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A NOTE ON THE DERIVATION OF RIGID-PLASTIC MODELS

JEAN-FRANÇOIS BABADJIAN AND GILLES A. FRANCFORT

Abstract. This note is devoted to a rigorous derivation of rigid-plasticity as the limit of elasto-plasticity when the elasticity tends to infinity.

1. Introduction

Small strain elasto-plasticity is formally modeled as follows. Consider a homogeneous elasto-plastic material occupying a volume $\Omega \subset \mathbb{R}^n$ with Hooke's law (elasticity tensor) $\mathbb{C}$. Assume that the body is subjected to a time-dependent loading process during a time interval $[0,T]$ with, say, $f(t)$ as body loads, $g(t)$ as surface loads on a part Γ_N of $\partial\Omega$, and $w(t)$ as displacement loads (hard device) on the complementary part Γ_D of $\partial\Omega$. Denoting by $Eu(t)$ the infinitesimal strain at t, that is, the symmetric part of the spatial gradient of the displacement field $u(t)$ at t, small strain elasto-plasticity requires that $Eu(t)$ decompose additively as

$$Eu(t) = e(t) + p(t) \text{ in } \Omega, \text{ with } u(t) = w(t) \text{ on } \Gamma_D$$

where $e(t)$ is the elastic strain and $p(t)$ the plastic strain. The elastic strain is related to the stress tensor $\sigma(t)$ through the constitutive law of linearized elasticity $\sigma(t) = \mathbb{C}e(t)$. In a quasi-static setting, the equilibrium equations read as

$$\operatorname{div}\sigma(t) + f(t) \quad \text{in } \Omega, \quad \sigma(t)\nu = g(t) \text{ on } \Gamma_N,$$

where ν denotes the outer unit normal to $\partial\Omega$. In plasticity, the stresses are constrained to remain below a yield stress at which permanent strains appear. Specifically, the deviatoric stress $\sigma_D(t)$ must belong to a fixed compact and convex subset K of the deviatoric (trace free) matrices

$$\sigma_D(t) \in K.$$

If $\sigma_D(t)$ lies inside the interior of K, the material behaves elastically ($p(t) = 0$). On the other hand, if $\sigma_D(t)$ reaches the boundary of K (called the yield surface), a plastic flow may develop, so that, after unloading, there will remain a non-trivial permanent plastic strain $p(t)$. Its evolution is described by the so-called flow rule

$$\dot{p}(t) \in N_K(\sigma_D(t))$$

where $N_K(\sigma_D(t))$ is the normal cone to K at $\sigma_D(t)$. By arguments of convex analysis, the flow rule can be equivalently written as Hill's principle of maximum plastic work

$$\sigma_D(t) : \dot{p}(t) = \max_{\tau_D \in K} \tau_D : \dot{p}(t) =: H(\dot{p}(t)),$$

where H is the support function of K, and $H(\dot{p}(t))$ identifies with the plastic dissipation.

In this self-contained note, we propose to show that rigid plasticity – that is the model where one formally sets $\mathbb{C} = \infty$ (and correspondingly $\dot{p}(t) = E\dot{u}(t)$, $\operatorname{div}\dot{u}(t) = 0$) in the system above – can be derived as an asymptotic limit of small strain elasto-plasticity as $\mathbb{C}$ actually gets larger and larger. Rigid-plastic models are particularly useful in order to compute analytical solutions in a plane-strain setting. Indeed, inside the plastic zone, the stress equations can be formally written

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as a non-linear hyperbolic system which is solved by the method of characteristics. The family of characteristics are the so-called *slip lines* along which some combinations of the stress remain constants, while the tangential velocities can jump. It thus seems appropriate to rigorously derive rigid-plasticity in order to investigate the hyperbolic structure of the equations. However, this later task falls outside the scope of the present work.

Notationwise, we denote by $\mathbb{M}^{n\times n}_{sym}$ the set of symmetric $n \times n$ matrices. If A and $B \in \mathbb{M}^{n\times n}_{sym}$, we use the Euclidean scalar product $A : B := \mathrm{tr}(AB)$ and the associated Euclidean norm $|A| := \sqrt{A : A}$. The subset $\mathbb{M}^{n\times n}_D$ of $\mathbb{M}^{n\times n}_{sym}$ stands for trace free symmetric matrices. If $A \in \mathbb{M}^{n\times n}_{sym}$, it can be orthogonally decomposed as

$$A = A_D + \frac{\mathrm{tr}\, A}{n} I,$$

where $A_D \in \mathbb{M}^{n\times n}_D$, and I is the identity matrix in $\mathbb{R}^n$. The notation $\odot$ stands for the symmetrized tensor product between vectors in $\mathbb{R}^n$, *i.e.*, if a and $b \in \mathbb{R}^n$, $(a \odot b)_{ij} = (a_i b_j + a_j b_i)/2$ for all $1 \leq i, j \leq n$. Note in particular that $\frac{1}{\sqrt{2}}|a||b| \leq |a \odot b| \leq |a||b|$.

The Lebesgue measure in $\mathbb{R}^n$ and the $(n-1)$-dimensional Hausdorff measure are denoted by $\mathcal{L}^n$ and $\mathcal{H}^{n-1}$, respectively. Given a locally compact set $E \subset \mathbb{R}^n$ and a Euclidean space X, we denote by $\mathcal{M}(E; X)$ (or simply $\mathcal{M}(E)$ if $X = \mathbb{R}$) the space of bounded Radon measures on E with values in X, endowed with the norm $\|\mu\|_{\mathcal{M}(E;X)} := |\mu|(E)$, where $|\mu| \in \mathcal{M}(E)$ is the variation of the measure μ. Moreover, if ν is a non-negative Radon measure over E, we denote by $d\mu/d\nu$ the Radon-Nikodym derivative of μ with respect to ν.

We use standard notation for Lebesgue and Sobolev spaces. In particular, for $1 \leq p \leq \infty$, the L^p-norms of the various quantities are denoted by $\| \cdot \|_p$. If $U \subset \mathbb{R}^n$ is an open set, the space $BD(U)$ of functions of bounded deformation in U is made of all functions $u \in L^1(U; \mathbb{R}^n)$ such that $Eu \in \mathcal{M}(U; \mathbb{M}^{n\times n}_{sym})$, where $Eu := (Du + Du^T)/2$ and Du is the distributional derivative of u. We refer to [14] for general properties of this space. Finally, $H(\mathrm{div}, U)$ stands for the Hilbert space of all $\tau \in L^2(U; \mathbb{M}^{n\times n}_{sym})$ such that $\mathrm{div}\, \tau \in L^2(U; \mathbb{R}^n)$.

