renormalized energy* *(with the same notation $\mathbb{W}$). This allows to check that the two variants of the renormalized energy coincide over stationary point processes thus answering a question raised in [RS13].*

3.2. **Almost monotonicity of the energy and truncation error.** The following is an immediate consequence of [PS14, Lemma 2.3], using the monotonicity of g. It shows the almost monotone character of the limit defining w_N, and provides at the same time an estimate of the error made when truncating the potentials at a level η.

Lemma 3.3. *For any $x_1, \dots, x_N \in \mathbb{R}^d$, letting $H'_{N,\eta}$ be as in (2.29), for any $1/2 > \eta > \tau > 0$ we have*

$$-CN\|\mu_V\|_{L^\infty}\eta^{\frac{d-s}{2}} \le \left(\int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla H'_{N,\tau}|^2 - Nc_{d,s}g(\tau)\right) - \left(\int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla H'_{N,\eta}|^2 - Nc_{d,s}g(\eta)\right) \le CN\|\mu_V\|_{L^\infty}\eta^{\frac{d-s}{2}} + c_{d,s} \sum_{i\neq j, |x_i - x_j|\le 2\eta} g(|x_i - x_j|).$$

where C depends only on d and s. In particular, sending $\tau \to 0$ yields that for all $\eta < 1/2$,

$$o_\eta(1)N \le Nc_{d,s}w_N(x_1,\dots,x_N) - \left(\int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla H'_{N,\eta}|^2 - Nc_{d,s}g(\eta)\right) \le o_\eta(1)N + c_{d,s}\sum_{i\neq j, |x_i-x_j|\le 2\eta} g(|x_i - x_j|). \tag{3.4}$$

where the term $o_\eta(1)$ goes to 0 when $\eta \to 0$ and is independent of the configuration.

Let us note that a lower estimate on the error of a similar form can also be obtained for finite N, however we will rather need such a lower estimate on the limiting renormalized energy, which will allow us to control the interaction due to close points by the truncation error $\mathcal{W} - \mathcal{W}_\eta$.

Lemma 3.4. *Let $E \in \mathcal{A}_m$ be such that $\mathcal{W}(E) < +\infty$. For any $1 > \eta > 0$ we have*

$$c_{d,s} \limsup_{\tau\to 0} \limsup_{R\to\infty} \frac{1}{R^d} \sum_{p\neq q \in \mathcal{C}\cap K_R, |p-q|\le\eta} (g(|p-q|+\tau) - g(\eta))_+ \le \mathcal{W}(E) - \mathcal{W}_\eta(E) + Cm^2\eta^{\frac{d-s}{2}}$$

where C depends only on s and d.

Proof. In [PS14, (2.29)] it is proven that for $0 < \tau < \eta < 1$, we have

$$-Cm^2\eta^{\frac{d-s}{2}} + \limsup_{R\to\infty} \frac{c_{d,s}}{R^d} \sum_{p\neq q\in\mathcal{C}\cap K_{R-3}} (g(|p-q|+\tau) - g(\eta))_+ \le \mathcal{W}_\tau(E) - \mathcal{W}_\eta(E). \tag{3.5}$$

It then suffices to let $\tau \to 0$ to conclude. □

The next important property of w_N and $\widetilde{\mathbb{W}}$ is that they control the number of points and their "discrepancies" (i.e. the difference between the number of points in a ball and the integral of the equilibrium measure over that ball), as well as the electric fields themselves.

3.3. **Coerciveness of the energy.** In view of the monotonicity properties of $\mathcal{W}$ and w_N, an upper bound on the renormalized energy of an electric field E translates into a bound on E_η in L^2 with weight $|y|^\gamma$, for any η small. This in turns easily implies a bound on E in L^p spaces according to the following lemma:

Lemma 3.5. *Let K be a compact set with piecewise C^1 boundary and let E be a vector field satisfying a relation of the form*

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\Big(\mathcal{C} - \mu\delta_{\mathbb{R}^d}\Big) \text{ in } K \times \mathbb{R}^k$$

where $\mathcal{C}$ is a point configuration in K and μ is a bounded measure on C_R, and let E_η be given by (2.27).

For any $0 < \eta < 1$, for any $p < p_{\max}$, we have

$$\|E\|_{L^p(K\times\mathbb{R}^k)} \le C\left(\int_{K\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2\right)^{1/2} + C_{p,\eta,d,s}\mathcal{C}(K). \tag{3.6}$$

with a constant $C_{p,\eta,d,s}$ depending only on p, η, d, s such that $C_{p,\eta,d,s} \to 0$ when $\eta \to 0$ and the other parameters are fixed, and a constant C depending on K, p, s, d.

Proof. Using Hölder's inequality, we note that $L^2_{|y|^\gamma}(K)$ embeds continuously into $L^p(K)$ for $1 < p < \min(2, \frac{2}{\gamma+1}) \le p_{\max}$. The lemma follows from observing that

$$\|E\|_{L^p(K\times\mathbb{R}^k)} \le \|E_\eta\|_{L^p(K\times\mathbb{R}^k)} + \|\nabla f_\eta\|_{L^p}\mathcal{C}(K).,$$

which follows from Minkowski inequality and the definition of E_η. □

We may now state a compactness result for electric fields.

Lemma 3.6. *For any compact set $K \subset \mathbb{R}^d$ with piecewise C^1 boundary, let $\{E_n\}_n$ be a sequence of vector fields in $L^p(K, \mathbb{R}^{d+k})$ such that*

$$-\,\mathrm{div}\,(|y|^\gamma E_n) = c_{d,s}\left(\mathcal{C}_n - \mu_n\delta_{\mathbb{R}^d}\right) \text{ in } K \times \mathbb{R}^k \tag{3.7}$$

for a certain sequence $\{\mathcal{C}_n\}_n$ of point configurations in K and $\{\mu_n\}_n$ of bounded functions on K. Assume that $\{\mathcal{C}_n\}_n$ converges to $\mathcal{C}$ and that $\{\mu_n\}_n$ converges to μ in $L^\infty(K)$. For any $\eta > 0$, if $\int_{K\times\mathbb{R}^k} |y|^\gamma |E_{n,\eta}|^2$ is bounded uniformly in n, then there exists a vector field E satisfying

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\left(\mathcal{C} - \mu\delta_{\mathbb{R}^d}\right) \text{ in } K \times \mathbb{R}^k$$

and such that for any $\eta > 0$, $z \in [0, +\infty]$

$$\int_{K\times[-z,z]^k} |y|^\gamma |E_\eta|^2 \le \liminf_{n+\infty} \int_{K\times[-z,z]^k} |y|^\gamma |E_{n,\eta}|^2 \tag{3.8}$$

and in the case $k = 1$, for any $z > 0$

$$\int_{K\times(\mathbb{R}\setminus(-z,z))} |y|^\gamma |E|^2 \le \liminf_{n+\infty} \int_{K\times(\mathbb{R}\setminus(-z,z))} |y|^\gamma |E_n|^2. \tag{3.9}$$

Proof. The sequence $\{E_{n,\eta}\}_n$ is bounded in $L^2_{|y|^\gamma}(K \times \mathbb{R}^k, \mathbb{R}^{d+k})$ hence we may find a vector field E such that up to extraction the sequence $\{E_n\}_n$ converges weakly to E in $L^2_{|y|^\gamma}(K \times \mathbb{R}^k, \mathbb{R}^{d+k})$. By Lemma 3.5, the convergence is also in L^p_{loc} for $p < p_{\max}$ hence in the sense of distributions, and we may take the limit in (3.7). Lower semi-continuity as in (3.8) and (3.9) is then a consequence of the weak convergence. □

Finally we state a compactness result for sequences of stationary electric processes with bounded energy.

Lemma 3.7. *Let $\{P^{\rm elec}_n\}_n$ be a sequence of stationary electric processes concentrated on $\mathcal{A}_1$ such that $\{\widetilde{\mathcal{W}}(P^{\rm elec}_n)\}_n$ is bounded. Then, up to extraction, the sequence $\{P^{\rm elec}\}_n$ converges weakly to a stationary electric process $P^{\rm elec}$ concentrated on $\mathcal{A}_1$ such that*

$$\widetilde{\mathcal{W}}(P^{\rm elec}) \le \liminf_{n\to\infty} \widetilde{\mathcal{W}}(P^{\rm elec}_n). \tag{3.10}$$

Proof. Up to extraction we may assume that the $\liminf_{n\to\infty}$ in (3.10) is actually a $\lim_{n\to\infty}$. It is clear that any weak limit point of $\{P^{\rm elec}_n\}_n$ is stationary and concentrated on $\mathcal{A}_1$. In view of Lemma 2.8 we have for any $R > 0$

$$\widetilde{\mathcal{W}}(P^{\rm elec}_n) = \mathbf{E}_{P^{\rm elec}_n}\left[\fint_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2\right] - c_{d,s} g(\eta),$$

but the function $E \mapsto \fint_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2$ is weakly lower semi-continuous as observed in the proof of Lemma 3.6, thus if $P^{\rm elec}$ is a weak limit point we have

$$\lim_{n\to\infty} \widetilde{\mathcal{W}}(P^{\rm elec}_n) \ge \widetilde{\mathcal{W}}(P^{\rm elec}).$$

Therefore it remains to show that there exists a converging subsequence. Using Lemma 2.8 as above and the boundedness of $\{\widetilde{\mathcal{W}}(P^{\rm elec}_n)\}_n$ we write for any $R, \eta > 0$

$$\mathbf{E}_{P^{\rm elec}_n}\left[\fint_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2\right] - c_{d,s} g(\eta) \le C.$$

From Lemma 3.5, we deduce that $\{\int \|E\|_{L^p(C_R)} dP^{\rm elec}_n\}_n$ is also bounded for any $p < p_{\max}$, and this for any hypercube C_R ($R > 0$). Using for example Lemma 2.1 in [SS12b] and the coerciveness of the L^p norm, we deduce as in [SS12b, Section 2, Step 1] that this implies existence of weak limit points for $\{P^{\rm elec}_n\}_n$. □

3.4. **Discrepancy estimates.** In this section we give estimates to control the discrepancy between the number of points in a domain and the expected number of points according to the background intensity, in terms of the energy. These estimates show that local non-neutrality of the configurations has an energy cost, which in turn implies that stationary point processes of finite energy must have small discrepancies. We then apply these considerations to coercivity properties of $\mathcal{W}$.

The first estimate is based on the following energy lower bound, proven in [PS14, Lemma 2.2] (there it is stated for balls, but the proof for cubes is identical). We let $C_R(x)$ be the hypercube of center x and sidelength R in $\mathbb{R}^d$ and we denote by $\mathcal{D}(x, R)$ the discrepancy between the number of points in $C_R(x)$ and its expected value

$$D(x, R) := \int_{C_R(x)} d\mathcal{C} - \int_{C_R(x)} \mu(y)\, dy.$$

Lemma 3.8. *[PS14, Lemma 2.2] Assume E satisfies a relation of the form*

$$-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\Big(\mathcal{C} - \mu\delta_{\mathbb{R}^d}\Big)$$

in some subset $U \subset \mathbb{R}^{d+k}$ for some $\mu \in L^\infty(U)$, with $\mathcal{C}$ a point configuration, and let E_η be associated as in (2.27). Then for any $0 < \eta < 1$, $R > 2$ and $x \in \mathbb{R}^d \times \{0\}$, denoting $\tilde{C}_R(x) = C_R(x) \times [-R/2, R/2]$, if $\tilde{C}_{2R}(x) \subset U$ we have

$$\int_{\tilde{C}_{2R}(x)} |y|^\gamma |E_\eta|^2 \ge C \frac{\mathcal{D}(x, R)^2}{R^s} \min\left(1, \frac{\mathcal{D}(x, R)}{R^d}\right), \tag{3.11}$$

for some C depending only on d, s and $\|\mu\|_{L^\infty}$.

For finite point configurations we get as a consequence of the previous lemma:

Lemma 3.9. *For any integer N and any $x_1, \dots, x_N \in \mathbb{R}^d$, we let w_N be as in (2.30) and $\bar{P}_{\nu_N}$ be $i_N(x_1 \dots, x_N)$ as in (1.10). Let us define $\overline{\mathcal{D}}_N(R)$ as* [1]

$$\overline{\mathcal{D}}_N(R)(x, \mathcal{C}) := \int_{C_R} d\mathcal{C} - \int_{C_R(N^{1/d}x)} d\mu'_V,$$

We have

$$\mathbf{E}_{\bar{P}_{\nu_N}} \left[\overline{\mathcal{D}}_N(R)^2 \min\left(1, \frac{\overline{\mathcal{D}}_N(R)}{R^d}\right) \right] \leq R^{d+s} \left(C_1 + C_2 w_N(x_1, \dots, x_N)\right) \tag{3.12}$$

with C_1, C_2 positive constants depending only on d and s.

Proof. Using the definition of P_{ν_N} and (3.11) we get (applying Fubini's identity in the first line) for any fixed $0 < \eta < 1$, with $E_N = \nabla H'_N$ is the electric field generated by the configuration (where H'_N is as in (2.24)):

$$\begin{aligned} \mathbf{E}_{\bar{P}_{\nu_N}} \left[\overline{\mathcal{D}}_N(R)^2 \min\left(1, \frac{\overline{\mathcal{D}}_N(R)}{R^d}\right) \right] &\leq \frac{CR^s}{N} \int_{N^{1/d}\Sigma} \int_{\theta_x \cdot \tilde{C}_{2R}} |y|^\gamma |E_{N,\eta}|^2 \\ &\leq \frac{CR^{d+s}}{N} \int_{\mathbb{R}^{d+k}} |y|^\gamma |E_\eta|^2 \leq \frac{CR^{d+s}}{N} \left(\int_{\mathbb{R}^{d+k}} |y|^\gamma |E_{N,\eta}|^2 - N c_{d,s} g(\eta) + N c_{d,s} g(\eta) \right) \end{aligned} \tag{3.13}$$

and we conclude by using (3.4). □

In the following we specialize to stationary point processes of intensity 1 but the corresponding result for a different intensity is easily deduced by scaling. We denote as previously by $\mathcal{N}(x, R) : \mathcal{X} \mapsto \mathbb{N}$ the number of points of a configuration in the hypercube $C_R(x)$ and by $\mathcal{D}(x, R)$ the discrepancy $\mathcal{D}(x, R) = \mathcal{N}(x, R) - R^d$. We note that in fact by stationarity their laws do not depend on x.

Lemma 3.10. *Let P be a stationary point process such that $\widetilde{\mathbb{W}}_1(P)$ is finite. Then P has intensity 1 i.e. $\mathbf{E}_P[\mathcal{N}(0, R)] = R^d$ for all $R > 0$. Moreover for any $R > 1$ it holds*

$$\mathbf{E}_P \left[\mathcal{D}^2(0, R)\right] \leq C(C + \widetilde{\mathbb{W}}_1(P)) R^{d+s} = o(R^{2d}) \tag{3.14}$$

with C a positive constant depending only on d and s. This implies for $R > 1$

$$\mathbf{E}_P \left[\mathcal{N}^2(0, R)\right] \leq R^{2d} + C(C + \widetilde{\mathbb{W}}_1(P)) R^{d+s}. \tag{3.15}$$

Proof. The first point of the lemma is an easy consequence of the second one, indeed from (3.14) we get using Jensen's inequality that $\mathbf{E}_P[\mathcal{D}(x, R)] = o(R^d)$. On the other hand the stationarity assumption implies that $\mathbf{E}_P[\mathcal{D}(0, R)] = R^d \mathbf{E}_P[\mathcal{D}(0, 1)]$ (for any $R > 0$) hence in fact $\mathbf{E}_P[\mathcal{D}(0, R)] = 0$ for any $R > 0$ which implies that P has intensity 1. We now turn to proving (3.14).

From Lemma 2.12 we know that we may find an electric process P^{elec} concentrated on $\mathcal{A}_1$ such that the push-forward of P^{elec} by Conf_1 is P, and satisfying $\widetilde{\mathcal{W}}(P^{\text{elec}}) = \widetilde{\mathbb{W}}_1(P)$. Set $\eta_0 = \frac{1}{4}$. By the monotonicity property (3.4) we see that

$$\int \mathcal{W}_{\eta_0}(E) dP^{\text{elec}}(E) \leq \widetilde{\mathcal{W}}(P^{\text{elec}}) + C \leq \widetilde{\mathbb{W}}_1(P) + C$$

[1] This is the correct object when dealing with $\bar{P}_{\nu_N}$ because the configurations have been translated.

with a constant C depending only on d, s.

In the case $k=1$, by stationarity and the definition of $\mathcal{W}$ we see that

$$\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{C_1\times\mathbb{R}^k}|y|^\gamma|E_{\eta_0}|^2\right]=\int \mathcal{W}_{\eta_0}(E)dP^{\mathrm{elec}}(E)+c_{d,s}g(\eta_0)\le \widetilde{\mathcal{W}}(P^{\mathrm{elec}})+C,$$

with a constant C depending only on d, s. Hence for any $R>0$ we may find $T\in(R,2R)$ such that

$$\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{C_1\times\{-T,T\}}|y|^\gamma|E_{\eta_0}|^2\right]\le \frac{1}{R}\left(\widetilde{\mathcal{W}}(P^{\mathrm{elec}})+C\right)=\frac{1}{R}\left(\widetilde{\mathbb{W}}_1(P)+C\right). \tag{3.16}$$

Letting $\check{C}_R$ be the hyperrectangle $C_R\times[-T,T]^k$ we have

$$\int_{\partial\check{C}_R}|y|^\gamma E_{\eta_0}\cdot\vec{\nu}=\int_{\check{C}_R}-\mathrm{div}\,(|y|^\gamma E_{\eta_0})=c_{d,s}(\mathcal{D}(0,R)+r_{\eta_0}), \tag{3.17}$$

where the point configuration is implicitly $\mathrm{Conf}_1(E)$ and where the error term r_{η_0} is bounded by n_{η_0}, the number of points of $\mathrm{Conf}_1(E)$ in an η_0-neighborhood of ∂C_R. We may see the η_0-neighborhood of ∂C_R as included in a disjoint union of $O(R^{d-1})$ hypercubes of sidelength 1 and by stationarity we have

$$\mathbf{E}_P[n_{\eta_0}^2]\le CR^{2d-2}\mathbf{E}_P\left[\mathcal{N}(0,1)^2\right]. \tag{3.18}$$

Taking the expectation of (3.17) against P^{elec} and using elementary inequalities and (3.18) we get

$$\mathbf{E}_P\left[\mathcal{D}(0,R)^2\right]\le C\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{\partial\check{C}_R}|y|^\gamma|E_{\eta_0}|^2\right]\left(\int_{\partial\check{C}_R}|y|^\gamma\right)+CR^{2d-2}\mathbf{E}_P\left[\mathcal{N}(0,1)^2\right]. \tag{3.19}$$

In the case $k=0$ we have

$$\int_{\partial\check{C}_R}|y|^\gamma=CR^{d-1}=CR^{s+1}, \tag{3.20}$$

whereas in the case $k=1$, recalling that $\gamma=s+2-d-k$ we easily compute that

$$\int_{\partial\check{C}_R}|y|^\gamma\le CR^{d-1}\int_0^T|y|^\gamma+CR^dT^\gamma\le CR^{d-1}R^{s+3-d-1}+CR^{d+s+2-d-1}=CR^{s+1}. \tag{3.21}$$

We may also split $\partial\check{C}_R$ as the disjoint union of

(1) $2d$ lateral faces of the type $[-R/2,-R/2]\times\dots\{\pm R/2\}\times\dots\times[-R/2,R/2]\times[-T,T]^k$,

(2) 0 (if $k=0$) or 2 (if $k=1$) faces of the type $C_R\times\{\pm T\}^k$.

For each of the $2d$ faces of the first type we may write using the stationarity of P^{elec}

(3.22)

$$\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{[-\frac{R}{2},\frac{R}{2}]\times\cdots\{\pm\frac{R}{2}\}\times\cdots\times[-\frac{R}{2},\frac{R}{2}]\times[-T,T]^k}|y|^\gamma|E_{\eta_0}|^2\right]=\frac{1}{R}\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{C_R\times[-T,T]^k}|y|^\gamma|E_{\eta_0}|^2\right]$$
$$\le\frac{1}{R}\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{C_R\times\mathbb{R}^k}|y|^\gamma|E_{\eta_0}|^2\right]=\frac{CR^d}{R}\left(\widetilde{\mathcal{W}}(P^{\mathrm{elec}})+C\right)\le CR^{d-1}\left(\widetilde{\mathbb{W}}_1(P)+C\right),$$

whereas for the second type of faces we have, using (3.16) and the stationarity of P^{elec}

$$\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{C_R\times\{-T,T\}^k}|y|^\gamma|E_{\eta_0}|^2\right] = R^d\mathbf{E}_{P^{\mathrm{elec}}}\left[\int_{C_1\times\{-T,T\}^k}|y|^\gamma|E_{\eta_0}|^2\right] \leq CR^{d-1}(\widetilde{\mathbb{W}}_1(P)+C) \tag{3.23}$$

Inserting (3.20) (if $k=0$) or (3.21) (if $k=1$), (3.22) and (3.23) (if $k=1$) into (3.19) we obtain

$$\mathbf{E}_P\left[\mathcal{D}(0,R)^2\right] \leq C\left(\widetilde{\mathbb{W}}_1(P)+C\right)R^{d-1+s+1} + CR^{2d-2}\mathbf{E}\left[\mathcal{N}(0,1)^2\right]. \tag{3.24}$$

The fact that $\mathbf{E}\left[\mathcal{N}(0,1)^2\right]$ is itself bounded by $C\left(\widetilde{\mathbb{W}}_1(P)+C\right)$ can be deduced from the previous discrepancy estimate (3.11). Let us denote by ψ_R the function $\psi_R : x \mapsto x^2\min(1,\frac{x}{R^d})$, dividing (3.11) by R^d we see that for any $E\in\mathcal{A}_1$ it holds, for R large enough,

$$\frac{1}{R^{d+s}}\psi_R(\mathcal{D}(0,R)) \leq \frac{C}{R^d}\int_{\tilde{C}_{2R}(x)}|y|^\gamma|E_{\eta_0}|^2$$

with the notation $\tilde{C}_{2R}$ as in Lemma 3.8. Taking as before the expectation under P^{elec} it yields

$$\frac{1}{R^{d+s}}\mathbf{E}_P[\psi_R(\mathcal{D}(0,R))] \leq C\mathbf{E}_{P^{\mathrm{elec}}}\left[\frac{1}{R^d}\int_{\tilde{C}_{2R}(x)}|y|^\gamma|E_{\eta_0}|^2\right]$$

By stationarity and the definition of $\mathcal{W}(E)$ we have

$$\mathbf{E}_{P^{\mathrm{elec}}}\left[\frac{1}{R^d}\int_{\tilde{C}_{2R}(0)}|y|^\gamma|E_{\eta_0}|^2\right] \leq C(\widetilde{\mathcal{W}}_{\eta_0}(P^{\mathrm{elec}})+C)$$

hence we get

$$\frac{1}{R^{d+s}}\mathbf{E}_P[\psi_R(\mathcal{D}(0,R))] \leq C(\widetilde{\mathbb{W}}_1(P)+C)$$

which gives a less accurate bound on the discrepancy than (3.14) but allows one to bound $\mathbf{E}\left[\mathcal{N}(0,1)^2\right]$ by $C\left(\widetilde{\mathbb{W}}_1(P)+C\right)$. Finally we get (3.14) from (3.24), and the bound (3.15) follows easily from (3.14). □

In particular, we observe that in the logarithmic cases (1.2), (1.3) the bound (3.14) yields

$$\mathbf{E}_P\left[\mathcal{D}(0,R)^2\right] \leq C(\widetilde{\mathbb{W}}_1(P)+C)R^d$$

hence the variance of the number of points for a point process of finite renormalized energy is comparable to that of a Poisson point process. It is unclear to us whether this estimate is sharp or not.

The discrepancy estimate (3.15) gives a uniform bound on the discrepancy in terms of the renormalized energy. The next lemma allows to control the number of points on small scales (in a more precise but non-uniform way) and is based instead on Lemma 3.4.

Lemma 3.11. *Let $\bar{P}^{\mathrm{elec}} \in \mathcal{P}_s(\Sigma\times L^p_{\mathrm{loc}}(\mathbb{R}^{d+k},\mathbb{R}^{d+k}))$ be such that $\overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\mathrm{elec}}) < +\infty$, and let $\bar{P}$ the underlying tagged point process i.e. the push-forward of $\bar{P}^{\mathrm{elec}}$ by $(x,E)\mapsto(x,\mathrm{Conf}_{\mu_V(x)}E)$.*

Then for any $0 < \tau < \eta^2/2$ *and* $\eta < 1$, *we have*

$$\frac{g(2\tau)}{\tau^d}\mathbf{E}_{\bar{P}}[(\mathcal{N}(0,\tau)^2-1)_+] + \mathbf{E}_{\bar{P}}\left[\sum_{p\neq q\in\mathcal{C}\cap C_1, |p-q|\le\eta^2/2} g(|p-q|)\right] \\ \le C\left(\overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\mathrm{elec}}) - \fint_{\Sigma}\widetilde{\mathcal{W}}_\eta(\bar{P}^{\mathrm{elec},x})dx\right) + C\eta^{d-s}, \tag{3.25}$$

for some constant $C > 0$ *depending only on* s, d.

Proof. We integrate the left-hand side in the result of Lemma 3.4 with respect to $\bar{P}^{\mathrm{elec},x}$ and obtain, by monotone convergence in τ and stationarity (cf. Lemma 2.8), that it is equal to

$$c_{d,s}\limsup_{\tau\to0}\mathbf{E}_{\bar{P}^{\mathrm{elec},x}}\left[\limsup_{R\to\infty}\sum_{p\neq q\in\mathcal{C}\cap C_R}(g(|p-q|+\tau)-g(\eta))_+\right] \\ = c_{d,s}\limsup_{\tau\to0}\mathbf{E}_{\bar{P}^x}\left[\sum_{p\neq q\in\mathcal{C}\cap C_1}(g(|p-q|+\tau)-g(\eta))_+\right].$$

Using again the monotone convergence theorem in $\tau\to0$, this is equal to

$$c_{d,s}\mathbf{E}_{\bar{P}^x}\left[\sum_{p\neq q\in\mathcal{C}\cap C_1}(g(|p-q|-g(\eta))_+\right].$$

Now we note that in all cases (1.2)–(1.3)–(1.4), there exists $C > 0$ depending only on s and d such that if $\tau < \eta^2/2$,

$$\sum_{p\neq q\in\mathcal{C}\cap C_1}(g(|p-q|-g(\eta))_+ \ge C\sum_{p\neq q\in\mathcal{C}\cap C_1,|p-q|<\eta^2/2} g(|p-q|) \\ \ge C\sum_{\vec{i}\in C_1\cap\tau\mathbb{Z}^d}\mathcal{N}(\vec{i},\tau)(\mathcal{C})(\mathcal{N}(\vec{i},\tau)(\mathcal{C})-1)g(2\tau) \ge \frac{C}{2}g(2\tau)\sum_{\vec{i}\in C_1\cap\tau\mathbb{Z}^d}((\mathcal{N}(\vec{i},\tau)^2(\mathcal{C})-1)_+$$

where we denote by $(\mathcal{N}(\vec{i},\tau)(\mathcal{C})$ the number of points of the configuration $\mathcal{C}$ in the hypercube of center $\vec{i}$ and sidelength, with $\vec{i}\in\tau\mathbb{Z}^d$ whose edges are parallel to the axes of $\mathbb{Z}^d$ and of sidelength τ.

Using stationarity again, we find that the expectation of this quantity is bounded below by a constant times the left-hand side in (3.25), and the result then follows from Lemma 3.4 integrated against $\bar{P}^{\mathrm{elec},x}$ and then against the normalized Lebesgue measure on Σ. □

3.5. **Minimality of the local energy.** As already mentioned, given a configuration $\mathcal{C}$ in a compact set K and an underlying (bounded, measurable) density μ on K, there exist many electric vector fields that are compatible with the configuration i.e. such that $-\mathrm{div}\,(|y|^\gamma E) = c_{d,s}(\mathcal{C}-\mu\delta_{\mathbb{R}^d})$. Indeed to any such vector field one may add any solution of $-\mathrm{div}\,(|y|^\gamma E) = 0$.

Since the configuration in a given compact set is finite there is however a natural choice, which we call the "local field", given by

$$E^{\mathrm{loc}} := \nabla H^{\mathrm{loc}},\ \text{with}\ H^{\mathrm{loc}} := c_{d,s}g * (\mathcal{C}-\mu\delta_{\mathbb{R}^d}\mathbf{1}_K). \tag{3.26}$$

The following lemma shows that among all possible electric fields for a finite point configurations, the local electric field defined by (3.26) has a smaller energy than any "screened" electric field. The reason is that E^{loc} is an $L^2_{|y|^\gamma}$ orthogonal projection of any generic compatible E onto gradients, and the projection decreases the $L^2_{|y|^\gamma}$ norm.

Lemma 3.12. *Let μ be a bounded measurable function on a compact set (with piecewise C^1 boundary) $K \subset \mathbb{R}^d$, $\mathcal{C}$ a point configuration and E^{loc} the local electric field as in* (3.26)*. Let $E \in L^p_{\text{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ be a vector field satisfying*

$$\begin{cases} -\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\,(\mathcal{C} - \mu\delta_{\mathbb{R}^d}) & \text{in } K \times \mathbb{R}^k \\ E \cdot \vec{\nu} = 0 & \text{on } \partial K \times \mathbb{R}^k. \end{cases} \tag{3.27}$$

Then, for any $0 < \eta < 1$ we have

$$\int_{\mathbb{R}^{d+k}} |y|^\gamma |E^{\text{loc}}_\eta|^2 \le \int_{K \times \mathbb{R}^k} |y|^\gamma |E_\eta|^2. \tag{3.28}$$

Proof. First we note that we may extend E by 0 outside of K and since $E \cdot \vec{\nu}$ is continuous across ∂K, no divergence is created there, and E satisfies

$$-\,\mathrm{div}\,(|y|^\gamma E) = c_{d,s}(\mathcal{C} - \mu\delta_{\mathbb{R}^d}) = -\mathrm{div}\,(|y|^\gamma E^{\text{loc}}) \quad \text{in } \mathbb{R}^{d+k}. \tag{3.29}$$

Second, we notice that if (3.27) holds we must have $\mathcal{C}(K) = \int_K \mu$, i.e. there is global neutrality of the charges in K. This global neutrality implies that H^{loc} as defined in (3.26) decays like $|\nabla g|$ i.e. like $|x|^{-s-1}$ as $|x| \to \infty$ in $\mathbb{R}^{d+k}$ and E^{loc} decreases like $|x|^{-s-2}$ (with the convention $s = 0$ in the cases (1.2)–(1.3)). If the right-hand side of (3.28) is infinite, then there is nothing to prove. If it is finite, given $M > 1$, and letting χ_M be a smooth nonnegative function equal to 1 in $C_M \times [-M, M]^k$ and 0 at distance ≥ 1 from $C_M \times [-M, M]$, we may write

$$\begin{aligned} \int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma |E_\eta|^2 &= \int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma |E_\eta - E^{\text{loc}}_\eta|^2 + \int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma |E^{\text{loc}}_\eta|^2 \\ &\quad + 2\int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma (E_\eta - E^{\text{loc}}_\eta) \cdot E^{\text{loc}}_\eta \\ &\ge \int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma |E^{\text{loc}}_\eta|^2 + 2\int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma (E_\eta - E^{\text{loc}}_\eta) \cdot (\nabla H^{\text{loc}}_\eta) \\ &= \int_{\mathbb{R}^{d+k}} \chi_M |y|^\gamma |E^{\text{loc}}_\eta|^2 + 2\int_{\mathbb{R}^{d+k}} H^{\text{loc}}_\eta |y|^\gamma (E_\eta - E^{\text{loc}}_\eta) \cdot \nabla \chi_M \end{aligned} \tag{3.30}$$

where we integrated by parts and used (3.29) to remove one of the terms. Letting $M \to \infty$, the last term tends to 0 by finiteness of the right-hand side of (3.28) and decay properties of H^{loc} and E^{loc}, and we obtain the result. □

4. Proof of the main results

In this section, we give the proof of Theorem 1 and its corollaries, assuming the results of Propositions 1.6 and 1.7, whose proof will occupy the main part of the paper.

4.1. Exponential tightness and goodness of the rate function.

Lemma 4.1. *The following holds:*

- *For any $\beta > 0$, the sequences $\{\mathfrak{P}_{N,\beta}\}_N$ and $\{\overline{\mathfrak{P}}_{N,\beta}\}_N$ are exponentially tight.*

- *The functionals* $\widetilde{\mathbb{W}}_1$ *(resp.* $\overline{\mathbb{W}}_{\mu_V}$*) are lower semi-continuous over stationary point processes (resp. tagged point processes), bounded below, and have compact sub-level sets.*
- *The functionals* $\mathcal{F}_\beta$ *and* $\overline{\mathcal{F}}_\beta$ *are good rate functions.*

Proof. The exponential tightness of $\{\overline{\mathfrak{P}}_{N,\beta}\}_N$ is an easy consequence of the fact that the total number of points in $N^{1/d}\Sigma$ is bounded by N. Indeed it implies that for $\overline{\mathfrak{P}}_{N,\beta}$-a.e. tagged point process $\bar{P}_N$ and for any $R > 0$

$$\mathbf{E}_{\bar{P}_N}\left[\mathcal{N}(0,R)\right] \leq CR^d$$

with a constant C depending only on V, d. Let us fix two increasing sequences $\{R_k\}_k$ and $\{M_k\}_k$ going to ∞. Markov's inequality implies that for any $N, k \geq 1$ for $\overline{\mathfrak{P}}_{N,\beta}$-a.e. tagged point process $\bar{P}_N$ we have

$$\bar{P}_N\left(\mathcal{N}(0,R_k) \geq M_k R_k^d\right) \leq \frac{C}{M_k}.$$

Using Lemma 2.6 and a simple extraction argument we see that

$$K := \bigcap_{k=1}^{+\infty} \left\{\bar{P}, \bar{P}\left(\mathcal{N}(0,R_k) \geq M_k R^d\right) \leq \frac{C}{M_k}\right\}$$

is a compact set in $\mathcal{P}(\Sigma \times \mathcal{X})$ which contains $\overline{\mathfrak{P}}_{N,\beta}$-a.e. tagged point process $\bar{P}_N$.

