Quasilinear problems involving a perturbation with quadratic growth in the gradient and a noncoercive zeroth order term

Boussad Hamour, François Murat

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Boussad Hamour [1] & François Murat [2]

Abstract

In this paper we consider the problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\mathrm{div}\,(A(x)Du) = H(x,u,Du) + f(x) + a_0(x)\,u \quad \text{in} \quad \mathcal{D}'(\Omega), \end{cases}$$

where Ω is an open bounded set of $\mathbb{R}^N$, $N \geq 3$, $A(x)$ is a coercive matrix with coefficients in $L^\infty(\Omega)$, $H(x,s,\xi)$ is a Carathéodory function which satisfies for some $\gamma > 0$

$$-c_0\,A(x)\,\xi\xi \leq H(x,s,\xi)\,\mathrm{sign}(s) \leq \gamma\,A(x)\,\xi\xi \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N,$$

f belongs to $L^{N/2}(\Omega)$ and $a_0 \geq 0$ to $L^q(\Omega)$, $q > N/2$. For f and a_0 sufficiently small, we prove the existence of at least one solution u of this problem which is moreover such that $e^{\delta_0|u|} - 1$ belongs to $H_0^1(\Omega)$ for some $\delta_0 \geq \gamma$ and satisfies an a priori estimate.

Résumé

Dans cet article nous étudions le problème

$$\begin{cases} u \in H_0^1(\Omega), \\ -\mathrm{div}\,(A(x)Du) = H(x,u,Du) + f(x) + a_0(x)\,u \quad \text{dans} \quad \mathcal{D}'(\Omega), \end{cases}$$

où Ω est un ouvert borné de $\mathbb{R}^N$, $N \geq 3$, $A(x)$ est une matrice coercive à coefficients $L^\infty(\Omega)$, $H(x,s,\xi)$ est une fonction de Carathédory qui satisfait pour un certain $\gamma > 0$

$$-c_0\,A(x)\,\xi\xi \leq H(x,s,\xi)\,\mathrm{sign}(s) \leq \gamma\,A(x)\,\xi\xi \quad \text{p.p. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N,$$

f appartient à $L^{N/2}(\Omega)$ et $a_0 \geq 0$ à $L^q(\Omega)$, $q > N/2$. Pour f et a_0 suffisamment petits, nous démontrons qu'il existe au moins une solution u de ce problème qui est de plus telle que $e^{\delta_0|u|} - 1$ appartient à $H_0^1(\Omega)$ pour un certain $\delta_0 \geq \gamma$ et satisfait une estimation a priori.

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1 Introduction

In this paper, we consider the quasilinear problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\mathrm{div}\,(A(x)Du) = H(x,u,Du) + f(x) + a_0(x)\,u \quad \text{in} \quad \mathcal{D}'(\Omega), \end{cases} \tag{1.1}$$

where Ω is a bounded open set of $\mathbb{R}^{\mathbb{N}}$, $N \geq 3$, where A is a coercive matrix with bounded measurable coefficients, where $H(x,s,\xi)$ is a Carathéodory function wich has quadratic growth in ξ, and more precisely which satisfies for some $\gamma > 0$ and $c_0 \geq 0$

$$-c_0\,A(x)\,\xi\xi \leq H(x,s,\xi)\,\mathrm{sign}(s) \leq \gamma\,A(x)\,\xi\xi, \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N, \tag{1.2}$$

where $f \in L^{N/2}(\Omega)$, $f \not\equiv 0$, and where $a_0 \in L^q(\Omega)$, $q > \frac{N}{2}$, with

$$a_0 \geq 0, \quad a_0 \not\equiv 0. \tag{1.3}$$

When f and a_0 are sufficiently small (and more precisely when f and a_0 satisfy the two smallness conditions (2.14) and (2.15), we prove in the present paper that problem (1.1) has a least one solution, which is moreover such that

$$e^{\delta_0|u|} - 1 \in H_0^1(\Omega), \tag{1.4}$$

with

$$\left\| \frac{e^{\delta_0|u|} - 1}{\delta_0} \right\|_{H_0^1(\Omega)} \leq Z_{\delta_0}, \tag{1.5}$$

where $\delta_0 \geq \gamma$ and Z_{δ_0} are two constants which depend only on the data of the problem (see (6.16), (6.17), (6.18) for the definitions of δ_0 and Z_{δ_0}).

The main originality of our result is the fact that we assume that a_0 satisfies (1.3), namely that a_0 is a nonnegative function.

Let us begin with some review of the literature.

Problem (1.1) has been studied in many papers in the case where $a_0 \leq 0$. Among these papers is a series of papers [8], [9], [10] and [11] by L. Boccardo, F. Murat and J.-P. Puel (see also the paper [23] by J.-M. Rokotoson), which are concerned with the case where

$$a_0(x) \leq -\alpha_0 < 0. \tag{1.6}$$

In these papers (which also consider nonlinear monotone operators and not only the linear operator $-\mathrm{div}\,(A(x)Du)$, the authors prove that when a_0 satisfies (1.6) and when f belongs to $L^q(\Omega)$, $q > \frac{N}{2}$, then there exists at least one solution of (1.1) which moreover belongs to $L^\infty(\Omega)$ and which satisfies some a priori estimates. The uniqueness of such a solution has been proved, under some further structure assumptions, by G. Barles and F. Murat in [4], by G. Barles, A.-P. Blanc, C. Georgelin and M. Kobylanski in [3] and by G. Barles and A. Porretta in [5].

The case where

$$a_0 = 0 \tag{1.7}$$

was considered, among others, by A. Alvino, P.-L. Lions and G. Trombetti in [1], by C. Maderna, C. Pagani and S. Salsa in [21], by V. Ferone and M.-R. Posteraro in [16], and by N. Grenon-Isselkou and J. Mossino in [17]. In these papers (which also consider nonlinear monotone operators), the authors prove that when a_0 satisfies (1.7) and when f belongs to $L^q(\Omega)$, $q > \frac{N}{2}$, with $\|f\|_{L^q(\Omega)}$ sufficiently small, then there exists at least one solution of (1.1) which moreover belongs to $L^\infty(\Omega)$ and which satisfies some a priori estimates.

The case where a_0 satisfies (1.7) but where f only belongs to $L^{N/2}(\Omega)$ for $N \geq 3$ (and no more to $L^q(\Omega)$ with $q > \frac{N}{2}$) was considered by V. Ferone and F. Murat in [13] (and in [14] in the nonlinear monotone case). These authors proved that when $\|f\|_{L^{N/2}(\Omega)}$ is sufficiently small, there exists at least one solution of (1.1) which is moreover such that $e^{\delta|u|} - 1 \in H_0^1(\Omega)$ for some $\delta > \gamma$, and that such a solution satisfies an a priori estimate. Similar results were obtained in the case where $f \in L^{N/2}(\Omega)$ by A. Dall'Aglio, D. Giachetti and J.-P. Puel in [12] for possibly unbounded domains when a_0 satisfies (1.6); in this case no smallness condition is required on f. Finally, in [15], V. Ferone and F. Murat considered (also in the case of nonlinear monotone operators) the case where a_0 satisfies $a_0 \leq 0$ and where f belongs to the Lorentz space $L^{N/2,\infty}(\Omega)$; in this case two smallness conditions should be fulfilled.

To finish with the case where a_0 satisfies $a_0 \leq 0$, let us quote the paper [22] by A. Porretta, where the author studies the asymptotic behaviour of the solution u of (1.1) when a_0 is a strictly positive constant which tends to zero, and proves that an ergodic constant appears at the limit $a_0 = 0$. Let us also mention the case where the nonlinearity $H(x, s, \xi)$ has the "good sign property", namely satisfies

$$-H(x, s, \xi) \operatorname{sign}(s) \geq 0. \tag{1.8}$$

In this case, when $a_0 \leq 0$ and when f belongs to $H^{-1}(\Omega)$, L. Boccardo, F. Murat and J.-P. Puel in [7] and A. Bensoussan, L. Boccardo and F. Murat in [6] proved the existence of at least one solution of (1.1) which belongs to $H_0^1(\Omega)$.

In contrast with the cases (1.6) and (1.7), the present paper is concerned with the case (1.3) where $a_0 \geq 0$ and $a_0 \neq 0$.

In this setting we are only aware of four papers, which are recent. In [20], L. Jeanjean and B. Sirakov proved the existence of at least two solutions of (1.1) (which moreover belong to $L^\infty(\Omega)$) when $A(x) = Id$, $H(x, s, \xi) = \mu|\xi|^2$, $\mu > 0$, $f \in L^q(\Omega)$, $q > \frac{N}{2}$, $f \geq 0$, and $a_0 \in L^q(\Omega)$, $a_0 \geq 0$, $a_0 \neq 0$, with $\|f\|_{L^q(\Omega)}$ and $\|a_0\|_{L^q(\Omega)}$ sufficiently small. In [2], D. Arcoya, C. De Coster, L. Jeanjean and K. Tanaka proved the existence of a continuum (u, λ) of solutions (with u which moreover belongs to $L^\infty(\Omega)$) when $A(x) = Id$, $H(x, s, \xi) = \mu(x)\,|\xi|^2$, with $\mu \in L^\infty(\Omega)$, $\mu(x) \geq \mu > 0$, $f \in L^q(\Omega)$, $q > \frac{N}{2}$, $f \geq 0$, $f \neq 0$ and $a_0(x) = \lambda a_0^\star(x)$ with $a_0^\star \in L^q(\Omega)$, $a_0^\star \geq 0$ and $a_0^\star \neq 0$; moreover, under some further conditions on f, these authors proved that this continuum is defined for $\lambda \in]-\infty\,, \lambda_0]$ with $\lambda_0 > 0$, and that there are at least two nonnegative solutions of (1.1) when $\lambda > 0$ is sufficiently small. In [24], in a similar setting, assuming only that $\mu(x) \geq 0$ but that the supports of μ and of $a_0^\star$ have a nonempty intersection and that $N \leq 5$, P. Souplet proved the existence of a continuum (u, λ) of solutions, and that there are at least two nonnegative solutions of (1.1) when $\lambda > 0$ is sufficiently small. In [19], L. Jeanjean and H. Ramos Quoirin proved the existence of two positive solutions (which moreover belong to $L^\infty(\Omega)$) when $A(x) = Id$,

$H(x,s,\xi)=\mu|\xi|^2$, $\mu>0$, $f\in L^q(\Omega)$, $q>\frac{N}{2}$, $f\geq 0$, $f\neq 0$, and $a_0\in C(\overline{\Omega})$ which can change sign with $a_0^+\neq 0$ and which satisfies the so called "thick zero set condition", when the first eigenvalue of the operator $-\Delta-(a_0+\mu f)$ in $H_0^1(\Omega)$ is positive.

With respect to the results obtained in the four latest papers, we prove in the present paper, as said above, the existence of (only) one solution of (1.1) in the case (1.3) ($a_0\geq 0$) when a_0 and f satisfy the two smallness conditions (2.14) and (2.15), but our result is obtained in the general case of a nonlinearity $H(x,s,\xi)$ which satisfies only (1.2), with $f\in L^{N/2}(\Omega)$ and with $a_0\in L^q(\Omega)$, $q>\frac{N}{2}$. Moreover, the method which allows us to prove this result continues *mutatis mutandis* to work in the nonlinear monotone case where the linear operator $-\mathrm{div}\,(A(x)Du)$ is replaced by a Leray-Lions operator $-\mathrm{div}\,(a(x,u,Du))$ working in $W_0^{1,p}(\Omega)$, for some $1<p<N$, and where the quasilinear term $H(x,u,Du)$ has p-growth in $|Du|$.

Let us now describe the contents of the present paper.

The precise statement of our result is given in Section 2 (Theorem 2.1), as well as the precise assumptions under which we are able to prove it. These conditions in particular include the two smallness conditions (2.14) and (2.15).

Our method for proving Theorem 2.1 is based on an equivalence result (see Theorem 3.5) that we state in Section 3 once we have introduced the functions $K_\delta(x,s,\zeta)$ and $g_\delta(s)$ (see (3.6) and (3.7)) and made some technical remarks on them. This result is very close to the equivalence result given in the paper [14] by V. Ferone and F. Murat.

This equivalence result implies that in order to prove the existence of a solution u of (1.1) which satisfies (1.4) and (1.5), it is equivalent to prove (see Theorem 3.8) the existence of a function w defined by (3.31), i.e.

$$w=\frac{1}{\delta_0}(e^{\delta_0|u|}-1)\,\mathrm{sign}(u), \tag{1.9}$$

which satisfies (3.33), i.e.

$$\begin{cases} w\in H_0^1(\Omega), \\ -\mathrm{div}(A(x)Dw)+K_{\delta_0}(x,w,Dw)\,\mathrm{sign}(w) \;= \\ =(1+\delta_0|w|)\,f(x)+a_0(x)\,w+a_0(x)\,g_{\delta_0}(w)\,\mathrm{sign}(w) \quad \text{in} \quad \mathcal{D}'(\Omega), \end{cases} \tag{1.10}$$

and the estimate (3.34), i.e.

$$\|w\|_{H_0^1(\Omega)}\leq Z_{\delta_0}, \tag{1.11}$$

(which is nothing but (1.5)).

Our goal thus becomes to prove Theorem 3.8, namely to prove the existence of a solution w which satisfies (1.10) and (1.11).

Problem (1.10) is very similar to problem (1.1), since it involves a term $-K_{\delta_0}(x,w,Dw)\,\mathrm{sign}(w)$ which has quadratic growth in Dw, as well as a zeroth order term $\delta_0\,|w|f(x)+$ $+\,a_0(x)\,w+a_0(x)\,g_{\delta_0}(w)\,\mathrm{sign}(w)$. But this problem is also very different from (1.1), since the term $-K_{\delta_0}(x,w,Dw)\,\mathrm{sign}(w)$ with quadratic growth has now the "good sign property" (see (1.8)),

since $K_{\delta_0}(x,s,\xi)$ satisfies
$$K_{\delta_0}(x,s,\xi) \geq 0,$$
while the zeroth order term is now no more a linear but a semilinear term with $|s|^{1+\theta}$ growth (see (6.20)) due to presence of the term $a_0(x)\, g_{\delta_0}(w)\,\mathrm{sign}(w)$.