2. The elasto-plastic model

We now consider a homogeneous elasto-plastic material with Hooke's law given by a fourth order tensor $\mathbb{C}$ satisfying the usual symmetry properties

$$\mathbb{C}_{ijkl} = \mathbb{C}_{jikl} = \mathbb{C}_{klij}, \quad \text{for all } 1 \leq i, j, k, l \leq n, \tag{2.1}$$

and the growth and coercity assumptions

$$\alpha|\xi|^2 \leq \mathbb{C}\xi : \xi \leq \beta|\xi|^2, \quad \text{for all } \xi \in \mathbb{M}^{n\times n}_{sym}, \tag{2.2}$$

where α and $\beta > 0$.

It occupies the domain Ω, a bounded and connected open subset of $\mathbb{R}^n$ with at least Lipschitz boundary (see Definition 2.1) and outer normal ν. Its boundary $\partial\Omega$ is split into the union of a Dirichlet part Γ_D which is non empty and open in the relative topology of $\partial\Omega$, a Neumann part $\Gamma_N := \partial\Omega \setminus \overline{\Gamma_D}$, and their common relative boundary denoted by $\partial_{\lfloor\partial\Omega}\Gamma_D$.

Standard plasticity is characterized by the fact that the deviatoric stress is constrained to stay in a fixed compact and convex subset $K \subset \mathbb{M}^{n\times n}_D$ of deviatoric matrices. We further assume that

$$B(0, c_*) \subset K \subset B(0, c^*), \tag{2.3}$$

where $0 < c_* < c^* < \infty$, and denote by

$$\mathcal{K} := \{\sigma \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) : \sigma_D(x) \in K \text{ for a.e. } x \in \Omega\}.$$

The support function of K, defined for any $p \in \mathbb{M}^{n\times n}_D$ by $H(p) := \sup_{\tau\in K} \tau : p$, satisfies, according to (2.3),

$$c_*|p| \le H(p) \le c^*|p|, \quad \text{for all } p \in \mathbb{M}^{n\times n}_{sym}.$$

On the Dirichlet part Γ_D of the boundary, the body is subjected to a hard device, *i.e.*, a boundary displacement which is the trace on Γ_D of a function $w \in AC([0,T]; H^1(\Omega;\mathbb{R}^n))$. In addition, the body is subjected to two types of forces: bulk forces $f \in AC([0,T]; L^n(\Omega;\mathbb{R}^n))$, and surface forces $g \in AC([0,T]; L^\infty(\Gamma_N;\mathbb{R}^n))$, the latter acting on the Neumann part Γ_N of the boundary. It is classical to assume a uniform safe load condition (see [12]) which ensures the existence of a plastically, as well as statically admissible state of stress π associated with the pair (f,g). Specifically, there exists $\pi \in AC([0,T]; L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ and some safety parameter $c > 0$ such that

$$\begin{cases} \pi_D(t,x) + B(0,c) \subset K \text{ for a.e. } x \in \Omega \text{ and all } t \in [0,T] \\ \operatorname{div} \pi(t) + f(t) = 0 \text{ in } \Omega, \quad \pi(t)\nu = g(t) \text{ on } \Gamma_N. \end{cases}$$

Given a boundary datum $\hat{w} \in H^1(\Omega;\mathbb{R}^n)$, we define the space of all kinematically admissible triples as

$$\begin{aligned} \mathcal{A}(\hat{w}) := \{(u,e,p) \in BD(\Omega) \times L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) \times \mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^{n\times n}_D) : \\ Eu = e + p \text{ in } \Omega,\ p = (\hat{w} - u)\odot\nu \text{ on } \Gamma_D\}, \end{aligned}$$

where we still denote by u the trace of u on $\partial\Omega$ (see [2]). We also define the space of all statically admissibles stresses as

$$\Sigma := \{\sigma \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym}) : \operatorname{div}\sigma \in L^n(\Omega;\mathbb{R}^n),\ \sigma\nu \in L^\infty(\Gamma_N;\mathbb{R}^n),\ \sigma_D \in L^\infty(\Omega;\mathbb{M}^{n\times n}_D)\},$$

where $\sigma\nu$ is the normal trace of $\sigma \in H(\operatorname{div},\Omega)$ which is well defined as an element of $H^{-1/2}(\Gamma_N;\mathbb{R}^n)$, the dual space of $H^{1/2}_{00}(\Gamma_N;\mathbb{R}^n)$.

Following [7, Section 6], we introduce the following class of domains for which a meaningful duality pairing between stresses and strains can be defined. Note that the class contains in particular $\mathcal{C}^2$-domains [10], as well as hypercubes where Γ_D is one of its faces [7, Section 6].

Definition 2.1. We say that Ω is admissible if for any $\sigma \in \Sigma$, and any $p \in \mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^{n\times n}_D)$, with $(u,e,p) \in \mathcal{A}(\hat{w})$ for some $\hat{w} \in H^1(\Omega;\mathbb{R}^n)$, $u \in BD(\Omega)$ and $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, the distribution defined for all $\varphi \in \mathcal{C}^\infty_c(\mathbb{R}^n)$ by

$$\begin{aligned} \langle[\sigma_D : p],\varphi\rangle := \int_\Omega \varphi\sigma : (E\hat{w} - e)\,dx - \int_\Omega \varphi \operatorname{div}\sigma\cdot(u-\hat{w})\,dx \\ - \int_\Omega \sigma : [(u-\hat{w})\odot\nabla\varphi]\,dx + \int_{\Gamma_N} \varphi\sigma\nu\cdot(u-\hat{w})\,d\mathcal{H}^{n-1} \end{aligned}$$

extends to a bounded Radon measure in $\mathbb{R}^n$ with $|[\sigma_D : p]| \le \|\sigma_D\|_\infty |p|$. In this case, its mass is given by

$$\langle\sigma_D, p\rangle := \langle[\sigma_D : p], 1\rangle = \int_\Omega \sigma : (E\hat{w} - e)\,dx - \int_\Omega \operatorname{div}\sigma\cdot(u-\hat{w})\,dx + \int_{\Gamma_N} \sigma\nu\cdot(u-\hat{w})\,d\mathcal{H}^{n-1}. \quad (2.4)$$