We may now be more precise in the description of a "typical" (up to very large deviations) limit point under $\overline{\mathfrak{P}}_{N,\beta}$. Inserting the formula (1.8)–(1.9) into (1.20), using the definition (1.20) and the control (1.21) on $K_{N,\beta}$, we obtain that for any β larger than any fixed $\beta_0 > 0$ and any $M > 0$

$$\begin{aligned}\mathbb{P}_{N,\beta}(w_N^{-1}([M,+\infty]) &\leq \frac{1}{K_{N,\beta}} e^{-\frac{1}{2}\beta M N} \int e^{-N\beta \sum_{i=1}^N \zeta(x_i)}\, dx_1 \dots dx_N \\ &\leq e^{C_\beta \beta N - \frac{1}{2}\beta M N} \left(\int_{\mathbb{R}^d} e^{-N\beta\zeta(x)}\, dx\right)^N\end{aligned}$$

Thanks to assumption (2.10), for N large enough the function $e^{-N\beta\zeta}$ is dominated in L^1 (indeed ζ behaves like $g(x) + \frac{1}{2}V - c$ as $|x| \to \infty$), and by the dominated convergence theorem we have

$$\lim_{N\to\infty} \int_{\mathbb{R}^d} e^{-N\beta\zeta(x)}\, dx = |\{\zeta = 0\}| = |\omega|, \tag{4.1}$$

hence we find

$$\frac{1}{N} \log \mathbb{P}_{N,\beta}(w_N^{-1}([M,+\infty]) \leq -\frac{1}{2}\beta(M - C) \tag{4.2}$$

for some constant C depending only on d, s, V and β_0. Let us define

$$K_N^M = i_N\left(w_N^{-1}([-\infty, M))\right) \text{ and } K^M := \cup_{N\geq 1} K_N^M.$$

Equation (4.2) implies that $\frac{1}{N}\log \overline{\mathfrak{P}}_{N,\beta}((K^M)^c) \leq -\frac{1}{2}\beta(M - C)$. On the other hand, Lemma 3.1 shows that any limit point of a sequence of finite configurations with bounded energy is stationary and has finite energy $\overline{\mathbb{W}}_{\mu_V}(\bar{P})$. This allows to restrict ourselves to studying the large deviations around tagged point processes $\bar{P}$ which are stationary and with finite energy. In particular, as a consequence of Lemma 3.10, $\bar{P}^x$ has intensity $\mu_V(x)$ for Lebesgue-a.e. $x \in \Sigma$.

Let us next prove the lower semi-continuity of $\widetilde{\mathbb{W}}_1$ (resp. $\overline{\mathbb{W}}_{\mu_V}$) on the space of stationary (resp. tagged stationary) point processes. Let $\{P_n\}_n$ be a sequence of stationary point processes converging to $P \in \mathcal{P}_s(\mathcal{X})$. We may assume that $\liminf_{n\to\infty} \widetilde{\mathbb{W}}_1(P_n) < +\infty$ otherwise there is nothing to prove, and up to extraction we may also assume that $\liminf_{n\to\infty} \widetilde{\mathbb{W}}_1(P_n) = \lim_{n\to\infty} \widetilde{\mathbb{W}}_1(P_n)$. By Lemma 2.12 there exists for each n a stationary electric process P_n^{elec} whose push-forward by Conf_1 is equal to P_n and such that $\widetilde{\mathcal{W}}(P_n^{\mathrm{elec}}) = \widetilde{\mathbb{W}}_1(P_n)$. The sequence $\{\widetilde{\mathcal{W}}(P_n^{\mathrm{elec}})\}$ is bounded, which together with Lemma 3.7 implies that up to extraction we have $P_n^{\mathrm{elec}} \to P^{\mathrm{elec}}$ for some electric process P^{elec} which is also stationary and it is easy to see that P^{elec} satisfies $\mathrm{Conf}_1 \# P^{\mathrm{elec}} = P$.

Moreover we know from Lemma 3.7 that $\liminf_{n\to\infty} \widetilde{\mathcal{W}}(P_n^{\mathrm{elec}}) \geq \widetilde{\mathcal{W}}(P^{\mathrm{elec}})$ but by assumption we have $\widetilde{\mathcal{W}}(P_n^{\mathrm{elec}}) = \widetilde{\mathbb{W}}_1(P_n)$ and by definition we have $\widetilde{\mathbb{W}}_1(P) \leq \widetilde{\mathcal{W}}(P^{\mathrm{elec}})$, hence

$$\widetilde{\mathbb{W}}_1(P) \leq \liminf_{n\to\infty} \widetilde{\mathbb{W}}_1(P_n)$$

which implies the lower semi-continuity of $\widetilde{\mathbb{W}}_1$. The lower semi-continuity of $\overline{\mathbb{W}}_{\mu_V}$ is a straightforward consequence. The fact that both are bounded below follows from the same fact known for $\mathcal{W}$.

To prove the compactness of sub-level sets for $\widetilde{\mathbb{W}}_1$ and $\overline{\mathbb{W}}_{\mu_V}$, they key point is to see that Lemma 3.10 implies uniform integrability of $\mathcal{N}(0, R)$ against point processes living on any sub-level set of the energy functional. Then using the compactness result of Lemma 2.6 we see that every sequence of point processes in a sub-level set is tight, hence the sub-level sets (being closed by lower semi-continuity) are compact.

Finally, it is known that the specific relative entropy ent is a good rate function (see e.g. [RAS09]), which also implies that $\overline{\mathsf{ent}}$ is and thus $\mathcal{F}_\beta$ and $\overline{\mathcal{F}}_\beta$ are good rate functions as the sum of two good rate functions. □

Goodness of the rate function implies in particular the existence of minimizers for $\mathcal{F}_\beta$ and $\overline{\mathcal{F}}_\beta$.

4.2. **Proof of Theorem 1 and Corollary 1.5.** From Propositions 1.6 and 1.7, the proof of Theorem 1 is standard.

Let $\bar{P}$ be in $\mathcal{P}_s(\Sigma \times \mathcal{X})$. Using the notation of (1.20) and (1.22), we have for any $\delta_1, \delta_2 > 0$

$$\begin{aligned}(4.3)\quad & \overline{\mathfrak{P}}_{N,\beta}\left(B(\bar{P}, \delta_1)\right) \\ &= \frac{1}{K_{N,\beta}} \left(\int_{\mathbb{R}} e^{-N\beta\zeta}\right)^N \int_{i_N(x_1,\dots,x_N)\in B(\bar{P},\delta_1)} \exp(-\frac{N}{2}\beta w_N(x_1,\dots,x_N)) d\mathbb{Q}_{N,\beta}(x_1,\dots,x_N) \\ &\geq \frac{1}{K_{N,\beta}} \left(\int_{\mathbb{R}} e^{-N\beta\zeta}\right)^N \exp(-\frac{\beta}{2}(\overline{\mathbb{W}}(\bar{P}) + \delta_2)) \\ &\quad \times \mathbb{Q}_{N,\beta}\Big(i_N(x_1,\dots,x_N) \in B(\bar{P}, \delta_1) \text{ and } w_N(x_1,\dots,x_N) \leq \overline{\mathbb{W}}(\bar{P}) + \delta_2\Big).\end{aligned}$$

Consequently for any $\delta_1, \delta_2 > 0$ we get

$$\begin{aligned}\frac{1}{N}\log \overline{\mathfrak{P}}_{N,\beta}\left(B(\bar{P}, \delta_1)\right) \geq \frac{1}{N}\log \bar{\mathfrak{Q}}_{N,\beta}\Big(B(\bar{P}, \delta_1) \text{ and } w_N(x_1,\dots,x_N) \leq \overline{\mathbb{W}}(\bar{P}) + \delta_2\Big) \\ - \frac{1}{N}\log K_{N,\beta} - \frac{\beta}{2}(\overline{\mathbb{W}}(\bar{P}) + \delta_2) + \log \int_{\mathbb{R}} e^{-N\beta\zeta}.\end{aligned}$$

Applying Proposition 1.7 and using (1.24) and (4.1), we get for any $\delta_2 > 0$

$$\lim_{\delta_1\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\overline{\mathfrak{P}}_{N,\beta}\left(B(\bar P, r)\right)+\frac{1}{N}\log K_{N,\beta} \geq -\int_\Sigma \mathsf{ent}[\bar P^x|\mathbf{\Pi}^1]dx-(\log|\omega|-|\Sigma|+1)+\log|\omega|-\frac{\beta}{2}(\overline{\mathbb{W}}(\bar P)+\delta_2). \tag{4.4}$$

More precisely Proposition 1.7 is only stated for $\bar P \in \mathcal{P}_{s,1}(\Sigma\times\mathcal{X})$ (with the restriction that the "global" intensity of $\bar P$ is 1) but if $\overline{\mathbb{W}}_{\mu_V}(\bar P)$ is finite we know from Lemma 3.10 that $\bar P$ is indeed in $\mathcal{P}_{s,1}(\Sigma\times\mathcal{X})$ because $\bar P^x$ must be a.e. of intensity $\mu_V(x)$, and otherwise (4.4) holds trivially since the right-hand side is $-\infty$. Now by sending $\delta_2\to 0$, we obtain

$$\lim_{\delta_1\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\overline{\mathfrak{P}}_{N,\beta}\left(B(\bar P,\delta_1)\right)+\frac{1}{N}\log K_{N,\beta}\geq -\int_\Sigma \mathsf{ent}[\bar P^x|\mathbf{\Pi}^1]dx-\frac{\beta}{2}\overline{\mathbb{W}}_{\mu_V}(\bar P)-(1-|\Sigma|). \tag{4.5}$$

On the other hand, returning to (4.3), we have by lower semi-continuity of $\overline{\mathbb{W}}_{\mu_V}$ over stationary processes as proven in Lemma 4.1 and by Lemma 3.1 and Proposition 1.6,

$$\limsup_{\delta_1\to 0}\lim_{N\to\infty}\frac{1}{N}\log\overline{\mathfrak{P}}_{N,\beta}\left(B(\bar P,\delta_1)\right)+\frac{1}{N}\log K_{N,\beta}\leq -\int_\Sigma \mathsf{ent}[\bar P^x|\mathbf{\Pi}^1]dx-\frac{\beta}{2}\overline{\mathbb{W}}_{\mu_V}(\bar P)-(1-|\Sigma|). \tag{4.6}$$

The exponential tightness proven in Lemma 4.1 allows one to pass from a weak formulation as in (4.5), (4.6) i.e. a large deviation inequality around a fixed $\bar P$ to any subset $A\subset\mathcal{P}(\Sigma\times\mathcal{X})$. We then get

$$\begin{aligned}-\inf_{\bar P\in\mathring{A}}&\left(-\int_\Sigma \mathsf{ent}[\bar P^x|\mathbf{\Pi}^1]dx+\frac{\beta}{2}\overline{\mathbb{W}}_{\mu_V}(\bar P)\right)-(1-|\Sigma|)\\ &\leq \liminf_{N\to\infty}\frac{1}{N}\left(\log\overline{\mathfrak{P}}_{N,\beta}(A)+\log K_{N,\beta}\right)\leq\limsup_{N\to\infty}\frac{1}{N}\left(\log\overline{\mathfrak{P}}_{N,\beta}(A)+\log K_{N,\beta}\right)\\ &\leq -\inf_{\bar P\in\bar A}\left(\int_\Sigma \mathsf{ent}[\bar P^x|\mathbf{\Pi}^1]dx+\frac{\beta}{2}\overline{\mathbb{W}}_{\mu_V}(\bar P)\right)-(1-|\Sigma|).\end{aligned} \tag{4.7}$$

Hence taking $A=\mathcal{P}(\Sigma\times\mathcal{X})$ we see that $\lim_{N\to\infty}\frac{1}{N}\log K_{N,\beta}$ exists and

$$\lim_{N\to\infty}\frac{1}{N}\log K_{N,\beta}=-\beta\inf\overline{\mathcal{F}}_\beta. \tag{4.8}$$

Finally, inserting (4.8) into (4.7) yields Theorem 1.

Combining (1.20) and (4.8) we immediately get the main results of Corollary 1.5. For (1.19) we use the following scaling result:

Lemma 4.2. *For any $m>0$ and P in $\mathcal{P}_s(\mathcal{X})$ of intensity m we have*

$$\mathsf{ent}[P|\mathbf{\Pi}^1]=m\,\mathsf{ent}[(\sigma_m P)|\mathbf{\Pi}^1]+1-m+m\log m. \tag{4.9}$$

Proof. Let us recall that the usual relative entropy $\mathrm{Ent}[\mu|\nu]$, where μ and ν are two probability measures on the same probability space is defined as $\int\log(\frac{d\mu}{d\nu})d\mu$ if μ is absolutely continuous with respect to ν and $+\infty$ otherwise). By a change of variable $\mathcal{C}\mapsto m^{1/d}\mathcal{C}$ (as in the definition

(2.43)) we get

$$\operatorname{Ent}[(\sigma_m P)_{|C_N}|\mathbf{\Pi}^1_{|C_N}] = \int_{\mathcal{C}\in\mathcal{X}(C_N)} \log\left[\frac{d(\sigma_m P)_{|C_N}(\mathcal{C})}{d\mathbf{\Pi}^1_{|C_N}(\mathcal{C})}\right] d(\sigma_m P)_{|C_N}(\mathcal{C})$$
$$= \int_{\mathcal{C}\in\mathcal{X}(C_{m^{-1}N})} \log\left[\frac{dP_{|C_{m^{-1}N}}(\mathcal{C})}{d\mathbf{\Pi}^1_{|C_N}(m^{1/d}\mathcal{C})}\right] dP_{|C_{m^{-1}N}}(\mathcal{C}). \tag{4.10}$$

hence

$$\operatorname{Ent}[(\sigma_m P)_{|C_N}|\mathbf{\Pi}^1_{|C_N}] = \operatorname{Ent}[P_{|C_{mN}}|\mathbf{\Pi}^1_{|C_{mN}}] + \int_{\mathcal{C}\in\mathcal{X}(C_{m^{-1}N})} \log\left[\frac{\mathbf{\Pi}^1_{|C_{m^{-1}N}}(\mathcal{C})}{\mathbf{\Pi}^1_{|C_N}(m^{1/d}\mathcal{C})}\right] dP_{|C_{m^{-1}N}}(\mathcal{C}),$$

thus we are left to compute the quotient of the densities $\frac{d\mathbf{\Pi}^1_{|C_{m^{-1}N}}(\mathcal{C})}{d\mathbf{\Pi}^1_{|C_N}(m^{1/d}\mathcal{C})}$. But the density of a Poisson point process depends only on the number of point of the configuration and if $\mathcal{C}$ is a point configuration with k points in $\mathcal{X}(C_{m^{-1}N})$, we have

$$\frac{d\mathbf{\Pi}^1_{|C_{m^{-1}N}}(\mathcal{C})}{d\mathbf{\Pi}^1_{|C_N}(m^{1/d}\mathcal{C})} = \frac{\frac{e^{-(m^{-1})N}(m^{-1}N)^k}{k!}}{\frac{e^{-N}(N)^k}{k!}} = e^{-(m^{-1}-1)N-k\log m}.$$

Since P has intensity m, the average number of points of a configuration under P in $\mathcal{X}(m^{-1}N)$ is N hence we get

$$\operatorname{Ent}[(\sigma_m P)_{|C_N}|\mathbf{\Pi}^1_{|C_N}] = \operatorname{Ent}[P_{|C_{m^{-1}N}}|\mathbf{\Pi}^1_{|C_{m^{-1}N}}] - (m^{-1}-1)N - N\log m.$$

Dividing the previous identity by N and taking the limit $N\to\infty$ yields, by definition of ent

$$\mathsf{ent}[(\sigma_m P)|\mathbf{\Pi}^1] = \frac{1}{m}\mathsf{ent}[P|\mathbf{\Pi}^1] - m^{-1} + 1 - \log m.$$

Consequently, if P is of intensity m, (4.9) holds. □

From Lemma 4.2 we observe that if $\bar{P}$ is a stationary tagged point process such that $\bar{P}^x$ has intensity $\mu_V(x)$ for Lebesgue-a.e. $x\in\Sigma$, then

$$\int_\Sigma \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx = \int_\Sigma \mu_V(x)\mathsf{ent}[(\sigma_{\mu_V(x)}\bar{P}^x)|\mathbf{\Pi}^1]dx + |\Sigma| - 1 + \int_\Sigma \mu_V(x)\log\mu_V(x)$$

which yields the scaling for the entropy term in (1.16) and (1.17), moreover the term $(|\Sigma|-1)$ cancels with that of (4.7).

4.3. **Properties of the limit Gibbs measure.** The large deviation principle of Theorem 1 deals with the empirical fields associated to the Gibbs measure $\mathbb{P}_{N,\beta}$, when averaging the random configurations over translations in the support of the equilibrium measure. A natural question is to ask about the behaviour of the Gibbsian point process itself, that is the push-forward $\mathbf{P}_{N,\beta}$ of $\mathbb{P}_{N,\beta}$ by the map $(x_1,\dots,x_N)\mapsto\sum_{i=1}^N\delta_{N^{1/d}x_i}$. The mere existence of limit points for $\{\mathbf{P}_{N,\beta}\}_N$ is unclear in general. Since we are not averaging over translations, we cannot use the discrepancy estimates as in Lemma 3.10 to bound the number of points in a given compact set.

In this section we recall some known results of convergence of the (non-averaged) Gibbsian point process and connect them with the minimization of the LDP rate function.

4.3.1. *Sine-beta processes.* In [VV09] Valko and Virag define a family indexed by $\beta > 0$ of point processes called "Sine-β processes" (by analogy with the usual sine-kernel, or Dyson sine process, known for $\beta = 2$) and prove the convergence of the Gibbsian point process associated to the (random) eigenvalues of β-matrix models to the Sine-β process.

For $x \in [-2, 2]$ let us denote by $\mathrm{Sine}_\beta(x)$ the Sine-β process of [VV09] rescaled to have intensity $\frac{1}{2\pi}\sqrt{4-x^2}$. For any $\beta > 0$, let $\mathbf{P}_{N,\beta}$ be the point process induced by pushing-forward the Gibbs measure corresponding to the case $d = 1$, $s = 0$ and $V(x) = x^2$ by the map $(x_1, \dots, x_N) \mapsto \sum_{i=1}^N \delta_{Nx_i}$. The following is an immediate consequence of [VV09, Theorem1]:

$$\text{For any } x \in (-2,2), \text{ for any } \beta > 0, \text{ we have } \theta_{Nx} \cdot \mathbf{P}_{N,\beta} \Longrightarrow \mathrm{Sine}_\beta(x). \tag{4.11}$$

The convergence $\Longrightarrow$ is proven in [VV09] "in law with respect to vague topology for the counting measure of the point process" which coincides with the notion of convergence used in this paper as explained in Remark 2.7.

We now give a proof of Corollary 1.3 i.e. the minimality of $\overline{\mathcal{F}}_\beta$ at $\overline{\mathrm{Sine}_\beta}$.

Proof. Let $F : \Sigma \times \mathcal{X} \to \mathbb{C}$ be a bounded continuous function. By definition we have

$$\mathbf{E}_{\overline{\mathfrak{P}}_{N,\beta}}\left[\int F(x,\mathcal{C})\, dP(x,\mathcal{C})\right] = \mathbf{E}_{\mathbb{P}_{N,\beta}}\left[\int_{[-2,2]} F(x, \theta_{Nx}\cdot\mathcal{C})dx\right] = \int_{[-2,2]} \mathbf{E}_{\mathbf{P}_{N,\beta}}\left[F(x,\theta_{Nx}\cdot\mathcal{C})\right] dx$$

From (4.11) we know that the sequence of functions $\{x \mapsto \mathbf{E}_{\mathbf{P}_{N,\beta}}\left[F(x,\theta_{Nx}\cdot\mathcal{C})\right]\}_N$ converges almost-everywhere on $[-2,2]$ to $x \mapsto \mathbf{E}_{\mathrm{Sine}_\beta(x)}[F(x,\mathcal{C})]$. Since F is bounded the dominated convergence theorem implies that

$$\lim_{N\to\infty} \mathbf{E}_{\overline{\mathfrak{P}}_{N,\beta}}\left[\int F(x,\mathcal{C})\, dP(x,\mathcal{C})\right] = \int_{[-2,2]} \mathbf{E}_{\mathrm{Sine}_\beta(x)}[F(x,\mathcal{C})] = \mathbf{E}_{\overline{\mathrm{Sine}_\beta}}[F(x,\mathcal{C})]. \tag{4.12}$$

Since this is true for any bounded continuous function on $\Sigma \times \mathcal{X}$ we get that the sequence of tagged point processes $\{\overline{\mathfrak{P}}_{N,\beta}\}_N$ converges to $\overline{\mathrm{Sine}_\beta}$, but the large deviation principle implies that if $\{\overline{\mathfrak{P}}_{N,\beta}\}_N$ converges the limit must be a minimizer of $\overline{\mathcal{F}}_\beta$.

The fact that the point process Sine_β itself minimizes $\mathcal{F}_\beta$ among stationary point processes of intensity 1 is then an easy consequence by scaling. □

4.3.2. *The Ginibre process.* In dimension $d = 2$, little is known about the asymptotic behaviour of the Gibbsian point processes except for the particular value $\beta = 2$. Again, we let $\mathbf{P}_{N,\beta}$ be the push-forward of $\mathbb{P}_{N,\beta}$ by the map $(x_1,\dots,x_N) \mapsto \sum_{i=1}^N \delta_{\sqrt{N}x_i}$

The following was proven by Ginibre [Gin65] (see also e.g. [HKPV09])

Proposition 4.3 (Ginibre)**.** *The point process $\mathbf{P}_{N,\beta}$ for $\beta = 2$ and $V(x) = |x|^2/2$ is determinantal with kernel*

$$K_N(x,y) := \frac{1}{\pi} e^{-\frac{|x_i|^2+|x_j|^2}{2}} \sum_{l=0}^{N-1} \frac{(x_i\bar{x_j})^l}{l!} \tag{4.13}$$

and has a limit Gin_2 *(called the Ginibre point process) which is the determinantal point process on $\mathbb{R}^2$ with kernel*

$$K_\infty(x,y) := \frac{1}{\pi} e^{-\frac{1}{2}|x|^2 - \frac{1}{2}|y|^2 + x\bar{y}}. \tag{4.14}$$

More recently Gin_2 has been identified as the limit of $\mathbf{P}_{N,\beta}$ for a wider class of potentials V – still at inverse temperature $\beta = 2$ – in [AHM11, Proposition 7.4.]. The convergence is proven for any potential V of class C^∞ (satisfying the growth conditions (2.3)) such that $\Delta V(0) = 1$ using a determinantal expression of $\mathbf{P}_{N,\beta}$.

The large deviation principle of Theorem 1 together with translation-invariance properties of the Ginibre ensemble imply Corollary 1.4.

We will rely on the following translation-invariance property, whose proof we postpone to Section 8.4.

Lemma 4.4. *Let $k \geq 0$ and f in Loc_k - see definition near (2.40). For all $\varepsilon > 0$, all integer $n \geq 0$ and all $u_n \in \mathbb{R}^d$ such that $C_k \cup (u_n + C_k) \subset B(0, \sqrt{(1-\varepsilon)n})$ we have*

$$\mathbf{E}_{\mathbf{P}_{n,2}}[f] - \mathbf{E}_{\mathbf{P}_{n,2}}[f(\theta_{u_n} \cdot)] = O\left(\exp(-\frac{\varepsilon^2}{2}n)\right) \quad \text{as } n \to \infty \tag{4.15}$$

uniformly on the choice of u_n.

We may now give a proof of Corollary 1.4.

Proof. Let F be a bounded continuous function in $\mathrm{Loc}_k(\mathcal{X})$. We have

$$\mathbf{E}_{\mathfrak{P}_{N,2}}\left[\int F(\mathcal{C})\, dP(x, \mathcal{C})\right] = \frac{1}{\pi N} \int_{B(0,\sqrt{N})} \mathbf{E}_{\mathbf{P}_{N,2}}\left[F(\theta_x \cdot \mathcal{C})\right] dx. \tag{4.16}$$

Let us denote by $A_{N,\varepsilon}$ the set

$$A_{N,\varepsilon} = \{x \in B(0, \sqrt{N}), (C_k \cup C_k + x) \subset B(0, \sqrt{(1-\varepsilon)N})\}.$$

Since k is fixed, we have $|A_{N,\varepsilon}| \sim \pi(1-\varepsilon)N$ as $N \to \infty$ and since F is bounded we have

$$\left|\frac{1}{\pi N} \int_{B(0,\sqrt{N})} \mathbf{E}_{\mathbf{P}_{N,2}}\left[F(\theta_x \cdot \mathcal{C})\right] dx - \frac{1}{\pi N} \int_{A_{N,\varepsilon}} \mathbf{E}_{\mathbf{P}_{N,2}}\left[F(\theta_x \cdot \mathcal{C})\right] dx\right| = O(\varepsilon). \tag{4.17}$$

From Lemma 4.4 we have

$$\mathbf{E}_{\mathbf{P}_{N,2}}\left[F(\theta_x \cdot \mathcal{C})\right] = \mathbf{E}_{\mathbf{P}_{N,2}}[F] + O(\exp(-\frac{\varepsilon^2}{2}N))$$

uniformly for $x \in A_{N,\varepsilon}$, so that

$$\frac{1}{\pi N} \int_{A_{N,\varepsilon}} \mathbf{E}_{\mathbf{P}_{N,2}}\left[F(\theta_x \cdot \mathcal{C})\right] dx = (1-\varepsilon)\mathbf{E}_{\mathbf{P}_{N,2}}[F(\mathcal{C})] + o(1).$$

But we know (from Proposition 4.3) that $\mathbf{E}_{\mathbf{P}_{N,2}}[F(\mathcal{C})]$ converges to $\mathbf{E}_{\mathrm{Gin}_2}[F(\mathcal{C})]$. Hence we have

$$\lim_{N\to\infty} \frac{1}{\pi N} \int_{A_{N,\varepsilon}} \mathbf{E}_{\mathbf{P}_{N,2}}\left[F(\theta_x \cdot \mathcal{C})\right] dx = (1-\varepsilon)\mathbf{E}_{\mathrm{Gin}_2}[F(\mathcal{C})]. \tag{4.18}$$

Combining (4.16), (4.17), (4.18) and letting $\varepsilon \to 0$ we obtain

$$\lim_{N\to\infty} \mathbf{E}_{\mathfrak{P}_{N,2}}\left[\int F(\mathcal{C})\, dP(x, \mathcal{C})\right] = \mathbf{E}_{\mathrm{Gin}_2}[F(\mathcal{C})] \tag{4.19}$$

for all continuous bounded local functions F. By Lemma 2.5 we know that local functions are dense in $\mathrm{Lip}_1(\mathcal{X})$, hence (4.19) is valid for any Lipschitz function F. This implies that $\mathfrak{P}_{N,2}$ converges to Gin_2, but the Large Deviation Principle of Theorem 1 implies that if $\mathfrak{P}_{N,2}$ has a limit it must be a minimizer of $\mathcal{F}_2$. □

5. Screening and regularization

In this section we enter the core of the proof, i.e. we describe important ingredients for the proof of Proposition 1.7, which rely on previous work, in particular the screening procedure introduced in [SS12b, SS12a, RS13, PS14]. The goal of this section is to introduce two operations on point configurations (say, in a given hypercube C_R) which we may roughly describe this way:

(1) The screening procedure Φ^{scr} takes "good" (also called "screenable") configurations and replace them by "better" configurations which are well-balanced (the number of points matches the volume) and for which there is a corresponding electric field supported in C_R with controlled energy. If the screening procedure encounters a "bad" configurations, it replaces it by "standard" configurations (at the cost of a loss of information).

(2) The regularization procedure Φ^{reg} takes a configuration and separates all the pair of points which are closer than a certain threshold τ.

5.1. The screening procedure.

When we get to the next section, we will want to construct point configurations by elementary blocks (hyperrectangles) and compute their energy additively in these blocks. One of the technical tricks borrowed from the original works above is that this may be done by gluing together electric fields whose normal components agree on the boundaries. More precisely, assume that space is partitioned into hyperrectangles $K \in \mathcal{K}$. We would like to construct a vector field E_K in each K such that

$$\begin{cases} -\mathrm{div}\,(|y|^\gamma E_K) = c_{d,s}\,(\mathcal{C}_K - \mu'_V \delta_{\mathbb{R}^d}) & \text{in } K \times \mathbb{R}^k \\ E_K \cdot \vec{\nu} = 0 & \text{on } \partial(K \times \mathbb{R}^k) \end{cases} \tag{5.1}$$

(where $\vec{\nu}$ is the outer unit normal to K) for some discrete set of points $\mathcal{C}_K \subset K$, and with

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |(E_K)_\eta|^2$$

well controlled (recall the definition (2.27)). Integrating the relation (5.1), we see that a compatibility condition must be satisfied in order for this equation to be solvable, i.e. that

$$\int_K d\mathcal{C} = \int_K d\mu'_V \tag{5.2}$$

in particular the partition must be made so that $\int_K d\mu'_V$ are integers.

When solving (5.1), we may take E_K to be a gradient, but we do not require it. Once the relations (5.1) are satisfied on each K, we may paste together the vector fields E_K into a unique vector field E, and the discrete sets of points $\mathcal{C}_K$ into a configuration $\mathcal{C}$. By (5.2) the cardinality of $\mathcal{C}$ will be equal to $\int_{\mathbb{R}^d} d\mu'_V$, which is exactly N. We will thus have obtained a configuration of N points, whose energy we will try to evaluate. The important fact is that the enforcement of the boundary condition $E_K \cdot \vec{\nu} = 0$ on each boundary ensures that

$$-\,\mathrm{div}\,(|y|^\gamma E) = c_{d,s}\left(\mathcal{C} - \mu'_V \delta_{\mathbb{R}^d}\right) \quad \text{in } \mathbb{R}^{d+k} \tag{5.3}$$

holds globally. Indeed, a vector field which is discontinuous across an interface has a distributional divergence concentrated on the interface equal to the jump of the normal derivative, i.e. here there is no extra divergence created across these interfaces. Even if the E_K's were gradients, the global E is in general no longer a gradient. This does not matter however, since the energy of the true electric field ∇H generated by the configuration $\mathcal{C}$ (and the background $-\mu'_V\delta_{\mathbb{R}^d}$) is necessarily smaller than that of E as seen in Lemma 3.12. This way

$$\int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla H'_{N,\eta}|^2 \le \sum_K \int_K |y|^\gamma |(E_K)_\eta|^2$$

and the energy has indeed become additive over the cells. This shows that to compute the w_N (recall (2.30)) associated with the configuration of N points $\mathcal{C}$, we may indeed relax the gradient constraint and evaluate the energy of the electric fields E_K constructed in each K. This explains why we need to find ways of obtaining vector fields E_K satisfying (5.1).

These vector fields will themselves be constructed from a given point configuration in each cell K, sampled at random via the law $\mathbb{Q}_{N,\beta}$ and two cases will occur. The first case occurs when the configuration in the cell K has an energy which is not too large, this "not too large energy" will be characterized by the fact that there exists a vector field E in the cell K such that $-\frac{1}{c_{d,s}}\mathrm{div}\,(|y|^\gamma E)+\mu'_V\delta_{\mathbb{R}^d}$ coincides with the configuration of points (i.e. the first equation in (5.1) is verified, but not necessarily the second), and whose $\int_K |y|^\gamma |E|^2$ is not too large (in a way that will be specified below). Such configurations will be called "screenable". Indeed, for them, the result of [PS14] ensures that we may modify the configuration in a thin layer near ∂K, and modify the vector field E a little bit as well, so that (5.1) is satisfied, and that the energy has not been changed very much. The second case is the case where there exists no such E of reasonable energy in the cell K. In that case the configuration is not screenable in the cell K, we will completely discard it and replace it by an artificial (frozen) configuration (typically a perturbation of a periodic one) whose energy is well controlled, but which has nothing to do with the original configuration. This will not matter in the end, because we will be able to show that such bad cells are rare for a typical configuration.

An important task will be later to estimate the volume in the space of configurations of the modified configurations that we obtain this way. In fact, what we need to produce above is not just one configuration, but a family of them whose volume is not too small.

5.1.1. *A preliminary construction.* The first lemma we state below concerns the construction of families of "artificial" configurations whose energy is well controlled. This will be used in two different ways: to fill up an empty space with points during the screening procedure, and also in the next section in order to replace "bad configurations" for which the screening procedure fails to apply.

Lemma 5.1. *Let $0 < \underline{m} \le \overline{m}$ be fixed, K be a hyperrectangle with sidelengths in $[R, 2R]$. There exists $\eta_0 > 0$ depending only on $d, \underline{m}, \overline{m}$ such that the following holds : let μ be a measurable function on K satisfying $\underline{m} \le \mu \le \overline{m}$ and such that $n_{K,\mu} := \int_K \mu$ is an integer, then there exists a family $\Phi^{\mathrm{gen}}(K,\mu)$ of configurations with $n_{K,\mu}$ points in K such that for any $\mathcal{C}^{\mathrm{gen}}$ in $\Phi^{\mathrm{gen}}(K,\mu)$, the following holds :*

1. *The distance between two points of $\mathcal{C}^{\mathrm{gen}}$ and between a point of $\mathcal{C}^{\mathrm{gen}}$ and ∂K is bounded below by η_0.*
2. *There exists E^{gen} satisfying*

$$\begin{cases} \mathrm{div}\,(|y|^\gamma E^{\mathrm{gen}}) = c_{d,s}\,(\mathcal{C}^{\mathrm{gen}} - \mu\delta_{\mathbb{R}^d}) & \text{in } K\times\mathbb{R}^k \\ E^{\mathrm{gen}}\cdot\vec{\nu} = 0 & \text{on } \partial K\times\mathbb{R}^k \end{cases} \tag{5.4}$$

and for any $\eta < \eta_0$,

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |E_\eta^{\mathrm{gen}}|^2 - c_{d,s} n_{K,\mu} g(\eta) \\ \leq C n_{K,\mu} + C R^{d+1-\gamma} \|\mu - m\|_{L^\infty(K)}^2 + C\left(n_{K,\mu} g(\eta)\right)^{\frac{1}{2}} R^{\frac{d+1-\gamma}{2}} \|\mu - m\|_{L^\infty(K)} \tag{5.5}$$

with a constant C depending only on $d, \overline{m}, \underline{m}$.

(3) *The volume of $\Phi^{\mathrm{gen}}(K,\mu)$ is bounded below by*

$$\mathbf{Leb}^{\otimes n_{K,\mu}}\left(\Phi^{\mathrm{gen}}(K,\mu)\right) \geq (n_{K,\mu}!) C^{n_{K,\mu}} \tag{5.6}$$

with a constant C depending only on $d, \overline{m}, \underline{m}$.

We postpone the proof of Lemma 5.1 to Section 8.

5.1.2. *The screening result.* We now state the screening result from [PS14], in a version rephrased for our needs. As mentioned above this result serves to modify a given electric vector field and the underlying point configuration, in such a way as to satisfy (5.1). We call this "screening" because the configuration is modified in such a way that the field generated by the cell can be taken to be zero outside of the cell, i.e. the configuration has no influence outside the cell.

The configuration and the field will only be modified in a thin layer near the boundary of a hyperrectangle K, and remain unchanged in an interior set denoted Old. To be accurate, we do not really need the original configuration to be defined in the whole K, but only in a subcube $C_R \subset K$, the configuration is then completed by hand until the whole K is covered with points. We also need the positions of the points added "by hand" in the layer near the boundary, which will be denoted New, to be flexible enough to have a nonzero volume of associated configurations in phase space. This is accomplished by letting the points move in small balls around their basic positions, which does not alter the estimates.

Proposition 5.2 (Screening)**.** *Let $\underline{m}, \overline{m} > 0$ be fixed.*

There exists $R_0 > 0$ universal, $\eta_0 > 0$ depending only on d and $\overline{m}$, there exists a constant C depending on $d, s, \underline{m}, \overline{m}$ such that the following holds.

Let $0 < \varepsilon < \frac{1}{2}$ and $0 < \eta < \eta_0$ be fixed. Let C_R be a hypercube of sidelength R for some $R > 0$ and let K be a hyperrectangle such that $C_R \subset K$. Let μ be a measurable function on K satisfying $\underline{m} \leq \mu \leq \overline{m}$ and such that $\int_K \mu$ is an integer. Let $m = \fint_K \mu$. Let $\mathcal{C}$ be a point configuration in C_R.

