Backward stochastic differential equations and stochastic control and applications to mathematical finance

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ÉCOLE DOCTORALE DE SCIENCES MATHÉMATIQUES DE PARIS CENTRE

THÈSE DE DOCTORAT

Discipline : Mathématiques Appliquées

Présentée par

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Equations différentielles stochastiques rétrogrades et contrôle stochastique et applications aux mathématiques financières

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Soutenue publiquement le **8 juillet 2015** devant le jury composé de :

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Remerciements

Je tiens tout d'abord à remercier mon directeur de thèse, Huyên Pham, pour m'avoir fait découvrir le monde de la recherche en mathématiques financières et en particulier le contrôle stochastique et les équations stochastiques rétrogrades. Je lui suis particulièrement reconnaissant pour sa grande disponibilité lors de ces trois années, ses nombreux conseils m'ont beaucoup apporté.
Je voudrais ensuite adresser tous mes remerciements à Saïd Hamadène et Marcel Nutz qui ont accepté de rapporter cette thèse. Je sais qu'ils ont des emplois du temps très chargés, et je leur suis extrêmement reconnaissant du temps qu'ils m'ont consacré.
Je remercie également Romuald Elie, Marie-Claire Quenez, Idriss Kharroubi et Nizar Touzi d'avoir accepté de participer au jury.
Je tiens à remercier les chercheurs avec qui j'ai eu la chance de collaborer lors de ces années de thèse : Andrea Cosso, Stéphane Goutte, Armand Ngoupeyou, Romuald Elie et Xavier Warin. Je remercie tout particulièrement Andrea Cosso et Marco Fuhrmann pour leur chaleureux accueil lors de mon passage à Milan.
Je remercie tous les amis qui m'ont entouré ces dernières années, ainsi que ma famille dont la confiance lors de mes nombreuses années d'étude a été très précieuse.
Je remercie aussi tous les doctorants que j'ai pu côtoyer pendant ces 4 années, Anne, Adrien, Amine, Clément, Jiatu, Guillaume, Lorick, Maude, Marc-Antoine, Nicolas, Oriane, Paul, Pietro, Roxana, Sophie, Thibaut, Thomas et Victor.
Je remercie enfin tous ceux qui oseront lire cette page, et je ne suis pas certain de leur conseiller de lire la suite. Il faut parfois savoir s'arrêter au bon moment.

Equations différentielles stochastiques rétrogrades et contrôle stochastique et applications aux mathématiques financières

Résumé : Cette thèse est constituée de deux parties pouvant être lues indépendamment.

Dans la première partie de la thèse, trois utilisations des équations différentielles stochastiques rétrogrades sont présentées.
Le premier chapitre est une application de ces équations au problème de couverture moyenne-variance dans un marché incomplet où des défauts multiples peuvent survenir. Nous faisons une hypothèse de densité conditionnelle sur les temps de défaut. Nous décomposons ensuite la fonction valeur en une suite de fonctions valeur entre deux défauts consécutifs et nous prouvons la forme quadratique de chacune d'entre elles. Enfin, nous illustrons nos résultats dans un cas particulier à 2 temps de défaut suivant des lois exponentielles indépendantes.

Les deux chapitres suivants sont des extensions de l'article [75].
Le deuxième chapitre est l'étude d'une classe d'équations différentielles stochastiques rétrogrades avec sauts négatifs et barrière supérieure. L'existence et l'unicité d'une solution minimale sont prouvées par double pénalisation sous des hypothèses de régularité sur l'obstacle. Cette méthode permet de résoudre le cas où le coefficient de diffusion est dégénéré. Nous montrons aussi, dans un cadre markovien adapté, le lien entre notre classe d'équations rétrogrades et des inégalités variationnelles non linéaires. En particulier, notre représentation d'équation rétrograde donne une formule de type Feynman-Kac pour les équations aux dérivées partielles associées à des jeux différentiels stochastiques de type contrôleur et stoppeur à somme nulle, où le contrôle affecte à la fois les termes dérives de volatilité. De plus, nous obtenons une formule duale du jeu de la solution minimale de l'équation rétrograde, ce qui donne une nouvelle représentation des jeux différentiels stochastiques contrôleur et stoppeur à somme nulle.

Le troisième chapitre est lié à l'incertitude de modèle, où l'incertitude affecte à la fois la volatilité et l'intensité. Ces problèmes de contrôle stochastiques sont associées à des équations intégro-différentielles aux dérivées partielles telles que la partie de saut est caractérisée par la mesure $\lambda(a,.)_a$ dépendant d'un paramètre a. Nous ne supposons pas que la famille $\lambda(a,.)_a$ est dominée. Nous obtenons une formule non linéaire de type Feynman-Kac à la fonction valeur associée à ces problèmes de contrôle. Pour cela, nous introduisons une classe d'équations différentielles stochastiques rétrogrades avec saut et une partie diffusive partiellement contrainte. Ici aussi le cas où le coefficient de diffusion est dégénéré est résolu.

Dans la seconde partie de la thèse, un problème de gestion actif-passif conditionnelle est résolu. Nous obtenons d'abord le domaine de définition de la fonction valeur associée au problème en identifiant la richesse minimale pour laquelle il existe une stratégie d'investissement admissible permettant de satisfaire la contrainte à maturité. Cette richesse minimal est identifiée comme une solution de viscosité d'une EDP. Nous montrons aussi que sa transformée de Fenschel-Legendre est une solution de viscosité d'une autre EDP, ce qui permet d'obtenir un schéma numérique avec une convergence plus rapide. Nous identifions ensuite la fonction valeur liée au problème d'intérêt comme une solution de viscosité d'une EDP sur son domaine de définition. Enfin, nous résolvons numériquement le problème en présentant des graphes de la richesse minimale, de la fonction valeur du problème et de la stratégie optimale.

Mots-clés : couverture moyenne-variance, équations différentielles stochastiques rétrogrades (EDSR) quadratiques, programmation dynamique, modèle défaut-densité, EDSR avec sauts contraints, EDSR réfléchies, changement de régime avec diffusion à saut, équation d'Hamilton-Jacobi-Bellman Isaacs, jeu contrôleur et stoppeur, contrôle optimal, problème de cible stochastique.

Backward stochastic differential equations and stochastic control and applications to mathematical fiance

Abstract : This thesis is divided into two parts that may be read independently.

In the first part, three uses of backward stochastic differential equations are presented.
The first chapter is an application of these equations to the mean-variance hedging problem in an incomplete market where multiple defaults can occur. We make a conditional density hypothesis on the default times. We then decompose the value function into a sequence of value functions between consecutive default times and we prove that each of them admits a quadratic form. Finally, we illustrate our results for a specific case where 2 default times follow independent exponential laws.

The two following applications are extensions of the paper [75].
The second chapter is the study of a class of backward stochastic differential equations with nonpositive jumps and upper barrier. Existence and uniqueness of a minimal solution are proved by a double penalization approach under regularity assumptions on the obstacle. This method allows us to solve the case where the diffusion coefficient is degenerate. We also show, in a suitable markovian framework, the connection between our class of backward stochastic differential equations and fully nonlinear variational inequalities. In particular, our backward equation representation provides a Feynman-Kac type formula for PDEs associated to general zero-sum stochastic differential controller-and-stopper games, where control affects both drift and diffusion term, and the diffusion coefficient can be degenerate. Moreover, we state a dual game formula of this backward equation minimal solution, which gives a new representation for zero-sum stochastic differential controller-and-stopper games.

The third chapter is linked to model uncertainty, where the uncertainty affects both volatility and intensity. This kind of stochastic control problems is associated to a fully nonlinear integro-partial differential equation, such that the measure $\lambda(a,.)_a$ characterizing the jump part depends on a parameter a. We do not assume that the family $\lambda(a,.)_a$ is dominated. We obtain a nonlinear Feynman-Kac formula for the value function associated to these control problems. To this aim, we introduce a class of backward stochastic differential equations with jumps and partially constrained diffusive part. Here the case where the diffusion coefficient is degenerate is solved as well.

In the second part, a conditional asset liability management problem is solved. We first derive the proper domain of definition of the value function associated to the problem by identifying the minimal wealth for which there exists an admissible investment strategy allowing to satisfy the constraint at maturity. This minimal wealth is identified as a solution of viscosity of a PDE. We also show that its Fenschel-Legendre transform is a solution of viscosity of another PDE, which allows to obtain a scheme with a faster convergence. We then identify the value function linked to the problem of interest as a solution of viscosity of a PDE on its domain of definition. Finally, we solve numerically

the problem and we provide graphs of the minimal wealth, of the value function of the problem and of the optimal strategy.

Keywords : Mean-variance hedging, Quadratic backward stochastic differential equation (BSDE), Dynamic programming, Default-density modelling, BSDE with constrained jumps, reflected BSDE, regime-switching jump-diffusion, Hamilton-Jacobi-Bellman Isaacs equation, controller-and-stopper game, optimal control, stochastic target problem .

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Chapitre 1

Introduction générale (en français)

1.1 Préliminaires sur les EDSRs classiques

La première partie de cette thèse est consacrée à différentes applications des Équations Différentielles Stochastiques Rétrogrades (EDSRs) liées au contrôle stochastique et aux mathématiques financières. Rappelons tout d'abord ce que désigne cette notion, en se bornant ici au cas réel. Notons $(\Omega, \mathcal{F}, \mathcal{P})$ un espace probabilisé muni d'un mouvement Brownien W (d-dimensionnel) dont la filtration naturelle et augmentée est notée $(\mathcal{F})_{t\geq 0}$. Une EDSR à horizon déterministe T s'écrit alors

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \mathcal{P} - p.s. \tag{1.1.1}$$

Les données sont ici :

1. La condition terminale ξ, qui est une variable aléatoire réelle $\mathcal{F}_T$-mesurable.
2. Le générateur $f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, supposé mesurable par rapport à $\mathbb{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$, notant $\mathcal{P}$ la tribu des évènements prévisibles.

Résoudre cette équation, c'est déterminer un couple de processus $\mathcal{F}_T$-adaptés $(Y_t, Z_t)_{0\leq t\leq T}$ vérifiant (1.1.1). Le mot *rétrograde* désigne le fait qu'ici c'est la condition terminale de l'équation qui est connue, soit $Y_T = \xi$, ce qui est la principale source de la complexité de ce problème. Or la solution doit être adaptée, un simple retournement du temps est donc ici inenvisageable. C'est pourquoi il faut chercher la solution sous la forme non pas d'un, mais de deux processus, le processus Z ayant pour but justement d'assurer l'adaptabilité de la solution.

Les EDSRs ont été introduites pour la première fois par Bismut dans le cas d'un générateur linéaire [11], mais l'article pionnier de la théorie telle qu'elle est formalisée aujourd'hui est celui de Pardoux et Peng[85], dans lequel est prouvé le théorème suivant.

Theorem 1.1.1. *Supposons que le générateur f est Lipschitz en (Y, Z) uniformément en (s, ω) et*

$$\mathbb{E}^{\mathbb{P}}[|\xi|^2 + \int_0^T |F(s, 0, 0)|^2 ds] < +\infty.$$

Alors l'EDSR (1.1.1) a une unique solution (Y, Z) telle que Z soit un processus de carré intégrable.

Après ce premier résultat général d'existence, une littérature toujours plus vaste, s'est attachée à affaiblir de plus en plus les hypothèses de ce théorème. Cet engouement s'explique en partie par le très grand nombre de champs d'applications de la théorie des EDSRs, comme notamment des problèmes de contrôle stochastique, de jeux stochastiques, ou des problèmes de gestion de portefeuille... Le lecteur pourra se référer à l'article [38] qui propose une revue détaillée des applications en finance. Cependant, c'est le lien extrêmement étroit qui existe entre la théorie des EDSRs et la théorie des Equations aux Dérivées Partielles (EDPs par la suite) qui demeure la raison principale de cet intérêt marqué de la communauté mathématique. Revenons maintenant sur cette connexion.

Considérons une classe d'EDSRs particulières, dites Markoviennes. Pour ces équations, l'aléatoire de la condition terminale et du générateur est supposé être entièrement généré par une certaine diffusion. Plus précisément, (Y, Z) est solution de

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \quad \mathbb{P} - p.s., \tag{1.1.2}$$

où f et g sont des fonctions déterministes et où $(X_t)_{0\leq t\leq T}$ est solution de l'EDS

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad 0 \leq t \leq T, \quad \mathbb{P} - p.s.$$

Soit alors l'EDP

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),\nabla u(t,x)\sigma(t,x)) = 0, \quad (t,x) \in [0,T) \times \mathbb{R} \\ &u(T,.) = g(.), \end{aligned} \tag{1.1.3}$$

où $\mathcal{L}$ est le générateur infinitésimal associé à la diffusion dont est solution X donné par

$$\mathcal{L}v(t,x) := \frac{1}{2}\text{Tr}[a(t,x)\nabla^2(t,x)] + b(t,x).\nabla(t,x),$$

où $a(t,x) := \sigma(t,x)'\sigma(t,x)$.

Si nous supposons que cette EDP possède une solution régulière, une simple application de la formule d'Itô montre que $(u(t,x), \nabla u(t,x)\sigma(t,x))$ est solution de l'EDSR (1.1.2). Ce résultat, qui n'est rien d'autre qu'une généralisation de la formule de Feynman-Kac, nous confère ainsi une interprétation probabiliste de l'EDP (1.1.3) et ouvre la voie de la simulation numérique de solutions d'EDPs par des méthodes probabilistes, qui ont comme grand avantage de ne pas (ou peu) souffrir de problèmes liés à la dimension. De telles méthodes ont fait l'objet de nombreux travaux, parmi lesquels nous pouvons citer Zhang [105], [106] et Bouchard et Touzi [16]. Dans la première partie de cette thèse figurent notamment une extension de ce résultat au cas des EDSRs réfléchies avec sauts contraints puis une autre dans un modèle de volatilité incertaine.

Notons par ailleurs que la théorie des EDSRs ne fournit une telle interprétation probabiliste que pour des EDPs dites quasi-linéaires, au sens où la dépendance en la Hessienne dans (1.1.3) ne peut être que linéaire. En effet, les termes faisant intervenir la Hessienne ne proviennent que de la variation quadratique de X dans la formule d'Itô. Cependant, vu l'extrême importance que de telles équations peuvent revêtir dans de nombreux domaines des mathématiques, de la physique ou encore de l'ingénierie, il est on ne peut plus naturel et désirable d'étendre les résultats ci-dessus à une classe plus grande d'EDPs.

En particulier, depuis la fin des années 1990, l'intérêt pour les EDSRs dites à croissance quadratique (au sens où le générateur est à croissance quadratique en Z) ont reçu une attention toute particulière, du fait de leur intérêt dans des problèmes liés aux mesures de risque dynamiques ou à la gestion de portefeuille avec contraintes, voir par exemple [40]. Ainsi, la question d'existence et d'unicité d'une solution dans le cas où le générateur vérifie

$$|f(t,y,z)| \leq |l(t)| + c(t)|y| + \frac{\delta}{2}|z|^2, \tag{1.1.4}$$

δ étant une constante positive et c et l des processus adaptés suffisamment intégrables, a d'abord été résolue par Kobylanski [76] dans le cas d'une condition terminale bornée. Une application de ce résultat est exposée dans cette thèse : la détermination d'une stratégie de couverture moyenne-variance sous risque de défauts multiples.

1.2 Couverture moyenne-variance sous risque de défauts multiples

La première application des EDSRs exposée dans cette thèse est l'étude d'une couverture moyenne-variance sous risque de défauts. Rappelons brièvement ce qu'est une couverture moyenne-variance. Soit $T > 0$ le temps de maturité et H_T un payoff. Notons $\mathcal{A}$ l'ensemble des stratégies admissibles de trading notées π et x le capital initial. Notant de plus $(X_t^{x,\pi})_{0\leq t\leq T}$ le processus de richesse correspondant, nous appellons la performance de la stratégie de trading :

$$J_0^H(x,\pi) = \mathbb{E}[(H_T - X_T^{x,\pi})^2] \tag{1.2.1}$$

et le problème de couverture moyenne-variance se formule ainsi :

$$V_0^H(x) = \inf_{\pi\in\mathcal{A}} J_0^H(x,\pi) \tag{1.2.2}$$

Ce problème a été introduit par Föllmer et Schweitzer [45], et depuis de nombreux auteurs ont développé cette approche. Nous renvoyons le lecteur à l'article [98] pour une revue détaillée de la littérature. Dans la plupart de ces articles, le problème est résolu en utilisant des filtrations continues, voir par exemple [91] et [97].

Cependant, notre modèle comporte un risque de défauts multiples, reprenant le formalisme de [62] et [63]. Une particularité importante est qu'ici le nombre de défauts n est fixé a priori, et nous associons à chaque défaut survenant à un temps τ_i une marque $L_i \in E \subset \mathbb{R}$.

La seule hypothèse faite sur les défauts est une hypothèse de densité : il existe un processus adapté α tel que pour toute fonction borélienne bornée f et pour tout temps $0 \leq t \leq T$,

$$\mathbb{E}[f(\tau,L)|\mathcal{F}_t] \;=\; \int f(\boldsymbol{\theta},\boldsymbol{l})\alpha_t(\boldsymbol{\theta},\boldsymbol{l})d\boldsymbol{\theta}\eta(d\boldsymbol{l}) \quad p.s., \tag{1.2.3}$$

où $d\boldsymbol{\theta} = d\theta_1 ... d\theta_n$ est la mesure de Lebesgue sur $\mathbb{R}^n$ et $\eta(d\boldsymbol{l})$ est une mesure de Borel sur E^n de la forme $\eta(d\boldsymbol{l}) = \eta_1(dl_1) \prod_{k=1}^{n-1} \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})$, où η_1 est une mesure de Borel positive sur E et, pour $1 \leq k \leq n-1$, $\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})$ un noyau de transition positif sur $E^k \times E$.
On est donc amené à considérer des n-uplets ordonnés de temps de défauts $\boldsymbol{\tau} = (\tau_1, ..., \tau_n) \in [0,T]^n$ associés à des n-uplets de marques $\boldsymbol{L} = (L_1, ..., L_n)$ et les évènements

$$\Omega_t^k := \{\tau_k \leq t < \tau_{k+1}\}, \quad 0 \leq t \leq T, \quad 0 \leq k \leq n$$

correspondants aux scenarii où k défauts ont été observés jusqu'à l'instant t. Le processus de l'actif de trading S considéré se décompose donc sous la forme

$$S_t = \sum_{k=0}^{n} 1_{\Omega_t^k} S_t^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k)$$

en notant $\boldsymbol{\tau}_k = (\tau_1, ..., \tau_k)$ et $\boldsymbol{L}_k = (L_1, ..., L_k)$. Les processus S^k vérifient les dynamiques, dans le scénario où $\boldsymbol{\tau}_k = \boldsymbol{\theta}_k$ et $\boldsymbol{L}_k = \boldsymbol{l}_k$:

$$dS_t^k(\boldsymbol{\tau}_k, \boldsymbol{l}_k) = S_t^k(\boldsymbol{\tau}_k, \boldsymbol{l}_k)(\mu_t^k(\boldsymbol{\tau}_k, \boldsymbol{l}_k)dt + \sigma_t^k(\boldsymbol{\tau}_k, \boldsymbol{l}_k)dW_t), \quad \theta_k \leq t \leq T,$$

où W est un mouvement brownien unidimensionnel et μ^k et σ^k vérifient les hypothèses usuelles.
De plus, dans ce modèle, chaque défaut induit un saut de l'actif. Nous nous munissons donc de processus $\gamma^k, 0 \leq k \leq n-1$ tels que

$$S_{\theta_{k+1}}^{k+1}(\boldsymbol{\theta}_{k+1}, \boldsymbol{l}_{k+1}) = S_{\theta_{k+1}^-}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\left(1 + \gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\right).$$

On est ainsi amené à décomposer les stratégies de trading π en n processus $(\pi^k)_{0 \leq k \leq n}$ tels que

$$\pi_t = \sum_{k=0}^{n} 1_{\Omega_{t-}^k} \pi_t^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k), \quad 0 \leq t \leq T.$$

De même, le payoff H_T se décompose sous la forme :

$$H_T = \sum_{k=0}^{n} 1_{\Omega_T^k} H_t^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k).$$

Enfin, (1.2.3) nous incite à définir par récurrence descendante, notant $\alpha^n = \alpha$:

$$\alpha_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = \int_t^\infty \int_E \alpha_t^{k+1}(\boldsymbol{\theta}_k, \theta_{k+1}, \boldsymbol{l}_k, l_{k+1}) d\theta_{k+1} \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}), \quad 0 \leq k \leq n-1.$$

On est alors amené à décomposer le problème de couverture moyenne-variance (1.2.1)-(1.2.2) associé aux stratégies de trading π à n sous-problèmes associés à chacun des π^k qui pourraient se reformuler chacun en "problème de couverture moyenne-variance entre le k-ème défaut et l'éventuel $k+1$-ème". Plus précisément, notant $\mathcal{A}^k$ l'ensemble des π^k admissibles, nous introduisons la famille de fonctions $(V^k)_{0 \leq k \leq n}$ définie récursivement par :

$$V^n(x, \boldsymbol{\theta}, \boldsymbol{l}) = \underset{\pi^n \in \mathcal{A}^n}{\operatorname{ess\,inf}} \mathbb{E}\left[(H_T^n - X_T^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l}))^2 \alpha_T(\boldsymbol{\theta}, \boldsymbol{l}) | \mathcal{F}_{\theta_n}\right] \tag{1.2.4}$$

et

$$V^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k) \quad = \quad \operatorname*{ess\,inf}_{\pi^k\in\mathcal{A}^k} \mathbb{E}[(H_T^k - X_T^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \tag{1.2.5}$$

$$\int_{\theta_k}^T\int_E V^{k+1}(X_{\theta_{k+1}}^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) \quad + \quad \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_{\theta_k}],$$

Puisqu'ici il y a n défauts au plus, le sous-problème associé à V^n est sans défaut et donc déjà résolu dans la littérature. Nous allons alors procéder par récurrence descendante sur k pour obtenir V^0 et une stratégie optimale associée $\pi^* = (\pi_k^*)_{0\leq k\leq n}$.
L'hérédité s'obtient en trois étapes :

- Nous supposons que chacun des sous-problèmes admet une décomposition quadratique de la forme V^k. Nous obtenons alors par programmation dynamique que les termes de cette éventuelle décomposition doivent vérifier un système d'EDSRs et un candidat pour une stratégie optimale.
- Ensuite, nous prouvons l'existence de solutions à ce système d'EDSRs. Notre preuve utilise des techniques d'EDSRs et est en ce sens une preuve "purement EDSR" qui est nouvelle dans la littérature. L'une des EDSRs du système est à croissance quadratique, ce qui nous a amené à utiliser le résultat de Kobylanski évoqué dans la section précédente pour cette preuve.
- Enfin, nous prouvons dans un théorème de vérification que le système d'EDSRs étudié a une unique solution qui induit une solution du problème (1.2.1)-(1.2.2). Ne reste plus qu'à vérifier que le candidat π^* est une stratégie admissible pour conclure.

On conclut ce chapitre par des applications numériques. Ici il y a 1 puis 2 défauts indépendants, chacun suivant une loi exponentielle. Les EDSRs deviennent alors des équations différentielles ordinaires, ce qui rend la simulation accessible. Cela nous permet d'obtenir des interprétations graphiques de l'incomplétude du marché et de la variance minimale du portefeuille d'investissement pour un capital donné.

Ce chapitre est tiré d'un article rédigé en collaboration avec Stéphane Goutte et Armand Ngoupeyou [24], à paraître dans Stochastic Analysis and Applications.

1.3 EDSRs réfléchies avec sauts négatifs, et jeux contrôleur et stoppeur

1.3.1 Articles sources

La deuxième application des EDSRs exposée dans cette thèse est l'étude d'EDSRs réfléchies avec sauts négatifs, et son application à des jeux type contrôleur et stoppeur.

Les EDSRs réfléchies sur un obstacle fixé ont été introduites par El Karoui, Kapoudjian, Pardoux, Peng et Quenez [37]. Il s'agit du premier cas d'EDSR avec contraintes, pour lesquelles on impose que la solution Y_t reste systématiquement au-dessus d'un obstacle S_t. Un processus croissant dont le but est de "pousser" la solution de l'EDSR vers le haut est introduit. Plus précisément, nous disons que le triplet de processus adaptés

(Y_t, Z_t, K_t) où K est un processus croissant, est solution de l'EDSR réfléchie sur l'obstacle S avec condition terminale ξ et générateur f si

$$\begin{aligned}
&Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t, \quad 0 \le t \le T, \quad \mathbb{P}-p.s.\\
&Y_t \ge S_t, \le 0 \le t \le T, \quad \mathbb{P}-p.s.\\
&\int_0^T (Y_s - S_s)dK_s = 0, \quad \mathbb{P}-p.s. \qquad (1.3.1)
\end{aligned}$$

La dernière condition dans (1.3.1) stipule que le processus croissant K est minimal au sens où il n'agit que lorsque Y touche l'obstacle. Elle permet d'obtenir l'unicité de la solution d'une telle équation. Dans [37], une preuve d'existence de solution est aussi proposée par pénalisation. De plus, il est prouvé que les EDSRs réfléchies fournissent une représentation probabiliste pour des EDPs quasi-linéaires avec un obstacle.
Considérons désormais l'équation non linéaire de type Hamilton-Jacobi-Bellman (HJB) suivante :

$$\frac{\partial v}{\partial t} + \sup_{a \in A} \left(\langle b(x,a), D_x v\rangle + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D_x^2 v) + f(x,a)\right) = 0, \qquad (1.3.2)$$

sur $[0,T) \times A$, où A est un sous-ensemble de $\mathbb{R}^q$, avec la condition terminale

$$v(T,x) = g(x), \quad x \in \mathbb{R}^d.$$

Il est bien connu, voir par exemple Pham [90] que cette équation est l'équation de programmation dynamique d'un problème de contrôle stochastique dont la fonction valeur est donnée par :

$$v(t,x) := \sup_\alpha \mathbb{E}[\int_t^T f(X_s^{t,x,\alpha}, \alpha_s)ds + g(X_T^{t,x,\alpha})]$$

où $X^{t,x,\alpha}$ est le processus d'état contrôlé partant au temps $t \in [0,T]$ de $x \in \mathbb{R}^d$ qui vérifie sur $[t,T]$ l'équation stochastique

$$X_s^{t,x,\alpha} = x + \int_t^s b(X_r^{t,x,\alpha}, \alpha_r)dr + \int_t^s \sigma(X_r^{t,x,\alpha}, \alpha_r)dW_r \qquad (1.3.3)$$

où α est un processus de contrôle prévisible à valeurs dans A. Remarquons que, si $\sigma(x)$ ne dépend pas de $a \in A$ et que $\sigma\sigma^\intercal(x)$ est inversible, alors l'équation de HJB précédente peut se réécrire :

$$\frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x)D_x^2 v) + F(x, \sigma^\intercal(x)D_x v) = 0, \qquad (1.3.4)$$

où $F(x,z) = \sup_{a\in A}[f(x,a) + \langle\theta(x,a), z\rangle]$ est la θ-transformée de Fenchel-Legendre de f et $\theta(x,a) = \sigma^{\intercal(x)}(\sigma\sigma^\intercal(x))^{-1}b(x,a)$ est une solution de $\sigma(x)\theta(x,a) = b(x,a)$. Nous déduisons alors des travaux de Pardoux et Peng [85, 86] que l'EDP semilinéaire (1.3.4) admet une formule de Feynman-Kac non linéaire à travers une équation différentielle stochastique progressive et rétrograde markovienne.

Le cas général, avec un coefficient de diffusion contrôlée $\sigma(x,a)$ éventuellement dégénéré a été résolu récemment par Kharroubi et Pham [75]. Mentionnons aussi qu'un premier pas avait été accompli par Soner, Touzi et Zhang [101], en utilisant cependant la théorie des EDSRs du second ordre (2EDSRs) plutôt que la théorie classique des EDSRs. Les 2EDSRs sont des EDSRs formulées avec une famille non dominée de mesures de probabilités singulières, leur théorie utilise donc des outils d'analyse quasi-sûre. D'autre part, dans [75], nous nous contentons d'étudier une EDSR avec sauts, où les sauts sont contraints d'être négatifs, formulée selon une unique mesure de probabilité, comme dans la théorie classique des EDSRs.

Présentons brièvement les résultats de l'article [75]. Le système progressif-rétrograde associé à l'équation HJB (1.3.2) est construit en introduisant le système d'équations progressives, partant au temps $t \in [0,T]$ de $(x,a) \in \mathbb{R}^d \times \mathbb{R}^q$:

$$\begin{aligned} X_s^{t,x,a} &= x + \int_t^s b(X_r^{t,x,a}, I_r^{t,a})dr + \int_t^s \sigma(X_r^{t,x,a}, I_r^{t,a})dW_r \quad t \le s \le T, \\ I_s^{t,a} &= a + \int_t^s \int_A (a' - I_{r-}^{t,a})\mu(dr,da') \quad t \le s \le T. \end{aligned}$$

Ces équations sont déduites des dynamiques de l'état contrôlé (1.3.3) en randomisant le processus d'état $X^{t,x,\alpha}$, c'est à dire en introduisant, à la place du contrôle α, un processus de saut pur I dirigé par une mesure aléatoire de Poisson μ sur $\mathbb{R}^+ \times A$ indépendante de W, avec une mesure d'intensité $\lambda(da)dt$, où λ est une mesure finie sur $(A, \mathcal{B}(A))$. W et μ sont définis sur un espace de probabilité filtré $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, où $\mathbb{F}$ est la complétion de la filtration générée par W et μ. Considérons désormais l'équation rétrograde. Comme attendu, elle est dirigée par le mouvement brownien W et la mesure aléatoire de Poisson μ, c'est à dire que c'est une EDSR avec sauts avec condition terminal $g(X_T^{t,x,a})$ et générateur $f(X_\cdot^{t,x,a}, I_\cdot^{t,a})$, ce qui est naturel vu l'expression de l'équation HJB. L'équation rétrograde est aussi caractérisée par une contrainte sur le composant de saut, ce qui s'avère être un point crucial de la théorie introduite dans [75] et requiert, comme les EDSRs réfléchies (voir par exemple (1.3.1)), la présence d'un processus croissant dans l'EDSR. Finalement, l'EDSR prend la forme suivante :

$$\begin{aligned} Y_s^{t,x,a} &= g(X_T^{t,x,a}) + \int_s^T f(X_r^{t,x,a}, I_r^{t,a})dr + K_T^{t,x,a} - K_s^{t,x,a} \\ &\quad - \int_s^T Z_s^{t,x,a} dW_r - \int_s^T \int_A L_r^{t,x,a}(a')\mu(dr,da'), \quad t \le s \le T, p.s. \end{aligned}$$

avec la contrainte de saut

$$L_s^{t,x,a}(a') \le 0, \quad d\mathbb{P} \otimes ds \otimes \lambda(da') p.p.$$

Remarquons que la présence du processus croissant K dans l'équation rétrograde ne garantit pas l'unicité de la solution. C'est pourquoi, comme dans la théorie des EDSRs réfléchies, les auteurs recherchent seulement dans [75] la solution minimale (Y,Z,L,K) de l'EDSR précédente, dans le sens où toute autre solution $(\bar{Y}, \bar{Z}, \bar{L}, \bar{K})$ est telle que $Y \le \bar{Y}$. L'existence de la solution minimale se prouve par pénalisation en utilisant le théorème de limite monotone de Peng [87].

La formule de Feynman-Kac non linéaire devient

$$v(t,x,a) := Y_t^{t,x,a}, \quad (t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q.$$

Observons que la fonction v ne devrait pas dépendre de a, mais de (t,x), conséquence de la contrainte de saut négatifs. En effet, si v est continue, alors

$$L_s^{t,x,a}(a') = v(s, X_s^{t,x,a}, a') - v(s, X_s^{t,x,a}, I_{s-}^{t,a}) \leq 0 \quad d\mathbb{P} \otimes ds \otimes \lambda(da')p.p.$$

dont nous déduisons que v ne dépend pas de a. Cependant, il n'est pas clair a priori que la fonction v est continue, c'est pourquoi, dans [75], la preuve rigoureuse requiert des arguments fins de solutions de viscosité et des hypothèses de régularité sur λ et A. comme celles que l'intérieur de A est connecté et que A est la fermeture de son intérieur. À la fin de [75], il est prouvé que la fonction v ne dépend pas de a dans l'intérieur de A et que la solution de viscosité de (1.3.2) admet la formule de représentation probabiliste suivante :

$$v(t,x) := Y_t^{t,x,a}, \quad (t,x) \in [0,T] \times \mathbb{R}$$

pour tout a dans l'intérieur de A. Cette formule ouvre de nouvelles perspectives pour des schémas probabilistes d'EDPs non linéaires, comme récemment montré par Kharroubi, Langrené et Pham [73].
Dans [75], une autre représentation probabiliste est proposée, appelée représentation duale, pour la solution v de (1.3.2). Plus précisément, soit $\mathcal{V}$ l'ensemble des processus prévisibles $\nu : \Omega \times [O,T] \times A \to (0,\infty)$ qui sont essentiellement bornés et introduisons la mesure de probabilité $\mathbb{P}^\nu$ équivalente à $\mathbb{P}$ sur $(\Omega, \mathcal{F}_T)$ avec densité de Radon-Nikodym :

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}}|_{\mathcal{F}_t} = \zeta_t^\nu := \mathcal{E}_t\Big(\int_0^\cdot \int_A (\nu_s(a) - 1)\tilde{\mu}(ds,da)\Big)$$

où $\mathcal{E}_t(.)$ est l'exponentielle de Doléans-Dade. Ici W reste un mouvement brownien sous $\mathbb{P}^\nu$, et l'effet de la mesure de probabilité $\mathbb{P}^\nu$, par le théorème de Girsanov, est de changer le compensateur $\lambda(da)dt$ de μ sous $\mathbb{P}$ en $\nu_t(a)\lambda(da)dt$ sous $\mathbb{P}^\nu$. La représentation duale s'écrit ainsi :

$$v(t,x) = Y_t^{t,x,a} = \operatorname*{ess\,sup}_{\nu \in \mathcal{V}} \mathbb{E}^\nu\big[g(X_T^{t,x,a}) + \int_t^T f(X_s^{t,x,a}, I_s^{t,a})ds | \mathcal{F}_t\big]$$

où est notée $\mathbb{E}^\nu$ l'espérance par rapport à $\mathbb{P}^\nu$.
Enfin, observons que que les outils utilisés dans [75] peuvent aussi être appliqués à d'autres problèmes de contrôles stochastiques, comme les problèmes de contrôles impulsionnels, voir par exemple [74].

1.3.2 Présentation du problème

Considérons l'équation d'Hamilton-Jacobi-Bellman-Isaacs (HJBI) :

$$\begin{aligned} \max\Big[-\frac{\partial v}{\partial t} - \sup_{a \in A}\big(b(x,a).D_x v + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D_x^2 v) + f(x,a)\big); \\ v - g\Big] &= 0, \end{aligned} \tag{1.3.5}$$

sur $[0,T) \times \mathbb{R}^d$, avec la condition terminale

$$v(T,x) = g(x), \quad x \in \mathbb{R}^d. \tag{1.3.6}$$

Dans Bayraktar et Huang [7], il est prouvé que cette équation est équation de programmation dynamique d'un jeu contrôleur et stoppeur à somme nulle dont les fonctions valeurs supérieure et inférieure sont données par

$$
\begin{aligned}
\bar{V}(t,x) &:= \inf_{\pi\in\Pi_{t,T}} \sup_{\alpha\in\mathcal{A}} \mathbb{E}\Big[\int_t^{\pi[\alpha]} f(X_s^{t,x,\alpha},\alpha_s)ds + g(X_{\pi[\alpha]}^{t,x,\alpha})\Big] \\
\underline{V}(t,x) &:= \sup_{\alpha\in\mathcal{A}} \inf_{\pi\in\mathcal{T}_{t,T}} \mathbb{E}\Big[\int_t^{\tau} f(X_s^{t,x,\alpha},\alpha_s)ds + g(X_\tau^{t,x,\alpha})\Big], \quad (t,x)\in[0,T]\times\mathbb{R}^d,
\end{aligned}
$$

où $X^{t,x,\alpha}$ est un processus de diffusion dans $\mathbb{R}^d$ vérifiant l'équation (1.3.3) contrôlé par un processus prévisible $\alpha\in\mathcal{A}$ à valeurs dans A, $\mathcal{T}_{t,T}$ est l'ensemble des temps d'arrêt à valeurs dans $[t,T]$ pour $0\le t\le T$, et $\Pi_{t,T}$ est l'ensemble des stratégies d'arrêt $\pi:\mathcal{A}\to\mathcal{T}_{t,T}$ satisfaisant une condition de non anticipation (voir la définition 3.1 de [7]). Il est montré dans [7] que ce jeu a une valeur, soit $\bar{V}=\underline{V}=v$, et que v est l'unique solution de viscosité de (1.3.5) - (1.3.6) satisfaisant une condition de croissance polynomiale.

Dans ce chapitre, nous prouvons que la fonction valeur v associée à l'équation HJBI (1.3.5) - (1.3.6) (nous considérons aussi des équations aux dérivées partielles plus générales de type HJBI) admet une représentation probabiliste (une formule de Feynman-Kac non linéaire) à travers une équation différentielle stochastique progressive et rétrograde. En particulier, en s'inspirant des preuves de [75] décrites précédemment et de la théorie des EDSRs réfléchies, nous introduisons une classe d'EDSR réfléchies à sauts négatifs et barrière supérieure. Comme dans le cas des EDSRs doublement réfléchies avec barrières supérieure et inférieure, liées aux jeux de Dynkin, notre classe d'EDSR implique l'introduction de deux processus croissants. Plus précisément, l'équation rétrograde a la forme suivante (nous étudions aussi des EDSRs plus générales, où le générateur f dépend aussi de $Y^{t,x,a}$ et $Z^{t,x,a}$, et même de la composante de saut dans le cas non markovien) :

$$
\begin{aligned}
Y_s^{t,x,a} &= g(X_T^{t,x,a}) + \int_s^T f(X_r^{t,x,a}, I_r^{t,a})dr + K_T^{t,x,a,+} - K_s^{t,x,a,+} \\
&\quad -(K_T^{t,x,a,-} - K_s^{t,x,a,-}) - \int_s^T Z_r^{t,x,a}dW_r \\
&\quad -\int_s^T \int_A L_r^{t,x,a}(a')\mu(dr,da'), \quad t\le s\le T, p.s.
\end{aligned}
$$

avec la contrainte de saut

$$
L_s^{t,x,a}(a') \le 0, \quad d\mathbb{P}\otimes ds\otimes\lambda(da') p.p.
$$

et la contrainte supérieure

$$
\begin{aligned}
Y_s^{t,x,a} &\le g(X_s^{t,x,a}), \quad t\le s\le T, p.s. \\
\int_t^T (g(X_s^{t,x,a}) - Y_{s-}^{t,x,a})dK_s^{t,x,a,-} &= 0, \quad p.s. \qquad (1.3.7)
\end{aligned}
$$

Notons que la présence du processus $K^{t,x,a,-}$ force la solution Y d'être sous l'obstacle supérieur $g(X_\cdot^{t,x,a})$. De plus, par la condition de Skorohod (1.3.7), $K^{t,x,a,-}$ est minimal. D'autre part, le processus $K^{t,x,a,+}$ est associé à la contrainte de saut, comme dans [75].

Pour garantir l'unicité de la solution, nous cherchons uniquement la solution minimale (Y, Z, L, K^+, K^-) de l'EDSR précédente dans le sens où toute autre solution $(\bar{Y}, \bar{Z}, \bar{L}, \bar{K}^+, \bar{K}^-)$ est telle que $Y \leq \bar{Y}$.
L'existence d'une solution minimale nécessite une hypothèse supplémentaire de régularité de la barrière supérieure, qui est l'équivalent dans notre contexte à la condition de Mokobodzki. Sous cette hypothèse, nous prouvons l'existence dans un contexte non markovien en utilisant une double pénalisation et un théorème de limite monotone pour les EDSRs avec sauts. Plus précisément, introduisons la suite d'EDSRs à sauts :

$$\begin{aligned} Y_s^{n,m,t,x,a} &= g(X_T^{t,x,a}) + \int_s^T f(X_r^{t,x,a}, I_r^{t,a})dr + K_T^{n,m,t,x,a,+} - K_s^{n,m,t,x,a,+} \\ &\quad -(K_T^{n,m,t,x,a,-} - K_s^{n,m,t,x,a,-}) - \int_s^T Z_r^{n,m,t,x,a} dW_r \\ &\quad - \int_s^T \int_A L_r^{n,m,t,x,a}(a')\mu(dr, da'), \quad t \leq s \leq T, p.s. \end{aligned}$$

pour $n, m \in \mathbb{N}$, où $K_s^{n,m,t,x,a,+}$ et $K_s^{n,m,t,x,a,-}$ sont les processus croissants définis par

$$K_s^{n,m,t,x,a,+} = m \int_s^T \int_A (L_r^{n,m,t,x,a})_+(a)\lambda(da)dr$$

et

$$K_s^{n,m,t,x,a,-} = n \int_s^T (g(X_r^{t,x,a}) - Y_r^{n,m,t,x,a})_- dr.$$

Ici nous utilisons les notations usuelles $f_+ = \max(f, 0)$ et $f_- = \max(-f, 0)$ pour les parties positives et négatives de f. La solution minimale de l'EDSR réfléchie avec sauts négatifs est obtenue en passant à la limite en n puis en m et en utilisant un théorème de limite monotone. Ce dernier est basé sur des estimations uniformes de $(Y^{n,m,t,x,a}, Z^{n,m,t,x,a}, L^{n,m,t,x,a}, K^{n,m,t,x,a,+}, K^{n,m,t,x,a,-})$, ce qui s'avère être la principale difficulté, principalement à cause de l'existence des processus $K^{n,m,t,x,a,+}$ et $K^{n,m,t,x,a,-}$. C'est ici que l'hypothèse de régularité sur la barrière supérieure intervient. Remarquons que l'ordre des limites importe ici, contrairement au cas des réflexions supérieure et inférieur associé aux jeux de Dynkin. En effet, nous n'avons pas de résultat de comparaison sur la composante de saut de la solution de l'EDSR, et a priori peu d'information sur la suite de processus croissants associés à la contrainte de saut, tandis qu'on peut exploiter des résultats de comparaison sur Y pour obtenir la monotonie de la suite de processus croissants associés à la barrière supérieure.
La formule de Feynman-Kac non linéaire s'avère être

$$v(t, x, a) = Y_t^{t,x,a}, \quad (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q.$$

Comme dans [75], il apparaît que v ne dépend pas de a dans l'intérieur de A, conséquence de la contrainte de sauts négatifs. Nous prouvons que v est une solution de viscosité de l'équation HJBI (1.3.5) avec la condition terminale (1.3.6). Nous étudions aussi des équations HJBI plus générales que (1.3.5), où le générateur $f(x, a, v, \sigma^\intercal D_x v)$ dépend aussi éventuellement de v et $D_x v$.

Enfin, nous prouvons une formule de représentation duale du jeu pour la solution minimale de notre EDSR, qui est inspirée de la représentation duale obtenue dans [75] et la formule de représentation duale de la Proposition 6.2 de [31]. Donnons une intuition de cette représentation duale. En plus de l'ensemble de mesures de probabilité $\mathbb{P}^\nu, \nu \in \mathcal{V}$ défini dans la sous-section précédente, introduisons l'ensemble Θ des facteurs d'actualisations, c'est à dire des processus progressivement mesurables $\theta : \Omega \times [0,T] \to \mathbb{R}_+$ qui sont essentiellement bornés. Alors la formule de de représentation duale devient, pour $s \in [0,T]$:

$$Y_s^{t,x,a} = \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \operatorname*{ess\,inf}_{\theta\in\Theta} \mathbb{E}^\nu\big[e^{-\int_s^T \theta_r dr} g(X_T^{t,x,a}) + \int_s^T e^{-\int_s^r \theta_u du}(f(X_r^{t,x,a}, I_r^{t,a}) + \theta_r g(X_r^{t,x,a}))dr \big| \mathcal{F}_s\big].$$

C'est une représentation originale de la fonction valeur d'un jeu contrôleur et stoppeur à somme nulle. Nous ne savons pas dans le cas général s'il est possible d'échanger les supremum et infimum dans la formule de représentation. Mais en prenant d'abord la limite par rapport à m puis par rapport à n dans la suite des équations pénalisées, nous obtenons un processus $\hat{Y}^{t,x,a}$ tel que $Y^{t,x,a} \leq \hat{Y}^{t,x,a}$, or il n'est pas clair si c'est une solution de l'EDSR et s'il est égal à $Y^{t,x,a}$. Cependant, $\hat{Y}^{t,x,a}$ admet la représentation, pour $s \in [t,T]$:

$$\hat{Y}_s^{t,x,a} = \operatorname*{ess\,inf}_{\theta\in\Theta} \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\big[e^{-\int_s^T \theta_r dr} g(X_T^{t,x,a}) + \int_s^T e^{-\int_s^r \theta_u du}(f(X_r^{t,x,a}, I_r^{t,a}) + \theta_r g(X_r^{t,x,a}))dr \big| \mathcal{F}_s\big].$$

Ce chapitre est tiré d'un article rédigé en collaboration avec Andrea Cosso et Huyên Pham [23], publié dans *Stochastic Processes and their Applications*.

1.3.3 Perspectives

Les jeux stochastiques contrôleur et stoppeur ont de nombreuses applications en mathématiques financières, par exemple la valorisation des options américaines sous contraintes, voir Karatzas et Kou [65] et Karatzas et Zamfirescu [67]. En effet, dans le cas sans contrainte, il est bien connu qu'il existe, en l'absence d'arbitrage, un unique prix d'option américaine qui s'avère être le supremum, sur tous les temps d'arrêt, de l'espérance de gain actualisée de l'option sous la mesure de martingale équivalente. En la présence de contraintes, cependant, il existe un intervalle $[h_1, h_2]$ de prix sans arbitrage. Selon [65] et [67], les extrémités de l'intervalle peuvent être caractérisées comme les fonctions supérieures et inférieures d'un jeu contrôleur et stoppeur à somme nulle. Un autre exemple d'application aux mathématiques financières est donné dans [8]. Il y est montré que le problème de minimisation de la probabilité de ruine avant la mort, quand le taux de consommation est stochastique et que l'individu peut investir dans un marché de type Black & Scholes, peut être reformulé en un jeu contrôleur et stoppeur.

Enfin, notons que la formule de représentation probabiliste obtenue suggère une nouvelle approche de schémas numériques probabilistes des équations HJBI par discrétisation et simulation d'EDSRs réfléchies avec sauts négatifs et obstacle supérieur. Une autre classe d'EDSRs que l'on pourrait étudier est celle des EDSRs réfléchies avec sauts négatifs et obstacle inférieur, qui est lié au problème sup sup sur le contrôle et le temps d'arrêt, autrement dit du temps d'arrêt optimal sous espérance non linéaire. La preuve de l'existence de la solution minimale par double pénalisation s'avère plus simple puisqu'on est amené à étudier la somme, plutôt que la différence, de deux processus croissants.

1.4 Représentation d'EDSR pour des problèmes de contrôle stochastique avec intensité contrôlée et non dominée

1.4.1 Articles sources

La troisième application des EDSRs exposée dans cette thèse est l'étude de la représentation d'EDSR pour des problèmes de contrôle stochastique avec intensité contrôlée et non dominée.

Ce chapitre est également une extension de [75]. Le lecteur est renvoyé à la section précédente pour une description des résultats de cet article. Un des résultats de [75] est l'obtention d'une formule de Feynman-Kac pour l'équation intégro-différentielle aux dérivées partielles non linéaire :

$$\frac{\partial v}{\partial t} + \sup_{a\in A}\Big[b(x,a).D_xv + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D_x^2v) + f(x,a,v,\sigma^\intercal(x,a)D_xv) \tag{1.4.1}$$
$$+\int_E (v(t,x+\beta(x,a,e)) - v(t,x) - \beta(x,a,e).D_xv(t,x))\lambda(de)\Big] \;=\; 0, \qquad \text{on } [0,T)\times\mathbb{R}^d,$$
$$v(T,x) \;=\; g(x), \qquad x\in\mathbb{R}^d,$$

où A est un sous-ensemble compact de $\mathbb{R}^q$, E un sous-ensemble borélien de $\mathbb{R}^k\backslash\{0\}$, et λ est une mesure positive σ-finie sur $(E,\mathcal{B}(E))$ telle que $\int_E(1\wedge|e|^2)\lambda(de)<\infty$.
Un cas particulier est l'équation HJB associée au modèle de volatilité incertaine en finance mathématique, qui prend la forme suivante :

$$\frac{\partial v}{\partial t} + G(D_x^2 v) \;=\; 0, \quad \text{on } [0,T)\times\mathbb{R}^d, \qquad v(T,x) \;=\; g(x), \quad x\in\mathbb{R}^d, \tag{1.4.2}$$

où $G(M)=\frac{1}{2}\sup_{c\in C}[cM]$ et C est un ensemble de matrices symétriques positives d'ordre d. Il est montré dans [89], que l'unique solution de viscosité de (1.4.2) est représentée en terme de ce qui est appelé un G-mouvement brownien B sous l'espérance non linéaire $\mathcal{E}(\cdot)$ de la façon suivante :

$$v(t,x) \;=\; \mathcal{E}\big(g(x+B_T-B_t)\big).$$

La simulation d'un G-mouvement brownien reste cependant un problème ouvert.

Nous nous intéressons dans ce chapitre à l'équation intégro-différentielle aux dérivées partielles non linéaire suivante :

$$\frac{\partial v}{\partial t} + \sup_{a\in A}\Big[b(x,a).D_xv + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D_x^2v) + f(x,a,v,\sigma^\intercal(x,a)D_xv) \tag{1.4.3}$$
$$+\int_E (v(t,x+\beta(x,a,e)) - v(t,x) - \beta(x,a,e).D_xv(t,x))\lambda(a,de)\Big] = 0, \quad \text{on } [0,T)\times\mathbb{R}^d,$$

$$v(T,x) = g(x), \qquad x \in \mathbb{R}^d,$$

où λ est un noyau de transition de $(A, \mathcal{B}(A))$ vers $(E, \mathcal{B}(E))$. Nous ne supposons pas que $(\lambda(a,\cdot))_{a\in A}$ est dominée. De plus, le coefficient de diffusion σ peut être dégénéré.
Une des motivations de l'étude de l'équation (1.4.3) vient des mathématiques financières, et de l'incertitude de modèle en particulier, quand l'incertitude affecte à la fois la volatilité et l'intensité. Ce sujet a été étudié à l'aide des 2EDSRs avec sauts dans [70] et [71]. Cependant, cette méthode ne traite pas le cas où la volatilité est dégénérée, contrairement à la nôtre. De plus, nous pourrions reprendre les arguments développés dans [72] et [72] pour obtenir un schéma numérique efficace pour l'équation (1.4.3).
L'incertitude de modèle est aussi liée à la théorie des processus de G-Lévy et, plus généralement, aux processus Lévy non linéaires, voir [54] et [83]. Dans ce cas particulier, l'équation intégro-différentielle aux dérivées partielles non linéaire associée prend la forme suivante :

$$\begin{aligned} &\frac{\partial v}{\partial t} + \sup_{(b,c,F)\in\Theta} \Big[b.D_x v + \frac{1}{2}\mathrm{tr}(cD_x^2 v) \qquad\qquad (1.4.4)\\ &+ \int_E \big(v(t,x+z) - v(t,x) - D_x v(t,x).z1_{\{|z|\le 1\}}\big)F(dz)\Big] \;=\; 0, \qquad \text{on } [0,T)\times\mathbb{R}^d,\\ &\qquad\qquad v(T,x) \;=\; g(x), \qquad x\in\mathbb{R}^d, \end{aligned}$$

où Θ est un ensemble de triplets de Lévy (b,c,F), $b \in \mathbb{R}^d$, c est une matrice symétrique positive d'ordre d et F est une mesure de Lévy measure sur $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Par [54] et [83], nous savons que l'unique solution de viscosité de l'équation (1.4.4) est représentée en terme de processus de Lévy non linéaire $\mathcal{X}$ sous l'espérance non linéaire $\mathcal{E}(\cdot)$ de la façon suivante :

$$v(t,x) \;=\; \mathcal{E}(g(x + \mathcal{X}_T - \mathcal{X}_t)).$$

Si nous sommes capables de décrire l'ensemble Θ à l'aide du paramètre a dans l'ensemble compact A d'un espace euclidien $\mathbb{R}^q$, alors (1.4.4) peut être réécrit sous la forme (1.4.3). Ainsi, v est aussi donnée par notre formule de représentation probabiliste, dans laquelle le processus progressive est éventuellement plus facile à simuler qu'un processus de Lévy non linéaire.

1.4.2 Présentation du problème

Pour résoudre (1.4.3), comme dans [75] et dans la section précédente, il nous faut introduire un problème de contrôle stochastique optimal dont une solution de l'équation (1.4.3) est la fonction valeur. Cependant, nous n'avons pas de référence dans la littérature pour cela, c'est pourquoi nous introduisons nous-mêmes un tel problème.
Décrivons brièvement comment ici. Soit $(\bar\Omega, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ un espace de probabilité complet vérifiant les conditions usuelles sur lequel est défini un mouvement brownien d-dimensionnel $\bar W = (\bar W_t)_{t\ge 0}$. Soit également $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t\ge 0}$ la complétion usuelle completion de la filtration naturelle générée par $\bar W$ et $\bar{\mathcal{A}}$ la classe des processus de contrôles α, qui sont $\bar{\mathbb{F}}$-prévisible et à valeurs dans A. Soit aussi Ω' l'espace canonique des processus ponctuels marqués sur $\mathbb{R}^+ \times E$ avec la filtration canonique continue à droite $\mathbb{F}'$ et la mesure aléatoire canonique π'. Considérons ensuite $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\ge0})$ défini par $\Omega := \bar\Omega \times \Omega'$, $\mathcal{F}$

$:= \bar{\mathcal{F}} \otimes \mathcal{F}'_\infty$, et $\mathcal{F}_t := \cap_{s>t} \bar{\mathcal{F}}_s \otimes \mathcal{F}'_s$. De plus, posons $W(\omega) := \bar{W}(\bar{\omega})$, $\pi(\omega, \cdot) := \pi'(\omega', \cdot)$, et $\mathcal{A} := \{\alpha\colon \alpha(\omega) = \bar{\alpha}(\bar{\omega}), \forall\, \omega \in \Omega, \text{ pour } \bar{\alpha} \in \bar{\mathcal{A}}\}$. Supposons que pour chaque $\alpha \in \mathcal{A}$ nous sommes capables de construire une mesure $\mathbb{P}^\alpha$ sur $(\Omega, \mathcal{F})$ telle que W est un mouvement brownien et π est une mesure aléatoire à valeurs entières avec compensateur $1_{\{t<T_\infty\}}\lambda(\alpha_t, de)dt$ sur $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\alpha)$, où T_∞ est le supremum des temps de sauts des processus ponctuels marqués associés à π. Considérons alors le problème de contrôle stochastique dont la fonction valeur est donnée par (notant $\mathbb{E}^\alpha$ l'espérance par rapport à $\mathbb{P}^\alpha$)

$$v(t,x) \quad := \quad \sup_{\alpha \in \mathcal{A}} \mathbb{E}^\alpha \Big[\int_t^T f(X_s^{t,x,\alpha}, \alpha_s) ds + g(X_T^{t,x,\alpha}) \Big], \tag{1.4.5}$$

où $X^{t,x,\alpha}$ suit la dynamique contrôlée sur $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\alpha)$:

$$dX_s^\alpha \quad = \quad b(X_s^\alpha, \alpha_s)ds + \sigma(X_s^\alpha, \alpha_s)dW_s + \int_E \beta(X_{s^-}^\alpha, \alpha_s, e)\tilde{\pi}(ds, de)$$

partant de x à l'instant t, avec $\tilde{\pi}(dt, de) = \pi(dt, de) - 1_{\{t<T_\infty\}}\lambda(\alpha_t, de)dt$ la mesure de martingale compensée de π. On s'attend à ce que ce soit le problème recherché, dans le sens où l'EDP (1.4.3) s'avère être l'équation de programmation dynamique du problème de contrôle stochastique dont la fonction valeur est donnée par (1.4.5).

Comme dans [75] et dans la section précédente, nous randomisons le contrôle. Pour ce faire, nous introduisons sur $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ un q-dimensionnel mouvement brownien $\bar{B} = (\bar{B}_t)_{t\geq 0}$, indépendant de $\bar{W}$. $\bar{\mathbb{F}}$ est désormais la complétion usuelle de la filtration naturelle générée par $\bar{W}$ et $\bar{B}$. Nous posons aussi $B(\omega) := \bar{B}(\bar{\omega})$, pour tout $\omega \in \Omega$, B est donc défini sur Ω. Puisque le contrôle est à valeurs dans l'ensemble compact $A \subset \mathbb{R}^q$, nous ne pouvons pas utiliser directement B pour randomiser le contrôle, il nous faut introduire une fonction qui envoie B sur A. C'est pourquoi nous supposon l'existence d'une surjection continue $h\colon \mathbb{R}^d \to A$. Alors, pour chaque $(t, x, \tilde{a}) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$, nous considérons l'équation différentielle stochastique progressive sur $\mathbb{R}^d \times \mathbb{R}^q$ suivante :

$$X_s \quad = \quad x + \int_t^s b(X_r, I_r)dr + \int_t^s \sigma(X_r, I_r)dW_r + \int_t^s \int_E \beta(X_{r^-}, I_r, e)\tilde{\pi}(dr, de), \tag{1.4.6}$$

$$I_s \quad = \quad h(\tilde{a} + B_s - B_t), \tag{1.4.7}$$

pour tout $t \leq s \leq T$, où $\tilde{\pi}(ds, de) = \pi(ds, de) - 1_{\{s<T_\infty\}}\lambda(I_s, de)ds$ est la mesure de martingale compensée de π, qui est une mesure aléatoire à valeurs entières de compensateur $1_{\{s<T_\infty\}}\lambda(I_s, de)ds$. Contrairement à [75] et à la section précédente, nous utilisons un mouvement brownien B pour randomiser le contrôle, plutôt qu'une mesure aléatoire de Poisson μ sur $\mathbb{R}_+ \times A$. D'une part, la mesure aléatoire de Poisson s'avère être plus adaptée à un ensemble compact A, puisque μ est déjà de support $\mathbb{R}_+ \times A$, donc nous n'avons pas à introduire de surjection h de $\mathbb{R}^q$ dans A, comme nous l'avons fait ici. D'autre part, le choix d'un mouvement brownien B est plus adapté pour obtenir un théorème de représentation de martingale pour notre modèle. En effet, contrairement au modèle de [75] ou à celui de la section précédente, l'intensité de la mesure π dépend du processus I, il est donc naturel d'obtenir une dépendance entre π et le bruit utilisé pour randomiser le contrôle. L'avantage de B par rapport à μ est que B est *orthogonal* à π, puisque B est continue (une défintion de l'orthogonalité entre une martingale et une mesure aléatoire

est donnée au bas de la page 183 de [59]). Grâce à l'orthogonalité nous pouvons obtenir un théorème de représentation de martingale dans notre contexte, ce qui est essentiel pour obtenir la formule de représentation de Feynman-Kac souhaitée.

Prêtons attention à l'équation à l'équation différentielle stochastique (1.4.6)-(1.4.7). Nous constatons que la partie de saut dans (1.4.6) n'est pas donnée, mais dépend de la solution via son intensité. Ceci rend non standard l'EDS (1.4.6)-(1.4.7). Ce type d'équations ont été d'abord étudiées dans [58] et apparaissent aussi dans la littérature, voir par exemple [9], [27], [28], [29] ou [42]. Cependant, dans [9], [27] et [28], λ est absolument continue par rapport à une mesure déterministe donnée sur $(E, \mathcal{B}(E))$, ce qui permet de résoudre (1.4.6)-(1.4.7) en se ramenant à une EDS standard, via un changement d'intensité "à la Girsanov". Par contre, dans ce chapitre, nous résolvons d'abord (1.4.7) pour tout $(t, \tilde{a}) \in [0, T] \times \mathbb{R}^q$, puis nous construisons une mesure de probabilité $\mathbb{P}^{t,\tilde{a}}$ sur $(\Omega, \mathcal{F})$ telle que la mesure aléatoire $\pi(ds, de)$ ait pour compensateur $\lambda(I_s^{t,\tilde{a}}, de)ds$, puis finalement nous résolvons (1.4.6). Dans l'appendice, nous prouvons aussi des propriétés de π et (X, I). Plus précisément, nous présentons une caractérisation de π en termes de transformées de Fourier et Laplace, ce qui montre que π est une mesure aléatoire de Poisson (aussi appélée mesure aléatoire de Cox) conditionnellement à $\sigma(I_s^{t,\tilde{a}}; s \geq 0)$. De plus, nous étudions les propriétés de Markov de (X, I).
L'EDSR correspondante est, comme attendu, dirigée par les mouvements browniens W et B, et par la mesure aléatoire π, c'est donc une EDSR avec sauts, de condition terminale $g(X_T^{t,x,\tilde{a}})$ et de générateur $f(X_\cdot^{t,x,\tilde{a}}, I_\cdot^{t,\tilde{a}}, y, z)$, comme l'indiquait l'équation (1.4.3). L'EDSR est aussi caractérisée par une contrainte sur la diffusion associée à B, qui s'avère cruciale et implique l'introduction d'un processus croissant dans l'EDSR. Finalement, pour tout $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, l'EDSR prend la forme suivante :

$$\begin{aligned} Y_s &= g(X_T^{t,x,\tilde{a}}) + \int_s^T f(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}, Y_r, Z_r)dr + K_T - K_s - \int_s^T Z_r dW_r \\ &\quad - \int_s^T V_r dB_r - \int_s^T \int_E U_r(e)\tilde{\pi}(dr, de), \qquad t \leq s \leq T, \ \mathbb{P}^{t,\tilde{a}} \ p.s. \end{aligned}$$

et

$$|V_s| = 0 \qquad ds \otimes d\mathbb{P}^{t,\tilde{a}} \ p.p.$$

Comme dans [75] et dans la section précédente, la présence du processus croissant K dans l'EDSR nous pousse à rechercher la solution minimale (Y, Z, V, U, K) de cette EDSR, au sens où pour toute autre solution $(\bar{Y}, \bar{Z}, \bar{V}, \bar{U}, \bar{K})$ nous avons nécessairement $Y \leq \bar{Y}$. L'existence de la solution minimale se montre également par pénalisation. Nous obtenons finalement la formule de Feynman-Kac non linéaire suivante :

$$v(t, x, \tilde{a}) := Y_t^{t,x,\tilde{a}}, \qquad (t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q.$$

De même que dans [75] et dans la section précédente, v ne dépend pas de $\tilde{a}$, mais uniquement de (t, x) et la preuve utilise des arguments de solutions de viscosité. Nous montrons aussi que v est l'unique solution de viscosité de (1.4.3), conséquence d'un théorème de comparaison prouvé en appendice. À cause de la présence de l'EDSR de la famille de mesures non dominées $(\lambda(a, \cdot))_{a \in A}$, nous n'avons pas trouvé dans la littérature de théorème de comparaison pour la solution de viscosité de notre équation (1.4.3). C'est pourquoi nous prouvons ce théorème en appendice, la preuve s'inspirant de l'article [4], en utilisant notamment le lemme de Jensen-Ishii pour les équations intégro-différentielles aux dérivées partielles.

Ce chapitre est tiré d'un article rédigé en collaboration avec Andrea Cosso [22], à paraître dans Annals of Applied Probability.

1.5 Gestion actif-passif conditionnelle

1.5.1 Articles sources

L'objectif de ce chapitre est d'obtenir une gestion actif-passif optimale, dans un contexte où le gestionnaire d'actif fait face à une contrainte sur la distribution de sa richesse à l'instant final. Plus précisément, l'investisseur doit payer, pour simplifier, une somme constante D_0 à maturité T et s'autorise à ne pas respecter cette contrainte avec une petite probabilité $1-p$. En pratique, quand un investisseur conservateur impose une contrainte presque sûre sur la valeur finale d'une stratégie d'investissement, il est plutôt amené à faire des choix trop prudents. Cela vient principalement du fait qu'il est trop coûteux de prendre un risque, puisque cela compliquera la nécessité de satisfaire la contrainte à maturité. Le principal objectif de ce chapitre est de quantifier l'effet d'un faible affaiblissement de cette contrainte en imposant seulement que la probabilité de succès à maturité soit supérieure à p, et de mesurer la dépendance en p de la gestion actif-passif optimale.

La théorie moderne de portefeuille en temps continue remonte à l'article pionnier de Merton [82], qui traite le cas d'un agent essayant de maximiser l'espérance de son utilité de la richesse terminale ou l'espérance de l'utilité de l'intégrale avec le temps de la consommation. Dans un modèle markovien, la stratégie optimale est caractérisée en terme de solution d'une équation correspondante de type Hamilton-Jacobi-Bellman, ou bien peut aussi être obtenue par des arguments de dualité, voir par exemple Karatzas, Lehoczky et Shreeve [64]. Ce modèle a été abondamment étudié, avec l'introduction de contraintes additionnelles : par exemple sur la stratégie d'investissement par Cvitanic et Karatzas dans [30], dans un portefeuille d'assurances avec contrainte presque sûre donnée par El Karoui, Jeanblanc et Lacoste dans [36] et avec des contraintes de diminution par Elie et Touzi dans [35]. Dans ce contexte, considérant la contrainte de battre un marché donné avec probabilité de succès donnée, ce problèmé a déjà été étudié par Boyle et Tian dans [18], par un argument de dualité, principalement inspiré de l'approche de Follmer et Leukert dans [44] pour des problèmes de couverture en quantile. Dans la littérature récente, une nouvelle approche introduite par Bouchard, Elie et Touzi dans [15] permet d'étudier ces problèmes a priori dynamiquement inconsistants avec une méthode dynamique.

La principale difficulté à considérer des contraintes écrites en termes de probabilité, c'est que la probabilité de succès p est imposée au temps 0, mais pour essayer d'obtenir un principe de programmation dynamique, il faut être capable de quantifier l'effet d'une telle contrainte à toute date intermédiaire t. La méthode pour traiter ce problème a été identifiée dans [15], où la probabilité dynamique de succès est vue comme un nouveau processus progressif contrôlé de martingale. Cette nouvelle variable permet de résoudre le problème dynamiquement d'une manière consistante. La résolution des problèmes de contrôle stochastique sous ce type de quantile a été plus spécifiquement étudiée dans [14], par un principe de programmation dynamique.

1.5.2 Présentation du problème

Nous considérons un investisseur qui peut à tout instant t choisir d'investir une proportion θ_t de son capital, avec un taux de consommation instantané c_t, qui est positif et majoré par une constante $\bar{c}$ donnée. Nous notons respectivement $\mathcal{A}$ et $\mathcal{C}$ les ensembles des stratégies admissibles d'investissement et de consommation. La dynamique de la richesse est donc la suivante :

$$dX_t^{\theta,c} = \theta_t X_t^{\theta,c} \frac{dS_t}{S_t} + c_t dt,$$

et nous imposons à la richesse de rester positive et de satisfaire la contrainte :

$$X_t \geq 0 \text{ pour tout } t \geq 0 \text{ p.s.} \qquad \text{et} \quad \mathbb{P}[X_T \geq D_0] \geq p \ , \tag{1.5.1}$$

Nous notons $\mathcal{A}_p(t,x)$ l'ensemble des stratégies admissibles d'investissement et de consommation dont le processus de richesse correspondant satisfait cette contrainte de couverture partielle.

Nous considérons un gestionnaire d'actif insensible au risque dont le taux d'actualisation subjectif est une constante donnée $\beta > 0$. Pour une richesse initiale donnée $x \geq 0$ et une probabilité p de succès, le gestionnaire d'actif souhaite résoudre le problème suivant d'investissement et de consommation sous la contrainte de couverture partielle (1.5.1) :

$$w(0,x,p) \ := \inf_{(c,\theta)\in\mathcal{A}_p(0,x)} \mathbb{E}\left[\int_0^T e^{-\beta t} c_t dt\right] \ ,$$

dont la version dynamique est la suivante :

$$w(t,x,p) \ := \inf_{(c,\theta)\in\mathcal{A}_p(t,x)} \mathbb{E}\left[\int_t^T e^{-\beta(s-t)} c_s ds\right] \ .$$

Nous devons d'abord déterminer le domaine de définition de la fonction w. Pour cela, nous introduisons la fonction de richesse minimale définie par :

$$u(t,p) = \inf\{x \geq 0 \,|\mathcal{A}_p(t,x) \neq \emptyset\}.$$

On en déduit que w est défini sur l'ensemble $\{(t,x,p) \in [0,T] \times \mathbb{R}_+ \times [0,1] | x \geq u(t,p)\}$. Nous introduisons une variable d'état contrôlée supplémentaire α, à valeurs dans $[0,1]$ et définie par :

$$P_t^{t,p,\alpha} = p, \quad dP_s^{t,p,\alpha} = \alpha_s dW_s^S, \ s \in [t,T],$$

Nous notons $\mathcal{B}$ l'ensemble de ces contrôles. Nous montrons ensuite que, notant $g(s,p) = 0 * \mathbf{1}_{s<T} + D_0 1_{p>0} \mathbf{1}_{s=T}$:

$$u(t,p) = \inf\{x \in \mathbb{R} \text{ s.t. } \exists(\theta,\alpha) \in \mathcal{A}\times\mathcal{B}, \forall s \in [t,T], X_s^{t,x,\theta,\bar{c}} \geq g(s, P_s^{t,p,\alpha})\}$$

Ainsi, notant $\mathcal{T}_{[t,T]}$ l'ensemble des temps d'arrêt à valeurs dans $[t,T]$, $u(.,P^{\alpha}_{.})$ vérifie le principe de programmation dynamique suivant :

– **(DP1)** Si $x > u(t,p)$, alors il existe $(\theta^*, \alpha^*) \in \mathcal{A} \times \mathcal{B}$ tel que

$$X_{\tau}^{t,x,\theta^*,\bar{c}} \geq u(\tau, P_{\tau}^{t,p,\alpha^*}) \text{ pour tout } \tau \in \mathcal{T}_{[t,T]}$$

– **(DP2)** Si $x < u(t,p)$, alors il existe $\tau^* \in \mathcal{T}_{[t,T]}$ tel que

$$\mathbb{P}[X_{\tau\wedge\tau^*}^{t,x,\theta,\bar{c}} > u(\tau, P_{\tau}^{t,p,\alpha})\mathbf{1}_{\tau<\tau^*} + g(\tau^*, P_{\tau^*}^{t,p,\alpha})\mathbf{1}_{\tau\geq\tau^*}] < 1$$

pour tout $\tau \in \mathcal{T}_{[t,T]}$ et $(\theta,\alpha) \in \mathcal{A} \times \mathcal{B}$.

Dès lors, notant

$$F^{\alpha,\theta}(z,a) := -\frac{\alpha^2}{2}a + \mu\theta z + \bar{c}$$

et

$$F(z,q,a) := \sup_{\{(\alpha,\theta)\in\mathbb{R}^2,\ \alpha q = \sigma\theta z\}} F^{\alpha,\theta}(z,a),$$

u est une solution de viscosité de

$$\min([-\frac{\partial\varphi}{\partial t}(t,p) + F(\varphi(t,p), \frac{\partial\varphi}{\partial p}(t,p), \frac{\partial^2\varphi}{\partial p^2}(t,p))], u) = 0$$

avec la condition terminale :

$$\varphi(T,p) = D_0 p.$$

De plus, nous montrons la condition de bord $u(.,0) = 0$, ce qui permet d'obtenir u numériquement.

Introduisant la transfomée duale de Fenchel-Legendre associée à u par rapport à la variable p :

$$v(t,q) = \sup_{p\in[0,1]} \{pq - u(t,p)\}, \qquad (t,q) \in [0,T] \times \mathbb{R}_+^*,$$

nous obtenons que v est solution de viscosité sur $[0,T) \times (0,\infty)$ de

$$\max(-\frac{\partial\varphi}{\partial t}(t,q) - \frac{\mu^2}{2\sigma^2}q^2\frac{\partial^2\varphi}{\partial q^2}(t,q) - \bar{c}, \varphi - q\frac{\partial\varphi}{\partial q}) = 0$$

avec la condition terminale :

$$v(T,q) = (q - D_0)^+.$$

Ce résultat permet un schéma pour u dont la convergence est rapide. Nous posons, pour $(x,z,q,a_{11},a_{12},a_{22}) \in \mathbb{R}^6$:

$$H^{\theta,\alpha,c}(x,z,q,a_{11},a_{12},a_{22}) := -(\mu\theta x + c)q - \frac{\sigma^2\theta^2x^2}{2}a_{11} - \alpha\sigma\theta x a_{12} - \frac{\alpha^2}{2}a_{22} - c + \beta z$$

et

$$H(x,z,q,a_{11},a_{12},a_{22}) := \sup_{(\theta,\alpha,c)\in\mathbb{R}^2\times[0,\bar{c}]} H^{\theta,\alpha,c}.$$

Nous obtenons alors que sur $int(u) := \{(t,x,p)|x > u(t,p)\}$, w est solution de viscosité de :

$$-\frac{\partial\varphi}{\partial t} + H(x,\varphi(t,x,p),\frac{\partial\varphi}{\partial x}(t,x,p),\frac{\partial^2\varphi}{\partial x^2}(t,x,p),\frac{\partial^2\varphi}{\partial x\partial p}(t,x,p),\frac{\partial^2\varphi}{\partial p^2}(t,x,p)) = 0.$$

Comme conséquence, nous obtenons que la consommation optimale est $c = 0$ quand $\frac{\partial w}{\partial x} > -1$ et $c = \bar{c}$ quand $\frac{\partial w}{\partial x} < -1$. Nous obtenons aussi la condition au bord suivante : pour $(t,p) \in [0,T]\times[0,1]$ tel que $u(t,p) > 0$,

$$\lim_{x\to u(t,p)^+} w(t,x,p) = \frac{\bar{c}}{\beta}(1-e^{\beta(t-T)}).$$

De plus,

$$w(t,x,1) = \inf_{(\theta,c)\in\mathcal{A}\times\mathcal{C} s.t. X_s^{t,x,\theta,c}\geq(D_0-\bar{c}(T-t))^+\ \forall s\geq t} \int_t^T e^{-\beta(s-t)}\mathbb{E}[c_s]ds \quad \forall 0\leq t\leq T, \quad x\geq(D_0-\bar{c}(T-t))^+$$

Ceci est un problème de type Merton qui peut être résolu numériquement. Nous pouvons ainsi obtenir w numériquement.
Nous proposons enfin des graphes de u, w et de la stratégie optimale à la fois pour c et θ à la date T. Nous voyons que la stratégie optimale en c consiste à ne pas investir sauf lorsque la frontière définie par u est proche.

Ce chapitre est tiré d'un article en cours d'écriture en collaboration avec Romuald Elie et Xavier Warin.

Chapitre 2

General introduction (in english)

2.1 Preliminaries on classical BSDEs

The first part of this thesis is dedicated to some applications of Backward Stochastic Differential Equations (BSDEs) linked to stochastic control and to financial mathematics. First we recall what this means, treating only the real case here. Set $(\Omega, \mathcal{F}, \mathcal{P})$ a probabilistic space equipped with a d-dimensional Brownian motion W whose natural filtration is denoted $(\mathcal{F})_{t\geq 0}$. A BSDE with deterministic terminal time T can then be written :

$$Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \mathcal{P} - p.s. \tag{2.1.1}$$

Here the datas are :

1. The terminal condition ξ, which is a real random variable $\mathcal{F}_T$-measurable.
2. The generator $f : [0, T] \times \Omega \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, which is $\mathbb{P} \times \mathcal{B}(\mathbb{R}) \times \mathcal{B}(\mathbb{R}^d)$-measurable, denoting $\mathcal{P}$ the tribe of predictable events.

Solving this equation is determining a couple of $\mathcal{F}_T$-adapted processes $(Y_t, Z_t)_{0\leq t\leq T}$ verifying (2.1.1). The word *backward* means that the terminal condition of the equation is known here, precisely $Y_T = \xi$, which is the main source of complexity of this problem. But the solution must be adapted, therefore we cannot compute a simple time change. That is why we have to look for the solution composed of two processes, instead of one, the process Z garanteeing the adaptability of the solution.

The BSDEs have been introduced for the first time by Bismut in the case of a linear generator [11], but the pioneering paper of the theory the way it is formulated nowadays is due to Pardoux and Peng[85], where the following theorem is proved.

Theorem 2.1.1. *Suppose that the generator f is Lipschitz in (Y, Z) uniformly in (s, ω) and*

$$\mathbb{E}^{\mathbb{P}}[|\xi|^2 + \int_0^T |F(s, 0, 0)|^2 ds] < +\infty.$$

Then the BSDE (2.1.1) has a unique solution (Y, Z) such that Z is a square integrable process.

After this first general existence result, many papers weakened the hypothesis of this theorem. This interest can be explained partially by the high number of fields of applications of the theory of BSDEs, such as stochastic control problems, stochastic games, portfolio management problems... The reader can refer to [38] which provides a detailed review of the applications in finance. However, the main reason of this interest by the mathematic community is the close link between the BSDEs theory and the Partial Differential Equations (PDEs). Let us describe this connexion now.

Consider a class of so called Markovian BSDEs. For these equations, the random part of the terminal condition and the generator is supposed to be entirely generated by some diffusion. More precisely, (Y, Z) is solution of

$$Y_t = g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T, \quad \mathbb{P} - p.s., \tag{2.1.2}$$

where f and g are deterministic functions and where $(X_t)_{0\le t\le T}$ is solution of the SDE

$$X_t = x + \int_0^t b(s, X_s)ds + \int_0^t \sigma(s, X_s)dW_s, \quad 0 \le t \le T, \quad \mathbb{P} - p.s.$$

Let now the PDE

$$\begin{aligned} &\frac{\partial u}{\partial t}(t,x) + \mathcal{L}u(t,x) + f(t,x,u(t,x),\nabla u(t,x)\sigma(t,x)) = 0, \quad (t,x) \in [0,T) \times \mathbb{R} \\ &u(T,.) = g(.), \end{aligned} \tag{2.1.3}$$

where $\mathcal{L}$ is the infinitesimal generator associated to the diffusion whose solution is X given by

$$\mathcal{L}v(t,x) := \frac{1}{2}\mathrm{Tr}[a(t,x)\nabla^2(t,x)] + b(t,x).\nabla(t,x),$$

where $a(t,x) := \sigma(t,x)'\sigma(t,x)$.

If we suppose that this PDE has a regular solution, a simple application of Itôo formula shows that $(u(t,x), \nabla u(t,x)\sigma(t,x))$ is solution of the BSDE (2.1.2). This result, which is a generalisation of the Feynman-Kac formula, provides a probabilistic interpretation of the PDE (2.1.3) and permits the numerical simulation of solutions of PDEs by probabilistic ways, which does not have much problems linked to dimension. Such methods have been studied in many papers, among them are Zhang [105], [106] and Bouchard and Touzi [16]. In the first part of this thesis, there are an extension of this result to the case of reflected BSDEs with constrained jumps and then an extension to an uncertainty volatility model.

In addition, note that the BSDEs theory only provides such a probabilistic representation for so call quasi-linear PDEs, in the sense that the dependence with the Hessian in (2.1.3) must be linear. Indeed, the terms depending on the Hessian comes only form the quadratic variation of X in Itô's formula. But the importance of such equations in many areas of mathematics, physic and engineering motivated researchers to extend the previous results to a wider class of PDEs.

In particular, since the end of the 1990's, the interest for the BSDEs so called with quadratic growth, in the sens where the generator has a quadratic growth in Z have been particularly studied because of their link to the measures of dynamic risks or to the portfolio management under constraints, see for example [40]. Therefore, the question of existence and unicity of a solution in the case the generator verifies

$$|f(t,y,z)| \leq |l(t)| + c(t)|y| + \frac{\delta}{2}|z|^2, \tag{2.1.4}$$

where δ is a nonnegative constant and c and l are adaptef processes integrable enough, have first been resolved by Kobylanski [76] in the case of a bounded terminal condition. An application of this result opens this thesis, the determination of a strategy of mean-variance hedging under multiple default risks.

2.2 Mean-variance hedging under multiple defaults risk

The first application of BSDEs presented in this this thesis is the study of mean-variance hedging under multiple defaults risk. Let us recall briefly what is mean-variance hedging. Let $T > 0$ be the terminal time and H_T a payoff. We denote $\mathcal{A}$ the set of the admissible strategies, which are denoted π, and x the initial capital. Denoting also $(X_t^{x,\pi})_{0\leq t\leq T}$ the correspondig wealth process, we call the performance of the trading strategy the following :

$$J_0^H(x,\pi) = \mathbb{E}[(H_T - X_T^{x,\pi})^2] \tag{2.2.1}$$

and the mean-variance hedging problem can be expressed as :

$$V_0^H(x) = \inf_{\pi\in\mathcal{A}} J_0^H(x,\pi) \tag{2.2.2}$$

This problem was introduced by Föllmer and Schweitzer in [45], and many papers have since followed and developed this approach. For a review of this literature, see [98]. In most of these papers, the problem has been solved using continuous filtration, for example in [91] and [97].

However, our model includes multiple default risks, using an approach introduced in [62] and [63]. An important peculiarity of the model is that the number of defaults is fixed to n a priori, and we associate to each default occuring at time τ_i a mark $L_i \in E \subset \mathbb{R}$.

The only hypothesis on the defaults is a density hypothesis, more precisely that there exists un adapted process α such that for any bounded Borel function f and any time $0 \leq t \leq T$,

$$\mathbb{E}[f(\tau,L)|\mathcal{F}_t] = \int f(\boldsymbol{\theta},\boldsymbol{l})\alpha_t(\boldsymbol{\theta},\boldsymbol{l})d\boldsymbol{\theta}\eta(d\boldsymbol{l}) \quad p.s., \tag{2.2.3}$$

where $d\boldsymbol{\theta} = d\theta_1...d\theta_n$ is the Lebesgue measure on $\mathbb{R}^n$ and $\eta(d\boldsymbol{l})$ is a Borel measure on E^n in the form $\eta(d\boldsymbol{l}) = \eta_1(dl_1)\prod_{k=1}^{n-1}\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})$, where η_1 is a nonnegative Borel measure on E and, for $1 \leq k \leq n-1$, $\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})$ is a nonnegative transition kernel on $E^k \times E$. Therefore we consider the ordered n-uples as default times $\boldsymbol{\tau} = (\tau_1,...,\tau_n) \in [0,T]^n$ associated to n-uples of marks $\boldsymbol{L} = (L_1,...,L_n)$ and the events

$$\Omega_t^k := \{\tau_k \leq t < \tau_{k+1}\}, \quad 0 \leq t \leq T, \quad 0 \leq k \leq n$$

corresponding to the scenarii where k defaults occured before time t. The trading asset S takes the decomposed form

$$S_t = \sum_{k=0}^{n} 1_{\Omega_t^k} S_t^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k),$$

where $\boldsymbol{\tau}_k = (\tau_1, ..., \tau_k)$ and $\boldsymbol{L}_k = (L_1, ..., L_k)$. The dynamics of the processes S^k are,in the case where $\boldsymbol{\tau}_k = \boldsymbol{\theta}_k$ and $\boldsymbol{L}_k = \boldsymbol{l}_k$:

$$dS^k_t(\boldsymbol{\tau}_k, \boldsymbol{l}_k) = S^k_t(\boldsymbol{\tau}_k, \boldsymbol{l}_k)(\mu^k_t(\boldsymbol{\tau}_k, \boldsymbol{l}_k)dt + \sigma^k_t(\boldsymbol{\tau}_k, \boldsymbol{l}_k)dW_t), \quad \theta_k \leq t \leq T,$$

where W is a one-dimensional Brownian motion and μ^k and σ^k verifies the usual hypothesis.
Moreover, in this model, every default may induce a jump in the assets portfolio. We therefore introduce processes $\gamma^k, 0 \leq k \leq n-1$ such that

$$S^{k+1}_{\theta_{k+1}}(\boldsymbol{\theta}_{k+1}, \boldsymbol{l}_{k+1}) = S^k_{\theta^-_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\left(1 + \gamma^k_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\right).$$

We also decompose the trading strategies π in n processes $(\pi^k)_{0\leq k\leq n}$ such that

$$\pi_t = \sum_{k=0}^{n} 1_{\Omega^k_{t-}} \pi^k_t(\boldsymbol{\tau}_k, \boldsymbol{L}_k), \quad 0 \leq t \leq T.$$

The payoff H_T is decomposed as well :

$$H_T = \sum_{k=0}^{n} 1_{\Omega^k_T} H^k_t(\boldsymbol{\tau}_k, \boldsymbol{L}_k).$$

Finally, (2.2.3) allows us to define by descending recurrence, denoting $\alpha^n = \alpha$:

$$\alpha^k_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = \int_t^\infty \int_E \alpha^{k+1}_t(\boldsymbol{\theta}_k, \theta_{k+1}, \boldsymbol{l}_k, l_{k+1})d\theta_{k+1}\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}), \quad 0 \leq k \leq n-1.$$

Hence we decompose the mean-variance hedging problem (2.2.1)-(2.2.2) associated to trading strategies π to n subproblems associated to each π^k which may be call "mean-variance hedging problem between the kth default and the hypothetical k + 1-h". More precisely, denoting $\mathcal{A}^k$ the set of admissible π^k, we introduce the family of functions $(V^k)_{0\leq k\leq n}$ recursively defined by :

$$V^n(x, \boldsymbol{\theta}, \boldsymbol{l}) \quad = \quad \operatorname*{ess\,inf}_{\pi^n \in \mathcal{A}^n} \mathbb{E}\left[(H^n_T - X^{n,x,\pi}_T(\boldsymbol{\theta}, \boldsymbol{l}))^2 \alpha_T(\boldsymbol{\theta}, \boldsymbol{l}) | \mathcal{F}_{\theta_n}\right] \tag{2.2.4}$$

and

$$V^k(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k) \quad = \quad \operatorname*{ess\,inf}_{\pi^k \in \mathcal{A}^k} \mathbb{E}[(H^k_T - X^{k,x,\pi}_T(\boldsymbol{\theta}_k, \boldsymbol{l}_k))^2 \alpha^k_T(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \tag{2.2.5}$$

$$\int_{\theta_k}^T \int_E V^{k+1}(X^{k,x,\pi}_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \quad + \quad \pi^k_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{l}_k).\gamma^k_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})d\theta_{k+1} | \mathcal{F}_{\theta_k}],$$

Since there are n defaults at most, the subproblem associated to V^n is without default and so already resolved in the litterature. Hence we will proceed by descending recurrence on k to obtain V^0 and an associated optimal strategy $\pi^* = (\pi^*_k)_{0\leq k\leq n}$.
The heredity is obtained in three steps :

- We suppose that each subproblem admits a quadratic decomposition of V^k. We obtain by dynamic programming that the terms of this hypothetical decomposition must verify a system of BSDEs and a candidate for an optimal strategy.

– Then, we prove the existence of solutions to this system of BSDEs. Our proof relies on BSDE technics and in this sense is "purely BSDE", which is new in the litterature. One of the BSDEs of the system is with quadratic growth, which has brought us to use the result of Kobylanski evoked above for this proof.
– Finally, we prove by a verification theorem that the considered system of BSDEs has a unique soltion which induces a solution of the problem (2.2.1)-(2.2.2). It just remaines to check that the candidate π^* is an admissible strategy to conclude.

We conclude this part by numerical applications. Here there are 1 and then 2 independent defaults, each following an exponential law. Hence, the BSDEs becomes ordinary differential equations, which renders the simulation suitable. It allows us to obtain graph interpretations of the incompleteness of the market and of the minimal variance of an investment portfolio with a given capital.

This chapter is based on a paper written in collaboration with Stéphane Goutte and Armand Ngoupeyou [24], to appear in Stochastic Analysis and Applications.

2.3 Reflected BSDEs with nonpositive jumps, and controller-and-stopper games

2.3.1 Background

The second application of BSDEs presented in this thesis is the study of reflected BSDEs with nonpositive jumps, and its application to controller-and-stopper games.

The reflected BSDEs on a fixed obstacle have been introduced by El Karoui, Kapoudjian, Pardoux, Penga and Quenez [37]. It was the first case of BSDE with constraints, where we force the solution Y_t to stay above an obstacle S_t. A nondecreasing process whose aim is to "push" upward the solution of the BSDE is introduced. More precisely, we say that the triplet of adapted processes (Y_t, Z_t, K_t) where K is a nondecreasing process, is solution to the reflected BSDE on the obstacle S with terminal condition ξ and generator f when

$$\begin{aligned}
&Y_t = \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s + K_T - K_t, \quad 0 \le t \le T, \quad \mathbb{P}-p.s. \\
&Y_t \ge S_t, \le 0 \le t \le T, \quad \mathbb{P}-p.s. \\
&\int_0^T (Y_s - S_s)dK_s = 0, \quad \mathbb{P}-p.s. \qquad (2.3.1)
\end{aligned}$$

The last condition in (2.3.1) means that the nondecreasing process K is minimal in the sense it acts only when Y hits the obstacle. It allows to obtain the uniqueness of the solution of such equation. In [37], a proof of existence of solution is also given by penalization. Besides, it is proved that the reflected BSDEs provide a probabilistic representation for quasi-linear PDEs with an obstacle.
Let us now consider the following fully nonlinear PDE of Hamilton-Jacobi-Bellman (HJB) type :

$$\frac{\partial v}{\partial t} + \sup_{a \in A} \big(\langle b(x,a), D_x v\rangle + \frac{1}{2}\text{tr}(\sigma\sigma^\intercal(x,a)D_x^2 v) + f(x,a)\big) = 0, \qquad (2.3.2)$$

on $[0,T) \times A$, where A is a subset of $\mathbb{R}^q$, together with the terminal condition

$$v(T,x) = g(x), \quad x \in \mathbb{R}^d.$$

As it is well-known, see for example Pham [90], the above equation is the dynamic programming equation of a stochastic control problem whose value function is given by :

$$v(t,x) := \sup_{\alpha} \mathbb{E}\Big[\int_t^T f(X_s^{t,x,\alpha}, \alpha_s)ds + g(X_T^{t,x,\alpha})\Big]$$

where $X^{t,x,\alpha}$ is the controlled state process starting at time $t \in [0,T]$ from $x \in \mathbb{R}^d$ which evolves on $[t,T]$ according to the stochastic equation

$$X_s^{t,x,\alpha} = x + \int_t^s b(X_r^{t,x,\alpha}, \alpha_r)dr + \int_t^s \sigma(X_r^{t,x,\alpha}, \alpha_r)dW_r \tag{2.3.3}$$

where α is a predictable control process valued in A. Notice that, if $\sigma(x)$ does not depend on $a \in A$ and $\sigma\sigma^{\intercal}(x)$ is of full rank, then the above HJB equation can be written as :

$$\frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}(\sigma\sigma^{\intercal}(x)D_x^2 v) + F(x, \sigma^{\intercal}(x)D_x v) = 0, \tag{2.3.4}$$

where $F(x,z) = \sup_{a\in A}[f(x,a) + \langle\theta(x,a), z\rangle]$ is the θ-Fenchel-Legendre transform of f and $\theta(x,a) = \sigma^{\intercal(x)}(\sigma\sigma^{\intercal}(x))^{-1}b(x,a)$ is a solution to $\sigma(x)\theta(x,a) = b(x,a)$. Then, from the seminal papers of Pardoux and Peng [85, 86], we know that the semilinear PDE (2.3.4) admits a nonlinear Feynman-Kac formula through a Markovian forward-backward stochastic differential equation.

