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Mean field type control with congestion

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Abstract

We analyze some systems of partial differential equations arising in the theory of mean field type control with congestion effects. We look for weak solutions. Our main result is the existence and uniqueness of suitably defined weak solutions, which are characterized as the optima of two optimal control problems in duality.

1 Introduction

In the recent years, an important research activity has been devoted to the study of stochastic differential games with a large number of players. In their pioneering articles [12, 13, 14], J-M. Lasry and P-L. Lions have introduced the notion of mean field games, which describe the asymptotic behavior of stochastic differential games (Nash equilibria) as the number N of players tends to infinity. In these models, it is assumed that the agents are all identical and that an individual agent can hardly influence the outcome of the game. Moreover, each individual strategy is influenced by some averages of functions of the states of the other agents. In the limit when $N \to +\infty$, a given agent feels the presence of the other agents through the statistical distribution of the states of the other players. Since perturbations of a single agent's strategy does not influence the statistical distribution of the states, the latter acts as a parameter in the control problem to be solved by each agent.

Another kind of asymptotic regime is obtained by assuming that all the agents use the same distributed feedback strategy and by passing to the limit as $N \to \infty$ before optimizing the common feedback. Given a common feedback strategy, the asymptotics are given by the McKean-Vlasov theory, [16, 20] : the dynamics of a given agent is found by solving a stochastic differential equation with coefficients depending on a mean field, namely the statistical distribution of the states, which may also affect the objective function. Since the feedback strategy is common to all agents, perturbations of the latter affect the mean field. Then, having each player optimize its objective function amounts to solving a control problem driven by the McKean-Vlasov dynamics. The latter is named control of McKean-Vlasov dynamics by R. Carmona and F. Delarue [8, 7] and mean field type control by A. Bensoussan et al, [3, 4]. When the dynamics of the players are independent stochastic processes, both mean field games and control of McKean-Vlasov dynamics naturally lead to a coupled system of partial differential equations, a forward Fokker-Planck equation and a backward Hamilton-Jacobi–Bellman equation. For mean field games, the coupled system of partial differential equations has been studied by Lasry and Lions in [12, 13, 14]. Besides, many important aspects of the mathematical theory developed by J-M. Lasry and P-L. Lions on MFG are not published in journals or books, but can be found in the videos of the lectures of P-L. Lions at Collège de France: see the web site of Collège de France, [15]. One can also see [11] for a brief survey, and we mention [17], a very nice article on weak solutions of Fokker-Planck equations and of MFG systems of partial differential equations.

The analysis of the system of partial differential equations arising from mean field type control can be performed with rather similar arguments as for MFG, see [2] for a work devoted to classical solutions.

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The class of MFG with congestion effects was introduced and studied in [15] in 2011, see also [1, 2] for some numerical simulations, to model situations in which the cost of displacement of the agents increases in the regions where the density is large. A striking fact is that in general, MFG with congestion cannot be cast into an optimal control problem driven by a partial differential equation, in contrast with simpler cases. In the present paper, we aim at studying mean field type control with congestion, in a setting in which classical solutions of the system of partial differential equations seem difficult to obtain. But, in contrast with MFG, mean field type control can genuinely be seen as a problem of optimal control of a partial differential equation. This will allow us to use techniques from the calculus of variations. Inspired by the works of Cardaliaguet et al, see [5, 6], we will introduce a pair of primal and dual optimization problems, leading to a suitable weak formulation of the system of partial differential equations for which there exists a unique solution. Note that [5] is devoted to some optimal transportation problems (i.e. finding the geodesics for a class of distances between probability measures), whereas [6] deals with some special cases of MFG with possibly degenerate diffusions to which the above mentioned techniques from the calculus of variations can be applied.

1.1 Model and assumptions

This paper is devoted to the analysis of the second order system

$$\frac{\partial u}{\partial t}(t,x) + \nu \Delta u(t,x) + H(x,m(t,x),Du(t,x)) + m(t,x)\frac{\partial H}{\partial m}(x,m(t,x),Du(t,x)) = 0, \tag{1.1}$$

$$\frac{\partial m}{\partial t}(t,x) - \nu \Delta m(t,x) + \mathrm{div}\Big(m(t,\cdot)\frac{\partial H}{\partial p}(\cdot,m(t,\cdot),Du(t,\cdot))\Big)(x) = 0, \tag{1.2}$$

with the initial and terminal conditions

$$m(0,x) = m_0(x) \quad \text{and} \quad u(T,x) = u_T(x). \tag{1.3}$$

Assumptions We now list the assumptions on the Hamiltonian H, the initial and terminal conditions m_0 and u_T. These conditions are supposed to hold in all what follows.

H1 The Hamiltonian $H : \mathbb{T}^d \times (0,+\infty) \times \mathbb{R}^d \to \mathbb{R}$ is of the form

$$H(x,m,p) = -\frac{|p|^\beta}{m^\alpha} + \ell(x,m), \tag{1.4}$$

with $1 < \beta \le 2$ and $0 \le \alpha < 1$, and where ℓ is continuous cost function that will be discussed below. It is clear that H is concave with respect to p. Calling β^* the conjugate exponent of β, i.e. $\beta^* = \beta/(\beta-1)$, it is useful to note that

$$H(x,m,p) = \inf_{\xi \in \mathbb{R}^d} \left(\xi \cdot p + L(x,m,\xi)\right), \tag{1.5}$$

$$L(x,m,\xi) = (\beta-1)\beta^{-\beta^*} m^{\frac{\alpha}{\beta-1}}|\xi|^{\beta^*} + \ell(x,m), \tag{1.6}$$

that L is convex with respect to ξ, and that

$$L(x,m,\xi) = \sup_{p \in \mathbb{R}^d} \left(-\xi \cdot p + H(x,m,p)\right). \tag{1.7}$$

H2 (conditions on the cost ℓ) The function $\ell : \mathbb{T}^d \times \mathbb{R}_+ \to \mathbb{R}$ is continuous with respect to both variables and continuously differentiable with respect to m if $m > 0$. We also assume that $m \mapsto m\ell(x,m)$ is strictly convex, and that there exist $q > 1$ and two positive constants C_1 and C_2 such that

$$\frac{1}{C_1}m^{q-1} - C_1 \le \ell(x,m) \le C_1 m^{q-1} + C_1, \tag{1.8}$$

$$\frac{1}{C_2}m^{q-1} - C_2 \le m\frac{\partial \ell}{\partial m}(x,m) \le C_2 m^{q-1} + C_2. \tag{1.9}$$

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Moreover, since we can always add a constant to L, we can assume that

$$\ell(x, m) \geq 0, \quad \forall x \in \mathbb{T}^d, \ \forall m \geq 0. \tag{1.10}$$

The convexity assumption on $m \mapsto m\ell(x, m)$ implies that $m \mapsto mH(x, m, p)$ is strictly convex with respect to m.

Moreover, we assume that there exists a constant $C_3 \geq 0$ such that

$$|\ell(x, m) - \ell(y, m)| \leq C_3(1 + m^{q-1})|x - y|. \tag{1.11}$$

H3 We assume that $\beta \geq q^*$.

H4 (initial and terminal conditions) We assume that m_0 is of class $\mathcal{C}^1$ on $\mathbb{T}^d$, that u_T is of class $\mathcal{C}^2$ on $\mathbb{T}^d$ and that $m_0 > 0$ and $\int_{\mathbb{T}^d} m_0(x)dx = 1$.

H5 ν is a positive number.

Remark 1.1. *Note that Assumption [H5] can be relaxed: all what follows can be generalized to degenerate diffusions, i.e. to the following system of PDEs:*

$$\frac{\partial u}{\partial t}(t, x) + A_{i,j}(x)\frac{\partial^2 u}{\partial x_i \partial x_j}(t, x) + H(x, m(t, x), Du(t, x)) + m(t, x)\frac{\partial H}{\partial m}(x, m(t, x), Du(t, x)) = 0, \tag{1.12}$$

$$\frac{\partial m}{\partial t}(t, x) - \nu\frac{\partial^2}{\partial x_i \partial x_j}(A_{i,j}(\cdot)m)(t, x) + \operatorname{div}\Big(m(t, \cdot)\frac{\partial H}{\partial p}(\cdot, m(t, \cdot), Du(t, \cdot))\Big)(x) = 0, \tag{1.13}$$

where $(A_{i,j})_{1 \leq i,j \leq d}(x) = \frac{1}{2}\Sigma(x)\Sigma^T(x)$ and Σ is a Lipschitz continous map from $\mathbb{T}^d$ to $\mathbb{R}^{d \times D}$ with D possibly smaller than d. The necessary modifications can easily be found in [6].

1.2 A heuristic justification of (1.1)-(1.3)

Consider a probability space $(\Omega, \mathcal{A}, \mathcal{P})$ and a filtration $\mathcal{F}^t$ generated by a d-dimensional standard Wiener process (W_t) and the stochastic process $(X_t)_{t \in [0,T]}$ in $\mathbb{R}^d$ adapted to $\mathcal{F}^t$ which solves the stochastic differential equation

$$dX_t = \xi_t \, dt + \sqrt{2\nu} \, dW_t \qquad \forall t \in [0, T], \tag{1.14}$$

given the initial state X_0 which is a random variable $\mathcal{F}^0$-measurable whose probability density is m_0. In (1.14), ξ_t is the control, which we take to be

$$\xi_t = v(t, X_t), \tag{1.15}$$

where $v(t, \cdot)$ is a continuous function on $\mathbb{T}^d$. As explained in [4], page 13, if the feedback function v is smooth enough, then the probability distribution m_t of X_t has a density with respect to the Lebesgue measure, $m_v(t, \cdot) \in \mathbb{P} \cap L^1(\mathbb{T}^d)$ for all t, and m_v is solution of the Fokker-Planck equation

$$\frac{\partial m_v}{\partial t}(t, x) - \nu\Delta m_v(t, x) + \operatorname{div}\Big(m_v(t, \cdot)v(t, \cdot)\Big)(x) = 0, \tag{1.16}$$

for $t \in (0, T]$ and $x \in \mathbb{T}^d$, with the initial condition

$$m_v(0, x) = m_0(x), \quad x \in \mathbb{T}^d. \tag{1.17}$$

We define the objective function

$$
\begin{aligned}
\mathcal{J}(v) &= \mathbb{E}\left[\int_0^T L(X_t, m_v(t, X_t), \xi_t)dt + u_T(X_T)\right] \\
&= \int_{[0,T] \times \mathbb{T}^d} L(x, m_v(t, x), v(t, x))m_v(t, x)dxdt + \int_{\mathbb{T}^d} u_T(x)m_v(T, x)dx.
\end{aligned}
\tag{1.18}
$$

The goal is to minimize $\mathcal{J}(v)$ subject to (1.16) and (1.17). Following A. Bensoussan, J. Frehse and P. Yam in [4], it can be seen that if there exists a smooth feedback function v^* achieving $\mathcal{J}(v^*) = \min \mathcal{J}(v)$ and such that $m_{v^*} > 0$ then

$$v^*(t,x) = \operatorname{argmin}_v \Big(L(x, m_{v^*}(t,x), v) + \nabla u(t,x) \cdot v(t,x) \Big)$$

and (m_{v^*}, u) solve (1.1), (1.2) and (1.3). The issue with the latter argument is that we do not know how to guarantee a priori that m will not vanish in some region of $(0,T) \times \mathbb{T}^d$. Hereafter, we propose a theory of weak solutions of (1.1)-(1.3), in order to cope with the cases when m may vanish.