We will prove Theorem 3.8 essentially by applying Schauder's fixed point theorem. But there are some difficulties to do it directly, since the term with quadratic growth $K_{\delta_0}(x,w,Dw)\,\mathrm{sign}(w)$ only belongs to $L^1(\Omega)$ in general. We therefore begin by defining an approximate problem (see (4.1)) where $K_\delta(x,w,Dw)$ is remplaced by its truncation at height k, namely $T_k(K_\delta(x,w,Dw))$, and we prove (see Theorem 4.1) that if f and a_0 satisfy the two smallness conditions (2.14) and (2.15), this approximate problem has at least one solution w_k which satisfies the a priori estimate

$$\|w_k\|_{H_0^1(\Omega)} \leq Z_{\delta_0}. \tag{1.12}$$

This result, which is proved in Section 4, is obtained by applying Schauder's fixed point theorem in a classical way.

We then pass to the limit as k tends to infinity and we prove in Section 5 that (for a subsequence of k) w_k tend to some $w^\star$ strongly in $H_0^1(\Omega)$ (see (5.2)) and that this $w^\star$ is a solution of (1.10) which also satisfies (1.11) (see End of the proof of Theorem 3.8).

This completes the proof of Theorem 3.8, and therefore proves Theorem 2.1, as announced.

This proof follows along the lines of the proof used by V. Ferone and F. Murat in [13] in the case where $a_0 = 0$. As mentionned above, this method can be applied *mutatis mutandis* to the nonlinear case where the linear operator $-\mathrm{div}(A(x)Du)$ is replaced by a Leray-Lions operator $-\mathrm{div}(a(x,u,Du))$ working in $W_0^1(\Omega)$ for some $1 < p < N$, and where the quasilinear term $H(x,u,Du)$ has p-growth in $|Du|$, as it was done in [14] by V. Ferone and F. Murat in this nonlinear setting when $a_0 = 0$. This will be the goal of our next paper [18].

2 Main result

In this paper we consider the following quasilinear problem

$$\begin{cases} u \in H_0^1(\Omega), \\ -\mathrm{div}\,(A(x)Du) = H(x,u,Du) + a_0(x)\,u + f(x) \quad \text{in} \quad \mathcal{D}'(\Omega), \end{cases} \tag{2.1}$$

where the set Ω satisfies (note that no regularity is assumed on the boundary of Ω)

$$\Omega \quad \text{is a bounded open subset of } \mathbb{R}^N,\ N \geq 3, \tag{2.2}$$

where the matrix A is a coercive matrix with bounded measurable coefficients, i.e.

$$\begin{cases} A \in (L^\infty(\Omega))^{N\times N}, \\ \exists \alpha > 0, \quad A(x)\,\xi\xi \geq \alpha|\xi|^2 \quad \text{a.e. } x \in \Omega, \quad \forall \xi \in \mathbb{R}^N, \end{cases} \tag{2.3}$$

where the function $H(x,s,\xi)$ is a Carathéodory function with quadratic growth in ξ, and more precisely satisfies

$$\begin{cases} H : \Omega \times \mathbb{R} \times \mathbb{R}^{\mathbb{N}} \to \mathbb{R} \text{ is a Carathéodory function such that} \\ -c_0\, A(x)\,\xi\xi \le H(x,s,\xi)\,\mathrm{sign}(s) \le \gamma\, A(x)\,\xi\xi, \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N, \\ \text{where } \gamma > 0 \text{ and } c_0 \ge 0, \end{cases} \tag{2.4}$$

where sign : $\mathbb{R} \to \mathbb{R}$ denotes the function defined by

$$\mathrm{sign}(s) = \begin{cases} +1 & \text{if } \; s > 0, \\ \;\;0 & \text{if } \; s = 0, \\ -1 & \text{if } \; s < 0, \end{cases} \tag{2.5}$$

where the coefficient a_0 satisfies

$$a_0 \in L^q(\Omega) \quad \text{for some} \quad q > \frac{N}{2}, \quad a_0 \ge 0, \quad a_0 \ne 0, \tag{2.6}$$

as well as the technical assumption (note that, since $\frac{N}{2} < \frac{2N}{6-N}$ when $3 \le N \le 6$ and since Ω is bounded, this assumption can be made without loss of generality once hypothesis (2.6) is assumed)

$$\frac{N}{2} < q < \frac{2N}{6-N} \quad \text{when} \quad 3 \le N \le 6, \tag{2.7}$$

and finally where

$$f \in L^{N/2}(\Omega), \quad f \ne 0. \tag{2.8}$$

Since $N \ge 3$, let 2^* be the Sobolev's exponent defined by

$$\frac{1}{2^*} = \frac{1}{2} - \frac{1}{N},$$

and let C_N be the Sobolev's constant defined as the best constant such that

$$\|\varphi\|_{2^*} \le C_N \|D\varphi\|_2, \quad \forall \varphi \in H_0^1(\Omega). \tag{2.9}$$

We claim that in view of (2.6) and (2.7), one has

$$0 < \frac{2^*}{q'} - 2 < 1, \tag{2.10}$$

where q' the Hölder's conjugate of the exponent q, i.e.

$$\frac{1}{q'} + \frac{1}{q} = 1;$$

indeed easy computations show that

$$0 < \frac{2^*}{q'} - 2 \iff q > \frac{N}{2},$$
$$\frac{2^*}{q'} - 2 < 1 \iff \frac{1}{q} > \frac{6-N}{2N},$$

where the latest inequality is satisfied when $N > 6$ and is equivalent to $q < \dfrac{2N}{6-N}$ when $N \leq 6$ (see (2.7)).

We now define the number θ by

$$\theta = \frac{2^*}{q'} - 2. \tag{2.11}$$

In view of (2.10) we have

$$0 < \theta < 1. \tag{2.12}$$

Since Ω is bounded, we equip the space $H_0^1(\Omega)$ with the norm

$$\|u\|_{H_0^1(\Omega)} = \|Du\|_{L^2(\Omega)^N}. \tag{2.13}$$

We finally assume that f and a_0 are sufficiently small (see Remark 2.2), and more precisely that

$$\alpha - C_N^2 \|a_0\|_{N/2} - \gamma C_N^2 \|f\|_{N/2} > 0, \tag{2.14}$$
$$\|f\|_{H^{-1}(\Omega)} \leq \frac{\theta}{1+\theta} \frac{(\alpha - C_N^2 \|a_0\|_{N/2} - \gamma C_N^2 \|f\|_{N/2})^{(1+\theta)/\theta}}{((1+\theta) G C_N^{2+\theta} \|a_0\|_q)^{1/\theta}}, \tag{2.15}$$

where the constant G is defined by (6.14).

Observe that in place of (2.14) we could as well have assumed that

$$\alpha - C_N^2 \|a_0\|_{N/2} - \gamma C_N^2 \|f\|_{N/2} \geq 0,$$

but that when equality takes places in the latest inequality, inequality (2.15) implies that $f = 0$, and then $u = 0$ is a solution of (2.1), so that the result of Theorem 2.1 becomes trivial.

Our main result is the following Theorem.

Theorem 2.1 *Assume that* (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) *hold true. Assume moreover that the two smallness conditions* (2.14) *and* (2.15) *hold true.*

Then there exist a constant δ_0 with $\delta_0 \geq \gamma$, and a constant Z_{δ_0}, which are defined in Lemma 6.2 (see (6.16), (6.17)*) and* (6.18)*), such that there exists at least one solution u of* (2.1) *which further satisfies*

$$(e^{\delta_0 |u|} - 1) \in H_0^1(\Omega), \tag{2.16}$$

with

$$\|e^{\delta_0 |u|} Du\|_{L^2(\Omega)^N} = \left\| \frac{e^{\delta_0 |u|} - 1}{\delta_0} \right\|_{H_0^1(\Omega)} \leq Z_{\delta_0}. \tag{2.17}$$

Our proof of Theorem 2.1 is based on an equivalence result (Theorem 3.5) which will be stated and proved in Section 3. This equivalence Theorem will allows us to replace proving Theorem 2.1 by proving Theorem 3.8 which is equivalent to Theorem 2.1.

Remark 2.2 In this Remark, we consider that the open set Ω, the matrix A and the function H are fixed (and therefore in particular that the constants $\alpha > 0$ and $\gamma > 0$ are fixed), and we consider the functions a_0 and f as parameters.

Our first set of assumptions on these parameters (assumptions (2.6) and (2.7)) is that a_0 belongs to $L^q(\Omega)$ with $q > \frac{N}{2}$ (and that $q < \frac{2N}{N-6}$ when $3 \leq N \leq 6$; as said above this assumption can be made without loss of generality). This first set of assumptions is essential to ensure (see (2.10)) that the exponent θ defined by (2.11) satisfies $0 < \theta < 1$ (see (2.12)). We also assume $a_0 \geq 0$ and $a_0 \neq 0$.

Our second set of assumptions on these parameters is made of the two smallness conditions (2.14) and (2.15).

Indeed, if, for example, a_0 is sufficiently small such that it satisfies

$$\alpha - C_N^2 \|a_0\|_{N/2} > 0,$$

then the two smallness conditions (2.14) and (2.15) are satisfied if $\|f\|_{N/2}$ (and therefore $\|f\|_{H^{-1}(\Omega)}$, since $L^{N/2}(\Omega) \subset H^{-1}(\Omega)$) is sufficiently small.

Similarly, if, for example, f is sufficiently small such that it satisfies

$$\alpha - \gamma C_N^2 \|f\|_{N/2} > 0,$$

then the two smallness conditions (2.14) and (2.15) are satisfied if $\|a_0\|_q$ is sufficiently small (which implies, since $L^q(\Omega) \subset L^{N/2}(\Omega)$, that $\|a_0\|_{N/2}$ is sufficiently small), since $\|a_0\|_q$ appears in the denominator of the right-hand side of (2.15). $\square$

Remark 2.3 The definitions of the two constants δ_0 and Z_{δ_0} which appear in Theorem 2.1 are given in (the technical) Appendix 6 (see Lemma 6.2). These definitions are based on the properties of the family of functions Φ_δ (see (6.13)) which look like convex parabolas (see Figure 2 and Remark 6.3): the constant δ_0 is the unique value of the parameter δ for which the function Φ_{δ_0} has a double zero, and Z_{δ_0} is the value of this double zero. The two smallness conditions (2.14) and (2.15) ensure that δ_0 satisfies $\delta_0 \geq \gamma$, a condition which is essential in our proof.

In Remark 3.9 we try to explain where the two smallness conditions (2.14) and (2.15) come from.

In Remark 3.10, we explain why we have chosen to state Theorem 3.8 with $\delta = \delta_0$ rather than with a fixed δ with $\gamma \leq \delta \leq \delta_0$. $\square$

Remark 2.4 In assumption (2.2) we have assumed that $N \geq 3$, because we use the Sobolev's embedding (2.9). All the proofs of the present paper can nevertheless be easily adapted to the cases where $N = 1$ and $N = 2$, providing similar results, by using the fact that $H_0^1(\Omega) \subset L^\infty(\Omega)$ when $N = 1$ and that $H_0^1(\Omega) \subset L^p(\Omega)$ for every $p < +\infty$ when $N = 2$, and by replacing assumption $q > \dfrac{N}{2}$ made in (2.6) when $N \geq 3$ by $q = 1$ when $N = 1$ and by $q > 1$ when $N = 2$, and the assumption made in (2.8) that $f \in L^{N/2}(\Omega)$ by $f \in L^1(\Omega)$ when $N = 1$ and by $f \in L^m(\Omega)$ with $m > 1$ when $N = 2$ (and also replacing the norm $\|f\|_{N/2}$ by the corresponding norm).

In assumption (2.6) and (2.8) we have assumed that $a_0 \neq 0$ and that $f \neq 0$. Indeed the case where $a_0 = 0$ has been treated by V. Ferone and F. Murat in [13], and in the case where $f = 0$, then $u = 0$ is a solution of (2.1) and the results of the present paper become trivial. $\square$

3 An equivalence result

The main results of this Section are Theorems 3.5 and 3.8. In contrast, Remarks 3.1, 3.2 and 3.4 and Lemma 3.3 can be considered as technical results.

Indeed, as said above, the proof of Theorem 2.1 is based on the equivalence result of Theorem 3.5 that we state and prove in this Section. This equivalence Theorem in particular implies that Theorem 3.8, which we state at the end of this Section, is equivalent to Theorem 2.1. Theorem 3.8 will be proved in Sections 4 and 5.

This Section also includes Remark 3.9 in which we try to explain where the two smallness conditions (2.14) and (2.15) come from, as well as Remark 3.10 where we explain why we have chosen to state Theorem 3.8 for $\delta = \delta_0$.

In this Section (as well as in the whole of the present paper) we always assume that

$$\delta > 0. \tag{3.1}$$

Let us first proceed with a formal computation.

