About the possibility of minimal blow up for Navier-Stokes solutions with data in $\dot{H}^s(R^3)$

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ABOUT THE POSSIBILITY OF MINIMAL BLOW UP FOR NAVIER-STOKES SOLUTIONS WITH DATA IN $\dot{H}^s(\mathbb{R}^3)$

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Abstract. Considering initial data in $\dot{H}^s$, with $\frac{1}{2} < s < \frac{3}{2}$, this paper is devoted to the study of possible blowing-up Navier-Stokes solutions such that $(T_*(u_0)-t)^{\frac{1}{2}(s-\frac{1}{2})}\,\|u\|_{\dot{H}^s}$ is bounded. Our result is in the spirit of the tremendous works of L. Escauriaza, G. Seregin, and V. Šverák and I. Gallagher, G. Koch, F. Planchon, where they proved there is no blowing-up solution which remain bounded in $L^3(\mathbb{R}^3)$. The main idea is that if such blowing-up solutions exist, they satisfy critical properties.

1. Introduction and statement of main result

We consider the Navier-Stokes system for incompressible viscous fluids evolving in the whole space $\mathbb{R}^3$. Denoting by u the velocity, a vector field in $\mathbb{R}^3$, by p in $\mathbb{R}$ the pressure function, the Cauchy problem for the homogeneous incompressible Navier-Stokes system is given by

$$(1)\qquad \left\{\begin{array}{ccc} \partial_t u + u\cdot\nabla u - \Delta u & = & -\nabla p \\ \operatorname{div} u & = & 0 \\ u_{|t=0} & = & u_0. \end{array}\right.$$

We recall a crucial property of the Navier-Stokes equation : the scaling invariance. Let us define the operator

$$(2)\qquad \begin{array}{c} \forall \alpha\in\mathbb{R}^+,\ \forall\lambda\in\mathbb{R}^+_*,\ \forall x_0\in\mathbb{R}^3,\ \ \Lambda^{\alpha}_{\lambda,x_0}\,u(t,x) \stackrel{\text{def}}{=} \dfrac{1}{\lambda^{\alpha}}u\Big(\dfrac{t}{\lambda^2},\dfrac{x-x_0}{\lambda}\Big). \\ \text{If } \ \alpha=1, \ \text{ we note } \Lambda^1_{\lambda,x_0}=\Lambda_{\lambda,x_0}. \end{array}$$

Clearly, if u is smooth solution of Navier-Stokes system on $[0,T]\times\mathbb{R}^3$ with pressure p associated with the initial data u_0, then, for any positive λ, the vector field and the pressure

$$u_\lambda \stackrel{\text{def}}{=} \Lambda_{\lambda,x_0}\,u \quad\text{and}\quad p_\lambda \stackrel{\text{def}}{=} \Lambda^2_{\lambda,x_0}\,p$$

is a solution of Navier-Stokes system on the interval $[0,\lambda^2 T]\times\mathbb{R}^3$, associated with the initial data

$$u_{0,\lambda} = \Lambda_{\lambda,x_0}\,u_0.$$

This leads to the definition of scaling invariant space.

Definition 1.1. *A Banach space X is said to be scaling invariant (or also critical), if its norm is invariant under the scaling transformation defined by $u\mapsto u_\lambda$*

$$||u_\lambda||_X = ||u||_X.$$

Let us give some exemples of critical spaces in dimension 3

$$\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)\hookrightarrow L^3(\mathbb{R}^3)\hookrightarrow \dot{B}^{-1+\frac{3}{p}}_{p,\infty}(\mathbb{R}^3)_{p<\infty}\hookrightarrow \mathcal{BMO}^{-1}(\mathbb{R}^3)\hookrightarrow \dot{B}^{-1}_{\infty,\infty}(\mathbb{R}^3).$$

The framework of this work is functional spaces which are above the natural scaling of Navier-Stokes equations. More precisely, our statements will take place in some Sobolev and Besov spaces, with a

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regularity index s such that $\frac{1}{2} < s < \frac{3}{2}\cdot$

Notations. We shall constantly be using the following simplified notations:

$$L^\infty_T(\dot{H}^s) \stackrel{\text{def}}{=} L^\infty([0,T],\dot{H}^s) \quad \text{and} \quad L^2_T(\dot{H}^{s+1}) \stackrel{\text{def}}{=} L^2([0,T],\dot{H}^{s+1}),$$

and the relevant function space we shall be working with in the sequel is

$$X^s_T \stackrel{\text{def}}{=} L^\infty_T(\dot{H}^s) \cap L^2_T(\dot{H}^{s+1}), \quad \text{endowed with the norm} \quad \|u\|^2_{X^s_T} \stackrel{\text{def}}{=} \|u\|^2_{L^\infty_T(\dot{H}^s)} + \|u\|^2_{L^2_T(\dot{H}^{s+1})}.$$

Let us start by recalling the local existence theorem for data in the Sobolev space $\dot{H}^s$.

Theorem 1.1. *Let u_0 be in $\dot{H}^s$, with $\frac{1}{2} < s < \frac{3}{2}\cdot$ Then there exists a time T and there exists a unique solution $NS(u_0)$ such that $NS(u_0)$ belongs to $L^\infty_T(\dot{H}^s) \cap L^2_T(\dot{H}^{s+1})$.*
Moreover, denoting by $T_*(u_0)$ the maximal time of existence of such a solution, there exists a positive constant c such that

$$T_*(u_0)\,\|u_0\|^{\sigma_s}_{\dot{H}^s} \geqslant c, \quad \text{with} \quad \sigma_s \stackrel{\text{def}}{=} \frac{1}{\frac{1}{2}(s-\frac{1}{2})}\cdot \tag{3}$$

Remark 1.1. Throughout this paper, we will adopt the useful notation $NS(u_0)$ to mean the maximal solution of the Navier-Stokes system, associated with the initial data u_0. Notice that our whole work relies on the hypothesis there exists some blowing up NS-solutions, e.g some NS-solutions with a finite lifespan $T_*(u_0)$. This is still an open question.

Remark 1.2. We point out that the infimum of the quantity $T_*(u_0)\,\|u_0\|^{\sigma_s}_{\dot{H}^s}$ exists and is positive (because of the constant c). It has been proved in [27] that there exists some intial data which reach this infimum and that the set of such data is compact, up to dilations and translations.

Remark 1.3. Theorem 1.1 implies there exists a constant $c > 0$, such that

$$(T_*(u_0) - t)\,\|NS(u_0)(t)\|^{\sigma_s}_{\dot{H}^s} \geqslant c, \tag{4}$$

and thus we get in particular the blow up of the $\dot{H}^s$-norm

$$\lim_{t\to T_*(u_0)} \|NS(u_0)(t)\|^{\sigma_s}_{\dot{H}^s} = +\infty.$$

Our motivation here is to wonder if there exist some Navier-Stokes solutions which stop living in finite time (e.g $T_*(u_0) < \infty$) and which blows up at a minimal rate, namely: there exists a positive constant M such that $(T_*(u_0) - t)\,\|NS(u_0)\|^{\sigma_s}_{\dot{H}^s} \leqslant M$. In others terms,

Question: Does there exist some blowing up NS-solutions such that $(T_*(u_0) - t)\,\|NS(u_0)\|^{\sigma_s}_{\dot{H}^s} \leqslant M$?
If yes, what do they look like ?

We assume an affirmative answer and we search to characterize such solutions.

Hypothesis $\mathcal{H}$: There exist some blowing up NS-solutions such that $(T_*(u_0) - t)\,\|NS(u_0)\|^{\sigma_s}_{\dot{H}^s} \leqslant M$.

Notice that a very close question to this one is to prove that

$$\text{If} \quad T_*(u_0) < \infty, \quad \text{does} \quad \limsup_{t\to T_*(u_0)} (T_*(u_0) - t)\,\|NS(u_0)(t)\|^{\sigma_s}_{\dot{H}^s} = +\infty \quad ?$$

We underline that this question about blowing-up Navier-Stokes solutions has been highly developed in the context of critical spaces, namely $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$ and $L^3(\mathbb{R}^3)$. Indeed, L. Escauriaza, G. Seregin and

V. Šverák showed in the fundamental work [13] that any "Leray-Hopf" weak solution which remains bounded in $L^3(\mathbb{R}^3)$ can not develop a singularity in finite time. Alternatively, it means that

$$\text{If } T_*(u_0) < +\infty, \quad \text{then } \limsup_{t \to T_*(u_0)} \|NS(u_0)(t)\|_{L^3} = +\infty. \tag{5}$$

I. Gallagher, G. Koch and F. Planchon revisited the above criteria in the context of mild Navier-Stokes solutions. They proved in [16] that strong solutions which remain bounded in $L^3(\mathbb{R}^3)$, do not become singular in finite time. To perform it, they develop an alternative viewpoint : the method of "critical elements" (or "concentration-compactness"), which was introduced by C. Kenig and F. Merle to treat critical dispersive equations. Recently, same authors extend the method in [17] to prove the same result in the case of the critical Besov space $\dot{B}_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3)$, with $3 < p, q < \infty$. Notice the work of J.-Y.Chemin and F. Planchon in [12], who gives the same answer in the case of the Besov space $\dot{B}_{p,q}^{-1+\frac{3}{p}}(\mathbb{R}^3)$, with $3 < p < \infty$, $q < 3$ and with an additional regularity assumption on the data. To conclude the non-exhaustive list of blow up results, we mention the work of C. Kenig and G. Koch who carried out in [21] such a program of critical elements for solutions in the simpler case $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$. More precisely, they proved for any data u_0 belonging to the smaller critical space $\dot{H}^{\frac{1}{2}}(\mathbb{R}^3)$,

$$\text{If } T_*(u_0) < +\infty, \quad \text{then } \lim_{t \to T_*(u_0)} \|NS(u_0)(t)\|_{\dot{H}^{\frac{1}{2}}} = +\infty. \tag{6}$$

In our case (remind : we consider Sobolev spaces $\dot{H}^s(\mathbb{R}^3)$ with $\frac{1}{2} < s < \frac{3}{2}$ which are non-invariant under the natural scaling of Navier-Stokes equations), we can not expect to prove our result in the same way, because of the scaling. Indeed, a similar proof leads us to define the critical quantity $M_c^{\sigma_s}$

$$M_c^{\sigma_s} = \sup\Big\{A > 0, \ \sup_{t < T_*(u_0)} (T_*(u_0) - t) \|NS(u_0)\|_{\dot{H}^s}^{\sigma_s} \leqslant A \ \Rightarrow T_*(u_0) = +\infty\Big\}.$$

But unfortunately, such a point of view makes no sense, owing to the meaning of $(T_*(u_0) - t)$ when $T_*(u_0) = +\infty$. We have to proceed in an other way and it may be removed by defining a new object $M_c^{\sigma_s}$

$$M_c^{\sigma_s} \stackrel{\text{def}}{=} \inf_{\substack{u_0 \in \dot{H}^s \\ T_*(u_0) < \infty}} \Big\{\limsup_{t \to T_*(u_0)} (T_*(u_0) - t) \|NS(u_0)(t)\|_{\dot{H}^s}^{\sigma_s}\Big\}.$$

Clearly, (4) implies that $M_c^{\sigma_s}$ exists and is positive. As we have decided to work under hypothesis $\mathcal{H}$, *a fortiori*, this implies that $M_c^{\sigma_s}$ is finite. The definition below is the key notion of critical solution in this context.

Definition 1.2. *(Sup-critical solution)*
Let u_0 be an element in $\dot{H}^s$. We say that $u = NS(u_0)$ is a sup-critical solution if $NS(u_0)$ satisfies the two following assumptions:

$$T_*(u_0) < \infty \quad \textit{and} \quad \limsup_{t \to T_*(u_0)} (T_*(u_0) - t) \|NS(u_0)(t)\|_{\dot{H}^s}^{\sigma_s} = M_c^{\sigma_s}.$$

A natural question is to know if such elements exist. The statement given below gives an affirmative answer and provides a general procedure to build some sup-critical solutions. Our main result follows.

Theorem 1.2. *(Key Theorem)*
Let us assume that there exists u_0 in $\dot{H}^s$ and M in $\mathbb{R}_*^+$ such that

$$T_*(u_0) < \infty \quad \textit{and} \quad (T_*(u_0) - t) \|NS(u_0)(t)\|_{\dot{H}^s}^{\sigma_s} \leqslant M.$$

Then, there exists $\Phi_0 \in \dot{H}^s \cap \dot{B}_{2,\infty}^{\frac{1}{2}}$ such that $\Phi \stackrel{\text{def}}{=} NS(\Phi_0)$ is a sup-critical solution, blowing up at time 1, such that

$$\sup_{\tau < 1} (1 - \tau) \|NS(\Phi_0)(\tau)\|_{\dot{H}^s}^{\sigma_s} = \limsup_{\tau \to 1} (1 - \tau) \|NS(\Phi_0)(\tau)\|_{\dot{H}^s}^{\sigma_s} = M_c^{\sigma_s}. \tag{7}$$

In addition, there exists a positive constant C such that

$$\text{and for any} \quad \tau < 1, \quad \|NS(\Phi_0)(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \leqslant C, \tag{8}$$

where the Besov norm (for regularity index $0 < \alpha < 1$) is defined by

$$\|u\|_{\dot{B}^{\alpha}_{2,\infty}} \stackrel{\text{def}}{=} \sup_{x \in \mathbb{R}^d} \frac{\|u(\cdot - x) - u\|_{L^2}}{|x|^{\alpha}}.$$

We postpone the proof of (7) of the Key Theorem 1.2 to the next section. The proof of (8) will be given in Section 5. We stress on the fact that (8) is somewhat close to a question raised by the paper of I. Gallagher, G. Koch and F. Planchon [17], in which they prove that for any initial data in the critical Besov space $\dot{B}^{-1+\frac{3}{p}}_{p,q}$, with $3 < p, q < \infty$, the NS-solution, (the lifespan of which is assumed finite) becomes unbounded at the blow-up time. Let us say a few words about the limit case $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$. We may wonder if the result holds in the limit case $q = \infty$. As far as the author is aware, the answer is still open. Actually, if it holds, *a fortiori* it holds in the smaller space $\dot{B}^{\frac{1}{2}}_{2,\infty}$, by vertue of the embedding $\dot{B}^{\frac{1}{2}}_{2,\infty} \hookrightarrow \dot{B}^{-1+\frac{3}{p}}_{p,\infty}$. In others terms, it would mean there is no blowing-up solution, bounded in the critical space $\dot{B}^{\frac{1}{2}}_{2,\infty}$. This is related to the concern of our paper since we build some blowing-up solutions bounded in this critical space, under the assumption of blow up at minimal rate. We mention the very interesting work of H. Jia and V. Šverák [20], where they prove that -1-homogeneous initial data generate global -1-homogeneous solutions. Unfortunately, the uniqueness of such solutions is not guaranteed.