The general case with possibly degenerate controlled diffusion coefficient $\sigma(x,a)$ associated to a fully nonlinear HJB equation, has only recently been completely solved by Kharroubi and Pham [75]. We also mention that a first step in this direction was made by Soner, Touzi, and Zhang [101], where however the theory of second-order BSDEs (2BSDEs) was used rather than the standard theory of backward stochastic differential equations. 2BSDEs are backward stochastic differential equations formulated under a nondominated family of singular probability measures, so that their theory relies on tools from quasi-sure analysis. On the other hand, according to [75], it is sufficient to consider a backward stochastic differential equation with jumps, where the jumps are constrained to be nonpositive, formulated under a single probability measure, as in the standard theory of BSDEs.

Let us give an idea of the results presented in [75]. the forward-backward system associated to the HJB equation (2.3.2) is constructed as follows : the forward equation, starting at time $t \in [0,T]$ from $(x,a) \in \mathbb{R}^d \times \mathbb{R}^q$ evolves on $[t,T]$ according to the system of equations :

$$\begin{aligned} X_s^{t,x,a} &= x + \int_t^s b(X_r^{t,x,a}, I_r^{t,a})dr + \int_t^s \sigma(X_r^{t,x,a}, I_r^{t,a})dW_r \quad t \le s \le T, \\ I_s^{t,a} &= a + \int_t^s \int_A (a' - I_{r-}^{t,a})\mu(dr, da') \quad t \le s \le T. \end{aligned}$$

Its form is deduced from the controlled state dynamics (2.3.3) randomizing the state process $X^{t,x,\alpha}$, i.e., introducing, in place of the control α, a pure-jump process I driven by a

Poisson random measure μ on $\mathbb{R}^+ \times A$ independent of W, with intensity measure $\lambda(da)dt$, where λ is a finite measure on $(A, \mathcal{B}(A))$. W and μ are defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F}$ is the completion of the natural filtration generated by W and μ. Regarding the backward equation, as expected, it is driven by the Brownian motion W and the Poisson random measure μ, namely it is a BSDE with jumps with terminal condition $g(X_T^{t,x,a})$ and generator $f(X_\cdot^{t,x,a}, I_\cdot^{t,a})$, as it is natural from the expression of the HJB equation. The backward equation is also characterized by a constraint on the jump component, which turns out to be a crucial aspect of the theory introduced in [75] and requires, as in the theory of reflected BSDEs (see for example (2.3.1)), the presence of an increasing process in the BSDE. In conclusion, the backward stochastic differential equation has the following form :

$$\begin{aligned} Y_s^{t,x,a} &= g(X_T^{t,x,a}) + \int_s^T f(X_r^{t,x,a}, I_r^{t,a})dr + K_T^{t,x,a} - K_s^{t,x,a} \\ &\quad - \int_s^T Z_s^{t,x,a} dW_r - \int_s^T \int_A L_r^{t,x,a}(a')\mu(dr, da'), \quad t \le s \le T, p.s. \end{aligned}$$

together with the jump constraint

$$L_s^{t,x,a}(a') \le 0, \quad d\mathbb{P} \otimes ds \otimes \lambda(da')p.p.$$

Notice that the presence of the increasing process K in the backward equation does not guarantee the uniqueness of the solution. For this reason, as in the theory of reflected BSDEs, in [75] the authors look only for the minimal solution (Y, Z, L, K) to the above BSDE, in the sense that for any other solution $(\bar{Y}, \bar{Z}, \bar{L}, \bar{K})$ we must have $Y \le \bar{Y}$. The existence of the minimal solution is based on a penalization approach and on the monotonic limit theorem of Peng [87].

The nonlinear Feynman-Kac formula becomes

$$v(t, x, a) := Y_t^{t,x,a}, \quad (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q.$$

Observe that the value function v should not depend on a, but only on (t, x), as a consequence of the nonpositive jump constraint. Indeed, if v is continuous, we have

$$L_s^{t,x,a}(a') = v(s, X_s^{t,x,a}, a') - v(s, X_s^{t,x,a}, I_{s-}^{t,a}) \le 0 \quad d\mathbb{P} \otimes ds \otimes \lambda(da')p.p.$$

from which we see that v does not depend on a. However, it is not clear a priori that the function v is continuous, therefore, in [75], the rigorous proof relies on fine viscosity solutions arguments and on mild conditions on λ and A, as the assumptions that the interior set of A is connected and that A is the closure of its interior. In the end, in [75], it is proved that the function v does not depend on the variable a in the interior of A and that the viscosity solution to equation (2.3.2) admits the following probabilistic representation formula :

$$v(t, x) := Y_t^{t,x,a}, \quad (t, x) \in [0, T] \times \mathbb{R}$$

for any a in the interior of A. This formula opens new perspectives for probabilistic schemes for fully nonlinear PDEs, as currently investigated in Kharroubi, Langrené and

Pham [73].
In [75], another probabilistic representation is also provided, called dual representation, for the solution v to (2.3.2). More precisely, let $\mathcal{V}$ be the set of predictable processes $\nu : \Omega \times [O,T] \times A \to (0,\infty)$ which are essentially bounded and consider the probability measure $\mathbb{P}^\nu$ equivalente to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ with Radon-Nikodym density :

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}}|_{\mathcal{F}_t} = \zeta_t^\nu := \mathcal{E}_t\Big(\int_0^\cdot \int_A (\nu_s(a) - 1)\tilde{\mu}(ds, da)\Big)$$

where $\mathcal{E}_t(.)$ is the Doléans-Dade exponential. Notice that W remains a Brownian motion under $\mathbb{P}^\nu$, and the effect of the probability measure $\mathbb{P}^\nu$, by Girsanov's Theorem, is to change the compensator $\lambda(da)dt$ of μ under $\mathbb{P}$ to $\nu_t(a)\lambda(da)dt$ under $\mathbb{P}^\nu$. The dual representation reads :

$$v(t,x) = Y_t^{t,x,a} = \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\big[g(X_T^{t,x,a}) + \int_t^T f(X_s^{t,x,a}, I_s^{t,a})ds|\mathcal{F}_t\big]$$

where $\mathbb{E}^\nu$ denotes the expectation with respect to $\mathbb{P}^\nu$.
Finally, we observe that the tools used in [75] can also be applied to other stochastic control problems, as impulse control problems, see for example [74].

2.3.2 Formulation of the problem

Let us consider the Hamilton-Jacobi-Bellman-Isaacs (HJBI) equation :

$$\begin{aligned}\max\Big[-\frac{\partial v}{\partial t} - \sup_{a\in A}\big(b(x,a).D_x v + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D_x^2 v) + f(x,a)\big); \\ v - g\Big] &= 0,\end{aligned} \tag{2.3.5}$$

on $[0,T) \times \mathbb{R}^d$, together with the terminal condition

$$v(T,x) = g(x), \quad x \in \mathbb{R}^d. \tag{2.3.6}$$

In Bayraktar and Huang [7], it is proved that the above equation is the dynamic programming equation of a zero-sum controller-and-stopper game, whose upper and lower value functions are given by :

$$\begin{aligned}\bar{V}(t,x) &:= \inf_{\pi\in\Pi_{t,T}} \sup_{\alpha\in\mathcal{A}} \mathbb{E}\Big[\int_t^{\pi[\alpha]} f(X_s^{t,x,\alpha}, \alpha_s)ds + g(X_{\pi[\alpha]}^{t,x,\alpha})\Big] \\ \underline{V}(t,x) &:= \sup_{\alpha\in\mathcal{A}} \inf_{\pi\in\mathcal{T}_{t,T}} \mathbb{E}\Big[\int_t^{\tau} f(X_s^{t,x,\alpha}, \alpha_s)ds + g(X_\tau^{t,x,\alpha})\Big], \quad (t,x) \in [0,T] \times \mathbb{R}^d,\end{aligned}$$

where $X^{t,x,\alpha}$ is a diffusion process in $\mathbb{R}^d$ satisfying equation (2.3.3) controlled by a predictable process $\alpha \in \mathcal{A}$ valued in A, $\mathcal{T}_{t,T}$ is the set of all stopping times valued in $[t,T]$ for $0 \le t \le T$, and $\Pi_{t,T}$ is the set of stopping strategies $\pi : \mathcal{A} \to \mathcal{T}_{t,T}$ satisfying a nonanticipative condition (see Definition 3.1 in [7]).It is shown in [7] that this game has a value, i.e., $\bar{V} = \underline{V} = v$, and that v is the unique viscosity solution to (2.3.5) - (2.3.6) satisfying a polynomial growth condition..

In this part, we prove that the value function v associated to the HJBI equation (2.3.5) - (2.3.6) we also consider more general partial differential equations of HJBI type) admits a probabilistic representation (nonlinear Feynman-Kac formula) through a forward-backward stochastic differential equation. In particular, inspired by the paper [75] recalled above and the standard theory of reflected BSDEs, we introduce a class of reflected backward stochastic differential equations with nonpositive jumps and upper barrier. As in the case of doubly reflected BSDEs with lower and upper obstacles, related to Dynkin's games, our BSDE formulation involves the introduction of two nondecreasing processes. More precisely, the backward equation has the following form (we also consider more general BSDEs in this part, with the generator f depending also on $Y^{t,x,a}$ and $Z^{t,x,a}$, and even on the jump component in the general non-Markovian case) :

$$\begin{aligned} Y_s^{t,x,a} &= g(X_T^{t,x,a}) + \int_s^T f(X_r^{t,x,a}, I_r^{t,a})dr + K_T^{t,x,a,+} - K_s^{t,x,a,+} \\ &\quad -(K_T^{t,x,a,-} - K_s^{t,x,a,-}) - \int_s^T Z_r^{t,x,a} dW_r \\ &\quad - \int_s^T \int_A L_r^{t,x,a}(a')\mu(dr, da'), \quad t \le s \le T, p.s. \end{aligned}$$

together with the jump constraint

$$L_s^{t,x,a}(a') \le 0, \quad d\mathbb{P} \otimes ds \otimes \lambda(da')p.p.$$

and the upper constraint

$$\begin{aligned} Y_s^{t,x,a} &\le g(X_s^{t,x,a}), \quad t \le s \le T, p.s. \\ \int_t^T (g(X_s^{t,x,a}) - Y_{s-}^{t,x,a})dK_s^{t,x,a,-} &= 0, \quad p.s. \end{aligned} \tag{2.3.7}$$

Notice that the presence of the increasing process $K^{t,x,a,-}$ forces the solution Y to be below the upper obstacle $g(X_\cdot^{t,x,a})$. Moreover, due to the Skorohod condition (2.3.7), $K^{t,x,a,-}$ acts in a minimal way. On the other hand, the increasing process $K^{t,x,a,+}$ is associated to the jump constraint, as in [75]. To guarantee uniqueness of the solution, we look only for the minimal solution (Y, Z, L, K^+, K^-) to the above BSDE, in the sense that for any other solution $(\bar{Y}, \bar{Z}, \bar{L}, \bar{K}^+, \bar{K}^-)$ we must have $Y \le \bar{Y}$.
The existence of a minimal solution necessitates an additional hypothesis of regularity on the upper barrier, which is equivalent in our context to Mokobodzki's conidtion. Under this hypothesis, we prove the existence in a general non-Markovian framework using double penalization approach and a monotonic limit theorem for BSDEs with jumps. More precisely, let us introduce the sequence of BSDEs with jumps :

$$\begin{aligned} Y_s^{n,m,t,x,a} &= g(X_T^{t,x,a}) + \int_s^T f(X_r^{t,x,a}, I_r^{t,a})dr + K_T^{n,m,t,x,a,+} - K_s^{n,m,t,x,a,+} \\ &\quad -(K_T^{n,m,t,x,a,-} - K_s^{n,m,t,x,a,-}) - \int_s^T Z_r^{n,m,t,x,a} dW_r \\ &\quad - \int_s^T \int_A L_r^{n,m,t,x,a}(a')\mu(dr, da'), \quad t \le s \le T, p.s. \end{aligned}$$

for $n, m \in \mathbb{N}$, where $K_s^{n,m,t,x,a,+}$ and $K_s^{n,m,t,x,a,-}$ are the increasing continuous processes defined by :

$$K_s^{n,m,t,x,a,+} = m \int_s^T \int_A (L_r^{n,m,t,x,a})_+(a)\lambda(da)dr$$

and

$$K_s^{n,m,t,x,a,-} = n\int_s^T (g(X_r^{t,x,a}) - Y_r^{n,m,t,x,a})_- dr.$$

Here we use the notation $f_+ = \max(f,0)$ and $f_- = \max(-f,0)$ to denote the positive and negative parts of f. The minimal solution to the reflected BSDE with nonpositive jumps is constructed by taking the limit first with respect to n and then with respect to m and using a monotonic limit theorem. This latter is based on uniform estimates for $(Y^{n,m,t,x,a}, Z^{n,m,t,x,a}, L^{n,m,t,x,a}, K^{n,m,t,x,a,+}, K^{n,m,t,x,a,-})$, which turn out to be the main issue, especially regarding the two increasing processes $K^{n,m,t,x,a,+}$ and $K^{n,m,t,x,a,-}$. Here intervenes the hypothesis on the regularity of the upper barrier. Note that the running order of the limits in the double penalization is crucial, in contrast with the case of upper and lower reflection (Dynkin's games). Indeed, we do not have comparison results on the jump component solution of a BSDE, and so a priori rather few information on the sequence of nondecreasing processes associated to the jump constraint, whereas one can exploit comparison results on the Y-component of a BSDE in order to derive useful monotonicity property for the sequence of nondecreasing processes associated to the upper obstacle.
The nonlinear Feynman-Kac formula turns out to be

$$v(t,x,a) = Y_t^{t,x,a}, \quad (t,x,a) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^q.$$

As in [75], it appears that v does not depend on a in the interior of A, as a consequence of the non positivity jumps constraint. We prove that v is a viscosity solution to the HJBI equation (2.3.5) and to the terminal condition (2.3.6). We also consider more general HJBI equations than (2.3.5), where the generator $f(x,a,v,\sigma^\intercal D_x v)$ may also depend on v and $D_x v$.

Finally, we prove a dual game representation formula for the minimal solution to our BSDE, which is inspired by the dual representation given in [75] and the representation formula of Proposition 6.2 in [31]. Let us give an idea of this dual representation formula. In addition to the set of probability measures $\mathbb{P}^\nu, \nu \in \mathcal{V}$ defined in the previous subsection, let Θ be the set of discount factors, i.e., progressively measurable processes $\theta : \Omega\times[0,T] \to \mathbb{R}_+$ which are essentially bounded. Then the dual representation formula becomes for $s \in [0,T]$:

$$Y_s^{t,x,a} = \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \operatorname*{ess\,inf}_{\theta\in\Theta} \mathbb{E}^\nu\big[e^{-\int_s^T \theta_r dr} g(X_T^{t,x,a}) + \int_s^T e^{-\int_s^r \theta_u du}(f(X_r^{t,x,a}, I_r^{t,a}) + \theta_r g(X_r^{t,x,a}))dr \big| \mathcal{F}_s\big].$$

This is an original representation for the value function of the stochastic zero-sum controller-and-stopper game. We do not know in general whether one can switch the essential infimum and supremum in the above representation formula. Actually, by taking first the limit with respect to m and then with respect to n in the doubly indexed penalized sequence, we end up with a process $\hat{Y}^{t,x,a}$ satisfying $Y^{t,x,a} \le \hat{Y}^{t,x,a}$, for which it is not clear whether it is a solution to a backward stochastic differential equation and whether it is equal or strictly greater $Y^{t,x,a}$. Nevertheless, $\hat{Y}^{t,x,a}$ admits the representation for s

$s \in [t, T]$:

$$\hat{Y}_s^{t,x,a} = \underset{\theta \in \Theta}{\text{ess inf}} \, \underset{\nu \in \mathcal{V}}{\text{ess sup}} \, \mathbb{E}^\nu \big[e^{-\int_s^T \theta_r dr} g(X_T^{t,x,a}) + \int_s^T e^{-\int_s^r \theta_u du} (f(X_r^{t,x,a}, I_r^{t,a}) + \theta_r g(X_r^{t,x,a})) dr | \mathcal{F}_s \big].$$

This chapter is based on a paper written in collaboration with Andrea Cosso and Huyên Pham [23], published in *Stochastic Processes and their Applications*.

2.3.3 Perspectives

Stochastic zero-sum controller-and-stopper games have been fruitfully employed in Mathematical Finance, for example in the valuation problem of American contingent claims under constraints, see Karatzas and Kou [65] and Karatzas and Zamfirescu [67]. Indeed, in the unconstrained case, it is well-known that there exists a single arbitrage-free price for the American contingent claim, which turns out to be the supremum, over all stopping times, of the claim's discounted expected value under the equivalent martingale measure. In the presence of constraints, instead, there is an entire interval $[h_1, h_2]$ of arbitrage free-prices. According to [65] and [67], the endpoints can be characterized as the lower and upper value functions of a zero-sum controller-and-stopper game. As another example of application in Mathematical Finance, we recall that in Bayraktar and Young [8] it is shown that the problem of minimizing the probability of lifetime ruin (namely the probability that the wealth reaches the value zero before the individual dies), when the rate of consumption is stochastic and when the individual can invest in a Black & Scholes financial market, may be reformulated as a controller-and-stopper game.

Finally, we point out that the probabilistic representation formula obtained suggests a new approach for probabilistic numerical schemes of HJBI equations by discretization and simulation of the reflected BSDE with nonpositive jumps and upper obstacle. Another class of BSDEs that might be studied is the refleected BSDEs with nonpositive jumps and lower obstacle, which is related to sup sup problem over control and stopping time, and in other words to optimal stopping under nonlinear expectation. Actually, the proof of existence of a minimal solution by a double penalization approach is more simple since it would involve the sum, instead of the difference, of two increasing processes.

2.4 BSDE representation for stochastic control problems with non dominated controlled intensity

2.4.1 Background

The third application to BSDEs presented in this thesis is the study of BSDE representation for stochastic control problems with non dominated controlled intensity.

This chapter is also an extension of [75]. We refer to the previous section for a description of the results of this paper. One of the results of [75] is to provide a probabilistic representation formula, known as nonlinear Feynman-Kac formula, for fully nonlinear integro-partial differential equations (IPDEs) of the following type :

$$\frac{\partial v}{\partial t} + \sup_{a \in A} \Big[b(x,a).D_x v + \frac{1}{2} \text{tr}(\sigma\sigma^\intercal(x,a) D_x^2 v) + f(x,a,v,\sigma^\intercal(x,a) D_x v) \tag{2.4.1}$$

$$+\int_E \big(v(t,x+\beta(x,a,e)) - v(t,x) - \beta(x,a,e).D_x v(t,x)\big)\lambda(de)\Big] \;=\; 0, \qquad \text{on } [0,T)\times\mathbb{R}^d,$$
$$v(T,x) \;=\; g(x), \qquad x\in\mathbb{R}^d,$$

where A is a compact subset of $\mathbb{R}^q$, E is a Borelian subset of $\mathbb{R}^k\backslash\{0\}$, and λ is a nonnegative σ-finite measure on $(E,\mathcal{B}(E))$ satisfying the integrability condition $\int_E(1\wedge|e|^2)\lambda(de)<\infty$. A special case is the Hamilton-Jacobi-Bellman equation associated to the uncertain volatility model in mathematical finance, which takes the following form :

$$\frac{\partial v}{\partial t} + G(D_x^2 v) \;=\; 0, \quad \text{on } [0,T)\times\mathbb{R}^d, \qquad v(T,x) \;=\; g(x), \quad x\in\mathbb{R}^d, \tag{2.4.2}$$

where $G(M)=\frac{1}{2}\sup_{c\in C}[cM]$ and C is a set of symmetric nonnegative matrices of order d. As described in [89], the unique viscosity solution to (2.4.2) is represented in terms of the so-called G-Brownian motion B under the nonlinear expectation $\mathcal{E}(\cdot)$ as follows :

$$v(t,x) \;=\; \mathcal{E}\big(g(x+B_T-B_t)\big).$$

It is however not clear how to simulate a G-Brownian motion.

In the present chapter, we study the following fully nonlinear integro-PDE of Hamilton-Jacobi-Bellman type :

$$\frac{\partial v}{\partial t} + \sup_{a\in A}\Big[b(x,a).D_x v + \frac{1}{2}\text{tr}\big(\sigma\sigma^{\intercal}(x,a)D_x^2 v\big) + f(x,a,v,\sigma^{\intercal}(x,a)D_x v\big) \tag{2.4.3}$$
$$+\int_E \big(v(t,x+\beta(x,a,e)) - v(t,x) - \beta(x,a,e).D_x v(t,x)\big)\lambda(a,de)\Big] = 0, \quad \text{on } [0,T)\times\mathbb{R}^d,$$
$$v(T,x) = g(x), \qquad x\in\mathbb{R}^d,$$

where λ is a transition kernel from $(A,\mathcal{B}(A))$ into $(E,\mathcal{B}(E))$. We do not assume that the family of measures $(\lambda(a,\cdot))_{a\in A}$ is dominated. Moreover, the diffusion coefficient σ can be degenerate.
A motivation to the study of equation (2.4.3) comes from mathematical finance and, in particular, from model uncertainty, when uncertainty affects both volatility and intensity. This topic was studied by means of second order BSDEs with jumps (2BSDEJs) in [70] and [71]. However, this method does not treat the case where the volatility is degenerate, contrary to ours. Moreover, by following the ideas of [72] and [72], we can obtain an efficient numerical scheme for equation (2.4.3).
Model uncertainty is also strictly related to the theory of G-Lévy processes and, more generally, of nonlinear Lévy processes, see [54] and [83]. In this case, the associated fully nonlinear integro-PDE takes the following form :

$$\frac{\partial v}{\partial t} + \sup_{(b,c,F)\in\Theta}\Big[b.D_x v + \frac{1}{2}\text{tr}\big(cD_x^2 v\big) \tag{2.4.4}$$
$$+\int_E \big(v(t,x+z) - v(t,x) - D_x v(t,x).z1_{\{|z|\le 1\}}\big)F(dz)\Big] \;=\; 0, \qquad \text{on } [0,T)\times\mathbb{R}^d,$$
$$v(T,x) \;=\; g(x), \qquad x\in\mathbb{R}^d,$$

where Θ denotes a set of Lévy triplets (b, c, F), $b \in \mathbb{R}^d$, c is a symmetric nonnegative matrix of order d and F is a Lévy measure on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. From [54] and [83], we know that the unique viscosity solution to equation (2.4.4) is represented in terms of the so-called nonlinear Lévy process $\mathcal{X}$ under the nonlinear expectation $\mathcal{E}(\cdot)$ as follows :

$$v(t,x) \;=\; \mathcal{E}(g(x + \mathcal{X}_T - \mathcal{X}_t)).$$

If we are able to describe the set Θ by means of a parameter a which lives in a compact set A of an Euclidean space $\mathbb{R}^q$, then (2.4.4) can be written in the form (2.4.3).Therefore, v is also given by our probabilistic representation formula, in which the forward process is possibly easier to simulate than a nonlinear Lévy process.

2.4.2 Formulation of the problem

To solve (2.4.3), as in [75] and in the previous section, we need to introduce a stochastic optimal control problem whose value function is a solution of equation (2.4.3). Unfortunately, we did not find any reference in the literature for this kind of stochastic control problem, that is why we intoduce ourselves such a problem.
We describe briefly how here. Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be a complete probability space satisfying the usual conditions on which a d-dimensional Brownian motion $\bar{W} = (\bar{W}_t)_{t\geq 0}$ is defined. Let also $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t\geq 0}$ denote the usual completion of the natural filtration generated by $\bar{W}$ and $\bar{\mathcal{A}}$ the class of control processes α, which are $\bar{\mathbb{F}}$-predictable and valued in A. Let also Ω' be the canonical space of the marked point process on $\mathbb{R}^+ \times E$ with canonical right-continuous filtration $\mathbb{F}'$ and canonical random measure π'. Then, consider $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ defined as $\Omega := \bar{\Omega} \times \Omega'$, $\mathcal{F} := \bar{\mathcal{F}} \otimes \mathcal{F}'_\infty$, and $\mathcal{F}_t := \cap_{s>t} \bar{\mathcal{F}}_s \otimes \mathcal{F}'_s$. Moreover, we set $W(\omega) := \bar{W}(\bar{\omega})$, $\pi(\omega, \cdot) := \pi'(\omega', \cdot)$, and $\mathcal{A} := \{\alpha\colon \alpha(\omega) = \bar{\alpha}(\bar{\omega}),\ \forall\, \omega \in \Omega,\ \text{for some } \bar{\alpha} \in \bar{\mathcal{A}}\}$. Suppose that for every $\alpha \in \mathcal{A}$ we are able to construct a measure $\mathbb{P}^\alpha$ on $(\Omega, \mathcal{F})$ such that W is a Brownian motion and π is an integer-valued random measure with compensator $1_{\{t<T_\infty\}}\lambda(\alpha_t, de)dt$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\alpha)$, where T_∞ denotes the supremum of the jump times of the marked point process associated to π. Then, consider the stochastic control problem with value function given by ($\mathbb{E}^\alpha$ denotes the expectation with respect to $\mathbb{P}^\alpha$)

$$v(t,x) \;:=\; \sup_{\alpha\in\mathcal{A}} \mathbb{E}^\alpha\Big[\int_t^T f(X_s^{t,x,\alpha}, \alpha_s)ds + g(X_T^{t,x,\alpha})\Big], \tag{2.4.5}$$

where $X^{t,x,\alpha}$ has the controlled dynamics on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\alpha)$:

$$dX_s^\alpha \;=\; b(X_s^\alpha, \alpha_s)ds + \sigma(X_s^\alpha, \alpha_s)dW_s + \int_E \beta(X_{s^-}^\alpha, \alpha_s, e)\tilde{\pi}(ds, de)$$

starting from x at time t, with $\tilde{\pi}(dt, de) = \pi(dt, de) - 1_{\{t<T_\infty\}}\lambda(\alpha_t, de)dt$ the compensated martingale measure of π. We expect it to be the researched problem, in the sense the PDE (2.4.3) turns out to be the dynamic programming equation of the stochastic control problem with value function formally given by (2.4.5).

As in [75] and in the previous section, we randomize the control. To do so, we introduce on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ a q-dimensional Brownian motion $\bar{B} = (\bar{B}_t)_{t\geq 0}$, independent of $\bar{W}$. Now $\bar{\mathbb{F}}$ denotes the usual completion of the natural filtration generated by $\bar{W}$ and $\bar{B}$. We also set $B(\omega) := \bar{B}(\bar{\omega})$, for all $\omega \in \Omega$, so that B is defined on Ω. Since the control lives in the compact set $A \subset \mathbb{R}^q$, we can not use directly B to randomize the control, but we need to map B on A. That is why we suppose the existence of a continuous surjection $h\colon \mathbb{R}^d \to A$. Then, for every $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, we consider the following forward stochastic differential equation in $\mathbb{R}^d \times \mathbb{R}^q$:

$$X_s = x + \int_t^s b(X_r, I_r)dr + \int_t^s \sigma(X_r, I_r)dW_r + \int_t^s \int_E \beta(X_{r^-}, I_r, e)\tilde{\pi}(dr, de), \quad (2.4.6)$$

$$I_s = h(\tilde{a} + B_s - B_t), \quad (2.4.7)$$

for all $t \leq s \leq T$, where $\tilde{\pi}(ds, de) = \pi(ds, de) - 1_{\{s<T_\infty\}}\lambda(I_s, de)ds$ is the compensated martingale measure of π, which is an integer-valued random measure with compensator $1_{\{s<T_\infty\}}\lambda(I_s, de)ds$. Unlike [75] and the previous section, we use a Brownian motion B to randomize the control, instead of a Poisson random measure μ on $\mathbb{R}_+ \times A$. From one hand, the Poisson random measure turns out to be more convenient to deal with a general compact set A, since μ is already supported by $\mathbb{R}_+ \times A$, so that we do not have to impose the existence of a continuous surjection h from $\mathbb{R}^q$ into A, as we did here. On the other hand, the choice of a Brownian motion B is more convenient to derive a martingale representation theorem for our model. Indeed, in contrast with [75] and with the previous section, the intensity of the measure π depends on the process I, therefore it is natural to obtain a dependence between π and the noise used to randomize the control. The advantage of B with respect to μ is given by the fact that B is *orthogonal* to π, since B is a continuous process (see the bottom of page 183 in [59] for a definition of orthogonality between a martingale and a random measure). Thanks to this orthogonality we are able to derive a martingale representation theorem in our context, which is essential for the derivation of our nonlinear Feynman-Kac representation formula.

Let us focus on the form of the stochastic differential equation (2.4.6)-(2.4.7). We observe that the jump part of the driving factors in (2.4.6) is not given, but depends on the solution via its intensity. This makes the SDE (2.4.6)-(2.4.7) nonstandard. These kinds of equations were firstly studied in [58] and have also been used in the financial literature, see for example [9], [27], [28], [29], [42]. However, in [9], [27], and [28], λ is absolutely continuous with respect to a given deterministic measure on $(E, \mathcal{B}(E))$, which allows to solve (2.4.6)-(2.4.7) bringing it back to a standard SDE, via a change of intensity "à la Girsanov". On the other hand, in this chapter, we shall tackle the above SDE solving firstly equation (2.4.7) for any $(t, \tilde{a}) \in [0, T] \times \mathbb{R}^q$, then constructing a probability measure $\mathbb{P}^{t,\tilde{a}}$ on $(\Omega, \mathcal{F})$ such that the random measure $\pi(ds, de)$ admits $\lambda(I_s^{t,\tilde{a}}, de)ds$ as compensator, and finally addressing (2.4.6). In the appendix, we also prove additional properties of π and (X, I). More precisely, we present a characterization of π in terms of Fourier and Laplace functionals, which shows that π is a conditionally Poisson random measure (also known as Cox random measure) relative to $\sigma(I_s^{t,\tilde{a}}; s \geq 0)$. Moreover, we study the Markov properties of the pair (X, I).

The corresponding backward stochastic differential equation is, as expected, driven by the Brownian motions W and B, and by the random measure π, so this is a BSDE with jumps, with terminal condition $g(X_T^{t,x,\tilde{a}})$ and generator $f(X_\cdot^{t,x,\tilde{a}}, I_\cdot^{t,\tilde{a}}, y, z)$, as it is natural from the expression of the HJB equation (2.4.3). The BSDE is also characterized by a constraint on the diffusive part relative to B, which turns out to be crucial and entails the presence of an increasing process in the BSDE. In conclusion, for any $(t,x,\tilde{a}) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$, the BSDE has the following form :

$$\begin{aligned} Y_s &= g(X_T^{t,x,\tilde{a}}) + \int_s^T f(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}, Y_r, Z_r)dr + K_T - K_s - \int_s^T Z_r dW_r \\ &\quad - \int_s^T V_r dB_r - \int_s^T \int_E U_r(e)\tilde{\pi}(dr,de), \qquad t \le s \le T,\ \mathbb{P}^{t,\tilde{a}}\ p.s. \end{aligned}$$

and

$$|V_s| \;=\; 0 \qquad\qquad ds \otimes d\mathbb{P}^{t,\tilde{a}}\ a.e.$$

As in [75] and in the previous section, the presence of the increasing process K in the BSDE makes us looking for the minimal solution (Y,Z,V,U,K) of this BSDE, in the sense that for any other solution $(\bar{Y},\bar{Z},\bar{V},\bar{U},\bar{K})$ we must have $Y \le \bar{Y}$. The existence of the minimal solution is also based on a penalization approach. We finally obtain the following nonlinear Feynman-Kac formula :

$$v(t,x,\tilde{a}) \;:=\; Y_t^{t,x,\tilde{a}}, \qquad (t,x,\tilde{a}) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q.$$

As in [75] and in the previous section, v does not depend on $\tilde{a}$, but only on (t,x) and the proof relies on viscosity solutions arguments. We also show that v is the unique viscosity solution of (2.4.3), as it follows from a comparison theorem proved in the appendix. Due to the presence of the non dominated family of measures $(\lambda(a,\cdot))_{a\in A}$, we did not find in literature a comparison theorem for viscosity solution to our equation (2.4.3). For this reason, we prove it in the appendix, even though the main ideas are already contained in the paper [4], in particular the remarkable Jensen-Ishii's lemma for integro-partial differential equations.

This chapter is based on a paper written in collaboration with Andrea Cosso [22], to appear in Annals of Applied Probability.

2.5 Conditional asset liability management

2.5.1 Background

The purpose of this chapter is the design of an optimal Asset Liability policy, in a framework where the asset manager faces a constraint on the distribution of its terminal wealth. More precisely, the investor requires to pay, for simplicity, a constant liability D_0 at maturity T and allows for this constraint to be violated with a given small probability $1-p$. In practice, whenever a conservative investor imposes an almost-sure constraint on the terminal value of an investment strategy, this leads to rather overcautious investment policies. This mainly comes from the fact that it is too costly to take some risk, since it will complicate the necessity of satisfying the constraint at maturity. The main objective of the chapter is to quantify the effect of relieving slightly this constraint by only imposing the probability of success at maturity to exceed p, and to measure the dependance in p on the optimal asset management policy.

The modern portfolio theory in continuous time goes back to the seminal paper of Merton [82], who considers an agent trying to maximize his expected utility from terminal wealth or expected time-integrated utility from consumption. In a Markovian framework, the optimal policy identifies in terms of the solution of the corresponding Hamilton-Jacobi-Bellman equation, or alternatively can be derived using duality arguments, see e.g. Karatzas, Lehoczky et Shreeve [64]. This framework has raised a large literature, with the introduction of additional constraints : e.g. on the investment policy by Cvitanic and Karatzas in [30], with a given almost sure constraint on a portfolio insurance by El Karoui, Jeanblanc and Lacoste in [36] or with drawdown constraints by Elie and Touzi in [35]. In this context, considering the constraint of beating a given benchmark with a given probability of success, this problem has already been studied by Boyle and Tian in [18], via a duality argument, mainly inspired from the approach of Follmer and Leukert in [44] for quantile hedging problems. In recent literature, a new approach introduced by Bouchard, Elie and Touzi in [15] allows to study these a priori dynamically inconsistent problems in a dynamic manner.

The main difficulty in considering constraints written in terms of probability, is that the probability of success p is imposed at time 0, but trying to build up a dynamic programming principle, one requires to be able to quantify the effect of such constraint at any given intermediate date t. The proper way to do this has been identified in [15], where the dynamic probability of success is viewed as a new forward controlled martingale process. This new variable allows to solve the problem in a dynamically consistent manner. The resolution of stochastic control problems under such quantile has been more specifically been studied in [14], via the derivation of a dynamic programming principle.

2.5.2 Formulation of the problem

We consider an investor, who can at any time t choose the investment policy θ_t, as well as the instantaneous rate c_t of additional endowment to the portfolio, which is non-negative and upper-bounded by a given constant $\bar{c}$. We denote respectively $\mathcal{A}$ and $\mathcal{C}$ the set of admissible portfolio strategies and of admissible consumption strategies. Hence the dynamics of the wealth is the following :

$$dX_t^{\theta,c} = \theta_t X_t^{\theta,c} \frac{dS_t}{S_t} + c_t dt,$$

and the wealth is contained to remain non-negative as well as to satisfy the constraint :

$$X_t \geq 0 \text{ for every } t \geq 0 \text{ a.s.} \qquad \text{and} \quad \mathbb{P}[X_T \geq D_0] \geq p \, , \tag{2.5.1}$$

We shall denote by $\mathcal{A}_p(t,x)$ the collection of all admissible consumption-investment strategies whose corresponding wealth process satisfies this partial hedging constraint.

We consider a risk neutral asset manager whose subjective discount factor is denoted by a constant $\beta > 0$. For a given initial wealth $x \geq 0$ and probability p of success, the asset manager wishes to solve the following endowment-investment problem under the partial hedging constraint (2.5.1) :

$$w(0,x,p) \; := \inf_{(c,\theta)\in\mathcal{A}_p(0,x)} \mathbb{E}\left[\int_0^T e^{-\beta t} c_t dt\right] .$$

whose dynamic version is the following :

$$w(t,x,p) \ := \ \inf_{(c,\theta)\in\mathcal{A}_p(t,x)} \mathbb{E}\left[\int_t^T e^{-\beta(s-t)}c_s ds\right] \ .$$

We first need to determine the proper domain of definition of this function w. To this aim, we introduce the minimal wealth function defined by :

$$u(t,p) = \inf\{x \geq 0 \,|\mathcal{A}_p(t,x) \neq \emptyset\}.$$

Therefore w is defined on $\{(t,x,p)|x \geq u(t,p)\}$.
We introduce an additional controlled state variable α, valued in $[0,1]$ and defined by :

$$P_t^{t,p,\alpha} = p, \quad dP_s^{t,p,\alpha} = \alpha_s dW_s^S, \ s \in [t,T],$$

We denote $\mathcal{B}$ the set of such controls. We then show that, denoting $g(s,p) = 0 * \mathbf{1}_{s<T} + D_0 1_{p>0}\mathbf{1}_{s=T}$:

$$u(t,p) = \inf\{x \in \mathbb{R} \text{ s.t. } \exists(\theta,\alpha) \in \mathcal{A}\times\mathcal{B}, \forall s \in [t,T], X_s^{t,x,\theta,\bar{c}} \geq g(s, P_s^{t,p,\alpha})\}$$

Therefore, denoting $\mathcal{T}_{[t,T]}$ the set of stopping times taking values in $[t,T]$, $u(.,P_.^{\alpha})$ satisfies the following dynamic programming principle :

- **(DP1)** If $x > u(t,p)$, then there exists $(\theta^*,\alpha^*) \in \mathcal{A}\times\mathcal{B}$ such that

$$X_\tau^{t,x,\theta^*,\bar{c}} \geq u(\tau, P_\tau^{t,p,\alpha^*}) \text{ for all } \tau \in \mathcal{T}_{[t,T]}$$

- **(DP2)** If $x < u(t,p)$, then there exists $\tau^* \in \mathcal{T}_{[t,T]}$ such that

$$\mathbb{P}[X_{\tau\wedge\tau^*}^{t,x,\theta,\bar{c}} > u(\tau, P_\tau^{t,p,\alpha})\mathbf{1}_{\tau<\tau^*} + g(\tau^*, P_{\tau^*}^{t,p,\alpha})\mathbf{1}_{\tau\geq\tau^*}] < 1$$

  for all $\tau \in \mathcal{T}_{[t,T]}$ and $(\theta,\alpha) \in \mathcal{A}\times\mathcal{B}$.

Hence, denoting

$$F^{\alpha,\theta}(z,a) := -\frac{\alpha^2}{2}a + \mu\theta z + \bar{c}$$

and

$$F(z,q,a) := \sup_{\{(\alpha,\theta)\in\mathbb{R}^2,\ \alpha q = \sigma\theta z\}} F^{\alpha,\theta}(z,a),$$

u is a viscosity solution of

$$\min([-\frac{\partial\varphi}{\partial t}(t,p) + F(\varphi(t,p), \frac{\partial\varphi}{\partial p}(t,p), \frac{\partial^2\varphi}{\partial p^2}(t,p))], u) = 0,$$

together with the terminal condition

$$\varphi(T,p) = D_0 p.$$

Besides, we show the boundary condition $u(.,0) = 0$ which allows to obtain u numerically.

Introducing the Fenchel-Legendre dual transform associated with u with respect to the p variable :

$$v(t,q) = \sup_{p\in[0,1]} \{pq - u(t,p)\}, \qquad (t,q) \in [0,T] \times \mathbb{R}_+^*,$$

we obtain that v is a viscosity solution on $[0,T) \times (0,\infty)$ of

$$\max(-\frac{\partial \varphi}{\partial t}(t,q) - \frac{\mu^2}{2\sigma^2} q^2 \frac{\partial^2 \varphi}{\partial q^2}(t,q) - \bar{c}, \varphi - q\frac{\partial \varphi}{\partial q}) = 0$$

with the terminal condition

$$v(T,q) = (q - D_0)^+.$$

This result provides a scheme for u with fast convergence. We set, for $(x,z,q,a_{11},a_{12},a_{22}) \in \mathbb{R}^6$:

$$H^{\theta,\alpha,c}(x,z,q,a_{11},a_{12},a_{22}) := -(\mu\theta x + c)q - \frac{\sigma^2\theta^2x^2}{2}a_{11} - \alpha\sigma\theta x a_{12} - \frac{\alpha^2}{2}a_{22} - c + \beta z$$

and

$$H(x,z,q,a_{11},a_{12},a_{22}) := \sup_{(\theta,\alpha,c)\in\mathbb{R}^2\times[0,\bar{c}]} H^{\theta,\alpha,c}.$$

We then obtain that on $int(u) := \{(t,x,p)|x > u(t,p)\}$, w is a viscosity solution of :

$$-\frac{\partial \varphi}{\partial t} + H(x, \varphi(t,x,p), \frac{\partial \varphi}{\partial x}(t,x,p), \frac{\partial^2 \varphi}{\partial x^2}(t,x,p), \frac{\partial^2 \varphi}{\partial x \partial p}(t,x,p), \frac{\partial^2 \varphi}{\partial p^2}(t,x,p)) = 0.$$

As a byproduct, we obtain that the optimal consumption is $c = 0$ whenever $\frac{\partial w}{\partial x} > -1$ and $c = \bar{c}$ whenever $\frac{\partial w}{\partial x} < -1$. We also obtain the following boundary condition : for $(t,p) \in [0,T] \times [0,1]$ such that $u(t,p) > 0$,

$$\lim_{x\to u(t,p)^+} w(t,x,p) = \frac{\bar{c}}{\beta}(1 - e^{\beta(t-T)}).$$

Besides,

$$w(t,x,1) = \inf_{(\theta,c)\in\mathcal{A}\times\mathcal{C} s.t. X_s^{t,x,\theta,c} \geq (D_0 - \bar{c}(T-t))^+ \ \forall s\geq t} \int_t^T e^{-\beta(s-t)}\mathbb{E}[c_s]ds \quad \forall 0 \leq t \leq T, \quad x \geq (D_0 - \bar{c}(T-t))^+$$

This is a Merton type problem which can be solved numerically. Hence we can also obtain w numerically.
We finally provide graphs of u, w and of the optimal strategy in both c and θ at time T. We see that the optimal strategy in c consists in not investing except close to the boundary defined by u.

This chapter is based on a work in progress in collaboration with Romuald Elie and Xavier Warin.

Chapitre 3

Mean-variance hedging under multiple defaults risk

3.1 Introduction

In this chapter, we study the problem of mean-variance hedging in a financial market model subject to defaults and contagion risk. We consider multiple default events ; such an event may correspond to a succession of crisis periods for a country or a succession of bad annual financial results for a firm, for example. Such defaults could induce loss or gain in asset prices. A classic modelling approach is to use an Itô process governed by some Brownian motion W for the asset price S and jumps appearing at random default times associated with a marked point process μ. The mean-variance hedging problem in this incomplete market framework may then be studied using stochastic control and dynamic programming methods in the global filtration $\mathbb{G}$ generated by W and μ. This leads, in principle, to Hamilton-Jacobi-Bellman integro-differential equations in a Markovian framework and, more generally, to backward stochastic differential equations (BSDEs) with jumps ; the derivation relies on a martingale representation under $\mathbb{G}$ with respect to W and μ, which holds under an intensity hypothesis on the defaults and the so-called immersion property (or $(\mathcal{H})$-hypothesis). Such an approach has been used in [61] for the multiple defaults case and in [49] for the mean-variance hedging problem under $\mathbb{G}$ for defaultable claims.

The mean-variance hedging problem was introduced in [45], and many papers have since followed and developed this approach (for a review of this literature, see [98]). In most of these papers, the problem has been solved using continuous filtration [91], [97]. The authors use the dual approach to prove the existence of the variance optimal measure (VOM). Moreover, they can write the solution to the primal problem using BSDEs, the existence of whose solutions can be deduced from the existence of the VOM. In the case of discontinuous filtration, the VOM is not always a measure of probability (see [1] for conditions) ; thus, the above mentioned approach cannot be used to solve the problem. Therefore, in general, in the case of discontinuous filtration, the authors assume that the VOM is a true probability measure, as in [77], and then deduce the solution to the primal problem using BSDEs. They then prove the existence of the solution to each BSDE using the VOM.

In a general model with discontinuous filtration generated by a continuous process and a discontinuous process, the author of [79] proved the existence of the solutions of the BSDEs for the mean variance problem under the assumption that its asset coefficients were adapted with respect to the continuous filtration $\mathbb{F}$. This strong assumption allowed the author not to assume that the VOM is a true probability and led him to directly solve the main BSDE without requiring any assumption regarding the VOM.

In this chapter, we also consider the case of a discontinuous filtration $\mathbb{G}$. Nevertheless, in this chapter, we address the more general case in which we do not assume both that the VOM is a true probability measure and that its asset coefficients are adapted with respect to the continuous filtration $\mathbb{F}$. In our model, jumps are generated by default times. Thus, we cannot use the same techniques as [79] because his strong assumption is not satisfied in our framework. Indeed, our asset coefficients depend on the jumps (defaults). Therefore, we use a different approach than the one mentioned previously.

We use an approach introduced in [62] and [63]. By viewing the global filtration $\mathbb{G}$ as a progressive enlargement of filtrations of the default-free filtration $\mathbb{F}$ generated by the Brownian motion W, with the default filtration generated based on random times, the basic concept is to split the global mean variance problem defined on $\mathbb{G}$ into sub-control problems in the reference filtration $\mathbb{F}$ that correspond to mean variance problems in default-free markets between two default times. More precisely, we derive a backwards recursive decomposition by starting from the mean variance problem in which all defaults have occurred and then working back towards the initial mean variance problem before any default. The primary objective is to connect this family of stochastic control problems in the filtration $\mathbb{F}$, and this is achieved by assuming the existence of a conditional density on the default times given the default-free information $\mathbb{F}$. Even if we use the approach developed in [62] to split our $\mathbb{G}$ problem, it is important to note that in [62], the authors solved the exponential utility maximisation case. For this stochastic control problem, they were obliged to solve a linear system of BSDEs, and they were able to apply a simplification by virtue of the morphism properties of the exponential function. In this chapter, we solve the mean-variance hedging problem, which requires us to solve a coupled system of non-linear (quadratic) BSDEs. Moreover, we can no longer apply any simplification techniques arising from the exponential function, and the theorem (verification theorem) that is necessary to relate or connect the solutions of each sub-control problem in $\mathbb{F}$ to the global problem in $\mathbb{G}$ becomes more difficult to prove.

Following the dynamic programming method, we show that between each default times, we must first obtain a characterisation of each dynamic version of the mean-variance hedging problem in the form of quadratic decompositions. These decompositions depend explicitly on the parameters and default times of our model. Second, we express the three terms that appear in these quadratic decompositions as solutions of three explicit BSDEs.

Then, beginning after the last default event and working backwards to the initial mean variance problem, we obtain, for each subset, a system of recursive coupled quadratic BSDEs.

We prove explicitly, in our first major contribution (Theorem 3.3.1), the existence and uniqueness of the solutions of these systems of quadratic BSDEs, which is not a trivial result, and we identify the optimal mean-variance hedging strategy. Indeed, recently, in

[60], the authors proved that the problem of mean-variance hedging for a general semi-martingale under the global filtration $\mathbb{G}$ can also be regarded as a solution of BSDEs. However, they did not provide an explicit form for these BSDEs and also did not prove the existence of the solution of these BSDEs. Moreover, because we have the explicit forms of the systems of BSDEs, we find an explicit formula for the optimal hedging strategy that solves our mean-variance hedging problem.

Then, in our second major contribution (Theorem 3.3.2), we prove that the solutions of each sub-control problem can be linked to the solution of our global mean-variance hedging problem by presenting a verification theorem. We also prove that the optimal hedging strategy for our $\mathbb{G}$ control problem can be deduced as the sequence of all optimal sub-control hedging strategies.

The final major contribution of this chapter is the numerical application of the mean-variance hedging problem to a multiple defaults case, which has, to the best of our knowledge, not previously been addressed.

The outline of this chapter is as follows : in Section 3.2, we introduce our model and the corresponding mean-variance hedging problem. We deduce the systems of BSDEs. Then, in Section 3.3, we present the solution to the mean-variance hedging problem. For this purpose, we first prove the existence of a solution to the recursive coupled system of quadratic BSDEs. Second, we provide a BSDE characterisation using a verification theorem and relate the solutions of each sub-control problem in $\mathbb{F}$ to our global control problem in $\mathbb{G}$. Finally, in Section 3.4, we present some numerical illustrations, most notably a multiple defaults case. We numerically recover certain theoretical results and obtain several financial interpretations of our model with respect to the value of the defaultable intensity or the size of the jumps.

3.2 Multiple defaults model

3.2.1 Market information

We define a probability space $(\Omega, \mathcal{G}, \mathcal{P})$ that is equipped with a reference filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ that satisfies the usual conditions and represents the default-free information concerning the market. In this section, we adopt the same model and notations used in [62]. Let $\boldsymbol{\tau} = (\tau_1, ..., \tau_n)$ be a vector of $n \in \mathbb{N}^*$ random times, and let $\boldsymbol{L} = (L_1, ..., L_n)$ be a vector of the n marks associated with $\boldsymbol{\tau}$ such that for any $1 \leq k \leq n$, L_k is a $\mathcal{G}$-measurable random variable that takes values in $E \subset \mathbb{R}$ and represents, for example, the loss given default at time τ_k. For $k = \{1, \ldots, n\}$, we denote by $\mathbb{D}^k = (\mathcal{D}^k_t)_{t\in[0,T]}$ the filtrations generated by the associated jump processes, where $\mathcal{D}^k_t = \tilde{\mathcal{D}}^k_{t+}$ and $\tilde{\mathcal{D}}^k_t = \sigma(1_{\tau_k\leq s}, s \leq t) \vee \sigma(L_k 1_{\tau_k \leq s}, s \leq t)$. Then, $\mathbb{G} = (\mathcal{G}_t)_{t\in[0,T]}$ will be the enlarged progressive filtration $\mathbb{F} \vee \mathbb{D}^1 \vee ... \vee \mathbb{D}^n$ that represents the structure of the global information available to investors over $[0, T]$. In other words, $\mathbb{G}$ is the smallest right-continuous filtration that contains $\mathbb{F}$ such that for any $1 \leq k \leq n$, τ_k is a $\mathbb{G}$ stopping time and L_k is $\mathcal{G}_{\tau_k}$-measurable. We assume that the default times are ordered (i.e., $\tau_1 \leq ... \leq \tau_n$) and can thus be valued in terms of Δ_n on $\{\tau_n < \infty\}$, where, for $k = 1, ..., n$, we have

$$\Delta_k := \left\{(\theta_1, ..., \theta_k) \in (\mathbb{R}_+)^k : \theta_1 \leq ... \leq \theta_k\right\}.$$

Thus, we do not distinguish specific credit names and observe only successive default times.

Remark 3.2.1.
We note that the general case of non-ordered multiple random times for $(\tau_1, ..., \tau_n)$ *(together with marks* $(L_1, ..., L_n)$*) can be derived from the case of successive random times by considering suitable auxiliary marks. Indeed, consider the corresponding ordered times, denoted by* $\hat{\tau}_1 \leq ... \leq \hat{\tau}_n$*, and the index mark valued in the range* $\{1, \ldots, n\}$*, denoted by* ι_k*, such that* $\hat{\tau}_k = \tau_{\iota_k}$ *for* $1 \leq k \leq n$*. Then, the progressive enlargement of the filtration of* $\mathbb{F}$ *using the successive random times* $\hat{\tau}_1, ..., \hat{\tau}_n$ *together with the marks* $\iota_1, L_{\iota_1}, ..., \iota_n, L_{\iota_n}$ *leads to the filtration* $\mathbb{G}$*.*

In the following, we assume that the n default times always occur before time T. For any $(\theta_1, ..., \theta_n) \in \Delta_n$, $(l_1, ..., l_n) \in E^n$, we use the notation $\boldsymbol{\theta} = (\theta_1, ..., \theta_n)$ and $\boldsymbol{l} = (l_1, ..., l_n)$; we also use the notation $\boldsymbol{\theta}_k = (\theta_1, ..., \theta_k)$ and $\boldsymbol{l}_k = (l_1, ..., l_k)$ for $0 \leq k \leq n$, with the convention that $\theta_0 = l_0 = \emptyset$. Similarly, we adopt the notation $\boldsymbol{\tau}_k = (\tau_1, ..., \tau_k)$ and $\boldsymbol{L}_k = (L_1, ..., L_k)$. Moreover, for $0 \leq t \leq T$, the set Ω_t^k denotes the event

$$\Omega_t^k \ := \ \{\tau_k \leq t < \tau_{k+1}\},$$

(where $\Omega_t^0 = \{t < \tau_1\}$ and $\Omega_t^n = \{\tau_n \leq t\}$) and represents the scenario in which k defaults occur before time t. We refer to Ω_t^k as the k-default scenario at time t. We similarly define $\Omega_{t-}^k = \{\tau_k < t \leq \tau_{k+1}\}$.

We begin by recalling some typical spaces. For $s \leq T$, $\mathcal{S}^\infty[s, T]$ is the Banach space of $\mathbb{R}$-valued càdlàg processes X such that there exists a constant C that satisfies

$$\|X\|_{\mathcal{S}^\infty[s,T]} := \operatorname*{ess\,sup}_{t \in [s,T]} |X_t| \ \leq C \ < +\infty.$$

Finally, the space BMO is the space of an $\mathbb{F}$-adapted martingale such that for any stopping times $0 \leq \sigma \leq \tau \leq T$, there exists a nonnegative constant $c > 0$ such that

$$\mathbb{E}\left[[M]_\tau - [M]_{\sigma^-} | \mathcal{G}_\sigma\right] \leq c;$$

then, $M = Z.W \in \mathrm{BMO}$. To simplify the notation, we write $Z \in \mathrm{BMO}$.

We now denote by $\mathcal{P}(\mathbb{F})$ the σ-algebra of $\mathbb{F}$-predictable measurable subsets on $\mathbb{R}_+ \times \Omega$, and we denote by $\mathcal{P}_\mathbb{F}(\Delta_k, E^k)$ the set of indexed $\mathbb{F}$-predictable processes $Z^k(.,.)$, i.e., processes such that the map $(t, \omega, \boldsymbol{\theta}_k, \boldsymbol{l}_k) \to Z_t^k(\omega, \boldsymbol{\theta}_k, \boldsymbol{l}_k)$ is $\mathcal{P}(\mathbb{F}) \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable. We also denote by $\mathcal{O}_\mathbb{F}(\Delta_k, E^k)$ the set of indexed $\mathbb{F}$-adapted processes $Z^k(.,.)$, i.e., processes such that for all $t \geq 0$, the map $(\omega, \boldsymbol{\theta}_k, \boldsymbol{l}_k) \to Z_t^k(\omega, \boldsymbol{\theta}_k, \boldsymbol{l}_k)$ is $\mathcal{F}_t \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable. In the following, we assume the *density hypothesis*, which is given by the following statement in the multiple defaults case :

Assumption 3.2.1 (*Density hypothesis*)**.** *There exists an* $\alpha \in \mathcal{O}_\mathbb{F}(\Delta_n, E^n)$ *such that for any bounded Borel function* f *on* $\Delta_n \times E^n$ *and* $0 \leq t \leq T$*, the following holds :*

$$\mathbb{E}[f(\tau, L)|\mathcal{F}_t] \ = \ \int_{\Delta_n \times E^n} f(\boldsymbol{\theta}, \boldsymbol{l}) \alpha_t(\boldsymbol{\theta}, \boldsymbol{l}) d\boldsymbol{\theta} \eta(d\boldsymbol{l}) \quad a.s., \tag{3.2.1}$$

where $d\boldsymbol{\theta} = d\theta_1 ... d\theta_n$ *is the Lebesgue measure on* $\mathbb{R}^n$ *and* $\eta(d\boldsymbol{l})$ *is a Borel measure on* E^n *in the form* $\eta(d\boldsymbol{l}) = \eta_1(dl_1) \prod_{k=1}^{n-1} \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})$*, where* η_1 *a nonnegative Borel measure on* E *and* $\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})$ *is a nonnegative transition kernel on* $E^k \times E$*.*

Remark 3.2.2. *The condition defined by* (3.2.1) *implies that if* α *is separable in the form* $\alpha_t(\boldsymbol{\theta}, \boldsymbol{l}) = \alpha_t^{\boldsymbol{\tau}}(\boldsymbol{\theta})\alpha_t^{\boldsymbol{L}}(\boldsymbol{l})$, *then the random times and marks are independent, given* $\mathcal{F}_t$.

We can now present a splitting formula in this progressively enlarged filtration :

Lemma 3.2.1. *Under the density hypothesis of Assumption* 3.2.1, *any* $\mathbb{G}$*-adapted process* $Z = (Z_t)_{0 \le t \le T}$ *has a decomposition in the form*

$$Z_t = \sum_{k=0}^{n} 1_{\Omega_t^k} Z_t^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k), \quad 0 \le t \le T,$$

where Z^k *lies in* $\mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$.

Démonstration. This is a consequence of Theorem 5.8 of [102]. The application to our model under the conditional hypothesis assumption is discussed in Sections 6 and 7.4 of [102]. □

3.2.2 Asset price model under default risk

The trading asset S is a $\mathbb{G}$-adapted process that takes, following Lemma 3.2.1, the decomposed form

$$S_t = \sum_{k=0}^{n} 1_{\Omega_t^k} S_t^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k), \tag{3.2.2}$$

where $S^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ (where $\boldsymbol{\theta}_k = (\theta_1, ..., \theta_k) \in \Delta_k$ and $\boldsymbol{l}_k = (l_1, ..., l_k) \in E^k$) is an indexed process in $\mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$ that is valued in $\mathbb{R}_+$ and represents the asset value in the k-default scenario, given the past default events $\boldsymbol{\tau}_k = \boldsymbol{\theta}_k$ and the marks at default $\boldsymbol{L}_k = \boldsymbol{l}_k$. Note that S_t is equal to the value S_t^k only on the set Ω_t^k, that is, only for $\tau_k \le t < \tau_{k+1}$. The dynamics of the indexed process S^k are given by

$$dS_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = S_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)(\mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dt + \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dW_t), \quad \theta_k \le t \le T, \tag{3.2.3}$$

where W is a one-dimensional $(P, \mathbb{F})$-Brownian motion and μ^k and σ^k are indexed processes in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ that are valued in $\mathbb{R}$. As in the one-default case, we adopt the usual no-arbitrage assumption that there exists an indexed risk premium process $\lambda^k \in \mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ such that for all $(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in \Delta_k \times E^k$,

$$\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\lambda_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = \mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k), \quad 0 \le t \le T. \tag{3.2.4}$$

Moreover, in this contagion risk model, each default time may induce a jump in the assets portfolio. This scenario is formalised by considering a family of indexed processes γ^k, where $0 \le k \le n-1$, in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k, E)$ and valued in $[-1, \infty)$. For $(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in \Delta_k \times E^k$ and $l_{k+1} \in E$, $\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})$ represents the vector of the relative jump size on the asset at time $t = \theta_{k+1} > \theta_k$ with a mark l_{k+1}, given the past default events $(\boldsymbol{\tau}_k, \boldsymbol{L}_k) = (\boldsymbol{\theta}_k, \boldsymbol{l}_k)$. In other words, we can write the following :

$$S_{\theta_{k+1}}^{k+1}(\boldsymbol{\theta}_{k+1}, \boldsymbol{l}_{k+1}) = S_{\theta_{k+1}^-}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\left(1 + \gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\right). \tag{3.2.5}$$

3.2.3 Strategy and wealth process

The trading strategy π is a $\mathbb{G}$-predictable process and can thus be decomposed in the form

$$\pi_t = \sum_{k=0}^{n} 1_{\Omega_{t-}^k} \pi_t^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k), \quad 0 \leq t \leq T, \tag{3.2.6}$$

where π^k is an indexed process in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ and $\pi^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ is valued in the closed set A^k of $\mathbb{R}$, which contains the zero element and represents the amount continuously invested in the asset in the k-default scenario, given the past default events $\boldsymbol{\tau}_k = \boldsymbol{\theta}_k$ and the marks at default $\boldsymbol{L}_k = \boldsymbol{l}_k$ for $(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in \Delta_k \times E^k$. We often identify the strategy π with the family $(\pi^k)_{0\leq k\leq n}$ given in 3.2.6, and we require the following integrability conditions : for all $\boldsymbol{\theta}_k \in \Delta_k$ and $\boldsymbol{l}_k \in E^k$,

$$\int_0^T |\pi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)|dt + \int_0^T |\pi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)|^2 dt < \infty, \quad a.s. \tag{3.2.7}$$

Given a trading strategy $\pi = (\pi^k)_{0\leq k\leq n}$, the corresponding wealth process is given by

$$X_t^{x,\pi} = \sum_{k=0}^{n} 1_{\Omega_t^k} X_t^{k,x,\pi}(\boldsymbol{\tau}_k, \boldsymbol{L}_k), \quad 0 \leq t \leq T, \tag{3.2.8}$$

where $X^{k,x,\pi}(\boldsymbol{\tau}_k, \boldsymbol{L}_k)$ (with $\boldsymbol{\theta}_k \in \Delta_k$ and $\boldsymbol{l}_k \in E^k$) is an indexed process in $\mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$ that represents the wealth controlled by $\pi^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ in the price process $S^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$, given the past default events $\boldsymbol{\tau}_k = \boldsymbol{\theta}_k$ and the marks at default $\boldsymbol{L}_k = \boldsymbol{l}_k$. From the dynamics given by (3.2.3) and under the conditions given by (3.2.7), this process is governed by the following equation :

$$dX_t^{k,x,\pi}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = \pi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)(\mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dt + \sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dW_t), \quad \theta_k \leq t \leq T. \tag{3.2.9}$$

Moreover, each default time induces a jump in the asset price process and thus also in the wealth process. From (3.2.5), this jump is given by

$$X_{\theta_{k+1}}^{k+1,x,\pi}(\boldsymbol{\theta}_{k+1}, \boldsymbol{l}_{k+1}) = X_{\theta_{k+1}^-}^{k,x,\pi}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}).$$

Ultimately, the payoff is a bounded $\mathcal{G}_T$-measurable random variable H_T that takes the following decomposed form :

$$H_T = \sum_{k=0}^{n} 1_{\Omega_T^k} H_t^k(\boldsymbol{\tau}_k, \boldsymbol{L}_k), \tag{3.2.10}$$

where $H_T^k(.,.)$ is $\mathcal{F}_T \otimes \mathcal{B}(\Delta_k) \otimes \mathcal{B}(E^k)$-measurable and represents the payoff when k defaults occur before maturity T.

3.2.4 The mean-variance problem

In our problem of *mean-variance hedging (MVH)*, the performance of an admissible trading strategy $\pi \in \mathcal{A}_{\mathbb{G}}$ implemented with initial capital $x \in \mathbb{R}$ is measured over the finite horizon T by

$$J_0^H(x, \pi) = \mathbb{E}[(H_T - X_T^{x,\pi})^2], \tag{3.2.11}$$

and the MVH problem is formulated as follows :

$$V_0^H(x) = \inf_{\pi \in \mathcal{A}_{\mathbb{G}}} J_0^H(x, \pi),$$

where $\mathcal{A}_{\mathbb{G}}$ is the set of admissible trading strategies, which is defined in Definition 3.2.1 below.

Value functions

We first define the set of admissible trading strategies for the multiple defaults case :

Definition 3.2.1. *For $0 \le k \le n$, $\mathcal{A}_{\mathbb{F}}^k$ denotes the set of indexed processes π^k in $\mathcal{P}_{\mathbb{F}}(\Delta_k, E^k)$ and valued in A^k that satisfy (3.2.7) such that*

$$\mathbb{E}\left[\int_{\theta_k}^T |\pi_s^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)|^2 ds\right] < \infty. \tag{3.2.12}$$

We then denote by $\mathcal{A}_{\mathbb{G}} = (\mathcal{A}_{\mathbb{F}}^k)_{0 \le k \le n}$ the set of admissible trading strategies $\pi = (\pi^k)_{0 \le k \le n}$.

Under the density hypothesis 3.2.1, let us define a family of auxiliary processes $\alpha^k \in \mathcal{O}_{\mathbb{F}}(\Delta_k, E^k)$, $0 \le k \le n$; this family of processes is related to the survival probability and is defined by recursive induction from $\alpha^n = \alpha$ as follows :

$$\alpha_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = \int_t^\infty \int_E \alpha_t^{k+1}(\boldsymbol{\theta}_k, \theta_{k+1}, \boldsymbol{l}_k, l_{k+1}) d\theta_{k+1} \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}), \tag{3.2.13}$$

for $0 \le k \le n-1$, such that $\mathbb{P}[\tau_{k+1} > t|\mathcal{F}_t] = \int_{\Delta_k \times E^k} \alpha_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) d\boldsymbol{\theta}_k \eta(d\boldsymbol{l}_k)$ and $\mathbb{P}[\tau_1 > t|\mathcal{F}_t] = \alpha_t^0$, where $d\boldsymbol{\theta}_k = d\theta_1 ... d\theta_k$, $\eta(d\boldsymbol{l}_k) = \eta_1(dl_1)...\eta_k(\boldsymbol{l}_{k-1}, dl_k)$. Given $\pi^k \in \mathcal{A}_{\mathbb{F}}^k$, we denote by $X^{k,x,\pi}(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ the controlled process solution to (3.2.9), starting from x at θ_k. We now present our model hypothesis :

Assumption 3.2.2. *For all $k \in \{0, 1, 2, \cdots, n\}$, $\forall(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in \Delta_k \bigcap [0, T] \times E^k$, there exist non-negative constants c, C, and δ such that*

- $\left(\sup_{t \in [\theta_k, T]} |\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)| + \sup_{t \in [\theta_k, T]} |\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)| + \sup_{t \in [\theta_k, T]} |\mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)| + \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\right) \le C$;
- $\inf_{t \in [\theta_k, T]} |\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)| \ge c$; *and*
- $\alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \ge \delta$.

  Moreover, we assume that the measure $\eta_k(dl_k)$ is also uniformly bounded.

The mean-variance hedging problem

The value function for the global mean variance $\mathbb{G}$ problem (3.2.11) is then given, in the multiple defaults case, through a backwards induction from the $\mathbb{F}$ problems :

$$V^n(x, \boldsymbol{\theta}, \boldsymbol{l}) = \operatorname*{ess\,inf}_{\pi^n \in \mathcal{A}_{\mathbb{F}}^n} \mathbb{E}\left[(H_T^n - X_T^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l}))^2 \alpha_T(\boldsymbol{\theta}, \boldsymbol{l}) | \mathcal{F}_{\theta_n}\right] \tag{3.2.14}$$

and

$$V^k(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k) = \operatorname*{ess\,inf}_{\pi^k \in \mathcal{A}_{\mathbb{F}}^k} \mathbb{E}[(H_T^k - X_T^{k,x,\pi}(\boldsymbol{\theta}_k, \boldsymbol{l}_k))^2 \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \tag{3.2.15}$$

$$\int_{\theta_k}^{T} \int_E V^{k+1}(X^{k,x,\pi}_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \pi^k_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{l}_k).\gamma^k_{\theta_{k+1}}(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})d\theta_{k+1}|\mathcal{F}_{\theta_k}],$$

where we recall that $\boldsymbol{\theta}_n = \boldsymbol{\theta}, \boldsymbol{l}_n = \boldsymbol{l}$, $\boldsymbol{\theta}_0 = \theta_0 = \emptyset$, and $\boldsymbol{l}_0 = l_0 = \emptyset$, and x denotes the capital at time θ_k.

Remark 3.2.3. *If there exists, for all $0 \le k \le n$, some $\pi^{k,*} \in \mathcal{A}^k_{\mathbb{F}}$ that attains the essential infimum in the previous equations, then the strategy $\pi^* = (\pi^{k,*})_{0\le k\le n} \in \mathcal{A}_{\mathbb{G}}$ is optimal for the MVH problem.*

3.3 Solution to the mean-variance hedging problem

We exploit the quadratic form of the mean-variance hedging problem to characterise, using dynamic programming methods, the solutions to the stochastic optimisation problems (3.2.14) and (3.2.15) in terms of a recursive system of indexed BSDEs with respect to the filtration $\mathbb{F}$. We use a verification approach that can be described as follows :

1. First, we formally derive the system of BSDEs associated with the $\mathbb{F}$-stochastic control problems (3.2.14) and (3.2.15) using the dynamic programming principle.
2. Second, we confirm the existence of the solutions of the corresponding system of coupled quadratic BSDEs (see Theorem 3.3.1) using BSDE techniques.
3. Finally, in a verification theorem (see Theorem 3.3.2), we prove that these BSDEs solutions are unique and present the solution to our mean-variance hedging problem. We also prove that the strategy identified in step 1 is optimal and admissible. Moreover, we prove that the assumption of the quadratic representation form of our value function is true.

Let us begin with step 1. For $t \in [\theta_n, T]$, $\nu^n \in \mathcal{A}^n_{\mathbb{F}}$, let us introduce the following set of controls coinciding with strategy ν^n through time t :

$$\mathcal{A}^n_{\mathbb{F}}(t, \nu^n) = \{\pi^n \in \mathcal{A}^n_{\mathbb{F}} : \pi^n_{.\wedge t} = \nu^n_{.\wedge t}\}.$$

We can now define the dynamic version of (3.2.14) by considering the family of $\mathbb{F}$-adapted processes :

$$V^n_t(x, \boldsymbol{\theta}, \boldsymbol{l}, \nu^n) = \operatorname*{ess\,inf}_{\pi^n \in \mathcal{A}^n_{\mathbb{F}}(t,\nu^n)} \mathbb{E}\left[(H^n_T - X^{n,x,\pi}_T(\boldsymbol{\theta}, \boldsymbol{l}))^2 \alpha_T(\boldsymbol{\theta}, \boldsymbol{l})|\mathcal{F}_t\right], \quad t \ge \theta_n, \tag{3.3.1}$$

such that $V^n_{\theta_n}(x, \boldsymbol{\theta}, \boldsymbol{l}, \nu^n) = V^n(x, \boldsymbol{\theta}, \boldsymbol{l})$ for any $\nu^n \in \mathcal{A}^n_{\mathbb{F}}$. According to the dynamic programming principle, the submartingale property holds on $\{V^n_t(x, \boldsymbol{\theta}, \boldsymbol{l}, \nu^n)\ , \theta_n \le t \le T\}$ for any $\nu^n \in \mathcal{A}^n_{\mathbb{F}}$, and if an optimal strategy exists for (3.3.1), then the martingale property holds on $\{V^n_t(x, \boldsymbol{\theta}, \boldsymbol{l}, \pi^{*,n}), \theta_n \le t \le T\}$ for some $\pi^{*,n} \in \mathcal{A}^n_{\mathbb{F}}$. Moreover, because we have adopted a quadratic minimisation approach, the value process $V^n_t(x, \boldsymbol{\theta}, \boldsymbol{l}, \nu^n)$ takes the quadratic decomposition form given by

$$V^n_t(x, \boldsymbol{\theta}, \boldsymbol{l}, \nu^n) = v^{n,\boldsymbol{\theta},\boldsymbol{l}}_t (X^{n,x,\pi}_t(\boldsymbol{\theta}, \boldsymbol{l}) - Y^{n,\boldsymbol{\theta},\boldsymbol{l}}_t)^2 + \xi^{n,\boldsymbol{\theta},\boldsymbol{l}}_t, \quad t \in [\theta_n, T].$$

We search for a triple $(v^{n,\boldsymbol{\theta},\boldsymbol{l}}, Y^{n,\boldsymbol{\theta},\boldsymbol{l}}, \xi^{n,\boldsymbol{\theta},\boldsymbol{l}})$ in the form

$$
(\mathbf{En}) \quad \begin{cases} \dfrac{dv_t^{n,\boldsymbol{\theta},\boldsymbol{l}}}{v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}} = -g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(1)}(v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}, \beta_t^{n,\boldsymbol{\theta},\boldsymbol{l}})dt + \beta_t^{n,\boldsymbol{\theta},\boldsymbol{l}}dW_t, \\ \\ dY_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = -g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(2)}(Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}, Z_t^{n,\boldsymbol{\theta},\boldsymbol{l}})dt + Z_t^{n,\boldsymbol{\theta},\boldsymbol{l}}dW_t, \\ \\ d\xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = -g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(3)}(\xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}}, R_t^{n,\boldsymbol{\theta},\boldsymbol{l}})dt + R_t^{n,\boldsymbol{\theta},\boldsymbol{l}}dW_t. \end{cases} \tag{3.3.2}
$$

Then, using the submartingale and martingale properties of the dynamic programming principle noted above and because $V_T^n(x,\boldsymbol{\theta},\boldsymbol{l},\nu^n) = \left(X_T^{n,x,\pi}(\boldsymbol{\theta},\boldsymbol{l}) - H_T^n(\boldsymbol{\theta},\boldsymbol{l})\right)^2 \alpha_T(\boldsymbol{\theta},\boldsymbol{l})$ by (3.3.1), we see from Itô calculus (see Proposition 3.5 of Goutte and Ngoupeyou [49] for more details) that the triple $(v^{n,\boldsymbol{\theta},\boldsymbol{l}}, Y^{n,\boldsymbol{\theta},\boldsymbol{l}}, \xi^{n,\boldsymbol{\theta},\boldsymbol{l}})$ satisfies (3.3.2) for all $t \in [\theta_n, T]$, with terminal conditions $v_T^{n,\boldsymbol{\theta},\boldsymbol{l}} = \alpha_T(\boldsymbol{\theta},\boldsymbol{l})$, $Y_T^{n,\boldsymbol{\theta},\boldsymbol{l}} = H_T^n(\boldsymbol{\theta},\boldsymbol{l})$ and $\xi_T^{n,\boldsymbol{\theta},\boldsymbol{l}} = 0$. The corresponding coefficients of the BSDEs are given by the following equations :

$$
g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(1)} = -\frac{\left(\mu^n(\boldsymbol{\theta},\boldsymbol{l}) + \sigma^n(\boldsymbol{\theta},\boldsymbol{l})\beta_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2}{(\sigma^n(\boldsymbol{\theta},\boldsymbol{l}))^2}, \quad g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(2)} = -\frac{\mu^n(\boldsymbol{\theta},\boldsymbol{l})}{\sigma^n(\boldsymbol{\theta},\boldsymbol{l})} Z_t^{n,\boldsymbol{\theta},\boldsymbol{l}} \quad \text{and} \quad g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(3)} = 0.
$$

We also find that the optimal strategy $\pi^{n,*}$ (such that $V_t^n(x,\boldsymbol{\theta},\boldsymbol{l},\pi^{n,*})$ is a true martingale) is given for all $t \in [\theta_n, T]$ by

$$
\pi_t^{n,*}(\boldsymbol{\theta},\boldsymbol{l}) = f_t^{n,\boldsymbol{\theta},\boldsymbol{l},1} X_t^{n,x,\pi^*}(\boldsymbol{\theta},\boldsymbol{l}) + f_t^{n,\boldsymbol{\theta},\boldsymbol{l},2},
$$

where

$$
f_t^{n,\boldsymbol{\theta},\boldsymbol{l},1} := -\frac{1}{\left(\sigma_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2}\left(\mu_t^{n,\boldsymbol{\theta},\boldsymbol{l}} + \sigma_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\beta_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)
$$

and

$$
f_t^{n,\boldsymbol{\theta},\boldsymbol{l},2} := \frac{1}{\left(\sigma_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2}\left[\sigma_t^{n,\boldsymbol{\theta},\boldsymbol{l}} Z_t^{n,\boldsymbol{\theta},\boldsymbol{l}} + Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(\mu_t^{n,\boldsymbol{\theta},\boldsymbol{l}} + \sigma_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\beta_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)\right].
$$

Hence, the optimal strategy is linear in X, which is also the case in the no-default model. Henceforth, we refer to this problem as the **(En)** problem.

Next, consider the problem defined by (3.2.15), and define the dynamic version in a similar manner by considering the value function process given by

$$
V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k) = \operatorname*{ess\,inf}_{\pi^k \in \mathcal{A}_{\mathbb{F}}^k(t,\nu^k)} \mathbb{E}[(H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - X_T^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \tag{3.3.3}
$$

$$
\int_t^T \int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_t]
$$

for $\theta_k \leq t \leq T$, where $\mathcal{A}_{\mathbb{F}}^k(t,\nu^k) = \{\pi^k \in \mathcal{A}_{\mathbb{F}}^k : \pi_{.\wedge t}^k = \nu_{.\wedge t}^k\}$ for $\nu^k \in \mathcal{A}_{\mathbb{F}}^k$, such that $V_{\theta_k}^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k) = V^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k)$. Similarly, we henceforth refer to this problem as the **(Ek)** problem for $k = 0,\ldots,n-1$. The dynamic programming principle for (3.3.3) formally implies that the process

$$
V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k)+\int_0^t \int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)+\pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}
$$

for $t \in [\theta_k, T]$ is a submartingale for any $\nu^k \in \mathcal{A}^k_{\mathbb{F}}$ and a true martingale for $\pi^{*,k}$ if it is an optimal strategy for (3.3.3). Again, because we have adopted a quadratic minimisation approach, the value process $V^k_t(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k, \nu^k)$ should take the quadratic decomposition form given by

$$V^k_t(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k, \nu^k) = v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,}_t (X^{k,x,\pi}_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t)^2 + \xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t \quad, \forall k = 0, \dots, n-1.$$

We also search for a triple $\left(v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, \xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)$ for all $k = 0, \dots, n-1$ in the form

$$(\mathbf{Ek}) \quad \begin{cases} \dfrac{dv^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t}{v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t} = -g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}_t (v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t, \beta^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t) dt + \beta^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t dW_t, \\ dY^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t = -g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}_t (Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t, Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t) dt + Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t dW_t, \\ d\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t = -g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(3)}_t (\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t, R^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t) dt + R^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t dW_t. \end{cases} \tag{3.3.4}$$

Then, using the submartingale and martingale properties of the dynamic programming principle described above and because $V^k_T(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k, \nu^k) = \left(X^{k,x,\pi}_T(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - H^k_T(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\right)^2 \alpha^k_T(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ by (3.3.3), we see from Itô calculus (again, see Proposition 3.5 of Goutte and Ngoupeyou [49] for more details) that the triple $\left(v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, \xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)$ satisfies (3.3.4) for all $t \in [\theta_k, T]$, with terminal conditions $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_T = \alpha^k_T(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$, $Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_T = H^k_T(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ and $\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_T = 0$. Moreover, the corresponding coefficients of the BSDEs are given by the following equations :

$$\begin{aligned} g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}_t &= \int_E (1 + v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \\ &- \frac{\left(\mu^k_t + \sigma^k_t \beta^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t + \int_E (1 + v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t) \gamma^k_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})\right)^2}{(\sigma^k_t)^2 + \int_E (1 + v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t)(\gamma^k_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}))^2 \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})}, \end{aligned}$$

$$\begin{aligned} g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}_t &= \beta^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t + \int_E U^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t (1 + v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \\ &+ \frac{\left(- \int_E U^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t (1 + v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t) \gamma^k_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) - \sigma^k_t Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t\right)}{(\sigma^k_t)^2 + \int_E (1 + v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t)(\gamma^k_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}))^2 \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})} \\ &\times \left(\mu^k_t + \sigma^k_t \beta^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t + \int_E (1 + v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t) \gamma^k_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})\right) \end{aligned}$$

and

$$\begin{aligned} g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(3)}_t &= v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t \Big[\int_E (U^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t)^2 (1 + v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) + (Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t)^2 \\ &- \frac{\left(- \int_E (1 + v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t) U^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t \gamma^k_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) - \sigma^k_t Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t\right)^2}{(\sigma^k_t)^2 + \int_E (1 + v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}_t)(\gamma^k_t(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}))^2 \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})}\Big], \end{aligned}$$

where

$$1+v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}=\frac{v^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}}{v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} \quad \text{and} \quad U^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}=Y^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}-Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.$$

The optimal strategy $\pi^{k,*}$ (such that $V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\pi^{k,*})$ is a true martingale) is given by

$$\begin{aligned}\pi_t^{k,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) &= \frac{1}{(\sigma_t^k)^2+\int_E(1+v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})^2\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})}\Big[\sigma_t^k Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}-K_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(\mu_t^k+\sigma_t^k\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\\ &+\frac{\int_E(X_t^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}-Y^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}})\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})}{v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}\Big],\end{aligned}$$

where $K_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} := X_t^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$. Again, we obtain the optimal strategy in a linear form with respect to X. Henceforth, we refer to this problem as the **(Ek)** problem for $k\in\{0,\dots,n-1\}$.

Remark 3.3.1. *For all* **(Ek)** *problems for $k\in\{0,1,\dots n\}$, we consider the time interval $[\theta_k,T]$. Hence, for the particular case in which we take the value function for $t=\theta_k$, we find that*

$$V_{t=\theta_k}^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k):=V^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k),$$

where we recall that x is the value of $X^{k,x,\pi}$ in θ_k, and thus, $X_{\theta_k}^{k,x,\pi}=x$.

Hence, **(Ek)** and **(En)** define a recursive system of families of BSDEs indexed by $(\boldsymbol{\theta},\boldsymbol{l})\in\Delta_n(T)\times E^n$, where $\Delta_n(T):=\Delta_n\cap[0,T]=\{(\theta_1,...,\theta_n)\in[0,T]^n : 0\le\theta_1<...<\theta_n\le T\}$.

The next portion of this part is devoted, first, to proving the existence of a solution to this system of BSDEs and, second, to demonstrating its uniqueness via a verification theorem related to the solution of the value functions 3.3.3 and 3.3.1.

3.3.1 Existence of a solution to the recursive system of BSDEs

The generators of our recursive system of BSDEs, (3.3.2) and (3.3.4), are not trivial, as the coefficients $g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$, $k\in\{0,\dots,n\}$ are not standard (coupled and quadratic). Moreover, we will prove the existence of the solution to the quadratic BSDE (i.e., the first one) under positivity constraints. Indeed, we will see that the first BSDE is related to the minimal variance of a pure investment problem (see Remark (3.3.2)).

Hence, we present a theorem to ensure that recursive BSDE solutions exist and remain in their own solution space for all $k\in\{0,1,\dots,n\}$.

Let us consider the family of probability measures $\{Q(\boldsymbol{\theta},\boldsymbol{l}),(\boldsymbol{\theta},\boldsymbol{l})\in\Delta_n(T)\times E^n\}$ such that the Radon-Nikodym density of $Q(\boldsymbol{\theta},\boldsymbol{l})$ with respect to P on $\mathcal{F}_T$ is given by

$$Z_T^Q(\boldsymbol{\theta},\boldsymbol{l}):=\frac{dQ(\boldsymbol{\theta},\boldsymbol{l})}{dP}|_{\mathcal{F}_T} = \exp\left[\int_{\theta_n}^T\frac{\mu_s^n(\boldsymbol{\theta},\boldsymbol{l})}{\sigma_s^n(\boldsymbol{\theta},\boldsymbol{l})}dW_s-\frac{1}{2}\int_{\theta_n}^T\left|\frac{\mu_s^n(\boldsymbol{\theta},\boldsymbol{l})}{\sigma_s^n(\boldsymbol{\theta},\boldsymbol{l})}\right|^2ds\right]. \qquad (3.3.5)$$

Theorem 3.3.1. *For all $k\in\{0,1,\dots,n\}$ and $t\in[\theta_k,T]$, we know the following :*

1. *There exists a couple solution $\left(v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)\in\mathcal{S}^\infty\times\mathrm{BMO}$ of the first BSDE of (3.3.4) (if $k\neq n$) or (3.3.2) (if $k=n$), and there exist constants δ_1^k and δ_2^k such that*

$$0<\delta_1^k\le v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\le\delta_2^k.$$

Moreover, for the case $k = n$, we have the following explicit solution :

$$v_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = \left(\mathbb{E}\left[\left(\frac{Z_T^Q(\boldsymbol{\theta},\boldsymbol{l})}{Z_t^Q(\boldsymbol{\theta},\boldsymbol{l})}\right)^2 \frac{1}{\alpha_T(\boldsymbol{\theta},\boldsymbol{l})}|\mathcal{F}_t\right]\right)^{-1}. \tag{3.3.6}$$

2. *There exists a couple solution $\left(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right) \in \mathcal{S}^\infty \times \mathrm{BMO}$ of the second BSDE of (3.3.4) (if $k \neq n$) or (3.3.2) (if $k = n$). Moreover, for the case $k = n$, we have the following explicit solution :*

$$Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = \mathbb{E}\left[\frac{Z_T^Q(\boldsymbol{\theta},\boldsymbol{l})}{Z_t^Q(\boldsymbol{\theta},\boldsymbol{l})} H_T^n(\boldsymbol{\theta},\boldsymbol{l}) \Big| \mathcal{F}_t\right] = \mathbb{E}^{Q(\boldsymbol{\theta},\boldsymbol{l})}\left[H_T^n(\boldsymbol{\theta},\boldsymbol{l}) \Big| \mathcal{F}_t\right]. \tag{3.3.7}$$

3. *There exists a couple solution $\left(\xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, R_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right) \in \mathcal{S}^\infty \times \mathrm{BMO}$ of the third BSDE of (3.3.4) (if $k \neq n$) or (3.3.2) (if $k = n$). Moreover, for the case $k = n$, we have the explicit solution $\xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = 0$ because the market is complete (i.e., we are considering the time after the last default).*

Démonstration. For each BSDE, we will proceed in a backwards recursive proof.

First BSDE : **(En) problem :** We recall that when $k = n$ (i.e., we are considering the time after the last default), the market is complete. Using (3.3.2) and (3.3.5), by Itô's formula, we find that $\left[\frac{\left(Z_t^Q(\boldsymbol{\theta},\boldsymbol{l})\right)^2}{v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}}\right]_{t\in[\theta_n,T]}$ is a $\mathbb{P}$ martingale. Using its terminal condition $v_T^{n,\boldsymbol{\theta},\boldsymbol{l}} = \alpha_T(\boldsymbol{\theta},\boldsymbol{l})$, we ultimately find, for all $t \in [\theta_n, T]$, that

$$v_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = \left(\mathbb{E}\left[\left(\frac{Z_T^Q(\boldsymbol{\theta},\boldsymbol{l})}{Z_t^Q(\boldsymbol{\theta},\boldsymbol{l})}\right)^2 \frac{1}{\alpha_T(\boldsymbol{\theta},\boldsymbol{l})}|\mathcal{F}_t\right]\right)^{-1}.$$

Moreover, under Assumption (3.2.2), the martingale $\frac{\mu^n(\boldsymbol{\theta},\boldsymbol{l})}{\sigma^n(\boldsymbol{\theta},\boldsymbol{l})}.W$ is a $\mathrm{BMO}(\mathbb{F})([\theta_n, \mathrm{T}])$ martingale. This implies that the family of probability measures $\{Q(\boldsymbol{\theta},\boldsymbol{l}), (\boldsymbol{\theta},\boldsymbol{l}) \in \Delta_n(T) \times E^n\}$, such that the Radon-Nikodym density of $Q(\boldsymbol{\theta},\boldsymbol{l})$ with respect to P is given by (3.3.5), satisfies the reverse Holder inequality $R_2(P)$. Hence, there exists a positive constant c_4 such that for all stopping times $\theta_n \leq \tau \leq T$, we have $\frac{\mathbb{E}[Z_T^Q(\boldsymbol{\theta},\boldsymbol{l})^2|\mathcal{F}_\tau]}{Z_\tau^Q(\boldsymbol{\theta},\boldsymbol{l})^2} \leq c_4$. Notably, this result implies that for all $t \in [\theta_n, T]$, $\frac{Z_t^Q(\boldsymbol{\theta},\boldsymbol{l})^2}{\mathbb{E}[Z_T^Q(\boldsymbol{\theta},\boldsymbol{l})^2|\mathcal{F}_t]} \geq \frac{1}{c_4} > 0$. We conclude, based on Assumption 3.2.2, that there exists a constant δ_1^n such that $v^{n,\boldsymbol{\theta},\boldsymbol{l}} \geq \delta_1^n$. Moreover, using Jensen's inequality and Assumption 3.2.2, there exists a positive constant δ_2^n such that for all $t \in [0, T]$, $v_t^{n,\boldsymbol{\theta},\boldsymbol{l}} \leq \delta_2^n$.