Remark 1.2. *Note that the system of partial differential equations that arises in a mean field game is*

$$\frac{\partial u}{\partial t}(t,x) + \nu \Delta u(t,x) + H(x, m(t,x), Du(t,x)) = 0,$$

with (1.2) and (1.3). To the best of our knowledge, for such a system with the Hamiltonian given in (1.4), the existence of a solution is an open problem except in the stationary case with $\beta = 2$ and $\alpha = 1$, see [10]; in the latter case, a very special trick can be used. Besides, the theory of weak solutions proposed below does not apply to MFG, because as explained above, MFG with congestion cannot be seen as an optimal control problem driven by a partial differential equation.

2   Two optimization problems

The first optimization is described as follows: consider the set $\mathcal{K}_0$:

$$\mathcal{K}_0 = \left\{ \phi \in \mathcal{C}^2([0,T] \times \mathbb{T}^d) : \phi(T, \cdot) = u_T \right\}$$

and the functional $\mathcal{A}$ on $\mathcal{K}_0$:

$$\mathcal{A}(\phi) = \inf_{\substack{m \in L^1((0,T) \times \mathbb{T}^d) \\ m \geq 0}} \mathcal{A}(\phi, m) \tag{2.1}$$

where

$$\mathcal{A}(\phi, m) = \int_0^T \int_{\mathbb{T}^d} m(t,x) \left(\frac{\partial \phi}{\partial t}(t,x) + \nu \Delta \phi(t,x) + H(x, m(t,x), D\phi(t,x)) \right) dx\,dt \tag{2.2}$$
$$+ \int_{\mathbb{T}^d} m_0(x) \phi(0,x) dx$$

with the convention that if $m = 0$ then $mH(x,m,p) = 0$. Then the first problem consists of maximizing

$$\sup_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi). \tag{2.3}$$

For the second optimization problem, we consider the set $\mathcal{K}_1$:

$$\mathcal{K}_1 = \left\{ \begin{array}{l} (m,z) \in L^1((0,T) \times \mathbb{T}^d) \times L^1((0,T) \times \mathbb{T}^d; \mathbb{R}^d) : \\ m \geq 0 \text{ a.e} \\ \dfrac{\partial m}{\partial t} - \nu \Delta m + \operatorname{div} z = 0, \\ m(0, \cdot) = m_0 \end{array} \right\} \tag{2.4}$$

where the boundary value problem is satisfied in the sense of distributions. We also define

$$\widetilde{L}(x, m, z) = \left\{ \begin{array}{ll} mL(x, m, \frac{z}{m}) & \text{if } m > 0 \\ 0 & \text{if } (m,z) = (0,0) \\ +\infty & \text{otherwise} \end{array} \right. . \tag{2.5}$$

Note that $(m, z) \mapsto \widetilde{L}(x, m, z)$ is LSC on $\mathbb{R} \times \mathbb{R}^d$. Using (1.6), (H2) and the results of [2] paragraph 3.2, it can be proved that $(m, z) \mapsto \widetilde{L}(x, m, z)$ is convex on $\mathbb{R} \times \mathbb{R}^d$, because $0 < \alpha < 1$. It can also be checked that

$$\widetilde{L}(x, m, z) = \begin{cases} \sup\limits_{p \in \mathbb{R}^d} \left(-z \cdot p + m H(x, m, p) \right) & \text{if } m > 0 \text{ or } (m, z) = (0, 0) \\ +\infty & \text{otherwise.} \end{cases} \tag{2.6}$$

Since L is bounded from below, $\int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x, m(t, x), z(t, x)) dx dt$ is well defined in $\mathbb{R} \cup \{+\infty\}$ for all $(m, z) \in \mathcal{K}_1$. We are interested in minimizing

$$\inf_{(m, z) \in \mathcal{K}_1} \mathcal{B}(m, z), \tag{2.7}$$

where if $\int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x, m(t, x), z(t, x)) dx dt < +\infty$,

$$\mathcal{B}(m, z) = \int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x, m(t, x), z(t, x)) dx dt + \int_{\mathbb{T}^d} m(T, x) u_T(x) dx, \tag{2.8}$$

and if not,

$$\mathcal{B}(m, z) = +\infty. \tag{2.9}$$

To give a meaning to the second integral in (2.8), we define $w(t, x) = \frac{z(t,x)}{m(t,x)}$ if $m(t, x) > 0$ and $w(t, x) = 0$ otherwise. From (1.6) and (1.8), we see that $\int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x, m(t, x), z(t, x)) dx dt < +\infty$ implies that $m^{1 + \frac{\alpha}{\beta - 1}} |w|^{\beta^*} \in L^1((0, T) \times \mathbb{T}^d)$, which implies that $m |w|^{\frac{\beta}{\beta - 1 + \alpha}} \in L^1((0, T) \times \mathbb{T}^d)$. In that case, the boundary value problem in (2.4) can be rewritten as follows:

$$\frac{\partial m}{\partial t} - \nu \Delta m + \operatorname{div}(m w) = 0, \qquad m(0, \cdot) = m_0, \tag{2.10}$$

and we can use the following Lemma which can be found in [6]:

Lemma 2.1. *If $(m, w) \in \mathcal{K}_1$ is such that $\int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x, m(t, x), z(t, x)) dx dt < +\infty$, then the map $t \mapsto m(t)$ for $t \in (0, T)$ and $t \mapsto m_0$ for $t < 0$ is Hölder continuous a.e. for the weak * topology of $\mathcal{P}(\mathbb{T}^d)$.*

Remark 2.1. *Following the proof of lemma 3.1 in [6], we see that the Hölder exponent in Lemma 2.1 is greater than or equal to $\min \left(\frac{1}{2}, \frac{1 - \alpha}{\beta} \right)$.*

This lemma implies that the measure $m(t)$ is defined for all t, so the second integral in (2.8) has a meaning.

Lemma 2.2.

$$\sup_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) = \min_{(m, z) \in \mathcal{K}_1} \mathcal{B}(m, z). \tag{2.11}$$

Moreover the latter minimum is achieved by a unique $(m^*, z^*) \in \mathcal{K}_1$, and $m^* \in L^q((0, T) \times \mathbb{T}^d)$.

Proof. Let us reformulate the optimization problem (2.3): take $E_0 = \mathcal{C}^2([0, T] \times \mathbb{T}^d)$ and $E_1 = \mathcal{C}^0([0, T] \times \mathbb{T}^d) \times \mathcal{C}^0([0, T] \times \mathbb{T}^d; \mathbb{R}^d)$. We define the functional $\mathcal{F}$ on E_0:

$$\mathcal{F}(\phi) = \chi_T(\phi) - \int_{\mathbb{T}^d} m_0(x) \phi(0, x) dx$$

where $\chi_T(\phi) = 0$ if $\phi|_{t = T} = u_T$ and $\chi_T(\phi) = +\infty$ otherwise. Let us also define the linear operator $\Lambda : E_0 \to E_1$ by

$$\Lambda(\phi) = \left(\frac{\partial \phi}{\partial t} + \nu \Delta \phi, D\phi \right).$$

For $(a, b) \in E_1$, let $\mathcal{G}(a, b)$ be defined by

$$\mathcal{G}(a, b) = - \inf_{\substack{m \in L^1((0,T) \times \mathbb{T}^d) \\ m \geq 0}} \int_0^T \int_{\mathbb{T}^d} m(t, x) \left(a(t, x) + H(x, m(t, x), b(t, x))\right) dx dt. \tag{2.12}$$

Note that the infimum with respect to m is in fact a minimum: indeed, from (1.4) and (1.8), we see that $m \mapsto \int_0^T \int_{\mathbb{T}^d} m(t, x) \left(a(t, x) + H(x, m(t, x), b(t, x))\right) dx dt$ is convex, coercive and continuous in the set $\{m \in L^q((0, T) \times \mathbb{T}^d) : m \geq 0\}$. Hence, this map is lower semi-continuous for the weak convergence in $\{m \in L^q((0, T) \times \mathbb{T}^d), m \geq 0\}$. On the other hand, since a minimizing sequence $(m_n)_{n \in \mathbb{N}}$ for (2.12) is bounded in $L^q((0, T) \times \mathbb{T}^d)$, we can extract a subsequence which converges weakly in $L^q((0, T) \times \mathbb{T}^d)$ to a nonnegative function. The weak limit achieves the minimum in (2.12). We now aim at characterizing the optimal m.

Let us first characterize

$$K(x, \gamma, p) = \min_{\mu \geq 0} \left(\mu \gamma + \mu H(x, \mu, p)\right), \tag{2.13}$$

which is nonpositive and concave with respect to (γ, p); since $\mu \mapsto \mu H(x, \mu, p)$ is $\mathcal{C}^1$, strictly convex on $\mathbb{R}_+$ and tends to $+\infty$ as $\mu \to +\infty$, we see that for any $x \in \mathbb{T}^d$, if $p \neq 0$ and $\gamma \in \mathbb{R}$, or if $p = 0$ and $\gamma + \ell(x, 0) \leq 0$, then there exists a unique $\mu = \psi(x, \gamma, p) \geq 0$ such that

$$\gamma + H(x, \mu, p) + \mu H_m(x, \mu, p) = 0.$$

Note that if $p = 0$ and $\gamma + \ell(x, 0) < 0$, then $\mu = \psi(x, \gamma, 0) > 0$ is characterized by $\gamma + \ell(x, \mu) + \mu \ell_m(x, \mu) = 0$. We extend ψ by 0 in the set $\{(\gamma, 0) : \gamma + \ell(x, 0) \geq 0\}$. Therefore,

$$K(x, \gamma, p) = \psi(x, \gamma, p)\gamma + \psi(x, \gamma, p)H(x, \psi(x, \gamma, p), p), \tag{2.14}$$

with the convention that $mH(x, m, p) = 0$ if $m = 0$.

We claim that the map $(\gamma, p) \mapsto \psi(x, \gamma, p)$ is continuous in $\mathbb{R} \times \mathbb{R}^d$. Indeed,

1. the continuity of $(\gamma, m, p) \mapsto \gamma + H(x, m, p) + \mu H_m(x, m, p)$ and the fact that this map is strictly increasing w.r.t. m in $(0, +\infty)$ implies that $(\gamma, p) \mapsto \psi(x, \gamma, p)$ is continuous in $\mathbb{R} \times \mathbb{R}^d \backslash \{0\}$.

2. Similarly, the continuity of $\gamma \mapsto \psi(x, \gamma, 0)$ stems from the continuity of the map $(\gamma, m) \mapsto \gamma + \ell(x, m) + m\ell_m(x, m)$ and its strictly increasing character w.r.t. m in $(0, +\infty)$.

3. Let us prove that if $(\tilde{\gamma}, \tilde{p})$ tends to $(\gamma, 0)$ with $\tilde{p} \neq 0$, then

   $$\lim_{(\tilde{\gamma}, \tilde{p}) \to (\gamma, 0)} \psi(x, \tilde{\gamma}, \tilde{p}) = \psi(x, \gamma, 0). \tag{2.15}$$

   (a) If $\gamma + \ell(x, 0) < 0$, then $\psi(x, \gamma, 0) > 0$ and we get (2.15) from the same argument as in point 1.

   (b) Consider the case $\gamma + \ell(x, 0) > 0$. Suppose that $(\tilde{\gamma}, \tilde{p})$ tends to $(\gamma, 0)$ with $\tilde{p} \neq 0$, and set $\tilde{\mu} = \psi(x, \tilde{\gamma}, \tilde{p}) > 0$. We see that

   $$(1 - \alpha)\tilde{\mu}^{-\alpha}|\tilde{p}|^\beta = \tilde{\gamma} + \frac{d}{dm}(m \mapsto m\ell(x, m))(\tilde{\mu}) > \tilde{\gamma} + \ell(x, 0) \to \gamma + \ell(x, 0).$$

   This implies that $\tilde{\mu}^\alpha|\tilde{p}|^{-\beta}$ is bounded as $(\tilde{\gamma}, \tilde{p}) \to (\gamma, 0)$, hence (2.15).