2. Existence of sup-critical solutions

The goal of this section is to give a partial proof of Key Theorem 1.2. It relies on the two Lemmas below.

Lemma 2.1. *(Existence of sup-critical solutions in $\dot{H}^s$)*
Let $(v_{0,n})_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^s$ such that

$$\tau^*(v_{0,n}) = 1 \quad \text{and} \quad \text{for any } \tau < 1, \quad (1-\tau)\,\|NS(v_{0,n})(\tau,\cdot)\|^{\sigma_s}_{\dot{H}^s} \leqslant M_c^{\sigma_s} + \varepsilon_n, \tag{9}$$

where ε_n is a generic sequence which tends to 0 when n goes to $+\infty$.
Then, there exists Ψ_0 in $\dot{H}^s$ such that $\Psi \stackrel{\text{def}}{=} NS(\Psi_0)$ is a sup-critical solution blowing up at time 1 and satisfies

$$\sup_{\tau<1}(1-\tau)\,\|NS(\Psi_0)(\tau)\|^{\sigma_s}_{\dot{H}^s} = \limsup_{\tau\to 1}(1-\tau)\,\|NS(\Psi_0)(\tau)\|^{\sigma_s}_{\dot{H}^s} = M_c^{\sigma_s}. \tag{10}$$

Moreover, the initial data of such element is a weak limit of the sequence $(v_{0,n})$ translated, e.g

$$\exists\ (x_{0,n})_{n\geqslant 0}, \quad v_{0,n}(\cdot + x_{0,n}) \rightharpoonup_{n\to+\infty} \Psi_0. \tag{11}$$

The proof of Lemma 2.1 will be the purpose of Section 4. It relies essentially on scaling argument and profile theory, which will be introduced in the next Section 3.

Lemma 2.2. *(Fluctuation estimates)*
Let $u = NS(u_0)$ be a NS-solution associated with a data $u_0 \in \dot{H}^s$, with $\frac{1}{2} < s < \frac{3}{2}$, such that

$$(T_*(u_0) - t)^{\frac{1}{\sigma_s}}\,\|NS(u_0)(t)\|_{\dot{H}^s} \leqslant M.$$

Then, the following estimates on the fluctuation part $B(u,u)(t) \stackrel{\text{def}}{=} u - e^{t\Delta}u_0$ yield

$$\text{for any} \quad s < s' < 2s - \frac{1}{2}, \quad (T_*(u_0) - t)^{\frac{1}{\sigma_{s'}}}\,\|B(u,u)(t)\|_{\dot{H}^{s'}} \leqslant F_{s'}(M^2) \tag{12}$$

Moreover, for the critical case $=\dfrac{1}{2}$, *we have*

$$\|B(u,u)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \leqslant C\, M^2. \tag{13}$$

The proof of this lemma is postpone to Section 8. It merely stems from product laws in Besov spaces, interpolation inequalities and from judicious splitting into low and high frequencies in the following sense

$$(T_* - t)2^{2j} \leqslant 1 \quad \text{and} \quad (T_* - t)2^{2j} \geqslant 1.$$

Remark 2.1. Let us point out that estimates of Lemma 2.2 do not hold if $0 < \alpha < \dfrac{1}{2}$, owing to low frequencies. Indeed, arguments similar to the ones used in the proof of Lemma 2.2 lead only to the following estimate

$$\|B(u,u)(t)\|_{\dot{B}^{\alpha}_{2,\infty}} \leqslant C\, M^2\, T_*(u_0)^{\frac{1}{2}(\alpha-\frac{1}{2})}.$$

Partial proof of Key Theorem 1.2
In all this text, we denote by (ε_n) a non increasing sequence, which tends to 0, when n tend to $+\infty$.

• Step 1 : Existence of sup-critical elements in $\dot{H}^s$, with $\dfrac{1}{2} < s < \dfrac{3}{2}$.

Let us consider the sequence $(M_c + \varepsilon_n)_{n\geqslant 0}$. By definition of M_c, there exists a sequence $(u_{0,n})$ belonging to $\dot{H}^s$, with a finite lifespan $T_*(u_{0,n})$, such that for any $t < T_*(u_{0,n})$:

$$\limsup_{t\to T_*(u_0)} (T_*(u_{0,n}) - t)\, \|NS(u_{0,n})\|^{\sigma_s}_{\dot{H}^s} \leqslant M_c^{\sigma_s} + \varepsilon_n.$$

By definition of $\limsup$, there exists a nondecreasing sequence of time t_n, converging to $T_*(u_0)$, such that

$$\forall t \geqslant t_n,\ (T_*(u_{0,n}) - t)\, \|NS(u_{0,n})(t,x)\|^{\sigma_s}_{\dot{H}^s} \leqslant M_c^{\sigma_s} + \varepsilon_n. \tag{14}$$

By rescaling, we consider the sequence

$$v_{0,n}(y) = \big(T_*(u_{0,n}) - t_n\big)^{\frac{1}{2}}\, NS(u_{0,n})\big(t_n, (T_*(u_{0,n}) - t_n)^{\frac{1}{2}}\, y\big).$$

and we have

$$\|v_{0,n}\|^{\sigma_s}_{\dot{H}^s} = \big(T_*(u_{0,n}) - t_n\big)\, \|NS(u_{0,n})(t_n)\|^{\sigma_s}_{\dot{H}^s}. \tag{15}$$

By vertue of (14), the sequence $(v_{0,n})_{n\geqslant 1}$ is bounded $\big($by $M_c^{\sigma_s} + \varepsilon_0\big)$ in the space $\dot{H}^s$. Moreover, such a sequence generates a Navier-Stokes solution, which keeps on living until the time $\tau^* = 1$ and satisfies

$$NS(v_{0,n})(\tau,y) = \big(T_*(u_{0,n}) - t_n\big)^{\frac{1}{2}}\, NS(u_{0,n})\big(t_n + \tau\, (T_*(u_{0,n}) - t_n)\, , (T_*(u_{0,n}) - t_n)^{\frac{1}{2}}\, y\big). \tag{16}$$

We introduce $\widetilde{t_n} = t_n + \tau\, \big(T_*(u_{0,n}) - t_n\big)$. Notice that, because of scaling, an easy computation yields

$$(1-\tau)\, \|NS(v_{0,n})(\tau)\|^{\sigma_s}_{\dot{H}^s} = \big(T_*(u_{0,n}) - \widetilde{t_n}\big)\, \|NS(u_{0,n})\big(\widetilde{t_n}\big)\|^{\sigma_s}_{\dot{H}^s}. \tag{17}$$

As $\widetilde{t_n} \geqslant t_n$ for any n (by definition of $\widetilde{t_n}$) we combine (17) with (14) and we get, for any $\tau \in [0,1[$,

$$(1-\tau)\|NS(v_{0,n})(\tau,x)\|^{\sigma_s}_{\dot{H}^s} \leqslant M_c^{\sigma_s} + \varepsilon_n.$$

The sequence $(v_{0,n})$ satisfies the hypothesis of Lemma 2.1. Applying it, we build a sup-critical solution $\Phi = NS(\Psi_0)$ in $\dot{H}^s$ which blows up at time 1, e.g

$$\limsup_{\tau\to 1} (1-\tau)\, \|NS(\Psi_0)(\tau)\|^{\sigma_s}_{\dot{H}^s} = M_c^{\sigma_s}.$$

This proves the first part of the statement of Theorem 1.2.

• Step 2 : Existence of sup-critical elements in $\dot{H}^s \cap \dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^{s'}$, with s and s' such that $s < s' < 2s - \dfrac{1}{2}\cdot$
This will be proved in Section 6. Notice that proving that $NS(\Psi_0)$ is bounded in the Besov space $\dot{B}^{\frac{1}{2}}_{2,\infty}$ is equivalent to prove that Ψ_0 belongs to $\dot{B}^{\frac{1}{2}}_{2,\infty}$, since, by vertue of Lemma 2.2, the fluctuation part is bounded in $\dot{B}^{\frac{1}{2}}_{2,\infty}$ and obviously we have

$$\|NS(\Psi_0)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \leqslant \|NS(\Psi_0)(t) \; - \; e^{t\Delta}\Psi_0\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \quad + \quad \|e^{t\Delta}\Psi_0\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}}.$$

The paper is structured as follows. In Section 3, we recall the main tools of this paper. Essentially, it deals with the profile theory of P. Gérard [18] and a structure lemma concerning a NS-solution associated with a sequence which satisfies hypothesis of profile theory. We also recall some basics facts on Besov spaces.
In Section 4, we are going to establish the proof of crucial Lemma 2.1, which provides the proof of the first part of Theorem 1.2 : there exists some sup-critical elements in $\dot{H}^s$. The second part of the proof is postponed in Section 6, where we build some sup-critical elements not only in $\dot{H}^s$, but also in others spaces, such as $\dot{B}^{\frac{1}{2}}_{2,\infty}$ and $\dot{B}^{s'}_{2,\infty}$, with $s < s' < 2s - \dfrac{1}{2}\cdot$ To carry out this, we need some estimates on the fluctuation part of the solution, which will be provided in Section 5.
Then in Section 7, we give an analogue sup-inf critical criteria. It turns out that among sup-critical solutions, there exists some of them which are sup-inf-critical in the sense of they reach the biggest infimum limit. Section 8 is devoted to the proof of Lemma 3.2, which gives the structure of a Navier-Stokes solution associated with a bounded sequence of data in $\dot{H}^s$. We recall to the reader that such structure result has been partially proved in [27], except for the orthogonality property of Navier-Stokes solution in $\dot{H}^s$-norm. As a result, we give the proof of such a property, after reminding the ideas of the complete proof.

3. Profile theory and Tool Box

We recall the fundamental result due to P. Gérard : the profile decomposition of a bounded sequence in the Sobolev space $\dot{H}^s$. The original motivation of this theory was the desciption, up to extractions, of the defect of compactness in Sobolev embeddings (see for instance the pionneering works of P.-L. Lions in [24], [25] and H. Brezis, J.-M. Coron in [6]. Here, we will use the theorem of P. Gérard [18], which gives, up to extractions, the structure of a bounded sequence of $\dot{H}^s$, with s between 0 and $\dfrac{3}{2}\cdot$
More precisely, the defect of compactness in the critical Sobolev embedding $\dot{H}^s \subset L^p$ is described in terms of a sum of rescaled and translated orthogonal profiles, up to a small term in L^p. For more details about the history of the profile theory, we refer the reader to the paper [27].

Theorem 3.1. *(Profile Theorem [18])*
Let $(u_{0,n})_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^s$. Then, up to an extraction:
- There exists a sequence vectors fields, called profiles $(\varphi^j)_{j\in\mathbb{N}}$ in $\dot{H}^s$.
- There exists a sequence of scales and cores $(\lambda_{n,j}, x_{n,j})_{n,j\in\mathbb{N}}$, such that, up to an extraction

$$\forall J \geqslant 0, \;\; u_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j(x) + \psi_n^J(x) \quad \textit{with} \quad \lim_{J\to+\infty} \limsup_{n\to+\infty} \|\psi_n^J\|_{L^p} = 0, \quad \textit{and} \quad p = \frac{6}{3-2s}\cdot$$

Where, $(\lambda_{n,j}, x_{n,j})_{n\in\mathbb{N},j\in\mathbb{N}^*}$ are sequences of $(\mathbb{R}^*_+ \times \mathbb{R}^3)^{\mathbb{N}}$ with the following orthogonality property: for every integers (j,k) such that $j \neq k$, we have

$$\textit{either} \lim_{n\to+\infty} \Big(\frac{\lambda_{n,j}}{\lambda_{n,k}} + \frac{\lambda_{n,k}}{\lambda_{n,j}}\Big) = +\infty \quad \textit{or} \quad \lambda_{n,j} = \lambda_{n,k} \quad \textit{and} \quad \lim_{n\to+\infty} \frac{|x_{n,j} - x_{n,k}|}{\lambda_{n,j}} = +\infty.$$

Moreover, for any $J \in \mathbb{N}$, we have the following orthogonality property

$$\|u_{0,n}\|^2_{\dot{H}^s} = \sum_{j=0}^{J} \|\varphi^j\|^2_{\dot{H}^s} + \|\psi^J_n\|^2_{\dot{H}^s} + \circ(1), \quad \text{when} \quad n \to +\infty. \tag{18}$$

Let us recall a structure Lemma, based on the crucial profils theorem of P. Gérard (see [18]). Let $(u_{0,n})$ be a bounded sequence in the Sobolev space $\dot{H}^s$, which profile decomposition is given by

$$u_{0,n}(x) = \sum_{j\in J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j(x) + \psi^J_n(x),$$

with the appropriate properties on the error term ψ^J_n. By vertue of orthogonality of scales and cores given by Theorem 3.1, we sort profiles according to their scales

$$u_{0,n}(x) = \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} \varphi^j(x - x_{n,j}) + \sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j(x) + \psi^J_n(x) \tag{19}$$

where for any $j \in \mathcal{J}_1$, for any $n \in \mathbb{N}$, $\lambda_{n,j} \equiv 1$.
Under these notations, we claim we have the following structure Lemma of the Navier-Stokes solutions, which proof will be provided in Section 8.