Assume that E is a vector field defined in $C_R \times \mathbb{R}^k$ and satisfies

$$-\mathrm{div}\left(|y|^\gamma E\right) = c_{d,s}\left(\mathcal{C} - \mu\delta_{\mathbb{R}^d}\right) \text{ in } C_R.$$

Letting E_η be associated to E as in (2.27), we define

$$M_{R,\eta} := \frac{1}{R^d} \int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2,$$

and in the case $k = 1$

$$e_{\varepsilon,R} := \frac{1}{\varepsilon^4 R^d} \int_{C_R\times\left(\mathbb{R}\setminus(-\frac{1}{2}\varepsilon^2 R, \frac{1}{2}\varepsilon^2 R)\right)} |y|^\gamma |E|^2, \tag{5.7}$$

and we assume the following inequalities are satisfied
(5.8)
$$R > \max\left(\frac{R_0}{\varepsilon^2}, \frac{CR_0 M_{R,\eta}}{\varepsilon^3}\right), \quad R > \begin{cases} \frac{CR_0 M_{R,\eta}^{1/2}}{\varepsilon^{d+3/2}} & \text{if } k=0 \\ \max(CR_0 M_{R,\eta}^{1/(1-\gamma)} \varepsilon^{\frac{-1-2d+\gamma}{1-\gamma}}, R_0 \varepsilon^{\frac{2\gamma}{1-\gamma}} e_{\varepsilon,R}^{1/(1-\gamma)}) & \text{if } k=1 \end{cases}.$$

Then there exists a (measurable) family $\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)$ *of point configurations in* K *and a partition of* K *as* $\mathrm{Old} \sqcup \mathrm{New}$ *with*

$$\mathrm{Int}_\varepsilon := \{x \in C_R, \mathrm{dist}(x, \partial C_R)\} \geq 2\varepsilon R\} \subset \mathrm{Old} \tag{5.9}$$

such that for any $\mathcal{C}^{\mathrm{scr}}$ *in* $\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)$ *we have*

1. *The configurations* $\mathcal{C}$ *and* $\mathcal{C}^{\mathrm{scr}}$ *coincide on* Old.
2. *For* $\eta < \eta_0$*, it holds that*

$$\sum_{x_i \neq x_j \in \mathcal{C}^{\mathrm{scr}}, |x_i - x_j| \leq 2\eta} g(x_i - x_j) = \sum_{x_i \neq x_j \in \mathcal{C}, |x_i - x_j| \leq 2\eta} g(x_i - x_j), \tag{5.10}$$

i.e. the contribution to the energy due to pairs of points which are 2η*-close is left unchanged. Moreover we have*

$$\min_{x \in \mathcal{C}^{\mathrm{scr}}} \mathrm{dist}(x, \partial K) \geq \eta_0 \tag{5.11}$$

$$\min_{x \in \mathcal{C}^{\mathrm{scr}} \cap \mathrm{New},\ y \in \mathcal{C}^{\mathrm{scr}}} |x - y| \geq \eta_0. \tag{5.12}$$

3. *There exists a vector field* $E^{\mathrm{scr}} \in L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ *satisfying*
   (a)

$$\begin{cases} -\mathrm{div}\ (|y|^\gamma E^{\mathrm{scr}}) = c_{d,s}\left(E^{\mathrm{scr}} - \mu \delta_{\mathbb{R}^d}\right) & \text{in } K \times \mathbb{R}^k \\ E^{\mathrm{scr}} \cdot \vec{\nu} = 0 & \text{on } \partial K \times \mathbb{R}^k, \end{cases} \tag{5.13}$$

   In particular the configuration $\mathcal{C}^{\mathrm{scr}}$ *has exactly* $\int_K \mu$ *points in* K.
   (b) *Letting* E^{scr}_η *be associated to* E^{scr} *as in* (2.27) *we have:*

$$\int_{K \times \mathbb{R}^k} |y|^\gamma |E^{\mathrm{scr}}_\eta|^2 \leq I + II + III \tag{5.14}$$

   with

$$I = \left(\int_{C_R \times \mathbb{R}^k} |y|^\gamma |E_\eta|^2\right)(1 + C\varepsilon) + Cg(\eta)\left((1 + M_{R,\eta})\varepsilon R^d + |K| - |C_R|\right) + Ce_{\varepsilon,R}\varepsilon R^d$$
$$II = CR^{d+1-\gamma}\|\mu - m\|^2_{L^\infty(K)}$$
$$III = \sqrt{I \cdot II}$$

   for some constant C *depending only on* $s, d, \underline{m}, \overline{m}$.

Moreover the number of points of $\mathcal{C}^{\mathrm{scr}}$ *in* New *is a constant* n_{New} *on* $\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)$ *and we have*

$$\mathbf{Leb}^{\otimes n_{\mathrm{New}}}\left(\Phi^{\mathrm{scr},\mathrm{New}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)\right) \geq (n_{\mathrm{New}})! c^{n_{\mathrm{New}}} \tag{5.15}$$

for a certain constant $c > 0$ *depending only on* $d, \overline{m}$*, where* $\Phi^{\mathrm{scr},\mathrm{New}}_{\varepsilon,\eta,R}$ *denotes the restriction of the configurations to the subset* $\mathrm{New} \subset K$.

Proof. The statement is based on a re-writing of [PS14, Proposition 6.1.] provided by a careful examination of its proof. First, let us assume that $\mu \equiv 1$. In that case we may apply directly [PS14, Proposition 6.1]. For the reader's convenience let us sketch that proof.

The first step is to find by a mean-value argument a good boundary, that is the boundary of a hypercube Old included in C_R and containing Int_ε, on which $\int |y|^\gamma |E_\eta|^2$ is not too large, more precisely controlled in terms of $M_{R,\eta}$. In the case where $k = 1$, i.e. the interaction potential is not coulombic and we need to use the extension representation, cf. Section 2.3, then we need to do the same "vertically" i.e. find by mean value a good height z such that $\int_{\mathrm{Old}\times\{-z,z\}} |y|^\gamma |E_\eta|^2$ is controlled in terms of $e_{\varepsilon,R}$.

The configuration $\mathcal{C}$ and the field E are kept unchanged inside Old. We then tile New := $K\backslash$Old by small hypercubes of sidelength $O(1)$ (and uniformly bounded below) and place one point near the center of each of these hypercubes (they may be chosen freely in a small ball near the center), see Figure 5.1.2. This way the new points are well separated by construction, and the distances between two points (of the configurations) in New or between a point (of the configuration) in New and a point (of the configuration) in Old is bounded below by $2\eta_0$. In particular if $\eta < \eta_0$ no new 2η-close pair has been created and property 2) holds.

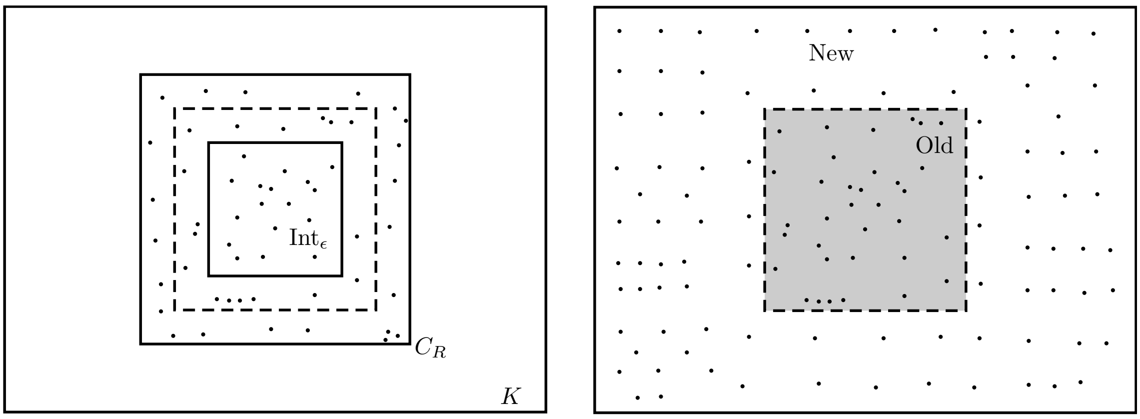

Figure 1. The original configuration (on the left) and the screened configuration (on the right). The dashed line corresponds to the good boundary. Proportions are distorted and Int_ε really contains most of the set K.

We then construct a global electric field on New as described at the beginning of the section by pasting together vector fields defined on each hypercube. For the global vector field to satisfy the relation (5.4), we need the normal components to be continuous (and not necessarily 0) across the interfaces. These normal components are chosen to agree with that of E on ∂Old and to be 0 on ∂K. As explained at the beginning of this section, the energy of the new electric field E^{scr} is then bounded by the energy of E in Old, plus the energy of the new E in all the added hypercubes of $K\backslash$Old. In [PS14], the construction is made for ∂K and ∂C_R at distance proportional to εR from each other. We may apply the result of [PS14] for such a K', and if ∂K is further away from C_R, then it suffices to tile $K\backslash K'$ and paste vector fields constructed exactly as in the proof of Lemma 5.1. In the end we obtain a global vector field E^{scr} satisfying item 3 (a). The energy due to the part $K\backslash K'$ is controlled just as in Lemma 5.1 by a constant times the number of points added there, i.e. $C(|K|-|K'|) \leq C(|K|-|C_R|)$.

More precisely, we may write

$$\int_{(K\setminus K')\times\mathbb{R}^k} (|y|^\gamma |E_\eta^{\mathrm{scr}}|^2 - c_{d,s} g(\eta)) \leq C(|K| - |C_R|).$$

The energy in K' is proven in [PS14] to be controlled in terms of $M_{\eta,R}$ and $e_{\varepsilon,R}$ as follows (taking $\eta = \eta'$ in [PS14, Proposition 6.1])

$$\int_{K'\times\mathbb{R}^k} |y|^\gamma |E_\eta^{\mathrm{scr}}|^2 \leq \left(\int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2\right)(1+C\varepsilon) + Cg(\eta)(1+M_{R,\eta})\varepsilon R^d + Ce_{\varepsilon,R}\varepsilon R^d.$$

Combining the two relations, we obtain

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |E_\eta^{\mathrm{scr}}|^2 \leq \left(\int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2\right)(1+C\varepsilon) + Cg(\eta)(1+M_{R,\eta})\varepsilon R^d + Ce_{\varepsilon,R}\varepsilon R^d + Cg(\eta)(|K| - |C_R|), \tag{5.16}$$

where C depends only on d, s.

For each of the hypercubes in New, we may move the point placed therein by a small distance $C = \frac{\eta_0}{4}$ without affecting the conclusions, as done in the proof of Lemma 5.1 or as explained in [PS14, Remark 6.7]. This way we obtain a set of configurations (and associated electric fields) satisfying all the requirements and whose n_{New}-dimensional volume is bounded below as in (5.15).

To conclude the proof, there remains to handle the fact that μ, in our setting, is not equal to 1 but may vary between $\underline{m}$ and $\overline{m}$. We start by treating the case where $\mu \equiv m$ is a constant function. We may then apply the scaling map σ_m as defined in (2.34) to the electric field E. By a change of variables, letting $E' := \sigma_m E$, we get

$$\int_{m^{1/d}(K_R\times[-R,R]^k)} |y|^\gamma |E'_{m^{1/d}\eta}|^2 = m^{-s/d} \int_{K_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2 = R^d M_{R,\eta}$$

and

$$\int_{m^{1/d}\left(K_R\times(\mathbb{R}\setminus(-\frac{1}{2}\varepsilon^2, \frac{1}{2}\varepsilon^2))\right)} |y|^\gamma |E'|^2 = m^{-s/d}\varepsilon^4 R^d e_{\varepsilon,R},$$

i.e. with obvious notation $M_{R,\eta}(E) = m^{1+s/d} M_{m^{1/d}R, m^{1/d}\eta}(E')$ and $e_{\varepsilon,R}(E) = m^{1+s/d} e_{\varepsilon, m^{1/d}R}(E')$. If the constant C is chosen large enough (depending on $\underline{m}, \overline{m}, d, s$) we see that inequalities (5.8) imply that the assumptions (6.2) of [PS14, Proposition 6.1] are satisfied by E' with R replaced by $m^{1/d}R$, η replaced by $m^{1/d}\eta$. We may then apply the result of [PS14, Proposition 6.1], i.e. what we have just outlined and get a family of configurations such that the desired conclusions are satisfied, up to a global scaling of all sets and distances by a factor $m^{1/d}$. In particular in view of (5.16) we control the energy by

$$\int_{m^{1/d}K\times\mathbb{R}^k} |y|^\gamma |E^{\mathrm{scr}'}_{m^{1/d}\eta}|^2 \leq \left(\int_{C_{m^{1/d}R}\times[-m^{1/d}R, m^{1/d}R]^k} |y|^\gamma |E'_{m^{1/d}\eta}|^2\right)(1+C\varepsilon)$$
$$+ Cg(m^{1/d}\eta)(1+m^{-1-s/d}M_{R,\eta}(E))\varepsilon m R^d + Cm^{-1-s/d} e_{\varepsilon,R} m R^d + Cmg(m^{1/d}\eta)(|K| - |C_R|)$$

We then apply the inverse map $\sigma_{m^{-1}}$ to this family of configurations and associated electric fields, and we obtain a family of configurations satisfying all the results of the proposition with μ replaced by m and for the energy bound

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |E_\eta^{\mathrm{scr}}|^2 \le \left(\int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2\right)(1+C\varepsilon) \\ + Cg(m^{1/d}\eta)(m^{s/d}+m^{-1}M_{R,\eta}(E))\varepsilon m R^d + Ce_{\varepsilon,R}R^d + Cm^{1+s/d}g(m^{1/d}\eta)(|K|-|C_R|)$$

thus in view of the exact form of g, (1.4) or (1.2)–(1.3), we obtain in all cases

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |E_\eta^{\mathrm{scr}}|^2 \le \left(\int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2\right)(1+C\varepsilon) \\ + Cg(\eta)(1+M_{R,\eta}(E))\varepsilon R^d + Ce_{\varepsilon,R}R^d + Cg(\eta)(|K|-|C_R|)$$

with a constant C which may now depend on $\underline{m},\overline{m}$. To get from a constant background μ to a variable μ we proceed as in the proof of Lemma 5.1 using Lemma 8.1. We obtain a family of configurations satisfying the desired conclusions. □

Let us now estimate how this procedure changes the volume of a set of configurations in phase-space.

Lemma 5.3. *Let* Int_ε *be as in* (5.9) *and* $\mathrm{Ext}_\varepsilon := C_R\backslash\mathrm{Int}_\varepsilon$. *Assume A is a (measurable) set of point configurations in $\mathcal{X}(C_R)$ such that each configuration of A has n points in C_R and* n_{int} *points in* Int_ε*, with n_{int} satisfying*

$$n_{K,\mu} - n_{\mathrm{int}} \le \frac{|\mathrm{Ext}_\varepsilon|}{2c}, \tag{5.17}$$

where c is the constant in (5.15). *Let us also assume that* $(\mathcal{C},\mu)$ *satisfies the conditions of Proposition 5.2 for all $\mathcal{C}$ in A. Then we have*

$$\log \mathbf{Leb}^{\otimes n_{K,\mu}}\left(\bigcup_{\mathcal{C}\in A}\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)\right) \ge \log \mathbf{Leb}^{\otimes n}(A) \\ + \log\left((n_{K,\mu}-n_{int})!\left(\frac{c}{|\mathrm{Ext}_\varepsilon|}\right)^{n_{K,\mu}-n_{\mathrm{int}}}\right) + (n_{K,\mu}-n)\log|\mathrm{Ext}_\varepsilon|. \tag{5.18}$$

Proof. Using the terminology of Proposition 5.2, we may partition A into

$$A = \cup_{n_{\mathrm{New}}=0}^{n_{K,\mu}}(A|n_{\mathrm{New}})$$

according to the number of points n_{New} that are created in $K\backslash C_R$ and in a thin layer (of width $\approx \varepsilon R$) close to ∂C_R. We denote by $A|n_{\mathrm{New}}$ the subset of A consisting of configurations for which n_{New} points are created.

We note that following the construction of Proposition 5.2, the number of points in Old (points which remain unchanged) is given by $n_{K,\mu}-n_{\mathrm{New}}$ by definition of n_{New}. By construction again, we have $\mathrm{Int}_\varepsilon \subset \mathrm{Old}$ which yields

$$n_{\mathrm{int}} \le n_{K,\mu} - n_{\mathrm{New}}.$$

Thus for each configuration in $A|n_{\mathrm{New}}$, when applying the construction, a number $n_{K,\mu}-n_{\mathrm{New}}$ of points are left untouched while the other ones i.e. $n-(n_{K,\mu}-n_{\mathrm{New}})$ points (all belonging to Ext_ε), are deleted and replaced by n_{New} points (up to permutation of indices) which live in some small balls in K. We may thus write, using (5.15)

$$\mathbf{Leb}^{\otimes n_{K,\mu}}\left(\bigcup_{\mathcal{C}\in(A|n_{\mathrm{New}})} \Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)\right) \geq \frac{\mathbf{Leb}^{\otimes n}(A|n_{\mathrm{New}})(n_{\mathrm{New}})! c^{n_{\mathrm{New}}}}{|\mathrm{Ext}_\varepsilon|^{n-(n_{K,\mu}-n_{\mathrm{New}})}}$$

But we have seen that $n_{\mathrm{New}} \leq n_{K,\mu} - n_{\mathrm{int}}$, while one may check that $x \mapsto x!\left(\frac{c}{|\mathrm{Ext}_\varepsilon|}\right)^x$ is decreasing as long as $x \leq \frac{|\mathrm{Ext}_\varepsilon|}{2c}$, so we may write

$$\mathbf{Leb}^{\otimes n_{K,\mu}}\left(\bigcup_{\mathcal{C}\in A|n_{\mathrm{New}}} \Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)\right) \\ \geq \mathbf{Leb}^{\otimes n}(A|n_{\mathrm{New}})|\mathrm{Ext}_\varepsilon|^{n_{K,\mu}-n}\left((n_{K,\mu}-n_{\mathrm{int}})!\frac{c}{|\mathrm{Ext}_\varepsilon|}\right)^{n_{K,\mu}-n_{\mathrm{int}}}.$$

Summing over n_{New} and taking the log yields the result. □

5.1.3. *Screenability.* The conditions (5.8) borrowed from [PS14] and which are sufficient for the screening result Proposition 5.2 to hold, provide (up to a condition on the number of points) the definition of "screenability", whose meaning we explained at the beginning of the section. Our main concern is then to prove the upper semi-continuity of the screening procedure, which forces us to go into its topological details.

Let $0 < \underline{m}, \overline{m} < +\infty$ be fixed, let η_0 be as in Lemma 5.2 (it depends only on d and $\overline{m}$). For any $R, M, e, \varepsilon > 0$ such that the following inequalities are satisfied

(5.19)
$$R > \max\left(\frac{R_0}{\varepsilon^2}, \frac{CR_0M}{\varepsilon^3}\right), \quad R > \begin{cases} \frac{CR_0M^{1/2}}{\varepsilon^{d+3/2}} & \text{if } k=0 \\ \max(CR_0M^{1/(1-\gamma)}\varepsilon^{\frac{-1-2d+\gamma}{1-\gamma}}, R_0\varepsilon^{\frac{2\gamma}{1-\gamma}}e^{1/(1-\gamma)}) & \text{if } k=1 \end{cases},$$

with the constants C, R_0 as in (5.8), and for any $0 < \eta < \eta_0$, for any configuration of points $\mathcal{C}$ in C_R and any bounded function μ on C_R we define $\mathcal{O}^{M,e,\varepsilon}_{R,\eta,+}(\mathcal{C},\mu)$ as the set of vector fields E such that

$$-\mathrm{div}\left(|y|^\gamma E\right) = c_{d,s}\left(\mathcal{C} - \mu\delta_{\mathbb{R}^d}\right) \text{ in } C_R$$

and such that

$$\frac{1}{R^d}\int_{C_R\times[-R,R]^k} |y|^\gamma |E_\eta|^2 \leq M \tag{5.20}$$

and in addition, in the case $k=1$,

$$\frac{1}{\varepsilon^4 R^d}\int_{C_R\times\left(\mathbb{R}\setminus(-\frac{1}{2}\varepsilon^2 R, \frac{1}{2}\varepsilon^2 R)\right)} |y|^\gamma |E|^2 \leq e. \tag{5.21}$$

The set $\mathcal{O}^{M,e,\varepsilon}_{R,\eta,-}(\mathcal{C},\mu)$ is defined in the same way except that the inequalities (5.20), (5.21) are taken to be strict.

We will denote by $\mathcal{O}_R(\mathcal{C},\mu)$ the set of vector fields E in C_R such that

$$-\mathrm{div}\left(|y|^\gamma E\right) = c_{d,s}\left(\mathcal{C} - \mu\delta_{\mathbb{R}^d}\right) \text{ in } C_R$$

without any condition on the energy.

Definition 5.4 (Screenability). *We denote by* $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ *(resp.* $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}$*) the set of screenable couples* $(\mathcal{C},\mu)$ *i.e. such that*

(1) $\mathcal{O}^{M,e,\varepsilon}_{R,\eta,+}(\mathcal{C},\mu)$ *(resp.* $\mathcal{O}^{M,e,\varepsilon}_{R,\eta,-}(\mathcal{C},\mu)$*) is not empty.*

(2) The number of points of $\mathcal{C}$ *in* C_R *is bounded above by*

$$\mathcal{N}(0,R) \le MR^d \text{ resp. } \mathcal{N}(0,R) < MR^d. \tag{5.22}$$

In the following we see $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ *as embedded into the product space* $\mathcal{X}(C_R)\times L^\infty(C_R)$ *endowed with the natural topology.*

Remark 5.5. *The condition* (5.22) *on the number of points is closed (resp. open for the second one) because* $\mathcal{C}\mapsto\mathcal{N}(0,R)(\mathcal{C})$ *is continuous on* $\mathcal{X}(C_R)$.

It is clear by definition that we have

$$\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-} \subset \mathcal{S}^{M,e,\varepsilon}_{R,\eta,+} \subset \mathcal{S}^{2M,2e,\varepsilon}_{R,\eta,-}. \tag{5.23}$$

Definition 5.6. *For any* $(\mathcal{C},\mu)\in\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ *we define* $F^{M,e,\varepsilon}_{R,\eta}$ *to be the "best screenable energy"*

$$F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu) = \inf\left\{\frac{1}{R^d}\int_{C_R\times[-R,R]^k}|y|^\gamma|E_\eta|^2,\quad E\in\mathcal{O}^{2M,2e,\varepsilon}_{R,\eta,-}(\mathcal{C},\mu)\right\}. \tag{5.24}$$

We extend the function $F^{M,e,\varepsilon}_{R,\eta}$ *by zero on the complement of* $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ *so that* $F^{M,e,\varepsilon}_{R,\eta} = F^{M,e,\varepsilon}_{R,\eta}\mathbf{1}_{\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}}$.

Remark 5.7. *It is easy to see that for any* $(\mathcal{C},\mu)$ *and any* E *in* $\mathcal{O}_R(\mathcal{C},\mu)$,

(1) If (5.21) *holds then*

$$F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu) \le \min\left(\frac{1}{R^d}\int_{C_R\times[-R,R]^k}|y|^\gamma|E_\eta|^2, M\right).$$

(2) If (5.21) *fails to hold then*

$$F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu) \le M.$$

Indeed, we note that we always have $F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu)\le M$: either $(\mathcal{C},\mu)\in\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$, in which case any element of $\mathcal{O}^{M,e,\varepsilon}_{R,\eta,+}(\mathcal{C},\mu)$ gives a test vector-field for $F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu)$ whose energy is less than M, or $(\mathcal{C},\mu)\notin\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ in which case $F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu)$ is defined to be 0. To prove the statements of the remark, it thus suffices to verify that if $\frac{1}{R^d}\int_{C_R\times[-R,R]^k}|y|^\gamma|E_\eta|^2\le M$ then $F^{M,e,\varepsilon}_{R,\eta}(\mathcal{C},\mu)\le\frac{1}{R^d}\int_{C_R\times[-R,R]^k}|y|^\gamma|E_\eta|^2$. But this is clear since in that case the configuration is in $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ and E is a test vector field for $F^{M,e,\varepsilon}_{R,\eta}$.

Lemma 5.8. *The set* $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}$ *(resp.* $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}$*) is closed (resp. open) in* $\mathcal{X}(C_R)\times L^\infty(C_R)$, *and the function* $F^{M,e,\varepsilon}_{R,\eta}$ *is upper semi-continuous on* $\mathcal{X}(C_R)\times L^\infty(C_R)$.

For that we need a lemma which proves the continuity of the energy, say the local one, with respect to the background density μ and with respect to the points.

Lemma 5.9. *Let* $R>0$ *and let* $\mathcal{C},\mathcal{C}'$ *be two configurations and* μ,μ' *be two bounded measurable functions on* C_R *as above. Let* $\tilde{E}$ *be the electric field generated by the algebraic difference of* $(\mathcal{C},\mu)$ *and* $(\mathcal{C}',\mu')$ *i.e.*

$$\tilde{E} := c_{d,s}\nabla g * \left(\mathcal{C}-\mathcal{C}'-(\mu-\mu')\delta_{\mathbb{R}^d}\right). \tag{5.25}$$

Then for any $\eta > 0$ the energy $\int_{C_R\times[-R,R]^k} |y|^\gamma |\tilde{E}_\eta|^2$ tends to 0 when $(\mathcal{C}', \mu')$ converges to $(\mathcal{C}, \mu)$ in $\mathcal{X}(C_R) \times L^\infty(C_R)$.

Proof. We recall that by Remark 2.1, letting $g_\eta(x) = \min(g(x), g(\eta))$ we have

$$\tilde{E}_\eta = \nabla g_\eta * (\mathcal{C} - \mathcal{C}') - \nabla g * ((\mu - \mu')\delta_{\mathbb{R}^d}).$$

To prove the result it suffices to prove that letting $H_1 := g_\eta * (\mathcal{C} - \mathcal{C}')$ and $H_2 := g * ((\mu-\mu')\mathbf{1}_{C_R}\delta_{\mathbb{R}^d})$, both $\int_{C_R\times[-R,R]^k} |y|^\gamma |\nabla H_1|^2$ and $\int_{C_R\times[-R,R]^k} |y|^\gamma|\nabla H_2|^2$ tend to 0 as $(\mathcal{C}', \mu')$ converges to $(\mathcal{C}, \mu)$ in $\mathcal{X}(C_R)\times L^\infty(C_R)$. But the number of points in C_R is locally constant for the topology on $\mathcal{X}(C_R)$, so we may assume that the distribution $\mathcal{C}-\mathcal{C}'$ is compactly supported and with total mass 0. Therefore H_1 (resp. ∇H_1) decays like $|x|^{-s-1}$ (resp. like $|x|^{-s-2}$) as $|x| \to \infty$ as noticed in the proof of Lemma 3.12. Integrating by parts we may thus write

$$\int_{\mathbb{R}^{d+k}} |y|^\gamma |\nabla H_1|^2 = \iint g_\eta(x-y)(\mathcal{C} - \mathcal{C}')(x)(\mathcal{C} - \mathcal{C}')(y)$$

and the desired result for H_1 follows by continuity of g_η. For H_2, we first notice that by integrability of g we have

$$|H_2| \le C\|\mu - \mu'\|_{L^\infty(C_R)} \tag{5.26}$$

where the constant C depends on R, and that

$$-\mathrm{div}\,(|y|^\gamma \nabla H_2) = c_{d,s}(\mu - \mu')\mathbf{1}_{C_R}\delta_{\mathbb{R}^d}$$

in view of (2.14). Let then χ be a smooth compactly supported positive function equal to 1 in $C_R \times [-R, R]$, and such that $|\nabla\chi| \le 1$. Integrating by parts, we have

$$\begin{aligned}\int_{\mathbb{R}^{d+k}} \chi^2 |y|^\gamma |\nabla H_2|^2 &= -\int_{\mathbb{R}^{d+k}} \chi^2 \mathrm{div}\,(|y|^\gamma\nabla H_2)H_2 - 2\int_{\mathbb{R}^{d+k}} \chi\nabla\chi\cdot\nabla H_2|y|^\gamma H_2 \\ &\le c_{d,s}\left|\int_{\mathbb{R}^{d+k}} \chi^2 H_2(\mu-\mu')\delta_{\mathbb{R}^d}\right| + \int_{\mathbb{R}^{d+k}} \chi|\nabla\chi||y|^\gamma|H_2||\nabla H_2|.\end{aligned}$$

From(5.26) we may thus write, using the Cauchy-Schwarz inequality,

$$\int_{\mathbb{R}^{d+k}} \chi^2|y|^\gamma|\nabla H_2|^2 \le C\|\mu-\mu'\|_{L^\infty}^2 + C\|\mu-\mu'\|_{L^\infty}\left(\int_{\mathbb{R}^{d+k}}\chi^2|y|^\gamma|\nabla H_2|^2\right)^{\frac12}\left(\int_{\mathbb{R}^{d+k}}|\nabla\chi|^2|y|^\gamma\right)^{\frac12}$$

therefore

$$\int_{C_R\times[-R,R]^k} |y|^\gamma|\nabla H_2|^2 \le \int_{\mathbb{R}^{d+k}}\chi^2|y|^\gamma|\nabla H_2|^2 \le C(\|\mu-\mu'\|_{L^\infty}^2 + \|\mu-\mu'\|_{L^\infty}^4),$$

with C depending on R. This completes the proof. □

Proof of Lemma 5.8. Let us first prove the upper semi-continuity of $F_{R,\eta}^{M,e,\varepsilon}$. Let $(\mathcal{C}, \mu)$ in $\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}$ be fixed and for any $(\mathcal{C}', \mu')$ in $\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}$ such that $\mathcal{C}'(\overline{C_R}) = \mathcal{C}(\overline{C_R})$ (i.e. $\mathcal{C}'$ and $\mathcal{C}$ have the same number of points in C_R), let $\tilde{E}$ be the vector field defined in (5.25). By definition of $\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}$ and $F_{R,\eta}^{M,e,\varepsilon}$, for any $\delta > 0$ we may find an electric field E in $\mathcal{O}_{R,\eta,-}^{2M,2e,\varepsilon}(\mathcal{C}, \mu)$ such that

$$\frac{1}{R^d}\int_{C_R\times[-R,R]^k} |y|^\gamma|E_\eta|^2 \le F_{R,\eta}^{M,e,\varepsilon}(\mathcal{C}, \mu) + \delta. \tag{5.27}$$

Then $E' := E^\delta + \tilde{E}$ satisfies

$$-\mathrm{div}\left(|y|^\gamma E'\right) = c_{d,s}\left(\mathcal{C}' - \mu'\delta_{\mathbb{R}^d}\right) \text{ in } C_R.$$

In view of Lemma 5.9, we easily deduce that if $(\mathcal{C}', \mu')$ is sufficiently close to $(\mathcal{C}, \mu)$ in $\mathcal{X}(\overline{C_R}) \times L^\infty(C_R)$, then

$$\frac{1}{R^d}\int_{C_R\times[-R,R]^k} |y|^\gamma |E'_\eta|^2 \le F_{R,\eta}^{M,e,\varepsilon}(\mathcal{C}, \mu) + 2\delta, \tag{5.28}$$

and E' is in $\mathcal{O}_{R,\eta,-}^{2M,2e,\varepsilon}(\mathcal{C}', \mu')$ provided $\delta > 0$ was small enough. In particular we have found a test vector-field E' which is in $\mathcal{O}_{R,\eta,-}^{2M,2e,\varepsilon}(\mathcal{C}', \mu')$ and satisfies (5.28) hence by definition of $F_{R,\eta}^{M,e,\varepsilon}$ we have

$$F_{R,\eta}^{M,e,\varepsilon}(\mathcal{C}', \mu') \le F_{R,\eta}^{M,e,\varepsilon}(\mathcal{C}, \mu) + 2\delta$$

if $(\mathcal{C}', \mu')$ is close enough to $(\mathcal{C}, \mu)$. Taking δ arbitrarily small this ensures that $F_{R,\eta}^{M,e,\varepsilon}$ is upper semi-continuous at any point $(\mathcal{C}, \mu)$ in $\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}$.

Following the same line of reasoning, together with Remark 5.5 we obtain that $\mathcal{S}_{R,\eta,-}^{M,e,\varepsilon}$ is open in $\mathcal{X}(\overline{C_R}) \times L^\infty(C_R)$. On the other hand the fact that $\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}$ is closed is a consequence of Lemma 3.6 together with Remark 5.5. This in turn ensures that $F_{R,\eta}^{M,e,\varepsilon} = F_{R,\eta}^{M,e,\varepsilon}\mathbf{1}_{\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}}$ is upper semi-continuous at any point. □

The next lemma shows that tagged point process $\bar{P}$ of finite energy have good properties: most configurations under $\bar{P}$ are "screenable" in the sense of Section 5.1.3, their energies are controlled by that of $\bar{P}$ and the truncation errors due to close pairs of points are small. These controls are then extended to point processes in small balls $B(\bar{P}, \nu)$ around $\bar{P}$. In the following when a couple $(x, \mathcal{C})$ is fixed the implicit background measure is $\mu_V(x)$ i.e.

$$(x, \mathcal{C}) \in \mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}/\mathcal{S}_{R,\eta,-}^{M,e,\varepsilon} \iff (\mathcal{C}, \mu_V(x)) \in \mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}/\mathcal{S}_{R,\eta,-}^{M,e,\varepsilon}.$$

Lemma 5.10. *Let $\bar{P}$ be a tagged point process in $\mathcal{P}_s(\Sigma \times \mathcal{X})$ such that $\overline{\mathbb{W}}_{\mu_V}(\bar{P})$ is finite. Then we have*

(1) For $\eta > 0$ small enough and any $e, \varepsilon > 0$,

$$\lim_{M,R\to\infty}\lim_{\nu\to 0}\inf_{\bar{Q}\in B(\bar{P},\nu)} \bar{Q}(\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}) = 1 \text{ and } \lim_{M,R\to\infty}\lim_{\nu\to 0}\inf_{\bar{Q}\in B(\bar{P},\nu)} \bar{Q}(\mathcal{S}_{R,\eta,-}^{M,e,\varepsilon}) = 1$$

where $M, R \to \infty$ in such a way that the conditions (5.19) are satisfied.

(2) For any $\eta, e, \varepsilon > 0$,

$$\limsup_{M,R\to\infty}\limsup_{\nu\to 0}\sup_{\bar{Q}\in B(\bar{P},\nu)} \int \left(F_{R,\eta}^{M,e,\varepsilon}(\mathcal{C}, \mu_V(x)) - c_{d,s}\mu_V(x) g(\eta)\right) d\bar{Q}(x, \mathcal{C}) \le \overline{\mathbb{W}}_{\mu_V}(\bar{P}) + C\eta^{\frac{d-s}{2}}. \tag{5.29}$$

(3) For any η, τ, with $0 < \tau < \eta^2/2 < 1$, any $x \in \mathbb{R}^d$, any $R > 0$

$$\limsup_{\eta \to 0} \limsup_{\tau \to 0} \left(\frac{g(2\tau)}{\tau^d} \mathbf{E}_{\bar{P}}[(\mathcal{N}(x,\tau)^2 - 1)_+] + \frac{1}{R^d} \mathbf{E}_{\bar{P}} \left[\left(\sum_{p \neq q \in \mathcal{C} \cap C_R, \tau \leq |p-q| \leq \eta^2/2} g(|p-q|) \right) \right] \right) = 0. \tag{5.30}$$

Note that we cannot directly extend (5.30) to a small ball around $\bar{P}$ because functions like $\mathcal{C} \mapsto (\mathcal{N}(x,\tau)^2 - 1)_+(\mathcal{C})$ are not bounded.