   (c) Finally, we consider the case when $\gamma = -\ell(x, 0) \leq 0$ and $(\tilde{\gamma}, \tilde{p}) \to (\gamma, 0)$ with $\tilde{p} \neq 0$; let us assume that for a subsequence, $\tilde{\mu}$ is bounded away from 0: passing to the limit in the identity

   $$\left(\frac{d}{dm}(m \mapsto m\ell(x, m))(\tilde{\mu}) - \ell(x, 0)\right) - (1 - \alpha)\tilde{\mu}^{-\alpha}|\tilde{p}|^\beta = \gamma - \tilde{\gamma},$$

   we obtain that

   $$\lim_{(\tilde{\gamma}, \tilde{p}) \to (\gamma, 0)} \frac{d}{dm}(m \mapsto m\ell(x, m))(\tilde{\mu}) = \ell(x, 0),$$

   which can happen only if $\tilde{\mu} \to 0$ and we reach a contradiction. Hence, (2.15) holds.

We have proved the continuity of ψ with respect to (γ, p). The continuity of ψ with respect to x follows from similar arguments, using the regularity assumptions on ℓ. Therefore, ψ is continuous in the set $\mathbb{T}^d \times \mathbb{R} \times \mathbb{R}^d$.

It is also useful to notice that

$$
\begin{aligned}
K(x, \gamma, p) &= \inf_{\mu \geq 0} \left(\mu(\gamma + H(x, \mu, p)) \right) \\
&= \inf_{\mu \geq 0} \left(\mu\gamma + \mu \inf_{\xi} \left(\xi \cdot p + L(x, \mu, \xi) \right) \right) \\
&= \inf_{(\mu, z) \in \mathbb{R} \times \mathbb{R}^d} \left(\mu\gamma + z \cdot p + \widetilde{L}(x, \mu, z) \right)
\end{aligned}
$$

and, from the Fenchel-Moreau theorem, see e.g. see [19], that

$$
\widetilde{L}(x, \mu, z) = \sup_{(\gamma, p) \in \mathbb{R} \times \mathbb{R}^d} \left(-\mu\gamma - z \cdot p + K(x, \gamma, p) \right). \tag{2.16}
$$

Remark 2.2. *Note that for all $0 \neq p \in \mathbb{R}^d$ and $\gamma \in \mathbb{R}$, the map $\mu \mapsto \mu\left(\gamma + H(x, \mu, p)\right)$ is strictly decreasing in some interval $[0, \bar{\mu}]$ where $\bar{\mu} > 0$ depends on p and γ, and that its derivative tends to $-\infty$ as $\mu \to 0^+$. Hence, if $\mu^* = 0$ achieves the minimum of $\mu \mapsto \mu\left(\gamma + H(x, \mu, p)\right)$, then p must be 0. Similarly, γ must be such that $\gamma + \ell(x, 0) \geq 0$.*

With (2.14), the optimality conditions for (2.12) yield that

$$
\mathcal{G}(a, b) = - \int_0^T \int_{\mathbb{T}^d} K(x, a(t, x), b(t, x)) dx dt. \tag{2.17}
$$

From (2.14) and the continuity of ψ, we see that $\mathcal{G}$ is continuous on E_1. We observe that

$$
\sup_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi) = - \inf_{\phi \in E_0} \left(\mathcal{F}(\phi) + \mathcal{G}(\Lambda(\phi)) \right). \tag{2.18}
$$

By choosing $\phi_0(t, \cdot) = u_T$, we see that $\mathcal{F}(\phi_0) < +\infty$, $\mathcal{G}(\Lambda(\phi_0)) < +\infty$ and that $\mathcal{G} \circ \Lambda$ is continuous at ϕ_0. We can thus apply Fenchel-Rockafellar duality theorem, see [19]:

$$
- \inf_{\phi \in E_0} \left(\mathcal{F}(\phi) + \mathcal{G}(\Lambda(\phi)) \right) = \min_{(m, z) \in E_1^*} \left(\mathcal{F}^*(\Lambda^*(m, z)) + \mathcal{G}^*(-m, -z) \right) \tag{2.19}
$$

where E_1^* is the topological dual of E_1 i.e. the set of Radon measures (m, z) on $(0, T) \times \mathbb{T}^d$ with values in $\mathbb{R} \times \mathbb{R}^d$. If E_0^* is the dual space of E_0, the operator $\Lambda^* : E_1^* \to E_0^*$ is the adjoint of Λ. The maps $\mathcal{F}^*$ and $\mathcal{G}^*$ are the Legendre-Fenchel conjugates of $\mathcal{F}$ and $\mathcal{G}$. Following [6], we check that

$$
\mathcal{F}^*(\Lambda^*(m, z)) = \begin{cases} \int_{\mathbb{T}^d} u_T(x) dm(T, x) & \text{if } \begin{cases} \dfrac{\partial m}{\partial t} - \nu \Delta m + \operatorname{div} z = 0, \\ m(0, \cdot) = m_0 \end{cases} \\ +\infty & \text{otherwise.} \end{cases}
$$

where the boundary value problem is understood in the sense of distributions.

On the other hand, from Rockafellar, [18] Theorem 5, and (2.16), see also [5], we see that

$$
\mathcal{G}^*(-m, -z) = \int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x, m^{\mathrm{ac}}(t, x), z^{\mathrm{ac}}(t, x)) dx dt + \int_0^T \int_{\mathbb{T}^d} \widetilde{L}_\infty \left(x, \frac{dm^{\mathrm{sing}}}{d\theta}, \frac{dz^{\mathrm{sing}}}{d\theta} \right) d\theta,
$$

where $(m^{\mathrm{ac}}, z^{\mathrm{ac}})$ and $(m^{\mathrm{sing}}, z^{\mathrm{sing}})$ respectively denote the absolutely continuous and singular parts of (m, z), θ is any measure with respect to which $(m^{\mathrm{sing}}, z^{\mathrm{sing}})$ is absolutely continuous, (for instance $m^{\mathrm{sing}} + |z^{\mathrm{sing}}|$, and $\widetilde{L}_\infty(x, \cdot)$ is the recession function of $\widetilde{L}(x, \cdot)$, i.e.

$$
\widetilde{L}_\infty(x, m, z) = \sup_{\lambda > 0} \frac{1}{\lambda} \widetilde{L}(x, \lambda m, \lambda z) = \begin{cases} 0 & \text{if } (m, z) = (0, 0), \\ +\infty & \text{otherwise.} \end{cases}
$$

Therefore,

$$\mathcal{G}^*(-m, -z) = \begin{cases} \int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x, m(t,x), z(t,x)) dx dt & \text{if } (m, z) \in L^1((0,T) \times \mathbb{T}^d) \times L^1((0,T) \times \mathbb{T}^d; \mathbb{R}^d) \\ +\infty & \text{otherwise.} \end{cases}$$

(2.20)

Hence,

$$\min_{(m,z) \in E_1^*} \left(\mathcal{F}^*(\Lambda^*(m,z)) + \mathcal{G}^*(-m, -z) \right) = \min_{(m,z) \in \mathcal{K}_1} \mathcal{B}(m, z).$$

and we obtain the desired result from (2.18) and (2.19).

Using (1.6), the strict convexity of $m \mapsto m\ell(x, m)$ assumed in (H2), the convexity of the map $(m > 0, z) \mapsto m^{1+\frac{\alpha}{\beta-1}} |\frac{z}{m}|^{\beta^*}$ (see [2] paragraph 3.2), and the convexity of $\mathcal{K}_1$, we obtain the uniqueness of m^* such that $(m^*, z^*) \in \mathcal{K}_1$ achieves a minimum of $\mathcal{B}$ for some z^*. Moreover, from the strict convexity of $(m, z) \mapsto m^{1+\frac{\alpha}{\beta-1}} |\frac{z}{m}|^{\beta^*}$ for $m > 0$, we deduce that $\frac{z^*}{m^*}$ is unique in $\{(t, x) : m^*(t, x) > 0\}$. Since $z^* = 0$ in $\{(t, x) : m^*(t, x) = 0\}$, the uniqueness of z^* follows. It is clear from (H2) that $m^* \in L^q((0, T) \times \mathbb{T}^d)$. $\square$

3 A priori estimates for a maximizing sequence of (2.1)

Let $M \in \mathbb{R}$ be the optimal value in (2.1). Take a maximizing sequence $(\phi_n)_{n \in N}$ for (2.1). For some $\epsilon > 0$, it can be chosen in such a way that
$$M - \epsilon < \mathcal{A}(\phi_n) \leq M.$$

From the definition of $\mathcal{A}$, we see that $\mathcal{A}(\phi_n) \leq \mathcal{A}(\phi_n, m_0)$. Hence,

$$\begin{aligned} M - \epsilon &\leq \int_0^T \int_{\mathbb{T}^d} m_0(x) \left(\frac{\partial \phi_n}{\partial t}(t,x) + \nu \Delta \phi_n(t,x) + H(x, m_0(x), D\phi_n(t,x)) \right) dx dt \\ &\quad + \int_{\mathbb{T}^d} m_0(x) \phi_n(0, x) dx \\ &= \int_0^T \int_{\mathbb{T}^d} \left(-\nu D m_0(x) \cdot D\phi_n(t,x) + m_0(x) H(x, m_0(x), D\phi_n(t,x)) \right) dx dt + \int_{\mathbb{T}^d} m_0(x) u_T(x) dx. \end{aligned}$$

which implies that $\|D\phi_n\|_{L^\beta((0,T) \times \mathbb{T}^d)}$ is bounded uniformly w.r.t. n.

Let m_n achieve $\mathcal{A}(\phi_n) = \mathcal{A}(\phi_n, m_n)$. Recall that m_n is unique.