Lemma 3.2. *(Profile decomposition of a sequence of Navier-Stokes solutions)*
Let $(u_{0,n})_{n\geqslant 0}$ be a bounded sequence of initial data in $\dot{H}^s$ which profile decomposition is given by

$$u_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j(x) + \psi^J_n(x).$$

Then, $\liminf_{n\geqslant 0} T_*(u_{0,n}) \geqslant \widetilde{T} \stackrel{\text{def}}{=} \inf_{j\in\mathcal{J}_1} T_*(\varphi^j)$ and for any $t < T_*(u_{0,n})$, we have

$$NS(u_{0,n})(t,x) = \sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t, x - x_{n,j}) \;+\; e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j(x) + \psi^J_n(x)\Big) \;+\; R^J_n(t,x) \tag{20}$$

where the remaining term R^J_n satisfies for any $T < \tilde{T}$, $\lim_{J\to+\infty} \lim_{n\to+\infty} \|R^J_n\|_{X^s_T} = 0$.
Moreover, we have the orthogonality property on the $\dot{H}^s$-norm for any $t < \tilde{T}$

$$\|NS(u_{0,n})(t)\|^2_{\dot{H}^s} = \sum_{j\in\mathcal{J}_1} \|NS(\varphi^j)(t)\|^2_{\dot{H}^s} \;+\; \Big\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j + \psi^J_n\Big)\Big\|^2_{\dot{H}^s} + \gamma^J_n(t). \tag{21}$$

with $\lim_{J\to+\infty} \limsup_{n\to+\infty} \sup_{t'<t} |\gamma^J_n(t')| = 0.$

For the convenience of the reader, we recall the usual definition of Besov spaces. We refer the reader to [1], from page 63, for a detailed presentation of the theory and analysis of homogeneous Besov spaces.

Definition 3.1. *Let s be in $\mathbb{R}$, (p, r) in $[1, +\infty]^2$ and u in $\mathcal{S}'$. A tempered distribution u is an element of the Besov space $\dot{B}^s_{p,r}$ if u satifies $\lim_{j\to\infty} ||\dot{S}_j u||_{L^\infty} = 0$ and*

$$\|u\|_{\dot{B}^s_{p,r}} \stackrel{\text{def}}{=} \Big(\sum_{j\in\mathbb{Z}} 2^{jrs}\, ||\dot{\Delta}_j u||^r_{L^p}\Big)^{\frac{1}{r}} < \infty,$$

where $\dot{\Delta}_j$ is a frequencies localization operator (called Littlewood-Paley operator), defined by

$$\dot{\Delta}_j u(\xi) \stackrel{\text{def}}{=} \mathcal{F}^{-1}\big(\varphi(2^{-j}|\xi|)\widehat{u}(\xi)\big),$$

with $\varphi \in \mathcal{D}([\frac{1}{2}, 2])$, such that $\displaystyle\sum_{j\in\mathbb{Z}} \varphi(2^{-j}t) = 1$, for any $t > 0$.

Remark 3.1. Notice that the characterization of Besov spaces with positive indices in terms of finite differences is equivalent to the above definition (cf [1]). In the case where the regularity index is between 0 and 1, one has the following property. Let s be in $]0, 1[$ and (p, r) in $[1, \infty]^2$. A constant C exists such that, for any $u \in \mathcal{S}'$,

$$C^{-1}\,\|u\|_{\dot{B}^s_{p,r}} \leqslant \Big\| \frac{\|u(\cdot - y) - u\|_{L^p}}{|y|^s} \Big\|_{L^r(\mathbb{R}^d; \frac{dy}{|y|^d})} \leqslant C\,\|u\|_{\dot{B}^s_{p,r}}. \tag{22}$$

Remark 3.2. Notice that $\dot{H}^s \subset \dot{B}^s_{2,2}$ and both spaces coincide if $s < \dfrac{3}{2}\cdot$

We recall an interpolation property in Besov spaces, which will be useful in the sequel.

Proposition 3.3. *A constant C exists which satisifes the following property. If s_1 and s_2 are real numbers such that $s_1 < s_2$ and $\theta \in]0, 1[$, then we have for any $p \in [1, +\infty]$*

$$\|u\|_{\dot{B}^{\theta\, s_1 + (1-\theta)\, s_2}_{p,1}} \;\leqslant\; C(s_1, s_2, \theta)\, \|u\|^{\theta}_{\dot{B}^{s_1}_{p,\infty}}\ \|u\|^{1-\theta}_{\dot{B}^{s_2}_{p,\infty}}.$$

4. Application of profile theory to sup-critical solutions

This section is devoted to the proof of Lemma 2.1. The statement given below is actually a bit stronger and clearly entails Lemma 2.1. We shall prove the following proposition.

Proposition 4.1. *Let $(v_{0,n})_{n\in\mathbb{N}}$ be a bounded sequence in $\dot{H}^s$ such that*

$$\tau^*(v_{0,n}) = 1 \quad \text{and} \quad \text{for any } \tau < 1, \quad (1-\tau)\,\|NS(v_{0,n})(\tau,)\|^{\sigma_s}_{\dot{H}^s} \leqslant\; M_c^{\sigma_s} + \varepsilon_n,$$

where ε_n is a generic sequence which tends to 0 when n goes to $+\infty$.
Then, up to extractions, we get the statements below
• the profile decomposition of such a sequence of data has a unique profile φ^{j_0} with constant scale such that $NS(\varphi^{j_0})$ is a sup-critical solution which blows up at time 1, e.g

$$\limsup_{\tau\to 1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|^{\sigma_s}_{\dot{H}^s} \;=\; M_c^{\sigma_s}. \tag{23}$$

• "The limsup is actually a sup"

$$\sup_{\tau<1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|^{\sigma_s}_{\dot{H}^s} \;=\; M_c^{\sigma_s}. \tag{24}$$

Proof. Let $(v_{0,n})_{n\geqslant 1}$ be a bounded sequence in $\dot{H}^s$, satisfiying the assumptions of Proposition 4.1. Therefore, $(v_{0,n})_{n\geqslant 1}$ has the profile decomposition below

$$v_{0,n}(x) = \sum_{\substack{j\in\mathcal{J}_1 \\ j\leqslant J}} \varphi^j(x - x_{n,j}) + \sum_{\substack{j\in\mathcal{J}_1^c \\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j}, x_{n,j}} \varphi^j(x) + \psi_n^J(x). \tag{25}$$

We denote by $\tau^*_{j_0} \stackrel{\text{def}}{=} \displaystyle\inf_{j\in\mathcal{J}_1} T_*(\varphi^j)$.

- Step 1 : we start by proving by a contradiction argument that $\tau^*_{j_0} = 1$.

We have already known by vertue of Lemma 3.2, that $\tau^*_{j_0} \leqslant 1$. Assuming that $\tau^*_{j_0} < 1$, we expect a contradiction. Moreover, orthogonal Estimate (21) can be bounded from below by

$$\|NS(v_{0,n})(\tau)\|^2_{\dot{H}^s} \geqslant \|NS(\varphi^{j_0})(\tau)\|^2_{\dot{H}^s} - |\gamma^J_n(\tau)|. \tag{26}$$

On the one hand, it seems clear by assumption that for any $\tau < \tau^*_{j_0}$, we have

$$(1-\tau^*_{j_0})^{\frac{2}{\sigma_s}} \leqslant (1-\tau)^{\frac{2}{\sigma_s}}.$$

On the other hand, hypothesis on $NS(v_{0,n})$ yields

$$(1-\tau)^{\frac{2}{\sigma_s}} \, \|NS(v_{0,n})(\tau)\|^2_{\dot{H}^s} \leqslant \; M_c^2 + \varepsilon_n.$$

Therefore, from the above remarks, we get

$$\|NS(v_{0,n})(\tau)\|^2_{\dot{H}^s} \leqslant \; \frac{M_c^2+\varepsilon_n}{(1-\tau^*_{j_0})^{\frac{2}{\sigma_s}}}. \tag{27}$$

Combining the above estimate with (26), we finally get, after multiplication by the factor $(\tau^*_{j_0}-\tau)^{\frac{2}{\sigma_s}}$,

$$\frac{M_c^2+\varepsilon_n}{(1-\tau^*_{j_0})^{\frac{2}{\sigma_s}}} \; (\tau^*_{j_0}-\tau)^{\frac{2}{\sigma_s}} \geqslant (\tau^*_{j_0}-\tau)^{\frac{2}{\sigma_s}} \, \|NS(\varphi^{j_0})(\tau)\|^2_{\dot{H}^s} - (\tau^*_{j_0}-\tau)^{\frac{2}{\sigma_s}} \, |\gamma^J_n(\tau)|. \tag{28}$$

Notice that $(\tau^*_{j_0}-\tau)^{\frac{2}{\sigma_s}}$ is always less than 1, which allows us to get rid of it in front of the remaining term $|\gamma^J_n(\tau)|$. In addition, applying (4) and hypothesis on the sequence ε_n, one has

$$\frac{M_c^2+\varepsilon_0}{(1-\tau^*_{j_0})^{\frac{2}{\sigma_s}}} \; (\tau^*_{j_0}-\tau)^{\frac{2}{\sigma_s}} \geqslant c \, - \, |\gamma^J_n(\tau)|.$$

We first choose $\tau = \tau_c$ such that $\tau_c < \tau^*_{j_0}$ and $\dfrac{M_c^2+\varepsilon_0}{(1-\tau^*_{j_0})^{\frac{2}{\sigma_s}}} \; (\tau^*_{j_0}-\tau_c)^{\frac{2}{\sigma_s}} = \dfrac{c}{4}\cdot$ Then, we take J and n large enough such that $|\gamma^J_n(\tau_c)| \leqslant \dfrac{c}{2}\cdot$ Therefore, we get a contradiction, which proves that $\tau^*_{j_0} = 1$.

- Step 2 : we prove here that $NS(\varphi^{j_0})$ is a sup-critical solution in $\dot{H}^s$.

Let us come back to Inequality (26), which we multiply by the factor $(1-\tau)^{\frac{2}{\sigma_s}}$. As we have shown that $\tau^*_{j_0} = 1$, hypothesis on $NS(v_{0,n})$ implies that for any $\tau < 1$,

$$M_c^2 + \varepsilon_n \geqslant (1-\tau)^{\frac{2}{\sigma_s}} \, \|NS(\varphi^{j_0})(\tau)\|^2_{\dot{H}^s} \; - \; |\gamma^J_n(\tau)|. \tag{29}$$

Our aim is to prove that the particular profile φ^{j_0} generates a sup-critical solution. If not, it means that

$$\exists \alpha_0>0, \forall \varepsilon>0, \; \exists \tau_\varepsilon, \text{ such that } 0<(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}} \; < \varepsilon \quad \text{and} \quad (1-\tau_\varepsilon)^{\frac{2}{\sigma_s}} \, \|NS(u_{0,n})(\tau_\varepsilon)\|^2_{\dot{H}^s} \geqslant M_c^2+\alpha_0.$$

Taking the above inequality at time τ_ε, one has

$$M_c^2 + \varepsilon_n \geqslant \; M_c^2 + \alpha_0 - \; |\gamma^J_n(\tau_\varepsilon)|.$$

Moreover, assumption on the remaining term γ^J_n implies that

$$\forall \eta>0, \; \exists \widetilde{J}(\eta) \in \mathbb{N}, \; \exists N_\eta \in \mathbb{N} \text{ such that } \forall J \geqslant \widetilde{J}(\eta), \; \forall n \geqslant N_\eta, \; |\gamma^J_n(\tau_\varepsilon)| \leqslant \eta.$$

Let $\eta > 0$. For any $J \geqslant \widetilde{J}(\eta)$ and for any $n \geqslant N_\eta$, we get at time τ_ε,

$$M_c^2 \geqslant M_c^2 + \alpha_0 \; - \; \eta.$$

Now, choosing η small enough (namely $\eta = \dfrac{\alpha_0}{2}$) we get a contradiction which proves that $NS(\varphi^{j_0})$ is a sup-critical solution. This concludes the proof of step 2 and thus the point (23) is proved.

• Step 3 : let us prove the point (24) of Proposition 4.1. The proof is a straightforward adaptation of the previous one. We shall use that $NS(\varphi^{j_0})$ is a sup-critical solution:

$$\limsup_{\tau\to 1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|^{\sigma_s}_{\dot H^s} = M_c^{\sigma_s}.$$

As we always have $\sup_{\tau<1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|^{\sigma_s}_{\dot H^s} \geqslant \limsup_{\tau\to 1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|^{\sigma_s}_{\dot H^s}$, we get a first inequality : $\sup_{\tau<1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|^{\sigma_s}_{\dot H^s} \geqslant M_c^{\sigma_s}$.
According to the previous computations, we have, for any $\tau<1$,

$$M_c^2+\varepsilon_n \geqslant (1-\tau)^{\frac{2}{\sigma_s}}\,\|NS(\varphi^{j_0})(\tau)\|^2_{\dot H^s} - |\gamma_n^J(\tau)|.$$

Hypothesis on the remaining term $|\gamma_n^J|$ implies that $\sup_{\tau<1}(1-\tau)\,\|NS(\varphi^{j_0})(\tau)\|^{\sigma_s}_{\dot H^s} \leqslant M_c^{\sigma_s}$, which provides the second desired inequality. This ends up the proof of (24).