For any $e \in L^2(\Omega;\mathbb{M}^{n\times n}_{sym})$, the elastic energy is

$$\mathcal{Q}(e) = \frac{1}{2}\int_\Omega \mathbb{C}e : e\,dx,$$

while, for any $p \in \mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^{n\times n}_D)$, the dissipation energy is the convex functional of measure (see [9, 6])

$$\mathcal{H}(p) := \int_{\Omega\cup\Gamma_D} H\left(\frac{dp}{d|p|}\right) d|p|.$$

If $p : [0,T] \to \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D)$, we define the total dissipation between times a and b by

$$\mathcal{V}_{\mathcal{H}}(p;[a,b]) := \sup\left\{ \sum_{i=1}^N \mathcal{H}(p(t_i) - p^\varepsilon(t_{i-1})) : N \in \mathbb{N},\ a = t_0 < t_1 < \cdots < t_N = b \right\}.$$

If additionally $p \in AC([0,T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D))$, then [4, Theorem 7.1] shows that

$$\mathcal{V}_{\mathcal{H}}(p;[a,b]) = \int_a^b \mathcal{H}(\dot p(s))\, ds.$$

We finally impose the following initial condition on the evolution: $(u_0, e_0, p_0) \in \mathcal{A}(w(0))$ with $\sigma_0 := \mathbb{C}e_0$ such that

$$\operatorname{div} \sigma_0 + f(0) = 0 \text{ in } \Omega, \quad \sigma_0 \nu = g(0) \text{ on } \Gamma_N, \quad (\sigma_0)_D \in \mathcal{K}.$$

The following existence result has been established in [4, 7].

Theorem 2.2. *Under the previous assumptions, there exist a quasi-static evolution, i.e. a mapping* $t \mapsto (u(t), e(t), p(t))$ *with the following properties*

$$u \in AC([0,T]; BD(\Omega)),\ \sigma,\ e \in AC([0,T]; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})),\ p \in AC([0,T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D)),$$

$$(u(0), e(0), p(0)) = (u_0, e_0, p_0),$$

and for all $t \in [0,T]$,

$$\begin{cases} Eu(t) = e(t) + p(t) \text{ in } \Omega, \\ p(t) = (w(t) - u(t)) \odot \nu \text{ on } \Gamma_D, \\ \sigma(t) = \mathbb{C}e(t) \text{ in } \Omega, \end{cases}$$

$$\begin{cases} \operatorname{div} \sigma(t) + f(t) = 0 \text{ in } \Omega, \\ \sigma(t)\nu = g(t) \text{ on } \Gamma_N, \\ \sigma_D(t) \in \mathcal{K}, \end{cases}$$

and for a.e. $t \in [0,T]$,

$$H(\dot p(t)) = [\sigma_D(t) : \dot p(t)] \text{ in } \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D). \tag{2.5}$$

Remark 2.3. Equation (2.5) is a measure-theoretic formulation of the usual flow rule of perfect plasticity. Using the definition (2.4) of duality, it can be equivalently written as an energy balance

$$\mathcal{Q}(e(t)) + \int_0^t \mathcal{H}(\dot p(s))\, ds = \mathcal{Q}(e_0) + \int_0^t \int_\Omega \sigma(s) : E\dot w(s)\, dx\, ds$$
$$+ \int_0^t \int_\Omega f(s) \cdot (\dot u(s) - \dot w(s))\, dx\, ds + \int_0^t \int_{\Gamma_N} g(s) \cdot (\dot u(s) - \dot w(s))\, d\mathcal{H}^{n-1}\, ds,$$

or equivalently, according to the safe-load condition,

$$\mathcal{Q}(e(t)) + \int_0^t \mathcal{H}(\dot p(s))\, ds - \int_0^t \langle \pi_D(s), \dot p(s)\rangle\, ds + \int_\Omega \pi(t) : (Ew(t) - e(t))\, dx$$
$$= \mathcal{Q}(e_0) + \int_\Omega \pi(0) : (Ew(0) - e_0)\, dx + \int_0^t \int_\Omega \sigma(s) : E\dot w(s)\, dx\, ds$$
$$+ \int_0^t \int_\Omega \dot\pi(s) : (Ew(s) - e(s))\, dx\, ds. \tag{2.6}$$

3. The rigid-plastic model

In order to derive the rigid-plastic model from elasto-plasticity, we assume that

$$\mathbb{C}^\varepsilon = \varepsilon^{-1}\mathbb{C}, \quad \text{where } \mathbb{C} \text{ satisfies (2.1) and (2.2),} \tag{3.1}$$

and $\varepsilon \to 0^+$. In addition, we suppose that the boundary data are compatible with rigid plasticity, that is

$$\operatorname{div} w(t) = 0 \text{ in } \Omega, \tag{3.2}$$

and, for simplicity, that the initial data satisfy

$$e_0 = \sigma_0 = 0. \tag{3.3}$$

Theorem 3.1. *Let u^ε, e^ε, p^ε and σ^ε be the solutions given by Theorem 2.2. There exist a subsequence (not relabeled), and functions $u \in AC([0,T]; BD(\Omega))$ and $\sigma \in L^2(0,T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym}))$ such that*

$$u^\varepsilon(t) \rightharpoonup u(t) \text{ weakly* in } BD(\Omega), \text{ for all } t \in [0,T],$$
$$\sigma^\varepsilon \rightharpoonup \sigma \text{ weakly in } L^2(0,T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})).$$

Denoting by $v := \dot u \in L^\infty_{w*}(0,T; BD(\Omega))$, then for a.e. $t \in [0,T]$, we have

$$\begin{cases} -\operatorname{div}\sigma(t) = f(t) \text{ in } \Omega, \\ \sigma(t)\nu = g(t) \text{ on } \Gamma_N, \\ \sigma(t) \in \mathcal{K}, \end{cases} \qquad \begin{cases} \operatorname{div} v(t) = 0 \text{ in } \Omega, \\ (\dot w(t) - v(t))\cdot\nu = 0 \text{ on } \Gamma_D, \\ H(Ev(t)) = [\sigma_D(t) : Ev(t)] \text{ in } \Omega\cup\Gamma_D. \end{cases} \tag{3.4}$$

The remaining of this paper is devoted to the proof of Theorem 3.1.

Remark 3.2. Although $Eu(t)$ is a measure *a priori* defined in Ω, we tacitly extend it by $(w(t) - u(t))\odot\nu$ on Γ_D so that $Eu(t) \in \mathcal{M}(\Omega\cup\Gamma_D; \mathbb{M}^{n\times n}_D)$.

