Non-standard Hilbert function and graded Betti numbers of powers of ideals

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Fonction de Hilbert non standard et nombres de Betti gradués des puissances d'idéaux

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Résumé

Fonction de Hilbert non standard et nombres de Betti gradués des puissances d'idéaux

En utilisant le concept des fonctions de partition , nous étudions le comportement asymptotique des nombres de Betti gradués des puissances d'idéaux homogènes dans un polynôme sur un corp.

Pour un $\mathbb{Z}$-graduer positif, notre résultat principal affirme que les nombres de Betti des puissances est codé par un nombre fini des polynômes. Plus précisément, $\mathbb{Z}^2$ peut être divisé en un nombre fini des régions telles que, dans chacun d'eux, $\dim_k \left(\mathrm{Tor}_i^S(I^t,k)_\mu\right)$ est un quasi-polynôme en (μ,t). Ce affine, dans une situation graduée, le résultat de Kodiyalam sur nombres de Betti des puissances dans [33].

La déclaration principale traite le cas des produits des puissances d'idéaux homogènes dans un algèbre $\mathbb{Z}^d$-graduée , pour un graduer positif, dans le sens de [37] et il est généralise également pour les filtrations I-good

.

Dans la deuxième partie, en utilisant la version paramétrique de l'algorithme de Barvinok, nous donnons une formule fermée pour les fonctions de Hilbert non-standard d'anneaux de polynômes, en petites dimensions.

Mots-clefs

Nombres de Betti, Fonction de Hilbert non standard , Fonction de partition vectorielle.

Abstract

Non-standard Hilbert function and graded Betti numbers of powers of ideals

Using the concept of vector partition functions, we investigate the asymptotic behavior of graded Betti numbers of powers of homogeneous ideals in a polynomial ring over a field.

For a positive $\mathbb{Z}$-grading, our main result states that the Betti numbers of powers is encoded by finitely many polynomials. More precisely, $\mathbb{Z}^2$ can be splitted into a finite number of regions such that, in each of them, $\dim_k\left(\mathrm{Tor}_i^S(I^t,k)_\mu\right)$ is a quasi-polynomial in (μ,t). This refines, in a graded situation, the result of Kodiyalam on Betti numbers of powers in [33].

The main statement treats the case of a power products of homogeneous ideals in a $\mathbb{Z}^d$-graded algebra, for a positive grading, in the sense of [37] and it is also generalizes to I-good filtrations .

In the second part , using the parametric version of Barvinok's algorithm, we give a closed formula for non-standard Hilbert functions of polynomial rings, in low dimensions.

Keywords

Betti numbers, Nonstandard Hilbert function, Vector partition function.

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CHAPTER 1

INTRODUCTION

The study of homological invariants of powers of ideals goes back, at least, to the work of Brodmann in the 70's and attracted a lot of attention these last two decades.

One of the most important results in this area is the result on the asymptotic linearity of Castelnuovo-Mumford regularity obtained by Kodiyalam [34] and Cutkosky, Herzog and Trung [19], independently. The proof of Cutkosky, Herzog and Trung further shows the eventual linearity in t of $\mathrm{end}\big(\mathrm{Tor}_i^S(I^t,k)\big) := \max\{\mu \mid \mathrm{Tor}_i^S(I^t,k)_\mu \neq 0\}$.

It is natural to investigate the asymptotic behavior of Betti numbers $\beta_i(I^t) := \dim_k \mathrm{Tor}_i^S(I^t,k)$ as t varies. In [39], Northcott and Rees already investigated the asymptotic behavior of $\beta_1^k(I^t)$. Later, using the Hilbert-Serre theorem, Kodiyalam [33, Theorem 1] proved that for any non-negative integer i and sufficiently large t, the i-th Betti number, $\beta_i^k(I^t)$, is a polynomial Q_i in t of degree at most the analytic spread of I minus one.

Recently, refining the result of [19] on end$\left(\mathrm{Tor}_i^S(I^t,k)\right)$, [6] gives a precise picture of the set of degrees γ such that $\mathrm{Tor}_i^S(I^t,A)_\gamma \neq 0$ when t varies in $\mathbb{N}$. The article [6] considers a polynomial ring $S = A[x_1,\dots,x_n]$ graded by a finitely generated abelian group G over a Noetherian ring A, see [6, Theorem 4.6].

When $A = k$ is a field and the ideal is generated by a single degree $d \in G$, it is proved that for any γ and any j, the function

$$\dim_k \mathrm{Tor}_i^S(I^t,k)_{\gamma+td}$$

is a polynomial in t for $t \gg 0$, See [6, Theorem 3.3] and [44].

My Thesis is here interested in the behavior of $\dim_k \mathrm{Tor}_i^S(I^t,k)_\gamma$ when I is an arbitrary graded ideal and $S = k[x_1,\dots,x_n]$ is a $\mathbb{Z}^p$-graded polynomial ring over a field k, for a positive grading in the sense of [37].

In the case of a positive $\mathbb{Z}$-grading, the result takes the following form:

Theorem 1.0.1. *(See Theorem 5.1.2). Let $S = k[x_1,\dots,x_n]$ be a positively graded polynomial ring over a field k and let I be a homogeneous ideal in S.*

There exist, $t_0, m, D \in \mathbb{Z}$, linear functions $L_i(t) = a_i t + b_i$, for $i = 0,\dots,m$, with a_i among the degrees of the minimal generators of I and $b_i \in \mathbb{Z}$, and polynomials $Q_{i,j} \in \mathbb{Q}[x,y]$ for $i = 1,\dots,m$ and $j \in 1,\dots,D$, such that, for $t \geq t_0$,

(i) $L_i(t) < L_j(t) \Leftrightarrow i < j$,

(ii) If $\mu < L_0(t)$ or $\mu > L_m(t)$, then $\mathrm{Tor}_i^S(I^t,k)_\mu = 0$.

(iii) If $L_{i-1}(t) \leq \mu \leq L_i(t)$ and $a_i t - \mu \equiv j \mod (D)$, then

$$\dim_k \mathrm{Tor}_i^S(I^t,k)_\mu = Q_{i,j}(\mu,t).$$

Our general result, Theorem 4.2.4, involves a finitely generated graded module M and a finite collection of graded ideals $I_1,\dots,I_s$. The grading is a positive $\mathbb{Z}^p$-grading, and a special type of finite decomposition of $\mathbb{Z}^{p+s}$ is described in such a way that in each region $\dim_k(\mathrm{Tor}_i^S(MI_1^{t_1}\dots I_s^{t_s},k)_\gamma)$ is a quasi-polynomial in $(\gamma,t_1,\dots,t_s)$, with respect to a lattice defined in terms of the degrees of generators of the ideals.

The central object in this study is the Rees modification $M\mathscr{R}_I$. This graded object admits a graded free resolution over a polynomial extension of S, from which we deduce a $\mathbb{Z}^{p+s}$-grading on Tor modules as in [6]. Investigating Hilbert series of modules for such a grading, using vector partition functions, leads to the results.

In the last chapter we will study the structure and dimension of each pieces of Tor module of I-good filtrations the main result takes the following form :

Theorem 1.0.2. *(See Theorem 5.2.3)Let $S = A[x_1,\cdots,x_1]$ be a graded algebra over a Noetherian local ring $(A,m)\subset S_0$. Let $\varphi=\{\varphi(n)\}_{n\geq 0}$ be a I-good filtration of ideals $\varphi(n)$ of R and $\varphi(1)=(f_1,f_2,...,f_r)$ with $deg\, f_i = d_i$ be $\mathbb{Z}$-homogenous ideal in S, and let $R=S[T_1,\dots,T_n]$ be a bigraded polynomial extension of S with $\deg(T_i)=(d_i,1)$ and $\deg(a)=(\deg(a),0)\in\mathbb{Z}\times\{0\}$ for all $a\in S$.*

(1)Then for all i,j :

$\mathrm{Tor}_i^A(\mathrm{Tor}_j^R(\mathscr{R}_\varphi,A),k)$ *is finitely generated $k[T_1,\dots,T_r]$-module .*

(2) There exist, $t_0,m,D\in\mathbb{Z}$, linear functions $L_i(t)=a_i t+b_i$, for $i=0,\dots,m$, with a_i among the degrees

of the minimal generators of I and $b_i \in \mathbb{Z}$, and polynomials $Q_{i,j} \in \mathbb{Q}[x,y]$ for $i = 1,\dots,m$ and $j \in 1,\dots,D$, such that, for $t \geq t_0$,

(i) $L_i(t) < L_j(t) \Leftrightarrow i < j$,

(ii) If $\mu < L_0(t)$ or $\mu > L_m(t)$, then $\operatorname{Tor}_i^S(\varphi(t),k)_\mu = 0$.

(iii) If $L_{i-1}(t) \leq \mu \leq L_i(t)$ and $a_i t - \mu \equiv j \mod (D)$, then

$$\dim_k \operatorname{Tor}_i^S(\varphi(t),k)_\mu = Q_{i,j}(\mu,t).$$

Other interest of my thesis is about Non-standard Hilbert functions, actually non-standard Hilbert functions first raised in the in Gabber's proof of Serre non-negativity conjecture [41]. It has been studied by several authors [21, 41, 42]. As it was noticed in [6] , the module $\oplus_t \operatorname{Tor}_i^S(I^t,k)$ for a homogeneous ideal I in graded ring S has the structure of a finitely generated graded module over a non-standard graded polynomial ring over k , from which one can deduce the behavior of $\operatorname{Tor}_i^S(I^t,k)$ when t varies.

It is also desirable to give closed formula for quasi-polynomials coming from a vector partition function. However, in general, such a formula doesn't exist . An algorithm that uses a continued fraction expansion and gives closed formula for the generating function corresponding to a two dimensional polytope was given by Barvinok. We use a parametric version of his algorithm and deduce the Hilbert function of polynomial ring $B = k[T_1,\dots,T_n]$ such that $\{\deg T_i | 1 \leqslant i \leqslant n\} = \{(d_j,1) | 1 \leqslant j \leqslant 4\}$ more precisely for the associated polytope $P(b = (b_1,b_2)) = \{x \in \mathbb{R}^r | Ax = b; x \geqslant 0\}$, where $A = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$, of the Hilbert function $HF(B,(b_1,b_2))$.The problem is that $P(b)$ is not full dimensional. To be able use the Barvinok algorithm, one should transform $P(b)$ to the full dimensional polytope Q which has the same lattice point as $P(b)$ .The following lemma gives us the complete information about Q:

Lemma 1.0.3. *(See Lemma 3.5.4)Let* $A = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ *for* $d_1 < d_2 < d_3 < d_4$ *then there is a one to one correspondence between the integer points of* $P(b)$ *and* $Q \subset \mathbb{R}^2$ *and we have the following about* Q*:*

1. $Q = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_1(d_2 - d_1) \leqslant 0;\ \ \lambda_2(d_2 - d_1) \leqslant 0;$
   $\lambda_1(d_1 - d_4) + \lambda_2(d_1 - d_3) + \frac{d_1 b_2 - b_1}{d_2 - d_1} \leqslant 0;\ \ \lambda_1(d_4 - d_2) + \lambda_2(d_3 - d_2) + \frac{b_1 - d_2 b_2}{d_2 - d_1} \leqslant 0\ \ for\ \ b_1, b_2 > 0\},$

2. *Q has the following vertices :*

   $Q_1 = (\frac{d_3 b_2 - b_1}{(d_2-d_1)(d_4-d_3)}, (\frac{b_1 - d_4 b_2}{(d_2-d_1)(d_4-d_3)})$,

   $Q_2 = (\frac{d_2 b_2 - b_1}{(d_2-d_1)(d_4-d_2)}, 0)$,

   $Q_3 = (0, \frac{d_2 b_2 - b_1}{(d_2-d_1)(d_3-d_2)})$,

   $Q_4 = (\frac{b_1 - d_1 b_2}{(d_2-d_1)(d_1-d_4)}, 0)$,

   $Q_5 = (0, \frac{b_1 - d_1 b_2}{(d_2-d_1)(d_1-d_3)})$,

   $Q_6 = (0, 0),$

3. *The generation function of Q in the first chamber* C_1 *is :*

   $$f_{C_1}(Q, \boldsymbol{x}) = \frac{1}{(1-x_1^{-1})(1-x_2^{-1})} + \frac{x_1^{\lceil s_1 \rceil}}{(1-x_1)(1-x_2^{-1})} - \frac{x_1^{\lceil s_1 \rceil} x_2^{-(\lceil a_0 s_1 \rceil + a_0 \lceil s_1 \rceil)}}{(1-x_2^{-1})(1-x_1 x_2^{-a_0})} +$$

   $$\frac{x_1^{(\lceil (a_0 a_1 + 1) s_1 \rceil - a_1 \lceil a_0 s_1 \rceil)} x_2^{-(a_0 \lceil (a_0 a_1 + 1) s_1 \rceil - (a_0 a_1 + 1) \lceil s_1 \rceil)}}{(1 - x_1 x_2^{-a_0})(1 - x_1^{a_1} x_2^{-(a_1 a_0 + 1)})} + \frac{x_2^{-\lceil s_2 \rceil}}{(1-x_2^{-1})(1-x_1 x_2^{-a_0})} -$$

   $$\frac{x_1^{(\lceil -a_1 s_2 \rceil + a_1 \lceil s_2 \rceil)} x_2^{-(a_0 \lceil -a_1 s_2 \rceil + (a_0 a_1 + 1) \lceil s_2 \rceil)}}{(1 - x_1 x_2^{-a_0})(1 - x_1^{a_1} x_2^{-(a_1 a_0 + 1)})}$$

Where $s_1 := \frac{b_1 - d_1 b_2}{(d_2-d_1)(d_1-d_4)}$ *and* $s_2 := \frac{b_1 - d_1 b_2}{(d_2-d_1)(d_1-d_3)}$.

Finally evaluating $f_{C_1}(Q, \mathbf{x})$ at $\mathbf{x} = (1,1)$ gives us $HF(B,(b_1,b_2))$. But our generating function has a pole at $\mathbf{x} = (1,1)$, so that we use the Yoshida et al[24] method to find the explicit formula of the Hilbert function of B from theorem 3.5.6 .

CHAPTER 2

PRELIMINARIES

2.1 commutative algebra

2.1.1 Graded Rings and Modules

Definition 2.1.1. A $\mathbb{N}$-graded ring R is a ring together with a decomposition (as abelian groups) $R = R_0 \oplus R_1 \oplus \ldots$ such that $R_m.R_n \subseteq R_{m+n}$ for all $m, n \in \mathbb{Z}_{\geq 0}$, and where R_0 is a subring (i.e. $1 \in R_0$). A $\mathbb{Z}$-graded ring is one where the decomposition is into $R = \oplus_{n \in \mathbb{Z}} R_n$. In either case, the elements of the subgroup R_n are called homogeneous of degree n.

Let R be a ring and $x_1, \ldots, x_n$ indeterminates over R. For $\mathbf{u} = (u_1, \cdots, u_n) \in \mathbb{N}^n$ let $\mathbf{x}^{\mathbf{u}} = x_1^{u_1} \cdots x_n^{u_n}$, then one can consider the polynomial ring $S = R[x_1, \ldots, x_n]$ as a graded ring by the total degree of polynomial where in this case the graded pieces can be write as

$$S_m = \{ \sum_{u_i \in \mathbb{N}^n; i=1}^{m} a_{u_i} x^{u_i} | a_{u_i} \in R \ \ and \ \ u_1 + \cdots + u_n = m \}$$

Definition 2.1.2. A graded R-module is an ordinary R-module M together with a decomposition $M = \oplus_{k \in \mathbb{Z}} M_k$ as abelian groups, such that $R_m.M_n \subseteq M_{m+n}$ for all $m \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$. Elements in one of these pieces are called homogeneous. Any $m \in M$ is thus uniquely a finite sum $\sum m_{n_i}$ where each $m_{n_i} \in M_{n_i}$ is homogeneous of degree n_i and $n_i \neq n_j$ for $i \neq j$.

In the category of graded R-modules, the morphisms of R-modules are the ones that preserve the grading. In other words, morphisms of graded modules take homogeneous elements to homogeneous elements of the same degree.

Definition 2.1.3. If M is a graded module, we set $M(n)$ for the same R-module but with the grading $M(n)_k = M_{n+k}$.

A graded homomorphism (of degree 0)between graded R-modules M, N is defined to be an R-module homomorphism sending M_n to N_n for any n.

Example 2.1.4.

1. If R is a graded ring, then R is a graded module over itself.

2. If S is any ring, then S can be considered as a graded ring with $S_0 = S$ and $S_i = 0$ for $i > 0$. Then a graded S-module is just a $\mathbb{Z}$-indexed collection of (ordinary) S-modules.

3. (The blow-up algebra ,also called Rees algebra) Let S be any ring, and let J be an ideal of S. We can make $R = S \oplus J \oplus J^2 \oplus \ldots$ (the blow-up algebra) into a graded ring, by defining the multiplication

from the one of S after noticing that $J^iJ^j \subseteq J^{i+j}$. Given any S-module M, there is a graded R-module $M \oplus JM \oplus J^2M \oplus \ldots$, where multiplication is defined in the obvious way. We thus get a functor from S-modules to graded R-modules.

Definition 2.1.5. Let R be a graded ring, M be a graded R-module and $N \subseteq M$ an R-submodule. N is called a graded submodule if the homogeneous components of elements in N are in N. Similarly, if $M = R$, a graded ideal is also called homogeneous ideal.

Proposition 2.1.6. *Let R be a graded ring, M a graded R-module and N a submodule of M. Then the followings are equivalent :*

(1) N is a graded R-module.

(2) $N = \sum_n N \bigcap M_n$.

(3) If $u = u_1 + \cdots + u_n \in N$ then $u_i \in N$ for $1 \leqslant i \leqslant n$.

(4) N has a homogeneous set of generators.

2.1.2 Rees filtration

The notion of Rees algebra is classically extended as follows :

Definition 2.1.7. Let R be a commutative ring and $\mathscr{I} = \{I_n\}_{n=0}^{\infty}$ a sequence of ideals of R. Then $\mathscr{I}$ is called filtration of R if,

(1) $I_0 = R$

(2) $I_n \supset I_{n+1} \quad \forall n \in \mathbb{N}$

(3) $I_n.I_m \subseteq I_{n+m}$.