Let us recall some notation and add a few words about profiles with constant scale. Thanks to Lemma 3.2 and obvious boundaries from below we get for any $\tau<\tau^*_{j_0} \overset{\text{def}}{=} \inf_{j\in\mathcal{J}_1} T_*(\varphi^j)=1$

$$\|NS(v_{0,n})(\tau)\|^2_{\dot H^s} \geqslant \sum_{j\in\mathcal{J}_1}\|NS(\varphi^j)(\tau)\|^2_{\dot H^s} - |\gamma_n^J(\tau)|. \tag{30}$$

Among profiles with a scale equal to 1 (e.g $j\in\mathcal{J}_1$), we distinguish profiles with a lifespan equal to $\tau^*_{j_0}=1$ and profiles with a lifespan τ^*_j strictly greater than 1. In other words, we consider the set

$$\tilde{\mathcal{J}}_1 \overset{\text{def}}{=} \{j\in\mathcal{J}_1 \mid \tau^*_j=1\}.$$

Therefore, for any $\tau<1$,

$$\begin{aligned}\|NS(v_{0,n})(\tau)\|^2_{\dot H^s} \geqslant \|NS(\varphi^{j_0})(\tau)\|^2_{\dot H^s} &+ \sum_{j\in\tilde{\mathcal{J}}_1,\, j\neq j_0}\|NS(\varphi^j)(\tau)\|^2_{\dot H^s}\\ &+ \sum_{j\in\mathcal{J}_1\setminus\tilde{\mathcal{J}}_1}\|NS(\varphi^j)(\tau)\|^2_{\dot H^s} - |\gamma_n^J(\tau)|,\end{aligned}$$

which be bounded from below once again by

$$\|NS(v_{0,n})(\tau)\|^2_{\dot H^s} \geqslant \|NS(\varphi^{j_0})(\tau)\|^2_{\dot H^s} + \sum_{j\in\tilde{\mathcal{J}}_1,\, j\neq j_0}\|NS(\varphi^j)(\tau)\|^2_{\dot H^s} - |\gamma_n^J(\tau)|, \tag{31}$$

since obviously the term $\sum_{j\in\mathcal{J}_1\setminus\tilde{\mathcal{J}}_1}\|NS(\varphi^j)(\tau)\|^2_{\dot H^s}$ is positive.

• Step 4 : in order to complete the proof of Lemma 2.1, we have to prove that there exists a unique profile with a lifespan $\tau^*_{j_0}=1$, namely $|\tilde{\mathcal{J}}_1|=1$. Once again, we assume that there exists at least two profiles in $\tilde{\mathcal{J}}_1$. We expect a contraction. Arguments of the proof are similar to the ones used in the step 2. We shall use the fact $(1-\tau)^{\frac{2}{\sigma_s}}\,\|NS(\varphi^j)(\tau)\|^2_{\dot H^s}$ can not be small as we want, by vertue of (4). Indeed, let us come back to Inequality (31). We have already proved that φ^{j_0} generates a sup-critical solution, blowing up at time 1. It means that for any $\varepsilon>0$, there exists a time τ_ε such that

$$0<(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}<\varepsilon \quad\text{and}\quad M_c^2-\varepsilon \leqslant (1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\,\|NS(\varphi^{j_0})(\tau_\varepsilon)\|^2_{\dot H^s} \leqslant M_c^2+\varepsilon.$$

Therefore, Inequality (31) becomes at time τ_ε

$$M_c^2+\varepsilon_n \geqslant M_c^2-\varepsilon + \sum_{j\in\tilde{\mathcal{J}}_1, j\neq j_0}(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\,\|NS(\varphi^j)(\tau_\varepsilon)\|^2_{\dot H^s} - |\gamma_n^J(\tau_\varepsilon)|. \tag{32}$$

By vertue of (4), there exists a universal constant $c > 0$ such that for any $j \in \tilde{\mathcal{J}}_1$ and $j \neq j_0$

$$(1-\tau)^{\frac{2}{\sigma_s}} \|NS(\varphi^j)(\tau)\|^2_{\dot H^s} \geqslant c^2. \tag{33}$$

As a result, taking the limit for n and J large enough, we infer that (still under the hypothesis $|\tilde{\mathcal{J}}_1| > 1$)

$$M_c^2 \geqslant M_c^2 - \varepsilon + (|\tilde{\mathcal{J}}_1| - 1)\, c^2 - \eta. \tag{34}$$

Choosing ε small enough, we get a contradiction and as a consequence, $|\tilde{\mathcal{J}}_1| = 1$. It means there exists a unique profile generating a sub-critical solution, blowing up at time 1. This completes the proof of Proposition 4.1, and thus the proof of Lemma 2.1. □

5. Fluctuation estimates in Besov spaces

This section is devoted to the proof of Lemma 2.2. We shall prove some estimates on the fluctuation part which is given by the bilinear form

$$B(u,u)(t) \stackrel{\text{def}}{=} NS(u_0)(t) - e^{t\Delta}u_0 = u - e^{t\Delta}u_0.$$

We distinguish the case $\dot B^{\frac12}_{2,\infty}$ from the case $\dot B^{s'}_{2,\infty}$, even if proves ideas are similar : we cut-off according low and high frequencies in the following sense :

$$(T_* - t)2^{2j} \leqslant 1 \quad \text{and} \quad (T_* - t)2^{2j} \geqslant 1.$$

Concerning high frequencies, we shall use the regularization effet of the Laplacian. Let us start by proving the critical part of Lemma 2.2.

Lemma 5.1. *Let $\dfrac12 < s < \dfrac32$ and $u_0 \in \dot H^s$. It exists a positive constant C_s such that*

$$\text{If} \quad T_*(u_0) < \infty \quad \text{and} \quad M_u \stackrel{\text{def}}{=} (T_*(u_0) - t)\, \|NS(u_0)(t)\|^{\sigma_s}_{\dot H^s} < \infty,$$

then, we have

$$\|u - e^{t\Delta}u_0\|_{\dot B^{\frac12}_{2,\infty}} < C_s\, M_u^2.$$

Proof. Duhamel formula gives

$$u - e^{t\Delta}u_0 \stackrel{\text{def}}{=} B(u,u) = -\int_0^t e^{(t-t')\Delta}\, \mathbb{P}(\operatorname{div}(u \otimes u)\, dt'. \tag{35}$$

By vertue of classsical estimates on the heat term (see for instance Lemma 2.4 in [1]), we have

$$\|\Delta_j e^{t\Delta}\, a\|_{L^2} \leqslant C\, e^{-ct\, 2^{2j}}\, \|\Delta_j a\|_{L^2}. \tag{36}$$

Therefore, the fluctuation part becomes

$$\begin{aligned} \|\Delta_j B(u,u)(t)\|_{L^2} &\lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\, 2^j\, \|\Delta_j(u\otimes u)(t')\|_{L^2}\, dt' \\ &\lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\, 2^j\, 2^{-j(2s-\frac32)}\, \|u \otimes u(t')\|_{\dot B^{2s-\frac32}_{2,\infty}}\, dt'. \end{aligned} \tag{37}$$

We infer thus, thanks to the product laws in Sobolev spaces

$$2^{\frac{j}{2}}\, \|\Delta_j B(u,u)(t)\|_{L^2} \lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\, 2^{j(3-2s)}\, \|u(t')\|^2_{\dot H^s}\, dt'. \tag{38}$$

By hypothesis, we have supposed that

$$M_u^2 \stackrel{\text{def}}{=} (T_*(u_0) - t)\, \|NS(u_0)(t)\|^{\sigma_s}_{\dot H^s} < \infty.$$

As a result,

$$
\begin{aligned}
2^{\frac{j}{2}}\,\|\Delta_j B(u,u)(t)\|_{L^2} &\leqslant C_s \int_0^t e^{-c(t-t')\,2^{2j}}\,2^{j(3-2s)}\,\frac{M_u^2}{(T_*(u_0)-t')^{\frac{2}{\sigma_s}}}\\
&= \int_0^t 1_{\{(T_*(u_0)-t')2^{2j}\leqslant 1\}}\,e^{-c(t-t')\,2^{2j}}\,2^{j(3-2s)}\,\frac{M_u^2}{(T_*(u_0)-t')^{\frac{2}{\sigma_s}}}\,dt'\\
&\quad+\int_0^t 1_{\{(T_*(u_0)-t')2^{2j}\geqslant 1\}}\,e^{-c(t-t')\,2^{2j}}\,2^{j(3-2s)}\,\frac{M_u^2}{(T_*(u_0)-t')^{\frac{2}{\sigma_s}}}\,dt'.
\end{aligned}
\tag{39}
$$

We apply Young inequality : in the first integral, we consider $L^\infty \star L^1$, whereas in the second one, we consider $L^1 \star L^\infty$ in order to use the regularization effect of the Laplacian.

(40)

$$
2^{\frac{j}{2}}\,\|\Delta_j B(u,u)(t)\|_{L^2} \leqslant C_s\,M_u^2 \int_{T_*(u_0)-2^{-2j}}^{T_*(u_0)} \frac{2^{j(3-2s)}\,dt'}{(T_*(u_0)-t')^{\frac{2}{\sigma_s}}} + C_s\,M_u^2 \int_0^t e^{-c(t-t')\,2^{2j}}\,2^{j(3-2s)}\,2^{2j(s-\frac12)}\,dt'.
$$

We recall that $\dfrac{2}{\sigma_s} \overset{\text{def}}{=} s-\dfrac12$ and $s-\dfrac12<1$. As a result,

$$
2^{\frac{j}{2}}\,\|\Delta_j B(u,u)(t)\|_{L^2} \leqslant C_s\,M_u^2\left(2^{j(2s-3)}\,2^{j(3-2s)} + \frac{1}{2^{2j}}\,2^{j(3-2s)}\,2^{2j(s-\frac12)}\right) \lesssim C_s\,M_u^2.
\tag{41}
$$

This concludes the proof on the fluctuation estimate in the critical case. □

The statement given below is a bit more general than the one of Lemma 2.2, which we deduce immediately by an interpoaltion argument (the same as given at the end of the proof of Theorem 1.2).

Lemma 5.2. *Let* $\dfrac12 < s < \dfrac32$ *and* $u_0 \in \dot H^s$. *It exists a positive constant* C_s *such that*

$$
\text{If}\quad T_*(u_0)<\infty \quad\text{and}\quad M_u \overset{\text{def}}{=} (T_*(u_0)-t)\,\|NS(u_0)(t)\|_{\dot H^s}^{\sigma_s}<\infty,
$$

then, we have for any $s<s'<2s-\dfrac12$

$$
(T_*(u_0)-t)^{\frac12(s'-\frac12)}\,\|u(t)-e^{t\Delta}u_0\|_{\dot B^{s'}_{2,\infty}}<\infty.
$$

Proof. Same arguments as above yield

$$
\|\Delta_j B(u,u)(t)\|_{L^2} \lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\,2^j\,2^{-j(2s-\frac32)}\,\|u\otimes u(t')\|_{\dot B^{2s-\frac32}_{2,\infty}}\,dt'.
\tag{42}
$$

Product laws in Sobolev spaces and hypothesis on u imply

$$
\begin{aligned}
2^{js'}\,\|\Delta_j B(u,u)(t)\|_{L^2} &\lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\,2^{j(\frac52-2s+s')}\,\|u(t')\|_{\dot H^s}^2\,dt'\\
&\lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\,2^{j(\frac52-2s+s')}\,\frac{C}{(T_*(u_0)-t')^{s-\frac12}}.
\end{aligned}
\tag{43}
$$

We split (the same cut off as before) according low and high frequencies. Concerning high frequencies, since $T_*(u_0)-t\leqslant T_*(u_0)-t'$, we get

$$
\begin{aligned}
2^{js'}\,\|\Delta_j B(u,u)(t)\,1_{\{(T_*-t)2^{2j}\geqslant 1\}}\|_{L^2} &\lesssim \int_0^t e^{-c(t-t')\,2^{2j}}\,2^{j(\frac52-2s+s')}\,\frac{C}{(T_*(u_0)-t)^{s-\frac12}}\,dt'\\
&\lesssim 2^{j(\frac12-2s+s')}\,\frac{C}{(T_*(u_0)-t)^{s-\frac12}}.
\end{aligned}
\tag{44}
$$

Choosing s' such that $\dfrac{1}{2} - 2s + s' < 0$, we get

$$2^{js'} \, \|\Delta_j B(u,u)(t) \, 1_{\{(T_*-t)2^{2j} \geqslant 1\}}\|_{L^2} \lesssim C \, \frac{(T_*(u_0)-t)^{\frac{1}{2}(-\frac{1}{2}+2s-s')}}{(T_*(u_0)-t)^{s-\frac{1}{2}}} = C \, (T_*(u_0)-t)^{-\frac{1}{2}(s'-\frac{1}{2})},$$

which yields the desired estimate, as far as high frequencies are concerned.
Concerning low frequencies, let us come back to the very beginning.

$$(45) \qquad \begin{aligned} 2^{js'} \, \|\Delta_j B(u,u)(t) \, 1_{\{(T_*(u_0)-t)2^{2j} \leqslant 1\}}\|_{L^2} &\lesssim 2^{j(s'-s)} \, 2^{js} \, \|\Delta_j B(u,u)\|_{L^2} \\ &\lesssim 2^{j(s'-s)} \, \|u(t) - e^{t\Delta} u_0\|_{\dot{B}^s_{2,\infty}}. \end{aligned}$$

As $\|u(t) - e^{t\Delta}u_0\|_{\dot{B}^s_{2,\infty}} \leqslant \dfrac{C}{(T_*(u_0)-t)^{\frac{1}{2}(s-\frac{1}{2})}}$, we infer that

$$2^{js'} \, \|\Delta_j B(u,u)(t) \, 1_{\{(T_*(u_0)-t)2^{2j} \leqslant 1\}} \, \|_{L^2} \lesssim 2^{j(s'-s)} \, \frac{C}{(T_*(u_0)-t)^{\frac{1}{2}(s-\frac{1}{2})}}.$$

Hypothesis of low frequencies implies

$$2^{js'} \, \|\Delta_j B(u,u)(t) \, 1_{\{(T_*(u_0)-t)2^{2j} \leqslant 1\}} \, \|_{L^2} \lesssim \frac{C}{(T_*(u_0)-t)^{\frac{1}{2}(s-\frac{1}{2})+\frac{1}{2}(s'-s)}} = \frac{C}{(T_*(u_0)-t)^{\frac{1}{2}(s'-\frac{1}{2})}}.$$

which ends up the proof for low frequency part. The proof of Lemma 5.2 is thus complete. $\square$