(Ek) problems : Now, assume that a solution exists for $\tilde{k} := k+1$ with $k \in \{0, 1, \ldots, n-1\}$ (our recursive hypothesis) ; we must demonstrate that this claim is still true for $\tilde{k} - 1 := k$. We will prove that the problem is equivalent to a BSDE problem with quadratic growth and bounded terminal conditions, allowing us to obtain the desired result using the results presented by Kobylanski in [76]. Hence, the proof is divided into two parts. First, we will present the results for a modified quadratic BSDE. Second, we will use the comparison theorem for quadratic BSDEs to demonstrate that the first component solution to the modified BSDE is nonnegative, thus concluding the proof. Let us define the modified BSDE for $k \in \{0, 1, \dots, n-1\}$ given by

$$dv_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = -\overline{g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}}(v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, \overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}})dt + \overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} dW_t, \tag{3.3.8}$$

with a generator given by

$$\begin{aligned} \overline{g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}} &= \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \\ &- \frac{\left(\mu_t^k |v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + \sigma_t^k \overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} + \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})\right)^2}{(\sigma_t^k)^2 |v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} (\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}))^2 \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})}. \end{aligned}$$

Using our recursive hypothesis that there exist constants δ_1^{k+1} and δ_2^{k+1} such that

$$0 < \delta_1^{k+1} \le v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \le \delta_2^{k+1}$$

and Assumption 3.2.2, we find that there exists a constant $C > 0$ such that

$$|\overline{g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}}| \le C\left[1 + |v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + |\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}|^2\right]. \tag{3.3.9}$$

Therefore, this coefficient exhibits quadratic growth (with respect to $\overline{\beta^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}$) and linear growth (with respect to $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$) ; according to the Kobylanski Theorem [76], there exists a pair solution $(v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, \overline{\beta^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}) \in \mathcal{S}^\infty \times \text{BMO}$ of this modified BSDE. Let us now identify a suitable lower bound on the coefficient $\overline{g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}}$. Let us first define the following :

$$e_t^k = \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}), \qquad h_t^k = 2\left(\frac{\mu_t^k}{\sigma_t^k} + \frac{\sigma_t^k e_t^k}{d_t^k}\right) \tag{3.3.10}$$

$$d_t^k = \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} (\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}))^2 \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \quad \text{and} \quad c_t^k = \frac{2\mu_t^k e_t^k}{d_t^k} + \left(\frac{\mu_t^k}{\sigma_t^k}\right)^2 \tag{3.3.11}$$

Using (3.3.9), we find that $-\overline{g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}} = K_t^0 + K_t^1 + K_t^2 + K_t^3$, where

$$K_t^0 = -\int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}),$$

$$K_t^1 = \frac{\left(\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + e_t^k\right)^2}{(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 |v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + d_t^k} \le \left(\frac{\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}{\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}\right)^2 |v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + \frac{2\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} |v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| e_t^k}{d_t^k} + \frac{(e_t^k)^2}{d_t^k},$$

$$K_t^2 = \frac{(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}})^2}{(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 |v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + d_t^k} \le \frac{|\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}|^2}{|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|}$$

and

$$K_t^3 = \frac{2\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} (\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} |v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + e_t^k)}{(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 |v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + d_t^k} \le 2\frac{\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}{\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} \overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} + 2\frac{\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} e_t^k}{d_t^k}.$$

Because the processes $\mu^k, \sigma^k, \gamma^k, v^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}$ are bounded, according to Assumption 3.2.2 and our recursive hypothesis at step $k+1$, we conclude that the processes h^k and c^k are also bounded. Using the expressions for K^0, K^1, K^2 and K^3, we obtain

$$-\overline{g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}} \leq \frac{|\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}|^2}{|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|} + c_t^k|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + h_t^k\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} + \frac{(e_t^k)^2}{d_t^k} - \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}).$$

Using Cauchy's inequality on the expression for e_t^k, we find

$$\begin{aligned}(e_t^k)^2 &= \left(\int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})\right)^2 \\ &\leq \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\left(\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\right)^2\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}),\end{aligned}$$

and we then obtain

$$\begin{aligned}&\frac{(e_t^k)^2}{d_t^k} - \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \\ =\ &\frac{\left(\int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})\right)^2}{\int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\left(\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\right)^2\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})} - \int_E v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \leq 0.\end{aligned}$$

Hence, we obtain a suitable lower bound $\bar{f}_t^k$ for the generator $\overline{g_t^{k,(1)}}$:

$$\overline{g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}} \geq \bar{f}_t^k := -\left(c_t^k|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| + h_t^k\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} + \frac{|\overline{\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}}|^2}{|v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|}\right).$$

Hence, if we now consider the BSDE

$$d\bar{Y}_t^k = \left(c_t^k\bar{Y}_t^k + h_t^k\bar{Z}_t^k + \frac{|\bar{Z}_t^k|^2}{\bar{Y}_t^k}\right)dt + \bar{Z}_t^k dW_t, \quad \bar{Y}_T^k = \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in (0,1),$$

then, according to Proposition 5.1 of [79], there exists a pair solution $(\bar{Y}^k, \bar{Z}^k) \in \mathcal{S}^\infty \times \mathrm{BMO}$ of the BSDE

$$d\bar{Y}_t^k = -\bar{f}_t^k dt + \bar{Z}_t^k dW_t, \quad \bar{Y}_T^k = \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k),$$

where $\bar{Y} \geq \delta_1^k$, and the coefficient $\bar{f}^k$ exhibits quadratic growth (with respect to $\bar{Z}^k$) and linear growth (with respect to $\bar{Y}^k$). Because $\overline{g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}} \geq \bar{f}^k$, when we now apply the comparison theorem of Kobylanski [76], the first component solution to the modified BSDE (3.3.8) yields

$$v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \geq \bar{Y}_t^k \geq \delta_1^k > 0.$$

Therefore, the modified BSDE is equivalent to the first BSDE of the **(Ek)** problem (3.3.4), and thus, we obtain the proof of the existence of the solution to this first BSDE.
Moreover, to obtain the upper bound δ_2^k on $v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$, we consider the terminal condition of the corresponding BSDE : $v_T^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) := \delta_2^k$. This proves that there exist constants δ_1^k and δ_2^k such that

$$0 < \delta_1^k \leq v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \leq \delta_2^k.$$

Second BSDE : **(En) problem :** Following the proof of the existence of the solution to the first BSDE for $k = n$ and (3.3.5), we obtain an explicit solution to the second BSDE, which is given by

$$Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = \mathbb{E}\left[\frac{Z_T^Q(\boldsymbol{\theta},\boldsymbol{l})}{Z_t^Q(\boldsymbol{\theta},\boldsymbol{l})} H_T^n(\boldsymbol{\theta},\boldsymbol{l}) \Big| \mathcal{F}_t\right]. \tag{3.3.12}$$

Because $H^n(\boldsymbol{\theta},\boldsymbol{l}) \in L^\infty$ for all $(\boldsymbol{\theta},\boldsymbol{l}) \in \Delta_n(T) \times E^n$ by assumption on the contingent claim, we find from (3.3.12) that $Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}} \in \mathcal{S}^\infty$. Moreover, we have the following representation theorem :

$$Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = H_T^n(\boldsymbol{\theta},\boldsymbol{l}) - \int_t^T Z_s^{n,\boldsymbol{\theta},\boldsymbol{l}} dW_s^{Q(\boldsymbol{\theta},\boldsymbol{l})}, \quad t \in [\theta_n, T], \tag{3.3.13}$$

where $W_s^{Q(\boldsymbol{\theta},\boldsymbol{l})} = W_s - \int_{\theta_n}^s \frac{\mu_u^n(\boldsymbol{\theta},\boldsymbol{l})}{\sigma_u^n(\boldsymbol{\theta},\boldsymbol{l})} du$ is a $Q(\boldsymbol{\theta},\boldsymbol{l})$ Brownian motion. For any stopping time $\theta_n \le \tau \le T$, according to (3.3.13), there exists a constant $d > 0$ such that

$$\mathbb{E}^{Q(\boldsymbol{\theta},\boldsymbol{l})}\left[\int_\tau^T \left(Z_s^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 ds | \mathcal{F}_\tau\right] \le \mathbb{E}^{Q(\boldsymbol{\theta},\boldsymbol{l})}\left[\left(H_T^{n,\boldsymbol{\theta},\boldsymbol{l}} - Y_\tau^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 |\mathcal{F}_\tau\right] \le d.$$

Thus, $Z^{n,\boldsymbol{\theta},\boldsymbol{l}}.W^{Q(\boldsymbol{\theta},\boldsymbol{l})}$ is a BMO martingale under the probability measure $Q(\boldsymbol{\theta},\boldsymbol{l})$, and therefore, $Z^{n,\boldsymbol{\theta},\boldsymbol{l}}.W$ is a BMO martingale under the probability measure P according to Theorem 3.3 of Kazamaki [69]. Therefore, we conclude that $Z^{n,\boldsymbol{\theta},\boldsymbol{l}} \in$ BMO.

(Ek) problems : Now, assume that a solution exists for $\tilde{k} := k+1$ with $k \in \{0, 1, \ldots, n-1\}$ (our recursive hypothesis) ; we must demonstrate that this claim still holds for $\tilde{k} - 1 := k$. We now wish to prove that $\left(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right) \in \mathcal{S}^\infty \times$ BMO for all $k \in \{0, 1, \ldots, n\}$. We can actually prove the existence of the solution to the second BSDE because the solution to the first one exists. Given the solution to the first BSDE, the coefficient of the second one is linear. Therefore, we can explicitly characterise the solution.

Step 1 : Preliminary results.

Given the explicit formula for the coefficient $g^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}$ in (3.3.4), we obtain

$$g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)} = a_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \kappa_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \Lambda_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k},$$

where

$$a_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} - \sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \frac{\left(\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \int_E \gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})(1+v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})\right)}{(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \int_E (1+v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})(\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}))^2 \eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})},$$

$$\begin{aligned}\kappa_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} &= -\int_E (1+v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1}) + \int_E (1+v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1}) \\ &\times \frac{\left(\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \int_E (1+v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})\right)}{(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \int_E (1+v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k})(\gamma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}))^2 \eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})}\end{aligned}$$

and

$$
\begin{aligned}
\Lambda_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} &= \int_E (1+v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) Y_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \\
&- \int_E (1+v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) Y_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} \gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1}) \\
&\times \frac{\left(\mu_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \int_E (1+v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) \gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})\right)}{\left(\sigma_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)^2 + \int_E (1+v_t^{J,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) \left(\gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})\right)^2 \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})}.
\end{aligned}
$$

Under Assumption 3.2.2 and the integrability conditions defined in 3.2.7, coefficients $\sigma^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$, $\mu^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ and γ^k are bounded. Moreover, from the solution to the first BSDE and the boundedness of the processes $v^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}$ and $Y_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}$ (recursive hypothesis), we find that the processes $v^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ are bounded for all $\boldsymbol{l}_k \in E^k$ and that $\beta^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ is a BMO martingale.
Thus, we deduce that the martingales $\Lambda^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$, $a^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ and $\kappa^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ are BMO martingales under the probability measure P. Let us define the probability measure $Q \sim P$, with a Radon-Nikodym density on $\mathcal{F}_T$ defined by $Z_T^Q = \mathcal{E}(a^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W)_T$. Because the martingale $a^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ is a BMO martingale, the process $Z_t^Q = \mathbb{E}\left[Z_T^Q|\mathcal{F}_t\right]$ is uniformly integrable, and from Theorem 3.3 of Kazamaki [69], the martingale $\kappa^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ is still a BMO martingale under the probability measure Q. Therefore, there exists a nonnegative constant c such that $\mathbb{E}^Q\left[\int_t^T |\kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|^2 ds|\mathcal{F}_t\right] \leq c$ for all $\theta_k \leq t \leq T$ and $k \in \{0,1,\ldots,n\}$.

Step 2 : Integrability of the adjoint process Γ :

For all $k \in \{0,1,\ldots,n\}$,

$$\widetilde{\Gamma}_t := \exp\left(\int_0^t \kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} ds\right).$$

We will prove that $\widetilde{\Gamma} \in \mathrm{L}^p(Q)$ for any $p > 1$ and $\delta > 0$:

$$
\begin{aligned}
\left|\frac{\widetilde{\Gamma}_T}{\widetilde{\Gamma}_t}\right|^p &= \exp\left(p\int_t^T \kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} ds\right) \leq \exp\left(\int_t^T \left(\delta(\kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \frac{p^2}{4\delta}\right) ds\right) \\
&\leq \exp\left(\frac{p^2}{4\delta}T\right) \exp\left(\delta\int_t^T (\kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 ds\right).
\end{aligned}
$$

Because there exists a nonnegative constant c such that

$$\mathbb{E}^Q\left[\int_t^T |\kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}|^2 ds|\mathcal{F}_t\right] \leq c,$$

we deduce from Proposition 3.5.1 in the Appendix that there exists a δ such that $0 \leq \delta \leq \frac{1}{c^2}$ and $\mathbb{E}^Q\left[\exp(\int_t^T \delta|(\kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2|ds)|\mathcal{F}_t\right] \leq \frac{1}{1-\delta c^2}$. Therefore, we conclude that there exists a nonnegative constant C_1 such that

$$\mathbb{E}^Q\left[\left|\frac{\widetilde{\Gamma}_T}{\widetilde{\Gamma}_t}\right|^p|\mathcal{F}_t\right] \leq C_1. \tag{3.3.14}$$

Step 3 : The solution of the BSDE.

Let us now define, for $k \in \{0, 1, \dots, n-1\}$,

$$Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \mathbb{E}^Q\left[\frac{1}{\widetilde{\Gamma}_t}\left(\widetilde{\Gamma}_T H_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \int_t^T \widetilde{\Gamma}_s \Lambda_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} ds\right) |\mathcal{F}_t\right], \quad \theta_k \le t \le T. \tag{3.3.15}$$

Because $\Gamma = Z^Q \widetilde{\Gamma}$, using the Bayes formula, equation (3.3.15) is equivalent to

$$Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \mathbb{E}\left[\frac{1}{\Gamma_t}\left(\Gamma_T H_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \int_t^T \Gamma_s \Lambda_s ds\right) |\mathcal{F}_t\right], \quad t \le T. \tag{3.3.16}$$

Moreover, because $\Lambda^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ is bounded and $H_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in \mathrm{L}^\infty$, there exists a nonnegative constant C such that

$$|Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| \le C\mathbb{E}^Q\left(\left[\left|\frac{\widetilde{\Gamma}_T}{\widetilde{\Gamma}_t}\right| + \int_t^T \left(\left|\frac{\widetilde{\Gamma}_s}{\widetilde{\Gamma}_t}\right|^2 + (\Lambda_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2\right) ds\right] \Big|\mathcal{F}_t\right).$$

Because the process $\Lambda^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W^Q$ is a BMO martingale under the probability measure Q, using (3.3.14), we find that there exists a constant $\bar{C} > 0$ such that

$$|Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}| \le \bar{C}, \quad t \le T.$$

Let us consider $Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$, defined by (3.3.15) ; then, the process

$$\widetilde{\Gamma}_t Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \int_0^t \Lambda_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \widetilde{\Gamma}_s ds = \mathbb{E}^Q\left[\widetilde{\Gamma}_T H_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \int_0^T \widetilde{\Gamma}_s \Lambda_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} ds|\mathcal{F}_t\right]$$

is a square-integrable Q martingale because H^k is bounded by assumption, $\Lambda^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ is a BMO martingale, and $\widetilde{\Gamma}$ satisfies (3.3.14). Therefore, according to the representation theorem, there exists a process $\bar{Z} \in \mathcal{H}^2$ such that $d(\widetilde{\Gamma}_t Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \int_0^t \widetilde{\Gamma}_s \Lambda_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} ds) = \bar{Z}_t dW_t^Q$. Setting $Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \frac{\bar{Z}}{\widetilde{\Gamma}}$ and using integration by parts, we find that

$$dY_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = -(\Lambda_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} a_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \kappa_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})dt + Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} dW_t^Q, \quad Y_T^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = H_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k).$$

Applying Itô's formula, we find that

$$d(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 = 2Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}[-(\Lambda_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \kappa_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})dt + Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.dW_t^Q] + (Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 dt,$$

and therefore, for any stopping time σ, we find that

$$\mathbb{E}^Q\left[\int_\sigma^T (Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 dt \Big|\mathcal{F}_\sigma\right] \le \mathbb{E}^Q\left[(H_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k))^2 + 2\int_\sigma^T 2Y_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(\Lambda_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} + \kappa_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} Y_s^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})ds \Big|\mathcal{F}_\sigma\right].$$

Because H^k and $Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ are bounded and because $\Lambda^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W^Q$ and $\kappa^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W^Q$ are BMO martingales under the probability measure Q, we conclude that $Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W^Q$ is a BMO martingale measure under Q. Then, $Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.W$ is a BMO martingale under the probability measure P, according to Theorem 3.3 of Kazamaki [69]. Therefore, we conclude that $(Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) \in \mathcal{S}^\infty \times \mathrm{BMO}$ is a solution of the second BSDE.

Third BSDE : **(En) problem :** Because $g_t^{n,\boldsymbol{\theta},\boldsymbol{l},(3)} \equiv 0$, we directly find that $\xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} \equiv 0$.

(Ek) problems : Now, assume that a solution exists for $\tilde{k} := k+1$ with $k \in \{0, 1, \dots, n-1\}$ (our recursive hypothesis) ; we must demonstrate that this claim still holds for $\tilde{k} - 1 := k$. It is sufficient to prove that $\left(\xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, R_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right) \in \mathcal{S}^\infty \times \mathrm{BMO}$. For all $k \in \{0, 1, \dots, n\}$, all terms that appear in the coefficient $g_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(3)}$ are bounded and $Z^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \in \mathrm{BMO}$, as demonstrated in the previous step ; we therefore conclude, using the representation theorem, that $(\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, R^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) \in \mathcal{S}^\infty \times \mathrm{BMO}$ for all $k \in \{0, 1, \dots, n\}$.

□

3.3.2 BSDE characterisation via verification theorem

We will now demonstrate that the triple solution $\left(v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, \xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)$ to the recursive system of indexed BSDEs, which appears in quadratic decomposition form, actually provides the solution to the global optimal investment problem in terms of the value functions V^k, $k \in \{0, 1, \dots, n\}$ in (3.3.1) and (3.3.3). As a byproduct, we will prove the existence of the optimal strategy $\pi^{k,*}$.

Theorem 3.3.2. *The value functions V^k , $k = 0, \dots, n$ defined in (3.3.1) and (3.3.3) are given, for all $t \in [\theta_k, T]$, by*

$$V_t^k(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k, \nu^k) = v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \left(X_t^{k,x,\pi}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \tag{3.3.17}$$

for all $x \in \mathbb{R}$, $(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in \Delta_k \times E^k$ and $\nu^k \in \mathcal{A}_{\mathbb{F}}^k$, where $(v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, \xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})$ is the unique solution to the recursive triple BSDE systems given for all $k = \{0, 1, \dots, n\}$ in (3.3.2) and (3.3.4).

In particular, the solution of the mean-variance hedging problem is given by

$$V_0^H(x) \;=\; \inf_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[(H_T - X_T^{x,\pi})^2\right] = v_0^0 (x - Y_0^0)^2 + \xi_0^0, \quad x \in \mathbb{R}, \tag{3.3.18}$$

where the triple (v^0, Y^0, ξ^0) is the solution of the recursive system of BSDEs : **(En)** *(3.3.2) and* **(Ek)** *(3.3.4), $k \in \{0, 1, \dots, n-1\}$.*

Moreover, there exists an optimal strategy $\pi^* := (\pi^{0,*}, \pi^{1,*}, \dots, \pi^{n,*})$ given by

$$\begin{aligned}
\pi_t^{k,*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \;=\;& \frac{1}{(\sigma_t^k)^2 + \int_E (1 + v_t^{J,k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) \gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1})^2 \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})} \Big[\sigma_t^k Z_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} - K_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(\mu_t^k + \sigma_t^k \beta_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) \\
&+ \frac{\int_E (X_t^{k,x,\pi}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} - Y^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}} v_t^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}) \gamma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) \eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})}{v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}} \Big]
\end{aligned} \tag{3.3.19}$$

where $K_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} := X_t^{k,x,\pi}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$. Moreover, for the problem concerning the scenario after the last default,

$$\pi_t^{n,*}(\boldsymbol{\theta}, \boldsymbol{l}) = \frac{1}{(\sigma_t^n)^2} \left[\sigma_t^n Z_t^{n,\boldsymbol{\theta},\boldsymbol{l}} - \left(X_t^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right) \left(\mu_t^n + \sigma_t^n \beta_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)\right]. \tag{3.3.20}$$

Remark 3.3.2. *Following (3.3.18), we offer some comments regarding the financial interpretation of our quadratic decomposition form :*

– *The process v^0 does not depend on the payoff H. Moreover, we have*

$$v_0^0 = V_0^0(1) := \inf_{\pi \in \mathcal{A}_{\mathbb{G}}} \mathbb{E}\left[X_T^{1,\pi}\right]^2.$$

Therefore, v^0 is related to the minimal variance of a pure investment on the asset S with an initial wealth of $x = 1$.

– *The process Y^0 is the quadratic approximation to the price of option H.*

– *The process ξ^0 represents the incompleteness of this market, as if the market is complete (as in the* **(En)** *problem), this process vanishes.*

Démonstration. **Step 1 :** We begin by proving, for all $k = \{0, 1, \ldots, n\}$, $t \in [\theta_k, T]$ and $\nu^k \in \mathcal{A}_{\mathbb{F}}^k$, that

$$v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \leq V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k). \tag{3.3.21}$$

Let us denote by D^k the process defined for all $k = \{0, \ldots, n-1\}$, $t \in [\theta_k, T]$ and $\nu^k \in \mathcal{A}_{\mathbb{F}}^k$ by

$$D_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k) \;:=\; v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \tag{3.3.22}$$

$$+\int_{\theta_k}^t \int_E \left(v_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_s^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \pi_s^k\gamma_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}) - Y_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)\eta(\boldsymbol{l}_k,dl_{k+1})ds$$

and $D_t^n(x,\boldsymbol{\theta},\boldsymbol{l},\nu^n) := v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}(X_t^{n,x,\pi}(\boldsymbol{\theta},\boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}})^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}}$.
Because D^k is a local submartingale (using Itô's formula and (3.3.4)), let (T_i) be a localising sequence of $\mathbb{F}$ stopping times valued in $[\theta_k, T]$ for D_t^k ; we then have, for all $\theta_k \leq t \leq s \leq T$,

$$D_{t\wedge T_i}^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k) \;\leq\; \mathbb{E}\left[D_{s\wedge T_i}^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k)|\mathcal{F}_t\right].$$

Now, using Definition 3.2.1 of the admissibility condition for ν^k, Assumption 3.2.2, and the fact that $Y^{n,\boldsymbol{\theta},\boldsymbol{l}}$ and $\xi^{n,\boldsymbol{\theta},\boldsymbol{l}}$ are square integrable and $v^{n,\boldsymbol{\theta},\boldsymbol{l}}$ is bounded, we find that the sequence $\left(D_{s\wedge T_i}^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k)\right)_i$ is uniformly integrable for $s \in [\theta_k, T]$, and thus, we obtain the submartingale property for D^k. Now, by writing this submartingale property between times t and T and recalling the terminal conditions of the three BSDEs, we obtain the expected results, which are, for all $\nu^k \in \mathcal{A}_{\mathbb{F}}^k$ and $k \in \{0, 1, \ldots, n-1\}$,

$$v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \;\leq\; \mathbb{E}\left[(H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - X_T^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|\mathcal{F}_t\right] \tag{3.3.23}$$

$$+\mathbb{E}\left[\int_t^T \int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_t\right],$$

and for $k = n$,

$$v_t^{n,\boldsymbol{\theta}_n,\boldsymbol{l}_n}(X_t^{n,x,\pi}(\boldsymbol{\theta},\boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta}_n,\boldsymbol{l}_n})^2 + \xi_t^{n,\boldsymbol{\theta}_n,\boldsymbol{l}_n} \;\leq\; \mathbb{E}\left[(H_T^n - X_T^{n,x,\pi}(\boldsymbol{\theta},\boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta},\boldsymbol{l})|\mathcal{F}_t\right] \tag{3.3.24}$$

Step 2 : We must now check that the trading strategy $\pi^* = (\pi^{k,*})_{k=0,\dots,n}$ is admissible in the sense of Definition 3.2.1. For improved readability, we neglect the dependence parameter $(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ for $\pi_t^{k,*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ and use the simpler notation $\pi_t^{k,*}$. We recall that $(D_t^k)_{t\in[0,T]}$, the local martingale (because we find this quantity using the optimal strategy π^*), is defined in (3.3.22) for all $k = \{0, \dots, n-1\}$ and $t \in [\theta_k, T]$ as

$$D_t^k(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k, \pi^{k,*}) = v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$$
$$+ \int_{\theta_k}^t \int_E \left(v_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_s^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \pi_s^{k,*}\gamma_s^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) - Y_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \right) \eta(\boldsymbol{l}_k, dl_{k+1}) ds.$$

Let (T_i) be a localising sequence of $\mathbb{F}$ stopping times valued in $[\theta_k, T]$ for the local martingale D_t^k; then, for any $i \in \mathbb{N}$,

$$D_{t\wedge T_i}^k(x, \boldsymbol{\theta}_k, \boldsymbol{l}_k, \pi^{k,*}) = v_{t\wedge T_i}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_{t\wedge T_i}^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y_{t\wedge T_i}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_{t\wedge T_i}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$$
$$+ \int_{\theta_k}^{t\wedge T_i} \int_E \left(v_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_s^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \pi_s^{k,*}\gamma_s^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) - Y_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \right) \eta(\boldsymbol{l}_k, dl_{k+1}) ds.$$

Because (D^k) is a local martingale (using Itô's formula and (3.3.4)), by taking the expectation value, we obtain

$$\mathbb{E}\left[v_{t\wedge T_i}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_{t\wedge T_i}^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y_{t\wedge T_i}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_{t\wedge T_i}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} | \mathcal{F}_{\theta_k} \right] = v_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_{\theta_k}^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \tag{3.3.25}$$
$$-\mathbb{E}\left[\int_{\theta_k}^{t\wedge T_i} \int_E \left(v_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_s^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \pi_s^{k,*}\gamma_s^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) - Y_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \right) \eta(\boldsymbol{l}_k, dl_{k+1}) ds | \mathcal{F}_{\theta_k} \right].$$

Through recursive backwards induction and the use of Theorem 3.2, we find, for all $k = \{0, \dots, n-1\}$, that $v_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_s^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \pi_s^{k,*}\gamma_s^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}) - Y_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_s^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ is positive for all $s \in [0, T]$. Hence, for all $t \in [\theta_k, T]$, we find that

$$\begin{aligned}\mathbb{E}\left[v_{t\wedge T_i}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_{t\wedge T_i}^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y_{t\wedge T_i}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_{t\wedge T_i}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} | \mathcal{F}_{\theta_k} \right] &\leq v_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_{\theta_k}^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \\ &\leq v_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(x - Y_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} < \infty. \end{aligned} \tag{3.3.26}$$

Using Theorem 3.3.1, we know that there exists a positive constant δ such that $v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \geq \delta$ for all $t \in [\theta_k, T]$. If we now let $i \to \infty$, it follows from Fatou's Lemma, in a similar manner to the proof of Proposition 3.2 in [79], that

$$\mathbb{E}\left[v_{t\wedge T_i}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_{t\wedge T_i}^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y_{t\wedge T_i}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_{t\wedge T_i}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} | \mathcal{F}_t \right] \geq \tilde{\delta}\left(\mathbb{E}\left[|X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k)|^2 \right] + 1 \right).$$

Hence, we find that there exist constants c_1 and c_2 such that

$$\mathbb{E}\left[|X_T^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k)|^2 \right] \leq c_1 \quad \text{and} \quad \mathbb{E}\left[\int_{\theta_k}^T |X_s^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k)|^2 ds \right] < c_2. \tag{3.3.27}$$

We must prove that this inequality implies Definition 3.2.1. Indeed, applying Itô's formula to $\left(X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \right)^2$ yields

$$d\left(X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \right)^2 = 2X_{t-}^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) dX_t^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + d\left[X^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \right]_t.$$

Using the dynamics of $X_t^{k,x,\pi*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ and letting $(T_i)_{i\in\mathbb{N}}$ be a localising time sequence, we obtain

$$x^2+\mathbb{E}\left[\int_{\theta_k}^{T\wedge T_i} |\pi_s^{k,*}|^2(\sigma_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2 ds\right] \leq \mathbb{E}\left[\left(X_{T\wedge T_i}^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2\right] - 2\mathbb{E}\left[\int_{\theta_k}^{T\wedge T_i} \pi_s^{k,*}\mu_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)X_s^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)ds\right]. \tag{3.3.28}$$

Because, by assumption, processes $\mu_s^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ and $\sigma_s^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ are bounded, we find that there is a constant $K_2 \leq (\sigma_s^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k))^2$ such that for all $s \in [0, T]$,

$$-2\pi_s^{k,*}\mu_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)X_s^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) \leq \frac{2}{K_2}|X_s^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2|\mu_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2 + \frac{K_2}{2}|\pi_s^{k,*}|^2. \tag{3.3.29}$$

Using (3.3.29) in (3.3.28) yields

$$\begin{aligned} x^2 + \mathbb{E}\left[\int_{\theta_k}^{T\wedge T_i} |\pi_s^{k,*}|^2(\sigma_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2 ds\right] \quad &\leq \quad \mathbb{E}\left[\left(X_{T\wedge T_i}^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2\right] \\ + \quad \mathbb{E}\left[\int_{\theta_k}^{T\wedge T_i} \frac{2}{K_2}|X_s^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2|\mu_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2 ds\right] &+ \mathbb{E}\left[\int_{\theta_k}^{T\wedge T_i} \frac{K_2}{2}|\pi_s^{k,*}|^2 ds\right]. \end{aligned}$$

Now, by applying Fatou's Lemma, we find that when i goes to infinity,

$$\begin{aligned} x^2 + \mathbb{E}\left[\int_{\theta_k}^{T} |\pi_s^{k,*}|^2(\sigma_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2 ds\right] \quad &\leq \quad \mathbb{E}\left[\left(X_T^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2\right] \qquad (3.3.30) \\ +\frac{2}{K_2}\mathbb{E}\left[\int_{\theta_k}^{T} |X_s^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2|\mu_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2 ds\right] &+ \frac{K_2}{2}\mathbb{E}\left[\int_{\theta_k}^{T} |\pi_s^{k,*}|^2 ds\right]. \end{aligned}$$

Moreover, because $K_2 \leq (\sigma_s^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k))^2$, we obtain

$$\frac{K_2}{2}\mathbb{E}\left[\int_{\theta_k}^{T} |\pi_s^{k,*}|^2 ds\right] \leq \mathbb{E}\left[\left(X_T^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)^2 - x^2\right] + \frac{2}{K_2}\mathbb{E}\left[\int_{\theta_k}^{T} |X_s^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2|\mu_s^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)|^2 ds\right].$$

Therefore, because $\mu_s^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ is bounded by assumption and based on (3.3.27), we conclude that (3.2.12) is satisfied, which implies that $\pi^{k,*}$ is admissible in the sense of Definition 3.2.1.

Step 3 : We must prove that the wealth process $X_t^{k,x,*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ associated with the strategy $\pi_t^{k,*}$ exists for all $k \in \{0, 1, \ldots, n\}$. First, we note that the optimal strategy (3.3.19) takes a linear form with respect to $X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ for $\theta_k \leq t \leq T$. Let us denote this linear form by

$$\pi_t^{k,*} = a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + d_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k), \quad \forall k \in \{0,1,\ldots,n\}.$$

Then, for $\theta_k \leq t \leq T$, substituting this expression into (3.2.9) yields

$$\begin{aligned} dX_t^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) \quad &= \quad \left(a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + d_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\right)\left(\mu_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dt + \sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dW_t\right) \\ &= \quad X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\left(a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\mu_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dt + a_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dW_t\right) \qquad (3.3.31) \\ &+ \quad \left(d_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\mu_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dt + d_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)dW_t\right). \end{aligned}$$

We recall that the solution for $\theta_k \le t \le T$ of the SDE given by

$$d\phi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = \phi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\left(a_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dt + a_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dW_t\right)$$

is $\phi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = \phi_{\theta_k}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\exp\left\{\int_{\theta_k}^T \left(a_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - \frac{1}{2}\left(a_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\right)^2\right)dt\right\}$.
Therefore, by setting $X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) := L_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\phi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ (where $dL_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) := \bar{\mu}_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dt + \bar{\sigma}_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dW_t$ and $L_{\theta_k}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = 1$) and applying integration by parts, we obtain, for all $\theta_k \le t \le T$,

$$\begin{aligned} dX_t^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) &= X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\left[a_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dt + a_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dW_t\right] \\ &+ \phi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\left[\left(\bar{\mu}_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + a_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\bar{\sigma}_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\right)dt + \bar{\sigma}_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)dW_t\right]. \end{aligned}$$

Hence, from (3.3.31), we obtain $\bar{\mu}_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = \frac{d_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\left(\mu_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - a_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\left(\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\right)^2\right)}{\phi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)}$ and $\bar{\sigma}_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) = \frac{d_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\sigma_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)}{\phi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)}$. Then, we deduce that $X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) := L_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)\phi_t^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)$ is a solution of the SDE (3.2.9).

Step 4 : We now wish to prove that the trading strategy $\pi^* = (\pi^{k,*})_{k=0,\dots,n}$ is optimal. Because the trading strategy $\pi^* = (\pi^{k,*})_{k=0,\dots,n}$ is admissible in the sense of Definition 3.2.1 and because the processes D^k are "true" martingales for $k = \{0, \dots, n\}$, we find that for all $(\boldsymbol{\theta}_k, \boldsymbol{l}_k) \in \Delta_k(T) \times E^k$, $x \in \mathbb{R}$, $t \in [\theta_k, T]$ and $k = \{0, 1, \dots, n-1\}$,

$$\begin{aligned} v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\left(X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} &= \mathbb{E}\left[(H_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k) - X_T^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k)|\mathcal{F}_t\right] \quad (3.3.32) \\ +\mathbb{E}\left[\int_t^T \int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi^*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k) + \pi_{\theta_{k+1}}^{k,*}(\boldsymbol{\theta}_k, \boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k, \boldsymbol{l}_k, l_{k+1}), \boldsymbol{\theta}_{k+1}, \boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k, dl_{k+1})d\theta_{k+1}|\mathcal{F}_t\right], \end{aligned}$$

and for $k = n$,

$$v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(X_t^{n,x,\pi^*}(\boldsymbol{\theta}, \boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = \mathbb{E}\left[(H_T^n(\boldsymbol{\theta}, \boldsymbol{l}) - X_T^{n,x,\pi^*}(\boldsymbol{\theta}, \boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta}, \boldsymbol{l})|\mathcal{F}_t\right] \quad (3.3.33)$$

where $X^{n,x,\pi^*}(\boldsymbol{\theta}, \boldsymbol{l})$ indicates that we use the strategy $\pi^* = (\pi^{k,*})_{k=0,\dots,n}$ to evaluate these wealth processes. Starting with $k = n$, let $F_t^n(\boldsymbol{\theta}, \boldsymbol{l})$ be the process given by

$$F_t^n(\boldsymbol{\theta}, \boldsymbol{l}) := \operatorname*{ess\,inf}_{\pi^n \in \mathcal{A}_{\mathbb{F}}^n(t,\nu^n)} \mathbb{E}\left[(H_T^n(\boldsymbol{\theta}, \boldsymbol{l}) - X_T^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta}, \boldsymbol{l}) - v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}((X_t^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l}))^2 - 2X_t^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l})Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}})|\mathcal{F}_t\right].$$

By the submartingale property given in (3.3.24), we find that

$$\begin{aligned} F_t^n(\boldsymbol{\theta}, \boldsymbol{l}) &:= \operatorname*{ess\,inf}_{\pi^n \in \mathcal{A}_{\mathbb{F}}^n(t,\nu^n)} \mathbb{E}\left[(H_T^n(\boldsymbol{\theta}, \boldsymbol{l}) - X_T^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta}, \boldsymbol{l}) - v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}((X_t^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l}))^2 - 2X_t^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l})Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}})|\mathcal{F}_t\right] \\ &\ge v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(X_t^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} - v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left((X_t^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l}))^2 - 2X_t^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l})Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right), \end{aligned}$$

and we conclude that

$$F_t^n(\boldsymbol{\theta}, \boldsymbol{l}) \ge v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}}. \quad (3.3.34)$$

Using the martingale property stated in (3.3.33), we obtain

$$v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(X_t^{n,x,\pi}(\boldsymbol{\theta}, \boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} = \mathbb{E}\left[(H_T^n(\boldsymbol{\theta}, \boldsymbol{l}) - X_T^{n,x,\pi^*}(\boldsymbol{\theta}, \boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta}, \boldsymbol{l})|\mathcal{F}_t\right].$$

By adding $-v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}((X_t^{n,x,\pi^*}(\boldsymbol{\theta},\boldsymbol{l}))^2 - 2X_t^{n,x,\pi^*}(\boldsymbol{\theta},\boldsymbol{l})Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}})$ to both sides of this last equation, we find that

$$v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}} \geq F_t^n(\boldsymbol{\theta},\boldsymbol{l}). \tag{3.3.35}$$

Using inequalities 3.3.34 and 3.3.35, we conclude that $F_t^n(\boldsymbol{\theta},\boldsymbol{l}) = v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}}$. By combining this expression with its definition, we finally obtain the first expected result :

$$\begin{aligned} V_t^n(x,\boldsymbol{\theta},\boldsymbol{l},\nu^n) &:= \operatorname*{ess\,inf}_{\pi^n\in\mathcal{A}_{\mathbb{F}}^n(t,\nu^n)} \mathbb{E}\left[(H_T^n(\boldsymbol{\theta},\boldsymbol{l}) - X_T^{n,x,\pi}(\boldsymbol{\theta},\boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta},\boldsymbol{l})|\mathcal{F}_t\right] \\ &= v_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\left(X_t^{n,x,\pi}(\boldsymbol{\theta},\boldsymbol{l}) - Y_t^{n,\boldsymbol{\theta},\boldsymbol{l}}\right)^2 + \xi_t^{n,\boldsymbol{\theta},\boldsymbol{l}}. \end{aligned}$$

Let us now consider $k = \{0,1,\ldots,n-1\}$ and assume that (3.3.17) holds true at step $k+1$. Then, in a similar manner as above, we observe that for any $t\in[\theta_k,T]$, $\pi^k \in \mathcal{A}_{\mathbb{F}}^k(t,\nu^k)$. Using the stated assumption regarding step $k+1$, we find that

$$V_{\theta_{k+1}}^{k+1}(x,(\boldsymbol{\theta}_k,\boldsymbol{l}_k) = v_{\theta_{k+1}}^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\left(x - Y_{\theta_{k+1}}^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)^2 + \xi_{\theta_{k+1}}^{k+1,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.$$

We set

$$\begin{aligned} F_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) &:= \operatorname*{ess\,inf}_{\pi^k\in\mathcal{A}_{\mathbb{F}}^k(t,\nu^k)} \mathbb{E}\Big[(H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - X_T^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) \\ &- v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}((X_t^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2 - 2X_t^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}) \\ &+ \int_t^T\int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_t\Big]. \end{aligned}$$

Therefore, again using the submartingale property stated in (3.3.23), we obtain

$$\begin{aligned} F_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) &\geq v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} \\ &- v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\left((X_t^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2 - 2X_t^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right) \\ &= v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\left(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}, \end{aligned}$$

and we thus conclude that

$$F_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) \geq v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\left(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}. \tag{3.3.36}$$

Using the martingale property of (3.3.32), we obtain

$$\begin{aligned} v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) - Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 + \xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \mathbb{E}\Big[(H_T^k(\boldsymbol{\theta},\boldsymbol{l}) - X_T^{k,x,\pi^*}(\boldsymbol{\theta},\boldsymbol{l}))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k) \\ + \int_t^T\int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) + \pi_{\theta_{k+1}}^{k,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_t\Big]. \end{aligned}$$

Hence, by adding $-v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\left((X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2 - 2X_t^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)$ to both sides of this last equation, we find that

$$v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\left(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)^2+\xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\geq F_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k). \qquad (3.3.37)$$

Using 3.3.36 and 3.3.37, we conclude that $F_t^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)=v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\left(Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}\right)^2+\xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$. By combining this expression with its definition, we finally obtain the second expected result, which is, for all $k\in\{0,1,\ldots,n-1\}$,

$$\begin{aligned}V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k) &= \underset{\pi^k\in\mathcal{A}_{\mathbb{F}}^k(t,\nu^k)}{\operatorname{ess\,inf}}\ \mathbb{E}[(H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-X_T^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)+\\ \int_t^T\int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) &+ \pi_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_t]\\ &= v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_t^{k,x}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2+\xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.\end{aligned}$$

Moreover, by now taking $t=\theta_k$ and using relations (3.3.32), (3.3.33) and (3.3.17), we obtain

$$\begin{aligned}V_{\theta_n}^n(x,\boldsymbol{\theta},\boldsymbol{l}) &= \mathbb{E}\left[(H_T^n(\boldsymbol{\theta},\boldsymbol{l})-X_T^{n,x,\pi^*}(\boldsymbol{\theta},\boldsymbol{l}))^2\alpha_T(\boldsymbol{\theta},\boldsymbol{l})|\mathcal{F}_{\theta_n}\right]\\ &= v_{\theta_n}^{n,\boldsymbol{\theta},\boldsymbol{l}}(X_{\theta_n}^{n,x,\pi^*}(\boldsymbol{\theta},\boldsymbol{l})-Y_{\theta_n}^{n,\boldsymbol{\theta},\boldsymbol{l}})^2+\xi_{\theta_n}^{n,\boldsymbol{\theta},\boldsymbol{l}}\end{aligned}$$

and

$$\begin{aligned}V_{\theta_k}^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k) &= \mathbb{E}[(H_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-X_T^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k))^2\alpha_T^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k)+\\ \int_{\theta_k}^T\int_E V_{\theta_{k+1}}^{k+1}(X_{\theta_{k+1}}^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k) &+ \pi_{\theta_{k+1}}^{k,*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k).\gamma_{\theta_{k+1}}^k(\boldsymbol{\theta}_k,\boldsymbol{l}_k,l_{k+1}),\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1})\eta_{k+1}(\boldsymbol{l}_k,dl_{k+1})d\theta_{k+1}|\mathcal{F}_{\theta_k}]\\ &= v_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(X_{\theta_k}^{k,x,\pi^*}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)-Y_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2+\xi_{\theta_k}^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}.\end{aligned}$$

These relations prove that $\pi^*=(\pi^{k,*})_{k=0,\ldots,n}$ is an optimal trading strategy.

Step 5 : For the verification theorem 3.3.2, we can write that, for $k=\{0,1,\ldots,n\}$, $H\equiv 0$ and $t\in[\theta_k,T]$,

$$V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k)=v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}X_t^{k,x,\pi}(\boldsymbol{\theta}_k,\boldsymbol{l}_k)^2.$$

Because the value process V^k is unique, we find that the process $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ is also unique.

$Y^{n,\boldsymbol{\theta},\boldsymbol{l}}$ is unique because of formula (3.3.7). Assume that $Y^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}$ is unique ; then, from (3.3.16) and because $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ and $v^{k+1,\boldsymbol{\theta}_{k+1},\boldsymbol{l}_{k+1}}$ are unique, we find that $Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ is also unique.

By (3.3.17), because $V_t^k(x,\boldsymbol{\theta}_k,\boldsymbol{l}_k,\nu^k)$, $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ and $Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ are unique, we find that $\xi^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$ is unique.

□

3.4 Numerical Applications

3.4.1 Study of a one-default case

We consider a special case in which there is only one default event and μ^0, σ^0 and γ^0 are constants ; $\mu^1(\theta,l)$ and $\sigma^1(\theta,l)$ are simply deterministic functions of θ, and the default time τ is independent of $\mathbb{F}$; thus, $\alpha_t(\theta, l)$ is simply a known deterministic function $\alpha(\theta)$ of $\theta \in \mathbb{R}^+$, and the survival probability $G(t) = \mathbb{P}[\tau > t|\mathcal{F}_t] = \mathbb{P}[\tau > t] = \int_t^\infty \alpha(\theta)d\theta$ is a deterministic function. We assume that the survival probability follows an exponential distribution with a constant default intensity λ. Thus, there is a constant $\lambda > 0$ such that $G(t) = e^{-\lambda t}$, and therefore, the density function is $\alpha(\theta) = \lambda e^{-\lambda\theta}$. Moreover, we suppose that $\gamma^0 > 0$ (loss at default), and we consider functions $\mu^1(\theta, l)$ and $\sigma^1(\theta, l)$ that have the form $\mu^1(\theta, l) = \mu^0 \left(\frac{\theta}{T}\right)$ and $\sigma^1(\theta, l) = \sigma^0 \left(2 - \frac{\theta}{T}\right)$ for all $\theta \in [0, T]$. This choice has the following economic interpretation : The ratio between the after-default and before-default rates of return is less than one, meaning that the asset is less competitive after the loss at default. Moreover, this ratio increases linearly with later default times : the after-default rate of return drops to zero when the default time occurs near the initial date, and it converges to the before-default rate of return when the default time occurs near the finite investment horizon. The interpretation for the volatility is similar but with the opposite relation : the ratio between the after-default and before-default volatilities is larger than one, meaning that the market is more volatile after default. Moreover, this ratio decreases linearly with later default times, converging to double the value (resp. the initial value) of the before-default volatility as the default time approaches the initial (resp. the terminal horizon) time. Moreover, in our model, H^0 and H^1 are constants such that $H^0 > H^1$. This corresponds to the payoffs for selling a basket default swap. This swap is a credit derivative contract, which provides to its buyer protection against default of the underlying asset. The protection buyer pays a premium. In return, the protection seller pays the buyer if the default occurs before maturity. Here, there is no mark, so we will not express the dependence in l. In this case, we have

$$\mathbb{E}\left[(Z_T^Q)^2\right] = \exp\left(-(T-\theta)\left(\frac{\mu^0}{\sigma^0(2\frac{T}{\theta}-1)}\right)^2\right),$$

which yields $v_t^{1,\theta} = \lambda \exp\left(-\lambda t + (T-\theta)\left(\frac{\mu^0}{\sigma^0(2\frac{T}{\theta}-1)}\right)^2\right)$. In this model, our system of BSDEs becomes a system of ordinary differential equations (ODEs) and has explicit solutions. For this example, we adopt another quadratic form, which is given by $V_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(x) = v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}x^2 - 2v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}x + v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(0)}$, where $k = \{0, 1\}$ (i.e., 0 for the before-default functions and 1 for the after-default functions). We can obtain the terms $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}$, $v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}$ and $v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(0)}$ using our classical quadratic decomposition form because we have $v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)} = v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}$, $Y^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = \frac{v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}}{v^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}}$ and $\xi_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k} = v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(0)} - (Y_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k})^2 v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}$.

Here, we will consider the particular time $t = \theta$. By applying dynamic programming to the corresponding value function V_t^0 given in (3.2.15), we find that in our Markovian framework, V^0 satisfies the Hamilton-Jacobi-Bellman equation given by

$$\frac{\partial V_t^0(x)}{\partial t} + \inf_{\pi \in \mathbb{R}^{[0,T]}} \left\{\mu^0 \pi_t \frac{\partial V_t^0(x)}{\partial x} + \frac{1}{2}\frac{\partial^2 V_t^0(x)}{\partial x^2}(\sigma^0)^2\pi_t^2 + V_t^{1,t}(x + \gamma^0\pi_t)\right\} = 0. \quad (3.4.1)$$

As we have quadratic decomposition forms of $V_t^{1,t}(x)$ and V^0 given by $V_t^{1,t}(x) = v_t^{1,t,(2)}x^2 - 2v_t^{1,t,(1)}x + v_t^{1,t,(0)}$ and $V_t^0(x) = v_t^{0,(2)}x^2 - 2v_t^{0,(1)}x + v_t^{0,(0)}$, respectively, we then determine that the optimal strategy is

$$\pi_t^{0,*} = \frac{-\gamma^0 \left(v_t^{1,t,(2)} x - v_t^{1,t,(1)}\right) + \mu^0 \left(-v_t^{0,(2)} x + v_t^{0,t,(1)}\right)}{(\sigma^0)^2 v_t^{0,(2)} + (\gamma^0)^2 v_t^{1,t,(2)}}.$$

Inserting this expression into (3.4.1), we obtain

$$\left(\frac{\partial v_t^{0,(2)}}{\partial t} + v_t^{1,t,(2)}\right) x^2 - 2\left(\frac{\partial v_t^{0,(1)}}{\partial t} + v_t^{1,t,(1)}\right) x + \left(\frac{\partial v_t^{0,(0)}}{\partial t} + v_t^{1,t,(0)}\right) = \frac{\left[\left(\mu^0 v_t^{0,(2)} + \gamma^0 v_t^{1,t,(2)}\right) x - \gamma^0 v_t^{1,t,(1)} - \mu^0 v_t^{1,t,(2)}\right]^2}{(\sigma^0)^2 v_t^{0,(2)} + (\gamma^0)^2 v_t^{1,t,(2)}}.$$

Then, by identifying the various coefficients in x, we obtain the following ODEs :

$$\begin{aligned}
\frac{\partial v_t^{0,(2)}}{\partial t} &= -v_t^{1,t,(2)} + \frac{\left(\mu^0 v_t^{0,(2)} + \gamma^0 v_t^{1,t,(2)}\right)^2}{(\sigma^0)^2 v_t^{0,(2)} + (\gamma^0)^2 v_t^{1,t,(2)}}, & v_T^{0,(2)} &= G(T) = e^{-\lambda T}, \\
\frac{\partial v_t^{0,(1)}}{\partial t} &= -v_t^{1,t,(1)} + \frac{\left(\mu^0 v_t^{0,(2)} + \gamma^0 v_t^{1,t,(2)}\right)\left(\mu^0 v_t^{0,(1)} + \gamma^0 v_t^{1,t,(1)}\right)}{(\sigma^0)^2 v_t^{0,(2)} + (\gamma^0)^2 v_t^{1,t,(2)}}, & v_T^{0,(1)} &= H_0 v_T^{0,(2)}, \\
\frac{\partial v_t^{0,(0)}}{\partial t} &= -v_t^{1,t,(0)} + \frac{\left(\mu^0 v_t^{0,(1)} + \gamma^0 v_t^{1,t,(1)}\right)^2}{(\sigma^0)^2 v_t^{0,(2)} + (\gamma^0)^2 v_t^{1,t,(2)}}, & v_T^{0,(0)} &= H_0^2 v_T^{0,(2)}.
\end{aligned}$$

The first ODE corresponds to the first BSDE in this Markovian framework. In fact, in this particular case, in which all coefficients and terminal conditions are deterministic, the predictable component β^0 of the pair solution $(v^{0,(2)}, \beta^0)$ to the first BSDE is equal to zero. Equivalently, the two last ODEs are related to the last two BSDEs in this particular setting. Therefore, we can numerically verify the characteristics of the triple $(v^{0,(2)}, Y^0, \xi^0)$ that appears in 3.3.4 and plot the solutions of the ODEs.

For the simulations, we take $\mu^0 = 0.2$, $\sigma^0 = 0.05$, $H_0 = 1.2$, $H_1 = 0.9$ and maturity $T = 1$. From Figure 3.1, we first find that there exists some $\delta, \bar{\delta} > 0$ such that $\delta \le v_t^0 \le \bar{\delta} \le 1$. This inequality verifies the result we proved in Theorem 3.3.1, point 1. Furthermore, from the quadratic decomposition form of V^0, we have

$$v_0^0 = V_0^0(1) = \min_{\pi \in \mathcal{A}} \mathbb{E}\left[X_T^{1,\pi}\right]^2.$$

Therefore, v^0 is related to the minimal variance of a portfolio investment on the asset S with initial wealth $x = 1$. Consequently, to understand the impact of asset parameters on the minimal variance, we must plot the coefficient v^0 with respect to time t. First, let us study the minimal variance with respect to the jump due to default. We recall that the variance of the portfolio is divided into two components, the continuous component driven by Brownian motion and the jump component driven by the default indicator process. In Figure 3.1, we clearly find that the minimal variance with no jump component $(\gamma = 0)$ is less than the minimal variance with a jump component. In other words, the component arising from the jump due to default increases the minimal variance.

We are also interested in understanding the variation of the minimal variance with respect to the intensity parameter. Hence, in Figure 3.2, we find that the minimal variance increases as the intensity parameter increases. This is an expected result because when the intensity increases, the corresponding probability of default also increases. Therefore, the occurrence of jumps increases, implying an increase in the variance.

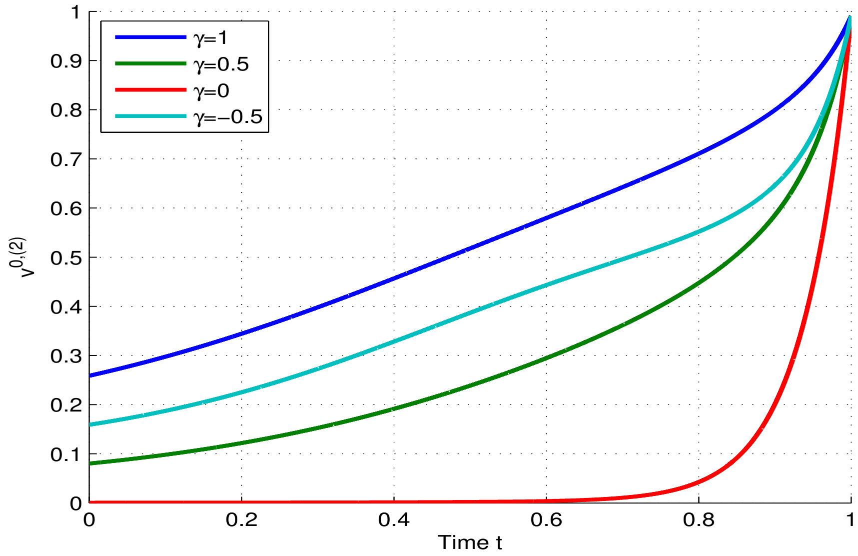

FIGURE 3.1 – v_t^0 as a function of time $t \in [0,T]$ with $T = 1$ and $\lambda = 0.01$ for various values of γ.

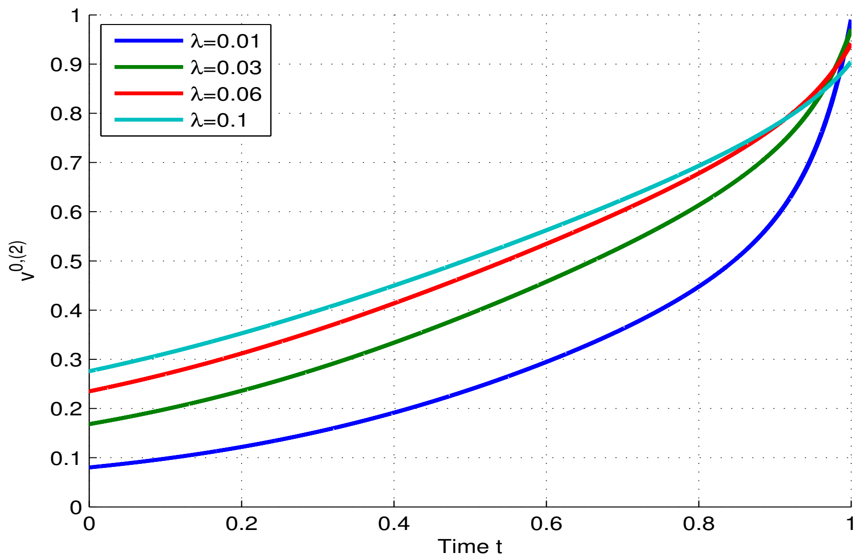

FIGURE 3.2 – v_t^0 as a function of time $t \in [0,T]$ with $T = 1$ and $\gamma = 0.5$ for various values of λ.

In Table 1, we observe that the values of the process Y_0^0 are quite stable with respect to λ for each value of γ. Moreover, they are decreasing with increasing λ, which is an expected result because a higher occurrence of jumps implies a lower price. For any fixed

Y_0^0	$\lambda = 0.01$	$\lambda = 0.03$	$\lambda = 0.06$	$\lambda = 0.1$
$\gamma = -0.5$	1.0413	1.0325	1.0280	1.0254
$\gamma = 0$	0.8847	0.8847	0.8847	0.8848
$\gamma = 0.5$	1.4343	1.4503	1.4579	1.4620
$\gamma = 1$	1.3145	1.3188	1.3203	1.3211

TABLE 3.1 – Y_0^0 as a function of γ for various values of λ.

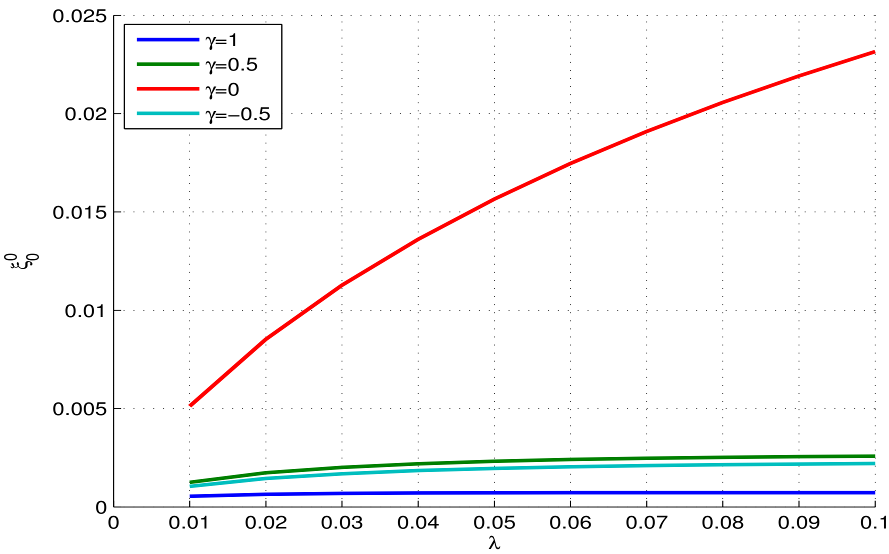

FIGURE 3.3 – ξ_0^0 as a function of γ for various values of λ.

λ, Y_0^0 is higher for $\gamma = 0.5$ than for $\gamma = 1$. This is because the difference in the payoffs, $H_1 - H_0$, is equal to 0.3, which is a low value compared with the difference in the chosen values for γ. We now recall that the process ξ^0 represents the incompleteness of the market. Hence, in Figure (3.3), we first observe that because the payoff exhibits a jump between values H_0 and H_1, if we consider a non-vanishing jump in the asset dynamics S (i.e., $\gamma \neq 0$), then the values of ξ_0^0 are quite close to zero. This result indicates that our hedging strategy covers the model well. By contrast, if we consider $\gamma = 0$, then the dynamics of the asset price S do not exhibit a jump when the default occurs, although the payoff still jumps ; we observe that the value of the process ξ increases with respect to the jump probability. Because we are considering a default risk model with a jump in the payoff, considering $\gamma = 0$ means that we must consider continuous asset dynamics S and must thus use a Black and Scholes hedging strategy. Hence, it is natural to obtain values of ξ^0 that are larger than those obtained in cases with $\gamma \neq 0$. As a financial example, if we assume that the payoff H is a CDO with multiple defaults, then assuming that S is a Black and Scholes model yields inferior results in terms of hedging compared with assuming that S is a CDS.

3.4.2 Study of a two-defaults case

We now consider another case, in which there are two default events. In this case, there are no marks and μ^0, σ^0 and γ^0 are also constants ; $\mu^1(\theta_1)$ and $\sigma^1(\theta_1)$ are deterministic functions of θ_1 ; $\mu^2(\theta_1,\theta_2)$ and $\sigma^2(\theta_1,\theta_2)$ are deterministic functions of (θ_1,θ_2) ; and the default times τ_1 and τ_2 are independent of $\mathbb{F}$. We assume that τ_1 and τ_2 are two independent random variables that follow the exponential distribution with two constant default intensities, λ_1 and λ_2. Here, then, $(\theta_1,\theta_2)=(\tau_1\wedge\tau_2,\tau_1\vee\tau_2)$, and therefore, $\alpha^1(\theta_1)=(\lambda_1+\lambda_2)\exp(-(\lambda_1+\lambda_2)\theta_1)$ and $\alpha(\theta_1,\theta_2)=\lambda_1\lambda_2(\exp(-\lambda_1\theta_1+\lambda_2\theta_2)+\exp(-\lambda_1\theta_2+\lambda_2\theta_1))$. Moreover, we consider the same constant $\gamma^0>0$ for both defaults, and we also consider functions $\mu^1(\theta_1)$ and $\sigma^1(\theta_1)$ that have the form $\mu^1(\theta_1)=\mu^0\left(\frac{\theta_1}{T}\right)$ and $\sigma^1(\theta_1)=\sigma^0\left(2-\frac{\theta_1}{T}\right)$ for all $\theta_1\in[0,T]$. As an extension of the previous case, for all $(\theta_1,\theta_2)\in\Delta_2$, we take $\mu^2(\theta_1,\theta_2)=\mu^0\left(\frac{\theta_1}{T}\right)\left(\frac{\theta_2}{T}\right)$ and $\sigma^1(\theta_1,\theta_2)=\sigma^0\left(2-\frac{\theta_1}{T}\right)\left(2-\frac{\theta_2}{T}\right)$. The economic interpretations are the same as in the previous case, except that there are now two defaults instead of one. In this case, we have

$$\mathbb{E}\left[(Z_T^Q)^2\right]=\exp\left(-(T-\theta_2)\left(\frac{\mu^0}{\sigma^0(2\frac{T}{\theta_1}-1)(2\frac{T}{\theta_2}-1)}\right)^2\right),$$

which yields

$$v_{\theta_2}^{2,\theta_1,\theta_2}=\lambda_1\lambda_2\left(\exp(-\lambda_1\theta_1+\lambda_2\theta_2)+\exp(-\lambda_1\theta_2+\lambda_2\theta_1))+(T-\theta_2)\left(\frac{\mu^0}{\sigma^0(2\frac{T}{\theta_1}-1)(2\frac{T}{\theta_2}-1)}\right)^2\right).$$

We consider three constant payoffs H_0, H_1 and H_2 such that $H_0>H_1>H_2$, which correspond to the payoffs for selling a basket default swap, as explained in the previous section. In this model, our system of BSDEs becomes a system of ordinary differential equations (ODEs) and has explicit solutions. For this example, we again adopt another quadratic form, which is given by $V_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k}(x)=v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(2)}x^2-2v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(1)}x+v_t^{k,\boldsymbol{\theta}_k,\boldsymbol{l}_k,(0)}$, where $k=\{0,1,2\}$.

Here, we consider the particular time $t=\theta_2$. By applying dynamic programming to the corresponding value function V_t^{1,θ_1} given in (3.2.15), we also find that V^1 satisfies the Hamilton-Jacobi-Bellman equation given by

$$\frac{\partial V_t^{1,\theta_1}(x)}{\partial t}+\inf_{\pi^1\in\mathbb{R}^{[0,T]}}\left\{\mu^0\frac{\theta_1}{T}\pi_t^1\frac{\partial V_t^{1,\theta_1}(x)}{\partial x}+\frac{1}{2}\frac{\partial^2 V_t^{1,\theta_1}(x)}{\partial x^2}(\sigma^0)^2(2-\frac{\theta_1}{T})^2(\pi_t^1)^2+V_t^{2,\theta_1,t}(x+\gamma^0\pi_t^1)\right\}=0.$$

Using the same method as in the one-default case, we obtain

$$\begin{aligned}
\frac{\partial v_t^{1,\theta_1,(2)}}{\partial t} &= -v_t^{2,\theta_1,t,(2)}+\frac{\left(\mu^0\frac{\theta_1}{T}v_t^{1,\theta_1,(2)}+\gamma^0v_t^{2,\theta_1,t,(2)}\right)^2}{(\sigma^0)^2(2-\frac{\theta_1}{T})^2v_t^{1,\theta_1,(2)}+(\gamma^0)^2v_t^{2,\theta_1,t,(2)}},\ v_T^{1,\theta_1,(2)}=(\lambda_1+\lambda_2)e^{-(\lambda_1+\lambda_2)(\theta_1+T)},\\
\frac{\partial v_t^{1,\theta_1,(1)}}{\partial t} &= -v_t^{2,\theta_1,t,(1)}+\frac{\left(\mu^0\frac{\theta_1}{T}v_t^{1,\theta_1,(2)}+\gamma^0v_t^{2,\theta_1,t,(2)}\right)\left(\mu^0\frac{\theta_1}{T}v_t^{1,\theta_1,(1)}+\gamma^0v_t^{2,\theta_1,t,(1)}\right)}{(\sigma^0)^2(2-\frac{\theta_1}{T})^2v_t^{1,\theta_1,(2)}+(\gamma^0)^2v_t^{2,\theta_1,t,(2)}},\ v_T^{1,\theta_1,(1)}=H_1v_T^{1,\theta_1,(2)},\\
\frac{\partial v_t^{1,\theta_1,(0)}}{\partial t} &= -v_t^{2,\theta_1,t,(0)}+\frac{\left(\mu^0\frac{\theta_1}{T}v_t^{1,\theta_1,(1)}+\gamma^0v_t^{2,\theta_1,t,(1)}\right)^2}{(\sigma^0)^2(2-\frac{\theta_1}{T})^2v_t^{1,\theta_1,(2)}+(\gamma^0)^2v_t^{2,\theta_1,t,(2)}},\qquad v_T^{1,\theta_1,(0)}=H_1^2v_T^{1,\theta_1,(2)}.
\end{aligned}$$

Considering the particular time $t = \theta_1$, we ultimately obtain the same ODEs for $v^{0,(2)}$, $v^{0,(1)}$ and $v^{0,(0)}$ as in the one-default case, except that in this case, we obtain $v_T^{0,(2)} = G(T) = e^{-(\lambda_1+\lambda_2)T}$.

For the simulations, we take $\mu^0 = 0.2$, $\sigma^0 = 0.05$, $\lambda_1 = 0.01$, $H_0 = 1.2$, $H_1 = 0.9$, $H_2 = 0.5$ and maturity $T = 1$.

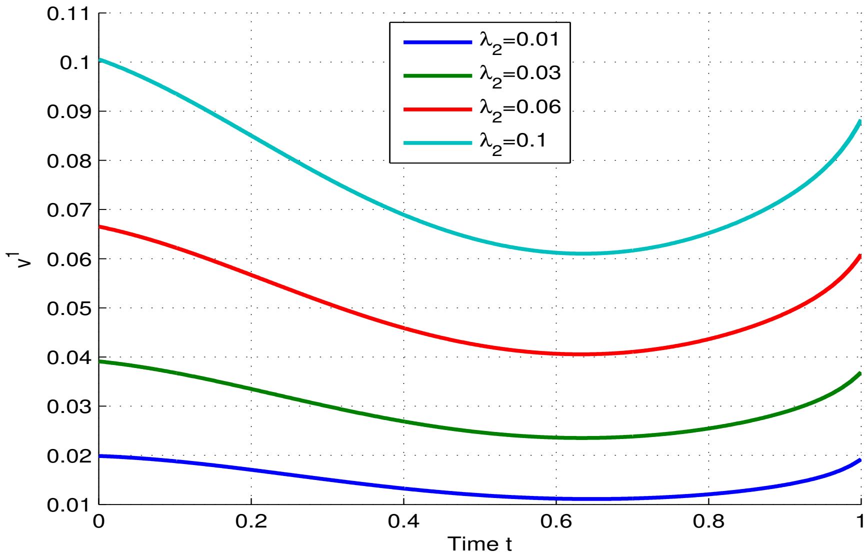

FIGURE 3.4 – $v_t^{1,t}$ as a function of time $t \in [0, T]$ with $\gamma = 0.5$ for various values of λ_2.

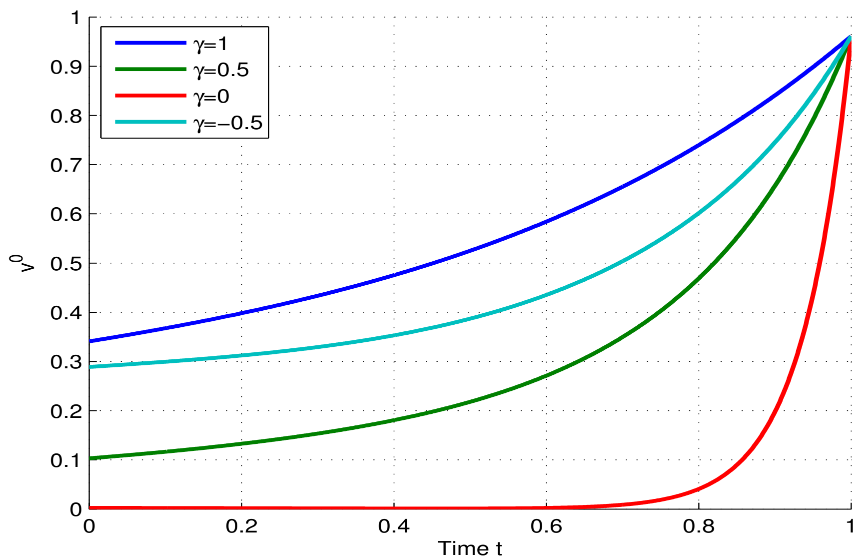

FIGURE 3.5 – v_t^0 as a function of time $t \in [0, T]$ with $\lambda_2 = 0.03$ for various values of γ.

In Table 2, we observe that in this case, the values of the process Y_0^0 are also quite

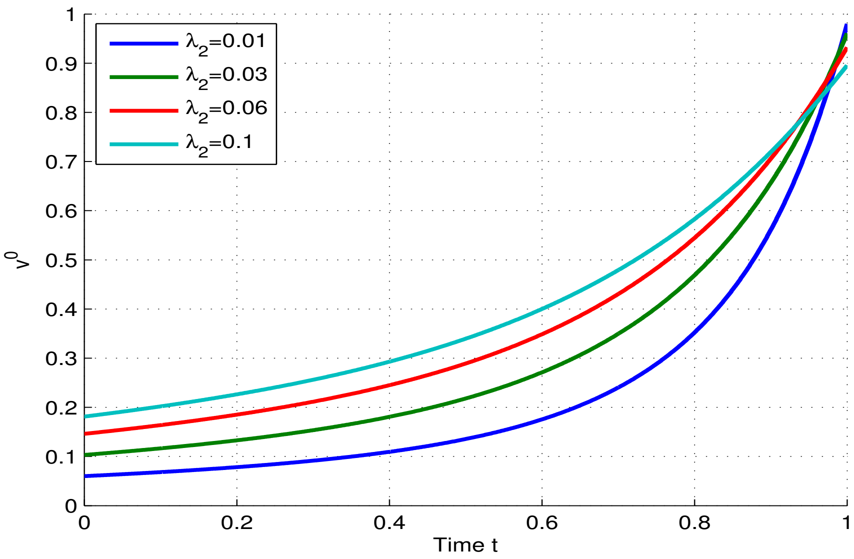

FIGURE 3.6 – v_t^0 as a function of time $t \in [0, T]$ with $\gamma = 0.5$ for various values of λ_2.

Y_0^0	$\lambda = 0.01$	$\lambda = 0.03$	$\lambda = 0.06$	$\lambda = 0.1$
$\gamma = -0.5$	1.1026	1.0971	1.0938	1.0914
$\gamma = 0$	0.8924	0.8884	0.8862	0.8847
$\gamma = 0.5$	1.3074	1.3107	1.3135	1.3155
$\gamma = 1$	1.2559	1.2572	1.2580	1.2585

TABLE 3.2 – Y_0^0 as a function of γ for various values of λ_2.

stable with respect to λ_2 for each value of γ. We observe in Figure 3.6 that v_t^0 is an increasing function of λ_2. The same interpretations hold for both Figure 3.1 and Figure 3.5. Moreover, in Figure 3.4, we observe that $v_t^{1,t}$ is initially decreasing and then increasing. We recall that $v_t^{1,t}$ is related to the minimal variance on the asset S at time t in the case when the first default occurred at time t. Therefore, there are two major effects on $v_t^{1,t}$: the variance of the underlying asset and the incompleteness of the market caused by the possible second default. This variance is decreasing for $t = \theta_1$ in our model, whereas the incompleteness of the market is increasing. When the time t is close to 0, meaning that the first default occurred early, the variance of the underlying asset is high and the incompleteness of the market is low, which explains why $v_t^{1,t}$ initially decreases. By contrast, when the default time is close to maturity at $T = 1$, the variance is low but the incompleteness of the market is high, causing $v_t^{1,t}$ to be increasing near maturity.

Moreover, one can also observe these dynamics in the various graphs of v_t^0. It is seen that in any case and for any value of γ and λ, $\partial v_t^0 / \partial t$ increases with time.

3.5 Appendix

Proposition 3.5.1. *Let A be an adapted increasing continuous process such that there exists a constant $C > 0$ that satisfies, for any $0 \leq s \leq t$,*

$$\mathbb{E}\big[(A_t - A_s)|\mathcal{F}_s\big] \leq C \; ;$$

then, this process A also satisfies

$$\mathbb{E}\big[\exp\big(\delta(A_t - A_s)\big)|\mathcal{F}_s\big] \leq \frac{1}{1-\delta C} \; , \quad \forall \, 0 < \delta < \frac{1}{C} \; .$$

Démonstration. Let A be an adapted increasing continuous process that satisfies $\mathbb{E}[(A_t - A_s)|\mathcal{F}_s] \leq C$. We first prove by iteration that $\mathbb{E}[(A_t - A_s)^p|\mathcal{F}_s] \leq p!C^p$ for any $p \in \mathbb{N}$. For this purpose, we assume that for $p \geq 2$, $\mathbb{E}[(A_t - A_s)^{p-1}|\mathcal{F}_s] \leq (p-1)!K^{p-1}$. Let us recall that for any adapted increasing continuous process A, we have $(A_t - A_s)^p = p\int_s^t (A_t - A_u)^{p-1} dA_u$ for $s \leq t$; consequently, we obtain

$$\begin{aligned} \mathbb{E}\big[(A_t - A_s)^p|\mathcal{F}_t\big] &= p\mathbb{E}\Big[\int_s^t (A_t - A_u)^{p-1} dA_u|\mathcal{F}_s\Big] = p\mathbb{E}\Big[\int_s^t \mathbb{E}[(A_t - A_u)^{p-1}|\mathcal{F}_u] dA_u|\mathcal{F}_s\Big] \\ &\leq (p-1)!C^{p-1}\mathbb{E}[A_t - A_s|\mathcal{F}_s] \leq p!C^p. \end{aligned}$$

Therefore, for any $0 < \delta < \frac{1}{C}$, we obtain $\mathbb{E}\Big[\sum_{p\geq 0} \frac{1}{p!}\delta^p (A_t - A_s)^p \Big| |\mathcal{F}_s\Big] \leq \sum_{p\geq 0} \delta^p C^p$, from which we conclude the expected result. □

Chapitre 4

Reflected BSDEs with nonpositive jumps, and controller-and-stopper games

4.1 Introduction

Backward stochastic differential equations (BSDEs), introduced in the seminal paper by Pardoux and Peng [85], have emerged over the last years as a major topic in probability, especially through its deep connection with nonlinear PDEs and associated probabilistic numerical methods, and stochastic control in mathematical finance. A solution to a standard BSDE on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ generated by an $\mathbb{R}^d$-valued Brownian motion W, is a pair of a progressively measurable process (Y, Z) satisfying :

$$Y_t \quad = \quad \xi + \int_t^T F(s, Y_s, Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T, \tag{4.1.1}$$

where the generator F is a progressively measurable function, and the terminal data ξ is $\mathcal{F}_T$-measurable. In the Markovian case where $\xi(\omega) = g(W_T(\omega))$, $F(t, \omega, y, z) = f^0(W_t(\omega), y, z)$, for some continuous functions g and f^0 on $\mathbb{R}^d$ and $\mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, it is well-known from [86] that BSDE (4.1.1) provides a Feynman-Kac formula to the semi-linear partial differential equation (PDE) :

$$\frac{\partial v}{\partial t} + \frac{1}{2}\mathrm{tr}(D_x^2 v) + f^0(x, v, D_x v) \quad = \quad 0, \quad \text{on } [0, T) \times \mathbb{R}^d, \tag{4.1.2}$$

with terminal condition $v(T, \cdot) = g$, through the relation : $Y_t = v(t, W_t)$, $0 \leq t \leq T$. We also notice that when the function f^0 is in the form : $f^0(x, z) = \sup_{a\in A}[f(x, a) + a.z]$, for some function f on $\mathbb{R}^d \times A$, with A compact set of $\mathbb{R}^d$, then the semi-linear PDE (4.1.2) is the Hamilton-Jacobi-Bellman equation for a stochastic control problem, where the controller can affect only the drift of the Brownian motion : $W_t + \int_0^t \alpha_s ds$, by a progressively measurable process α valued in A, and with a running gain function f. The extension of a standard BSDE driven by a Brownian motion and an independent Poisson random measure was considered in [103] and [3], and is shown to be related in a Markovian framework to semi-linear integro-PDE.

The notion of reflected BSDE was introduced by El Karoui et al. [37], and consists in the addition (resp. subtraction) of a nondecreasing process to the standard BSDE (4.1.1) in order to keep the solution Y above (resp. below) a lower (resp. upper) obstacle, and chosen in a minimal way via the so-called Skorohod condition. Existence and uniqueness results for reflected BSDEs under general assumptions on the obstacle have been investigated in several papers, among others [50], [78], [88]. We also mention works by [53] and [41] for reflected BSDEs driven by Brownian motion and Poisson random measure. An important application of reflected BSDE is its connection to optimal stopping problems and its associated variational inequalities in the Markovian case.

The extension to fully nonlinear PDE, motivated in particular by uncertain volatility model and more generally by stochastic control problem where control can affect both drift and diffusion terms of the state process, generated important recent developments. Soner, Touzi and Zhang [101] introduced the notion of second order BSDEs (2BSDEs), whose basic idea is to require that the solution verifies the equation $\mathbb{P}^\alpha$ a.s. for every probability measure in a non dominated class of mutually singular measures. This theory is closely related to the notion of nonlinear and G-expectation of Peng [89]. Alternatively, Kharroubi and Pham [75], following [74], introduced the notion of BSDE with nonpositive jumps. The basic idea was to constrain the jumps-component solution to the BSDE driven by Brownian motion and Poisson random measure, to remain nonpositive, by adding a nondecreasing process in a minimal way. A key feature of this class of BSDEs is its formulation under a single probability measure in contrast with 2BSDEs, thus avoiding technical issues in quasi-sure analysis, and its connection with fully nonlinear HJB equation when considering a Markovian framework with a simulatable regime switching diffusion process, defined as a randomization of the controlled state process. This approach opens new perspectives for probabilistic scheme for fully nonlinear PDEs as currently investigated in [73].

In this chapter, we define a class of reflected BSDEs with nonpositive jumps and upper obstacle. As in the case of doubly reflected BSDEs with lower and upper obstacles, related to Dynkin games, our BSDE formulation involves the introduction of two nondecreasing processes, one corresponding to the nonpositive jump constraint and added in a minimal way, and the other associated to the upper reflection, satisfying the Skorohod condition, and acting in the opposite direction. The first aim of this chapter is to prove the existence and uniqueness of a minimal solution to reflected BSDEs with nonpositive jumps and upper obstacle. We use a double penalization approach, and the main issue is to obtain uniform estimates on both penalized nondecreasing processes associated on one hand to the nonpositive jumps constraint and on the other hand to the upper obstacle. This is achieved under some regularity assumptions on the upper obstacle. It is worth mentioning that the running order of the limits in the double penalization is crucial, in contrast with the case of upper and lower reflection. Indeed, we do not have comparison results on the jump-component solution of a BSDE, and so a priori rather few information on the sequence of nondecreasing processes associated to the jump constraint, whereas one can exploit comparison results on the Y-component of a BSDE in order to derive useful monotonicity property for the sequence of nondecreasing processes associated to the upper obstacle. Once, we get uniform estimates, we conclude by a monotonic convergence theorem for BSDEs. We also prove a dual game representation formula for the minimal solution to our BSDE, in terms of equivalent probability measures and discount processes.

The main motivation for considering such class of upper-reflected BSDEs with nonpositive jumps arises from a zero-sum stochastic differential game between a controller and a stopper : the controller can manipulate a state process X^α in $\mathbb{R}^d$ through the selection of the control α valued in A, while the stopper has the right to choose the duration of the game via a stopping time τ. The stopper would like to minimize his expected cost :

$$\mathbb{E}\Big[\int_0^\tau f(X_t^\alpha, \alpha_t)dt + g(X_\tau^\alpha)\Big], \tag{4.1.3}$$

over all choices of τ, while the controller plays against him by maximizing (4.1.3) over all choices of α. Controller-and-stopper game problem was studied in [66] when the state process X^α is a one-dimensional diffusion, in [68] by a martingale approach and in [51] by BSDE methods, but only when the drift is controlled. General existence results for optimal actions and saddle point were recently obtained in [84] in a non Markovian and non dominated framework by exploiting the theory of nonlinear expectations. We also mention the recent papers [80], [81] where the authors considered 2BSDE with reflection, in connection with optimal stopping and Dynkin game under nonlinear expectation. In the Markovian case where both drift $b(X^\alpha, \alpha)$ and diffusion term $\sigma(X^\alpha, \alpha)$ of the state process X^α are controlled (hence in a non dominated framework), the recent paper [7] proved the existence of the game value, by a comparison principle for the associated Hamilton-Jacobi-Bellman Isaacs equation :

$$\begin{aligned} \max\Big[-\frac{\partial v}{\partial t} - \sup_{a\in A}\big(b(x,a).D_x v + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D_x^2 v) + f(x,a)\big); \qquad & \qquad (4.1.4) \\ v - g\Big] \;=\; 0, \;\; \text{on } [0,T)\times\mathbb{R}^d. & \end{aligned}$$

Our second main result is to connect the minimal solution to our reflected BSDE with nonpositive jumps to a general Markovian controller-and-stopper game problem through the HJB Isaacs equation (4.1.4). We follow the idea in [12] and [75] by a randomization of the state process X^α, and thus consider a regime switching forward diffusion process X with drift $b(X_t, I_t)$ and diffusion coefficient $\sigma(X_t, I_t)$, where I_t is a pure jump process associated to the Poisson random measure driving the BSDE. The minimal solution Y_t to the reflected BSDE with nonpositive jumps, with terminal data $\xi = g(X_T)$, upper obstacle $U_t = u(t, X_t)$, and generator $f(X_t, I_t, Y_t, Z_t)$, is written in this Markovian framework as : $Y_t = v(t, X_t, I_t)$ for some deterministic function v. It appears as in [75] that actually v does not depend on a in the interior of A as a consequence of the non positivity jumps constraint, and we show that v is a viscosity solution to the general HJB Isaacs equation (4.1.4) where the generator $f(x, a, v, \sigma^\intercal D_x v)$ may depend also on v and $D_x v$.

The rest of the chapter is organized as follows. Section 4.2 gives a detailed formulation of reflected BSDE with nonpositive jumps and upper obstacle. Section 4.3 is devoted to the existence of a minimal solution to our BSDE by a double penalization approach. We derive in Section 4.4 a dual game representation formula for the BSDE minimal solution. Section 4.5 makes the connection of the minimal BSDE-solution to fully nonlinear variational inequalities of HJB Isaacs type. We conclude in Section 4.6 by indicating some possible extensions to our paper. Finally, in the appendix, we recall some useful comparison results for BSDE with jumps, and state a monotonic convergence theorem, which extends to the jump case the result in [88].

4.2 Reflected BSDE with nonpositive jumps

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space on which are defined a d-dimensional Brownian motion $W = (W_t)_{t \geq 0}$ and a Poisson random measure μ on $\mathbb{R}_+ \times A$, where A is a compact subset of $\mathbb{R}^q$, endowed with its Borel σ-field $\mathcal{B}(A)$. We assume that W and μ are independent, and μ has an intensity measure $\lambda(da)dt$ for some finite measure λ on $(A, \mathcal{B}(A))$. We set $\tilde{\mu}(dt, da) = \mu(dt, da) - \lambda(da)dt$ the compensated martingale measure associated to μ, and denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ the completion of the natural filtration generated by W and μ.

We fix a finite time duration $T < \infty$ and we denote by $\mathcal{P}$ the σ-field of $\mathbb{F}$-predictable subsets of $\Omega \times [0, T]$. Let us introduce some additional notations. We denote by :

- $\mathbf{L^p}(\mathcal{F}_t)$, $p \geq 1$, $0 \leq t \leq T$, the set of $\mathcal{F}_t$-measurable random variables X such that $\mathbb{E}|X|^p < \infty$.
- $\mathbf{S^2}$ the set of real-valued càdlàg adapted processes $Y = (Y_t)_{0 \leq t \leq T}$ such that

$$\|Y\|_{\mathbf{S^2}}^2 := \mathbb{E}\Big[\sup_{0 \leq t \leq T} |Y_t|^2\Big] < \infty.$$

- $\mathbf{L^p(0,T)}$, $p \geq 1$, the set of real-valued adapted processes $(\phi_t)_{0 \leq t \leq T}$ such that

$$\|\phi\|_{\mathbf{L^p(0,T)}}^p := \mathbb{E}\Big[\int_0^T |\phi_t|^p dt\Big] < \infty.$$

- $\mathbf{L^p(W)}$, $p \geq 1$, the set of $\mathbb{R}^d$-valued $\mathcal{P}$-measurable processes $Z = (Z_t)_{0 \leq t \leq T}$ such that

$$\|Z\|_{\mathbf{L^p(W)}}^p := \mathbb{E}\Big[\Big(\int_0^T |Z_t|^2 dt\Big)^{\frac{p}{2}}\Big] < \infty.$$

- $\mathbf{L^p}(\tilde{\mu})$, $p \geq 1$, the set of $\mathcal{P} \otimes \mathcal{B}(A)$-measurable maps $L \colon \Omega \times [0, T] \times A \to \mathbb{R}$ such that

$$\|L\|_{\mathbf{L^p}(\tilde{\mu})}^p := \mathbb{E}\Big[\Big(\int_0^T \int_A |L_t(a)|^2 \lambda(da) dt\Big)^{\frac{p}{2}}\Big] < \infty.$$

- $\mathbf{L^2}(\lambda)$ the set of $\mathcal{B}(A)$-measurable maps $\ell \colon A \to \mathbb{R}$ such that

$$|\ell|_{\mathbf{L^2}(\lambda)}^2 := \int_A |\ell(a)|^2 \lambda(da) < \infty.$$

- $\mathbf{K^2}$ the set of nondecreasing predictable processes $K = (K_t)_{0 \leq t \leq T} \in \mathbf{S^2}$ with $K_0 = 0$, so that

$$\|K\|_{\mathbf{S^2}}^2 \quad = \quad \mathbb{E}|K_T|^2.$$

We are then given three objects :

1. A *terminal condition* $\xi \in \mathbf{L^2}(\mathcal{F}_T)$.
2. A *generator function* $F : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{L^2}(\lambda) \to \mathbb{R}$, which is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbf{L^2}(\lambda))$-measurable map, satisfying :

(i) The square integrability condition :

$$\mathbb{E}\Big[\int_0^T |F(t,0,0,0)|^2 dt\Big] \quad < \quad \infty.$$

(ii) The uniform Lipschitz condition :

$$|F(t,y,z,\ell) - F(t,y',z',\ell')| \quad \leq \quad C_F\big(|y-y'| + |z-z'| + |\ell-\ell'|_{\mathbf{L^2}(\lambda)}\big) \ ,$$

for all $t \in [0,T]$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, and $\ell, \ell' \in \mathbf{L^2}(\lambda)$, where C_F is some positive constant.

(iii) The monotonicity condition :

$$F(t,y,z,\ell) - F(t,y,z,\ell') \quad \leq \quad \int_A (\ell(a)-\ell'(a))\gamma(t,y,z,\ell,\ell',a)\lambda(da), \qquad (4.2.1)$$

for all $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, and $\ell, \ell' \in \mathbf{L^2}(\lambda)$, where $\gamma : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{L^2}(\lambda) \times \mathbf{L^2}(\lambda) \times A \to \mathbb{R}$ is a $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}^d) \otimes \mathcal{B}(\mathbf{L^2}(\lambda)) \otimes \mathcal{B}(\mathbf{L^2}(\lambda)) \otimes \mathcal{B}(A)$-measurable map satisfying : $0 \leq \gamma(t,y,z,\ell,\ell',a) \leq C_\gamma$, for all $t \in [0,T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\ell, \ell' \in \mathbf{L^2}(\lambda)$, and $a \in A$, for some positive constant C_γ.

3. An *upper barrier* $U \in \mathbf{S^2}$ satisfying $U_T \geq \xi$, almost surely.

Let us now consider our problem of reflected BSDE with nonpositive jumps. We say that a quintuple $(Y, Z, L, K^+, K^-) \in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2} \times \mathbf{K^2}$ is a solution to the upper-reflected BSDE with nonpositive jumps with data (ξ, F, U) if the following relation holds :

$$\begin{aligned} Y_t \quad &= \quad \xi + \int_t^T F(s, Y_s, Z_s, L_s)ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) \qquad (4.2.2) \\ &\quad - \int_t^T Z_s dW_s - \int_t^T \int_A L_s(a)\mu(ds,da), \qquad 0 \leq t \leq T, \ a.s. \end{aligned}$$

together with the jump constraint

$$L_t(a) \quad \leq \quad 0 \ , \qquad d\mathbb{P} \otimes dt \otimes \lambda(da) \ \ a.e. \qquad (4.2.3)$$

and the upper constraint

$$Y_t \quad \leq \quad U_t \ , \qquad 0 \leq t \leq T, \ a.s. \qquad (4.2.4)$$

$$\int_0^T (U_{t^-} - Y_{t^-})dK_t^- \quad = \quad 0 \ , \qquad a.s. \qquad (4.2.5)$$

We look for the *minimal solution* (Y, Z, L, K^+, K^-), in the sense that for any other solution $(\tilde{Y}, \tilde{Z}, \tilde{L}, \tilde{K}^+, \tilde{K}^-)$ to the reflected BSDE with nonpositive jumps (4.2.2)-(4.2.3)-(4.2.4)-(4.2.5), it must hold that $Y \leq \tilde{Y}$.

Remark 4.2.1. We have chosen to formulate the BSDE (4.2.2) directly in terms of the random measure μ instead of the compensated random measure $\tilde{\mu}$ since we dealt with finite intensity measure $\lambda(A) < \infty$. Of course, one can formulate equivalently the BSDE (4.2.2) in terms of $\tilde{\mu}$ by changing the generator F to :

$$\tilde{F}(t,y,z,\ell) \quad = \quad F(t,y,z,\ell) - \int_A \ell(a)\lambda(da).$$

In this case, the monotonicity condition (4.2.1) for $\tilde{F}$ holds with a measurable map $\tilde{\gamma}$ satisfying : $-1 \leq \tilde{\gamma}(t, y, z, \ell, \ell', a) \leq C_{\tilde{\gamma}}$, for all $t \in [0, T]$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\ell, \ell' \in \mathbf{L^2}(\lambda)$, and $a \in A$, for some positive constant $C_{\tilde{\gamma}}$. This condition is consistent with the assumption required in comparison Theorem 4.2 in [92]. □

Remark 4.2.2. *Uniqueness of the minimal solution.* Uniqueness of a minimal solution holds in the following sense : if (Y, Z, L, K^+, K^-) and $(Y, \tilde{Z}, \tilde{L}, \tilde{K}^+, \tilde{K}^-)$ are minimal solutions to (4.2.2)-(4.2.3)-(4.2.4)-(4.2.5), then $Y = Y'$, $Z = Z'$, $L = L'$, and $K^+ - K^- = \tilde{K}^+ - \tilde{K}^-$. As a matter of fact, the uniqueness of the Y component is clear by definition. Then, denoting by $K := K^+ - K^-$, and $\tilde{K} := \tilde{K}^+ - \tilde{K}^-$, which are predictable finite variation processes, we have

$$\begin{aligned} &\int_0^t [F(s, Y_s, Z_s, L_s) - F(s, Y_s, \tilde{Z}_s, \tilde{L}_s)]ds + K_t - \tilde{K}_t \\ &+ \int_0^t (\tilde{Z}_s - Z_s)dW_s + \int_0^t \int_A (\tilde{L}_s(a) - L_s(a))\mu(ds, da) \quad = \quad 0, \end{aligned}$$

for all $t \in [0, T]$, almost surely. The uniqueness of $Z = \tilde{Z}$ follows by identifying the Brownian part and the finite variation part, while the uniqueness of $(L, K) = (\tilde{L}, \tilde{K})$ is obtained by identifying the predictable part, and by recalling that the jumps of μ are totally inaccessible. □

The main feature in this class of BSDEs is to consider a reflection constraint on Y in addition to the nonpositive jump constraint as already studied in [74] and [75]. Moreover, we deal with an upper barrier U associated to a nondecreasing process K^-, which is subtracted in (4.2.2) from the nondecreasing process K^+ associated to the nonpositive constrained jumps. In order to ensure that the problem of getting a minimal solution to (4.2.2)-(4.2.3)-(4.2.4)-(4.2.5) is well-posed, and similarly as in [75], we make the assumption that there exists a supersolution to the BSDE with nonpositive jumps, namely :

(H0) There exists $(\bar{Y}, \bar{Z}, \bar{L}, \bar{K}^+) \in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2}$ satisfying the BSDE with nonpositive jumps :

$$\begin{aligned} \bar{Y}_t \quad = \quad & \xi + \int_t^T F(s, \bar{Y}_s, \bar{Z}_s, \bar{L}_s)ds + \bar{K}^+_T - \bar{K}^+_t \\ & - \int_t^T \bar{Z}_s dW_s - \int_t^T \int_A \bar{L}_s(a)\mu(ds, da)\,, \qquad 0 \leq t \leq T, \ a.s. \end{aligned} \tag{4.2.6}$$

and

$$\bar{L}_t(a) \quad \leq \quad 0\,, \qquad d\mathbb{P} \otimes dt \otimes \lambda(da) \ \ a.e. \tag{4.2.7}$$

We shall see later in the Markovian case (see Remark 4.5.2) how this condition **(H0)** is directly satisfied.