Remark 3.3. In contrast with the framework of classical elasto-plasticity, that of rigid plasticity only involves the velocity field, and not the displacement field itself. As expressed above, time is merely a parameter, although the associated measurability properties of the various fields are obtained through the limit process $\varepsilon \searrow 0$ and would be difficult to obtain directly from the limit formulation.

3.1. A priori estimates.

In this section all constants are independent of ε. We start with an estimate of the stress. Since $\sigma^\varepsilon_D(t) \in K$ in Ω, and K is bounded by (2.3), we first deduce that

$$\sup_{t\in[0,T]} \|\sigma^\varepsilon_D(t)\|_\infty \le C. \tag{3.5}$$

The following result allows us to bound the hydrostatic stress.

Lemma 3.4. *There exists a bounded sequence $(c^\varepsilon)_{\varepsilon>0}$ in $L^2(0,T)$ such that for each $\varepsilon > 0$,*

$$\int_0^T \left\| \frac{\operatorname{tr}\sigma^\varepsilon(t)}{n} + c^\varepsilon(t)\right\|_2^2 dt \le C.$$

Proof. Since the mapping $t \mapsto \sigma^\varepsilon(t)$ belongs to $L^2(0,T; H(\operatorname{div},\Omega))$, there is a sequence $(\sigma^\varepsilon_k)_{k\in\mathbb{N}}$ of $H(\operatorname{div},\Omega)$-valued simple functions such that $\sigma^\varepsilon_k \to \sigma^\varepsilon$ strongly in $L^2(0,T; H(\operatorname{div},\Omega))$ as $k \to +\infty$. For all $k \in \mathbb{N}$ and all $t \in [0,T]$, we have

$$\nabla\left(\frac{\operatorname{tr}\sigma^\varepsilon_k(t)}{n}\right) = \operatorname{div}\sigma^\varepsilon_k(t) - \operatorname{div}(\sigma^\varepsilon_k)_D(t) \text{ in } \Omega$$

which leads to

$$\int_0^T \left\|\nabla\left(\frac{\operatorname{tr}\sigma^\varepsilon_k(t)}{n}\right)\right\|^2_{H^{-1}(\Omega;\mathbb{R}^n)} dt \le \int_0^T \|\operatorname{div}\sigma^\varepsilon_k(t)\|^2_{H^{-1}(\Omega;\mathbb{R}^n)}\, dt + \int_0^T \|(\sigma^\varepsilon_k)_D(t)\|_2^2\, dt.$$

Since $\operatorname{div}\sigma^\varepsilon_k \to \operatorname{div}\sigma^\varepsilon$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^n))$ and $-\operatorname{div}\sigma^\varepsilon = f \in L^2(0,T;L^2(\Omega;\mathbb{R}^n))$, we deduce that the first integral in the right-hand-side of the previous inequality is uniformly bounded with respect to ε and k. The second integral is bounded as well since $(\sigma^\varepsilon_k)_D \to \sigma^\varepsilon_D$ in $L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_D))$, and $(\sigma^\varepsilon_D)_{\varepsilon>0}$ is uniformy bounded in that space in view of (3.5). Consequently, there exists a constant $C>0$ (independent of k and ε) such that

$$\int_0^T \left\|\nabla\left(\frac{\operatorname{tr}\sigma^\varepsilon_k(t)}{n}\right)\right\|^2_{H^{-1}(\Omega;\mathbb{R}^n)} dt \le C.$$

Next, according to [8, Corollary 2.1] (see also [13, Lemma 9] in the case of smooth boundaries), for each $\varepsilon>0$, $k\in\mathbb{N}$ and $t\in[0,T]$, there exists some $c^\varepsilon_k(t)\in\mathbb{R}$ such that

$$\left\|\frac{\operatorname{tr}\sigma^\varepsilon_k(t)}{n} + c^\varepsilon_k(t)\right\|_2 \le C_\Omega \left\|\nabla\left(\frac{\operatorname{tr}\sigma^\varepsilon_k(t)}{n}\right)\right\|_{H^{-1}(\Omega;\mathbb{R}^n)},$$

for some constant $C_\Omega>0$ only depending on Ω. Note that, since the mapping $t\mapsto \operatorname{tr}\sigma^\varepsilon_k(t)$ is a simple $L^2(\Omega))$-valued function, $t\mapsto c^\varepsilon_k(t)$ is a simple real-valued measurable function as well. Additionally,

$$\int_0^T \left\|\frac{\operatorname{tr}\sigma^\varepsilon_k(t)}{n} + c^\varepsilon_k(t)\right\|_2^2 dt \le C, \tag{3.6}$$

where $C>0$ is again independent of k and ε. Setting $\hat\sigma^\varepsilon_k := \sigma^\varepsilon_k + c^\varepsilon_k\, I$ yields

$$\int_0^T \|\hat\sigma^\varepsilon_k(t)\|^2_{H(\operatorname{div},\Omega)}\, dt \le C,$$

and thus,

$$\int_0^T \|\hat\sigma^\varepsilon_k(t)\nu\|^2_{H^{-1/2}(\Gamma_N;\mathbb{R}^n)}\, dt \le C.$$

Using that $\sigma^\varepsilon_k\nu \to \sigma^\varepsilon\nu = g$ in $L^2(0,T;H^{-1/2}(\Gamma_N;\mathbb{R}^n))$ and that $g\in L^2(0,T;L^2(\Gamma_N;\mathbb{R}^n))$, we obtain

$$\begin{aligned}\int_0^T |c^\varepsilon_k(t)|^2\, dt \|\nu\|^2_{H^{-1/2}(\Gamma_N;\mathbb{R}^n)} \\ \le \int_0^T \|\hat\sigma^\varepsilon_k(t)\nu\|^2_{H^{-1/2}(\Gamma_N;\mathbb{R}^n)}\, dt + \int_0^T \|\sigma^\varepsilon_k(t)\nu\|^2_{H^{-1/2}(\Gamma_N;\mathbb{R}^n)}\, dt \le C,\end{aligned} \tag{3.7}$$

for some constant $C>0$, independent of k and ε. Therefore, the sequence $(c^\varepsilon_k)_{k\in\mathbb{N}}$ is bounded in $L^2(0,T)$ and a subsequence converges weakly in that space to some $c^\varepsilon\in L^2(0,T)$. Passing to the lower limit in (3.6) implies that

$$\int_0^T \left\|\frac{\operatorname{tr}\sigma^\varepsilon(t)}{n} + c^\varepsilon(t)\right\|_2^2 dt \le C,$$

while (3.7) shows that $(c^\varepsilon)_{\varepsilon>0}$ is bounded in $L^2(0,T)$. □