6. Existence of sup-critical solutions bounded in $\dot{B}^{\frac{1}{2}}_{2,\infty}$

This section is devoted to complete the proof of Theorem 1.2, namely the part concerning the $\dot{B}^{\frac{1}{2}}_{2,\infty}$-norm of the sup-critical solutions. We have already built some sup-critical elements in the space $\dot{H}^s$. It turns out that, starting from this statement, we shall prove that data generating a sup-critical element are not only in $\dot{H}^s$, but also in some others spaces such as $\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{B}^{s'}_{2,\infty}$, with s' satisfiying the condition given below, which stems from the proof of Lemma 2.2.
The statement given below is actually a bit stronger than the one we want to prove, since we are going to catch some sup-critical solutions not only in $\dot{B}^{\frac{1}{2}}_{2,\infty}$ (as claimed by Theorem 1.2) but also in $\dot{B}^{s'}_{2,\infty}$. The main idea to get such information on the regularity is to focus on the fluctuation part which is more regular than the solution itself. Notice that, in all this section, we use regularity index s' satisfying

$$s < s' < 2s - \frac{1}{2}.$$

Theorem 6.1. *There exists a data $\Phi_0 \in \dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^s \cap \dot{B}^{s'}_{2,\infty}$, such that $T_*(\Phi_0) < \infty$ and*

$$\sup_{t < T_*(\Phi_0)} (T_*(\Phi_0) - t) \, \|NS(\Phi_0)(t)\|^{\sigma_s}_{\dot{H}^s} = \limsup_{t \to T_*(\Phi_0)} (T_*(\Phi_0) - t) \, \|NS(\Phi_0)\|^{\sigma_s}_{\dot{H}^s} = M_c^{\sigma_s},$$

$$\textit{and for any} \quad t < T_*(\Phi_0), \quad \|NS(\Phi_0)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} < \infty.$$

Proof. The idea of the proof is to start with the existence of sup-sup-critical elements in $\dot{H}^s$. Indeed, we have proved previously that there exists a data $\Psi_0 \in \dot{H}^s$, such that $\Psi \stackrel{\text{def}}{=} NS(\Psi_0)$ is sup-critical. Therefore, by definition of lim sup, there exists a sequence $t_n \nearrow T_*(\Psi_0)$ such that

$$\lim_{n \to +\infty} (T_*(\Psi_0) - t_n) \, \|NS(\Psi_0)(t_n)\|^{\sigma_s}_{\dot{H}^s} = M_c^{\sigma_s}.$$

Let us introduce as before the rescaled sequence

$$v_{0,n}(y) = \big(T_*(\Psi_0) - t_n\big)^{\frac{1}{2}} \, NS(\Psi_0)(t_n, \big(T_*(\Psi_0) - t_n\big)^{\frac{1}{2}} y).$$

Such a sequence generates a solution which keeps on living until the time 1 and satisfies

$$\|v_{0,n}\|_{\dot{H}^s}^{\sigma_s} = \big(T_*(\Psi_0) - t_n\big)\,\|NS(\Psi_{0,n})(t_n)\|_{\dot{H}^s}^{\sigma_s}. \tag{46}$$

In the sake of simplicity, we note

$$\tau_n \stackrel{\text{def}}{=} T_*(\Psi_0) - t_n.$$

Previous computations imply that $(v_{0,n})$ is a bounded sequence of $\dot{H}^s$. Now, inspired by the idea of Y. Meyer (fluctuation-tendancy method, [26]), we decomposed the sequence $(v_{0,n})$ into

$$v_{0,n}(y) \stackrel{\text{def}}{=} v_{0,n}(y) - \tau_n^{\frac{1}{2}}\, e^{t_n\Delta}\Psi_0(\tau_n^{\frac{1}{2}}\, y) \; + \; \tau_n^{\frac{1}{2}}\, e^{t_n\Delta}\Psi_0(\tau_n^{\frac{1}{2}}\, y), \tag{47}$$

where we have

$$v_{0,n}(y) \stackrel{\text{def}}{=} \tau_n^{\frac{1}{2}}\, NS(\Psi_0)(t_n, \tau_n^{\frac{1}{2}}\, y)$$

It follows

$$v_{0,n}(y) \stackrel{\text{def}}{=} \tau_n^{\frac{1}{2}}\, \underbrace{\Big(NS(\Psi_0)(t_n,\cdot\,) - e^{t_n\Delta}\Psi_0\Big)}_{B(\Psi,\Psi)(t_n) = \text{fluctuation part}}(\tau_n^{\frac{1}{2}}\, y) \; + \; \tau_n^{\frac{1}{2}} \underbrace{e^{t_n\Delta}\Psi_0}_{\text{tendancy part}} (\tau_n^{\frac{1}{2}}\, y). \tag{48}$$

Lemma 6.2. *The rescaled fluctuation part* $\phi_n \stackrel{\text{def}}{=} \tau_n^{\frac{1}{2}}\, B(\Psi,\Psi)(t_n, \tau_n^{\frac{1}{2}}\,\cdot\,)$ *is bounded in* $\dot{H}^s \cap \dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{B}^{s'}_{2,\infty}$.

Proof. Indeed, concerning the $\dot{B}^{\frac{1}{2}}_{2,\infty}$-norm, we use firstly the scaling invariance of this norm and then we apply Lemma 2.2, which gives

$$\sup_n \|\phi_n\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} = \sup_n \|NS(\Psi_0)(t_n,\cdot\,) - e^{t_n\Delta}\Psi_0\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} < \infty. \tag{49}$$

Concerning the $\dot{H}^s$-norm, we apply successively the following arguments : scaling, triangular inequality and the fact that $NS(\Psi_0)$ is a sup-critical element in $\dot{H}^s$.

$$\begin{aligned} \|\phi_n\|_{\dot{H}^s}^{\sigma_s} &= \tau_n\, \|NS(\Psi_0)(t_n,\cdot\,) - e^{t_n\Delta}\Psi_0\|_{\dot{H}^s}^{\sigma_s} \\ &\lesssim \tau_n\, \|NS(\Psi_0)(t_n,\cdot\,)\|_{\dot{H}^s}^{\sigma_s} + \tau_n\, \|e^{t_n\Delta}\Psi_0\|_{\dot{H}^s}^{\sigma_s} \\ &\lesssim \Big(M_c + \frac{1}{n}\Big)^{\sigma_s} + \tau_n\, \|\Psi_0\|_{\dot{H}^s}^{\sigma_s} < \infty. \end{aligned} \tag{50}$$

Therefore, $\sup_n \|\phi_n\|_{\dot{H}^s}^{\sigma_s} < \infty$.

Concerning the $\dot{B}^{s'}_{2,\infty}$-norm, scaling argument combinig with Lemma 2.2 yields

$$\|\phi_n\|^{\sigma_{s'}}_{\dot{B}^{s'}_{2,\infty}} = \tau_n\, \|NS(\Psi_0)(t_n,\cdot\,) - e^{t_n\Delta}\Psi_0\|^{\sigma_{s'}}_{\dot{B}^{s'}_{2,\infty}}. \tag{51}$$

This concludes the proof of this Lemma 6.2. □

By vertue of profile theory, we perform a profile decomposition of the sequence ϕ_n in the Sobolev space $\dot{H}^s$. But in this decomposition, there is only left profiles with constant scale, as Lemma below will prove it. The idea is clear. As ϕ_n is bounded in the Besov space $\dot{H}^s \cap \dot{B}^{\frac{1}{2}}_{2,\infty}$, big scales vanish. Likewise, the fact that ϕ_n is bounded in the Besov space $\dot{H}^s \cap \dot{B}^{s'}_{2,\infty}$ implies that small scales vanish. That is the point in the Lemma below.

Lemma 6.3. • *If* (f_n) *is a bounded sequence in* $\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^s$ *and if* $\limsup_{n\to+\infty} \|f_n\|_{\dot{B}^s_{2,\infty}} = L > 0$*, then there is no big scales in the profile decomposition of the sequence* f_n *in* $\dot{H}^s$*.*

• *If* (f_n) *is a bounded sequence in* $\dot{B}^{s'}_{2,\infty} \cap \dot{H}^s$*, with* $s' > s > \dfrac{1}{2}$ *and if* $\limsup_{n\to+\infty} \|f_n\|_{\dot{B}^s_{2,\infty}} = L > 0$*, then there is no small scales in the profile decomposition of the sequence* f_n *in* $\dot{H}^s$*.*

Proof. We only proof the first part of the Lemma. The other one is similar. If $\limsup_{n\to+\infty}\|f_n\|_{\dot B^s_{2,\infty}}=L>0$, it means there exists an extraction $\varphi(n)$ such that $\|f_{\varphi(n)}\|_{\dot B^s_{2,\infty}}\geqslant\frac{L}{2}\cdot$ Otherwise, for any subsequence of (f_n), we would have

$$\|f_{\varphi(n)}\|_{\dot B^s_{2,\infty}}<\frac{L}{2}\quad\text{and thus,}\quad\lim_{n\to+\infty}\|f_{\varphi(n)}\|_{\dot B^s_{2,\infty}}\leqslant\frac{L}{2}\cdot$$

As a result, we would have $\limsup_{n\to+\infty}\|f_n\|_{\dot B^s_{2,\infty}}\leqslant\frac{L}{2}<L$, which is wrong by hypothesis. Moreover, by definition of the Besov norm, we can find a sequence $(k_n)_{n\in\mathbb{Z}}$, such that

$$\lim_{n\to+\infty}2^{k_ns}\|\Delta_{k_n}f_{\varphi(n)}\|_{L^2}=\|f_{\varphi(n)}\|_{\dot B^s_{2,\infty}}.\tag{52}$$

Therefore, $\lim_{n\to+\infty}2^{k_ns}\|\Delta_{k_n}f_{\varphi(n)}\|_{L^2}\geqslant\frac{L}{2}$.

Let us introduce the scale $\lambda_n\stackrel{\text{def}}{=}2^{-k_n}$. As (up to extraction) $2^{k_ns}\|\Delta_{k_n}f_{\varphi(n)}\|_{L^2}\geqslant\frac{L}{2}$, then one has

$$2^{k_n(s-\frac12)}\,\|f_{\varphi(n)}\|_{\dot B^{\frac12}_{2,\infty}}\geqslant\frac{L}{2}\cdot$$

Hence, the infimum limit of the sequence k_n is not $-\infty$, otherwise, the term $2^{k_n(s-\frac12)}$ would tend to 0 and thus $L=0$ (since the sequence $\|f_{\varphi(n)}\|_{\dot B^{\frac12}_{2,\infty}}$ is bounded by hypothesis), which is false by hypothesis. Therefore, $\lambda_n\nrightarrow+\infty$: big scales are excluded from the profile decomposition of the sequence f_n. This concludes the proof of Lemma 6.3. □

Continuation of the proof of Theorem 6.1.

Let us come back to the proof of sup-critical element in the Besov space $\dot B^{\frac12}_{2,\infty}\cap\dot B^{s'}_{2,\infty}$. Firstly, we check that ϕ_n satisfies hypothesis of Lemma 6.3. As it was already checked previously, ϕ_n is bounded in $\dot B^{\frac12}_{2,\infty}\cap\dot H^s\cap\dot B^{s'}_{2,\infty}$. Concerning assumption $\limsup_{n\to+\infty}\|\phi_n\|_{\dot B^s_{2,\infty}}>0$, by scaling argument, one has

$$\begin{aligned}\|\phi_n\|^{\sigma_s}_{\dot B^s_{2,\infty}}&=\tau_n\,\|NS(\Psi_0)(t_n,\cdot)-e^{t_n\Delta}\Psi_0\|^{\sigma_s}_{\dot B^s_{2,\infty}}=(T_*(\Psi_0)-t_n)\|NS(\Psi_0)(t_n,\cdot)-e^{t_n\Delta}\Psi_0\|^{\sigma_s}_{\dot B^s_{2,\infty}}\\&\geqslant(T_*(\Psi_0)-t_n)\|NS(\Psi_0)(t_n,\cdot)\|^{\sigma_s}_{\dot B^s_{2,\infty}}\;-\;(T_*(\Psi_0)-t_n)\|\Psi_0\|^{\sigma_s}_{\dot H^s}.\end{aligned}\tag{53}$$

Obviously, the term $(T_*(\Psi_0)-t_n)\|\Psi_0\|^{\sigma_s}_{\dot H^s}$ tends to 0 when n goes to $+\infty$. By vertue of (4) and [23], there exists a constant $c>0$ such that $(T_*(\Psi_0)-t_n)\|NS(\Psi_0)(t_n,\cdot)\|^{\sigma_s}_{\dot B^s_{2,\infty}}\geqslant c$. Therefore,

$$\limsup_{n\to+\infty}\|\phi_n\|_{\dot B^s_{2,\infty}}>0$$

and thus profile decomposition of ϕ_n in the space $\dot H^s$ is reduced to (with notations of Theorem 3.1)

$$\phi_n=\sum_{j\geqslant0}^{J}V^j(\cdot-x_{n,j})\,+\,r_n^J.\tag{54}$$

Moreover, as the sequence ϕ_n is bounded in $\dot B^{\frac12}_{2,\infty}\cap\dot B^{s'}_{2,\infty}$, profiles V^j belong also to $\dot B^{\frac12}_{2,\infty}\cap\dot B^{s'}_{2,\infty}$. That's the crucial point in the proof. Indeed, each profile V^j can be seen as a translated (by $x_{n,j}$) weak limit of the sequence ϕ_n. As a result, we get immediately

$$\|V^j\|_{\dot B^{\frac12}_{2,\infty}}\leqslant\liminf_{n\to+\infty}\|\phi_n\|_{\dot B^{\frac12}_{2,\infty}}<\infty\quad\text{and}\quad\|V^j\|_{\dot B^{s'}_{2,\infty}}\leqslant\liminf_{n\to+\infty}\|\phi_n\|_{\dot B^{s'}_{2,\infty}}<\infty.$$

Let us come back to the sequence $(v_{0,n})$ defined by

$$v_{0,n}\stackrel{\text{def}}{=}\phi_n\;+\;\tau_n^{\frac12}\;e^{t_n\Delta}\Psi_0(\tau_n^{\frac12}\,\cdot).$$