The optimality conditions for m_n in (2.1) are

$$\frac{\partial \phi_n}{\partial t}(t,x) + \nu \Delta \phi_n(t,x) + H(x, m_n(t,x), D\phi_n(t,x)) + m_n(t,x) H_m((x, m_n(t,x), D\phi_n(t,x)) \geq 0 \quad \text{a.e.}, (3.1)$$

$$0 = \begin{cases} \int_0^T \int_{\mathbb{T}^d} m_n(t,x) \left(\frac{\partial \phi_n}{\partial t}(t,x) + \nu \Delta \phi_n(t,x) \right) dx dt \\ + \int_0^T \int_{\mathbb{T}^d} m_n(t,x) \left(H(x, m_n(t,x), D\phi_n(t,x)) + m_n(t,x) H_m(x, m_n(t,x), D\phi_n(t,x)) \right) dx dt. \end{cases}$$

(3.2)

From (3.2), we deduce that

$$\mathcal{A}(\phi_n) = -\int_0^T \int_{\mathbb{T}^d} m_n^2(t,x) H_m((x, m_n(t,x), D\phi_n(t,x)) dx dt + \int_{\mathbb{T}^d} m_0(x) \phi_n(0, x) dx.$$

From (H1) and (H2),

$$m^2 H_m(x, m, p) \geq \frac{1}{C_2} m^q - C_2 m + m^{1-\alpha} |p|^\beta.$$

Hence,

$$\int_0^T \int_{\mathbb{T}^d} \left(\frac{1}{C_2} m_n^q(t,x) - C_2 m_n(t,x) + m_n^{1-\alpha}(t,x) |D\phi_n(t,x)|^\beta \right) dx dt - \int_{\mathbb{T}^d} m_0(x) \phi_n(0, x) dx$$
$$\leq -M + \epsilon.$$

(3.3)

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On the other hand, (3.1), (H1) and (H2) imply that for some constant $C_4 > 0$,

$$\frac{\partial \phi_n}{\partial t}(t, x) + \nu \Delta \phi_n(t, x) + C_4 m_n^{q-1}(t, x) + C_4 \geq 0, \quad \text{a.e.} \tag{3.4}$$

Multiplying (3.4) by $(\phi_n^+)^{q*-1} e^{-\lambda(T-t)}$ for λ large enough, integrating in $(\tau, T) \times \mathbb{T}^d$, we obtain that

$$\|\phi_n^+\|_{L^\infty(0,T;L^{q^*}(\mathbb{T}^d))} \leq C \left(1 + \|m_n\|_{L^q((0,T)\times\mathbb{T}^d)}^{q-1} + \|u_T^+\|_{L^{q^*}(\mathbb{T}^d)} \right), \tag{3.5}$$

and that

$$\|D((\phi_n^+)^{q^*/2})\|_{L^2((0,T)\times\mathbb{T}^d)}^2 \leq C \left(1 + \|m_n\|_{L^q((0,T)\times\mathbb{T}^d)}^{q} + \|u_T^+\|_{L^{q^*}(\mathbb{T}^d)}^{q^*} \right).$$

Remark 3.1. *Note that the latter estimate does not hold with a degenerate diffusion as in Remark 1.1, but it will not be used hereafter.*

This implies that

$$\int_{\mathbb{T}^d} m_0(x)\phi_n^+(0, x)dx \leq C \left(\|m_n\|_{L^q((0,T)\times\mathbb{T}^d)}^{q-1} + 1 \right). \tag{3.6}$$

Combining (3.3) and (3.6), we obtain that

$$\int_0^T \int_{\mathbb{T}^d} \left(\frac{1}{C_2} m_n^q(t, x) - C_2 m_n(t, x) + m_n^{1-\alpha}(t, x)|D\phi_n(t, x)|^\beta \right) dxdt + \int_{\mathbb{T}^d} m_0(x)\phi_n^-(0, x)dx$$
$$\leq -M + \epsilon + C \left(\|m_n\|_{L^q((0,T)\times\mathbb{T}^d)}^{q-1} + 1 \right).$$

The latter and (3.6) yield

$$\int_0^T \int_{\mathbb{T}^d} \left(m_n^q(t, x) + m_n^{1-\alpha}(t, x)|D\phi_n(t, x)|^\beta \right) dxdt + \int_{\mathbb{T}^d} |\phi_n(0, x)|dx \leq C. \tag{3.7}$$

Let $\bar{m}(t, x) = 1_{\tau < t}$: since $\mathcal{A}(\phi_n, \bar{m}) \geq \mathcal{A}(\phi_n)$, we obtain that

$$
\begin{aligned}
M - \epsilon &\leq \int_0^t \int_{\mathbb{T}^d} \left(\frac{\partial \phi_n}{\partial t}(\tau, x) + \nu \Delta \phi_n(\tau, x) + H(x, 1, D\phi_n(\tau, x)) \right) dxdt \\
&\quad + \int_{\mathbb{T}^d} m_0(x)\phi_n(0, x)dx \\
&= \int_0^t \int_{\mathbb{T}^d} H(x, 1, D\phi_n(\tau, x))dxd\tau + \int_{\mathbb{T}^d} \phi_n(t, x)dx + \int_{\mathbb{T}^d} (m_0(x) - 1)\phi_n(0, x)dx.
\end{aligned}
$$

This implies that $t \mapsto \int_{\mathbb{T}^d} \phi_n(t, x)dx$ is bounded from below uniformly w.r.t. n. Combining with (3.5), we get that ϕ_n^- is bounded in $L^\infty(0, T; L^1(\mathbb{T}^d))$, and finally that ϕ_n is bounded in $L^\infty(0, T; L^1(\mathbb{T}^d))$. Finally, setting $\gamma_n^1 = -\ell(\cdot, m_n) - m_n \frac{\partial \ell}{\partial m}(\cdot, m_n)$ and $0 \leq \gamma_n^2 = (1-\alpha) \frac{|D\phi_n|^\beta}{m_n^\alpha} 1_{\{m_n > 0\}}$, we see that the sequence $(\gamma_n^1)_n$ is bounded in $L^{q^*}((0,T) \times \mathbb{T}^d)$, and that

$$\frac{\partial \phi_n}{\partial t} + \nu \Delta \phi_n \geq \gamma_n^1 + \gamma_n^2.$$

Integrating the latter on $(0, T) \times \mathbb{T}^d$, we obtain that

$$\int_0^T \int_{\mathbb{T}^d} \gamma_n^2(t, x)dxdt \leq \int_{\mathbb{T}^d} u_T(x)dx - \int_{\mathbb{T}^d} \phi_n(0, x)dx - \int_0^T \int_{\mathbb{T}^d} \gamma_n^1(t, x)dxdt,$$

which implies that the sequence of positive function $(\gamma_n^2)_n$ is bounded in $L^1((0, T) \times \mathbb{T}^d)$.
To summarize, we have proven the following lemma:

Lemma 3.1. *The maximizing sequence $(\phi_n)_{n\in\mathbb{N}}$ introduced at the beginning of § 3.1 is uniformly bounded in $L^\beta(0,T;W^{1,\beta}(\mathbb{T}^d))$ and in $L^\infty(0,T;L^1(\mathbb{T}^d))$.*
Noting m_n the nonnegative function achieving $\mathcal{A}(\phi_n,m_n) = \mathcal{A}(\phi_n)$, the sequence $(m_n)_{n\in\mathbb{N}}$ is uniformly bounded in $L^q((0,T)\times\mathbb{T}^d)$.
Calling $\gamma_n(t,x) = -H(x,m_n(t,x),D\phi_n(t,x)) - m_n(t,x)H_m(x,m_n(t,x),D\phi_n(t,x))$, with the convention that $H(x,m,p) + mH_m(x,m,p) = \ell(x,0)$ if $m = 0$ and $p = 0$, we can split γ_n as follows: $\gamma_n = \gamma_n^1 + \gamma_n^2$, where $\gamma_n^1 = -\ell(\cdot,m_n) - m_n\frac{\partial\ell}{\partial m}(\cdot,m_n)$ and $\gamma_n^2 = (1-\alpha)\frac{|D\phi_n|^\beta}{m_n^\alpha}\mathbb{1}_{\{m_n>0\}}$. The sequence $(\gamma_n^1)_{n\in\mathbb{N}}$ is uniformly bounded in $L^{q^*}((0,T)\times\mathbb{T}^d)$. The function γ_n^2 is nonnegative and the sequence $(\gamma_n^2)_{n\in\mathbb{N}}$ is uniformly bounded in $L^1((0,T)\times\mathbb{T}^d)$.

4 A relaxed problem

4.1 Definition and first properties

Let $\mathcal{K}$ be the set of pairs (ϕ,γ) such that

- $\phi \in L^\beta(0,T;W^{1,\beta}(\mathbb{T}^d)) \cap L^\infty(0,T;\mathcal{M}(\mathbb{T}^d))$

- $\gamma \in \mathcal{M}([0,T]\times\mathbb{T}^d)$ and $\gamma^{\mathrm{ac}} = \gamma^1 + \gamma^2$, with $\gamma^1 \in L^{q^*}((0,T)\times\mathbb{T}^d)$ is non positive, γ^2 is non negative and $\gamma^2 \in L^1((0,T)\times\mathbb{T}^d)$, $\gamma^{\mathrm{sing}} \in \mathcal{M}_+([0,T]\times\mathbb{T}^d)$

-
$$\frac{\partial\phi}{\partial t} + \nu\Delta\phi \geq \gamma, \quad \text{and} \quad \phi|_{t=T} \leq u_T. \tag{4.1}$$

It is clear that $\mathcal{K}$ is convex. The following lemma implies that ϕ has a trace in a very weak sense:

Lemma 4.1. *Consider $(\phi,\gamma) \in \mathcal{K}$. For any Lipschitz continuous map $\xi : \mathbb{T}^d \to \mathbb{R}$, the map $t \mapsto \int_{\mathbb{T}^d}\xi(x)\phi(t,x)dx$ has a BV representative on $[0,T]$. Moreover, if we note $\int_{\mathbb{T}^d}\xi(x)\phi(t^+,x)dx$ its right limit at $t \in [0,T)$, then the map $\xi \mapsto \int_{\mathbb{T}^d}\xi(x)\phi(t^+,x)dx$ can be extended to a bounded linear form on $\mathcal{C}^0(\mathbb{T}^d)$.*

Proof. Consider first a nonnegative and Lipschitz continuous function $\xi : \mathbb{T}^d \to \mathbb{R}_+$; the following identity holds in the sense of distributions:

$$-\frac{d}{dt}\left(t \mapsto \int_{\mathbb{T}^d}\xi(x)\phi(t,x)dx\right) - \nu\int_{\mathbb{T}^d}\nabla\phi(t,x)\cdot\nabla\xi(x)dx \geq \int_{\mathbb{T}^d}\xi(x)\gamma^{\mathrm{ac}}(t,x)dx,$$

The second and third integral in the latter inequality belong to $L^1((0,T))$. From this, we deduce that $t \mapsto \int_{\mathbb{T}^d}\xi(x)\phi(t,x)dx$ has a BV representative.
If now ξ is a Lipschitz continuous function that may change sign, $\xi : \mathbb{T}^d \to \mathbb{R}$, then we write $\xi = \xi^+ - \xi^-$ and use the above argument separately for ξ^+ and ξ^-: we still obtain that $t \mapsto \int_{\mathbb{T}^d}\xi(x)\phi(t,x)dx = \int_{\mathbb{T}^d}\xi^+(x)\phi(t,x)dx - \int_{\mathbb{T}^d}\xi^-(x)\phi(t,x)dx$ has a BV representative.
The continuity of $\xi \mapsto \int_{\mathbb{T}^d}\xi(x)\phi(t^+,x)dx$ comes from the fact that $\phi \in L^\infty(0,T;\mathcal{M}(\mathbb{T}^d))$. $\square$

Thanks to Lemma 4.1, we may define the concave functional J on $\mathcal{K}$ by

$$J(\phi,\gamma) = \int_0^T\int_{\mathbb{T}^d}K(x,\gamma^{\mathrm{ac}}(t,x),D\phi(t,x))dxdt + \int_{\mathbb{T}^d}m_0(x)\phi(0^+,x)dx, \tag{4.2}$$

and the relaxed optimization problem:

$$\sup_{(\phi,\gamma)\in\mathcal{K}} J(\phi,\gamma). \tag{4.3}$$

In (4.2), note that $(x,t) \mapsto K(x,\gamma^{\mathrm{ac}}(t,x),D\phi(t,x))$ is a measurable nonpositive function, so the first integral is meaningful and has a value in $[-\infty,0]$.
Note also that, from (2.14), it is possible to restrict ourselves to the pairs $(\phi,\gamma) \in \mathcal{K}$ such that $\gamma^{\mathrm{ac}}(t,x) \leq -\ell(x,0)$ for almost every (t,x) such that $D\phi(t,x) = 0$. Noting $\widetilde{\mathcal{K}}$ the set

$$\widetilde{\mathcal{K}} = \{(\phi,\gamma) \in \mathcal{K} : \gamma^{\mathrm{ac}}(t,x) \leq -\ell(x,0) \text{ for almost every } (t,x) \text{ s.t. } D\phi(t,x) = 0\},$$

we have

$$\sup_{(\phi,\gamma)\in\mathcal{K}} J(\phi,\gamma) = \sup_{(\phi,\gamma)\in\widetilde{\mathcal{K}}} J(\phi,\gamma). \tag{4.4}$$