Proof. As a consequence of Lemma 2.12 and (2.45) we know that since $\overline{\mathbb{W}}_{\mu_V}(\bar{P})$ is finite we may find a tagged random electric field $\bar{P}^{\mathrm{elec}}$ in $\mathcal{P}_s\left(\Sigma \times L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})\right)$ such that

$$\forall x \in \Sigma,\ \mathrm{Conf}_{\mu_V(x)} \# \bar{P}^{\mathrm{elec},x} = \bar{P}^x, \qquad \overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\mathrm{elec}}) \leq \overline{\mathbb{W}}_{\mu_V}(\bar{P}),$$

where $\mathrm{Conf}_{\mu_V(x)} \# \bar{P}^{\mathrm{elec},x}$ denotes the push-forward of $\bar{P}^{\mathrm{elec},x}$ by $\mathrm{Conf}_{\mu_V(x)}$. By stationarity, in view of Lemma 2.8 we have for all $R > 0$

$$\int \left(\frac{1}{R^d} \int_{C_R \times \mathbb{R}^k} |y|^\gamma |E_\eta|^2 - c_{d,s} \mu_V(x) g(\eta) \right) d\bar{P}^{\mathrm{elec}}(x, E) = \fint_\Sigma \widetilde{\mathcal{W}}_\eta(\bar{P}^{\mathrm{elec},x}) dx \tag{5.31}$$

and by Markov's inequality we see that for any $M, R > 0$ we have for η small enough

$$\bar{P}^{\mathrm{elec}} \left(\frac{1}{R^d} \int_{C_R \times \mathbb{R}^k} |y|^\gamma |E_\eta|^2 \geq M \right) \leq \frac{\overline{\mathbb{W}}_{\mu_V}(\bar{P}) + c_{d,s} g(\eta)}{M}. \tag{5.32}$$

On the other hand we have $\bar{P}^{\mathrm{elec}}$-almost surely

$$\lim_{R \to \infty} \int_{C_1 \times (\mathbb{R} \setminus (-\varepsilon^2 R, \varepsilon^2 R))^k} |y|^\gamma |E_\eta|^2 = 0$$

which in turn implies, by stationarity (Lemma 2.8 again) that for any $e > 0$

$$\lim_{R \to \infty} \bar{P}^{\mathrm{elec}} \left(\frac{1}{R^d} \int_{C_R \times (\mathbb{R} \setminus (-\varepsilon^2 R, \varepsilon^2 R))^k} |y|^\gamma |E_\eta|^2 \geq e \right) = 0. \tag{5.33}$$

Finally from Lemma 3.10 we see that $\mathbf{E}_{\bar{P}}\left[\mathcal{N}(0,R)^2\right] \leq C R^{2d}$ with a constant C depending only on $\bar{P}$ hence by Markov's inequality we have

$$\bar{P}\left(\mathcal{N}(0,R) \geq M R^d \right) \leq \frac{C}{M^2} \tag{5.34}$$

uniformly on R.

Combining (5.32), (5.33) and (5.34) yields that $\lim_{M,R \to \infty} \bar{P}(\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+}) = 1$ and also, in view of (5.23), $\lim_{M,R \to \infty} \bar{P}(\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}) = 1$. Let us emphasize that although we do not need to satisfy the conditions (5.19), we may require them to be satisfied. Since $\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}$ is open, $\mathbf{1}_{\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}}$ is lower semi-continuous, hence

$$\lim_{\nu \to 0} \inf_{\bar{Q} \in B(\bar{P}, \nu)} \bar{Q}(\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-}) \geq \bar{P}(\mathcal{S}^{M,e,\varepsilon}_{R,\eta,-})$$

and the first item of the lemma follows using again (5.23) to handle $\bar{Q}(\mathcal{S}^{M,e,\varepsilon}_{R,\eta,+})$.

To prove the second point, according to Lemma 2.12 and (2.45) we may consider for any $\delta > 0$ a random tagged electric field $\bar{P}^{\mathrm{elec},\delta}$ in $\mathcal{P}_s\left(\Sigma \times L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})\right)$ such that

$$\forall x \in \Sigma,\ \mathrm{Conf}_{\mu_V(x)}\#\bar{P}^{\mathrm{elec},\delta,x} = \bar{P}^x, \qquad \overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\mathrm{elec},\delta}) \le \overline{\mathbb{W}}_{\mu_V}(\bar{P}) + \delta.$$

For $\eta > 0$ small enough, we get with Lemma 3.4

$$\fint_\Sigma \widetilde{\mathcal{W}}_\eta(\bar{P}^{\mathrm{elec},\delta,x})dx \le \overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\mathrm{elec},\delta}) + C\eta^{\frac{d-s}{2}}. \tag{5.35}$$

We still have for any $R > 0$, by stationarity,

$$\begin{aligned}\int \left(\frac{1}{R^d}\int_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2 - c_{d,s}\mu_V(x)g(\eta)\right) d\bar{P}^{\mathrm{elec},\delta}(x,E) &= \fint_\Sigma \widetilde{\mathcal{W}}_\eta(\bar{P}^{\mathrm{elec},\delta,x})dx \\ &\le \overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\mathrm{elec},\delta}) + C\eta^{\frac{d-s}{2}} \le \overline{\mathcal{W}}_{\mu_V}(\bar{P}) + \delta + C\eta^{\frac{d-s}{2}}\end{aligned} \tag{5.36}$$

where we have used Lemma 3.4. Using Remark 5.7 we see that

$$\begin{aligned}\int F^{M,e,\varepsilon}_{R,\eta} d\bar{P} \le \int &\left(\frac{1}{R^d}\int_{C_R\times\mathbb{R}^k} |y|^\gamma |E_\eta|^2\right) d\bar{P}^{\mathrm{elec},\delta}(x,E) \\ &+ M\bar{P}^{\mathrm{elec},\delta}\left(\int_{\mathbb{R}^d\times(\mathbb{R}^k\setminus(-\varepsilon^2R,\varepsilon^2R))} |y|^\gamma|E_\eta|^2 \ge e\right).\end{aligned}$$

Together with (the analogue of) (5.33) and the upper semi-continuity of $F^{M,e,\varepsilon}_{R,\eta}$, the last two relations yield the second item of the lemma, taking $\delta \to 0$, and $R \to \infty$ in (5.35).

We turn to the third item: for $\eta > 0$ and $\tau < \eta^2/2$ we have by Lemma 3.11 that

$$\begin{aligned}\fint_\Sigma \widetilde{\mathcal{W}}_\eta(\bar{P}^{\mathrm{elec},\delta,x})dx + C\left(\frac{g(2\tau)}{\tau^d}\mathbf{E}_{\bar{P}}[(\mathcal{N}(0,\tau)^2-1)_+] + \mathbf{E}_{\bar{P}}\left[\sum_{p\neq q\in\mathcal{C}\cap C_1,\tau\le|p-q|\le\eta^2/2} g(|p-q|)\right]\right) \\ \le \overline{\mathcal{W}}_{\mu_V}(\bar{P}^{\mathrm{elec},\delta}) + C\eta^{\frac{d-s}{2}}\end{aligned} \tag{5.37}$$

and $\mathcal{N}(0,\tau)$ can be replaced by $\mathcal{N}(x,\tau)$ for any $x \in \mathbb{R}^d$ and C_1 by an average over C_R, by stationarity of $\bar{P}$ (cf. Lemma 2.8). Letting $\eta \to 0$ gives the result. ☐

5.2. **Regularization of point configurations.** The singularity of the interaction kernel g has been dealt with via the truncation procedure at the level η, which renormalizes the energy by truncating the short distance interactions. As $\eta \to 0$ the truncated energy converges for any configuration to the renormalized energy w_N (cf (2.30)). However to prove the Large Deviation Principle we need to have a *uniform* control on the error made by truncating. This is what allows us to obtain a conclusion of the type of Varadhan's Integral Lemma (e.g. [DZ10, Theorem 4.3.1.]). Note that this difficulty already appears for example in [BG99] at the leading order of the LDP. In view of Lemma 3.3 to control the truncation, we need to control pairs of η-close points. We will do this in two steps: by separating points by a minimum distance τ with $\tau < \eta^2/2$ and then by estimating the interaction of pairs of points whose distance is between τ and η via Lemma 3.11. Let us note that since the screening procedure already erases "bad" (non-screenable) configurations and replace them by configurations for which there is no pair of points at distance $\le \eta$ we only need to apply this regularization to screenable configurations.

5.2.1. *The regularization procedure.* When $l > 0$ is fixed, for any $\vec{i} \in l\mathbb{Z}^d$ by "the hypercube of center $\vec{i}$" we mean the closed hypercube of sidelength l of center $\vec{i}$ and whose edges are parallel to the axes of $\mathbb{Z}^d$, and we identify such a hypercube with its center. If $\vec{i} \in l\mathbb{Z}^d$ and $r > 0$ we let again $\mathcal{N}(\vec{i}, r)$ denote the number of points in the hypercube of sidelength $r > 0$ centered at $\vec{i}$.

The purpose of the following lemma is to "regularize" a point configuration by spacing out the points that are too close to each other, while remaining close to the original configuration. This operation generates a certain volume of configurations $\mathcal{C}^{\text{reg}}$ ("reg" as "regularized") for which we control the contribution of the energy due to pair of close points.

Lemma 5.11. *For any $\tau \in (0,1)$ and any hyperrectangle K whose sidelengths are in $[R, 2R]$ there exists a measurable multivalued function $\Phi^{\text{reg}}_{\tau,R}$ mapping $\mathcal{X}(K)$ into the set of subsets of $\mathcal{X}(K)$ such that any configuration $\mathcal{C}^{\text{reg}}$ in $\Phi^{\text{reg}}_{\tau,R}(\mathcal{C})$ has the same number of points as $\mathcal{C}$ and satisfies*

(1) The distance to the original configuration goes to zero when $\tau \to 0$ (uniformly on $\mathcal{C}^{\text{reg}}$)

$$\sup\{d_{\mathcal{X}}(\mathcal{C}, \mathcal{C}^{\text{reg}}) \mid \mathcal{C}^{\text{reg}} \in \Phi^{\text{reg}}_{\tau,R}(\mathcal{C})\} \underset{\tau \to 0}{\longrightarrow} 0.$$

(2) For any finite configuration $\mathcal{C}$ and any $\mathcal{C}^{\text{reg}} \in \Phi^{\text{reg}}_{\tau,R}(\mathcal{C})$ we have for any $\eta \geq 8\tau$

$$\sum_{x_i \neq x_j \in \mathcal{C}^{\text{reg}}, |x_i - x_j| \leq \eta} g(x_i - x_j) \leq C g(\tau) \Big(\sum_{\vec{i} \in 6\tau\mathbb{Z}^d} \Big(\mathcal{N}(\vec{i}, 12\tau)^2(\mathcal{C}) - 1 \Big)_+ + \sum_{x_i \neq x_j \in \mathcal{C}, \tau \leq |x_i - x_j| \leq 2\eta} g(x_i - x_j) \Big) \tag{5.38}$$

where C is a universal constant (depending only on d).

(3) For any integer n_K and any set A of configurations with n_K points, we have:

$$\log \mathbf{Leb}^{\otimes n_K} \Big(\bigcup_{\mathcal{C} \in A} \Phi^{\text{reg}}_{\tau,R}(\mathcal{C}) \Big) \geq \log \mathbf{Leb}^{\otimes n_K}(A) - C \fint_{\mathcal{C} \in A} \sum_{\vec{i} \in 6\tau\mathbb{Z}^d} \mathcal{N}(\vec{i}, 6\tau)(\mathcal{C}) \log \mathcal{N}(\vec{i}, 6\tau)(\mathcal{C}) \tag{5.39}$$

where C is a universal constant (depending only on d).

Proof. **Definition of the regularization procedure.**
For any $\tau > 0$ and $\mathcal{C} \in \mathcal{X}(K)$ we consider two categories of hypercubes in $6\tau\mathbb{Z}^d$:

- $S_\tau(\mathcal{C})$ is the set of hypercubes $\vec{i} \in 6\tau\mathbb{Z}^d$ such that $\mathcal{C}$ has at most one point in $\vec{i}$ and no point in the adjacent hypercubes.
- $T_\tau(\mathcal{C})$ is the set of the hypercubes that are not in $S_\tau(\mathcal{C})$ and that contain at least one point of $\mathcal{C}$.

We define $\varphi_\tau(\mathcal{C})$ to be the following configuration: the points of $\mathcal{C}$ that belong to some hypercube of $S_\tau(\mathcal{C})$ are left unchanged, whereas for any $\vec{i} \in T_\tau(\mathcal{C})$ we replace the configuration $\mathcal{C} \cap \vec{i}$ by a well-separated configuration in a smaller hypercube. More precisely we consider the lattice $3\mathcal{N}(\vec{i}, 6\tau)^{-1/d}\tau\mathbb{Z}^d$ translated so that the origin coincides with the point $\vec{i} \in 6\tau\mathbb{Z}^d$ and place $\mathcal{N}(\vec{i}, 6\tau)$ points on this lattice in such a way that they are all contained in the hypercube of sidelength 3τ and center $\vec{i}$ (a simple argument of volume shows that this is indeed possible, the precise way of disposing points is not important - it is easy to see that one may do it in a measurable fashion). This defines a measurable function $\varphi_{\tau,R} : \mathcal{X}(K) \to \mathcal{X}(K)$ (see Figure 2).

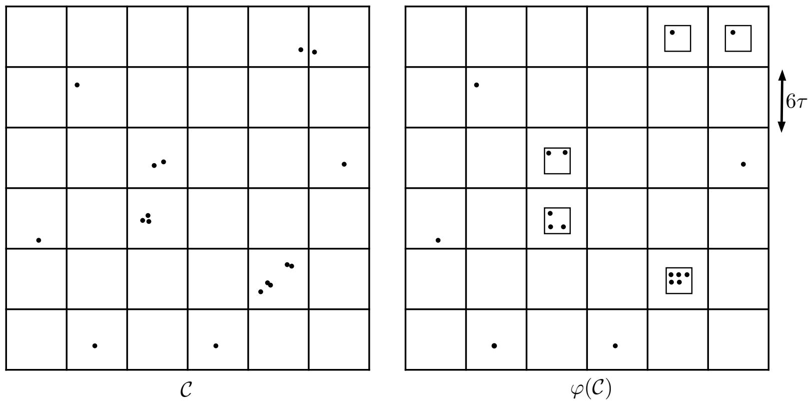

FIGURE 2. Effect of the regularization. On the right are shown the smaller hypercubes in which the new configurations are created for $\vec{i} \in T_\tau(\mathcal{C})$.

We then define $\Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C})$ to be the set of configurations that are obtained from $\mathcal{C}$ the following way: the points of $\mathcal{C}$ that belong to some hypercube of $S_\tau(\mathcal{C})$ are left unchanged and for any $\vec{i} \in T_\tau(\mathcal{C})$ we allow the points of $\varphi_{\tau,R}(\mathcal{C}) \cap \vec{i}$ to move arbitrarily (and independently) within a radius $\mathcal{N}(\vec{i}, 6\tau)^{-1/d}\tau$. We claim that $\Phi^{\mathrm{reg}}_{\tau,R}$ has the three desired properties.

A. Distance estimate.

The first claim of the lemma is easy to check: since for any $\mathcal{C}^{\mathrm{reg}} \in \Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C})$ the configurations $\mathcal{C}$ and $\mathcal{C}^{\mathrm{reg}}$ have the same number of points in every hypercube of $6\tau\mathbb{Z}^d$ it implies that every point of $\mathcal{C}$ is moved by a distance at most $O(\tau)$ (with a constant depending only on the dimension) which in view of the definition (2.39) of $d_{\mathcal{X}}$ yields $d_{\mathcal{X}}(\mathcal{C}, \mathcal{C}^{\mathrm{reg}}) = O(\tau)$ uniformly for $\mathcal{C}^{\mathrm{reg}} \in \Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C})$ (it really depends only on the number of points of $\mathcal{C}$ in K).

B. Truncation estimate.

To prove the second point let us distinguish three types of pairs of points $x_i, x_j \in \mathcal{C}^{\mathrm{reg}}$ which might satisfy $|x_i - x_j| \le \eta$:

(1) The pairs of points x_i, x_j belonging to some hypercube of $T_\tau(\mathcal{C})$.
(2) The pairs of points x_i, x_j belonging to two adjacent hypercubes of $T_\tau(\mathcal{C})$.
(3) The pairs of points x_i, x_j such that $|x_i - x_j| \le \eta$ but neither of the two previous cases holds.

To bound the contributions of the first type of pairs, let us observe that in any hypercube $\vec{i} \in T_\tau(\mathcal{C})$ the sum of pairwise interactions is bounded above by

$$\sum_{x_i \neq x_j \in \mathcal{C}^{\mathrm{reg}} \cap \vec{i}} g(x_i - x_j) \le C g(\tau) (\mathcal{N}(\vec{i}, 6\tau)^2(\mathcal{C}) - 1)_+. \tag{5.40}$$

Indeed by construction the point configuration $\mathcal{C}^{\mathrm{reg}}$ in a hypercube $\vec{i} \in T_\tau(\mathcal{C})$ consists in a subset of the lattice $3\mathcal{N}(\vec{i}, 6\tau)^{-1/d}\tau\mathbb{Z}^d$ where each point has been allowed to move within a ball of radius $\mathcal{N}(\vec{i}, 6\tau)^{-1/d}\tau$. The minimal distance between points is hence at least $2\mathcal{N}(\vec{i}, 6\tau)^{-1/d}\tau$ and moreover a simple combinatorial argument shows that for each point of $\mathcal{C}^{\mathrm{reg}}$ in this hypercube $\vec{i}$ there is $O(r^{d-1})$ other points at distance $r\mathcal{N}(\vec{i}, 6\tau)^{-1/d}\tau$ hence we have

$$\sum_{x_i \neq x_j \in \mathcal{C}^{\mathrm{reg}} \cap \vec{i}} g(x_i - x_j) = \mathcal{N}(\vec{i}, 6\tau)\ O\left(\int_2^{O\left(\mathcal{N}(\vec{i}, 6\tau)^{1/d}\right)} g(r\mathcal{N}(\vec{i}, 6\tau) - 1/d\tau) r^{d-1} dr \right),$$

which in turn yields (5.40)[2], since the sum is obviously zero when $\mathcal{N}(\vec{i}, 6\tau) = 1$.

For the second type of pairs, let us denote by $Z_{\tau,0}$ the set of $\vec{i} \in \mathbb{Z}^d$ whose coordinates belong to $\{0, 1\}$ and by $n_{\vec{i},2\tau,\vec{i_0}}$ the number of points in $\vec{i} + \vec{i_0}$ where $\vec{i} \in 12\tau\mathbb{Z}^d$. We notice that the points x_i and x_j belong to the same hypercube in either $12\tau\mathbb{Z}^d$ or one of the translates $12\tau\mathbb{Z}^d + 6\tau\vec{i_0}$ for some $\vec{i_0} \in Z_{\tau,0}$. Observing that the distance between the points in $\mathcal{C}^{\mathrm{reg}}$ which belong to different hypercubes is bounded below by 6τ, we find that the total contribution of the second type of pairs is bounded by

$$Cg(\tau) \sum_{\vec{i_0} \in Z_{\tau,0}} \sum_{\vec{i} \in 12\tau\mathbb{Z}^d} (n^2_{\vec{i},2\tau,\vec{i_0}} - 1)_+.$$

We may simplify the previous expression by extending the size of the hypercube in which we count the points and bound the total contribution of the second type of pairs by

$$Cg(\tau) \sum_{\vec{i} \in 6\tau\mathbb{Z}^d} \left(\mathcal{N}(\vec{i}, 12\tau)^2(\mathcal{C}) - 1\right)_+.$$

Finally the contribution of the third type of pairs is easily bounded by

$$\sum_{x_i, x_j \in \mathcal{C}, \tau \leq |x_i - x_j| \leq \eta + 8\tau} Cg(x_i - x_j),$$

indeed any such two points live in non-adjacent hypercubes hence were at distance $|x_i - x_j| \geq 12\tau$ in $\mathcal{C}$ and their distance is at worst reduced by 8τ during the regularization (then one discusses according to whether g is logarithmic or satisfies (1.4)).

C. Volume loss estimate.

Finally we turn to the volume consideration. The fibers of $\Phi^{\mathrm{reg}}_{\tau,R}$ have a simple description: we have $\Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C}) = \Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C}')$ only if $\mathcal{C}' \cap \vec{i} = \mathcal{C} \cap \vec{i}$ for $\vec{i} \in S_\tau(\mathcal{C})$ and $\mathcal{N}(\vec{i}, 6\tau)(\mathcal{C}') = \mathcal{N}(\vec{i}, 6\tau)(\mathcal{C})$ for $\vec{i} \in T_\tau(\mathcal{C})$ (these conditions are sufficient once symmetrized with respect to the roles of $\mathcal{C}$ and $\mathcal{C}'$). For a given configuration $\mathcal{C}$ in K with N points this describes a submanifold of $(\mathbb{R}^d)^N$ of co-dimension $\#S_\tau(\mathcal{C})$. The volume of a fiber is bounded by

$$\left(\sum_{\vec{i} \in T_\tau(\mathcal{C})} \mathcal{N}(\vec{i}, 6\tau)\right)! \left(\tau^d\right)^{\sum_{\vec{i} \in T_\tau(\mathcal{C})} \mathcal{N}(\vec{i}, 6\tau)}$$

whereas the volume of $\Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C})$ is given by

$$\left(\sum_{\vec{i} \in T_\tau(\mathcal{C})} \mathcal{N}(\vec{i}, 6\tau)\right)! \prod_{\vec{i} \in T_\tau(\mathcal{C})} \left(\frac{\tau}{\mathcal{N}(\vec{i}, 6\tau)^{1/d}}\right)^{\mathcal{N}(\vec{i}, 6\tau) d}$$

which after taking the logarithm yields the volume comparison of equation (5.39). □

[2] Another way of seeing (5.40) is to recall that according to the "first-order" results (1.6) the minimal energy of N points in a fixed compact set grows as N^2 and (5.40) follows by scaling.

5.2.2. *Effect on the energy.* We argue that the regularization procedure at scale τ has a negligible influence on the screened energy e.g. for configurations obtained by the screening procedure of Proposition 5.2. Let K and μ, $\mathcal{C}$ satisfying the assumptions in Proposition 5.2 and let $\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)$ be the set of configurations generated by the screening procedure of Proposition 5.2. For any $\mathcal{C}^{\mathrm{scr}}$ in $\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)$ let E^{scr} be the corresponding screened vector field (5.13). Let $\Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C}^{\mathrm{scr}})$ be the set of configurations generated by the regularization procedure applied to $\mathcal{C}^{\mathrm{scr}}$. The following holds

Lemma 5.12. *For any $\mathcal{C}^{\mathrm{reg}}$ in $\Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C}^{\mathrm{scr}})$ there exists a vector field $E^{\mathrm{reg}} \in L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ satisfying*

(1)

$$\begin{cases} -\mathrm{div}\,(|y|^\gamma E^{\mathrm{reg}}) = c_{d,s}\,(\mathcal{C}^{\mathrm{reg}} - \mu\delta_{\mathbb{R}^d}) & \text{in } K\times\mathbb{R}^k \\ E^{\mathrm{reg}}\cdot\vec{\nu} = 0 & \text{on } \partial K\times\mathbb{R}^k, \end{cases} \tag{5.41}$$

(2) Letting E^{reg}_η be associated to E^{reg} as in (2.27) we have

$$\int_{K\times\mathbb{R}^k} |y|^\gamma |E^{\mathrm{reg}}_\eta|^2 \le \left(\int_{K\times\mathbb{R}^k} |y|^\gamma |E^{\mathrm{scr}}_\eta|^2\right)(1+o_\tau(1)) \tag{5.42}$$

where the error term $o_\tau(1)$ goes to zero as $\tau\to 0$, depending only on η and R.

Proof. Let g^{Neu} be the unique solution with mean zero to

$$\begin{cases} -\mathrm{div}\,(|y|^\gamma \nabla g^{\mathrm{Neu}}) = c_{d,s}\left(\delta_0 - \frac{1}{|K|}\delta_{\mathbb{R}^d}\right) & \text{in } K\times\mathbb{R}^k \\ \nabla g^{\mathrm{Neu}}\cdot\vec{\nu} = 0 & \text{on } \partial K\times\mathbb{R}^k, \end{cases}$$

and let g^{Neu}_η be the truncated kernel at scale η as above. For any $\mathcal{C}^{\mathrm{scr}}$ in $\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C},\mu)$ and any $\mathcal{C}^{\mathrm{reg}}$ in $\Phi^{\mathrm{reg}}_{\tau,R}(\mathcal{C}^{\mathrm{scr}})$ let us consider the vector field $\tilde{E}$ generated by the difference $\mathcal{C}^{\mathrm{reg}} - \mathcal{C}^{\mathrm{scr}}$ with Neumann boundary conditions on ∂K

$$\tilde{E}(x) := \int \nabla g^{\mathrm{Neu}}(x-p)(\mathcal{C}^{\mathrm{reg}} - \mathcal{C}^{\mathrm{scr}})(p).$$

Since the regularization procedure preserves the number of points, it is clear that $E^{\mathrm{reg}} := E^{\mathrm{scr}} + \tilde{E}$ satisfies (5.41). To bound its energy we proceed as in the proof of Lemma 5.9 and it is enough to bound $\int_{\mathbb{R}^{k+d}} |y|^\gamma |\tilde{E}_\eta|^2$ by a $o_\tau(1)$. Integrating by parts we are left to bound

$$\iint g^{\mathrm{Neu}}_\eta(x-y)(\mathcal{C}^{\mathrm{reg}} - \mathcal{C}^{\mathrm{scr}})(x)(\mathcal{C}^{\mathrm{reg}} - \mathcal{C}^{\mathrm{scr}})(y).$$

By construction there is no point of $\mathcal{C}^{\mathrm{scr}}$ or $\mathcal{C}^{\mathrm{reg}}$ closer than some constant $\eta_0 > 0$ to ∂K, and g^{Neu}_η is uniformly continuous at distance $\ge \eta_0$ from ∂K. Moreover there is the same number of points in $\mathcal{C}^{\mathrm{reg}}$ and $\mathcal{C}^{\mathrm{scr}}$, this number is at most $C||\mu||_\infty R^d$, and the minimal connection distance between the points of $\mathcal{C}^{\mathrm{scr}}$ and $\mathcal{C}^{\mathrm{reg}}$ is then bounded by $C||\mu||_\infty R^d\tau$ because each point of $\mathcal{C}^{\mathrm{scr}}$ has been moved by a distance at most $C\tau$ during the regularization (see Item 1 of Lemma 5.11). We may then bound

$$\iint g^{\mathrm{Neu}}_\eta(x-y)(\mathcal{C}^{\mathrm{reg}} - \mathcal{C}^{\mathrm{scr}})(x)(\mathcal{C}^{\mathrm{reg}} - \mathcal{C}^{\mathrm{scr}})(y) = O(\tau)$$

with a $O(\tau)$ depending only on $R, d, ||\mu||_\infty$, but independent of $\mathcal{C}^{\mathrm{reg}}$ and $\mathcal{C}^{\mathrm{scr}}$. □

5.3. **Conclusion.** We may now combine the previous ingredients to accomplish the program stated at the beginning of the section.

For any point configuration $\mathcal{C}$ in $\mathcal{X}(C_R)$, any hyperrectangle K containing C_R and any bounded measure μ on K such that $n_{K,\mu} := \int_K \mu$ is an integer we define a family $\Phi^{\mathrm{mod}}(\mathcal{C}, \mu, K)$ (depending on the other parameters $\eta, \varepsilon, M, R, \tau$) of point configurations which are contained in K and have $n_{K,\mu}$ points the following way :

(1) If $(\mathcal{C}, \mu)$ **is** *screenable* i.e. is in $\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}$ then we let $\Phi^{\mathrm{mod}}(\mathcal{C}, \mu)$ be the image by $\Phi^{\mathrm{reg}}_{\tau,R}$ of the family $\Phi^{\mathrm{scr}}_{\varepsilon,\eta,R}(\mathcal{C}, \mu)$ of point configurations in K obtained by applying Proposition 5.2 to any electric field $E \in \mathcal{O}_{R,\eta,-}^{2M,2,\varepsilon}(\mathcal{C}, \mu)$ such that

$$\frac{1}{R^d} \int_{C_R \times [-R,R]^k} |y|^\gamma |E_\eta|^2 \leq F_{R,\eta}^{M,1,\varepsilon}(\mathcal{C}, \mu) + \varepsilon.$$

By Proposition 5.2 together with Lemma 5.12, to any of these point configurations is associated a *screened* and *regularized* electric field whose energy is bounded in terms of $\frac{1}{R^d} \int_{C_R \times \mathbb{R}^k} |y|^\gamma |E_\eta|^2$, hence in terms of $F_{R,\eta}^{M,1,\varepsilon}(\mathcal{C}, \mu) + \varepsilon$ as in (5.14) and (5.42).

(2) If $(\mathcal{C}, \mu)$ is **not** in $\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}$ then we let $\Phi^{\mathrm{mod}}(\mathcal{C}, \mu, K)$ be the family of configurations $\Phi^{\mathrm{gen}}(K, \mu)$ defined in Lemma 5.1. By the conclusions of Lemma 5.1, to any of these point configurations is associated an electric field whose energy is bounded as in (5.5) and which vanishes outside K.

Let us evaluate the effect of this operation on the volume of configurations i.e. we compare the volume of a certain set of configurations in C_R with the volume of the resulting configurations after applying Φ^{mod}. We distinguish between the cases of a set of *screenable* configurations and a set of *non-screenable* configurations.

Lemma 5.13. *Let R, K, μ be as above. Assume A is a (measurable) set of point configurations in $\mathcal{X}(C_R)$ such that each configuration of A has n points in C_R and n_{int} points in Int_ε.*

(1) If $(\mathcal{C}, \mu)$ is in $\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}$ for all $\mathcal{C} \in A$ and (5.17) holds then

$$\begin{aligned}(5.43)\quad \log \mathbf{Leb}^{\otimes n_{K,\mu}} \left(\bigcup_{\mathcal{C} \in A} \Phi^{\mathrm{mod}}(\mathcal{C}, \mu) \right) &\geq \log \mathbf{Leb}^{\otimes n}(A) \\ &+ \log \left((n_{K,\mu} - n_{int})! \left(\frac{c}{|\mathrm{Ext}_\varepsilon|} \right)^{n_{K,\mu} - n_{\mathrm{int}}} \right) + (n_{K,\mu} - n) \log |\mathrm{Ext}_\varepsilon| \\ &- C \fint_{\mathcal{C} \in A} \sum_{\vec{i} \in 6\tau\mathbb{Z}^d} \mathcal{N}(\vec{i}, 12\tau) \log \mathcal{N}(\vec{i}, 12\tau).\end{aligned}$$

(2) If $(\mathcal{C}, \mu)$ is not in $\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}$ for all $\mathcal{C} \in A$ then

$$(5.44)\quad \log \mathbf{Leb}^{\otimes n_{K,\mu}} \left(\bigcup_{\mathcal{C} \in A} \Phi^{\mathrm{mod}}(\mathcal{C}, \mu, K) \right) \geq \log \mathbf{Leb}^{\otimes n}(A) + \log \left((n_{K,\mu}! C^{n_{K,\mu}} |R^d|^{-n} \right).$$

Proof. The bound (5.43) follows from combining (5.18) with (5.39) whereas (5.44) follows directly from (5.6). □

6. Construction of configurations

This section is devoted to the proof of Proposition 1.7 by expliciting a set of compatible configurations with a large enough asymptotic (logarithmic) volume. To do so, we follow the strategy initiated in the previous section, i.e. first partition (some subset of) $\mathbb{R}^d$ into hyperrectangles K such that $\int_K \mu'_V$ is an integer (here and in the following we mostly deal with quantities defined at the blown-up scale $N^{1/d}$). Each hyperrectangle K will contain a hypercube translate of C_R such that $|K| - |C_R|$ is small and each hypercube will contain a point configuration. We want the global configurations (when considering all hyperrectangles together) to approximate (after averaging over translations) a given tagged point process $\bar{P}$. To do so, we will draw the point configurations in each hypercube jointly at random according to a (slightly modified) Poisson point process, and standard large deviations results will allow us to show that the correct ones end up occupying enough volume in phase space i.e. that sufficiently many of the (averaged) point configurations ressemble $\bar{P}$.

Then these configurations drawn "abstractly" at random using Sanov's theorem are modified as described in the previous section by screening-then-regularizing the parts for which it is possible to do so, and replacing the "bad" parts by "standard" configurations constructed by hand. This will allow to eventually obtain a global configuration with N points whose energy can be computed additively with respect to the hyperrectangles. At each step we need to check that the transformations imposed to the configurations do not alter much their phase-space volume, their energy, and keep them close to the given tagged process $\bar{P}$.

One of the additional technical difficulties is that the density of the equilibrium measure μ_V is in general not bounded from below near the boundary $\partial\Sigma$ and that its support Σ cannot be exactly tiled by hyperrectangles. To deal with this, we follow the construction made in [PS14] which consists in removing a thin layer near the boundary, and in placing in that layer some "frozen configuration" constructed by hand where the points are only free to move within small balls. We will later have to show again that the contributions to the energy and to the volume of this thin layer are negligible.

In the following we always assume that $\bar{P}$ is a stationary tagged point processes with finite energy $\overline{\mathbb{W}}_{\mu_V}(\bar{P})$ (otherwise Proposition 1.7 reduces to Proposition 1.6).

6.1. **Subdividing the domain.** We start the construction as in [PS14, Section 7] : we divide the domain between a neighborhood of the boundary, where the density is not bounded below and which must be treated "by hand", and a large interior. We recall that Σ is the support of the equilibrium measure μ_V. We let $\Sigma' = N^{1/d}\Sigma$ (which depends on N) be its blown-up and $\mu'_V(x') = \mu_V(N^{-1/d}x')$ the blow-up of the equilibrium measure, and recall that its density is bounded above by $\overline{m}$.

For convenience we recall the construction of [PS14, Section 7]. For $t > 0$ we define the tubular neighborhood of $\partial\Sigma'$ and its boundary to be

$$\Sigma'_t = \{x \in \Sigma', \mathrm{dist}(x, \partial\Sigma') > t\} \qquad \Gamma_t = \{x \in \Sigma', \mathrm{dist}(x, \partial\Sigma') = t\}.$$

Since (2.6) holds, Γ_t is C^1 for $t < t_c$ small enough.