4.3 Existence and approximation by double penalization

This section is devoted to the existence of the minimal solution to (4.2.2)-(4.2.3)-(4.2.4)-(4.2.5). We use a penalization approach and introduce the doubly indexed sequence of

BSDEs with jumps :

$$\begin{aligned} Y_t^{n,m} &= \xi + \int_t^T F(s, Y_s^{n,m}, Z_s^{n,m}, L_s^{n,m})ds + K_T^{n,m,+} - K_t^{n,m,+} - (K_T^{n,m,-} - K_t^{n,m,-}) \\ &\quad - \int_t^T Z_s^{n,m} dW_s - \int_t^T \int_A L_s^{n,m}(a)\mu(ds, da), \end{aligned} \tag{4.3.1}$$

for $n, m \in \mathbb{N}$, where $K^{n,m,+}$ and $K^{n,m,-}$ are the nondecreasing continuous processes in $\mathbf{K^2}$ defined by

$$K_t^{n,m,+} = m \int_0^t \int_A (L_s^{n,m}(a))_+ \lambda(da)ds, \qquad K_t^{n,m,-} = n \int_0^t (U_s - Y_s^{n,m})_- ds.$$

Here we use the notation $f_+ = \max(f, 0)$ and $f_- = \max(-f, 0)$ to denote the positive and negative parts of f. Notice that this penalized BSDE can be written as

$$Y_t^{n,m} = \xi + \int_t^T F_{n,m}(s, Y_s^{n,m}, Z_s^{n,m}, L_s^{n,m})ds - \int_t^T Z_s^{n,m} dW_s - \int_t^T \int_A L_s^{n,m}(a)\mu(ds, da),$$

with a generator $F_{n,m}$ given by

$$F_{n,m}(t, y, z, \ell) = F(t, y, z, \ell) + m \int_A (\ell(a))_+ \lambda(da) - n(U_t - y)_-, \quad a.s.$$

for $(t, y, z, \ell) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{L^2}(\lambda)$. Observe that the generator $F_{n,m}$ satisfies the assumptions of square integrability and uniform Lipschitzianity, which ensure by Lemma 2.4 in [103] the existence and uniqueness of a solution $(Y^{n,m}, Z^{n,m}, L^{n,m}) \in \mathbf{S^2} \times \mathbf{L^2}(\mathbf{W}) \times \mathbf{L^2}(\tilde{\mu})$ to the BSDE with jumps (4.3.1). Notice also that $F_{n,m}$ satisfies the monotonicity condition (4.2.1), is increasing in m for any fixed n, and decreasing in n for any fixed m. Thus, by the comparison Theorem 4.7.1, we deduce that $(Y^{n,m})_{n,m}$ inherits the same property :

$$Y^{n+1,m} \leq Y^{n,m} \leq Y^{n,m+1}, \quad \forall\, n, m \in \mathbb{N}. \tag{4.3.2}$$

We shall first fix m, and let n to infinity, and then let m to infinity (the order of the limits is important here, see Remark 4.3.2). The key point, as in the case of doubly reflected BSDEs related to Dynkin games, is to deal with the difference of the nondecreasing processes $K^{n,m+}$ and $K^{n,m,-}$, and the main difficulty is to prove their convergence towards respectively the nondecreasing processes K^+ and K^-, which appear in the minimal solution to the reflected BSDE with nonpositive jumps we are looking for. We have to impose some regularity conditions on the upper barrier process that will be precised later.

For fixed m, let us now consider the reflected BSDE with jumps :

$$\begin{aligned} Y_t^m &= \xi + \int_t^T F_m(s, Y_s^m, Z_s^m, L_s^m)ds - (K_T^{m,-} - K_t^{m,-}) \\ &\quad - \int_t^T Z_s^m dW_s - \int_t^T \int_A L_s^m(a)\mu(ds, da), \quad 0 \leq t \leq T, \ a.s. \end{aligned} \tag{4.3.3}$$

and

$$Y_t^m \leq U_t, \quad 0 \leq t \leq T, \ a.s. \tag{4.3.4}$$

$$\int_0^T (U_{t^-} - Y_{t^-}^m) dK_t^{m,-} = 0, \quad a.s. \tag{4.3.5}$$

where

$$F_m(t,y,z,\ell) \quad = \quad F(t,y,z,\ell) + m\int_A (\ell(a))_+ \lambda(da)\ , \qquad a.s. \tag{4.3.6}$$

for $(t,y,z,\ell) \in [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{L^2}(\lambda)$. We know from Theorem 4.2 in [53] that there exists a unique solution $(Y^m, Z^m, L^m, K^{m,-}) \in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2}$ to the reflected BSDE with jumps (4.3.3)-(4.3.4)-(4.3.5).

Remark 4.3.1. Note that in [53] the existence of $(Y^m, Z^m, L^m, K^{m,-})$ is proved using a fixed point argument and not through the penalized sequence $(Y^{n,m}, Z^{n,m}, L^{n,m})$, except for the particular case where the generator $F_{n,m}(t,\omega)$ does not depend on y, z, ℓ, see Theorem 4.1 and Remark 4.1(i) in [53]. The reason is that in [53] the authors do not impose any monotonicity condition on the generator F and therefore they do not have at disposal a comparison theorem for BSDEs with jumps. Nevertheless, under our monotonicity condition (4.2.1) and by means of the comparison Theorem 4.7.1, the existence of $(Y^m, Z^m, L^m, K^{m,-})$ can be proved via the penalized sequence $(Y^{n,m}, Z^{n,m}, L^{n,m})$. This program is carried out in [41], Theorem 5.1, even though under the additional hypothesis that the barrier U is a $\mathcal{P}$-measurable process. More precisely, it can be shown that Y^m is obtained as the decreasing limit of $Y^{n,m}$ when n goes to infinity :

$$Y_t^m \quad = \quad \lim_{n\to\infty} \downarrow Y_t^{n,m}, \quad 0 \le t \le T, a.s.$$

and this convergence also holds in $\mathbf{L^2(0,T)}$. Furthermore, $(Z^{n,m}, L^{n,m})$ converges weakly to (Z^m, L^m) in $\mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu})$, and we have the strong convergence

$$(Z^{n,m}, L^{n,m}) \quad \to \quad (Z^m, L^m) \quad \text{in } \ \mathbf{L^p(W)} \times \mathbf{L^p}(\tilde{\mu}), \quad \text{ as } \ n \to \infty,$$

for any $p \in [1,2)$, while

$$K_t^{n,m,-} \quad \rightharpoonup \quad K_t^{m,-} \quad \text{weakly in } \mathbf{L^2}(\mathcal{F}_t), \quad \text{ as } \ n \to \infty$$

for all $0 \le t \le T$. □

We first derive the following important property on the sequence of nondecreasing processes $(K^{m,-})$.

Lemma 4.3.1. *The sequence of processes* $(K^{m,-})_m$ *satisfies :*

$$K_t^{m,-} - K_s^{m,-} \quad \le \quad K_t^{m+1,-} - K_s^{m+1,-}, \qquad 0 \le s \le t \le T, \ a.s., \ \ \forall m \in \mathbb{N}. \tag{4.3.7}$$

Proof. By definition of $K^{n,m,-}$, and from (4.3.2), we clearly have for all $n, m \in \mathbb{N}$:

$$K_t^{n,m,-} - K_s^{n,m,-} \quad \le \quad K_t^{n,m+1,-} - K_s^{n,m+1,-}, \qquad 0 \le s \le t \le T, \ a.s.$$

Thus, by passing to the (weak) limit as n goes to infinity, we get the required result. □

By (4.3.2), we see that $(Y^m)_m$ is a nondecreasing sequence : $Y^m \le Y^{m+1}$, and we denote :

$$\underline{Y}_t \quad := \quad Y_t^0, \qquad 0 \le t \le T,$$

which thus provides a lower bound for the sequences (Y^m) and $(Y^{n,m})$:

$$\underline{Y}_t \;\leq\; Y_t^m \;\;\leq\;\; Y_t^{n,m}, \quad 0 \leq t \leq T, \; \forall n,m \in \mathbb{N}. \tag{4.3.8}$$

Moreover, under condition **(H0)**, we observe that the quintuple $(\bar{Y}, \bar{Z}, \bar{L}, \bar{K}^+, \bar{K}^-)$ satisfies $\int_A (\bar{L}_t(a))_+ \lambda(da) = 0 \; dt \otimes d\mathbb{P}$ a.e. so that

$$F_{n,m}(t, \bar{Y}_t, \bar{Z}_t, \bar{L}_t) \;\;\leq\;\; F(\bar{Y}_t, \bar{Z}_t, \bar{L}_t), \quad dt \otimes d\mathbb{P} \; a.e.$$

By the comparison Theorem 4.7.1, we then get an upper bound for the sequences (Y^m) and $(Y^{n,m})$:

$$Y_t^m \;\leq\; Y_t^{n,m} \;\;\leq\;\; \bar{Y}_t, \quad 0 \leq t \leq T, \forall n,m \in \mathbb{N}. \tag{4.3.9}$$

By standard arguments, we now state some estimates on the doubly indexed sequence $(Y^{n,m}, Z^{n,m}, L^{n,m}, K^{n,m,+})$ expressed in terms of $(K^{n,m,-})$.

Lemma 4.3.2. *Let assumption* **(H0)** *hold. Then there exists a positive constant C, such that for all $n, m \in \mathbb{N}$,*

$$\begin{aligned} & \|Y^{n,m}\|^2_{\mathbf{S^2}} + \|Z^{n,m}\|^2_{\mathbf{L^2(W)}} + \|L^{n,m}\|^2_{\mathbf{L^2(\tilde{\mu})}} + \|K^{n,m,+}\|^2_{\mathbf{S^2}} \\ \leq\;\; & C\Big(\mathbb{E}|\xi|^2 + \mathbb{E}\int_0^T |F(s,0,0,0)|^2 ds + \|\underline{Y}\|^2_{\mathbf{S^2}} + \|\bar{Y}\|^2_{\mathbf{S^2}} + \|K^{n,m,-}\|^2_{\mathbf{S^2}}\Big). \end{aligned} \tag{4.3.10}$$

Proof. In what follows we shall denote by $C > 0$ a generic positive constant depending only on T, $\lambda(A)$, and the Lipschitz constant of F, which may vary from line to line. Proceeding as in the proof of Lemma 3.3 in [75], we apply Itô's formula to $|Y_s^{n,m}|^2$ between t and T, and get after some rearrangement :

$$\begin{aligned} & \mathbb{E}|Y_t^{n,m}|^2 + \|Z^{n,m} 1_{[t,T]}\|^2_{\mathbf{L^2(W)}} + \|L^{n,m} 1_{[t,T]}\|^2_{\mathbf{L^2(\tilde{\mu})}} \\ = \;\; & \mathbb{E}|\xi|^2 + 2\mathbb{E}\int_t^T Y_s^{n,m} F(s, Y_s^{n,m}, Z_s^{n,m}, L_s^{n,m}) ds - 2\mathbb{E}\int_t^T \int_A Y_{s^-}^{n,m} L_s^{n,m}(a) \lambda(da) ds \\ & + 2\mathbb{E}\int_t^T Y_s^{n,m} dK_s^{n,m,+} - 2\mathbb{E}\int_t^T Y_s^{n,m} dK_s^{n,m,-}. \end{aligned} \tag{4.3.11}$$

By the linear growth condition on F, the inequality $ab \leq a^2/2 + b^2/2$, and recalling that $\lambda(A) < \infty$, we get

$$\begin{aligned} & 2\mathbb{E}\int_t^T Y_s^{n,m} F(s, Y_s^{n,m}, Z_s^{n,m}, L_s^{n,m}) ds - 2\mathbb{E}\int_t^T \int_A Y_{s^-}^{n,m} L_s^{n,m}(a) \lambda(da) ds \\ \leq & C\mathbb{E}\int_t^T |Y_s^{n,m}|^2 ds + \frac{1}{2}\mathbb{E}\int_0^T |F(s,0,0,0)|^2 ds + \frac{1}{2}\|Z^{n,m} 1_{[t,T]}\|^2_{\mathbf{L^2(W)}} + \frac{1}{2}\|L^{n,m} 1_{[t,T]}\|^2_{\mathbf{L^2(\tilde{\mu})}}. \end{aligned} \tag{4.3.12}$$

From the bounds (4.3.8)-(4.3.9) on $Y^{n,m}$: $\underline{Y} \leq Y^{n,m} \leq \bar{Y}$, and thanks to the inequality $2ab \leq a^2/\alpha + \alpha b^2$ for any constant $\alpha > 0$, we have

$$\begin{aligned} & 2\mathbb{E}\int_t^T Y_s^{n,m} dK_s^{n,m,+} - 2\mathbb{E}\int_t^T Y_s^{n,m} dK_s^{n,m,-} \\ \leq \;\; & \frac{1}{\alpha}\Big(\|\underline{Y}\|^2_{\mathbf{S^2}} + \|\bar{Y}\|^2_{\mathbf{S^2}}\Big) + \alpha\mathbb{E}|K_T^{n,m,+} - K_t^{n,m,+}|^2 + \alpha\mathbb{E}|K_T^{n,m,-} - K_t^{n,m,-}|^2 \end{aligned}$$

$$\leq \frac{1}{\alpha}\Big(\|\underline{Y}\|^2_{\mathbf{S^2}} + \|\bar{Y}\|^2_{\mathbf{S^2}}\Big) + 3\alpha\mathbb{E}|K_T^{n,m,-} - K_t^{n,m,-}|^2 + 2\alpha\mathbb{E}|K_T^{n,m} - K_t^{n,m}|^2,$$

where we set $K_t^{n,m} := K_t^{n,m,+} - K_t^{n,m,-}$, so that $\mathbb{E}|K_T^{n,m,+} - K_t^{n,m,+}|^2 \leq 2\mathbb{E}|K_T^{n,m} - K_t^{n,m}|^2 + 2\mathbb{E}|K_T^{n,m,-} - K_t^{n,m,-}|^2$. Together with (4.3.12) and (4.3.11), this yields :

$$\begin{aligned} & \mathbb{E}|Y_t^{n,m}|^2 + \frac{1}{2}\|Z^{n,m}1_{[t,T]}\|^2_{\mathbf{L^2(W)}} + \frac{1}{2}\|L^{n,m}1_{[t,T]}\|^2_{\mathbf{L^2(\tilde{\mu})}} \\ \leq\ & C\mathbb{E}\int_t^T |Y_s^{n,m}|^2 ds + \mathbb{E}|\xi|^2 + \frac{1}{2}\mathbb{E}\int_0^T |F(s,0,0,0)|^2 ds + \frac{1}{\alpha}\Big(\|\underline{Y}\|^2_{\mathbf{S^2}} + \|\bar{Y}\|^2_{\mathbf{S^2}}\Big) \\ \text{Â}\qquad & + 3\alpha\mathbb{E}|K_T^{n,m,-} - K_t^{n,m,-}|^2 + 2\alpha\mathbb{E}|K_T^{n,m} - K_t^{n,m}|^2. \end{aligned} \tag{4.3.13}$$

Now, from the relation (4.3.1), we have

$$\begin{aligned} K_T^{n,m} - K_t^{n,m} &= Y_t^{n,m} - \xi - \int_t^T F(s, Y_s^{n,m}, Z_s^{n,m}, L_s^{n,m}) ds \\ &\quad + \int_t^T Z_s^{n,m} dW_s + \int_t^T \int_A L_s^{n,m}(a)\mu(ds,da), \end{aligned}$$

so that by the linear growth condition on F :

$$\begin{aligned} \mathbb{E}|K_T^{n,m} - K_t^{n,m}|^2 \leq\ & C\Big(\mathbb{E}|\xi|^2 + \mathbb{E}\int_0^T |F(s,0,0,0)|^2 ds + \mathbb{E}|Y_t^{n,m}|^2 \\ & + \mathbb{E}\int_t^T |Y_s^{n,m}|^2 ds + \|Z^{n,m}1_{[t,T]}\|^2_{\mathbf{L^2(W)}} + \|L^{n,m}1_{[t,T]}\|^2_{\mathbf{L^2(\tilde{\mu})}}\Big). \end{aligned} \tag{4.3.14}$$

By choosing $\alpha > 0$ such that $2\alpha C \leq 1/4$, and plugging this estimate of $\mathbb{E}|K_T^{n,m} - K_t^{n,m}|^2$ into (4.3.13), we get for all $0 \leq t \leq T$:

$$\begin{aligned} & \frac{3}{4}\mathbb{E}|Y_t^{n,m}|^2 + \frac{1}{4}\|Z^{n,m}1_{[t,T]}\|^2_{\mathbf{L^2(W)}} + \frac{1}{4}\|L^{n,m}1_{[t,T]}\|^2_{\mathbf{L^2(\tilde{\mu})}} \\ \leq\ & C\mathbb{E}\int_t^T |Y_s^{n,m}|^2 ds + \frac{5}{4}\mathbb{E}|\xi|^2 + \frac{3}{4}\mathbb{E}\int_0^T |F(s,0,0,0)|^2 ds \\ & + \frac{1}{\alpha}\Big(\|\underline{Y}\|^2_{\mathbf{S^2}} + \|\bar{Y}\|^2_{\mathbf{S^2}}\Big) + 3\alpha\mathbb{E}|K_T^{n,m,-} - K_t^{n,m,-}|^2 \\ \leq\ & C\Big(\|\underline{Y}\|^2_{\mathbf{S^2}} + \|\bar{Y}\|^2_{\mathbf{S^2}} + \mathbb{E}|\xi|^2 + \mathbb{E}\int_0^T |F(s,0,0,0)|^2 ds\Big) + 12\alpha\|K^{n,m,-}\|^2_{\mathbf{S^2}}, \end{aligned} \tag{4.3.15}$$

where we used again the bounds $\underline{Y} \leq Y^{n,m} \leq \bar{Y}$ and the inequality $\mathbb{E}|K_T^{n,m,-} - K_t^{n,m,-}|^2 \leq 4\mathbb{E}|K_T^{n,m,-}|^2$. This proves, taking $t = 0$ in (4.3.15), the required estimate (4.3.10) for $(Z^{n,m}, L^{n,m})$, and also for $K^{n,m,+}$ by (4.3.14), and recalling that $\mathbb{E}|K_T^{n,m,+}|^2 \leq 2\mathbb{E}|K_T^{n,m}|^2 + 2\mathbb{E}|K_T^{n,m,-}|^2$. Finally, the estimate for $\|Y^{n,m}\|_{\mathbf{S^2}}$ in (4.3.10) follows as usual from the relation (4.3.1), Burkholder-Davis-Gundy inequality, and the estimates for $(Z^{n,m}, L^{n,m}, K^{n,m,+})$.
□

The key point is now to obtain a uniform estimate on $K^{n,m,-}$, and consequently uniform estimates on $(Y^{n,m}, Z^{n,m}, L^{n,m}, K^{n,m,+})$ in view of Lemma 4.3.2. Let us introduce the following set of probability measures. For $m \in \mathbb{N}$, let $\mathcal{V}_m$ be the set of $\mathcal{P} \otimes \mathcal{B}(A)$-measurable processes valued in $(0, m]$, $\mathcal{V} = \cup_m \mathcal{V}_m$, and given $\nu \in \mathcal{V}$, consider the probability measure $\mathbb{P}^\nu$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ with Radon-Nikodym density :

$$\frac{d\mathbb{P}^\nu}{d\mathbb{P}}\Big|_{\mathcal{F}_t} = \zeta_t^\nu := \mathcal{E}_t\Big(\int_0^\cdot \int_A (\nu_s(a) - 1)\tilde{\mu}(ds,da)\Big),$$

where $\mathcal{E}_t(\cdot)$ is the Doléans-Dade exponential. Indeed, since $\nu \in \mathcal{V}$ is essentially bounded, and $\lambda(A) < \infty$, it is known that ζ^ν is a uniformly integrable martingale (see e.g. Lemma 4.1 in [75]), and so defines a probability measure $\mathbb{P}^\nu$. Moreover, $\zeta_T^\nu \in \mathbf{L^p}(\mathcal{F}_T)$ for any $p \geq 1$. Notice that the Brownian motion W remains a Brownian motion W under $\mathbb{P}^\nu$, while the effect of the probability measure $\mathbb{P}^\nu$, by Girsanov's theorem, is to change the compensator $\lambda(da)dt$ of μ under $\mathbb{P}$ to $\nu_t(a)\lambda(da)dt$ under $\mathbb{P}^\nu$. We then denote by $\tilde{\mu}^\nu(dt, da)$ $:= \mu(dt, da) - \nu_t(a)\lambda(da)dt$ the compensated martingale measure of μ under $\mathbb{P}^\nu$.

Inspired by [52] (see also [31]), we make the following regularity assumption on the upper barrier :

(H1) There exists a nonincreasing sequence of processes $(U^k)_k$ such that :

(i) $\lim_{k\to\infty} U_t^k = U_t$, for all $0 \leq t \leq T$, a.s..

(ii) For any $k \in \mathbb{N}$, U^k is in the form :

$$U_t^k \;=\; U_0^k + \int_0^t \upsilon_s^k ds + \int_0^t \vartheta_s^k dW_s, \qquad 0 \leq t \leq T,\ a.s.$$

where $(\upsilon^k)_k \subset \mathbf{L^2(0,T)}$ and $(\vartheta^k)_k \subset \mathbf{L^2(W)}$.

(iii) There exists some $p > 2$ such that :

$$\sup_{k\in\mathbb{N}} \int_0^T \mathbb{E}\Big[\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\big[\sup_{t\leq s\leq T} (|U_s^k|^p + |\upsilon_s^k|^p + |\vartheta_s^k|^p)|\mathcal{F}_t\big]\Big]dt$$
$$+ \int_0^T \mathbb{E}\Big[\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\big[\sup_{t\leq s\leq T} |F(s,0,0,0)|^p|\mathcal{F}_t\big]\Big]dt \;<\; \infty.$$

We shall see later in the Markovian framework how Assumption **(H1)** is automatically satisfied, see Remark 4.5.3. The following key lemma states a uniform estimate for $K^{n,m,-}$ under condition **(H1)**.

Lemma 4.3.3. *Under condition* **(H1)**, *we have*

$$\sup_{n,m\in\mathbb{N}} \left\|K^{n,m,-}\right\|_{\mathbf{S^2}} \;<\; \infty.$$

Proof. Let $(U^k)_k$ be in the form as in assumption **(H1)**(ii) and consider for positive integers n, m, k, the difference $\bar{Y}^{n,m,k} := Y^{n,m} - U^k$, which is then expressed in backward form as :

$$\begin{aligned}\bar{Y}_t^{n,m,k} \;&=\; \xi - U_T^k + \int_t^T (F(s, Y_s^{n,m}, Z_s^{n,m}, L_s^{n,m}) + \upsilon_s^k)ds \\ &\quad + m\int_t^T \int_A (L_s^{n,m}(a))_+\lambda(da)ds - n\int_t^T (U_s - U_s^k - \bar{Y}_s^{n,m,k})_- ds \\ &\quad - \int_t^T (Z_s^{n,m} - \vartheta_s^k)dW_s - \int_t^T \int_A L_s^{n,m}(a)\mu(ds,da). \end{aligned} \tag{4.3.16}$$

Now, by the Lipschitz condition of F in (y, z), and the monotonicity condition (4.2.1) of F in ℓ, we have for all $n, m \in \mathbb{N}$:

$$\begin{aligned}F(t, Y_t^{n,m}, Z_t^{n,m}, L_t^{n,m}) \;&=\; F(t,0,0,0) + \alpha_t^{n,m} Y_t^{n,m} + \beta_t^{n,m}.Z_t^{n,m} \\ &\quad + \int_A \gamma_t^{n,m}(a) L_t^{n,m}(a)\lambda(da) - \delta_t^{n,m},\end{aligned}$$

for some sequence of bounded predictable processes $(\alpha^{n,m})$ valued in $\mathbb{R}$, $(\beta^{n,m})$ valued in $\mathbb{R}^d$, uniformly bounded in n,m, a nonnegative sequence of predictable process $(\delta^{n,m})$, and a nonnegative sequence of bounded $\mathcal{P}\otimes\mathcal{B}(A)$-measurable maps $(\gamma^{n,m})$, uniformly bounded in n,m. Plug this decomposition of F into (4.3.16), and let us consider the process $\{\Gamma^{n,m}_{ts}, t\leq s\leq T\}$ of dynamics :

$$d\Gamma^{n,m}_{ts} \;=\; \Gamma^{n,m}_{ts}[(\alpha^{n,m}_s - n)ds + \beta^{n,m}_s dW_s], \quad t\leq s\leq T, \quad \Gamma^{n,m}_{tt} = 1,$$

and given explicitly by :

$$\Gamma^{n,m}_{ts} \;=\; e^{-n(s-t)}e^{\int_t^s \alpha^{n,m}_u du}M^{n,m}_{ts}, \quad M^{n,m}_{ts} \;=\; \frac{\mathcal{E}_s\big(\int_0^\cdot \beta^{n,m}_u dW_u\big)}{\mathcal{E}_t\big(\int_0^\cdot \beta^{n,m}_u dW_u\big)}, \; t\leq s\leq T,$$

where $\mathcal{E}_t(\cdot)$ is the Doléans-Dade exponential. Since $\beta^{n,m}$ is a bounded process, we see that $\{M^{n,m}_{ts}, t\leq s\leq T\}$ is a uniformly integrable martingale, with $M^{n,m}_{tT}\in\mathbf{L}^{\mathbf{p}}(\mathcal{F}_T)$ for any $p\geq 1$. By applying Itô's formula to the product $\{\Gamma^{n,m}_{ts}\bar Y^{n,m,k}_s, t\leq s\leq T\}$, we then obtain :

$$\begin{aligned}
\bar Y^{n,m,k}_t \;=\;& \Gamma^{n,m}_{tT}(\xi - U^k_T) + \int_t^T \Gamma^{n,m}_{ts}(F(s,0,0,0) + \alpha^{n,m}_s U^k_s + \beta^{n,m}_s\vartheta^k_s + \upsilon^k_s)ds\\
&+ \int_t^T \Gamma^{n,m}_{ts}\big[n\bar Y^{n,m,k}_s - n(U_s - U^k_s - \bar Y^{n,m,k}_s)_- - \delta^{n,m}_s\big]ds\\
&+ \int_t^T\int_A \Gamma^{n,m}_{ts}\big[\gamma^{n,m}_s(a)L^{n,m}_s(a) + m(L^{n,m}_s(a))_+ - \nu_s(a)L^{n,m}_s(a)\big]\lambda(da)ds\\
&- \int_t^T \Gamma^{n,m}_{ts}(Z^{n,m}_s - \vartheta^k_s + \bar Y^{n,m,k}_s\beta^{n,m}_s)dW_s - \int_t^T\int_A \Gamma^{n,m}_{ts}L^{n,m}_s(a)\tilde\mu^\nu(ds,da),
\end{aligned}$$

for any $\nu\in\mathcal{V}$, where we introduced the compensated measure $\tilde\mu^\nu$ of μ under $\mathbb{P}^\nu$. By choosing $\nu = \nu^{n,m,\varepsilon}\in\mathcal{V}$ defined by : $\nu^{n,m,\varepsilon}_t(a) = (\gamma^{n,m}_t(a) + m)1_{\{L^{n,m}_t(a)\geq 0\}} + (\gamma^{n,m}_t(a) + \varepsilon)1_{\{L^{n,m}_t(a)<0\}}$, for some arbitrary $\varepsilon>0$, we see that :

$$\gamma^{n,m}_t(a)L^{n,m}_t(a) + m(L^{n,m}_t(a))_+ - \nu^{n,m}_t(a)L^{n,m}_t(a) \;=\; -\varepsilon L^{n,m}_t(a)1_{\{L^{n,m}_t(a)<0\}}.$$

Observe also that

$$n\bar Y^{n,m,k}_t - n(U_t - U^k_t - \bar Y^{n,m,k}_t)_- - \delta^{n,m}_s \;\leq\; 0, \quad 0\leq t\leq T, \; a.s.$$

since $U\leq U^k$, and $\delta^{n,m}\geq 0$. Recalling that $\xi\leq U_T\leq U^k_T$, the explicit expression of $\Gamma^{n,m}$, and the fact that $(\alpha^{n,m})$, $(\beta^{n,m})$ are uniformly bounded in (t,ω,n,m), we then get the existence of some positive constant C such that :

$$\begin{aligned}
\bar Y^{n,m,k}_t \;\leq\;& C\int_t^T e^{-n(s-t)}M^{n,m}_{ts}(|F(s,0,0,0)| + |U^k_s| + |\vartheta^k_s| + |\upsilon^k_s|)ds \qquad (4.3.17)\\
&- \varepsilon\int_t^T\int_A \Gamma^{n,m}_{ts}L^{n,m}_s(a)1_{\{L^{n,m}_s(a)<0\}}\lambda(da)ds\\
&- \int_t^T \Gamma^{n,m}_{ts}(Z^{n,m}_s - \vartheta^k_s + \bar Y^{n,m,k}_s\beta^{n,m}_s)dW_s - \int_t^T\int_A \Gamma^{n,m}_{ts}L^{n,m}_s(a)\tilde\mu^{\nu^{n,m,\varepsilon}}(ds,da),
\end{aligned}$$

for any $n,m,k\in\mathbb{N}\setminus\{0\}$, $\varepsilon>0$. Denote by $S^{n,m,k}_t = \int_0^t \Gamma^{n,m}_{0s}(Z^{n,m}_s - \vartheta^k_s + \bar Y^{n,m,k}_s\beta^{n,m}_s)dW_s$, $0\leq t\leq T$, which is a $\mathbb{P}^\nu$-local martingale, for any $\nu\in\mathcal{V}$, by recalling that W remains a Brownian motion under $\mathbb{P}^\nu$. From Burkholder-Davis-Gundy, Bayes formula, Cauchy-Schwarz, and Doob inequalities, we have

$$\begin{aligned}
& \mathbb{E}^\nu[\sup_{0\leq t\leq T}|S_t^{n,m,k}|] \\
\leq \;& C\mathbb{E}^\nu[\sqrt{< S^{n,m,k} >_T}] = C\mathbb{E}^\nu[\sqrt{\int_0^T |\Gamma_{0t}^{n,m}|^2|Z_t^{n,m} - \vartheta_t^k + \bar{Y}_t^{n,m,k}\beta_t^{n,m}|^2 dt}] \\
\leq \;& C\mathbb{E}\Big[\zeta_T^\nu \sup_{0\leq t\leq T}\Gamma_{0t}^{n,m}\sqrt{\int_0^T |Z_t^{n,m} - \vartheta_t^k + \bar{Y}_t^{n,m,k}\beta_t^{n,m}|^2 dt}\Big] \\
\leq \;& C\Big(\mathbb{E}[|\zeta_T^\nu|^4]\mathbb{E}[\sup_{0\leq t\leq T}|\Gamma_{0t}^{n,m}|^4]\Big)^{\frac{1}{4}}\sqrt{\mathbb{E}[\int_0^T |Z_t^{n,m} - \vartheta_t^k + \bar{Y}_t^{n,m,k}\beta_t^{n,m}|^2 dt]} \\
\leq \;& C\Big(\mathbb{E}[|\zeta_T^\nu|^4]\mathbb{E}[|M_{0T}^{n,m}|^4]\Big)^{\frac{1}{4}}\sqrt{\mathbb{E}[\int_0^T |Z_t^{n,m} - \vartheta_t^k + \bar{Y}_t^{n,m,k}\beta_t^{n,m}|^2 dt]} \\
< \;& \infty, \qquad (4.3.18)
\end{aligned}$$

where we used the fact that $\alpha^{n,m}$, $\beta^{n,m}$ are bounded processes, $Z^{n,m}$, ϑ^k lie in $\mathbf{L^2(W)}$, and $\bar{Y}^{n,m,k}$ in $\mathbf{L^2(0,T)}$. Therefore, $S^{n,m,k}$ is a uniformly $\mathbb{P}^\nu$-integrable martingale for any $\nu \in \mathcal{V}$, and similarly we show that $\int_0^t\int_A \Gamma_{ts}^{n,m}L_s^{n,m}(a)\tilde{\mu}^\nu(ds,da)$ is a $\mathbb{P}^\nu$-martingale. Hence, by taking conditional expectation with respect to $\mathbb{P}^{\nu^{n,m,\varepsilon}}$ into (4.3.17), we have for all $n, m, k \in \mathbb{N}\setminus\{0\}$, $\varepsilon > 0$:

$$\begin{aligned}
\bar{Y}_t^{n,m,k} \;\leq\;& \frac{C}{n}\mathbb{E}^{\nu^{n,m,\varepsilon}}\Big[\sup_{t\leq s\leq T} M_{ts}^{n,m}(|F(s,0,0,0)| + |U_s^k| + |\vartheta_s^k| + |\upsilon_s^k|)|\mathcal{F}_t\Big] \\
& -\varepsilon\, \mathbb{E}^{\nu^{n,m,\varepsilon}}\Big[\int_t^T\int_A \Gamma_{ts}^{n,m}L_s^{n,m}(a)1_{\{L_s^{n,m}(a)<0\}}\lambda(da)ds|\mathcal{F}_t\Big] \\
\leq\;& \frac{C}{n}\operatorname*{ess\,sup}_{\nu\in\mathcal{V}}\mathbb{E}^\nu\Big[\sup_{t\leq s\leq T} M_{ts}^{n,m}(|F(s,0,0,0)| + |U_s^k| + |\vartheta_s^k| + |\upsilon_s^k|)|\mathcal{F}_t\Big] \qquad (4.3.19) \\
& +\varepsilon\, \mathbb{E}\Big[\frac{\zeta_T^{\nu^{n,m,\varepsilon}}}{\zeta_t^{\nu^{n,m,\varepsilon}}}\int_t^T\int_A \Gamma_{ts}^{n,m}|L_s^{n,m}(a)|\lambda(da)ds|\mathcal{F}_t\Big], \quad 0\leq t\leq T,
\end{aligned}$$

from Bayes formula. Now, for $\varepsilon \leq m$, we see that $\nu^{n,m,\varepsilon} \leq \bar{\nu}^{n,m} := \gamma^{n,m} + m$, and so :

$$0 \leq \frac{\zeta_T^{\nu^{n,m,\varepsilon}}}{\zeta_t^{\nu^{n,m,\varepsilon}}} \;\leq\; \frac{\zeta_T^{\bar{\nu}^{n,m}}}{\zeta_t^{\bar{\nu}^{n,m}}}\exp\Big(\int_t^T\int_A \bar{\nu}_s^{n,m}(a)\lambda(da)ds\Big). \qquad (4.3.20)$$

This shows that

$$\lim_{\varepsilon\to 0}\varepsilon\, \mathbb{E}\Big[\frac{\zeta_T^{\nu^{n,m,\varepsilon}}}{\zeta_t^{\nu^{n,m,\varepsilon}}}\int_t^T\int_A \Gamma_{ts}^{n,m}|L_s^{n,m}(a)|\lambda(da)ds|\mathcal{F}_t\Big] \;=\; 0, \quad 0\leq t\leq T, \qquad (4.3.21)$$

and so by sending ε to zero into (4.3.19) :

$$\begin{aligned}
& (U_t^k - Y_t^{n,m})_- \;=\; (\bar{Y}_t^{n,m,k})_+ \\
\leq\;& \frac{C}{n}\operatorname*{ess\,sup}_{\nu\in\mathcal{V}}\mathbb{E}^\nu\Big[\sup_{t\leq s\leq T} M_{ts}^{n,m}(|F(s,0,0,0)| + |U_s^k| + |\vartheta_s^k| + |\upsilon_s^k|)|\mathcal{F}_t\Big] \\
\leq\;& \frac{C}{n}\operatorname*{ess\,sup}_{\nu\in\mathcal{V}}\mathbb{E}^\nu\Big[\sup_{t\leq s\leq T}|M_{ts}^{n,m}|^{\frac{p}{p-2}} + \sup_{t\leq s\leq T}(|F(s,0,0,0)|^{\frac{p}{2}} + |U_s^k|^{\frac{p}{2}} + |\vartheta_s^k|^{\frac{p}{2}} + |\upsilon_s^k|^{\frac{p}{2}})|\mathcal{F}_t\Big]
\end{aligned}$$

for all $0 \leq t \leq T$, and $p > 2$, by Young inequality. Recall that W is a Brownian motion under $\mathbb{P}^\nu$, and so $\{M_{ts}^{n,m}, t \leq s \leq T\}$ is a martingale under $\mathbb{P}^\nu$, for any $\nu \in \mathcal{V}$. By Doob's inequality, we then have with $q = p/(p-2) > 1$:

$$\begin{aligned}
\mathbb{E}^\nu\Big[\sup_{t\leq s\leq T} |M_{ts}^{n,m}|^q|\mathcal{F}_t\Big] &\leq \Big(\frac{q}{q-1}\Big)^q \mathbb{E}^\nu\big[|M_{tT}^{n,m}|^q|\mathcal{F}_t\big] \\
&\leq \Big(\frac{q}{q-1}\Big)^q \exp\big(q(q-1)\|\beta\|_\infty^2(T-t)\big),
\end{aligned}$$

where $\|\beta\|_\infty$ is a uniform bound of $(\beta^{n,m})$, hence independent of n, m and $\nu \in \mathcal{V}$. We then deduce that

$$\begin{aligned}
&(U_t^k - Y_t^{n,m})_- \\
\leq\ & \frac{C}{n}\Big(1 + \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\Big[\sup_{t\leq s\leq T}\big(|F(s,0,0,0)|^{\frac{p}{2}} + |U_s^k|^{\frac{p}{2}} + |\vartheta_s^k|^{\frac{p}{2}} + |\upsilon_s^k|^{\frac{p}{2}}\big)|\mathcal{F}_t\Big]\Big)
\end{aligned}$$

for all $0 \leq t \leq T$, $n, m, k \in \mathbb{N}\setminus\{0\}$. By Cauchy-Schwarz inequality, we then obtain :

$$\begin{aligned}
&\mathbb{E}\Big[n\int_0^T (U_t^k - Y_t^{n,m})_- dt\Big]^2 \\
\leq\ & C\Big(1 + \int_0^T \mathbb{E}\Big[\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\Big[\sup_{t\leq s\leq T}\big(|F(s,0,0,0)|^p + |U_s^k|^p + |\vartheta_s^k|^p + |\upsilon_s^k|^p\big)|\mathcal{F}_t\Big]dt\Big).
\end{aligned}$$

By taking $p > 2$ as in Assumption **(H1)**(iii), and then sending k to infinity in the l.h.s. of the above inequality, we get the required uniform estimate on $K^{n,m,-}$. □

Corollary 4.3.1. *Let assumptions* **(H0)** *and* **(H1)** *hold. Then, we have*

$$\sup_{m\in\mathbb{N}}\Big(\|Y^m\|_{\mathbf{S^2}} + \|Z^m\|_{\mathbf{L^2(W)}} + \|L^m\|_{\mathbf{L^2}(\tilde\mu)} + \|K^{m,+}\|_{\mathbf{S^2}} + \|K^{m,-}\|_{\mathbf{S^2}}\Big) \;<\; \infty,$$

where $K_t^{m,+} := m\int_0^t\int_A \big(L_s^m(a)\big)_+\lambda(da)ds$*.*

Proof. From the bounds (4.3.8) and (4.3.9), we already have the uniform estimate for $\|Y^m\|_{\mathbf{S^2}}$. Moreover, by Lemmata 4.3.2 and 4.3.3, we have the uniform estimates :

$$\sup_{n,m\in\mathbb{N}}\Big(\|Z^{n,m}\|_{\mathbf{L^2(W)}} + \|L^{n,m}\|_{\mathbf{L^2}(\tilde\mu)} + \|K^{n,m,+}\|_{\mathbf{S^2}} + \|K^{n,m,-}\|_{\mathbf{S^2}}\Big) \;<\; \infty,$$

We deduce that the weak limits $(Z^m, L^m, K^{m,-})$ of $(Z^{m,n}, L^{m,n}, K^{n,m,-})$ when n goes to infinity, are also uniformly bounded in $\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde\mu)\times\mathbf{S^2}$. From the strong convergence of $L^{n,m}$ to L^m in $\mathbf{L^p}(\tilde\mu)$, $1 \leq p < 2$, we see by definition of $K^{n,m,+}$ and $K^{m,+}$ that $K_T^{n,m,+}$ converges strongly to $K_T^{m,+}$ in $\mathbf{L^p}(\mathcal{F}_T)$, when n goes to infinity. Moreover, since $(K_T^{n,m,+})_n$ is uniformly bounded in $\mathbf{L^2}(\mathcal{F}_T)$, it also converges weakly to $K_T^{m,+}$ in $\mathbf{L^2}(\mathcal{F}_T)$. It follows that $(K^{m,+})_m$ inherits from $(K^{n,m,+})_{n,m}$ the uniform estimate in $\mathbf{S^2}$. □

We can now state the main result of this section as a consequence of the monotonic convergence theorem stated in Appendix 4.7.2, which extends to the Brownian-Poisson filtration framework the result of Peng and Xu [88].

Theorem 4.3.1. *Let assumptions* **(H0)** *and* **(H1)** *hold. Then there exists a minimal solution* $(Y, Z, L, K^+, K^-) \in \mathbf{S^2}\times\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde\mu)\times\mathbf{K^2}\times\mathbf{K^2}$ *to the reflected BSDE with nonpositive jumps* (4.2.2)-(4.2.3)-(4.2.4)-(4.2.5)*, where :*

(i) *Y is the increasing limit of $(Y^m)_m$.*

(ii) *(Z, L) is the strong (resp. weak) limit of $(Z^m, L^m)_m$ in $\mathbf{L^p(W)} \times \mathbf{L^p}(\tilde{\mu})$, with $p \in [1, 2)$, (resp. in $\mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu})$).*

(iii) *K_t^+ is the weak limit of $(K_t^{m,+})_m$ in $\mathbf{L^2}(\mathcal{F}_t)$, and K_t^- is the strong limit of $(K_t^{m,-})_m$ in $\mathbf{L^2}(\mathcal{F}_t)$, for any $0 \leq t \leq T$.*

Proof. We already know that $(Y^m)_m$ is a nondecreasing sequence in $\mathbf{S^2}$, which converges to some Y, which satisfies $\underline{Y} \leq Y \leq \bar{Y}$ from (4.3.8) and (4.3.9), and so lies in $\mathbf{S^2}$. By Lemma 4.3.1 and Corollary 4.3.1, we then see that the sequence $(Y^m, Z^m, L^m, K^{m,+}, K^{m,-})_m$ solution to the BSDE (4.3.3) satisfies all the conditions of the monotonic limit Theorem 4.7.3. This provides the existence of $(Z, L, K^+, K^-) \in \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2} \times \mathbf{K^2}$ as in the assertions (ii) and (iii) of Theorem 4.3.1 such that the quintuple (Y, Z, L, K^+, K^-) solves (4.2.2).

From the strong convergence in $\mathbf{L^1}(\tilde{\mu})$ of $(L^m)_m$ to L, and since $\lambda(A) < \infty$, we have

$$\mathbb{E}\Big[\int_0^T \int_A \big(L_t^m(a)\big)_+ \lambda(da)dt\Big] \quad \longrightarrow \quad \mathbb{E}\Big[\int_0^T \int_A \big(L_t(a)\big)_+ \lambda(da)dt\Big],$$

as m goes to infinity. Moreover, since $K_T^{m,+} = m\int_0^T (L_t(a))_+ \lambda(da)dt$ is bounded in m in $\mathbf{L^2}(\mathcal{F}_T)$, this implies that

$$\mathbb{E}\Big[\int_0^T \int_A \big(L_t(a)\big)_+ \lambda(da)dt\Big] \quad = \quad 0,$$

which means that the constraint (4.2.3) is satisfied. The upper reflection (4.2.4) is obviously satisfied from (4.3.4) and by sending m to infinity. Let us now check the Skorohod reflecting condition (4.2.5). We recall from (4.3.5) that $\int_0^T (U_{t^-} - Y_{t^-}^m) dK_t^{m,-} = 0$. Together with the fact that $U_{t^-} - Y_{t^-}^m \geq U_{t^-} - Y_{t^-} \geq 0$, this yields $\int_0^T (U_{t^-} - Y_{t^-}) dK_t^{m,-} = 0$. Since $(K_t^{m,-})_m$ converges strongly to K_t^- in $\mathbf{L^2}(\mathcal{F}_t)$ for all t, and by Lemma 4.3.1, this implies that the measure $dK^{m,-}$ converges weakly to dK^-, and so $\int_0^T (U_{t^-} - Y_{t^-}) dK_t^- = 0$ a.s.

It remains to prove the minimality condition. Let $(\tilde{Y}, \tilde{Z}, \tilde{L}, \tilde{K}^+, \tilde{K}^-)$ be another solution to the reflected BSDE with nonpositive jumps (4.2.2)-(4.2.3)-(4.2.4)-(4.2.5). We then see that $\int_0^t \int_A (\tilde{L}_s(a))_+ \lambda(da)ds = 0$, and thus $F(t, \tilde{Y}_t, \tilde{Z}_t, \tilde{L}_t) = F_m(t, \tilde{Y}_t, \tilde{Z}_t, \tilde{L}_t)$, for $0 \leq t \leq T$. From the comparison Theorem 4.7.2, we deduce that $Y_t^m \leq \tilde{Y}_t$, $0 \leq t \leq T$. Taking the limit with respect to m, this proves the minimality condition : $Y_t \leq \tilde{Y}_t$, $0 \leq t \leq T$. □

Remark 4.3.2. The order of the limits : first let n to infinity, and then let m to infinity, is crucial in our approach. Indeed, by sending first n to infinity, we get a nondecreasing sequence of processes $(K^{m,-})_m$ (see Lemma 4.3.1), which is a required property for applying the monotonic convergence theorem in Theorem 4.3.1. On the other hand, if we would first let m to infinity in the double sequence $(Y^{n,m}, Z^{n,m}, L^{n,m}, K^{n,m,+}, K^{n,m,-})$, then we would obtain a minimal solution $(\hat{Y}^n, \hat{Z}^n, \hat{K}^{n,+})$ to the BSDE with nonpositive jumps :

$$\begin{aligned}
\hat{Y}_t^n \quad &= \quad \xi + \int_t^T F(s, \hat{Y}_s^n, \hat{Z}_s^n, \hat{L}_s^n)ds - n\int_t^T (U_s - \hat{Y}_s^n)_- ds + \hat{K}_T^{n,+} - \hat{K}_t^{n,+} \\
&\quad - \int_t^T \hat{Z}_s^n dW_s - \int_t^T \int_A \hat{L}_s^n(a)\mu(ds, da), \quad 0 \leq t \leq T, \qquad (4.3.22)
\end{aligned}$$

$$\hat{L}^n_t(a) \quad \leq \quad 0, \quad d\mathbb{P}\otimes dt\otimes\lambda(da) \ a.e.$$

and $(\hat{Y}^n)_n$ is a nonincreasing sequence, converging to some $\hat{Y} \geq Y$ by (4.3.2). But neither $K^{n,+}$, which is the weak limit of $K^{n,m,+}$, as m goes to infinity, nor $K^{n,-}_t := n\int_0^t (U_s - \hat{Y}^n_s)_- ds$, satisfy monotonicity properties in n, which prevents to apply the monotonic convergence theorem to the sequence $(\hat{Y}^n, \hat{Z}^n, \hat{K}^{n,+}, \hat{K}^{n,-})_n$, and thus to identify $\hat{Y} = Y$ as the minimal solution to the reflected BSDE with nonpositive jumps. This differs from the case of doubly reflected BSDEs where one can send indifferently first m or n to infinity. □

4.4 Dual game representation

In this section, we consider the case where the generator $F(t,\omega)$ does not depend on y, z, ℓ, and we provide a dual game representation of the minimal solution to the reflected BSDE with nonpositive jumps in terms of a family of equivalent probability measures and discount factors. In addition to the set of probability measures $\mathbb{P}^\nu$, $\nu \in \mathcal{V} = \cup_m \mathcal{V}_m$ defined in the previous section, let us introduce for any $n \in \mathbb{N}$, the set Θ_n of $\mathbb{F}$-progressively measurable processes valued in $[0,n]$, and set $\Theta = \cup_n \Theta_n$, which shall represent the set of discount processes. Inspired by Proposition 6.2 in [31] and the dual representation in Section 4 of [75], we prove an explicit representation formula for the minimal solution to the reflected BSDE with nonpositive jumps.

Proposition 4.4.1. (i) *For any $n \in \mathbb{N}$ and $m \in \mathbb{N}\setminus\{0\}$, the solution to the penalized BSDE (4.3.1) admits the following dual representation formula :*

$$Y^{n,m}_t \; = \; \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} \operatorname*{ess\,inf}_{\theta\in\Theta_n} G_t(\nu,\theta) \; = \; \operatorname*{ess\,inf}_{\theta\in\Theta_n} \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} G_t(\nu,\theta),$$

for all $0 \leq t \leq T$, where

$$G_t(\nu,\theta) \; := \; \mathbb{E}^\nu\Big[e^{-\int_t^T \theta_s ds}\xi + \int_t^T e^{-\int_t^s \theta_r dr}(F(s) + \theta_s U_s)ds \mid \mathcal{F}_t\Big].$$

(ii) *Under assumptions* **(H0)** *and* **(H1)**, *the minimal solution to the reflected BSDE with nonpositive jumps (4.2.2)-(4.2.3)-(4.2.4)-(4.2.5) is explicitly represented as :*

$$Y_t \; = \; \operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \operatorname*{ess\,inf}_{\theta\in\Theta} G_t(\nu,\theta), \quad 0 \leq t \leq T. \tag{4.4.1}$$

Proof. (i) Fix $n \in \mathbb{N}$ and $m \in \mathbb{N}\setminus\{0\}$. For $\theta \in \Theta$, by applying Itô's rule to the product of the processes $e^{-\int_0^{\cdot}\theta_s ds}$ and $Y^{n,m}$ in (4.3.1), and by introducing the compensated measure $\tilde{\mu}^\nu(dt,da)$ under $\mathbb{P}^\nu$ for $\nu \in \mathcal{V}$, we obtain :

$$\begin{aligned} Y^{n,m}_t \; = \; & e^{-\int_t^T \theta_s ds}\xi + \int_t^T e^{-\int_t^s \theta_r dr}(F(s) + \theta_s U_s)ds \\ & + \int_t^T \int_A e^{-\int_t^s \theta_r dr}(m(L^{n,m}_s(a))_+ - \nu_s(a) L^{n,m}_s(a))\lambda(da)ds \\ & - \int_t^T e^{-\int_t^s \theta_r dr}(n(U_s - Y^{n,m}_s)_- + \theta_s(U_s - Y^{n,m}_s))ds \end{aligned}$$

$$
- \int_t^T e^{-\int_t^s \theta_r dr} Z_s^{n,m} dW_s - \int_t^T \int_A e^{-\int_t^s \theta_r dr} L_s^{n,m}(a) \tilde{\mu}^{\nu}(ds, da).
$$

By same arguments as in (4.3.18) (see also Lemma 4.2 in [75]), we can check that the $\mathbb{P}^{\nu}$ local martingales $\{\int_t^s e^{-\int_t^u \theta_r dr} Z_u^{n,m} dW_u, t \leq s \leq T\}$ and $\{\int_t^s \int_A e^{-\int_t^u \theta_r dr} L_u^{n,m}(a) \tilde{\mu}^{\nu}(du, da), t \leq s \leq T\}$ are actually uniformly integrable $\mathbb{P}^{\nu}$-martingales, so that by taking conditional expectation under $\mathbb{P}^{\nu}$:

$$
\begin{aligned}
Y_t^{n,m} \;=\; & G_t(\nu, \theta) + \mathbb{E}^{\nu}\Big[\int_t^T \int_A e^{-\int_t^s \theta_r dr}(m(L_s^{n,m}(a))_+ - \nu_s(a) L_s^{n,m}(a))\lambda(da)ds \\
& - \int_t^T e^{-\int_t^s \theta_r dr}(n(U_s - Y_s^{n,m})_- + \theta_s(U_s - Y_s^{n,m}))ds|\mathcal{F}_t\Big] \qquad (4.4.2)
\end{aligned}
$$

and this relation holds for any $\nu \in \mathcal{V}$, and $\theta \in \Theta$. Now, observe that for any $\nu \in \mathcal{V}_m$, hence valued in $(0, m]$, we have

$$
m(L_t^{n,m}(a))_+ - \nu_t(a) L_t^{n,m}(a) \quad \geq \quad 0, \quad 0 \leq t \leq T, \; a \in A, \; a.s.
$$

and for $\nu = \nu^{\varepsilon} \in \mathcal{V}_m$ defined by : $\nu_t^{\varepsilon}(a) = m 1_{\{L_t^{n,m}(a) \geq 0\}} + \varepsilon 1_{\{L_t^{n,m}(a) < 0\}}$, for arbitrary $\varepsilon \in (0, m]$, we have

$$
m(L_t^{n,m}(a))_+ - \nu_t^{\varepsilon}(a) L_t^{n,m}(a) \quad = \quad -\varepsilon L_t^{n,m}(a) 1_{\{L_t^{n,m}(a) < 0\}}, \quad 0 \leq t \leq T, \; a \in A, \; a.s.
$$

Similarly, for any $\theta \in \Theta_n$, hence valued in $[0, n]$, we have

$$
n(U_t - Y_t^{n,m})_- + \theta_t(U_t - Y_t^{n,m}) \quad \geq \quad 0, \quad 0 \leq t \leq T, \; a.s.
$$

and for $\theta^* \in \Theta_n$ defined by : $\theta_t^* = n 1_{\{Y_t^{n,m} \geq U_t\}}$, we have

$$
n(U_t - Y_t^{n,m})_- + \theta_t^*(U_t - Y_t^{n,m}) \quad = \quad 0, \quad 0 \leq t \leq T, \; a.s.
$$

Therefore, by (4.4.2), we get

$$
\begin{aligned}
G_t(\nu, \theta^*) \;\leq\; Y_t^{n,m} \;&=\; G_t(\nu^{\varepsilon}, \theta^*) + \varepsilon R_t^{n,m,\varepsilon}(\theta^*), \quad \forall \nu \in \mathcal{V}_m, \qquad (4.4.3) \\
&\leq\; G_t(\nu^{\varepsilon}, \theta) + \varepsilon R_t^{n,m,\varepsilon}(\theta), \\
&\leq\; G_t(\nu^{\varepsilon}, \theta) + \varepsilon R_t^{n,m,\varepsilon}(0), \quad \forall \theta \in \Theta_n, \qquad (4.4.4)
\end{aligned}
$$

for all $\varepsilon \in (0, m]$, where we set :

$$
R_t^{n,m,\varepsilon}(\theta) \quad := \quad \mathbb{E}^{\nu^{\varepsilon}}\Big[\int_t^T \int_A e^{-\int_t^s \theta_r dr} |L_s^{n,m}(a)| \lambda(da) ds | \mathcal{F}_t\Big].
$$

For fixed m, and by viewing the BSDE (4.3.1) as a penalized BSDE in n for the upper-reflected BSDE with generator F_m in (4.3.6), we have by standard arguments based on Itô's lemma, uniform estimates in n for $(Y^{n,m}, Z^{n,m}, L^{n,m})$ in $\mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu})$ (see Theorem 4.2 in [41]). Actually, these arguments show that for all $0 \leq t \leq T$, there exists some real-valued $\mathcal{F}_t$-measurable random variable C_t^m such that

$$
\sup_{n \in \mathbb{N}} \mathbb{E}\Big[\int_t^T \int_A |L_s^{n,m}(a)|^2 \lambda(da) ds | \mathcal{F}_t\Big] \quad \leq \quad C_t^m. \qquad (4.4.5)
$$

Moreover, since $\nu^\varepsilon \leq m$, we see as in (4.3.20) that $\zeta_T^{\nu^\varepsilon}/\zeta_t^{\nu^\varepsilon} \leq e^{m(T-t)\lambda(A)}\zeta_T^m/\zeta_t^m$, where ζ^m is the Radon-Nikodym density of $d\mathbb{P}^\nu/d\mathbb{P}$ for $\nu = m$. Thus, by Cauchy-Schwarz inequality, there exists some real-valued $\mathcal{F}_t$-measurable random variable $\tilde{C}_t^m$ such that

$$\sup_{n\in\mathbb{N}} R_t^{n,m,\varepsilon}(0) \quad \leq \quad \tilde{C}_t^m, \tag{4.4.6}$$

for all $\varepsilon \in (0, m]$. Now, by (4.4.3), we have : $\operatorname*{ess\,inf}_{\theta\in\Theta_n} \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} G_t(\nu,\theta) \leq Y_t^{n,m}$, and by (4.4.4), we get :

$$Y_t^{n,m} \quad \leq \quad \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} \operatorname*{ess\,inf}_{\theta\in\Theta_n} G_t(\nu,\theta) + \varepsilon R_t^{n,m,\varepsilon}(0).$$

By (4.4.6), we see in particular that $\varepsilon R_t^{n,m,\varepsilon}(0) \to 0$ a.s. as ε goes to zero. Since we always have $\operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} \operatorname*{ess\,inf}_{\theta\in\Theta_n} G_t(\nu,\theta) \leq \operatorname*{ess\,inf}_{\theta\in\Theta_n} \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} G_t(\nu,\theta)$, this shows that

$$\begin{aligned} Y_t^{n,m} \quad = \quad \lim_{\varepsilon\to 0} G_t(\nu^\varepsilon,\theta^*) \quad &= \quad \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} \operatorname*{ess\,inf}_{\theta\in\Theta_n} G_t(\nu,\theta) \\ &= \quad \operatorname*{ess\,inf}_{\theta\in\Theta_n} \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} G_t(\nu,\theta), \end{aligned} \tag{4.4.7}$$

i.e. $(\nu^\varepsilon, \theta^*) \in \mathcal{V}_m \times \Theta_n$ is an ε-saddle point for $G_t(\nu,\theta)$.

(ii) By sending m to infinity into (4.4.7), and recalling that $Y^m = \lim_n Y^{n,m}$, we get :

$$Y_t^m \quad = \quad \operatorname*{ess\,inf}_{\theta\in\Theta} \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} G_t(\nu,\theta) \;\geq\; \operatorname*{ess\,sup}_{\nu\in\mathcal{V}_m} \operatorname*{ess\,inf}_{\theta\in\Theta} G_t(\nu,\theta). \tag{4.4.8}$$

On the other hand, for arbitrary $n_0 \in \mathbb{N}$, we see that for any $\theta \in \Theta_{n_0}$ and any $n \geq n_0$:

$$n(U_t - Y_t^{n,m})_- + \theta_t(U_t - Y_t^{n,m}) \quad \geq \quad 0, \quad 0 \leq t \leq T, \; a.s.,$$

which implies, from (4.4.2),

$$\begin{aligned} Y_t^{n,m} \quad \leq \quad & G_t(\nu,\theta) \\ & + \mathbb{E}^\nu\Big[\int_t^T \int_A e^{-\int_t^s \theta_r dr}\big(m(L_s^{n,m}(a))_+ - \nu_s(a)L_s^{n,m}(a)\big)\lambda(da)ds\big|\mathcal{F}_t\Big], \end{aligned} \tag{4.4.9}$$

for any $\nu \in \mathcal{V}$, $\theta \in \Theta_{n_0}$, and $n \geq n_0$. Now note that, since $L^{n,m} \to L^m$ strongly in $\mathbf{L}^\mathbf{p}(\tilde{\mu})$, $p \in [1,2)$, then, up to a subsequence, $L^{n,m} \to L^m$ $d\mathbb{P} \otimes dt \otimes \lambda(da)$ almost everywhere. Moreover, as already recalled in step (i) of the proof, we have uniform estimates in n for $(L^{n,m}) \in \mathbf{L}^\mathbf{2}(\tilde{\mu})$, namely, from (4.4.5) with $t = 0$,

$$\sup_{n\in\mathbb{N}} \mathbb{E}\Big[\int_0^T \int_A |L_s^{n,m}(a)|^2\lambda(da)ds\Big] \quad \leq \quad C_0^m, \tag{4.4.10}$$

for some positive constant C_0^m. Then, sending n to infinity in (4.4.9) we obtain, from Lebesgue's dominated convergence theorem,

$$\begin{aligned} Y_t^m \quad \leq \quad & G_t(\nu,\theta) \\ & + \mathbb{E}^\nu\Big[\int_t^T \int_A e^{-\int_t^s \theta_r dr}\big(m(L_s^m(a))_+ - \nu_s(a)L_s^m(a)\big)\lambda(da)ds\big|\mathcal{F}_t\Big], \end{aligned} \tag{4.4.11}$$

for any $\nu \in \mathcal{V}$, $\theta \in \Theta_{n_0}$. Since $\Theta = \cup_n \Theta_n$, from the arbitrariness of n_0 we conclude that (4.4.11) remains true for all $\theta \in \Theta$. Take $\tilde{\nu}^\varepsilon \in \mathcal{V}_m$ defined by : $\tilde{\nu}_t^\varepsilon(a) = m 1_{\{L_t^m(a) \geq 0\}} + \varepsilon 1_{\{L_t^m(a) < 0\}}$, for arbitrary $\varepsilon \in (0, m]$, so that

$$m(L_t^m(a))_+ - \nu_t^\varepsilon(a) L_t^m(a) \quad = \quad -\varepsilon L_t^m(a) 1_{\{L_t^m(a)<0\}}, \quad 0 \leq t \leq T, \ a \in A, \ a.s.,$$

and thus by (4.4.11) :

$$Y_t^m \quad \leq \quad G_t(\tilde{\nu}^\varepsilon, \theta) + \varepsilon \tilde{R}_t^{m,\varepsilon}(\theta) \quad \leq \quad G_t(\tilde{\nu}^\varepsilon, \theta) + \varepsilon \tilde{R}_t^{m,\varepsilon}(0), \qquad \forall \theta \in \Theta, \tag{4.4.12}$$

for all $\varepsilon \in (0, m]$, where we set :

$$\tilde{R}_t^{m,\varepsilon}(\theta) \quad := \quad \mathbb{E}^{\tilde{\nu}^\varepsilon}\Big[\int_t^T \int_A e^{-\int_t^s \theta_r dr} |L_s^m(a)| \lambda(da) ds | \mathcal{F}_t \Big].$$

Using again the uniform estimate (4.4.10) and the fact that, up to a subsequence, $L^{n,m} \to L^m$ $d\mathbb{P} \otimes dt \otimes \lambda(da)$ a.e., we obtain, from (4.4.5) and Lebesgue's dominated convergence theorem,

$$\mathbb{E}\Big[\int_t^T \int_A |L_s^m(a)|^2 \lambda(da) ds | \mathcal{F}_t \Big] \quad \leq \quad C_t^m.$$

Moreover, as in step (i) of the proof, since $\tilde{\nu}^\varepsilon \leq m$ we see that $\zeta_T^{\tilde{\nu}^\varepsilon} / \zeta_t^{\tilde{\nu}^\varepsilon} \leq e^{m(T-t)\lambda(A)} \zeta_T^m / \zeta_t^m$. Thus, by Cauchy-Schwarz inequality, it follows that, for all $\varepsilon \in (0, m]$,

$$\tilde{R}_t^{m,\varepsilon}(0) \quad \leq \quad \tilde{C}_t^m,$$

with the same real-valued $\mathcal{F}_t$-measurable random variable $\tilde{C}_t^m$ as in (4.4.6). Then, from (4.4.12) we get

$$Y_t^m \quad \leq \quad \underset{\nu \in \mathcal{V}_m}{\text{ess sup}} \, \underset{\theta \in \Theta}{\text{ess inf}} \, G_t(\nu, \theta) + \varepsilon \tilde{C}_t^m,$$

for all $\varepsilon \in (0, m]$. By sending ε to zero, and combining with (4.4.8), we obtain :

$$\begin{aligned} Y_t^m \quad &= \quad \underset{\theta \in \Theta}{\text{ess inf}} \, \underset{\nu \in \mathcal{V}_m}{\text{ess sup}} \, G_t(\nu, \theta) \\ &= \quad \underset{\nu \in \mathcal{V}_m}{\text{ess sup}} \, \underset{\theta \in \Theta}{\text{ess inf}} \, G_t(\nu, \theta). \end{aligned} \tag{4.4.13}$$

Finally, by sending m to infinity into (4.4.13), we obtain the dual relation (4.4.1) for $Y = \lim_m Y^m$. □

Remark 4.4.1. We don't know in general if one can switch in (4.4.1) the essential infimum and supremum. Actually, by considering $\hat{Y}^n = \lim_m Y^{n,m}$ the minimal solution to the BSDE with nonnegative jumps (4.3.22), one could show by similar arguments as in the second part (ii) of Proposition 4.4.1 that :

$$\hat{Y}_t^n \quad = \quad \underset{\theta \in \Theta_n}{\text{ess inf}} \, \underset{\nu \in \mathcal{V}}{\text{ess sup}} \, G_t(\nu, \theta) \quad = \quad \underset{\nu \in \mathcal{V}}{\text{ess sup}} \, \underset{\theta \in \Theta_n}{\text{ess inf}} \, G_t(\nu, \theta),$$

so that $\hat{Y} := \lim_n \hat{Y}^n$ satisfies :

$$\hat{Y}_t \quad = \quad \underset{\theta \in \Theta}{\text{ess inf}} \, \underset{\nu \in \mathcal{V}}{\text{ess sup}} \, G_t(\nu, \theta).$$

However, as pointed out in Remark 4.3.2, we cannot conclude whether $\hat{Y}_t$ is equal or strictly greater than Y_t. □

4.5 Connection with HJB Isaacs equation for controller-and-stopper games

In this section, we show how the minimal solution to our class of reflected BSDEs with nonpositive jumps provides a probabilistic representation (hence a Feynman-Kac formula) to fully nonlinear variational inequalities of Hamilton-Jacobi-Bellman (HJB) Isaacs type arising in a controller/stopper game, when considering a suitable Markovian framework.

4.5.1 The Markovian framework

We are given two measurable functions $b : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^d$ and $\sigma : \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}^{d\times d}$ and we introduce the forward Markov regime-switching process (X, I) in $\mathbb{R}^d \times \mathbb{R}^q$ governed by :

$$dX_t = b(X_t, I_t)dt + \sigma(X_t, I_t)dW_t \tag{4.5.1}$$

$$dI_t = \int_A (a - I_{t^-})\mu(dt, da). \tag{4.5.2}$$

Therefore, the coefficients b and σ, appearing in the dynamics of the diffusion process X, change according to the pure jump process I, which is associated to the Poisson random measure μ on $\mathbb{R}_+ \times A$. We make the following standard assumption on the forward coefficients b and σ :

(HFC) There exists a constant C such that

$$|b(x, a) - b(x', a')| + |\sigma(x, a) - \sigma(x', a')| \le C(|x - x'| + |a - a'|),$$

for all $x, x' \in \mathbb{R}^d$ and $a, a' \in \mathbb{R}^q$.

It is well-known that under hypothesis **(HFC)** there exists a unique solution $(X^{t,x,a}, I^{t,a})$ $= (X_s^{t,x,a}, I_s^{t,a})_{t\le s\le T}$ to (4.5.1)-(4.5.2) starting from $(x, a) \in \mathbb{R}^d \times \mathbb{R}^q$ at time $s = t \in [0, T]$. Furthermore, we have the standard estimates : for all $p \ge 2$, there exists some constant C_p such that

$$\mathbb{E}\Big[\sup_{t\le s\le T}(|X_s^{t,x,a}|^p + |I_s^{t,a}|^p)\Big] \le C_p(1 + |x|^p + |a|^p), \tag{4.5.3}$$

for all $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$.

Remark 4.5.1. Notice that the constant C_p in (4.5.3) depends only on p, T, and the growth linear condition of b, σ in **(HFC)**. Since the dynamics (4.5.1) of X is not changed by the change of probability measure $\mathbb{P}^\nu$, $\nu \in \mathcal{V}$ (recall that W remains a Brownian motion under $\mathbb{P}^\nu$), we then see that for all $p \ge 2$:

$$\mathbb{E}^\nu\Big[\sup_{s\le r\le T}(|X_r^{t,x,a}|^p + |I_r^{t,a}|^p)|\mathcal{F}_s\Big] \le C_p(1 + |X_s^{t,x,a}|^p + |I_s^{t,a}|^p), \quad t \le s \le T,$$

for all $\nu \in \mathcal{V}$, and thus :

$$\int_t^T \mathbb{E}\Big[\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\big[\sup_{s\le r\le T}(|X_r^{t,x,a}|^p + |I_r^{t,a}|^p)|\mathcal{F}_s\big]\Big]ds \le C_p(1 + |x|^p + |a|^p), \tag{4.5.4}$$

for all $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$. □

Regarding the reflected BSDE with nonpositive jumps, the terminal condition, the generator function, and the barrier are given respectively by some continuous functions $g : \mathbb{R}^d \to \mathbb{R}$, $f : \mathbb{R}^d \times \mathbb{R}^q \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$, and $u : [0,T] \times \mathbb{R}^d \to \mathbb{R}$. We make the following assumptions on the BSDE coefficients :

(HBC)

(i) The functions g, $f(\cdot,\cdot,0,0)$ and u satisfy a polynomial growth condition :

$$\sup_{x\in\mathbb{R}^d, a\in\mathbb{R}^q} \frac{|f(x,a,0,0)|}{1+|x|^h+|a|^h} + \sup_{t\in[0,T], x\in\mathbb{R}^d} \frac{|g(x)|+|u(t,x)|}{1+|x|^h} \quad < \quad \infty,$$

for some $h \geq 0$.

(ii) There exists some constant C such that :

$$|f(x,a,y,z) - f(x,a,y',z')| \quad \leq \quad C(|y-y'|+|z-z'|),$$

for all $x \in \mathbb{R}^d$, $a \in \mathbb{R}^q$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$.

(iii) $u(T,x) \geq g(x)$, for all $x \in \mathbb{R}^d$, and there exists a nonincreasing sequence of functions $(u^k)_k$ lying in $C^{1,2}([0,T] \times \mathbb{R}^d)$, and converging pointwisely to u such that the following polynomial growth condition holds

$$\sup_{k\in\mathbb{N}} \sup_{t\in[0,T], x\in\mathbb{R}^d} \frac{\left|\frac{\partial u^k}{\partial t}(t,x)\right| + |D_x u^k(t,x)| + |D_x^2 u^k(t,x)|}{1+|x|^h} \quad < \quad \infty,$$

for some $h \geq 0$.

In this Markovian framework, the reflected BSDE with nonpositive jumps (4.2.2)-(4.2.3)-(4.2.4)-(4.2.5) takes the form :

$$\begin{aligned} Y_t \quad = \quad & g(X_T) + \int_t^T f(X_s, I_s, Y_s, Z_s) ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) \\ & - \int_t^T Z_s dW_s - \int_t^T \int_A L_s(a) \mu(ds, da), \quad 0 \leq t \leq T, \ a.s. \end{aligned} \tag{4.5.5}$$

with

$$L_t(a) \quad \leq \quad 0\,, \qquad d\mathbb{P} \otimes dt \otimes \lambda(da) \ \ a.e. \tag{4.5.6}$$

and

$$Y_t \quad \leq \quad u(t, X_t)\,, \qquad 0 \leq t \leq T, \ a.s. \tag{4.5.7}$$

$$\int_0^T (u(t,X_t) - Y_{t^-}) dK_t^- \quad = \quad 0\,, \qquad a.s. \tag{4.5.8}$$

Notice that under **(HFC)** and **(HBC)** the terminal condition $\xi(\omega) = g(X_T(\omega))$, the generator $F(t,\omega,y,z,\ell) = f(X_t(\omega), I_{t^-}(\omega), y, z)$, and the barrier $U_t(\omega) = u(t, X_t(\omega))$ clearly satisfy the standing assumptions 1-4 in Section 2. Let us now discuss about conditions **(H0)** and **(H1)** in the two following remarks.

Remark 4.5.2. Condition **(H0)** is satisfied in our Markovian framework. Actually, it is shown in Lemma 5.1 in [75] that under **(HFC)** and **(HBC)**(i), (ii), there exists for any initial condition $(t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$, a solution $\{(\bar Y_s^{t,x,a}, \bar Z_s^{t,x,a}, \bar L_s^{t,x,a}, \bar K_s^{t,x,a,+}), t \leq s \leq T\}$ to the BSDE with nonpositive jumps (4.2.6)-(4.2.7) when $(X,I) = \{(X_s^{t,x,a}, I_s^{t,a}), t \leq s \leq T\}$, with $\bar Y_s^{t,x,a} = \bar v(s, X_s^{t,x,a})$ for some deterministic function $\bar v$ on $[0,T] \times \mathbb{R}^d$ satisfying the polynomial growth condition :

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{|\bar v(t,x)|}{1+|x|^r} \quad < \quad \infty$$

for some $r \geq 2$. Such solution is constructed by Itô's lemma from a smooth supersolution to

$$\begin{aligned} -\frac{\partial \bar v}{\partial t} - \sup_{a\in A}[\mathcal{L}^a \bar v + f(\cdot, a, \bar v, \sigma^\intercal(\cdot,a) D_x \bar v)] \quad &\geq \quad 0, \quad \text{on } [0,T) \times \mathbb{R}^d \\ \bar v(T,x) \quad &\geq \quad g(x), \qquad x \in \mathbb{R}^d, \end{aligned}$$

where

$$\mathcal{L}^a \varphi \quad = \quad b(x,a).D_x\varphi + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a) D_x^2 \varphi),$$

which can be chosen equal to $\bar v(t,x) = \bar C e^{\rho(T-t)}(1+|x|^r)$, with $r = \max(2,h)$, for $\bar C$ and ρ positive large enough. □

Remark 4.5.3. We also observe that assumption **(H1)** is satisfied in the present framework. More precisely, given an initial condition $(t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$, let us consider the process U^k, $k \in \mathbb{N}$, defined by :

$$U_s^k \quad := \quad u^k(s, X_s^{t,x,a}), \quad t \leq s \ \leq T.$$

By Itô's formula, U^k is in the form of condition **(H1)**(ii), with

$$\begin{aligned} \upsilon_s^k \quad &= \quad \frac{\partial u^k}{\partial t}(s, X_s^{t,x,a}) + b(X_s^{t,x,a}, I_s^{t,a}).D_x u^k(s, X_s^{t,x,a}) \\ &\quad + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(X_s^{t,x,a}, I_s^{t,a}) D_x^2 u^k(s, X_s^{t,x,a})), \\ \vartheta_s^k \quad &= \quad D_x u^k(s, X_s^{t,x,a})^\intercal \sigma(X_s^{t,x,a}, I_s^{t,a}), \end{aligned}$$

for all $t \leq s \leq T$, a.s., and we clearly see from **(HFC)**, **(HBC)**(iii), and (4.5.3) that

$$\mathbb{E}\Big[\int_t^T |\upsilon_s^k|^2 ds\Big] + \mathbb{E}\Big[\int_t^T |\vartheta_s^k|^2 ds\Big] \quad < \quad \infty.$$

Moreover, by using (4.5.4), and again from the polynomial growth conditions on b, σ, F and u^k in **(HFC)**, **(HBC)**, there exists some $p > 2$ such that

$$\begin{aligned} &\sup_{k\in\mathbb{N}} \int_t^T \mathbb{E}\Big[\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\big[\sup_{s\leq r\leq T} (|U_r^k|^p + |\upsilon_r^k|^p + |\vartheta_r^k|^p)|\mathcal{F}_s\big]\Big] ds \\ &\quad + \int_t^T \mathbb{E}\Big[\operatorname*{ess\,sup}_{\nu\in\mathcal{V}} \mathbb{E}^\nu\big[\sup_{s\leq r\leq T} |f(X_r^{t,x,a}, I_r^{t,a}, 0, 0)|^p|\mathcal{F}_s\big]\Big] ds \quad \leq \quad C_p(1+|x|^p+|a|^p). \end{aligned}$$

for all $(t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$. □

From Theorem 4.3.1, we get, for any initial condition $(t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$, the existence of a minimal solution $\{(Y_s^{t,x,a}, Z_s^{t,x,a}, L_s^{t,x,a}, K_s^{t,x,a,+}, K_s^{t,x,a,-}), t \le s \le T\}$ to the Markovian reflected BSDE with nonpositive jumps (4.5.5)-(4.5.6)-(4.5.7)-(4.5.8) when $(X,I) = \{(X_s^{t,x,a}, I_s^{t,a}), t \le s \le T\}$. Moreover, as we shall see in the next paragraph, this minimal solution is written in this Markovian context as : $Y_s^{t,x,a} = v(s, X_s^{t,x,a}, I_s^{t,a})$, where v is a real-valued deterministic function defined on $[0,T] \times \mathbb{R}^d \times \mathbb{R}^q$ by

$$v(t,x,a) \;:=\; Y_t^{t,x,a}, \qquad (t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q. \tag{4.5.9}$$

We aim at proving that this function v does not depend actually on the argument a in the interior of A, and is connected to the fully nonlinear variational inequality of HJB Isaacs type :

$$\max\Big[-\frac{\partial v}{\partial t} - \sup_{a\in A}\big(\mathcal{L}^a v + f(\cdot,a,v,\sigma^{\intercal}(\cdot,a)D_x v)\big); v-u\Big] \;=\; 0, \;\; \text{on } [0,T) \times \mathbb{R}^d \tag{4.5.10}$$

$$v(T,x) \;=\; g(x), \quad x \in \mathbb{R}^d. \tag{4.5.11}$$

4.5.2 Viscosity property of the penalized BSDE

Let us consider the Markovian penalized BSDE associated to (4.5.5)-(4.5.6)-(4.5.7)-(4.5.8)

$$\begin{aligned} Y_t^{n,m} \;=\;& g(X_T) + \int_t^T f(X_s, I_s, Y_s^{n,m}, Z_s^{n,m})ds \\ &+ m\int_t^T\int_A (L_s^{n,m}(a))_+\lambda(da)ds - n\int_t^T (u(s,X_s) - Y_s^{n,m})_- ds \\ &- \int_t^T Z_s^{n,m} dW_s - \int_t^T\int_A L_s^{n,m}(a)\mu(ds,da), \quad 0 \le t \le T, \end{aligned} \tag{4.5.12}$$

and denote by $\{(Y_s^{n,m,t,x,a}, Z_s^{n,m,t,x,a}, L_s^{n,m,t,x,a}), t \le s \le T\}$ the unique solution to (4.5.12) when $(X,I) = \{(X_s^{t,x,a}, I_s^{t,a}), t \le s \le T\}$ for any initial condition $(t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$. From the Markov property of the jump-diffusion process (X,I), we recall from [3] that $Y_s^{n,m,t,x,a} = v^{n,m}(s, X_s^{t,x,a}, I_s^{t,a})$, $t \le s \le T$, for some deterministic function $v^{n,m}$ defined on $[0,T] \times \mathbb{R}^d \times \mathbb{R}^q$ by

$$v^{n,m}(t,x,a) \;:=\; Y_t^{n,m,t,x,a}, \qquad (t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q. \tag{4.5.13}$$

Next, for fixed m, let us consider the limiting BSDE of (4.5.12) as n goes to infinity, that is the reflected BSDE :

$$\begin{aligned} Y_t^m \;=\;& g(X_T) + \int_t^T f(X_s, I_s, Y_s^m, Z_s^m)ds + m\int_t^T\int_A (L_s^m(a))_+\lambda(da)ds \\ &- (K_T^{m,-} - K_t^{m,-}) - \int_t^T Z_s^m dW_s - \int_t^T\int_A L_s^m(a)\mu(ds,da), \qquad 0 \le t \le T, \ a.s. \end{aligned} \tag{4.5.14}$$

and

$$Y_t^m \;\le\; u(t,X_t), \qquad 0 \le t \le T, \ a.s. \tag{4.5.15}$$

$$\int_0^T (u(t,X_t) - Y_{t^-}^m)dK_t^{m,-} \;=\; 0, \qquad a.s. \tag{4.5.16}$$

and denote by $\{(Y_s^{m,t,x,a}, Z_s^{m,t,x,a}, L_s^{m,t,x,a}, K_s^{m,t,x,a,+}), t \leq s \leq T\}$ the unique solution to (4.5.14)-(4.5.15)-(4.5.16) when $(X, I) = \{(X_s^{t,x,a}, I_s^{t,a}), t \leq s \leq T\}$ for any initial condition $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$. Since $Y^{n,m,t,x,a}$ converges to $Y^{m,t,x,a}$ as n goes to infinity, we see from (4.5.13) that $Y^{m,t,x,a}$ may be written as $Y_s^{m,t,x,a} = v^m(s, X_s^{t,x,a}, I_s^{t,a})$, $t \leq s \leq T$, where v^m is the deterministic function defined on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ by :

$$v^m(t, x, a) \;:=\; \lim_{n\to\infty} v^{n,m}(t, x, a) \;=\; Y_t^{m,t,x,a}, \quad (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q. \tag{4.5.17}$$

From the convergence of $Y^{m,t,x,a}$ to the minimal solution $Y^{t,x,a}$, when m goes to infinity, as stated in Theorem 4.3.1, we deduce that $Y^{t,x,a}$ has indeed the form $Y_s^{t,x,a} = v(s, X_s^{t,x,a}, I_s^{t,a})$, with a deterministic function v defined as the pointwise (nondecreasing) limit of $(v^m)_m$:

$$v(t, x, a) \;:=\; \lim_{m\to\infty} v^m(t, x, a) \;=\; Y_t^{t,x,a}, \quad (t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q. \tag{4.5.18}$$

From the bounds (4.3.8)-(4.3.9), we have for all $m \in \mathbb{N}$: $\underline{v}(t, x, a) \leq v^m(t, x, a) \leq \bar{v}(t, x)$, $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, where $\underline{v} := v^0$ is associated to the reflected BSDE Y^m for $m = 0$, and $\bar{v}$ is the supersolution as defined in Remark 4.5.2. By the polynomial growth condition on $\bar{v}$, and also on $\underline{v}$ (see e.g. Lemma 3.2 in [34]), we deduce that v^m, and thus also v by passing to the limit, satisfy a polynomial growth condition : there exist some positive constant C and some $p \geq 2$, such that, for all $m \in \mathbb{N}$:

$$|v^m(t, x, a)| + |v(t, x, a)| \leq C(1 + |x|^p + |a|^p), \tag{4.5.19}$$

for all $(t, x, a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$. As expected, for fixed m, the function $v^m = v^m(t, x, a)$ associated to the reflected BSDE with jumps (4.5.14)-(4.5.15)-(4.5.16) is connected to the integro-differential variational inequality :

$$\begin{aligned}\max\Big[-\frac{\partial v^m}{\partial t} - b(x,a).D_x v^m - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D_x^2 v^m) - f(x,a,v^m,\sigma^\intercal(x,a)D_x v^m) \quad (4.5.20)\\ -\int_A (v^m(t,x,a') - v^m(t,x,a))\lambda(da') - m\int_A (v^m(t,x,a') - v^m(t,x,a))_+\lambda(da');\\ v^m(t,x,a) - u(t,x)\Big] \;=\; 0,\end{aligned}$$

for $(t, x, a) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}^q$, together with the terminal condition :

$$v^m(T, x, a) \;=\; g(x), \qquad (x, a) \in \mathbb{R}^d \times \mathbb{R}^q. \tag{4.5.21}$$

More precisely, we have the following result, which may be proved by extending to the multidimensional case Lemma 3.1 and Theorem 3.4 of [34], and by using Theorem 4.7.1 as comparison theorem for BSDEs with jumps.

Proposition 4.5.1. *Let assumptions* **(HFC)** *and* **(HBC)** *hold. The function v^m in (4.5.17) is a continuous viscosity solution to (4.5.20)-(4.5.21), i.e., it is continuous on $[0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, a viscosity supersolution (resp. subsolution) to (4.5.21), i.e.*

$$v^m(T, x, a) \;\geq\; (resp.\ \leq)\; g(x)$$

for any $(x, a) \in \mathbb{R}^d \times \mathbb{R}^q$, and a viscosity supersolution (resp. subsolution) to (4.5.20), i.e.

$$\max\Big[-\frac{\partial \varphi}{\partial t}(t, x, a) - b(x, a).D_x\varphi(t, x, a) - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x, a)D_x^2\varphi(t, x, a)) \tag{4.5.22}$$

$$-f(x,a,v^m(t,x,a),\sigma^\intercal(x,a)D_x\varphi(t,x,a)) - \int_A (\varphi(t,x,a') - \varphi(t,x,a))\lambda(da')$$
$$-m\int_A (\varphi(t,x,a') - \varphi(t,x,a))_+\lambda(da') \; ; \; v^m(t,x,a) - u(t,x)] \quad \geq \quad (\textit{resp.} \; \leq) \quad 0$$

for any $(t,x,a) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^q$ *and any* $\varphi \in C^{1,2}([0,T] \times (\mathbb{R}^d \times \mathbb{R}^q))$ *such that*

$$(v^m - \varphi)(t,x,a) \quad = \quad \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v^m - \varphi) \quad (\textit{resp.} \max_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v^m - \varphi)). \quad (4.5.23)$$

Remark 4.5.4. Notice that

$$v^m(t,x,a) \quad \leq \quad u(t,x), \qquad \text{for all } (t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q. \qquad (4.5.24)$$

Indeed, for any $(t,x,a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$, since $Y_s^{m,t,x,a} = v^m(s, X_s^{t,x,a}, I_s^{t,a})$, $t \leq s \leq T$, we deduce, from (4.5.15) that

$$\mathbb{E}\Big[\frac{1}{s-t}\int_t^s (v^m(r, X_r^{t,x,a}, I_r^{t,a}) - u(r, X_r^{t,x,a}))dr\Big] \quad \leq \quad 0$$

for all $t < s \leq T$. Since $(X^{t,x,a}, I^{t,a})$ is càdlàg, in particular it is right-continuous at time t. Therefore, (4.5.24) follows from the continuity of v^m and u. □

4.5.3 HJB Isaacs equation

This paragraph is devoted to the derivation of the equation satisfied in the viscosity sense by the function v in (4.5.18), by passing to the limit, as m goes to infinity, in the equation satisfied by v^m. The first step is to prove that v does not depend on a, which is basically a consequence of the nonpositive jump constraint :

$$L_s^{t,x,a}(a') \quad = \quad v(s, X_s^{t,x,a}, a') - v(s, X_s^{t,x,a}, I_{s^-}^{t,a}) \;\leq\; 0, \qquad d\mathbb{P} \otimes ds \otimes \lambda(da') \; a.e.$$

providing that the function v is continuous. However, as we do not know a priori that the function v is continuous, we shall rely on (discontinuous) viscosity solutions arguments as in [75], and make the following conditions on the set A and the intensity measure λ :

(HA) The interior set $\mathring{A}$ of A is connex, and $A = \mathrm{Adh}(\mathring{A})$, the closure of its interior.

(Hλ)

(i) The measure λ supports the whole set $\mathring{A}$: for any $a \in \mathring{A}$ and any open neighborhood $\mathcal{O}$ of a in $\mathbb{R}^q$ we have $\lambda(\mathcal{O} \cap \mathring{A}) > 0$.

(ii) The boundary of A : $\partial A = A\backslash\mathring{A}$, is negligible with respect to λ, i.e., $\lambda(\partial A) = 0$.

Proposition 4.5.2. *Let assumptions* **(HFC)**, **(HBC)**, **(HA)**, *and* **(Hλ)** *hold. Then the function v does not depend on the variable a on $[0,T) \times \mathbb{R}^d \times \mathring{A}$:*

$$v(t,x,a) \quad = \quad v(t,x,a'), \quad a, a' \in \mathring{A}, \qquad (4.5.25)$$

for all $(t,x) \in [0,T) \times \mathbb{R}^d$.

Proof. The proof borrows most arguments from section 5.3 in [75], and we only report here the main steps and the points to be modified. First, we see from (4.5.24), and sending m to infinity that :

$$v \quad \leq \quad u \quad \text{on } [0,T] \times \mathbb{R}^d \times \mathbb{R}^q. \tag{4.5.26}$$

We next show that the function v is a viscosity supersolution to :

$$-|D_a v(t,x,a)| \quad = \quad 0, \quad (t,x,a) \in [0,T) \times \mathbb{R}^d \times \mathring{A}, \tag{4.5.27}$$

i.e., for any $(t,x,a) \in [0,T) \times \mathbb{R}^d \times \mathring{A}$ and any function $\varphi \in C^{1,2}([0,T] \times (\mathbb{R}^d \times \mathbb{R}^q))$ such that $(v-\varphi)(t,x,a) = \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v-\varphi)$, we have

$$-|D_a\varphi(t,x,a)| \quad \geq \quad 0, \quad \text{i.e.} \quad D_a\varphi(t,x,a) \;=\; 0.$$

Indeed, let $(t,x,a) \in [0,T) \times \mathbb{R}^d \times \mathring{A}$ and $\varphi \in C^{1,2}([0,T] \times (\mathbb{R}^d \times \mathbb{R}^q))$ such that $0 = (v-\varphi)(t,x,a) = \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v-\varphi)$. We distinguish two cases.
(i) $v(t,x,a) = u(t,x)$. From (4.5.26), we have

$$\varphi(t,x,a') \quad \leq \quad v(t,x,a') \quad \leq \quad u(t,x), \qquad \forall\, a' \in \mathbb{R}^q$$

and $\varphi(t,x,a) = v(t,x,a) = u(t,x)$. It follows that $\varphi(t,x,a) = \max_{a'\in\mathbb{R}^q}\varphi(t,x,a')$, which yields : $D_a\varphi(t,x,a) = 0$, since $a \in \mathring{A}$.
(ii) $v(t,x,a) < u(t,x)$. We may assume, without loss of generality, that φ satisfies the polynomial growth condition $\sup_{(t,x,a)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}\frac{|\varphi(t,x,a)|}{1+|x|^p+|a|^p} < \infty$, with p as in (4.5.19). Then, for any $\varepsilon > 0$, consider the test function

$$\varphi^\varepsilon(t',x',a') \quad = \quad \varphi(t',x',a') - \varepsilon\big(|t'-t|^2 + |x'-x|^{2p} + |a'-a|^{2p}\big),$$

for all $(t',x',a') \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$. Since $\varphi^\varepsilon(t,x,a) = \varphi(t,x,a)$ and $\varphi^\varepsilon \leq \varphi$, with equality if and only if $(t',x',a') = (t,x,a)$, we see that

$$(v-\varphi^\varepsilon)(t,x,a) \quad = \quad \text{strict} \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v-\varphi^\varepsilon).$$

From the continuity and the growth conditions of v^m and φ, we see that there exists a bounded sequence $(t_m,x_m,a_m)_m$ (we omit the dependence on ε) in $[0,T] \times \mathbb{R}^d \times \mathbb{R}^q$ such that

$$(v^m-\varphi^\varepsilon)(t_m,x_m,a_m) \quad = \quad \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v^m-\varphi^\varepsilon).$$

By standard arguments, we obtain, up to a subsequence,

$$(t_m,x_m,a_m,v^m(t_m,x_m,a_m)) \quad \overset{m\to\infty}{\longrightarrow} \quad (t,x,a,v(t,x,a)).$$

From the viscosity supersolution property of v^m to (4.5.22) at (t_m,x_m,a_m), we find

$$\begin{aligned}
-\frac{\partial\varphi^\varepsilon}{\partial t}(t_m,x_m,a_m) - \mathcal{L}^{a_m}\varphi^\varepsilon(t_m,x_m,a_m) \\
-f(x_m,a_m,v^m(t_m,x_m,a_m),\sigma^\intercal(x_m,a_m)D_x\varphi^\varepsilon(t_m,x_m,a_m)) \\
-\int_A(\varphi^\varepsilon(t_m,x_m,a')-\varphi^\varepsilon(t_m,x_m,a_m))\lambda(da')
\end{aligned}$$

$$-m \int_A \big(\varphi^\varepsilon(t_m, x_m, a') - \varphi^\varepsilon(t_m, x_m, a_m)\big)_+ \lambda(da') \;\; \geq \;\; 0.$$

By sending m to infinity, and then ε to zero, we conclude as in the proof of Lemma 5.3 in [75] that : $\int_A \big(\varphi(t,x,a') - \varphi(t,x,a)\big)_+ \lambda(da') = 0$, which means under **(H$\lambda$)** that $\varphi(t,x,a)$ $= \max_{a' \in \mathbb{R}^q} \varphi(t,x,a')$, i.e., $D_a\varphi(t,x,a) = 0$.