As a consequence of the previous result and of (3.5), we deduce that

$$\int_0^T \|\sigma^\varepsilon(t)\|_2^2\, dt \le C. \tag{3.8}$$

Next, according to the energy balance (2.6), [4, Lemma 3.2], assumptions (3.2)–(3.3), and Cauchy-Schwarz inequality, we infer that

$$\begin{aligned}\frac{1}{2}\int_\Omega \mathbb{C}^\varepsilon e^\varepsilon(t):e^\varepsilon(t)\,dx &\le \int_\Omega \pi(t):(e^\varepsilon(t)-Ew(t))\,dx+\int_\Omega \pi(0):Ew(0)\,dx\\ &\quad+\int_0^t\int_\Omega \sigma^\varepsilon_D(s):E\dot w(s)\,dx\,ds+\int_0^t\int_\Omega \dot\pi(s):(Ew(s)-e^\varepsilon(s))\,dx\,ds\\ &\le C\left(\sup_{t\in[0,T]}\|\pi(t)\|_2+\int_0^T\|\dot\pi(s)\|_2\,ds\right)\left(\sup_{t\in[0,T]}\|e^\varepsilon(t)\|_2+\sup_{t\in[0,T]}\|Ew(t)\|_2\right)\\ &\qquad+\sup_{t\in[0,T]}\|\sigma^\varepsilon_D(t)\|_\infty\int_0^T\|E\dot w(s)\|_2\,ds,\end{aligned}$$

which implies, according to the assumption (3.1) on $\mathbb{C}^\varepsilon$ together with Young's inequality, that

$$\sup_{t\in[0,T]}\|e^\varepsilon(t)\|_2\le C\sqrt{\varepsilon}. \tag{3.9}$$

Using again the energy balance (2.6), Cauchy-Schwarz inequality and (3.9), we find that

$$\begin{aligned}\int_0^t \mathcal{H}(\dot p^\varepsilon(s))\,ds-\int_0^t\langle \pi_D(s),\dot p^\varepsilon(s)\rangle\,ds &\le \int_\Omega \pi(t):(e^\varepsilon(t)-Ew(t))\,dx+\int_\Omega\pi(0):Ew(0)\,dx\\ &\quad+\int_0^t\int_\Omega\sigma^\varepsilon_D(s):E\dot w(s)\,dx\,ds+\int_0^t\int_\Omega\dot\pi(s):(Ew(s)-e^\varepsilon(s))\,dx\,ds\le C.\end{aligned}$$

Applying [4, Lemma 3.2] again yields

$$\int_0^T\|\dot p^\varepsilon(s)\|_{\mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^{n\times n}_D)}\,ds\le C, \tag{3.10}$$

and thus

$$\sup_{t\in[0,T]}\|p^\varepsilon(t)\|_{\mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^{n\times n}_D)}\le C. \tag{3.11}$$

For the displacement, Poincaré-Korn's inequality (see [14, Chap. 2, Rmk. 2.5(ii)]) yields

$$\begin{aligned}\|u^\varepsilon(t)\|_{BD(\Omega)} &\le c\left(\int_{\Gamma_D}|u^\varepsilon(t)|\,d\mathcal{H}^{n-1}+\|Eu^\varepsilon(t)\|_{\mathcal{M}(\Omega;\mathbb{M}^{n\times n}_{sym})}\right)\\ &\le c\left(\int_{\Gamma_D}|w(t)|\,d\mathcal{H}^{n-1}+\int_{\Gamma_D}|u^\varepsilon(t)-w(t)|\,d\mathcal{H}^{n-1}+\|Eu^\varepsilon(t)\|_{\mathcal{M}(\Omega;\mathbb{M}^{n\times n}_{sym})}\right)\\ &\le c\left(\|w(t)\|_{L^1(\Gamma_D;\mathbb{R}^n)}+\|p^\varepsilon(t)\|_{\mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^{n\times n}_D)}+\|e^\varepsilon(t)\|_2\right)\le C, \qquad (3.12)\end{aligned}$$

where we have used (3.9) and (3.11) in the last inequality.

3.2. **Convergences.** According to the stress estimate (3.8), there exist a subsequence (not relabeled) and $\sigma\in L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym}))$ such that

$$\sigma^\varepsilon\rightharpoonup\sigma \text{ weakly in } L^2(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_{sym})). \tag{3.13}$$

Consequently, since for all $t\in[0,T]$, we have $-\operatorname{div}\sigma^\varepsilon(t)=f(t)$ in Ω and $\sigma^\varepsilon(t)\nu=g(t)$ on Γ_N, we infer that for a.e. $t\in[0,T]$,

$$-\operatorname{div}\sigma(t)=f(t)\text{ in }\Omega,\quad \sigma(t)\nu=g(t)\text{ on }\Gamma_N.$$

In addition, since $\sigma^\varepsilon_D(t)\in\mathcal{K}$ for all $t\in[0,T]$, then

$$\sigma_D(t)\in\mathcal{K}\text{ for a.e. }t\in[0,T].$$

We then apply Helly's selection principle (see [11, Theorem 3.2]) which ensures, thanks to (3.10), the existence of a further subsequence (independent of time and still not relabeled) such that

$$p^\varepsilon(t) \rightharpoonup p(t) \text{ weakly* in } \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D), \text{ for all } t \in [0,T], \tag{3.14}$$

for some $p \in BV([0,T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D))$.