Lemma 4.2. *For any* $(\phi,\gamma) \in \widetilde{\mathcal{K}}$ *such that*

$$\int_0^T \int_{\mathbb{T}^d} K(x,\gamma^{\mathrm{ac}}(t,x),D\phi(t,x))dxdt > -\infty,$$

for any $(m,z) \in \mathcal{K}_1$ *such that* $m \in L^q((0,T)\times\mathbb{T}^d)$ *and*

$$\int_0^T \int_{\mathbb{T}^d} \tilde{L}(x,m(t,x),z(t,x))dxdt < +\infty,$$

the following holds:
for almost any $t \in (0,T)$,

$$-\int_{\mathbb{T}^d} m(T,x)u_T(x)dx + \int_{\mathbb{T}^d} m(t,x)\phi(t,x)dx + \int_{s=t}^T \int_{\mathbb{T}^d} m(s,x)\left(\gamma^{\mathrm{ac}}(s,x) + H(x,m(s,x),D\phi(s,x))\right)dxds$$

$$\leq \int_{s=t}^T \int_{\mathbb{T}^d} \tilde{L}(x,m(s,x),z(s,x))dxds, \tag{4.5}$$

$$-\int_{\mathbb{T}^d} m(t,x)\phi(t,x)dx + \int_{\mathbb{T}^d} m_0(x)\phi(0^+,x)dx + \int_{s=0}^t \int_{\mathbb{T}^d} m(s,x)\left(\gamma^{\mathrm{ac}}(s,x) + H(x,m(s,x),D\phi(s,x))\right)dxds$$

$$\leq \int_{s=0}^t \int_{\mathbb{T}^d} \tilde{L}(x,m(s,x),z(s,x))dxds, \tag{4.6}$$

and the meaning of $\int_{\mathbb{T}^d} m(t,x)\phi(t,x)dx$ *will be explained in the proof.*
Moreover, if

$$-\int_{\mathbb{T}^d} m(T,x)u_T(x)dx + \int_{\mathbb{T}^d} m_0(x)\phi(0^+,x)dx + \int_{s=0}^T \int_{\mathbb{T}^d} m(s,x)\left(\gamma^{\mathrm{ac}}(s,x) + H(x,m(s,x),D\phi(s,x))\right)dxds$$

$$= \int_{s=0}^T \int_{\mathbb{T}^d} \tilde{L}(x,m(s,x),z(s,x))dxds, \tag{4.7}$$

then $z(t,x) = m(t,x)H_p(x,m(t,x),D\phi(t,x))$ *holds almost everywhere.*

Remark 4.1. *Before proving Lemma 4.2, note that the integrals of*
$(s,x) \mapsto m(s,x)\left(\gamma^{\mathrm{ac}}(s,x) + H(x,m(s,x),D\phi(s,x))\right)$ *in (4.5)-(4.7) have a meaning in* $(-\infty,+\infty]$, *because*
$m(s,x)\left(\gamma^{\mathrm{ac}}(s,x) + H(x,m(s,x),D\phi(s,x))\right) \geq K(x,\gamma^{\mathrm{ac}}(s,x),D\phi(s,x))$ *a.e. and from the hypothesis of*
Lemma 4.2, the latter is integrable in $(0,T)\times\mathbb{T}^d$ *with a nonpositive integral.*

Proof. We first extend m to $[-1,T+1]$ by setting $m(t) = m_0$ for $t \leq 0$ and $m(t) = m(T)$ for $t > T$. Note that $m(T)$ is well defined from Lemma 2.1. Similarly, we extend z to $[-1,T+1]$ by setting $z(t) = 0$ if $t \notin (0,T)$.

Consider a regularizing kernel $\eta_\epsilon(t,x) = \epsilon^{-d-\delta}h_1(\frac{t}{\epsilon^\delta})h_2(\frac{x}{\epsilon})$ where h_1 is a smooth even and nonnegative function supported in $[-1/2,1/2]$ such that $\int_{\mathbb{R}} h_1(t)dt = 1$, h_2 is a smooth symmetric nonnegative function supported in $[-1/2,1/2]^d$ such that $\int_{\mathbb{R}^d} h_2(x)dx = 1$, and δ will be chosen later. We define $m_\epsilon = \eta_\epsilon \star m$ and $z_\epsilon = \eta_\epsilon \star z$ in $(-1/2,T+1/2)\times\mathbb{T}^d$. We can see that in $(0,T)\times\mathbb{T}^d$,

$$\frac{\partial m_\epsilon}{\partial t} - \nu\Delta\left(m_\epsilon \mathbb{1}_{t\in(0,T)}\right) + \mathrm{div}z_\epsilon = \mathrm{div}R_\epsilon \tag{4.8}$$

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where $R_\epsilon = \eta_\epsilon \star \left(Dm\, 1_{t\in(0,T)}\right) - 1_{t\in(0,T)} D(\eta_\epsilon \star m)$. From [9], we know that $R_\epsilon \to 0$ in $L^q((0,T)\times\mathbb{T}^d)$ as $\epsilon \to 0$, because $m \in L^q((0,T)\times\mathbb{T}^d)$.

We know that $\phi(T) \le u_T$. Thus $\int_{\mathbb{T}^d} \phi(T,x)m_\epsilon(T,x)dx \le \int_{\mathbb{T}^d} u_T(x)m_\epsilon(T,x)dx$ and the latter integral converges to $\int_{\mathbb{T}^d} u_T(x)m(T,x)dx$ in view of Lemma 2.1.

We know that $m_\epsilon \to m$ in $L^q((0,T)\times\mathbb{T}^d)$ and that $\phi \in L^\beta((0,T)\times\mathbb{T}^d)$. From Assumption (H3), this implies that $\phi m_\epsilon \to \phi m$ in $L^1((0,T)\times\mathbb{T}^d)$.

From Lemma 4.1, we know that out of a countable set, $\phi(t^-) = \phi(t^+) = \phi(t)$ and that $\int_{\mathbb{T}^d} \phi(t,x)m_\epsilon(t,x)dx$ is well defined.

From the latter two observations, up to the extraction of a subsequence, we may assume that

$$\int_{\mathbb{T}^d} \phi(t,x)m_\epsilon(t,x)dx \to \int_{\mathbb{T}^d} \phi(t,x)m(t,x)dx$$

as $\epsilon \to 0$, with $\phi(t^-) = \phi(t^+) = \phi(t)$, for almost all $t \in (0,T)$. Let $t \in (0,T)$ be such that the latter is true.

Since $(\phi,\gamma) \in \mathcal{K}$,

$$
\begin{aligned}
&\int_t^T \int_{\mathbb{T}^d} \left(\phi(s,x)\frac{\partial m_\epsilon}{\partial t}(s,x) + \nu D\phi \cdot D(1_{s\in(0,T)}m_\epsilon(s,x))\right) dxds \\
&\quad - \int_{\mathbb{T}^d} u_T(x)m_\epsilon(T,x)dx + \int_{\mathbb{T}^d} \phi(t,x)m_\epsilon(t,x)dx \\
&\le - \int_t^T \int_{\mathbb{T}^d} \gamma^{\mathrm{ac}}(s,x)m_\epsilon(s,x)dxds
\end{aligned}
\tag{4.9}
$$

On the other hand, from (4.8),

$$
\begin{aligned}
&\int_t^T \int_{\mathbb{T}^d} \left(\phi(s,x)\frac{\partial m_\epsilon}{\partial t}(s,x) + \nu D\phi \cdot D(1_{s\in(0,T)}m_\epsilon(s,x))\right) dxds \\
&= - \int_t^T \int_{\mathbb{T}^d} D\phi(s,x) \cdot R_\epsilon(s,x)dxds + \int_t^T \int_{\mathbb{T}^d} D\phi(s,x) \cdot z_\epsilon(s,x)dxds
\end{aligned}
\tag{4.10}
$$

We deduce from (4.9) and (4.10) that

$$
\begin{aligned}
&- \int_t^T \int_{\mathbb{T}^d} D\phi(s,x) \cdot R_\epsilon(s,x)dxds + \int_t^T \int_{\mathbb{T}^d} D\phi(s,x) \cdot z_\epsilon(s,x)dxds \\
&- \int_t^T \int_{\mathbb{T}^d} m_\epsilon(s,x)H(x,m_\epsilon(s,x),D\phi(s,x))dsds - \int_{\mathbb{T}^d} u_T(x)m_\epsilon(T,x)dx + \int_{\mathbb{T}^d} \phi(t,x)m_\epsilon(t,x)dx \\
&\le - \int_t^T \int_{\mathbb{T}^d} \gamma^{\mathrm{ac}}(s,x)m_\epsilon(s,x)dxds - \int_t^T \int_{\mathbb{T}^d} m_\epsilon(s,x)H(x,m_\epsilon(s,x),D\phi(s,x))dsds.
\end{aligned}
\tag{4.11}
$$

Finally, we use the fact that

$$
\begin{aligned}
&\int_t^T \int_{\mathbb{T}^d} \left(D\phi(s,x) \cdot z_\epsilon(s,x) - m_\epsilon(s,x)H(x,m_\epsilon(s,x),D\phi(s,x))\right) dxds \\
&\ge - \int_t^T \int_{\mathbb{T}^d} \tilde{L}(x,m_\epsilon(s,x),z_\epsilon(s,x))dxds.
\end{aligned}
\tag{4.12}
$$

Combining this with (4.11), we find that

$$
\begin{aligned}
&- \int_{\mathbb{T}^d} u_T(x)m_\epsilon(T,x)dx + \int_{\mathbb{T}^d} \phi(t,x)m_\epsilon(t,x)dx \\
&\le - \int_t^T \int_{\mathbb{T}^d} m_\epsilon(s,x)\left(\gamma^{\mathrm{ac}}(s,x) + H(x,m_\epsilon(s,x),D\phi(s,x))\right) dsds \\
&\quad + \int_t^T \int_{\mathbb{T}^d} \tilde{L}(x,m_\epsilon(s,x),z_\epsilon(s,x))dxds + \int_t^T \int_{\mathbb{T}^d} D\phi(s,x) \cdot R_\epsilon(s,x)dxds.
\end{aligned}
\tag{4.13}
$$

1. We have seen that the first line of (4.13) tends to $-\int_{\mathbb{T}^d} u_T(x)m(T,x)dx + \int_{\mathbb{T}^d} \phi(t,x)m(t,x)dx$.

2. From Remark 4.1, or more precisely the facts that $\int_0^T \int_{\mathbb{T}^d} K(x,\gamma^{\mathrm{ac}}(t,x),D\phi(t,x))dxdt > -\infty$ and that

$$\gamma^{\mathrm{ac}}(s,x) + H(x,m_\epsilon(s,x),D\phi(s,x))m_\epsilon(s,x) \geq K(x,\gamma^{\mathrm{ac}}(s,x),D\phi(s,x)),$$

we can use Fatou lemma and get that

$$\int_t^T \int_{\mathbb{T}^d} (\gamma^{\mathrm{ac}}(s,x) + H(x,m(s,x),D\phi(s,x)))\, m(s,x)dxds$$

$$\leq \liminf_{\epsilon\to 0} \int_t^T \int_{\mathbb{T}^d} (\gamma^{\mathrm{ac}}(s,x) + H(x,m_\epsilon(s,x),D\phi(s,x)))\, m_\epsilon(s,x)dxds.$$

3. $\int_t^T \int_{\mathbb{T}^d} D\phi(s,x) \cdot R_\epsilon(s,x)dxds$ tends to 0 because R_ϵ tends to 0 in $L^q((0,T)\times\mathbb{T}^d)$ and $D\phi \in L^\beta((0,T)\times\mathbb{T}^d))$ with $\beta \geq q^*$.