If u is a solution of

$$\begin{cases} -\text{div}\,(A(x)Du) = H(x,u,Du) + f(x) + a_0(x)\,u \quad \text{in} \quad \Omega, \\ \\ u = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.2}$$

and if we formally define the function w_δ by

$$w_\delta = \frac{1}{\delta}\left(e^{\delta|u|} - 1\right)\text{sign}(u), \tag{3.3}$$

where the function sign is defined by (2.5), we have, at least formally,

$$\begin{cases} e^{\delta|u|} = 1 + \delta|w_\delta|, \qquad |u| = \dfrac{1}{\delta}\log(1+\delta|w_\delta|), \qquad \text{sign}(u) = \text{sign}(w_\delta), \\ \\ Dw_\delta = e^{\delta|u|}Du, \quad A(x)Dw_\delta = e^{\delta|u|}A(x)Du, \\ \\ -\text{div}\,(A(x)Dw_\delta) = -\delta e^{\delta|u|}\,A(x)DuDu\,\text{sign}(u) - e^{\delta|u|}\big(\text{div}\,(A(x)Du)\big), \end{cases} \tag{3.4}$$

and therefore w_δ is, at least formally, a solution of

$$\begin{cases} -\text{div}\,(A(x)Dw_\delta) = \\ = -\delta e^{\delta|u|}A(x)DuDu\,\text{sign}(u) + e^{\delta|u|}H(x,u,Du) + e^{\delta|u|}\,f(x) + e^{\delta|u|}\,a_0(x)\,u = \\ = -K_\delta(x,w_\delta,Dw_\delta)\,\text{sign}(w_\delta) + \\ +(1+\delta|w_\delta|)\,f(x) + \,a_0(x)\,w_\delta + a_0(x)\,g_\delta\,(w_\delta)\,\text{sign}(w_\delta) \quad \text{in} \quad \Omega, \\ w_\delta = 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{3.5}$$

whenever the functions $K_\delta : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ and $g_\delta : \mathbb{R} \to \mathbb{R}$ are defined by the formulas

$$\begin{cases} K_\delta(x,t,\zeta) = \\ = \dfrac{\delta}{1+\delta|t|}\,A(x)\zeta\zeta - (1+\delta|t|)H(x, \dfrac{1}{\delta}\log(1+\delta|t|)\,\text{sign}(t), \dfrac{\zeta}{1+\delta|t|})\,\text{sign}(t), \\ \text{a.e. } x \in \Omega, \quad \forall t \in \mathbb{R}, \quad \forall \zeta \in \mathbb{R}^N, \end{cases} \tag{3.6}$$

and

$$g_\delta(t) = -|t| + \frac{1}{\delta}(1+\delta|t|)\log(1+\delta|t|), \quad \forall t \in \mathbb{R}, \tag{3.7}$$

which is equivalent to

$$t + g_\delta(t)\,\text{sign}(t) = (1+\delta|t|)\frac{1}{\delta}\log(1+\delta|t|)\,\text{sign}(t), \quad \forall t \in \mathbb{R}. \tag{3.8}$$

Conversely, if w_δ is a solution of (3.5), and if we formally define the function u by

$$u = \frac{1}{\delta}\log(1+\delta|w_\delta|)\,\text{sign}(w_\delta), \tag{3.9}$$

the same formal computation easily shows that u is a solution of (3.2).

The main goal of this Section is to transform this formal equivalence into a mathematical result, namely Theorem 3.5. We begin with three remarks on the functions K_δ and g_δ, namely Remarks 3.1, 3.2 and 3.4.

Remark 3.1 Observe that, because of the inequality (2.4) on the function H, and because of the coercivity (2.3) of the matrix A, one has

$$\begin{cases} (c_0+\delta)A(x)\zeta\zeta \geq \dfrac{c_0+\delta}{(1+\delta|t|)}A(x)\zeta\zeta = \\ \\ = \dfrac{\delta}{(1+\delta|t|)}A(x)\zeta\zeta + (1+\delta|t|)\dfrac{c_0}{(1+\delta|t|)^2}A(x)\zeta\zeta \geq \\ \\ \geq K_\delta(x,t,\zeta) \geq \\ \\ \geq \dfrac{\delta}{(1+\delta|t|)}A(x)\zeta\zeta - (1+\delta|t|)\dfrac{\gamma}{(1+\delta|t|)^2}A(x)\zeta\zeta = \\ \\ = \dfrac{(\delta-\gamma)}{(1+\delta|t|)}A(x)\zeta\zeta \geq -|\delta-\gamma|\,A(x)\zeta\zeta, \\ \\ \text{a.e. } x\in\Omega, \quad \forall t\in\mathbb{R}, \quad \forall\zeta\in\mathbb{R}^N, \quad \forall\delta>0. \end{cases} \tag{3.10}$$

When $\delta \geq \gamma$, this computation in particular implies that

$$(c_0+\delta)A(x)\zeta\zeta \geq K_\delta(x,t,\zeta) \geq 0 \quad \text{a.e. } x\in\Omega, \quad \forall t\in\mathbb{R}, \quad \forall\zeta\in\mathbb{R}^N \quad \text{if} \quad \delta\geq\gamma. \tag{3.11}$$

□

Remark 3.2 In this technical Remark we prove that the functions $K_\delta(x,w,Dw)$ and $K_\delta(x,w,Dw)\,\mathrm{sign}(w)$ are correctly defined and are measurable functions when $w \in H^1(\Omega)$, and we prove their continuity with respect to the almost everywhere convergence of w and Dw (see Lemma 3.3).

Note that the function $K_\delta(x,t,\zeta)$ defined by (3.6) and the function $K_\delta(x,t,\zeta)\,\mathrm{sign}(t)$ are not Carathéodory functions, because their definitions involve the function $\mathrm{sign}(t)$, which it is not a Carathéodory function since is not continuous at $t=0$. This lack of continuity in $t=0$ is however the only obstruction for the functions $K_\delta(x,t,\zeta)$ and $K_\delta(x,t,\zeta)\,\mathrm{sign}(t)$ to be Carathéodory functions, and

$$\begin{cases} \text{for every } w\in H^1(\Omega), \\ \\ \text{the functions } K_\delta(x,w,Dw) \text{ and } K_\delta(x,w,Dw)\,\mathrm{sign}(w) \text{ are well defined} \\ \\ \text{and are measurable functions,} \end{cases} \tag{3.12}$$

as it immediately results from the two formulas

$$\begin{cases} K_\delta(x,t,\zeta) = \\ = \dfrac{\delta}{1+\delta|t|}\,A(x)\zeta\zeta - (1+\delta|t|)\,H(x,\dfrac{1}{\delta}\log(1+\delta|t|)\,\mathrm{sign}(t),\dfrac{\zeta}{1+\delta|t|})\,\mathrm{sign}(t)\; = \\ = \dfrac{\delta}{1+\delta|t|}\,A(x)\zeta\zeta + \\ + \chi_{\{t<0\}}(t)\,(1+\delta|t|)\,H(x,-\dfrac{1}{\delta}\log(1+\delta|t|),\dfrac{\zeta}{1+\delta|t|}) - \chi_{\{t=0\}}(t)\,0 + \\ - \chi_{\{t>0\}}(t)\,(1+\delta|t|)\,H(x,\dfrac{1}{\delta}\log(1+\delta|t|),\dfrac{\zeta}{1+\delta|t|}), \\ \text{a.e. } x\in\Omega, \quad \forall t\in\mathbb{R}, \quad \forall\zeta\in\mathbb{R}^N, \end{cases} \tag{3.13}$$

$$\begin{cases} K_\delta(x,t,\zeta)\,\mathrm{sign}(t) = \\ = \mathrm{sign}(t)\,\dfrac{\delta}{1+\delta|t|}\,A(x)\zeta\zeta - (1+\delta|t|)\,H(x,\dfrac{1}{\delta}\log(1+\delta|t|)\,\mathrm{sign}(t),\dfrac{\zeta}{1+\delta|t|})\; = \\ = -\,\chi_{\{t<0\}}(t)\,\dfrac{\delta}{1+\delta|t|}\,A(x)\zeta\zeta + \chi_{\{t=0\}}(t)\,0 + \chi_{\{t>0\}}(t)\,\dfrac{\delta}{1+\delta|t|}\,A(x)\zeta\zeta + \\ - \chi_{\{t<0\}}(t)\,(1+\delta|t|)\,H(x,-\dfrac{1}{\delta}\log(1+\delta|t|),\dfrac{\zeta}{1+\delta|t|}) - \chi_{\{t=0\}}(t)\,H(x,0,\zeta) + \\ - \chi_{\{t>0\}}(t)\,(1+\delta|t|)\,H(x,\dfrac{1}{\delta}\log(1+\delta|t|),\dfrac{\zeta}{1+\delta|t|}), \\ \text{a.e. } x\in\Omega, \quad \forall t\in\mathbb{R}, \quad \forall\zeta\in\mathbb{R}^N. \end{cases} \tag{3.14}$$

Moreover, in view of (3.10) one has

$$K_\delta(x,w,Dw)\in L^1(\Omega), \quad K_\delta(x,w,Dw)\,\mathrm{sign}(w)\in L^1(\Omega), \quad \forall w\in H^1(\Omega), \quad \forall\delta>0. \tag{3.15}$$

□

One also has the following convergence result.

Lemma 3.3 *Consider a sequence w_n such that*

$$w_n\in H^1(\Omega),\ w\in H^1(\Omega), \quad w_n\to w \quad \text{a.e. in } \Omega, \quad Dw_n\to Dw \quad \text{a.e. in } \Omega. \tag{3.16}$$

Then

$$\begin{cases} K_\delta(x,w_n,Dw_n)\to K_\delta(x,w,Dw) \quad \text{a.e. in } \Omega, \\ K_\delta(x,w_n,Dw_n)\,\mathrm{sign}(w_n)\to K_\delta(x,w,Dw)\,\mathrm{sign}(w) \quad \text{a.e. in } \Omega. \end{cases} \tag{3.17}$$

Proof On the first hand, we have the following almost everywhere convergences

$$\begin{cases} \dfrac{\delta}{1+\delta|w_n|}A(x)Dw_nDw_n \to \dfrac{\delta}{1+\delta|w|}A(x)DwDw \quad \text{a.e.} \quad \text{in} \quad \Omega, \\ (1+\delta|w_n|)H(x,-\dfrac{1}{\delta}\log(1+\delta|w_n|),\dfrac{Dw_n}{1+\delta|w_n|}) \to \\ \to (1+\delta|w|)H(x,-\dfrac{1}{\delta}\log(1+\delta|w|),\dfrac{Dw}{1+\delta|w|}) \quad \text{a.e.} \quad \text{in} \quad \Omega, \\ H(x,0,Dw_n) \to H(x,0,Dw) \quad \text{a.e.} \quad \text{in} \quad \Omega, \\ (1+\delta|w_n|)H(x,\dfrac{1}{\delta}\log(1+\delta|w_n|),\dfrac{Dw_n}{1+\delta|w_n|}) \to \\ \to (1+\delta|w|)H(x,\dfrac{1}{\delta}\log(1+\delta|w|),\dfrac{Dw}{1+\delta|w|}) \quad \text{a.e.} \quad \text{in} \quad \Omega. \end{cases} \tag{3.18}$$

On the other hand, for almost every x fixed in the set $\{y \in \Omega : w(y) > 0\}$, the assertion $w_n(x) \to w(x)$ implies, since $w(x) > 0$, that one has $w_n(x) > 0$ for n sufficiently large (depending on x), and therefore that, for some $n > n^\star(x)$, one has

$$\begin{cases} \chi_{\{w_n<0\}}(w_n(x)) = 0 = \chi_{\{w<0\}}(w(x)) \quad \text{for} \quad n > n^\star(x), \\ \chi_{\{w_n=0\}}(w_n(x)) = 0 = \chi_{\{w=0\}}(w(x)) \quad \text{for} \quad n > n^\star(x), \\ \chi_{\{w_n>0\}}(w_n(x)) = 1 = \chi_{\{w>0\}}(w(x)) \quad \text{for} \quad n > n^\star(x). \end{cases} \tag{3.19}$$

These convergences and (3.18) imply that

$$\begin{cases} K_\delta(x,w_n,Dw_n) \to K_\delta(x,w,Dw) \quad \text{a.e.} \quad \text{in} \quad \{y \in \Omega : w(y) > 0\}, \\ K_\delta(x,w_n,Dw_n)\,\text{sign}(w_n) \to K_\delta(x,w,Dw)\,\text{sign}(w) \quad \text{a.e.} \quad \text{in} \quad \{y \in \Omega : w(y) > 0\}. \end{cases}$$

The same proof gives the similar result in the set $\{y \in \Omega : w(y) < 0\}$.

The proof in the set $\{y \in \Omega : w(y) = 0\}$ is a little bit more delicate. Let us first observe that for $w \in H^1(\Omega)$ one has

$$Dw = 0 \quad \text{a.e.} \quad \text{in} \quad \{y \in \Omega : w(y) = 0\}, \tag{3.20}$$

and since inequality (2.4) on the function H implies that

$$H(x,s,0) = 0 \quad \text{a.e. } x \in \Omega, \quad \forall s \in \mathbb{R},\ s \neq 0, \tag{3.21}$$

and therefore by continuity in s that

$$H(x,0,0) = 0 \quad \text{a.e. } x \in \Omega. \tag{3.22}$$

Then, on the first hand, theses results and formulas (3.13) and (3.14) imply that

$$K_\delta(x,w,Dw) = K_\delta(x,w,Dw)\,\text{sign}(w) = 0 \quad \text{a.e.} \quad \text{in} \quad \{y \in \Omega : w(y) = 0\}. \tag{3.23}$$

On the other hand, in view of (3.20) and (3.22), the four functions which appear in the four limits in (3.18) vanish almost everywhere in the set $\{y \in \Omega : w(y) = 0\}$. Even if we do not know anything about the pointwise limits of the functions $\chi_{\{w_n<0\}}(w_n(x))$, $\chi_{\{w_n=0\}}(w_n(x))$ and $\chi_{\{w_n>0\}}(w_n(x))$ in the set $\{y \in \Omega : w(y) = 0\}$, this fact and formulas (3.13) and (3.14) prove that

$$\begin{cases} K_\delta(x, w_n, Dw_n) \to 0 \quad \text{a.e.} \quad \text{in} \quad \{y \in \Omega : w(y) = 0\}, \\ K_\delta(x, w_n, Dw_n)\,\text{sign}(w_n) \to 0 \quad \text{a.e.} \quad \text{in} \quad \{y \in \Omega : w(y) = 0\}. \end{cases} \tag{3.24}$$

From (3.23) and (3.24) we deduce that

$$\begin{cases} K_\delta(x, w_n, Dw_n) \to K_\delta(x, w, Dw) \quad \text{a.e.} \quad \text{in} \quad \{y \in \Omega : w(y) = 0\}, \\ K_\delta(x, w_n, Dw_n)\,\text{sign}(w_n) \to K_\delta(x, w, Dw)\,\text{sign}(w) \quad \text{a.e.} \quad \text{in} \quad \{y \in \Omega : w(y) = 0\}. \end{cases} \tag{3.25}$$

This completes the proof of (3.17). □

Remark 3.4 Observe that the function $g_\delta(s)\,\text{sign}(s)$ is a Carathéodory function since in view of (6.4) this function is continuous at $s = 0$. This allows one to define $g_\delta(w)\,\text{sign}(w)$ as a measurable function for every $w \in H^1(\Omega)$. □

The main result of this Section is the following equivalence Theorem.

Theorem 3.5 *Assume that* (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) *hold true, and let* $\delta > 0$ *be fixed. Let the functions* K_δ *and* g_δ *be defined by* (3.6) *and* (3.7).