Pick $1 > \underline{m} > 0$ a small number. By assumption (2.8), if $\alpha > 0$ in that assumption, rescaling by $N^{1/d}$, if $\mathrm{dist}(x, \partial\Sigma') \geq \frac{N^{1/d}}{c_1^{1/\alpha}} \underline{m}^{1/\alpha}$ where c_1 is the constant in (2.8), then $\mu'_V(x) \geq \underline{m}$. Thus we may find

$$T = T(N) \in \left[\frac{N^{1/d}}{c_1^{1/\alpha}} \underline{m}^{1/\alpha}, \frac{N^{1/d}}{c_1^{1/\alpha}} \underline{m}^{1/\alpha} + c \right], \tag{6.1}$$

for a constant c depending on $\underline{m}$, such that : $\mathcal{N}(T) := \mu'_V(\Sigma'_T) \in \mathbb{N}$, and $\mu'_V \geq \underline{m}$ in Σ'_T. We note that we may have taken $\underline{m}$ small enough so that $T < t_c$ and

$$\mathcal{H}^{d-1}(\partial\Sigma'_t) \leq 2\mathcal{H}^{d-1}(\partial\Sigma') = O(N^{\frac{d-1}{d}}) \quad \text{for all } t \leq T. \tag{6.2}$$

If $\alpha = 0$ in assumption (2.8) then μ'_V is bounded below by a positive constant on its support and we simply take $T = 0$ (of course (6.2) holds also in this case). By (6.1) the quantity $N^{-1/d}T$ tends to $\frac{\underline{m}^{1/\alpha}}{c_1^{1/\alpha}}$ as $N \to \infty$. In the following we let $r_{\underline{m}} := \frac{\underline{m}^{1/\alpha}}{c_1^{1/\alpha}}$, $\Sigma_{\underline{m}} := \{x \in \Sigma, \mathrm{dist}(x, \partial\Sigma) \geq r_{\underline{m}}\}$ and $\Sigma'_{\underline{m}} := \{x \in \Sigma', \mathrm{dist}(x, \partial\Sigma') \geq N^{1/d} r_{\underline{m}}\}$.

In the region Σ'_T we have the lower bound $\mu'_V \geq \underline{m}$ and there is no degeneracy. We now tile Σ'_T by hypercubes whose size is large but independent of N. The next lemma is a straightforward modification of [SS12a, Lemma 6.5].

Lemma 6.1 (Tiling the interior of the domain)**.** *There exists a constant $C_0 > 0$ depending on $\underline{m}, \overline{m}$ such that, given any $R > 1$, there exists for any $N \in \mathbb{N}^*$ a collection $\mathcal{K}_N$ of closed hyperrectangles in Σ'_T with disjoint interiors, whose sidelengths are between R and $R + C_0/R$, and which are such that*

$$\left\{x \in \Sigma'_T : d(x, \partial\Sigma'_T) \geq C_0 R\right\} \subset \bigcup_{K \in \mathcal{K}_N} K := \Sigma'_{\mathrm{int}}, \tag{6.3}$$

$$\forall K \in \mathcal{K}_N, \quad \int_K \mu'_V \in \mathbb{N}. \tag{6.4}$$

Moreover, an inspection of the proof allows us to observe that the hyperrectangles have their axes parallel to those of $\mathbb{R}^d$.

Let us enumerate the elements of $\mathcal{K}_N$ as $K_1, \ldots, K_{m_{N,R}}$ where $m_{N,R}$ is the number of hyperrectangles in $\mathcal{K}_N$. For any hyperrectangle K_i in $\mathcal{K}_N$ we denote by x_i the center of K_i and by $\overline{C}_i$ the closed hypercube of sidelength R contained in K_i whose center is x_i and whose axes are parallel to those of K_i, we also let $N_i := \int_{K_i} \mu'_V$ (which is an integer by construction). Since μ'_V is bounded above by $\overline{m}$ and below by $\underline{m}$ on Σ'_{int} and since K_i has its sidelengths in $[R, R + C_0/R]$ we have

$$C_1 R^d \leq N_i \leq C_2 R^d \tag{6.5}$$

with constants $C_1, C_2 > 0$ depending only on $\underline{m}, \overline{m}$. For any hypercube $\overline{C}_i$ we denote

$$\mathrm{Int}_{\varepsilon,i} := \{x \in \overline{C}_i \mid \mathrm{dist}(x, \partial\overline{C}_i) \geq 2\varepsilon R^d\}$$

as defined in (5.9) and by $\mathrm{Ext}_{\varepsilon,i}$ its complement in $\overline{C}_i$. We denote by $\mathcal{N}_i$ (resp. $\mathcal{N}_i^{\mathrm{int}}$) the function "number of points of a configuration in $\overline{C}_i$" (resp. in $\mathrm{Int}_{\varepsilon,i}$).

In the following lemma we collect some useful estimates about the quantities related to the tiling.

Lemma 6.2. *We have for any $R > 0$:*

(1)

$$\lim_{R\to\infty}\lim_{N\to\infty}\frac{R^d}{N}m_{N,R}=|\Sigma_{\underline{m}}|. \tag{6.6}$$

More precisely we have

$$\frac{R^d}{N}m_{N,R}=|\Sigma_{\underline{m}}|(1+o_{N\to\infty}(1))(1+O_{R\to\infty}(R^{-2})). \tag{6.7}$$

(2)

$$|K_i|=R^d+O(R^{d-2}) \tag{6.8}$$

$$N_i=R^d\mu_V'(x_i)+o_{N\to\infty}(1)R^d+O(R^{d-2}) \tag{6.9}$$

where $O(R^{d-2})$ depends only on $\underline{m},\overline{m}$.

(3)

$$|\mathrm{Ext}_{\varepsilon,i}|=R^d(1-(1-2\varepsilon)^d)=2d\varepsilon R^d+O(\varepsilon^2)R^d \tag{6.10}$$

where the $O(\varepsilon^2)$ depends only on ε.

Proof. From (6.3) we see that $|\Sigma'_T-\Sigma'_{\mathrm{int}}|$ is bounded by $|\{x\in\Sigma'_T,\mathrm{dist}(x,\partial\Sigma'_T)\le C_0R\}|$. From (6.2) we see that

$$|\{x\in\Sigma'_T,\mathrm{dist}(x,\partial\Sigma'_T)\le C_0R\}|=O(RN^{\frac{d-1}{d}})=o(|\Sigma'_T|)$$

because $|\Sigma'_T|$ is of order N. This implies

$$|\Sigma'_{\mathrm{int}}|\sim|\Sigma'_T| \text{ when } N\to\infty. \tag{6.11}$$

By construction the $m_{N,R}$ hyperrectangles partition Σ'_{int} and have sidelengths in $[R,R+C_0/R]$ hence the following holds

$$m_{N,R}R^d\le|\Sigma'_{\mathrm{int}}|\le m_{N,R}\left(R+\frac{C_0}{R}\right)^d, \tag{6.12}$$

in particular

$$m_{N,R}R^d=|\Sigma'_{\mathrm{int}}|\left(1+O_{R\to\infty}\left(R^{-2}\right)\right).$$

Moreover we have from (6.1) and by definition

$$\lim_{N\to\infty}\frac{|\Sigma'_T|}{N}=\lim_{N\to\infty}\frac{|\Sigma'_{\underline{m}}|}{N}=|\Sigma_{\underline{m}}|. \tag{6.13}$$

The three relations (6.11), (6.12), (6.13) easily yield (6.6) and (6.7).

The bound (6.8) holds because by construction the sidelengths of K_i are between R and $R+C_0/R$ with a constant C_0 depending only on $\underline{m}$). To get (6.9) we use the fact that $|K_i|=R^d+O(R^{d-2})$ and from the Hölder condition (2.7) we see that $\|\mu_V'(x)-\mu_V'(x_i)\|_{L^\infty(K_i)}$ tends to 0 as $N\to\infty$ (depending only on the size of K_i, hence on R and $\underline{m}$).

The bound (6.10) follows immediately from the definitions.

□

From now on, until Section 6.4 we work only in Σ'_{int} defined in (6.3), which we recall is a disjoint union of hyperrectangles (see Figure 3, where the region in grey corresponds to Σ'_{int}).

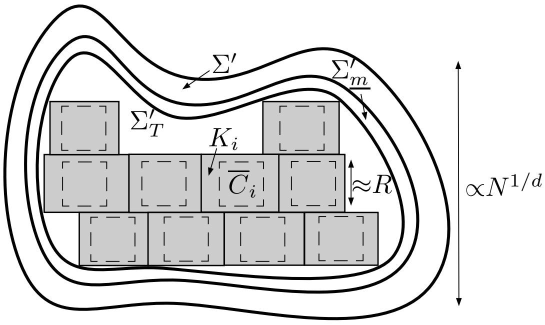

FIGURE 3. The tiling of Σ'.

6.2. **Generating approximating microstates.** This step is devoted to presenting an argument in the spirit of Sanov's theorem in order to generate "abstractly" a whole family of point configurations in Σ'_{int} whose continuous and discrete averages over translations are close to some fixed tagged point process. The proof of Lemma 6.3 follows the same line as the proof of Proposition 1.6 and is given in Section 7.

For any $\bar{P}$ in $\mathcal{P}_s(\Sigma \times \mathcal{X})$ we let $\bar{P}_{\underline{m}}$ be the tagged point process induced by restricting the "tag" coordinates to $\Sigma_{\underline{m}} \subset \Sigma$ i.e.

$$\bar{P}_{\underline{m}} := \frac{1}{|\Sigma_{\underline{m}}|} \int_{\Sigma_{\underline{m}}} \bar{P}^x dx.$$

Since $|\Sigma - \Sigma_{\underline{m}}| \to 0$ as $\underline{m} \to 0$, if F is a measurable function on $\Sigma \times \mathcal{X}$ such that F is $L^1(\bar{P})$, then

$$\int F d\bar{P}_{\underline{m}} \to \int F d\bar{P} \quad \text{as } \underline{m} \to 0$$

and this convergence is uniform for $F \in \text{Lip}_1(\Sigma \times \mathcal{X})$. This also implies that for any $r > 0$ we have

$$B(\bar{P}_{\underline{m}}, r/2) \subset B(\bar{P}, r) \subset B(\bar{P}_{\underline{m}}, 2r)$$

for $\underline{m}$ small enough (depending on r).

The following lemma says that the discrete space average as well as the continuum space average of randomly chosen configurations occupy a volume in $B(\bar{P}_m, \varepsilon)$ which is given by the entropy of $\bar{P}$. We will prove it in Section 7 together with Proposition 1.6.

Lemma 6.3. *Let $(\mathbf{C}_1, \dots, \mathbf{C}_{m_{N,R}})$ be $m_{N,R}$ independent Poisson point processes of intensity 1 on each hypercube $\overline{C}_i$ conditioned so that the total number of points is equal to*

$$N_{\text{int}} := \mu'_V(\Sigma'_{\text{int}}) = \sum_{i=1}^{m_{N,R}} N_i. \tag{6.14}$$

We define $\dot{\mathfrak{M}}_{N,R}$ as the law of the following random variable in $\Sigma'_{\text{int}} \times \mathcal{X}$:

$$\frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} \delta_{(N^{-1/d} x_i, \theta_{x_i} \cdot \mathbf{C}_i)}. \tag{6.15}$$

Moreover let $\mathbf{C}$ *be the point process obtained as the union of the point processes* $\mathbf{C}_i$ *i.e.*

$$\mathbf{C} := \sum_{i=1}^{m_{N,R}} \mathbf{C}_i$$

as a sum of random measures, and let us define $\widehat{\mathfrak{M}}_{N,R}$ *as the law of the random variable in* $\Sigma'_{\underline{m}} \times \mathcal{X}$

$$\frac{1}{N|\Sigma_{\underline{m}}|} \int_{\Sigma'_{\underline{m}}} \delta_{(N^{-1/d}x, \theta_x \cdot \mathbf{C})} dx.$$

Let us denote by $\bar{P}_{\underline{m},R}$ *the law induced by* $\bar{P}_{\underline{m}}$ *in the hypercube* C_R*, i.e. the push-forward of* $\bar{P}_{\underline{m}}$ *by the map* $(x, \mathcal{C}) \mapsto (x, \mathcal{C} \cap C_R)$*. Finally let*

$$r_{\mu_V, \underline{m}} := -\frac{\mu_V(\Sigma_{\underline{m}})}{|\Sigma_{\underline{m}}|} \log \frac{\mu_V(\Sigma_{\underline{m}})}{|\Sigma_{\underline{m}}|} + \frac{\mu_V(\Sigma_{\underline{m}})}{|\Sigma_{\underline{m}}|} - 1. \tag{6.16}$$

Then for any $\bar{P} \in \mathcal{P}_{s,1}(\Sigma \times \mathcal{X})$ *the following inequality holds :*

$$\liminf_{R\to\infty} \frac{1}{R^d} \lim_{\nu\to 0} \liminf_{N\to\infty} \frac{1}{m_{N,R}} \log \dot{\mathfrak{M}}_{N,R}(B(\bar{P}_{\underline{m},R}, \nu)) \geq -\fint_{\Sigma_{\underline{m}}} \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1] dx - r_{\mu_V,\underline{m}}, \tag{6.17}$$

moreover, for any $\delta > 0$ *we have*

$$\liminf_{R\to\infty} \frac{1}{R^d} \lim_{\nu\to 0} \liminf_{N\to\infty} \frac{1}{m_{N,R}} \log(\dot{\mathfrak{M}}_{N,R}, \widehat{\mathfrak{M}}_{N,R}) \left(B(\bar{P}_{\underline{m},R}, \nu) \times B(\bar{P}_{\underline{m}}, \delta)\right) \tag{6.18}$$
$$\geq -\fint_{\Sigma_{\underline{m}}} \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1] dx - r_{\mu_V,\underline{m}}$$

where by $(\dot{\mathfrak{M}}_{N,R}, \widehat{\mathfrak{M}}_{N,R})$ *we denote the joint law of* $\dot{\mathfrak{M}}_N$ *and* $\widehat{\mathfrak{M}}_N$ *(with the natural coupling).*

6.3. **Regularizing and screening microstates.** In this subsection we take the approximating microstates introduced in Lemma 6.3 and apply to them the screening-then-regularization procedure described in Section 5.3.

We obtain the following (recall N_{int} is defined in (6.14)) :

Lemma 6.4. *Let* $\bar{P} \in \mathcal{P}_{s,1}(\Sigma \times \mathcal{X})$*. Given* $\delta_1, \eta, \varepsilon, M, R, \tau, \nu, N$ *positive with* (5.8) *satisfied, there exists a set (depending on the parameters)* A^{mod} *of point configurations in* Σ'_{int} *which are of the form* $\mathcal{C}^{\text{mod}} = \sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\text{mod}}$ *where* $\mathcal{C}_i^{\text{mod}}$ *is a configuration in* K_i *and such that the following holds :*

(1) For any $\mathcal{C}^{\text{mod}}$ *in* A^{mod}*, if* η *is small enough,* ε *small enough,* R, M *large enough satisfying* (5.19)*,* τ, ν *small enough and* N *large enough then*

$$\frac{1}{|\Sigma'_{\underline{m}}|} \int_{\Sigma'_{\underline{m}}} \delta_{(N^{-1/d}x, \theta_x \cdot \mathcal{C}^{\text{mod}})} dx \in B(\bar{P}_{\underline{m}}, \frac{3\delta_1}{4}). \tag{6.19}$$

(2)

$$\lim_{\eta\to 0} \lim_{M,R\to\infty} \lim_{\tau\to 0} \lim_{\nu\to 0} \lim_{N\to\infty} \sup_{\mathcal{C}^{\text{mod}} \in A^{\text{mod}}} \frac{1}{N} \left(\sum_{x_i \neq x_j \in \mathcal{C}^{\text{mod}}, |x_i - x_j| \leq \eta} g(x_i - x_j) \right) = 0. \tag{6.20}$$

(3) For any $\mathcal{C}^{\mathrm{mod}} \in A^{\mathrm{mod}}$ there exists an electric field E^{mod} satisfying

$$\begin{cases} \operatorname{div}(|y|^\gamma E^{\mathrm{mod}}) = c_{d,s}\left(\mathcal{C}^{\mathrm{mod}} - \mu'_V \delta_{\mathbb{R}^d}\right) & \text{in } \Sigma'_{\mathrm{int}} \\ E^{\mathrm{mod}} \cdot \vec{\nu} = 0 & \text{on } \partial\Sigma'_{\mathrm{int}} \end{cases} \tag{6.21}$$

and

$$\limsup_{\varepsilon\to 0, M,R\to\infty,\tau,\nu\to 0, N\to\infty} \left(\frac{1}{|\Sigma'_{\mathrm{int}}|} \int_{\Sigma'_{\mathrm{int}}\times\mathbb{R}^k} |y|^\gamma |E^{\mathrm{mod}}_\eta|^2 - \int F^{M,1,\varepsilon}_{R,\eta} d\bar{P}_{\underline{m}} \right) \le 0. \tag{6.22}$$

(4) There is a good volume of such microstates

$$\liminf_{\varepsilon\to 0} \liminf_{R,M\to\infty} \liminf_{\tau,\nu\to 0} \liminf_{N\to\infty} \frac{1}{N_{\mathrm{int}}} \log \frac{\mathbf{Leb}^{N_{\mathrm{int}}}}{|\Sigma'_{\mathrm{int}}|^{N_{\mathrm{int}}}} \left(A^{\mathrm{mod}}\right) \ge -\int_{\Sigma_{\underline{m}}} \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1] - |\Sigma_{\underline{m}}| r_{\mu_V,\underline{m}}. \tag{6.23}$$

Proof. Let $\bar{P} \in \mathcal{P}_{s,1}(\Sigma \times \mathcal{X})$ of finite energy and $\delta_1 > 0$ be given (as in the statement of Proposition 1.7). For any δ and ν, let us write the conditions for a point configuration $\mathcal{C} := \sum_{i=1}^{m_{N,R}} \mathcal{C}_i$

$$\frac{1}{|\Sigma'_{\underline{m}}|} \int_{\Sigma'_{\underline{m}}} \delta_{(N^{-1/d}x, \theta_x \cdot \mathcal{C})} dx \in B(\bar{P}_{\underline{m}}, \delta) \tag{6.24}$$

and

$$\frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} \delta_{(N^{-1/d}x_i, \theta_{x_i}\cdot \mathcal{C}_i)} \in B(\bar{P}_{\underline{m},R}, \nu). \tag{6.25}$$

By Lemma 6.3 we know that given N, R, δ, ν there exists a set A^{abs} ("abs" as "abstract" because we generate them abstractly - and not by hand - using Sanov theorem as explained in the previous section) of configurations $\mathcal{C}^{\mathrm{abs}} = \sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\mathrm{abs}}$ (understood of a sum of measures) with N_{int} points, where $\mathcal{C}_i^{\mathrm{abs}}$ is a point configuration in the hypercube $\overline{C}_i$, such that

(6.26)

$$\liminf_{R\to\infty} \liminf_{\nu\to 0} \liminf_{N\to\infty} \frac{1}{m_{N,R}R^d} \log \frac{\mathbf{Leb}^{N_{\mathrm{int}}}}{(m_{N,R}R^d)^{N_{\mathrm{int}}}} \left(\{\mathcal{C}^{\mathrm{abs}} \in A^{\mathrm{abs}}, (6.24) \text{ and } (6.25) \text{ hold}\} \right) \ge -\fint_{\Sigma_{\underline{m}}} \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1] - r_{\mu_V,\underline{m}}.$$

To see how Lemma 6.3 yields (6.26) it suffices to note that the law of the N_{int}-points point process $\mathbf{C}$ of Lemma 6.3 coincides with the law of the point process induced by the N_{int}-th product of the normalized Lebesgue measure on $\cup_{i=1}^{m_{N,R}} \overline{C}_i$, and then (6.18) gives (6.26).

We let A^{mod} be the set of configurations obtained after applying the procedure described in Section 5. More precisely, for each $\mathcal{C}^{\mathrm{abs}}$ in A^{abs} we decompose $\mathcal{C}^{\mathrm{abs}}$ as $\sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\mathrm{abs}}$ where $\mathcal{C}_i^{\mathrm{abs}}$ is a point configuration in $\overline{C}_i$, and for any $i = 1 \dots m_{N,R}$ we let $\Phi_i^{\mathrm{mod}}(\mathcal{C}^{\mathrm{abs}})$ be the set of configurations obtained after screening-then-regularizing $\mathcal{C}_i^{\mathrm{abs}}$ by the map Φ^{mod} (in the following, for good definition, we have to translate back $\overline{C}_i$ and the other quantities by a vector x_i)

$$\Phi_i^{\mathrm{mod}}(\mathcal{C}) := \Phi^{\mathrm{mod}}(\theta_{x_i} \cdot \mathcal{C}_i^{\mathrm{abs}}, \mu'_V(x_i + \cdot), \theta_{x_i} \cdot K_i).$$

We then let $\overline{\Phi^{\mathrm{mod}}}(\mathcal{C}^{\mathrm{abs}})$ be the set of global configurations obtained as the product of the $\Phi_i^{\mathrm{mod}}(\mathcal{C}^{\mathrm{abs}})$

$$\overline{\Phi^{\mathrm{mod}}}(\mathcal{C}^{\mathrm{abs}}) := \prod_{i=1}^{m_{N,R}} \Phi_i^{\mathrm{mod}}(\mathcal{C}^{\mathrm{abs}})$$

and A^{mod} ("mod" as "modified") is finally defined as the image of A^{abs} by $\overline{\Phi^{\mathrm{mod}}}$.

Let us now check that A^{mod} satisfies the properties of items 1 to 4.

6.3.1. *Dealing with the variation of μ_V.*

Lemma 6.5. *We have for any $N, R, \underline{m}$ and $i \in 1 \dots m_{N,R}$*

$$\|\mu'_V(x_i) - \mu'_V\|_{L^\infty(\overline{C}_i)} \le C \left(\frac{R}{N^{1/d}}\right)^{\kappa} \tag{6.27}$$

for some $\kappa > 0$ with a constant C depending only on d, V and $\underline{m}$. In particular for any $M' > M$ and $e' > e$, for any $R, \varepsilon, \eta > 0$ we have for N large enough (depending on the other parameters but not on the configuration):

$$(\mathcal{C}_i, \mu'_V(x'_i)) \in \mathcal{S}_{R,\eta,+}^{M,e,\varepsilon} \Longrightarrow (\mathcal{C}_i, \mu'_V) \in \mathcal{S}_{R,\eta,+}^{M',e',\varepsilon}, \tag{6.28}$$

moreover if $(\mathcal{C}_i, \mu'_V(x'_i))$ and $(\mathcal{C}_i, \mu'_V)$ are in $\mathcal{S}_{R,\eta,+}^{M,e,\varepsilon}$ we have

$$F_{R,\eta}^{M',e',\varepsilon}(\mathcal{C}_i, \mu'_V) \le F_{R,\eta}^{M,e,\varepsilon}(\mathcal{C}_i, \mu'_V(x'_i))(1 + o(1)) + o(1) \tag{6.29}$$

where the terms $o(1)$ tend to zero when $N \to \infty$ depending only on R and $\underline{m}$ not on the configuration $\mathcal{C}_i$ nor on $i = 1 \dots m_{N,R}$.

Proof. The bound (6.27) follows immediatly from the Hölder assumption (2.7) on the density of μ_V and the definition of μ'_V as the blown-up quantity associated to μ_V. The two controls (6.28) and (6.29) follow then from Lemma 5.8, and the fact that they are uniform (independent of the configurations) results from the proofs of Lemmas 5.8 and 5.9 (the energy of the "difference electric field" can be expressed in terms of $\|\mu'_V(x_i) - \mu'_V\|$ and using the Cauchy-Schwarz inequality together with (6.27) is enough to conclude). □

An important consequence is the following : if $\mathcal{C}$ is a finite configuration whose discrete average (over translations) is close to $\bar{P}$, then most of the configurations in the discrete average are screenable. Indeed by construction, configurations in A^{abs} verify (6.24) and (6.25). In particular, combining Item 1 of Lemma 5.10 and Lemma 6.5 we see that

$$\lim_{M,R\to\infty} \lim_{\nu\to 0} \lim_{N\to\infty} \inf_{\mathcal{C}^{\mathrm{abs}}\in A^{\mathrm{abs}}} \frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} \delta_{(N^{-1/d}x, \mathcal{C}_i^{\mathrm{abs}})}(\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}) = 1. \tag{6.30}$$

6.3.2. *Distance to $\bar{P}_{\underline{m}}$.* To prove the first item of Lemma 6.4 we claim that the screening-then-regularizing procedure preserves the closeness of the continuous average to $\bar{P}_{\underline{m}}$ (however in general it does not preserve that of the discrete average). For that purpose we have to distinguish between hyperrectangles where the configuration is screenable (where the configuration is only modified in a thin layer or by moving points by a distance at most τ) and hyperrectangles where it is not (where the configuration is then completely modified).

Let $\mathcal{C}^{\text{mod}} = \sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\text{mod}}$ be in A^{mod} (where $\mathcal{C}_i^{\text{mod}}$ is the point configuration in the hyperrectangle K_i), we may find $\mathcal{C}^{\text{abs}} = \sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\text{abs}}$ in A^{abs} such that equation (6.24) holds and for any $i = 1 \dots m_{N,R}$

$$\mathcal{C}_i^{\text{mod}} \in \theta_{-x_i} \cdot \Phi_i^{\text{mod}}(\mathcal{C}_i^{\text{abs}})$$

i.e. $\mathcal{C}^{\text{mod}}$ has been obtained from $\mathcal{C}^{\text{abs}}$ by screening-then-regularizing.

We want to show that the continuous average

$$\frac{1}{|\Sigma'_{\underline{m}}|} \int_{\Sigma'_{\underline{m}}} \delta_{(N^{-1/d}x, \theta_x \cdot \mathcal{C}^{\text{mod}})} dx \in \mathcal{P}(\Sigma \times \mathcal{X})$$

satisfies (6.19).

We claim that we may evaluate the distance between the continuous averages of $\mathcal{C}^{\text{abs}}$ and $\mathcal{C}^{\text{mod}}$ in terms of the distance between the configurations in each hypercube K_i:

$$d_{\mathcal{P}(\Sigma\times\mathcal{X})} \left(\frac{1}{|\Sigma'_{\underline{m}}|} \int_{\Sigma'_{\underline{m}}} \delta_{(N^{-1/d}x, \theta_x \cdot \mathcal{C}^{\text{abs}})}, \frac{1}{|\Sigma'_{\underline{m}}|} \int_{\Sigma'_{\underline{m}}} \delta_{(N^{-1/d}x, \theta_x \cdot \mathcal{C}^{\text{mod}})} \right) \leq C \sum_{i=1}^{m_{N,R}} d_{\mathcal{X}(K_i)} \left(\mathcal{C}_i^{\text{abs}}, \mathcal{C}_i^{\text{mod}} \right) + \frac{\delta_1}{3} + o_{R,N\to\infty}(1) \tag{6.31}$$

for a certain constant C depending only on δ_1. The proof of (6.31) is elementary and we sketch it below.

First, from the approximation property of Lipschitz functions on $\mathcal{X}$ by local functions (see Lemma 2.5, Item 3) and the definition (2.37) of the distance on $\mathcal{P}(\Sigma \times \mathcal{X})$ we see that there exists $k \geq 0$ large enough such that for any tagged point process $\bar{Q}$ in $\mathcal{P}(\Sigma \times \mathcal{X})$, if $\bar{Q}_k$ denotes the push-forward of $\bar{Q}$ by $(x, \mathcal{C}) \mapsto (x, \mathcal{C} \cap C_k)$ (in other words, the point process induced on $\Sigma \times C_k$), we have

$$d_{\mathcal{P}(\Sigma\times\mathcal{X})}(\bar{Q}, \bar{Q}_k) \leq \frac{\delta_1}{6}. \tag{6.32}$$

This means that when comparing two point processes we can localize the configurations to some hypercube of fixed size up to a small uniform error and in the following we let k be an integer such that (6.32) holds. Hence in order to evaluate the distance between the two continuous averages we may reduce ourselves to evaluate the distance between their projection on $\mathcal{P}(\Sigma \times \mathcal{X}(C_k))$ up to an error $\frac{\delta}{6} + \frac{\delta}{6}$ according to (6.32). It amounts to testing the averages against Lipschitz functions $F \in \text{Lip}_1(\Sigma \times \mathcal{X})$ such that $F(x, \mathcal{C}) = F(x, \mathcal{C} \cap C_K)$ for any $(x, \mathcal{C})$. The continuous average (over translates in $\Sigma_{\underline{m}}$) of such a function F can be compared to its discrete average on the hypercubes K_i up to an error comparable to the fraction of the volume $\Sigma'_{\underline{m}}$ which is at distance less than k of $\Sigma'_{\underline{m}} \setminus \Sigma'_{\text{int}}$. By Lemma 6.2 we see that this fraction is $o(1)$ as $R, N \to \infty$.

Now for any $i = 1 \dots m_{N,R}$ we want to evaluate $d_{\mathcal{X}(K_i)} \left(\mathcal{C}_i^{\text{abs}}, \mathcal{C}_i^{\text{mod}} \right)$. We denote by I_1 the set of indices $i = 1 \dots m_{N,R}$ such that $(\mathcal{C}_i^{\text{abs}}, \mu'_V)$ is in $\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}$ and I_2 the set of indices $i = 1 \dots m_{N,R}$ such that $(\mathcal{C}_i^{\text{abs}}, \mu'_V)$ is **not** in $\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}$. Let us recall that the distance $d_{\mathcal{X}(K_i)}$ has been defined in (2.38) by testing against Lipschitz functions which are bounded by 1 in sup-norm. Consequently if $i \in I_2$ we have

$$d_{\mathcal{X}(K_i)} \left(\mathcal{C}_i^{\text{abs}}, \mathcal{C}_i^{\text{mod}} \right) \leq 2|K_i| \leq CR^d \tag{6.33}$$

which is the maximal distance between two configurations of $\mathcal{X}(K_i)$. On the other hand if $i \in I_1$ we have

$$d_{\mathcal{X}(K_i)}\left(\mathcal{C}_i^{\text{abs}}, \mathcal{C}_i^{\text{mod}}\right) \leq 2(|K_i| - |\overline{C}_i|) + 2(|\overline{C}_i| - |\text{Int}_\varepsilon^i|) + CR^d\tau, \tag{6.34}$$

where we denote $\text{Int}_{\varepsilon,i} := \{x \in \overline{C}_i \mid \text{dist}(x, \partial\overline{C}_i) \geq 2\varepsilon R^d\}$ and C is a constant. This is because the configurations $\mathcal{C}_i^{\text{abs}}$ and $\mathcal{C}_i^{\text{mod}}$ may differ completely on $K_i \backslash \text{Int}_\varepsilon^i$, but in Int_ε^i the screening procedure has not modified $\mathcal{C}_i^{\text{abs}}$ (according to Item 1 of Proposition 5.2) and the only modification is due to the regularization procedure which moves the points by at most $C\tau$ (and there are at most CR^d points in Int_ε^i). Using (6.10) and (6.8) we get

$$d_{\mathcal{X}(K_i)}\left(\mathcal{C}_i^{\text{abs}}, \mathcal{C}_i^{\text{mod}}\right) \leq CR^d(\varepsilon + \tau) + O(R^{d-2}). \tag{6.35}$$

Combining (6.31), (6.33) and (6.35) we have

$$\begin{aligned} d_{\mathcal{P}(\Sigma\times\mathcal{X})}\left(\frac{1}{|\Sigma'_{\underline{m}}|}\int_{\Sigma'_{\underline{m}}} \delta_{\left(N^{-1/d}x, \theta_x\cdot\mathcal{C}^{\text{abs}}\right)}, \frac{1}{|\Sigma'_{\underline{m}}|}\int_{\Sigma'_{\underline{m}}} \delta_{\left(N^{-1/d}x, \theta_x\cdot\mathcal{C}^{\text{mod}}\right)}\right) \\ \leq \frac{\#I_1}{m_{N,R}}\left(CR^d(\varepsilon+\tau) + O(R^{d-2})\right) + \frac{\#I_2}{m_{N,R}}CR^d. \end{aligned} \tag{6.36}$$

Using (6.30) we see that $\frac{\#I_2}{m_{N,R}} = o(1)$ and $\frac{\#I_1}{m_{N,R}} = 1 - o(1)$ when $M, R \to \infty, \nu \to 0, N \to \infty$ uniformly in A^{abs}. Combined with (6.36) it yields (6.19) when the parameters are sent to their limit as described in Item 1 of Lemma 6.4.

6.3.3. *Truncation error.* We turn to the second item of Lemma 6.4 which bounds the truncation error of the configuration that we construct.

Let $\mathcal{C}^{\text{mod}} = \sum_{i=1}^{m_{N,R}} \mathcal{C}_i^{\text{mod}}$ be in A^{mod} and let $\mathcal{C}^{\text{abs}}$ such that $\mathcal{C}^{\text{mod}}$ has been obtained from $\mathcal{C}^{\text{abs}}$ by screening-then-regularizing. By construction (see (5.11) in Item 2 of Proposition 5.2 in the case of a screenable configuration and Item 1 of Lemma 5.1 in the case of a non-screenable configuration) we have

$$\min_{i=1\ldots m_{N,R}} \min_{x\in\mathcal{C}_i^{\text{mod}}} \text{dist}(x, \partial K_i) \geq \eta_0$$

for some $\eta_0 > 0$ depending only on $d, \underline{m}, \overline{m}$. Therefore if η is small enough (depending only on $d, \underline{m}, \overline{m}$) the only pair of points in $\mathcal{C}^{\text{mod}}$ at distance less than η are included in some hyperrectangle K_i. We denote by I_1 the set of indices $i = 1 \ldots m_{N,R}$ such that $(\mathcal{C}_i^{\text{abs}}, \mu'_V)$ is in $\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}$ and I_2 the set of indices $i = 1 \ldots m_{N,R}$ such that $(\mathcal{C}_i^{\text{abs}}, \mu'_V)$ is not in $\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}$.

If $i \in I_2$ the configuration $\mathcal{C}_i^{\text{mod}}$ is by construction (see Lemma 5.1) made of points which are well-separated by the same constant η_0 hence there is no pair of points at distance less than η in K_i for $i \in I_2$ and for η small enough (depending only on $d, \underline{m}, \overline{m}$).

If $i \in I_1$ we know by construction that the only pair of points at distance less than η are in $\overline{C}_i$ (the points in $K_i \backslash \overline{C}_i$ are well-separated, see (5.12)). We may apply Item 2 of Lemma 5.11

to bound the truncation error in $\overline{C}_i$ in terms of the points of $\mathcal{C}_i^{\mathrm{abs}}$, for any $\eta \geq 8\tau$ it holds:

$$\sum_{x_i \neq x_j \in \mathcal{C}_i^{\mathrm{mod}}, |x_i - x_j| \leq \eta} g(x_i - x_j) \\ \leq Cg(\tau) \left(\Big(\sum_{\vec{i} \in 6\tau\mathbb{Z}^d} \mathcal{N}(\vec{i}, 12\tau)^2(\mathcal{C}_i^{\mathrm{abs}}) - 1 \Big)_+ + \sum_{x_i \neq x_j \in \mathcal{C}_i^{\mathrm{abs}}, \tau \leq |x_i - x_j| \leq 2\eta} g(x_i - x_j) \right)$$

where C is a universal constant (depending only on d). Since $i \in I_1$ the condition (5.22) in the definition of the screenability implies that $\mathcal{C}_i^{\mathrm{abs}}$ has at most MR^d points hence we may also write the previous equation as

$$\sum_{x_i \neq x_j \in \mathcal{C}_i^{\mathrm{mod}}, |x_i - x_j| \leq \eta} g(x_i - x_j) \leq Cg(\tau) \Big(\sum_{\vec{i} \in \tau\mathbb{Z}^d} (\mathcal{N}(\vec{i}, 12\tau)^2(\mathcal{C}_i^{\mathrm{abs}}) - 1)_+ \wedge M^2 R^{2d}) \\ + \sum_{x_i \neq x_j \in \mathcal{C}_i^{\mathrm{abs}}, \tau \leq |x_i - x_j| \leq 2\eta} g(x_i - x_j) \wedge MR^d g(\tau) \Big). \tag{6.37}$$

This re-writing is only technical, and meant to replace the functions depending on the number of points by bounded functions, which can now be tested against the convergence of point processes.