Finally, by arguing exactly as in Lemma 5.4 and Proposition 5.2 of [75], we obtain under the additional condition **(HA)** the non dependence of v on $a \in \mathring{A}$ from the viscosity supersolution property to (4.5.27). □

From Proposition 4.5.2, we can define by misuse of notation the function v on $[0,T) \times \mathbb{R}^d$ by :

$$v(t,x) \;\; = \;\; v(t,x,a), \quad (t,x) \in [0,T) \times \mathbb{R}^d,$$

for any $a \in \mathring{A}$, and we see that v satisfies a polynomial growth condition when x goes to infinity by (4.5.19). We finally state the viscosity property of v to the HJB Isaacs type equation (4.5.10)-(4.5.11). Recall the definition of lower semicontinuous envelope v_*, and upper semicontinuous envelope v^* :

$$v_*(t,x) \;\; = \;\; \liminf_{\substack{(t',x') \to (t,x) \\ t' < T}} v(t',x') \qquad \text{and} \qquad v^*(t,x) \;\; = \;\; \limsup_{\substack{(t',x') \to (t,x) \\ t' < T}} v(t',x'),$$

for all $(t,x) \in [0,T] \times \mathbb{R}^d$.

Theorem 4.5.1. *Let assumptions* **(HFC)**, **(HBC)**, **(HA)**, *and* **(Hλ)** *hold. Then v is a viscosity solution to* (4.5.10)-(4.5.11) *in the sense that it verifies :*
(i) Viscosity supersolution property :

$$v_*(T,x) \;\; \geq \;\; g(x), \tag{4.5.28}$$

for any $x \in \mathbb{R}^d$, and

$$\max\Big[-\frac{\partial \varphi}{\partial t}(t,x) - \sup_{a \in A}\Big(\mathcal{L}^a\varphi(t,x) + f\big(x,a,v_*(t,x),\sigma^\intercal(x,a)D_x\varphi(t,x)\big)\Big); \quad v_*(t,x) - u(t,x)\Big] \;\; \geq \;\; 0 \tag{4.5.29}$$

for any $(t,x) \in [0,T) \times \mathbb{R}^d$ and any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ such that $(v_* - \varphi)(t,x) = \min_{[0,T]\times\mathbb{R}^d}(v_* - \varphi)$
(ii) Viscosity subsolution property :

$$v^*(T,x) \;\; \leq \;\; g(x), \tag{4.5.30}$$

for any $x \in \mathbb{R}^d$, and

$$\max\Big[-\frac{\partial \varphi}{\partial t}(t,x) - \sup_{a \in A}\Big(\mathcal{L}^a\varphi(t,x) + f\big(x,a,v^*(t,x),\sigma^\intercal(x,a)D_x\varphi(t,x)\big)\Big); \quad v^*(t,x) - u(t,x)\Big] \;\; \leq \;\; 0 \tag{4.5.31}$$

for any $(t,x) \in [0,T) \times \mathbb{R}^d$ and any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ such that $(v^* - \varphi)(t,x) = \max_{[0,T]\times\mathbb{R}^d}(v^* - \varphi)$.

Proof. The proof is quite similar to the proof detailed in Section 5.4 of [75], and we report only the main arguments and the points to be modified with respect to the proof in [75].

• *Viscosity supersolution property* (4.5.29) : Since v is the pointwise limit of the nondecreasing sequence of continuous functions (v^m), and recalling (4.5.25), we know (see e.g. [2]) that v is lower semicontinuous and so :

$$v(t,x) \;=\; v_*(t,x) \;=\; \lim_{m\to\infty} v^m(t,x,a), \qquad \forall (t,x,a) \in [0,T]\times\mathbb{R}^d\times\mathring{A}.$$

Fix now $(t,x) \in [0,T)\times\mathbb{R}^d$, and let $\varphi \in C^{1,2}([0,T]\times\mathbb{R}^d)$ such that $(v_*-\varphi)(t,x) = \min_{[0,T]\times\mathbb{R}^d}(v_*-\varphi)$. We already know from (4.5.26) that $v_* \leq u$, and so distinguish two cases :

(1) $v_*(t,x) = u(t,x)$, then the viscosity supersolution property of v at (t,x) is obviously satisfied.

(2) We have $v(t,x) = v_*(t,x) < u(t,x)$. We may assume, without loss of generality, that φ satisfies $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\frac{|\varphi(t,x)|}{1+|x|^p} < \infty$, with p as in (4.5.19). Then, take $a \in \mathring{A}$ and consider, for any $\varepsilon > 0$, the test function

$$\varphi^\varepsilon(t',x',a') \;=\; \varphi(t',x') - \varepsilon\big(|t'-t|^2 + |x'-x|^{2p} + |a'-a|^{2p}\big),$$

for all $(t',x',a') \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^q$. Proceeding as in the proof of Proposition 4.5.2, step (ii), we can find a bounded sequence $(t_m,x_m,a_m)_m$ (we omit the dependence on ε) in $[0,T]\times\mathbb{R}^d\times\mathbb{R}^q$ such that

$$(v^m-\varphi^\varepsilon)(t_m,x_m,a_m) \;=\; \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v^m-\varphi^\varepsilon)$$

and, up to a subsequence,

$$(t_m,x_m,a_m,v^m(t_m,x_m,a_m)) \;\overset{m\to\infty}{\longrightarrow}\; (t,x,a,v(t,x)).$$

Therefore, recalling that $v(t,x) < u(t,x)$ and using the continuity of u, we see that $v^m(t_m,x_m,a_m) < u(t_m,x_m)$ for m large enough. As a consequence, from the viscosity supersolution property (4.5.22) of v^m at (t_m,x_m,a_m) with the test function φ^ε, we then get :

$$\begin{aligned}
&-\frac{\partial\varphi^\varepsilon}{\partial t}(t_m,x_m,a_m) - \mathcal{L}^{a_m}\varphi^\varepsilon(t_m,x_m,a_m)\\
&-f(x_m,a_m,v^m(t_m,x_m,a_m),\sigma^\intercal(x_m,a_m)D_x\varphi^\varepsilon(t_m,x_m,a_m))\\
&-\int_A(\varphi^\varepsilon(t_m,x_m,a') - \varphi^\varepsilon(t_m,x_m,a_m))\lambda(da')\\
&-m\int_A(\varphi^\varepsilon(t_m,x_m,a') - \varphi^\varepsilon(t_m,x_m,a_m))_+\lambda(da') \;\geq\; 0.
\end{aligned}$$

By sending firstly m to infinity, and afterwards ε to zero, then using that a is arbitrary in $\mathring{A}$, together with the continuity of the coefficients b, σ, and f in the variable a, we obtain the required viscosity supersolution inequality :

$$-\frac{\partial\varphi}{\partial t}(t,x) - \sup_{a\in A}\Big(\mathcal{L}^a\varphi(t,x) + f(x,a,v_*(t,x),\sigma^\intercal(x,a)D_x\varphi(t,x))\Big) \;\geq\; 0.$$

• *Viscosity subsolution property* (4.5.31) : By (4.5.26), we have : $v^* \leq u$ on $[0,T) \times \mathbb{R}^d$, and so it remains to show the viscosity subsolution property of v to :

$$-\frac{\partial v}{\partial t} - \sup_{a \in A} \Big(\mathcal{L}^a v(t,x) + f(x,a,v(t,x),\sigma^\intercal(x,a) D_x v(t,x)) \Big) \quad \leq \quad 0.$$

This follows by same arguments as in [75] from the viscosity subsolution property of v^m to :

$$-\frac{\partial v^m}{\partial t} - b(x,a).D_x v^m - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a) D_x^2 v^m) - f(x,a,v^m,\sigma^\intercal(x,a)D_x v^m)$$
$$- \int_A (v^m(t,x,a') - v^m(t,x,a))\lambda(da') - m\int_A \big(v^m(t,x,a') - v^m(t,x,a)\big)_+ \lambda(da') \quad \leq \quad 0,$$

and by sending m to infinity under **(H**λ**)**(ii).

• Finally, the viscosity supersolution and subsolution inequalities (4.5.28), (4.5.30) are proved by same arguments as in [75]. □

Remark 4.5.5. *Zero-sum controller/stopper game*
Let us consider the particular and important case where the generator $f(x,a)$ does not depend on (y,z), and $u(t,x) = g(x)$. In this case, the nonlinear variational inequality (4.5.10)-(4.5.11) is the HJB Isaacs equation associated to the following zero-sum controller-and-stopper game : let us introduce the controlled diffusion process in $\mathbb{R}^d$

$$dX_s^\alpha \quad = \quad b(X_s^\alpha,\alpha_s)ds + \sigma(X_s^\alpha,\alpha_s)dW_s, \tag{4.5.32}$$

where the control $\alpha \in \mathcal{A}$ is an $\mathbb{F}^W$-progressively measurable process, valued in A, affecting both drift and diffusion coefficient, possibly degenerate. Here $\mathbb{F}^W$ denotes the natural filtration generated by the Brownian motion W. Notice that the laws $\mathbb{P}^\alpha$ of X^α under $\mathbb{P}$, for α varying in $\mathcal{A}$, belong to a non dominated set of probability measures. Given $(t,x) \in [0,T] \times \mathbb{R}^d$, and $\alpha \in \mathcal{A}$, we denote by $\{X_s^{t,x,\alpha}, t \leq s \leq T\}$ the solution to (4.5.32) starting from x at $s = t$. Let us also define $\mathcal{T}_{t,T}$ as the set of all $\mathbb{F}^W$-stopping times valued in $[t,T]$ for $0 \leq t \leq T$, and consider $\Pi_{t,T}$ the set of stopping strategies $\pi : \mathcal{A} \mapsto \mathcal{T}_{t,T}$ satisfying a non-anticipative condition as defined in [7]. The upper and lower value functions of the controller/stopper game are given by :

$$\overline{V}(t,x) \quad := \quad \inf_{\pi \in \Pi_{t,T}} \sup_{\alpha \in \mathcal{A}} \mathbb{E}\Big[\int_t^{\pi[\alpha]} f(X_s^{t,x,\alpha},\alpha_s)ds + g(X_{\pi[\alpha]}^{t,x,\alpha})\Big],$$
$$\underline{V}(t,x) \quad := \quad \sup_{\alpha \in \mathcal{A}} \inf_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}\Big[\int_t^{\tau} f(X_s^{t,x,\alpha},\alpha_s)ds + g(X_{\tau}^{t,x,\alpha})\Big], \qquad (t,x) \in [0,T] \times \mathbb{R}^d.$$

It is shown in [7] that this game has a value, i.e., $\overline{V} = \underline{V} = V$, and that V is the unique viscosity solution to (4.5.10)-(4.5.11) satisfying a polynomial growth condition. By combining this result with Theorem 4.5.1, this shows that $v = V$. In other words, we have provided a representation of HJB Isaacs equation, arising in zero-sum controller/stopper game, including control on possibly degenerate diffusion coefficient, in terms of minimal solution to reflected BSDE with nonpositive jumps. Furthermore, by combining with the dual game representation in Proposition 4.4.1, we obtain an original representation for the value function of the controller-and-stopper game :

$$\inf_{\pi \in \Pi_{0,T}} \sup_{\alpha \in \mathcal{A}} \mathbb{E}\Big[\int_0^{\pi[\alpha]} f(X_t^\alpha,\alpha_t)dt + g(X_{\pi[\alpha]}^\alpha)\Big] \;=\; \sup_{\alpha \in \mathcal{A}} \inf_{\tau \in \mathcal{T}_{0,T}} \mathbb{E}\Big[\int_0^{\tau} f(X_t^\alpha,\alpha_t)dt + g(X_\tau^\alpha)\Big]$$

$$= \sup_{\nu \in \mathcal{V}} \inf_{\theta \in \Theta} \mathbb{E}^{\nu}\Big[\int_0^T e^{-\int_0^t \theta_s ds}(f(X_t, I_t) + \theta_t g(X_t))dt + e^{-\int_0^T \theta_t dt} g(X_T)\Big].$$

□

4.6 Conclusion

We introduced in this chapter a class of reflected BSDEs with nonpositive jumps and upper obstacle, and showed in the Markov case its connection with fully nonlinear variational inequalities arising typically in controller-and-stopper games with control both on drift and diffusion term. Such representation suggests an original approach for probabilistic numerical schemes of HJB Isaacs equations by discretization and simulation of this reflected BSDE with nonpositive jumps. From a theoretical point of view, an open problem is to relate this class of BSDEs to general controller-and-stopper games in the non Markovian case. A variation of our class of BSDEs would be to consider reflected BSDEs with nonpositive jumps and lower obstacle, which is related to $\sup \sup$ problem over control and stopping time, and in other words to optimal stopping under nonlinear expectation. Actually, the proof of existence of a minimal solution by a double penalization approach is simpler since it would involve the sum (instead of the difference) of two nondecreasing processes. Another possible extension is the class of doubly reflected BSDEs with nonpositive jumps motivated by Dynkin games under nonlinear expectation (see [81]).

4.7 Appendix

4.7.1 Comparison theorems for sub and supersolutions to BSDEs with jumps

We provide in this section two comparison theorems for BSDEs with jumps. We first recall a comparison theorem for sub and supersolutions to BSDEs driven by the Brownian motion W and the Poisson random measure μ, for which we refer to Theorem 4.2 in [92] (see also Section 4.3 in [92] and Theorem 2.5 in [95]).

Theorem 4.7.1. *Let $\xi^1, \xi^2 \in \mathbf{L^2}(\mathcal{F}_T)$ be two terminal conditions and let $F^1, F^2 : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^d \times \mathbf{L^2}(\lambda) \to \mathbb{R}$ be two generators satisfying the assumptions 2.(i)-(iii) of Section 2. Let $(Y^1, Z^1, L^1, K^{1,-}) \in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2}$ satisfying*

$$\begin{aligned} Y_t^1 &= \xi^1 + \int_t^T F^1(s, Y_s^1, Z_s^1, L_s^1)ds - (K_T^{1,-} - K_t^{1,-}) \\ &\quad - \int_t^T Z_s^1 dW_s - \int_t^T \int_A L_s^1(a)\mu(ds, da), \qquad 0 \le t \le T, \ a.s. \end{aligned} \tag{4.7.1}$$

and $(Y^2, Z^2, L^2, K^{2,+}) \in \mathbf{S^2} \times \mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2}$ satisfying

$$\begin{aligned} Y_t^2 &= \xi^2 + \int_t^T F^2(s, Y_s^2, Z_s^2, L_s^2)ds + K_T^{2,+} - K_t^{2,+} \\ &\quad - \int_t^T Z_s^2 dW_s - \int_t^T \int_A L_s^2(a)\mu(ds, da), \qquad 0 \le t \le T, \ a.s. \end{aligned} \tag{4.7.2}$$

If $F^1(t,Y^1_t,Z^1_t,L^1_t)\le F^2(t,Y^1_t,Z^1_t,L^1_t)$ *(resp.* $F^1(t,Y^2_t,Z^2_t,L^2_t)\le F^2(t,Y^2_t,Z^2_t,L^2_t)$*),* $d\mathbb{P}\otimes dt$ *a.e., and* $\xi^1\le\xi^2$ *a.s., then*

$$Y^1_t\le Y^2_t,\qquad 0\le t\le T,\ a.s.$$

We now state a comparison theorem between a Skorohod solution and a Skorohod supersolution, both driven by the Brownian motion W and the Poisson random measure μ. This slightly extends Theorem 5.2 in [41].

Theorem 4.7.2. *Let* $\xi^1,\xi^2\in\mathbf{L^2}(\mathcal{F}_T)$ *be two terminal conditions and let* $F^1,F^2:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d\times\mathbf{L^2}(\lambda)\to\mathbb{R}$ *be two generators satisfying assumptions 2.(i)-(iii) of Section 2. Let* $(Y^1,Z^1,L^1,K^{1,-})\in\mathbf{S^2}\times\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde\mu)\times\mathbf{K^2}$ *satisfying*

$$\begin{aligned}Y^1_t &= \xi^1+\int_t^T F^1(s,Y^1_s,Z^1_s,L^1_s)ds-(K^{1,-}_T-K^{1,-}_t)\\ &\quad -\int_t^T Z^1_s dW_s-\int_t^T\int_A L^1_s(a)\mu(ds,da)\ ,\qquad 0\le t\le T,\ a.s.\end{aligned}\tag{4.7.3}$$

and

$$\begin{aligned}Y^1_t &\le U_t\ ,\qquad 0\le t\le T,\ a.s.\\ \int_0^T(U_{t^-}-Y^1_{t^-})dK^{1,-}_t &= 0\ ,\qquad a.s.\end{aligned}$$

Furthermore, let $(Y^2,Z^2,L^2,K^{2,+},K^{2,-})\in\mathbf{S^2}\times\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde\mu)\times\mathbf{K^2}\times\mathbf{K^2}$ *satisfying*

$$\begin{aligned}Y^2_t &= \xi^2+\int_t^T F^2(s,Y^2_s,Z^2_s,L^2_s)ds+K^{2,+}_T-K^{2,+}_t-(K^{2,-}_T-K^{2,-}_t)\\ &\quad -\int_t^T Z^2_s dW_s-\int_t^T\int_A L^2_s(a)\mu(ds,da)\ ,\qquad 0\le t\le T,\ a.s.\end{aligned}\tag{4.7.4}$$

and

$$\begin{aligned}Y^2_t &\le U_t\ ,\qquad 0\le t\le T,\ a.s.\\ \int_0^T(U_{t^-}-Y^2_{t^-})dK^{2,-}_t &= 0\ ,\qquad a.s.\end{aligned}$$

If $\xi^1\le\xi^2$ *a.s. and* $F^1(t,Y^1_t,Z^1_t,L^1_t)\le F^2(t,Y^1_t,Z^1_t,L^1_t)$*,* $d\mathbb{P}\otimes dt$ *a.e., then*

$$Y^1_t\le Y^2_t,\qquad 0\le t\le T,\ a.s.$$

Proof. Consider the following penalized BSDEs :

$$\begin{aligned}Y^{n,1}_t &= \xi^1+\int_t^T F^1(s,Y^{n,1}_s,Z^{n,1}_s,L^{n,1}_s)ds-n\int_t^T(U_s-Y^{n,1}_s)^-ds\\ &\quad -\int_t^T Z^{n,1}_s dW_s-\int_t^T\int_A L^{n,1}_s(a)\mu(ds,da)\end{aligned}$$

and

$$Y^{n,2}_t = \xi^2+\int_t^T F^2(s,Y^{n,2}_s,Z^{n,2}_s,L^{n,2}_s)ds+K^{2,+}_T-K^{2,+}_t-n\int_t^T(U_s-Y^{n,2}_s)^-ds$$

$$-\int_t^T Z_s^{n,2}dW_s - \int_t^T \int_A L_s^{n,2}(a)\mu(ds,da),$$

for all $0 \le t \le T$, almost surely. By comparison Theorem 4.7.1 we get $Y_t^{n,1} \le Y_t^{n,2}$, for all $n \in \mathbb{N}$. Recalling Remark 4.3.1, we have that $Y_t^{n,1}$ converges to Y_t^1. It remains to prove the convergence of $Y_t^{n,2}$ towards Y_t^2.

Set $\tilde{Y}^{n,2} := Y^{n,2} + K^{2,+}$, $\tilde{U} := U + K^{2,+}$, $\tilde{\xi}^2 := \xi^2 + K_T^{2,+}$, and $\tilde{F}^2(t,y,z,\ell) := F^2(t, y - K_t^{2,+}, z, \ell)$, for all $0 \le t \le T$, $y \in \mathbb{R}$, $z \in \mathbb{R}^d$, $\ell \in \mathbf{L^2}(\lambda)$, almost surely. Then

$$\begin{aligned}\tilde{Y}_t^{n,2} &= \tilde{\xi}^2 + \int_t^T \tilde{F}^2(s, \tilde{Y}_s^{n,2}, Z_s^{n,2}, L_s^{n,2})ds - n\int_t^T (\tilde{U}_s - \tilde{Y}_s^{n,2})^- ds \\ &\quad - \int_t^T Z_s^{n,2}dW_s - \int_t^T \int_A L_s^{n,2}(a)\mu(ds,da),\end{aligned}$$

for all $0 \le t \le T$, almost surely. Note that $\tilde{\xi}^2$ verifies the square integrability condition and $\tilde{F}^2$ satisfies assumptions 2.(i)-(iii) of Section 2. Moreover, $\tilde{U}_T \in \mathbf{S^2}$ and $\tilde{U}_T \ge \tilde{\xi}^2$, almost surely. Now, again from Remark 4.3.1, we have that $\tilde{Y}^{n,2}$ converges to $\tilde{Y}^2 = Y^2 + K^{2,+}$, and hence $Y^{n,2}$ converges to Y^2. □

4.7.2 Monotonic limit theorem for BSDEs with jumps

We state a monotonic limit theorem for BSDEs driven by the Brownian motion W and the Poisson random measure μ. This extends the monotonic limit Theorem 3.1 in [88] to the jump case.

Theorem 4.7.3. *Let $(Y^m, Z^m, L^m, K^{m,+}, K^{m,-})_m$ be a sequence in $\mathbf{S^2} \times \mathbf{L^2}(\mathbf{W}) \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2} \times \mathbf{K^2}$, with $K^{m,+}$ continuous, solution to :*

$$\begin{aligned}Y_t^m &= \xi + \int_t^T F(s, Y_s^m, Z_s^m, L_s^m)ds + K_T^{m,+} - K_t^{m,+} - (K_T^{m,-} - K_t^{m,-}) \quad (4.7.5)\\ &\quad - \int_t^T Z_s^m dW_s - \int_t^T \int_A L_s^m(a)\mu(ds,da), \qquad 0 \le t \le T,\ a.s.\end{aligned}$$

such that

$$\sup_{m\in\mathbb{N}} \left(\|Y^m\|_{\mathbf{S^2}} + \|Z^m\|_{\mathbf{L^2(W)}} + \|L^m\|_{\mathbf{L^2}(\tilde{\mu})} + \|K^{m,+}\|_{\mathbf{S^2}} + \|K^{m,-}\|_{\mathbf{S^2}} \right) < \infty, \quad (4.7.6)$$

and $(Y^m)_m$ converges increasingly to $Y \in \mathbf{S^2}$. Suppose also that the sequence $(K^{m,-})_m$ satisfies :

$$K_t^{m,-} - K_s^{m,-} \le K_t^{m+1,-} - K_s^{m+1,-}, \qquad 0 \le s \le t \le T,\ a.s. \qquad (4.7.7)$$

for all $m \in \mathbb{N}$. Then there exists $(Z, L, K^+, K^-) \in \mathbf{L^2}(\mathbf{W}) \times \mathbf{L^2}(\tilde{\mu}) \times \mathbf{K^2} \times \mathbf{K^2}$ such that

$$\begin{aligned}Y_t &= \xi + \int_t^T F(s, Y_s, Z_s, L_s)ds + K_T^+ - K_t^+ - (K_T^- - K_t^-) \qquad (4.7.8)\\ &\quad - \int_t^T Z_s dW_s - \int_t^T \int_A L_s(a)\mu(ds,da), \qquad 0 \le t \le T,\ a.s.\end{aligned}$$

Here (Z, L) is the strong (resp. weak) limit of $(Z^m, L^m)_m$ in $\mathbf{L^p}(\mathbf{W}) \times \mathbf{L^p}(\tilde{\mu})$, with $p \in [1,2)$, (resp. in $\mathbf{L^2}(\mathbf{W}) \times \mathbf{L^2}(\tilde{\mu})$). Furthermore, K_t^+ is the weak limit of $(K_t^{m,+})_m$ in $\mathbf{L^2}(\mathcal{F}_t)$, and $(K_t^{m,-})_m$ converges strongly up to K_t^- in $\mathbf{L^2}(\mathcal{F}_t)$, for any $0 \le t \le T$.

Proof. *Step 1. Limit BSDE.* From the boundedness condition (4.7.6) and the Hilbert structure of $\mathbf{L^2(W)}\times\mathbf{L^2}(\tilde{\mu})\times\mathbf{L^2(0,T)}$, there exists a subsequence, $(Z^{m_k}, L^{m_k}, F(\cdot, Y^{m_k}, Z^{m_k}, L^{m_k}))_k$ which converges weakly to some $(Z, L, G) \in \mathbf{L^2(W)}\times\mathbf{L^2}(\tilde{\mu})\times\mathbf{L^2(0,T)}$. Thus, for each stopping time $\tau \leq T$, the following weak convergences hold in $\mathbf{L^2}(\mathcal{F}_\tau)$ as $k \to \infty$:

$$\begin{aligned} \int_0^\tau F(s, Y_s^{m_k}, Z_s^{m_k}, L_s^{m_k})ds &\rightharpoonup \int_0^\tau G(s)ds, \\ \int_0^\tau Z_s^{m_k} dW_s &\rightharpoonup \int_0^\tau Z_s dW_s, \\ \int_0^\tau \int_A L_s^{m_k}(a)\mu(ds, da) &\rightharpoonup \int_0^\tau \int_A L_s(a)\mu(ds, da). \end{aligned}$$

From (4.7.7), there exists $K^- \in \mathbf{K^2}$, such that K_t^- is the strong limit of $(K_t^{m_k,-})_k$ in $\mathbf{L^2}(\mathcal{F}_t)$ for all $0 \leq t \leq T$. In particular, $K_\tau^{m_k,-} \rightharpoonup K_\tau^-$. Moreover, since

$$\begin{aligned} K_\tau^{m_k,+} &= Y_0^{m_k} - Y_\tau^{m_k} + K_\tau^{m_k,-} - \int_0^\tau F(s, Y_s^{m_k}, Z_s^{m_k}, L_s^{m_k})ds \\ &\quad + \int_0^\tau Z_s^{m_k} dW_s + \int_0^\tau \int_A L_s^{m_k}(a)\mu(ds, da). \end{aligned}$$

we also have the weak convergence in $\mathbf{L^2}(\mathcal{F}_\tau)$

$$\begin{aligned} K_\tau^{m_k,+} \rightharpoonup K_\tau^+ &:= Y_0 - Y_\tau + K_\tau^- - \int_0^\tau G(s)ds \\ &\quad + \int_0^\tau Z_s dW_s + \int_0^\tau \int_A L_s(a)\mu(ds, da), \end{aligned}$$

as $k \to \infty$. Note that $\mathbb{E}[(K_T^+)^2] < \infty$ and for any two stopping times $0 \leq \sigma \leq \tau \leq T$, we have $K_\sigma^+ \leq K_\tau^+$ since $K_\sigma^{m,+} \leq K_\tau^{m,+}$. From this it follows that K^+ is an increasing process. Observe now that we have obtained the following decomposition for Y :

$$Y_t = Y_0 - \int_0^\tau G(s)ds - K_t^+ + K_t^- + \int_0^t Z_s dW_s + \int_0^t \int_A L_s(a)\mu(ds, da). \quad (4.7.9)$$

Since the processes $K^{m_k,+}$ and $K^{m_k,-}$ are predictable, we deduce that K^+ and K^- are also predictable. Besides, by Lemmas 3.1 and 3.2 of [88], K^+, K^- and Y are càdlàg processes. Thus, in the above decomposition of Y in (4.7.9), the components Z and L are unique. As a matter of fact, the uniqueness of Z follows by identifying the Brownian parts and finite variation parts. The uniqueness of L is then obtained by identifying the predictable parts and by recalling that the jumps of μ are totally inaccessible. From the uniqueness of (Z, L), it follows that the whole sequence $(Z^m, L^m)_m$ converges weakly to (Z, L) in $\mathbf{L^2(W)} \times \mathbf{L^2}(\tilde{\mu})$.

Step 2. Properties of the process K^+. We establish that the contribution of the jumps of K^+ is mainly concentrated within a finite number of intervals with sufficiently small total length. More precisely, we apply Lemma 2.3 in [87] to K^+. Consequently, as in Lemma 2.3 in [87], for any $\delta, \varepsilon > 0$, there exists a finite number of pairs of stopping times (σ_k, τ_k), $k = 0, \ldots, N$, with $0 < \sigma_k \leq \tau_k \leq T$, such that all the intervals $(\sigma_k, \tau_k]$ are disjoint and

$$\mathbb{E}\sum_{k=0}^N (\tau_k - \sigma_k) \geq T - \frac{\varepsilon}{2}, \qquad \mathbb{E}\sum_{k=0}^N \sum_{\sigma_k < t \leq \tau_k} |\Delta K_t^+|^2 \leq \frac{\varepsilon\delta}{3}. \quad (4.7.10)$$

We should note that in [87] the filtration is Brownian, therefore it is continuous, and hence each stopping time σ_k can be approximated by a sequence of announceable stopping times. In our case the stopping times σ_k's are constructed as the successive times of jumps of the predictable process K^+ with size bigger than some given positive level, therefore each σ_k is a predictable stopping time and the approximation of σ_k by announceable stopping times is again possible. We can thus argue exactly the same way as in Lemma 2.3 in [87] to derive both estimates in (4.7.10).

Step 3. Strong convergence. By applying Itô's formula to $|Y^m_t - Y_t|^2$ on a subinterval $(\sigma, \tau]$, with $0 \leq \sigma \leq \tau \leq T$, two stopping times, and recalling that $K^{m,+}$ is continuous, we obtain :

$$\begin{aligned}
\mathbb{E}|Y^m_\tau - Y_\tau|^2 \;&=\; \mathbb{E}|Y^m_\sigma - Y_\sigma|^2 + \mathbb{E}\int_\sigma^\tau |Z^m_s - Z_s|^2 ds + \mathbb{E}\int_\sigma^\tau\int_A |L^m_s(a) - L_s(a)|^2\lambda(da)ds \\
&+ 2\mathbb{E}\int_\sigma^\tau (Y^m_s - Y_s)(G(s) - F(s, Y^m_s, Z^m_s, L^m_s))ds \\
&+ \mathbb{E}\sum_{t\in(\sigma,\tau]} |\Delta K^+_t - \Delta K^-_t + \Delta K^{m,-}_t|^2 \\
&+ 2\mathbb{E}\int_{(\sigma,\tau]} (Y^m_{s^-} - Y_{s-})dK^+_s - 2\mathbb{E}\int_{(\sigma,\tau]} (Y^m_{s^-} - Y_{s-})dK^-_s \\
&- 2\mathbb{E}\int_{(\sigma,\tau]} (Y^m_s - Y_s)dK^{m,+}_s + 2\mathbb{E}\int_{(\sigma,\tau]} (Y^m_{s^-} - Y_{s-})dK^{m,-}_s \\
&+ 2\mathbb{E}\int_{(\sigma,\tau]}\int_A (Y^m_s - Y_s)(L^m_s(a) - L_s(a))\lambda(da)ds. \qquad (4.7.11)
\end{aligned}$$

Now, let us write

$$\begin{aligned}
\int_{(\sigma,\tau]} (Y^m_{s^-} - Y_{s-})dK^+_s \;&=\; \int_{(\sigma,\tau]} (Y^m_{s^-} + \Delta K^{m,-}_s - Y_{s-} + \Delta K^+_s - \Delta K^-_s)dK^+_s \\
&- \sum_{t\in(\sigma,\tau]} (\Delta K^+_t)^2 + \sum_{t\in(\sigma,\tau]} \Delta K^+_t \Delta(K^-_s - K^{m,-}_s),
\end{aligned}$$

and observe that

$$\int_{(\sigma,\tau]} (Y^m_{s^-} - Y_{s-})d(K^-_s - K^{m,-}_s) \leq 0, \quad \text{and} \quad \int_{(\sigma,\tau]} (Y^m_s - Y_s)dK^{m,+}_s \leq 0.$$

Therefore, by using the inequality $2ab \geq -2b^2 - a^2/2$, we obtain from (4.7.11)

$$\begin{aligned}
&\mathbb{E}\int_\sigma^\tau |Z^m_s - Z_s|^2 ds + \frac{1}{2}\mathbb{E}\int_\sigma^\tau\int_A |L^m_s(a) - L_s(a)|^2\lambda(da)ds \qquad (4.7.12)\\
\leq\;& \mathbb{E}|Y^m_\tau - Y_\tau|^2 + 2\lambda(A)\mathbb{E}\int_\sigma^\tau |Y^m_s - Y_s|^2 ds \\
&+ 2\mathbb{E}\int_\sigma^\tau |Y^m_s - Y_s||G(s) - F(s, Y^m_s, Z^m_s, L^m_s)|ds \\
&-2\mathbb{E}\int_{(\sigma,\tau]} (Y^m_{s^-} + \Delta K^{m,-}_s - Y_{s-} + \Delta K^+_s - \Delta K^-_s)dK^+_s + 2\mathbb{E}\sum_{t\in(\sigma,\tau]} |\Delta K^+_t|^2 \\
&-2\mathbb{E}\sum_{t\in(\sigma,\tau]} \Delta K^+_t \Delta(K^-_s - K^{m,-}_s) - \mathbb{E}\sum_{t\in(\sigma,\tau]} |\Delta K^+_t - \Delta K^-_t + \Delta K^{m,-}_t|^2, \\
\leq\;& \mathbb{E}|Y^m_\tau - Y_\tau|^2 + 2\lambda(A)\mathbb{E}\int_\sigma^\tau |Y^m_s - Y_s|^2 ds
\end{aligned}$$

$$+2\mathbb{E}\int_{\sigma}^{\tau}|Y_s^m - Y_s||G(s) - F(s, Y_s^m, Z_s^m, L_s^m)|ds$$
$$-2\mathbb{E}\int_{(\sigma,\tau]}(Y_{s^-}^m + \Delta K_s^{m,-} - Y_{s^-} + \Delta K_s^+ - \Delta K_s^-)dK_s^+ + \mathbb{E}\sum_{t\in(\sigma,\tau]}|\Delta K_t^+|^2.$$

by using the inequality $2a^2 - 2ab - (a-b)^2 \le a^2$. We know that the first two terms on the right-hand side of (4.7.12) converge to zero as $m \to \infty$. The third term also tends to zero since $(G(\cdot) - F(\cdot, Y^m, Z^m, L^m))_m$ is bounded in $\mathbf{L^2(0,T)}$, and so by Cauchy-Schwarz inequality

$$\mathbb{E}\int_0^T |Y_s^m - Y_s||G(s) - F(s, Y_s^m, Z_s^m, L_s^m)|ds \quad \to \quad 0, \qquad \text{as } m \to \infty.$$

For the fourth term, since $K^{m,-}$ is predictable, the predictable projection of Y^m is ${}^pY_t^m = Y_{t^-}^m + \Delta K_t^{m,-}$. Similarly, from (4.7.9) and since K^+ and K^- are predictable processes, we see that ${}^pY_t = Y_{t^-} - \Delta K_t^+ + \Delta K_t^-$. By the dominated convergence theorem, we obtain

$$\lim_{m\to\infty}\mathbb{E}\int_{(\sigma,\tau]}(Y_{s^-}^m + \Delta K_s^{m,-} - Y_{s^-} + \Delta K_s^+ - \Delta K_s^-)dK_s^+ \quad = \quad 0.$$

For the last term in (4.7.12), we exploit the results in (4.7.10), regarding the contribution of the jumps of K^+. More precisely, we apply estimate (4.7.12) for each $\sigma = \sigma_k$ and $\tau = \tau_k$, with σ_k, τ_k defined in Step 2, and then take the sum over $k = 0, \dots, N$. It follows that

$$\sum_{k=0}^N \mathbb{E}\int_{\sigma_k}^{\tau_k}|Z_s^m - Z_s|^2 ds + \frac{1}{2}\sum_{k=0}^N \mathbb{E}\int_{\sigma_k}^{\tau_k}\int_A |L_s^m(a) - L_s(a)|^2\lambda(da)ds$$
$$\le \sum_{k=0}^N \mathbb{E}|Y_{\tau_k}^m - Y_{\tau_k}|^2 + 2\lambda(A)\mathbb{E}\int_0^T |Y_s^m - Y_s|^2 ds$$
$$+2\mathbb{E}\int_0^T |Y_s^m - Y_s||G(s) - F(s, Y_s^m, Z_s^m, L_s^m)|ds + \sum_{k=0}^N \mathbb{E}\sum_{t\in(\sigma_k,\tau_k]}|\Delta K_t^+|^2$$
$$-2\sum_{k=0}^N \mathbb{E}\int_{(\sigma_k,\tau_k]}(Y_{s^-}^m + \Delta K_s^{m,-} - Y_{s^-} + \Delta K_s^+ - \Delta K_s^-)dK_s^+.$$

From the above convergence results, we deduce that

$$\limsup_{m\to\infty}\left(\sum_{k=0}^N \mathbb{E}\int_{\sigma_k}^{\tau_k}|Z_s^m - Z_s|^2 ds + \frac{1}{2}\sum_{k=0}^N \mathbb{E}\int_{\sigma_k}^{\tau_k}\int_A |L_s^m(a) - L_s(a)|^2\lambda(da)ds\right)$$
$$\le \sum_{k=0}^N \mathbb{E}\sum_{t\in(\sigma_k,\tau_k]}|\Delta K_t^+|^2 \le \frac{\varepsilon\delta}{3}.$$

Therefore, following the same steps as in the proof of Theorem 2.1 in [87], we deduce that the sequences $(Z^m)_m$ and $(L^m)_m$ converge in measure, respectively, to Z and L. Since they are bounded, respectively, in $\mathbf{L^2(W)}$ and $\mathbf{L^2}(\tilde{\mu})$, they are uniformly integrable in $\mathbf{L^p(W)}$ and $\mathbf{L^p}(\tilde{\mu})$, for any $p \in [1,2)$. Thus, $(Z^m)_m$ and $(L^m)_m$ converge strongly to Z and L in $\mathbf{L^p(W)}$ and $\mathbf{L^p}(\tilde{\mu})$, respectively.

By the Lipschitz condition on F, we also have the strong convergence in $\mathbf{L^p(0,T)}$ of $(F(\cdot, Y^m, Z^m, L^m))_m$ to $F(\cdot, Y, Z, L)$. Since $G(\cdot)$ is the weak limit of $(F(\cdot, Y^m, Z^m, L^m))_m$ in $\mathbf{L^2(0,T)}$, we deduce that $G(\cdot) = F(\cdot, Y, Z, L)$. Therefore we obtain that (Y, Z, L, K^+, K^-) satisfies the BSDE (4.7.8). □

Chapitre 5

BSDE representation for stochastic control problems with non dominated controlled intensity

5.1 Introduction

Recently, [75] introduced a new class of backward stochastic differential equations (BSDEs) with nonpositive jumps in order to provide a probabilistic representation formula, known as nonlinear Feynman-Kac formula, for fully nonlinear integro-partial differential equations (IPDEs) of the following type (we use the notation $x.y$ to denote the scalar product in $\mathbb{R}^d$) :

$$\frac{\partial v}{\partial t} + \sup_{a \in A} \Big[b(x,a).D_x v + \frac{1}{2}\mathrm{tr}(\sigma\sigma^{\intercal}(x,a)D_x^2 v) + f(x,a,v,\sigma^{\intercal}(x,a)D_x v) \tag{5.1.1}$$

$$+ \int_E (v(t,x+\beta(x,a,e)) - v(t,x) - \beta(x,a,e).D_x v(t,x))\lambda(de) \Big] \; = \; 0, \qquad \text{on } [0,T) \times \mathbb{R}^d,$$

$$v(T,x) \; = \; g(x), \qquad x \in \mathbb{R}^d,$$

where A is a compact subset of $\mathbb{R}^q$, E is a Borelian subset of $\mathbb{R}^k \backslash \{0\}$, and λ is a nonnegative σ-finite measure on $(E, \mathcal{B}(E))$ satisfying the integrability condition $\int_E (1 \wedge |e|^2)\lambda(de) < \infty$. Notice that the case $f = f(x,a)$ is particularly relevant, as (5.1.1) turns out to be the Hamilton-Jacobi-Bellman equation of a stochastic control problem where the state process is a jump-diffusion with drift b, diffusion coefficient σ (possibly degenerate), and jump size β, which are all controlled ; a special case is the Hamilton-Jacobi-Bellman equation associated to the uncertain volatility model in mathematical finance, which takes the following form :

$$\frac{\partial v}{\partial t} + G(D_x^2 v) \; = \; 0, \quad \text{on } [0,T) \times \mathbb{R}^d, \qquad v(T,x) \; = \; g(x), \quad x \in \mathbb{R}^d, \tag{5.1.2}$$

where $G(M) = \frac{1}{2}\sup_{c \in C}[cM]$ and C is a set of symmetric nonnegative matrices of order d. As described in [89], the unique viscosity solution to (5.1.2) is represented in terms of the so-called G-Brownian motion B under the nonlinear expectation $\mathcal{E}(\cdot)$ as follows :

$$v(t,x) \; = \; \mathcal{E}\big(g(x + B_T - B_t)\big).$$

It is however not clear how to simulate G-Brownian motion. On the other hand, when C can be identified with a compact subset A of a Euclidean space $\mathbb{R}^q$, we have the probabilistic representation formula presented in [75], which can be implemented numerically as shown in [72] and [73]. We recall that the results presented in [75] were generalized to the case of controller-and-stopper games in [23] and to non-Markovian stochastic control problems in [48].

In the present paper, our aim is to generalize the results presented in [75] providing a probabilistic representation formula for the unique viscosity solution to the following fully nonlinear integro-PDE of Hamilton-Jacobi-Bellman type :

$$\begin{aligned}\frac{\partial v}{\partial t} + \sup_{a\in A}\Big[b(x,a).D_x v + \frac{1}{2}\mathrm{tr}(\sigma\sigma^{\intercal}(x,a)D_x^2 v) + f(x,a,v,\sigma^{\intercal}(x,a)D_x v) \qquad (5.1.3)\\ + \int_E (v(t,x+\beta(x,a,e)) - v(t,x) - \beta(x,a,e).D_x v(t,x))\lambda(a,de)\Big] = 0, \quad \text{on } [0,T)\times\mathbb{R}^d,\\ v(T,x) = g(x), \qquad x\in\mathbb{R}^d,\end{aligned}$$

where λ is a transition kernel from $(A,\mathcal{B}(A))$ into $(E,\mathcal{B}(E))$, namely $\lambda(a,\cdot)$ is a nonnegative measure on $(E,\mathcal{B}(E))$ for every $a\in A$ and $\lambda(\cdot,E')$ is a Borel measurable function for every $E'\in\mathcal{B}(E)$. We do not assume that the family of measures $(\lambda(a,\cdot))_{a\in A}$ is dominated. Moreover, the diffusion coefficient σ can be degenerate.

A motivation to the study of equation (5.1.3) comes from mathematical finance and, in particular, from model uncertainty, when uncertainty affects both volatility and intensity. This topic was studied by means of second order BSDEs with jumps (2BSDEJs) in [70] and [71], to which we refer for the wellposedness of these kinds of backward equations, see also [101] ; however, notice that, with respect to [71], we are able to treat PDEs with degenerate diffusion coefficient ; moreover, as in [75], the advantage of our probabilistic representation might be the development of an efficient numerical scheme for equation (5.1.3), as it was done in [72] and [73] for equation (5.1.1) starting from the representation derived in [75]. Model uncertainty is also strictly related to the theory of G-Lévy processes and, more generally, of nonlinear Lévy processes, see [54] and [83]. In this case, the associated fully nonlinear integro-PDE, which naturally generalizes equation (5.1.2), takes the following form :

$$\begin{aligned}\frac{\partial v}{\partial t} + \sup_{(b,c,F)\in\Theta}\Big[b.D_x v + \frac{1}{2}\mathrm{tr}(cD_x^2 v) \qquad (5.1.4)\\ + \int_E (v(t,x+z) - v(t,x) - D_x v(t,x).z1_{\{|z|\le 1\}})F(dz)\Big] \;=\; 0, \quad \text{on } [0,T)\times\mathbb{R}^d,\\ v(T,x) \;=\; g(x), \qquad x\in\mathbb{R}^d,\end{aligned}$$

where Θ denotes a set of Lévy triplets (b,c,F) ; here b is a vector in $\mathbb{R}^d$, c is a symmetric nonnegative matrix of order d, and F is a Lévy measure on $(\mathbb{R}^d,\mathcal{B}(\mathbb{R}^d))$. From [54] and [83], we know that the unique viscosity solution to equation (5.1.4) is represented in terms of the so-called nonlinear Lévy process $\mathcal{X}$ under the nonlinear expectation $\mathcal{E}(\cdot)$ as follows :

$$v(t,x) \;=\; \mathcal{E}(g(x+\mathcal{X}_T-\mathcal{X}_t)).$$

If we are able to describe the set Θ by means of a parameter a which lives in a compact set A of an Euclidean space $\mathbb{R}^q$, then (5.1.4) can be written in the form (5.1.3). Therefore,

v is also given by our probabilistic representation formula, in which the forward process is possibly easier to simulate than a nonlinear Lévy process.

More generally, we expect that the viscosity solution v to equation (5.1.3), when $f = f(x,a)$, should represent the value function of a stochastic control problem where, roughly speaking, the state process X is a jump-diffusion process, which has the peculiarity that we may control the dynamics of X changing its jump intensity, other than acting on the coefficients b, σ, and β of the SDE solved by X. We refer to this problem as a stochastic optimal control problem with (non dominated) controlled intensity. Unfortunately, we did not find any reference in the literature for this kind of stochastic control problem. For this reason, and also because it will be useful to understand the general idea behind the derivation of our nonlinear Feynman-Kac formula, we describe it here, even if only formally. Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be a complete probability space satisfying the usual conditions on which a d-dimensional Brownian motion $\bar{W} = (\bar{W}_t)_{t\geq 0}$ is defined. Let $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t\geq 0}$ denote the usual completion of the natural filtration generated by $\bar{W}$ and $\bar{\mathcal{A}}$ the class of control processes α, i.e., of $\bar{\mathbb{F}}$-predictable processes valued in A. Let also Ω' be the canonical space of the marked point process on $\mathbb{R}_+ \times E$ (see Section 5.2 below for a definition), with canonical right-continuous filtration $\mathbb{F}'$ and canonical random measure π'. Then, consider $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ defined as $\Omega := \bar{\Omega} \times \Omega'$, $\mathcal{F} := \bar{\mathcal{F}} \otimes \mathcal{F}'_\infty$, and $\mathcal{F}_t := \cap_{s>t} \bar{\mathcal{F}}_s \otimes \mathcal{F}'_s$. Moreover, we set $W(\omega) := \bar{W}(\bar{\omega})$, $\pi(\omega, \cdot) := \pi'(\omega', \cdot)$, and $\mathcal{A} := \{\alpha\colon \alpha(\omega) = \bar{\alpha}(\bar{\omega}), \forall\, \omega \in \Omega, \text{ for some } \bar{\alpha} \in \bar{\mathcal{A}}\}$. Suppose that for every $\alpha \in \mathcal{A}$ we are able to construct a measure $\mathbb{P}^\alpha$ on $(\Omega, \mathcal{F})$ such that W is a Brownian motion and π is an integer-valued random measure with compensator $1_{\{t<T_\infty\}}\lambda(\alpha_t, de)dt$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\alpha)$, where T_∞ denotes the supremum of the jump times of the marked point process associated to π. Then, consider the stochastic control problem with value function given by ($\mathbb{E}^\alpha$ denotes the expectation with respect to $\mathbb{P}^\alpha$)

$$v(t,x) \;:=\; \sup_{\alpha\in\mathcal{A}} \mathbb{E}^\alpha\Big[\int_t^T f(X_s^{t,x,\alpha}, \alpha_s)ds + g(X_T^{t,x,\alpha})\Big], \tag{5.1.5}$$

where $X^{t,x,\alpha}$ has the controlled dynamics on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^\alpha)$

$$dX_s^\alpha \;=\; b(X_s^\alpha, \alpha_s)ds + \sigma(X_s^\alpha, \alpha_s)dW_s + \int_E \beta(X_{s^-}^\alpha, \alpha_s, e)\tilde{\pi}(ds, de)$$

starting from x at time t, with $\tilde{\pi}(dt, de) = \pi(dt, de) - 1_{\{t<T_\infty\}}\lambda(\alpha_t, de)dt$ the compensated martingale measure of π. As mentioned above, even if we do not address this problem here, we expect that the above partial differential equation (5.1.3) turns out to be the dynamic programming equation of the stochastic control problem with value function formally given by (5.1.5). Having this in mind, we can now begin to describe the intuition, inspired by [74] and [75], behind the derivation of our Feynman-Kac representation formula for the HJB equation (5.1.3) in terms of a forward backward stochastic differential equation (FBSDE).

The fundamental idea concerns the *randomization* of the control, which is achieved introducing on $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ a q-dimensional Brownian motion $\bar{B} = (\bar{B}_t)_{t\geq 0}$, independent of $\bar{W}$. Now $\bar{\mathbb{F}}$ denotes the usual completion of the natural filtration generated by $\bar{W}$ and $\bar{B}$. We also set $B(\omega) := \bar{B}(\bar{\omega})$, for all $\omega \in \Omega$, so that B is defined on Ω. Since the control lives in the compact set $A \subset \mathbb{R}^q$, we can not use directly B to randomize the control, but we need to map B on A. More precisely, we shall assume the existence of a continuous surjection $h\colon \mathbb{R}^d \to A$. Then, for every $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, we consider the forward stochastic differential equation in $\mathbb{R}^d \times \mathbb{R}^q$:

$$
\begin{aligned}
X_s &= x + \int_t^s b(X_r, I_r)dr + \int_t^s \sigma(X_r, I_r)dW_r + \int_t^s \int_E \beta(X_{r^-}, I_r, e)\tilde{\pi}(dr, de), && (5.1.6)\\
I_s &= h(\tilde{a} + B_s - B_t), && (5.1.7)
\end{aligned}
$$

for all $t \leq s \leq T$, where $\tilde{\pi}(ds, de) = \pi(ds, de) - 1_{\{s<T_\infty\}}\lambda(I_s, de)ds$ is the compensated martingale measure of π, which is an integer-valued random measure with compensator $1_{\{s<T_\infty\}}\lambda(I_s, de)ds$. Unlike [75], we used a Brownian motion B to randomize the control, instead of a Poisson random measure μ on $\mathbb{R}_+ \times A$. From one hand, the Poisson random measure turns out to be more convenient to deal with a general compact set A, since μ is already supported by $\mathbb{R}_+ \times A$, so that we do not have to impose the existence of a continuous surjection h from the entire space $\mathbb{R}^q$ onto A, as we did here. On the other hand, the choice of a Brownian motion B is more convenient to derive a martingale representation theorem for our model. Indeed, in contrast with [75], the intensity of the measure π depends on the process I, therefore it is natural to expect a dependence between π and the noise used to randomize the control. The advantage of B with respect to μ is given by the fact that B is *orthogonal* to π, since B is a continuous process (see the bottom of page 183 in [59] for a definition of orthogonality between a martingale and a random measure). Thanks to this orthogonality we are able to derive a martingale representation theorem in our context, which is essential for the derivation of our nonlinear Feynman-Kac representation formula.

Let us focus on the form of the stochastic differential equation (5.1.6)-(5.1.7). We observe that the jump part of the driving factors in (5.1.6) is not given, but depends on the solution via its intensity. This makes the SDE (5.1.6)-(5.1.7) nonstandard. These kinds of equations were firstly studied in [58] and have also been used in the financial literature, see e.g. [9], [27], [28], [29], [42]. Notice that in [9], [27], and [28], λ is absolutely continuous with respect to a given deterministic measure on $(E, \mathcal{B}(E))$, which allows to solve (5.1.6)-(5.1.7) bringing it back to a standard SDE, via a change of intensity "à la Girsanov". On the other hand, in the present paper, we shall tackle the above SDE solving firstly equation (5.2.2) for any $(t, \tilde{a}) \in [0, T] \times \mathbb{R}^q$, then constructing a probability measure $\mathbb{P}^{t,\tilde{a}}$ on $(\Omega, \mathcal{F})$ such that the random measure $\pi(ds, de)$ admits $\lambda(I_s^{t,\tilde{a}}, de)ds$ as compensator, and finally addressing (4.5.1). In the appendix, we also prove additional properties of π and (X, I). More precisely, we present a characterization of π in terms of Fourier and Laplace functionals, which shows that π is a conditionally Poisson random measure (also known as doubly stochastic Poisson random measure or Cox random measure) relative to $\sigma(I_s^{t,\tilde{a}}; s \geq 0)$. Moreover, we study the Markov properties of the pair (X, I).

Regarding the backward stochastic differential equation, as expected, it is driven by the Brownian motions W and B, and by the random measure π, namely it is a BSDE with jumps with terminal condition $g(X_T^{t,x,\tilde{a}})$ and generator $f(X_\cdot^{t,x,\tilde{a}}, I_\cdot^{t,\tilde{a}}, y, z)$, as it is natural from the expression of the HJB equation (5.1.3). The backward equation is also characterized by a constraint on the diffusive part relative to B, which turns out to be crucial and entails the presence of an increasing process in the BSDE. In conclusion, for any $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, the backward stochastic differential equation has the following form :

$$\begin{aligned} Y_s &= g(X_T^{t,x,\tilde{a}}) + \int_s^T f(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}, Y_r, Z_r)dr + K_T - K_s - \int_s^T Z_r dW_r \\ &\quad - \int_s^T V_r dB_r - \int_s^T \int_E U_r(e)\tilde{\pi}(dr, de), \qquad t \le s \le T,\ \mathbb{P}^{t,\tilde{a}}\ a.s. \end{aligned} \tag{5.1.8}$$

and

$$|V_s| \;=\; 0 \qquad\qquad ds \otimes d\mathbb{P}^{t,\tilde{a}}\ a.e. \tag{5.1.9}$$

We refer to (5.1.8)-(5.1.9) as backward stochastic differential equation with jumps and partially constrained diffusive part. Notice that we could omit the term $\int_s^T V_r dB_r$ in equation (5.1.8) (together with the constraint (5.1.9)), since V is required to be zero ; however, we keep it to recall that the solution to (5.1.8)-(5.1.9) has to be adapted to the filtration generated by W, $\tilde{\pi}$, and also B. We also observe that the presence of the increasing process K in the backward equation does not guarantee the uniqueness of the solution. For this reason, we look only for the minimal solution (Y, Z, V, U, K) to the above BSDE, in the sense that for any other solution $(\bar{Y}, \bar{Z}, \bar{V}, \bar{U}, \bar{K})$ we must have $Y \le \bar{Y}$. The existence of the minimal solution is based on a penalization approach as in [75]. We can now write down the nonlinear Feynman-Kac formula :

$$v(t, x, \tilde{a}) \;:=\; Y_t^{t,x,\tilde{a}}, \qquad (t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q.$$

Observe that the function v should not depend on $\tilde{a}$, but only on (t, x). The function v turns out to be independent of the variable $\tilde{a}$ as a consequence of the constraint (5.1.9). Indeed, if v (and also h) were regular enough, then, for any $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, we would have

$$V_s^{t,x,\tilde{a}} \;=\; D_h v(s, X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}}) D_{\tilde{a}} h(\tilde{a} + B_s - B_t) \;=\; 0, \qquad ds \otimes d\mathbb{P}^{t,\tilde{a}}\ a.e.$$

This would imply (see Subsection 5.4.2) that v does not depend on its last argument. However, we do not know in general if the function v is so regular in order to justify the previous passages. Therefore, the rigorous proof relies on viscosity solutions arguments. In the end, we prove that the function v does not depend on the variable $\tilde{a}$. Moreover, v is a viscosity solution to (5.1.3). Actually, v is the unique viscosity solution to (5.1.3), as it follows from the comparison theorem proved in the Appendix. Notice that, due to the presence of the non dominated family of measures $(\lambda(a, \cdot))_{a \in A}$, we did not find in the literature a comparison theorem for viscosity solution to our equation (5.1.3). For this reason, we prove it in the Appendix, even though the main ideas are already contained in the paper [4], in particular the remarkable Jensen-Ishii's lemma for integro-partial differential equations.

The rest of the chapter is organized as follows. Section 5.2 introduces some notations and studies the construction of the solution to the forward equation (5.1.6)-(5.1.7). Section 5.3 gives a detailed formulation of the BSDE with jumps and partially constrained diffusive part. In particular, Subsection 5.3.1 is devoted to the existence of the minimal solution to our BSDE by a penalization approach. Section 5.4 makes the connection between the minimal solution to our BSDE and equation (5.1.3). In the Appendix, we prove a martingale representation theorem for our model, we collect some properties of the random measure π and of the pair (X, I), and we prove a comparison theorem for equation (5.1.3).

5.2 Notations and preliminaries

Let $(\bar{\Omega}, \bar{\mathcal{F}}, \bar{\mathbb{P}})$ be a complete probability space satisfying the usual conditions on which are defined a d-dimensional Brownian motion $\bar{W} = (\bar{W}_t)_{t\geq 0}$ and an independent q-dimensional Brownian motion $\bar{B} = (\bar{B}_t)_{t\geq 0}$. We will always assume that $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t\geq 0}$ is the usual completion of the natural filtration generated by $\bar{W}$ and $\bar{B}$. Let us introduce some additional notations.

(i) Ω' is the set of sequences $\omega' = (t_n, e_n)_{n\in\mathbb{N}} \subset (0,\infty] \times E_\Delta$, where $E_\Delta = E \cup \{\Delta\}$ and Δ is an external point of E. Moreover $t_n < \infty$ if and only if $e_n \in E$, and when $t_n < \infty$ then $t_n < t_{n+1}$. Ω' is equipped with the canonical marked point process $(T'_n, \alpha'_n)_{n\in\mathbb{N}}$, with associated canonical random measure π', defined as

$$T'_n(\omega') \;=\; t_n, \qquad \alpha'_n(\omega') \;=\; e_n$$

and

$$\pi'(\omega', dt, de) \;=\; \sum_{n\in\mathbb{N}} 1_{\{T'_n(\omega')<\infty\}} \delta_{(T'_n(\omega'), \alpha'_n(\omega'))}(dt, de),$$

where δ_x denotes the Dirac measure at point x. Set $T'_\infty := \lim_n T'_n$. Finally, define $\mathbb{F}' = (\mathcal{F}'_t)_{t\geq 0}$ as $\mathcal{F}'_t = \cap_{s>t}\mathcal{G}'_s$, where $\mathbb{G}' = (\mathcal{G}'_t)_{t\geq 0}$ is the canonical filtration, given by $\mathcal{G}'_t = \sigma(\pi'(\cdot, F) \colon F \in \mathcal{B}([0,t]) \otimes \mathcal{B}(E))$.

(ii) $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0})$ is such that $\Omega := \bar{\Omega} \times \Omega'$, $\mathcal{F} := \bar{\mathcal{F}} \otimes \mathcal{F}'_\infty$, and $\mathcal{F}_t := \cap_{s>t}\bar{\mathcal{F}}_s \otimes \mathcal{F}'_s$. Moreover, we set $W(\omega) := \bar{W}(\bar{\omega})$, $B(\omega) := \bar{B}(\bar{\omega})$, and $\pi(\omega, \cdot) := \pi'(\omega', \cdot)$. Finally, we set also $T_n(\omega) := T'_n(\omega')$, $\alpha_n(\omega) := \alpha'_n(\omega')$, and $T_\infty(\omega) := T'_\infty(\omega')$.

Let $\mathcal{P}_\infty$ denote the σ-field of $\mathbb{F}$-predictable subsets of $\mathbb{R}_+ \times \Omega$. We recall that a random measure π on $\mathbb{R}_+ \times E$ is a transition kernel from $(\Omega, \mathcal{F})$ into $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E))$, satisfying $\pi(\omega, \{0\} \times E) = 0$ for all $\omega \in \Omega$; moreover, an integer-valued random measure π on $\mathbb{R}_+ \times E$ is an optional and $\mathcal{P}_\infty \otimes \mathcal{B}(E)$-$\sigma$-finite, $\mathbb{N} \cup \{+\infty\}$-valued random measure such that $\pi(\omega, \{t\} \times E) \leq 1$ for all $(t, \omega) \in [0,T] \times \Omega$, see Definition 1.13, Chapter II, in [59].

Let A be a compact subset of some Euclidean space $\mathbb{R}^q$. We are given some measurable functions $b \colon \mathbb{R}^d \times A \to \mathbb{R}^d$, $\sigma \colon \mathbb{R}^d \times A \to \mathbb{R}^{d\times d}$, and $\beta \colon \mathbb{R}^d \times A \times E \to \mathbb{R}^d$, where E is a Borelian subset of $\mathbb{R}^k \backslash \{0\}$, equipped with its Borel σ-field $\mathcal{B}(E)$. Moreover, let λ be a transition kernel from $(A, \mathcal{B}(A))$ into $(E, \mathcal{B}(E))$, namely $\lambda(a, \cdot)$ is a nonnegative measure on $(E, \mathcal{B}(E))$ for every $a \in A$ and $\lambda(\cdot, E')$ is a Borel measurable function for every $E' \in \mathcal{B}(E)$. Furthermore, we assume that there exists a continuous surjection $h \colon \mathbb{R}^d \to A$.

Remark 5.2.1. (i) The existence of such a function h is guaranteed whenever A is connected and locally connected, this is indeed a consequence of the Hahn-Mazurkiewicz theorem (see, e.g., Theorem 6.8 in [96]).
(ii) In the sequel we use the notation $\tilde{a}$ (resp. a) to denote a generic element in the domain $\mathbb{R}^q$ (image A) of h. □

For any $t \in [0,T]$ and $(x, \tilde{a}) \in \mathbb{R}^d \times \mathbb{R}^q$, we consider the forward stochastic differential equation in $\mathbb{R}^d \times \mathbb{R}^q$:

$$X_s \;=\; x + \int_t^s b(X_r, I_r)dr + \int_t^s \sigma(X_r, I_r)dW_r + \int_t^s \int_E \beta(X_{r^-}, I_r, e)\tilde{\pi}(dr, de), \tag{5.2.1}$$

$$I_s = h(\tilde{a} + B_s - B_t), \tag{5.2.2}$$

for all $t \le s \le T$, where $\tilde{\pi}(ds, de) = \pi(ds, de) - \lambda(I_s, de)ds$ is the compensated martingale measure of π, which is an integer-valued random measure with compensator $\lambda(I_s, de)ds$.

As noticed in the introduction, the above SDE (5.2.1)-(5.2.2) is nonstandard, in the sense that the jump part of the driving factors in (5.2.1) is not given, but depends on the solution via its intensity. When the intensity λ is absolutely continuous with respect to a given deterministic measure on $(E, \mathcal{B}(E))$, as in [9], [27], and [28], we can obtain (5.2.1)-(5.2.2) starting from a standard SDE via a change of intensity "à la Girsanov". On the other hand, in the present paper, we shall tackle the above SDE solving firstly equation (5.2.2), then constructing the random measure $\pi(ds, de)$, and finally addressing (5.2.1). The nontrivial part is the construction of π, which is essentially based on Theorem 3.6 in [55], and also on similar results in [42], Theorem 5.1, and [29], Theorem A.4. Let us firstly introduce the following assumptions on the forward coefficients.

(HFC)

(i) There exists a constant C such that

$$|b(x,a) - b(x',a')| + |\sigma(x,a) - \sigma(x',a')| \le C(|x - x'| + |a - a'|),$$

for all $x, x' \in \mathbb{R}^d$ and $a, a' \in A$.

(ii) There exists a constant C such that

$$\begin{aligned} |\beta(x,a,e)| &\le C(1 + |x|)(1 \wedge |e|), \\ |\beta(x,a,e) - \beta(x',a',e)| &\le C(|x - x'| + |a - a'|)(1 \wedge |e|), \end{aligned}$$

for all $x, x' \in \mathbb{R}^d$, $a, a' \in A$, and $e \in E$.

(iii) The following integrability condition holds :

$$\sup_{a \in A} \int_E (1 \wedge |e|^2) \lambda(a, de) < \infty.$$

Inspired by [58], we give the definition of weak solution to equation (5.2.1)-(5.2.2).

Definition 5.2.1. *A **weak solution** to equation (5.2.1)-(5.2.2) with initial condition $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ is a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ satisfying :*

(i) *$\mathbb{P}(d\omega) = \bar{\mathbb{P}}(d\bar{\omega}) \otimes \mathbb{P}'(\bar{\omega}, d\omega')$, for some transition kernel $\mathbb{P}'$ from $(\bar{\Omega}, \bar{\mathcal{F}})$ into $(\Omega', \mathcal{F}'_\infty)$.*

(ii) *Under $\mathbb{P}$, π is an integer-valued random measure on $\mathbb{R}_+ \times E$ with $\mathbb{F}$-compensator $1_{\{s < T_\infty\}} \lambda(I_s, de)ds$ and associated compensated martingale measure given by $\tilde{\pi}(ds, de) = \pi(ds, de) - 1_{\{s < T_\infty\}} \lambda(I_s, de)ds$.*

(iii) *We have*

$$\begin{aligned} X_s &= x + \int_t^s b(X_r, I_r)dr + \int_t^s \sigma(X_r, I_r)dW_r + \int_t^s \int_E \beta(X_{r^-}, I_r, e)\tilde{\pi}(dr, de), \\ I_s &= h(\tilde{a} + B_s - B_t), \end{aligned}$$

for all $t \le s \le T$, $\mathbb{P}$ almost surely. Moreover, $(X_s, I_s) = (x, h(\tilde{a}))$ for $s < t$, and $(X_s, I_s) = (X_T, I_T)$ for $s > T$.

Consider a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ satisfying condition (i) of Definition 5.2.1. For every $(t, \tilde{a}) \in [0, T] \times \mathbb{R}^q$ let us denote $I^{t,\tilde{a}} = \{I_s^{t,\tilde{a}},\ s \geq 0\}$ the unique process on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ satisfying $I_s^{t,\tilde{a}} = h(\tilde{a} + B_s - B_t)$ on $[t, T]$, with $I_s^{t,\tilde{a}} = h(\tilde{a})$ for $s < t$ and $I_s^{t,\tilde{a}} = I_T^{t,\tilde{a}}$ for $s > T$. We notice that the notation $I^{t,\tilde{a}}$ can be misleading, since $\tilde{a}$ is not the initial point of $I^{t,\tilde{a}}$ at time t, indeed $I_t^{t,\tilde{a}} = h(\tilde{a})$. Now we proceed to the construction of a probability measure on $(\Omega, \mathcal{F})$ for which conditions (i) and (ii) of Definition 5.2.1 are satisfied. This result is based on Theorem 3.6 in [55], and we borrow also some ideas from [42], Theorem 5.1, and [29], Theorem A.4.

Lemma 5.2.1. *Under assumption* **(HFC)***, for every $(t, \tilde{a}) \in [0, T] \times \mathbb{R}^q$ there exists a unique probability measure on $(\Omega, \mathcal{F})$, denoted by $\mathbb{P}^{t,\tilde{a}}$, satisfying conditions (i) and (ii) of Definition 5.2.1, and also condition (ii)′ given by :*

(ii)′ *$1_{\{s<T_\infty\}}\lambda(I_s^{t,\tilde{a}}, de)ds$ is the $(\bar{\mathcal{F}} \otimes \mathcal{F}'_s)_{s\geq 0}$-compensator of π.*

Proof. The proof is essentially based on Theorem 3.6 in [55], after a reformulation of our problem in the setting of [55], which we now detail. Let $\hat{\mathbb{F}} = (\hat{\mathcal{F}}_s)_{s\geq 0}$ where $\hat{\mathcal{F}}_s := \bar{\mathcal{F}} \otimes \mathcal{F}'_s$. Notice that in $\hat{\mathcal{F}}_s$ we take $\bar{\mathcal{F}}$ instead of $\bar{\mathcal{F}}_s$. Indeed, in [55] the σ-field $\bar{\mathcal{F}}$ represents the past information and is fixed throughout (we come back to this point later). Take $(t, \tilde{a}) \in [0, T] \times \mathbb{R}^q$ and consider the process $I^{t,\tilde{a}} = (I_s^{t,\tilde{a}})_{s\geq 0}$. Set

$$\nu(\omega, F) \quad = \quad \int_F 1_{\{s<T_\infty(\omega)\}}\lambda(I_s^{t,\tilde{a}}(\omega), de)ds$$

for any $\omega \in \Omega$ and any $F \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$. Now we show that ν satisfies the properties required in order to apply Theorem 3.6 in [55]. In particular, since λ is a transition kernel, we see that ν is a transition kernel from $(\Omega, \mathcal{F})$ into $(\mathbb{R}_+ \times E, \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E))$; moreover, $\nu(\omega, \{0\} \times E) = 0$ for all $\omega \in \Omega$, therefore ν is a random measure on $\mathbb{R}_+ \times E$. Furthermore, for every $E' \in \mathcal{B}(E)$, the process $\nu((0, \cdot] \times E') = (\nu((0, s] \times E'))_{s\geq 0}$ is $\hat{\mathbb{F}}$-predictable, hence ν is an $\hat{\mathbb{F}}$-predictable random measure. In addition, $\nu(\{s\} \times E) \leq 1$, indeed ν is absolutely continuous with respect to the Lebesgue measure ds and therefore $\nu(\{s\} \times E) = 0$. Finally, we see by definition that $\nu([T_\infty, \infty) \times E) = 0$. In conclusion, it follows from Theorem 3.6 in [55] that there exists a unique probability measure on $(\Omega, \mathcal{F})$, denoted by $\mathbb{P}^{t,\tilde{a}}$, satisfying condition (i) of Definition 5.2.1, and for which ν is the $\hat{\mathbb{F}}$-compensator of π, i.e., the process

$$\left(\nu((0, s \wedge T_n] \times E') - \pi((0, s \wedge T_n] \times E')\right)_{s\geq 0} \tag{5.2.3}$$

is a $(\mathbb{P}^{t,\tilde{a}}, \hat{\mathbb{F}})$-martingale, for any $E' \in \mathcal{B}(E)$ and any $n \in \mathbb{N}$. Therefore condition (ii)′ is also satisfied.

To conclude, we need to prove that ν is also the $\mathbb{F}$-compensator of π. Since ν is an $\mathbb{F}$-predictable random measure, it follows from (2.6) in [55] that it remains to prove that the process (5.2.3) is a $(\mathbb{P}^{t,\tilde{a}}, \mathbb{F})$-martingale. We solve this problem reasoning as in [42], Theorem 5.1, point (iv). Basically, for every $T \in \mathbb{R}_+$ we repeat the above construction with $\bar{\mathcal{F}}_T$ in place of $\bar{\mathcal{F}}$, changing what in [55] is called the past information. More precisely, let $T \in \mathbb{R}_+$ and define $\hat{\mathbb{F}}^T = (\hat{\mathcal{F}}_s^T)_{s\geq 0}$, where $\hat{\mathcal{F}}_s^T := \bar{\mathcal{F}}_T \otimes \mathcal{F}'_s$. Let

$$\nu^T(\omega, F) \quad = \quad \int_F 1_{\{s\leq T\}} 1_{\{s<T_\infty(\omega)\}}\lambda(I_s^{t,\tilde{a}}(\omega), de)ds.$$

Proceeding as before, we conclude that there exists a unique probability measure on $(\Omega, \bar{\mathcal{F}}_T \otimes \mathcal{F}'_\infty)$, denoted by $\mathbb{P}^{t,\tilde{a},T}$, whose restriction to $(\bar{\Omega}, \bar{\mathcal{F}}_T)$ coincides with the restriction of $\bar{\mathbb{P}}$ to this measurable space, and for which ν^T is the $\hat{\mathbb{F}}^T$-compensator of π, i.e.,

$$\left(\nu^T((0, s \wedge T_n] \times E') - \pi((0, s \wedge T_n] \times E')\right)_{s \geq 0}$$

is a $(\mathbb{P}^{t,\tilde{a},T}, \hat{\mathbb{F}}^T)$-martingale, for any $E' \in \mathcal{B}(E)$ and any $n \in \mathbb{N}$. This implies that $\nu^T((0, T \wedge T_n] \times E') - \pi((0, T \wedge T_n] \times E')$ is $\hat{\mathcal{F}}^T_T$-measurable, and therefore $\mathcal{F}_T$-measurable. Notice that

$$\nu^T((0, s \wedge T_n] \times E') \quad = \quad \nu((0, s \wedge T \wedge T_n] \times E'),$$

hence $\nu((0, T \wedge T_n] \times E') - \pi((0, T \wedge T_n] \times E')$ is $\mathcal{F}_T$-measurable. As $T \in \mathbb{R}_+$ was arbitrary, we see that the process (5.2.3) is $\mathbb{F}$-adapted. Since (5.2.3) is a $(\mathbb{P}^{t,\tilde{a}}, \hat{\mathbb{F}})$-martingale, with $\mathcal{F}_s \subset \hat{\mathcal{F}}_s$, then it is also a $(\mathbb{P}^{t,\tilde{a}}, \mathbb{F})$-martingale. In other words, ν is the $\mathbb{F}$-compensator of π. □

Remark 5.2.2. Notice that, under assumption **(HFC)** and if in addition λ satisfies the integrability condition (which implies the integrability condition **(HFC)**(iii)) :

$$\sup_{a \in A} \int_E \lambda(a, de) \quad < \quad \infty, \tag{5.2.4}$$

then $T_\infty = \infty$, $\mathbb{P}^{t,\tilde{a}}$ a.s., and the compensator ν is given by

$$\nu(\omega, F) \quad = \quad \int_F \lambda(I_s^{t,\tilde{a}}(\omega), de) ds$$

for any $F \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$ and for $\mathbb{P}^{t,\tilde{a}}$ almost every $\omega \in \Omega$. Indeed, for any $T \geq 0$, we have (we denote by $\mathbb{E}^{t,\tilde{a}}$ the expectation with respect to $\mathbb{P}^{t,\tilde{a}}$)

$$\mathbb{E}^{t,\tilde{a}}\Big[\sum_{n \in \mathbb{N}} 1_{\{T_n \leq T\}}\Big] \;=\; \mathbb{E}^{t,\tilde{a}}\Big[\int_0^T \int_E \pi(ds, de)\Big] \;=\; \mathbb{E}^{t,\tilde{a}}\Big[\int_0^T \int_E \nu(ds, de)\Big].$$

Therefore

$$\mathbb{E}^{t,\tilde{a}}\Big[\sum_{n \in \mathbb{N}} 1_{\{T_n \leq T\}}\Big] \;=\; \mathbb{E}^{t,\tilde{a}}\Big[\int_0^T \int_E 1_{\{s < T_\infty\}} \lambda(I_s^{t,\tilde{a}}, de) ds\Big] \;\leq\; T \sup_{a \in A} \int_E \lambda(a, de) \;<\; \infty,$$

where we used condition (5.2.4). Hence, $\mathbb{P}^{t,\tilde{a}}$ a.s.,

$$\sum_{n \in \mathbb{N}} 1_{\{T_n \leq T\}} < \infty, \qquad \forall T \geq 0.$$

From the arbitrariness of T, this implies that $T_\infty = \infty$, $\mathbb{P}^{t,\tilde{a}}$ almost surely. □

Lemma 5.2.2. *Under assumption* **(HFC)**, *for every $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ there exists a unique (up to indistinguishability) process $X^{t,x,\tilde{a}} = \{X_s^{t,x,\tilde{a}},\ s \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{t,\tilde{a}})$, solution to (5.2.1) on $[t, T]$, with $X_s^{t,x,\tilde{a}} = x$ for $s < t$ and $X_s^{t,x,\tilde{a}} = X_T^{t,x,\tilde{a}}$ for $s > T$. Moreover, for any $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ there exists a positive constant $\tilde{C}$ such that*

$$\mathbb{E}^{t,\tilde{a}}\Big[\sup_{t \leq s \leq T} |X_s^{t,x,\tilde{a}}|^2\Big] \;\leq\; \tilde{C}(1 + |x|^2), \tag{5.2.5}$$

where $\tilde{C}$ depends only on T, $|b(0,0)|$, $|\sigma(0,0)|$, $\sup_{a \in A} |a|$, $\sup_{a \in A} \int_E (1 \wedge |e|^2) \lambda(a, de)$, and the Lipschitz constants of b, σ.

Proof. Since hypotheses (14.15) and (14.22) in [57] are satisfied under **(HFC)**, the claim follows from Theorem 14.23 in [57]. Concerning estimate (5.2.5), taking the square in (5.2.1) (using the standard inequality $(x_1+\cdots+x_4)^2 \leq 4(x_1^2+\cdots+x_4^2)$, for any $x_1,\ldots,x_4 \in \mathbb{R}$) and then the supremum, we find

$$\sup_{t\leq u\leq s}|X_u^{t,x,\tilde{a}}|^2 \;\leq\; 4|x|^2 + 4\sup_{t\leq u\leq s}\bigg|\int_t^u b(X_r^{t,x,\tilde{a}},I_r^{t,\tilde{a}})dr\bigg|^2 + 4\sup_{t\leq u\leq s}\bigg|\int_t^u \sigma(X_r^{t,x,\tilde{a}},I_r^{t,\tilde{a}})dW_r\bigg|^2$$
$$+\,4\sup_{t\leq u\leq s}\bigg|\int_t^u\int_E \beta(X_{r^-}^{t,x,\tilde{a}},I_r^{t,\tilde{a}},e)\tilde{\pi}(dr,de)\bigg|^2. \tag{5.2.6}$$

Notice that, from Cauchy-Schwarz inequality we have

$$\mathbb{E}^{t,\tilde{a}}\bigg[\sup_{t\leq u\leq s}\bigg|\int_t^u b(X_r^{t,x,\tilde{a}},I_r^{t,\tilde{a}})dr\bigg|^2\bigg] \;\leq\; T\,\mathbb{E}^{t,\tilde{a}}\bigg[\int_t^s \big|b(X_r^{t,x,\tilde{a}},I_r^{t,\tilde{a}})\big|^2 dr\bigg]. \tag{5.2.7}$$

Moreover, from Burkholder-Davis-Gundy inequality there exists a positive constant $\bar{C}$ such that

$$\mathbb{E}^{t,\tilde{a}}\bigg[\sup_{t\leq u\leq s}\bigg|\int_t^u \sigma(X_r^{t,x,\tilde{a}},I_r^{t,\tilde{a}})dW_r\bigg|^2\bigg] \;\leq\; \bar{C}\,\mathbb{E}^{t,\tilde{a}}\bigg[\int_t^s \mathrm{tr}\big(\sigma\sigma^{\intercal}(X_r^{t,x,\tilde{a}},I_r^{t,\tilde{a}})\big)dr\bigg]. \tag{5.2.8}$$

Similarly, since the local martingale $M_u = \int_t^u\int_E \beta(X_{r^-}^{t,x,\tilde{a}},I_r^{t,\tilde{a}},e)\tilde{\pi}(dr,de)$, $t\leq u\leq s$, is such that $[M]_u = \int_t^u\int_E|\beta(X_{r^-}^{t,x,\tilde{a}},I_r^{t,\tilde{a}},e)|^2\pi(dr,de)$, from Burkholder-Davis-Gundy inequality we obtain

$$\begin{aligned}
&\mathbb{E}^{t,\tilde{a}}\bigg[\sup_{t\leq u\leq s}\bigg|\int_t^u\int_E \beta(X_{r^-}^{t,x,\tilde{a}},I_r^{t,\tilde{a}},e)\tilde{\pi}(dr,de)\bigg|^2\bigg]\\
&\leq\; \bar{C}\,\mathbb{E}^{t,\tilde{a}}\bigg[\int_t^s\int_E \big|\beta(X_{r^-}^{t,x,\tilde{a}},I_r^{t,\tilde{a}},e)\big|^2\pi(dr,de)\bigg]\\
&=\; \bar{C}\,\mathbb{E}^{t,\tilde{a}}\bigg[\int_t^s\int_E \big|\beta(X_{r^-}^{t,x,\tilde{a}},I_r^{t,\tilde{a}},e)\big|^2\lambda(I_r^{t,\tilde{a}},de)dr\bigg].
\end{aligned} \tag{5.2.9}$$

In conclusion, taking the expectation in (5.2.6) and using (5.2.7)-(5.2.8)-(5.2.9), we find (denoting by $\tilde{C}$ a generic positive constant depending only on T, $|b(0,0)|$, $|\sigma(0,0)|$, $\sup_{a\in A}|a|$, $\sup_{a\in A}\int_E(1\wedge|e|^2)\lambda(a,de)$, and the Lipschitz constants of b, σ)

$$\mathbb{E}^{t,\tilde{a}}\Big[\sup_{t\leq u\leq s}|X_u^{t,x,\tilde{a}}|^2\Big] \leq 4|x|^2 + \tilde{C}\bigg(1+\int_t^s \mathbb{E}^{t,\tilde{a}}\Big[\sup_{t\leq u\leq r}|X_u^{t,x,\tilde{a}}|^2\Big]dr\bigg).$$

Then, applying Gronwall's lemma to the map $r\mapsto \mathbb{E}^{t,\tilde{a}}[\sup_{t\leq u\leq r}|X_u^{t,x,\tilde{a}}|^2]$, we end up with estimate (5.2.5). □

5.3 BSDE with jumps and partially constrained diffusive part

Our aim is to derive a probabilistic representation formula, also called nonlinear Feynman-Kac formula, for the following nonlinear IPDE of HJB type :

$$-\frac{\partial u}{\partial t}(t,x) - \sup_{a\in A}\big(\mathcal{L}^a u(t,x) + f(x,a,u,\sigma^{\intercal}(x,a)D_x u)\big) \;=\; 0, \quad (t,x)\in[0,T)\times\mathbb{R}^d \tag{5.3.1}$$

$$u(T,x) = g(x), \quad x \in \mathbb{R}^d, \tag{5.3.2}$$

where

$$\begin{aligned}\mathcal{L}^a u(t,x) &= b(x,a).D_x u(t,x) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D_x^2 u(t,x)) \\ &+ \int_E (u(t,x+\beta(x,a,e)) - u(t,x) - \beta(x,a,e).D_x u(t,x))\lambda(a,de),\end{aligned}$$

for all $(t,x,a) \in [0,T] \times \mathbb{R}^d \times A$. Let us firstly introduce some additional notation. Fix a finite time horizon $T < \infty$ and set $\mathcal{P}_T$ the σ-field of $\mathbb{F}$-predictable subsets of $[0,T]\times\Omega$. For any $(t,\tilde{a}) \in [0,T] \times \mathbb{R}^q$, we denote :

- $\mathbf{L^p_{t,\tilde{a}}}(\mathcal{F}_s)$, $p \geq 1$, $s \geq 0$, the set of $\mathcal{F}_s$-measurable random variables X such that $\mathbb{E}^{t,\tilde{a}}[|X|^p] < \infty$.
- $\mathbf{S^2_{t,\tilde{a}}}$ the set of real-valued càdlàg adapted processes $Y = (Y_s)_{t\leq s\leq T}$ such that

$$\|Y\|^2_{\mathbf{S^2_{t,\tilde{a}}}} := \mathbb{E}^{t,\tilde{a}}\Big[\sup_{t\leq s\leq T}|Y_s|^2\Big] < \infty.$$

- $\mathbf{L^p_{t,\tilde{a}}}(\mathbf{t},\mathbf{T})$, $p \geq 1$, the set of real-valued adapted processes $(\phi_s)_{t\leq s\leq T}$ such that

$$\|\phi\|^p_{\mathbf{L^p_{t,\tilde{a}}}(\mathbf{t},\mathbf{T})} := \mathbb{E}^{t,\tilde{a}}\Big[\int_t^T |\phi_s|^p ds\Big] < \infty.$$

- $\mathbf{L^p_{t,\tilde{a}}}(\mathbf{W})$, $p \geq 1$, the set of $\mathbb{R}^d$-valued $\mathcal{P}_T$-measurable processes $Z = (Z_s)_{t\leq s\leq T}$ such that

$$\|Z\|^p_{\mathbf{L^p_{t,\tilde{a}}}(\mathbf{W})} := \mathbb{E}^{t,\tilde{a}}\Big[\Big(\int_t^T |Z_s|^2 ds\Big)^{\frac{p}{2}}\Big] < \infty.$$

- $\mathbf{L^p_{t,\tilde{a}}}(\mathbf{B})$, $p \geq 1$, the set of $\mathbb{R}^q$-valued $\mathcal{P}_T$-measurable processes $V = (V_s)_{t\leq s\leq T}$ such that

$$\|V\|^p_{\mathbf{L^p_{t,\tilde{a}}}(\mathbf{B})} := \mathbb{E}^{t,\tilde{a}}\Big[\Big(\int_t^T |V_s|^2 ds\Big)^{\frac{p}{2}}\Big] < \infty.$$

- $\mathbf{L^p_{t,\tilde{a}}}(\tilde{\pi})$, $p \geq 1$, the set of $\mathcal{P}_T \otimes \mathcal{B}(E)$-measurable maps $U\colon [t,T]\times\Omega\times E \to \mathbb{R}$ such that

$$\|U\|^p_{\mathbf{L^p_{t,\tilde{a}}}(\tilde{\pi})} := \mathbb{E}^{t,\tilde{a}}\Big[\Big(\int_t^T \int_E |U_s(e)|^2 \lambda(I_s^{t,\tilde{a}}, de) ds\Big)^{\frac{p}{2}}\Big] < \infty.$$

- $\mathbf{K^2_{t,\tilde{a}}}$ the set of nondecreasing predictable processes $K = (K_s)_{t\leq s\leq T} \in \mathbf{S^2_{t,\tilde{a}}}$ with $K_t = 0$, so that

$$\|K\|^2_{\mathbf{S^2_{t,\tilde{a}}}} = \mathbb{E}^{t,\tilde{a}}[|K_T|^2].$$

Remark 5.3.1. *Equivalence relation in* $\mathbf{L^p_{t,\tilde{a}}}(\tilde{\pi})$. When $U^1, U^2 \in \mathbf{L^p_{t,\tilde{a}}}(\tilde{\pi})$, with $U^1 = U^2$ we mean $\|U^1 - U^2\|_{\mathbf{L^p_{t,\tilde{a}}}(\tilde{\pi})} = 0$, i.e., $U^1 = U^2$ $ds \otimes d\mathbb{P}^{t,\tilde{a}} \otimes \lambda(I_s^{t,\tilde{a}}, de)$ a.e. on $[t,T]\times\Omega\times E$, where $ds \otimes d\mathbb{P}^{t,\tilde{a}} \otimes \lambda(I_s^{t,\tilde{a}}, de)$ is the measure on $([t,T]\times\Omega\times E, \mathcal{B}(t,T)\otimes\mathcal{F}\otimes\mathcal{B}(E))$ given by

$$ds \otimes d\mathbb{P}^{t,\tilde{a}} \otimes \lambda(I_s^{t,\tilde{a}}, de)(F) = \mathbb{E}^{t,\tilde{a}}\Big[\int_t^T \int_E 1_F(s,\omega,e)\lambda(I_s^{t,\tilde{a}}(\omega), de) ds\Big],$$

for all $F \in \mathcal{B}(t,T)\otimes\mathcal{F}\otimes\mathcal{B}(E)$. See also the beginning of Section 3 in [25]. □

The probabilistic representation formula is given in terms of the following BSDE with jumps and partially constrained diffusive part, for any $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, $\mathbb{P}^{t,\tilde{a}}$ a.s.,

$$\begin{aligned} Y_s &= g(X_T^{t,x,\tilde{a}}) + \int_s^T f(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}, Y_r, Z_r)dr + K_T - K_s - \int_s^T Z_r dW_r \\ &\quad - \int_s^T V_r dB_r - \int_s^T \int_E U_r(e)\tilde{\pi}(dr, de), \qquad t \leq s \leq T \end{aligned} \tag{5.3.3}$$

and

$$|V_s| = 0 \qquad ds \otimes d\mathbb{P}^{t,\tilde{a}} \; a.e. \tag{5.3.4}$$

We look for the minimal solution $(Y, Z, V, U, K) \in \mathbf{S^2_{t,\tilde{a}}} \times \mathbf{L^2_{t,\tilde{a}}(W)} \times \mathbf{L^2_{t,\tilde{a}}(B)} \times \mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi}) \times \mathbf{K^2_{t,\tilde{a}}}$ to (5.3.3)-(5.3.4), in the sense that for any other solution $(\bar{Y}, \bar{Z}, \bar{V}, \bar{U}, \bar{K}) \in \mathbf{S^2_{t,\tilde{a}}} \times \mathbf{L^2_{t,\tilde{a}}(W)} \times \mathbf{L^2_{t,\tilde{a}}(B)} \times \mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi}) \times \mathbf{K^2_{t,\tilde{a}}}$ to (5.3.3)-(5.3.4) we must have $Y \leq \bar{Y}$. We impose the following assumptions on the terminal condition $g\colon \mathbb{R}^d \to \mathbb{R}$ and on the generator $f\colon \mathbb{R}^d \times A \times \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}$.

(HBC) There exists some continuity modulus ρ (namely $\rho\colon [0, \infty) \to [0, \infty)$ is continuous, nondecreasing, subadditive, and $\rho(0) = 0$) and a constant C such that

$$|f(x, a, y, z) - f(x', a', y', z')| + |g(x) - g(x')| \leq \rho(|x - x'| + |a - a'|) + C(|y - y'| + |z - z'|)$$

for all $(x, a, y, z), (x', a', y', z') \in \mathbb{R}^d \times A \times \mathbb{R} \times \mathbb{R}^d$.

Proposition 5.3.1. *Let assumptions* **(HFC)** *and* **(HBC)** *hold. For any* $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, *there exists at most one minimal solution on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{t,\tilde{a}})$ *to the BSDE* (5.3.3)-(5.3.4).