As it has been already underlined previously, the term $\gamma_n \stackrel{\text{def}}{=} \tau_n^{\frac{1}{2}}\, e^{t_n\Delta}\Psi_0(\tau_n^{\frac{1}{2}}\,\cdot)$ tends to 0 in $\dot{H}^s$-norm (and thus in L^p-norm, by Sobolev embedding) since

$$\|\tau_n^{\frac{1}{2}}\, e^{t_n\Delta}\Psi_0(\tau_n^{\frac{1}{2}}\,\cdot)\|_{\dot{H}^s}^{\sigma_s} = \tau_n\,\|\, e^{t_n\Delta}\Psi_0\|_{\dot{H}^s}^{\sigma_s} \leqslant \tau_n\,\|\,\Psi_0\|_{\dot{H}^s}^{\sigma_s}. \tag{55}$$

Combining the profile decomposition of (ϕ_n) with the definition of $(v_{0,n})$, we finally get

$$v_{0,n} = \sum_{j\geqslant 0}^{J} V^j(\cdot - x_{n,j}) \,+\, r_n^J \,+\, \gamma_n,$$

with $\lim_{J\to+\infty}\limsup_{n\to+\infty}\|r_n^J\|_{L^p} = 0$ and $\lim_{n\to+\infty}\|\gamma_n\|_{L^p} = 0$. By vertue of Lemma 3.2, one has for any $\tau < 1$

$$NS(v_{0,n})(\tau) = \sum_{j\geqslant 0}^{J} NS(V^j)(\tau, \cdot - x_{n,j}) + e^{\tau\Delta}(r_n^J + \gamma_n) + R_n^J(\tau).$$

By definition of the sequence $(v_{0,n})$, $NS(v_{0,n})$ is given by

$$NS(v_{0,n})(\tau,\cdot) = \big(T_*(\Psi_0) - t_n\big)^{\frac{1}{2}}\, NS(\Psi_0)\big(t_n + \tau\,\big(T_*(\Psi_0) - t_n\big)\,,\, \big(T_*(\Psi_0) - t_n\big)^{\frac{1}{2}}\cdot\big).$$

Once again, we denote $\widetilde{t_n} = t_n + \tau\,\big(T_*(\Psi_0) - t_n\big)$ and one has

$$(1-\tau)\,\|NS(v_{0,n})(\tau,\cdot)\|_{\dot{H}^s}^{\sigma_s} = \big(T_*(\Psi_0) - \widetilde{t_n}\big)\,\|NS(\Psi_0)\big(\widetilde{t_n},\cdot\big)\|_{\dot{H}^s}^{\sigma_s}.$$

As $\widetilde{t_n} \geqslant t_n$ for any n, we get

$$(1-\tau)\|NS(v_{0,n})(\tau)\|_{\dot{H}^s}^{\sigma_s} = (T_*(\Psi_0) - \widetilde{t_n})\|NS(\Psi_0)(\widetilde{t_n})\|_{\dot{H}^s}^{\sigma_s} \leqslant M_c^{\sigma_s} + \frac{2}{n}\cdot$$

Hence, Proposition 4.1 implies there exists some a unique profile Φ_0 in $\dot{B}^{\frac{1}{2}}_{2,\infty}\,\cap\,\dot{H}^s\,\cap\dot{B}^{s'}_{2,\infty}$ such that the NS-solution genrated by this profile is a sup-critical solution. As Φ_0 belongs to $\dot{B}^{\frac{1}{2}}_{2,\infty}$, Lemma 2.2 implies that $NS(\Phi_0)$ is bounded in the same space. This ends up the proof of Theorem 6.1.
Hence, we claim that the proof of Theorem 1.2 is over. Indeed, this stems from an interpolation argument. By vertue of Proposition 3.3, we have for any $s < s_1 < s'$

$$\|\Phi_0\|_{\dot{H}^{s_1}} \leqslant \|\Phi_0\|_{\dot{B}^{s_1}_{2,1}} \leqslant \|\Phi_0\|^{\theta}_{\dot{B}^{s}_{2,\infty}}\,\|\Phi_0\|^{1-\theta}_{\dot{B}^{s'}_{2,\infty}} \leqslant \|\Phi_0\|^{\theta}_{\dot{H}^{s}}\,\|\Phi_0\|^{1-\theta}_{\dot{B}^{s'}_{2,\infty}}. \tag{56}$$

This concludes the proof of Theorem 1.2. □

7. Another notion of critical solution

In this section, we wonder if among sup-critical solutions, we can find some of them which reach the biggest infimum limit of the quantity $(T_*(u_0) - t)\,\|NS(u_0)(t)\|_{\dot{H}^s}^{\sigma_s}$. We define the following set $\mathcal{E}_c$ by

$$\begin{aligned}\mathcal{E}_c \stackrel{\text{def}}{=}\Big\{ & u_0 \in \dot{B}^{\frac{1}{2}}_{2,\infty}\cap\dot{H}^s\cap\dot{B}^{s'}_{2,\infty} \quad \text{such that } T_*(u_0) < \infty\ ;\\ & \sup_{t<T_*(u_0)}\ (T_*(u_0) - t)\,\|NS(u_0)(t)\|_{\dot{H}^s}^{\sigma_s} = \limsup_{t\to T_*(u_0)}(T_*(u_0) - t)\,\|NS(u_0)(t)\|_{\dot{H}^s}^{\sigma_s} = M_c^{\sigma_s}\ ;\\ & \text{for any}\quad t < T_*(u_0),\quad \|NS(u_0)(t)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} < \infty \quad\text{and}\quad (T_*(u_0) - t)\,\|NS(u_0)(t)\|^{\sigma_{s'}}_{\dot{B}^{s'}_{2,\infty}} < \infty\Big\}.\end{aligned}$$

Let us introduce the following quantity $m_c^{\sigma_s}$

$$m_c^{\sigma_s} \stackrel{\text{def}}{=} \sup_{u_0\,\in\,\mathcal{E}_c}\big\{\liminf_{t\to T_*(u_0)}(T_*(u_0) - t)\,\|NS(u_0)(t)\|_{\dot{H}^s}^{\sigma_s}\big\}.$$

Definition 7.1. *(sup-inf-critical solution)*
A solution $u = NS(u_0)$ is said to be a sup-inf-critical solution if u_0 belongs to $\mathcal{E}_c$ and

$$\liminf_{t\to T_*(u_0)}(T_*(u_0) - t)\,\|NS(u_0)(t)\|_{\dot{H}^s}^{\sigma_s} = m_c^{\sigma_s}. \tag{57}$$

Notice we need to look for such elements among sup-critical solutions, otherwise the definition of $m_c^{\sigma_s}$ would be meaningless. We claim that there exist such elements.

Lemma 7.1. *There exists some elements belonging to $\mathcal{E}_c$, which are sup-inf-critical.*

Proof. By definition of $m_c^{\sigma_s}$, we can find a sequence $(u_{0,n}) \in \dot{H}^s$ and a sequence $t_n \nearrow T_*(u_{0,n}) \equiv T_*$ (we can assume this, up to a rescaling) such that

$$m_c - \varepsilon_n \leqslant (T_* - t_n)^{\frac{1}{\sigma_s}} \|NS(u_{0,n})(t_n)\|_{\dot{H}^s} \leqslant m_c + \varepsilon_n \tag{58}$$

and

$$\text{For any} \quad t \geqslant t_n, \quad m_c - \varepsilon_n \leqslant (T_* - t)^{\frac{1}{\sigma_s}} \|NS(u_{0,n})(t)\|_{\dot{H}^s}. \tag{59}$$

Assume in addition that the sequence $(u_{0,n})$ belongs to the set $\mathcal{E}_c$. As a consequence, we have

$$\text{For any} \quad t \geqslant t_n \quad, m_c - \varepsilon_n \leqslant (T_* - t)^{\frac{1}{\sigma_s}} \|NS(u_{0,n})(t)\|_{\dot{H}^s} \leqslant M_c + \varepsilon_n. \tag{60}$$

Considering the rescaled sequence

$$v_{0,n}(y) = \big(T_* - t_n\big)^{\frac{1}{2}} NS(u_{0,n})\big(t_n, (T_* - t_n)^{\frac{1}{2}} y\big).$$

Hence, $v_{0,n}$ satisfies properties below by scaling argument

$$\begin{aligned} \|v_{0,n}\|_{\dot{H}^s}^{\sigma_s} = \big(T_* - t_n\big) \|NS(u_{0,n})(t_n)\|_{\dot{H}^s}^{\sigma_s}, \quad \|v_{0,n}\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} = \|NS(u_{0,n})(t_n)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} \\ \text{and} \quad \|v_{0,n}\|^{\sigma_{s'}}_{\dot{B}^{s'}_{2,\infty}} = \big(T_* - t_n\big) \|NS(u_{0,n})(t_n)\|^{\sigma_{s'}}_{\dot{B}^{s'}_{2,\infty}}. \end{aligned} \tag{61}$$

Combining (58) with the fact that $(u_{0,n})$ belongs to $\mathcal{E}_c$, we infer that the sequence $(v_{0,n})_{n\geqslant 1}$ is bounded in $\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^s \cap \dot{B}^{s'}_{2,\infty}$. Moreover, concerning the Navier-Stokes solution generated by such a data $NS(v_{0,n})$, we know that it keeps on living until the time $\tau^* = 1$ and satisfies once again (with $\widetilde{t_n} = t_n + \tau\big(T_* - t_n\big)$)

$$(1-\tau)^{\frac{1}{\sigma_s}} \|NS(v_{0,n})(\tau)\|_{\dot{H}^s} = (T_* - \widetilde{t_n})^{\frac{1}{\sigma_s}} \|NS(u_{0,n})(\widetilde{t_n})\|_{\dot{H}^s}. \tag{62}$$

As $\widetilde{t_n} \geqslant t_n$ for any n, we infer that for any $\tau < 1$

$$(1-\tau)^{\frac{1}{\sigma_s}} \|NS(v_{0,n})(\tau)\|_{\dot{H}^s} \geqslant m_c - \varepsilon_n.$$

Let us sum up information we have on the sequence $v_{0,n}$. Firstly, the lifespan of the Navier-Stokes associated with the sequence $v_{0,n}$ is equal to 1. Then,

$$\limsup_{\tau\to 1}(1-\tau)^{\frac{1}{\sigma_s}} \|NS(v_{0,n})(\tau)\|_{\dot{H}^s} = \limsup_{\widetilde{t_n}\to T_*} (T_* - \widetilde{t_n})^{\frac{1}{\sigma_s}} \|NS(u_{0,n})(\widetilde{t_n})\|_{\dot{H}^s},$$

which implies, thanks to (60) and definition of M_c , that for any $\tau < 1$,

$$\limsup_{\tau\to 1} (1-\tau)^{\frac{1}{\sigma_s}} \|NS(v_{0,n})(\tau)\|_{\dot{H}^s} = M_c \quad \text{and} \quad \|NS(v_{0,n})(\tau)\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} = \|NS(u_{0,n})(\widetilde{t_n})\|_{\dot{B}^{\frac{1}{2}}_{2,\infty}} < \infty.$$

In addition,

$$(1-\tau)^{\frac{1}{\sigma_{s'}}} \|NS(v_{0,n})(\tau)\|_{\dot{B}^{s'}_{2,\infty}} = (T_* - \widetilde{t_n})^{\frac{1}{\sigma_{s'}}} \|NS(u_{0,n})(\widetilde{t_n})\|_{\dot{B}^{s'}_{2,\infty}} < \infty. \tag{63}$$

To summerize, from the minimizing sequence $(u_{0,n})$ of the set $\mathcal{E}_c$, we build another sequence $(v_{0,n})$ (the rescaled sequence of $(u_{0,n})$) which also belongs to the set $\mathcal{E}_c$. Moreover, as the sequence $(v_{0,n})$ is bounded in the spaces $\dot{B}^{\frac{1}{2}}_{2,\infty} \cap \dot{H}^s \cap \dot{B}^{s'}_{2,\infty}$ and satisfies $\limsup_{n\to+\infty} \|v_{0,n}\|_{\dot{B}^s_{2,\infty}} < \infty$, Lemma 6.3 implies that profile decomposition in $\dot{H}^s$ of such a sequence is reduced, up to extractions, to a sum of translated profiles and a remaining term (under notations of Theorem 3.1)

$$v_{0,n} = \sum_{j\in\mathcal{J}_1} \varphi^j(\cdot - x_{n,j}) + \psi_n^J.$$

By vertue of Theorem 3.2, combining with Proposition 4.1, we infer there exists only one profile φ^{j_0} which blows up at time 1 and such that

$$NS(v_{0,n})(\tau,\cdot)=NS(\varphi^{j_0})(\tau,\cdot-x_{n,j_0})\ +\sum_{\substack{j\in\mathcal{J}_1,j\neq j_0\\ \tau_*^j>1}}NS(\varphi^j)(\cdot-x_{n,j})+\ e^{\tau\Delta}\psi_n^J(\cdot)\ +\ R_n^J(\tau,\cdot). \tag{64}$$

By orthogonality, we have

$$\|NS(v_{0,n})(\tau)\|_{\dot H^s}^2\geqslant\|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}^2\ +\sum_{\substack{j\in\mathcal{J}_1,j\neq j_0\\ \tau_*^j>1}}\|NS(\varphi^j)(\tau)\|_{\dot H^s}^2\ ++\|e^{\tau\Delta}\psi_n^J\|_{\dot H^s}^2\ +|\gamma_n^J(\tau)|. \tag{65}$$

We want to prove that $\liminf_{\tau\to1}\,(1-\tau)^{\frac{1}{\sigma_s}}\,\|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}\geqslant m_c$. By definition of m_c, this will imply that $\liminf_{\tau\to1}\,(1-\tau)^{\frac{1}{\sigma_s}}\,\|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}=m_c$. Let us assume that is not the case. Therefore,

$$\exists\alpha_0>0,\forall\varepsilon>0,\ \exists\tau_\varepsilon,\ \text{ such that }\ 0<(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\ <\varepsilon\text{ and }(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\,\|NS(u_{0,n})(\tau_\varepsilon)\|_{\dot H^s}^2\leqslant m_c^2-\alpha_0.$$