4. Finally, from the convexity of $\tilde{L}$ with respect to (m,z) and from (1.11), we see that

$$\limsup_{\epsilon\to 0} \int_t^T \int_{\mathbb{T}^d} \tilde{L}(x,m_\epsilon(s,x),z_\epsilon(s,x))dxds \leq \int_t^T \int_{\mathbb{T}^d} \tilde{L}(x,m(s,x),z(s,x))dxds.$$

From (4.13) and all the points above, we deduce (4.5).
Similarly as for (4.13), we obtain that

$$-\int_{\mathbb{T}^d} \phi(t,x)m_\epsilon(t,x)dx + \int_{\mathbb{T}^d} \phi(0,x)m_\epsilon(0,x)dx$$

$$\leq -\int_0^t \int_{\mathbb{T}^d} \gamma^{\mathrm{ac}}(s,x)m_\epsilon(s,x)dxds - \int_0^t \int_{\mathbb{T}^d} m_\epsilon(s,x)H(x,m_\epsilon(s,x),D\phi(s,x))dxds \qquad (4.14)$$

$$+\int_0^t \int_{\mathbb{T}^d} \tilde{L}(x,m_\epsilon(s,x),z_\epsilon(s,x))dxds + \int_0^t \int_{\mathbb{T}^d} D\phi(s,x)\cdot R_\epsilon(s,x)dxds,$$

in which $\int_{\mathbb{T}^d} \phi(0,x)m_\epsilon(0,x)dx$ has a meaning from Lemma 4.1. We claim that $\int_{\mathbb{T}^d} \phi(0,x)m_\epsilon(0,x)dx \to \int_{\mathbb{T}^d} \phi(0,x)m_0(x)dx$ as $\epsilon \to 0$: indeed, let $\zeta \in (0,1)$ be the Hölder exponent in Lemma 2.1: in view of Remark 2.1, we know that $\zeta \geq \min(1/2,(1-\alpha)/\beta)$; calling $h_{2,\epsilon}(x) = \epsilon^{-d}h_2(\frac{x}{\epsilon})$ and $h_{1,\epsilon}(t) = \epsilon^{-\delta}h_1(\frac{t}{\epsilon^\delta})$, we get that for all $x \in \mathbb{T}^d$,

$$|m_\epsilon(0,x) - (h_{2,\epsilon}\star m_0)(x)| = \left|\int\int (m(s,y)-m_0(y))h_{1,\epsilon}(-s)h_{2,\epsilon}(x-y)dsdy\right|$$

$$\leq C\int s^{\delta\zeta-d-1}h_{1,\epsilon}(-s)ds \leq C\epsilon^{\delta\zeta-d-1}.$$

Choosing δ large enough, (i.e. such that $\delta\zeta - d - 1 > 0$) and using the fact that m_0 is $\mathcal{C}^1$,

$$\lim_{\epsilon\to 0} \|h_{2,\epsilon}\star m_0 - m_0\|_{L^\infty(\mathbb{T}^d)} = 0.$$

Therefore, $\lim_{\epsilon\to 0}\|m_\epsilon(0,\cdot) - m_0\|_{L^\infty(\mathbb{T}^d)} = 0$. The claim follows from the continuity stated in Lemma 4.1.
The fact that $\int_{\mathbb{T}^d} \phi(0,x)m_\epsilon(0,x)dx \to \int_{\mathbb{T}^d} \phi(0,x)m_0(x)dx$ and the arguments above imply (4.6).
Let us now suppose that (4.7) holds: then the inequalities in (4.5) and (4.6) are equalities, for almost all t. For $\sigma > 0$ let us introduce the set

$$E_\sigma(t) = \left\{ \begin{array}{l} (s,y) \in [t,T]\times\mathbb{T}^d \;: \\ m(s,y)H(x,m(s,y),D\phi(s,y)) - \tilde{L}(y,m(s,y),z(s,y)) \leq z(s,y)\cdot D\phi(s,y) - \sigma \end{array} \right\}.$$

If $|E_\sigma(t)| > 0$, then for $\epsilon > 0$ small enough, $|E_{\epsilon,\sigma}(t)| > |E_\sigma(t)|/2$, where

$$E_{\epsilon,\sigma}(t) = \left\{ \begin{array}{l} (s,y) \in [t,T]\times\mathbb{T}^d \;: \\ m_\epsilon(s,y)H(x,m_\epsilon(s,y),D\phi(s,y)) - \tilde{L}(y,m_\epsilon(s,y),z_\epsilon(s,y)) \leq z_\epsilon(s,y)\cdot D\phi(s,y) - \frac{\sigma}{2} \end{array} \right\}.$$

Then (4.12) becomes

$$\int_t^T \int_{\mathbb{T}^d} \left(D\phi(s,x) \cdot z_\epsilon(s,x) - m_\epsilon(s,x) H(x, m_\epsilon(s,x), D\phi(s,x)) \right) dx ds$$
$$\geq - \int_t^T \int_{\mathbb{T}^d} \tilde{L}(x, m_\epsilon(s,x), z_\epsilon(s,x)) dx ds + \frac{\sigma}{4} |E_\sigma(t)|, \tag{4.15}$$

which implies that

$$- \int_{\mathbb{T}^d} m(T,x) u_T(x) dx + \int_{\mathbb{T}^d} m(t,x) \phi(t,x) dx + \int_{s=t}^T \int_{\mathbb{T}^d} m(s,x) \left(\gamma^{\mathrm{ac}}(s,x) + H(x, m(s,x), D\phi(s,x)) \right) dx ds$$
$$\leq \int_{s=t}^T \int_{\mathbb{T}^d} \tilde{L}(x, m(s,x), z(s,x)) dx ds - \frac{\sigma}{4} |E_\sigma(t)|,$$

in contradiction with the fact that there is an equality in (4.5).

Hence, $m(s,y) H(x, m(s,y), D\phi(s,y)) - \tilde{L}(y, m(s,y), z(s,y)) = z(s,y) \cdot D\phi(s,y)$ holds almost everywhere. In view of (2.6), this shows that $z(s,y) = m(s,y) H_p(x, m(s,y), D\phi(s,y))$ a.e. in $\{m > 0\}$. Furthermore, from (2.5) and the fact that $\int_0^T \int_{\mathbb{T}^d} \tilde{L}(x, m(t,x), z(t,x)) dx dt < +\infty$, we see that $z(s,y) = 0$ a.e. in $\{m = 0\}$. Hence, $z(s,y) = m(s,y) H_p(x, m(s,y), D\phi(s,y))$ a.e. in $(0,T) \times \mathbb{T}^d$. $\square$

Proposition 4.1.

$$\sup_{(\phi,\gamma) \in \mathcal{K}} J(\phi, \gamma) = \sup_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi).$$

Proof. It is clear that $\forall \phi \in \mathcal{K}_0$, $-\infty < \mathcal{A}(\phi) \leq \sup_{(\phi,\gamma) \in \mathcal{K}} J(\phi, \gamma)$: indeed consider m achieving $\mathcal{A}(\phi, m) = \mathcal{A}(\phi)$, i.e. $m = \psi \left(\frac{\partial \phi}{\partial t} + \nu \Delta \phi, D\phi \right)$. We know that $\int_0^T \int_{\mathbb{T}^d} m^{1-\alpha} |D\phi|^\beta < +\infty$ and that $\int_0^T \int_{\mathbb{T}^d} m^q < \infty$. Let us take $\gamma = \gamma^1 + \gamma^2$ where $\gamma^1 = -\ell(\cdot, m) - m \frac{\partial \ell}{\partial m}(\cdot, m)$ and $\gamma^2 = \frac{(1-\alpha)}{m^\alpha} |D\phi|^\beta 1_{m>0}$. It is easy to check that $(\phi, \gamma) \in \mathcal{K}$ and that $\mathcal{A}(\phi) = J(\phi, \gamma)$. Hence $\sup_{(\phi,\gamma) \in \mathcal{K}} J(\phi, \gamma) \geq \sup_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi)$.

For the reverse inequality, consider $(\phi, \gamma) \in \widetilde{\mathcal{K}}$ such that $J(\phi, \gamma) > -\infty$. This implies that

$$\int_0^T \int_{\mathbb{T}^d} K(x, \gamma^{\mathrm{ac}}(s,x), D\phi(s,x)) dx ds > -\infty.$$

Let (m^*, z^*) be the pair of functions achieving (2.7), see Lemma 2.2. From Lemma 4.2,

$$J(\phi, \gamma) \leq \int_0^T \int_{\mathbb{T}^d} m^*(t,x) \left(\gamma^{\mathrm{ac}}(t,x) + H(x, m^*(t,x), D\phi(t,x)) \right) dx dt + \int_{\mathbb{T}^d} m_0(x) \phi(0^+, x) dx$$
$$\leq \int_{s=0}^T \int_{\mathbb{T}^d} \tilde{L}(x, m^*(s,x), z^*(s,x)) dx ds + \int_{\mathbb{T}^d} m^*(T,x) u_T(x) dx$$
$$= \mathcal{B}(m^*, z^*).$$

Hence,

$$J(\phi, \gamma) \leq \min_{(m,z) \in \mathcal{K}_1} \mathcal{B}(m,z) = \sup_{\phi \in \mathcal{K}_0} \mathcal{A}(\phi).$$

and we conclude using (4.4). $\square$

4.2 Existence of a solution of the relaxed problem

Proposition 4.2. *The relaxed problem (4.3) has at least a solution $(\phi, \gamma) \in \widetilde{\mathcal{K}}$.*

Proof.

Step 1 Consider the maximizing sequence ϕ_n for problem (2.3) described in Lemma 3.1 and call m_n the function such that $\mathcal{A}(\phi_n) = \mathcal{A}(\phi_n, m_n)$. Also, let the functions γ_n^1 and γ_n^2 be defined as in Lemma 3.1. We know that

$$\frac{\partial \phi_n}{\partial t} + \nu \Delta \phi_n = \gamma_n \geq \gamma_n^1 + \gamma_n^2,$$

and $(\phi_n, \gamma_n) \in \widetilde{K}$. The definition of γ_n^1 ensures that $\gamma_n^1 \leq -\ell(\cdot, 0)$.