Next according to (3.9), we have that

$$e^\varepsilon \to 0 \text{ strongly in } L^\infty(0,T; L^2(\Omega; \mathbb{M}^{n\times n}_{sym})). \tag{3.15}$$

Finally, as a consequence of the displacement estimate (3.12), for each $t \in [0,T]$, there exists a further subsequence $(u^{\varepsilon_j}(t))_{j\in\mathbb{N}}$ (now possibly depending on t) such that $u^{\varepsilon_j}(t) \rightharpoonup u(t)$ weakly* in $BD(\Omega)$, for some $u(t) \in BD(\Omega)$. Note that by (3.14)–(3.15), for a.e. $t \in [0,T]$, one has $Eu(t) = p(t)$ in Ω and, by [4, Lemma 2.1], $p(t) = (w(t) - u(t)) \odot \nu$ on Γ_D which shows that $u(t)$ is uniquely determined, and thus that the full sequence

$$u^\varepsilon(t) \rightharpoonup u(t) \text{ weakly* in } BD(\Omega), \text{ for all } t \in [0,T]. \tag{3.16}$$

In particular, since $Eu(t) = p(t) \in \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D)$, we also deduce that

$$\operatorname{div} u(t) = 0 \text{ in } \Omega, \quad (w(t) - u(t)) \cdot \nu = 0 \text{ on } \Gamma_D. \tag{3.17}$$

3.3. **Flow rule.** According to the energy balance (2.6) and the fact that the plastic strain $p^\varepsilon \in AC([0,T]; \mathcal{M}(\overline{\Omega}; \mathbb{M}^{n\times n}_D))$, we can integrate by parts in time, so that for all $t \in [0,T]$,

$$\begin{aligned}
\mathcal{V}_{\mathcal{H}}(p^\varepsilon; [0,t]) &+ \int_\Omega \pi(t) : (Ew(t) - e^\varepsilon(t))\,dx - \langle \pi_D(t), p^\varepsilon(t)\rangle \\
&\le \int_\Omega \pi(0) : Ew(0)\,dx - \langle \pi_D(0), p_0\rangle + \int_0^t \int_\Omega \sigma^\varepsilon_D(s) : E\dot w(s)\,dx\,ds \\
&\qquad + \int_0^t \int_\Omega \dot\pi(s) : (Ew(s) - e^\varepsilon(s))\,dx\,ds - \int_0^t \langle \dot\pi_D(s), p^\varepsilon(s)\rangle\,ds.
\end{aligned}$$

Since by (3.14)–(3.16) $p^\varepsilon(t) \rightharpoonup Eu(t)$ weakly* in $\mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D)$ for a.e. $t \in [0,T]$, Reshetnyak lower semicontinuity theorem, (3.13), (3.15), (3.16) and the definition (2.4) of duality ensures that

$$\begin{aligned}
\mathcal{V}_{\mathcal{H}}(Eu; [0,t]) &+ \int_\Omega \pi(t) : Ew(t)\,dx - \langle \pi_D(t), Eu(t)\rangle \\
&\le \int_\Omega \pi(0) : Ew(0)\,dx - \langle \pi_D(0), Eu_0\rangle + \int_0^t \int_\Omega \sigma_D(s) : E\dot w(s)\,dx\,ds \\
&\qquad + \int_0^t \int_\Omega \dot\pi(s) : Ew(s)\,dx\,ds - \int_0^t \langle \dot\pi_D(s), Eu(s)\rangle\,ds.
\end{aligned} \tag{3.18}$$

We now show the converse inequality. Since $\sigma_D \in L^1(0,T; L^2(\Omega; \mathbb{M}^{n\times n}_D))$, while $u - w \in L^1(0,T; L^{\frac{n}{n-1}}(\Omega; \mathbb{R}^n))$, and $u - w \in L^1(0,T; L^1(\Gamma_N; \mathbb{R}^n))$, [5, Lemma 7.5] implies the existence of a subdivision $0 = t_0 < t_1 < \cdots < t_k = t$ of the time interval $[0,t]$ such that

$$\sum_{i=1}^k \chi_{[t_{i-1},t_i[}(\sigma_D(t_i), u(t_i) - w(t_i), u(t_i) - w(t_i)) \to (\sigma_D, u - w, u - w)$$

and

$$\sum_{i=1}^k \chi_{[t_{i-1},t_i[}(\sigma_D(t_{i-1}), u(t_{i-1}) - w(t_{i-1}), u(t_{i-1}) - w(t_{i-1})) \to (\sigma_D, u - w, u - w)$$

strongly in $L^1(0,T;L^2(\Omega;\mathbb{M}^{n\times n}_D))\times L^1(0,T;L^{\frac{n}{n-1}}(\Omega;\mathbb{R}^n))\times L^1(0,T;L^1(\Gamma_N;\mathbb{R}^n))$, as $\max_{1\le i\le k}(t_i-t_{i-1})\to 0$. According to Proposition 3.9 in [7] and to the fact that Ω is admissible, we infer that for each $1\le i\le k$,

$$\begin{aligned}\mathcal{H}(Eu(t_i)-Eu(t_{i-1}))&\ge\langle\sigma_D(t_i),Eu(t_i)-Eu(t_{i-1})\rangle\\ &=\int_\Omega\sigma_D(t_i):(Ew(t_i)-Ew(t_{i-1}))\,dx+\int_\Omega f(t_i)\cdot(u(t_i)-u(t_{i-1})-w(t_i)+w(t_{i-1}))\,dx\\ &\qquad+\int_{\Gamma_N}g(t_i)\cdot(u(t_i)-u(t_{i-1})-w(t_i)+w(t_{i-1}))\,d\mathcal{H}^{n-1}.\end{aligned}$$

Summing up for $i=1,\dots,k$, and performing discrete integration by parts yields

$$\begin{aligned}\mathcal{V}_\mathcal{H}(Eu,[0,t])&\ge\sum_{i=1}^k\int_{t_{i-1}}^{t_i}\int_\Omega\sigma_D(t_i):E\dot w(s)\,dx\,ds\\ &-\sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}}\int_\Omega\dot f(s)\cdot(u(t_i)-w(t_i))\,dx\,ds-\sum_{i=1}^{k-1}\int_{t_i}^{t_{i+1}}\int_{\Gamma_N}\dot g(s)\cdot(u(t_i)-w(t_i))\,d\mathcal{H}^{n-1}\,ds\\ &+\int_\Omega f(t)\cdot(u(t)-w(t))\,dx+\int_{\Gamma_N}g(t)\cdot(u(t)-w(t))\,d\mathcal{H}^{n-1}\\ &-\int_\Omega f(t_1)\cdot(u_0-w(0))\,dx-\int_{\Gamma_N}g(t_1)\cdot(u_0-w(0))\,d\mathcal{H}^{n-1}.\end{aligned}$$