Let $\mathscr{I}$ be a filtration of R then we define the Rees algebra and associated graded ring associated to $\mathscr{I}$ by

$$\mathscr{R}_{\mathscr{I}} = \oplus_{n\geqslant 0} I_n \quad , \quad gr_{\mathscr{I}} = \oplus_{n\geqslant 0} \frac{I_n}{I_{n+1}}$$

One of the most impotent example is when a filtration $\mathscr{I}$ is given by a power of ideal I, in this case if I is generated by $\{f_1, \cdots, f_m\} \subset R$ then Rees algebra can be described as subring of the graded polynomial ring $R[t]$ and denoted by $R[It]$. One can define the R-algebra surjective homomorphism as follows :

$$\psi : R[T_1, \cdots T_n] \to \psi R[It] \quad , \quad \psi(T_i) = f_i t$$

Remark 2.1.8. If R is a ring with a filtration $\mathscr{I}$ given by powers of an ideal I , then $R[It]$ is a Noetherian. If a Rees ring with respect to a filtration $\mathscr{I}$ is Noetherian, then $\mathscr{I}$ is called a Noetherian filtration . The following proposition gives some equivalent conditions.

Proposition 2.1.9. *Let R be a ring with a filtration$\mathscr{I} = \{I_n\}_{n=0}^{\infty}$. The following conditions are equivalent:*

(i) $\mathscr{I}$ is Noetherian.

(ii) R is Noetherian, and $\mathscr{R}_{\mathscr{I}}$ is finitely generated over R.

(iii) R is Noetherian, and $\mathscr{R}_{\mathscr{I}+}$ is finitely generated over R.

2.1.3 Graded Free Resolution

Definition 2.1.10. Let M be a finite module over local ring (S, m, k) , the following exact sequence is called a **minimal free resolution**

$$\star \quad \cdots F_n \longrightarrow^{d_n} F_{n-1} \longrightarrow^{d_{n-1}} \cdots \longrightarrow F_0 \longrightarrow^{d} \longrightarrow M \longrightarrow 0$$

if it statisfies the following conditions :

(1) F_i are free S-module for all $i \in \mathbb{N}$

(2) $d_i(F_i) \subset mF_{i-1}$

(3)$\overline{d} : F_0 \bigotimes k \longrightarrow M \bigotimes k$ is an isomorphism

The Hilbert Syzygy theorem says that every module M over a polynomial ring S over a field has a free resolution with length at most the number of variables.

Theorem 2.1.11. *(Hilbert's Syzygy Theorem). Let M be a finitely generated graded module over the polynomial ring $S = k[x_1, x_2, \ldots, x_n]$. Then there exists a minimal free resolution:*

$$0 \to F_s \to \ldots \to F_1 \to F_0 \to M \to 0$$

with $s \leqslant n$ and the rank of the F_i's in any minimal resolution only depends on M.

We can choose all the generators of various syzygy modules to be homogeneous and we can define the generators of the free modules in a way that all the maps are of degree zero.

Furthermore, in each step, if we choose a minimal generating set for the syzygy modules, we get a minimal free resolution of M. In this way, we can write

$$F_i = \bigoplus_{j \in \mathbb{Z}} S(-j)^{\beta_{i,j}}$$

for some natural numbers $\beta_{i,j}$. These numbers form a set of invariants of M as a graded S-module and we can also obtain them as the homological invariants $\beta_{i,j}(M) = \dim_k \mathrm{Tor}_i^S(M,k)_j$. These numbers are called graded Betti numbers.

Indeed, a minimal free resolution is an invariant associated to a graded module over a ring graded by the natural numbers $\mathbb{N}$, or more generally by $\mathbb{N}^n$. The information provided by free resolutions is a refinement of the information provided by the Hilbert polynomial and Hilbert function.

Proposition 2.1.12. *If* $\mathbb{F}: \ldots F_1 \to F_0$ *is the minimal free resolution of a finitely generated graded* S*-module* M*, and* k *denotes the residue field* $S/\mathfrak{m}$*, then any minimal set of homogeneous generators of* F_i *contains precisely* $\dim_k \operatorname{Tor}_i^S(k, M)_j$ *generators of degree j.*

Proposition 2.1.13. *Let* $\{\beta_{i,j}\}$ *be the graded Betti numbers of a finitely generated* S*-module. If* $\beta_{i,j} = 0$ *for all* $j \leq d$ *then* $\beta_{i+1,j} = 0$ *for all* $j \leq d$.

2.1.4 Hilbert functions and Hibert seies

We can define graded modules similarly to the classical $\mathbb{Z}$-graded case. When $G = \mathbb{Z}^d$ and the grading is positive, (generalized) Laurent series are associated to finitely generated graded modules:

Definition 2.1.14. The Hilbert function of a finitely generated module M over a positively graded polynomial ring is the map:

$$\begin{array}{rccc} HF(M;-): & \mathbb{Z}^d & \longrightarrow & \mathbb{N} \\ & \mu & \longmapsto & \dim_k(M_\mu). \end{array}$$

The Hilbert series of M is the Laurent series:

$$H(M;t) = \sum_{\mu \in \mathbb{Z}^d} \dim_k(M_\mu)\, t^\mu.$$

Remark 2.1.15. By [37, 8.8], if S is positively graded by $\mathbb{Z}^d$, then the semigoup $Q = \deg(\mathbb{N}^n)$ can be embedded in $\mathbb{N}^d$. Hence, after such a change of embedding, the above Hilbert series are Laurent series in the usual sense.

We recall that the support of a $\mathbb{Z}^d$-graded module N is

$$\operatorname{Supp}_{\mathbb{Z}^d}(N) := \{\mu \in \mathbb{Z}^d | N_\mu \neq 0\},$$

and use the abbreviated notations $\mathbb{Z}[t] := \mathbb{Z}[t_1, \ldots, t_d]$ for $t = (t_1, \ldots, t_d)$ and $t^\mu := t_1^{\mu_1} \cdots t_d^{\mu_d}$ for $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{Z}^d$.

Proposition 2.1.16. *Let $S = k[x_1, \ldots, x_n]$ be a positively graded $\mathbb{Z}^d$-graded polynomial ring over the field k. Then the followings hold:*

1. *The Hilbert series of $S(-\mu)$ is the development in Laurent series of the rational function*

$$H(S(-\mu); t) = \frac{t^{\mu}}{\prod_{i=1}^{n}(1 - t^{\mu_i})}.$$

where $\mu_i = \deg(x_i)$.

2. *If M is a finitely generated graded S-module, setting $\Sigma_M := \cup_{\ell} \mathrm{Supp}_{\mathbb{Z}^d}(\mathrm{Tor}_{\ell}^{R}(M, k))$ and*

$$\kappa_M(t) := \sum_{a \in \Sigma_M} \left(\sum_{\ell} (-1)^{\ell} \dim_k (\mathrm{Tor}_{\ell}^{R}(M, k))_a \right) t^a,$$

one has $H(M; t) = \kappa_M(t) H(S; t)$.

2.1.5 Closures of ideals

Here we recall some basic fact of closures about ideal froms [49] which is usefull for the last chapter.

2.1.5.1 Integral Closure

Definition 2.1.17. Let $R \subseteq S$ be rings. An element $f \in S$ is integral over R if f satisfies a monic polynomial equation

$$f^n + a_1 f^{n-1} + \ldots + a_n = 0$$

with coefficients in R. The integral closure of R in S is the set of all elements of S integral over R, it turns out to be a subring of S.

The ring R is integrally closed in S if all elements of S that are integral over R actually belong to R. The ring R is normal if it is integrally closed in the ring obtained from R by inverting all non-zerodivisors.

Instead of a ring, the integral closure of an ideal is defined as follows:

Definition 2.1.18. Let R be a ring and I be an ideal of R. An element $z \in R$ is integral over I if it satisfies the following equation.

$$z^n + a_1 z^{n-1} + \cdots + a_n = 0 \qquad a_i \in I^i$$

The set of all integral elements over I is called integral closure of I and denoted by $\overline{I}$

2.1.5.2 Ratliff-Rush closure

Definition 2.1.19. Let A be a Noetherian local ring and $I \subset A$ an ideal with grade$(I) > 0$. The Ratliff-Rush closure of I is the ideal:

$$\widetilde{I} := \bigcup_{n \geqslant 1} I^{n+1} : I^n.$$

One of basic facts about Ratliff-Rush closure of powers of ideals is the following:

Theorem 2.1.20. *Let I be an ideal containing regular elements. Then there exists an integer n_0 such that $\widetilde{I^n} = I^n$ for $n \geqslant n_0$.*

So we can define an invariant for I $\widetilde{\rho}(I) := min\{n_0 \geqslant 0 | \widetilde{I^n} = I^n \quad for \quad all \quad n \geqslant n_0\}$ and it can be calculated by the following lemma.

Lemma 2.1.21. *Let (A, m) be a local Noetherian ring and let $I \subset A$ be an ideal of* grade$(I) > 0$. *Suppose that $\left[H^1_{R+}(\mathscr{R})\right]_0 = 0$. Then $a^1_{R+}(\mathscr{R}) + 1 = \widetilde{\rho}(I)$. Where $\mathscr{R} = \oplus_{n \geqslant 0} I^n$ and $a^1_{R+} = sup\{n \in \mathbb{Z} | \left[H^1_{R+}(\mathscr{R})\right]_n \neq 0\}$.*

2.1.6 Spectral sequence

In this section, we are going to collect some necessary notations and terminologies about spectral sequences from [52].

Definition 2.1.22. A spectral sequence in an abelian category A is a collection of the following data:

1. A family $\{E^r_{p,q}\}$ defined for all $p,q \in \mathbb{Z}$.
2. Maps $d^r_{p,q} : E^r_{p,q} \to E^r_{p-r,q+r-1}$ are differentials in the sense that $d^r d^r = 0$.
3. There is isomorphisms between $E^{r+1}_{p,q}$ and homology of $E^r_{*,*}$ at the spot $E^r_{p,q}$:

$$E^{r+1}_{p,q} \cong \frac{\ker(d^r_{p,q})}{Im(d^r_{p+r,q-r+1})}$$

Definition 2.1.23. 1. A spectral sequence is called bounded if for each n there are only finitely many nonzero terms of total degree n in $E^n_{*,*}$, more precisely there is an n_0 such that $E^n_{p,q} = E^{n+1}_{p,q}$ for all $p,q \in \mathbb{Z}$ and $n \geqslant n_0$. We represent the stable value of $E^n_{p,q}$ by $E^\infty_{p,q}$.

2. A bounded spectral sequence is called converges to a given family $\{H_n\}$ of objects of an abelian category A, if we have a following filtration for each H_n :

$$0 = F_s H_n \subseteq \cdots \subseteq F_{p-1} H_n \subseteq F_p H_n \subseteq F_{p+1} H_n \subseteq \cdots F_t H_n = H_n$$

and we are given isomorphisms :

$$E^\infty_{p,q} = \frac{F_p H_{p+q}}{F_{p-1} H_{p+q}}$$

and we write $E^r_{p,q} \Rightarrow H_{p+q}$ to represent this fact .

2.1.6.1 Spectral Sequence of Double Complex

Let $C = C_{*,*}$ be a double complex in an abelian category A with total complex $Tot(C)_n = \oplus_{i+j=n} C_{i,j}$. We define two filtrations on the total complex by rows and columns.

$$
\begin{array}{ccccccccc}
 & & \cdots & & \cdots & & \cdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{p-1,q+1} & \longleftarrow & C_{p,q+1} & \longleftarrow & C_{p+1,q+1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{p-1,q} & \longleftarrow & C_{p,q} & \longleftarrow & C_{p+1,q} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & C_{p-1,q-1} & \longleftarrow & C_{p,q-1} & \longleftarrow & C_{p+1,q-1} & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots & &
\end{array}
$$

More precisely we define two filtrations ${}^{I}F_p(Tot(C))$ and ${}^{II}F_p(Tot(C))$ as follows:

$$
{}^{I}F^n_p(Tot(C)) = \bigoplus_{m \geqslant p} C_{m,n-m}
$$

$$
{}^{II}F^n_p(Tot(C)) = \bigoplus_{m \geqslant p} C_{n-m,m}
$$

Theorem 2.1.24. *Let C be a first quadrant double complex in an abelian category A. There are two spectral sequences ${}^{'}E^n$ and ${}^{''}E^n$ determined by ${}^{I}F_p(Tot(C))$ and ${}^{II}F_p(Tot(C))$ filtrations respectively with*

$$
{}^{'}E^0_{p,q} = C_{p,q} \quad , \quad {}^{'}E^1_{p,q} = H^q_{vert}(C_{p,*}) \quad , \quad {}^{'}E^2_{p,q} = H^p_{hor} H^q_{vert}(C) \ ,
$$

$$
{}^{''}E^0_{p,q} = C_{q,p} \quad , \quad {}^{''}E^1_{p,q} = H^q_{hor}(C_{*,p}) \quad , \quad {}^{''}E^2_{p,q} = H^p_{ver} H^q_{hor}(C)
$$

Both spectral sequences ${}^{'}E^n$ and ${}^{''}E^n$ converge to $H_{p+q}(Tot(C))$.

One of the applications of these different spectral sequences with the same convergence could be for computing Tor modules, more precisely here we recall the Base-change theorem about Tor modules which is useful in the last chapter .

Theorem 2.1.25. *Let $f : R \to S$ be a ring map. Then there is a first quadrant homology spectral sequence*

$$E^2_{p,q} = \mathrm{Tor}^S_p(\mathrm{Tor}^R_q(A,S),B) \Rightarrow \mathrm{Tor}^R_{p+q}(A,B)$$

for every R-module A and S-module B.

2.1.7 Castelnuovo-Mumford Regularity

One of the most important invariants which measures the complexity of a coherent sheaf $\mathscr{F}$, on $\mathbb{P}^r$ is the Castelnuovo-Mumford regularity. It was first introduced by Mumford in [38, Chapter 14] as how much one has to twist a coherent sheaf $\mathscr{F}$ in order for the higher cohomology to vanish. Alternatively a coherent sheaf $\mathscr{F}$ is called m-regular if,

$$\mathrm{H}^i(\mathbb{P}^r, \mathscr{F} \otimes \mathbb{P}^r(m-i)) = 0$$

for all $i > 0$ then $\mathrm{reg}(\mathscr{F}) = \min\{m \in \mathbb{Z} | \mathscr{F} \text{ is } m-\text{regular}\}$.

A related idea in commutative algebra was given by Eisenbud and Goto [26]. Let $R = k[x_1, \cdots, x_n]$ be a polynomial ring over a field k and M a finitely generated graded R-module. Assume M has a minimal graded free resolution as:

$$\mathbb{F}_\bullet : 0 \to F_t \to \ldots \to F_2 \to F_1 \to F_0 \to M \to 0$$

Set

$$a_i(M) := max\{\mu | H^i_{\mathbf{m}}(M)_\mu \neq 0\}$$

if $H^i_{\mathbf{m}}(M) \neq 0$ and $a_i(M) := \infty$ else,

$$b_i(M) := max\{\mu | \mathrm{Tor}_i^R(M,k)_\mu \neq 0\}$$

if $\mathrm{Tor}_i^R(M,k) \neq 0$ and $b_i(M) := \infty$ else, then the Castelnuovo-Mumford regularity of M is defined as:

$$reg(M) := max_i\{a_i(M) + i\} = max_i\{b_i(M) - i\}.$$

Central motivation of current thesis concerns the remarkable result about linearity behavior of regularity of powers of graded ideal I in R due to Kodiyalam [34], Coutkosky,Herzog and Trung [19] , Trung and Wang [48] and Chardin [17] . We state it as:

Theorem 2.1.26. *Let A be a standard graded Noetherian algebra . If I is a graded ideal and $M \neq 0$ a finitely generated graded A-module, then there exists n_1 and b such that*

$$\mathrm{reg}(I^n M) = nd + b, \qquad \forall n \geqslant n_1$$

with

$$d := \min\{\mu | \ \exists m \geqslant 1, \quad (I_{\leqslant \mu}) I^{m-1} M = I^m M\}$$

Similar question studied for ideal sheaves, but the behavior is much more complicated than for graded ideals. Let I be a graded ideal generated in degree at most $d$in the standard algebra over a field and $\mathbf{m} = A_+$. We denote the saturation of I with respect to $\mathbf{m}$ by $I^{sat} := \bigcup_n (I : \mathbf{m}^n)$, it has been shown in [19] that the regularity of $(I^n)^{sat}$ is in several cases not eventually linear but at least Cutkosky, Ein and Lazarsfeld proved in [] that the following limit exists

$$s(I) = \lim_{n\to\infty} \frac{\mathrm{reg}(I^n)^{sat}}{n}$$

where $s(I)$ equals the invesrse of a Seshadri constant.

2.2 Discrete Geometry

2.2.1 Polytopes

There are two notions of polytopes (**H**-polytope and **V**-polytope) where their equivalence have been proved in the main theorem of polytopes (see 2.2.3) but first we look at their definitions and related concepts .

Definition 2.2.1. 1. A **hyperplane** in $\mathbb{R}^d$ is a set $H := \{x \in \mathbb{R}^d | a_1 x_1 + \cdots + a_d x_d = b\}$.

2. A **convex combination** of finite points $q_1, \cdots, q_t$ is a set

$$com(\{q_1, \cdots, q_t\}) := \{\mathbf{x} \in \mathbb{R}^d | \sum_{i=1}^{t} \lambda_i q_i \quad for \quad \lambda_i \geqslant 0 \quad , \sum \lambda_i = 1\}$$

The set of all convex combination of the points $q_1, \cdots, q_t$ is called **convex hull** of $q_1, \cdots, q_t$ and it denoted by $conv(q_1, \cdots, q_t)$.

3. The minkowski sum of two sets $P, Q \subseteq \mathbb{R}^d$ is defined as:

$$P + Q = \{x + y | x \in P, y \in Q\}$$

Definition 2.2.2. 1. A **H**-polyhedron in $\mathbb{R}^d$ denotes as the intersection of closed halfspaces in $\mathbb{R}^d$ in the following form

$$P = \{x \in \mathbb{R}^d | Ax \leqslant b\}$$

where $A \in \mathbb{R}^{(n \times d)}$ and $b \in \mathbb{R}^n$. A bounded **H**-polyhedron is called **H**-polytope.

2. A **V**-polyhedron in $\mathbb{R}^d$ denotes as the convex hull of a finite number of points and it is of form $P = conv(q_1, \cdots, q_t)$. A bounded **V**-polyhedron is called **V**-polytope.

Theorem 2.2.3. *[53] Every* ***V****-polytope has a description by inequalities as* ***H****-polytope and every* ***H****-polytope is the convex hull of minimal number of finitely many points .*

Based on the above theorem, we will only use the term polytope for both notions of polytops.

2.2.1.1 Faces of Polytopes

Definition 2.2.4. Let $P \subseteq \mathbb{R}^d$ be convex polytope. The inequality $\mathbf{c.x} \leqslant c_0$ is called valid for P if all points $\mathbf{x} \in P$ satisfy this inequality. The hyperplane $H = \{\mathbf{x} \in \mathbb{R}^d | \mathbf{c.x} = c_0\}$ is called supporting hyperplane of P. A face of P is set of form $F = P \cap H$ and the dimension of a face of P is the dimension of its affine hull.