In view of Item 3 of Lemma 5.10, setting $\bar{Q} = \frac{1}{m_{N,R}} \sum_{i=1}^{m_{N,R}} \delta_{(N^{-1/d} x_i, \mathcal{C}_i^{\mathrm{abs}})}$ for any $\mathcal{C}^{\mathrm{abs}} \in A^{\mathrm{abs}}$ (which is by assumption included in $B(\bar{P}_{\underline{m}}, \nu)$) we have for any η, τ, with $0 < \tau < \eta^2/2 < 1$

$$\limsup_{\eta \to 0} \limsup_{\tau \to 0} \limsup_{\nu \to 0} \sup_{\mathcal{C}^{\mathrm{abs}} \in A^{\mathrm{abs}}} \Big[\frac{g(2\tau)}{\tau^d} \mathbf{E}_{\bar{Q}}[(\mathcal{N}(x, \tau)^2 - 1)_+ \wedge M^2 R^{2d}] \\ + \frac{1}{R^d} \mathbf{E}_{\bar{Q}} \Big(\Big(\sum_{p \neq q \in \mathcal{C} \cap C_R, \tau \leq |p-q| \leq \eta^2/2} g(|p - q|) \Big) \wedge MR^d g(\tau) \Big) \Big] = 0.$$

In particular both of the expectations must go to zero. Writing $\bar{Q}$ explicitly, this implies that

$$\lim_{\eta \to 0} \lim_{M,R \to \infty} \lim_{\tau \to 0} \lim_{\nu \to 0} \lim_{N \to \infty} \sup_{\mathcal{C}^{\mathrm{abs}} \in A^{\mathrm{abs}}} \frac{1}{N} \sum_{i=1}^{m_{N,R}} Cg(\tau) \Big(\sum_{\vec{i} \in \tau\mathbb{Z}^d} (\mathcal{N}(\vec{i}, 12\tau)^2(\mathcal{C}_i^{\mathrm{abs}}) - 1)_+ \wedge M^2 R^{2d}) \\ + \sum_{x_i \neq x_j \in \mathcal{C}_i^{\mathrm{abs}}, \tau \leq |x_i - x_j| \leq 2\eta} g(x_i - x_j) \wedge MR^d g(\tau) \Big) = 0$$

which when combined with (6.37) proves (6.20) because the sum on $i = 1, \ldots, m_{N,R}$ bounds of course the sum on $i \in I_1$.

6.3.4. *Energy.* We want to control the energy of electric fields associated to the configurations in A^{mod}.

First we associate to any $\mathcal{C}^{\mathrm{mod}} \in A^{\mathrm{mod}}$ a screened electric field E^{mod} satisfying (6.21). As explained in Section 5.3 we know by definition that for $\mathcal{C}^{\mathrm{mod}} \in A^{\mathrm{mod}}$, for any $i = 1 \ldots m_{N,R}$ there exists an vector field E_i^{mod} such that

$$\begin{cases} \operatorname{div}(|y|^\gamma E_i^{\mathrm{mod}}) = c_{d,s}(\mathcal{C}_i^{\mathrm{mod}} - \mu'_V \delta_{\mathbb{R}^d}) & \text{in } K_i \times \mathbb{R}^k \\ E_i^{\mathrm{mod}} \cdot \vec{\nu} = 0 & \text{on } \partial K_i \times \mathbb{R}^k \end{cases}$$

Setting $E^{\mathrm{mod}} = \sum_i E_i^{\mathrm{mod}} \mathbf{1}_{K_i \times \mathbb{R}^k}$ provides the vector field mentioned in Item 3 of Lemma 6.4 which satisfies (6.21).

We now turn to bound its energy, refering again to Section 5.3. Let $\mathcal{C}^{\mathrm{abs}} \in A^{\mathrm{abs}}$ be such that $\mathcal{C}^{\mathrm{mod}}$ is obtained from $\mathcal{C}^{\mathrm{abs}}$ after screening/regularizing. We denote again by I_1 the set of indices $i = 1 \dots m_{N,R}$ such that $(\mathcal{C}_i^{\mathrm{abs}}, \mu'_V)$ is in $\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}$ and I_2 the set of indices $i = 1 \dots m_{N,R}$ such that $(\mathcal{C}_i^{\mathrm{abs}}, \mu'_V)$ is not in $\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}$.

For $i \in I_1$ the energy is bounded as in (5.14) (after screening) and (5.42) (after regularization). For $i \in I_2$ it is bounded as in (5.5).

At this point, we insert the information on μ'_V provided by (2.7) with (2.9). This ensures that

$$\left\| \mu'_V - \fint_{K_i} \mu'_V \right\|_{L^\infty(K_i)} \le C R^\kappa N^{-\kappa/d}$$

for a constant depending only on μ_V. Moreover, we have $0 \le |K_i| - |\overline{C}| = O(R^{d-2})$ as stated in (6.8). Inserting these estimates into the bounds of (5.5) and (5.14), and combining with (5.42) in the case $i \in I_1$ we find

$$(6.38)\quad \frac{1}{R^d} \int_{\Sigma'_{\mathrm{int}} \times \mathbb{R}^k} |y|^\gamma |E_\eta^{\mathrm{mod}}|^2 \le \Big(\sum_{i \in I_1} (F_{R,\eta}^{M,1,\varepsilon}(\mathcal{C}_i^{\mathrm{abs}}, \mu'_V) + \varepsilon)(1 + C\varepsilon) + C g(\eta)((1+M)\varepsilon + o_R(1)) + Ce\varepsilon + o_N(1) + \frac{C}{R^d} \sum_{i \in I_2} N_i g(\eta) \Big)(1 + o_\tau(1))$$

where the term $o_N(1)$ tends to 0 as $N \to \infty$, keeping the other parameters fixed, $o_R(1)$ tends to 0 as $R \to \infty$ independently of the other parameters and $o_\tau(1)$ is as in (5.42).

Using Lemma 6.5 together with the upper semi-continuity of $F_{R,\eta}^{M,1,\varepsilon} \mathbf{1}_{\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}}$ we obtain

$$(6.39)\quad \limsup_{\varepsilon \to 0} \limsup_{M,R \to \infty} \limsup_{\tau,\nu \to 0} \limsup_{N \to \infty} \frac{1}{m_{N,R}} \sum_{i \in I_1} F_{R,\eta}^{M,1,\varepsilon}(\mathcal{C}_i^{\mathrm{abs}}, \mu'_V(x_i)) \le \int \mathbf{1}_{\mathcal{S}_{R,\eta,+}^{M,1,\varepsilon}} F_{R,\eta}^{M,1,\varepsilon} d\bar{P}_{\underline{m}}.$$

Moreover by (6.41) again and $N_i = O(R^d)$, the term $\frac{C}{R^d} \sum_{i \in I_2} N_i g(\eta)$ is $o(m_{N,R})$ when η is fixed. Let us also recall that $m_{N,R} \approx \frac{N}{R^d}$ (cf. (6.6)).

Combining (6.38) and (6.39) we get (6.22). ☐

6.3.5. *Control on the volume loss.* We now wish to bound the volume loss between the set A^{abs} of microstates generated "abstractly" and the set A^{mod} of configurations obtained after modification by the screening-and-regularizing procedure.

For each configuration $\mathcal{C}^{\mathrm{abs}}$ in A^{abs} we keep the distinction between $i \in I_1$ and $i \in I_2$ as above. From Lemma 5.13 we see that the difference of volume between A^{mod} and A^{abs} is bounded below as follows

$$(6.40)\quad \log \mathbf{Leb}^{\otimes N_{\mathrm{int}}} \left(A^{\mathrm{mod}}\right) - \log \mathbf{Leb}^{\otimes N_{\mathrm{int}}} \left(A^{\mathrm{abs}}\right) \ge \fint_{\mathcal{C}^{\mathrm{abs}} \in A^{\mathrm{abs}}} \sum_{i \in I_1} \log \left((N_i - \mathcal{N}_i^{\mathrm{int}})! \left(\frac{c}{|\mathrm{Ext}_\varepsilon|} \right)^{N_i - \mathcal{N}_i^{\mathrm{int}}} \right) + (N_i - \mathcal{N}_i) \log |\mathrm{Ext}_\varepsilon| - C \sum_{i \in I_1} \sum_{\vec{i} \in 6\tau\mathbb{Z}^d} \mathcal{N}(\vec{i}, 6\tau) \log \mathcal{N}(\vec{i}, 6\tau) + \sum_{i \in I_2} \log \left(N_i! C^{N_i} |\overline{C}_i|^{-\mathcal{N}_i} \right) d\mathcal{C}.$$

We note that from (6.30), we have

$$\lim_{M,R\to\infty}\lim_{\nu\to 0}\lim_{N\to\infty}\frac{\#I_1}{m_{N,R}}=1 \qquad \limsup_{M,R\to\infty}\lim_{\nu\to 0}\lim_{N\to\infty}\frac{\#I_2}{m_{N,R}}=0. \tag{6.41}$$

In order to apply Lemma 5.13 however we need to check that the condition (5.17) holds for $i\in I_1$ i.e. that

$$N_i-\mathcal{N}_i^{\mathrm{int}}\le\frac{|\mathrm{Ext}_\varepsilon|}{2c}, \tag{6.42}$$

is satisfied for any $i\in I_1$, which can be achieved by taking c small enough. Indeed we have $N_i\le CR^d$ with a constant C depending only on $\underline{m},\overline{m}$ as observed in (6.5). Hence up to changing c in (6.40) into

$$c_1=\min(c_0\varepsilon,c) \tag{6.43}$$

where c_0 depends only on d, we see that we can always satisfy (6.42).

The integrand in (6.40) may be bounded below using Stirling's estimate as follows

$$\begin{aligned}\sum_{i\in I_1}\log\left((N_i-\mathcal{N}_i^{\mathrm{int}})!\left(\frac{c_1}{|\mathrm{Ext}_\varepsilon|}\right)^{N_i-\mathcal{N}_i^{\mathrm{int}}}\right)+(N_i-\mathcal{N}_i)\log|\mathrm{Ext}_\varepsilon| \\ \ge\sum_{i\in I_1}(N_i-\mathcal{N}_i^{\mathrm{int}})\log(N_i-\mathcal{N}_i^{\mathrm{int}})-(N_i-\mathcal{N}_i^{\mathrm{int}})-(N_i-\mathcal{N}_i^{\mathrm{int}})\log|\mathrm{Ext}_\varepsilon| \\ +(N_i-\mathcal{N}_i)\log|\mathrm{Ext}_\varepsilon|+(N_i-\mathcal{N}_i^{\mathrm{int}})\log c_1-C\sum_{\vec{i}\in 6\tau\mathbb{Z}^d}\mathcal{N}(\vec{i},6\tau)\log\mathcal{N}(\vec{i},6\tau)\end{aligned} \tag{6.44}$$

(with c_1 as in (6.43)) on the one hand, and on the other hand

$$\sum_{i\in I_2}\log\left(N_i!C^{N_i}|\overline{C}_i|^{-\mathcal{N}_i}\right)\ge\sum_{i\in I_2}N_i\log N_i-\mathcal{N}_i\log|\overline{C}_i|-N_i(1-\log C). \tag{6.45}$$

We now turn to studying the terms in (6.44), (6.45) which relies on estimating the quantities $\mathcal{N}_i$ and $\mathcal{N}_i^{\mathrm{int}}$.

Let $\mathcal{D}^{\mathrm{int}}(x,\mathcal{C})$ be the discrepancy quantity $\mathcal{N}^{\mathrm{int}}(\mathcal{C})-\mu_V(x)|\mathrm{Int}_\varepsilon|$. If $i\in I_1$ the quantity $\mathcal{D}^{\mathrm{int}}(x_i,\mathcal{C}_i):=\mathcal{N}_i^{\mathrm{int}}-\mu'_V(x_i)|\mathrm{Int}_\varepsilon|$ is bounded since the uniform bound (5.22) on the number of points holds for $i\in I_1$. We may then pass to the limit $\nu\to 0$ using (6.25)

$$\begin{aligned}\lim_{\nu\to 0}\frac{1}{m_{N,R}}\sum_{i\in I_1}\left|\mathcal{N}_i^{\mathrm{int}}-\mu'_V(x_i)|\mathrm{Int}_\varepsilon|\right|&\le\int\left|\mathcal{D}^{\mathrm{int}}(x,\mathcal{C})\right|d\bar{P}_{\underline{m},R}(x,\mathcal{C}) \\ &=\fint_{\Sigma_{\underline{m}}}dx\int\left|\mathcal{D}^{\mathrm{int}}(x,\mathcal{C})\right|d\bar{P}^x_R(\mathcal{C}).\end{aligned} \tag{6.46}$$

The discrepancy estimates of Lemma 3.10, more precisely (3.14), show that

$$\int_{x\in\Sigma_{\underline{m}}}dx\int\left|\mathcal{D}^{\mathrm{int}}(x,\mathcal{C})\right|d\bar{P}^x_R(\mathcal{C})=O(R^{\frac{1}{2}(d+s)})$$

as $R\to\infty$ with a bound depending only on $\bar{P}$. Inserting the previous estimate in (6.46) we obtain that (since $s<d$)

$$\lim_{\nu\to 0}\frac{1}{m_{N,R}}\sum_{i\in I_1}\left(\mathcal{N}_i^{\mathrm{int}}-\mu'_V(x_i)|\mathrm{Int}_\varepsilon|\right)=O(R^{\frac{1}{2}(d+s)})=o(R^d) \tag{6.47}$$

as $R \to \infty$ with a $o(R^d)$ depending only on $\bar{P}$. Arguing similarly we also get

$$\lim_{\nu\to 0}\left(\frac{1}{m_{N,R}}\sum_{i\in I_1}\mu_V'(x_i)R^d - \mathcal{N}_i\right) = O(R^{d-\frac{1}{2}(d-s)}), \tag{6.48}$$

The same also holds for I_2 and using the fact that $m_{N,R} \approx \frac{N}{R^d}$ we get

$$\lim_{\nu\to 0}\left(\sum_{i\in I_2}\mu_V'(x_i)R^d - \mathcal{N}_i\right) = O(NR^{-\frac{1}{2}(d-s)}). \tag{6.49}$$

Next, from (6.9) and the definition of Int, Ext, we may write

$$N_i = \mu_V'(x_i)|\mathrm{Int}_\varepsilon| + \mu_V'(x_i)|\mathrm{Ext}_\varepsilon| + o_{N\to\infty}(1)R^d + O(R^{d-2}),$$

hence, in view of (6.47) we have in the limit $\nu \to 0$ (depending on R)

$$\frac{1}{m_{N,R}}\sum_{i\in I_1}\frac{(N_i - \mathcal{N}_i^{\mathrm{int}})}{|\mathrm{Ext}_\varepsilon|} = \frac{1}{|\mathrm{Ext}_\varepsilon|}\Big(o(R^d) + o_{N\to\infty}(1)R^d\Big) + \frac{1}{m_{N,R}}\sum_{i\in I_1}\mu_V'(x_i). \tag{6.50}$$

Since $\int \mu_V(x) = 1$ by definition and since from (6.41) we have $\#I_1 \approx m_{N,R}$ it holds that

$$\frac{1}{m_{N,R}}\sum_{i\in I_1}\mu_V'(x_i) = 1 + o(1)$$

as $\underline{m} \to 0$, $M, R \to \infty$, $\nu \to 0$. Moreover when ε is fixed and $R, N \to \infty$ we have by (6.10)

$$\frac{1}{|\mathrm{Ext}_\varepsilon|}\Big(o(R^d) + o_{N\to\infty}(1)R^d\Big) = o(1).$$

Finally we get that

$$\lim_{\underline{m}\to 0}\lim_{\varepsilon\to 0}\lim_{M,R\to\infty}\lim_{\nu\to 0}\lim_{N\to\infty}\sup_{\mathcal{C}\in A^{\mathrm{abs}}}\frac{1}{\#I_1}\sum_{i\in I_1}\frac{(N_i - \mathcal{N}_i^{\mathrm{int}})}{|\mathrm{Ext}_\varepsilon|} = 1 + o(1). \tag{6.51}$$

We may now write using Jensen's inequality

$$\sum_{i\in I_1}(N_i - \mathcal{N}_i^{\mathrm{int}})\log(N_i - \mathcal{N}_i^{\mathrm{int}}) - (N_i - \mathcal{N}_i^{\mathrm{int}})\log|\mathrm{Ext}_\varepsilon|$$
$$\geq \#I_1|\mathrm{Ext}_\varepsilon|\left(\frac{1}{\#I_1}\sum_{i\in I_1}\frac{(N_i - \mathcal{N}_i^{\mathrm{int}})}{|\mathrm{Ext}_\varepsilon|}\right)\log\left(\frac{1}{\#I_1}\sum_{i\in I_1}\frac{(N_i - \mathcal{N}_i^{\mathrm{int}})}{|\mathrm{Ext}_\varepsilon|}\right).$$

and using (6.51) together with (6.6) and (6.10) we get that

$$\sum_{i\in I_1}(N_i - \mathcal{N}_i^{\mathrm{int}})\log(N_i - \mathcal{N}_i^{\mathrm{int}}) - (N_i - \mathcal{N}_i^{\mathrm{int}})\log|\mathrm{Ext}_\varepsilon| \geq -C\varepsilon N \tag{6.52}$$

for some constant $C > 0$ depending only on the dimension and $\bar{P}$. This settles the first terms of the right-hand side in (6.44). For the next one, in view of (6.48) we have

$$\lim_{\nu\to 0}\log|\mathrm{Ext}_{\varepsilon,R}|\left(\frac{1}{\#I_1}\sum_{i\in I_1}\mu_V'(x_i)R^d - \mathcal{N}_i\right) = o(R^d)$$

which together with (6.9) yields that $\sum_{i\in I_1}(N_i - \mathcal{N}_i)\log|\mathrm{Ext}_\varepsilon|$ is a $o(N)$ when $M, R \to \infty$ and $N \to \infty$.

Similarly we may write that

$$\frac{1}{\#I_1}\sum_{i\in I_1}(N_i - \mathcal{N}_i^{\mathrm{int}}) = \frac{1}{\#I_1}\sum_{i\in I_1}\left((N_i - \mathcal{N}_i) + \mathcal{N}_i^{\mathrm{ext}}\right) = o(R^d) + \frac{1}{\#I_1}\sum_{i\in I_1}\mathcal{N}_i^{\mathrm{ext}}$$

and arguing as above we see that

$$\frac{1}{\#I_1}\sum_{i\in I_1}\mathcal{N}_i^{\mathrm{ext}} \le \overline{m}|\mathrm{Ext}_\varepsilon| + o(R^d),$$

which together with the choice (6.43) of c_1 yields that

$$\sum_{i\in I_1}(N_i - \mathcal{N}_i^{\mathrm{int}})(\log c_1 - 1) \ge -CN\varepsilon|\log\varepsilon|$$

with a constant C depending only on $\bar{P}, d, \overline{m}$. Concerning the sum on $i \in I_1$ we are left to control the terms $\sum_{\vec{i}\in 6\tau\mathbb{Z}^d}\mathcal{N}(\vec{i}, 6\tau)\log\mathcal{N}(\vec{i}, 6\tau)$, which are treated as in Section 6.3.3: they are uniformly bounded because of (5.22) which allows us to take the limit $\nu \to 0$ and Item 3 of Lemma 5.30 together with the trivial bound $n\log n \le (n^2-1)_+$ ensures that

$$\frac{1}{m_{N,R}}\sum_{i\in I_1}\sum_{\vec{i}\in 6\tau\mathbb{Z}^d}\mathcal{N}(\vec{i}, 6\tau)\log\mathcal{N}(\vec{i}, 6\tau) = o(1)$$

as $M, R \to \infty, \tau \to 0, \nu \to 0, N \to \infty$.

We now treat the terms for $i \in I_2$. In order to control (6.45) we follow the same line as above. First we decompose the difference as

$$N_i\log N_i - \mathcal{N}_i\log|\bar{C}_i| = (N_i - \mathcal{N}_i)\log R^d + N_i\log\frac{N_i}{R^d}.$$

The sum of $(N_i - \mathcal{N}_i)\log R^d$ is bounded as above thanks to (6.49) and (6.9) which yield

$$\sum_{i\in I_2}(N_i - \mathcal{N}_i)\log R^d = o(N) \text{ in the limit } R \to \infty, \tau \to 0, N \to \infty.$$

On the other hand, the second term is bounded using (6.5)

$$\sum_{i\in I_2}N_i\log\frac{N_i}{R^d} \ge -C\#I_2R^d$$

and with (6.41) we finally get that

$$\sum_{i\in I_2}N_i\log N_i - \mathcal{N}_i\log|\bar{C}_i| = o(N) \text{ in the limit } M, R \to \infty, \tau \to 0, N \to \infty.$$

The last term $\sum_{i\in I_2}N_i(1-\log C)$ is easily bounded because $N_i \le CR^d$ and $\#I_2 = o(m_{N,R})$ hence

$$\sum_{i\in I_2}N_i(1-\log C) = o(N).$$

Inserting all these controls into (6.44), (6.45), using (6.6) and the fact that $|\Sigma_{\underline{m}}| \to |\Sigma|$ as $\underline{m} \to 0$, we get from (6.40) that

$$\liminf_{\underline{m}\to 0} \liminf_{\varepsilon\to 0} \liminf_{M,R\to\infty} \liminf_{\tau\to 0} \liminf_{\nu\to 0} \liminf_{N\to\infty} \frac{1}{N}\left(\log \mathbf{Leb}^{N_{\mathrm{int}}}\left(A^{\mathrm{mod}}\right) - \log \mathbf{Leb}^{N_{\mathrm{int}}}\left(A^{\mathrm{abs}}\right)\right) \geq 0. \tag{6.53}$$

Combining (6.53) with (6.26) and (6.6) we conclude that (6.23) holds.

6.4. **Completing the construction and conclusion.** Once the construction has been made in Σ'_{int}, there remains to complete it in the thin layer $\Sigma' \backslash \Sigma'_{\mathrm{int}}$ by placing "frozen" points there. That precise construction was already made in [PS14, Proposition 7.3, Step 3], where the following is shown :

Lemma 6.6. *There exists a family A^{ext} depending on N, R, η, of point configurations with $N - N_{\mathrm{int}}$ points in $\Sigma' \backslash \Sigma'_{\mathrm{int}}$ and which satisfy the following.*

(1) For any $\mathcal{C}^{\mathrm{ext}}$ in A^{ext}, the distance between two points of $\mathcal{C}^{\mathrm{ext}}$ or between a point of $\mathcal{C}^{\mathrm{ext}}$ and the $\partial(\Sigma' \backslash \Sigma'_{\mathrm{int}})$ is bounded below by $\eta_0 > 0$ depending only on d and $\overline{m}$.

(2) For any $\mathcal{C}^{\mathrm{ext}}$ in A^{ext}, there exists a vector field $E^{\mathrm{ext}} \in L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ such that

$$\begin{cases} -\mathrm{div}\left(|y|^\gamma E^{\mathrm{ext}}\right) = c_{d,s}\left(\mathcal{C}^{\mathrm{ext}} - \mu'_V\right) & \text{in } \Sigma' \backslash \Sigma'_{\mathrm{int}} \\ E^{\mathrm{ext}} \cdot \vec{\nu} = 0 & \text{on } \partial(\Sigma' \backslash \Sigma'_{\mathrm{int}}) \end{cases}.$$

(3) The vector field E^{ext} described above satisfies

$$\int_{(\Sigma' \backslash \Sigma'_{\mathrm{int}}) \times \mathbb{R}^k} |y|^\gamma |E^{\mathrm{ext}}_\eta|^2 \leq C(|\Sigma'| - |\Sigma'_{\mathrm{int}}|). \tag{6.54}$$

(4) The volume of A^{ext} is bounded below the following way:

$$\mathbf{Leb}^{\otimes(N-N_{\mathrm{int}})}\left(A^{\mathrm{ext}}\right) \geq C^{(N-N_{\mathrm{int}})}(N-N_{\mathrm{int}})! \tag{6.55}$$

for some C depending only on d, s and $\overline{m}$.

We may now finish the proof of Proposition 1.7. Let $\bar{P}, \delta_1, \delta_2$ be given as in the statement of the proposition, and let N, η, R, ε be given. Let us consider any configuration $\mathcal{C}^{\mathrm{mod}}$ in Σ'_{int} and electric field E^{mod} provided by Lemma 6.4 for these parameters, and any configuration $\mathcal{C}^{\mathrm{ext}}$ and associated electric field E^{ext} provided by Lemma 6.6. We may then consider the total configuration $\mathcal{C}^{\mathrm{tot}} := \mathcal{C}^{\mathrm{mod}} + \mathcal{C}^{\mathrm{ext}}$ (the sum is again in the sense of measures) and the total vector field $E^{\mathrm{tot}} := E^{\mathrm{mod}} \mathbf{1}_{\Sigma'_{\mathrm{int}}} + E^{\mathrm{ext}} \mathbf{1}_{\Sigma' \backslash \Sigma'_{\mathrm{int}}}$.

A. Distance to $\bar{P}$.
In view of (6.19) together with the fact that $|\Sigma'| - |\Sigma'_{\underline{m}}| = o(N)$ as $\underline{m} \to 0$ it is easy to conclude that

$$\boxed{\frac{1}{|\Sigma| N} \int_{\Sigma'} \delta_{(N^{-1/d} x, \theta_x \cdot \mathcal{C}^{\mathrm{tot}})} dx \in B(\bar{P}, \delta_1),} \tag{6.56}$$

for $\underline{m}$ small enough, ε small enough, M, R large enough, ν small enough and N large enough. To see it we may choose k large enough so that (6.32) holds with $\frac{\delta_1}{10}$ and argue as in the proof of item 1 : the configurations $(\theta_x \cdot \mathcal{C}^{\mathrm{tot}}) \cap C_k$ and $(\theta_x \cdot \mathcal{C}^{\mathrm{mod}}) \cap C_k$ coincide on

$$\left\{x \in \Sigma_{\mathrm{int}}, d\left(x, \partial \Sigma_{\mathrm{int}}\right) \geq C k^{1/d}\right\}$$

which represents a fraction $1-o(1)$ (when $R,N\to\infty$) of Σ_{int}, hence of $\Sigma'_{\underline{m}}$ so that

$$d\left(\frac{1}{|\Sigma'_{\underline{m}}|}\int_{\Sigma'_{\underline{m}}}\delta_{(N^{-1/d}x,\theta_x\cdot\mathcal{C}^{\mathrm{tot}})},\bar{P}\right)\le\frac{\delta_1}{10}+\frac{3\delta_1}{4}+o(1).$$

To conclude it suffices to observe (see Remark 7.1 for a precise statement) that since the difference of volume $|\Sigma'|-|\Sigma'_{\underline{m}}|=o(N)$ as $\underline{m}\to 0$, the continuous averages of a given configuration over both domains lie at distance $o(1)$ uniformly as $\underline{m}\to 0$.

B. Energy.

We have

$$\begin{cases}-\mathrm{div}\,(|y|^\gamma E^{\mathrm{tot}})=c_{d,s}\left(\mathcal{C}^{\mathrm{tot}}-\mu'_V\delta_{\mathbb{R}^d}\right) & \text{in } \mathbb{R}^{d+k}\\ E^{\mathrm{tot}}=0 & \text{in } \mathbb{R}^{d+k}\backslash(\Sigma'\times\mathbb{R}^k).\end{cases}$$

Moreover from (6.22) and (6.54), since $|\Sigma'|-|\Sigma'_{\mathrm{int}}|=o(N)$ as $\underline{m}\to 0$, we see that the energy of E^{tot} satisfies for any η small enough

$$\limsup_{\underline{m}\to 0}\limsup_{\varepsilon\to 0}\limsup_{M,R\to\infty}\limsup_{\tau,\nu\to 0}\limsup_{N\to\infty}\left(\frac{1}{N|\Sigma|}\int_{\mathbb{R}^{d+k}}|y|^\gamma|E^{\mathrm{tot}}|^2-\int F_{R,\eta}^{M,1,\varepsilon}d\bar{P}\right)\le 0. \tag{6.57}$$

Moreover since the points added in $\Sigma\backslash\Sigma'_{\mathrm{int}}$ are well-separated from item 1 of Lemma 6.6, we keep (6.20).

Combining Lemma 3.12 and (6.57) we see that for every point configuration obtained this way, the associated electric field as in (2.29) satisfies

$$\limsup_{\eta\to0,\underline{m}\to0,\varepsilon\to0,M,R\to\infty,\tau,\nu\to0 N\to\infty}\left[\left(\frac{1}{N}\int_{\mathbb{R}^{d+k}}|y|^\gamma|\nabla H'_{N,\eta}|^2-c_{d,s}g(\eta)\right)\right.$$
$$\left.-\int\left(F_{R,\eta}^{M,1,\varepsilon}-c_{d,s}g(\eta)\mu_V(x)\right)d\bar{P}\right]\le 0.$$

Using the definition (2.30), (3.4) from Lemma 3.3, the fact that (6.20) holds, and (5.29) from Lemma 5.10 we get that given $\delta_1,\delta_2>0$ we may obtain (6.56) and

$$\boxed{w_N(\mathcal{C}^{\mathrm{tot}})\le\overline{\mathbb{W}}_{\mu_V}(\bar{P})+\delta_2} \tag{6.58}$$

by choosing η small enough, $\underline{m}$ small enough, ε small enough, M,R large enough, ν,τ small enough and N large enough.

C. Volume.

We are left to bound below the volume of configurations $A^{\mathrm{tot}}:=\{\mathcal{C}^{\mathrm{tot}}\}$ that we have constructed, and connect this volume with a large enough probability for $\bar{\mathfrak{Q}}_{N,\beta}$ as in (1.26). We may bound the volume of A^{tot} as follows

$$\frac{\mathbf{Leb}^{\otimes N}\left(A^{\mathrm{tot}}\right)}{|\Sigma'|^N}\ge\binom{N}{N^{\mathrm{int}}}\frac{\mathbf{Leb}^{\otimes N^{\mathrm{int}}}\left(A^{\mathrm{mod}}\right)}{|\Sigma'|^{N^{\mathrm{int}}}}\frac{\mathbf{Leb}^{\otimes(N-N^{\mathrm{int}})}\left(A^{\mathrm{ext}}\right)}{|\Sigma'|^{N-N^{\mathrm{int}}}}$$

which yields by taking the log and using (6.23) and (6.55)

$$\frac{1}{N}\log\frac{\mathbf{Leb}^{\otimes N}\left(A^{\mathrm{tot}}\right)}{|\Sigma'|^N}\ge\frac{1}{N}\log\binom{N}{N^{\mathrm{int}}}+\frac{N^{\mathrm{int}}}{N}\log\frac{|\Sigma'_{\mathrm{int}}|}{|\Sigma'|}-\int_{\Sigma_{\underline{m}}}\mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx$$
$$-(\log|\Sigma|-|\Sigma|+1)-\frac{1}{N}(N-N^{\mathrm{int}})\log|\Sigma'|+\frac{1}{N}(N-N^{\mathrm{int}})\log C+\frac{1}{N}\log(N-N^{\mathrm{int}})!-r$$

with an error r going to zero as $\eta \to 0, \underline{m} \to 0, \varepsilon \to 0, M, R \to \infty, \tau, \nu \to 0, N \to \infty$. The terms of combinatorial nature may be re-arranged and bounded below by Stirling's formula

$$\frac{1}{N} \log \binom{N}{N^{\text{int}}} + \frac{1}{N} \log(N - N^{\text{int}})! \geq \log N + \frac{1}{N}(N^{\text{int}} - N) - \frac{N^{\text{int}}}{N} \log N^{\text{int}}.$$

Moreover we have $(N - N^{\text{int}}) \log |\Sigma'| = (N - N^{\text{int}}) \log N + (N - N^{\text{int}}) \log |\Sigma|$. Let us also observe that $\frac{1}{N}(N^{\text{int}} - N) = o(1)$ and $\frac{|\Sigma'_{\text{int}}|}{|\Sigma'|} \to 1$ when $\underline{m} \to 0, R \to \infty, N \to \infty$. We thus have

$$\frac{1}{N} \log \frac{\mathbf{Leb}^{\otimes N}\left(A^{\text{tot}}\right)}{|\Sigma'|^N} \geq -\int_\Sigma \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx - (\log |\Sigma| - |\Sigma| + 1) \\ + \log N - \frac{N^{\text{int}}}{N} \log N^{\text{int}} - (N - N^{\text{int}}) \log N + o(1).$$

Since $N^{\text{int}} \leq N$ we finally get

$$\frac{1}{N} \log \frac{\mathbf{Leb}^{\otimes N}\left(A^{\text{tot}}\right)}{|\Sigma'|^N} \geq -\int_\Sigma \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx - (\log |\Sigma| - |\Sigma| + 1) + o(1). \tag{6.59}$$

Conclusion. To complete the proof of Proposition 1.7 we need to link the preceding construction with a large enough volume of "good" events for $\bar{\mathfrak{Q}}_{N,\beta}$. This is done by conditioning $\mathbb{Q}_{N,\beta}$ into having all N points in Σ, the resulting conditional expectation (after scaling) is equal in law to $\frac{\mathbf{Leb}^{\otimes N}}{|\Sigma'|^N}$. The probability of this event is bounded below by

$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbb{Q}_{N,\beta} \left(N \text{ points in } \Sigma\right) \geq \log \frac{|\Sigma|}{|\omega|} \tag{6.60}$$

because $\mathbb{Q}_{N,\beta}$ is essentially the (N times product of the) normalized Lebesgue measure on ω. More precisely $\mathbb{Q}_{N,\beta}$ is the N times product of the measure $\frac{e^{-N\beta\zeta(x)}dx}{\int e^{-N\beta\zeta(x)}dx}$, but we know that ζ vanishes on ω and is positive outside ω and moreover from (4.1) we know that $\int e^{-N\beta\zeta(x)} dx \to_{N\to\infty} |\omega|$ (see equation (7.18) and the proof of the lower bound after it for more details).

In view of (6.56) and (6.58) and using (6.60) we have (with the notations of Proposition 1.7):

$$\begin{aligned} \liminf_{N \to \infty} \frac{1}{N} \log \bar{\mathfrak{Q}}_{N,\beta} \left(B(\bar{P}, \delta_1) \cap T_{N,\delta_2}(\bar{P})\right) &\geq -\int_\Sigma \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx - (\log |\Sigma| - |\Sigma| + 1) \\ &\quad + \liminf_{N \to \infty} \frac{1}{N} \log \mathbb{Q}_{N,\beta} \left(N \text{ points in } \Sigma\right) \\ &\geq -\int_\Sigma \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx - (\log |\omega| - |\Sigma| + 1), \end{aligned}$$

which, in view of (1.24), concludes the proof of Proposition 1.7.

7. Proof of the LDPs for the reference measure

In this section, we prove Proposition 1.6 and Lemma 6.3. Proposition 1.6 is a "process-level" (or type 3) LDP, whereas Lemma 6.3 is closer to a (type 2) Sanov-like large deviation result.

In order to prove Proposition 1.6 we rely on a similar result, Proposition 7.5 below proved in [GZ93] with the Poisson process $\mathbf{\Pi}^1$ as reference measure instead of $\bar{\mathfrak{Q}}_{N,\beta}$. What we have to do is then to show that the result remains true when perturbing away from the Poisson case, which will take several steps. The proof of Lemma 6.3, on the other hand, relies on the classical Sanov's theorem whose adaptation to our setting is very similar to the previous one. We believe that some (if not all) of these variations around classical results belong to folklore knowledge but we provide a proof for the sake of completeness.