From (65), we deduce that

$$\begin{aligned}(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\,\|NS(v_{0,n})(\tau_\varepsilon)\|_{\dot H^s}^2&=(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\,\|NS(\varphi^{j_0})(\tau_\varepsilon)\|_{\dot H^s}^2\ +\ (1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\Big\{\sum_{\substack{j\in\mathcal{J}_1,j\neq j_0\\ \tau_*^j>1}}\|NS(\varphi^j)(\tau_\varepsilon)\|_{\dot H^s}^2\\&\quad+\ \|e^{\tau_\varepsilon\Delta}\psi_n^J\|_{\dot H^s}^2\ +\ |\gamma_n^J(\tau_\varepsilon)|\Big\}.\end{aligned}$$

By hypothesis, $(1-\tau_\varepsilon)^{\frac{1}{\sigma_s}}\,\|NS(v_{0,n})(\tau_\varepsilon)\|_{\dot H^s}\geqslant m_c-\varepsilon_n$, and $1-\tau_\varepsilon\leqslant1$. Hence, we get

$$\big(m_c-\varepsilon_n\big)^2\leqslant\ m_c^2-\alpha_0\ +\ (1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\Big\{\sum_{\substack{j\in\mathcal{J}_1,j\neq j_0\\ \tau_*^j>1}}\sup_{\tau\in[0,1]}\|NS(\varphi^j)(\tau)\|_{\dot H^s}^2\ +\ \|\psi_n^J\|_{\dot H^s}^2\Big\}\ +|\gamma_n^J(\tau_\varepsilon)|. \tag{66}$$

On the one hand, as profiles φ^j have a lifespan $\tau_*^j>1$, the quantity $\sup_{\tau\in[0,1]}\|NS(\varphi^j)(\tau)\|_{\dot H^s}^2$ is finite. On the other hand, by vertue of profile decomposition of the sequence $(v_{0,n})$, we have obviously that $\|\psi_n^J\|_{\dot H^s}^2\leqslant\|v_{0,n}\|_{\dot H^s}^2$. As we have proved that $(v_{0,n})$ is an element of the set $\mathcal{E}_c$, we get in particular that $\sup_{\tau<1}(1-\tau)^{\frac{1}{\sigma_s}}\,\|NS(v_{0,n})(\tau)\|_{\dot H^s}=M_c$, which leads to (at $\tau=0$) $\|v_{0,n}\|_{\dot H^s}\leqslant M_c$. Finally, for all τ_ε,

$$(1-\tau_\varepsilon)^{\frac{2}{\sigma_s}}\Big\{\sum_{\substack{j\in\mathcal{J}_1,j\neq j_0\\ \tau_*^j>1}}\sup_{\tau\in[0,1]}\|NS(\varphi^j)(\tau)\|_{\dot H^s}^2\ +\ \|\psi_n^J\|_{\dot H^s}^2\Big\}\leqslant\frac{\alpha_0}{4},$$

we get

$$\big(m_c-\varepsilon_n\big)^2\leqslant\ m_c^2-\alpha_0\ +\ \frac{\alpha_0}{4}+|\gamma_n^J(\tau_\varepsilon)|. \tag{67}$$

Now, by assumption of γ_n^J, we take the limit for n and J large enough, and we get

$$m_c^2\leqslant m_c^2\ -\ \frac{3\,\alpha_0}{4}\ +\ \frac{\alpha_0}{4}, \tag{68}$$

which is obviously absurd. Thus, we have proved that

$$\liminf_{\tau\to1}\ (1-\tau)^{\frac{1}{\sigma_s}}\,\|NS(\varphi^{j_0})(\tau)\|_{\dot H^s}=m_c.$$

This concludes the proof of Lemma 7.1. □

8. Structure Lemma for Navier-Stokes solutions with bounded data

The sequence $(v_{0,n})_{n\geqslant 0}$ be a bounded sequence of initial data in $\dot{H}^s$. Thanks to Theorem 3.1, $(v_{0,n})_{n\geqslant 0}$ can be written as follows, up to an extraction

$$v_{0,n}(x) = \sum_{j=0}^{J} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j(x) + \psi_n^J(x),$$

which can be written as follows

$$v_{0,n}(x) = \sum_{\substack{j\in\mathcal{J}_1\\ j\leqslant J}} \varphi^j(x-x_{n,j}) + \sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j(x) + \psi_n^J(x). \tag{69}$$

Let $\eta > 0$ be the parameter of rough cutting off frequencies. We define by $w_\eta(x)$ and $w_{c_\eta}(x)$ the elements which Fourier transform is given by

$$\widehat{w_\eta}(\xi) = \widehat{w}(\xi) 1_{\{\frac{1}{\eta}\leqslant|\xi|\leqslant\eta\}} \quad \text{and} \quad \widehat{w_{c_\eta}}(\xi) = \widehat{w}(\xi)\big(1 - 1_{\{\frac{1}{\eta}\leqslant|\xi|\leqslant\eta\}}\big). \tag{70}$$

After rough cutting off frequencies with respect to the notations (70) and sorting profiles supported in the annulus $1_{\{\frac{1}{\eta}\leqslant|\xi|\leqslant\eta\}}$ according to their scale (thanks to the orthogonality property of scales and cores, given by Theorem 3.1). We get the following profile decomposition

$$\begin{aligned} v_{0,n}(x) &= \sum_{j\in\mathcal{J}_1} \varphi^j(x-x_{n,j}) + \sum_{j\in\mathcal{J}_0} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j_\eta(x) + \sum_{j\in\mathcal{J}_\infty} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j_\eta(x) + \psi^J_{n,\eta}(x) \\ &\text{where} \quad \psi^J_{n,\eta}(x) \overset{\text{def}}{=} \sum_{\substack{j\in\mathcal{J}_1^c\equiv\mathcal{J}_0\cup\mathcal{J}_\infty\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} V^j_{c_\eta}(x) + \psi^J_n(x), \end{aligned} \tag{71}$$

for any j in $\mathcal{J}_1 \subset J$, $\lambda_{n,j} = 1$, for any j in $\mathcal{J}_0$, $\lim_{n\to+\infty} \lambda_{n,j} = 0$ and for any j in $\mathcal{J}_\infty$, $\lim_{n\to+\infty} \lambda_{n,j} = +\infty$. As mentionned in the introduction, the whole Lemma 3.2 has been already proved in [27], except for the orthogonality property of the Navier-stokes solution associated with such a sequence of initial data. Therefore, we refer the reader to [27] for details of the proof and here, we focus on the "Pythagore property". Let us recall the notations

$$U^0_{n,\eta} \overset{\text{def}}{=} \sum_{j\in\mathcal{J}_0} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j_\eta \quad \text{and} \quad U^\infty_{n,\eta} \overset{\text{def}}{=} \sum_{j\in\mathcal{J}_\infty} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}} \varphi^j_\eta.$$

We recall some properties on profiles with small and large scale and remaining term. We refer the reader to [27] to the proof of the two propositions below.

Proposition 8.1.

For any $s_1 < s$, *for any* $\eta > 0$, *for any* $j \in \mathcal{J}_0$, *(e.g* $\lim_{n\to+\infty} \lambda_{n,j} = 0$*), then* $\lim_{n\to+\infty} \big\|U^0_{n,\eta}\big\|_{\dot{H}^{s_1}} = 0.$

For any $s_2 > s$, *for any* $\eta > 0$, *for any* $j \in \mathcal{J}_\infty$, *(e.g* $\lim_{n\to+\infty} \lambda_{n,j} = +\infty$*), then* $\lim_{n\to+\infty} \big\|U^\infty_{n,\eta}\big\|_{\dot{H}^{s_2}} = 0.$

Concerning the remaining term, we can show it tends to 0, thanks to Lebesgue Theorem.

Proposition 8.2.

$$\lim_{J\to+\infty} \lim_{\eta\to+\infty} \limsup_{n\to+\infty} \|\psi^J_{n,\eta}\|_{L^p} = 0.$$

Continuation of Proof of Lemma 3.2. By vertue of (20) in Lemma 3.2, it seems clear that for any $t < \tilde{T}$

$$\|NS(v_{0,n})(t,\cdot)\|^2_{\dot H^s} = \Big\|\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j})\Big\|^2_{\dot H^s} + \Big\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j(x) + \psi_n^J\Big)\Big\|^2_{\dot H^s}$$
$$+ \|R_n^J(t,\cdot)\|^2_{\dot H^s} + 2\Big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j}) \mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j(x) + \psi_n^J\Big)\Big)_{\dot H^s}$$
$$+ 2\Big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j}) \mid R_n^J\Big)_{\dot H^s} + 2\Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j(x) + \psi_n^J\Big) \mid R_n^J\Big)_{\dot H^s}.$$

Therefore, proving (21) is equivalent to prove Propositions 8.3 and 8.4 below. Both of them essentially stem from the orthogonality of cores and a compactness argument.

Proposition 8.3. *Let $\varepsilon > 0$. Then, for any $t \in [0, \tilde{T} - \varepsilon]$,*

$$\Big\|\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j})\Big\|^2_{\dot H^s} = \sum_{j\in\mathcal{J}_1} \big\|NS(\varphi^j)(t,\cdot)\big\|^2_{\dot H^s} + \gamma_{n,\varepsilon}(t), \tag{72}$$

with $\lim\limits_{n\to+\infty} \sup\limits_{t\in[0,\tilde{T}-\varepsilon]} |\gamma_{n,\varepsilon}(t)| = 0.$

Proof. Once again, we developp the square of $\dot H^s$-norm and we get for any $t < \tilde{T}$

$$\Big\|\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j})\Big\|^2_{\dot H^s} = \sum_{j\in\mathcal{J}_1} \big\|NS(\varphi^j)(t,\cdot - x_{n,j})\big\|^2_{\dot H^s}$$
$$+ 2 \sum_{\substack{(j,k)\in\mathcal{J}_1\times\mathcal{J}_1\\ j\neq k}} \Big(\Lambda^s NS(\varphi^j)(t,\cdot - x_{n,j}) \mid \Lambda^s NS(\varphi^k)(t,\cdot - x_{n,k})\Big)_{L^2},$$

where $\Lambda = \sqrt{-\Delta}$. Let $\varepsilon > 0$. Then, for any t in $[0, \widetilde{T} - \varepsilon]$, we get

$$\Big\|\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j})\Big\|^2_{\dot H^s} = \sum_{j\in\mathcal{J}_1} \big\|NS(\varphi^j)(t,\cdot)\big\|^2_{\dot H^s} + 2 \sum_{\substack{(j,k)\in\mathcal{J}_1\times\mathcal{J}_1\\ j\neq k}} \Gamma^{s,j,k}_{\varepsilon,n},$$

where $\Gamma^{s,j,k}_{\varepsilon,n} \stackrel{\text{def}}{=} \Big(\Lambda^s NS(\varphi^j)(t,\cdot - x_{n,j}) \mid \Lambda^s NS(\varphi^k)(t,\cdot - x_{n,k})\Big)_{L^2}$.

We denote by

$$K_\varepsilon^J \stackrel{\text{def}}{=} \bigcup_{j\in J} \Lambda^s NS(\varphi^j)([0, \widetilde{T} - \varepsilon]).$$

By vertue of the continuity of the map $t \in [0, \widetilde{T} - \varepsilon] \mapsto \Lambda^s NS(\varphi^j)(t,\cdot) \in L^2$, we deduce that K_ε^J is compact (and thus precompact) in L^2. It means that it can be covered by a finite open ball with an arbitrarily radius $\alpha > 0$. Let α be a positive radius. There exists an integer N_α, and there exists $(\theta_\ell)_{1\leqslant\ell\leqslant N_\alpha}$ some elements of $\mathcal{D}(\mathbb{R}^3)$, such that

$$K_\varepsilon^J \subset \bigcup_{\ell=1}^{N_\alpha} B(\theta_\ell, \alpha). \tag{73}$$

Let us come back to the proof of 8.3. Thanks to the previous remark, we approach each profil $\Lambda^s NS(\varphi^j)(t,\cdot)$ (resp. $\Lambda^s NS(\varphi^k)(t,\cdot)$) by a smooth function: e.g there exists a integer $\ell \in$

$\{1,\cdots N_\alpha\}$ and there exists a function $\theta_{\ell(j,t)}$ (resp. $\theta_{\ell(k,t)}$) in $\mathcal{D}(\mathbb{R}^3)$ and we get

$$
\begin{aligned}
\Gamma^{s,j,k}_{\varepsilon,n} &= \Big(\Lambda^s NS(\varphi^j)(t,\cdot-x_{n,j}) - \theta_{\ell(j,t)}(\cdot-x_{n,j}) \mid \Lambda^s NS(\varphi^k)(t,\cdot-x_{n,k}) - \theta_{\ell(k,t)}(\cdot-x_{n,k})\Big)_{L^2} \\
&+ \big(\Lambda^s NS(\varphi^j)(t,\cdot-x_{n,j}) - \theta_{\ell(j,t)}(\cdot-x_{n,j}) \mid \theta_{\ell(k,t)}(\cdot-x_{n,k})\big)_{L^2} \\
&+ \Big(\theta_{\ell(j,t)}(\cdot-x_{n,j}) \mid \Lambda^s NS(\varphi^k)(t,\cdot-x_{n,k}) - \theta_{\ell(k,t)}(\cdot-x_{n,k})\Big)_{L^2} \\
&+ \big(\theta_{\ell(j,t)}(\cdot-x_{n,j}) \mid \theta_{\ell(k,t)}(\cdot-x_{n,k})\big)_{L^2}.
\end{aligned} \tag{74}
$$