2.2.1.2 Gale Diagrams

Let n points $v_1, \cdots, v_n$ be in $\mathbb{R}^{d-1}$ whose affine hull has dimension $d-1$. Consider the matrix

$$A := \begin{pmatrix} 1 & 1 & \cdots & 1 \\ v_1 & v_2 & \cdots & v_n \end{pmatrix}$$

the kernel of A is defined as:

$$\ker(A) := \{x \in \mathbb{R}^n | Ax = \mathbf{0}\}$$

Definition 2.2.5. Let $x \in \mathbb{R}^n$ and $v_1, \cdots, v_n$ vectors in $\mathbb{R}^n$.

(1) If $\sum_{i=1}^n v_i x i = \mathbf{0}$ and $\sum_{i=1}^n x_i = \mathbf{0}$ then x is called an affine dependence relation on $v_1, \cdots, v_n$.

(2) If $\sum_{i=1}^n v_i x i = \mathbf{0}$ then x is called a linear dependence relation on $v_1, \cdots, v_n$.

(3) If $x = \mathbf{0}$ is the only solution of $\sum_{i=1}^n v_i x i = \mathbf{0}$ and $\sum_{i=1}^n x_i = \mathbf{0}$ then $v_1, \cdots, v_n$ is called affinely independent.

Let $B_1, \cdots, B_{n-d} \in \mathbb{R}^n$ be a basis for the vector space $\ker(A)$ and put them as the columns of $n \times (n-d)$ matrix.

$$B := \begin{pmatrix} B_1 & B_2 & \cdots & B_{n-d} \end{pmatrix}$$

Definition 2.2.6. Let $\mathbf{G} = \{b_1, \cdots, b_n\} \subset \mathbb{R}^{n-d}$ be the n ordered rows of B . Then $\mathbf{G}$ is called a Gale transform of $\{v_1, \cdots, v_n\}$. The Gale diagrams of $\{v_1, \cdots, v_n\}$ is the vector configuration $\mathbf{G}$ in $\mathbb{R}^{n-d}$.

Example 2.2.7. Let

$$A = \begin{pmatrix} d_1 & \dots & d_n \\ 1 & \dots & 1 \end{pmatrix}$$

be a $2 \times n$-matrix with entries in $\mathbb{N}$ such that $d_1 \leq \ldots \leq d_n$. By computing a basis for the kernel of A, we have:

$$B = \begin{pmatrix} d_2 - d_3 & d_2 - d_4 & \cdots & d_2 - d_n \\ d_3 - d_1 & d_4 - d_1 & \cdots & d_n - d_1 \\ d_1 - d_2 & 0 & \cdots & 0 \\ 0 & d_1 - d_2 & \cdots & 0 \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \cdots & d_1 - d_2 \end{pmatrix}$$

.

2.2.2 Lattices

A lattice is an additive subgroup of $\mathbb{R}^n$, here we recall some basic definitions and algorithms about the lattice which is usefull for the thesis.

Definition 2.2.8. consider $< .,. >$ as an inner product.

(1) The dual Λ^* of a lattice Λ is the lattice of vectors v such that $< v, w >\in \mathbb{Z}$ where $w \in \{\mathbf{x} \in \mathbb{R}^n | < \mathbf{x}, \mathbf{s} >\in \mathbb{Z} \quad for \ all \ \mathbf{s} \in \Lambda\}$.

(2) A lattice Λ is called integral if $< v, w >\in \mathbb{Z}$ for all $v, w \in \Lambda$.

(3) A lattice Λ is called unimodular when $\Lambda^* = \Lambda$.

(4) The volume of a lattice Λ is the volume of it's fundamental domain which is $\det(A)$ where A is the matrix of $\mathbb{Z}$-basis of Λ.

It is well-known that if U_1 and U_2 are two $m \times n$-matrices and L_1 and L_2 are the corresponding lattice generated by columns of the matrices U_i, we can apply the following algorithm to find the generators of $L_1 \cap L_2$:

Algorithm.

- Given basis $U_1 = (u_1, \dots, u_n)$ and $V_2 = (v_1, \dots, v_n)$.
- Take dual of U_i by $D(U_i) = U_i \left(U_i^t U_i\right)^{-1} = \left(U_i^t\right)^{-1}$.
- Set K the matrix of adjunction of $D(U_1)$ and $D(U_2)$.
- Compute Hermite normal form of K, say H.
- Compute dual of H.

End of algorithm.

CHAPTER 3

NON STANDARD HILBERT FUNCTION

3.1 Grading over abelian group

Let $S = k[x_1, \ldots, x_n]$ be a polynomial ring over field k. We first make clear our definition of grading.

Definition 3.1.1. Let G be an abelian group. A G-grading of S is a group homomorphism $\deg : \mathbb{Z}^n \longrightarrow G$ and $\deg(x^u) := \deg(u)$ for a monomial $x^u = x_1^{u_1} \ldots x_n^{u_n} \in S$. An element $\sum c_u x^u \in S$ is homogeneous of degree $\mu \in G$ if $\deg(u) = \mu$ whenever $c_u \neq 0$ and an ideal $I \subset S$ is homogeneous if every polynomial in I is a sum of homogeneous polynomials under the given grading.

Theorem 3.1.2. *(See [37, Theorem 8.6]) Let* $Q = \deg(\mathbb{N}^n)$ *and* $L = \ker(\deg)$ *of the above group homomorphism . Then the following canditions are equivalent for a polynomial ring S graded by G.*

(1) There exists $\mu \in Q$ such that the vector space S_μ is finite-dimensional.

(2) The only polynomials of degree zero are the constants ; i.e., $S_0 = k$.

(3) For all $\mu \in G$, the k-vector space S_μ is finite-dimensional.

(4) For all finitely generated graded modules M and degree $\mu \in G$, the k-vector space M_μ is finite-dimensional.

(5) The only nonnegative vector in the lattice L is 0 ; i.e., $L \cap \mathbb{N}^n = \{0\}$.

(6) The semigroup Q has no units, and no variable x_i has degree zero .

Definition 3.1.3. [37] If the equivalent conditions of the above theorem hold for a torsion-free abelian group G then we call grading by G **positive**, and for the polynomial ring $S = k[x_1,\dots,x_n]$ which is graded by G we say that S is a positively graded polynomial ring.

Remark 3.1.4. There is a two important cases about the image of group homomorphism $\deg : \mathbb{Z}^n \longrightarrow G$ where it has torsion or not. In the case where image of deg has a torsion it can be happen that associated primes of G-graded S-module M are not graded but in the other the associated primes of M are graded by following proposition.

Proposition 3.1.5. *(See [37, Theorem 8.11]) Let $S = k[x_1,\dots,x_n]$ be a polynomial ring over field k graded by a torsion-free abelian group G and let M be a G-graded S-module. If P be an associated prime of M then P is homogeneous and $P = ann(m)$ where m is homogeneous element of M.*

3.2 Vector Partition functions

We first recall the definition of quasi-polynomials. Let $d \geq 1$ and Λ be a lattice in $\mathbb{Z}^d$.

Definition 3.2.1. [3] A function $f : \mathbb{Z}^d \to \mathbb{Q}$ is a quasi-polynomial with respect to Λ if there exists a list of polynomials $Q_i \in \mathbb{Q}[T_1, \ldots, T_d]$ for $i \in \mathbb{Z}^d/\Lambda$ such that $f(s) = Q_i(s)$ if $s \equiv i \mod \Lambda$.

Notice that $\mathbb{Z}^d/\Lambda$ has $|\det(\Lambda)|$ elements, and that when $d = 1$, $\Lambda = q\mathbb{Z}$ for some $q > 0$, in which case f is also called a quasi-polynomial of period q.

Now assume that a positive grading of S by $\mathbb{Z}^d$ with $Q := \deg(\mathbb{N}^n) \subseteq \mathbb{N}^d$ is given and that Q spans a subgroup of rank d in $\mathbb{Z}^d$. In other words, the matrix $A = (a_{i,j})$ representing $\deg : \mathbb{Z}^n \to \mathbb{Z}^d$ is a $d \times n$-matrix of rank d with entries in $\mathbb{N}$. Let $a_j := (a_{1,j}, \ldots, a_{d,j})$ and

$$\begin{array}{rccc} \varphi_A : & \mathbb{N}^d & \longrightarrow & \mathbb{N} \\ & u & \longrightarrow & \#\left\{\lambda \in \mathbb{N}^n | A.\lambda = u\right\}. \end{array}$$

Equivalently, $\varphi_A(u)$ is the coefficient of t^u in the formal power series $\prod_{i=1}^n \frac{1}{(1-t^{a_i})}$.

Notice that φ_A vanishes outside of $\mathrm{Pos}(A) := \{\sum \lambda_i a_i \in \mathbb{R}^n | \lambda_i \geq 0, 1 \leq i \leq n\}$.

Blakley showed in [10] that $\mathbb{N}^d$ can be decomposed into a finite number of parts, called chambers, in such a way that φ_A is a quasi-polynomial of degree $n - d$ in each chamber. Later, Sturmfels in [47] investigated these decompositions and the differences of polynomials from one piece to another.

Here we briefly introduce the basic facts and necessary terminology of vector partition functions, specially the chambers and the polynomials (quasi-polynomials) obtained from vector partition functions corresponding to a matrix A. For more details about the vector partition function, we refer the reader to [10, 13, 47].

Definition 3.2.2. [53] A polyhedral complex $\Im$ is a finte collection of polyhedra in $\mathbb{R}^d$ such that

1. the empty polyhedron is in $\Im$,

2. if $P \in \Im$, then all the faces of P are also in $\Im$,

3. the intersection $P \cap Q$ of two polyhedra $P, Q \in \Im$ is a face of both of P and of Q.

Definition 3.2.3. [22] A vector configuration in $\mathbb{R}^m$ is a finite set $\mathbf{A} = (p_j : j \in J)$ of labeled vectors $p_j \in \mathbb{R}^m$. Its rank in the same as its rank as a set of vectors. A subconfiguration is any (labeled) subset of it .

For any subset C of the label set J we will associate the followings :

$$\text{Cone}_{\mathbf{A}} := \{\sum_{j \in C} \lambda_j p_j | \lambda_j \geqslant 0, \forall j \in C\}$$

and

$$\text{relint}_{\mathbf{A}} := \{\sum_{j \in C} \lambda_j p_j | \lambda_j > 0, \forall j \in C\}$$

The above definitions help us to understand the following definition of polyhedral subdivision of a set of vectors.

Definition 3.2.4. [22] A polyhedral subdivision of a vector configuration $\mathbf{A}$ is a collection $\Re$ of subconfigurations of $\mathbf{A}$ in $\mathbb{R}^d$ that satisfies the following conditions :

1. If $C \in \Re$ then all the faces of C are also in $\Re$,

2. $\bigcup_{C \in \Re} \text{Cone}(C) \supseteq \text{Cone}(\mathbf{A})$,

3. $\text{relint}(C) \cap \text{relint}(C^{'}) \neq \emptyset \quad \text{for} \quad C, C^{'} \in \Re$ implies that $C = C^{'}$

Remark 3.2.5. 1. The elements of a polyhedral subdivision $\Re$ are called cells. Cells of the same rank as **A** are maximal. Cells of rank 1 are called rays of $\Re$. A **triangulation** of **A** is a polyhedral subdivision whose cells are simplices.

2. A subdivision $\Re$ refines another one $\Re^{'}$ (written $\Re \leqslant \Re^{'}$) if every face of $\Re$ is a subset of some face of $\Re^{'}$

In the following we recall the definition of the chamber complex of given set $A = \{a_1, \dots, a_n\}$ of non-zero vectors in $\mathbb{R}^d$ follows from [9].

Definition 3.2.6. The chamber complex $\Gamma(A)$ of A is defined to be the coarsest polyhedral complex that covers $\text{Pos}(A)$ and that refines all triangulations of A.

Note that the chamber complex is a polyhedral subdivision of a vector configuration contaning **A** strictly possible. Now for a given point $x_0 \in \text{Pos}(A)$ we can associate the unique cell $\Gamma(A, x_0)$ of $\Gamma(A)$ which is containing x_0 . This can be written:

$$\Gamma(A, x_0) = \bigcap \{\text{relint}_{A'} \,|A^{'} \subseteq A, \;\; x_0 \in \text{relint}_{A'}\}$$

If $\sigma \subseteq \{1, \dots, n\}$ is such that the a_i's for $i \in \sigma$ are linearly independent, we will say that σ is independent. We set $A_\sigma := (a_i)_{i\in\sigma}$ and denote by Λ_σ the $\mathbb{Z}$-module with base the columns of A_σ and $\partial Pos(A_\sigma)$ the boundary of $\text{Pos}(A_\sigma)$. When σ has d elements (*i.* e. is a maximal independent set), Λ_σ is a sublattice of $\mathbb{Z}^d$.

Let $\sum_A$ be the set of all simplicial cones whose extremal rays are generated by d-linearly independent column vectors of A . Then, following [23, end of section 3] the maximal chambers C of the chamber complex of A are the connected components of $Pos(A) - \bigcup_{\ell \in \sum_A} \partial \ell$. These chambers are open and convex.

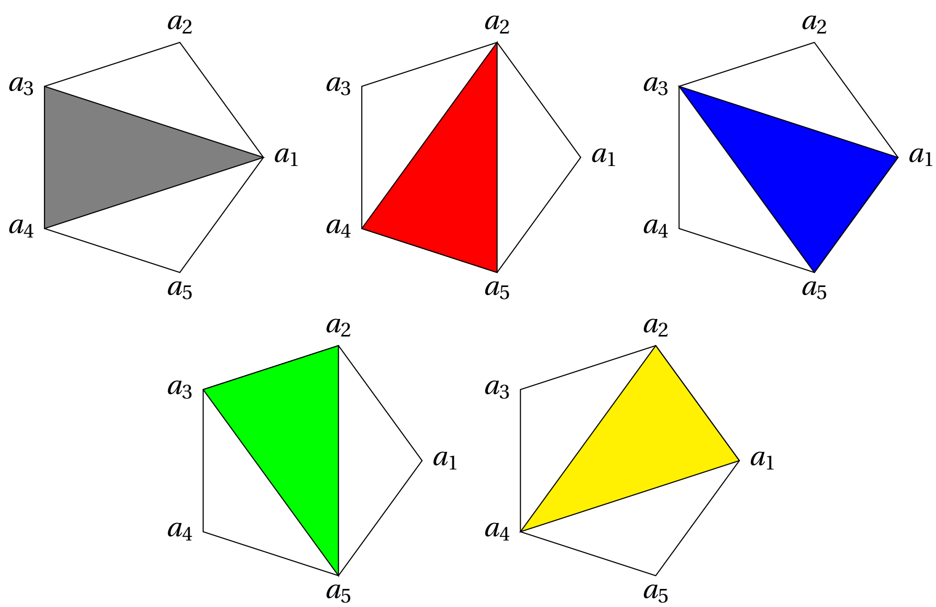

Figure 3.1: Triangulation of pentagonal cone for a 2-dimensional slice

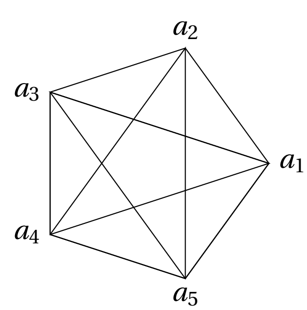

Figure 3.2: The chamber complex with its 11-maximal chambers

Associated to each maximal chamber C there is an index set $\Delta(C) := \{\sigma \subset \{1, \ldots, n\} \mid C \subseteq \mathrm{Pos}(A_\sigma)\}$ and $\sigma \in \Delta(C)$ is called non-trivial if $G_\sigma := \mathbb{Z}^d / \Lambda_\sigma \neq 0$, equivalently if $\det(\Lambda_\sigma) \neq \pm 1$ (G_σ is finite because

$C \subseteq \mathrm{Pos}(A_\sigma)$).

Now, we are ready to state the vector partition function theorem, which relies on the chamber decomposition of $\mathrm{Pos}(A) \subseteq \mathbb{N}^d$.

Theorem 3.2.7. *(See [47, Theorem 1]) For each chamber C of maximal dimension in the chamber complex of A, there exist a polynomial P of degree $n-d$, a collection of polynomials Q_σ and functions $\Omega_\sigma : G_\sigma \setminus \{0\} \to \mathbb{Q}$ indexed by non-trivial $\sigma \in \Delta(C)$ such that, if $u \in \mathbb{N}A \cap \overline{C}$,*

$$\varphi_A(u) = P(u) + \sum\{\Omega_\sigma([u]_\sigma).Q_\sigma(u) : \sigma \in \Delta(C), [u]_\sigma \neq 0\}$$

where $[u]_\sigma$ denotes the image of u in G_σ. Furthermore, $\deg(Q_\sigma) = \#\sigma - d$.

Corollary 3.2.8. *For each chamber C of maximal dimension in the chamber complex of A, there exists a collection of polynomials Q_τ for $\tau \in \mathbb{Z}^d/\Lambda$ such that*

$$\varphi_A(u) = Q_\tau(u), \text{ if } u \in \mathbb{N}A \cap \overline{C} \text{ and } u \in \tau + \Lambda_C.$$

where $\Lambda_C = \cap_{\sigma \in \Delta(C)} \Lambda_\sigma$

Proof. The class τ of u modulo Λ determines $[u]_\sigma$ in $G_\sigma = \mathbb{Z}^d/\Lambda_\sigma$. The term of the right-hand side of the equations in the above theorem is a polynomial determined by $[u]_\sigma$, hence by τ. □

Notice that setting Λ for the intersection of the lattices Λ_σ with σ maximal, the class of u mod Λ determines the class of u mod Λ_C, hence the corollary holds with Λ in place of Λ_C.

It is important to know about the relation between the partion function associated to a list of vectors in $\mathbb{Z}^s$ and partion function associated to it's sublist, because it help us to find some recursive formula to compute the partion functions. Let X be a list of vectors $a^1, \cdots, a^n$ in $\mathbb{Z}^s - \{0\}$, in general for any aublis V of X one has the following

$$\varphi_X(u) = \varphi_{X-\{V\}}(u) * \varphi_V(u)$$

where $*$ denotes discrete convolution, more precisely $(g * f)(u) = \sum_{\mu \in \mathbb{Z}^s} g(u-\mu) f(\mu)$.

Lemma 3.2.9. *Let X be a list of vectors $x^1, \cdots, x^n$ in $\mathbb{Z}^s - \{0\}$ then the following recursive formula hols for the vector partition function $\varphi_X(u)$*

$$\varphi_X(u) = \sum_{j=0}^{\infty} \varphi_{X-\{x^i\}}(u - jx^i)$$

Proof. Let $V = x^i$ be a sublist of X then by using the above formula

$$\begin{aligned} \varphi_X(u) &= \varphi_{X-\{x^i\}}(u) * \varphi_{\{x^i\}}(u) \\ &= \textstyle\sum_{\mu \in \mathbb{Z}^s} \varphi_{X-\{x^i\}}(u-\mu) \varphi_{\{x^i\}}(\mu) \\ &= \textstyle\sum_{j=0}^{\infty} \varphi_{X-\{x^i\}}(u - jx^i) \end{aligned}$$

□

By the above lemma we can do the new proof for the Hilbert function of standard graded polynomial rings.