If u *is any solution of* (2.1) *which satisfies*

$$(e^{\delta|u|} - 1) \in H_0^1(\Omega), \tag{3.26}$$

then the function w_δ *defined by* (3.3), *namely by*

$$w_\delta = \frac{1}{\delta}\left(e^{\delta|u|} - 1\right)\text{sign}(u),$$

satisfies

$$\begin{cases} w_\delta \in H_0^1(\Omega), \\ -\text{div}\,(A(x)Dw_\delta) + K_\delta(x, w_\delta, Dw_\delta)\,\text{sign}(w_\delta) = \\ = (1 + \delta|w_\delta|)\,f(x) + a_0(x)\,w_\delta + a_0(x)\,g_\delta(w_\delta)\,\text{sign}(w_\delta) \quad \text{in} \quad \mathcal{D}'(\Omega). \end{cases} \tag{3.27}$$

Conversely, if w_δ *is any solution of* (3.27), *then the function* u *defined by* (3.9), *namely by*

$$u = \frac{1}{\delta}\log(1 + \delta|w_\delta|)\,\text{sign}(w_\delta),$$

is a solution of (2.1) *which satisfies* (3.26).

Remark 3.6 Every term of the equation in (3.27) has a meaning in the sense of distributions: indeed the first term of the left-hand side of the equation in (3.27) belongs to $H^{-1}(\Omega)$; on the other hand, the four other terms of the equation are measurable functions (see Remarks 3.2 and 3.4); the second term of the left-hand side of the equation in (3.27) belongs to $L^1(\Omega)$ in view of (3.10), while the three terms of the right-hand side of the equation in (3.27) can be proved to belong to $(L^{2^\star}(\Omega))'$ (see e.g. the proof of (3.40) in Remark 3.9 and the proof of (4.8) in the proof of Lemma 4.2). □

Remark 3.7 Observe that the equivalence Theorem 3.5 holds true without assuming the two smallness conditions (2.14) and (2.15); moreover one could even have removed in (2.3) the assumption that the matrix A is coercive, and still obtain the same equivalence result.

Note however that Theorem 3.5 is an equivalence result which does not proves neither the existence of a solution of (2.1) nor the existence of a solution of (3.27), but which assumes as an hypothesis either the existence of a solution of (2.1) which also satisfies (3.26), or the existence of a solution of (3.27). □

Proof of Theorem 3.5 Define the function $\hat{f}$ by

$$\hat{f}(x) = f(x) + a_0(x)\, u(x), \tag{3.28}$$

In view of (3.3) and of the definition (3.7) of $g_\delta(s)$, one has (see (3.8) and (3.4))

$$\begin{cases} (1+\delta|w_\delta|)\, f(x) + a_0(x)\, w_\delta + a_0(x)\, g_\delta(w_\delta)\, \mathrm{sign}(w_\delta) = \\ = (1+\delta|w_\delta|)\left(f(x) + a_0(x)\, \dfrac{1}{\delta}\, \log(1+\delta|w_\delta|)\, \mathrm{sign}(w_\delta)\right) = \\ = (1+\delta|w_\delta|)\,(f(x) + a_0(x)\, u(x)) = (1+\delta|w_\delta|)\hat{f}(x). \end{cases} \tag{3.29}$$

Then Theorem 3.5 becomes an immediate application of Proposition 1.8 of [13], once one observes that

$$\hat{f} \in L^{N/2}(\Omega); \tag{3.30}$$

such is the case in the setting of Theorem 3.5: indeed $\hat{f} = f + a_0 u$, where f belongs to $L^{N/2}(\Omega)$ by assumption (2.8), and where $a_0 u$ also belongs to $L^{N/2}(\Omega)$, since a_0 is assumed to belong to $L^q(\Omega)$, $q > \frac{N}{2}$ (see assumption (2.6)), while u belongs to $L^r(\Omega)$ for every $r < +\infty$, since by (3.26) $(e^{\delta|u|} - 1)$ is assumed to belong to $H_0^1(\Omega)$, hence in particular to $L^1(\Omega)$, which implies that $e^{\delta|u|}$ belongs to $L^1(\Omega)$. Theorem 3.5 is therefore proved. □

From the equivalence Theorem 3.5 one immediately deduces, setting

$$w = \frac{1}{\delta_0}(e^{\delta_0|u|} - 1)\, \mathrm{sign}(u) \tag{3.31}$$

and equivalently

$$u = \frac{1}{\delta_0}\log(1+\delta_0|w|)\, \mathrm{sign}(w), \tag{3.32}$$

that Theorem 2.1 is equivalent to the following Theorem.

Theorem 3.8 *Assume that* (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) *hold true. Assume moreover that the two smallness conditions* (2.14) *and* (2.15) *hold true.*

Then there exist a constant δ_0 with $\delta_0 \geq \gamma$, and a constant Z_{δ_0}, which are defined in Lemma 6.2 (see (6.16), (6.17) *and* (6.18)*), such that there exists at least one solution w of*

$$\begin{cases} w \in H_0^1(\Omega), \\ -\mathrm{div}(A(x)Dw) + K_{\delta_0}(x, w, Dw)\,\mathrm{sign}(w) = \\ = (1 + \delta_0|w|)\, f(x) + a_0(x)\, w + a_0(x)\, g_{\delta_0}(w)\,\mathrm{sign}(w) \quad in \quad \mathcal{D}'(\Omega), \end{cases} \tag{3.33}$$

which satisfies

$$\|w\|_{H_0^1(\Omega)} \leq Z_{\delta_0}. \tag{3.34}$$

The rest of this paper will therefore be devoted to the proof of Theorem 3.8. This will be done in two steps: first, in Section 4 we will prove the existence of a solution satisfying (3.34) for a problem which approximates (3.33), see Theorem 4.1; second, in Section 5, we will pass to the limit in this approximate problem and prove that for a subsequence the limit satisfies (3.33) and (3.34).

Remark 3.9 In this Remark we assume that (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) hold true. We also assume that the two smallness conditions (2.14) and (2.15) hold true, and we try to explain how these two conditions come from an "a priori estimate" that one can obtain on the solutions of (3.27).

If w_δ is any solution of (3.27), using $T_k(w_\delta) \in H_0^1(\Omega) \cap L^\infty(\Omega)$ as test function, where $T_k : \mathbb{R} \to \mathbb{R}$ is the usual truncation at height k defined by

$$T_k(s) = \begin{cases} -k & \text{if} \quad s \leq -k, \\ \;\; s & \text{if} \quad -k \leq s \leq +k, \\ +k & \text{if} \quad +k \leq s, \end{cases} \tag{3.35}$$

one has

$$\begin{cases} \displaystyle\int_\Omega A(x)Dw_\delta\, DT_k(w_\delta) + \int_\Omega K_\delta(x, w_\delta, Dw_\delta)|T_k(w_\delta)| = \\ \displaystyle = \int_\Omega f(x)\, T_k(w_\delta) + \int_\Omega \delta f(x)|w_\delta|\, T_k(w_\delta) + \int_\Omega a_0(x)\, w_\delta\, T_k(w_\delta) + \\ \displaystyle + \int_\Omega a_0(x)\, g_\delta(w_\delta)\, |T_k(w_\delta)|. \end{cases} \tag{3.36}$$

From now on we assume in this Remark that δ satisfies

$$\gamma \leq \delta \leq \delta_1, \tag{3.37}$$

where δ_1 is defined by (6.11) (note that one has $\gamma < \delta_1$, see (6.12)).

Since $\delta \geq \gamma$ by (3.37), we deduce from (3.11) that $K_\delta(x, s, \zeta) \geq 0$, and therefore that the second term of the left-hand side of (3.36) is nonnegative. We then use in the first term of the left-hand side of (3.36) the fact that the matrix A is coercive (see (2.3)), in the first term of the right-hand side of (3.36) the fact that $f \in L^{N/2}(\Omega)$ (see (2.8)), which implies that $f \in H^{-1}(\Omega)$, in the second and in the third terms of the right-hand side of (3.36) Hölder's inequality with

$$\frac{1}{\frac{N}{2}} + \frac{1}{2^\star} + \frac{1}{2^\star} = 1,$$

and finally in the fourth term of the right-hand side of (3.36) the second statement of (6.19), namely

$$0 \le g_\delta(t) < G|t|^{1+\theta}, \quad \forall t \in \mathbb{R},\ t \neq 0, \quad \forall \delta,\ 0 < \delta \le \delta_1, \tag{3.38}$$

(note that here we use $\delta \le \delta_1$, see (3.37)) and Hölder's inequality with

$$\frac{1}{q} + \frac{1+\theta}{2^\star} + \frac{1}{2^\star} = 1$$

(which results from the definition (2.11) of θ). This allows us to obtain an estimate on $T_k(w_\delta)$, in wich we pass to the limit in k to get

$$\begin{cases} \alpha\|Dw_\delta\|_2^2 < \|f\|_{H^{-1}(\Omega)}\|Dw_\delta\|_2 + \delta\|f\|_{N/2}\|w_\delta\|_{2^\star}^2 + \|a_0\|_{N/2}\|w_\delta\|_{2^\star}^2 + G\|a_0\|_q\|w_\delta\|_{2^\star}^{2+\theta}, \\ \text{if } w_\delta \neq 0, \end{cases} \tag{3.39}$$

(note that in view of (3.38), inequality (3.39) is actually a strict inequality). Using Sobolev's inequality (2.9) and dividing by $\|Dw_\delta\|_2$ this implies that (even in the case where $w_\delta = 0$)

$$\alpha\|Dw_\delta\|_2 < \|f\|_{H^{-1}(\Omega)} + \delta C_N^2\|f\|_{N/2} + C_N^2\|a_0\|_{N/2} + GC_N^{2+\theta}\|a_0\|_q\|Dw_\delta\|_2^{1+\theta}. \tag{3.40}$$

In view of the definition (6.13) of the function Φ_δ (see also Figure 2), we have proved that if w_δ is any solution of (3.27), one has

$$\Phi_\delta(\|Dw_\delta\|_2) > 0 \quad \text{if} \quad \gamma \le \delta \le \delta_1. \tag{3.41}$$

But by the definition of δ_0, one has

$$\Phi_\delta(X) > 0, \quad \forall X \ge 0, \quad \forall \delta,\ 0 < \delta \le \delta_1,$$

and therefore inequality (3.41) does not imply anything on $\|Dw_\delta\|_2$ when δ satisfies $\delta_0 < \delta \le \delta_1$. In contrast, when $\delta < \delta_0$, the strict inequality (3.41) implies that

$$\text{either} \quad \|Dw_\delta\|_2 < Y_\delta^- \quad \text{or} \quad \|Dw_\delta\|_2 > Y_\delta^+ \quad \text{if} \quad \delta < \delta_0, \tag{3.42}$$

where $Y_\delta^- < Y_\delta^+$ are the two distinct zeros of the function Φ_δ (see Remarks 6.3 and 6.5 and Figure 2), while when $\delta = \delta_0$, the strict inequality (3.41) implies that

$$\text{either} \quad \|Dw_{\delta_0}\|_2 < Z_{\delta_0} \quad \text{or} \quad \|Dw_{\delta_0}\|_2 > Z_{\delta_0} \quad \text{if} \quad \delta = \delta_0. \tag{3.43}$$

Inequalities (3.42) and (3.43) are not a priori estimates, since they do not imply any bound on $\|Dw_\delta\|_2$. Nevertheless these inequalities exclude the closed interval $[Y_\delta^-, Y_\delta^+]$ or the point Z_{δ_0} for $\|Dw_\delta\|_2$, and they give the hope to prove the existence of a fixed point in the set $\|Dw_\delta\|_2 \le Y_\delta^-$, when $\delta < \delta_0$, or in the set $\|Dw_{\delta_0}\|_2 \le Z_{\delta_0}$, when $\delta = \delta_0$.

These inequalities also explain where the two smallness conditions (2.14) and (2.15) come from. Indeed (see Remark 6.3), these two smallness conditions imply that the value δ_0 of δ for which Φ_δ has a double zero satisfies $\delta_0 \geq \gamma$, which is the case where, as said just above, some hope is allowed. $\square$

Remark 3.10 In the present paper, we have chosen to prove the existence of a function w which is a solution of (3.33) (or in other terms of a function $w = w_{\delta_0}$ which is a solution of (3.27) with $\delta = \delta_0$) which satisfies $\|w\|_{H_0^1(\Omega)} \leq Z_{\delta_0}$ (see (3.34)). When $\gamma < \delta_0$, i.e. when the inequality (2.15) is a strict inequality, we could as well have chosen to prove that for any fixed δ with $\gamma \leq \delta < \delta_0$, there exists a function $\hat{w}_\delta$ which is a solution of (3.27) which satisfies $\|\hat{w}_\delta\|_{H_0^1(\Omega)} \leq Y_\delta^-$, where Y_δ^- is the smallest zero of the function Φ_δ (see Remark 6.5 and Figure 2): indeed the proofs made in Sections 4 and 5 continue to work in this framework and allow one to prove this result.