Passing to the limit as $\max_{1\le i\le k}(t_i-t_{i-1})\to 0$, and invoking the dominated convergence theorem yields

$$\begin{aligned}\mathcal{V}_\mathcal{H}(Eu,[0,t])&\ge\int_0^t\int_\Omega\sigma_D(s):E\dot w(s)\,dx\,ds\\ &-\int_0^t\int_\Omega\dot f(s)\cdot(u(s)-w(s))\,dx\,ds-\int_0^t\int_{\Gamma_N}\dot g(s)\cdot(u(s)-w(s))\,d\mathcal{H}^{n-1}\,ds\\ &+\int_\Omega f(t)\cdot(u(t)-w(t))\,dx+\int_{\Gamma_N}g(t)\cdot(u(t)-w(t))\,d\mathcal{H}^{n-1}\\ &-\int_\Omega f(0)\cdot(u_0-w(0))\,dx-\int_{\Gamma_N}g(0)\cdot(u_0-w(0))\,d\mathcal{H}^{n-1},\end{aligned}$$

and using the definition (2.4) of duality

$$\begin{aligned}\mathcal{V}_\mathcal{H}(Eu;[0,t])&+\int_\Omega\pi(t):Ew(t)\,dx-\langle\pi_D(t),Eu(t)\rangle\\ &\ge\int_\Omega\pi(0):Ew(0)\,dx-\langle\pi_D(0),Eu_0\rangle+\int_0^t\int_\Omega\sigma_D(s):E\dot w(s)\,dx\,ds\\ &\qquad+\int_0^t\int_\Omega\dot\pi(s):Ew(s)\,dx\,ds-\int_0^t\langle\dot\pi_D(s),Eu(s)\rangle\,ds.\end{aligned}$$

Thus, combining with (3.18) leads to the equality in the previous inequality, or still, integrating by parts with respect to time

$$\begin{aligned}\mathcal{V}_\mathcal{H}(Eu;[0,t])&=\langle\pi_D(t),Eu(t)\rangle-\langle\pi_D(0),Eu_0\rangle\\ &+\int_0^t\int_\Omega(\sigma_D(s)-\pi_D(s)):E\dot w(s)\,dx\,ds-\int_0^t\langle\dot\pi_D(s),Eu(s)\rangle\,ds.\end{aligned}\tag{3.19}$$

According to [4, Lemma 3.2], for all $0 \le t_1 \le t_2 \le T$,

$$\begin{aligned} c\|Eu(t_2) - Eu(t_1)\|_{\mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^{n\times n}_D)} &\le \mathcal{H}(Eu(t_2) - Eu(t_1)) - \langle \pi_D(t_2), Eu(t_2) - Eu(t_1)\rangle \\ &\le \mathcal{V}_{\mathcal{H}}(Eu, [t_1, t_2]) - \langle \pi_D(t_2), Eu(t_2) - Eu(t_1)\rangle. \end{aligned}$$

In view of (3.19), we get that

$$\begin{aligned} c\|Eu(t_2) - Eu(t_1)\|_{\mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^{n\times n}_D)} &\le \langle \pi_D(t_2) - \pi_D(t_1), Eu(t_1)\rangle \\ &+ \int_{t_1}^{t_2}\int_\Omega (\sigma_D(s) - \pi_D(s)) : E\dot w(s)\, dx\, ds - \int_{t_1}^{t_2} \langle \dot\pi_D(s), Eu(s)\rangle\, ds. \end{aligned}$$

Since $Eu = p$ and $p \in BV([0,T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D))$, we get that $Eu \in L^\infty_{w*}(0, T; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D))$, and thus

$$\begin{aligned} c\|Eu(t_2) - Eu(t_1)\|_{\mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^{n\times n}_D)} \le \int_{t_1}^{t_2} \Big\{ &\|Eu(t_1)\|_{\mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^{n\times n}_D)} \|\dot\pi_D(s)\|_\infty \\ &+ (\|\pi_D(s)\|_2 + \|\sigma_D(s)\|_2)\|E\dot w(s)\|_2 + \|\dot\pi_D(s)\|_\infty \|Eu(s)\|_{\mathcal{M}(\Omega\cup\Gamma_D;\mathbb{M}^{n\times n}_D)} \Big\}\, ds. \end{aligned}$$

The integrand being sommable, it ensures that the strain $Eu \in AC([0,T]; \mathcal{M}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D))$, and by the Poincaré-Korn inequality that $u \in AC([0,T]; BD(\Omega))$. Thus, integrating by part with respect to time and space in the energy equality (3.19),

$$\begin{aligned} \int_0^t \mathcal{H}(E\dot u(s))\, ds = \mathcal{V}_{\mathcal{H}}(Eu, [0,t]) &= \int_0^t \int_\Omega \sigma_D(s) : E\dot w(s)\, dx\, ds \\ &+ \int_0^t\int_\Omega f(s)\cdot(\dot u(s) - \dot w(s))\, dx\, ds + \int_0^t \int_{\Gamma_N} g(s)\cdot(\dot u(s) - \dot w(s))\, d\mathcal{H}^{n-1}\, ds, \end{aligned}$$

and deriving this equality with respect to time yields, thanks to (2.4), for a.e. $t \in [0,T]$,

$$\mathcal{H}(E\dot u(t)) = \langle \sigma_D(t), E\dot u(t)\rangle.$$

Since, by [7, Proposition 3.9], $H(E\dot u(t)) \ge [\sigma_D(t) : E\dot u(t)]$ in $\mathcal{M}(\Omega \cup \Gamma_D)$, we finally deduce that $H(E\dot u(t)) = [\sigma_D(t) : E\dot u(t)]$ in $\mathcal{M}(\Omega \cup \Gamma_D)$.

Denoting by $v = \dot u$ the velocity, we proved that $v \in L^\infty_{w*}(0,T; BD(\Omega))$, and recalling (3.17), we have for a.e. $t \in [0,T]$,

$$\operatorname{div} v(t) = 0 \text{ in } \Omega, \quad (\dot w(t) - v(t))\cdot\nu = 0 \text{ on } \Gamma_D,$$

and

$$H(Ev(t)) = [\sigma_D(t) : Ev(t)] \text{ in } \Omega \cup \Gamma_D.$$

4. Uniqueness and regularity issues for the stress with a Von Mises yield criterion

We now specialize to the case where $K := \{\tau_D \in \mathbb{M}^{n\times n}_D : |\tau_D| \le 1\}$. In such a setting, it is known (see [3]) when elasto-plasticity is considered the stress field is unique and belongs to $H^1_{\rm loc}(\Omega; \mathbb{M}^{n\times n}_{sym})$. These properties fail in the case of rigid-plasticity as demonstrated below.