Proposition 3.2.10. *Let $S = k[x_1, \cdots, x_n]$ be a polynomial ring over a field k and $\deg x_i = 1$ for $1 \leqslant i \leqslant n$. Then*

$$HF(S; m) = \binom{m+n-1}{n-1}$$

Proof. First we can translate the Hilbert function of S into partition function associated to X_n where it consists of repetitions n-times of 1, then by using induction on n. For the case of n=1 is clear. Now by lemma3.2.9 we have

$$HF(S;m) = \varphi_{X_n}(m) = \sum_{j=0}^{\infty} \varphi_{X_{n-1}}(m-j)$$

we now use our assumption that the formula is true for $n-1$ and Pascal formula

$$\varphi_{X_n}(m) = \sum_{j=0}^{\infty} \binom{m-j+n-2}{n-2} = \sum_{j=0}^{\infty} \left\{ \binom{m-j+n-1}{n-1} - \binom{m-j+n-2}{n-1} \right\} = \binom{m+n-1}{n-1}$$

□

3.3 Lattice points problem and Barvinok algorithm

Definition 3.3.1. [7] A rational polyhedron $P \subset \mathbb{R}^d$ is the set of solutions of a finite system of linear inequalities with integer coefficient :

$$P = \{x \in \mathbb{R}^d :< c_i, x > \leq \beta_i \; for \; i = 1, \cdots, m\}$$

A bounded rational polyhedron is called a polytope. A polytope $P \subset \mathbb{R}^d$ is called integer polytope if its vertices are points from $\mathbb{Z}^d$

First we recall the definition of a polytope and a very classical of G.Pick[1899] for a two-dimensional polygone :

Proposition 3.3.2. *[G.Pick] Suppose that $P \subset \mathbb{R}^2$ is an integer polygon. Then the number of integer points inside P is :*

$$|P \cap \mathbb{Z}^2| = area(P) + \frac{|\partial P \cap \mathbb{Z}^2|}{2} + 1$$

One of important generalizations of the Pick formula is the theorem of **Ehrhart** which shows the polynomial behavior of lattice point enumeration independ of the dimension.

Theorem 3.3.3. *[Ehrhart 1977] Let $P \subset \mathbb{R}^d$ be a polytope where its vertices have a rational coordinate. Let $nP = \{nx | x \in P\}$ for a positive integer n then the function $\#(nP \cap \mathbb{Z}^d)$ is a quasi-polynomial of degree* $\dim P$ *. Further more if C is an integer where CP is an integer polytope, then C is a period of $\#(nP \cap \mathbb{Z}^d)$. In the particular case where P is an integer polytope , the Ehrhart polynomial is as follows :*

$$\#(nP \cap \mathbb{Z}^d) = q(n) \qquad where \quad q(n) = vol(P)n^d + a_{d-1}n^{d-1} + \cdots + a_1 x + 1.$$

In the general case for any rational polyhedron $P \subset \mathbb{R}^n$ we consider following generationg function:

$$f(P,\mathbf{x}) = \sum_{m \in P \cap \mathbb{Z}^n} \mathbf{x}^m$$

where $m = (m_1, \cdots, m_n)$ and $\mathbf{x}^m = x_1^{m_1} \cdots x_n^{m_n}$.

3.3.1 Valuations and generating functions of rational polyhedra

To explain berifly the Barvinok method first we should define the vertex cone and generating function associated to each polytope.

Definition 3.3.4. For a set $A \subset \mathbb{R}^d$, the indicator function of A defined by

$$[A] : \mathbb{R}^d \to \mathbb{R}$$

$$[A](x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$$

Write the vector space $\Im(\mathbb{R}^d)$ over $\mathbb{Q}$ generated by the indicator functions of all polyhedra inside $\mathbb{R}^d$ also $\Im_c(\mathbb{R}^d)$ ($\Im_k(\mathbb{R}^d)$)denote for the subspace spanned by the indicator functions of polytopes (respectively. cones) in $\mathbb{R}^d$.

Remark 3.3.5. The pointwise multiplication of indicator functions make's $\Im(\mathbb{R}^d)$ a commutative algebra and $\Im_c(\mathbb{R}^d)$,$\Im_k(\mathbb{R}^d)$ are subalgebra of $\Im(\mathbb{R}^d)$.

Definition 3.3.6. A linear transformation $\Psi : \Im(\mathbb{R}^d) \to V$ where V is a vector space over $\mathbb{Q}$, is called valuation.

Theorem 3.3.7. *[7] There is a map Φ which , to each rational polyhedron $P \subset \mathbb{R}^d$ associates a rational function $f(P;\boldsymbol{x})$ in the d complex variables $\boldsymbol{x} \in \mathbb{C}^d$ such that the following properties are stisfied:*

(1) The map Φ is a valuation.

(2) If $u+P$ is a translation of P by an integer vector $u \in \mathbb{Z}^d$, then

$$f(u+P;\boldsymbol{x}) = \boldsymbol{x}^u f(P;\boldsymbol{x}).$$

(3) We have

$$f(P,\boldsymbol{x}) = \sum_{m \in P \cap \mathbb{Z}^n} \boldsymbol{x}^m$$

for any $\boldsymbol{x} \in \mathbb{C}^d$ such that the series converges absolutely.

(4) If P contains a straight line then $f(P,\boldsymbol{x}) \equiv 0$.

Definition 3.3.8. [7] Let $P \subset \mathbb{R}^d$ be a polyhedron and let $v \in P$ be a vertex of P. The tangent cone $K = cone(P, v)$ of P at v is defiend as follows:

suppose that

$$P = \{x \in \mathbb{R}^d :< c_i, x > \leqslant \beta_i \ for \ i = 1, \cdots, m\}$$

is a representation of P, where $c_i \in \mathbb{R}^d$ and $\beta_i \in \mathbb{R}$. Let $I_v = \{i :< c_i, v >= \beta_i\}$ be the set of constraints that are active on v. Then

$$K = cone(P, v) = \{x \in \mathbb{R}^d :< c_i, x > \leqslant \beta_i \ for \ i \in I_v\}$$

Lemma 3.3.9. *[4]Let $K \subset \mathbb{R}^d$ be a pointed rational cone. Then $K = co(w_1, \cdots, w_n)$ for some $w_1, \cdots, w_n \in \mathbb{Z}^d - \{0\}$. Let us define*

$$W_K = \{\boldsymbol{x} \in \mathbb{C}^d | \ |\boldsymbol{x}^{w_i}| < 1 \ \ for \ \ i = 1, \cdots, n\}.$$

Then W_K is a non-empty open set and for every $\boldsymbol{x} \in W_K$, the series $\sum_{m \in K \cap \mathbb{Z}^n} \boldsymbol{x}^m$ converges to a rational function $f(K, \boldsymbol{x})$ of the type

$$f(K, \boldsymbol{x}) = \sum_{i \in I} \epsilon_i \frac{\boldsymbol{x}^{v_i}}{(1 - \boldsymbol{x}^{u_{i1}}) \cdots (1 - \boldsymbol{x}^{u_{id}})}$$

where $\epsilon_i \in \{1, -1\}$, $v_i \in \mathbb{Z}^d$ and $u_{ij} \in \mathbb{Z}^d - \{0\}$ for all i and j.

Now by Brion's theorem, the generating function of the polytope P is equal to the sum of the generating functions of its vertex cones, more precisely

$$f(P; \mathbf{x}) = \sum_{m \in P \cap \mathbb{Z}^n} \mathbf{x}^m = \sum_{v \in \Omega(P)} f(K; \mathbf{v})$$

where $\Omega(P)$ is the set of vertices of P.

Example 3.3.10. [55]Consider the quadrilateral with the vertex $V_1 = (0,0)$, $V_2 = (5,0)$, $V_3 = (4,2)$, and $V_4 = (0,2)$. Then wen obtain the following rational functions for each vertex :

$$f(K,V_1) = \frac{1}{(1-x_1)(1-x_2)} \quad , \quad f(K,V_1) = \frac{(x_1^5 + x_1^4 x_2)}{(1-x_1^{-1})(1-x_2^2 x_1^{-1})}$$

$$f(K,V_3) = \frac{(x_1^4 x_2 + x_1^4 x_2^2)}{(1-x_1^{-1})(1-x_1 x_2^{-2})} \quad , \quad f(K,V_4) = \frac{x_2^2}{(1-x_2^{-1})(1-x_1)}$$

3.3.2 Decomposing a rational cone into unimodular cones

In order to finding the generation function of arbitrary pointed cones Stanly give the formula by using a triangulation of a rational cone into simplicial cones but instead in 1994 Barvinok proved the general fact that every rational poolyhedral cone can be triangulated into uimodular cones as follows:

Theorem 3.3.11. *[7] Fix the dimension d . Then , there exists a polynomial time algorithm, for a rational polyhedral cone $K \subset \mathbb{R}^d$, which computes unimodular cones K_i , $i \in I = \{1,\cdots,r\}$, and numbers $\epsilon \in \{-1,1\}$ such that*

$$[K] = \sum_{i\in I} \epsilon [K_i].$$

Remark 3.3.12. By having the above decomposition of cones we can write $f(K,\mathbf{x}) = \sum_{i\in I} \epsilon_i f(K_i,\mathbf{x})$, as we have an explicit formula for the unimodular case so we can calculate an explicit formula for rational cones , it is a key idea of the section3.5.

Theorem 3.3.13. *[7] Fix the dimension d . There exists a polynomial time algorithm, for a rational polyhedron $P \subset \mathbb{R}^d$,*

$$P = \{x \in \mathbb{R}^d :< c_i, x > \leqslant \beta_i \; for \; i = 1,\cdots,m\} \quad where \quad c_i \in \mathbb{Z}^d \quad and \quad \beta_i \in \mathbb{Q}$$

computes the generation function $f(P,\boldsymbol{x}) = \sum_{m\in P\cap\mathbb{Z}^n} \boldsymbol{x}^m$ as follows

$$f(P,\boldsymbol{x}) = \sum_{i\in I} \epsilon_i \frac{\boldsymbol{x}^{a_i}}{(1-\boldsymbol{x}^{b_{i1}})\cdots(1-\boldsymbol{x}^{b_{id}})}$$

where $\epsilon_i \in \{1, -1\}$, $a_i \in \mathbb{Z}^d$ *and* $b_{i1}, \cdots, b_{id}$ *form a basis of* $\mathbb{Z}^d$ *for each* i.

Suppose that the vectors c_i *for* $1 \leqslant i \leqslant m$ *are fixed and the* β_i *vary in such a way that the combinatorial structure of polyhedron* $P = P(\beta)$ *stays the same . Then the exponents* b_{ij} *in the denominators remain the same , whereas the exponents* $a_i = a_i(\beta)$ *in the numerator change with* $\beta \in \mathbb{Q}^m$ *as*

$$a_i = \sum_{i=1}^{d} \lfloor l_{ij}(\beta) \rfloor b_{ij}$$

Where the $l_{ij} : \mathbb{Q}^m :\rightarrow \mathbb{Q}$ *are linear functions. If* β *is such that* $P(\beta)$ *is an integer polytope, then* $l_{ij} \in \mathbb{Z}$ *for each pair i,j .*

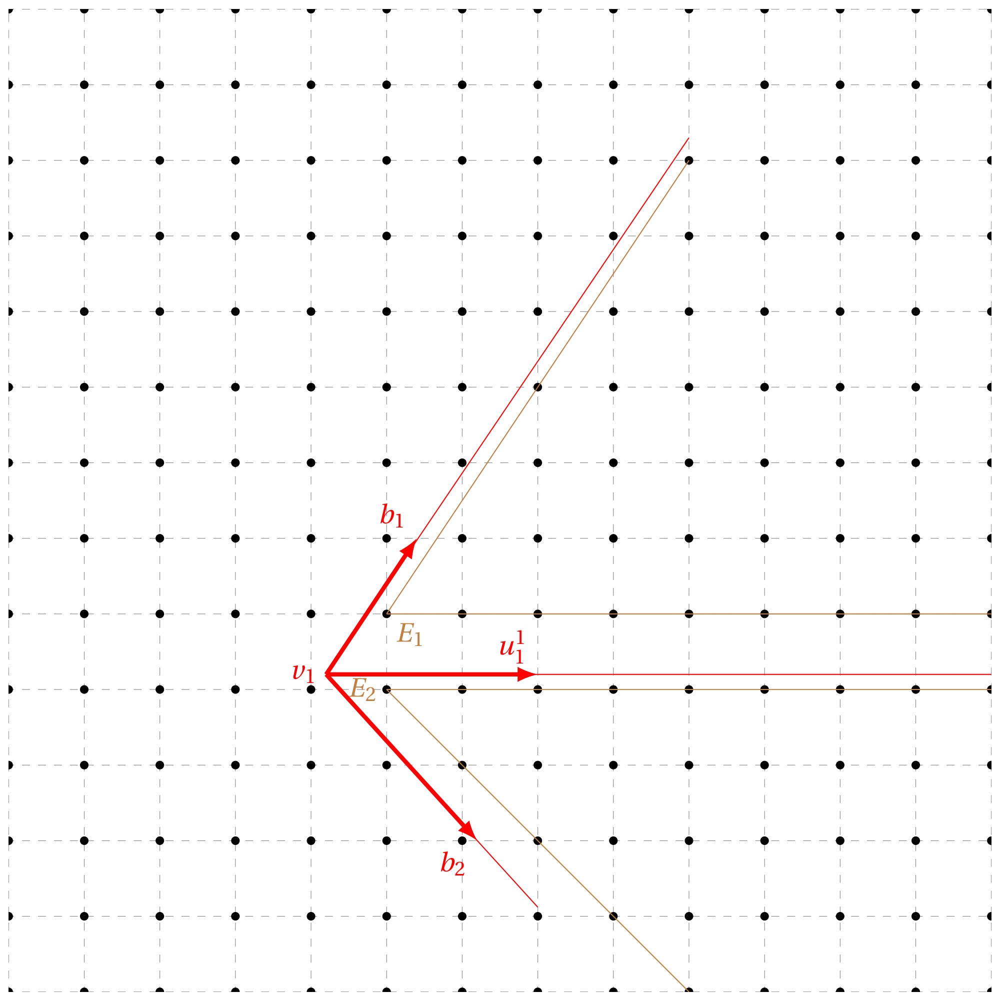

Figure 3.3: Unimodular decomposition.

3.3.3 Decomposition of two-dimensional cones and continued fraction

Here we recall the algorithm of continued fraction expansion of a real number r as follows:

we can write r as $r = \lfloor r \rfloor + \rho(r)$ where $0 \leqslant \rho(r) < 1$ and let $r_0 = \lfloor r \rfloor$,

if $\rho(r) = 0$ we stop the algorithm ,

if not we put a as $\frac{1}{\rho(r)}$ then we repeat as above. At the end we represent by it's continued fraction as

$r = [r_0; r_1, \cdots, r_n, \cdots]$

Example 3.3.14. Let $r = \frac{-42}{10}$ then by the above we write

$$\frac{-42}{10} = -5 + \frac{1}{1 + \frac{1}{3+\frac{1}{1}}}$$

so we can write the expansion of r as $[-5; 1, 3, 1]$.

Theorem 3.3.15. *[4]Let $K \subset \mathbb{R}^2$ be a cone generated by vectors $(1,0)$ and (q,p) where q and p are coprime positive integers. Let $\frac{p}{p} = [a_0, \cdots, a_n]$, we define the cone K_i for $i = -1, 0, \cdots, n$ as follows :*

consider

$$\frac{p_i}{q_i} = [a_0, \cdots, a_i] \qquad for \quad i = 0, \cdots, n.$$

Let K_{-1} be the cone generated by $(1,0)$ and $(0,1)$, K_0 as the cone generated by $(0,1)$ and $(1, p_0)$, and K_i as the cone generated by (q_{i-1}, p_{i-1}) and (q_i, p_i) for $i = 1, \cdots, n$. Then K_i are unimodular cones for $i = -1, 0, 1, \cdots, n$ and we can write K as follows :

$$[K] = \begin{cases} \sum_{i=-1}^{n} (-1)^{i+1} [K_i] \quad if \quad n \quad is \quad odd \\ \\ \sum_{i=-2}^{n} (-1)^{i+1} [K_i] \quad if \quad n \quad is \quad even \end{cases}$$

where K_{-2} is the ray emanating from the origin in the direction of (q_n, p_n).

Remark 3.3.16. It is not hard to see that an arbitrary 2-dimensional cones can be represented as a combination of two 2-dimensional cones each of them generated by $(1,0)$ and some other integer vector, and one 1-dimensional cone.

(1) In the above theorem if $q < 0$, we let K_1 be the cone generated by $(1,0)$ and $(-q, -p)$ then the indicators $[K]$ and $[K_1]$ differ by a halfplane and some boundary rays.

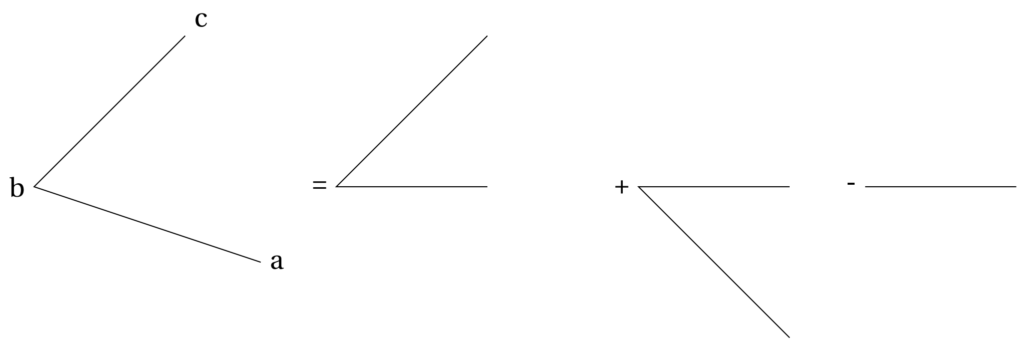

Figure 3.4: changing into the cone generated by $(1,0)$ and some other integer vector

(2) I the above theorem if $p<0$ we let K_1 be the cone generated by $(1,0)$ and $(q,-p)$ then $f(K_1,(x_1,x_2)) = f(K_2,(x_1,x_2^{-1}))$.