In this framework, if we define, for any fixed δ with $\gamma \leq \delta < \delta_0$, the function $\hat{u}_\delta$ by

$$\hat{u}_\delta = \frac{1}{\delta}\log(1 + \delta|\hat{w}_\delta|)\,\mathrm{sign}(\hat{w}_\delta) \tag{3.44}$$

(compare with the definition (3.32) of u, where $\delta = \delta_0$), the existence of a solution $\hat{w}_\delta$ of (3.27) which satisfies $\|\hat{w}_\delta\|_{H_0^1(\Omega)} \leq Y_\delta^-$ proves (see the equivalence Theorem 3.5) that $\hat{u}_\delta$ defined by (3.44) is a solution of (2.1) which satisfies

$$\|e^{\delta|\hat{u}_\delta|} D\hat{u}_\delta\|_2 = \|D\hat{w}_\delta\|_2 \leq Y_\delta^-.$$

But the function u which is defined by (3.32) from the function w given by Theorem 3.8 satisfies (2.1) (by the equivalence Theorem 3.5 with $\delta = \delta_0$) and (see (3.4) again)

$$\|e^{\delta_0|u|} Du\|_2 = \|Dw\|_2 \leq Z_{\delta_0}.$$

When $\delta < \delta_0$, we therefore have

$$\|e^{\delta|u|} Du\|_2 \leq \|e^{\delta_0|u|} Du\|_2 \leq Z_{\delta_0}, \quad \text{if} \quad \delta < \delta_0,$$

and therefore $e^{\delta|u|} - 1 \in H_0^1(\Omega)$. By the equivalence Theorem 3.5, the function $\overline{w}_\delta$ defined from u by

$$\overline{w}_\delta = \frac{1}{\delta}(e^{\delta|u|} - 1)\,\mathrm{sign}(u) \tag{3.45}$$

is a solution of (3.27). Moreover, since $D\overline{w}_\delta = e^{\delta|u|}Du$ and since $Z_{\delta_0} < Y_\delta^+$ for $\delta < \delta_0$ (see (6.25)), one has in particular

$$\|D\overline{w}_\delta\|_2 < Y_\delta^+, \quad \text{if} \quad \delta < \delta_0,$$

which implies by (3.42) that $\overline{w}_\delta$ satisfies

$$\|D\overline{w}_\delta\|_2 < Y_\delta^-. \tag{3.46}$$

Therefore the result of Theorem 3.8 (which is concerned with the case $\delta = \delta_0$) provides us with a function w, and then with a function u defined by (3.32), and finally with a function $\overline{w}_\delta$ defined by (3.45) which is a solution of (3.27) and which satisfies (3.46). This function $\overline{w}_\delta$ is a solution $\hat{w}_\delta$ of (3.27) which satisfies $\|\hat{w}_\delta\|_{H_0^1(\Omega)} \leq Y_\delta^-$. Therefore the result of Theorem 3.8 (where $\delta = \delta_0$) also provides us for every δ with $\gamma \leq \delta < \delta_0$ with a proof of the result stated in the second paragraph of the present Remark. $\square$

4 Existence of a solution for an approximate problem

In this Section we introduce an approximation (see (4.1)) of problem (3.33). Under the two smallness conditions (2.14) and (2.15), we prove by applying Schauder's fixed point theorem that this approximate problem has at least one solution which satisfies the estimate (3.34).

Let δ_0 be defined by (6.16), (6.17) and (6.18). For any $k > 0$, we consider the approximate problem of finding a solution w_k of (compare with (3.33))

$$\begin{cases} w_k \in H_0^1(\Omega), \\ -\text{div}\,(A(x)Dw_k) + T_k(K_{\delta_0}(x, w_k, Dw_k))\,\text{sign}_k(w_k) \;= \\ = (1 + \delta_0|w_k|)\,f(x) + a_0(x)\,w_k + a_0(x)\,g_{\delta_0}(w_k)\,\text{sign}(w_k) \quad \text{in} \quad \mathcal{D}'(\Omega), \end{cases} \tag{4.1}$$

where T_k is the usual truncation at height k defined by (3.35) and where $\text{sign}_k : \mathbb{R} \to \mathbb{R}$ is the approximation of the function sign which is defined by

$$\text{sign}_k(s) = \begin{cases} ks, & \text{if} \quad |s| \le \dfrac{1}{k}, \\ \text{sign}(s), & \text{if} \quad |s| \ge \dfrac{1}{k}. \end{cases} \tag{4.2}$$

Theorem 4.1 *Assume that* (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) *hold true. Assume moreover the two smallness conditions that* (2.14) *and* (2.15) *hold true. Let* δ_0 *be defined in Lemma 6.2 (see* (6.16) *and* (6.17)*), and let* $k > 0$ *be fixed.*

Then there exists at least one solution of (4.1) *such that*

$$\|w_k\|_{H_0^1(\Omega)} \le Z_{\delta_0}, \tag{4.3}$$

where Z_{δ_0} *is defined in Lemma* 6.2 *(see* (6.16), (6.17) *and* (6.18)*).*

The proof of Theorem 4.1 consists in applying Schauder's fixed point theorem. First we prove the two following lemmas.

Lemma 4.2 *Assume that* (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) *hold true. Let* $k > 0$ *be fixed.*

Then, for any $w \in H_0^1(\Omega)$*, there exists a unique solution* W *of the following semilinear problem*

$$\begin{cases} W \in H_0^1(\Omega), \\ -\text{div}(A(x)DW) + T_k(K_{\delta_0}(x, w, Dw))\,\text{sign}_k(W) \;= \\ = (1 + \delta_0|w|)\,f(x) + a_0(x)\,w + a_0(x)\,g_{\delta_0}(w)\,\text{sign}(w) \quad in \quad \mathcal{D}'(\Omega). \end{cases} \tag{4.4}$$

Moreover W *satisfies*

$$\alpha\|DW\|_2 \le \|f\|_{H^{-1}(\Omega)} + \delta_0 C_N^2\|f\|_{N/2}\|Dw\|_2 + C_N^2\|a_0\|_{N/2}\|Dw\|_2 + GC_N^{2+\theta}\|a_0\|_q\|Dw\|_2^{1+\theta}, \tag{4.5}$$

where C_N *and* G *are the constants given by* (2.9) *and* (6.14)*.*

Proof Problem (4.4) is of the form

$$\begin{cases} W \in H_0^1(\Omega), \\ -\mathrm{div}\,(A(x)DW) + \hat{b}(x)\,\mathrm{sign}_k(W) = \widehat{F}(x) \quad \text{in} \quad \mathcal{D}'(\Omega), \end{cases} \tag{4.6}$$

where $\hat{b}(x)$ and $\hat{F}(x)$ are given. Since $\hat{b}(x) = T_k(K_{\delta_0}(x, w, Dw))$ belongs to $L^\infty(\Omega)$ and is non-negative in view of (3.11) and of $\delta_0 \geq \gamma$ (see (6.16)), since the function sign_k is continuous and nondecreasing, and since $\widehat{F}$ belong to $H^{-1}(\Omega)$ (see e.g. the computation which allows one to obtain (4.8) below), this problem has a unique solution.

Since $W \in H_0^1(\Omega)$, the use of W as a test function in (4.4) is licit. Since $T_k(K_\delta(x, t, \zeta))$ is nonnegative, this gives

$$\begin{cases} \displaystyle\int_\Omega A(x)\, DW DW\, dx \leq \\ \displaystyle\leq \int_\Omega (1 + \delta_0|w|)\, f(x)\, W dx \,+ \int_\Omega a_0(x)\, w\, W dx \,+ \int_\Omega a_0(x)\, g_{\delta_0}(w)\, \mathrm{sign}(w)\, W dx. \end{cases} \tag{4.7}$$

As in the computation made in Remark 3.9 to obtain the inequality (3.39), we use in (4.7) the coercivity (2.3) of the matrix A, Hölder's inequality with $\dfrac{1}{\frac{N}{2}} + \dfrac{1}{2^\star} + \dfrac{1}{2^\star} = 1$, inequality (6.20) on g_{δ_0}, $\dfrac{1}{q} + \dfrac{1+\theta}{2^\star} + \dfrac{1}{2^\star} = 1$ (which results from the definition (2.11) of θ), and finally Sobolev's inequality (2.9). We obtain

$$\begin{cases} \alpha\|DW\|_2^2 \leq \|f\|_{H^{-1}(\Omega)}\|DW\|_2 + \delta_0\|f\|_{N/2}\|w\|_{2^\star}\|W\|_{2^\star} + \|a_0\|_{N/2}\|w\|_{2^\star}\|W\|_{2^\star} + \\ +G\,\|a_0\|_q\|w\|_{2^\star}^{1+\theta}\|W\|_{2^\star} \leq \\ \leq |f\|_{H^{-1}(\Omega)}\|DW\|_2 + \delta_0 C_N^2\|f\|_{N/2}\|Dw\|_2\|DW\|_2 + C_N^2\|a_0\|_{N/2}\|Dw\|_2\|DW\|_2 + \\ +\,GC_N^{2+\theta}\,\|a_0\|_q\|Dw\|_2^{1+\theta}\|DW\|_2, \end{cases} \tag{4.8}$$

which immediately implies (4.5). □

Lemma 4.3 *Assume that* (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) *hold true. Let* $k > 0$ *be fixed.*
Let w_n *be a sequence such that*

$$w_n \rightharpoonup w \quad \textit{in } H_0^1(\Omega) \quad \textit{weakly} \quad \textit{and} \quad \textit{a.e.} \quad \textit{in} \quad \Omega. \tag{4.9}$$

Define W_n *as the unique solution of* (4.4) *for* $w = w_n$, *i.e.*

$$\begin{cases} W_n \in H_0^1(\Omega), \\ -\mathit{div}\,(A(x)DW_n) + T_k(K_{\delta_0}(x, w_n, Dw_n))\,\mathrm{sign}_k(W_n) = \\ = (1 + \delta_0|w_n|)\, f(x) + a_0(x)\, w_n + a_0(x)\, g_{\delta_0}(w_n)\,\mathrm{sign}(w_n) \quad \mathit{in} \quad \mathcal{D}'(\Omega). \end{cases} \tag{4.10}$$

Assume moreover that for a subsequence, still denoted by n, and for some $W^\star \in H_0^1(\Omega)$, W_n satisfies

$$W_n \rightharpoonup W^\star \quad \text{in} \quad H_0^1(\Omega) \quad \textit{weakly} \quad \textit{and} \quad \textit{a.e.} \quad \textit{in} \quad \Omega. \tag{4.11}$$

Then for the same subsequence one has

$$W_n \to W^\star \quad \textit{in} \quad H_0^1(\Omega) \quad \textit{strongly.} \tag{4.12}$$

Proof Since $W_n - W^\star \in H_0^1(\Omega)$, the use of $(W_n - W^\star)$ as test function in (4.10) is licit. This gives

$$\begin{cases} \displaystyle\int_\Omega A(x)D(W_n - W^\star)D(W_n - W^\star)\,dx = \\ \displaystyle = -\int_\Omega A(x)DW^\star D(W_n - W^\star)\,dx + \\ \displaystyle -\int_\Omega T_k\left(K_{\delta_0}(x, w_n, Dw_n)\right)\,\mathrm{sign}_k(W_n)\,(W_n - W^\star)\,dx + \\ \displaystyle +\int_\Omega (1+\delta_0|w_n|)f(x)\,(W_n - W^\star)\,dx \; + \int_\Omega a_0(x)\,w_n(W_n - W^\star)\,dx + \\ \displaystyle +\int_\Omega a_0(x)\,g_{\delta_0}(w_n)\,\mathrm{sign}(w_n)\,(W_n - W^\star)\,dx. \end{cases} \tag{4.13}$$

We claim that every term of the right-hand of (4.13) tends to zero as n tends to infinity.

For the first term, we just use the fact that $W_n - W^\star$ tends to zero in $H_0^1(\Omega)$ weakly.

For the second term, we use the fact $T_k(K_{\delta_0}(x, w_n, Dw_n))\,\mathrm{sign}_k(W_n)$ is bounded in $L^\infty(\Omega)$, since k is fixed, while $W_n - W^\star$ tends to zero in $L^1(\Omega)$ strongly.

For the last three terms we observe that, since w_n and W_n respectively converge almost everywhere to w and to W^* (see (4.9) and (4.11)), we have

$$\begin{cases} (1+\delta_0|w_n|)\,f(x)\,(W_n - W^\star) \to 0 & \text{a.e. in } \Omega, \\ a_0(x)\,w_n\,(W_n - W^\star) \to 0 & \text{a.e. in } \Omega, \\ a_0(x)\,g_{\delta_0}(w_n)\,\mathrm{sign}(w_n)\,(W_n - W^\star) \to 0 & \text{a.e. in } \Omega. \end{cases} \tag{4.14}$$

We will now prove that each of the three sequences which appear in (4.14) are equiintegrable. Together with (4.14), this will imply that these sequences converge to zero in $L^1(\Omega)$ strongly, and this will prove that the three last terms of the right-hand side of (4.13) tend to zero as n tends to infinity.

In order to prove that the sequence $(1+\delta_0|w_n|)\, f(x)\, (W_n - W^\star)$ is equiintegrable, we use Hölder's inequality with $\dfrac{1}{2^\star}+\dfrac{1}{\frac{N}{2}}+\dfrac{1}{2^\star}=1$. For any measurable set E, $E\subset\Omega$, we have

$$\left\{\begin{array}{l}\displaystyle\int_E |(1+\delta|w_n|)\, f(x)\ (W_n - W^\star)|\, dx \le \\[2ex] \displaystyle\le \left(\int_E |f(x)|^{N/2} dx\right)^{2/N} \|(1+\delta_0|w_n|)\|_{2^\star}\, \|W_n - W^\star\|_{2^\star} \le c\left(\int_E |f(x)|^{N/2}\, dx\right)^{2/N},\end{array}\right.$$

where c denotes a constant which is independent of n.

Proving that the sequence $a_0(x)\, w_n\, (W_n - W^\star)$ is equiintegrable is similar, since for any measurable set E, $E\subset\Omega$, we have

$$\left\{\begin{array}{l}\displaystyle\int_E |a_0(x)\, w_n\, (W_n - W^\star)|\, dx \le \\[2ex] \displaystyle\le \left(\int_E |a_0(x)|^{N/2}\, dx\right)^{2/N} \|w_n\|_{2^\star}\, \|W_n - W^\star\|_{2^\star} \le c\left(\int_E |a_0(x)|^{N/2}\, dx\right)^{2/N}.\end{array}\right.$$

Finally, in order to prove that the sequence $a_0(x)\, g_{\delta_0}(w_n)\, \mathrm{sign}(w_n)\, (W_n - W^\star)$ is equiintegrable, we use as in (4.8) inequality (6.20) and Hölder's inequality with $\dfrac{1}{q}+\dfrac{1+\theta}{2^\star}+\dfrac{1}{2^\star}=1$; for any measurable set E, $E\subset\Omega$, we have

$$\left\{\begin{array}{l}\displaystyle\int_E |a_0(x)\, g_{\delta_0}(w_n)\, \mathrm{sign}(w_n)\, (W_n - W^\star)|\, dx \le \int_E |a_0(x)|\, G\, |w_n|^{1+\theta}\, |W_n - W^\star|\, dx \le \\[2ex] \displaystyle\le \left(\int_E |a_0(x)|^q\, dx\right)^{1/q} G\, \|w_n\|_{2^\star}^{1+\theta}\, \|W_n - W^\star\|_{2^\star} \le c\left(\int_E |a_0(x)|^q\, dx\right)^{1/q}.\end{array}\right.$$

We have proved that the right-hand side of (4.13) tends to zero. Since the matrix A is coercive (see (2.3)), this proves that W_n tends to $W^\star$ in $H_0^1(\Omega)$ strongly. Lemma 4.3 is proved. □

Proof of Theorem 4.1 Recall that in this Theorem $k>0$ is fixed.