Proof. Let (Y, Z, V, U, K) and $(\tilde{Y}, \tilde{Z}, \tilde{V}, \tilde{U}, \tilde{K})$ be two minimal solutions to (5.3.3)-(5.3.4). The uniqueness of the Y component is clear by definition. Regarding the other components, taking the difference between the two backward equations we obtain

$$\begin{aligned} 0 &= \int_t^s (f(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}, Y_r, Z_r) - f(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}, Y_r, \tilde{Z}_r))dr + K_s - \tilde{K}_s - \int_t^s (Z_r - \tilde{Z}_r)dW_r \\ &\quad - \int_t^s (V_r - \tilde{V}_r)dB_r - \int_t^s \int_E (U_r(e) - \tilde{U}_r(e))\tilde{\pi}(dr, de), \end{aligned}$$

for all $t \leq s \leq T$, $\mathbb{P}^{t,\tilde{a}}$ almost surely. Identifying the Brownian and finite variation parts, recalling that W and B are independent, we deduce $Z = \tilde{Z}$ and $V = \tilde{V}$. Therefore, we obtain the identity

$$\int_t^s \int_E (U_r(e) - \tilde{U}_r(e))\pi(dr, de) = \int_t^s \int_E (U_r(e) - \tilde{U}_r(e))\lambda(I_r^{t,\tilde{a}}, de)dr + K_s - \tilde{K}_s,$$

where the right-hand side is a predictable process, therefore it has no totally inaccessible jumps (see, e.g., Proposition 2.24, Chapter I, in [59]) ; on the other hand, the left-hand side is a pure-jump process with totally inaccessible jumps, unless $U = \tilde{U}$. As a consequence, we must have $U = \tilde{U}$, from which it follows that $K = \tilde{K}$. □

To guarantee the existence of the minimal solution to (5.3.3)-(5.3.4) we shall need the following result.

Lemma 5.3.1. *Let assumptions* **(HFC)** *and* **(HBC)** *hold. Then, for any initial condition* $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, *there exists a solution* $\{(\bar{Y}_s^{t,x,\tilde{a}}, \bar{Z}_s^{t,x,\tilde{a}}, \bar{V}_s^{t,x,\tilde{a}}, \bar{U}_s^{t,x,\tilde{a}}, \bar{K}_s^{t,x,\tilde{a}}),\ t \leq s \leq T\}$ *on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{t,\tilde{a}})$ *to the BSDE* (5.3.3)-(5.3.4), *with* $\bar{Y}_s^{t,x,\tilde{a}} = \bar{v}(s, X_s^{t,x,\tilde{a}})$ *for some deterministic function* $\bar{v}$ *on* $[0, T] \times \mathbb{R}^d$ *satisfying a linear growth condition*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} \frac{|\bar{v}(t, x)|}{1 + |x|} < \infty.$$

Proof. The proof can be done along the lines of the proof of Lemma 5.1 in [75], but for the fact that here we look for a function $\bar{v}$ satisfying a linear growth condition, rather than a more general polynomial growth condition. For this reason, we consider the mollifier $\eta(x) = \bar{c}\exp(1/(|x|^2 - 1))1_{\{|x|<1\}}$, where $\bar{c} > 0$ is such that $\int_{\mathbb{R}^d} \eta(x)dx = 1$, and we introduce the smooth function

$$\bar{v}(t, x) = \bar{C}e^{\rho(T-t)}\Big(1 + \int_{\mathbb{R}^d} \eta(x - y)|y|dy\Big), \qquad \forall\, (t, x) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q,$$

for some positive constants $\bar{C}$ and ρ. We can now proceed as in Lemma 5.1 in [75] to conclude that, for $\bar{C}$ and ρ large enough, the function $\bar{v}$ is a classical supersolution to (5.3.1)-(5.3.2). □

5.3.1 Existence of the minimal solution by penalization

In this section we prove the existence of the minimal solution to (5.3.3)-(5.3.4). We use a penalization approach and introduce the indexed sequence of BSDEs with jumps, for any $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, $\mathbb{P}^{t,\tilde{a}}$ a.s.,

$$\begin{aligned} Y_s^n &= g(X_T^{t,x,\tilde{a}}) + \int_s^T f(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}, Y_r^n, Z_r^n)dr + K_T^n - K_s^n - \int_s^T Z_r^n dW_r \\ &\quad - \int_s^T V_r^n dB_r - \int_s^T \int_E U_r^n(e)\tilde{\pi}(dr, de), \qquad t \leq s \leq T, \end{aligned} \tag{5.3.5}$$

for $n \in \mathbb{N}$, where K^n is the nondecreasing continuous process defined by

$$K_s^n = n\int_t^s |V_r^n|dr, \qquad t \leq s \leq T.$$

Proposition 5.3.2. *Under assumptions* **(HFC)** *and* **(HBC)**, *for every* $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ *and every* $n \in \mathbb{N}$ *there exists a unique solution* $(Y^{n,t,x,\tilde{a}}, Z^{n,t,x,\tilde{a}}, V^{n,t,x,\tilde{a}}, U^{n,t,x,\tilde{a}}) \in \mathbf{S}^2_{\mathbf{t},\tilde{\mathbf{a}}} \times \mathbf{L}^2_{\mathbf{t},\tilde{\mathbf{a}}}(\mathbf{W}) \times \mathbf{L}^2_{\mathbf{t},\tilde{\mathbf{a}}}(\mathbf{B}) \times \mathbf{L}^2_{\mathbf{t},\tilde{\mathbf{a}}}(\tilde{\pi})$ *on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{t,\tilde{a}})$ *satisfying the BSDE with jumps* (5.3.5).

Proof. As usual, the proof is based on a fixed point argument. More precisely, let us consider the function $\Phi\colon \mathbf{L}^2_{\mathbf{t},\tilde{\mathbf{a}}}(\mathbf{t},\mathbf{T}) \times \mathbf{L}^2_{\mathbf{t},\tilde{\mathbf{a}}}(\mathbf{W}) \times \mathbf{L}^2_{\mathbf{t},\tilde{\mathbf{a}}}(\mathbf{B}) \times \mathbf{L}^2_{\mathbf{t},\tilde{\mathbf{a}}}(\tilde{\pi}) \to \mathbf{L}^2_{\mathbf{t},\tilde{\mathbf{a}}}(\mathbf{t},\mathbf{T}) \times \mathbf{L}^2_{\mathbf{t},\tilde{\mathbf{a}}}(\mathbf{W}) \times \mathbf{L}^2_{\mathbf{t},\tilde{\mathbf{a}}}(\mathbf{B}) \times \mathbf{L}^2_{\mathbf{t},\tilde{\mathbf{a}}}(\tilde{\pi})$, mapping (Y', Z', V', U') to (Y, Z, V, U) defined by

$$\begin{aligned} Y_s &= g(X_T^{t,x,\tilde{a}}) + \int_s^T f_n(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}, Y_r', Z_r', V_r')dr - \int_s^T Z_r dW_r \\ &\quad - \int_s^T V_r dB_r - \int_s^T \int_E U_r(e)\tilde{\pi}(dr, de), \end{aligned} \tag{5.3.6}$$

where

$$f_n(x, a, y, z, v) = f(x, a, y, z) + n|v|.$$

More precisely, the quadruple (Y, Z, V, U) is constructed as follows : we consider the martingale $M_s = \mathbb{E}^{t,\tilde{a}}[g(X_T^{t,x,\tilde{a}}) + \int_t^T f_n(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}, Y'_r, Z'_r, V'_r)dr|\mathcal{F}_s]$, which is square integrable under the assumptions on g and f. From the martingale representation Theorem 5.5.1, we deduce the existence and uniqueness of $(Z, V, U) \in \mathbf{L^2_{t,\tilde{a}}(W)} \times \mathbf{L^2_{t,\tilde{a}}(B)} \times \mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi})$ such that

$$M_s \;=\; M_t + \int_t^s Z_r dW_r + \int_t^s V_r dB_r + \int_t^s \int_E U_r(e)\tilde{\pi}(dr, de). \tag{5.3.7}$$

We then define the process Y by

$$\begin{aligned} Y_s \;&=\; \mathbb{E}^{t,\tilde{a}}\Big[g(X_T^{t,x,\tilde{a}}) + \int_s^T f_n(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}, Y'_r, Z'_r, V'_r)dr\Big|\mathcal{F}_s\Big] \\ &=\; M_s - \int_t^s f_n(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}, Y'_r, Z'_r, V'_r)dr. \end{aligned}$$

By using the representation (5.3.7) of M in the previous relation, and noting that $Y_T = g(X_T^{t,x,\tilde{a}})$, we see that Y satisfies (5.3.6). Using the conditions on g and f, we deduce that Y lies in $\mathbf{L^2_{t,\tilde{a}}(t,T)}$, and also in $\mathbf{S^2_{t,\tilde{a}}}$. Hence, Φ is a well-defined map. We then see that $(Y^{n,t,x,\tilde{a}}, Z^{n,t,x,\tilde{a}}, V^{n,t,x,\tilde{a}}, U^{n,t,x,\tilde{a}})$ is a solution to the penalized BSDE (5.3.5) if and only if it is a fixed point of Φ. To this end, for any $\alpha > 0$ let us introduce the equivalent norm on $\mathbf{L^2_{t,\tilde{a}}(t,T)} \times \mathbf{L^2_{t,\tilde{a}}(W)} \times \mathbf{L^2_{t,\tilde{a}}(B)} \times \mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi})$:

$$\|(Y, Z, V, U)\|_\alpha \;:=\; \mathbb{E}^{t,\tilde{a}}\Big[\int_t^T e^{\alpha(s-t)}\Big(|Y_s|^2 + |Z_s|^2 + |V_s|^2 + \int_E |U_s(e)|^2\lambda(I_s^{t,\tilde{a}}, de)\Big)ds\Big].$$

It can be shown, proceeding along the same lines as in the classical case (for which we refer, e.g., to Theorem 6.2.1 in [90]), that there exists $\bar{\alpha} > 0$ such that Φ is a contraction on $\mathbf{L^2_{t,\tilde{a}}(t,T)} \times \mathbf{L^2_{t,\tilde{a}}(W)} \times \mathbf{L^2_{t,\tilde{a}}(B)} \times \mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi})$ endowed with the equivalent norm $\|\cdot\|_{\bar{\alpha}}$. Then, the claim follows from the Banach-Caccioppoli fixed-point theorem. □

We can now prove our main result of this section. Firstly, we need the following two lemmata.

Lemma 5.3.2. *Suppose that assumptions* **(HFC)** *and* **(HBC)** *hold. Then, for every* $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$*, we have, for all* $n \in \mathbb{N}$*,*

$$Y_s^{n,t,x,\tilde{a}} \;\leq\; Y_s^{n+1,t,x,\tilde{a}} \;\leq\; \hat{Y}_s^{t,x,\tilde{a}}$$

for all $0 \leq s \leq T$*,* $\mathbb{P}^{t,\tilde{a}}$ *a.s., where* $(\hat{Y}^{t,x,\tilde{a}}, \hat{Z}^{t,x,\tilde{a}}, \hat{V}^{t,x,\tilde{a}}, \hat{U}^{t,x,\tilde{a}}, \hat{K}^{t,x,\tilde{a}}) \in \mathbf{S^2_{t,\tilde{a}}} \times \mathbf{L^2_{t,\tilde{a}}(W)} \times \mathbf{L^2_{t,\tilde{a}}(B)} \times \mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi}) \times \mathbf{K^2_{t,\tilde{a}}}$ *on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{t,\tilde{a}})$ *is a generic solution to the BSDE (5.3.3)-(5.3.4). In particular, the sequence* $(Y^{n,t,x,\tilde{a}})_n$ *is upper bounded by* $\bar{Y}^{t,x,\tilde{a}}$ *introduced in Lemma 5.3.1.*

Proof. Fix $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ and $n \in \mathbb{N}$, and observe that

$$f_n(x, a, y, z, v) \;\leq\; f_{n+1}(x, a, y, z, v),$$

for all $(x, a, y, z, v) \in \mathbb{R}^d \times A \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^q$. Then, the inequality $Y_s^{n,t,x,\tilde{a}} \leq Y_s^{n+1,t,x,\tilde{a}}$, for all $0 \leq s \leq T$, $\mathbb{P}^{t,\tilde{a}}$ a.s., follows from the comparison Theorem A.1 in [75]. We should notice that Theorem A.1 in [75] is designed for BSDE with jumps driven by a Wiener process and a Poisson random measure, while in our case we have a general random measure π. Nevertheless, Theorem A.1 in [75] can be proved proceeding along the same lines as in

[75] to encompass this more general case.
Similarly, since $\int_t^s |\hat{V}_r^{t,x,\tilde{a}}|dr = 0$, it follows that $(\hat{Y}^{t,x,\tilde{a}}, \hat{Z}^{t,x,\tilde{a}}, \hat{V}^{t,x,\tilde{a}}, \hat{U}^{t,x,\tilde{a}}, \hat{K}^{t,x,\tilde{a}})$ solves the BSDE (5.3.3) with generator f_n, for any $n \in \mathbb{N}$, other than with generator f. Therefore, we can again apply the (generalized version, with the random measure π in place of the Poisson random measure, of the) comparison Theorem A.1 in [75], from which we deduce the claim. □

Lemma 5.3.3. *Under assumptions* **(HFC)** *and* **(HBC)**, *there exists a positive constant C such that, for all $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ and $n \in \mathbb{N}$,*

$$\begin{aligned} &\|Y^{n,t,x,\tilde{a}}\|^2_{\mathbf{S^2_{t,\tilde{a}}}} + \|Z^{n,t,x,\tilde{a}}\|^2_{\mathbf{L^2_{t,\tilde{a}}(W)}} + \|V^{n,t,x,\tilde{a}}\|^2_{\mathbf{L^2_{t,\tilde{a}}(B)}} + \|U^{n,t,x,\tilde{a}}\|^2_{\mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi})} + \|K^{n,t,x,\tilde{a}}\|^2_{\mathbf{S^2_{t,\tilde{a}}}} \\ &\leq C\Big(\mathbb{E}^{t,\tilde{a}}[|g(X_T^{t,x,\tilde{a}})|^2] + \mathbb{E}^{t,\tilde{a}}\Big[\int_t^T |f(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}}, 0, 0)|^2 ds\Big] + \|\bar{v}(\cdot, X_\cdot^{t,x,\tilde{a}})\|^2_{\mathbf{S^2_{t,\tilde{a}}}}\Big), \quad (5.3.8) \end{aligned}$$

where $\bar{v}$ is the function introduced in Lemma 5.3.1.

Proof. The proof is very similar to the proof of Lemma 3.3 in [75], so it is not reported. We simply recall that the claim follows applying Itô's formula to $|Y_s^{n,t,x,\tilde{a}}|^2$ between t and T, and exploiting Gronwall's lemma and Burkholder-Davis-Gundy inequality in an usual way. □

Theorem 5.3.1. *Under assumptions* **(HFC)** *and* **(HBC)**, *for every $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ there exists a unique minimal solution $(Y^{t,x,\tilde{a}}, Z^{t,x,\tilde{a}}, V^{t,x,\tilde{a}}, U^{t,x,\tilde{a}}, K^{t,x,\tilde{a}}) \in \mathbf{S^2_{t,\tilde{a}}} \times \mathbf{L^2_{t,\tilde{a}}(W)} \times \mathbf{L^2_{t,\tilde{a}}(B)} \times \mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi}) \times \mathbf{K^2_{t,\tilde{a}}}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{t,\tilde{a}})$ to the BSDE with jumps and partially constrained diffusive part (5.3.3)-(5.3.4), where :*

(i) *$Y^{t,x,\tilde{a}}$ is the increasing limit of $(Y^{n,t,x,\tilde{a}})_n$.*
(ii) *$(Z^{t,x,\tilde{a}}, V^{t,x,\tilde{a}}, U^{t,x,\tilde{a}})$ is the weak limit of $(Z^{n,t,x,\tilde{a}}, V^{n,t,x,\tilde{a}}, U^{n,t,x,\tilde{a}})_n$ in $\mathbf{L^2_{t,\tilde{a}}(W)} \times \mathbf{L^2_{t,\tilde{a}}(B)} \times \mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi})$.*
(iii) *$K_s^{t,x,\tilde{a}}$ is the weak limit of $(K_s^{n,t,x,\tilde{a}})_n$ in $\mathbf{L^2_{t,\tilde{a}}}(\mathcal{F}_s)$, for any $t \leq s \leq T$.*

Proof. Let $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$ be fixed. From Lemma 5.3.2 it follows that $(Y^{n,t,x,\tilde{a}})_n$ converges increasingly to some adapted process $Y^{t,x,\tilde{a}}$. We see that $Y^{t,x,\tilde{a}}$ satisfies the integrability condition $\mathbb{E}^{t,\tilde{a}}[\sup_{t\leq s\leq T} |Y_s^{t,x,\tilde{a}}|^2] < \infty$ as a consequence of the uniform estimate for $(Y^{n,t,x,\tilde{a}})_n$ in Lemma 5.3.3 and Fatou's lemma. Moreover, by Lebesgue's dominated convergence theorem, the convergence also holds in $\mathbf{L^2_{t,\tilde{a}}(t, T)}$. Next, by the uniform estimates in Lemma 5.3.3, the sequence $(Z^{n,t,x,\tilde{a}}, V^{n,t,x,\tilde{a}}, U^{n,t,x,\tilde{a}})_n$ is bounded in the Hilbert space $\mathbf{L^2_{t,\tilde{a}}(W)} \times \mathbf{L^2_{t,\tilde{a}}(B)} \times \mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi})$. Then, we can extract a subsequence which weakly converges to some $(Z^{t,x,\tilde{a}}, V^{t,x,\tilde{a}}, U^{t,x,\tilde{a}})$ in $\mathbf{L^2_{t,\tilde{a}}(W)} \times \mathbf{L^2_{t,\tilde{a}}(B)} \times \mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi})$. Thanks to the martingale representation Theorem 5.5.1, for every stopping time $t \leq \tau \leq T$, the following weak convergences hold in $\mathbf{L^2_{t,\tilde{a}}}(\mathcal{F}_\tau)$, as $n \to \infty$,

$$\int_t^\tau Z_s^{n,t,x,\tilde{a}} dW_s \rightharpoonup \int_t^\tau Z_s^{t,x,\tilde{a}} dW_s, \qquad \int_t^\tau V_s^{n,t,x,\tilde{a}} dB_s \rightharpoonup \int_t^\tau V_s^{t,x,\tilde{a}} dB_s,$$
$$\int_t^\tau \int_E U_s^{n,t,x,\tilde{a}}(e)\tilde{\pi}(ds, de) \rightharpoonup \int_t^\tau \int_E U_s^{t,x,\tilde{a}}(e)\tilde{\pi}(ds, de).$$

Since

$$K_\tau^{n,t,x,\tilde{a}} = Y_t^{n,t,x,\tilde{a}} - Y_\tau^{n,t,x,\tilde{a}} - \int_t^\tau f(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}}, Y_s^{n,t,x,\tilde{a}}, Z_s^{n,t,x,\tilde{a}}) ds + \int_t^\tau Z_s^{n,t,x,\tilde{a}} dW_s$$

$$+\int_t^\tau V_s^{n,t,x,\tilde a}dB_s + \int_t^\tau \int_E U_s^{n,t,x,\tilde a}(e)\tilde\pi(ds,de),$$

we also have the following weak convergence in $\mathbf{L^2_{t,\tilde a}}(\mathcal{F}_\tau)$, as $n\to\infty$,

$$\begin{aligned} K_\tau^{n,t,x,\tilde a} \rightharpoonup K_\tau^{t,x,\tilde a} &:= Y_t^{t,x,\tilde a} - Y_\tau^{t,x,\tilde a} - \int_t^\tau f(X_s^{t,x,\tilde a}, I_s^{t,\tilde a}, Y_s^{t,x,\tilde a}, Z_s^{t,x,\tilde a})ds \\ &\quad + \int_t^\tau Z_s^{t,x,\tilde a}dW_s + \int_t^\tau V_s^{t,x,\tilde a}dB_s + \int_t^\tau\int_E U_s^{t,x,\tilde a}(e)\tilde\pi(ds,de). \end{aligned}$$

Since the process $(K_s^{n,t,x,\tilde a})_{t\le s\le T}$ is nondecreasing and predictable and $K_t^{n,t,x,\tilde a}=0$, the limit process $K^{t,x,\tilde a}$ remains nondecreasing and predictable with $\mathbb{E}^{t,\tilde a}[|K_T^{t,x,\tilde a}|^2]<\infty$ and $K_t^{t,x,\tilde a}=0$. Moreover, by Lemma 2.2 in [87], $K^{t,x,\tilde a}$ and $Y^{t,x,\tilde a}$ are càdlàg, therefore $Y^{t,x,\tilde a}\in \mathbf{S^2_{t,\tilde a}}$ and $K^{t,x,\tilde a}\in\mathbf{K^2_{t,\tilde a}}$. In conclusion, we have

$$\begin{aligned} Y_t^{t,x,\tilde a} &= g(X_T^{t,x,\tilde a}) + \int_t^T f(X_s^{t,x,\tilde a}, I_s^{t,\tilde a}, Y_s^{t,x,\tilde a}, Z_s^{t,x,\tilde a})ds + K_T^{t,x,\tilde a} - K_t^{t,x,\tilde a} \\ &\quad - \int_t^T Z_s^{t,x,\tilde a}dW_s - \int_t^T V_s^{t,x,\tilde a}dB_s - \int_t^T\int_E U_s^{t,x,\tilde a}(e)\tilde\pi(ds,de). \end{aligned}$$

It remains to show that the diffusion constraint (5.3.4) is satisfied. To this end, we consider the functional $F\colon \mathbf{L^2_{t,\tilde a}}(\mathbf{B})\to\mathbb{R}$ given by

$$F(V) := \mathbb{E}^{t,\tilde a}\Big[\int_t^T |V_s|ds\Big], \qquad \forall\, V\in\mathbf{L^2_{t,\tilde a}}(\mathbf{B}).$$

Notice that $F(V^{n,t,x,\tilde a}) = \mathbb{E}^{t,\tilde a}[K_T^{n,t,x,\tilde a}]/n$, for any $n\in\mathbb{N}$. From estimate (5.3.8), we see that $F(V^{n,t,x,\tilde a})\to 0$ as $n\to\infty$. Since F is convex and strongly continuous in the strong topology of $\mathbf{L^2_{t,\tilde a}}(\mathbf{B})$, then F is lower semicontinuous in the weak topology of $\mathbf{L^2_{t,\tilde a}}(\mathbf{B})$, see, e.g., Corollary 3.9 in [19]. Therefore, we find

$$F(V^{t,x,\tilde a}) \le \liminf_{n\to\infty} F(V^{n,t,x,\tilde a}) = 0,$$

which implies the validity of the diffusion constraint (5.3.4). Hence, $(Y^{t,x,\tilde a}, Z^{t,x,\tilde a}, V^{t,x,\tilde a}, U^{t,x,\tilde a}, K^{t,x,\tilde a})$ is a solution to the BSDE with jumps and partially constrained diffusive part (5.3.3)-(5.3.4). From Lemma 5.3.2, we also see that $Y^{t,x,\tilde a}=\lim Y^{n,t,x,\tilde a}$ is the minimal solution to (5.3.3)-(5.3.4). Finally, the uniqueness of the solution $(Y^{t,x,\tilde a}, Z^{t,x,\tilde a}, V^{t,x,\tilde a}, U^{t,x,\tilde a}, K^{t,x,\tilde a})$ follows from Proposition 5.3.1. □

5.4 Nonlinear Feynman-Kac formula

We know from Theorem 5.3.1 that, under **(HFC)** and **(HBC)**, there exists a unique minimal solution $(Y^{t,x,\tilde a}, Z^{t,x,\tilde a}, V^{t,x,\tilde a}, U^{t,x,\tilde a}, K^{t,x,\tilde a})$ on $(\Omega,\mathcal{F},\mathbb{F},\mathbb{P}^{t,\tilde a})$ to (5.3.3)-(5.3.4). As we shall see below, this minimal solution admits the representation $Y_s^{t,x,\tilde a}=v(s,X_s^{t,x,\tilde a},I_s^{t,\tilde a})$, where $v\colon[0,T]\times\mathbb{R}^d\times A\to\mathbb{R}$ is the deterministic function defined as

$$v(t,x,h(\tilde a)) := Y_t^{t,x,\tilde a}, \qquad (t,x,\tilde a)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^q. \tag{5.4.1}$$

Our aim is to prove that the function v given by (5.4.1) does not depend on its last argument and that it is related to the fully nonlinear partial differential equation of HJB type (5.3.1)-(5.3.2). Notice that we do not know a priori whether the function v is continuous. Therefore, we shall adopt the definition of discontinuous viscosity solution to (5.3.1)-(5.3.2). Firstly, we impose the following conditions on h and A.

(HA) There exists a compact set $A_h \subset \mathbb{R}^q$ such that $h(A_h) = A$. Moreover, the interior set $\mathring{A}_h$ of A_h is connected, and $A_h = \mathrm{Cl}(\mathring{A}_h)$, the closure of its interior. Furthermore, $h(\mathring{A}_h) = \mathring{A}$.

We also impose some conditions on λ, which will imply the validity of a comparison theorem for viscosity sub and supersolutions to the fully nonlinear IPDE of HJB type (5.3.1)-(5.3.2) and also for penalized IPDE (5.4.5)-(5.4.6). To this end, let us define, for every $\delta > 0$ and $(t, x, a) \in [0, T] \times \mathbb{R}^d \times A$,

$$I_a^{1,\delta}(t, x, \varphi) = \int_{E\cap\{|e|\leq\delta\}} (\varphi(t, x + \beta(x, a, e)) - \varphi(t, x) - \beta(x, a, e).D_x\varphi(t, x))\lambda(a, de),$$

for any $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$, and

$$I_a^{2,\delta}(t, x, q, u) = \int_{E\cap\{|e|>\delta\}} (u(t, x + \beta(x, a, e)) - u(t, x) - \beta(x, a, e).q)\lambda(a, de),$$

for any $q \in \mathbb{R}^d$ and any locally bounded function u. Let us impose the following conditions on $I_a^{1,\delta}$ and $I_a^{2,\delta}$.

(Hλ)

(i) For any $(t, x) \in [0, T] \times \mathbb{R}^d$, we have

$$\sup_{a\in A} \int_{E\cap\{|e|\leq\delta\}} (1 \wedge |e|^2)\lambda(a, de) \overset{\delta\to 0^+}{\longrightarrow} 0.$$

(ii) Let $\varphi \in C^{1,2}([0, T]\times\mathbb{R}^d)$. If the sequence $\{(t_k, x_k, a_k)\}_k \subset [0, T]\times\mathbb{R}^d\times A$ converges to (t^*, x^*, a^*) as k goes to infinity, then

$$\lim_{k\to\infty} I_{a_k}^{1,\delta}(t_k, x_k, \varphi) = I_{a^*}^{1,\delta}(t^*, x^*, \varphi),$$

for any $\delta > 0$.

(iii) Let $u\colon [0, T] \times \mathbb{R}^d \to \mathbb{R}$ be usc (resp. lsc) and locally bounded. If the sequence $\{(t_k, x_k, q_k, a_k)\}_k \subset [0, T] \times \mathbb{R}^d \times \mathbb{R}^d \times A$ converges to (t^*, x^*, q^*, a^*) and $u(t_k, x_k) \to u(t^*, x^*)$, as k goes to infinity, then

$$\limsup_{k\to\infty} I_{a_k}^{2,\delta}(t_k, x_k, q_k, u) \leq I_{a^*}^{2,\delta}(t^*, x^*, q^*, u)$$
$$\Big(\text{resp. } \liminf_{k\to\infty} I_{a_k}^{2,\delta}(t_k, x_k, q_k, u) \geq I_{a^*}^{2,\delta}(t^*, x^*, q^*, u)\Big)$$

for any $\delta > 0$.

Remark 5.4.1. Assumption **(Hλ)** is required for the proof of the comparison Theorem 5.5.2 (as well as for the comparison theorem to equation (5.4.5)-(5.4.6)). Notice that conditions (i)-(ii)-(iii) are inspired by the fourth and fifth Assumptions (NLT) in [4]. We also observe that, whenever $I_a^{1,\delta}$ and $I_a^{2,\delta}$ do not depend on a, then **(Hλ)**(i)-(ii) are consequences of Lebesgue's dominated convergence theorem, while **(Hλ)**(iii) follows from Fatou's lemma. □

For a locally bounded function u on $[0,T) \times \mathbb{R}^k$, we define its lower semicontinuous (lsc for short) envelope u_*, and upper semicontinuous (usc for short) envelope u^*, by

$$u_*(t,\xi) = \liminf_{\substack{(s,\eta)\to(t,\xi) \\ s<T}} u(s,\xi) \quad \text{and} \quad u^*(t,\xi) = \limsup_{\substack{(s,\eta)\to(t,\xi) \\ s<T}} u(s,\xi)$$

for all $(t,\xi) \in [0,T] \times \mathbb{R}^k$.

Definition 5.4.1. (*Viscosity solution to* (5.3.1)-(5.3.2))

(i) *A lsc (resp. usc) function u on $[0,T] \times \mathbb{R}^d$ is called a* ***viscosity supersolution*** *(resp.* ***viscosity subsolution****) to* (5.3.1)-(5.3.2) *if*

$$u(T,x) \;\geq\; (resp. \;\leq)\; g(x)$$

for any $x \in \mathbb{R}^d$, and

$$-\frac{\partial \varphi}{\partial t}(t,x) - \sup_{a\in A}\big(\mathcal{L}^a\varphi(t,x) + f(x,a,u(t,x),\sigma^\intercal(x,a)D_x\varphi(t,x))\big) \;\geq\; (resp. \;\leq)\; 0$$

for any $(t,x) \in [0,T) \times \mathbb{R}^d$ and any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ such that

$$(u-\varphi)(t,x) \;=\; \min_{[0,T]\times\mathbb{R}^d}(u-\varphi) \quad (resp. \;\max_{[0,T]\times\mathbb{R}^d}(u-\varphi)).$$

(ii) *A locally bounded function u on $[0,T) \times \mathbb{R}^d$ is called a* ***viscosity solution*** *to* (5.3.1)-(5.3.2) *if u_* is a viscosity supersolution and u^* is a viscosity subsolution to* (5.3.1)-(5.3.2).

We can now state the main result of this paper.

Theorem 5.4.1. *Assume that conditions* **(HFC)**, **(HBC)**, **(HA)**, *and* **(Hλ)** *hold. Then, the function v in* (5.4.1) *does not depend on the variable a on $[0,T) \times \mathbb{R}^d \times \mathring{A}$:*

$$v(t,x,a) \;=\; v(t,x,a'), \qquad \forall\, a,a' \in \mathring{A},$$

for all $(t,x) \in [0,T) \times \mathbb{R}^d$. Let us then define by misuse of notation the function v on $[0,T) \times \mathbb{R}^d$ by

$$v(t,x) \;=\; v(t,x,a), \qquad (t,x) \in [0,T) \times \mathbb{R}^d,$$

for any $a \in \mathring{A}$. Then v is a viscosity solution to (5.3.1)-(5.3.2).

The rest of the paper is devoted to the proof of Theorem 5.4.1.

5.4.1 Viscosity property of the penalized BSDE

For every $n \in \mathbb{N}$, let us introduce the deterministic function v_n defined on $[0,T] \times \mathbb{R}^d \times A$ by

$$v_n(t,x,h(\tilde{a})) \;:=\; Y_t^{n,t,x,\tilde{a}}, \qquad (t,x,\tilde{a}) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q, \tag{5.4.2}$$

where $(Y^{n,t,x,\tilde{a}}, Z^{n,t,x,\tilde{a}}, V^{n,t,x,\tilde{a}}, U^{n,t,x,\tilde{a}})$ is the unique solution to the BSDE with jumps (5.3.5), see Proposition 5.3.2. As we shall see in Proposition 5.4.1, the identification $Y_s^{n,t,x,\tilde{a}} = v_n(s, X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})$ holds. Therefore, sending n to infinity, it follows from the convergence results of the penalized BSDE, Theorem 5.3.1, that the minimal solution to the BSDE with jumps and partially constrained diffusive part (5.3.3)-(5.3.4) can be written as $Y_s^{t,x,\tilde{a}} = v(s, X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})$, $t \leq s \leq T$, where v is the deterministic function defined in (5.4.1).

Now, notice that, from the uniform estimate (5.3.8), the linear growth conditions of g, f, and $\bar{v}$, estimate (5.2.5), and the compactness of A, it follows that v_n, and thus also v by passing to the limit, satisfies the following linear growth condition : there exists some positive constant C_v such that, for all $n \in \mathbb{N}$,

$$|v_n(t,x,a)| + |v(t,x,a)| \ \leq \ C_v(1+|x|), \qquad \forall\,(t,x,a) \in [0,T]\times\mathbb{R}^d\times A. \tag{5.4.3}$$

As expected, for every $n \in \mathbb{N}$, the function v_n in (5.4.2) is related to a parabolic semi-linear penalized IPDE. More precisely, let us introduce the function $v_n^h\colon [0,T]\times\mathbb{R}^d\times\mathbb{R}^q \to \mathbb{R}$ given by

$$v_n^h(t,x,\tilde{a}) \ := \ v_n(t,x,h(\tilde{a})), \qquad (t,x,\tilde{a}) \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^q. \tag{5.4.4}$$

Then, the function v_n^h is related to the semi-linear penalized IPDE :

$$\begin{aligned} -\frac{\partial v_n^h}{\partial t}(t,x,\tilde{a}) - \mathcal{L}^{h(\tilde{a})} v_n^h(t,x,\tilde{a}) & \\ -f\big(x,h(\tilde{a}),v_n^h(t,x,\tilde{a}),\sigma^{\intercal}(x,h(\tilde{a}))D_x v_n^h(t,x,\tilde{a})\big) & \qquad\qquad\qquad (5.4.5)\\ -\frac{1}{2}\mathrm{tr}\big(D_{\tilde{a}}^2 v_n^h(t,x,\tilde{a})\big) - n\big|D_{\tilde{a}} v_n^h(t,x,\tilde{a})\big| &= 0, \quad \text{on } [0,T)\times\mathbb{R}^d\times\mathbb{R}^q,\\ v_n^h(T,\cdot,\cdot) &= g, \quad \text{on } \mathbb{R}^d\times\mathbb{R}^q. \qquad (5.4.6) \end{aligned}$$

Let us provide the definition of discontinuous viscosity solution to equation (5.4.5)-(5.4.6).

Definition 5.4.2. *(Viscosity solution to (5.4.5)-(5.4.6))*

(i) *A lsc (resp. usc) function u on $[0,T]\times\mathbb{R}^d\times\mathbb{R}^q$ is called a **viscosity supersolution** (resp. **viscosity subsolution**) to (5.4.5)-(5.4.6) if*

$$u(T,x,\tilde{a}) \ \geq \ (resp. \ \leq)\ g(x)$$

for any $(x,\tilde{a}) \in \mathbb{R}^d\times\mathbb{R}^q$, and

$$\begin{aligned} -\frac{\partial \varphi}{\partial t}(t,x,\tilde{a}) - \mathcal{L}^{h(\tilde{a})}\varphi(t,x,\tilde{a}) - f\big(x,h(\tilde{a}),u(t,x,\tilde{a}),\sigma^{\intercal}(x,h(\tilde{a}))D_x\varphi(t,x,\tilde{a})\big) \\ -\frac{1}{2}\mathrm{tr}\big(D_{\tilde{a}}^2\varphi(t,x,\tilde{a})\big) - n\big|D_{\tilde{a}}\varphi(t,x,\tilde{a})\big| \ \geq \ 0 \quad (resp. \quad \leq \ 0) \end{aligned}$$

for any $(t,x,\tilde{a}) \in [0,T)\times\mathbb{R}^d\times\mathbb{R}^q$ and any $\varphi \in C^{1,2}([0,T]\times(\mathbb{R}^d\times\mathbb{R}^q))$ such that

$$(u-\varphi)(t,x,\tilde{a}) \ = \ \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(u-\varphi) \qquad (resp.\ \max_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(u-\varphi)). \tag{5.4.7}$$

(ii) *A locally bounded function u on $[0,T)\times\mathbb{R}^d\times\mathbb{R}^q$ is called a **viscosity solution** to (5.4.5)-(5.4.6) if u_* is a viscosity supersolution and u^* is a viscosity subsolution to (5.4.5)-(5.4.6).*

Then, we have the following result, which states that the penalized BSDE with jumps (5.3.5) provides a viscosity solution to the penalized IPDE (5.4.5)-(5.4.6).

Proposition 5.4.1. *Let assumptions* **(HFC)**, **(HBC)**, **(HA)**, *and* **(Hλ)** *hold. Then, the function v_n^h in (5.4.4) is a viscosity solution to (5.4.5)-(5.4.6). Moreover, v_n^h is continuous on $[0,T]\times\mathbb{R}^d\times\mathbb{R}^q$.*

Proof We divide the proof into three steps.

Step 1. Identification $Y_s^{n,t,x,\tilde a} = v_n(s, X_s^{t,x,\tilde a}, I_s^{t,\tilde a}) = v_n^h(s, X_s^{t,x,\tilde a}, \tilde a + B_s - B_t)$. Inspired by the proof of Theorem 4.1 in [39], we shall prove the identification $Y_s^{n,t,x,\tilde a} = v_n(s, X_s^{t,x,\tilde a}, I_s^{t,\tilde a})$ using the Markovian property of (X, I) studied in Appendix 5.5.2 and the construction of $(Y^{n,t,x,\tilde a}, Z^{n,t,x,\tilde a}, V^{n,t,x,\tilde a}, U^{n,t,x,\tilde a})$ based on Proposition 5.3.2. More precisely, for any $(t, x, \tilde a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, from Proposition 5.3.2 we know that there exists a sequence $(Y^{n,k,t,x,\tilde a}, Z^{n,k,t,x,\tilde a}, V^{n,k,t,x,\tilde a}, U^{n,k,t,x,\tilde a}) \in \mathbf{L^2_{t,\tilde a}(t,T)} \times \mathbf{L^2_{t,\tilde a}(W)} \times \mathbf{L^2_{t,\tilde a}(B)} \times \mathbf{L^2_{t,\tilde a}}(\tilde\pi)$, converging to $(Y^{n,t,x,\tilde a}, Z^{n,t,x,\tilde a}, V^{n,t,x,\tilde a}, U^{n,t,x,\tilde a})$ in $\mathbf{L^2_{t,\tilde a}(t,T)} \times \mathbf{L^2_{t,\tilde a}(W)} \times \mathbf{L^2_{t,\tilde a}(B)} \times \mathbf{L^2_{t,\tilde a}}(\tilde\pi)$, such that $(Y^{n,0,t,x,\tilde a}, Z^{n,0,t,x,\tilde a}, V^{n,0,t,x,\tilde a}, U^{n,0,t,x,\tilde a}) \equiv (0,0,0,0)$ and

$$\begin{aligned} Y_s^{n,k+1,t,x,\tilde a} &= g(X_T^{t,x,\tilde a}) + \int_s^T f(X_r^{t,x,\tilde a}, I_r^{t,\tilde a}, Y_r^{n,k,t,x,\tilde a}, Z_r^{n,k,t,x,\tilde a}) dr + n \int_s^T |V_r^{n,k,t,x,\tilde a}| dr \\ &\quad - \int_s^T Z_r^{n,k+1,t,x,\tilde a} dW_r - \int_s^T V_r^{n,k+1,t,x,\tilde a} dB_r - \int_s^T \int_E U_r^{n,k+1,t,x,\tilde a}(e) \tilde\pi(dr, de), \end{aligned}$$

for all $t \le s \le T$, $\mathbb{P}^{t,\tilde a}$ almost surely. Let us define $v_{n,k}(t, x, \tilde a) := Y_t^{n,k,t,x,\tilde a}$. We begin noting that, for $k = 0$ we have

$$Y_s^{n,1,t,x,\tilde a} = \mathbb{E}^{t,\tilde a}\Big[g(X_T^{t,x,\tilde a}) + \int_s^T f(X_r^{t,x,\tilde a}, I_r^{t,\tilde a}, 0, 0) dr \Big| \mathcal{F}_s \Big].$$

Then, we see from Proposition 5.5.3 that $Y_s^{n,1,t,x,\tilde a} = v_{n,1}(s, X_s^{t,x,\tilde a}, I_s^{t,\tilde a})$, $d\mathbb{P}^{t,\tilde a} \otimes ds$ almost everywhere. Proceeding as in Lemma 4.1 of [39] (in particular, relying on Theorem 6.27 in [21]), we also deduce that there exist Borel measurable functions $\tilde z_{n,1}$ and $\tilde v_{n,1}$ such that, respectively, $Z_s^{n,1,t,x,\tilde a} = \tilde z_{n,1}(s, X_{s^-}^{t,x,\tilde a}, I_s^{t,\tilde a})$ and $V_s^{n,1,t,x,\tilde a} = \tilde v_{n,1}(s, X_{s^-}^{t,x,\tilde a}, I_s^{t,\tilde a})$, $d\mathbb{P}^{t,\tilde a} \otimes ds$ almost everywhere. Since $Z^{n,1,t,x,\tilde a} \in \mathbf{L^2_{t,\tilde a}(W)}$ and $V^{n,1,t,x,\tilde a} \in \mathbf{L^2_{t,\tilde a}(B)}$, we notice that

$$\mathbb{E}^{t,\tilde a}\Big[\int_t^T |\tilde z_{n,1}(s, X_{s^-}^{t,x,\tilde a}, I_s^{t,\tilde a})|^2 ds \Big] < \infty, \quad \mathbb{E}^{t,\tilde a}\Big[\int_t^T |\tilde v_{n,1}(s, X_{s^-}^{t,x,\tilde a}, I_s^{t,\tilde a})|^2 ds \Big] < \infty. \tag{5.4.8}$$

Let us now prove the inductive step : consider $k \ge 1$ and suppose that $Y_s^{n,k,t,x,\tilde a} = v_{n,k}(s, X_s^{t,x,\tilde a}, I_s^{t,\tilde a})$, $Z_s^{n,k,t,x,\tilde a} = \tilde z_{n,k}(s, X_{s^-}^{t,x,\tilde a}, I_s^{t,\tilde a})$, and $V_s^{n,k,t,x,\tilde a} = \tilde v_{n,k}(s, X_{s^-}^{t,x,\tilde a}, I_s^{t,\tilde a})$, $d\mathbb{P}^{t,\tilde a} \otimes ds$ a.e., with $\mathbb{E}^{t,\tilde a}[\int_t^T |\tilde z_{n,k}(s, X_{s^-}^{t,x,\tilde a}, I_s^{t,\tilde a})|^2 ds] < \infty$ and $\mathbb{E}^{t,\tilde a}[\int_t^T |\tilde v_{n,k}(s, X_{s^-}^{t,x,\tilde a}, I_s^{t,\tilde a})|^2 ds] < \infty$. Then, we have

$$\begin{aligned} Y_s^{n,k+1,t,x,\tilde a} &= \mathbb{E}^{t,\tilde a}\Big[g(X_T^{t,x,\tilde a}) + \int_s^T f(X_r^{t,x,\tilde a}, I_r^{t,\tilde a}, v_{n,k}(r, X_{r^-}^{t,x,\tilde a}, I_r^{t,\tilde a}), \tilde z_{n,k}(r, X_{r^-}^{t,x,\tilde a}, I_r^{t,\tilde a})) dr \\ &\quad + n \int_s^T |\tilde v_{n,k}(r, X_{r^-}^{t,x,\tilde a}, I_r^{t,\tilde a})| dr \Big| \mathcal{F}_s \Big]. \end{aligned}$$

Using again Proposition 5.5.3 (notice that, by a monotone class argument, we can extend Proposition 5.5.3 to Borel measurable functions verifying an integrability condition of the type (5.4.8)) we see that $Y_s^{n,k+1,t,x,\tilde a} = v_{n,k+1}(s, X_s^{t,x,\tilde a}, I_s^{t,\tilde a})$, $d\mathbb{P}^{t,\tilde a} \otimes ds$ almost everywhere. Now, we notice that it can be shown that $\mathbb{E}[\sup_{t \le s \le T} |Y_s^{n,k,t,x,\tilde a} - Y_s^{n,t,x,\tilde a}|] \to 0$, as k tends to infinity (e.g., proceeding as in Remark (b) after Proposition 2.1 in [39]). Therefore, $v_{n,k}(t, x, \tilde a) \to v_n(t, x, \tilde a)$ as k tends to infinity, for all $(t, x, \tilde a) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, from which it follows the validity of the identification $Y_s^{n,t,x,\tilde a} = v_n(s, X_s^{t,x,\tilde a}, I_s^{t,\tilde a}) = v_n^h(s, X_s^{t,x,\tilde a}, \tilde a + B_s - B_t)$, $d\mathbb{P}^{t,\tilde a} \otimes ds$ almost everywhere.

Step 2. Viscosity property of v_n^h. We shall divide the proof into two substeps.

Step 2a. v_n^h is a viscosity solution to (5.4.5). We now prove the viscosity supersolution property of v_n^h to (5.4.5). A similar argument would show that v_n^h it is a viscosity subsolution to (5.4.5). Let $(\bar t, \bar x, \bar{\tilde a}) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^q$ and $\varphi \in C^{1,2}([0,T] \times (\mathbb{R}^d \times \mathbb{R}^q))$ such that

$$0 \;=\; ((v_n^h)_* - \varphi)(\bar t, \bar x, \bar{\tilde a}) \;=\; \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q} ((v_n^h)_* - \varphi). \tag{5.4.9}$$

Let us proceed by contradiction, assuming that

$$\begin{aligned} -\frac{\partial\varphi}{\partial t}(\bar t, \bar x, \bar{\tilde a}) - \mathcal{L}^{h(\bar{\tilde a})}\varphi(\bar t, \bar x, \bar{\tilde a}) - f\big(\bar x, h(\bar{\tilde a}), \varphi(\bar t, \bar x, \bar{\tilde a}), \sigma^\intercal(\bar x, h(\bar{\tilde a})) D_x\varphi(\bar t, \bar x, \bar{\tilde a})\big) \\ -\frac{1}{2}\mathrm{tr}\big(D^2_{\tilde a}\varphi(\bar t, \bar x, \bar{\tilde a})\big) - n\big|D_{\tilde a}\varphi(\bar t, \bar x, \bar{\tilde a})\big| \;=:\; -2\varepsilon \;<\; 0. \end{aligned}$$

Using the continuity of b, σ, β, f, and h, we find $\delta > 0$ such that

$$\begin{aligned} -\frac{\partial\varphi}{\partial t}(t, x, \tilde a) - \mathcal{L}^{h(\tilde a)}\varphi(t, x, \tilde a) - f\big(x, h(\tilde a), \varphi(t, x, \tilde a), \sigma^\intercal(x, h(\tilde a)) D_x\varphi(t, x, \tilde a)\big) \\ -\frac{1}{2}\mathrm{tr}\big(D^2_{\tilde a}\varphi(t, x, \tilde a)\big) - n\big|D_{\tilde a}\varphi(t, x, \tilde a)\big| \;\le\; -\varepsilon. \end{aligned} \tag{5.4.10}$$

for any $(t, x, \tilde a) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$ with $|t - \bar t|, |x - \bar x|, |\tilde a - \bar{\tilde a}| < \delta$. Define

$$\tau := \inf\big\{r \ge \bar t \colon |X_r^{\bar t, \bar x, \bar{\tilde a}} - \bar x| > \delta,\; |B_r - B_{\bar t}| > \delta\big\} \wedge (\bar t + \delta) \wedge T.$$

Since $X^{\bar t, \bar x, \bar{\tilde a}}$ is càdlàg, it is in particular right-continuous at time $\bar t$. Therefore, $\tau > \bar t$, $\mathbb{P}^{\bar t, \bar{\tilde a}}$ almost surely. Then, an application of Itô's formula to $(r - \bar t)\varphi(r, X_r^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_r - B_{\bar t})$ between $\bar t$ and τ, using also (5.4.10), yields

$$\begin{aligned} &(\tau - \bar t)\varphi(\tau, X_\tau^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_\tau - B_{\bar t}) \;\ge\; \int_{\bar t}^{\tau} \varphi(r, X_r^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_r - B_{\bar t}) dr \\ &- n\int_{\bar t}^{\tau} (r - \bar t)|D_{\tilde a}\varphi(r, X_r^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_r - B_{\bar t})| dr + \int_{\bar t}^{\tau} (r - \bar t) D_{\tilde a}\varphi(r, X_r^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_r - B_{\bar t}) dB_r \\ &- \int_{\bar t}^{\tau} (r - \bar t) f\big(X_r^{\bar t, \bar x, \bar{\tilde a}}, I_r^{\bar t, \bar{\tilde a}}, \varphi(r, X_r^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_r - B_{\bar t}), \sigma^\intercal D_x\varphi(r, X_r^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_r - B_{\bar t})\big) dr \\ &+ \varepsilon(\tau - \bar t)^2 + \int_{\bar t}^{\tau} (r - \bar t)(D_x\varphi(r, X_r^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_r - B_{\bar t}))^\intercal \sigma(X_r^{\bar t, \bar x, \bar{\tilde a}}, I_r^{\bar t, \bar{\tilde a}}) dW_r \\ &+ \int_{\bar t}^{\tau}\int_E (r - \bar t)\big(\varphi(r, X_{r^-}^{\bar t, \bar x, \bar{\tilde a}} + \beta, \bar{\tilde a} + B_r - B_{\bar t}) - \varphi(r, X_{r^-}^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_r - B_{\bar t})\big)\tilde\pi(dr, de). \end{aligned} \tag{5.4.11}$$

Applying Itô's formula to $(r - \bar t)Y_r^{n, \bar t, \bar x, \bar{\tilde a}}$ between $\bar t$ and τ, using (5.3.5) and the identification $Y_r^{n, \bar t, \bar x, \bar{\tilde a}} = v_n^h(r, X_r^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_r - B_{\bar t})$, we find

$$\begin{aligned} (\tau - \bar t) v_n^h(\tau, X_\tau^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_\tau - B_{\bar t}) \;&=\; \int_{\bar t}^{\tau} Y_r^{n, \bar t, \bar x, \bar{\tilde a}} dr - n\int_{\bar t}^{\tau} (r - \bar t)|V_r^{n, \bar t, \bar x, \bar{\tilde a}}| dr \\ - \int_{\bar t}^{\tau} (r - \bar t) f\big(X_r^{\bar t, \bar x, \bar{\tilde a}}, I_r^{\bar t, \bar{\tilde a}}, Y_r^{n, \bar t, \bar x, \bar{\tilde a}}, Z_r^{n, \bar t, \bar x, \bar{\tilde a}}\big) dr &+ \int_{\bar t}^{\tau} (r - \bar t) Z_r^{n, \bar t, \bar x, \bar{\tilde a}} dW_r \\ + \int_{\bar t}^{\tau} (r - \bar t) V_r^{n, \bar t, \bar x, \bar{\tilde a}} dB_r + \int_{\bar t}^{\tau}\int_E (r - \bar t) U_r^{n, \bar t, \bar x, \bar{\tilde a}}(e)\tilde\pi(dr, de). \end{aligned} \tag{5.4.12}$$

Plugging (5.4.12) into (5.4.11), we obtain

$$(\tau - \bar t)\big(\varphi(\tau, X_\tau^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_\tau - B_{\bar t}) - v_n^h(\tau, X_\tau^{\bar t, \bar x, \bar{\tilde a}}, \bar{\tilde a} + B_\tau - B_{\bar t})\big) \tag{5.4.13}$$

$$
\begin{aligned}
&\geq \int_{\bar t}^{\tau} \big(\varphi(r, X_r^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t}) - Y_r^{n,\bar t,\bar x,\bar a}\big)dr + \varepsilon(\tau - \bar t)^2 \\
&- \int_{\bar t}^{\tau} (r-\bar t) f\big(X_r^{\bar t,\bar x,\bar a}, I_r^{\bar t,\bar a}, \varphi(r, X_r^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t}), \sigma^{\intercal} D_x\varphi(r, X_r^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t})\big)dr \\
&+ \int_{\bar t}^{\tau} (r-\bar t) f\big(X_r^{\bar t,\bar x,\bar a}, I_r^{\bar t,\bar a}, Y_r^{n,\bar t,\bar x,\bar a}, Z_r^{n,\bar t,\bar x,\bar a}\big)dr - n\int_{\bar t}^{\tau}(r-\bar t)|D_{\bar a}\varphi(r, X_r^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t})|dr \\
&+ n\int_{\bar t}^{\tau}(r-\bar t)|V_r^{n,\bar t,\bar x,\bar a}|dr + \int_{\bar t}^{\tau}(r-\bar t)(D_x\varphi(r, X_r^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t}))^{\intercal}\sigma(X_r^{\bar t,\bar x,\bar a}, I_r^{\bar t,\bar a})dW_r \\
&- \int_{\bar t}^{\tau}(r-\bar t)Z_r^{n,\bar t,\bar x,\bar a}dW_r + \int_{\bar t}^{\tau}(r-\bar t)D_{\bar a}\varphi(r, X_r^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t})dB_r \\
&- \int_{\bar t}^{\tau}(r-\bar t)V_r^{n,\bar t,\bar x,\bar a}dB_r - \int_{\bar t}^{\tau}\int_E (r-\bar t)U_r^{n,\bar t,\bar x,\bar a}(e)\tilde\pi(dr,de) \\
&+ \int_{\bar t}^{\tau}\int_E (r-\bar t)\big(\varphi(r, X_{r^-}^{\bar t,\bar x,\bar a} + \beta, \bar a + B_r - B_{\bar t}) - \varphi(r, X_{r^-}^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t})\big)\tilde\pi(dr,de).
\end{aligned}
$$

Let us introduce the predictable processes $\alpha\colon [\bar t, T]\times\Omega\to\mathbb{R}$, $\beta\colon [\bar t, T]\times\Omega\to\mathbb{R}^d$, and $\gamma\colon [\bar t, T]\times\Omega\to\mathbb{R}^q$ given by

$$
\begin{aligned}
\alpha_r &= 1 - (r-\bar t)\frac{f(X_{r^-}^{\bar t,\bar x,\bar a}, I_{r^-}^{\bar t,\bar a}, \varphi, \sigma^{\intercal}D_x\varphi) - f(X_{r^-}^{\bar t,\bar x,\bar a}, I_{r^-}^{\bar t,\bar a}, Y_{r^-}^{n,\bar t,\bar x,\bar a}, \sigma^{\intercal}D_x\varphi)}{\varphi(r, X_{r^-}^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t}) - Y_{r^-}^{n,\bar t,\bar x,\bar a}} 1_{\{\varphi\neq Y_{r^-}^{n,\bar t,\bar x,\bar a}\}}, \\
\beta_r &= -(r-\bar t)\frac{f(X_{r^-}^{\bar t,\bar x,\bar a}, I_{r^-}^{\bar t,\bar a}, Y_{r^-}^{n,\bar t,\bar x,\bar a}, \sigma^{\intercal}D_x\varphi) - f(X_{r^-}^{\bar t,\bar x,\bar a}, I_{r^-}^{\bar t,\bar a}, Y_{r^-}^{n,\bar t,\bar x,\bar a}, Z_r^{n,\bar t,\bar x,\bar a})}{|\sigma^{\intercal}D_x\varphi(r, X_{r^-}^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t}) - Z_r^{n,\bar t,\bar x,\bar a}|}\cdot \\
&\quad\cdot \frac{\sigma^{\intercal}D_x\varphi(r, X_{r^-}^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t}) - Z_r^{n,\bar t,\bar x,\bar a}}{|\sigma^{\intercal}D_x\varphi(r, X_{r^-}^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t}) - Z_r^{n,\bar t,\bar x,\bar a}|} 1_{\{\sigma^{\intercal}D_x\varphi(r, X_{r^-}^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t})\neq Z_r^{n,\bar t,\bar x,\bar a}\}}, \\
\gamma_r &= -n(r-\bar t)\frac{|D_{\bar a}\varphi(r, X_{r^-}^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t})| - |V_r^{n,\bar t,\bar x,\bar a}|}{|D_{\bar a}\varphi(r, X_{r^-}^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t}) - V_r^{n,\bar t,\bar x,\bar a}|}\cdot \\
&\quad\cdot \frac{D_{\bar a}\varphi(r, X_{r^-}^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t}) - V_r^{n,\bar t,\bar x,\bar a}}{|D_{\bar a}\varphi(r, X_{r^-}^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t}) - V_r^{n,\bar t,\bar x,\bar a}|} 1_{\{D_{\bar a}\varphi(r, X_{r^-}^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t})\neq V_r^{n,\bar t,\bar x,\bar a}\}},
\end{aligned}
$$

for all $\bar t \leq r \leq T$. Notice that α, β, and γ are bounded. Consider now the probability measure $\hat{\mathbb{P}}^{\bar t,\bar a}$ equivalent to $\mathbb{P}^{\bar t,\bar a}$ on $(\Omega, \mathcal{F}_T)$, with Radon-Nikodym density given by

$$
\frac{d\hat{\mathbb{P}}^{\bar t,\bar a}}{d\mathbb{P}^{\bar t,\bar a}}\bigg|_{\mathcal{F}_r} = \mathcal{E}_r\Big(-\int_{\bar t}^{\cdot}\beta_u dW_u - \int_{\bar t}^{\cdot}\gamma_u dB_u\Big)
$$

for all $\bar t \leq r \leq T$, where $\mathcal{E}(\cdot)$ is the Doléans-Dade exponential. Notice that the stochastic integrals with respect to $\tilde\pi$ in (5.4.13) remain martingales with respect to $\hat{\mathbb{P}}^{\bar t,\bar a}$, while the effect of the measure $\hat{\mathbb{P}}^{\bar t,\bar a}$ is to render the processes $W_r - W_{\bar t} + \int_{\bar t}^r \beta_u du$ and $B_r - B_{\bar t} + \int_{\bar t}^r \gamma_u du$ Brownian motions. As a consequence, applying Itô's formula to $\exp(-\int_{\bar t}^r \alpha_u du)(r-\bar t)(\varphi - v_n^h)(r, X_r^{\bar t,\bar x,\bar a}, \bar a + B_r - B_{\bar t})$ between $\bar t$ and τ, using (5.4.13), and taking the expectation $\hat{\mathbb{E}}^{\bar t,\bar a}$ with respect to $\hat{\mathbb{P}}^{\bar t,\bar a}$, we end up with (recalling that $v_n^h \geq (v_n^h)_*$)

$$
\begin{aligned}
&\hat{\mathbb{E}}^{\bar t,\bar a}\Big[e^{-\int_{\bar t}^{\tau}\alpha_u du}(\tau-\bar t)(\varphi - (v_n^h)_*)(\tau, X_\tau^{\bar t,\bar x,\bar a}, \bar a + B_\tau - B_{\bar t})\Big] \\
&\geq \hat{\mathbb{E}}^{\bar t,\bar a}\Big[e^{-\int_{\bar t}^{\tau}\alpha_u du}(\tau-\bar t)(\varphi - v_n^h)(\tau, X_\tau^{\bar t,\bar x,\bar a}, \bar a + B_\tau - B_{\bar t})\Big] \geq \varepsilon\,\hat{\mathbb{E}}^{\bar t,\bar a}\big[(\tau-\bar t)^2\big].
\end{aligned}
$$

Since $\tau > \bar t$, $\mathbb{P}^{\bar t,\bar a}$ a.s., it follows that $\tau > \bar t$, $\hat{\mathbb{P}}^{\bar t,\bar a}$ a.s., therefore $\hat{\mathbb{E}}^{\bar t,\bar a}[(\tau-\bar t)^2] > 0$. This implies that there exists $B \in \mathcal{F}_\tau$ such that $(\varphi - (v_n^h)_*)(\tau, X_\tau^{\bar t,\bar x,\bar a}, \bar a + B_\tau - B_{\bar t}))1_B > 0$ and

$\hat{\mathbb{P}}^{\bar{t},\bar{\tilde{a}}}(B) > 0$. This is a contradiction with (5.4.9).

Step 2b. v_n^h is a viscosity solution to (5.4.6). As in step 2a, we shall only prove the viscosity supersolution property of v_n^h to (5.4.6), since the viscosity subsolution of v_n^h to (5.4.6) can be proved similarly. Let $(\bar{x}, \bar{\tilde{a}}) \in \mathbb{R}^d \times \mathbb{R}^q$. Our aim is to show that

$$(v_n^h)_*(T, \bar{x}, \bar{\tilde{a}}) \;\geq\; g(\bar{x}). \tag{5.4.14}$$

Notice that there exists $(t_k, x_k, \tilde{a}_k)_k \subset [0, T) \times \mathbb{R}^d \times \mathbb{R}^q$ such that

$$(t_k, x_k, \tilde{a}_k, v_n^h(t_k, x_k, \tilde{a}_k)) \overset{k\to\infty}{\longrightarrow} (\bar{t}, \bar{x}, \bar{\tilde{a}}, (v_n^h)_*(\bar{t}, \bar{x}, \bar{\tilde{a}})).$$

Recall that $v_n^h(t_k, x_k, \tilde{a}_k) = Y_{t_k}^{n,t_k,x_k,\tilde{a}_k}$ and

$$\begin{aligned} Y_{t_k}^{n,t_k,x_k,\tilde{a}_k} &= \mathbb{E}^{t_k,\tilde{a}_k}\big[g(X_T^{t_k,x_k,\tilde{a}_k})\big] + \int_{t_k}^T \mathbb{E}^{t_k,\tilde{a}_k}\big[f(X_s^{t_k,x_k,\tilde{a}_k}, I_s^{t_k,\tilde{a}_k}, Y_s^{n,t_k,x_k,\tilde{a}_k}, Z_s^{n,t_k,x_k,\tilde{a}_k})\big]ds \\ &\quad + n\int_{t_k}^T \mathbb{E}^{t_k,\tilde{a}_k}\big[|V_s^{n,t_k,x_k,\tilde{a}_k}|\big]ds. \end{aligned} \tag{5.4.15}$$

Now we observe that, from classical convergence results of diffusion processes with jumps, see, e.g., Theorem 4.8, Chapter IX, in [59], we have that the law of $(X^{t',x',\tilde{a}'}, I^{t',\tilde{a}'})$ weakly converges to the law of $(X^{t,x,\tilde{a}}, I^{t,\tilde{a}})$. As a consequence, we obtain

$$\mathbb{E}^{t_k,\tilde{a}_k}\big[g(X_T^{t_k,x_k,\tilde{a}_k})\big] \overset{k\to\infty}{\longrightarrow} g(\bar{x}).$$

Moreover, from estimate (5.2.5) and (5.3.8), it follows by Lebesgue's dominated convergence theorem that the two integrals in time in (5.4.15) go to zero as $k \to \infty$. In conclusion, letting $k \to \infty$ in (5.4.15) we deduce that $(v_n^h)_*(T, \bar{x}, \bar{\tilde{a}}) = g(\bar{x})$, therefore (5.4.14) holds. Notice that, from this proof, we also have that, for any $(x, \tilde{a}) \in \mathbb{R}^d \times \mathbb{R}^q$, $v_n^h(t', x', \tilde{a}') \to v_n^h(T, x, \tilde{a}) = g(x)$, as $(t', x', \tilde{a}') \to (T, x, \tilde{a})$, with $t' < T$. In other words, v_n^h is continuous at T.

Step 3. Continuity of v_n^h on $[0,T] \times \mathbb{R}^d \times \mathbb{R}^q$. The continuity of v_n^h at T was proved in step 2b. On the other hand, the continuity of v_n^h on $[0, T) \times \mathbb{R}^d \times \mathbb{R}^q$ follows from the comparison theorem for viscosity solutions to equation (5.4.5)-(5.4.6). We notice, however, that a comparison theorem for equation (5.4.5)-(5.4.6) does not seem to be at disposal in the literature. Indeed, Theorem 3.5 in [3] applies to semilinear PDEs in which a Lévy measure appears, instead in our case λ depends on a. We can not even apply our comparison Theorem 5.5.2, designed for equation (5.3.1)-(5.3.2), since in Theorem 5.5.2 the variable a is a parameter while in equation (5.4.5) is a state variable. Nevertheless, we observe that, under assumption **(Hλ)** we can easily extend Theorem 3.5 in [3] to our case and, since the proof is very similar to that of Theorem 3.5 in [3], we do not prove it here to alleviate the presentation. □

5.4.2 The non dependence of the function v on the variable a

In the present subsection, our aim is to prove that the function v does not depend on the variable a. This is indeed a consequence of the constraint (5.3.4) on the component V of equation (5.3.3). If v (and also h) were smooth enough, then, for any $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$, we could express the process $V^{t,x,\tilde{a}}$ as follows (we use the notations $h(\tilde{a}) =$

$(h_i(\tilde{a}))_{i=1,\ldots,q}$, $D_{\tilde{a}}h(\tilde{a}) = (D_{\tilde{a}_j}h_i(\tilde{a}))_{i,j=1,\ldots,q}$, and finally $D_h v$ to denote the gradient of v with respect to its last argument)

$$V_s^{t,x,\tilde{a}} \;=\; D_h v(s, X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}}) D_{\tilde{a}} h(\tilde{a} + B_s - B_t), \qquad t \leq s \leq T.$$

Therefore, from the constraint (5.3.4) we would find

$$\mathbb{E}^{t,\tilde{a}}\bigg[\int_t^{t+\delta} |D_h v(s, X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}}) D_{\tilde{a}} h(\tilde{a} + B_s - B_t)| ds\bigg] \;=\; 0,$$

for any $\delta > 0$. By sending δ to zero in the above equality divided by δ, we would obtain

$$|D_h v(t, x, h(\tilde{a})) D_{\tilde{a}} h(\tilde{a})| \;=\; 0.$$

Let us consider the function $v^h \colon [0,T] \times \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}$ given by

$$v^h(t,x,\tilde{a}) \;:=\; v(t,x,h(\tilde{a})), \qquad (t,x,\tilde{a}) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q. \tag{5.4.16}$$

Then $|D_{\tilde{a}} v^h| \equiv 0$, so that the function v^h is constant with respect to $\tilde{a}$. Since $h(\mathbb{R}^q) = A$, we have that v does not depend on the variable a on A.

Unfortunately, we do not know if v is regular enough in order to justify the above passages. Therefore, we shall rely on viscosity solutions techniques to derive the non dependence of v on the variable a. To this end, let us introduce the following first-order PDE :

$$-\, |D_{\tilde{a}} v^h(t,x,\tilde{a})| \;=\; 0, \qquad (t,x,\tilde{a}) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^q. \tag{5.4.17}$$

Lemma 5.4.1. *Let assumptions* **(HFC)**, **(HBC)**, **(HA)**, *and* **(Hλ)** *hold. The function v^h in (5.4.16) is a viscosity supersolution to (5.4.17): for any $(t,x,\tilde{a}) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^q$ and any function $\varphi \in C^{1,2}([0,T] \times (\mathbb{R}^d \times \mathbb{R}^q))$ such that*

$$(v^h - \varphi)(t,x,\tilde{a}) \;=\; \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q} (v^h - \varphi)$$

we have

$$-\,|D_{\tilde{a}}\varphi(t,x,\tilde{a})| \;\geq\; 0.$$

Proof. We know that v^h is the pointwise limit of the nondecreasing sequence of functions $(v_n^h)_n$. By continuity of v_n^h, the function v^h is lower semicontinuous and we have (see, e.g., page 91 in [2]) :

$$v^h(t,x,\tilde{a}) \;=\; v_*^h(t,x,\tilde{a}) \;=\; \liminf_{n\to\infty}{}_* \, v_n^h(t,x,\tilde{a}),$$

for all $(t,x,\tilde{a}) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^q$, where

$$\liminf_{n\to\infty}{}_* \, v_n^h(t,x,\tilde{a}) \;=\; \liminf_{\substack{n\to\infty \\ (t',x',\tilde{a}')\to(t,x,\tilde{a}) \\ t'<T}} v_n^h(t',x',\tilde{a}'), \qquad (t,x,\tilde{a}) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^q.$$

Let $(t,x,\tilde{a}) \in [0,T) \times \mathbb{R}^d \times \mathbb{R}^q$ and $\varphi \in C^{1,2}([0,T] \times (\mathbb{R}^d \times \mathbb{R}^q))$ such that

$$(v^h - \varphi)(t,x,\tilde{a}) \;=\; \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q} (v^h - \varphi).$$

We may assume, without loss of generality, that this minimum is strict. Up to a suitable negative perturbation of φ for large values of x and $\tilde{a}$, we can assume, without loss of generality, that there exists a bounded sequence $(t_n, x_n, \tilde{a}_n) \in [0,T] \times \mathbb{R}^d \times \mathbb{R}^q$ such that

$$(v_n^h - \varphi)(t_n, x_n, \tilde{a}_n) \;=\; \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q} (v_n^h - \varphi).$$

Then, it follows that, up to a subsequence,

$$(t_n, x_n, \tilde{a}_n, v_n^h(t_n, x_n, \tilde{a}_n)) \;\longrightarrow\; (t, x, \tilde{a}, v^h(t, x, \tilde{a})), \qquad \text{as } n \to \infty. \tag{5.4.18}$$

Now, from the viscosity supersolution property of v_n^h at $(t_n, x_n, \tilde{a}_n)$ with the test function φ, we have

$$\begin{aligned}
-\frac{\partial \varphi}{\partial t}(t_n, x_n, \tilde{a}_n) - f(x_n, h(\tilde{a}_n), v_n^h(t_n, x_n, \tilde{a}_n), \sigma^\intercal(x_n, h(\tilde{a}_n)) D_x \varphi(t_n, x_n, \tilde{a}_n)) \\
-\mathcal{L}^{h(\tilde{a}_n)}\varphi(t_n, x_n, \tilde{a}_n) - \frac{1}{2}\mathrm{tr}(D_{\tilde{a}}^2 \varphi(t_n, x_n, \tilde{a}_n)) - n|D_{\tilde{a}}\varphi(t_n, x_n, \tilde{a}_n)| \;\geq\; 0,
\end{aligned}$$

which implies

$$\begin{aligned}
|D_{\tilde{a}}\varphi(t_n, x_n, \tilde{a}_n)| \;\leq\; \frac{1}{n}\Big(-\frac{\partial \varphi}{\partial t}(t_n, x_n, \tilde{a}_n) - \mathcal{L}^{h(\tilde{a}_n)}\varphi(t_n, x_n, \tilde{a}_n) \\
- f(x_n, h(\tilde{a}_n), v_n^h(t_n, x_n, \tilde{a}_n), \sigma^\intercal(x_n, h(\tilde{a}_n)) D_x \varphi(t_n, x_n, \tilde{a}_n)) - \frac{1}{2}\mathrm{tr}(D_{\tilde{a}}^2 \varphi(t_n, x_n, \tilde{a}_n))\Big).
\end{aligned}$$

Sending n to infinity, we get from (5.4.18) and the continuity of b, σ, β, f, and h :

$$|D_{\tilde{a}}\varphi(t, x, \tilde{a})| \;=\; 0,$$

from which the claim follows. □

We can now state the main result of this subsection.

Proposition 5.4.2. *Let assumptions* **(HFC)**, **(HBC)**, **(HA)**, *and* **(Hλ)** *hold. Then, the function* v *in* (5.4.1) *does not depend on its last argument on* $[0,T) \times \mathbb{R}^d \times \mathring{A}$ *:*

$$v(t, x, a) \;=\; v(t, x, a'), \qquad a, a' \in \mathring{A},$$

for any $(t, x) \in [0,T) \times \mathbb{R}^d$.

Proof. From Lemma 5.4.1, we have that v^h is a viscosity supersolution to the first-order PDE :

$$-|D_{\tilde{a}} v^h(t, x, \tilde{a})| \;=\; 0, \qquad (t, x, \tilde{a}) \in [0,T) \times \mathbb{R}^d \times \mathring{A}_h,$$

where A_h was introduced in assumption **(HA)**. Then, from Proposition 5.2 in [75] we conclude that v^h does not depend on the variable $\tilde{a}$ in $\mathring{A}_h$:

$$v^h(t, x, \tilde{a}) \;=\; v^h(t, x, \tilde{a}'), \qquad (t, x) \in [0,T) \times \mathbb{R}^d,\ \tilde{a}, \tilde{a}' \in \mathring{A}_h.$$

Since, from assumption **(HA)** we have $h(\mathring{A}_h) = \mathring{A}$, we deduce the claim. □

5.4.3 Viscosity properties of the function v

From Proposition 5.4.2, by misuse of notation, we can define the function v on $[0,T)\times\mathbb{R}^d$ by

$$v(t,x) \;=\; v(t,x,a), \qquad (t,x)\in[0,T)\times\mathbb{R}^d,$$

for some $a\in\mathring{A}$. Since $h(\mathring{A}_h)=\mathring{A}$, we also have

$$v(t,x) \;=\; v^h(t,x,\tilde a), \qquad (t,x,\tilde a)\in[0,T)\times\mathbb{R}^d,$$

for some $\tilde a\in\mathring{A}_h$. Moreover, from estimate (5.4.3) we deduce the linear growth condition for v :

$$\sup_{(t,x)\in[0,T)\times\mathbb{R}^d}\frac{|v(t,x)|}{1+|x|} \;<\; \infty. \tag{5.4.19}$$

The present subsection is devoted to the remaining part of the proof of Theorem 5.4.1, namely that v is a viscosity solution to (5.3.1)-(5.3.2).

Proof of the viscosity supersolution property to (5.3.1). We know that v is the pointwise limit of the nondecreasing sequence of functions $(v_n^h)_n$, so that v is lower semicontinuous and we have

$$v(t,x) \;=\; v_*(t,x) \;=\; \liminf_{n\to\infty}{}_* \, v_n^h(t,x,\tilde a), \tag{5.4.20}$$

for all $(t,x,\tilde a)\in[0,T)\times\mathbb{R}^d\times\mathring{A}_h$. Let $(t,x)\in[0,T)\times\mathbb{R}^d$ and $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d)$ such that

$$(v-\varphi)(t,x) \;=\; \min_{[0,T]\times\mathbb{R}^d}(v-\varphi).$$

From the linear growth condition (5.4.19) on v, we can assume, without loss of generality, that φ satisfies $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}|\varphi(t,x)|/(1+|x|)<\infty$. Fix some $\tilde a\in\mathring{A}_h$ and define, for any $\varepsilon>0$, the test function

$$\varphi^\varepsilon(t',x',\tilde a') \;=\; \varphi(t',x')-\varepsilon\big(|t'-t|^2+|x'-x|^2+|\tilde a'-\tilde a|^2\big),$$

for all $(t',x',\tilde a')\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^q$. Notice that $\varphi^\varepsilon\le\varphi$ with equality if and only if $(t',x',\tilde a')=(t,x,\tilde a)$, therefore $v-\varphi^\varepsilon$ has a strict global minimum at $(t,x,\tilde a)$. From the linear growth condition on the continuous functions v_n^h and φ, there exists a bounded sequence $(t_n,x_n,\tilde a_n)_n$ (we omit the dependence in ε) in $[0,T)\times\mathbb{R}^d\times\mathbb{R}^q$ such that

$$(v_n^h-\varphi^\varepsilon)(t_n,x_n,\tilde a_n) \;=\; \min_{[0,T]\times\mathbb{R}^d\times\mathbb{R}^q}(v_n^h-\varphi^\varepsilon).$$

By standard arguments, we obtain that, up to a subsequence,

$$\big(t_n,x_n,\tilde a_n,v_n^h(t_n,x_n,\tilde a_n)\big) \;\longrightarrow\; \big(t,x,\tilde a,v(t,x)\big), \qquad \text{as } n\to\infty.$$

Now, from the viscosity supersolution property of v_n^h at $(t_n,x_n,\tilde a_n)$ with the test function φ_ε, we have

$$-\frac{\partial\varphi^\varepsilon}{\partial t}(t_n,x_n,\tilde a_n)-f\big(x_n,h(\tilde a_n),v_n^h(t_n,x_n,\tilde a_n),\sigma^\intercal(x_n,h(\tilde a_n))D_x\varphi^\varepsilon(t_n,x_n,\tilde a_n)\big)$$
$$-\mathcal{L}^{h(\tilde a_n)}\varphi^\varepsilon(t_n,x_n,\tilde a_n)-\frac{1}{2}\mathrm{tr}\big(D^2_{\tilde a}\varphi^\varepsilon(t_n,x_n,\tilde a_n)\big)-n\big|D_{\tilde a}\varphi^\varepsilon(t_n,x_n,\tilde a_n)\big| \;\ge\; 0.$$

Therefore

$$
\begin{aligned}
-\frac{\partial \varphi^{\varepsilon}}{\partial t}(t_n, x_n, \tilde{a}_n) - f\big(x_n, h(\tilde{a}_n), v_n^h(t_n, x_n, \tilde{a}_n), \sigma^{\intercal}(x_n, h(\tilde{a}_n)) D_x \varphi^{\varepsilon}(t_n, x_n, \tilde{a}_n)\big) \\
- \mathcal{L}^{h(\tilde{a}_n)} \varphi^{\varepsilon}(t_n, x_n, \tilde{a}_n) - \frac{1}{2}\mathrm{tr}\big(D_{\tilde{a}}^2 \varphi^{\varepsilon}(t_n, x_n, \tilde{a}_n)\big) \;\geq\; 0.
\end{aligned}
$$

Sending n to infinity in the above inequality, we obtain, from the definition of φ^{ε},

$$
-\frac{\partial \varphi^{\varepsilon}}{\partial t}(t, x, \tilde{a}) - \mathcal{L}^{h(\tilde{a})} \varphi^{\varepsilon}(t, x, \tilde{a}) - f\big(x, h(\tilde{a}), v(t,x), \sigma^{\intercal}(x, h(\tilde{a})) D_x \varphi^{\varepsilon}(t, x, \tilde{a})\big) + \varepsilon \;\geq\; 0.
$$

Sending ε to zero, recalling that $\varphi^{\varepsilon}(t, x, \tilde{a}) = \varphi(t, x)$, we find

$$
-\frac{\partial \varphi}{\partial t}(t, x) - \mathcal{L}^{h(\tilde{a})} \varphi(t, x) - f\big(x, h(\tilde{a}), v(t,x), \sigma^{\intercal}(x, h(\tilde{a})) D_x \varphi(t, x)\big) \;\geq\; 0.
$$

Since $\tilde{a} \in \mathring{A}_h$ and $h(\mathring{A}_h) = \mathring{A}$, the above equation can be rewritten in an equivalent way as follows

$$
-\frac{\partial \varphi}{\partial t}(t, x) - \mathcal{L}^{a} \varphi(t, x) - f\big(x, a, v(t,x), \sigma^{\intercal}(x, a) D_x \varphi(t, x)\big) \;\geq\; 0,
$$

where a is arbitrarily chosen in $\mathring{A}$. As a consequence, using assumption **(HA)** and the continuity of the coefficients b, σ, β, and f in the variable a, we end up with

$$
-\frac{\partial \varphi}{\partial t}(t, x) - \sup_{a \in A}\Big[\mathcal{L}^{a} \varphi(t, x) + f\big(x, a, v(t,x), \sigma^{\intercal}(x, a) D_x \varphi(t, x)\big)\Big] \;\geq\; 0,
$$

which is the viscosity supersolution property. □

Proof of the viscosity subsolution property to (5.3.1). Since v is the pointwise limit of the nondecreasing sequence $(v_n^h)_n$, we have (see, e.g., page 91 in [2]) :

$$
v^*(t, x) \;=\; \limsup_{n\to\infty}{}_{*}\, v_n^h(t, x, \tilde{a}), \tag{5.4.21}
$$

for all $(t, x, \tilde{a}) \in [0, T) \times \mathbb{R}^d \times \mathring{A}_h$, where

$$
\limsup_{n\to\infty}{}_{*}\, v_n^h(t, x, \tilde{a}) \;=\; \limsup_{\substack{n\to\infty \\ (t',x',\tilde{a}')\to(t,x,\tilde{a}) \\ t'<T,\, \tilde{a}'\in\mathring{A}_h}} v_n^h(t', x', \tilde{a}'), \qquad (t, x, \tilde{a}) \in [0, T) \times \mathbb{R}^d \times \mathbb{R}^q.
$$

Let $(t, x) \in [0, T) \times \mathbb{R}^d$ and $\varphi \in C^{1,2}([0, T] \times \mathbb{R}^d)$ such that

$$
(v^* - \varphi)(t, x) \;=\; \max_{[0,T]\times\mathbb{R}^d}(v^* - \varphi).
$$

We may assume, without loss of generality, that this maximum is strict and that φ satisfies a linear growth condition $\sup_{(t,x)\in[0,T]\times\mathbb{R}^d} |\varphi(t, x)|/(1 + |x|) < \infty$. Fix $\tilde{a} \in \mathring{A}_h$ and consider a sequence $(t_n, x_n, \tilde{a}_n)_n$ in $[0, T) \times \mathbb{R}^d \times \mathring{A}_h$ such that

$$
(t_n, x_n, \tilde{a}_n, v_n(t_n, x_n, \tilde{a}_n)) \;\longrightarrow\; (t, x, \tilde{a}, v^*(t, x)), \qquad \text{as } n \to \infty.
$$

Let us define for $n \geq 1$ the function $\varphi_n \in C^{1,2}([0, T] \times (\mathbb{R}^d \times \mathbb{R}^q))$ by

$$
\varphi_n(t', x', \tilde{a}') \;=\; \varphi(t', x') + n\big(|t' - t_n|^2 + |x' - x_n|^2\big),
$$

for all $(t',x',\tilde a') \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^q$. From the linear growth condition on v_n^h and φ, we can find a sequence $(\bar t_n,\bar x_n,\bar{\tilde a}_n)_n$ in $[0,T)\times\mathbb{R}^d\times A_h$ such that

$$(v_n^h-\varphi_n)(\bar t_n,\bar x_n,\bar{\tilde a}_n) \;=\; \max_{[0,T]\times\mathbb{R}^d\times A_h}(v_n^h-\varphi_n).$$

By standard arguments, we obtain that, up to a subsequence,

$$n\big(|\bar t_n-t_n|^2+|\bar x_n-x_n|^2\big) \;\overset{n\to\infty}{\longrightarrow}\; 0.$$

As a consequence, up to a subsequence, we have

$$(\bar t_n,\bar x_n,\bar{\tilde a}_n) \;\overset{n\to\infty}{\longrightarrow}\; (t,x,\bar{\tilde a}),$$

for some $\bar{\tilde a}\in A_h$. Now, from the viscosity subsolution property of v_n^h at $(\bar t_n,\bar x_n,\bar{\tilde a}_n)$ with the test function φ_n, we have :

$$-\frac{\partial\varphi_n}{\partial t}(\bar t_n,\bar x_n,\bar{\tilde a}_n)-f\big(\bar x_n,h(\bar{\tilde a}_n),v_n^h(\bar t_n,\bar x_n,\bar{\tilde a}_n),\sigma^\intercal(\bar x_n,h(\bar{\tilde a}_n))D_x\varphi_n(\bar t_n,\bar x_n,\bar{\tilde a}_n)\big)$$
$$-\mathcal{L}^{h(\bar{\tilde a}_n)}\varphi_n(\bar t_n,\bar x_n,\bar{\tilde a}_n)-\frac{1}{2}\mathrm{tr}\big(D^2_{\tilde a}\varphi_n(\bar t_n,\bar x_n,\bar{\tilde a}_n)\big)-n\big|D_{\tilde a}\varphi_n(\bar t_n,\bar x_n,\bar{\tilde a}_n)\big| \;\le\; 0.$$

Therefore, using the definition of φ_n,

$$-\frac{\partial\varphi_n}{\partial t}(\bar t_n,\bar x_n,\bar{\tilde a}_n)-\mathcal{L}^{h(\bar{\tilde a}_n)}\varphi_n(\bar t_n,\bar x_n,\bar{\tilde a}_n)-f\big(\bar x_n,h(\bar{\tilde a}_n),v_n^h,\sigma^\intercal D_x\varphi_n(\bar t_n,\bar x_n,\bar{\tilde a}_n)\big) \;\le\; 0.$$

Sending n to infinity in the above inequality, we obtain

$$-\frac{\partial\varphi}{\partial t}(t,x)-\mathcal{L}^{h(\bar{\tilde a})}\varphi(t,x)-f\big(x,h(\bar{\tilde a}),v^*(t,x),\sigma^\intercal(x,h(\bar{\tilde a}))D_x\varphi(t,x)\big) \;\le\; 0.$$

Setting $a'=h(\bar{\tilde a})$, the above equation can be rewritten in an equivalent way as follows

$$-\frac{\partial\varphi}{\partial t}(t,x)-\mathcal{L}^{a'}\varphi(t,x)-f\big(x,a',\sigma^\intercal(x,a')D_x\varphi(t,x)\big) \;\le\; 0.$$

As a consequence, we have

$$-\frac{\partial\varphi}{\partial t}(t,x)-\sup_{a\in A}\Big[\mathcal{L}^a\varphi(t,x)+f\big(x,a,\sigma^\intercal(x,a)D_x\varphi(t,x)\big)\Big] \;\le\; 0,$$

which is the viscosity subsolution property. □

Proof of the viscosity supersolution property to (5.3.2). Let $x\in\mathbb{R}^d$. From (5.4.20), we can find a sequence $(t_n,x_n,\tilde a_n)_n$ valued in $[0,T)\times\mathbb{R}^d\times\mathbb{R}^q$ such that

$$\big(t_n,x_n,\tilde a_n,v_n^h(t_n,x_n,\tilde a_n)\big) \;\longrightarrow\; \big(T,x,\tilde a,v_*(T,x)\big), \qquad \text{as } n\to\infty,$$

for some $\tilde a\in\mathring{A}_h$. Since the sequence $(v_n^h)_n$ is nondecreasing and $v_n^h(T,\cdot,\cdot)=g$, we have

$$v_*(T,x) \;\ge\; \lim_{n\to\infty} v_1^h(t_n,x_n,\tilde a_n) \;=\; g(x).$$

□

Proof of the viscosity subsolution property to (5.3.2). Let $x \in \mathbb{R}^d$. From (5.4.21), for every $\varepsilon > 0$ and $\tilde{a} \in \mathring{A}_h$ there exist $N \in \mathbb{N}$ and $\delta > 0$ such that

$$|v_n^h(t', x', \tilde{a}') - v^*(T, x)| \;\leq\; \varepsilon, \tag{5.4.22}$$

for all $n \geq N$ and $|t' - T|, |x' - x|, |\tilde{a}' - \tilde{a}| \leq \delta$, with $t' < T$ and $\tilde{a}' \in \mathring{A}_h$. Now, we recall that $v_n^h(T, x, \tilde{a}) = g(x)$, therefore, from the continuity of v_n^h, for every $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that

$$|v_n^h(t', x', \tilde{a}') - g(x)| \;\leq\; \varepsilon, \tag{5.4.23}$$

for all $|t' - T|, |x' - x|, |\tilde{a}' - \tilde{a}| \leq \delta_n$, with $\tilde{a}' \in \mathring{A}_h$. Combining (5.4.22) with (5.4.23), we end up with

$$v^*(T, x) \;\leq\; g(x) + 2\varepsilon.$$

From the arbitrariness of ε, we get the claim. $\square$

5.5 Appendix

5.5.1 Martingale representation theorem

We present here a martingale representation theorem, which is one of the fundamental result to derive our nonlinear Feynman-Kac representation formula. It is indeed a direct consequence of Theorem 4.29, Chapter III, in [59], which is however designed for local (instead of square integrable) martingales.

Theorem 5.5.1. *Let $(t, \tilde{a}) \in [0, T] \times \mathbb{R}^q$ and $M = (M_s)_{t \leq s \leq T}$ be a càdlàg square integrable $\mathbb{F}$-martingale, with M_t constant. Then, there exist $Z \in \mathbf{L^2_{t,\tilde{a}}(W)}$, $V \in \mathbf{L^2_{t,\tilde{a}}(B)}$, and $U \in \mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi})$ such that*

$$M_s \;=\; M_t + \int_t^s Z_r dW_r + \int_t^s V_r dB_r + \int_t^s \int_E U_r(e) \tilde{\pi}(dr, de),$$

for all $t \leq s \leq T$, $\mathbb{P}^{t,\tilde{a}}$ almost surely.

Proof. Since M is a local martingale, we know from Theorem 4.29, Chapter III, in [59], that

$$M_s \;=\; M_t + \int_t^s Z_r dW_r + \int_t^s V_r dB_r + \int_t^s \int_E U_r(e) \tilde{\pi}(dr, de),$$

for some predictable processes $(Z_s)_{t \leq s \leq T}$, $(V_s)_{t \leq s \leq T}$, and $(U_s)_{t \leq s \leq T}$, satisfying

$$\mathbb{E}^{t,\tilde{a}}\bigg[\int_t^{T \wedge \tau_n^Z} |Z_s|^2 ds\bigg] \;<\; \infty, \qquad \mathbb{E}^{t,\tilde{a}}\bigg[\int_t^{T \wedge \tau_n^V} |V_s|^2 ds\bigg] \;<\; \infty,$$

$$\mathbb{E}^{t,\tilde{a}}\bigg[\bigg(\int_t^{T \wedge \tau_n^U} \int_E |U_s(e)|^2 \pi(ds, de)\bigg)^{1/2}\bigg] \;<\; \infty,$$

for all $n \in \mathbb{N}$, where $(\tau_n^Z)_{n \in \mathbb{N}}$, $(\tau_n^V)_{n \in \mathbb{N}}$, and $(\tau_n^U)_{n \in \mathbb{N}}$ are nondecreasing sequences of $\mathbb{F}$-stopping times valued in $[t, T]$, converging pointwise $\mathbb{P}^{t,\tilde{a}}$ a.s. to T. It remains to show that $Z \in \mathbf{L^2_{t,\tilde{a}}(W)}$, $V \in \mathbf{L^2_{t,\tilde{a}}(B)}$, and $U \in \mathbf{L^2_{t,\tilde{a}}}(\tilde{\pi})$. This is indeed a consequence of Theorem 4.1.d in [56]. $\square$

5.5.2 Characterization of π and Markov property of (X, I)

In the following lemma, inspired by the results concerning Poisson random measures (see, e.g., Proposition 1.12, Chapter XII, in [93]), we present a characterization of π in terms of Fourier and Laplace functionals. This shows that π is a conditionally Poisson random measure (also known as doubly stochastic Poisson random measure or Cox random measure) relative to $\sigma(I_z; z \geq 0)$.