3.4 Hilbert functions of non-standard bigraded rings

Let $S = k[y_1,\dots,y_m]$ be a $\mathbb{Z}^{d-1}$-graded polynomial ring over a field and let $I = (f_1,\dots,f_n)$ be a graded ideal, with f_i homogeneous of degree d_i. To get information about the behavior of i-syzygy module of I^t as t varies, we pass to Rees algebra $\mathscr{R}_I = \oplus_{t\geq 0} I^t$ which is a $(\mathbb{Z}^{d-1}\times\mathbb{Z})$-graded algebra such that $(\mathscr{R}_I)_{(\mu,n)} = (I^n)_\mu$.

Recall that $\mathscr{R}_I$ is a graded quotient of $R := S[x_1,\dots,x_n]$ with grading extended from the one of S by setting $\deg(a) := (\deg(a),0)$ for $a\in S$ and $\deg(x_j) := (d_j,1)$ for all j. As noticed in [6], if $\mathbb{G}_\bullet$ is a $\mathbb{Z}^d$-graded free R-resolution of $\mathscr{R}_I$, then, setting $B := k[x_1,\dots,x_n] = R/(y_1,\dots,y_m)$,

$$\mathrm{Tor}_i^S(I^t,k)_\mu = H_i(\mathbb{G}_\bullet\otimes_R B)_{(\mu,t)}.$$

The complex $\mathbb{G}_\bullet\otimes_R B$ is a $\mathbb{Z}^d$-graded complex of free S-modules. Its homology modules are therefore finitely generated $\mathbb{Z}^d$-graded S-modules, on which we will apply results derived from the ones on vector partition functions describing the Hilbert series of S.

As an example, we describe the chamber complex associated to the matrix corresponding to the degrees $(d_i, 1)$, when $d = 1$ (i.e. $d_i \in \mathbb{N}$).

Lemma 3.4.1. *Let*

$$A = \begin{pmatrix} d_1 & \dots & d_n \\ 1 & \dots & 1 \end{pmatrix}$$

be a $2 \times n$*-matrix with entries in* $\mathbb{N}$ *such that* $d_1 \leq \ldots \leq d_n$. *Then the chambers corresponding to* $\operatorname{Pos}(A)$ *are positive polyhedral cones* (Δ) *where* Δ *is generated by* $\{(d_i, 1), (d_{i+1}, 1)\}$ *for all* $d_i \neq d_{i+1}$ *where* i *runs over* $\{1, \ldots, n\}$.

Proof. Since any arbitrary pair $\{(d_i, 1), (d_j, 1)\}$ makes an independent set whenever $d_i \neq d_j$, the common refinement consists of disjoint union of open convex polyhedral cones generated by $\{(d_i, 1), (d_{i+1}, 1)\}$ for all $i = 1, \ldots, n$ s.t.$d_i \neq d_{i+1}$.

□

Now we are ready to prove the main result of this section.

Proposition 3.4.2. *Let* $B = k[T_1, \ldots, T_n]$ *be a bigraded polynomial ring over field* k *with* $\deg(T_i) = (d_i, 1)$. *Assume that the number of distinct* d_i*'s is* $r \geq 2$. *Then there exist a finite index sublattice* L *of* $\mathbb{Z}^2$ *and collections of polynomials* Q_{ij} *of degree* $n-2$ *for* $1 \leq i \leq r-1$ *and* $1 \leq j \leq s$ *such that for any* $(\mu, \nu) \in \mathbb{Z}^2 \cap R_i$ *and* $\overline{(\mu, \nu)} \equiv g_j \mod L$ *in* $\mathbb{Z}^2 / L := \{g_1, \ldots, g_s\}$,

$$HF(B, (\mu, \nu)) = Q_{ij}(\mu, \nu)$$

where R_i *is the convex polyhedral cone generated by linearly independent vectors* $\{(d_i, 1), (d_{i+1}, 1)\}$.

Furthermore, $Q_{ij}(\mu,\nu) = Q_{ij}(\mu',\nu')$ *if* $\mu - \nu d_i \equiv \mu' - \nu' d_i \mod (\det(L))$.

Proof. Let

$$A = \begin{pmatrix} d_1 & \dots & d_n \\ 1 & \dots & 1 \end{pmatrix}$$

be a $2 \times n$-matrix corresponding to degrees of T_i.

The Hilbert function in degree $\mathbf{u} = (\mu,\nu)$ is the number of monomials $T_1^{\alpha_1} \dots T_n^{\alpha_n}$ such that $\alpha_1(d_1,1) + \dots + \alpha_n(d_n,1) = (\mu,\nu)$. This equation is equivalent to the system of linear equations

$$A.\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} \mu & \nu \end{pmatrix}.$$

In this sense $HF(B,(\mu,\nu))$ will be the value of vector partition function at (μ,ν). Assume that (μ,ν) belongs to the chamber R_i which is the convex polyhedral cone generated by $\{(d_i,1),(d_{i+1},1)\}$. By 3.2.8, we know that for $(\mu,\nu) \in R_i$ and $(\mu,\nu) \equiv g_j \mod (\det L)$,

$$\varphi_A(\mu,\nu) = Q_{ij}(\mu,\nu). \tag{3.4.1}$$

□

Notice that in the proposition 3.4.2, if moreover we suppose that $d_i \neq d_j$ for all $i \neq j$, then the Hilbert function in degree (μ,ν) will also be the number of possible ways to reach from $(0,0)$ to (μ,ν) in $\mathbb{Z}^2$ but it is not necessarily correct when we have equalities between some of degrees. For example if one has $d_i = d_{i+1} < d_{i+2}$, so the independent sets of vectors $\{(d_i,1),(d_{i+2},1)\}$ and $\{(d_{i+1},1),(d_{i+2},1)\}$ generate the same chamber and the number of possible ways to reach from $(0,0)$ to (μ,ν) is less than $\mathrm{HF}(B,(\mu,\nu))$.

In the following example, we are going to give a formula for Hilbert function of a non-standard graded polynomial ring in the case of three indeterminates which is a special case of formula done by Xu in [54].

Example 3.4.3. Let $B = k[T_1, T_2, T_3]$ be a polynomial ring over field k and $\deg T_i = (d_i, 1)$ for $1 \leq i \leq 3$ such that $d_i - d_{i+1} \gneqq 0$ for $i = 1, 2$. Set $Y_{ij} = d_i - d_j$ and suppose that $\gcd(Y_{12}, Y_{13}, Y_{23}) = 1$. Then there exist f_{ij}, g_{ij} such that

$$
\begin{aligned}
f_{12} Y_{23} + g_{12} Y_{23} &= \gcd(Y_{23}, Y_{13}) \quad \gcd(f_{12} Y_{13} + g_{12} Y_{23}, Y_{12}) = 1,\\
f_{13} Y_{12} + g_{13} Y_{23} &= \gcd(Y_{12}, Y_{23}) \quad \gcd(f_{13} Y_{12} + g_{13} Y_{23}, Y_{13}) = 1,\\
f_{23} Y_{13} + g_{23} Y_{12} &= \gcd(Y_{13}, Y_{12}) \quad \gcd(f_{23} Y_{13} + g_{23} Y_{12}, Y_{23}) = 1,
\end{aligned}
$$

with

$$
\begin{aligned}
\left(f_{12} Y_{13} + g_{12} Y_{23}\right)^{-1} \left(f_{12} Y_{13} + g_{12} Y_{23}\right) &\equiv 1 \mod Y_{12}\\
\left(f_{13} Y_{12} + g_{13} Y_{23}\right)^{-1} \left(f_{13} Y_{12} + g_{13} Y_{23}\right) &\equiv 1 \mod Y_{13}\\
\left(f_{23} Y_{13} + g_{23} Y_{12}\right)^{-1} \left(f_{23} Y_{13} + g_{23} Y_{12}\right) &\equiv 1 \mod Y_{23}.
\end{aligned}
$$

and $f_{12}, g_{12}, f_{13}, g_{13}, f_{23}$ and g_{23} can be calculated by an Euclidean algorithm.

Our chambers are regions

$$\Omega_i = \{(\mu, \nu) \mid \nu d_i > \mu > \nu d_{i+1}\}$$

for $i = 1, 2$.

Then for (n_1, n_2) belonging to the positive cone generated by vectors $\{(d_1, 1), (d_2, 1), (d_3, 1)\}$. When $(n_1, n_2)^t \in \overline{\Omega_1} \cap \mathbb{Z}^2$, it is proved in [54, Theorem 4.3] that

$$
\begin{aligned}
\mathrm{HF}(B, (n_1, n_2)) &= \tfrac{n_2 d_1 - n_1}{Y_{12} Y_{13}} + 1\\
&- \left\{ \tfrac{(f_{12} Y_{13} + g_{12} Y_{23})^{-1} (n_2 (f_{12} d_1 + g_{12} d_2) - n_1 (f_{12} + g_{12})}{Y_{12}} \right\}\\
&- \left\{ \tfrac{(f_{13} Y_{12} + g_{13} Y_{23})^{-1} (n_2 (f_{13} d_1 + g_{13} d_3) - n_1 (f_{13} + g_{13}))}{Y_{13}} \right\}.
\end{aligned}
$$

3.5 Explicit formula for non-standard Hilbert function

In this section we want to give an explicit formula for non standard Hilbert functions of polynimial ring in low dimensions by the using theory of lattice points in the convex polytope.

3.5.1 Variable Polytopes of partition function

Let e_i be the standard basis of the space $\mathbb{R}^r$ for $1 \leqslant i \leqslant r$ and linear map $f : \mathbb{R}^r \to \mathbb{R}^2$ defined by $f(e_i) = v_i$. Let $a \in \mathbb{R}^2$, we define the following convex polytope:

$$P(a) := f^{-1}(a) \cap \mathbb{R}^r_{\geqslant 0} = \{x \in \mathbb{R}^r \,|\, Ax = a; x \geqslant 0\}$$

Where A is the matrix of f .

Proposition 3.5.1. *[16] Let $A = \{a_1, \cdots, a_n\}$ be a set of vectors in $\mathbb{R}^d$. If b is in the interior of* $\mathrm{Pos}(A) := \{\sum_{i=1}^n \lambda_i a_i \in \mathbb{R}^d | \lambda_i \geq 0, 1 \leq i \leq n\}$*, the polytope $P(b)$ has dimension $n - d$.*

Definition 3.5.2. Let $A = \{a_1, \cdots, a_n\}$ be a set of vectors in $\mathbb{R}^d$.

1. Let $b \in \mathrm{Pos}(A)$. A basis $a_{i_1}, \cdots, a_{i_s}$ extracted from A with respect to which b has positive coordinares called b-positive.

2. A point $c \in \mathrm{Pos}(A)$ is called strongly regular if there is no sublist $Y \subset A$ lying in a proper vector subspace , such that $c \in \mathrm{Pos}(Y)$. A point in $\mathrm{Pos}(A)$ is called strongly singular if it is not strongly regular .

For using the Barvinok algorithm on the variable polytope $P(b)$ we need to know about it's structures as vertices and faces which done by the following theorem .

Theorem 3.5.3. *[16] Let $b \in \mathrm{Pos}(A)$ be a strongly regular point . Then*

(i) The vertices of $P(b)$ are of the form $P_Y(b) = \{x \in \mathbb{R}^r | Yx = a; x \geqslant 0\}$ with Y a b-positive basis.

(ii) The faces of $P(b)$ are of the form $P_Z(b)$ where Z runs over the subsets of A containing a b-positive basis. $|Z| - d$ is the dimension of $P_Z(b)$ and positive basis in Z correspond to vertices of $P_Z(b)$

Now we consider the polytope $P(b)$ such that A is $\begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ for $d_1 < d_2 < d_3 < d_4$ then by the method of Barvinok we want to calculate the generating function of the tangent cones of the polytopes depending on b, before that we should mention that the polytope $P(b)$ associated to the matrix A is not full dimensional so to use the Barvinok metod we need to transform $P(b)$ to polytope Q which is full dimensional and the integer points of Q are in one-to-one correspondence to the integer points of $P(b)$. The following procedure describes how it can be done:

1. let $P = \{x \in \mathbb{R}^n | Ax = a, Bx \leqslant b\}$ be a polytope related to full row-rank $d \times n$ matrix A.
2. Find the generators $\{g_1, \cdots, g_{n-d}\}$ of the integer null-space of A.
3. Find integer solution x_0 to $Ax = a$.
4. Substituting the general integer solution $x = x_0 + \sum_{i=1}^{n-d} \beta_i g_i$ into the inequalities $Bx \leqslant b$.
5. By Substitution of (4) we arrive at a new system $C\beta \leqslant c$ which defines the new polytope $Q = \{\beta \in \mathbb{R}^{n-d} | C\beta \leqslant c\}$.

By using above procedure we will associat to the polytope $P(b)$ the full dimensional polytope Q in the following lemma:

Lemma 3.5.4. *Let* $A = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ *for* $d_1 < d_2 < d_3 < d_4$ *then there is a one to one correspondence to the integer points of* $P(b)$ *and* $Q \subset \mathbb{R}^2$ *and we have the followings about* Q*:*

1. $Q = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 | \lambda_1(d_2 - d_1) \leqslant 0;\ \ \lambda_2(d_2 - d_1) \leqslant 0;$
   $\lambda_1(d_1 - d_4) + \lambda_2(d_1 - d_3) + \frac{d_1 b_2 - b_1}{d_2 - d_1} \leqslant 0;\ \ \lambda_1(d_4 - d_2) + \lambda_2(d_3 - d_2) + \frac{b_1 - d_2 b_2}{d_2 - d_1} \leqslant 0\ \ for\ \ b_1, b_2 > 0\}$,

2. *with the following vertices :*

   $q_1 = (\frac{d_3 b_2 - b_1}{(d_2 - d_1)(d_4 - d_3)}, (\frac{b_1 - d_4 b_2}{(d_2 - d_1)(d_4 - d_3)})$,

   $q_2 = (\frac{d_2 b_2 - b_1}{(d_2 - d_1)(d_4 - d_2)}, 0)$,

   $q_3 = (0, \frac{d_2 b_2 - b_1}{(d_2 - d_1)(d_3 - d_2)})$,

   $q_4 = (\frac{b_1 - d_1 b_2}{(d_2 - d_1)(d_1 - d_4)}, 0)$,

   $q_5 = (0, \frac{b_1 - d_1 b_2}{(d_2 - d_1)(d_1 - d_3)})$,

   $q_6 = (0, 0)$,

3. *The generation function of* Q *in the first chamber* C_1*:*

$$f_{C_1}(Q, \boldsymbol{x}) = \frac{1}{(1 - x_1^{-1})(1 - x_2^{-1})} + \frac{x_1^{\lceil s_1 \rceil}}{(1 - x_1)(1 - x_2^{-1})} - \frac{x_1^{\lceil s_1 \rceil} x_2^{-(\lceil a_0 s_1 \rceil + a_0 \lceil s_1 \rceil)}}{(1 - x_2^{-1})(1 - x_1 x_2^{-a_0})} +$$

$$\frac{x_1^{(\lceil (a_0 a_1 + 1) s_1 \rceil - a_1 \lceil a_0 s_1 \rceil)} x_2^{-(a_0 \lceil (a_0 a_1 + 1) s_1 \rceil - (a_0 a_1 + 1) \lceil s_1 \rceil)}}{(1 - x_1 x_2^{-a_0})(1 - x_1^{a_1} x_2^{-(a_1 a_0 + 1)})} + \frac{x_2^{-\lceil s_2 \rceil}}{(1 - x_2^{-1})(1 - x_1 x_2^{-a_0})} -$$

$$\frac{x_1^{(\lceil -a_1 s_2\rceil + a_1\lceil s_2\rceil)} x_2^{-(a_0\lceil -a_1 s_2\rceil + (a_0 a_1+1)\lceil s_2\rceil)}}{(1-x_1 x_2^{-a_0})(1-x_1^{a_1} x_2^{-(a_1 a_0+1)})}$$

Where $s_1 := \frac{b_1 - d_1 b_2}{(d_2-d_1)(d_1-d_4)}$ *and* $s_2 := \frac{b_1 - d_1 b_2}{(d_2-d_1)(d_1-d_3)}$.

Proof. (1) It is not hard to see that a generators of integer null-space of $A = \begin{pmatrix} d_1 & d_2 & d_3 & d_4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$ are $\overrightarrow{g_1} = (d_2 - d_4, d_4 - d_1, 0, d_1 - d_2)$ and $\overrightarrow{g_2} = (d_2 - d_3, d_3 - d_1, d_1 - d_2, 0)$ and $(\frac{b_1 - d_2 b_2}{(d_1-d_2)}, \frac{d_1 b_2 - b_1}{(d_1-d_2)}, 0, 0)$ is the solution of linear system then considering the five steps in which describe before this lemma we can get supporting half planes of Q. (2) can easily calculated from (1). As we suppose that $b \in C_1$ then only vertices Q_6, Q_4 and Q_5 are active, then we associate to each one a tangent cone as follows:

$$\text{Cone}(Q, q_5) = co((1,0),(d_3 - d_1, d_1 - d_4)),$$

$$\text{Cone}(Q, q_4) = co((0,1),(d_1 - d_3, d_4 - d_1)),$$

$$\text{Cone}(Q, q_4) = co((-1,0),(0,-1)).$$

The first two tangent cones are necessarily unimodular so as our cone are 2-dimensional we can decompose it to unimodular cone by continued fraction. For simplicity of calculations suppose that $\frac{d_4-d_1}{d_3-d_1} = [a_0; a_1] = a_0 + \frac{1}{a_1}$ then

$$[\text{Cone}(Q, q_5)] = [K_{-1} = co((1,0),(0,1))] + [K_0 = co((0,1),(1,a_0))] + [K_1 = co((1,a_0),(a_1, a_0 a_1 + 1))]$$

using the 3.3.13 and 3.3.16 we can decompose the generating function of Cone(Q, q_5)

$$f(\text{Cone}(Q, q_5), (x_1, x_2)) = f(K_{-1}, (x_1, x_2^{-1})) + f(K_0, (x_1, x_2^{-1})) + f(K_1, (x_1, x_2^{-1}))$$

we can decompose Cone(Q, q_4) in the same way $[\text{Cone}(Q, q_4)] = [S_{-1} = co((1,0),(0,1))] + [S_0 = co((0,1),(1,a_0))] + [S_1 = co((1,a_0),(a_1, a_0 a_1 + 1))] - [S_2 = co((1,0),(0,1))]$

$$f(\text{Cone}(Q, q_4), (x_1, x_2)) = f(S_0, (x_1, x_2^{-1})) + f(K_1, (x_1, x_2^{-1}))$$

So (3) can be achieved from 3.3.13. □

Remark 3.5.5. In the above polytope Q when we fix a chamber there are some vertices which are inactive, more precisely if the point $b = (b_1, b_2)$ is in the first chamber then $b_1 - d_1 b_2 \geqslant 0$ and $d_2 b_2 - b_1 \geqslant 0$ so only vertices Q_6, Q_5, Q_4 are active.