The three first terms in the right-hand side of the above estimate tend uniformly (in time) to 0, by vertue of Cauchy-Schwarz and the translation-invariance of the $\dot{H}^s$-norm (we just perform the estimate for the first term, the others are similar). For any $t\in[0,\tilde{T}-\varepsilon]$

$$
\begin{aligned}
&\Big(\Lambda^s NS(\varphi^j)(t,\cdot-x_{n,j}) - \theta_{\ell(j,t)}(\cdot-x_{n,j}) \mid \Lambda^s \big(NS(\varphi^k)(t,\cdot-x_{n,k}) - \theta_{\ell(k,t)}(\cdot-x_{n,k})\big)\Big)_{L^2} \\
&\qquad\qquad \leqslant \|\Lambda^s NS(\varphi^j)(t) - \theta_{\ell(j,t)}\|_{L^2}\ \|\Lambda^s NS(\varphi^k)(t) - \theta_{\ell(k,t)}\|_{L^2} \\
&\qquad\qquad \leqslant \alpha^2.
\end{aligned} \tag{75}
$$

Therefore, for any $\alpha>0$, we have

$$
\sup_{t\in[0,\tilde{T}-\varepsilon]} \Big(\Lambda^s NS(\varphi^j)(t,\cdot-x_{n,j}) - \theta_{\ell(j,t)}(\cdot-x_{n,j}) \mid \Lambda^s NS(\varphi^k)(t,\cdot-x_{n,k}) - \theta_{\ell(k,t)}(\cdot-x_{n,k})\Big)_{L^2} \leqslant \alpha^2. \tag{76}
$$

For the last term $\big(\theta_{\ell(j,t)}(\cdot-x_{n,j}) \mid \theta_{\ell(k,t)}(\cdot-x_{n,k})\big)_{L^2}$, we have

$$
\big(\theta_{\ell(j,t)}(\cdot-x_{n,j}) \mid \theta_{\ell(k,t)}(\cdot-x_{n,k})\big)_{L^2} = \int_{\mathbb{R}^3} \theta_{\ell(j,t)}(x)\,\theta_{\ell(k,t)}(x+x_{n,j}-x_{n,k})\,dx.
$$

It follows immediately that the above term tends to 0, when n tend to $+\infty$, by vertue of Lebesgue theorem combining with the orthogonality property of cores(e.g. $\lim_{n\to\infty}|x_{n,j}-x_{n,k}|=+\infty$). To sum up, we have proved that $\Gamma^{s,j,k}_{\varepsilon,n}$ tends to 0 when n tends to $+\infty$, uniformly in time. This concludes the proof of Proposition 8.3.

□

Concerning the crossed-terms in the profile decomposition, we have to prove they are also negligable, uniformly in time. That is the point in the following proposition.

Proposition 8.4. *Let $\varepsilon>0$, We denote by*

$$
I_n(t,\cdot) \stackrel{\text{def}}{=} \Big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot-x_{n,j}) \mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j(x)+\psi^J_n\Big)\Big)_{\dot{H}^s},
$$

then, one has
$$
\lim_{J\to+\infty}\lim_{\eta\to+\infty}\lim_{n\to+\infty}\sup_{t\in[0,\tilde{T}-\varepsilon]} I_n(t,\cdot)=0, \tag{77}
$$

$$
\lim_{J\to+\infty}\lim_{n\to+\infty}\sup_{t\in[0,\tilde{T}-\varepsilon]} \Big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot-x_{n,j}) \mid R^J_n(t)\Big)_{\dot{H}^s}=0, \tag{78}
$$

$$
\lim_{J\to+\infty}\lim_{\eta\to+\infty}\lim_{n\to+\infty}\sup_{t\in[0,\tilde{T}-\varepsilon]} \Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j(x)+\psi^J_n\Big) \mid R^J_n(t)\Big)_{\dot{H}^s}=0. \tag{79}
$$

Proof. Let us start by proving (77). We shall use once again an approximation argument. Let us define

$$
\Lambda^J_\varepsilon \stackrel{\text{def}}{=} \bigcup_{j\in J} NS(\varphi^j)([0,\widetilde{T}-\varepsilon]).
$$

By vertue of the continuity of the map $t \in [0, \widetilde{T}-\varepsilon] \mapsto NS(\varphi^j)(t,\cdot) \in \dot{H}^s$, we deduce that Λ_ε^J is compact (and thus precompact) in $\dot{H}^s$. It means that it can be covered by a finite open ball with an arbitrarily radius $\beta > 0$. Let β be a positive radius. There exists an integer N_β, and there exists $(\chi_\ell)_{1\leqslant \ell \leqslant N_\beta}$ some elements of $\mathcal{D}(\mathbb{R}^3)$, such that

$$\Lambda_\varepsilon^J \subset \bigcup_{\ell=1}^{N_\beta} B(\chi_\ell, \beta). \tag{80}$$

Let us come back to the proof of (77). Same arguments as previously imply there exists an integer $\ell \in \{1 \cdots N_\beta\}$ and a smooth function $\chi_{\ell(t,j)}$ in $\mathcal{D}(\mathbb{R}^3)$ such that

$$\begin{aligned} I_n(t,\cdot) &\stackrel{\text{def}}{=} \Big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j}) \mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi_n^J\Big)\Big)_{\dot{H}^s} \\ &= \Big(\sum_{j\in\mathcal{J}_1} NS(\varphi^j)(t,\cdot - x_{n,j}) - \chi_{\ell(t,j)}(\cdot - x_{n,j}) \mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi_n^J\Big)\Big)_{\dot{H}^s} \\ &\quad + \Big(\sum_{j\in\mathcal{J}_1} \chi_{\ell(t,j)}(\cdot - x_{n,j}) \mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi_n^J\Big)\Big)_{\dot{H}^s}. \end{aligned} \tag{81}$$

As $\big\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi_n^J\Big)\big\|_{\dot{H}^s} \leqslant \|v_{0,n}\|_{\dot{H}^s}$, we infer that

$$I_n(t,\cdot) \leqslant |\mathcal{J}_1|\,\beta\,\big\|v_{0,n}\big\|_{\dot{H}^s} + \Big(\sum_{j\in\mathcal{J}_1} \chi_{\ell(t,j)}(\cdot - x_{n,j}) \mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi_n^J\Big)\Big)_{\dot{H}^s} \tag{82}$$

Concerning the second part of above inequality, we shall use the splitting with respect to the parameter of cut off η. We refer the reader to the beginning of this section for notations.

$$\Big(\sum_{j\in\mathcal{J}_1} \chi_{\ell(t,j)}(\cdot - x_{n,j}) \mid e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}} \Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j + \psi_n^J\Big)\Big)_{\dot{H}^s} = I_n^1(t,\cdot) + I_n^2(t,\cdot) + I_n^3(t,\cdot),$$

$$\text{where}\quad I_n^1(t,\cdot) = \sum_{j\in\mathcal{J}_1} \big(\chi_{\ell(t,j)}(\cdot - x_{n,j}) \mid e^{t\Delta}U^0_{n,\eta}\big)_{\dot{H}^s} \quad ; \quad I_n^2(t,\cdot) = \sum_{j\in\mathcal{J}_1} \big(\chi_{\ell(t,j)}(\cdot - x_{n,j}) \mid e^{t\Delta}U^\infty_{n,\eta}\big)_{\dot{H}^s}$$
$$\text{and}\quad I_n^3(t,\cdot) = \sum_{j\in\mathcal{J}_1} \big(\chi_{\ell(t,j)}(\cdot - x_{n,j}) \mid e^{t\Delta}\psi^J_{n,\eta}\big)_{\dot{H}^s}.$$

Let us start with $I_n^1(t,\cdot)$. One has

$$\begin{aligned} |I_n^1(t,\cdot)| &\leqslant |\mathcal{J}_1|\, \|\chi_{\ell(t,j)}\|_{\dot{H}^{2s-s_1}}\, \|e^{t\Delta}U^0_{n,\eta}\|_{\dot{H}^{s_1}} \\ &\leqslant |\mathcal{J}_1|\, \|\chi_{\ell(t,j)}\|_{\dot{H}^{2s-s_1}}\, \|U^0_{n,\eta}\|_{\dot{H}^{s_1}}. \end{aligned}$$

Proposition 8.1 (for η and $j \in \mathcal{J}_1$ fixed) implies thus $\lim\limits_{n\to+\infty} \sup\limits_{t\in[0,\tilde{T}-\varepsilon]} |I_n^1(t,\cdot)| = 0$.

Concerning profiles with large scale, the proof is similar and we get for any $t \in [0, \tilde{T}-\varepsilon]$

$$|I_n^2(t,\cdot)| \leqslant |\mathcal{J}_1|\, \|\chi_{\ell(t,j)}\|_{\dot{H}^{2s-s_2}}\, \big\|U^\infty_{n,\eta}(x)\big\|_{\dot{H}^{s_2}}. \tag{83}$$

Once again, Proposition 8.1 implies the result : $\lim_{n\to+\infty}\sup_{t\in[0,\tilde T-\varepsilon]}|I^2_n(t,\cdot)|=0.$

Concerning the last term I^3_n, Hölder inequality with $\dfrac{1}{p}+\dfrac{1}{p'}=1$ yields

$$\begin{aligned}|I^3_n(t,\cdot)| &\leqslant \big|\big(\Lambda^{2s}\,\chi_{\ell(t,j)}\mid e^{t\Delta}\psi^J_{n,\eta}(\cdot+x_{n,j})\big)_{L^2}\big|\\ &\leqslant \|\Lambda^{2s}\,\chi_{\ell(t,j)}\|_{L^{p'}}\,\|e^{t\Delta}\psi^J_{n,\eta}(\cdot+x_{n,j})\|_{L^p}\big).\end{aligned}$$

By translation invariance of the L^p-norm and estimate on the heat equation, we get

$$|I^3_n(t,\cdot)| \leqslant \|\Lambda^{2s}\,\chi_{\ell(t,j)}\|_{L^{p'}}\,\|\psi^J_{n,\eta}\|_{L^p}. \tag{84}$$

Obviously the term $\|\psi^J_{n,\eta}\|_{\dot H^s}$ is bounded by profiles hypothesis and the term $\|\Lambda^{2s}\,\chi\chi_{\ell(t,j)}\|_{L^{p'}}$ is bounded too, since the function χ is as regular as we need. By vertue of Proposition 8.2, the term $\|\psi^J_{n,\eta}\|_{L^p}$ is small in the sense of for any $\varepsilon>0$, there exists an integer $N_0\in N$, such that for any $n\geqslant N_0$, there exists $\tilde\eta>0$ and $\tilde J\geqslant 0$, such that for any $\eta\geqslant\tilde\eta$ and for any $J\geqslant\tilde J$, we have $\|\psi^J_{n,\eta}\|_{L^p}\leqslant\varepsilon$. As a result, we get for any

$$\lim_{J\to+\infty}\lim_{\eta\to+\infty}\lim_{n\to+\infty}\sup_{t\in[0,\tilde T-\varepsilon]}|I^3_n(t,\cdot)|=0.$$

This ends up the proof of estimate (77).

Concerning the proof of (78) and (79), the proof is very close in both cases and relies on the fact that the error term R^J_n tends to 0 in the $L^\infty_T(\dot H^s)$-norm. For any $t\in[0,\tilde T-\varepsilon]$, we have

$$\begin{aligned}\big|\big(\sum_{j\in\mathcal{J}_1}NS(\varphi^j)(t,\cdot-x_{n,j})\mid R^J_n\big)_{\dot H^s}\big| &\leqslant \sum_{j\in\mathcal{J}_1}\big|\big(NS(\varphi^j)(t,\cdot)\mid R^J_n(t,\cdot+x_{n,j})\big)_{\dot H^s}\big|\\ &\leqslant |\mathcal{J}_1|\,\|NS(\varphi^j)(t,\cdot)\|_{L^\infty_T(\dot H^s)}\,\|R^J_n(t,\cdot)\|_{L^\infty_T(\dot H^s)}.\end{aligned} \tag{85}$$

Obviously, the term $\|NS(\varphi^j)(t,\cdot)\|_{L^\infty_T(\dot H^s)}$ is bounded since $t\in[0,\tilde T-\varepsilon]$. As a result, Lemma 3.2 implies that

$$\lim_{J\to+\infty}\lim_{n\to+\infty}\sup_{t\in[0,\tilde T-\varepsilon]}\mid\big(\sum_{j\in\mathcal{J}_1}NS(\varphi^j)(t,\cdot-x_{n,j})\mid R^J_n\big)_{\dot H^s}\mid=0.$$

As far as estimate (79) is concerned, the idea is the same. For any $t\in[0,\tilde T-\varepsilon]$,

(86)

$$\begin{aligned}\Big|\Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j(x)+\psi^J_n\Big)\mid R^J_n\Big)_{\dot H^s}\Big| &\leqslant \Big|\Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j(x)+\psi^J_n\Big)\mid R^J_n\Big)_{\dot H^s}\Big|\\ &\leqslant \Big\|e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j(x)+\psi^J_n\Big)\Big\|_{L^\infty_T(\dot H^s)}\|R^J_n\|_{L^\infty_{\tilde T-\varepsilon}(\dot H^s)}\\ &\leqslant \|U^0_{n,\eta}+U^\infty_{n,\eta}+\psi^J_{n,\eta}\|_{\dot H^s}\,\|R^J_n\|_{L^\infty_{\tilde T-\varepsilon}(\dot H^s)}.\end{aligned}$$

Thanks to profile decomposition (71), we get

$$\|U^0_{n,\eta}+U^\infty_{n,\eta}+\psi^J_{n,\eta}\|^2_{\dot H^s}\leqslant\|v_{0,n}\|^2_{\dot H^s}+\circ(1). \tag{87}$$

Thus, finally we get

$$\Big|\Big(e^{t\Delta}\Big(\sum_{\substack{j\in\mathcal{J}_1^c\\ j\leqslant J}}\Lambda^{\frac{3}{p}}_{\lambda_{n,j},x_{n,j}}\varphi^j(x)+\psi^J_n\Big)\mid R^J_n\Big)_{\dot H^s}\Big|\leqslant C\left(\|v_{0,n}\|^2_{\dot H^s}+\circ(1)\right)\|R^J_n\|_{L^\infty_{\tilde T-\varepsilon}(\dot H^s)}. \tag{88}$$

We end up the proof as before, thanks to the hypothesis on R^J_n. This completes the proof of Proposition 8.4 and thus Lemma 3.2. □

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