Proposition 5.5.1 (Fourier and Laplace functionals of π)**.** *Assume that* **(HFC)** *holds and fix* $(t, \tilde{a}) \in [0, T] \times \mathbb{R}^q$*. Let* $\ell \colon \mathbb{R}_+ \times E \to \mathbb{R}$ *be a* $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(E)$*-measurable function such that* $\int_0^\infty \int_E |\ell_u(e)| \lambda(I_u^{t,\tilde{a}}, de) du < \infty$*,* $\mathbb{P}^{t,\tilde{a}}$ *a.s., then, for every* $s \leq \infty$*,*

$$\mathbb{E}^{t,\tilde{a}}\Big[e^{i\int_0^s \int_E \ell_u(e) \pi(du,de)} \Big| \sigma(I_z^{t,\tilde{a}}; z \geq 0)\Big] \; = \; e^{\int_0^s \int_E (e^{i\ell_u(e)} - 1) \lambda(I_u^{t,\tilde{a}}, de) du}, \qquad \mathbb{P}^{t,\tilde{a}} \; a.s.$$

If ℓ *is nonnegative, then the following equality holds :*

$$\mathbb{E}^{t,\tilde{a}}\Big[e^{-\int_0^s \int_E \ell_u(e) \pi(du,de)} \Big| \sigma(I_z^{t,\tilde{a}}; z \geq 0)\Big] \; = \; e^{-\int_0^s \int_E (1 - e^{-\ell_u(e)}) \lambda(I_u^{t,\tilde{a}}, de) du}, \qquad \mathbb{P}^{t,\tilde{a}} \; a.s.$$

In particular, if $(F_k)_{1 \leq k \leq n}$*, with* $n \in \mathbb{N}\backslash\{0\}$*, is a finite sequence of pairwise disjoint Borel measurable sets from* $\mathbb{R}_+ \times E$*, with* $\int_{F_k} \lambda(I_u^{t,\tilde{a}}, de) du < \infty$*,* $\mathbb{P}^{t,\tilde{a}}$ *a.s., then*

$$\mathbb{E}^{t,\tilde{a}}\Big[e^{i\sum_{k=1}^n \theta_k \pi(F_k)} \Big| \sigma(I_z^{t,\tilde{a}}; z \geq 0)\Big] \; = \; \prod_{k=1}^n e^{\int_{F_k} (e^{i\theta_k} - 1) \lambda(I_u^{t,\tilde{a}}, de) du}, \qquad \mathbb{P}^{t,\tilde{a}} \; a.s.$$

for all $\theta_1, \ldots, \theta_n \in \mathbb{R}$*. In other words,* $\pi(F_1), \ldots, \pi(F_n)$ *are conditionally independent relative to* $\sigma(I_z^{t,\tilde{a}}; z \geq 0)$*.*

Proof. Let $J_s = \int_0^s \int_E \ell_u(e) \pi(du, de)$, for any $s \geq 0$, and define

$$\phi(s) \; = \; \mathbb{E}^{t,\tilde{a}}\big[e^{iJ_s} \big| \sigma(I_z^{t,\tilde{a}}; z \geq 0)\big], \qquad \forall \, s \geq 0.$$

Applying Itô's formula to the process e^{iJ_s}, we find

$$e^{iJ_s} \; = \; 1 + \int_0^s \int_E e^{iJ_{u^-}} (e^{i\ell_u(e)} - 1) \pi(du, de).$$

Taking the conditional expectation with respect to $\sigma(I_u^{t,\tilde{a}}; u \geq 0)$, we get

$$\begin{aligned} \mathbb{E}^{t,\tilde{a}}\big[e^{iJ_s} \big| \sigma(I_z^{t,\tilde{a}}; z \geq 0)\big] \; &= \; 1 + \mathbb{E}^{t,\tilde{a}}\Big[\int_0^s \int_E e^{iJ_{u^-}} (e^{i\ell_u(e)} - 1) \lambda(I_u^{t,\tilde{a}}, de) du \Big| \sigma(I_z^{t,\tilde{a}}; z \geq 0)\Big] \\ &= \; 1 + \int_0^s \int_E \mathbb{E}^{t,\tilde{a}}\big[e^{iJ_{u^-}} \big| \sigma(I_z^{t,\tilde{a}}; z \geq 0)\big] (e^{i\ell_u(e)} - 1) \lambda(I_u^{t,\tilde{a}}, de) du. \end{aligned}$$

In terms of ϕ this reads

$$\phi(s) \; = \; 1 + \int_0^s \phi(u^-) \psi(u) du, \qquad \mathbb{P}^{t,\tilde{a}} \; a.s.,$$

where

$$\psi(u) \; = \; \int_E (e^{i\ell_u(e)} - 1) \lambda(I_u^{t,\tilde{a}}, de), \qquad \mathbb{P}^{t,\tilde{a}} \; a.s.$$

Notice that ψ belongs to $\mathbf{L^1}(\mathbb{R}_+)$, as a consequence of the integrability condition on f. We see then that ϕ is continuous, so that

$$\phi(s) \;=\; e^{\int_0^s \psi(u)du}, \qquad \mathbb{P}^{t,\tilde{a}}\,a.s.,$$

which yields the first formula of the lemma. The second formula is proved similarly. □

We shall now study the Markov properties of the pair (X, I) in the following two propositions.

Proposition 5.5.2. *Under assumption* **(HFC)***, for every* $(t, x, \tilde{a}) \in [0, T]\times\mathbb{R}^d\times\mathbb{R}^q$ *the stochastic process* $(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})_{s\geq 0}$ *on* $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{t,\tilde{a}})$ *is Markov with respect to* $\mathbb{F}$*: for every* $r, s \in \mathbb{R}_+$, $r \leq s$, *and for every Borel measurable and bounded function* $h\colon \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}$ *we have*

$$\mathbb{E}^{t,\tilde{a}}\big[h(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})|\mathcal{F}_r\big] \;=\; \mathbb{E}^{t,\tilde{a}}\big[h(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})|\sigma(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})\big], \qquad \mathbb{P}^{t,\tilde{a}}\,a.s.$$

Proof. Fix $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$. Notice that it is enough to show the Markov property for $t \leq r \leq s \leq T$. Therefore, let $r \in [t, T]$ and consider, on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{t,\tilde{a}})$, the following equation for $\tilde{X}$:

$$\begin{aligned}\tilde{X}_s \;&=\; X_r^{t,x,\tilde{a}} + \int_r^s b(\tilde{X}_u, I_u^{t,\tilde{a}})du + \int_r^s \sigma(\tilde{X}_u, I_u^{t,\tilde{a}})dW_u \\ &\quad + \int_r^s\int_E \beta(\tilde{X}_{u^-}, I_u^{t,\tilde{a}}, e)\tilde{\pi}(du, de),\end{aligned} \tag{5.5.1}$$

for all $s \in [r, T]$, $\mathbb{P}^{t,\tilde{a}}$ a.s., where $\tilde{\pi}(du, de) = \pi(du, de) - 1_{\{u<T_\infty\}}\lambda(I_u^{t,\tilde{a}}, de)du$. Under assumption **(HFC)**, it is known (see, e.g., Theorem 14.23 in [57]) that there exists a unique solution to equation (5.5.1), which is clearly given by the process $(X_s^{t,x,\tilde{a}})_{s\in[r,T]}$. We recall that this solution is constructed using an iterative procedure, which relies on a recursively defined sequence of processes $(\tilde{X}^{(n)})_n$, see, e.g., Lemma 14.20 in [57]. More precisely, we set $\tilde{X}^{(0)} \equiv 0$ and then we define $\tilde{X}^{(n+1)}$ from $\tilde{X}^{(n)}$ as follows :

$$\begin{aligned}\tilde{X}_s^{(n+1)} \;&=\; X_r^{t,x,\tilde{a}} + \int_r^s b(\tilde{X}_u^{(n)}, I_u^{t,\tilde{a}})du + \int_r^s \sigma(\tilde{X}_u^{(n)}, I_u^{t,\tilde{a}})dW_u \\ &\quad + \int_r^s\int_E \beta(\tilde{X}_{u^-}^{(n)}, I_{u^-}^{t,\tilde{a}}, e)\tilde{\pi}(du, de),\end{aligned}$$

for all $s \in [r, T]$, $\mathbb{P}^{t,\tilde{a}}$ a.s., for every $n \in \mathbb{N}$. It can be shown that $\tilde{X}^{(n)}$ converges uniformly towards the solution $X^{t,x,\tilde{a}}$ of (5.5.1) on $[r, T]$, $\mathbb{P}^{t,\tilde{a}}$ a.s., namely $\sup_{s\in[r,T]}|\tilde{X}_s^{(n)} - X_s^{t,x,\tilde{a}}| \to 0$ as n tends to infinity, $\mathbb{P}^{t,\tilde{a}}$ almost surely. This shows that $X_s^{t,x,\tilde{a}}$ (and also $(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})$) is $\tilde{\mathbb{F}}$-adapted, where $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_s)_{s\in[r,T]}$ is the augmentation of the filtration $\tilde{\mathbb{G}} = (\tilde{\mathcal{G}}_s)_{s\in[r,T]}$ given by :

$$\tilde{\mathcal{G}}_s \;=\; \sigma(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}) \vee \mathcal{F}_{[r,s]}^W \vee \mathcal{F}_{[r,s]}^B \vee \mathcal{F}_{[r,s]}^\pi,$$

where $\mathcal{F}_{[r,s]}^W = \sigma(W_u - W_r; r \leq u \leq s)$, $\mathcal{F}_{[r,s]}^B = \sigma(B_u - B_r; r \leq u \leq s)$, and $\mathcal{F}_{[r,s]}^\pi = \sigma(\pi(F); F \in \mathcal{B}([r, s])\otimes\mathcal{B}(E))$. Since $\mathcal{F}_{[r,s]}^W$ and $\mathcal{F}_{[r,s]}^B$ are independent with respect to $\mathcal{F}_r$, it is enough to prove that $\mathcal{F}_{[r,s]}^\pi$ and $\mathcal{F}_r$ are conditionally independent relative to $\sigma(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})$. To prove this, take $C \in \mathcal{F}_r$ and a $\mathcal{B}(\mathbb{R}_+)\otimes\mathcal{B}(E)$-measurable function $\ell\colon \mathbb{R}_+ \times E \to \mathbb{R}$ such that $\int_0^\infty\int_E |\ell_u(e)|\lambda(I_u^{t,\tilde{a}}, de)du < \infty$, $\mathbb{P}^{t,\tilde{a}}$ almost surely. Then, the claim follows if we prove that

$$\mathbb{E}^{t,\tilde{a}}\Big[e^{i\theta_1 1_C + i\theta_2\int_r^s\int_E \ell_u(e)\pi(du,de)}\Big|\sigma(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})\Big] \tag{5.5.2}$$

$$= \mathbb{E}^{t,\tilde a}\big[e^{i\theta_1 1_C}\big|\sigma(X_r^{t,x,\tilde a}, I_r^{t,\tilde a})\big]\mathbb{E}^{t,\tilde a}\Big[e^{i\theta_2\int_r^s\int_E \ell_u(e)\pi(du,de)}\Big|\sigma(X_r^{t,x,\tilde a}, I_r^{t,\tilde a})\Big], \qquad \mathbb{P}^{t,\tilde a}\ a.s.,$$

for all $\theta_1, \theta_2 \in \mathbb{R}$. Firstly, let us prove that 1_C and $\int_r^s\int_E \ell_u(e)\pi(du,de)$ are conditionally independent relative to $\sigma(I_z^{t,\tilde a}; z \geq r)$, i.e.,

$$\begin{aligned} &\mathbb{E}^{t,\tilde a}\Big[e^{i\theta_1 1_C + i\theta_2\int_r^s\int_E \ell_u(e)\pi(du,de)}\Big|\sigma(I_z^{t,\tilde a}; z\geq r)\Big] \\ &= \mathbb{E}^{t,\tilde a}\big[e^{i\theta_1 1_C}\big|\sigma(I_z^{t,\tilde a}; z\geq r)\big]e^{\int_r^s\int_E (e^{i\ell_u(e)\theta_2}-1)\lambda(I_u^{t,\tilde a},de)du}, \qquad \mathbb{P}^{t,\tilde a}\ a.s. \end{aligned} \tag{5.5.3}$$

Proceeding as in Proposition 5.5.1, let $J_s = \int_r^s\int_E \ell_u(e)\pi(du,de)$ and

$$\phi(s) = \mathbb{E}^{t,\tilde a}\big[e^{i\theta_1 1_C + i\theta_2 J_s}\big|\sigma(I_z^{t,\tilde a}; z\geq r)\big], \qquad \forall\, s \geq r.$$

Applying Itô's formula to the process e^{iJ_s}, we find

$$\begin{aligned} &\mathbb{E}^{t,\tilde a}\big[e^{i\theta_1 1_C + i\theta_2 J_s}\big|\sigma(I_z^{t,\tilde a}; z\geq r)\big] = \mathbb{E}^{t,\tilde a}\big[e^{i\theta_1 1_C}\big|\sigma(I_z^{t,\tilde a}; z\geq r)\big] \\ &+ \mathbb{E}^{t,\tilde a}\Big[\int_r^s\int_E e^{i\theta_1 1_C + i\theta_2 J_{u^-}}(e^{i\ell_u(e)\theta_2}-1)\lambda(I_u^{t,\tilde a},de)du\Big|\sigma(I_z^{t,\tilde a}; z\geq 0)\Big] \\ &= \mathbb{E}^{t,\tilde a}\big[e^{i\theta_1 1_C}\big|\sigma(I_z^{t,\tilde a}; z\geq r)\big] \\ &+ \int_r^s\int_E \mathbb{E}^{t,\tilde a}\big[e^{i\theta_1 1_C + i\theta_2 J_{u^-}}\big|\sigma(I_z^{t,\tilde a}; z\geq r)\big](e^{i\ell_u(e)\theta_2}-1)\lambda(I_u^{t,\tilde a},de)du. \end{aligned}$$

In terms of ϕ this reads

$$\phi(s) = 1 + \int_r^s \phi(u^-)\psi(u)du, \qquad \mathbb{P}^{t,\tilde a}\ a.s.,$$

where

$$\psi(u) = \int_E (e^{i\ell_u(e)\theta_2}-1)\lambda(I_u^{t,\tilde a},de), \qquad \mathbb{P}^{t,\tilde a}\ a.s.$$

Notice that ψ belongs to $\mathbf{L}^1(\mathbb{R}_+)$, as a consequence of the integrability condition on f. We see then that ϕ is continuous, so that

$$\phi(s) = \mathbb{E}^{t,\tilde a}\big[e^{i\theta_1 1_C}\big|\sigma(I_z^{t,\tilde a}; z\geq r)\big]e^{\int_r^s \psi(u)du}, \qquad \mathbb{P}^{t,\tilde a}\ a.s.,$$

which yields (5.5.3). Let us come back to (5.5.2). We have, using (5.5.3),

$$\mathbb{E}^{t,\tilde a}\Big[e^{i\theta_1 1_C + i\theta_2\int_r^s\int_E \ell_u(e)\pi(du,de)}\Big|\sigma(X_r^{t,x,\tilde a}, I_r^{t,\tilde a})\Big] = \mathbb{E}^{t,\tilde a}[Y_1Y_2|\sigma(X_r^{t,x,\tilde a}, I_r^{t,\tilde a})],$$

where

$$\begin{aligned} Y_1 &= \mathbb{E}^{t,\tilde a}\big[e^{i\theta_1 1_C}\big|\sigma(I_z^{t,\tilde a}; z\geq r)\vee\sigma(X_r^{t,x,\tilde a}, I_r^{t,\tilde a})\big], \\ Y_2 &= \mathbb{E}^{t,\tilde a}\Big[e^{i\theta_2\int_r^s\int_E \ell_u(e)\pi(du,de)}\Big|\sigma(I_z^{t,\tilde a}; z\geq r)\vee\sigma(X_r^{t,x,\tilde a}, I_r^{t,\tilde a})\Big]. \end{aligned}$$

Since $(I_z^{t,\tilde a})_{z\geq 0}$ is Markov with respect to $\mathbb{F}$, we have that $\mathcal{F}_r$ and $\sigma(I_z^{t,\tilde a}; z \geq r)$ are independent relative to $\sigma(I_r^{t,\tilde a})$. Therefore, Y_1 can be written as

$$Y_1 = \mathbb{E}^{t,\tilde a}\big[e^{i\theta_1 1_C}\big|\sigma(X_r^{t,x,\tilde a}, I_r^{t,\tilde a})\big].$$

It follows that Y_1 is $\sigma(X_r^{t,x,\tilde a}, I_r^{t,\tilde a})$-measurable, so that

$$\mathbb{E}^{t,\tilde a}\Big[e^{i\theta_1 1_C + i\theta_2\int_r^s\int_E \ell_u(e)\pi(du,de)}\Big|\sigma(X_r^{t,x,\tilde a}, I_r^{t,\tilde a})\Big] = Y_1\mathbb{E}^{t,\tilde a}[Y_2|\sigma(X_r^{t,x,\tilde a}, I_r^{t,\tilde a})], \qquad \mathbb{P}^{t,\tilde a}\ a.s.,$$

which proves (5.5.2). □

Proposition 5.5.3. *Under assumption* **(HFC)***, the family* $(\Omega, \mathcal{F}, (X^{t,x,\tilde{a}}, I^{t,\tilde{a}}), \mathbb{P}^{t,\tilde{a}})_{t,x,\tilde{a}}$ *is Markovian with respect to* $\mathbb{F}$ *and satisfies, for every* $(t, x, \tilde{a}) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q$*,* $r, s \in \mathbb{R}_+$ *with* $r \leq s$*, and for every Borel measurable and bounded function* $h\colon \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}$*,*

$$\mathbb{E}^{t,\tilde{a}}\big[h(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})|\mathcal{F}_r\big] \;=\; \int_{\mathbb{R}^d\times\mathbb{R}^q} h(x',\tilde{a}')p(r,(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}), s, dx'd\tilde{a}'), \qquad \mathbb{P}^{t,\tilde{a}}\ a.s. \quad (5.5.4)$$

where p *is the Markovian transition function given by*

$$p(r,(x',\tilde{a}'),s,\Gamma) \;=\; \mathbb{P}^{r,\tilde{a}'}((X_s^{r,x',\tilde{a}'}, I_s^{r,\tilde{a}'}) \in \Gamma),$$

for every $r, s \in \mathbb{R}_+$*,* $r \leq s$*,* $(x', \tilde{a}') \in \mathbb{R}^d \times \mathbb{R}^q$*, and every Borelian set* $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^q$*.*

Remark 5.5.1. For the proof of Proposition 5.5.3 we shall need to consider simultaneously two distinct solutions $\{(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}}),\ s \geq 0\}$ and $\{(X_s^{t',x',\tilde{a}'}, I_s^{t',\tilde{a}'}),\ s \geq 0\}$, for $(t, x, \tilde{a}), (t', x', \tilde{a}') \in [0,T]\times\mathbb{R}^d\times\mathbb{R}^q$. According to Lemma 5.2.2, $\{(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}}),\ s \geq 0\}$ is defined on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{t,\tilde{a}})$ and $\{(X_s^{t',x',\tilde{a}'}, I_s^{t',\tilde{a}'}),\ s \geq 0\}$ on $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P}^{t',\tilde{a}'})$, respectively. However, we can construct a single probability space supporting both solutions. More precisely, we can construct a single probability space supporting both the random measure with compensator given by $1_{\{s<T_\infty\}}\lambda(I_s^{t,\tilde{a}}, de)ds$ and the random measure with compensator $1_{\{s<T_\infty\}}\lambda(I_s^{t',\tilde{a}'}, de)ds$, proceeding as follows.

Let Ω'' be a copy of Ω', with corresponding canonical marked point process denoted by $(T''_n, \alpha''_n)_{n\in\mathbb{N}}$, canonical random measure π'', $T''_\infty := \lim_n T''_n$, and filtration $\mathbb{F}'' = (\mathcal{F}'_s)_{t\geq 0}$. Define $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}} = (\hat{\mathcal{F}}_t)_{t\geq 0})$ with $\hat{\Omega} := \Omega \times \Omega''$, $\hat{\mathcal{F}} := \mathcal{F} \otimes \mathcal{F}''_\infty$, and $\hat{\mathcal{F}}_t := \cap_{s>t}\mathcal{F}_s \otimes \mathcal{F}''_s$. Moreover, set $\hat{W}(\hat{\omega}) := W(\omega)$, $\hat{B}(\hat{\omega}) := B(\omega)$, $\hat{\pi}'(\hat{\omega}, \cdot) := \pi(\omega, \cdot)$, and $\hat{\pi}''(\hat{\omega}, \cdot) := \pi''(\omega'', \cdot)$. Set also $\hat{T}'_\infty(\hat{\omega}) := T_\infty(\omega)$ and $\hat{T}''_\infty(\hat{\omega}) := T''_\infty(\omega'')$. Let $\mathbb{P}^{t,\tilde{a},t',\tilde{a}'}$ be the probability measure on $(\hat{\Omega}, \hat{\mathcal{F}})$ given by $\mathbb{P}^{t,\tilde{a},t',\tilde{a}'}(d\hat{\omega}) = \bar{\mathbb{P}}(d\bar{\omega}) \otimes \mathbb{P}'^{,t,\tilde{a}}(\bar{\omega}, d\omega') \otimes \mathbb{P}''^{,t',\tilde{a}'}(\bar{\omega}, d\omega'')$. Finally, set $(\hat{X}^{t,x,\tilde{a}}, \hat{I}^{t,\tilde{a}})(\hat{\omega}) := (X^{t,x,\tilde{a}}, I^{t,\tilde{a}})(\bar{\omega}, \omega')$ and $(\hat{X}^{t',x',\tilde{a}'}, \hat{I}^{t',\tilde{a}'})(\hat{\omega}) := (X^{t',x',\tilde{a}'}, I^{t',\tilde{a}'})(\bar{\omega}, \omega'')$. Then $(\hat{X}^{t,x,\tilde{a}}, \hat{I}^{t,\tilde{a}})$ solves (5.2.1)-(5.2.2) on $[t, T]$ starting from $(x, \tilde{a})$ at t, and $(\hat{X}^{t',x',\tilde{a}'}, \hat{I}^{t',\tilde{a}'})$ solves (5.2.1)-(5.2.2) on $[t', T]$ starting from $(x', \tilde{a}')$ at time t'. □

Proof (of Proposition 5.5.3). We begin noting that from Proposition 5.5.2 the left-hand side of (5.5.4) is equal to $\mathbb{E}^{t,\tilde{a}}[h(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})|\sigma(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})]$, $\mathbb{P}^{t,\tilde{a}}$ almost surely. Let us now divide the proof into two steps.
Step 1. $(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})$ *is a discrete random variable.* Suppose that

$$(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}) \;=\; \sum_{i\geq 1}(x_i, \tilde{a}_i)1_{\Gamma_i},$$

for some $(x_i, \tilde{a}_i) \in \mathbb{R}^d \times \mathbb{R}^q$ and a Borel partition $(\Gamma_i)_{i\geq 1}$ of $\mathbb{R}^d \times \mathbb{R}^q$ satisfying $\mathbb{P}(\Gamma_i) > 0$, for any $i \geq 1$. In this case, (5.5.4) becomes

$$\mathbb{E}^{t,\tilde{a}}\big[h(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})|\sigma(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})\big] \;=\; \sum_{i\geq 1} 1_{\Gamma_i}\mathbb{E}^{r,\tilde{a}_i}\big[h(X_s^{r,x_i,\tilde{a}_i}, I_s^{r,\tilde{a}_i})\big], \qquad \mathbb{P}^{t,\tilde{a}}\ a.s. \quad (5.5.5)$$

Now notice that the process $(\hat{X}_s^{t,x,\tilde{a}}1_{\Gamma_i})_{s\geq r}$ satisfies on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \mathbb{P}^{t,\tilde{a},r,\tilde{a}_i})$ (using the same notation as in Remark 5.5.1)

$$\hat{X}_s^{t,x,\tilde{a}}1_{\Gamma_i} \;=\; x_i1_{\Gamma_i} + \int_r^s b_i(\hat{X}_u^{t,x,\tilde{a}}1_{\Gamma_i}, \hat{I}_u^{t,\tilde{a}}1_{\Gamma_i})dr + \int_r^s \sigma_i(\hat{X}_u^{t,x,\tilde{a}}1_{\Gamma_i}, \hat{I}_u^{t,\tilde{a}}1_{\Gamma_i})d\hat{W}_u$$

$$+ \int_r^s \int_E \beta(\hat{X}_{u^-}^{t,x,\tilde{a}} 1_{\Gamma_i}, \hat{I}_{u^-}^{t,\tilde{a}} 1_{\Gamma_i}, e) \tilde{\hat{\pi}}_i(du, de),$$

with $b_i = b1_{\Gamma_i}$, $\sigma_i = \sigma 1_{\Gamma_i}$, and $\tilde{\hat{\pi}}_i$ is the compensated martingale measure associated to the random measure $\hat{\pi}_i$, which has $1_{\Gamma_i}\lambda(\hat{I}_{s^-}^{t,\tilde{a}} 1_{\Gamma_i}, de)ds$, $s \geq r$, as compensator. Similarly, the process $(\hat{X}_s^{r,x_i,\tilde{a}_i} 1_{\Gamma_i})_{s \geq r}$ satisfies on $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{F}}, \mathbb{P}^{t,\tilde{a},r,\tilde{a}_i})$

$$\begin{aligned}
\hat{X}_s^{r,x_i,\tilde{a}_i} 1_{\Gamma_i} &= x_i 1_{\Gamma_i} + \int_r^s b_i(\hat{X}_u^{r,x_i,\tilde{a}_i} 1_{\Gamma_i}, \hat{I}_u^{r,\tilde{a}_i} 1_{\Gamma_i}) dr + \int_r^s \sigma_i(\hat{X}_u^{r,x_i,\tilde{a}_i} 1_{\Gamma_i}, \hat{I}_u^{r,\tilde{a}_i} 1_{\Gamma_i}) d\hat{W}_u \\
&\quad + \int_r^s \int_E \beta(\hat{X}_{u^-}^{r,x_i,\tilde{a}_i} 1_{\Gamma_i}, \hat{I}_{u^-}^{r,\tilde{a}_i} 1_{\Gamma_i}, e) \tilde{\hat{\pi}}_i'(du, de),
\end{aligned}$$

where $\tilde{\hat{\pi}}_i'$ is the compensated martingale measure associated to the random measure $\hat{\pi}_i'$, which has $1_{\Gamma_i}\lambda(\hat{I}_{s^-}^{r,\tilde{a}_i} 1_{\Gamma_i}, de)ds$, $s \geq r$, as compensator. Since the two processes $(\hat{I}_s^{t,\tilde{a}} 1_{\Gamma_i})_{s \geq r}$ and $(\hat{I}_s^{r,\tilde{a}_i} 1_{\Gamma_i})_{s \geq r}$ have the same law, we see that $(\hat{X}_s^{t,x,\tilde{a}} 1_{\Gamma_i})_{s \geq r}$ and $(\hat{X}_s^{r,x_i,\tilde{a}_i} 1_{\Gamma_i})_{s \geq r}$ solve the same equation, and, from uniqueness, they have the same law, as well. This implies (denoting $\mathbb{E}^{t,\tilde{a},r,\tilde{a}_i}$ the expectation with respect to $\mathbb{P}^{t,\tilde{a},r,\tilde{a}_i}$)

$$\mathbb{E}^{t,\tilde{a},r,\tilde{a}_i}\big[h(\hat{X}_s^{t,x,\tilde{a}}, \hat{I}_s^{t,\tilde{a}}) 1_{\Gamma_i}\big] = \mathbb{E}^{t,\tilde{a},r,\tilde{a}_i}\big[h(\hat{X}_s^{r,x_i,\tilde{a}_i}, \hat{I}_s^{r,\tilde{a}_i}) 1_{\Gamma_i}\big].$$

Notice that

$$\mathbb{E}^{t,\tilde{a},r,\tilde{a}_i}\big[h(\hat{X}_s^{t,x,\tilde{a}}, \hat{I}_s^{t,\tilde{a}}) 1_{\Gamma_i}\big] = \mathbb{E}^{t,\tilde{a}}\big[h(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}}) 1_{\Gamma_i}\big]$$

and

$$\begin{aligned}
\mathbb{E}^{t,\tilde{a},r,\tilde{a}_i}\big[h(\hat{X}_s^{r,x_i,\tilde{a}_i}, \hat{I}_s^{r,\tilde{a}_i}) 1_{\Gamma_i}\big] &= \mathbb{E}^{t,\tilde{a},r,\tilde{a}_i}\big[\mathbb{E}^{t,\tilde{a},r,\tilde{a}_i}\big[h(\hat{X}_s^{r,x_i,\tilde{a}_i}, \hat{I}_s^{r,\tilde{a}_i}) 1_{\Gamma_i} \big| \mathcal{F}_r\big]\big] \\
&= \mathbb{E}^{t,\tilde{a},r,\tilde{a}_i}\big[\mathbb{E}^{t,\tilde{a},r,\tilde{a}_i}\big[h(\hat{X}_s^{r,x_i,\tilde{a}_i}, \hat{I}_s^{r,\tilde{a}_i}) \big| \mathcal{F}_r\big] 1_{\Gamma_i}\big] \\
&= \mathbb{E}^{t,\tilde{a},r,\tilde{a}_i}\big[\mathbb{E}^{t,\tilde{a},r,\tilde{a}_i}\big[h(\hat{X}_s^{r,x_i,\tilde{a}_i}, \hat{I}_s^{r,\tilde{a}_i})\big] 1_{\Gamma_i}\big] \\
&= \mathbb{E}^{t,\tilde{a}}\big[\mathbb{E}^{r,\tilde{a}_i}\big[h(X_s^{r,x_i,\tilde{a}_i}, I_s^{r,\tilde{a}_i})\big] 1_{\Gamma_i}\big].
\end{aligned}$$

In other words, we have

$$\mathbb{E}^{t,\tilde{a}}\big[h(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}}) 1_{\Gamma_i}\big] = \mathbb{E}^{t,\tilde{a}}\big[\mathbb{E}^{r,\tilde{a}_i}\big[h(X_s^{r,x_i,\tilde{a}_i}, I_s^{r,\tilde{a}_i})\big] 1_{\Gamma_i}\big],$$

from which (5.5.5) follows.

Step 2. General case. From estimate (5.2.5), we see that $(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})$ is square integrable, so that there exists a sequence $(X_r^{t,x,\tilde{a},n}, I_r^{t,\tilde{a},n})_n$ of square integrable discrete random variables converging to $(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})$ pointwise $\mathbb{P}^{t,\tilde{a}}$ a.s. and in $L^2(\Omega, \mathcal{F}, \mathbb{P}^{t,\tilde{a}}; \mathbb{R}^d \times \mathbb{R}^q)$. The sequence $(X_r^{t,x,\tilde{a},n}, I_r^{t,\tilde{a},n})_n$ can be chosen in such a way that $(X_r^{t,x,\tilde{a},n+1}, I_r^{t,\tilde{a},n+1})$ is a better approximation of $(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})$ than $(X_r^{t,x,\tilde{a},n}, I_r^{t,\tilde{a},n})$, in other words such that $\sigma(X_r^{t,x,\tilde{a},n}, I_r^{t,\tilde{a},n}) \subset \sigma(X_r^{t,x,\tilde{a},n+1}, I_r^{t,\tilde{a},n+1})$. Let us denote $(X_s^{t,x,\tilde{a},n}, I_s^{t,\tilde{a},n})$ the solution to (5.2.1)-(5.2.2) starting at time r from $(X_r^{t,x,\tilde{a},n}, I_r^{t,\tilde{a},n})$. Notice that, from classical convergence results of diffusion processes with jumps (see, e.g., Theorem 4.8, Chapter IX, in [59]), it follows that $(X_s^{t,x,\tilde{a},n}, I_s^{t,\tilde{a},n})$ converges weakly to $(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})$. From Step 1, for any n we have

$$\mathbb{E}^{t,\tilde{a}}\big[h(X_s^{t,x,\tilde{a},n}, I_s^{t,\tilde{a},n}) \big| \sigma(X_r^{t,x,\tilde{a},n}, I_r^{t,\tilde{a},n})\big] = p\big(r, (X_r^{t,x,\tilde{a},n}, I_r^{t,\tilde{a},n}), s, h\big), \qquad \mathbb{P}^{t,\tilde{a}} \text{ a.s.} \quad (5.5.6)$$

where

$$p(r, (x', \tilde{a}'), s, h) = \mathbb{E}^{r,\tilde{a}'}\big[h(X_s^{r,x',\tilde{a}',n}, I_s^{r,\tilde{a}',n})\big],$$

for every $r, s \in \mathbb{R}_+$, $r \leq s$, $(x', \tilde{a}') \in \mathbb{R}^d \times \mathbb{R}^q$, and every Borel measurable and bounded function $h \colon \mathbb{R}^d \times \mathbb{R}^q \to \mathbb{R}$. Let us suppose that h is bounded and continuous. Since the sequence $(\mathbb{E}^{t,\tilde{a}}[h(X_s^{t,x,\tilde{a},n}, I_s^{t,\tilde{a},n})|\sigma(X_r^{t,x,\tilde{a},n}, I_r^{t,\tilde{a},n})])_n$ is uniformly bounded in $L^2(\Omega, \mathcal{F}, \mathbb{P}^{t,\tilde{a}})$, there exists a subsequence $(\mathbb{E}^{t,\tilde{a}}[h(X_s^{t,x,\tilde{a},n_k}, I_s^{t,\tilde{a},n_k})|\sigma(X_r^{t,x,\tilde{a},n_k}, I_r^{t,\tilde{a},n_k})])_k$ which converges weakly to some $Z \in L^2(\Omega, \mathcal{F}, \mathbb{P}^{t,\tilde{a}})$. For any $N \in \mathbb{N}$ and $\Gamma_N \in \sigma(X_r^{t,x,\tilde{a},N}, I_r^{t,\tilde{a},N})$, we have, by definition of conditional expectation,

$$\mathbb{E}^{t,\tilde{a}}\big[\mathbb{E}^{t,\tilde{a}}\big[h(X_s^{t,x,\tilde{a},n_k}, I_s^{t,\tilde{a},n_k})|\sigma(X_r^{t,x,\tilde{a},n_k}, I_r^{t,\tilde{a},n_k})\big]1_{\Gamma_N}\big] \;=\; \mathbb{E}^{t,\tilde{a}}\big[h(X_s^{t,x,\tilde{a},n_k}, I_s^{t,\tilde{a},n_k})1_{\Gamma_N}\big],$$

for all $n_k \geq N$. Letting $k \to \infty$, we deduce

$$\mathbb{E}^{t,\tilde{a}}\big[Z 1_{\Gamma_N}\big] \;=\; \mathbb{E}^{t,\tilde{a}}\big[h(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})1_{\Gamma_N}\big].$$

Since $\sigma(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}) = \vee_n \sigma(X_r^{t,x,\tilde{a},n}, I_r^{t,\tilde{a},n})$, it follows that

$$Z \;=\; \mathbb{E}^{t,\tilde{a}}[h(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})|\sigma(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})], \qquad \mathbb{P}^{t,\tilde{a}}\ a.s.$$

Notice that every convergent subsequence of $(\mathbb{E}^{t,\tilde{a}}[h(X_s^{t,x,\tilde{a},n}, I_s^{t,\tilde{a},n})|\sigma(X_r^{t,x,\tilde{a},n}, I_r^{t,\tilde{a},n})])_n$ has to converge to $\mathbb{E}^{t,\tilde{a}}[h(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})|\sigma(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})]$, so that the whole sequence converges. On the other hand, when h is bounded and continuous, it follows again from classical convergence results of diffusion processes with jumps (see, e.g., Theorem 4.8, Chapter IX, in [59]), that $p = p(r, (x', \tilde{a}'), s, h)$ is continuous in $(x', \tilde{a}')$. Since $(X_r^{t,x,\tilde{a},n}, I_r^{t,\tilde{a},n})_n$ converges pointwise $\mathbb{P}^{t,\tilde{a}}$ a.s. to $(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})$, letting $n \to \infty$ in (5.5.6) we obtain

$$\mathbb{E}^{t,\tilde{a}}\big[h(X_s^{t,x,\tilde{a}}, I_s^{t,\tilde{a}})|\sigma(X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}})\big] \;=\; p(r, (X_r^{t,x,\tilde{a}}, I_r^{t,\tilde{a}}), s, h), \qquad \mathbb{P}^{t,\tilde{a}}\ a.s. \tag{5.5.7}$$

for any h bounded and continuous. Using a monotone class argument, we conclude that (5.5.7) remains true for any h bounded and Borel measurable. □

5.5.3 Comparison theorem for equation (5.3.1)-(5.3.2)

We shall prove a comparison theorem for viscosity sub and supersolutions to the fully nonlinear IPDE of HJB type (5.3.1)-(5.3.2). Inspired by Definition 2 in [4], we begin recalling the following result concerning an equivalent definition of viscosity super and subsolution to (5.3.1)-(5.3.2), whose standard proof is not reported.

Lemma 5.5.1. *Let assumption* **(HFC)**, **(HBC)**, *and* **(Hλ)** *hold. A locally bounded and lsc (resp. usc) function u on $[0,T] \times \mathbb{R}^d$ is a viscosity supersolution (resp. viscosity subsolution) to (5.3.1)-(5.3.2) if and only if*

$$u(T, x) \;\geq\; (resp.\ \leq)\ g(x)$$

for any $x \in \mathbb{R}^d$, and, for any $\delta > 0$,

$$-\frac{\partial \varphi}{\partial t}(t,x) - \sup_{a \in A}\Big[b(x,a).D_x\varphi(t,x) + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x,a)D_x^2\varphi(t,x)) + I_a^{1,\delta}(t,x,\varphi)$$
$$+ I_a^{2,\delta}(t,x,D_x\varphi(t,x),u) + f(x,a,u(t,x),\sigma^\intercal(x,a)D_x\varphi(t,x))\Big] \;\geq\; (resp.\ \leq)\ 0,$$

for any $(t,x) \in [0,T) \times \mathbb{R}^d$ and any $\varphi \in C^{1,2}([0,T] \times \mathbb{R}^d)$ such that

$$(u - \varphi)(t,x) \;=\; \min_{[0,T]\times\mathbb{R}^d}(u-\varphi) \quad \Big(resp.\ \max_{[0,T]\times\mathbb{R}^d}(u-\varphi)\Big).$$

As in [4], see Definition 4, for the proof of the comparison theorem it is useful to adopt another equivalent definition of viscosity solution to equation (5.3.1)-(5.3.2), see Lemma 5.5.2 below, where we mix test functions and sub/superjets. We first recall the definition of sub and superjets.

Definition 5.5.1. *Let $u\colon [0,T]\times\mathbb{R}^d\to\mathbb{R}$ be a lsc (resp. usc) function.*
(i) *We denote by $\mathcal{P}^{2,-}u(t,x)$ (resp. $\mathcal{P}^{2,+}u(t,x)$) the* ***parabolic subjet*** *(resp.* ***parabolic superjet****) of u at $(t,x)\in[0,T)\times\mathbb{R}^d$, as the set of triples $(p,q,M)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$ (we denote by $\mathbb{S}^d$ the set of $d\times d$ symmetric matrices) satisfying*

$$\begin{aligned} u(s,y) \ \geq \ (resp.\ \leq) \ & u(t,x)+p(s-t)+q.(y-x)+\frac{1}{2}(y-x).M(y-x) \\ & + o(|s-t|+|y-x|^2), \qquad\qquad \text{as } (s,y)\to(t,x). \end{aligned}$$

(ii) *We denote by $\bar{\mathcal{P}}^{2,-}u(t,x)$ (resp. $\bar{\mathcal{P}}^{2,+}u(t,x)$) the* ***parabolic limiting subjet*** *(resp.* ***parabolic limiting superjet****) of u at $(t,x)\in[0,T)\times\mathbb{R}^d$, as the set of triples $(p,q,M)\in\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$ such that*

$$(p,q,M) \ = \ \lim_{n\to\infty}(p_n,q_n,M_n)$$

with $(p_n,q_n,M_n)\in\mathcal{P}^{2,-}u(t_n,x_n)$ (resp. $\mathcal{P}^{2,+}u(t_n,x_n)$), where

$$(t,x,u(t,x)) \ = \ \lim_{n\to\infty}(t_n,x_n,u(t_n,x_n)).$$

Lemma 5.5.2. *Let assumption* **(HFC)**, **(HBC)**, *and* **(Hλ)** *hold. A locally bounded and lsc (resp. usc) function u on $[0,T]\times\mathbb{R}^d$ is a viscosity supersolution (resp. viscosity subsolution) to 1(5.3.1)-(5.3.2) if and only if*

$$u(T,x) \ \geq \ (resp.\ \leq)\ g(x)$$

for any $x\in\mathbb{R}^d$, and, for any $\delta>0$,

$$\begin{aligned} -p-\sup_{a\in A}\Big[& b(x,a).q+\frac{1}{2}\mathrm{tr}(\sigma\sigma^{\intercal}(x,a)M)+I_a^{1,\delta}(t,x,\varphi) \\ & + I_a^{2,\delta}(t,x,q,u)+f(x,a,u(t,x),\sigma^{\intercal}(x,a)q)\Big] \ \geq \ (resp.\ \leq) \ \ 0, \end{aligned}$$

for any $(t,x)\in[0,T)\times\mathbb{R}^d$, $(p,q,M)\in\bar{\mathcal{P}}^{2,-}u(t,x)$ (resp. $(p,q,M)\in\bar{\mathcal{P}}^{2,+}u(t,x)$), and any $\varphi\in C^{1,2}([0,T]\times\mathbb{R}^d)$, with $\frac{\partial\varphi}{\partial t}(t,x)=p$, $D_x\varphi(t,x)=q$, and $D_x^2\varphi(t,x)\leq M$ (resp. $D_x^2\varphi(t,x)\geq M$), such that

$$(u-\varphi)(t,x) \ = \ \min_{[0,T]\times\mathbb{R}^d}(u-\varphi) \quad (resp.\ \max_{[0,T]\times\mathbb{R}^d}(u-\varphi)).$$

Proof. The proof can be done along the lines of the proof of Proposition 1 in [4]. □

We can now state the main result of this appendix.

Theorem 5.5.2. *Assume that* **(HFC)**, **(HBC)**, *and* **(Hλ)** *hold. Let u be a usc viscosity subsolution to (5.3.1)-(5.3.2) and w a lsc viscosity supersolution to (5.3.1)-(5.3.2), satisfying a linear growth condition*

$$\sup_{(t,x)\in[0,T]\times\mathbb{R}^d}\frac{|u(t,x)|+|w(t,x)|}{1+|x|} \ < \ \infty. \tag{5.5.8}$$

If $u(T,x)\leq w(T,x)$ for all $x\in\mathbb{R}^d$, then $u\leq w$ on $[0,T]\times\mathbb{R}^d$.

Proof We shall argue by contradiction, assuming that

$$\sup_{[0,T]\times\mathbb{R}^d} (u - w) \;\; > \;\; 0. \tag{5.5.9}$$

Step 1. For some $\rho > 0$ to be chosen later, set

$$\tilde{u}(t,x) \; = \; e^{\rho t}u(t,x), \qquad \tilde{w}(t,x) \; = \; e^{\rho t}w(t,x), \qquad (t,x) \in [0,T] \times \mathbb{R}^d.$$

Then, we see that $\tilde{u}$ (resp. $\tilde{w}$) is a viscosity subsolution (resp. supersolution) to the following equation :

$$\rho\tilde{v} - \frac{\partial \tilde{v}}{\partial t} - \sup_{a\in A}\left(\mathcal{L}^a\tilde{v} + \tilde{f}(\cdot, a, \tilde{v}, \sigma^{\intercal}(\cdot,a)D_x\tilde{v})\right) \;\; = \;\; 0, \qquad \text{on } [0,T) \times \mathbb{R}^d, \tag{5.5.10}$$

$$\tilde{v}(T,x) \;\; = \;\; \tilde{g}(x), \quad x \in \mathbb{R}^d, \tag{5.5.11}$$

where

$$\tilde{f}(t,x,a,y,z) \; = \; e^{\rho t}f(x,a,e^{-\rho t}y, e^{-\rho t}z), \qquad \tilde{g}(x) \; = \; e^{\rho T}g(x),$$

for all $(t,x,a,y,z) \in [0,T] \times \mathbb{R}^d \times A \times \mathbb{R} \times \mathbb{R}^d$.
Step 2. Denote, for all $(t,s,x,y) \in [0,T]^2 \times \mathbb{R}^{2d}$, and for any $n \in \mathbb{N}\backslash\{0\}$ and $\gamma > 0$,

$$\Phi_{n,\gamma}(t,s,x,y) \; = \; \tilde{u}(t,x) - \tilde{w}(s,y) - n\frac{|t-s|^2}{2} - n\frac{|x-y|^2}{2} - \gamma(|x|^2 + |y|^2).$$

By the linear growth assumption on u and w, for each n and γ, there exists $(t_{n,\gamma}, s_{n,\gamma}, x_{n,\gamma}, y_{n,\gamma}) \in [0,T]^2 \times \mathbb{R}^{2d}$ attaining the maximum of $\Phi_{n,\gamma}$ on $[0,T]^2 \times \mathbb{R}^{2d}$. Using standard techniques from the theory of viscosity solutions, we see that, for each γ, there exists $(t_\gamma, x_\gamma) \in [0,T] \times \mathbb{R}^d$ such that

$$(t_{n,\gamma}, s_{n,\gamma}, x_{n,\gamma}, y_{n,\gamma}) \overset{n\to\infty}{\longrightarrow} (t_\gamma, t_\gamma, x_\gamma, x_\gamma), \tag{5.5.12}$$

$$n|x_{n,\gamma} - x_\gamma|^2 + n|y_{n,\gamma} - y_\gamma|^2 \overset{n\to\infty}{\longrightarrow} 0, \tag{5.5.13}$$

$$\tilde{u}(t_{n,\gamma}, x_{n,\gamma}) - \tilde{w}(s_{n,\gamma}, y_{n,\gamma}) \overset{n\to\infty}{\longrightarrow} \tilde{u}(t_\gamma, x_\gamma) - \tilde{w}(s_\gamma, y_\gamma). \tag{5.5.14}$$

We also notice that, proceeding by contradiction, we can prove that, if γ is small enough, then $t_\gamma < T$, so that $t_{n,\gamma}, s_{n,\gamma} < T$, up to a subsequence. Finally, we derive a useful inequality. More precisely, for any $\xi, \xi' \in \mathbb{R}^d$, from the maximum property $\Phi_{n,\gamma}(t_{n,\gamma}, s_{n,\gamma}, x_{n,\gamma} + d, y_{n,\gamma} + d') \leq \Phi_{n,\gamma}(t_{n,\gamma}, s_{n,\gamma}, x_{n,\gamma}, y_{n,\gamma})$ we get

$$\begin{aligned}
&\tilde{u}(t_{n,\gamma}, x_{n,\gamma} + d) - \tilde{u}(t_{n,\gamma}, x_{n,\gamma}) - nd.(x_{n,\gamma} - y_{n,\gamma}) \\
&\leq \; \tilde{w}(s_{n,\gamma}, y_{n,\gamma} + d') - \tilde{w}(s_{n,\gamma}, y_{n,\gamma}) - nd'.(x_{n,\gamma} - y_{n,\gamma}) \\
&+ n\frac{|d - d'|^2}{2} + \gamma(|x_{n,\gamma} + d|^2 - |x_{n,\gamma}|^2 + |y_{n,\gamma} + d'|^2 - |y_{n,\gamma}|^2).
\end{aligned} \tag{5.5.15}$$

Step 3. We shall apply the nonlocal Jensen-Ishii's lemma (see Lemma 1 in [4]). To this end, let $\gamma \in (0, \gamma^*]$ and define

$$\varphi_n(t,s,x,y) = n\frac{|t-s|^2}{2} + n\frac{|x-y|^2}{2} + \gamma(|x|^2 + |y|^2) - \Phi_{n,\gamma}(t_{n,\gamma}, s_{n,\gamma}, x_{n,\gamma}, y_{n,\gamma}),$$

for all $(t,s,x,y) \in \mathbb{R}^{2+2d}$ and for any $n \in \mathbb{N}\backslash\{0\}$. Then $(t_n, s_n, x_n, y_n) := (t_{n,\gamma}, s_{n,\gamma}, x_{n,\gamma}, y_{n,\gamma})$ is a zero global maximum point for $\tilde{u}(t,x) - \tilde{w}(s,y) - \varphi_n(t,s,x,y)$ on $[0,T]^2 \times \mathbb{R}^{2d}$. Set

$$
\begin{aligned}
(p_n, q_n) &:= \Big(\frac{\partial \varphi_n}{\partial t}(t_n, s_n, x_n, y_n), D_x\varphi_n(t_n, s_n, x_n, y_n)\Big), \\
(-p'_n, -q'_n) &:= \Big(\frac{\partial \varphi_n}{\partial s}(t_n, s_n, x_n, y_n), D_y\varphi_n(t_n, s_n, x_n, y_n)\Big).
\end{aligned}
$$

Then, for any $\hat{r} > 0$, it follows from the nonlocal Jensen-Ishii's lemma that there exists $\hat{\alpha}(\hat{r}) > 0$ such that, for any $0 < \alpha \leq \hat{\alpha}(\hat{r})$, we have : there exist sequences (for simplicity, we omit the dependence on α) $(t_{n,k}, s_{n,k}, x_{n,k}, y_{n,k}) \to (t_n, s_n, x_n, y_n)$, $(t_{n,k}, s_{n,k}, x_{n,k}, y_{n,k}) \in [0,T)^2 \times \mathbb{R}^{2d}$, $(p_{n,k}, p'_{n,k}, q_{n,k}, q'_{n,k}) \to (p_n, p'_n, q_n, q'_n)$, matrices $N_{n,k}, N'_{n,k} \in \mathbb{S}^d$, with $(N_{n,k}, N'_{n,k})$ converging to some $(M_{n,\alpha}, M'_{n,\alpha})$, and a sequence of functions $\varphi_{n,k} \in C^{1,2}([0,T]^2 \times \mathbb{R}^{2d})$ such that :

(i) $(t_{n,k}, s_{n,k}, x_{n,k}, y_{n,k})$ is a global maximum point of $\tilde{u} - \tilde{w} - \varphi_{n,k}$;

(ii) $\tilde{u}(t_{n,k}, x_{n,k}) \to \tilde{u}(t_n, x_n)$ and $\tilde{w}(s_{n,k}, y_{n,k}) \to \tilde{w}(s_n, y_n)$, as k tends to infinity ;

(iii) $(p_{n,k}, q_{n,k}, N_{n,k}) \in \mathcal{P}^{2,+}\tilde{u}(t_{n,k}, x_{n,k})$, $(p'_{n,k}, q'_{n,k}, N'_{n,k}) \in \mathcal{P}^{2,-}\tilde{w}(s_{n,k}, y_{n,k})$, and

$$
\begin{aligned}
(p_{n,k}, q_{n,k}) &:= \Big(\frac{\partial \varphi_{n,k}}{\partial t}(t_{n,k}, s_{n,k}, x_{n,k}, y_{n,k}), D_x\varphi_{n,k}(t_{n,k}, s_{n,k}, x_{n,k}, y_{n,k})\Big), \\
(-p'_{n,k}, -q'_{n,k}) &:= \Big(\frac{\partial \varphi_{n,k}}{\partial s}(t_{n,k}, s_{n,k}, x_{n,k}, y_{n,k}), D_y\varphi_{n,k}(t_{n,k}, s_{n,k}, x_{n,k}, y_{n,k})\Big);
\end{aligned}
$$

(iv) The following inequalities hold (we denote by I the $2d \times 2d$ identity matrix and by $D^2_{(x,y)}\varphi_{n,k}$ the Hessian matrix of $\varphi_{n,k}$ with respect to (x,y))

$$
-\frac{1}{\alpha} I \leq \begin{pmatrix} N_{n,k} & 0 \\ 0 & -N'_{n,k} \end{pmatrix} \leq D^2_{(x,y)}\varphi_{n,k}(t_{n,k}, s_{n,k}, x_{n,k}, y_{n,k}). \tag{5.5.16}
$$

(v) $\varphi_{n,k}$ converges uniformly in $\mathbb{R}^{2+2d}$ and in $C^2(B_{\hat{r}}(t_n, s_n, x_n, y_n))$ (where $B_{\hat{r}}(t_n, s_n, x_n, y_n)$ is the ball in $\mathbb{R}^{2+2d}$ of radius $\hat{r}$ and centered at (t_n, s_n, x_n, y_n)) towards $\psi_{n,\alpha} := R^\alpha[\varphi_n](\cdot, (p_n, p'_n, q_n, q'_n))$, where, for any $\xi \in \mathbb{R}^{2+2d}$,

$$
R^\alpha[\varphi_n](z,\xi) := \sup_{|z'-z|\leq 1}\Big\{\varphi_n(z') - \xi.(z'-z) - \frac{|z'-z|^2}{2\alpha}\Big\}, \qquad \forall\, z \in \mathbb{R}^{2+2d}.
$$

Then, from Lemma 5.5.2 and the viscosity subsolution property to (5.5.10)-(5.5.11) of $\tilde{u}$, we have :

$$
\begin{aligned}
\rho\tilde{u}(t_{n,k}, x_{n,k}) - p_{n,k} - \sup_{a\in A}\Big[& b(x_{n,k}, a).q_{n,k} + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x_{n,k}, a)N_{n,k}) \\
& + I_a^{1,\delta}(t_{n,k}, x_{n,k}, \varphi_{n,k}(\cdot, s_{n,k}, \cdot, y_{n,k})) + I_a^{2,\delta}(t_{n,k}, x_{n,k}, q_{n,k}, \tilde{u}) \\
& + \tilde{f}(t_{n,k}, x_{n,k}, a, \tilde{u}(t_{n,k}, x_{n,k}), \sigma^\intercal(x_{n,k}, a)q_{n,k})\Big] \leq 0.
\end{aligned}
$$

On the other hand, from the viscosity supersolution property to (5.5.10)-(5.5.11) of $\tilde{w}$, we have :

$$
\rho\tilde{w}(s_{n,k}, y_{n,k}) - p'_{n,k} - \sup_{a\in A}\Big[b(y_{n,k}, a).q'_{n,k} + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(y_{n,k}, a)N'_{n,k})
$$

$$+I_a^{1,\delta}(s_{n,k},y_{n,k},-\varphi_{n,k}(t_{n,k},\cdot,x_{n,k},\cdot)) + I_a^{2,\delta}(s_{n,k},y_{n,k},q'_{n,k},\tilde{w})$$
$$+\tilde{f}(s_{n,k},y_{n,k},a,\tilde{w}(s_{n,k},y_{n,k}),\sigma^\intercal(y_{n,k},a)q'_{n,k})\Big] \ \geq \ 0.$$

For every $k \in \mathbb{N}\backslash\{0\}$, consider $a_k \in A$ such that

$$\begin{aligned} &\rho\tilde{u}(t_{n,k},x_{n,k}) - p_{n,k} - b(x_{n,k},a_k).q_{n,k} - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x_{n,k},a_k)N_{n,k}) \\ &-I_{a_k}^{1,\delta}(t_{n,k},x_{n,k},\varphi_{n,k}(\cdot,s_{n,k},\cdot,y_{n,k})) - I_{a_k}^{2,\delta}(t_{n,k},x_{n,k},q_{n,k},\tilde{u}) \\ &-\tilde{f}(t_{n,k},x_{n,k},a_k,\tilde{u}(t_{n,k},x_{n,k}),\sigma^\intercal(x_{n,k},a_k)q_{n,k}) \ \leq \ \frac{1}{k}. \end{aligned} \tag{5.5.17}$$

From the compactness of A, we can suppose that $a_k \to a_\infty \in A$, up to a subsequence. Moreover, for every $a \in A$ we have

$$\begin{aligned} &\rho\tilde{w}(s_{n,k},y_{n,k}) - p'_{n,k} - b(y_{n,k},a).q'_{n,k} - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(y_{n,k},a)N'_{n,k}) \\ &-I_a^{1,\delta}(s_{n,k},y_{n,k},-\varphi_{n,k}(t_{n,k},\cdot,x_{n,k},\cdot)) - I_a^{2,\delta}(s_{n,k},y_{n,k},q'_{n,k},\tilde{w}) \\ &-\tilde{f}(s_{n,k},y_{n,k},a,\tilde{w}(s_{n,k},y_{n,k}),\sigma^\intercal(y_{n,k},a)q'_{n,k}) \ \geq \ 0. \end{aligned} \tag{5.5.18}$$

Set $r^* := 2\sup_{(a,e)\in A\times(E\cap\{|e|\leq\delta\})}(|\beta(x^*,a,e)| \vee |\beta(y^*,a,e)|)$, where from (5.5.12) we define $(x^*,y^*) := \lim_{n\to\infty}(x_n,y_n)$, and $\alpha^* := \hat{\alpha}(r^*)$. Notice that for all $n \in \mathbb{N}\backslash\{0\}$ we have $\sup_{(a,e)\in A\times(E\cap\{|e|\leq\delta\})}(|\beta(x_n,a,e)|\vee|\beta(y_n,a,e)|) < r^*$, up to a subsequence. Therefore, sending k to infinity, we get $\varphi_{n,k} \to \psi_{n,\alpha}$, as k tends to infinity, uniformly in $C^2(B_{r^*}(t_n,s_n,x_n,y_n))$ for any $0 < \alpha \leq \alpha^*$. Moreover, from assumption **(Hλ)(iii)** we have

$$\begin{aligned} &\limsup_{k\to\infty}\int_{E\cap\{|e|>\delta\}}(\tilde{u}(t_{n,k},x_{n,k}+\beta(x_{n,k},a_k,e)) - \tilde{u}(t_{n,k},x_{n,k}) - \beta(x_{n,k},a_k,e).q_{n,k})\lambda(a_k,de) \\ &\leq \int_{E\cap\{|e|>\delta\}}(\tilde{u}(t_n,x_n+\beta(x_n,a_\infty,e)) - \tilde{u}(t_n,x_n) - \beta(x_n,a_\infty,e).q_n)\lambda(a_\infty,de). \end{aligned}$$

Therefore, from (5.5.17) we obtain

$$\begin{aligned} &\rho\tilde{u}(t_n,x_n) - p_n - b(x_n,a_\infty).q_n - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x_n,a_\infty)M_{n,\alpha}) - I_{a_\infty}^{1,\delta}(t_n,x_n,\psi_{n,\alpha}(\cdot,s_n,\cdot,y_n)) \\ &-I_{a_\infty}^{2,\delta}(t_n,x_n,q_n,\tilde{u}) - \tilde{f}(t_n,x_n,a_\infty,\tilde{u}(t_n,x_n),\sigma^\intercal(x_n,a_\infty)q_n) \ \leq \ 0. \end{aligned}$$

A fortiori, if we take the supremum over $a \in A$ we conclude

$$\begin{aligned} &\rho\tilde{u}(t_n,x_n) - p_n - \sup_{a\in A}\Big[b(x_n,a).q_n + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x_n,a)M_{n,\alpha}) + I_a^{1,\delta}(t_n,x_n,\psi_{n,\alpha}(\cdot,s_n,\cdot,y_n)) \\ &+ I_a^{2,\delta}(t_n,x_n,q_n,\tilde{u}) + \tilde{f}(t_n,x_n,a,\tilde{u}(t_n,x_n),\sigma^\intercal(x_n,a)q_n)\Big] \ \leq \ 0, \end{aligned} \tag{5.5.19}$$

for any $0 < \alpha \leq \alpha^*$. On the other hand, letting k to infinity in (5.5.18) for every fixed $a \in A$, and then taking the supremum, we end up with

$$\begin{aligned} &\rho\tilde{w}(s_n,y_n) - p'_n - \sup_{a\in A}\Big[b(y_n,a).q'_n + \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(y_n,a)M'_{n,\alpha}) + I_a^{1,\delta}(s_n,y_n,-\psi_{n,\alpha}(t_n,\cdot,x_n,\cdot)) \\ &+ I_a^{2,\delta}(s_n,y_n,q'_n,\tilde{w}) + \tilde{f}(s_n,y_n,a,\tilde{w}(s_n,y_n),\sigma^\intercal(y_n,a)q'_n)\Big] \ \geq \ 0, \end{aligned} \tag{5.5.20}$$

for any $0 < \alpha \leq \alpha^*$. Moreover, from (5.5.16) we have

$$-\frac{1}{\alpha}I \;\leq\; \begin{pmatrix} M_{n,\alpha} & 0 \\ 0 & -M'_{n,\alpha} \end{pmatrix} \;\leq\; D^2_{(x,y)}\psi_{n,\alpha}(t_n,s_n,x_n,y_n) \tag{5.5.21}$$

and by direct calculation

$$D^2_{(x,y)}\psi_{n,\alpha}(t_n,s_n,x_n,y_n) \;=\; D^2_{(x,y)}\varphi_n(t_n,s_n,x_n,y_n) + o(1), \qquad \text{as } \alpha \to 0^+. \tag{5.5.22}$$

Step 4. From (5.5.19), for any n, consider $a_n \in A$ such that

$$\begin{aligned}&\rho\tilde{u}(t_n,x_n) - p_n - b(x_n,a_n).q_n - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x_n,a_n)M_{n,\alpha}) - I^{1,\delta}_{a_n}(t_n,x_n,\psi_{n,\alpha}(\cdot,s_n,\cdot,y_n))\\ &- I^{2,\delta}_{a_n}(t_n,x_n,q_n,\tilde{u}) - \tilde{f}(t_n,x_n,a_n,\tilde{u}(t_n,x_n),\sigma^\intercal(x_n,a_n)q_n) \;\leq\; \frac{1}{n}.\end{aligned} \tag{5.5.23}$$

On the other hand, from (5.5.20) we deduce that

$$\begin{aligned}&\rho\tilde{w}(s_n,y_n) - p'_n - b(y_n,a_n).q'_n - \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(y_n,a_n)M'_{n,\alpha}) - I^{1,\delta}_{a_n}(s_n,y_n,-\psi_{n,\alpha}(t_n,\cdot,x_n,\cdot))\\ &- I^{2,\delta}_{a_n}(s_n,y_n,q'_n,\tilde{w}) - \tilde{f}(s_n,y_n,a_n,\tilde{w}(s_n,y_n),\sigma^\intercal(y_n,a_n)q'_n) \;\geq\; 0.\end{aligned} \tag{5.5.24}$$

By subtracting (5.5.24) to (5.5.23), we obtain :

$$\begin{aligned}\rho(\tilde{u}(t_n,x_n) - \tilde{w}(s_n,y_n)) \;\leq\;& \frac{1}{n} + p_n - p'_n + \Delta F_n + \Delta I^{1,\delta}_n + \Delta I^{2,\delta}_n\\ &+ b(x_n,a_n).q_n - b(y_n,a_n).q'_n\\ &+ \frac{1}{2}\mathrm{tr}(\sigma\sigma^\intercal(x_n,a_n)M_{n,\alpha} - \sigma\sigma^\intercal(y_n,a_n)M'_{n,\alpha}),\end{aligned} \tag{5.5.25}$$

where

$$\begin{aligned}\Delta F_n \;&=\; \tilde{f}(t_n,x_n,a_n,\tilde{u}(t_n,x_n),\sigma^\intercal(x_n,a_n)q_n) - \tilde{f}(s_n,y_n,a_n,\tilde{w}(s_n,y_n),\sigma^\intercal(y_n,a_n)q'_n),\\ \Delta I^{1,\delta}_n \;&=\; I^{1,\delta}_{a_n}(t_n,x_n,\psi_{n,\alpha}(\cdot,s_n,\cdot,y_n)) - I^{1,\delta}_{a_n}(s_n,y_n,-\psi_{n,\alpha}(t_n,\cdot,x_n,\cdot)),\\ \Delta I^{2,\delta}_n \;&=\; I^{2,\delta}_{a_n}(t_n,x_n,q_n,\tilde{u}) - I^{2,\delta}_{a_n}(s_n,y_n,q'_n,\tilde{w}).\end{aligned}$$

We have

$$p_n - p'_n \;=\; \frac{\partial\varphi_n}{\partial t}(t_n,s_n,x_n,y_n) + \frac{\partial\varphi_n}{\partial s}(t_n,s_n,x_n,y_n) \;=\; 0.$$

By the uniform Lipschitz property of b with respect to x, and (5.5.13), we see that

$$\begin{aligned}&\lim_{n\to\infty}(b(x_n,a_n).q_n - b(y_n,a_n).q'_n)\\ &= \lim_{n\to\infty}(b(x_n,a_n).D_x\varphi_n(t_n,x_n,y_n) + b(y_n,a_n).D_y\varphi_n(t_n,x_n,y_n)) \;=\; 0.\end{aligned}$$

Regarding the trace term in (5.5.25), by the uniform Lipschitz property of σ with respect to x, (5.5.21), (5.5.22), and (5.5.13), we obtain

$$\limsup_{n\to\infty}\limsup_{\alpha\to 0^+}\mathrm{tr}(\sigma\sigma^\intercal(x_n,a_n)M_{n,\alpha} - \sigma\sigma^\intercal(y_n,a_n)M'_{n,\alpha}) \;\leq\; 0.$$

Moreover, from assumption **(HBC)** and (5.5.13)-(5.5.14), we find

$$\lim_{n\to\infty}|\Delta F_n| = 0.$$

Concerning the integral term $\Delta I_n^{1,\delta}$, we have, for some $\vartheta', \vartheta'' \in (0,1)$,

$$\Delta I_n^{1,\delta} = \int_{E\cap\{|e|\leq\delta\}} [D_x^2\psi_{n,\alpha}(t_n, s_n, x_n + \vartheta'\beta(x_n, a_n, e), y_n)\beta(x_n, a_n, e).\beta(x_n, a_n, e) \\ + D_y^2\psi_{n,\alpha}(t_n, s_n, x_n, y_n + \vartheta''\beta(y_n, a_n, e))\beta(x_n, a_n, e).\beta(x_n, a_n, e)]\lambda(a_n, de).$$

Therefore, using (5.5.22) we see that there exists a positive constant C_n', depending only on (x_n, y_n), the Lipschitz constant of β, and on $\sup_{\vartheta',\vartheta''\in[0,1]} |D_x^2\varphi_n(t_n, s_n, x_n+\vartheta'\beta(x_n, a_n, e), y_n)| \vee |D_y^2\varphi_n(t_n, s_n, x_n, y_n + \vartheta''\beta(y_n, a_n, e))|$, such that

$$\limsup_{\alpha\to 0^+} |\Delta I_n^{1,\delta}| \leq C_n' \int_{E\cap\{|e|\leq\delta\}} (1 \wedge |e|^2)\lambda(a_n, de).$$

Finally, it remains to consider the integral term $\Delta I_n^{2,\delta}$. Integrating inequality (5.5.15), with $d = \beta(x_n, a_n, e)$ and $d' = \beta(y_n, a_n, e)$, we find

$$I_{a_n}^{2,\delta}(t_n, x_n, q_n, \tilde{u}) \leq I_{a_n}^{2,\delta}(s_n, y_n, q_n', \tilde{w}) + n\int_{E\cap\{|e|>\delta\}} \frac{|\beta(x_n, a_n, e) - \beta(y_n, a_n, e)|^2}{2}\lambda(a_n, de) \\ + \gamma\int_{E\cap\{|e|>\delta\}} (|x_n + \beta(x_n, a_n, e)|^2 - |x_n|^2)\lambda(a_n, de) \\ + \gamma\int_{E\cap\{|e|>\delta\}} (|y_n + \beta(y_n, a_n, e)|^2 - |y_n|^2)\lambda(a_n, de).$$

Then, it follows from assumption **(HFC)** that there exists a positive constant C'', such that

$$I_{a_n}^{2,\delta}(t_n, x_n, q_n, \tilde{u}) \leq I_{a_n}^{2,\delta}(s_n, y_n, q_n', \tilde{w}) + nC''\frac{|x_n - y_n|^2}{2} + \gamma C''.$$

In conclusion, taking the $\limsup_{n\to\infty} \limsup_{\delta\to 0^+} \limsup_{\alpha\to 0^+}$ in both sides of (5.5.25), we see that the we get the required contradiction for γ small enough. $\square$

Chapitre 6

Conditional asset liability management

6.1 Introduction

The purpose of this chapter is the design of optimal Asset Liability policy, in a framework where the asset manager faces a constraint on the distribution of its terminal wealth. More precisely, the investor requires to pay for simplicity a constant liability D_0 at maturity T and allows for this constraint to be violated with a given small probability $1-p$. In practice, whenever a conservative investor imposes an almost-sure constraint on the terminal value of an investment strategy, this leads to rather overcautious investment policies. This mainly comes from the fact that it is too costly to take some risk, since it will complicate the necessity of satisfying the constraint at maturity. The main objective of the chapter is to quantify the effect of relieving slightly this constraint by only imposing the probability of success at maturity to exceed p, and to measure the dependence in p on the optimal asset management policy.

The modern portfolio theory in continuous time goes back to the seminal paper of Merton [82], who consider an agent trying to maximize his expected utility from terminal wealth or expected time-integrated utility from consumption. In a Markovian framework, the optimal policy identifies in terms of the solution of the corresponding Hamilton-Jacobi-Bellman equation, or alternatively can be derived using duality arguments, see e.g. Karatzas, Lehoczky et Shreeve [64]. This framework has raised a large literature, with the introduction of additional constraints : e.g. on the investment policy by Cvitanic and Karatzas in [30], with a given almost sure portfolio insurance by El Karoui, Jeanblanc and Lacoste in [36] or with drawdown constraints by Elie and Touzi in [35]. In this context, considering the constraint of beating a given benchmark with a given probability of success, this problem has already been studied by Boyle and Tian in [18], via a duality argument, mainly inspired from the approach of Follmer and Leukert in [44] for quantile hedging problems. In the recent literature, a new approach introduced by Bouchard, Elie and Touzi in [15] allows to study these a priori dynamically inconsistent problems in a dynamic manner.

The main difficulty in considering constraints written in terms of probability, is that the probability of success p is imposed at time 0, but trying to build up a dynamic programming principle, one requires to be able to quantify the effect of such constraint at any given intermediate date t. The proper way to do this has been identified in [15], where the dynamic probability of success is viewed as a new forward controlled martingale process. This new variable allows to solve the problem in a dynamically consistent manner. The resolution of stochastic control problems under such quantile has been more specifically been studied in [14], via the derivation of a dynamic programming principle. The objective of this chapter is to provide a realistic financial application of such methodology in the framework of asset liability management, via the derivation of a proper converging numerical approximation procedure.

More specifically, we consider an investor, who can at any time t choose the investment policy θ_t, as well as the instantaneous rate c_t of additional endowment to the portfolio, which is non-negative and upper-bounded by a given constant $\bar{c}$. Hence the dynamics of the wealth is the following :

$$dX_t^{\theta,c} = \theta_t X_t^{\theta,c} \frac{dS_t}{S_t} + c_t dt,$$

and the wealth is contained to remain non-negative as well as to satisfy the constraint :

$$\mathbb{P}[X_T^{\theta,c} \geq D_0] \geq p\,.$$

Under theses constraints, the objective of the risk-neutral investor is to minimize the total discounted amount of additional endowment required in the portfolio by solving

$$\inf_{c,\theta} \mathbb{E}\left[\int_0^T e^{-\beta t} c_t dt\right]\ .$$

The solution of this problem rewrites as $w(0,x,p)$, where the function w is identified hereafter as the unique viscosity solution of a corresponding HJB equation.

The main difficulty consists first in deriving the proper domain of definition of this function, by identifying the minimal wealth $u(t,p)$, for which there exists an admissible investment strategy (c,θ), allowing to satisfy the constraint at maturity T. The function u identifies to the unique viscosity solution of a non linear variational inequality, which can be solved numerically. It is worth noticing that the Fenchel transform of u in the p variable solves a simpler linear variational inequality. We observe that function $u(t,.)$ is convex in p and the convexity decreases slowly to a linear limit as t converges to the maturity T. We also exhibit empirical numerical results for the more complex case, where the investor faces additional fixed constraints on the investment strategy θ.

As for the optimal policy (c,θ) of the investor, we observe that no extra endowment c is required whenever the current wealth x at time t is far enough from the minimal wealth function $u(t,p)$. In terms of investment strategy θ, we observe that the investor does not take any risk by cutting the financial market investment θ whenever the current wealth is sufficiently high, but prefers investing on the financial market for lower wealth, anticipating that he may get closer to the minimal admissible wealth $u(t,p)$ and hereby require to add extra endowment c to his portfolio.

The chapter is organized as follows : Section 6.2 is dedicated to the proper formulation of the problem as well as its reformulation as a stochastic control problem with almost sure constraints. Section 6.3 is focused on the determination of the minimal admissible wealth function u allowing the existence of a portfolio strategy satisfying the constraint. Section 6.4 provides a characterization of the value function of the asset liability problem, whereas Section 6.5 presents the numerical approximating scheme and discusses the obtained numerical results.

Notations : We also denote, for $u \in \mathbb{R}^n$, $B_\delta(u)$ the ball of radius η around u, ∂X the frontier of a set X and $\mathcal{T}_{[t,T]}$ the set of stopping times taking values in $[t,T]$.

6.2 Problem formulation

Throughout this chapter, we consider a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ endowed with a Brownian motion $W = \{W_t, 0 \leq t \leq T\}$ with values in $\mathbb{R}$, and we denote by $\mathbb{F} := \{\mathcal{F}_t, t \geq 0\}$.

The financial market consists of a non-risky asset, with process normalized to unity, and one risky asset with price process defined by the Black and Scholes model :

$$dS_t \quad = \quad S_t\left(\mu dt + \sigma dW_t\right) \ ,$$

where $\sigma > 0$ is the volatility parameter, and $\mu \in \mathbb{R}$ is the drift of the financial asset.

The normalization of the non-risky asset to unity is as usual a reduction of the model obtained by taking this asset as a numéraire. Hence, all amounts are evaluated in terms of their discounted values.

6.2.1 Endowment-investment strategy and partial hedging constraint

We now introduce the set of admissible income-investment strategies, whose induced wealth process satisfies the following constraints

$$X_t \geq 0 \text{ for every } t \geq 0 \text{ a.s.} \qquad \text{and} \quad \mathbb{P}[X_T \geq D_0] \geq p \ , \tag{6.2.1}$$

where D_0 is a constant (discounted) liability at maturity T and $p \in [0,1]$ is the probability parameter of success.

In order to satisfy the quantile liability constraint, the asset manager can invest on the risky asset S as well as bringing new endowment.

A portfolio strategy is an $\mathbb{F}-$adapted process $\theta = \{\theta_t,\ 0 \leq t \leq T\}$, with values in $\mathbb{R}$, satisfying the integrability condition

$$\int_0^T |\theta_t X_t|^2 dt \ < \ \infty \quad \text{for all} \quad T > 0\,. \tag{6.2.2}$$

We denote $\mathcal{A}$ the set of such portfolio strategies.

A consumption strategy is an $\mathbb{F}-$adapted process $c = \{c_t,\ t \geq 0\}$, with values in $[0, \bar{c}]$, where $\bar{c}$ is a given maximum rate of endowment. We denote $\mathcal{C}$ the set of such consumption strategies.

Here, θ_t and c_t denote respectively the proportion of wealth invested in the risky asset and the running endowment at time t. By the self-financing condition, the wealth process induced by such a pair (c, θ) is defined by

$$X_s^{t,x,\theta,c} = x + \int_t^s \theta_u X_u^{t,x,\theta,c} \frac{dS_u}{S_u} + \int_t^s c_u du, \quad t \leq s \leq T \tag{6.2.3}$$

where x is some given initial capital at time t. We shall denote by $\mathcal{A}_p(t, x)$ the collection of all such consumption-investment strategies whose corresponding wealth process satisfies the partial hedging constraint (6.2.1).

6.2.2 The optimal endowment-investment problem

We consider a risk neutral asset manager whose subjective discount factor is denoted by a constant $\beta > 0$. For a given initial wealth $x \geq 0$ and probability p of success, the asset manager wishes to solve the following endowment-investment problem under the partial hedging constraint (6.2.1) :

$$w(0, x, p) := \inf_{(c,\theta) \in \mathcal{A}_p(0,x)} \mathbb{E}\left[\int_0^T e^{-\beta t} c_t dt\right] .$$

Remark 6.2.1. For $p = 0$, the partial hedging constraint is automatically satisfied, so that the optimal policy is obviously given by $(c, \theta) = (0, 0)$ and $w(0, ., 0) = 0$ on $\mathbb{R}^+$. Similarly, observe that, for $p = 1$, the partial hedging constraint becomes an almost sure classical one.

We now introduce the dynamic version of the problem as

$$w(t, x, p) := \inf_{(c,\theta) \in \mathcal{A}_p(t,x)} \mathbb{E}\left[\int_t^T e^{-\beta(s-t)} c_s ds\right] .$$

6.3 The minimal admissible wealth

6.3.1 Definition and viscosity solution property

We introduce an additional controlled state variable, valued in $[0, 1]$ and defined by :

$$P_t^{t,p,\alpha} = p, \quad dP_s^{t,p,\alpha} = \alpha_s dW_s, \ s \in [t, T],$$

where the additional control α is an $\mathbb{F}$-progressively measurable $\mathbb{R}$-valued $\mathbb{P}$-a.s. square integrable process. We denote $\mathcal{B}$ the set of such controls.

The minimal p-admissible wealth at time t is given by

$$\begin{aligned} & u(t,p) \\ = \; & \inf\{x \geq 0 \text{ s.t. } \exists (\theta, c) \in \mathcal{A} \times \mathcal{C}, \forall s \in [t, T], X_s^{t,x,\theta,c} \geq 0 \text{ and } \mathbb{P}[X_T^{t,x,\theta,c} \geq D_0] \geq p\} \\ = \; & \inf\{x \in \mathbb{R} \text{ s.t. } \exists \theta \in \mathcal{A}, \forall s \in [t, T], X_s^{t,x,\theta,\bar{c}} \geq 0 \text{ and } \mathbb{P}[X_T^{t,x,\theta,\bar{c}} \geq D_0] \geq p\} \\ = \; & \inf\{x \in \mathbb{R} \text{ s.t. } \exists (\theta, \alpha) \in \mathcal{A} \times \mathcal{B}, \forall s \in [t, T], X_s^{t,x,\theta,\bar{c}} \geq 0 \end{aligned}$$

$$\text{and } \mathbf{1}\{x+\bar{c}(T-t)+\int_0^T \theta_u X_u^{t,x,\theta,\bar{c}} \frac{dS_u}{S_u} \geq D_0\} \geq p + \int_t^T \alpha_s dW_s\}$$
$$= \inf\{x \in \mathbb{R} \text{ s.t. } \exists(\theta,\alpha) \in \mathcal{A}\times\mathcal{B}, \forall s \in [t,T], X_s^{t,x,\theta,\bar{c}} \geq 0 \text{ and } X_T^{t,x,\theta,\bar{c}} \geq D_0 \mathbf{1}_{P_T^{t,p,\alpha}>0}\}$$

where the third equality follows from Paragraph 3.5 of [15].
Denoting $g(s,p) = 0 * \mathbf{1}_{s<T} + D_0 1_{p>0} \mathbf{1}_{s=T}$, we finally obtain

$$u(t,p) = \inf\{x \in \mathbb{R} \text{ s.t. } \exists(\theta,\alpha) \in \mathcal{A}\times\mathcal{B}, \forall s \in [t,T], X_s^{t,x,\theta,\bar{c}} \geq g(s, P_s^{t,p,\alpha})\}$$

Therefore, by Example 2.1 of [17], $u(., P_.^{\alpha})$ satisfies the following dynamic programming principle :

– **(DP1)** If $x > u(t,p)$, then there exists $(\theta^*, \alpha^*) \in \mathcal{A}\times\mathcal{B}$ such that

$$X_\tau^{t,x,\theta^*,\bar{c}} \geq u(\tau, P_\tau^{t,p,\alpha^*}) \text{ for all } \tau \in \mathcal{T}_{[t,T]}$$

– **(DP2)** If $x < u(t,p)$, then there exists $\tau^* \in \mathcal{T}_{[t,T]}$ such that

$$\mathbb{P}[X_{\tau\wedge\tau^*}^{t,x,\theta,\bar{c}} > u(\tau, P_\tau^{t,p,\alpha})\mathbf{1}_{\tau<\tau^*} + g(\tau^*, P_{\tau^*}^{t,p,\alpha})\mathbf{1}_{\tau\geq\tau^*}] < 1$$

for all $\tau \in \mathcal{T}_{[t,T]}$ and $(\theta,\alpha) \in \mathcal{A}\times\mathcal{B}$.
We set, for $(z,q,a) \in \mathbb{R}^3$:

$$F^{\alpha,\theta}(z,a) := -\frac{\alpha^2}{2}a + \mu\theta z + \bar{c}$$

and

$$F(z,q,a) := \sup_{\{(\alpha,\theta)\in\mathbb{R}^2,\ \alpha q = \sigma\theta z\}} F^{\alpha,\theta}(z,a).$$

However, since $\mathbb{R}^2$ is not bounded, the operator F is not necessarily continuous and we shall have to relax it and consider its lower semicontinuous and upper semicontinuous envelopes F_* and F^* on $\mathbb{R}^3$. The Dynamic Programming Principle leads to the PDE :

$$\min([-\frac{\partial\varphi}{\partial t}(t,p) + F(\varphi(t,p), \frac{\partial\varphi}{\partial p}(t,p), \frac{\partial^2\varphi}{\partial p^2}(t,p))], \varphi) = 0 \quad (6.3.1)$$

Besides, the terminal condition is

$$\varphi(T,p) = D_0 p. \quad (6.3.2)$$

Definition 6.3.1. – *The lower semicontinuous envelope of the function u is defined on* $[0,T]\times[0,1]$ *by*

$$u_*(t,x) = \liminf_{(t',p')\in[0,T)\times(0,1)\to(t,p)} u(t',p')$$

– *The upper semicontinuous envelope of the function* u *is defined on* $[0,T]\times[0,1]$ *by*

$$u^*(t,x) = \limsup_{(t',p')\in[0,T)\times(0,1)\to(t,p)} u(t',p')$$

Theorem 6.3.1. *u is a viscosity solution to* (6.3.1) *in the sense it verifies :*

(i) *Viscosity supersolution property :*

$$\min([-\frac{\partial\varphi}{\partial t}(t,p)+F^*(\varphi(t,p),\frac{\partial\varphi}{\partial p}(t,p),\frac{\partial^2\varphi}{\partial p^2}(t,p))],u)\geq 0$$

for any $(t,p)\in[0,T)\times(0,1)$ *and any* $\varphi\in C^{1,2}([0,T)\times(0,1))$ *such that*

$$(u_*-\varphi)(t,p)=\inf_{[0,T]\times(0,1)}u_*-\varphi.$$

(ii) *Viscosity subsolution property :*

$$\min([-\frac{\partial\varphi}{\partial t}(t,p)+F(\varphi(t,p),\frac{\partial\varphi}{\partial p}(t,p),\frac{\partial^2\varphi}{\partial p^2}(t,p))],u)\leq 0$$

for any $(t,p)\in[0,T)\times(0,1)$ *and any* $\varphi\in C^{1,2}([0,T)\times(0,1))$ *such that*

$$(u^*-\varphi)(t,p)=\sup_{[0,T]\times(0,1)}u^*-\varphi.$$

Proof We adapt the proof of Theorem 2.1 in [15].
Supersolution property on $[0,T)\times(0,1)$
Let $(\bar{t},\bar{p})\in[0,T)\times(0,1)$ and φ be a smooth function such that

$$0=(u_*-\varphi)(\bar{t},\bar{p})<(u_*-\varphi)(t,p),\quad(\bar{t},\bar{p})\neq(t,p)\in[0,T)\times(0,1)\tag{6.3.3}$$

$u\geq 0$ by definition and therefore $u_*\geq 0$. It thus suffices to show that

$$-\frac{\partial\varphi}{\partial t}(t,p)+F^*(\varphi(t,p),\frac{\partial\varphi}{\partial p}(t,p),\frac{\partial^2\varphi}{\partial p^2}(t,p))\geq 0$$

Assume to the contrary that

$$-\frac{\partial\varphi}{\partial t}(t,p)+F^*(\varphi(t,p),\frac{\partial\varphi}{\partial p}(t,p),\frac{\partial^2\varphi}{\partial p^2}(t,p))=-2\epsilon<0.\tag{6.3.4}$$

By smoothness of φ there exists $\delta>0$ such that :

$$-\frac{\partial\varphi}{\partial t}(t,p)-\frac{\alpha^2}{2}\frac{\partial^2\varphi}{\partial p^2}(t,p)+\mu\theta x+\bar{c}\leq-\epsilon<0\tag{6.3.5}$$

for any $(\alpha,\theta,x,t,p)\in\mathbb{R}^3\times[0,T]\times(0,1)$ such that $|t-\bar{t}|<\delta$, $|p-\bar{p}|<\delta$, $|x-\varphi(t,p)|<\delta$ and

$$|\sigma\theta x-\alpha\frac{\partial\varphi}{\partial p}(t,p)|<\delta$$

Let $\partial_p B_\delta(\bar{t},\bar{p}):=\{\bar{t}+\delta\}\times[\bar{p}-\delta,\bar{p}+\delta]\cup[\bar{t},\bar{t}+\delta)\times\{\bar{p}-\delta,\bar{p}+\delta\}$ and observe that, since $(\bar{t},\bar{p})$ is a strict minimum of $u_*-\varphi$ on $[0,T]\times(0,1)$,

$$\xi=\min_{\partial_p B_\delta(\bar{t},\bar{p})}u_*-\varphi>0.\tag{6.3.6}$$

We now show that (6.3.5) and (6.3.6) lead to a contradiction to **(DP1)**. Let (t_n,p_n) be a sequence in $[0,T)\times(0,1)$ which converges to $(\bar{t},\bar{p})$ and such that $u(t_n,p_n)\to u_*(\bar{t},\bar{p})$. Set $x_n=u(t_n,p_n)+n^{-1}$ and observe that

$$\gamma_n:=x_n-\varphi(t_n,p_n)\to 0.\tag{6.3.7}$$

For each $n \geq 1$, $x_n > u(t_n, p_n) \geq 0$. Then, by **(DP1)** there exists some $(\theta^n, \alpha^n) \in \mathcal{A} \times \mathcal{B}$ such that, denoting $X^n := X^{t_n, x_n, \theta_n, \bar{c}}$,

$$X^n_\tau \geq u(\tau, P^{t_n, p_n, \alpha^n}_\tau) \quad \forall \tau \in \mathcal{T}_{[t_n, T]}$$

We now define the stopping times

$$\tau^o_n := \inf\{s \geq t_n : (s, P^{t_n,p_n,\alpha_n}_s) \notin B_\delta(\bar{t}, \bar{p})\}, \quad \tau_n := \inf\{s \geq t_n : |X^n_s - \varphi(s, P^{t_n,p_n,\alpha_n}_s)| \geq \delta\} \wedge \tau^o_n$$

and set

$$A_n := \{s \in [t_n, \tau_n] : -\frac{\partial \varphi}{\partial t}(s, P^{t_n,p_n,\alpha_n}_s) - \frac{(\alpha^n_s)^2}{2}\frac{\partial^2 \varphi}{\partial p^2}(s, P^{t_n,p_n,\alpha_n}_s) + \mu\theta^n_s X^n_s + \bar{c} > -\epsilon\} \tag{6.3.8}$$

By (6.3.5), the process

$$\psi^n_s = \sigma\theta^n_s X^n_s - \alpha^n_s \frac{\partial \varphi}{\partial p}(s, P^{t_n,p_n,\alpha_n}_s) \text{ satisfies } |\psi^n_s| > \delta \text{ for } s \in A_n. \tag{6.3.9}$$

By definition of (θ^n, α^n),

$$X^n_{t\wedge\tau_n} \geq u(t \wedge \tau_n, P^{t_n,p_n,\alpha_n}_{t\wedge\tau_n}), \quad t \geq t_n.$$

Using the definition of ξ and τ^n, this implies that

$$\begin{aligned} X^n_{t\wedge\tau_n} &\geq \varphi(t \wedge \tau_n, P^{t_n,p_n,\alpha_n}_{t\wedge\tau_n}) + (\xi \mathbf{1}_{\{\tau^o_n = \tau_n\}} + \delta \mathbf{1}_{\{\tau^o_n > \tau_n\}})\mathbf{1}_{\{t=\tau_n\}} \\ &\geq \varphi(t \wedge \tau_n, P^{t_n,p_n,\alpha_n}_{t\wedge\tau_n}) + (\xi \wedge \delta)\mathbf{1}_{\{t=\tau_n\}}, \quad t \geq t_n. \end{aligned}$$

Since φ is smooth, it follows from Itô's Lemma, (6.3.7), the definition of ψ^n and (6.3.8) that

$$\begin{aligned} -(\xi \wedge \delta)\mathbf{1}_{\{t<\tau_n\}} &\leq \gamma_n - (\xi \wedge \delta) + \int_{t\wedge t_n}^{\tau^n} \mu\theta^n_u X^n_u du + \int_{t\wedge t_n}^{\tau^n} \bar{c} du - \int_{t\wedge t_n}^{\tau^n} \frac{\partial \varphi}{\partial t}(u, P^{t_n,p_n,\alpha_n}_u) du \\ &\quad - \int_{t\wedge t_n}^{\tau_n} \frac{(\alpha^n_u)^2}{2}\frac{\partial^2 \varphi}{\partial p^2}(u, P^{t_n,p_n,\alpha_n}_u) du + \int_{t\wedge t_n}^{\tau^n} \psi^n_u dW_u \\ &\leq M^n_t := \gamma_n - \xi \wedge \delta + \int_{t\wedge t_n}^{\tau^n} b^n_u du + \int_{t\wedge t_n}^{\tau^n} \psi^n_u dW_u \end{aligned} \tag{6.3.10}$$

where we set

$$b^n_u := (\mu\theta^n_u X^n_u + \bar{c} - \frac{\partial \varphi}{\partial t}(u, P^{t_n,p_n,\alpha_n}_u) - \frac{(\alpha^n_u)^2}{2}\frac{\partial^2 \varphi}{\partial p^2}(u, P^{t_n,p_n,\alpha_n}_u))\mathbf{1}_{A_n}(u).$$

Let L^n be the exponential local martingale defined by $L^n_{t_n} = 1$ and, for $s \geq t_n$,

$$dL^n_s = -L^n_s b^n_s (\psi^n_s)^{-1} dW(s),$$

which is well defined by (6.3.9).
By Itô's formula and (6.3.10), we see that $L^n M^n$ is a local martingale which is bounded from below by the submartingale $-(\xi \wedge \delta)L^n$. Then $L^n M^n$ is a supermartingale, and it follows from (6.3.10) that $M^n_{t_n}$ is nonnegative, therefore

$$0 \leq \mathbb{E}[L^n_{\tau_n} M^n_{\tau_n}] \leq L^n_{t_n} M^n_{t_n} = M^n_{t_n} = \gamma_n - (\xi \wedge \delta),$$

which contradicts (6.3.7) for n large enough.
Subsolution property on $[0,T)\times(0,1)$
Let $(\bar t,\bar p)\in[0,T)\times(0,1)$ and φ be a smooth function such that

$$0=(u^*-\varphi)(\bar t,\bar p)>(u^*-\varphi)(t,p),\quad (\bar t,\bar p)\neq(t,p)\in[0,T)\times(0,1) \tag{6.3.11}$$

Assume to the contrary that

$$\min\Big(\sup_{\{\alpha\in\mathbb{R},\theta\in\mathbb{R},\ \alpha\frac{\partial\varphi}{\partial p}(\bar t,\bar p)=\sigma\theta\varphi(\bar t,\bar p)\}}\big[-\frac{\partial\varphi}{\partial t}(\bar t,\bar p)-\frac{\alpha^2}{2}\frac{\partial^2\varphi}{\partial p^2}(\bar t,\bar p)+\mu\theta\varphi(\bar t,\bar p)+\bar c\big],\varphi(\bar t,\bar p)\Big)=4\epsilon>0.$$