Now we are able to optain the explicit formula for non-standard Hilbert function by above lemma.

Theorem 3.5.6. *Let $B = k[T_1, \ldots, T_4]$ be a graded polynomial ring over a field k with $\deg T_i = (d_i, 1)$ and $d_i \neq d_j$ for $1 \leqslant i \leqslant 4$. Suppose that $\frac{d_4-d_1}{d_3-d_1} = a_0 + \frac{1}{a_1}$ and set $s_1 := \frac{b_1-d_1b_2}{(d_2-d_1)(d_1-d_4)}$ and $s_2 := \frac{b_1-d_1b_2}{(d_2-d_1)(d_1-d_3)}$. Then the Hilber function of B at degree $(b_1, b_2) \in C_1$ given by following formula:*

$$HF(B;(b_1,b_2)) = \left\{\frac{(a_0+2)(a_0+1)}{2(a_0-1)} + \frac{4a_0(a_0+2)-3a_0^2}{12} + \frac{a_0(a_0+2)}{2(a_0-1)}\right\} + \left\{\frac{(a_0\lceil s_1\rceil+2)(a_0\lceil s_1\rceil+1)}{2(a_0-1)} + \frac{a_0(a_0\lceil s_1\rceil+2)}{2(a_0-1)} + \frac{4a_0(a_0+2)-3a_0^2}{12}\right\}$$

$$\left\{\frac{4a_0(a_0+2)-3a_0^2}{6} + \frac{a_0(a_0\lceil s_1\rceil+2\lceil a_0s_1\rceil+1)(a_0\lceil s_1\rceil+2\lceil a_0s_1\rceil+1)}{2(a_0-1)} + \frac{a_0(a_0+2)(a_0+1)}{2(a_0-1)} + \frac{a_0(a_0+2)}{2(a_0-1)} + (a_0+2)(a_0\lceil s_1\rceil+2\lceil a_0s_1\rceil) + \frac{a_0(a_0\lceil s_1\rceil+2\lceil a_0s_1\rceil)}{2(a_0-1)}\right\} +$$

$$\left\{\frac{(4a_0(a_0+2)-3a_0^2)(2a_0a_1+3)}{12(2a_0a_1+1)} + \frac{(a_0\lceil(a_0a_1+1)s_1\rceil)(a_0\lceil(a_0a_1+1)s_1\rceil+1)}{2(a_0-1)} + \frac{(4(a_0a_1+2)(a_0a_1+4)-3(a_0a_1+2)^2)((2a_0-1))}{12(a_0-1)} +\right.$$

$$\frac{(a_0a_1\lceil s_1\rceil+2\lceil s_1\rceil+a_0a_1+2)(a_0a_1\lceil s_1\rceil+2\lceil s_1\rceil+a_0a_1+1)(a_0^2a_1+2a_0-1)}{(a_0-1)(a_0a_1+1)} + \frac{(a_0a_1\lceil s_1\rceil+2\lceil s_1\rceil+a_0a_1+2)a_0}{2(a_0-1)} +$$

$$\frac{(a_0a_1\lceil s_1\rceil+2\lceil s_1\rceil+a_0a_1+2)((a_0\lceil(a_0a_1+1)s_1\rceil))}{1} + \frac{(a_0a_1\lceil s_1\rceil+2\lceil s_1\rceil+a_0a_1+2)(a_0a_1+2)}{2(a_0a_1+2)} + \frac{(a_0^2\lceil(a_0a_1+1)s_1\rceil)}{2(a_0-1)}$$

$$\left.\frac{a_0(a_0a_1+2))}{2(a_0-1)(a_0a_1+1)} + \frac{(a_0\lceil(a_0a_1+1)s_1\rceil)(a_0a_1+2)}{2(a_0a_1+1)}\right\} + \left\{\frac{a_0(a_0+2)(a_0+1)}{2(a_0-1)} + \frac{a_0(2\lceil s_2\rceil)(2\lceil s_2\rceil+1)}{2(a_0-1)} + \frac{4a_0(a_0+2)-3a_0^2}{6}\right.$$

$$\frac{a_0(a_0+2)}{2(a_0-1)} + \frac{2(a_0+2)\lceil s_2\rceil}{1} + \frac{2a_0\lceil s_2\rceil}{2(a_0-1)}\Bigg\} + \Bigg\{\frac{(a_0a_1+1)(a_0a_1+2)(a_0^2a_1+2a_0-1)}{2(a_0-1)(a_0a_1+1)} + \frac{(4a_0(a_0+2)-3a_0^2)(2a_0a_1+3)}{12(2a_0a_1+1)} +$$

$$\frac{(2a_0-1)(4(a_0a_1+2)(a_0a_1+4)-3(a_0a_1+2)^2)((2a_0-1))}{12(a_0-1)} + \frac{(a_0\lceil a_1s_2\rceil+2\lceil s_2\rceil+a_0a_1\lceil s_2\rceil)(a_0\lceil a_1s_2\rceil+2\lceil s_2\rceil+a_0a_1\lceil s_2\rceil+1)(a_0^2a_1+2a_0-1)}{2(a_0-1)(a_0a_1+1)}$$

$$\frac{a_0(a_0a_1+2)}{2(a_0-1)} + \frac{(a_0a_1+2)^2}{2(a_0a_1+1)} + \frac{(a_0a_1+2)(a_0\lceil a_1s_2\rceil+2\lceil s_2\rceil+a_0a_1\lceil s_2\rceil)}{1} + \frac{a_0(a_0a_1+2))}{2(a_0-1)(a_0a_1+1)} + \frac{a_0(a_0\lceil a_1s_2\rceil+\lceil s_2\rceil+a_0a_1\lceil s_2\rceil)}{2(a_0-1}$$

$$\frac{(a_0a_1+2)(a_0\lceil a_1s_2\rceil+2\lceil s_2\rceil+a_0a_1\lceil s_2\rceil)}{2(a_0a_1+2)}\Bigg\}.$$

Proof. If $b=(b_1,b_2)\in C_1$ then it is clear from 3.5.4 that $HF(B,(b_1,b_2)) = f_{C_1}(Q,\mathbf{(1,1)})$ however f has a pole at $\mathbf{x}=(1,1)$, it is analytic at $\mathbf{x}=(1,1)$. Because of cancelation of the coefficients of negative powers in the Luarent series at $\mathbf{x}=(1,1)$ so the value at $\mathbf{x}=(1,1)$ it is the sum of the cofficients of the constant term in Laurent series of each term in 3.5.4(3), to be able of computions first we change f from multivariate to univariate by following the Yoshida at al.[24] method.

Choose the vector $\eta=(a_0,2)$ which is not orthogonal to any generators of the vertex cones of Q and variable substituition $x_i=(s+1)^{\eta_i}$ for $i=1,2$ then we obtain:

$$f_{C_1}(Q,((s+1)^{a_0}),(s+1)^2)) = \frac{(s+1)^{a_0+2}}{(1-(s+1)^{a_0})(1-(s+1)^2)} - \frac{(s+1)^{a_0\lceil s_1\rceil+2}}{(1-(s+1)^{a_0})(1-(s+1)^2)} + \frac{(s+1)^{a_0+2-2\lceil a_0s_1\rceil-a_0\lceil s_1\rceil}}{(1-(s+1)^{a_0})(1-(s+1)^2)} +$$

$$\frac{(s+1)^{-a_0\lceil(a_0a_1+1)s_1\rceil+a_0a_1\lceil s_1\rceil+2\lceil s_1\rceil+a_0a_1+a_0+2}}{(1-(s+1)^{a_0})(1-(s+1)^{a_0a_1+2})} - \frac{(s+1)^{a_0+2}+2\lceil s_2\rceil}{(1-(s+1)^{a_0})(1-(s+1)^2)} + \frac{(s+1)^{-a_0\lceil a_1s_2\rceil-a_0a_1\lceil s_2\rceil-2\lceil s_2\rceil+a_0a_1+a_0+2}}{(1-(s+1)^{a_0})(1-(s+1)^{a_0a_1+2})}$$

Now we using following general expansions:

$$(s+1)^n \equiv 1+ns+\frac{n(n-1)}{2}s^2 \qquad mod \quad s^3$$

and

$$\frac{1}{s+2} \equiv \frac{1}{2}-\frac{1}{4}s+\frac{1}{4}s^2 \qquad mod \quad s^3$$

we obtain:

$$\frac{(s+1)^{a_0+2}}{(1-(s+1)^{a_0})(1-(s+1)^2)} \equiv (1+(a_0+2)s+\frac{(a_0+2)(a_0+1)s^2}{2})(\frac{1}{a_0-1}+\frac{a_0 s}{2(a_0-1)}+\frac{(4a_0(a_0+2)-3a_0^2)s^2}{12})$$

,

$$\frac{(s+1)^{a_0\lceil s_1\rceil+2}}{(1-(s+1)^{a_0})(1-(s+1)^2)} \equiv (1+(a_0\lceil s_1\rceil+2)s+(\frac{(a_0\lceil s_1\rceil+2)(a_0\lceil s_1\rceil+1)}{2})s^2)(\frac{1}{a_0-1}+\frac{a_0 s}{2(a_0-1)}+\frac{(4a_0(a_0+2)-3a_0^2)s^2}{12})$$

,

$$\frac{(s+1)^{a_0+2-2\lceil a_0 s_1\rceil-a_0\lceil s_1\rceil}}{(1-(s+1)^{a_0})(1-(s+1)^2)} \equiv (1+(a_0+2)s+\frac{(a_0+2)(a_0+1)s^2}{2})(\frac{1}{a_0-1}+\frac{a_0 s}{2(a_0-1)}+\frac{(4a_0(a_0+2)-3a_0^2)s^2}{12})$$

$$(1+(a_0\lceil s_1\rceil+2\lceil a_0 s_1\rceil)s+\frac{(1+a_0\lceil s_1\rceil+2\lceil a_0 s_1\rceil)(a_0\lceil s_1\rceil+2\lceil a_0 s_1\rceil)}{2}s^2)(\frac{1}{a_0-1}+\frac{a_0 s}{2(a_0-1)}+\frac{(4a_0(a_0+2)-3a_0^2)s^2}{12})$$

,

$$\frac{(s+1)^{-a_0\lceil(a_0a_1+1)s_1\rceil+a_0a_1\lceil s_1\rceil+2\lceil s_1\rceil+a_0a_1+a_0+2}}{(1-(s+1)^{a_0})(1-(s+1)^{a_0a_1+2})} \equiv (1+(a_0a_1\lceil s_1\rceil+2\lceil s_1\rceil+a_0a_1+a_0+2)s+$$

$$\frac{(a_0a_1\lceil s_1\rceil + 2\lceil s_1\rceil + a_0a_1 + a_0 + 2)(a_0a_1\lceil s_1\rceil + 2\lceil s_1\rceil + a_0a_1 + a_0 + 1)}{2}s^2)$$

$$(\frac{1}{a_0-1}+\frac{a_0 s}{2(a_0-1)}+\frac{(4a_0(a_0+2)-3a_0^2)s^2}{12})(1+(a_0\lceil(a_0a_1+1)s_1\rceil)s+\frac{(a_0\lceil(a_0a_1+1)s_1\rceil)(a_0\lceil(a_0a_1+1)s_1\rceil+1)}{2}s^2)$$

$$(\frac{1}{(a_0a_1+1)}+\frac{(a_0a_1+2)}{2(a_0a_1+1)}s+\frac{4(a_0a_1+2)(a_0a_1+4)-3(a_0a_1+1)^2}{12}s^2)$$

,

$$\frac{(s+1)^{a_0+2}+2\lceil s_2\rceil}{(1-(s+1)^{a_0})(1-(s+1)^2)} \equiv (1+(a_0+2)s+\frac{(a_0+2)(a_0+1)}{2}s^2)(\frac{1}{a_0-1}+\frac{a_0}{2(a_0-1)}s+\frac{4a_0(a_0+2)-3a_0^2}{12}s^2)$$

$$(1+2\lceil s_2\rceil s+\frac{(2\lceil s_2\rceil)(2\lceil s_2\rceil+1)}{2}s^2)$$

and

$$\frac{(s+1)^{-a_0\lceil a_1s_2\rceil-a_0a_1\lceil s_2\rceil-2\lceil s_2\rceil+a_0a_1+a_0+2}}{(1-(s+1)^{a_0})(1-(s+1)^{a_0a_1+2})} \equiv (1+(a_0a_1+2)s+\frac{(a_0a_1+2)(a_0a_1+1)}{2}s^2)(\frac{1}{a_0-1}+\frac{a_0}{2(a_0-1)}+\frac{4a_0(a_0+2)-3a_0^2}{12}s^2)$$

$$(\frac{1}{a_0a_1+1}+\frac{a_0a_1+2}{2(a_0a_1+2)}s+\frac{4(a_0a_1+2)(a_0a_1+4)-3(a_0a_1+2)^2}{12}s^2)$$

$$(1+(a_0\lceil a_1s_2\rceil+a_0a_1\lceil s_2\rceil+2\lceil s_2\rceil)s+\frac{(a_0\lceil a_1s_2\rceil+a_0a_1\lceil s_2\rceil+2\lceil s_2\rceil)(a_0\lceil a_1s_2\rceil+a_0a_1\lceil s_2\rceil+2\lceil s_2\rceil+1)}{2}s^2)$$

So the number of lattice points in the polytope Q is given by the sum of the coefficients of s^2 which is the final formula. □

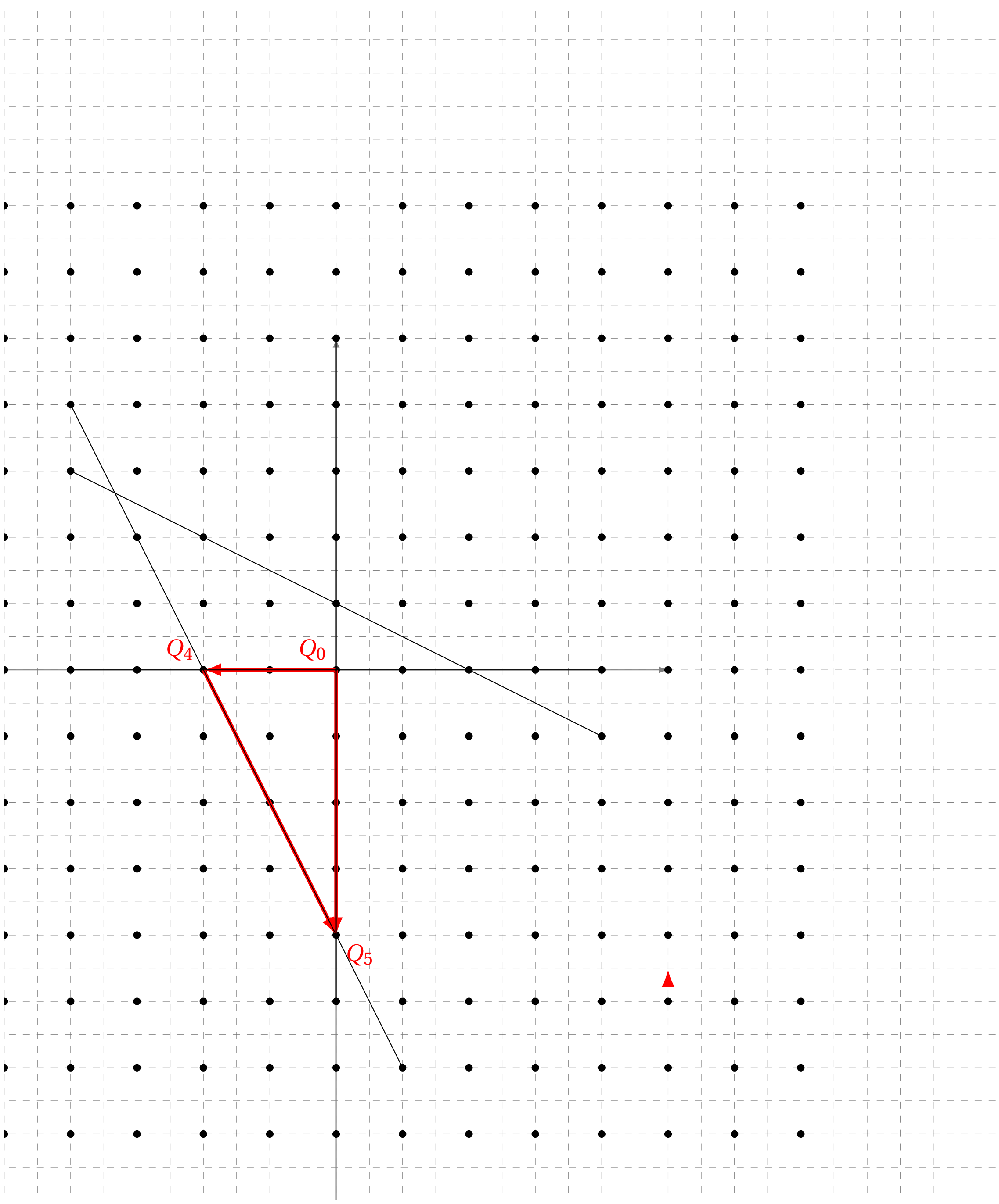

Figure 3.5: transformed polytope in $\mathbb{R}^2$ related to the first chamber

CHAPTER 4

ASSYMPTOTIC BEHAVIOR BETTI NUMBER OF POWERS OF IDEALS

4.1 Kodiyalam Polynomials

Let R be a Noetherian local ring with maximal ideal m and residue field k. Let I be a proper ideal of R. Kodiyalam in [33] proved the polynomial behavior of Betti number and Bass number of a finitely generated graded module as follows :

Theorem 4.1.1. *Let $\mathscr{S} = \bigoplus_{n\geqslant 0}\mathscr{S}_n$ be a Noetherian graded ring generated as an $\mathscr{S}_0$-algebra by $\mathscr{S}_1$ and with $\mathscr{S}_0$ local with maximal ideal m. Let $\mathscr{M} = \bigoplus_{n\geqslant 0}\mathscr{M}_n$ be a finitely generated graded $\mathscr{S}$-module. Then both $\beta_i^{\mathscr{S}_0}(\mathscr{M}_n)$ and $\mu^i_{\mathscr{S}_0}(\mathscr{M}_n)$ are polynomials for $n \gg 0$ and for any $i \geqslant 0$. The degrees of those polynomials are at most* $\dim(\frac{\mathscr{M}}{m\mathscr{M}}) - 1$.

In the case where $\mathscr{M} = \bigoplus_{n\geqslant 0} I^n$ the polynomials $\mathscr{P}_i(k) = \beta_i(\frac{R}{I^k}) = \dim \operatorname{Tor}_i^R(\frac{R}{m}, \frac{R}{I^k})$ are called the Kodiyalam Polynomials of I for $k \gg 0$ and $i \geqslant 0$.
One of the above theorem's outcomes is about projective and injective dimensions of $\frac{R}{I^k}$ which is proved first by Brodmann[12] .