Consider the ball B of $H_0^1(\Omega)$ defined by

$$B = \{w \in H_0^1(\Omega) : \|Dw\|_2 \le Z_{\delta_0}\}, \tag{4.15}$$

where Z_{δ_0} is defined from δ_0 by (6.18).

Consider also the mapping $S : H_0^1(\Omega) \to H_0^1(\Omega)$ defined by

$$S(w) = W, \tag{4.16}$$

where for every $w \in H_0^1(\Omega)$, W is the unique solution of (4.4) (see Lemma 4.2) .

We will apply Schauder's fixed point theorem in the Hilbert space $H_0^1(\Omega)$ to the mapping S and to the ball B.

First step In this step we prove that S maps B into itself.

Indeed by Lemma 4.2, $W = S(w)$ satisfies (4.5); therefore, when $\|Dw\|_2 \le Z_{\delta_0}$, one has, in view of the definition (6.13) the function Φ_δ and of the property (6.17) of Z_{δ_0},

$$\begin{cases} \alpha\|DW\|_2 \le \\ \le \|f\|_{H^{-1}(\Omega)} + \delta_0 C_N^2 \|f\|_{N/2}\|Dw\|_2 + C_N^2\|a_0\|_{N/2}\|Dw\|_2 + GC_N^{2+\theta}\|a_0\|_q\|Dw\|_2^{1+\theta} \le \\ \le \|f\|_{H^{-1}(\Omega)} + \delta_0 C_N^2 \|f\|_{N/2}\, Z_{\delta_0} + C_N^2\|a_0\|_{N/2} Z_{\delta_0} + GC_N^{2+\theta}\|a_0\|_q\, Z_{\delta_0}^{1+\theta} = \\ = \alpha Z_{\delta_0} + \Phi_{\delta_0}(Z_{\delta_0}) = \alpha Z_{\delta_0}, \end{cases} \tag{4.17}$$

i.e. $\|DW\|_2 \le Z_{\delta_0}$, or in other terms $W \in B$, which proves that $S(B) \subset B$.

Second step In this step we prove that S is continuous from $H_0^1(\Omega)$ strongly into $H_0^1(\Omega)$ strongly.

For this we consider a sequence such that

$$w_n \in B, \quad w_n \to w \quad \text{in} \quad H_0^1(\Omega) \quad \text{strongly}, \tag{4.18}$$

and we define W_n as $W_n = S(w_n)$, i.e. as the solution of (4.10).

The functions w_n belong to B, and therefore the functions W_n belong to B in view of the first step. We can therefore extract a subsequence, still denoted by n, such that for some $W^\star \in H_0^1(\Omega)$,

$$W_n \rightharpoonup W^\star \quad \text{in } H_0^1(\Omega) \quad \text{weakly} \quad \text{and} \quad \text{a.e.} \quad \text{in} \quad \Omega. \tag{4.19}$$

We can moreover assume that for a further subsequence, still denoted by n,

$$w_n \rightharpoonup w \quad \text{a.e.} \quad \text{in} \quad \Omega \quad \text{and} \quad Dw_n \rightharpoonup Dw \quad \text{a.e.} \quad \text{in} \quad \Omega. \tag{4.20}$$

Since the assumptions of Lemma 4.3 are satisfied by the subsequences w_n and W_n, the subsequence W_n converges to $W^\star$ in $H_0^1(\Omega)$ strongly.

We now pass to the limit in equation (4.10) as n tends to infinity by using the fact that $\mathrm{sign}_k(s)$ and $g_{\delta_0}(s)\,\mathrm{sign}(s)$ are Carathéodory functions, and the first result of (3.17) as far as $T_k(K_{\delta_0}(x, w_n, Dw_n))$ is concerned (this point is the only point of the proof of Theorem 4.1 where the assumption of strong $H_0^1(\Omega)$ convergence in (4.18), or more exactly its consequence (4.20), is used). This implies that $W^\star$ is a solution of (4.4). Since the solution of (4.4) is unique, one has $W^\star = S(w)$.

In view of the fact that $W^\star$ is uniquely determined, we conclude that it was not necessary to extract a subsequence in (4.19) and (4.20), and that the whole sequence $W_n = S(w_n)$ converges in $H_0^1(\Omega)$ strongly to $W^\star = S(w)$. This proves the continuity of the application S.

Third step In this step we prove that $S(B)$ is precompact in $H_0^1(\Omega)$.

For this we consider a sequence $w_n \in B$ and we define W_n as $W_n = S(w_n)$; in other terms W_n is the solution of (4.10). Since w_n and W_n belong to B, they are bounded in $H_0^1(\Omega)$, and we can extract a subsequence, still denoted by n, such that

$$w_n \rightharpoonup w \quad \text{in} \quad H_0^1(\Omega) \quad \text{weakly} \quad \text{and} \quad \text{a.e.} \quad \text{in} \quad \Omega,$$

$$W_n \rightharpoonup W^\star \quad \text{in} \quad H_0^1(\Omega) \quad \text{weakly} \quad \text{and} \quad \text{a.e.} \quad \text{in} \quad \Omega.$$

Since w_n and W_n satisfies the assumptions of Lemma 4.3, we have

$$W_n \to W^\star \quad \text{in} \quad H_0^1(\Omega) \quad \text{strongly}.$$

This proves that $S(B)$ is precompact in $H_0^1(\Omega)$ (note that in contrast with the second step, we do not need here to prove that $W^\star = S(w)$).

End of the proof of Theorem 4.1 We have proved that the application S and the ball B satisfy the assumptions of Schauder's fixed point theorem. Therefore there exists at least one $w_k \in B$ such that $S(w_k) = w_k$. This proves Theorem 4.1. □

5 Proof of Theorem 3.8

Theorem 4.1 asserts that for for every $k > 0$ fixed there exists at least a solution w_k of (4.1) which satisfies (4.3). We can therefore extract a subsequence, still denoted by k, such that for some $w^\star \in H_0^1(\Omega)$

$$w_k \rightharpoonup w^\star \quad \text{in} \quad H_0^1(\Omega) \quad \text{weakly} \quad \text{and} \quad \text{a.e.} \quad \text{in} \quad \Omega, \tag{5.1}$$

where $w^\star$ satisfies

$$\|w^\star\|_{H_0^1(\Omega)} \le Z_{\delta_0}, \tag{5.2}$$

i.e. (3.34).

In this Section we will first prove that for this subsequence

$$w_k \to w^\star \quad \text{in} \quad H_0^1(\Omega) \quad \text{strongly}, \tag{5.3}$$

and then that $w^\star$ is a solution of (3.33) (which satisfies (3.34)). This will prove Theorem 3.8.

To prove (5.3), we use a technique which traces back to [6] (see also [13]).

For $n > 0$, we define $G_n : \mathbb{R} \to \mathbb{R}$ as the remainder of the truncation at height n, namely

$$G_n(s) = s - T_n(s), \quad \forall s \in \mathbb{R} \tag{5.4}$$

where T_n is the truncation at height n defined by (3.35), or in other terms

$$G_n(s) = \begin{cases} s + n & \text{if} \quad s \le -n, \\ 0 & \text{if} \quad -n \le s \le n, \\ s - n & \text{if} \quad s \ge n. \end{cases} \tag{5.5}$$

First we prove the two following Lemmas.

Lemma 5.1 *Assume that* (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) *hold true. Assume moreover that the two smallness conditions* (2.14) *and* (2.15) *hold true. Let w_k be a solution of* (4.1). *Assume finally that the subsequence w_k satisfies* (5.1).

Then for this subsequence we have

$$\limsup_{k\to+\infty} \int_\Omega |DG_n(w_k)|^2 dx \to 0 \quad as \quad n \to +\infty. \tag{5.6}$$

Proof Since $G_n(w_k) \in H_0^1(\Omega)$, the use of $G_n(w_k)$ as test function in (4.1) is licit. This gives

$$\begin{cases} \displaystyle\int_\Omega A(x)Dw_k DG_n(w_k)\,dx + \int_\Omega T_k(K_{\delta_0}(x, w_k, Dw_k))\,\mathrm{sign}_k(w_k)\,G_n(w_k)\,dx = \\ \displaystyle = \int_\Omega \Big((1+\delta_0|w_k|)\,f(x) + a_0(x)\,w_k + a_0(x)\,g_{\delta_0}(w_k)\,\mathrm{sign}(w_k)\Big)\,G_n(w_k)\,dx. \end{cases} \tag{5.7}$$

Using the coercivity (2.3) of the matrix A, we have for the first term of (5.7)

$$\int_\Omega A(x)Dw_k DG_n(w_k)\,dx = \int_\Omega A(x)DG_n(w_k)DG_n(w_k)\,dx \geq \alpha \int_\Omega |DG_n(w_k|^2\,dx. \tag{5.8}$$

On the other hand, since

$$\mathrm{sign}_k(s)\,G_n(s) = |\mathrm{sign}_k(s)|\,|G_n(s)| \geq 0, \qquad \forall s \in \mathbb{R},$$

and since $T_k(K_{\delta_0}(x, w_k, Dw_k)) \geq 0$ in view of (3.11) and of $\delta_0 \geq \gamma$ (see (6.16)), we have

$$\int_\Omega T_k(K_{\delta_0}(x, w_k, Dw_k))\,\mathrm{sign}_k(w_k)\,G_n(w_k)\,dx \geq 0. \tag{5.9}$$

Finally, we observe that, by a proof which is similar to the one that we used in the proof of Lemma 4.3, we have

$$\begin{cases} \Big((1+\delta_0|w_k|)\,f(x) + a_0(x)\,w_k + a_0(x)\,g_{\delta_0}(w_k)\,\mathrm{sign}(w_k)\Big)\,G_n(w_k) \to \\ \to \Big((1+\delta_0|w^\star|)\,f(x) + a_0(x)\,w^\star + a_0(x)\,g_{\delta_0}(w^\star)\,\mathrm{sign}(w^\star)\Big)\,G_n(w^\star) \\ \text{in} \quad L^1(\Omega) \quad \text{strongly,} \end{cases} \tag{5.10}$$

since the functions in the left-hand side of (5.10) converge almost everywhere in Ω and are equi-integrable.

Together with (5.7), the three results (5.8), (5.9) and (5.10) imply that

$$\begin{cases} \displaystyle\limsup_{k\to+\infty} \alpha \int_\Omega |DG_n(w_k|^2\,dx \leq \\ \displaystyle\leq \int_\Omega \Big((1+\delta_0|w^\star|)\,f(x) + a_0(x)\,w^\star + a_0(x)\,g_{\delta_0}(w^\star)\,\mathrm{sign}(w^\star)\Big)\,G_n(w^\star)\,dx. \end{cases} \tag{5.11}$$

But since $|G_n(w^\star)| \leq |w^\star|$ and since $G_n(w^\star) = 0$ in the set $\{|w^\star| \leq n\}$, the right-hand side of (5.11) is bounded from above by

$$\int_{\{|w^\star|\geq n\}} \Big((1+\delta_0|w^\star|)\,|f(x)| + a_0(x)\,|w^\star| + a_0(x)\,|g_{\delta_0}(w^\star)|\Big)\,|w^\star|\,dx, \tag{5.12}$$

which tends to zero when n tends to infinity because the integrand belongs to $L^1(\Omega)$.

This prove (5.6). $\square$

Lemma 5.2 *Assume that* (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) *hold true. Assume moreover that the two smallness conditions* (2.14) *and* (2.15) *hold true. Let* w_k *be a solution of* (4.1). *Assume finally that the subsequence* w_k *satisfies* (5.1).

Then for this subsequence we have for every $n > 0$ *fixed*

$$T_n(w_k) \to T_n(w^\star) \quad in \quad H_0^1(\Omega) \quad strongly \quad as \quad k \to +\infty. \tag{5.13}$$

Proof In this proof n is fixed. We define

$$z_k = T_n(w_k) - T_n(w^\star), \tag{5.14}$$

and we fix a C^1 function $\psi : \mathbb{R} \to \mathbb{R}$ such that

$$\psi(0) = 0, \quad \psi'(s) - (c_0 + \delta_0)\,|\psi(s)| \geq 1/2, \quad \forall s \in \mathbb{R}, \tag{5.15}$$

where c_0 is the constant which appears in the left-hand side of assumption (2.4) on H; there exist such functions ψ: indeed an example is

$$\psi(s) = s \exp\left(\frac{(c_0 + \delta_0)^2}{4} s^2\right).$$

First step Since $z_k \in H_0^1(\Omega) \cap L^\infty(\Omega)$, and since $\psi(0) = 0$, the function $\psi(z_k)$ belongs to $H_0^1(\Omega) \cap L^\infty(\Omega)$. The use of $\psi(z_k)$ as test function in (4.1) is therefore licit. This gives

$$\begin{cases} \displaystyle\int_\Omega A(x) Dw_k Dz_k\, \psi'(z_k) dx + \int_\Omega T_k(K_{\delta_0}(x, w_k, Dw_k))\, \mathrm{sign}_k(w_k)\, \psi(z_k)\, dx \;= \\ \displaystyle = \int_\Omega \Big((1 + \delta_0 |w_k|)\, f(x) + a_0(x)\, w_k + a_0(x)\, g_{\delta_0}(w_k)\, \mathrm{sign}(w_k)\Big)\, \psi(z_k)\, dx. \end{cases} \tag{5.16}$$

Since

$$Dw_k = DT_n(w_k) + DG_n(w_k) = Dz_k + DT_n(w^\star) + DG_n(w_k), \tag{5.17}$$

the first term of the left-hand side of (5.16) reads as

$$\begin{cases} \displaystyle\int_\Omega A(x) Dw_k Dz_k\, \psi'(z_k)\, dx = \int_\Omega A(x) Dz_k Dz_k\, \psi'(z_k)\, dx \,+ \\ \displaystyle + \int_\Omega A(x) DT_n(w^\star) Dz_k\, \psi'(z_k)\, dx \;+ \int_\Omega A(x) DG_n(w_k) Dz_k\, \psi'(z_k)\, dx. \end{cases} \tag{5.18}$$