7.1. **Two comparison lemmas.** We start by introducing some notions that will allow to replace point processes by equivalent ones.

Definition 7.1. *Let (X, d_X) be a metric space and let $\{R_N\}_N$ and $\{R'_N\}_N$ be two coupled sequences of random variables with value in X, defined on some probability spaces $\{(\Omega_N, \mathcal{B}_N, \pi_N)\}_N$. For any $\delta > 0$ we say that $\{R_N\}_N$ and $\{R'_N\}_N$ are eventually almost surely (e.a.s.) δ-close when for N large enough we have*

$$\pi_N\left(d_X(R_N, R'_N) \geq \delta\right) = 0.$$

It two sequences are e.a.s. δ-close for any $\delta > 0$ we say that they are eventually almost surely equivalent (e.a.s.e.).

Let us emphasize that being eventually almost surely equivalent is strictly stronger than the usual convergence in probability. It is also easily seen to be stronger than the classical notion of "exponential equivalence" (see [DZ10, Section 4.2.2]) and thanks to that, large deviation principles may be transfered from one sequence to the other.

Lemma 7.2. *If the sequences $\{R_N\}_N$ and $\{R'_N\}_N$ are eventually almost surely equivalent and an LDP with good rate function holds for $\{R_N\}_N$, then the same LDP holds for $\{R'_N\}_N$.*

Proof. This is a straightforward consequence of [DZ10, Theorem 4.2.13]. □

A first example is given by the averages of a configuration over (translations in) two close sequences of sets.

Remark 7.3. *Let $\{V_N\}_N, \{W_N\}_N$ be two sequences of Borel sets in $\mathbb{R}^d$ of bounded Lebesgue measure and let f be a bounded measurable function on $\mathcal{X}$. Then for any configuration $\mathcal{C} \in \mathcal{X}$ we have*

$$\frac{1}{|W_N|}\left|\int_{W_N} f(\theta_x \cdot \mathcal{C})dx - \int_{V_N} f(\theta_x \cdot \mathcal{C})dx\right| \leq \frac{|W_N \triangle V_N|}{|W_N|}\|f\|_\infty, \tag{7.1}$$

where $\triangle$ denotes the symmetric difference between sets. In particular if P is a point process and $\lim_{N\to\infty} \frac{|W_N \triangle V_N|}{|W_N|} = 0$, the random variables obtained as the push-forward of P by the maps

$$\mathcal{C} \mapsto \frac{1}{|W_N|}\int_{W_N} \delta_{\theta_x \cdot \mathcal{C}} dx \text{ and } \mathcal{C} \mapsto \frac{1}{|V_N|}\int_{V_N} \delta_{\theta_x \cdot \mathcal{C}} dx$$

are eventually almost surely equivalent.

The point (7.1) is straightforward. To get e.a.s. δ-closeness it suffices to recall that the distance between point processes is defined in (2.37) by testing against functions in $\mathrm{Lip}_1(\mathcal{X})$ (which are in particular bounded in sup-norm). In a similar spirit we have

Lemma 7.4. *Let P be a point process in $\mathbb{R}^d$ and $\{\Lambda_N\}_N$ be a sequence of Borel sets of $\mathbb{R}^d$ of finite Lebesgue measure, such that*

$$\forall k \in \mathbb{N}, \quad |\{x \in \Lambda_N, d(x, \partial\Lambda_N) \geq k\}| = o(|\Lambda_N|).$$

In particular the assumption holds when $\Lambda_N = N^{1/d}\Lambda$ where Λ is a compact set with Lipschitz boundary.

Let us denote by R_N, resp. R'_N the push-forward of P by the map $\mathcal{C} \mapsto \frac{1}{|\Lambda_N|}\int_{\Lambda_N} \delta_{\theta_x \cdot \mathcal{C}} dx$, resp. $\mathcal{C} \mapsto \frac{1}{|\Lambda_N|}\int_{\Lambda_N} \delta_{\theta_x \cdot (\mathcal{C}\cap\Lambda_N)} dx$. Then the sequences $\{R_N\}_N$ and $\{R'_N\}_N$ are e.a.s.e.

Proof. Let us observe that the operation of taking the intersection with Λ_N affects only a small portion of the translates, indeed we have for any $k \geq 1$

$$(\theta_x \cdot \mathcal{C}) \cap C_k = (\theta_x \cdot (\mathcal{C} \cap \Lambda_N))) \cap C_k$$

for all x such that $d(x, \partial\Lambda_N) \geq k^{1/d}$. Thus, combining the uniform approximation of functions in $\mathrm{Lip}_1(\mathcal{X})$ by bounded local functions as in Lemma 2.5 and the definition (2.37) of $d_{\mathcal{P}(\mathcal{X})}$ as testing against functions in $\mathrm{Lip}_1(\mathcal{X})$ we get that for any $\delta > 0$ there exists $k \geq 1$ such that

$$d_{\mathcal{P}(\mathcal{X})}(R_N, R'_N) \leq \delta + \frac{|x \in \Lambda, d(x, \partial\Lambda_N) \geq k|}{|\Lambda_N|}, \ P\text{-almost surely.} \tag{7.2}$$

By assumption the second term in the right-hand side is $o(1)$ when $N \to \infty$ hence R_N, R'_N are e.a.s. 2δ-close and this holds for any $\delta > 0$. □

7.2. **Continuous average, proof of Proposition 1.6.** We now turn to the proof of the large deviation result for $\bar{\mathfrak{Q}}_{N,\beta}$ stated in Proposition 1.6. We start by recalling the following fundamental large deviation principle for empirical fields.

Proposition 7.5 (Georgii-Zessin). *Let $\{\Lambda_N\}_N$ be a fixed sequence of cubes increasing to $\mathbb{R}^d$ and let R_N be the push-forward of $\mathbf{\Pi}^1$ by the map*

$$\mathcal{C} \mapsto \frac{1}{|\Lambda_N|}\int_{\Lambda_N} \delta_{\theta_x \cdot \mathcal{C}} dx.$$

Then $\{R_N\}_N$ satisfies a large deviation principle at speed $|\Lambda_N|$ with rate function $\mathsf{ent}[\cdot|\mathbf{\Pi}^1]$.

This is a consequence of [GZ93, Theorem 3.1] together with [GZ93, Remark 2.4] to get rid of the periodization used in their definition of R_N (see also [FO88]). One could also adapt the method of [RAS09, Chapter 6] from the discrete case (point processes on $\mathbb{Z}^d$) to the case of point processes on $\mathbb{R}^d$, where the Gärtner-Ellis theorem (see [DZ10, Section 4.5.3]) is used by establishing the existence of a pressure and studying its Legendre-Fenchel transform. We now need to extend the result to our setting.

7.2.1. *Extension to Lipschitz boundaries.* In this first step we extend the LDP of Proposition 7.5 to more general shapes of $\{\Lambda_N\}_N$.

Lemma 7.6. *Let Λ be a compact set of $\mathbb{R}^d$ with a non-empty interior and a Lipschitz boundary, and let $\Lambda_N := N^{1/d}\Lambda$. Let R_N be the push-forward of $\mathbf{\Pi}^1$ by the map*

$$\mathcal{C} \mapsto \frac{1}{|\Lambda_N|}\int_{\Lambda_N} \delta_{\theta_x \cdot \mathcal{C}} dx.$$

Then $\{R_N\}_N$ satisfies a large deviation principle at speed $N|\Lambda|$ with rate function $\mathsf{ent}[\cdot|\mathbf{\Pi}^1]$.

Proof. In the following every hypercube is such that its edges are parallel to the axes of $\mathbb{R}^d$. Let N be given. Let us consider the hypercubes centered at the points of $\Lambda \cap \frac{1}{n}\mathbb{Z}^d$ and of sidelength $\frac{1}{n}$, and remove those that are centered at points in

$$A_n := \left\{ x \in \Lambda \cap \frac{1}{n}\mathbb{Z}^d, d(x, \partial\Lambda) \leq 2c_d n^{-1/d} \right\},$$

where c_d is the distance between the center of the unit hypercube in dimension d and any vertex of this hypercube. Since the boundary of Λ is Lipschitz, we have $\lim_{n\to\infty} |A_n| = 0$, so that the total volume lost when removing the boundary hypercubes is less than $2^{-N}|\Lambda|$ for n large enough. In other words, we have found a family of $m = m(N)$ hypercubes $\{\Lambda^{(i,N)}\}_{i=1}^{m(N)}$ included in Λ and such that $|\Lambda| - |\cup_{i=1}^m \Lambda^{(i,N)}| \leq 2^{-N}|\Lambda|$.

For any N we may then define $\tilde{\Lambda}^{(N)}$ as the hypercube of center 0 and such that

$$|\tilde{\Lambda}^{(N)}| = \sum_{i=1}^m |\Lambda^{(i,N)}| = m|\Lambda^{(1,N)}| \geq |\Lambda| - 2^{-N}|\Lambda|. \tag{7.3}$$

There exists a measurable bijection $\Phi_N : \cup_{i=1}^m \Lambda^{(i,N)} \to \tilde{\Lambda}^{(N)}$ which is a translation on each hypercube $\Lambda^{(i,N)}$ $(i = 1, \dots, m)$.

Next, we let R_N be as before the push-forward of $\mathbf{\Pi}^1$ by the map $\mathcal{C} \mapsto \frac{1}{|\Lambda_N|}\int_{\Lambda_N} \delta_{\theta_x \cdot \mathcal{C}} dx$ and R'_N be the push-forward of $\mathbf{\Pi}^1$ by the map

$$\mathcal{C} \mapsto \frac{1}{Nm|\Lambda^{(1,N)}|}\int_{\cup_{i=1}^m N^{1/d}\Lambda^{(i,N)}} \delta_{\theta_x \cdot \mathcal{C}} dx.$$

Finally, from any configuration of points $\mathcal{C}$ on $\cup_{i=1}^m N^{1/d}\Lambda^{(i,N)}$ we get by applying $x \mapsto N^{1/d}\Phi_N(N^{-1/d}(x))$ a configuration in $N^{1/d}\tilde{\Lambda}^{(N)}$, which by abusing notation we denote again by $\Phi_N(\mathcal{C})$. We denote by R''_N the push-forward of $\mathbf{\Pi}^1_{|\Lambda_N}$ by:

$$\mathcal{C} \mapsto \frac{1}{N|\tilde{\Lambda}^{(N)}|}\int_{\Lambda_N} \delta_{\theta_{\Phi_N(x)} \cdot \Phi_N(\mathcal{C})} dx.$$

We impose that the random variables R_N, R'_N, R''_N are coupled together the natural way.

It is easily seen that the push-forward of $\mathbf{\Pi}^1_{|\Lambda_N}$ – or more precisely of the process induced on the subset $\cup_{i=1}^m N^{1/d}\Lambda^{(i,N)}$ – by the map $\mathcal{C} \mapsto \Phi_N(\mathcal{C})$ is equal in law to $\mathbf{\Pi}^1_{N^{1/d}\tilde{\Lambda}^{(N)}}$. The sequence of hypercubes $\{N^{1/d}\tilde{\Lambda}^{(N)}\}_N$ satisfies the hypothesis of Proposition 7.5 hence a Large Deviation Principle holds at speed $|\Lambda|N$ for the sequence $\{R''_N\}_N$ (the fact that we consider the push-forward of $\mathbf{\Pi}^1_{|\Lambda_N}$ instead of that of $\mathbf{\Pi}^1$ is irrelevant thanks to Lemma 7.4). To show that the same principle holds for $\{R_N\}_N$ it is enough to show that the two sequences are e.a.s. equivalent in the sense of Definition 7.1.

Let us first observe that the sequences $\{R_N\}_N$ and $\{R'_N\}$ are e.a.s.e. because as a consequence of (7.3) the tiling of Λ_N by the hypercubes $\cup_{i=1}^m N^{1/d}\Lambda^{(i,N)}$ only misses a $o(1)$ fraction of the volume of Λ_N and e.a.s. equivalence is then a consequence of Lemma 7.3.

As for the pair of sequences $\{R'_N\}_N$ and $\{R''_N\}_N$, let us observe that for any $k \geq 1$ we have

$$(\theta_x \cdot \mathcal{C}) \cap C_k = \left(\theta_{\Phi_N(x)} \cdot \Phi_N(\mathcal{C})\right) \cap C_k$$

for any x in one of the tiling hypercubes $\cup_{i=1}^m N^{1/d}\Lambda^{(i,N)}$ except for the points that are near the boundary of their hypercube - those such that

$$d\left(x, \cup_{i=1}^m \partial N^{1/d}\Lambda^{(i,N)}\right) \leq |C_k|^{1/d}.$$

For any k the fraction of volume of points in the hypercube that are close to the boundary in the previous sense is negligible as $N \to \infty$. Arguing as in the proof of Lemma 7.4 gives the result. □

7.2.2. *Tagged point processes.* We now recast the result of Lemma 7.6 in the context of tagged point processes (as defined in Section 2.4) which necessitates to replace the specific relative entropy ent by its analogue with tags.

Lemma 7.7. *Let Λ be a compact set of $\mathbb{R}^d$ with C^1 boundary and non-empty interior and let $\bar{R}_N$ be the push-forward of $\mathbf{\Pi}^1$ by the map*

$$\mathcal{C} \mapsto \frac{1}{|\Lambda|} \int_\Lambda \delta_{(x, \theta_{N^{1/d}x} \cdot \mathcal{C})} dx.$$

Then $\{\bar{R}_N\}_N$ satisfies a large deviation principle at speed N with rate function

$$\bar{P} \mapsto \int_\Lambda \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx.$$

Proof. **Upper bound.** Let $\bar{P}$ be a stationary tagged point process. We claim that

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbf{\Pi}^1(\bar{R}_N \in B(\bar{P}, \varepsilon)) \leq - \int_\Lambda \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx. \tag{7.4}$$

Let us observe that the "forgetful" map $\varphi : \mathcal{P}(\Lambda \times \mathcal{X}) \to \mathcal{P}(\mathcal{X})$ obtained by pushing forward the map $(x, \mathcal{C}) \mapsto \mathcal{C}$ is continuous. This yields

$$\limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbf{\Pi}^1(\bar{R}_N \in B(\bar{P}, \varepsilon)) \leq \limsup_{\varepsilon \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \mathbf{\Pi}^1 \left(\varphi(\bar{R}_N) \in B(\varphi(\bar{P}), \varepsilon) \right).$$

The definition of φ implies that $R_N := \varphi(\bar{R}_N)$ is the push-forward of $\mathbf{\Pi}^1$ by the map $\mathcal{C} \mapsto \frac{1}{N|\Lambda|} \int_{\Lambda_N} \delta_{\theta_x \cdot \mathcal{C}} dx$. From Lemma 7.6 we know that an LDP holds for R_N at speed $|\Lambda| N$ with rate function $\mathsf{ent}[\cdot | \mathbf{\Pi}^1]$ (or equivalently at speed N with rate function $|\Lambda| \mathsf{ent}[\cdot | \mathbf{\Pi}^1]$) hence the right-hand side is bounded by $|\Lambda| \mathsf{ent}[\varphi(\bar{P}) | \mathbf{\Pi}^1]$. Now let us note that $\varphi(\bar{P}) = \frac{1}{|\Lambda|} \int_\Lambda \bar{P}^x dx$ (where $\bar{P}^x$ is the disintegration of $\bar{P}$ with respect to the first coordinate of $\Sigma \times \mathcal{X}$ and where the integral is understood in the Gelfand-Pettis sense), but the relative specific entropy is affine hence we have

$$\mathsf{ent}[\varphi(\bar{P}) | \mathbf{\Pi}^1] = \frac{1}{|\Lambda|} \int_\Lambda \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx$$

and this shows the upper bound of the lemma.

Lower bound. Let $\bar{P}$ be a tagged point process. We want to prove that

$$\liminf_{\varepsilon \to 0} \liminf_{N \to \infty} \frac{1}{N} \log \mathbf{\Pi}^1(\bar{R}_N \in B(\bar{P}, \varepsilon)) \geq - \int_\Lambda \mathsf{ent}[\bar{P}^x | \mathbf{\Pi}^1] dx. \tag{7.5}$$

For any $\varepsilon > 0$ we claim that there exists a covering of Λ by compact sets $A_1, \dots, A_M \subset \Lambda$ of pairwise disjoint interiors such that each set A_i has a Lipschitz boundary and such that, denoting by $R_N^{(i)}$ the push-forward of $\mathbf{\Pi}^1$ by $\mathcal{C} \mapsto \frac{1}{N|A_i|} \int_{N^{1/d} A_i} \delta_{\theta_x \cdot \left(\mathcal{C} \cap N^{1/d} A_i \right)} dx$ (we impose that the random variables $\bar{R}_N$ and $R_N^{(1)}, \dots, R_N^{(M)}$ are coupled together in the natural way), the following holds for any δ small enough and N large enough:

$$\bigcap_{i=1}^M \left\{ R_N^{(i)} \in B \left(\fint_{A_i} \bar{P}^x dx, \delta \right) \right\} \subset \left\{ \bar{R}_N \in B(\bar{P}, \varepsilon) \right\}. \tag{7.6}$$

This is shown by the following successive approximations.

(1) By definition of the topology of weak convergence, the ball $B(\bar{P}, \varepsilon)$ contains a certain open set of the type
$$\bigcap_{i \in I} \left\{ \left| \int F_i(x, \mathcal{C})(d\bar{R}_N - d\bar{P}) \right| \leq \delta_1 \right\}$$
for a finite family of continuous functions $F_i \in C^0(\Lambda \times \mathcal{X})$.

(2) A standard application of the Stone-Weierstrass theorem implies that each function F_i can be approximated in sup-norm by a finite sum $\sum_j f_{i,j} g_{i,j}$ where $f_{i,j}$ are continuous functions on Λ and $g_{i,j}$ are continuous functions on $\mathcal{X}$.

(3) We may then approximate each $f_{i,j}$ by step functions on Λ with a common partition $\{A_1, \dots, A_M\}$ for all functions $f_{i,j}$. Each set in the partition can be chosen to be either a hypercube or the intersection of a hypercube with Λ so that they all have a Lipschitz boundary. At this point (7.6) is seen to hold for some $\delta > 0$ small enough, only with the random variable $R_N'^{(i)}$ instead of $R_N^{(i)}$, where $R_N'^{(i)}$ is the push-forward of $\mathbf{\Pi}^1$ by the map $\mathcal{C} \mapsto \fint_{N^{1/d} A_i} \delta_{\theta_x \cdot \mathcal{C}} dx$.

(4) We may also approximate each function $g_{i,j}$ by a bounded local function in $\mathrm{Loc}_k(\mathcal{X})$ (as in Lemma 2.5) with the same k for all functions $g_{i,j}$. This allows us to argue as in Lemma 7.4 to neglect the points that are close to the boundary between two elements of the partition, hence passing from $R_N'^{(i)}$ to $R_N^{(i)}$.

Since the $A_1, \dots, A_M$ are pairwise disjoint (up to a boundary of zero Lebesgue measure), the events $\left\{ R_N^{(i)} \in B\left(\fint_{A_i} \bar{P}^x dx, \delta \right) \right\}$ are globally independent so that (7.6) yields
$$\frac{1}{N} \log \mathbf{\Pi}^1 \left(\{ \bar{R}_N \in B(\bar{P}, \varepsilon) \} \right) \geq \frac{1}{N} \sum_{i=1}^M \log \mathbf{\Pi}^1 \left(\left\{ R_N^{(i)} \in B\left(\fint_{A_i} \bar{P}^x dx, \delta \right) \right\} \right).$$
Moreover, for any $i = 1 \dots M$, we have
$$\begin{aligned} \liminf_{\delta \to 0} \liminf_{N \to \infty} \frac{1}{|A_i| N} \log \mathbf{\Pi}^1 \left(\left\{ R_N^{(i)} \in B\left(\fint_{A_i} \bar{P}^x dx, \delta \right) \right\} \right) &= -\mathsf{ent}\left[\fint_{A_i} \bar{P}^x dx \Big| \mathbf{\Pi}^1 \right] \\ &= -\fint_{A_i} \mathsf{ent}\left[\bar{P}^x | \mathbf{\Pi}^1 \right] dx \end{aligned}$$
by the large deviation principle of Lemma 7.6 and by the fact that $\mathsf{ent}[\cdot|\mathbf{\Pi}^1]$ is affine. This finally implies that
$$\liminf_{N \to \infty} \frac{1}{N} \log \mathbf{\Pi}^1 \left(\{ \bar{R}_N \in B(\bar{P}, \varepsilon) \} \right) \geq \sum_{i=1}^M \int_{A_i} \mathsf{ent}\left[\bar{P}^x | \mathbf{\Pi}^1 \right] dx = -\int_\Lambda \mathsf{ent}\left[\bar{P}^x | \mathbf{\Pi}^1 \right] dx$$
for any $\varepsilon > 0$, which implies (7.5).

Conclusion. From (7.4) and (7.5) we get a weak LDP for the sequence $\{\bar{R}_N\}_N$. The full LDP is obtained by observing that $\{\bar{R}_N\}_N$ is exponentially tight, a fact for which we only sketch the (elementary) proof : for any integer M we may find an integer $T(M)$ large enough such that a point process has less than $T(M)$ points in C_M expect for a fraction $\leq \frac{1}{M}$ of the configurations, with $\bar{R}_N$-probability bounded below (when $N \to \infty$) by $1 - e^{-NM}$. The union (on $N \geq N_0$ large enough) of such events has a large $\bar{R}_N$-probability (bounded below by $1 - e^{-NM}$ when $N \to \infty$) and is easily seen to be compact. □

7.2.3. *From Poisson to Bernoulli.* **Modification in the lower bound.** From Lemma 7.4 we know that the large deviation principle of Lemma 7.7 is still true when restricting the Poisson point process to $N^{1/d}\Lambda$. If we consider an N-point Bernoulli point process on $N^{1/d}\Lambda$ instead of the restriction of a Poisson point process as the reference measure i.e. if we constrain $\mathbf{\Pi}^1$ into having a fixed number of points in $N^{1/d}\Lambda$ then the LDP is modified. The large deviation upper bound holds but the large deviation lower bound ceases to be true in general, for the limit point processes might have large excesses of points with non-negligible probability e.g. in the case of the Poisson point process itself. Let us recall that we denote by $\mathcal{P}_{s,1}(\Lambda \times \mathcal{X})$ the set of stationary tagged point processes (with space coordinate taken in Λ) such that the integral on $x \in \Lambda$ of the intensity of the disintegration measure $\bar{P}^x$ (which is by assumption a stationary point process) is 1.

In what follows, when a set Λ is fixed if M, N are integers we denote by $\mathbf{B}_{M,N}$ the Bernoulli point process with M points in $N^{1/d}\Lambda$ and we let $\mathbf{B}_N := \mathbf{B}_{N,N}$ for any integer N. We want to prove

Lemma 7.8. *Let Λ be a compact set of $\mathbb{R}^d$ with C^1 boundary and non-empty interior and let $\bar{S}_N$ be the push-forward of $\mathbf{B}_N$ by the map*

$$\mathcal{C} \mapsto \frac{1}{N|\Lambda|}\int_{N^{1/d}\Lambda} \delta_{(N^{-1/d}x,\theta_x\cdot\mathcal{C})}dx.$$

Then for any $A \subset \mathcal{P}_s(\Lambda \times \mathcal{X})$ we have:

$$\left(-\inf_{\bar{P}\in\mathring{A}\cap\mathcal{P}_{s,1}(\Lambda\times\mathcal{X})}\int_\Lambda \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx\right) - (\log|\Lambda| - |\Lambda| + 1) \le \liminf_{N\to\infty}\frac{1}{N}\log \bar{S}_N(A)$$
$$\le \limsup_{N\to\infty}\frac{1}{N}\log\bar{S}_N(A) \le \left(-\inf_{\bar{P}\in\bar{A}}\int_\Lambda \mathsf{ent}[\bar{P}^x|\mathbf{\Pi}^1]dx\right) - (\log|\Lambda| - |\Lambda| + 1). \tag{7.7}$$

Let us emphasize that in the lower bound of (7.7) the *infimum* is taken on the restriction $\mathring{A}\cap\mathcal{P}_{s,1}(\Lambda\times\mathcal{X})$.

Variations of the domain and the number of points. For M, N integers we denote by $\bar{S}_{M,N}$ the push-forward of $\mathbf{B}_{M,N}$ by the map

$$\mathcal{C} \mapsto \frac{1}{|\Lambda|}\int_\Lambda \delta_{(x,\theta_{N^{1/d}x}\cdot\mathcal{C})}dx.$$

Let us observe that $\bar{S}_{N,N} = \bar{S}_N$ as defined in Lemma 7.8. The following lemma allows us to handle the variations of the number of points.

Lemma 7.9. *Let $L = L_N$ and $M = M_N$ be two sequences depending on N with $L \ge \max(M,N)$ and let $l := \limsup_{N\to\infty}\max(|\frac{N}{L}-1|, |\left(\frac{N}{L}\right)^{1/d}-1|, |\frac{M}{N}-1|, |\frac{N}{M}-1|)$.*

Let $\bar{P}$ be a stationary tagged point process. The following holds:

$$\lim_{\delta\to 0}\limsup_{N\to\infty}\frac{1}{N}\log\bar{S}_{M,N}\left(B(\bar{P},\delta)\right) \le \limsup_{N\to\infty}\frac{1}{N}\log\bar{S}_L\left(B(\bar{P},O(l))\right) + O(l).$$

Proof. The probability of the point process $\mathbf{B}_L$ having exactly M points in $N^{1/d}\Lambda$ is given by $\left(\frac{N}{L}\right)^M\left(1-\frac{N}{L}\right)^{L-M}\binom{L}{M}$ and conditionally to this event $\mathbf{B}_L$ induces a point process on $N^{1/d}\Lambda$ which is equal in law to $\mathbf{B}_{M,N}$. Moreover it easy to see from the definitions that for any $\mathcal{C}$ in

$\mathcal{X}(N^{1/d}\Lambda)$ we have

$$(7.8)\quad d_{\mathcal{P}(L^{1/d}\Lambda\times\mathcal{X})}\left(\frac{1}{L|\Lambda|}\int_{L^{1/d}\Lambda}\delta_{(L^{-1/d}x,\theta_x\cdot\mathcal{C})}dx, \frac{1}{N|\Lambda|}\int_{N^{1/d}\Lambda}\delta_{(N^{-1/d}x,\theta_x\cdot\mathcal{C})}dx\right) = O\left(\left|(N/L)^{1/d}-1\right| + |N/L-1|\right) = O(l)$$

Indeed for any $F \in \mathrm{Lip}_1(\Lambda\times\mathcal{X})$ we may write

$$(7.9)\quad \frac{1}{L|\Lambda|}\int_{L^{1/d}\Lambda}F(L^{-1/d}x,\theta_x\cdot\mathcal{C})dx - \frac{1}{N|\Lambda|}\int_{N^{1/d}\Lambda}F(N^{-1/d}x,\theta_x\cdot\mathcal{C})dx = \frac{1}{L|\Lambda|}\int_{L^{1/d}\Lambda}\left(F(L^{-1/d}x,\theta_x\cdot\mathcal{C}) - F(N^{-1/d}x,\theta_x\cdot\mathcal{C})\right)dx + \left(\frac{1}{N|\Lambda|}-\frac{1}{L|\Lambda|}\right)\int_{N^{1/d}\Lambda}F(N^{-1/d}x,\theta_x\cdot\mathcal{C})dx.$$

We have, for any $x\in N^{1/d}\Lambda$

$$|F(L^{-1/d}x,\theta_x\cdot\mathcal{C}) - F(N^{-1/d}x,\theta_x\cdot\mathcal{C})| \le C|L^{-1/d}-N^{-1/d}|N^{1/d} = C\left((N/L)^{1/d}-1\right)$$

and on the other hand F is bounded by 1, which together with (7.9) and the fact that $N\le L$ yields (7.8).

Conditioning $\mathbf{B}_L$ to have exactly M points in $N^{1/d}\Lambda$ we get

$$(7.10)\quad \log \bar{S}_{M,N}\left(B(\bar{P},\delta)\right) \le \log \bar{S}_L\left(B(\bar{P},\delta+O(l))\right) - \left(M\log(\frac{N}{L}) + (L-M)\log(1-\frac{N}{L})\right).$$

By definition of l we have $\liminf_{N\to\infty}\frac{1}{N}\left(M\log(\frac{N}{L})+(L-M)\log(1-\frac{N}{L})\right) = O(l)+O(l^2) = O(l)$, hence taking the limit $N\to\infty, \delta\to 0$ in (7.10) yields the lemma. □

We now turn to the proof of Lemma 7.8.

Proof. In what follows $\bar{R}_N$ will denote the push-forward of $\mathbf{\Pi}^1_{|N^{1/d}\Lambda}$ by the map

$$\mathcal{C}\mapsto\frac{1}{|\Lambda|}\int_\Lambda\delta_{(x,\theta_{N^{1/d}x}\cdot\mathcal{C})}dx.$$

To establish the upper bound of (1.25) it is enough to condition $\mathbf{\Pi}^1_{|N^{1/d}\Lambda}$ into having exactly N points. The conditional expectation is then equal in law to $\mathbf{B}_N$ so that

$$(7.11)\quad \frac{1}{N}\log\bar{R}_N(A) \ge \frac{1}{N}\log\bar{S}_N(A) + \frac{1}{N}\log\mathbf{\Pi}^1_{|N^{1/d}\Lambda}(N\text{ points}) = \frac{1}{N}\log\bar{S}_N(A)+\frac{1}{N}\log e^{-N|\Lambda|}\frac{1}{N!}(N|\Lambda|)^N = \frac{1}{N}\log\bar{S}_N(A) + N(\log|\Lambda|-|\Lambda|+1)+o_{N\to\infty}(1)$$

hence the upper bound of (1.25) follows from the LDP upper bound of Lemma 7.7.

We now turn to the lower bound in (1.25). Let us denote by $\#\Lambda_N$ the number of points of a configuration in $N^{1/d}\Lambda$ and by $\#\partial\Lambda_N$ the number of points in a 2-tubular neighborhood of $\partial\Lambda_N$. Let χ be a non-negative smooth function compactly supported in the unit ball of $\mathbb{R}^d$ such that $\int\chi = 1$ and let us denote by $\tilde{\chi}$ the continuous function on $\mathcal{X}$ obtained by testing

χ against the point configurations (seen as Radon measures). If $\mathcal{C}$ is a point configuration in $N^{1/d}\Lambda$ we have

$$\frac{1}{N|\Lambda|}\int_{N^{1/d}\Lambda}\tilde{\chi}(\mathcal{C})\delta_{(N^{-1/d}x,\theta_x\cdot\mathcal{C})}dx = \frac{\#\Lambda_N}{N|\Lambda|} + O\left(\frac{\#\partial\Lambda_N}{N|\Lambda|}\right). \tag{7.12}$$

Moreover for all $\bar{P} \in \mathcal{P}_{s,1}(\Lambda\times\mathcal{X})$ we have by definition of the intensity $\int \tilde{\chi}(\mathcal{C})d\bar{P}(x,\mathcal{C}) = \frac{1}{|\Lambda|}$. It implies that for all $\varepsilon > 0$

$$\begin{aligned}\lim_{\delta\to 0}\lim_{N\to\infty}\frac{1}{N}\log\bar{R}_N\left(B(\bar{P},\delta)\right) \\ \leq \lim_{\delta\to 0}\lim_{N\to\infty}\frac{1}{N}\log\bar{R}_N\left(\left\{\bar{Q}\in B(\bar{P},\delta)\mid \left|\int\tilde{\chi}(\mathcal{C})d\bar{Q}(x,\mathcal{C}) - \frac{1}{|\Lambda|}\right|\leq\varepsilon\right\}\right).\end{aligned} \tag{7.13}$$

We now observe that under a Poisson point process $\mathbf{\Pi}^1$ there are at most $\frac{N}{\log\log N}$ points near the boundary $\partial\Lambda_N$ with overwhelming probability :

$$\lim_{N\to\infty}\frac{1}{N}\log\mathbf{\Pi}^1\left(\left\{\frac{\#\partial\Lambda_N}{N|\Lambda|}\geq\frac{1}{\log\log N}\right\}\right) = -\infty.$$

It means in particular that in the right-hand side of (7.13) we may neglect the (intersection with the) event $\{\frac{\#\partial\Lambda_N}{N|\Lambda|}\geq\frac{1}{\log\log N}\}$ since this event has a logarithmically negligible probability. We may then neglect the $O(\frac{\#\partial\Lambda_N}{N|\Lambda|})$ error term in (7.12) and replace the right-hand side of (7.13) by

$$\lim_{N\to\infty}\frac{1}{N}\log\bar{R}_N\left(B(\bar{P},\delta)\cap\left|\frac{\#\Lambda_N}{|\Lambda_N|}-\frac{1}{|\Lambda|}\right|\leq\varepsilon\right).$$

In the previous equation and in the rest of the proof we make a slight abuse of notation since $\bar{R}_N$ is the push-forward of $\mathbf{\Pi}^1$ by a certain map.

Next, up to replacing ε by $|\Lambda|\varepsilon$ let us write this term as

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N}\log\bar{R}_N\left(B(\bar{P},\delta)\cap\left|\frac{\#\Lambda_N}{N}-1\right|\leq\varepsilon\right) \\ = \lim_{N\to\infty}\frac{1}{N}\log\sum_{\frac{M}{N}=1-\varepsilon}^{\frac{M}{N}=\frac{1}{1+\varepsilon}}\bar{S}_{M,N}\left(B(\bar{P},\delta)\right)\mathbf{\Pi}^1_{\Lambda_N}(M\text{ points in }\Lambda_N).\end{aligned} \tag{7.14}$$

The previous expression is obtained by applying the law of the total probability with respect to the possible values of $\#\Lambda_N$, and observing that the conditional expectation of $\bar{R}_N$ knowing the event $\{\#\Lambda_N = M\}$ is equal in law to $\bar{S}_{M,N}$.