Theorem 4.1.2. *Let R be a Noetherian local ring with maximal ideal m. Let M be a finitely generated R-module and $\mathscr{M} = \bigoplus_{n\geqslant 0} I^n M$. Then both $pr_R(\frac{M}{I^n M})$ and $id_R(\frac{M}{I^n M})$ stabilizes for $k \geqslant 0$.*

Example 4.1.3. [20] Let $I = (x^3, x^2 - yz, y^4 + xz^3, xy - z^2) \subset R = k[x, y, z]$, the Kodiyalam polynomials of I are as follows:

$$P_1(I)(k) = (k+1)^2 \quad , \quad P_2(I)(k) = (\frac{5}{2}k + \frac{7}{2})k \quad , \quad P_3(I)(k) = \frac{3}{2}k(k+1) \quad .$$

Kodiyalam asked in [34] that " is it true that polynomials $P_i(I)(n)$ for $\gg 0$ either vanishes or has degree $l(I) - 1$? " , then J. Herzog and V. Welker proved in [20] the following result about degrees of $P_i(I)$

Proposition 4.1.4. $l - 1 = \deg P_1(I)(k) \;\geqslant P_2(I)(k) \;\geqslant\; \cdots \;\geqslant P_n(I)(k)$.

In fact, the Kodiyalam polynomials is explained asymptotic behavior of total Betti numbers of powers of ideals. More generally, in the next section, we will study the asymptotic behavior of graded Betti numbers of powers of homogeneous ideals.

4.2 The general case

Before studying the graded Betti numbers of powers of ideals let me recall the result in [6] about the important case where the generators of the ideal I have the same degree.

Theorem 4.2.1. *Let $R = S[T_1,\dots,T_r]$ be a $G\times G'$-graded polynomial extension of S with $\deg_{G\times G'}(a)\in G\times 0$ for all $a\in S$ and $\deg_{G\times G'}(T_j)\in 0\times G'$ for all j. Let $\mathbb{M}$ be a finitely generated $G\times G'$-graded R-module and let i be an integer. Assume that $i=0$ or A is a Noetherian ring. Then*

1. *There exists a finite subset $\Delta_i\subseteq G$ such that, for any t, $\operatorname{Tor}_i^S(\mathbb{M}_{(*,t)},A)_\delta = 0$ for all $\delta\notin\Delta_i$.*

2. *Assume that $G'=\mathbb{Z}^s$. For $\delta\in\Delta_i$, $\operatorname{Tor}_i^S(\mathbb{M}_{(*,t)},A)_\delta=0$ for $t\gg 0$ or $\operatorname{Tor}_i^S(\mathbb{M}_{(*,t)},A)_\delta\neq 0$ for $t\gg0$. If, furthermore, $A\to k$ is a ring homomorphism to a field k, then for any j, the function*

$$\dim_k \operatorname{Tor}_j^A(\operatorname{Tor}_i^S(\mathbb{M}_{(*,t)},A)_\delta,k)$$

is polynomial in the t_is for $t\gg0$, and the function

$$\dim_k \operatorname{Tor}_i^S(\mathbb{M}_{(*,t)},k)_\delta$$

is polynomial in the t_is for $t\gg0$.

Now we return to the main result on Betti numbers of powers of ideals. We can treat without any further effort the case of a collection of graded ideals and include a graded module M. Hence, we will study the behaviour of $\dim_k \operatorname{Tor}_i^R(MI_1^{t_1}\cdots I_s^{t_s},k)_\mu$ for $\mu\in\mathbb{Z}^p$ and $t\gg0$. To this aim, we first use the important fact that the module

$$B_i := \oplus_{t_1,\dots,t_p}\operatorname{Tor}_i^R(MI_1^{t_1}\cdots I_s^{t_s},k)$$

is a finitely generated $(\mathbb{Z}^p \times \mathbb{Z}^s)$-graded ring, over $k[T_{i,j}]$ setting $\deg(T_{i,j}) = (\deg(f_{i,j}), e_i)$ with e_i the i-th canonical generator of $\mathbb{Z}^s$ and $f_{i,j}$'s minimal generators of I_i. Hence, $\mathrm{Tor}_i^R(MI_1^{t_1}\cdots I_s^{t_s}, k)_\mu = (B_i)_{\mu, t_1e_1+\cdots+t_se_s}$.

The following result applied to B_i will then give the asymptotic behavior of Betti numbers. In the particular case of one $\mathbb{Z}$-graded ideal, we will use it to give a simple description of this eventual behaviour.

Let $\varphi : \mathbb{Z}^n \to \mathbb{Z}^d$ with $\varphi(\mathbb{N}^n) \subseteq \mathbb{N}^d$ be a positive $\mathbb{Z}^d$-grading of $R := k[T_{i,j}]$. Set $\mathbb{Z}^n := \sum_{i=1}^n \mathbb{Z}e_i$, let E be the set of d-tuples $e = (e_{i_1}, ..., e_{i_d})$ such that $(\varphi(e_{i_1}), ..., \varphi(e_{i_d}))$ generates a lattice Λ_e in $\mathbb{Z}^d$, and set

$$\Lambda := \cap_{e\in E}\Lambda_e, \qquad s_\Lambda : \mathbb{Z}^d \xrightarrow{can} \mathbb{Z}^d/\Lambda\ .$$

Denote by C_i, $i \in F$, the maximal cells in the chamber complex associated to φ. One has

$$\overline{C_i} = \{\xi \mid H_{i,j}(\xi) \geqslant 0,\ 1 \leqslant j \leqslant d\}$$

where $H_{i,j}$ is a linear form in $\xi \in \mathbb{Z}^d$.

Proposition 4.2.2. *With notations as above, let B be a finitely generated $\mathbb{Z}^d$-graded R-module. There exist convex sets of dimension d in $\mathbb{R}^d$ of the form*

$$\Delta_u = \{x \mid H_{i,j}(x) \geqslant a_{u,i,j}, \forall (i,j) \in G_u\} \subseteq \mathbb{R}^d$$

for $u \in U$, U finite, with $a_{u,i,j} = H_{i,j}(a)$ for $a \in \cup_\ell \mathrm{Supp}_{\mathbb{Z}^d}(\mathrm{Tor}_\ell^R(B,k))$, $G_u \subset F \times \{1,\ldots,d\}$ and polynomials $P_{u,\tau}$ for $u \in U$ and $\tau \in \mathbb{Z}^d/\Lambda$ such that:

$$\dim_k(B_\xi) = P_{u,s_\Lambda(\xi)}(\xi), \quad \forall \xi \in \Delta_u,$$

and $\dim_k(B_\xi) = 0$ *if* $\xi \notin \cup_{u\in U}\Delta_u$.

Proof. By Proposition 2.1.16, there exists a polynomial $\kappa_B(t_1,\ldots,t_d)$ with integral coefficients such that

$$H(B;t) = \kappa_B(t)H(R;t)$$

and $\kappa_B(t) = \sum_{a\in A} c_a t^a$ with $A \subset \cup_\ell \operatorname{Supp}_{\mathbb{Z}^d}(\operatorname{Tor}_\ell^R(B,k))$. Let $D_i := \cup_j \{x \mid H_{i,j}(x) = 0\}$ be the minimal union of hyperplanes containing the border of C_i. The union C of the convex sets $\overline{C_i} + a$ can be decomposed into a finite union of convex sets Δ_u, each $u \in U$ corresponding to one connected component of $C \setminus \bigcup_{i,a}(D_i + a)$. (Notice that $\mathbb{R}^d \setminus \bigcup_{i,a}(D_i + a)$ has finitely many connected components, which are convex sets of the form of Δ_u, and that we may drop the ones not contained in C as the dimension of B_ξ is zero for ξ not contained in any $\overline{C_i} + a$.) We define u as the set of pairs (i,a) such that $(C_i + a)\bigcap \Delta_u \neq \emptyset$, and remark that if $(i,a) \in u$ then $(j,a) \notin u$ for $j \neq i$.

If $\xi \notin \bigcup_i \overline{C_i} + a$, then $\dim_k R_{\xi-a} = 0$, while if $\xi \in \overline{C_i} + a$ then it follows from Corollary 3.2.8 that there exist polynomials $Q_{i,\tau}$ such that

$$\dim_k R_{\xi-a} = Q_{i,\tau}(\xi - a) \quad \text{if } \xi - a \equiv \tau \mod (\Lambda).$$

Hence, setting $Q'_{i,a,\tau}(\xi) := Q_{i,\tau+a}(\xi - a)$, one gets the conclusion with

$$P_{u,\tau} = \sum_{(i,a)\in u} c_a Q'_{i,a,\tau}.$$

□

Remark 4.2.3. The above proof shows that if one has a finite collection of modules B_i, setting $A := \cup_{i,\ell} \operatorname{Supp}_{\mathbb{Z}^d} \operatorname{Tor}_\ell^R(B_i,k)$, there exist convex polyhedral cones Δ_u as above on which any B_i has its Hilbert function given by a quasi-polynomial with respect to the lattice Λ.

Let $S = k[y_1,\ldots,y_m]$ be a $\mathbb{Z}^p$-graded polynomial ring over a field. Assume that $\deg(y_j) \in \mathbb{N}^p$ for any j, and let $I_i = (f_{i,1},\ldots,f_{i,r_i})$ be ideals, with $f_{i,j}$ homogeneous of degree $d_{i,j}$.

Consider $R := k[T_{i,j}]_{1\leq i\leq s, 1\leq j\leq r_i}$, set $\deg(T_{i,j}) = (\deg(f_{i,j}), e_i)$, with e_i the i-th canonical generator of $\mathbb{Z}^s$ and the induced grading $\varphi : \mathbb{Z}^{r_1+\cdots+r_s} \to \mathbb{Z}^d := \mathbb{Z}^p \times \mathbb{Z}^s$ of R.

Denote as above by Λ the lattice in $\mathbb{Z}^d$ associated to φ, by $s_\Lambda : \mathbb{Z}^d \to \mathbb{Z}^d/\Lambda$ the canonical morphism and by C_i, for $i \in F$, the maximal cells in the chamber complex associated to φ. One has $\overline{C_i} = \{(\mu, t) \mid H_{i,j}(\mu, t) \geqslant 0, 1 \leqslant j \leqslant d\}$ where $H_{i,j}$ is a linear form in $(\mu, t) \in \mathbb{Z}^p \times \mathbb{Z}^s = \mathbb{Z}^d$.

Theorem 4.2.4. *In the situation above, there exist a finite number of polyhedral convex cones*

$$\Delta_u = \{(\mu, t) | H_{i,j}(\mu, t) \geqslant a_{u,i,j}, (i, j) \in G_u\} \subseteq \mathbb{R}^d,$$

polynomials $P_{\ell,u,\tau}$ for $u \in U$ and $\tau \in \mathbb{Z}^d/\Lambda$ such that, for any ℓ,

$$\dim_k(\mathrm{Tor}^S_\ell(MI_1^{t_1} \dots I_s^{t_s}, k)_\mu) = P_{\ell,u,s_\Lambda}(\mu, t), \quad \forall (\mu, t) \in \Delta_u,$$

and $\dim_k(\mathrm{Tor}^S_\ell(MI_1^{t_1} \dots I_s^{t_s}, k)_\mu) = 0$ *if* $(\mu, t) \notin \cup_{u \in U} \Delta_u$.

Furthermore, for any (u, i, j), $a_{u,i,j} = H_{i,j}(b)$, for some

$$b \in \bigcup_{i,\ell} \mathrm{Supp}_{\mathbb{Z}^d} \mathrm{Tor}^R_\ell(\mathrm{Tor}^S_i(M\mathscr{R}_{I_1,\dots,I_s}, R), k).$$

Proof. It has been presented in [6] that $B_i := \oplus_{t_1,\dots,t_t} \mathrm{Tor}^S_i(MI_1^{t_1} \cdots I_s^{t_s}, k)$ is a finitely generated $\mathbb{Z}^d$-graded module over R. As $B_i \neq 0$ for only finitely many i, the conclusion follows from proposition4.2.2 and remark4.2.3. □

From above results, it can be concluded that $\mathbb{R}^d$ could be decomposed in a finite union of convex polyhedral cones Δ_u on which, for any ℓ, the dimension of $\mathrm{Tor}^S_\ell(MI_1^{t_1} \dots I_m^{t_s}, k)_\mu$, as a function of $(\mu, t) \in \mathbb{Z}^{p+s}$ is a quasi-polynomial with respect to a lattice determined by the degrees of the generators of the ideals $I_1, \dots, I_s$.

This general finiteness statement may lead to pretty complex decompositions in general, that depend on the number of ideals and on arithmetic properties of the sets of degrees of generators. This complexity is reflected both by the covolume of Λ as defined above and by the number of simplicial chambers in the chamber complex associated to φ.

4.3 The case of one graded ideal on a positively $\mathbb{Z}$-graded ring

We now explain in detail an important special case: one ideal in a positively $\mathbb{Z}$-graded polynomial ring over a field. We will use the following elementary lemma.

Lemma 4.3.1. *For a strictly increasing sequence $d_1 < \cdots < d_r$, and points of coordinates $(\beta_1^j, \beta_2^j) \in \mathbb{R}^2$ for $1 \le j \le N$, consider the half-lines $L_i^j := \{(\beta_1^j, \beta_2^j) + \lambda(d_i, 1),\ \lambda \in \mathbb{R}_{\ge 0}\}$ and set $L_i^j(t) := L_i^j \cap \{y = t\}$. Then there exist a positive integer t_0 and permutations σ_i, for $i = 1, \cdots, r$, in the permutation group S_N such that, for all $t \ge t_0$, the following properties are satisfied :*

(1) $L_i^{\sigma_i(1)}(t) \leqslant L_i^{\sigma_i(2)}(t) \leqslant \cdots \leqslant L_i^{\sigma_i(N)}(t)$ for $1 \leqslant i \leqslant r$,

(2) $L_i^{\sigma_i(N)}(t) \leqslant L_{i+1}^{\sigma_i(1)}(t)$.

Moreover t_0 can be taken as the biggest second coordinate of the intersection points of all pairs of half lines.

Proof. If two half-lines L_i^j and L_u^v intersect at a unique point $A(x_A, y_A)$, then

$$y_A = \frac{\det\begin{pmatrix} \beta_1^v & d_u \\ \beta_2^v & 1 \end{pmatrix} - \det\begin{pmatrix} \beta_1^j & d_i \\ \beta_2^j & 1 \end{pmatrix}}{d_i - d_u}.$$

Choose t_0 as the max of y_A, A running over the intersection points. For $t \in [t_0, +\infty[$ the ordering of the intersection points $L_i^j(t)$ on the line $\{y = t\}$ is independent of t. Furthermore, as the d_i's are strictly increasing (2) holds, which shows (1) as the ordering is independent of t.

□

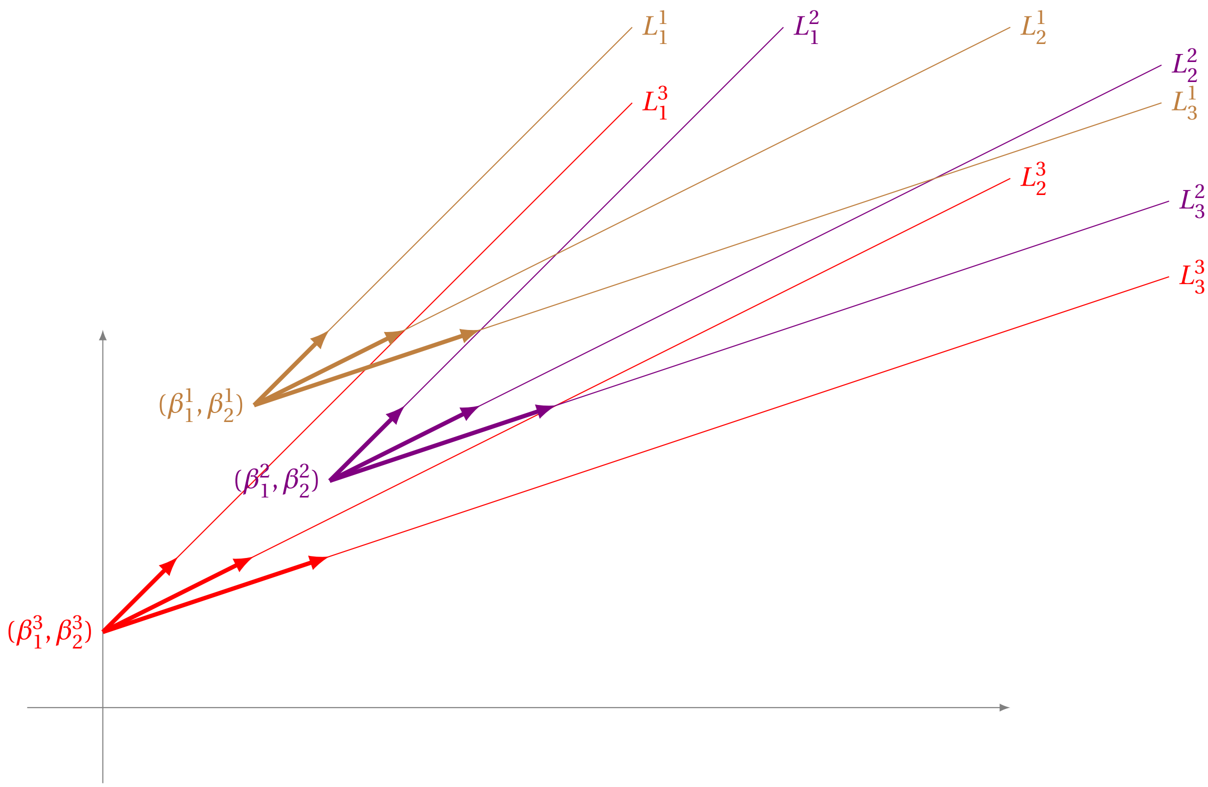

Figure 4.1: 3-Shifts.