On the other hand, splitting Ω into $\Omega = \{|w_k| > n\} \cup \{|w_k| \leq n\}$, the second term of the left-hand side of (5.16) reads as

$$\begin{cases} \displaystyle\int_\Omega T_k(K_{\delta_0}(x, w_k, Dw_k))\, \mathrm{sign}_k(w_k)\, \psi(z_k)\, dx \;= \\ \displaystyle = \int_{\{|w_k| > n\}} T_k(K_{\delta_0}(x, w_k, Dw_k))\, \mathrm{sign}_k(w_k)\, \psi(z_k)\, dx \,+ \\ \displaystyle + \int_{\{|w_k| \leq n\}} T_k(K_{\delta_0}(x, w_k, Dw_k))\, \mathrm{sign}_k(w_k)\, \psi(z_k)\, dx. \end{cases} \tag{5.19}$$

For what concerns the first term of the right-hand of (5.19), we claim that

$$\int_{\{|w_k|>n\}} T_k(K_{\delta_0}(x,w_k,Dw_k))\,\mathrm{sign}_k(w_k)\,\psi(z_k)\,dx \geq 0\,; \tag{5.20}$$

indeed in $\{|w_k|>n\}$, the integrand is nonnegative since on the first hand $T_k(K_{\delta_0}(x,w_k,Dw_k)) \geq 0$ in view of (3.11) and of $\delta_0 \geq \gamma$ (see (6.16)), and since on the other hand one has

$$\mathrm{sign}_k(w_k)\,\psi(z_k) \geq 0 \quad \text{in} \quad \{|w_k|>n\}; \tag{5.21}$$

indeed since $\mathrm{sign}(s)$ and $\mathrm{sign}_k(s)$ have the same sign, it is equivalent either to prove (5.21) or to prove that

$$\mathrm{sign}(w_k)\,\psi(z_k) \geq 0 \quad \text{in} \quad \{|w_k|>n\}; \tag{5.22}$$

but in $\{|w_k|>n\}$ one has $z_k = n\,\mathrm{sign}(w_k) - T_n(w^\star)$, and therefore $\mathrm{sign}(z_k) = \mathrm{sign}(w_k)$; this implies that

$$\mathrm{sign}(w_k)\,\psi(z_k) = \mathrm{sign}(z_k)\,\psi(z_k) = |\psi(z_k)| \quad \text{in} \quad \{|w_k|>n\},$$

which proves (5.22).

For what concerns the second term of the right-hand side of (5.19), we observe that, in view of (3.11) and of $\delta_0 \geq \gamma$ (see(6.16)), we have

$$\begin{cases} |T_k(K_{\delta_0}(x,w_k,Dw_k))\,\mathrm{sign}_k(w_k)\,\psi(z_k)| \leq |K_{\delta_0}(x,w_k,Dw_k)||\psi(z_k)| \leq \\ \leq (c_0+\delta_0)|\psi(z_k)|\,A(x)Dw_kDw_k. \end{cases} \tag{5.23}$$

Since in view of (5.17) one has

$$Dw_k = Dz_k + DT_n(w^\star) \quad \text{in} \quad \{|w_k| \leq n\},$$

we obtain

$$\begin{cases} \displaystyle\int_{\{|w_k|\leq n\}} T_k(K_{\delta_0}(x,w_k,Dw_k))\,\mathrm{sign}_k(w_k)\,\psi(z_k)\,dx \geq \\ \displaystyle\geq -\int_{\{|w_k|\leq n\}} (c_0+\delta_0)|\psi(z_k)|\,A(x)Dw_kDw_k\,dx = \\ \displaystyle= -\int_{\{|w_k|\leq n\}} (c_0+\delta_0)|\psi(z_k)|\,A(x)(Dz_k+DT_n(w^\star))(Dz_k+DT_n(w^\star))\,dx \geq \\ \displaystyle\geq -\int_\Omega (c_0+\delta_0)|\psi(z_k)|\,A(x)(Dz_k+DT_n(w^\star))(Dz_k+DT_n(w^\star))\,dx \geq \\ \displaystyle\geq -\int_\Omega (c_0+\delta_0)|\psi(z_k)|\,A(x)Dz_kDz_k\,dx + \\ \displaystyle-\int_\Omega (c_0+\delta_0)|\psi(z_k)| \\ \qquad \Big(A(x)DT_n(w^\star)Dz_k + A(x)Dz_k\,DT_n(w^\star) + A(x)DT_n(w^\star)DT_n(w^\star)\Big)\,dx. \end{cases} \tag{5.24}$$

From (5.16), (5.18), (5.19), (5.20) and (5.24) we deduce that

$$\begin{cases} \displaystyle\int_\Omega A(x)Dz_k Dz_k \left(\psi'(z_k) - (c_0+\delta_0)|\psi(z_k)|\right) dx \le \\ \le -\displaystyle\int_\Omega A(x)DT_n(w^\star)Dz_k\, \psi'(z_k)dx - \int_\Omega A(x)DG_n(w_k)Dz_k\, \psi'(z_k)\, dx + \\ + \displaystyle\int_\Omega (c_0+\delta_0)|\psi(z_k)| \\ \qquad \Big(A(x)DT_n(w^\star)Dz_k + A(x)Dz_k DT_n(w^\star) + A(x)DT_n(w^\star)DT_n(w^\star)\Big)\, dx + \\ + \displaystyle\int_\Omega \Big((1+\delta_0|w_k|)\, f(x) + a_0(x)\, w_k + a_0(x)\, g_{\delta_0}(w_k)\,\mathrm{sign}(w_k)\Big)\, \psi(z_k)\, dx. \end{cases} \tag{5.25}$$

Second step We claim that each term of the right-hand side of (5.25) tends to zero as k tends to infinity. Since $\psi'(z_k) - (c_0+\delta)\,|\psi(z_k)| \ge 1/2$ by (5.15), and since the matrix A is coercive (see (2.3)), this will imply that

$$z_k \to 0 \quad \text{in} \quad H_0^1(\Omega) \quad \text{strongly},$$

or in other terms (see the definition (5.14) of z_k) that

$$T_n(w_k) \to T_n(w^\star) \quad \text{in} \quad H_0^1(\Omega) \quad \text{strongly} \quad \text{as} \quad k \to +\infty,$$

which is nothing but (5.13). Lemma 5.2 will therefore be proved whenever the claim will be proved.

In order to prove the claim let us recall that in view of (5.1) and of the definition (5.14) of z_k one has

$$z_k \rightharpoonup 0 \quad \text{in} \quad H_0^1(\Omega) \quad \text{weakly}, \quad L^\infty(\Omega) \quad \text{weakly star} \quad \text{and} \quad \text{a.e.} \quad \text{in} \quad \Omega \quad \text{as} \quad k \to +\infty.$$

Since $\psi(0) = 0$, this implies that $\psi(z_k)$ tends to zero almost everywhere in Ω and in $L^\infty(\Omega)$ weakly star as k tends to infinity, which in turn implies that

$$Dz_k\, \psi'(z_k) = D\psi(z_k) \rightharpoonup 0 \quad \text{in} \quad L^2(\Omega)^N \quad \text{weakly} \quad \text{as} \quad k \to +\infty.$$

This implies that the first term of the right-hand side of (5.25) tends to zero as k tends to infinity.

For the second term of the right-hand side of of (5.25) we observe that

$$A(x)DG_n(w_k)Dz_k = A(x)DG_n(w_k)(DT_n(w_k) - DT_n(w^\star)) = -A(x)DG_n(w_k)DT_n(w^\star),$$

and that by Lebesgue's dominated convergence theorem

$$DT_n(w^\star)\, \psi'(z_k) \to DT_n(w^\star)\, \psi'(0) \quad \text{in} \quad L^2(\Omega)^N \quad \text{strongly} \quad \text{as} \quad k \to +\infty,$$

while $DG_n(w_k)$ tends to $DG_n(w^\star)$ weakly in $L^2(\Omega)^N$. Since $A(x)DG_n(w^\star)DT_n(w^\star) = 0$ almost everywhere, the second term of the right-hand side of (5.25) tends to zero.

For the third term of the right-hand side of (5.25), we observe that

$$(c_0+\delta_0)|\psi(z_k)|A(x)DT_n(w^\star)\to 0 \quad \text{in } L^2(\Omega)^N \quad \text{strongly} \quad \text{as} \quad k\to+\infty$$

by Lebesgue's dominated convergence theorem, since $\psi(z_k)$ is bounded in $L^\infty(\Omega)$ and since $\psi(z_k)$ tends almost everywhere to zero because $\psi(0)=0$. Since Dz_k is bounded in $L^2(\Omega)^N$, this implies that the first part of this third term tends to zero. A similar proof holds true for the two others parts of this third term.

Finally the fourth term of the right-hand side of (5.25) tends to zero by a proof which is similar to the one that we used in the proof of Lemma 4.3, since the integrand converges almost everywhere to zero and is equiintegrable.

The claim made at the beginning of the second step is proved. This completes the proof of Lemma 5.2. □

End of the proof of Theorem 3.8

First step Since we have

$$w_k-w^\star=T_n(w_k)+G_n(w_k)-T_n(w^\star)-G_n(w^\star),$$

and since by Lemma 5.2 (see (5.13)) we have

$$\|T_n(w_k)-T_n(w^\star)\|_{H_0^1(\Omega)}\to 0 \quad \text{as} \quad k\to+\infty \quad \text{for every} \quad n>0 \text{ fixed},$$

while by Lemma 5.1 (see (5.6)) we have

$$\limsup_{n\to+\infty}\limsup_{k\to+\infty}\|G_n(w_k)\|_{H_0^1(\Omega)}=0,$$

and while we have

$$\limsup_{n\to+\infty}\|G_n(w^\star)\|_{H_0^1(\Omega)}=0,$$

since $w^\star\in H_0^1(\Omega)$, we conclude that

$$w_k\to w^\star \quad \text{in} \quad H_0^1(\Omega) \quad \text{strongly} \quad \text{as} \quad k\to+\infty, \tag{5.26}$$

which is nothing but (5.3).

Second step Let us now pass to the limit in (4.1) as k tends to infinity. This is easy for the first term of the left-hand side of (4.1) as well as for the three terms of the right-hand side of (4.1), which pass to the limit in $(L^{2^\star}(\Omega))'$ strongly by a proof which is similar to the one that we used in the proof of Lemma 4.3.

It remains to pass to the limit in the second term of the left-hand side of (4.1), namely in

$$T_k\big(K_{\delta_0}(x,w_k,Dw_k)\big)\,\mathrm{sign}_k(w_k).$$

We first observe that in view of (3.11) and of $\delta_0 \geq \gamma$ (see (6.16)), we have

$$|T_k\big(K_{\delta_0}(x, w_k, Dw_k)\big)\,\mathrm{sign}_k(w_k)| \leq |K_{\delta_0}(x, w_k, Dw_k)| \leq (c_0 + \delta_0)\,\|A\|_\infty |Dw_k|^2 \quad \text{a.e. in } \Omega,$$

which implies that the functions $T_k\big(K_{\delta_0}(x, w_k, Dw_k)\big)\,\mathrm{sign}_k(w_k)$ are equiintegrable since Dw_k converges strongly to $Dw^\star$ in $L^2(\Omega)^N$.

Extracting if necessary a subsequence, still denoted by k, such that

$$Dw_k \to Dw^\star \quad \text{a.e. in } \Omega,$$

we claim that

$$T_k\big(K_{\delta_0}(x, w_k, Dw_k)\big)\,\mathrm{sign}_k(w_k) \to K_{\delta_0}(x, w^\star, Dw^\star)\,\mathrm{sign}(w^\star) \quad \text{a.e. in } \Omega. \tag{5.27}$$

On the first hand we use the first part of (3.17), which asserts that

$$K_{\delta_0}(x, w_k, Dw_k) \to K_{\delta_0}(x, w^\star, Dw^\star) \quad \text{a.e. in } \Omega,$$

and the fact that for every $s \in \mathbb{R}$

$$T_k(s_k) \to s \quad \text{if} \quad k \to +\infty \quad \text{when} \quad s_k \to s,$$

to deduce that

$$T_k\big(K_{\delta_0}(x, w_k, Dw_k)\big) \to K_{\delta_0}(x, w^\star, Dw^\star) \quad \text{a.e. in } \Omega. \tag{5.28}$$

On the other hand we use the fact that

$$\mathrm{sign}_k(w_k) \to \mathrm{sign}(w^\star) \quad \text{a.e. in } \{y \in \Omega : w^\star(y) \neq 0\},$$

which together with (5.28) proves the convergence (5.27) in the set $\{y \in \Omega : w^\star(y) \neq 0\}$.

Finally, as far as the convergence in the set $\{y \in \Omega : w^\star(y) = 0\}$ is concerned, convergence (5.28), the fact that (see (3.23))

$$K_{\delta_0}(x, w^\star, Dw^\star) = 0 \quad \text{a.e. in } \{y \in \Omega : w^\star(y) = 0\},$$

and the fact that $|\mathrm{sign}_k(s)| \leq 1$ for every $s \in \mathbb{R}$ together prove that

$$T_k\big(K_{\delta_0}(x, w_k, Dw_k)\big)\,\mathrm{sign}_k(w_k) \to 0 = K_{\delta_0}(x, w^\star, Dw^\star)\,\mathrm{sign}(w^\star) \quad \text{a.e. in } \{y \in \Omega : w^\star(y) = 0\}.$$

This completes the proof of (5.27).

The equiintegrability and the almost everywhere convergence of $T_k\big(K_{\delta_0}(x, w_k, Dw_k)\big)\,\mathrm{sign}_k(w_k)$ then imply that

$$T_k\big(K_{\delta_0}(x, w_k, Dw_k)\big)\,\mathrm{sign}_k(w_k) \to K_{\delta_0}(x, w^\star, Dw^\star)\,\mathrm{sign}(w^\star) \quad \text{in } L^1(\Omega) \text{ strongly.}$$

This proves that $w^\star$ satisfies (3.33). Since $w^\star$ also satisfies (3.34) (see (5.2)), Theorem 3.8 is proved. □

6 Appendix

In this Appendix, we give an estimate of the function g_δ defined by (3.7) (see Lemma 6.1), and the definitions of the constants δ_0 and Z_{δ_0} which appear in Theorem 2.1 (see Lemma 6.2).