We bound the $O(\varepsilon N)$ terms in the sum by their maximum to get

(7.15)

$$\begin{aligned}\limsup_{N\to\infty}\frac{1}{N}\log\sum_{\frac{M}{N}=1-\varepsilon}^{\frac{M}{N}=1+\varepsilon}\bar{S}_{M,N}\left(B(\bar{P},\delta)\right) \leq \limsup_{N\to\infty}\frac{1}{N}\log\max_{M=N(1-\varepsilon),\dots,N(1+\varepsilon)}\log\bar{S}_{M,N}\left(B(\bar{P},\delta)\right) \\ + \limsup_{N\to\infty}\frac{1}{N}\max_{M=N(1-\varepsilon),\dots,N(1+\varepsilon)}\log\mathbf{\Pi}^1_{\Lambda_N}(M\text{ points in }\Lambda_N).\end{aligned}$$

Applying lemma 7.9 with $|\frac{M}{N} - 1| \leq \varepsilon$ and $L = N(1+\varepsilon)$ we get

$$\lim_{\delta \to 0} \limsup_{N \to \infty} \frac{1}{N} \log \bar{S}_{M,N}\left(B(\bar{P}, \delta)\right) \leq \limsup_{N\to\infty} \frac{1}{N} \log \bar{S}_L\left(B(\bar{P}, O(1))\right) + o(1), \tag{7.16}$$

where $O(1), o(1)$ hold when $\varepsilon \to 0$, whereas an elementary computation yields for $\mathbf{\Pi}^1_{\Lambda_N}$

$$\limsup_{N\to\infty} \frac{1}{N} \max_{M=N(1-\varepsilon),\dots,N(1+\varepsilon)} \log \mathbf{\Pi}^1_{\Lambda_N}(M \text{ points in } \Lambda_N) \leq (\log|\Lambda| - |\Lambda| + 1) + O(\varepsilon). \tag{7.17}$$

Combining (7.15), (7.16), (7.17), using that $L = N(1+\varepsilon)$ and letting $\varepsilon \to 0$ we obtain

$$\lim_{\delta\to 0} \lim_{N\to\infty} \frac{1}{N} \log \bar{R}_N\left(B(\bar{P}, \delta)\right) \leq \liminf_{\delta\to 0} \liminf_{N\to\infty} \frac{1}{N} \bar{S}_N\left(B(\bar{P}, \delta)\right) + (\log|\Lambda| - |\Lambda| + 1).$$

The lower-bound for $\bar{S}_N$ is now a consequence of the LDP lower bound obtained for $\bar{R}_N$ in Lemma 7.7. This completes the proof of Lemma 7.8. □

7.2.4. *From Bernoulli to $\bar{\mathfrak{Q}}_{N,\beta}$.* We now wish to extend the large deviation principle to the case of the point process $\bar{\mathfrak{Q}}_{N,\beta}$, defined as the push-forward of $\mathbb{Q}_{N,\beta}$ by i_N, cf. (1.10) and (1.22). Let us observe that the probability measure $\mathbb{Q}_{N,\beta}$ has a constant density on ω^N since by definition ζ vanishes on ω, and that its (marginal) density tends to zero like $\exp(-\beta N \zeta(x))$ outside ω. Hence we expect $\bar{\mathfrak{Q}}_{N,\beta}$ to behave like a Bernoulli point process with roughly N points on $N^{1/d}\omega$ (which would correspond to the case where $\zeta = +\infty$ outside ω). We may now turn to proving Proposition 1.6.

Proof. **Lower bound.** The lower bound of (1.25) is obtained by conditioning the points to all fall inside $N^{1/d}\Sigma$. Denote by $\#\Sigma_N$ the number of points in $N^{1/d}\Sigma$, by definition of $\mathbb{Q}_{N,\beta}$ we have

$$\mathbb{Q}_{N,\beta}\left(\{\#\Sigma_N = N\}\right) = \left(\frac{|\Sigma|}{\int_{\mathbb{R}^d} e^{-\beta N \zeta(x)} dx}\right)^N.$$

It is easy to deduce using (4.1) that

$$\frac{1}{N} \log \mathbb{Q}_{N,\beta}\left(\{\#\Sigma_N = N\}\right) = \log \frac{|\Sigma|}{|\omega|} + o(1). \tag{7.18}$$

Conditionally to $\#\Sigma_N = N$, the point process generated by $\mathbb{Q}_{N,\beta}$ is equal in law to an N-point Bernoulli process in $N^{1/d}\Sigma$ hence we have, with the notation $\bar{S}_N$ of the previous paragraph (the reference set is now $\Lambda = \Sigma$)

$$\frac{1}{N} \log \bar{\mathfrak{Q}}_{N,\beta}(A) \geq \frac{1}{N} \log \bar{S}_N(A) - \frac{1}{N} \log \mathbb{Q}_{N,\beta}\left(\{\#\Sigma_N = N\}\right).$$

Using the LDP lower bound for $\bar{S}_N$ proven in Lemma 7.8 together with (7.18) we get the lower bound for $\bar{\mathfrak{Q}}_{N,\beta}$.

Upper bound. The law of total probabilities yields (abusing notation as in the proof of the lower bound of Lemma 7.8)

$$\limsup_{N\to\infty} \frac{1}{N} \log \bar{\mathfrak{Q}}_{N,\beta}(A) \leq \limsup_{N\to\infty} \frac{1}{N} \log \sum_{k=0}^{N} \bar{\mathfrak{Q}}_{N,\beta}(A \cap \#\Sigma_N = k) \mathbb{Q}_{N,\beta}(\#\Sigma_N = k)$$

Conditionally to $\#\Sigma_N = k$ the point process generated by $\mathbb{Q}_{N,\beta}$ is equal in law to a Bernoulli point process with $k \leq N$ points in $N^{1/d}\Sigma$ and the LDP upper bound of Lemma 7.8 allows us to bound each term, so that the upper bound follows from that of Lemma 7.8. More precisely it is easy to see that with overwhelming probability the number of points $\#\Sigma_N$ tends to infinity as e.g. $\sqrt{N}$ so that we may bound

$$\frac{1}{N}\log\sum_{k=0}^{N}\bar{\mathfrak{Q}}_{N,\beta}(A\cap\#\Sigma_N=k)\mathbb{Q}_{N,\beta}(\#\Sigma_N=k)$$
$$\leq\frac{1}{N}\log\sum_{k=\sqrt{N}}^{N}\bar{\mathfrak{Q}}_{N,\beta}(A\cap\#\Sigma_N=k)\mathbb{Q}_{N,\beta}(\#\Sigma_N=k).$$

Bounding $\mathbb{Q}_{N,\beta}(\#\Sigma_N=k)$ by 1 and the terms $\frac{1}{N}\log\bar{\mathfrak{Q}}_{N,\beta}(A\cap\#\Sigma_N=k)$ by $\frac{1}{k}\log\bar{\mathfrak{Q}}_{N,\beta}(A\cap\#\Sigma_N=k)$ and using Lemma 7.8 we get the result. □

7.3. **Discrete average, proof of Lemma 6.3.** In this section we give the proof of Lemma 6.3. The line of reasoning is analogous to the continuous case and we will only sketch the argument. Let us first forget about the condition on the total number of points (i.e. we consider independent Poisson point processes) and about the tags (i.e. the coordinate in Σ'_{int}), then there holds for any fixed R a Large Deviation Principle for $\dot{\mathfrak{M}}_{N,R}$ at speed $m_{N,R}$ with rate function $\mathrm{Ent}[\cdot|\mathbf{\Pi}^1_{|C_R}]$. This is a consequence of the classical Sanov theorem (see [DZ10, Section 6.2]) since in this case the random variables $\theta_{x_i}\cdot\mathbf{C}_i$ are independent and identically distributed Poisson point processes on each hypercube. Taking the limit $R\to\infty$ yields, in view of the asymptotics (6.6) on $m_{N,R}$ and the definition (1.13) of the specific relative entropy,

$$\lim_{R\to\infty}\liminf_{\nu\to 0}\liminf_{N\to\infty}\frac{1}{N}\log\dot{\mathfrak{M}}_{N,R}(B(P_{\underline{m}|C_R},\varepsilon))\geq-\mathsf{ent}\left[P_{\underline{m}}|\mathbf{\Pi}^1\right]. \tag{7.19}$$

We may then extend this LDP to the context of tagged point processes by following essentially the same argument as in the proof of the continuous case.

We then argue as before in the proof of LDP lower bound for the continuous average in the Bernoulli case. To condition the point process into having $N_{\mathrm{int}}\approx N\mu_V(\Sigma_{\underline{m}})$ points in $\Sigma'_{\mathrm{int}}\approx\Sigma'_{\underline{m}}$ modifies the LDP lower bound obtained from Sanov's theorem by a quantity which is an adaptation of (7.17) in this setting

$$\limsup_{N\to\infty}\frac{1}{N|\Sigma_{\underline{m}}|}\log\left(e^{-N|\Sigma_{\underline{m}}|}\frac{\left(N|\Sigma_{\underline{m}}|\right)^{N\mu_V(\Sigma_{\underline{m}})}}{\left(N\mu_V(\Sigma_{\underline{m}})\right)!}\right),$$

hence the constant $r_{\mu_V,\underline{m}}$ in (6.17). This settles the first point of Lemma 6.3.

The second point follows from the first one by elementary manipulations. The main argument is that if one knows that a discrete average of large hypercubes is very close to some point process P, then the continuous average of much smaller hypercubes is also close to P since it can be re-written using the discrete average up to a small error. More precisely for any fixed $\delta>0$ establishing that a point process is in $B(P,\delta)$ can be done by testing against local functions in Loc_k for some k large enough (because of the topology on $\mathcal{X}$ and the approximation Lemma 2.5). For R,N large enough an overwhelming majority of all translates of C_k by a point in $\Sigma'_{\underline{m}}$ is included in one of the hypercubes $\overline{C}_i$ ($i=1\dots m_{N,R}$) (this follows from the definitions and (6.6)).

For any such local function $f\in\mathrm{Loc}_k$ we have

$$\frac{1}{|\Sigma'_{\underline{m}}|}\int_{\Sigma'_{\underline{m}}}f(\theta_x\cdot\mathcal{C})\approx\frac{1}{m_{N,R}}\sum_{i=1}^{m_{N,R}}\frac{1}{R^d}\int_{\bar{C}_i}f(\theta_x\cdot\mathcal{C})dx, \tag{7.20}$$

which allows us to pass from the assumption that the discrete average (in the right-hand side of (7.20)) of a configuration is close to P to the fact that the continuous average (in the left-hand side of (7.20)) is close to P. These considerations are easily adapted to the situation of tagged point processes.

8. Additional proofs

We collect here the proofs of various lemmas used in the course of the paper.

8.1. **Proof of Lemma 2.5.** The first point ($\mathcal{X}$ is a Polish space) is well-known, see e.g. [DVJ08, Proposition 9.1.IV]). It is easy to see that $d_{\mathcal{X}}$ is a well-defined distance (the only point to check is the separation property). It is also clear that any sequence converging for $d_{\mathcal{X}}$ converges for the topology on $\mathcal{X}$. Conversely, let $\{\mu_n\}_n$ be a sequence in $\mathcal{X}$ which converges vaguely to μ and let $\varepsilon > 0$. There exists an integer K such that $\sum_{k\geq K} \frac{1}{2^k} \leq \frac{\varepsilon}{2}$ so we might restrict ourselves to the first K terms in the series defining $d_{\mathcal{X}}(\mu_n, \mu)$. For each $k = 1, \dots, K$ and for any n, let $\mu_{n,k}$ and μ_k be the restriction to the hypercube C_k of each term (and of the limit). For any $k = 1, \dots, K$ the sequence of masses $(\mu_{n,k}(C_k))_{k\geq 1}$ is an integer sequence and up to passing to a common subsequence by a standard diagonal argument, we may assume that for each k the sequence $\{\mu_{n,k}(C_k)\}_{k\geq 1}$ is either constant or diverging to $+\infty$. We may then restrict ourselves to the terms k for which the sequence is constant $\equiv N_k$. By compactness we may then assume that the N_k points of the configuration converge to some N_k-uple $x_1 \dots x_{N_k}$ of points in C_k. It is easy to see that N_k must be equal to $\mu_k(C_k)$ and that the points $x_1 \dots x_{N_k}$ must correspond to the points of the configuration μ_k. This implies the convergence in the sense of $d_{\mathcal{X}}$. From any sequence $\{\mu_n\}_n$ which converges weakly to μ we may extract a subsequence which converges to μ in the sense of $d_{\mathcal{X}}$ (and the converse is true), which ensures that $d_{\mathcal{X}}$ is compatible with the topology on $\mathcal{X}$.

We now prove the approximation property stated in the third point of Lemma 2.5. By density it is enough to prove the second part of the statement i.e. the (uniform) approximation of Lipschitz functions by local functions. Let F be in $\mathrm{Lip}_1(\mathcal{X})$ and $\delta > 0$. From the definition (2.39) of $d_{\mathcal{X}}$ we see that there exists k such that if two configurations $\mathcal{C}, \mathcal{C}'$ coincide on C_k then $d_{\mathcal{X}}(\mathcal{C}, \mathcal{C}') \leq \delta$. We let $f_k := F(\mathcal{C} \cap C_k)$. By definition f is a local function in Loc_k, we have chosen k such that $d_{\mathcal{X}}(\mathcal{C}, \mathcal{C} \cap C_k) \leq \delta$ for any configuration $\mathcal{C}$ and since by assumption F is 1-Lipschitz we have

$$|F(\mathcal{C}) - f(\mathcal{C})| = |F(\mathcal{C}) - F(\mathcal{C} \cap C_k)| \leq d_{\mathcal{X}}(\mathcal{C}, \mathcal{C} \cap C_k) \leq \delta,$$

and k here depends only on δ, which concludes the proof of Lemma 2.5.

8.2. **Proof of Lemma 2.11.** Let us denote as in Section 2.2 by $X = (x, y)$ the coordinates in $\mathbb{R}^d \times \mathbb{R}^k$. We also recall that $\gamma \in (-1, 1)$. Let E_1 and E_2 be elements of $\mathcal{A}_m$ such that $\mathrm{Conf}_m E_1 = \mathrm{Conf}_m E_2$. Then we have $E_1 - E_2 = \nabla u$ where u solves $-\mathrm{div}\,(|y|^\gamma \nabla u) = 0$. We can also observe that $\nabla_x u$ (where ∇_x denote the vector of derivatives in the x directions only, is also a solution to the same equation (this should be understood component by component). This is a divergence form equation with a weight $|y|^\gamma$ which belongs to the so-called Muckenhoupt class A_2. The result of [FKS82, Theorem 2.3.12] then says that there exists $\lambda > 0$ such that for $0 < r < R$,

$$\mathrm{osc}(\nabla_x u, B(X, r)) \leq C \left(\frac{1}{\int_{B(X,R)} |y|^\gamma} \int_{B(X,R)} |y|^\gamma |\nabla_x u|^2 \right)^{1/2} (r/R)^\lambda, \tag{8.1}$$

where $\mathrm{osc}(u, B(X,r)) = \max_{B(X,r)} u - \min_{B(X,r)} u$. We note that the condition that $\mathcal{W}(E_1)$ and $\mathcal{W}(E_2)$ imply without difficulty that

$$\limsup_{R\to\infty} \frac{1}{R^d} \int_{K_R\times\mathbb{R}^d} |y|^\gamma |\nabla u|^2 < +\infty. \tag{8.2}$$

Applying (8.1) to X which belongs to a fixed compact set, and inserting (8.2) we find that

$$\mathrm{osc}(\nabla_x u, B(X,r)) \le C\left(R^{-(d+1+\gamma)}R^d\right)^{1/2}(r/R)^\lambda$$

in the case $k=1$, and respectively

$$\mathrm{osc}(\nabla_x u, B(X,r)) \le C\left(R^{-d}R^d\right)^{1/2}(r/R)^\lambda$$

in the case $k=0$. In both cases, letting $R\to\infty$, we deduce that $\mathrm{osc}(\nabla_x u, B(X,r)) = 0$, which means that $\nabla_x u$ is constant on every compact set of $\mathbb{R}^{d+k}$.

In the case $k=0$, this concludes the proof that u is affine, and then E_1 and E_2 differ by a constant vector.

In the case $k=1$, this implies that u is an affine function of x, for each given y. We may thus write $u(x,y) = a(y)\cdot x + b(y)$. Inserting into the equation $\mathrm{div}\,(|y|^\gamma \nabla u) = 0$, we find that $\partial_y(|y|^\gamma(a'(y)x + b'(y)) = 0$, i.e. $a'(y)x + b'(y) = \frac{c(x)}{|y|^\gamma}$. But the fact that $\int_{\mathbb{R}} |y|^\gamma |\partial_y u|^2\, dy$ is convergent implies that $\int \frac{c(x)^2}{|y|^\gamma}\, dy$ must be, which implies that $c(x) = 0$ and thus $\partial_y u = 0$. This means that $u(x,y) = f(x)$. But then again $\int |y|^\gamma |\nabla u|^2\, dy$ is convergent so we must have $\nabla f(x) = 0$ and u is constant. Thus $E_1 = E_2$ as claimed.

In the case $k=1$, it follows that $\mathbb{W}_m(\mathcal{C})$ (if it is not infinite) becomes an inf over a singleton, hence is achieved.

Let us now turn to the case $k=0$ (in that case we note that we must have $s = d-2$ or (1.3)). Let $E\in\mathcal{A}_m$ be such that $\mathrm{Conf}_m E = \mathcal{C}$ and $\mathcal{W}(E) < \infty$ (if it exists), and let c be a constant vector in $\mathbb{R}^d$, then

$$\fint_{K_R} |E_\eta + c|^2 - mc_{d,s}g(\eta) = \fint_{K_R} |E_\eta|^2 - mc_{d,s}g(\eta) + |c|^2 + 2c\cdot \fint_{K_R} E_\eta. \tag{8.3}$$

We claim that $\fint_{K_R} E_\eta$ is bounded independently of η and R. So the right-hand side of (8.3) is a quadratic function of c, with fixed quadratic coefficients and linear and constant coefficients which are bounded with respect to R and η. A little bit of convex analysis implies that $c\mapsto \mathcal{W}(E+c)$ being a limsup (over R and η) of such functions is strictly convex, coercive and locally Lipschitz, hence it achieves its minimum for a unique c. This means that the infimum defining $\mathbb{W}_m$ is a uniquely achieved minimum.

To conclude the proof, we just need to justify that $\fint_{K_R} E_\eta$ is bounded independently of η and R. We may write

$$\fint_{K_R} E_\eta = \fint_{K_R} E_1 + \fint_{K_R} ((\nabla f_1 - \nabla f_\eta) * \mathcal{C},$$

where f_η is as in (2.20). Because we are in the case $s = d-2$ or (1.3), ∇f_η and ∇f_1 are integrable and we may check that $\int_{K_R} ((\nabla f_1 - \nabla f_\eta) * \mathcal{C}$ is bounded by $C\mathcal{C}(K_R)$ where C is independent of R and η. But since $\mathcal{W}(E) < \infty$ and $E\in\mathcal{A}_m$, we have $\lim_{R\to\infty} \frac{1}{|K_R|}\mathcal{C}(K_R) = m$

(cf. [PS14, Lemma 2.1]). It follows that

$$\left| \fint_{K_R} E_\eta \right| \le C(1 + \mathcal{W}_1(E) + m)$$

and by almost monotonicity of $\mathcal{W}$ (Lemma 3.4) the claim follows.

8.3. **Proof of Lemma 2.12.** Let $\mathcal{X}_1^o$ be the image of $\mathcal{A}_1$ by $\mathcal{X}_1$ i.e. the set of point configuration of "mean density" 1 for which one can define a corresponding electric field. Let $\mathcal{C} \mapsto E(\mathcal{C})$ be a measurable map from $\mathcal{X}_1^o$ to $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$ such that for any $\mathcal{C}$ we have $\mathcal{X}_1(E(\mathcal{C})) = \mathcal{C}$ and $\mathcal{W}(E(\mathcal{C})) = \mathbb{W}_1(\mathcal{C})$ (such a map can be chosen measurable because the set of electric fields E satisfying these conditions is closed, since it is a singleton according to Lemma 2.11).

For any $\mathcal{C}$ in $\mathcal{X}_1^o$ let us define the following sequence of random electric fields

$$P^{\mathrm{elec}}_{k,\mathcal{C}} := \fint_{C_k} \delta_{\theta_x \cdot E(\mathcal{C})}.$$

We claim that if $\mathbb{W}_1(\mathcal{C})$ is finite then the sequence $\{P^{\mathrm{elec}}_{k,\mathcal{C}}\}_k$ is relatively compact in $\mathcal{P}(L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k}))$ for the weak topology on $L^p_{\mathrm{loc}}(\mathbb{R}^{d+k}, \mathbb{R}^{d+k})$. Indeed for any integer m we have

$$\int \left[\int_{C_m} |y|^\gamma |E_\eta|^2 \right] dP^{\mathrm{elec}}_{k,\mathcal{C}} = \frac{1}{|C_k|} \int_{C_k} dx \int_{C_m} |y|^\gamma |\theta_x \cdot E_\eta|^2 \le \frac{1}{|C_k|} \int_{C_{k+m}} |y|^\gamma |E_\eta|^2 \tag{8.4}$$

and by definition of $\mathcal{W}$ we have

$$\lim_{k\to\infty} \frac{1}{|C_k|} \int_{C_{k+m}} |y|^\gamma |E_\eta|^2 = \mathcal{W}_\eta(E) + c_{d,s} g(\eta). \tag{8.5}$$

This implies that the sequence $\left\{ \int \left[\int_{C_m} |y|^\gamma |E_\eta|^2 \right] dP^{\mathrm{elec}}_{k,\mathcal{C}} \right\}_k$ is bounded. Arguing as in the proof of Lemma 3.7 we get the existence of a limit point for $\{P^{\mathrm{elec}}_{k,\mathcal{C}}\}_k$.

Let $\mathcal{C} \mapsto P^{\mathrm{elec}}_{\infty,\mathcal{C}}$ be a measurable choice of a weak limit point (see e.g. [Coh72]) on the (measurable) set $\{\mathcal{C}, \mathbb{W}_1(\mathcal{C}) \text{ is finite}\}$. It is easy to see that $P^{\mathrm{elec}}_{\infty,\mathcal{C}}$ is stationary (since we average $\theta_x \cdot E(\mathcal{C})$ on large hypercubes), concentrated on $\mathcal{A}_1$. Moreover, in view of (8.4)–(8.5) we have

$$\begin{aligned} \mathcal{W}_\eta(E^\varepsilon(\mathcal{C})) &\ge \liminf_{k\to\infty} \int \left[\int_{C_m} |y|^\gamma |E_\eta|^2 \right] dP^{\mathrm{elec}}_{k,\mathcal{C}}(E) - c_{d,s} g(\eta) \\ &= \int \left[\int_{C_m} |y|^\gamma |E_\eta|^2 \right] dP^{\mathrm{elec}}_{\infty,\mathcal{C}}(E) - c_{d,s} g(\eta) \end{aligned}$$

and the right-hand side is $\widetilde{\mathcal{W}}_\eta(P^{\mathrm{elec}}_{\infty,\mathcal{C}})$ by stationarity of $P^{\mathrm{elec}}_{\infty,\mathcal{C}}$ and Lemma 2.8. Letting $\eta \to 0$ we deduce that $\widetilde{\mathcal{W}}(P^{\mathrm{elec}}_{\infty,\mathcal{C}}) \le \mathcal{W}(E)(\mathcal{C}))$.

Since $\widetilde{\mathbb{W}}_1(P)$ is finite, $\mathbb{W}_1(\mathcal{C})$ is finite P-a.s. and we may define the probability measure

$$P^{\mathrm{elec}}_\infty := \int \delta_{P^{\mathrm{elec}}_{\infty,\mathcal{C}}} dP(\mathcal{C}).$$

We check that

- P^{elec}_∞ is stationary, because $P^{\mathrm{elec}}_{\infty,\mathcal{C}}$ is stationary for P-almost every $\mathcal{C}$.

- We have

$$\widetilde{\mathcal{W}}(P^{\mathrm{elec}}_{\infty}) = \int \widetilde{\mathcal{W}}(P^{\mathrm{elec}}_{\infty,\mathcal{C}})dP(\mathcal{C}) \leq \int \mathcal{W}(E(\mathcal{C}))dP(\mathcal{C}) \leq \int (\mathbb{W}_1(\mathcal{C}))dP(\mathcal{C}) \leq \widetilde{\mathbb{W}}_1(P) + \varepsilon.$$

- The push-forward of $P^{\mathrm{elec}}_{\infty}$ by Conf_1 is P, because this the case of $\int \delta_{P^{\mathrm{elec}}_{k,c}} dP(\mathcal{C})$ for all $k \geq 1$.

Hence we get that

$$\widetilde{\mathbb{W}}_1(P) \geq \min\{\widetilde{\mathcal{W}}(P^{\mathrm{elec}}) \mid P^{\mathrm{elec}} \text{ is stationary and the push-forward of } P^{\mathrm{elec}} \text{ by } \mathrm{Conf}_1 \text{ is } P\}.$$

The reverse inequality is obvious by definition of $\widetilde{\mathbb{W}}_1$.

8.4. **Proof of Lemma 4.4.** Let $k > 0$, $\varepsilon > 0$ and N be fixed. Let us consider $u_N \in \mathbb{R}^d$ such that $C_k \cup (u_N + C_k) \subset B(0, \sqrt{(1-\varepsilon)N})$. We first bound the number of points in $(C_k \cup C_k + u_N)$ with overwhelming probability, uniformly on the choice of u_N. For $k_N = N^{1/2+1/10}$ we have:

$$\mathbb{P}_{N,2}\left(\mathcal{N}(0,k) + \mathcal{N}(u_n,k) \geq k_N\right) = o\left(N^{-N}\right), \tag{8.6}$$

uniformly on the choice of u_N. This can be deduced e.g. from discrepancy estimates as in Lemma 3.8, which imply that

$$\mathbb{P}_{N,2}\left(\mathcal{N}(0,k) + \mathcal{N}(u_N,k) \geq k_N\right)$$

is bounded above by $\exp(-k_N^2)$ pour $k_N \gg \sqrt{N}$ (and uniformly on the choice of u_N). We may then neglect the event $\{\mathcal{N}(0,k) + \mathcal{N}(u_N,k) \geq k_N\}$ which contributes only with order $o(N^{-N})$ to (4.15).

Conditioning on the number of points in $C_k \cup (u_N + C_k)$ we may then restrict ourselves to quantify the translation-invariance of $\rho_{(N,2),k}$ (k-point correlation function of the determinantal point process $\mathbf{P}_{N,2}$) for all $k \leq k_N$. The determinantal nature of $\mathbf{P}_{N,2}$ implies that

$$\rho_{(N,2),k} = \det\left[K_N(x_i,x_j)\right]_{1\leq i,j\leq k}. \tag{8.7}$$

We know that $\rho_{(N,2),k}$ converges the correlation function $\rho_{(\infty,2),k}$ of Gin_2 which are translation-invariant (see e.g. [HKPV09, 4.3.7]). Thus we are left to bound the difference between $\rho_{(N,2),k}$ and $\rho_{(\infty,2),k}$. Let us compare the kernels K_∞ and K_N :

$$\begin{aligned} K_N(x_i,x_j) &= \frac{1}{\pi} e^{-\frac{|x_i|^2+|x_j|^2}{2}} \sum_{l=0}^{N-1} \frac{(x_i\bar{x_j})^l}{l!} = \frac{1}{\pi} e^{-\frac{|x_i|^2+|x_j|^2}{2}} \left((e^{x_i\bar{x_j}}) - \sum_{l=N}^{+\infty} \frac{(x_i\bar{x_j})^l}{l!} \right) \\ &= K_\infty(x_i,x_j) - \frac{1}{\pi} e^{-\frac{|x_i|^2+|x_j|^2}{2}} \left(\sum_{l=N}^{+\infty} \frac{(x_i\bar{x_j})^l}{l!} \right). \end{aligned} \tag{8.8}$$

To bound the error term let us observe that

$$\begin{aligned} \left| \frac{1}{\pi} e^{-\frac{|x_i|^2+|x_j|^2}{2}} \left(\sum_{l=N}^{+\infty} \frac{(x_i\bar{x_j})^l}{l!} \right) \right| &\leq \frac{1}{\pi} e^{-|x_i\bar{x_j}|} \sum_{l=N}^{+\infty} \frac{1}{l!} \left(\frac{|x_i\bar{x_j}|}{N} \right)^l N^l \\ &\leq \frac{1}{\pi} e^{-|x_i\bar{x_j}|} \left(\frac{|x_i\bar{x_j}|}{N} \right)^N \left(\sum_{l=N}^{+\infty} \frac{1}{l!} N^l \right). \end{aligned} \tag{8.9}$$

We may now use the well-known equivalent

$$\sum_{l=N}^{+\infty} \frac{1}{l!} N^l \sim \frac{e^N}{2}.$$

We deduce that

$$\frac{1}{\pi} e^{-|x_i \bar{x_j}|} \left(\frac{|x_i \bar{x_j}|}{N} \right)^N \left(\sum_{l=N}^{+\infty} \frac{1}{l!} N^l \right) \leq C \exp\left(-N \left(\frac{|x_i \bar{x_j}|}{N} - \log \frac{|x_i \bar{x_j}|}{N} - 1 \right) \right). \tag{8.10}$$

It is elementary that $\log(1-t) \leq t + \frac{t^2}{2}$ for all $t \in [0,1]$. We deduce that for all x_i, x_j in the disk of radius $\sqrt{(1-\varepsilon)N}$, since $1 - \frac{|x_i \bar{x_j}|}{N} \geq \varepsilon$, we have

$$\left| \frac{1}{\pi} e^{-\frac{|x_i|^2 + |x_j|^2}{2}} \left(\sum_{l=N}^{+\infty} \frac{(x_i \bar{x_j})^l}{l!} \right) \right| \leq C \exp(-\frac{\varepsilon^2}{2} N). \tag{8.11}$$

We thus obtain $K_N(x_i, x_j) = K_\infty(x_i, x_j) + O(\exp(-\frac{\varepsilon^2}{2} N))$ uniformly for x_i, x_j in the disk of radius $\sqrt{(1-\varepsilon)N}$.

An explicit computation yields ($\mathfrak{S}_k$ denotes the group of permutation of k elements and s the signature morphism)

$$\begin{aligned} \det\left[K_N(x_i, x_j)\right] &= \sum_{\sigma \in \mathfrak{S}_k} s(\sigma) \prod_{i=1}^k \left(K_\infty(x_i, x_{\sigma(j)}) + O(\exp(-\frac{\varepsilon^2}{2} N)) \right) \\ &= \det\left[K_\infty(x_i, x_j)\right] + k! \times R_N \end{aligned} \tag{8.12}$$

with an error term R_N satisfying

$$\begin{aligned} |R_N| &\leq \sum_{l=0}^{k-1} \binom{N}{k} \left(\sup_{i,j} |K_N(x_i, x_j)| \right)^l \times O(\exp(-\frac{\varepsilon^2}{2} N))^{k-l} \\ &= \left(\sup_{i,j} |K_N(x_i, x_j)| + O(\exp(-\frac{\varepsilon^2}{2} N)) \right)^k - \sup_{i,j} |K_N(x_i, x_j)| \end{aligned}$$

but K_N is uniformly bounded by 1 so that for $k \leq k_N$ we have

$$|R_N| \leq O(\exp(-\frac{\varepsilon^2}{2} N)). \tag{8.13}$$

Since $k! \leq (k_N)!$ is bounded above by

$$\left(N^{\frac{1}{2} + \frac{1}{10}} \right)! \ll \exp(N^{\frac{1}{2} + \frac{1}{5}}). \tag{8.14}$$

Combining (8.12), (8.13) and (8.14) yields

$$\left| \det\left[K_n(x_i, x_j)\right]_{1 \leq i,j \leq k} - \det\left[K_\infty(x_i, x_j)\right]_{1 \leq i,j \leq k} \right| = O(\exp(-\frac{\varepsilon^2}{2} N)). \tag{8.15}$$

Equation (8.15) together with the invariance property of $\rho_{(\infty,2),k}$ concludes the proof of the lemma.

8.5. **Proof of Lemma 5.1.** Let $m = \fint_K \mu$ be the average of μ over K. We may (see e.g. [PS14, Lemma 6.3]) partition K into $n_{K,\mu}$ hyperrectangles $\mathcal{R}_i$, which all have volume $1/m$, and whose sidelengths are in $[2^{-d}m^{-1/d}, 2^d m^{1/d}]$. In each of these hyperrectangles we solve

$$\begin{cases} \operatorname{div}(|y|^\gamma \nabla h_i) = c_{d,s}(\delta_{X_i} - m\delta_{\mathbb{R}^d}) & \text{in } \mathcal{R}_i \times [-1,1]^k \\ \nabla h_i \cdot \vec{\nu} = 0 & \text{on } \partial(\mathcal{R}_i \times [-1,1]^k) \end{cases} \tag{8.16}$$

According to [PS14, Lemma 6.5], if $X_i \subset \mathbb{R}^d \times \{0\}$ is at distance $\le 2^{-(d+1)}m^{-1/d}$ from the center p_i of $\mathcal{R}_i$ then we have

$$\lim_{\eta \to 0} \left| \int_{\mathcal{R}_i \times [-1,1]^k} |y|^\gamma |\nabla (h_i)_\eta|^2 - c_{d,s} g(\eta) \right| \le C$$

where C depends only on d and m. We may then define $E_i = \nabla h_i \mathbf{1}_{\mathcal{R}_i \times [-1,1]^k}$, and by compatibility of the normal components, the vector field $E^{\text{gen}} = \sum_i E_i$ satisfies

$$\begin{cases} \operatorname{div}(|y|^\gamma E^{\text{gen}}) = c_{d,s}(\sum_i \delta_{X_i} - m\delta_{\mathbb{R}^d}) & \text{in } K \times \mathbb{R}^k \\ E^{\text{gen}} \cdot \vec{\nu} = 0 & \text{on } \partial(K \times \mathbb{R}^k) \end{cases} \tag{8.17}$$

and if $\eta < \eta_0 < 2^{-(d+2)}m^{-1/d}$,

$$\int_{K \times \mathbb{R}^k} |y|^\gamma |E^{\text{gen}}_\eta|^2 - c_{d,s} n_{K,\mu} g(\eta) \le C n_{K,\mu} \tag{8.18}$$

with C depending only on d and m. The last step is to rectify for the error made by replacing μ by m. For that, we use the following

Lemma 8.1 ([PS14], Lemma 6.4). *Let K_R be a hyperrectangle whose sidelengths are in $[R, 2R]$, and μ a bounded measurable function such that $\int_{K_R} \mu$ is an integer, and let $m = \fint_{K_R} \mu$. The solution (unique up to constant) to*

$$\begin{cases} \operatorname{div}(|y|^\gamma \nabla h) = c_{d,s}(\mu - m)\delta_{\mathbb{R}^d} & \text{in } K_R \times [-R,R]^k \\ \nabla h \cdot \vec{\nu} = 0 & \text{on } \partial(K_R \times [-R,R]^k), \end{cases}$$

exists and satisfies

$$\int_{K_R \times [-R,R]^k} |y|^\gamma |\nabla h|^2 \le C R^{d+1-\gamma} \|\mu - m\|^2_{L^\infty(K_R)}. \tag{8.19}$$

Applying this lemma provides a function h, and we let

$$\hat{E} = E + \nabla h \mathbf{1}_{K \times [-R,R]^k}.$$

It is obvious that $\hat{E}$ solves

$$\begin{cases} \operatorname{div}(|y|^\gamma \hat{E}) = c_{d,s}(\sum_i \delta_{X_i} - \mu\delta_{\mathbb{R}^d}) & \text{in } K \times \mathbb{R}^k \\ \hat{E} \cdot \vec{\nu} = 0 & \text{on } \partial(K \times \mathbb{R}^k). \end{cases} \tag{8.20}$$

Combining (8.19) and (8.18) and using the Cauchy-Schwarz inequality, we obtain

$$\int_{K \times \mathbb{R}^k} |y|^\gamma |\hat{E}_\eta|^2 \le c_{d,s} n_{K,\mu}(g(\eta) + C) + C R^{d+1-\gamma} \|\mu - m\|^2_{L^\infty(K_R)} \\ + C\,(n_{K,\mu} g(\eta))^{\frac{1}{2}} R^{\frac{d+1-\gamma}{2}} \|\mu - m\|_{L^\infty(K_R)}.$$

Letting then $\mathcal{R}(K,\mu)$ be the family of configurations $\{X_i\}_{i=1}^{n_{K,\mu}}$ above where each X_i varies in $B(p_i, 2^{-(d+1)}m^{-1/d})$ (p_i being the center of $\mathcal{R}_i$), and with all possible permutations of the labels, we have thus obtained that for every $\mathcal{C} \in \mathcal{R}(K,\mu)$ the desired results hold.

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