Now we are ready to give a specific description of $\mathrm{Tor}_i^S(I^t, k)$ in the case of a $\mathbb{Z}$-graded ideal. Let $E := \{e_1, \dots, e_s\}$ with $e_1 < \cdots < e_s$ be a set of positive integers. For ℓ from 1 to $s-1$, let

$$\Omega_\ell := \{a\begin{pmatrix} e_\ell \\ 1 \end{pmatrix} + b\begin{pmatrix} e_{\ell+1} \\ 1 \end{pmatrix},\ (a,b) \in \mathbb{R}^2_{\geq 0}\}$$

be the closed cone spanned by $\binom{e_\ell}{1}$ and $\binom{e_{\ell+1}}{1}$. For integers $i \neq j$, let $\Lambda_{i,j}$ be the lattice spanned by $\binom{e_i}{1}$ and $\binom{e_j}{1}$ and

$$\Lambda_\ell := \cap_{i \leq \ell < j} \Lambda_{i,j}.$$

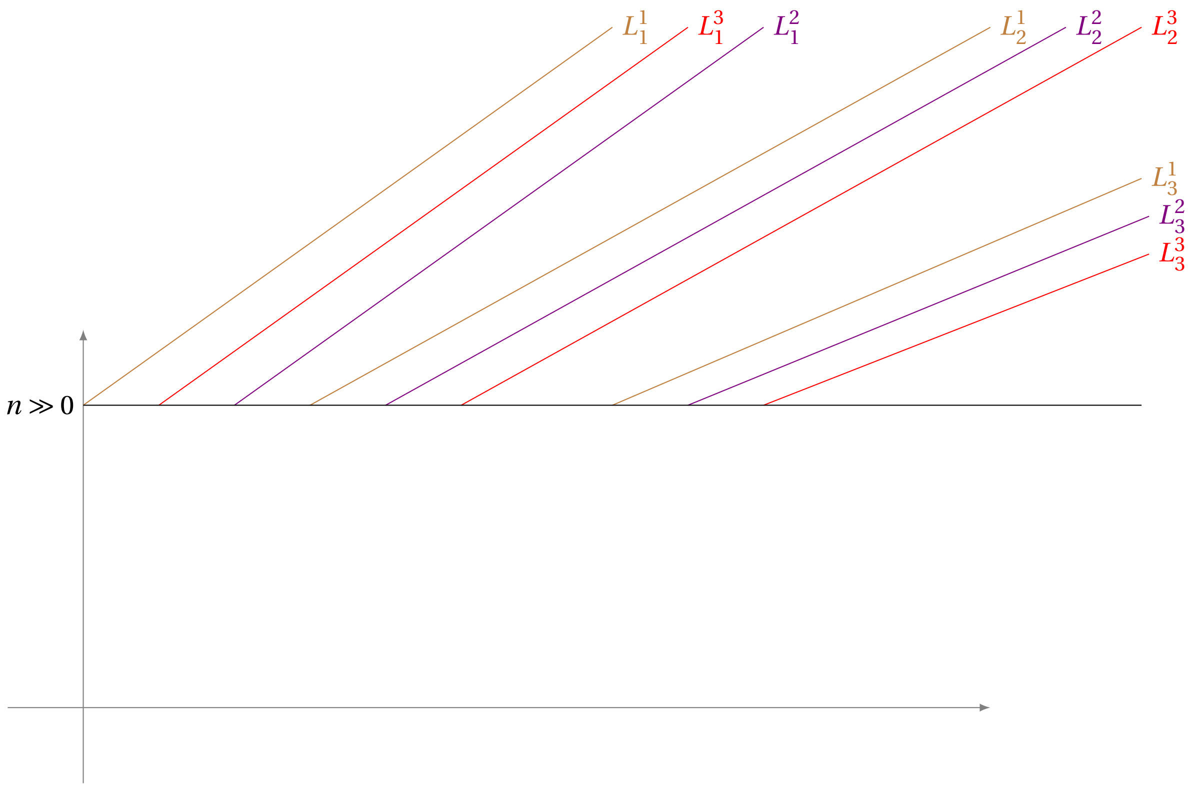

Figure 4.2: regions when n is sufficiently large.

,

$$\Lambda := \cap_{i<j} \Lambda_{i,j} \quad with \quad \Delta = \det(\Lambda)$$

In the case $E := \{d_1, \dots, d_r\}$, $e_1 = d_1$ and $e_s = d_r$, and, if $s \geq 2$, it follows from Theorem 3.2.7 that

(i) $\dim_k B_{\mu,t} = 0$ if $(\mu, t) \notin \Omega := \cup_\ell \Omega_\ell$,

(ii) $\dim_k B_{\mu,t}$ is a quasi-polynomial with respect to the lattice Λ_ℓ for $(\mu, t) \in \Omega_\ell$.

Notice further that $\Lambda := \cap_{i<j} \Lambda_{i,j}$ is a sublattice of Λ_ℓ for any ℓ.

Proposition 4.3.2. *In the above situation, if M is a finitely generated graded B-module, there exist t_0, N and $L_i(t) := a_i t + b_i$ for $i = 1, \dots, N$ with $b_i \in \mathbb{Z}$ and $\{a_1, \dots, a_N\} = E$ such that for $t \geq t_0$:*

(i) $L_i(t) < L_j(t) \Leftrightarrow i < j$,

(ii) $M_{\mu,t} = 0$ *if* $\mu < L_1(t)$ *or* $\mu > L_N(t)$,

(iii) For $t \geq t_0$ and $L_i(t) \leq \mu \leq L_{i+1}(t)$ $1 \leq i < N$, $\dim_k(M_{\mu,t})$ is a quasi-polynomials $Q_i(\mu, t)$ with respect to the lattice Λ.

Proof. By Proposition 2.1.16, there exists a polynomial $P(x, y)$ with integral coefficients such that

$$H(M;(x, y)) = P(x, y)H(B;(x, y))$$

of the form $P(x, y) = \sum_{(a,b)\in A} c_{a,b} x^a y^b$ with $A \subset \cup_\ell \operatorname{Supp}_{\mathbb{Z}^2}(\operatorname{Tor}^R_\ell(M, k))$. Write :

$$A = \{(\beta^1_1, \beta^1_2), \dots, (\beta^N_1, \beta^N_2)\}.$$

Now let $L^j_i(t) := d_i t + b_j$ be the half-line parallel to the vector $(d_i, 1)$ and passing through the point (β^j_1, β^j_2) for $1 \leq i, j \leq N$. Then by description before proposition item(i) follows directly from 4.3.1 (i) and item(ii) from the fact that $M_{(\mu,t)} = 0$ unless $(\mu, t) \in \bigcup_{i=1}^N (\beta^i_1, \beta^i_2) + \Omega$.

To prove (iii), following 4.3.1 we can consider two type of intervals as follows:

$$I^j_i := [L^{\sigma_i(j)}_i(n), L^{\sigma_i(j+1)}_i(n)] \quad for \quad j < N$$

and

$$I_i^N := [L_i^{\sigma_i(N)}(n), L_{i+1}^{\sigma_{i+1}(1)}(n)] \quad for \quad i < r$$

And wite $I_{N_p+q} := I_p^{\sigma_p(q)}$,$L_{N_p+q} = L_p^{\sigma_p(q)}$for $0 \leq q < N$ then for any degree (α, n) in the support of M there is two cases :

case I. if $\alpha \in I_i^j$, then (α, n) belongs to i-th chamber of shifts $\{(\beta_1^{\sigma_j(1)}, \beta_2^{\sigma_j(1)}), \dots, (\beta_1^{\sigma_i(j)}, \beta_2^{\sigma_i(j)})\}$ and for the other shifts (α, n) belongs to $(i-1)$-th chambers.

case II. if $\alpha \in I_i^N$, then (α, n) belongs to i-th chamber for all of the shifts.

Then by Proposition 3.4.2 there exist polynomials Q_{ij} such that $\dim B_{(\mu,t)} = Q_{ij}((\mu, t))$, if $\mu - td_i \equiv j \mod (\Delta)$.

By setting $\widetilde{Q}_{ik}^j = c_{(\beta_1^j, \beta_2^j)} Q_{i,(k-\beta_1^j+\beta_2^j d_i)}(x - \beta_1^j, y - \beta_2^j)$ we can conclud that:

if $\alpha \in I_i^j$, then

$$dim_k(M_{\alpha,t}) = \sum_{c=1}^{j} \widetilde{Q}_{i,(\alpha-td_i)}^c(\alpha, n) + \sum_{c=j+1}^{N} \widetilde{Q}_{(i-1),(\alpha-td_{i-1})}^c(\alpha, t)$$

□

Theorem 4.3.3. *Let $S = k[x_1, \dots, x_n]$ be a positively graded polynomial ring over a field k and let I be a homogeneous ideal in S.*

There exist, $t_0, m, D \in \mathbb{Z}$*, linear functions* $L_i(t) = a_i t + b_i$*, for* $i = 0, \dots, m$*, with* a_i *among the degrees of the minimal generators of* I *and* $b_i \in \mathbb{Z}$*, and polynomials* $Q_{i,j} \in \mathbb{Q}[x, y]$ *for* $i = 1, \dots, m$ *and* $j \in 1, \dots, D$*, such that, for* $t \geq t_0$*,*

(i) $L_i(t) < L_j(t) \Leftrightarrow i < j$*,*

(ii) If $\mu < L_0(t)$ *or* $\mu > L_m(t)$*, then* $\operatorname{Tor}_i^S(I^t, k)_\mu = 0$.

(iii) If $L_{i-1}(t) \leq \mu \leq L_i(t)$ *and* $a_i t - \mu \equiv j \mod (D)$*, then*

$$\dim_k \operatorname{Tor}_i^S(I^t, k)_\mu = Q_{i,j}(\mu, t).$$

Proof. We know from [6] that $M := \operatorname{Tor}_i^S(I^t, k)$ is a finitely generated $\mathbb{Z}^2$-graded module over R. Then it follows from Proposition 4.3.2 □

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CHAPTER 5

GRADED BETTI NUMBERS OF HILBERT FILTRATIONS

5.1 structure of Tor module of Rees algebra

Let $S = A[x_1,\dots,x_n]$ be a graded algebra over a commutative noetherian local ring $S_0 = (A,m)$ with residue field k and set $R = S[T_1,\dots,T_r]$ and $B = k[T_1,\dots,T_r]$. We set $\deg(T_i) = (d_i,1)$ and extended the grading from S to R by setting $\deg(x_i) = (\deg(x_i),0)$. In this section we use the two important following fact that were already at the center of the work [6] . The first one is that $\mathrm{Tor}_i^R(M\mathscr{R}_I,B)$ is a finitely generated graded B-module. The second is that :

$$\mathrm{Tor}_i^R(M\mathscr{R}_I,B)_{(\mu,t)} = \mathrm{Tor}_i^S(MI^t,k)_\mu$$

.

In particular, it provides a B-structure on $\oplus_t \operatorname{Tor}_i^S(MI^t, k)$ making it a finitely generated B-module. Slightly more generally, it was showed in [6] that the following holds.

Theorem 5.1.1. *[6] Let $S = A[x_1,\ldots,x_n]$ be a $\mathbb{G}$-graded algebra over Notherian local ring (A, m, k). Let $I = (f_1, f_2, ..., f_r)$ with $deg\, f_i = d_i$ be $\mathbb{G}$-homogenous ideal in S, and let $R = S[T_1,\ldots,T_n]$ be a bigraded polynomial extension of S with $\deg(T_i) = (d_i, 1)$ and $\deg(a) = (\deg(a), 0) \in \mathbb{G}\times\{0\}$ for all $a \in S$. Let M be a finitely generated $\mathbb{G}$-graded S-module. Then for all i, j:*

1. $\operatorname{Tor}_i^A(\operatorname{Tor}_j^R(M\mathscr{R}_I, A), k)$ *is a finitely generated* $k[T_1,\ldots,T_r]$*-module .*

2. $\operatorname{Tor}_i^R(M\mathscr{R}_I, k)$ *is a finitely generated* $k[T_1,\ldots,T_r]$*-module .*

Theorem 5.1.2. *In the above situation if I is a homogeneous ideal in S and $\mathbb{G} = \mathbb{Z}$.*

There exist, $t_0, m, D \in \mathbb{Z}$, linear functions $L_i(t) = a_i t + b_i$, for $i = 0,\ldots,m$, with a_i among the degrees of the minimal generators of I and $b_i \in \mathbb{Z}$, and polynomials $Q_{i,j} \in \mathbb{Q}[x, y]$ for $i = 1,\ldots,m$ and $j \in 1,\ldots,D$, such that, for $t \geq t_0$,

(i) $L_i(t) < L_j(t) \Leftrightarrow i < j$,

(ii) If $\mu < L_0(t)$ *or* $\mu > L_m(t)$, *then* $\operatorname{Tor}_i^S(I^t, k)_\mu = 0$.

(iii) If $L_{i-1}(t) \leq \mu \leq L_i(t)$ *and* $a_i t - \mu \equiv j \mod (D)$, *then*

$$\dim_k \operatorname{Tor}_i^S(I^t, k)_\mu = Q_{i,j}(\mu, t).$$

Proof. By the above the theorem we know that $\mathrm{Tor}_i^R(M\mathscr{R}_I, k)$ is finitely generated $k[T_1, \ldots, T_r]$-module then the result follows from proposition 4.3.2.

□

5.2 structure of Tor module of Hilbert filtrations

To study blowup algebras, Northcott and Rees defined the notion of reduction of an ideal I in a commutative ring R. An ideal $J \subseteq I$ is a reduction of I if there exists r such that $JI^r = I^{r+1}$ (equivalently this hold for $r \gg 0$) . An impotent fact about reduction ideal J of I is that it is equivalent to ask that

$$\mathscr{R}_J = \oplus_n J^n \to \mathscr{R}_I = \oplus_n I^n$$

is a finite morphism. Okon and Ratliff in [40] extended the above notion of reduction to the case of filtrations by setting the following definition:

Definition 5.2.1. If R is a ring and I and J be ideals in R, then:

(1) A filtration $\varphi = \{\varphi(n)\}_{n\geq 0}$ on R is a decreasing sequence of ideals $\varphi(n)$ of R such that $\varphi(0) = R$and $\varphi(m)\varphi(n) \subseteq \varphi(m+n)$ for all nonnegative integers m and n.

(2) If φ and γ are filtrations on R, then $\varphi = \gamma$ in case $\varphi(n) = \gamma(n)$ for all $n \geq 0$, and $\varphi \leq \gamma$ in case $\varphi(n) \subseteq \gamma(n)$ for all $n \geq 0$.

(3) If φ and γ are filtrations on R, then φ is a reduction of γ in case $\varphi \leq \gamma$ and there exists a positive integer d such that $\gamma(n) = \sum_{i=0}^{d} \varphi(n-i)\gamma(i)$ for all $n \geq 1$.

(4) Let I be an ideal of R and φ is filtration on R, then φ is called I-good filtration if $I\varphi_i \subseteq \varphi_{i+1}$ for all $i \geq 0$ and $\varphi_{n+1} = I\varphi_n$ for all $n \gg 0$.

(5) Let γ be I-good filtration , then a J-good filtration φ is called good reduction of γ if it is a reduction in the sense of (3).

In opposite to the ideal case, minimal reduction of a filtration does not exist in general. But Hoa and Zarzuela showed in [32] the existence of a minimal reduction for I-good filtrations as follows :

Proposition 5.2.2. *Let φ and γ are filtrations on R, then φ is the minimal reduction of a good filtration γ if and only if $\varphi = \{J^n\}_{n\geqslant 0}$, where J is a minimal reduction of γ_1). In particular, a minimal reduction of γ do exist.*

If $\varphi = \{\varphi(n)\}_{n\geq 0}$ is an I-good filtration on R, then $\mathscr{R}_\varphi$ is a finite $\mathscr{R}_I$-module(See [11, Theorem III.3.1.1]), that is why we are interested about I-good filtration to generalize the previous results. The following theorem explain the structure of Tor module of I-good filtrations:

Theorem 5.2.3. *Let $S = A[x_1, \cdots, x_n]$ be a graded algebra over a Noetherian local ring $(A, \boldsymbol{m}, k) \subset S_0$. Let $\varphi = \{\varphi(n)\}_{n\geq 0}$ be an I-good filtration of ideals $\varphi(n)$ of R and $\varphi(1) = (f_1, f_2, ..., f_r)$ with $deg f_i = d_i$ be $\mathbb{Z}$-homogenous ideal in S, and let $R = S[T_1, \ldots, T_n]$ be a bigraded polynomial extension of S with $\deg(T_i) = (d_i, 1)$ and $\deg(a) = (\deg(a), 0) \in \mathbb{Z} \times \{0\}$ for all $a \in S$.*

(1)Then for all i:

$\mathrm{Tor}_i^R(\mathscr{R}_\varphi, k)$ *is a finitely generated $k[T_1, \ldots, T_r]$-module.*

(2) There exist, $t_0, m, D \in \mathbb{Z}$*, linear functions* $L_i(t) = a_i t + b_i$*, for* $i = 0, \dots, m$*, with* a_i *among the degrees of the minimal generators of* I *and* $b_i \in \mathbb{Z}$*, and polynomials* $Q_{i,j} \in \mathbb{Q}[x, y]$ *for* $i = 1, \dots, m$ *and* $j \in 1, \dots, D$*, such that, for* $t \geq t_0$*,*

(i) $L_i(t) < L_j(t) \Leftrightarrow i < j$*,*

(ii) If $\mu < L_0(t)$ *or* $\mu > L_m(t)$*, then* $\mathrm{Tor}_i^S(\varphi(t), k)_\mu = 0$.

(iii) If $L_{i-1}(t) \leq \mu \leq L_i(t)$ *and* $a_i t - \mu \equiv j \mod (D)$*, then*

$$\dim_k \mathrm{Tor}_i^S(\varphi(t), k)_\mu = Q_{i,j}(\mu, t).$$

Proof. Let $F_\bullet$ be a $\mathbb{Z} \times \mathbb{Z}$-graded minimal free resolution of $\mathscr{R}_\varphi$ over R. Each F_i is of of finite rank due to the Noetherianity of A . The graded stanf $F_\bullet^t := (F_\bullet)_{*,t}$ is a $\mathbb{Z}$-graded free resolution of $\varphi(t)$ over $S = R_{(*,0)}$. Thus,

$$\mathrm{Tor}_i^S(\varphi(t), k) = H_i(F_\bullet^t \otimes_S k).$$

Moreover, taking homology respects the graded structure, and therefore,

$$H_i(F_\bullet^t \otimes_S k) = H_i(F_\bullet \otimes_R R/\mathbf{m} + \mathbf{n}R)_{(*,t)},$$

where $\mathbf{n} = (x_1, \dots, x_n)$ is the homogeneous irrelevant ideal in S. So it follws that $\mathrm{Tor}_j^R(\mathscr{R}_I, k)$ is finitely generated graded $k[T_1, \dots, T_r]$-module .The second fact comes from proposition 4.3.2.

□

This in particular applies to the ideals when ever $(A, \mathbf{m}, k)$ is local Noetherian ring and S be a graded local Noetherian algebra over A:

- If I be a graded ideal of S and S be analytically unramified ring without nilpotent elements then $\varphi(n) = \overline{I^n}$ is I-good filtration then result follows from theorem 5.2.3.
- If I be a graded ideal of S then by theorem 2.1.20 the filtration $\varphi(n) = \widetilde{I^n}$ is I-good filtration then result follows from theorem 5.2.3.

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