Up to the extraction of a subsequence, we may assume that $\phi_n \rightharpoonup \phi$ in $L^\beta(0, T, W^{1,\beta}(\mathbb{T}^d))$, $m_n \rightharpoonup m$ in $L^q((0, T) \times \mathbb{T}^d)$, $\gamma_n^1 \rightharpoonup \gamma^1$ in $L^{q^*}((0, T) \times \mathbb{T}^d)$, with $\gamma^1 \leq -\ell(\cdot, 0)$ a.e., $\gamma_n^2 \rightharpoonup \tilde{\gamma}^2$ in $\mathcal{M}((0, T) \times \mathbb{T}^d) *$. It is clear that $\tilde{\gamma}^2$ is a positive Radon measure that we write $\tilde{\gamma}^2 = \gamma^2 + \gamma^{\text{sing}}$. We see that

$$\frac{\partial \phi}{\partial t} + \nu \Delta \phi \geq \gamma = \gamma^1 + \gamma^2.$$

Thus for any function ξ in $Lip(\mathbb{T}^d)$, the function $t \mapsto \int_{\mathbb{T}^d} \xi(x)\phi(t, x)dx$ has a BV representative. Moreover, since $t \mapsto \zeta_n(t) = \int_{\mathbb{T}^d} \xi(x)\phi_n(t, x)dx$ is bounded in $L^\infty((0, T))$ by $C\|\xi\|_{L^\infty(\mathbb{T}^d)}$, where C is independent of ξ, we can assume (up to the extraction of a subsequence) that $\zeta_n \rightharpoonup \zeta$ in $L^\infty((0, T))$ weak *, and $\|\zeta\|_{L^\infty((0,T))} \leq C\|\xi\|_{L^\infty(\mathbb{T}^d)}$. Since for all smooth function $\psi : [0, T] \to \mathbb{R}$, we know that $\int_0^T \int_{\mathbb{T}^d} \xi(x)\phi_n(t, x)\psi(t)dxdt$ tends to $\int_0^T \int_{\mathbb{T}^d} \xi(x)\phi(t, x)\psi(t)dxdt$, we see that $\zeta = \int_{\mathbb{T}^d} \xi(x)\phi(\cdot, x)dx$. Therefore $\left| \int_{\mathbb{T}^d} \xi(x)\phi(t, x)dx \right| \leq C\|\xi\|_{L^\infty(\mathbb{T}^d)}$. This shows that $(\phi, \gamma) \in \mathcal{K}$.

Step 2 It can be proved that the map $(\phi, \gamma) \in \mathcal{K} \mapsto -\int_0^T \int_{\mathbb{T}^d} K(x, -\gamma^{\text{ac}}(t, x), -D\phi(t, x))dxdt$ is the restriction to $\mathcal{K}$ of the convex conjugate of the map

$$(\mu, z) \in E_1 \mapsto \int_0^T \int_{\mathbb{T}^d} \tilde{L}(x, \mu(t, x), z(t, x))dxdt.$$

Therefore, the map $(\phi, \gamma) \in \mathcal{K} \mapsto \int_0^T \int_{\mathbb{T}^d} K(x, \gamma^{\text{ac}}(t, x), D\phi(t, x))dxdt$ is upper semi-continuous for the weak * topology of $L^\beta(0, T; W^{1,\beta}(\mathbb{T}^d)) \times \mathcal{M}$, and therefore,

$$\int_0^T \int_{\mathbb{T}^d} K(x, \gamma^{\text{ac}}(t, x), D\phi(t, x))dxdt \geq \limsup_{n \to \infty} \int_0^T \int_{\mathbb{T}^d} K(x, \gamma_n(t, x), D\phi(t, x))dxdt. \tag{4.16}$$

Step 3 Let $\zeta_n = \int_{\mathbb{T}^d} m_0(x)\phi_n(t, x)dx$ and $\zeta = \int_{\mathbb{T}^d} m_0(x)\phi(t, x)dx$. We have seen that $\zeta_n \rightharpoonup \zeta$ in $L^\infty((0, T))$ weak *; from the inequation satisfied by ϕ_n,

$$\frac{d\zeta_n}{dt}(t) - \nu \int_{T^d} Dm_0(x)D\phi_n(t, x)dx - \int_{\mathbb{T}^d} (\gamma_n^1(t, x) + \gamma_n^2(t, x))m_0(x)dx \geq 0,$$

which implies that

$$\frac{d\zeta_n}{dt}(t) - \nu \int_{T^d} Dm_0(x)D\phi_n(t, x)dx - \int_{\mathbb{T}^d} \gamma_n^1(t, x)m_0(x)dx \geq 0.$$

Thanks to the a priori bounds on ϕ_n and γ_n^1, this implies that

$$\zeta_n(t) \geq \zeta_n(0) - C(t^{\frac{1}{\beta^*}} + t^{\frac{1}{q}}), \quad \forall t \in [0, T].$$

Letting n tend to $+\infty$,

$$\zeta(t) \geq \limsup_{n \to \infty} \zeta_n(0) - C(t^{\frac{1}{\beta^*}} + t^{\frac{1}{q}}), \quad \text{a.e. } t \in [0, T].$$

Hence $\zeta(0) \geq \limsup_{n \to \infty} \zeta_n(0)$, i.e. $\int_{\mathbb{T}^d} m_0(x)\phi(t, x)dx \geq \limsup_{n \to \infty} \int_{\mathbb{T}^d} m_0(x)\phi_n(t, x)dx$.

Step 4 By combining the results of steps 2 and 3, we see that

$$J(\phi, \gamma) \geq \limsup_{n \to \infty} \mathcal{A}(\phi_n) = \sup_{\tilde{\phi} \in \mathcal{K}_0} \mathcal{A}(\tilde{\phi}) = \sup_{(\tilde{\phi}, \tilde{\gamma}) \in \mathcal{K}} J(\tilde{\phi}, \tilde{\gamma}),$$

Using the fact that $\gamma^1 \leq -\ell(\cdot, 0)$ a.e. and (2.14), we can always decrease γ^2 where $D\phi = 0$ in such a way that $(\phi, \gamma) \in \widetilde{\mathcal{K}}$ and the value of $J(\phi, \gamma)$ is preserved. $\square$

5   Weak solutions to the system of PDEs

Definition 5.1. *A pair $(\phi, m) \in L^\beta(0, T; W^{1,\beta}(\mathbb{T}^d)) \times L^q((0, T) \times \mathbb{T}^d)$ with $m \geq 0$ almost everywhere, is a weak solution of (1.1)-(1.3) if*

1. $m^{-\alpha}|D\phi|^\beta 1_{m>0} \in L^1((0, T) \times \mathbb{T}^d)$, $D\phi(t, x) = 0$ *a.e.   in the region* $\{(t, x) : m(t, x) = 0\}$, $m^{1-\alpha}|D\phi|^\beta \in L^1((0, T) \times \mathbb{T}^d)$ *and*

$$\int_0^T \int_{\mathbb{T}^d} m(t, x) L\left(x, m(t, x), H_p(x, m(t, x), D\phi(t, x))\right) dxdt < +\infty.$$

2. *The following:*

$$\frac{\partial \phi}{\partial t}(t, x) + \nu \Delta \phi(t, x) + H(x, m(t, x), D\phi(t, x)) + m(t, x) H_m(x, m(t, x), D\phi(t, x)) \geq 0,$$
$$\phi(T, x) \leq u_T(x),$$
(5.1)

holds in the sense of distributions, with the convention that

$$H(x, m, p) + m H_m(x, m, p) = \ell(x, 0) \quad if \quad m = 0 \ and \ p = 0.$$
(5.2)

3. *The following:*

$$\frac{\partial m}{\partial t}(t, x) - \nu \Delta m(t, x) + \mathrm{div}(m(t, \cdot) H_p(\cdot, m(t, \cdot), D\phi(t, \cdot)))(x) = 0,$$
$$m(0, x) = m_0(x)$$
(5.3)

holds in the sense of distributions

4. *There exists a constant C such that for each function $\xi \in Lip(\mathbb{T}^d)$,*

$$\| \int_{\mathbb{T}^d} \xi(x) \phi(t, x) dx \|_{L^\infty(0, T)} \leq C \|\xi\|_{L^\infty(\mathbb{T}^d)}.$$

5. *The following identity holds*

$$0 = \int_0^T \int_{\mathbb{T}^d} m(t, x) L(x, m(t, x), H_p(x, m(t, x), D\phi(t, x))) dxdt + \int_0^T \int_{\mathbb{T}^d} m^2(t, x) H_m(x, m(t, x), D\phi(t, x)) dxdt$$
$$+ \int_{\mathbb{T}^d} m(T, x) u_T(x) dx - \int_{\mathbb{T}^d} m_0(x) \phi(0^+, x) dx$$
(5.4)

Theorem 5.1. *Let $(m^*, z^*) \in \mathcal{K}_1$ be the minimizer of (2.7) and $(\phi^*, \gamma^*) \in \widetilde{\mathcal{K}}$ be a maximizer of (4.3). Then (ϕ^*, m^*) is a weak solution of (1.1)-(1.3), and $z^*(t, x) = m^*(t, x) H_p(x, m^*(t, x), D\phi^*(t, x))$, $\gamma^{*, \mathrm{ac}}(t, x) \geq -H(x, m^*(t, x), D\phi^*(t, x)) - m^*(t, x) H_m(x, m^*(t, x), D\phi^*(t, x))$ almost everywhere. Conversely, any weak solution $(\bar{\phi}, \bar{m})$ of (1.1)-(1.3) is such that the pair $(\bar{m}, \bar{m} H_p(x, \bar{m}, D\bar{\phi}))$ is the minimizer of (2.7) and $(\bar{\phi}, \bar{\gamma})$ is a maximizer of (4.3), with*

$$\bar{\gamma}(t, x) = -H(x, \bar{m}(t, x), D\bar{\phi}(t, x)) - \bar{m}(t, x) H_m(x, \bar{m}(t, x), D\bar{\phi}(t, x)),$$

with the convention (5.2).

Proof. Let $(m^*, z^*) \in \mathcal{K}_1$ be the minimizer of (2.7), (which implies that $m^* \in L^q((0, T) \times \mathbb{T}^d)$ and that $\int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x, m^*(t, x), z^*(t, x)) dxdt < +\infty$) and $(\phi^*, \gamma^*) \in \widetilde{\mathcal{K}}$ be a maximizer of (4.3). We know that $\int_0^T \int_{\mathbb{T}^d} K(x, \gamma^{*, \mathrm{ac}}(t, x), D\phi^*(t, x)) dxdt > -\infty$. Hence,

$$\int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x, m^*(t, x), z^*(t, x)) dxdt + \int_{\mathbb{T}^d} m^*(T, x) u_T(x) dx$$
$$= \int_0^T \int_{\mathbb{T}^d} K(x, \gamma^{*, \mathrm{ac}}(t, x), D\phi^*(t, x)) dxdt + \int_{\mathbb{T}^d} m_0(x) \phi^*(0, x) dx$$
$$\leq \int_0^T \int_{\mathbb{T}^d} m^*(t, x) \left(\gamma^{*, \mathrm{ac}}(t, x) + H(x, m^*(t, x), D\phi^*(t, x))\right) dxdt + \int_{\mathbb{T}^d} m_0(x) \phi^*(0, x) dx$$

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But from Lemma 4.2 and especially (4.5)-(4.6), we see that the latter inequality is in fact an equality; then, from the last part of Lemma 4.2, this yields that $z^*(t,x) = m^*(t,x)H_p(x,m^*(t,x),D\phi^*(t,x))$ and that

$$\widetilde{L}(x,m^*(t,x),z^*(t,x)) = \widetilde{L}(x,m^*(t,x),m^*(t,x)H_p(x,m^*(t,x),D\phi^*(t,x)))$$

almost everywhere. We have proved that $m^*H_p(\cdot,m^*,D\phi^*) \in L^1((0,T)\times\mathbb{T}^d)$ and that (5.3) holds. Moreover

$$\int_0^T \int_{\mathbb{T}^d} K(x,\gamma^{*,\mathrm{ac}}(t,x),D\phi^*(t,x))dxdt = \int_0^T \int_{\mathbb{T}^d} m^*(t,x)\left(\gamma^{*,\mathrm{ac}}(t,x) + H(x,m^*(t,x),D\phi^*(t,x))\right)dxdt.$$

This implies that

$$K(x,\gamma^{*,\mathrm{ac}}(t,x),D\phi^*(t,x)) = m^*(t,x)\left(\gamma^{*,\mathrm{ac}}(t,x) + H(x,m^*(t,x),D\phi^*(t,x))\right) \tag{5.5}$$

almost everywhere. In view of Remark 2.2, this implies that $D\phi^*(t,x) = 0$ almost everywhere in the region where $m^*(t,x) = 0$, and that

$$\gamma^{*,\mathrm{ac}}(t,x) = -H(x,m^*(t,x),D\phi^*(t,x)) - m^*(t,x)H_m(x,m^*(t,x),D\phi^*(t,x)), \qquad \text{a.e. where } m^*(t,x) > 0$$

and with the convention (5.2), we see that

$$\gamma^{*,\mathrm{ac}}(t,x) \geq -H(x,m^*(t,x),D\phi^*(t,x)) - m^*(t,x)H_m(x,m^*(t,x),D\phi^*(t,x)), \qquad \text{a.e..}$$

These observation imply that $(m^*)^{-\alpha}|D\phi^*|^\beta 1_{m^*>0} \in L^1((0,T)\times\mathbb{T}^d)$. Using also (5.5), we see that $(m^*)^{1-\alpha}|D\phi^*|^\beta \in L^1((0,T)\times\mathbb{T}^d)$. From the inequalities satisfied by $\gamma^{*,\mathrm{ac}}$, we see that (5.1) holds. Finally, (5.4) is obtained by combining all the points above. We have proved that (ϕ^*,m^*) is a weak solution in the sense of Definition 5.1.