We may find $(\tilde\alpha,\tilde\theta)\in\mathbb{R}^2$ and $\delta>0$ such that

$$\tilde\alpha\frac{\partial\varphi}{\partial p}(\bar t,\bar p)=\sigma\tilde\theta\varphi(\bar t,\bar p) \tag{6.3.12}$$

and

$$-\frac{\partial\varphi}{\partial t}(t,p)-\frac{\tilde\alpha^2}{2}\frac{\partial^2\varphi}{\partial p^2}(t,p)+\mu\tilde\theta x+\bar c\geq 2\epsilon \tag{6.3.13}$$

for any $(x,t,p)\in\mathbb{R}\times[0,T]\times(0,1)$ such that $|t-\bar t|<\delta$, $|p-\bar p|<\delta$ and $|x-\varphi(t,p)|<\delta$. Since $\varphi(\bar t,\bar p)\geq 4\epsilon$ and $g=0$ on $[0,T)\times[0,1]$, we may assume without loss of generality that

$$x-g(t,p)=x>2\epsilon \tag{6.3.14}$$

for any $(x,t,p)\in\mathbb{R}\times[0,T]\times(0,1)$ such that $|t-\bar t|<\delta$, $|p-\bar p|<\delta$ and $|x-\varphi(t,p)|<\delta$.
We define $\bar\theta$ and $\bar\alpha$ by

$$\bar\alpha_t(x):=\tilde\alpha\frac{x}{\varphi(\bar t,\bar p)}\text{ and }\bar\theta_t(p):=\frac{\tilde\alpha}{\sigma\varphi(\bar t,\bar p)}\frac{\partial\varphi}{\partial p}(t,p),\ (x,t,p)\in\mathbb{R}\times[0,T)\times(0,1) \tag{6.3.15}$$

Clearly, $\forall(x,p)\in\mathbb{R}\times(0,1),\bar\alpha(x)\in\mathcal{B}$ and $\bar\theta(p)\in\mathcal{A}$. Besides, by (6.3.12), $\bar\alpha_{\bar t}(\varphi(\bar t,\bar p))=\tilde\alpha$ and $\bar\theta_{\bar t}(\bar p)=\tilde\theta$. Therefore for sufficiently small $\delta>0$,

$$-\frac{\partial\varphi}{\partial t}(t,p)-\frac{\bar\alpha_t(x)^2}{2}\frac{\partial^2\varphi}{\partial p^2}(t,p)+\mu\bar\theta_t(p)x+\bar c\geq\epsilon \tag{6.3.16}$$

for any $(x,t,p)\in\mathbb{R}\times[0,T]\times(0,1)$ such that $|t-\bar t|<\delta$, $|p-\bar p|<\delta$, $|x-\varphi(t,p)|<\delta$.
Observe that, since $(\bar t,\bar p)$ is a strict maximizer in (6.3.11), we have

$$-\xi:=\max_{\partial_pB_\delta(\bar t,\bar p)}u^*-\varphi<0, \tag{6.3.17}$$

where $\partial_pB_\delta(\bar t,\bar p):=\{\bar t+\delta\}\times[\bar p-\delta,\bar p+\delta]\cup[\bar t,\bar t+\delta)\times\{\bar p-\delta,\bar p+\delta\}$ denotes the parabolic boundary of $B_\delta(\bar t,\bar p)$.
We now show that (6.3.16) and (6.3.17) lead to a contradiction of **(DP2)**. Let (t_n,p_n) be a sequence in $[0,T)\times(0,1)$ which converges to $(\bar t,\bar p)$ and such that $u(t_n,p_n)\to u^*(\bar t,\bar p)$. Set $x_n=u(t_n,p_n)-n^{-1}$. Since $\varphi(\bar t,\bar p)\geq 4\epsilon$, $x_n>0$ for n large enough. Without loss of generality, we can assume that $|x_n-\varphi(t_n,p_n)|\leq\delta$ for each n. Observe that

$$\gamma_n:=x_n-\varphi(t_n,p_n)\to 0. \tag{6.3.18}$$

Let $X^n := X^{t_n, x_n, \theta^n, \bar{c}}$ denote the solution of (6.2.3) starting from $t = t_n$ associated to the initial condition x_n, the running endowment $\bar{c}$ and the Markovian controls

$$\alpha_t^n := \bar{\alpha}_t(X_t^n) \quad \text{and} \quad \theta_t^n := \bar{\theta}_t(X_t^n, P_t^{t_n, p_n, \alpha^n}), \quad t_n \le t \le T. \tag{6.3.19}$$

We next define the stopping times

$$\tau_n^o := \inf\{s \ge t_n : (s, P_s^{t_n, p_n, \alpha^n}) \notin B_\delta(\bar{t}, \bar{p})\}, \quad \tau_n := \inf\{s \ge t_n : |X_s^n - \varphi(s, P_s^{t_n, p_n, \alpha^n})| \ge \delta\} \wedge \tau_n^o$$

Note that X^n is well defined on $[t_n, \tau_n]$. Besides, by (6.3.16), (6.3.18) and a standard comparison theorem implies that $X_s^n - \varphi(s, P_s^{t_n, p_n, \alpha^n}) \ge \delta$ on $\{s \ge t_n : |X_s^n - \varphi(s, P_s^{t_n, p_n, \alpha^n})| \ge \delta\}$ for n large enough. Since $u \le u^* \le \varphi$, we then deduce from (6.3.17) and the definition of τ^n that

$$\begin{aligned}
X_{t\wedge\tau_n}^n - u(t\wedge\tau_n, P_{t\wedge\tau_n}^{t_n, p_n, \alpha^n}) &\ge \mathbf{1}_{\tau_n < \tau_n^o}\{X_{t\wedge\tau_n}^n - \varphi(t\wedge\tau_n, P_{t\wedge\tau_n}^{t_n, p_n, \alpha^n})\} \\
&\quad + \mathbf{1}_{\tau_n = \tau_n^o}\{X_{t\wedge\tau_n}^n - u^*(t\wedge\tau_n, P_{t\wedge\tau_n}^{t_n, p_n, \alpha^n})\} \\
&= \delta\mathbf{1}_{\tau_n < \tau_n^o} + \mathbf{1}_{\tau_n = \tau_n^o}\{X_{t\wedge\tau_n}^n - u^*(t\wedge\tau_n, P_{t\wedge\tau_n}^{t_n, p_n, \alpha^n})\} \\
&\ge \delta\mathbf{1}_{\tau_n < \tau_n^o} + \mathbf{1}_{\tau_n = \tau_n^o}\{X_{t\wedge\tau_n}^n + \xi - \varphi(t\wedge\tau_n, P_{t\wedge\tau_n}^{t_n, p_n, \alpha^n})\} \\
&\ge \delta \wedge \xi + \mathbf{1}_{\tau_n = \tau_n^o}\{X_{t\wedge\tau_n}^n - \varphi(t\wedge\tau_n, P_{t\wedge\tau_n}^{t_n, p_n, \alpha^n})\}
\end{aligned}$$

We continue by using Itô's formula :

$$\begin{aligned}
X_{t\wedge\tau_n}^n - u(t\wedge\tau_n, P_{t\wedge\tau_n}^{t_n, p_n, \alpha^n}) \ge\ & \delta\wedge\xi + \mathbf{1}_{\tau_n = \tau_n^o}(\gamma_n + \int_{t_n}^{\tau^n} \mu\theta_u^n X_u^n du + \int_{t_n}^{\tau^n} \bar{c} du - \int_{t_n}^{\tau^n} \frac{\partial\varphi}{\partial t}(u, P_u^{t_n, p_n, \alpha^n}) du \\
& - \int_{t_n}^{\tau^n} \frac{(\alpha_u^n)^2}{2} \frac{\partial^2\varphi}{\partial p^2}(u, P_u^{t_n, p_n, \alpha^n}) du + \int_{t_n}^{\tau^n} \psi_u^n dW_u)
\end{aligned}$$

where

$$\psi_s^n = \sigma\theta_s^n X_s^n - \alpha_s^n \frac{\partial\varphi}{\partial p}(s, P_s^{t_n, p_n, \alpha^n}), \quad t_n \le s \le \tau_n.$$

By (6.3.16) the drift term is greater than ϵ and by (6.3.15) and (6.3.19), $\psi^n \equiv 0$.
Since $\xi, \delta > 0$ and $\gamma_n \to 0$, this implies that for sufficiently large n,

$$X_{t\wedge\tau_n}^n > u(t\wedge\tau_n, P_{t\wedge\tau_n}^{t_n, p_n, \alpha^n}), \quad t \ge t_n.$$

Moreover, by definition of τ_n and by (6.3.14), $X^n - g(., P^{t_n, p_n, \alpha^n}) = X^n > 2\epsilon$ on $[t_n, \tau_n]$. Recalling that the initial position of the process X^n is $x_n = u(t_n, p_n) - n^{-1} < u(t_n, p_n)$, this is clearly in contradiction with **(DP2)**. □

6.3.2 Boundary condition

Theorem 6.3.2. *1. $u^*(., 0) = 0$ on $[0, T)$ and $u_*(., 0) = 0$ on $[0, T]$.*

2. $u_*(., 1)$ is a viscosity supersolution of (6.3.1) in the following sense :

$$\min\big(-\frac{\partial\varphi}{\partial t}(t, 1) + F^*(\varphi(t, 1), \frac{\partial\varphi}{\partial p}(t, 1), \frac{\partial^2\varphi}{\partial p^2}(t, 1)), \varphi(t, 1)\big) \ge 0$$

for any $t \in [0, T)$ and any $\varphi \in C^{1,2}([0, T) \times (0, 1]$ such that

$$(u_* - \varphi)(t, 1) = \inf_{[0,T]\times(0,1]} u_* - \varphi.$$

Proof We adapt the proof of Theorem 3.1 in [15].
The endpoint $p = 0$
Let $\bar{t} \in [0, T)$ and φ be a smooth function such that

$$0 = (u^* - \varphi)(\bar{t}, 0) > (u^* - \varphi)(t, p), \quad (\bar{t}, 0) \neq (t, p) \in [0, T) \times [0, 1)$$

By following the arguments in the proof of the subsolution property of u^* on $[0, T) \times (0, 1)$, we obtain that

$$\min \Big(\sup_{\{(\alpha,\theta)\in\mathbb{R}^2,\ \alpha\frac{\partial\varphi}{\partial p}(\bar{t},0)=\sigma\theta\varphi(\bar{t},0)\}} \Big[-\frac{\partial \varphi}{\partial t}(\bar{t}, 0) - \frac{\alpha^2}{2}\frac{\partial^2 \varphi}{\partial p^2}(\bar{t}, 0) + \mu\theta\varphi(\bar{t}, 0) + \bar{c}\Big], \quad (6.3.20)$$
$$\varphi(\bar{t}, 0)\Big) \leq 0$$

Let φ be a smooth function on $[0, T]$ and $\bar{t} \in [0, T]$ be such that

$$0 = u^*(\bar{t}, 0) - \varphi(\bar{t}) > u^*(t, 0) - \varphi(t), \quad \bar{t} \neq t \in [0, T) \qquad (6.3.21)$$

By definition, $u^*(\bar{t}, 0) \geq 0$. We now assume that

$$u^*(\bar{t}, 0) > 0$$

and work towards a contradiction. Define

$$\psi_k(p) = -k \int_{1-p}^{1} \frac{e^{2k}}{e^{k(r+1)} - e^{2k+1}} dr, \quad k > 0$$

and $\varphi_k(t, p) = \varphi(t) + (t - \bar{t})^2 + \psi_k(p)$. Observe that

$$\frac{k}{2(e-1)} \leq \psi_k'(p) = -k\frac{e^{2k}}{e^{k(2-p)} - e^{2k+1}} \leq 2k \quad \text{for } k \text{ large enough} \qquad (6.3.22)$$
$$\psi_k''(p) = -k^2 \frac{e^{4k-pk}}{(e^{k(2-p)} - e^{2k+1})^2} < 0 \quad \text{for all } k > 0 \qquad (6.3.23)$$

Let (t_k, p_k) be a maximizer of $u^* - \varphi_k$ on $[0, T] \times [0, 1]$. Then,

$$\begin{aligned} u^*(\bar{t}, 0) - \varphi(\bar{t}) &= (u^* - \varphi_k)(\bar{t}, 0) \\ &\leq (u^* - \varphi_k)(t_k, p_k) \\ &= u^*(t_k, p_k) - \varphi(t_k) - (t_k - \bar{t})^2 - \psi_k(p_k) \\ &\leq u^*(t_k, p_k) - \varphi(t_k) - (t_k - \bar{t})^2 - \frac{k}{2(e-1)}p_k \end{aligned}$$

where the last inequality follows from (6.3.22) for k large enough and the fact that $\psi_k(0) = 0$. Since $u^* \leq D_0$ by construction and φ is bounded, this implies that the sequence (t_k, p_k) is bounded and therefore converges to some (t_*, p_*) up to a subsequence. Clearly, $p_* = 0$ since otherwise we would have $kp_k \to \infty$. By (6.3.21), this implies that

$$\begin{aligned} u^*(\bar{t}, 0) - \varphi(\bar{t}) &\leq \limsup_{k\to\infty} (u^* - \varphi_k)(t_k, p_k) \\ &\leq u^*(t_*, 0) - \varphi(t_*) - (t_* - \bar{t})^2 + \limsup_{k\to\infty} \frac{-k}{2(e-1)}p_k \end{aligned}$$

$$\leq \quad u^*(\bar{t},0) - \varphi(\bar{t}) - (t_* - \bar{t})^2 + \limsup_{k\to\infty} \frac{-k}{2(e-1)} p_k$$

This shows that, after possibly passing to a subsequence,

$$(t_k, p_k) \to (\bar{t}, 0), \quad u^*(t_k, p_k) \to u^*(\bar{t}, 0), \quad \text{and} \quad \varphi_k(t_k, p_k) \to \varphi(\bar{t}).$$

Hence, since $u^*(\bar{t},0) = \varphi(\bar{t}) > 0$, we have $u^*(t_k, p_k) > 0$ and $\varphi_k(t_k, p_k) > 0$ for all k, after possibly passing to a subsequence. Then, it follows from the subsolution property of u^* that for all k,

$$\sup_{\{(\alpha,\theta)\in\mathbb{R}^2,\ \alpha\frac{\partial\varphi_k}{\partial p}(t_k,p_k) = \sigma\theta\varphi_k(t_k,p_k)\}} [-\frac{\partial\varphi_k}{\partial t}(t_k,p_k) - \frac{\alpha^2}{2}\frac{\partial^2\varphi_k}{\partial p^2}(t_k,p_k) + \mu\theta\varphi_k(t_k,p_k) + \bar{c}] \leq 0$$

Therefore,

$$\sup_{\alpha\in\mathbb{R}} [-\frac{\partial\varphi_k}{\partial t}(t_k,p_k) - \frac{\alpha^2}{2}\frac{\partial^2\varphi_k}{\partial p^2}(t_k,p_k) + \mu\frac{\alpha}{\sigma}\frac{\partial\varphi_k}{\partial p}(t_k,p_k) + \bar{c}] \leq 0.$$

Since by (6.3.23), $\frac{\partial^2\varphi_k}{\partial p^2}(t_k,p_k) = \psi_k''(p) < 0$, this inequality does not hold and therefore $u^*(.,0) = 0$ on $[0,T) \times [0,1)$.
Hence, we can find a sequence $(t_n, p_n) \in [0,T) \times (0,1)$ such that $(t_n,p_n) \to (T,0)$ and $0 \leq u^*(t_n,p_n) \leq 1/n$ for all $n \geq 0$, which shows that $u_*(T,0) = 0$.
The endpoint $p = 1$ *on* $[0,T)$
Adapt in the straightforward way the proof of the supersolution property of u_* on $[0,T)\times(0,1)$.

6.3.3 Terminal condition

Proposition 6.3.1. *For any* $t \in [0,T)$*,* $u^*(t,.)$ *is convex on* $[0,1]$*.*

Proof Let $t \in [0,T]$. Since, by definition, for any $t' \in [0,T]$, $u(t',.)$ is non decreasing and nonnegative, $u^*(t,.)$ is also non decreasing and nonnegative. Therefore it suffices to show that $u^*(t,.)$ is convex on $\{p \in [0,1] | u^*(t',p) > 0\} := U_p$. Let $p \in U_p$, by Theorem 6.3.1 and Theorem 6.3.2,

$$\sup_{\{(\alpha,\theta)\in\mathbb{R}^2,\ \alpha\frac{\partial\varphi}{\partial p}(t,p) = \sigma\theta\varphi(t,p)\}} [-\frac{\partial\varphi}{\partial t}(t,p) - \frac{\alpha^2}{2}\frac{\partial^2\varphi}{\partial p^2}(t,p) + \mu\theta\varphi(t,p) + \bar{c}] \leq 0$$

for any $\varphi \in C^{1,2}([0,T) \times (0,1))$ such that

$$(u^* - \varphi)(t,p) = \sup_{[0,T]\times(0,1)} u^* - \varphi.$$

Let φ be such a smooth function, therefore $\varphi(t,p) = u^*(t,p) > 0$. Then,

$$\sup_{\alpha\in\mathbb{R}} [-\frac{\partial\varphi}{\partial t}(t,p) - \frac{\alpha^2}{2}\frac{\partial^2\varphi}{\partial p^2}(t,p) + \mu\frac{\alpha}{\sigma}\frac{\partial\varphi}{\partial p}(t,p) + \bar{c}] \leq 0.$$

This implies that $\frac{\partial^2\varphi}{\partial p^2}(t,p) \geq 0$. The convexity then follows from the same arguments as in Proposition 5.2 of [32]. □

Theorem 6.3.3. *u is a viscosity solution to (6.3.2) in the sense it verifies :*

1. *$u_*(T,p) \geq pD_0$ for all $p \in [0,1]$.*
2. *$u^*(T,p) \leq pD_0$ for all $p \in [0,1]$.*

Hence, $u_*(T,p) = u^*(T,p) = pD_0$ for all $p \in [0,1]$.

Proof It follows from Theorem 6.3.2 and Proposition 6.3.1 that, for all $(t,p) \in [0,T] \times [0,1]$,

$$u^*(t,p) \leq pu^*(t,1) + (1-p)u^*(t,0) \leq pD_0.$$

On the other hand, given a sequence (t_n,p_n) in $[0,T) \times (0,1)$ such that $(t_n,p_n) \to (T,p)$ and $u(t_n,p_n) \to u_*(T,p)$, we can find $(\theta_n,\alpha_n,c_n) \in \mathcal{A} \times \mathcal{B} \times \mathcal{C}$ such that, denoting $x_n = u(t_n,p_n) + 1/n \to u_*(T,p)$ and $X^n = X^{t_n,x_n,\theta_n,c_n}$,

$$\mathbf{1}_{\{X_T^n \geq D_0\}} \geq P_T^{t_n,p_n,\alpha_n}.$$

Besides, we can assume without loss of generality that $X_T^n \leq 2D_0$ for all n. This implies that $X_T^n \geq P_T^{t_n,p_n,\alpha_n} D_0$. Taking the expectation and recalling that P^{t_n,p_n,α_n} is a bounded martingale, we get $\mathbb{E}[X_T^n] \geq p_n D_0$. Passing to the limit, since $X_T^n \leq 2D_0$, we obtain that $u_*(T,p) \geq pD_0$. □

Corollary 6.3.1.

$$\forall p \in (0,1), \lim_{t \to T-} u(t,p) = D_0 p \neq D_0 = u(T,p).$$

Therefore u is discontinuous on $\{T\} \times (0,1)$.

6.3.4 Continuousness

The following proposition is admitted.

Proposition 6.3.2. *Let V (resp. U) be a nonnegative lower-semicontinuous (resp. upper-semicontinuous) bounded map on $[0,T] \times (0,1)$. Assume that V (resp. U) is a supersolution (resp. subsolution) of (6.3.1) on $[0,T) \times (0,1)$ such that $V(T,.)$ (resp. $U(T,.)$) is a supersolution (resp. subsolution) of (6.3.2) on $(0,1)$. Then, $V \geq U$ on $[0,T] \times (0,1)$.*

The following theorem is a consequence of the Proposition 6.3.2 and the fact that by definition $0 \leq u \leq D_0$.

Theorem 6.3.4. *$u_* = u^*$ is continuous on $[0,T] \times [0,1)$ and is the unique viscosity solution of (6.3.1)-(6.3.2) in the class of nonnegative and bounded functions.*

6.3.5 The Fenschel-Legendre dual transform

For sake of clarity, we extend u to $[0,T] \times \mathbb{R}$ by setting

$$u(.,p) := 0 \text{ for } p < 0 \quad \text{and} \quad u(.,p) := \infty \text{ for } p > 1. \tag{6.3.24}$$

We introduce the Fenchel-Legendre dual transform associated with u with respect to the p variable :

$$v(t,q) := \sup_{p\in\mathbb{R}}\{pq - u(t,p)\}, \qquad (t,q) \in [0,T]\times\mathbb{R}.$$

Note that by (6.3.24) and since $u(.,0) = 0$ on $[0,T]$ by definition of u,

$$v(.,q) = \infty \text{ for } q < 0 \quad \text{and} \quad v(.,q) = \sup_{p\in[0,1]}\{pq - u(t,p)\} \text{ for } q > 0 \tag{6.3.25}$$

Theorem 6.3.5. *v is a viscosity solution on $[0,T)\times(0,\infty)$ of*

$$\max\left(-\frac{\partial\varphi}{\partial t}(t,q) - \frac{\mu^2}{2\sigma^2}q^2\frac{\partial^2\varphi}{\partial q^2}(t,q) - \bar{c}, \varphi - q\frac{\partial\varphi}{\partial q}\right) = 0 \tag{6.3.26}$$

with the terminal condition

$$v(T,q) = (q - D_0)^+. \tag{6.3.27}$$

Proof

First note that the fact that v is upper-semicontinuous on $[0,T]\times(0,\infty)$ follows from the lower-semicontinuity of $u_* = u$ and (6.3.25) which allows to reduce the computation of the sup to the compact set $[0,1]$. Moreover, the boundary condition (6.3.27) is an immediate consequence of the point 1 in Theorem 6.3.1 and (6.3.25) again.
We now turn to the PDE characterization inside the domain. We only prove the subsolution part. Let φ be a smooth function with bounded derivatives and $(\bar{t},\bar{q}) \in [0,T]\times(0,\infty)$ be a local maximizer of $v - \varphi$ such that $(v-\varphi)(\bar{t},\bar{q}) = 0$. As shown in the page 17 of [15], we can reduce to the case where the map $q \to \varphi(.,q)$ is strictly convex. Let $\tilde{\varphi}$ be the Fenchel transform of φ with respect to q, i.e.

$$\tilde{\varphi}(t,p) = \sup_{q\in\mathbb{R}}\{pq - \varphi(t,q)\}. \tag{6.3.28}$$

Since φ is strictly convex in q and smooth on its domain, $\tilde{\varphi}$ is strictly convex in p and smooth on its domain, see e.g. [94]. Moreover, we have

$$\begin{aligned}
\varphi(t,q) &= \sup_{p\in\mathbb{R}}\{pq - \tilde{\varphi}(t,p)\} \\
&= J(t,q)p - \tilde{\varphi}(t,J(t,q)) \quad \text{on } (0,T)\times(0,\infty)\subset \text{int}(\text{dom}(\varphi))
\end{aligned} \tag{6.3.29}$$

where $q \to J(.,q)$ denotes the inverse of $p \to D_p\tilde{\varphi}(.,p)$, recall that $\tilde{\varphi}$ is strictly convex in p.
We now deduce from the assumption $\bar{q} > 0$ and (6.3.25) that we can find $\bar{p} > 0$ such that $v(\bar{t},\bar{q}) = \bar{p}\bar{q} - u(\bar{t},\bar{p})$ which, by using the very definition of $(\bar{t},\bar{p},\bar{q})$ and v, implies that

$$(\bar{t},\bar{p}) \text{ is a local minimizer of } u - \tilde{\varphi} \text{ such that } (u-\tilde{\varphi})(\bar{t},\bar{p}) = 0 \tag{6.3.30}$$

and

$$\varphi(\bar{t},\bar{q}) = \sup_{p\in\mathbb{R}}\{p\bar{q} - \tilde{\varphi}(\bar{t},p)\} = \bar{p}\bar{q} - \tilde{\varphi}(\bar{t},\bar{p}) \quad \text{with } \bar{p} = J(\bar{t},\bar{q}), \tag{6.3.31}$$

where the last equality follows from (6.3.29) and the strict convexity of the map $p \to p\bar{q} - \tilde{\varphi}(\bar{t},p)$ in the domain of $\tilde{\varphi}$.
We conclude the proof by discussing three alternative cases depending on the value of $\bar{p}$.

1. If $\bar{p} \in (0,1)$ then (6.3.30) implies that $\tilde{\varphi}$ verifies the hypothesis of (i) of Theorem 6.3.1 and the required result follows by exploiting the link between the derivatives of $\tilde{\varphi}$ and the derivatives of its p-Fenchel transform φ, which can be deduced from (6.3.28) and (6.3.29).
2. If $\bar{p} = 0$, using that $u(.,0) = 0$, we can conclude as in 1. above.
3. If $\bar{p} = 1$, using that $u(.,1) = (D_0 - \bar{c}(T-t))_+$, we can argue as in the first case.

□

6.4 Back to the control problem of interest

Once the minimal admissible wealth function u is known, we can rewrite the control problem of interest as follows

$$\begin{aligned} w(t,x,p) &= \inf_{(\theta,c)\in\mathcal{A}\times\mathcal{C}} \{\int_t^T e^{-\beta(s-t)}\mathbb{E}[c_s]ds, \text{ with } \forall s \in [t,T],\ X_s^{t,x,\theta,c} \geq 0 \text{ and } \mathbb{P}[X_T^{t,x,\theta,c} \geq D_0] \geq p\} \\ &= \inf_{(\theta,\alpha,c)\in\mathcal{A}\times\mathcal{B}\times\mathcal{C}} \{\int_t^T e^{-\beta(s-t)}\mathbb{E}[c_s]ds, \text{ with } \forall s \in [t,T],\ X_s^{t,x,\theta,c} \geq g(s, P_s^{t,p,\alpha})\} \end{aligned}$$

Definition 6.4.1. *For $(t,x,p) \in [0,T] \times \mathbb{R}_+ \times [0,1]$, $(\theta,\alpha,c) \in \mathcal{A}\times\mathcal{B}\times\mathcal{C}$ is suitable relatively to (t,x,p) when*

$$\forall s \in [t,T],\ X_s^{t,x,\theta,c} \geq g(s, P_s^{t,p,\alpha})$$

We denote $(\mathcal{A}\times\mathcal{B}\times\mathcal{C})(t,x,p)$ the set of suitable controls relatively to (t,x,p).

Therefore

$$w(t,x,p) = \inf_{(\theta,\alpha,c)\in(\mathcal{A}\times\mathcal{B}\times\mathcal{C})(t,x,p)} \int_t^T e^{-\beta(s-t)}\mathbb{E}[c_s]ds$$

Remark 6.4.1. *For $x < u(t,p)$, $(\mathcal{A}\times\mathcal{B}\times\mathcal{C})(t,x,p) = \emptyset$.*

Hence we introduce the following sets.

Definition 6.4.2.

$$\begin{aligned} int(u) &:= \{(t,x,p) \in [0,T) \times \mathbb{R}_+ \times (0,1) : x > u(t,p)\} \\ \partial(u) &:= \{(t,x,p) \in [0,T) \times \mathbb{R}_+ \times (0,1) : x = u(t,p)\} \end{aligned}$$

Definition 6.4.3. *For any $(t,x,p) \in int(u) \cup \partial(u)$,*

$$\begin{aligned} w_*(t,x,p) &:= \liminf_{(t',x',p')\in int(u)\to(t,x,p)} w(t',x',p') \\ w^*(t,x,p) &:= \limsup_{(t',x',p')\in int(u)\to(t,x,p)} w(t',x',p') \end{aligned}$$

Remark 6.4.2. *By definition of w, for any $(t,x,p) \in int(u) \cup \partial(u)$,*

$$0 \leq w(t,x,p) \leq \bar{c}e^{\beta t} \int_t^T e^{-\beta s} ds = \frac{\bar{c}}{\beta}(1 - e^{\beta(t-T)}) \tag{6.4.1}$$

Therefore $w(T,.,.) = w^*(T,.,.) = w_*(T,.,.) = 0$ on $\{(x,p) \in \mathbb{R}_+ \times (0,1); x \geq u(T,p)\} = [D_0,\infty) \times (0,1)$.

Remark 6.4.3. *For $x \geq D_0$, $w(.,x,.) \equiv 0$. Besides, $w(.,.,0) \equiv 0$.*

We set, for $(x, z, q, a_{11}, a_{12}, a_{22}) \in \mathbb{R}^6$:

$$H^{\theta,\alpha,c}(x, z, q, a_{11}, a_{12}, a_{22}) := -(\mu\theta x + c)q - \frac{\sigma^2\theta^2 x^2}{2} a_{11} - \alpha\sigma\theta x a_{12} - \frac{\alpha^2}{2} a_{22} - c + \beta z$$

and

$$H(x, z, q, a_{11}, a_{12}, a_{22}) := \sup_{(\theta,\alpha,c)\in\mathbb{R}^2\times[0,\bar{c}]} H^{\theta,\alpha,c}.$$

Since $\mathbb{R}^2 \times [0, \bar{c}]$ is not bounded, the operator H is not necessarily continuous and we shall have to relax it and consider its lower semicontinuous and upper semicontinuous envelopes H_* and H^* on We now show that on the domain $int(u)$, w solves the PDE :

$$-\frac{\partial\varphi}{\partial t} + H(x, \varphi(t,x,p), \frac{\partial\varphi}{\partial x}(t,x,p), \frac{\partial^2\varphi}{\partial x^2}(t,x,p), \frac{\partial^2\varphi}{\partial x\partial p}(t,x,p), \frac{\partial^2\varphi}{\partial p^2}(t,x,p)) = 0 \quad (6.4.2)$$

Theorem 6.4.1. *w is a viscosity solution to (6.4.2) in the sense it verifies :*

(i) *Viscosity supersolution property :*

$$-\frac{\partial\varphi}{\partial t} + H^*(x, \varphi(t,x,p), \frac{\partial\varphi}{\partial x}(t,x,p), \frac{\partial^2\varphi}{\partial x^2}(t,x,p), \frac{\partial^2\varphi}{\partial x\partial p}(t,x,p), \frac{\partial^2\varphi}{\partial p^2}(t,x,p)) \geq 0$$

for any $(t, x, p) \in [0, T) \times \mathbb{R}_+^* \times (0, 1)$ such that $x \geq u(t, p)$ and any $\varphi \in C^{1,2,2}([0, T] \times \mathbb{R}_+^* \times (0, 1))$ such that

$$(w_* - \varphi)(t, x, p) = \inf_{int(u)\cup\partial(u)} w_* - \varphi$$

(ii) *Viscosity subsolution property :*

$$-\frac{\partial\varphi}{\partial t} + H(x, \varphi(t,x,p), \frac{\partial\varphi}{\partial x}(t,x,p), \frac{\partial^2\varphi}{\partial x^2}(t,x,p), \frac{\partial^2\varphi}{\partial x\partial p}(t,x,p), \frac{\partial^2\varphi}{\partial p^2}(t,x,p)) \leq 0$$

for any $(t, x, p) \in [0, T) \times \mathbb{R}_+ \times (0, 1)$ such that $x > u(t, p)$ and any $\varphi \in C^{1,2,2}([0, T] \times \mathbb{R}_+ \times (0, 1))$ such that

$$(w^* - \varphi)(t, x, p) = \sup_{int(u)\cup\partial(u)} w^* - \varphi$$

We first need to provide a dynamic programming principle for our control problem with the following lemma.

Lemma 6.4.1. *Fix $(t, x, p) \in int(u)$ and let $\mathcal{U}$ be a set of stopping times and let $\{\tau^\nu, \nu \in \mathcal{U}\}$ be a family of stopping times with values in $[t, T]$. Then,*

$$e^{-\beta t} w(t, x, p) \geq \sup_{\nu\in\mathcal{U}} \sup_{(\theta,\alpha,c)\in(\mathcal{A}\times\mathcal{B}\times\mathcal{C})(t,x,p)} \mathbb{E}[e^{-\beta\tau^\nu} w_*(\tau^\nu, X_{\tau^\nu}^{t,x,\theta,c}, P_{\tau^\nu}^{t,p,\alpha}) + \int_t^{\tau^\nu} e^{-\beta u} c_u du]$$

$$e^{-\beta t} w(t, x, p) \leq \sup_{\nu\in\mathcal{U}} \inf_{(\theta,\alpha,c)\in(\mathcal{A}\times\mathcal{B}\times\mathcal{C})(t,x,p)} \mathbb{E}[e^{-\beta\tau^\nu} w^*(\tau^\nu, X_{\tau^\nu}^{t,x,\theta,c}, P_{\tau^\nu}^{t,p,\alpha}) + \int_t^{\tau^\nu} e^{-\beta u} c_u du]$$

Proof *First inequality*
Let $(t,x,p) \in int(u)$, $\nu \in \mathcal{U}$, and $(\theta,\alpha,c) \in (\mathcal{A}\times\mathcal{B}\times\mathcal{C})(t,x,p)$. By the flow property, $X_s^{t,x,\theta,c} = X_s^{\tau^n, X_{\tau^\nu}^{t,x,\theta,c},\theta,c}$. Then, by definition of w,

$$\begin{aligned} e^{-\beta t}\int_t^T e^{-\beta(s-t)}\mathbb{E}[c_s]ds &= \int_t^{\tau^\nu} e^{-\beta s}\mathbb{E}[c_s]ds + e^{-\beta\tau^\nu}\int_{\tau^\nu}^T e^{-\beta(s-\tau^\nu)}\mathbb{E}[c_s]ds \\ &\geq \int_t^{\tau^\nu} e^{-\beta s}\mathbb{E}[c_s]ds + \mathbb{E}[e^{-\beta\tau^\nu} w(\tau^\nu, X_{\tau^\nu}^{t,x,\theta,c}, P_{\tau^\nu}^{t,p,\alpha})] \\ &\geq \inf_{(\theta,\alpha,c)\in(\mathcal{A}\times\mathcal{B}\times\mathcal{C})(t,x,p)} \mathbb{E}[e^{-\beta\tau^\nu} w_*(\tau^\nu, X_{\tau^\nu}^{t,x,\theta,c}, P_{\tau^\nu}^{t,p,\alpha}) + \int_t^{\tau^\nu} e^{-\beta u}c_u du] \end{aligned}$$

and the result follows from the arbitrariness of $\nu \in \mathcal{U}$ and $(\theta,\alpha,c) \in (\mathcal{A}\times\mathcal{B}\times\mathcal{C})(t,x,p)$.
Second inequality
It follows from Lemma 6.6.2 that the set $\{(t,x,p,\theta,\alpha,c) \in [0,T]\times\mathbb{R}_*^+\times(0,1)\times\mathcal{A}\times\mathcal{B}\times\mathcal{C} : (\theta,\alpha,c) \in (\mathcal{A}\times\mathcal{B}\times\mathcal{C})(t,x,p)\}$ is an analytic set. Clearly, $(t,c) \to \int_t^T e^{-\beta(s-t)}\mathbb{E}[c_s]ds$ is Borel measurable and therefore upper semianalytic. It thus follows from Proposition 7.50 in [10] that, for each $\epsilon > 0$, we can find an analytically measurable triple of maps $(\hat{\alpha}^\epsilon, \hat{\theta}^\epsilon, \hat{c}^\epsilon) \in (\mathcal{A}\times\mathcal{B}\times\mathcal{C})(t,x,p)$ such that $\int_t^T e^{-\beta(s-t)}\mathbb{E}[\hat{c}_s^\epsilon]ds \leq w(t,x,p)+\epsilon$. Since analytically measurable maps are also universally measurable, it follows from Lemma 7.27 in [10] that, for any probability measure m on $[0,T]\times\mathbb{R}_*^+\times[0,1]$, we can find a Borel measurable triple of map $(\hat{\theta}_m^\epsilon, \hat{\alpha}_m^\epsilon, \hat{c}_m^\epsilon) : (t,x,p) \in [0,T]\times\mathbb{R}_*^+\times[0,1] \to (\mathcal{A}\times\mathcal{B}\times\mathcal{C})(t,x,p)$ such that $\int_t^T e^{-\beta(s-t)}\mathbb{E}[\hat{c}_m^\epsilon(t,x,p))]ds \leq w(t,x,p)+\epsilon \leq w^*(t,x,p)+\epsilon$ for m-almost every element of $[0,T]\times\mathbb{R}_*^+\times[0,1]$. Let us now fix $(\theta_1,\alpha_1,c_1) \in (\mathcal{A}\times\mathcal{B}\times\mathcal{C})(\bar{t},\bar{x},\bar{p})$ for some $(\bar{t},\bar{x},\bar{p}) \in$ int(u). Let m be the measure induced by $(\tau, X_\tau^{\bar{t},\bar{x},\theta_1,c_1}, P_\tau^{\bar{t},\bar{p},\alpha_1})$ on $[0,T]\times\mathbb{R}_*^+\times[0,1]$. Since $(\theta_1,\alpha_1,c_1) \in (\mathcal{A}\times\mathcal{B}\times\mathcal{C})(\bar{t},\bar{x},\bar{p})$ $\mathbb{P}$-a.s.,

$$(\hat{\theta}_m^\epsilon, \hat{\alpha}_m^\epsilon, \hat{c}_m^\epsilon)(\tau, X_\tau^{\bar{t},\bar{x},\theta_1,c_1}, P_\tau^{\bar{t},\bar{p},\alpha_1}) \in (\mathcal{A}\times\mathcal{B}\times\mathcal{C})(\tau, X_\tau^{\bar{t},\bar{x},\theta_1,c_1}, P_\tau^{\bar{t},\bar{p},\alpha_1})$$

and

$$\int_t^T e^{-\beta(s-t)}\mathbb{E}[\hat{c}_m^\epsilon(\tau, X_\tau^{\bar{t},\bar{x},\theta_1,c_1}, P_\tau^{\bar{t},\bar{p},\alpha_1})]ds \leq w^*(t,x,p)+\epsilon \quad \mathbb{P}\text{-a.s.}$$

Moreover it follows from Lemma 2.1 of [99] that we can find $(\theta_2^\epsilon, \alpha_2^\epsilon, c_2^\epsilon) \in \mathcal{A}\times\mathcal{B}\times\mathcal{C}$ such that

$$(\theta_2^\epsilon, \alpha_2^\epsilon, c_2^\epsilon)\mathbf{1}_{[\tau,T]} = (\hat{\theta}_m^\epsilon, \hat{\alpha}_m^\epsilon, \hat{c}_m^\epsilon)(\tau, X_\tau^{\bar{t},\bar{x},\theta_1,c_1}, P_\tau^{\bar{t},\bar{p},\alpha_1})\mathbf{1}_{[\tau,T]} \quad dt\times d\mathbb{P}\text{-}a.e.$$

This implies that $(\theta^\epsilon,\alpha^\epsilon,c^\epsilon) = (\theta_1,\alpha_1,c_1)\mathbf{1}_{[\bar{t},\tau)} + (\theta_2^\epsilon,\alpha_2^\epsilon,c_2^\epsilon)\mathbf{1}_{[\tau,T]} \in (\mathcal{A}\times\mathcal{B}\times\mathcal{C})(\bar{t},\bar{x},\bar{p})$ and

$$e^{-\beta\tau}\mathbb{E}[\int_\tau^T e^{-\beta(s-\tau)}\mathbb{E}[c_2^\epsilon]|(\tau, X_\tau^{\bar{t},\bar{x},\theta_1,c_1}, P_\tau^{\bar{t},\bar{p},\alpha_1})] \leq e^{-\beta\tau}w^*(\tau, X_\tau^{\bar{t},\bar{x},\theta_1,c_1}, P_\tau^{\bar{t},\bar{p},\alpha_1}) + e^{-\beta\tau}\epsilon$$

and therefore

$$\begin{aligned} e^{-\beta t}w(t,x,p) &\leq \mathbb{E}[e^{-\beta\tau}\mathbb{E}[\int_\tau^T e^{-\beta(s-\tau)}\mathbb{E}[c_2^\epsilon]|(\tau, X_\tau^{\bar{t},\bar{x},\theta_1,c_1}, P_\tau^{\bar{t},\bar{p},\alpha_1})] + \int_t^\tau e^{-\beta u}c_1(u)du] \\ &\leq \mathbb{E}[e^{-\beta\tau}w^*(\tau, X_\tau^{\bar{t},\bar{x},\theta_1,c_1}, P_\tau^{\bar{t},\bar{p},\alpha_1}) + \int_t^\tau e^{-\beta u}c_1(u)du] + \epsilon \end{aligned}$$

The required result then follows from the arbitrariness of $(\theta_1,\alpha_1,c_1) \in (\mathcal{A}\times\mathcal{B}\times\mathcal{C})(\bar{t},\bar{x},\bar{p})$ and $\epsilon > 0$. □

Proof of Theorem 6.4.1 We adapt the proof of Theorem 3.1 in [14].

Subsolution property on $[0,T)\times\mathbb{R}_+\times(0,1)$

Let $(\bar t,\bar x,\bar p)\in int(u)$ and φ be a smooth function such that

$$0=(w^*-\varphi)(\bar t,\bar x,\bar p)>(w^*-\varphi)(t,x,p),\quad (\bar t,\bar x,\bar p)\neq(t,x,p)\in int(u)\cup\partial(u) \tag{6.4.3}$$

We argue by contradiction and assume that the subsolution property does not hold at $(\bar t,\bar x,\bar p)$ for φ, i.e.,

$$\sup_{(\theta,\alpha,c)\in\mathbb{R}^2\times[0,\bar c]}\Big[-\frac{\partial\varphi}{\partial t}-(\mu\theta x+c)\frac{\partial\varphi}{\partial x}-\frac{\sigma^2\theta^2x^2}{2}\frac{\partial^2\varphi}{\partial x^2}-\alpha\sigma\theta x\frac{\partial^2\varphi}{\partial x\partial p}-\frac{\alpha^2}{2}\frac{\partial^2\varphi}{\partial p^2}-c+\beta\varphi\Big](\bar t,\bar x,\bar p)>0$$

By smoothness of φ and continuousness of u, there exists $(\tilde\theta,\tilde\alpha,\tilde c)\in\mathbb{R}^2\times[0,\bar c]$ and $\delta>0$ such that $T>\bar t+\delta$, $B:=[\bar t-\delta,\bar t+\delta]\times[\bar x-\delta,\bar x+\delta]\times[\bar p-\delta,\bar p+\delta]\subset int(u)$, and for any $(t,x,p)\in B$,

$$\Big[-\frac{\partial\varphi}{\partial t}-(\mu\tilde\theta x+\tilde c)\frac{\partial\varphi}{\partial x}-\frac{\sigma^2\tilde\theta^2x^2}{2}\frac{\partial^2\varphi}{\partial x^2}-\tilde\alpha\sigma\tilde\theta x\frac{\partial^2\varphi}{\partial x\partial p}-\frac{\tilde\alpha^2}{2}\frac{\partial^2\varphi}{\partial p^2}-\tilde c+\beta\varphi\Big](t,x,p)\geq 0 \tag{6.4.4}$$

Let (t_n,x_n,p_n) be a sequence in B such that $w(t_n,x_n,p_n)\to w^*(\bar t,\bar x,\bar p)$ and $(t_n,x_n,p_n)\to(\bar t,\bar x,\bar p)$. Denote $X^n=X^{t_n,x_n,\theta^n,c^n}$ and $P^n=P^{t_n,p_n,\alpha^n}$ where $(\theta^n,\alpha^n,c^n)\in(\mathcal{A}\times\mathcal{B}\times\mathcal{C})(t_n,x_n,p_n)$. We now define

$$\tau^n:=\inf\{s\geq t_n:(s,X^n_s,P^n_s)\notin B\}\wedge T.$$

Since $B\subset int(u)$, we can assume without loss of generality that $(\theta^n,\alpha^n,c^n)=(\tilde\theta,\tilde\alpha,\tilde c)$ on $t\in[t_n,\tau^n]$. Therefore an application of Itô's formula to $e^{-\beta\tau^n}\varphi(\tau^n,X^n_{\tau^n},P^n_{\tau^n})$ yields, by (6.4.4),

$$\begin{aligned}
& e^{-\beta\tau^n}\varphi(\tau^n,X^n_{\tau^n},P^n_{\tau^n})=e^{-\beta t_n}\varphi(t_n,x_n,p_n)+\int_{t_n}^{\tau^n}e^{-\beta u}\frac{\partial\varphi}{\partial t}(u,X^n_u,P^n_u)du\\
- & \beta\int_{t_n}^{\tau^n}e^{-\beta u}\varphi(u,X^n_u,P^n_u)du+\tilde\alpha\int_{t_n}^{\tau^n}e^{-\beta u}\frac{\partial\varphi}{\partial p}(u,X^n_u,P^n_u)dW_u\\
+ & \int_{t_n}^{\tau^n}(\mu\tilde\theta_uX^n_u+\tilde c_u)e^{-\beta u}\frac{\partial\varphi}{\partial x}(u,X^n_u,P^n_u)du+\int_{t_n}^{\tau^n}\sigma\tilde\theta_uX^n_ue^{-\beta u}\frac{\partial\varphi}{\partial x}(u,X^n_u,P^n_u)dW_u\\
+ & \frac{1}{2}\int_{t_n}^{\tau^n}\sigma^2\tilde\theta_u^2(X^n_u)^2e^{-\beta u}\frac{\partial^2\varphi}{\partial x^2}(u,X^n_u,P^n_u)du+\frac{\tilde\alpha^2}{2}\int_{t_n}^{\tau^n}e^{-\beta u}\frac{\partial^2\varphi}{\partial p^2}(u,X^n_u,P^n_u)du\\
+ & \tilde\alpha\sigma\int_{t_n}^{\tau^n}\tilde\theta_uX^{\bar t,\bar x,\tilde\theta,\tilde c}_ue^{-\beta u}\frac{\partial^2\varphi}{\partial x\partial p}(u,X^n_u,P^n_u)du\\
\leq & e^{-\beta t_n}\varphi(t_n,x_n,p_n)-\int_{t_n}^{\tau^n}e^{-\beta u}\tilde c_udu+\tilde\alpha\int_{t_n}^{\tau^n}e^{-\beta u}\frac{\partial\varphi}{\partial p}(u,X^n_u,P^n_u)dW_u\\
+ & \int_{t_n}^{\tau^n}\sigma\tilde\theta_uX^n_ue^{-\beta u}\frac{\partial\varphi}{\partial x}(u,X^n_u,P^n_u)dW_u
\end{aligned}$$

Then,

$$\begin{aligned}
e^{-\beta t_n}\varphi(t_n,x_n,p_n) &\geq \mathbb{E}[e^{-\beta\tau^n}\varphi(\tau^n,X^n_{\tau^n},P^n_{\tau^n})]+\int_{t_n}^{\tau^n}e^{-\beta u}\mathbb{E}[\tilde c_u]du\\
&\geq \mathbb{E}[e^{-\beta\tau^n}w^*(\tau^n,X^n_{\tau^n},P^n_{\tau^n})+\int_{t_n}^{\tau^n}e^{-\beta u}\tilde c_udu]+\xi\mathbb{E}[e^{-\beta\tau^n}]
\end{aligned} \tag{6.4.5}$$

where, defining $A = \partial B \subset int(u)$,

$$\xi := \min_A(\varphi - w^*) > 0$$

Let $\epsilon = \inf_{n\in\mathbb{N}} \xi\mathbb{E}[e^{-\beta\tau^n}] > \xi e^{-\beta T} > 0$. Since $(\varphi - w)(t_n, x_n, p_n) \to (\varphi - w^*)(\bar{t}, \bar{x}, \bar{p}) = 0$, there exists $n \in \mathbb{N}$ such that $|e^{-\beta t_n}\varphi(t_n, x_n, p_n) - e^{-\beta t_n} w(t_n, x_n, p_n)| < \epsilon$. Then by (6.4.5),

$$e^{-\beta t_n} w(t_n, x_n, p_n) > e^{-\beta t_n}\varphi(t_n, x_n, p_n) - \epsilon > \mathbb{E}[e^{-\beta\tau^n} w_*(\tau^n, X^n_{\tau^n}, P^n_{\tau^n}) + \int_{t_n}^{\tau^n} e^{-\beta u}\tilde{c}_u du]$$

which contradicts Lemma 6.4.1.

Supersolution property on $[0, T) \times \mathbb{R}^*_+ \times (0, 1)$
This proof avoids delicate limit arguments of the supersolution derivation in [99] and in [100].
Let φ be a smooth function such that $(\bar{t}, \bar{x}, \bar{p}) \in int(u) \cup \partial(u)$ achieves a strict minimum (equal to 0) of $w_* - \varphi$ on $int(u) \cup \partial(u)$. We proceed by contradiction, assuming that

$$\sup_{(\theta,\alpha,c)\in\mathbb{R}^2\times[0,\bar{c}]} \Big[-\frac{\partial\varphi}{\partial t} - (\mu\theta x + c)\frac{\partial\varphi}{\partial x} - \frac{\sigma^2\theta^2 x^2}{2}\frac{\partial^2\varphi}{\partial x^2} - \alpha\sigma\theta x\frac{\partial^2\varphi}{\partial x\partial p} - \frac{\alpha^2}{2}\frac{\partial^2\varphi}{\partial p^2} - c + \beta\varphi\Big](\bar{t}, \bar{x}, \bar{p}) < 0$$

By smoothness of φ, there exists $\delta > 0$ such that $T > \bar{t} + \delta$, and, denoting $O := (\bar{t}-\delta, \bar{t}+\delta)\times(\bar{x}-\delta, \bar{x}+\delta)\times(\bar{p}-\delta, \bar{p}+\delta)$, for any $(t, x, p) \in O$ and for any $(\theta, \alpha, c) \in \mathbb{R}^2\times[0, \bar{c}]$,

$$\Big[-\frac{\partial\varphi}{\partial t} - (\mu\theta x + c)\frac{\partial\varphi}{\partial x} - \frac{\sigma^2\theta^2 x^2}{2}\frac{\partial^2\varphi}{\partial x^2} - \alpha\sigma\theta x\frac{\partial^2\varphi}{\partial x\partial p} - \frac{\alpha^2}{2}\frac{\partial^2\varphi}{\partial p^2} - c + \beta\varphi\Big](t, x, p) \leq 0 \quad (6.4.6)$$

Let (t_n, x_n, p_n) be a sequence in $O \cap int(u)$ such that $w(t_n, x_n, p_n) \to w_*(\bar{t}, \bar{x}, \bar{p})$ and $(t_n, x_n, p_n) \to (\bar{t}, \bar{x}, \bar{p})$. For each n, since $(t_n, x_n, p_n) \in int(u)$, there exists $(\theta^n, \alpha^n, c^n) \in (\mathcal{A}\times\mathcal{B}\times\mathcal{C})(x_n, t_n, p_n)$. Denote $X^n = X^{t_n,x_n,\theta^n,c^n}$ and $P^n = P^{t_n,p_n,\alpha^n}$. We now define

$$\tau^n := \inf\{s \geq t_n : (s, X^n_s, P^n_s) \notin O \cap (\text{int}(u) \cup \partial(u))\} \wedge T.$$

Since $(\theta^n, \alpha^n, c^n) \in (\mathcal{A}\times\mathcal{B}\times\mathcal{C})(x_n, t_n, p_n)$, $(\tau^n, X^n_{\tau^n}, P^n_{\tau^n}) \in \text{int}(u) \cup \partial(u)$, and therefore $(\tau^n, X^n_{\tau^n}, P^n_{\tau^n}) \in \partial O$. An application of Itô's formula to $e^{-\beta\tau^n}\varphi(\tau^n, X^n_{\tau^n}, P^n_{\tau^n})$ yields, recalling (6.4.6),

$$e^{-\beta t_n}\varphi(t_n, x_n, p_n) \leq \mathbb{E}[e^{-\beta\tau^n} w_*(\tau^n, X^n_{\tau^n}, P^n_{\tau^n}) + \int_{t_n}^{\tau^n} e^{-\beta u}\tilde{c}_u du] - \xi\mathbb{E}[e^{-\beta\tau^n}] \quad (6.4.7)$$

where, defining $A = \partial O$,

$$-\xi := \min_A(w_* - \varphi) < 0$$

Let $\epsilon = \inf_{n\in\mathbb{N}} \xi\mathbb{E}[e^{-\beta\tau^n}] > \xi e^{-\beta T} > 0$. Since $(\varphi - w)(t_n, x_n, p_n) \to (\varphi - w_*)(\bar{t}, \bar{x}, \bar{p}) = 0$, there exists $n \in \mathbb{N}$ such that $|e^{-\beta t_n}\varphi(t_n, x_n, p_n) - e^{-\beta t_n} w(t_n, x_n, p_n)| < \epsilon$. Then by (6.4.7),

$$e^{-\beta t_n} w(t_n, x_n, p_n) < e^{-\beta t_n}\varphi(t_n, x_n, p_n) + \epsilon \leq \mathbb{E}[e^{-\beta\tau^n} w_*(\tau^n, X^n_{\tau^n}, P^n_{\tau^n}) + \int_{t_n}^{\tau^n} e^{-\beta u}\tilde{c}_u du]$$

which contradicts Lemma 6.4.1.

□

The following corollaries follow directly from Theorem 6.4.1.

Corollary 6.4.1. *The optimal consumption is* $c = 0$ *whenever* $\frac{\partial w}{\partial x} > -1$ *and* $c = \bar{c}$ *whenever* $\frac{\partial w}{\partial x} < -1$.

Corollary 6.4.2. *For* p *such that* $u(t,p) = 0$, *denoting* $\bar{p} = \max\{p \in [0,1] | u(t,p) = 0\}$, w_* *is the viscosity supersolution of :*

$$\sup_{\alpha \in \mathbb{R}, c \in [0,\bar{c}]} [-\frac{\partial \varphi}{\partial t} - c(\frac{\partial \varphi}{\partial x} + 1) - \frac{\alpha^2}{2}\frac{\partial^2 \varphi}{\partial p^2} + \beta\varphi](t,0,p) = 0 \quad \forall p \in [0,\bar{p}]$$

For $u(t,p) > 0$ it is optimal to invest the maximum by definition of the minimal wealth.

Proposition 6.4.1. *Let* $(t,p) \in [0,T] \times [0,1]$ *such that* $u(t,p) > 0$. *Then*

$$\lim_{x \to u(t,p)^+} w(t,x,p) = \frac{\bar{c}}{\beta}(1 - e^{\beta(t-T)}). \tag{6.4.8}$$

Proof Let $(t,p) \in [0,T] \times [0,1]$ such that $u(t,p) > 0$. By (6.4.1),

$$\forall x > u(t,p),\ w(t,x,p) \leq \frac{\bar{c}}{\beta}(1 - e^{\beta(t-T)}).$$

We suppose that

$$\epsilon := \frac{\bar{c}}{\beta}(1 - e^{\beta(t-T)}) - \liminf_{x \to u(t,p)^+} w(t,x,p) > 0.$$

Therefore, by definition of w, there exist $x \in [u(t,p), u(t,p) + \epsilon/4]$ and $(\theta, \alpha, c) \in (\mathcal{A} \times \mathcal{B} \times \mathcal{C})(t,x,p)$ such that

$$\int_t^T e^{-\beta(u-t)} c_u du \leq \liminf_{x \to u(t,p)^+} w(t,x,p) + \frac{\epsilon}{2} = \frac{\bar{c}}{\beta}(1 - e^{\beta(t-T)}) - \frac{\epsilon}{2} \tag{6.4.9}$$

We now define

$$h(t') := \int_t^{t'} e^{-\beta(u-t)}(\bar{c} - c_u) du, \quad t' \in [t,T].$$

Note that by (6.4.9), $h(T) \geq \epsilon/2$. We now suppose $h(T) \leq x$. Let $\tilde{\theta} \in \mathcal{B}$ such that

$$\tilde{\theta}_{t'} X_{t'}^{t,x-h(T),\tilde{\theta},\bar{c}} = \theta_{t'} X_{t'}^{t,x,\theta,c}, \quad t' \in [t,T].$$

Therefore we obtain that, for $s \in [t,T]$,

$$\begin{aligned} X_s^{t,x-h(T),\tilde{\theta},\bar{c}} &= x - h(T) + \int_t^s \tilde{\theta}_u X_u^{t,x-h(T),\tilde{\theta},\bar{c}} + \int_t^s \bar{c} du \\ &= x - h(T) + \int_t^s \theta_u X_u^{t,x,\theta,c} du + \int_t^s c_u du + h(s) \\ &= X_s^{t,x,\theta,c} + h(s) - h(T). \end{aligned}$$

Hence, since $h(T) \leq x$ and $(\theta, \alpha, c) \in (\mathcal{A} \times \mathcal{B} \times \mathcal{C})(t,x,p)$,

$$\forall s \in [t,T], X_s^{t,x-h(T),\tilde{\theta},\bar{c}} \geq h(s) - h(T) \quad \text{and} \quad \mathbb{P}[X_T^{t,x-h(T),\tilde{\theta},\bar{c}} \geq D_0] \geq p.$$

Therefore, by definition of u, recalling that $x \in [u(t,p), u(t,p)+\epsilon/4]$ and $h(T) \geq \epsilon/2$,

$$u(t,p) \leq x - h(T) \leq u(t,p) - \frac{\epsilon}{2}$$

which is a contradiction.
If $h(T) > x$, we may choose $\tilde{c} \in \mathcal{C}$ such that $c \leq \tilde{c}$ and

$$\tilde{h}(T) := \int_t^T e^{-\beta(u-t)}(\tilde{c}_u - c_u)du \in (x - u(t,p), x].$$

Adapting the previous proof we obtain

$$u(t,p) \leq x - \tilde{h}(T) < u(t,p)$$

which is the required contradiction. □

The following proposition is admitted.

Proposition 6.4.2. *Let V (resp. U) be a nonnegative lower-semicontinuous (resp. upper-semicontinuous) bounded map on $int(u) \cup \partial(u)$. Assume that V (resp. U) is a supersolution (resp. subsolution) of* (6.4.2) *on $int(u)$ and that $V \geq U$ on $\partial(u)$. Assume further that for all $(x,p) \in \times\mathbb{R}_+ \times (0,1), U(T,x,p) = 0$. Then, $V \geq U$ on $int(u) \cup \partial(u)$.*

The following theorem is a consequence of Proposition 6.4.2 and of the boundedness of w.

Theorem 6.4.2. *$w_* = w^*$ is continuous on $int(u)$ and is the unique viscosity solution of* (6.4.2) *on $int(u)$ and of* (6.4.8) *on $\partial(u)$ which nullifies at time T in the class of nonnegative and bounded functions.*

Remark 6.4.4.

$$w(t,x,1) = \inf_{(\theta,c)\in\mathcal{A}\times\mathcal{C} s.t. X_s^{t,x,\theta,c} \geq (D_0 - \bar{c}(T-t))^+ \; \forall s \geq t} \int_t^T e^{-\beta(s-t)}\mathbb{E}[c_s]ds \quad \forall 0 \leq t \leq T, \quad x \geq (D_0 - \bar{c}(T-t))^+$$

This is a Merton type problem which shall be solved numerically.

6.5 Numerical resolution of the PDEs

In this section we provide some numerical methods to compute the continuous viscosity solution of equation (6.3.1) u with the boundary conditions obtained previously and the continuous viscosity solution of (6.4.2) w, together with its boundary conditions as well.
In a first subsection we give a scheme to solve the equation (6.3.1) by using a finite difference implicit-explicit scheme : an explicit treatment of the obstacle is used, while an implicit treatment of the PDE is used. This implicit treatment permits to get an unconditionnal stable resolution method. The order of the method is one in both time and space. We prove that the scheme converge towards the viscosity solution of the problem. In order to check the efficiency of the resolution, we next solve the equation (6.3.26) obtained by the Fenchel approach and use the solution calculated to estimate the function u. We use an implicit scheme to solve equation (6.3.26). It is easy to show that it converges

towards the unique viscosity solution of equation (6.3.26). Then we can compare numerically the two approaches and we show that the obtained solution are equal. With the solved PDE and the obtained optimal control, we can test the method numerically by Monte carlo and show that we are able to target a probability of success.
We then give a numerical finite difference scheme (without proving its convergence) to tackle the difficult non convex case where the portfolio is constrained. Because the initial condition is discontinuous, see [17], we have to approximate it by a continuous function. We give some numerical examples for the u function obtained with different levels of constraints on the portfolio.
In the last subsection we are back to the estimation of the function w. The 2 dimensional PDE obtained is degenerated and no finite difference scheme can be used. For this case we decide to use Semi Lagrangian methods that are known to converge towards the (unique) viscosity solution of the equation (6.4.2). We give the different boundary conditions used : some of them need the resolution of a one dimensional Merton type HJB equation that is solved by finite difference methods. Because the function w is defined above the function u, the domain of resolution is time varying and non rectangular. In order to follow the boundaries accurately we have to refine a lot the meshes and we decided to use a low order method explained in [20] to solve the problem. We then give the obtained solution at the end of the resolution period and show that the investment area corresponds to a thin layer near some boundaries of the domain.

6.5.1 Minimal wealth problem

Direct approach

Reverting the problem in time, the admissible wealth function u satisfies :

$$\min_{\kappa\in\{0,1\}}\left(\sup_{\alpha\mathbb{R}}(1-\kappa)\mathcal{L}_\alpha u(t,p)+\kappa u(t,p)\right)=0,$$
$$\mathcal{L}_\alpha u(t,p)=\frac{\partial u}{\partial t}(t,p)-\frac{\alpha^2}{2}\frac{\partial^2 u}{\partial p^2}(t,p)+\frac{\mu\alpha}{\sigma}\frac{\partial u(t,p)}{\partial p}+\bar{c}$$

with $u(x,0)=pD_0$, $u(0,t)=0$, $u(1,t)=(D_0-\bar{c}t)^+$.
First the control is truncated by K such that we solve :

$$\min_{\kappa\in\{0,1\}}\left(\sup_{\alpha\in[-K,K]}(1-\kappa)\mathcal{L}_\alpha u(t,p)+\kappa u(t,p)\right)=0 \tag{6.5.1}$$

Equation (6.5.1) admits a comparison theorem easily proved by classical arguments with truncated controls. Besides a viscosity solution of equation (6.5.1) exists and corresponds to the solution of an optimization problem with α truncated. So there exists a unique viscosity solution of equation (6.5.1) and this solution is continuous.
Using a time discretization $t_n=n\Delta t$ with $T=N\Delta t$ and a space dicretization $x_i=i\Delta x$ with $1=I\Delta x$, we denote $u_i^n=u(t_n,x_i)$ and a discrete form of the equation (6.5.1) is given by the following scheme, setting $\kappa_i^0=0$ for $i=0$ to I :

- Calculate the solution u^{n+1} of

$$\sup_{\alpha\in[-K,K]}(1-\kappa_i^n)\left[u_i^{n+1}(1+\Delta t\frac{\alpha^2}{\Delta x^2}+|\frac{\mu\alpha\Delta t}{\sigma\Delta x}|)-u_{i-1}^{n+1}\Delta t(\frac{\alpha^2}{2\Delta x^2}+\frac{\mu(\alpha)^+}{\sigma\Delta x})\right.$$

$$-u_{i+1}^{n+1}\Delta t(\frac{\alpha^2}{2\Delta x^2}+\frac{\mu(\alpha)^-}{\sigma\Delta x})-u_i^n+\bar{c}\Delta t]+\kappa_i^n u_i^{n+1}=0 \quad (6.5.2)$$

– Then

$$\alpha^{n+1}=\arg\max_{\alpha}((1-\kappa_i^n)[u_i^{n+1}(1+\Delta t\frac{\alpha^2}{\Delta x^2}+|\frac{\mu\alpha\Delta t}{\sigma\Delta x}|)-u_{i-1}^{n+1}\Delta t(\frac{\alpha^2}{2\Delta x^2}+\frac{\mu(\alpha)^+}{\sigma\Delta x}) -u_{i+1}^{n+1}\Delta t(\frac{\alpha^2}{2\Delta x^2}+\frac{\mu(\alpha)^-}{\sigma\Delta x}))\quad (6.5.3)$$

– Update κ^{n+1} by :

$$\kappa^{n+1}=\arg\min_{\kappa\in\{0,1\}}(1-\kappa)[u_i^{n+1}(1+\Delta t\frac{(\alpha^{n+1})^2}{\Delta x^2}+|\frac{\mu\alpha^{n+1}\Delta t}{\sigma\Delta x}|)-u_{i-1}^{n+1}\Delta t(\frac{(\alpha^{n+1})^2}{2\Delta x^2}+\frac{\mu(\alpha^{n+1})^+}{\sigma\Delta x}) -u_{i+1}^{n+1}\Delta t(\frac{(\alpha^{n+1})^2}{2\Delta x^2}+\frac{\mu(\alpha^{n+1})^-}{\sigma\Delta x})-u_i^n+\bar{c}\Delta t]+\kappa u_i^{n+1}\quad (6.5.4)$$

where $x^+=\max(0,x)$ and $x^-=\max(0,-x)$. Notice that the discretization of $\frac{\partial u(t,p)}{\partial p}$ is chosen such that it is monotone. At the boundary $u_0^{n+1}=0$ and $u_I^{n+1}=D_0$.

Remark 6.5.1. *Expliciting the constraint, we guess that the scheme is of order one in time. Therefore, high order time schemes such as Crank Nicholson schemes are useless.*

We denote $A(\kappa,\alpha)$ the matrix such that, for a vector $v:=(v_i)_{i=0,I}$,

$$(A(\kappa,\alpha)v)_i = (1-\kappa_i)[v_i(1+\Delta t\frac{\alpha^2}{\Delta x^2}+|\frac{\mu\alpha\Delta t}{\sigma\Delta x}|)-v_{i-1}\Delta t(\frac{\alpha^2}{2\Delta x^2}+\frac{\mu(\alpha)^+}{\sigma\Delta x}) -v_{i+1}\Delta t(\frac{\alpha^2}{2\Delta x^2}+\frac{\mu(\alpha)^-}{\sigma\Delta x})]+\kappa_i v_i.$$

We propose the algorithm 1 in order to solve the equation (6.5.2) at each time step.

Algorithm 1 Fixed point iteration algorithm at a time step n

```
Initialize κ^0
for k = 0,1,2.. until convergence do
  α^k = arg max_α (A(κ^n, α)v^k − κ^n u^n + κ^n c̄Δt)
  Solve A(κ^n, α^k)v^{k+1} = κ^n u^n − κ^n c̄Δt
  if |v_i^{k+1} − v_i^k| < ε then
    set u^{n+1} = v^{k+1}, break from iteration
  end if
end for
```

Theorem 6.5.1. *The algorithm 1 converges to the unique solution u^{n+1} of the scheme 6.5.2.*

Proof

First we prove that $(v^k)_k$ is bounded. For $k\in\mathbb{N}$, recalling that v^k is nonnegative, we denote i^* the index such that $v_{i^*}^k=||v^k||_\infty$. Note that $\kappa_{i^*}^n=0$, then :

$$(1+\Delta t\frac{(\alpha_{i^*}^{k-1})^2}{\Delta x^2}+|\frac{\mu\alpha_{i^*}^{k-1}\Delta t}{\sigma\Delta x}|)v_{i^*}^{k,l}=v_{i^*-1}^{k,l}\Delta t(\frac{(\alpha_{i^*}^{k-1})^2}{2\Delta x^2}+\frac{\mu(\alpha_{i^*}^{k-1})^+}{\sigma\Delta x})$$

$$+v^{k,l}_{i^*+1}\Delta t(\frac{(\alpha^{k-1}_{i^*})^2}{2\Delta x^2}+\frac{\mu(\alpha^{k-1}_{i^*})^-}{\sigma\Delta x})]+u^n_{i^*}-\bar{c}\Delta t,$$

so

$$(1+\Delta t\frac{(\alpha^{k-1}_{i^*})^2}{\Delta x^2}+|\frac{\mu\alpha^{k-1}_{i^*}\Delta t}{\sigma\Delta x}|)v^{k,l}_{i^*}\leq v^{k,l}_{i^*}\Delta t(\frac{(\alpha^{k-1}_{i^*})^2}{2\Delta x^2})+\frac{\mu(\alpha^{k-1}_{i^*})^+}{\sigma\Delta x})$$
$$+v^{k,l}_{i^*}\Delta t(\frac{(\alpha^{k-1}_{i^*})^2}{2\Delta x^2})+\frac{\mu(\alpha^{k-1}_{i^*})^-}{\sigma\Delta x})]+u^n_{i^*}$$

and $||v^{k,l}||_\infty\leq||u^n||_\infty\leq D_0$.
The A matrix is a M matrix, so $A^{-1}>0$. First fixing l, as done in [47] :

$$\begin{aligned}A(\kappa^n,\alpha^k)(v^{k+1}-v^k) &= -A(\kappa^n,\alpha^k)v^k+\kappa^n u^n-\kappa^n\bar{c}\Delta t,\\ &= [A(\kappa^n,\alpha^{k-1})v^k-\kappa^n u^n+\kappa^n\bar{c}\Delta t]-[A(\kappa^n,\alpha^k)v^k)-\kappa^n u^n+\kappa^n\bar{c}\Delta t],\\ &\leq 0,\end{aligned}$$

because α^k maximizes $[A(\kappa^n,\alpha)v^k)-\kappa^n u^n+\kappa^n\bar{c}\Delta t]$. Then using $A^{-1}>0$, $v^{k+1}-v^k\leq 0$ and the sequence is decreasing. Because it is bounded, it is converging.
Suppose that there are two solutions v and w associated to the controls α_v and α_w, then

$$\begin{aligned}A(\kappa^n,\alpha_v)(v-w) &= [A(\kappa^n,\alpha_w)w-\kappa^n u^n+\kappa^n\bar{c}\Delta t]-[A(\kappa^n,\alpha_v)w-\kappa^n u^n+\kappa^n\bar{c}\Delta t]\\ &\geq 0\end{aligned}$$

because α_w maximizes $A(\kappa^n,\alpha)w-\kappa^n u^n+\kappa^n\bar{c}\Delta t$. Then $v\geq w$. Inverting v and w, we obtain that $v=w$.

Theorem 6.5.2. *The scheme* (6.5.2)-(6.5.3)-(6.5.4) *converges to the unique solution of equation* (6.5.1)*.*

First the scheme is l_∞ stable since the solution is bounded by D_0 as shown above. We now show that the scheme is consistent. Suppose u is regular such that $u^n_i=u(t^n,x_i)$ is solution of the scheme.

– If $\kappa^{n+1}_i=0$, then

$$|\sup_\alpha\mathcal{L}_\alpha u(t^{n+1},x_i)|=|\sup_\alpha[(\frac{A(0,\alpha)u^{n+1}-u^n}{\Delta t})_i+\bar{c}]-\sup_\alpha\mathcal{L}_\alpha u(t^{n+1},x_i)|$$
$$\leq\sup_\alpha|(\frac{A(0,\alpha)u^{n+1}-u^n}{\Delta t})_i+\bar{c}-\mathcal{L}_\alpha u(t^{n+1},x_i)|\leq O(\Delta t)+O(\Delta x)$$

using the fact that the max is taken on a bounded set. Using equation (6.5.4), we get that

$$u^{n+1}_i\geq[u^{n+1}_i(1+\Delta t\frac{(\alpha^{n+1})^2}{\Delta x^2}+|\frac{\mu\alpha^{n+1}\Delta t}{\sigma\Delta x}|)-u^{n+1}_{i-1}\Delta t(\frac{(\alpha^{n+1})^2}{2\Delta x^2}+\frac{\mu(\alpha^{n+1})^+}{\sigma\Delta x})$$
$$-u^{n+1}_{i+1}\Delta t(\frac{(\alpha^{n+1})^2}{2\Delta x^2}+\frac{\mu(\alpha^{n+1})^-}{\sigma\Delta x})-u^n_i+\bar{c}\Delta t]$$

and using equations (6.5.2) and (6.5.3) we get that $u^{n+1}_i\geq 0$. Then $|\min_{\kappa\in\{0,1\}}\left(\sup_\alpha(1-\kappa)\mathcal{L}_\alpha u(t^{n+1},x_i)+\kappa u(t^{n+1},x_i)\right)|\leq O(\Delta t)+O(\Delta x)$.

– If $\kappa_i^{n+1} = 1$, then $u_i^{n+1} = 0$, and

$$0 \leq [u_i^{n+1}(1 + \Delta t\frac{(\alpha^{n+1})^2}{\Delta x^2} + |\frac{\mu\alpha^{n+1}\Delta t}{\sigma\Delta x}|) - u_{i-1}^{n+1}\Delta t(\frac{(\alpha^{n+1})^2}{2\Delta x^2} + \frac{\mu(\alpha^{n+1})^+}{\sigma\Delta x}) - u_{i+1}^{n+1}\Delta t(\frac{(\alpha^{n+1})^2}{2\Delta x^2} + \frac{\mu(\alpha^{n+1})^-}{\sigma\Delta x}) - u_i^n + \bar{c}\Delta t]$$

So

$$\mathcal{L}_{\alpha^{n+1}}u(t^{n+1}, x_i) = \mathcal{L}_{\alpha^{n+1}}u(t^{n+1}, x_i) - [(\frac{A(0,\alpha^{n+1})u^{n+1} - u^n}{\Delta t})_i + \bar{c}] + [(\frac{A(0,\alpha^{n+1})u^{n+1} - u^n}{\Delta t})_i + \bar{c}]$$

and

$$\sup_\alpha \mathcal{L}_\alpha u(t^{n+1}, x_i) \geq O(\Delta t)$$

Therefore we get

$$|\min_{\kappa\in\{0,1\}}(\sup_\alpha(1-\kappa)\mathcal{L}_\alpha u(t^{n+1}, x_i) + \kappa u(t^{n+1}, x_i))| \leq O(\Delta t)$$

and the consistency is checked.

So using [6] it is converging towards the viscosity solution of (6.5.1).

The fenchel approach

It is also possible to solve (6.3.26) to get the Fenchel transform of the admissible wealth. First, reverting time, $h(t,y) = v(t,x)$ where $q = \log(x)$ solves :

$$\max[\frac{\partial h}{\partial t}(t,q) - \frac{\mu^2}{2\sigma^2}(\frac{\partial^2 h}{\partial q^2}(t,q) - \frac{\partial h}{\partial q}(t,q)) - \bar{c}, h - \frac{\partial h}{\partial q}] = 0, \quad (6.5.5)$$

with the initial condition

$$h(0,q) = (q - D_0)^+, \quad (6.5.6)$$

and the boundary condition $h(0,0) = 0$.

The discretization scheme is then

$$\max_{\kappa\in\{0,1\}^{I+1}}((1-\kappa_i)[h_i^{n+1}(1 + \Delta t\frac{\mu^2}{\sigma^2}(\frac{1}{\Delta x^2} + \frac{1}{2\Delta x})) - h_{i-1}^{n+1}\Delta t(\frac{\mu^2}{2\sigma^2})(\frac{1}{\Delta x^2} + \frac{1}{\Delta x}) - h_{i+1}^{n+1}\Delta t\frac{\mu^2}{2\sigma^2\Delta x^2} - h_i^n - \bar{c}\Delta t] + \kappa_i[h_i^{n+1}(1 + \frac{1}{\Delta x}) - h_{i-1}^{n+1}\frac{1}{\Delta x}]) = 0. \quad (6.5.7)$$

It can be solved using an algorithm similar to algorithm 1. Besides the scheme is monotone, consistent, l_∞ stable and we get the following convergence theorem :

Theorem 6.5.3. *The scheme (6.5.7) converges to the unique viscosity solution of (6.5.5).*

Remark 6.5.2. *In the direct approach, we had to truncate the control in order to solve the problem. In the Fenchel approach, this truncation is replaced by a truncation of the domain. The size D of the domain is chosen such that the Fenchel transform of the calculated solution is nearly independent of D. In the sequel, the right boundary condition chosen is $\max(\frac{\partial h}{\partial t}(t,D) - \bar{c}, h(t,D) - \frac{\partial h}{\partial q}(t,D)) = 0$.*

Numerical results

We choose $\mu = 0.1$, $\sigma = 0.2$, $T = 1$, $D_0 = 1$, $\bar{c} = 0.1$. We take $N = 800$ for the number of time steps, $I = 800$ for the number of meshes. On figure (6.1), we give the Fenchel transform h of the minimal wealth.

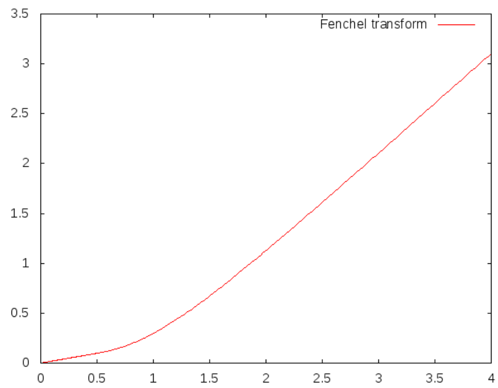

FIGURE 6.1 – Fenchel transform of the minimal wealth

By taking the Fenchel transform of h we get an estimation $\hat{u}$ of the minimal wealth u. On figure (6.2), we compute a direct estimation of u by solving (6.5.2) and $\hat{u}$: the two curves are nearly identical.

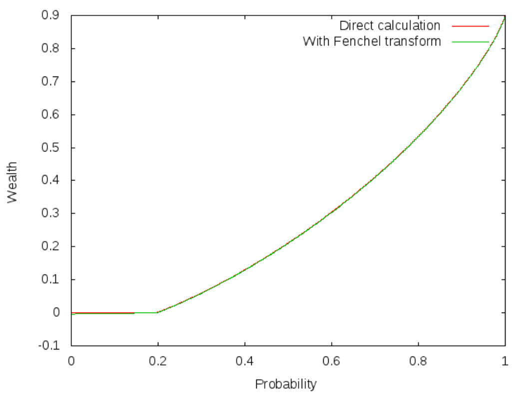

FIGURE 6.2 – Minimal wealth by direct calculation and using the Fenchel transform.

The minimal wealth function is interesting but we may try to check if we are able to simulate the optimal strategy. In optimization, $\theta(t, p)$ and the optimal values $u(t, p)$ are stored. In a simulation phase, we simulate the strategy for $u(0, p) > 0$:

- the wealth is initialized with the minimal wealth associated to the initial probability,
- at each time step, given the wealth, the probability level $\tilde{p}$ is obtained by inverting $p \longrightarrow u(t, p)$, $\theta(t, \tilde{p})$ is obtained by interpolation and used to update the portfolio composition.

In the table 6.5.1, we simulate the optimal strategy with 1000 time steps (optimization and simulation) and 500000 trajectories in simulation. For a high probability target, results obtained in simulation perfectly match the target.

Target probability	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.
Probability reached	0.18	0.28	0.37	0.47	0.58	0.68	0.78	0.90	1.

TABLE 6.1 – Simulate the optimal strategy and compute the probability of success.

Possible extension

A case of interest is the special case where the portfolio is constrained : we may consider the case where $\theta \in [0, \theta_M]$. In this case we guess using [17] that the value function u is solution of

$$\min_{\kappa\in\{0,1\}} \Big(\sup_{\{(\alpha,\theta)\in\mathbb{R}\times[0,1]\ \alpha\frac{\partial u}{\partial p}(t,p)=\sigma\theta u(t,p)\}} (1-\kappa)\Big[\frac{\partial u}{\partial t}(t,p) - \frac{\alpha^2}{2}\frac{\partial^2 u}{\partial p^2}(t,p) + \frac{\mu\alpha}{\sigma}\frac{\partial u(t,p)}{\partial p} + \bar{c}\Big] + \kappa u(t,p)\Big) = 0,$$

with the initial condition :

$$u(0,p) = \begin{cases} D_0, \text{if } p \neq 0, \\ 0, \text{if } p = 0 \end{cases}$$

The solution u is no longer continuous nor convex and no finite difference scheme can handle the discontinuity. In this case we choose to approximate the initial condition with the following

$$u(0,p) = \begin{cases} D_0, \text{if } p \geq \hat{p}, \\ D_0\frac{p}{\hat{p}}, \text{if } p < \hat{p}. \end{cases}$$

We propose the following scheme :

– Calculate u^{n+1} solution of

$$(1-\kappa_i^n) \max_{\substack{\alpha\frac{u^n(i+1)-u^n(i-1)}{2\Delta x} = \sigma\theta u^n(i), \\ \theta\in[0,\theta_M],\, \alpha\in\mathbb{R}}} [u_i^{n+1}(1+\Delta t\frac{\alpha^2}{\Delta x^2}) - u_{i-1}^{n+1}\Delta t\frac{\alpha^2}{2\Delta x^2} - u_{i+1}^{n+1}\Delta t\frac{\alpha^2}{2\Delta x^2} + \mu\theta u_i^{n+1} - u_i^n - \bar{c}\Delta t] + \kappa_i^n[u_i^{n+1}] = 0,$$

– Then

$$(\alpha^{n+1},\theta^{n+1}) = \arg\max_{\substack{\theta\in[0,\theta_M],\, \alpha\in\mathbb{R} \\ \alpha\frac{u^n(i+1)-u^n(i-1)}{2\Delta x} = \sigma\theta u^n(i)}} (1-\kappa_i^n)[u_i^{n+1}(1+\Delta t\frac{\alpha^2}{\Delta x^2}) - u_{i-1}^{n+1}\Delta t\frac{\alpha^2}{2\Delta x^2} - u_{i+1}^{n+1}\Delta t\frac{\alpha^2}{2\Delta x^2} + \mu\theta u_i^{n+1} - u_i^n - \bar{c}\Delta t] + \kappa_i^n[u_i^{n+1}]$$

– Update κ^{n+1} by :

$$\kappa^{n+1} = \arg\min_{\kappa\in\{0,1\}}(1-\kappa)[u_i^{n+1}(1+\Delta t\frac{(\alpha^{n+1})^2}{\Delta x^2}) - u_{i-1}^{n+1}\Delta t\frac{(\alpha^{n+1})^2}{2\Delta x^2} - u_{i+1}^{n+1}\Delta t\frac{(\alpha^{n+1})^2}{2\Delta x^2} + \mu\theta^{n+1}u_i^{n+1} - u_i^n - \bar{c}\Delta t] + \kappa u_i^{n+1}$$

Remark 6.5.3. *In order to be able to solve this scheme by an iterative procedure using a fixed point iteration as in algorithm 1, we had to explicit the constraint in the max.*

We took $\hat{p} = 0.01$, $N = 400$, $I = 2000$ so the scheme converges numerically.

The numerical solution is given on figure (6.3).

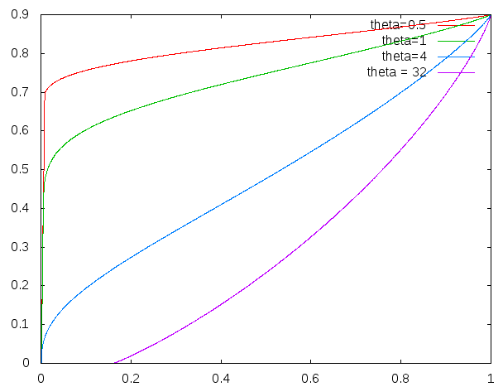

FIGURE 6.3 – Minimal weath function for constrained portfolio.

6.5.2 The function w

Scheme

We are now interested in the resolution of (6.4.2) where $x \geq u(t,p)$. The PDE is degenerated and in order to have a monotone scheme we propose to use Camilli Falcone's scheme [20]. Let us introduce $Y_t = (X_t, P_t)^t$, a time discretization $t_n = n\Delta t$, $w^n(t, Y_{t^n}) = w(t^n, X_{t^n}, P_{t^n})$. Camilli Falcone's scheme is the following one :

$$w^{n+1}(y) = \inf_{(\theta,\alpha,c)\in[\theta_{min},\theta_{max}]\times\mathbb{R}\times[0,\bar{c}]} \Bigg[\frac{1}{2} \Bigg(I_{\Delta x} w^n (y + \begin{pmatrix} \mu\theta y_1 + c \\ 0 \end{pmatrix} \Delta t + \begin{pmatrix} \sigma\theta x \\ \alpha \end{pmatrix} \sqrt{\Delta t}) + I_{\Delta x} w^n (y + \begin{pmatrix} \mu\theta y_1 + c \\ 0 \end{pmatrix} \Delta t - \begin{pmatrix} \sigma\theta x \\ \alpha \end{pmatrix} \sqrt{\Delta t}) \Bigg) - \beta \Delta t w^n(y) + c\Delta t \Bigg] \quad (6.5.8)$$

with $w^0 = 0$ as initial condition, and $I_{\Delta x}$ is a linear interpolator on a grid with meshes of size Δx.

The boundary conditions are $w^n((.,0)^t) = 0$, $w^n((D_0,.)^t) = 0$ and $w^n((u(t^n,.),.)^t) = \frac{\bar{c}}{\beta}(1 - e^{\beta(t-T)})$ if $u(t^n,p) \neq 0$. Note that if $u(t^n,p) = 0$, due to the advection term, no special boundary condition is required.

At last, for $x \in [(D_0 - \bar{c}t^n)^+, D_0]$, $w^n((x,1)^t) = \hat{w}(t_n, x)$ where

$$\max_{\theta,c\in[0,\bar{c}]} \frac{\partial \hat{w}}{\partial t}(t,x) - (\mu\theta + c)\frac{\partial \hat{w}}{\partial x} - \frac{\sigma^2\theta^2 x^2}{2}\frac{\partial^2 \hat{w}}{\partial x^2} - c + \beta\hat{w} = 0, \quad (6.5.9)$$

with $\hat{w}(0, D_0) = 0$, $\hat{w}(t, (D_0 - \bar{c}t)^+) = \frac{\bar{c}}{\beta}(1 - e^{\beta(t-T)})$. In order to solve (6.5.9), we use the following discrete scheme :

$$\max_{\theta,c\in[0,\bar{c}]} \Bigg(w_i^{n+1}(1 + \Delta t(\frac{\sigma^2\theta^2x^2}{\Delta x^2} + \beta + \frac{|\mu\theta x + c|}{\Delta x}) - w_{i+1}^{n+1}\Delta t(\frac{(\mu\theta x + c)^+}{\Delta x}) + \frac{\sigma^2\theta^2x^2}{2\Delta x^2}) -$$

$$w_{i-1}^{n+1}\Delta t(\frac{(\mu\theta x + c)^-}{\Delta x}) + \frac{\sigma^2\theta^2 x^2}{2\Delta x^2}) - c\Delta t - w_i^n\Bigg) = 0$$

Note that no truncation is necessary for θ because the optimal value maximizing the previous equation is bounded. The scheme used is an iteration policy combining two nests : an outer nest iterating on the optimal c and an internal nest iterating on the optimal θ for a given c.
The scheme (6.5.8) is known to converge towards the unique viscosity solution of equation (6.4.2) where the controls are bounded ([5, 20, 33]).

Numerical results

We keep the same parameters as in section 6.5.1 and $\beta = 0.05$. On figure 6.4, we give the solution numerically calculated for $\hat{w}$ at time $t = T$ taking a number of time steps equal to $N = 400$, and a mesh size Δx equal to $\bar{c}\Delta t$. This choice of mesh size imposes the computation on previous calculated points, even if the resolution domain changes with time.

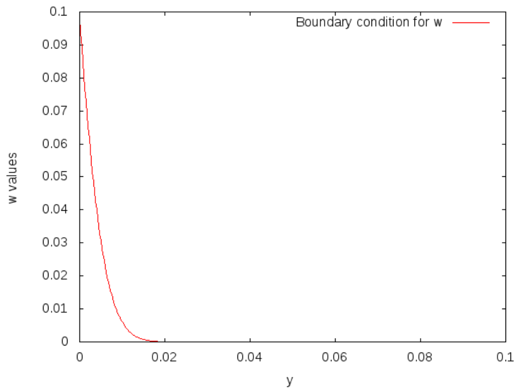

FIGURE 6.4 – Boundary condition at $t = T$ as a function of $y = x - D_0 + \bar{c}T$ for $y \in [0, \bar{c}T]$

All solution calculated on boundaries are stored at each time step and an interpolator in time and space is used to recover the values used in the problem (6.5.8).
On figure 6.5, we give the value function w taking a number of time steps $N = 200$, a number of meshes I in x and p equal to 1600. We impose $\theta_{min} = -20$, $\theta_{max} = 20$ and we discretize the possible θ values with a thin mesh in order to estimate the optimal control in θ. The very high value taken by n is due to the very slow spacial convergence of the scheme. In order to get converged results we had to parallelize the scheme on 192 cores running during two days using the methodology developped in [104].

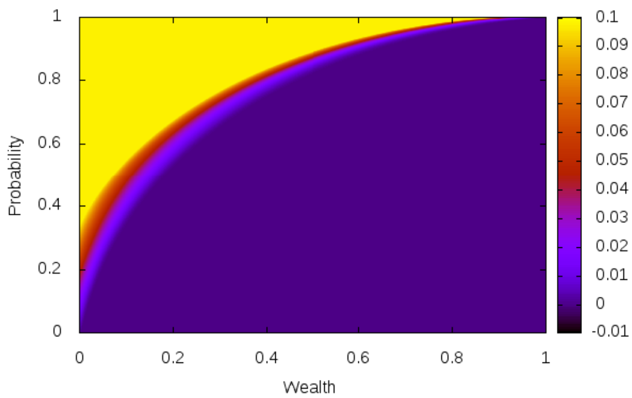

FIGURE 6.5 – Value function w at time $t = T$.

The optimal strategy in c consists in not investing except close to the boundary defined by the minimal wealth as shown in figure 6.6 and the optimal strategy in θ is given on figure 6.7.

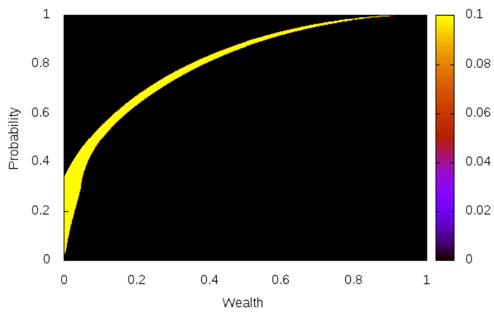

FIGURE 6.6 – Optimal strategy in c at time $t = T$.

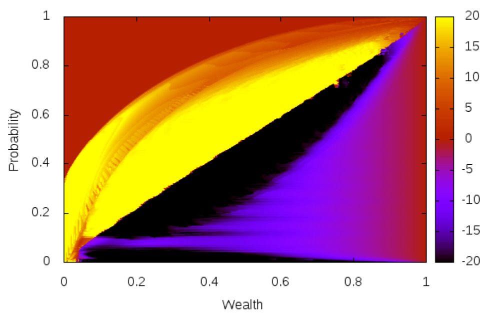

FIGURE 6.7 – Optimal strategy in θ at time $t = T$.

6.6 Appendix

The following lemmas follows exactly from the same arguments as in the proof of Lemma 3.1 of [99]

Lemma 6.6.1. *The set* $\{(t, x, p, \alpha, \theta, c) \in [0, T] \times \mathbb{R}_*^+ \times [0, 1] \times \mathcal{A} \times \mathcal{B} \times \mathcal{C} : (\alpha, \theta, c) \in (\mathcal{A} \times \mathcal{B} \times \mathcal{C})(t, x, p)\}$ *is a Borel set in* $[0, T] \times \mathbb{R}_*^+ \times [0, 1] \times \mathcal{A} \times \mathcal{B} \times \mathcal{C}$.

Lemma 6.6.2. *For any probability measure* m *on* $([0, T] \times \mathbb{R}_*^+ \times [0, 1], \mathcal{B}([0, T] \times \mathbb{R}_*^+ \times [0, 1]))$, *there exists a Borel measurable map* ϕ_m *such that* $\phi_m(t, x, p) \in (\mathcal{A} \times \mathcal{B} \times \mathcal{C})(t, x, p)$ *for m-a.e.* $(t, x, p) \in [0, T] \times \mathbb{R}_*^+ \times [0, 1]$.

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