Example 4.1. Let us consider a two-dimensional body occupying the square $\Omega = (0,1)^2$ in its reference configuration (the generalization to the n-dimensional case is obvious). We also assume that the boundary conditions are of pure Dirichlet type with a rigid body motion $\dot w(x) = Ax + b$ (where $A \in \mathbb{M}^{n\times n}$ is such that $A^T = -A^T$, and $b \in \mathbb{R}^n$) as boundary datum.

Then, defining $v(x) = Ax+b$ for all $x \in \Omega$ ensures that $Ev = 0$ in Ω. In particular, all equations on v are satisfied. Now define the stress as

$$\sigma(x) = \begin{pmatrix} f(x_2) & c \\ c & g(x_1) \end{pmatrix}$$

where $c \in \mathbb{R}$, f, $g \in L^\infty(0,1)$ so that $\operatorname{div}\sigma = 0$ in Ω. In particular

$$\sigma_D(x) = \begin{pmatrix} \frac{f(x_2)-g(x_1)}{2} & c \\ c & \frac{g(x_1)-f(x_2)}{2} \end{pmatrix}$$

and $|\sigma_D(x)|^2 \leq 2c^2+|f(x_2)|^2+|g(x_1)|^2$ for a.e. $x \in \Omega$. Assuming that $\sqrt{2c^2 + \|f\|^2_\infty + \|g\|^2_\infty} < 1/2$, we deduce that the one parameter family $\sigma^\lambda := \lambda\sigma$ still satisfies $\operatorname{div}\sigma^\lambda = 0$ and $|\sigma^\lambda_D| < 1$ in Ω provided that $|\lambda| \leq 2$.

In general, a certain amount of uniqueness holds true as shown below. It uses a notion of precise representative for the stress field first introduced in [1] (see also [4]).

Proposition 4.2. *Let (σ^1, v^1), $(\sigma^2, v^2) \in L^2(\Omega; \mathbb{M}^{n\times n}_{sym}) \times BD(\Omega)$ be two solutions of the rigid-plastic model (3.4) at a given time $t = t_0$. Then,*

- *There exist two $|Ev^1|$-measurable functions $\hat\sigma^1_D$ and $\hat\sigma^2_D \in L^\infty_{|Ev^1|}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D)$ such that $\hat\sigma^1_D = \sigma^1$ and $\hat\sigma^2_D = \sigma^2_D$ $\mathcal{L}^n$-a.e. in $\Omega \cup \Gamma_D$, and*
$$\hat\sigma^1_D = \hat\sigma^2_D \quad |Ev^1|\text{-a.e. in } \Omega \cup \Gamma_D;$$
- *There exist two $|Ev^2|$-measurable functions $\tilde\sigma^1_D$ and $\tilde\sigma^2_D \in L^\infty_{|Ev^2|}(\Omega \cup \Gamma_D; \mathbb{M}^{n\times n}_D)$ such that $\tilde\sigma^1_D = \sigma^1$ and $\tilde\sigma^2_D = \sigma^2_D$ $\mathcal{L}^n$-a.e. in $\Omega \cup \Gamma_D$, and*
$$\tilde\sigma^1_D = \tilde\sigma^2_D \quad |Ev^2|\text{-a.e. in } \Omega \cup \Gamma_D.$$

Proof. Since (σ^1, v^1), (σ^2, v^2) are two solutions of the rigid-plastic model (3.4), the following inequalities in $\mathcal{M}(\Omega \cup \Gamma_D)$ hold true

$$[\sigma^1_D : Ev^1] = |Ev^1| \geq [\sigma^2_D : Ev^1], \quad [\sigma^2_D : Ev^2] = |Ev^2| \geq [\sigma^1_D : Ev^2].$$

As a consequence,

$$[(\sigma^1_D - \sigma^2_D) : Ev^1] \geq 0, \quad [(\sigma^2_D - \sigma^1_D) : Ev^2] \geq 0,$$

and thus,

$$[(\sigma^1_D - \sigma^2_D) : (Ev^1 - Ev^2)] \geq 0.$$

In addition, by definition (2.4) of the duality pairing, the total mass of the measure on the left-hand side of the previous inequality is given by

$$\langle \sigma^1_D - \sigma^2_D, Ev^1 - Ev^2 \rangle = 0.$$

It thus follows that

$$[(\sigma^1_D - \sigma^2_D) : Ev^1] = 0, \quad [(\sigma^2_D - \sigma^1_D) : Ev^2] = 0,$$

or still that

$$[\sigma^1_D : Ev^1] = |Ev^1| = [\sigma^2_D : Ev^1], \quad [\sigma^2_D : Ev^2] = |Ev^2| = [\sigma^1_D : Ev^2]. \tag{4.1}$$

Arguing as in [4], since $\mathcal{L}^n$ and $E^s v^1$ are mutually singular Borel measures, it is possible to find two disjoint Borel sets A and $B \subset \Omega \cup \Gamma_D$ such that $A \cup B = \Omega \cup \Gamma_D$, and $\mathcal{L}^n(B) = |E^s v^1|(A) = 0$. Then, defining (for $i = 1, 2$)

$$\hat\sigma^i_D := \begin{cases} \sigma^i_D & \mathcal{L}^n\text{-a.e. in } A, \\ \dfrac{dEv^1}{d|Ev^1|} & |E^s v^1|\text{-a.e. in } B, \end{cases}$$

it follows that $\hat\sigma^1_D$ and $\hat\sigma^2_D \in L^\infty_{|Ev^1|}(\Omega\cup\Gamma_D;\mathbb{M}^{n\times n}_D)$, and

$$\hat\sigma^1_D : \frac{dEv^1}{d|Ev^1|}|Ev^1| = [\sigma^1_D : Ev^1] = |Ev^1| = [\sigma^2_D : Ev^1] = \hat\sigma^2_D : \frac{dEv^1}{d|Ev^1|}|Ev^1|.$$

By definition, we have that $\hat\sigma^1_D = \hat\sigma^2_D$ $|E^s v^1|$-a.e. in $\Omega\cup\Gamma_D$. In addition, taking the absolutely continuous part in (4.1) yields (see [4, 7]),

$$\sigma^1_D : E^a v^1 = [\sigma^1_D : Ev^1]^a = |E^a v^1| = [\sigma^2_D : Ev^1]^a = \sigma^2_D : E^a v^1.$$

Thus $\sigma^1_D = \sigma^2_D$ $\mathcal{L}^n$-a.e. in $\{|E^a v^1| > 0\}$ and finally $\hat\sigma^1_D = \hat\sigma^2_D$ $|Ev^1|$-a.e. in $\Omega\cup\Gamma_D$ as requested. □

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