Suppose now that $(\bar\phi,\bar m)$ is a weak solution in the sense of Definition 5.1. Let us choose $\bar\gamma(t,x) = -H(x,\bar m(t,x),D\bar\phi(t,x)) - \bar m(t,x)H_m(x,\bar m(t,x),D\bar\phi(t,x))$ always with the convention (5.2), (we split $\bar\gamma$ as follows: $\bar\gamma = \bar\gamma^1 + \bar\gamma^2$, where $\bar\gamma^1 = -\ell(\cdot,\bar m) - \bar m\frac{\partial\ell}{\partial m}(\cdot,\bar m)$, $\bar\gamma^2 = (1-\alpha)\frac{|D\bar\phi|^\beta}{\bar m^\alpha}1_{\{\bar m>0\}}$) and $\bar z(t,x) = \bar m(t,x)H_p(x,\bar m(t,x),D\bar\phi(t,x))$. It is clear that $(\bar\phi,\bar\gamma) \in \widetilde{\mathcal{K}}$ and that $(\bar m,\bar z) \in \mathcal{K}_1$.

From the definition of $\bar\gamma$, we see that a.e., $\bar m(t,x) = \mathrm{argmin}_{\mu\geq0}\left(\mu\bar\gamma(t,x) + \mu H(x,\mu,D\bar\phi(t,x))\right)$. Moreover, since a.e., $H_p(x,\bar m(t,x),D\bar\phi(t,x)) = \mathrm{argmin}_\xi\left(\xi\cdot D\bar\phi(t,x) + L(x,\bar m(t,x),\xi)\right)$, we see that a.e. $(\bar m(t,x),\bar z(t,x)) = \mathrm{argmin}_{(\mu,z)}\left(\mu\bar\gamma(t,x) + z\cdot D\bar\phi(t,x) + \widetilde{L}(x,\mu,z)\right)$, i.e.

$$K(x,\bar\gamma(t,x),D\bar\phi(t,x)) = \bar m(t,x)\bar\gamma(t,x) + \bar z(t,x)\cdot D\bar\phi(t,x) + \widetilde{L}(x,\bar m(t,x),\bar z(t,x)). \tag{5.6}$$

Note also that $0 \leq \int_0^T \int_{\mathbb{T}^d} \bar m(t,x)\bar\gamma^2(t,x)dxdt < +\infty$ and that

$$\int_0^T \int_{\mathbb{T}^d} |\bar z(t,x)\cdot D\bar\phi(t,x)|dxdt \leq C\int_0^T \int_{\mathbb{T}^d} \bar m^{1-\alpha}(t,x)|D\bar\phi(t,x)|^\beta dxdt < +\infty.$$

Let (m^*,z^*) be the solution of (2.7). From Lemma 4.2,

$$\mathcal{B}(m^*,z^*) = \int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x,m^*(t,x),z^*(t,x))dxdt + \int_{\mathbb{T}^d} m^*(T,x)u_T(x)dx$$

$$\geq \int_0^T \int_{\mathbb{T}^d} m^*(t,x)\left(\bar\gamma(t,x) + H(x,m^*(t,x),D\bar\phi(t,x))\right)dxdt + \int_{\mathbb{T}^d} m_0(x)\bar\phi(0,x)dx.$$

Hence,

$$\mathcal{B}(m^*,z^*) \geq \int_0^T \int_{\mathbb{T}^d} K(x,\bar\gamma(t,x),D\bar\phi(t,x))dxdt + \int_{\mathbb{T}^d} m_0(x)\bar\phi(0,x)dx$$

$$= \int_0^T \int_{\mathbb{T}^d} \bar m(t,x)\left(\bar\gamma(t,x) + H(x,\bar m(t,x),D\bar\phi(t,x))\right)dxdt + \int_{\mathbb{T}^d} m_0(x)\bar\phi(0,x)dx,$$

where the first line comes from (5.6) and the definition of K, and the last line comes from the definition of $\bar{\gamma}$.

Finally, from the latter inequality and (5.4), we deduce that

$$\mathcal{B}(m^*, z^*) \geq \int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x, \bar{m}(t,x), \bar{z}(t,x))dxdt + \int_{\mathbb{T}^d} \bar{m}(T,x)u_T(x)dx$$
$$= \mathcal{B}(\bar{m}, \bar{z}).$$

Therefore $(\bar{m}, \bar{z})$ achieves the minimum in (2.7) and $\bar{m} = m^*$, $\bar{z} = z^*$.

It remains to prove that $(\bar{\phi}, \bar{\gamma})$ achieves the maximum in (4.3). We deduce from Lemma 2.2, Proposition 4.1 and the latter point that

$$\max_{(\phi,\gamma)\in\mathcal{K}} J(\phi, \gamma) = \mathcal{B}(\bar{m}, \bar{z}) = \int_0^T \int_{\mathbb{T}^d} \widetilde{L}(x, \bar{m}(t,x), \bar{z}(t,x))dxdt + \int_{\mathbb{T}^d} \bar{m}(T,x)u_T(x)dx$$
$$= -\int_0^T \int_{\mathbb{T}^d} \bar{m}^2(t,x)H_m(x, \bar{m}(t,x), D\bar{\phi}(t,x))dxdt + \int_{\mathbb{T}^d} \bar{m}_0(x)\bar{\phi}(0^+, x)dx,$$

where the last line comes from (5.4) and the definition of $\bar{z}$. But, using the definition of $\bar{\gamma}$,

$$-\bar{m}^2(t,x)H_m(x, \bar{m}(t,x), D\bar{\phi}(t,x)) = \bar{m}(t,x)\bar{\gamma}(t,x) + \bar{m}(t,x)H(x, \bar{m}(t,x), D\bar{\phi}(t,x))$$
$$= \bar{m}(t,x)\bar{\gamma}(t,x) + \bar{z}(t,x) \cdot D\bar{\phi}(t,x) + \widetilde{L}(x, \bar{m}(t,x), \bar{z}(t,x))$$
$$= K(x, \bar{\gamma}(t,x), D\bar{\phi}(t,x))$$

where the second line is obtained using the definition of $\bar{z}$ and the third line is obtained using (5.6). Combining the latter two observations, we see that

$$\max_{(\phi,\gamma)\in\mathcal{K}} J(\phi, \gamma) = \int_0^T \int_{\mathbb{T}^d} K(x, \bar{\gamma}(t,x), D\bar{\phi}(t,x))dxdt + \int_{\mathbb{T}^d} \bar{m}_0(x)\bar{\phi}(0^+, x)dx = J(\bar{\phi}, \bar{\gamma}),$$

which concludes the proof. $\square$

Theorem 5.2. *There exists a unique weak solution of (1.1)-(1.3).*

Proof. Existence is a direct consequence of Theorem 5.1, Proposition 4.2 and Lemma 2.2.

From Theorem 5.1, we also see that any weak solution $(\bar{\phi}, \bar{m})$ of (1.1)-(1.3) is such that the pair $(\bar{m}, \bar{m}H_p(x, \bar{m}, D\bar{\phi}))$ is the minimizer of (2.7). Thus $(\bar{m}, \bar{m}H_p(x, \bar{m}, D\bar{\phi}))$ is unique.

Consider now two weak solutions $(\bar{\phi}_1, \bar{m})$ and $(\bar{\phi}_2, \bar{m})$ of (1.1)-(1.3). We know that $\bar{m}H_p(x, \bar{m}, D\bar{\phi}_1) = \bar{m}H_p(x, \bar{m}, D\bar{\phi}_2)$: this implies that $|D\bar{\phi}_1(t,x)|^{\beta-2}D\bar{\phi}_1(t,x) = |D\bar{\phi}_2(t,x)|^{\beta-2}D\bar{\phi}_2(t,x)$ at almost every (t,x) such that $\bar{m}(t,x) > 0$: hence at almost every (t,x) such that $\bar{m}(t,x) > 0$, $|D\bar{\phi}_1(t,x)| = |D\bar{\phi}_2(t,x)|$ and finally $D\bar{\phi}_1(t,x) = D\bar{\phi}_2(t,x)$. Moreover, $D\bar{\phi}_1(t,x) = D\bar{\phi}_2(t,x) = 0$ at almost every (t,x) such that $\bar{m}(t,x) = 0$. Therefore, $D\bar{\phi}_1(t,x) = D\bar{\phi}_2(t,x)$ at almost every (t,x). This means that $\bar{\phi}_1 - \bar{\phi}_2$ only depends on t, and that $\bar{\gamma}_1 = \bar{\gamma}_2$ a.e., where $\bar{\gamma}_i(t,x) = -H(x, \bar{m}(t,x), D\bar{\phi}_i(t,x)) - \bar{m}(t,x)H_m(x, \bar{m}(t,x), D\bar{\phi}_i(t,x))$, using convention (5.2).

Using (5.4), we deduce from the previous points that $\bar{\phi}_1(t = 0^+, \cdot) = \bar{\phi}_2(t = 0^+, \cdot)$ a.e. in $\mathbb{T}^d$.

We set $\bar{\gamma} = \bar{\gamma}_1 = \bar{\gamma}_2 \in L^1((0,T) \times \mathbb{T}^d)$ and $\bar{z} = \bar{m}H_p(x, \bar{m}, D\bar{\phi}_1) = \bar{m}H_p(x, \bar{m}, D\bar{\phi}_2)$. Going back to the proof of Lemma 4.2, we see that for almost all t, both $(\bar{\phi}_1, \bar{\gamma})$ and $(\bar{\phi}_2, \bar{\gamma})$ achieve the equality in (4.6) with $(m, z) = (\bar{m}, \bar{z})$. From the previous points, this implies that

$$\int_{\mathbb{T}^d} \bar{m}(t,x)\bar{\phi}_1(t,x)dx = \int_{\mathbb{T}^d} \bar{m}(t,x)\bar{\phi}_2(t,x)dx$$

for almost every t. Since $\bar{\phi}_1(t,x) - \bar{\phi}_2(t,x)$ only depends on t and $\bar{m}(t)$ is a probability measure, the latter implies that $\phi_1 = \phi_2$ holds almost everywhere. $\square$

Acknowledgements

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