6.1 An estimate for the function $\mathbf{g}_\delta$

Lemma 6.1 *For $\delta > 0$, let $g_\delta : \mathbb{R} \to \mathbb{R}$ be the function defined by (3.7), i.e. by*

$$g_\delta(t) = -|t| + \frac{1}{\delta}(1 + \delta|t|) \log(1 + \delta|t|), \quad \forall t \in \mathbb{R}. \tag{6.1}$$

Then, for every λ and $\delta_\star$ with

$$0 < \lambda < 1, \quad 0 < \delta_\star < +\infty, \tag{6.2}$$

there exists a constant $C(\lambda)$ which depends only on λ, with

$$0 < C(\lambda) \leq \sup\left\{1\,,\ \frac{2^{1+\lambda}}{\lambda e}\right\}, \tag{6.3}$$

such that

$$0 \leq g_\delta(t) \leq \delta_\star^\lambda C(\lambda)|t|^{1+\lambda}, \quad \forall t \in \mathbb{R}, \quad \forall \delta,\ 0 < \delta \leq \delta_\star. \tag{6.4}$$

Moreover

$$0 \leq g_\delta(t) < \delta_\star^\lambda C(\lambda)|t|^{1+\lambda}, \quad \forall t \in \mathbb{R},\ t \neq 0, \quad \forall \delta,\ 0 < \delta \leq \delta_\star. \tag{6.5}$$

Proof Let $g : \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$g(\tau) = -\tau + (1 + \tau) \log(1 + \tau), \quad \forall \tau \geq 0.$$

Since $g(0) = 0$ and $g'(\tau) \geq 0$, one has

$$g(\tau) \geq 0, \quad \forall \tau \geq 0. \tag{6.6}$$

On the other hand, since $\log(1 + \tau) < \tau$ for $\tau > 0$, one has $g(\tau) < \tau^2 \leq \tau^{1-\lambda}\tau^{1+\lambda}$ for $\tau > 0$, and therefore for $0 < \lambda < 1$ and for every $m > 0$

$$g(\tau) < m^{1-\lambda}\tau^{1+\lambda}, \quad \forall \tau,\ 0 < \tau \leq m. \tag{6.7}$$

One has also

$$\frac{g(\tau)}{\tau^{1+\lambda}} < \frac{(1 + \tau) \log(1 + \tau)}{\tau^{1+\lambda}} = \left(\frac{1 + \tau}{\tau}\right)^{1+\lambda} \frac{\log(1 + \tau)}{(1 + \tau)^\lambda}, \quad \forall \tau > 0,$$

and therefore

$$\frac{g(\tau)}{\tau^{1+\lambda}} < \left(\frac{1 + m}{m}\right)^{1+\lambda} \frac{\log(1 + \tau)}{(1 + \tau)^\lambda} \quad \forall \tau \geq m > 0.$$

But the function $\dfrac{\log(1 + \tau)}{(1 + \tau)^\lambda}$ reaches its maximum for τ_0 defined by $(1 + \tau_0) = e^{1/\lambda}$, hence

$$\frac{\log(1 + \tau)}{(1 + \tau)^\lambda} \leq \frac{1}{\lambda e}, \quad \forall \tau \geq 0.$$

This implies that for $0 < \lambda < 1$ and for every $m > 0$

$$g(\tau) < \left(\frac{1+m}{m}\right)^{1+\lambda} \frac{1}{\lambda e} \ \tau^{1+\lambda}, \quad \forall \tau \geq m, \tag{6.8}$$

which, with (6.6) and (6.7), implies that for $0 < \lambda < 1$ and for every $m > 0$

$$0 \leq g(\tau) < \sup \left\{ m^{1-\lambda} \, , \, \left(\frac{1+m}{m}\right)^{1+\lambda} \frac{1}{\lambda e} \right\} \tau^{1+\lambda}, \quad \forall \tau > 0, \tag{6.9}$$

or in other terms that for every $\lambda, \ 0 < \lambda < 1$,

$$0 \leq g(\tau) < C(\lambda) \, \tau^{1+\lambda}, \quad \forall \tau > 0, \tag{6.10}$$

for some constant $C(\lambda)$, with (take $m = 1$)

$$0 < C(\lambda) \leq \sup \left\{ 1 \, , \, \frac{2^{1+\lambda}}{\lambda e} \right\},$$

which is nothing but (6.3).

Since

$$g_\delta(t) = \frac{1}{\delta} g(\delta |t|), \quad \forall t \in \mathbb{R},$$

one deduces from (6.10) that g_δ satisfies

$$\begin{cases} 0 \leq g_\delta(t) \leq \delta^\lambda C(\lambda) |t|^{1+\lambda}, \quad \forall t \in \mathbb{R}, \quad \forall \delta > 0, \\ \\ 0 \leq g_\delta(t) < \delta^\lambda C(\lambda) |t|^{1+\lambda}, \quad \forall t \in \mathbb{R}, \ t \neq 0, \quad \forall \delta > 0, \end{cases}$$

which proves (6.4) and (6.5) with a constant $C(\lambda)$ which satisfies (6.3). □

6.2 Definition of δ_0 and Z_{δ_0}

The goal of this Subsection is to define the constants δ_0 and Z_{δ_0} which appear in Theorem 2.1. We will prove the following result.

Lemma 6.2 *Assume that* (2.2), (2.3), (2.4), (2.6), (2.7) and (2.8) hold true. Assume moreover that the two smallness conditions (2.14) *and* (2.15) *hold true.*

Let δ_1 be the number defined by

$$\delta_1 = \frac{\alpha - C_N^2 \|a_0\|_{N/2}}{C_N^2 \|f\|_{N/2}}. \tag{6.11}$$

One has

$$\delta_1 > \gamma. \tag{6.12}$$

For $\delta \geq 0$, let $\Phi_\delta : \mathbb{R}^+ \to \mathbb{R}$ (see Figure 2) be the function defined by

$$\Phi_\delta(X) = G C_N^{2+\theta} \|a_0\|_q X^{1+\theta} - (\alpha - C_N^2 \|a_0\|_{N/2} - \delta C_N^2 \|f\|_{N/2}) X + \|f\|_{H^{-1}(\Omega)}, \tag{6.13}$$

where θ is defined by (2.11) *(note that $0 < \theta < 1$ in view of* (2.12)*) and where G is the constant defined by*

$$G = \left(\frac{\alpha - C_N^2 \|a_0\|_{N/2}}{C_N^2 \|f\|_{N/2}} \right)^{\theta} C(\theta), \tag{6.14}$$

with C_N the best constant in the Sobolev's inequality (2.9) *and $C(\theta)$ the constant which appears in* (6.4)*(see also* (6.3)*).*

Then, for $0 \leq \delta \leq \delta_1$, the function Φ_δ has a unique minimizer Z_δ on $\mathbb{R}^+$, which is given by

$$Z_\delta = \left(\frac{\alpha - C_N^2 \|a_0\|_{N/2} - \delta C_N^2 \|f\|_{N/2}}{(1+\theta) G C_N^{2+\theta} \|a_0\|_q} \right)^{1/\theta}, \quad for \quad 0 \leq \delta \leq \delta_1. \tag{6.15}$$

Moreover, there exists a unique number δ_0 such that

$$\gamma \leq \delta_0 < \delta_1, \tag{6.16}$$

and

$$\Phi_{\delta_0}(Z_{\delta_0}) = 0. \tag{6.17}$$

This number is the number δ_0 which appear in Theorem 2.1, and Z_{δ_0} is then defined from δ_0 through formula (6.15)*, namely by*

$$Z_{\delta_0} = \left(\frac{\alpha - C_N^2 \|a_0\|_{N/2} - \delta_0 C_N^2 \|f\|_{N/2}}{(1+\theta) G C_N^{2+\theta} \|a_0\|_q} \right)^{1/\theta}. \tag{6.18}$$

Remark 6.3 Let us explain the meaning of the results stated in Lemma 6.2.

As we will see in the proof of Lemma 6.2 (see also Figure 2), the function Φ_δ is the restriction to $\mathbb{R}^+$ of a function which looks like a convex parabola. This function attains its minimum at a unique point Z_δ, and for δ which satisfies $\delta < \delta_1$ with δ_1 given by (6.11), one has $Z_\delta > 0$.

The smallness condition (2.14) is equivalent to the fact that $\delta_1 > \gamma$, and the smallness condition (2.15) to the fact that the minimum $\Phi_\gamma(Z_\gamma)$ of Φ_γ is nonpositive. For $\delta = \delta_1$, the minimum $\Phi_{\delta_1}(Z_{\delta_1})$ of Φ_{δ_1} is equal to $\|f\|_{H^{-1}(\Omega)}$, which is strictly positive. Therefore it can be proved that there exists some δ_0 with $\gamma \leq \delta_0 < \delta_1$ (see (6.16)) such that the minimum $\Phi_{\delta_0}(Z_{\delta_0})$ of Φ_{δ_0} is equal to zero (see (6.17)), or in other terms such that the function Φ_{δ_0} has a double zero in Z_{δ_0}. Moreover, when $\gamma < \delta_0$, for every δ with $\gamma \leq \delta < \delta_0$, the function Φ_δ has two distinct zeros Y_δ^- and Y_δ^+ with $Y_\delta^- < Y_\delta^+$ which satisfy $0 < Y_\delta^- < Z_{\delta_0} < Y_\delta^+$ (see (6.25) in Remark 6.5). □

Remark 6.4 In the present paper we use Lemma 6.1 with $\lambda = \theta$ defined by (2.11) (note that $0 < \theta < 1$ in view of (2.12)) and with $\delta_\star = \delta_1$ defined by (6.11). Using the fact that G defined by (6.14) is nothing but $G = \delta_1^\theta C(\theta)$, inequalities (6.4) and (6.5) imply that

$$\begin{cases} 0 \leq g_\delta(t) \leq \delta_1^\theta C(\theta) |t|^{1+\theta} = G|t|^{1+\theta}, \quad \forall t \in \mathbb{R}, \quad \forall \delta,\ 0 < \delta \leq \delta_1, \\ 0 \leq g_\delta(t) < \delta_1^\theta C(\theta) |t|^{1+\theta} = G|t|^{1+\theta}, \quad \forall t \in \mathbb{R},\ t \neq 0, \quad \forall \delta,\ 0 < \delta \leq \delta_1. \end{cases} \tag{6.19}$$

In particular for $\delta = \delta_0$ defined by (6.16) and (6.17) one has

$$0 \leq g_{\delta_0}(t) \leq G|t|^{1+\theta}, \quad \forall t \in \mathbb{R}. \tag{6.20}$$

□

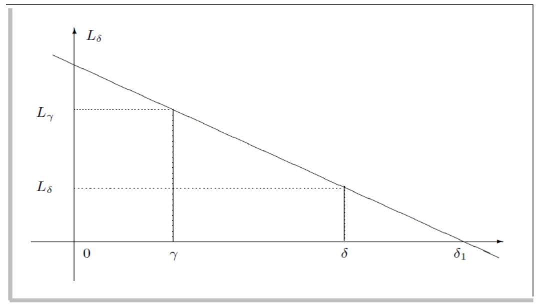

Figure 1: The graph of the straight line L_δ

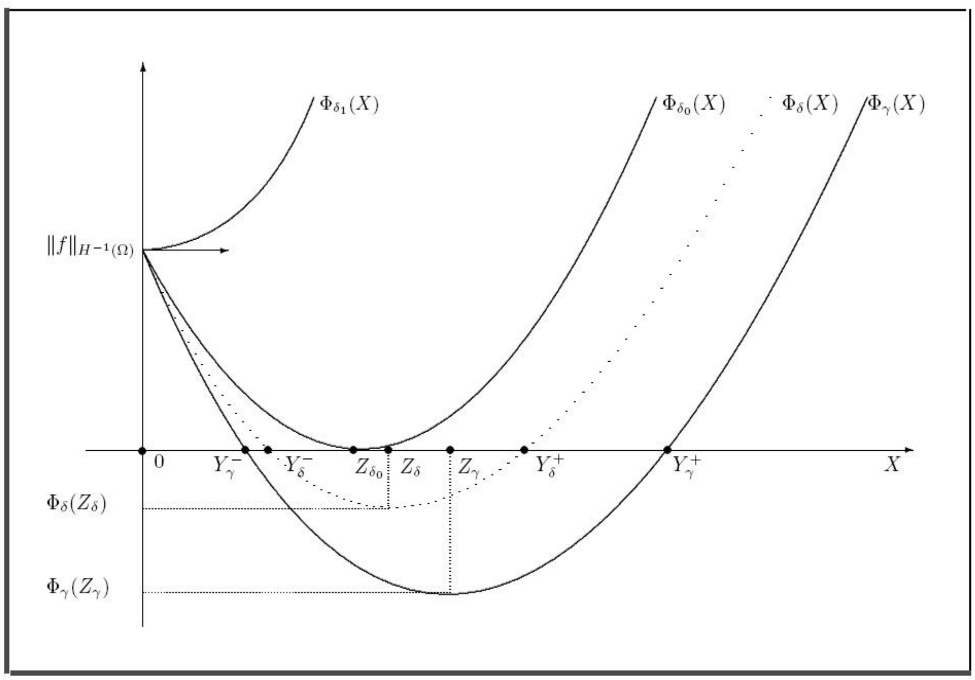

Figure 2: The graphs of the functions $\Phi_\delta(X)$ for $\delta = \gamma$, $\gamma < \delta < \delta_0, \delta = \delta_0$ and $\delta = \delta_1$

Proof of Lemma 6.2 For $\delta \geq 0$, let L_δ be the constant defined by (see Figure 1)

$$L_\delta = \alpha - C_N^2 \|a_0\|_{N/2} - \delta C_N^2 \|f\|_{N/2}, \tag{6.21}$$

where C_N is the best constant in the Sobolev's inequality (2.9). Note that L_δ is decreasing with respect to δ.

Since δ_1 is defined by (6.11), one has $L_{\delta_1} = 0$. On the other hand, the first smallness condition (2.14) is nothing but $L_\gamma > 0$. Since L_δ is decreasing in δ, one has $\delta_1 > \gamma$, i.e. (6.12).

Let us now study the family of functions $\Phi_\delta : \mathbb{R}^+ \to \mathbb{R}$ defined by (6.13), i.e., in view of the definition (6.21) of L_δ, by

$$\Phi_\delta(X) = GC_N^{2+\theta}\|a_0\|_q X^{1+\theta} - L_\delta X + \|f\|_{H^{-1}(\Omega)}, \quad \forall X \geq 0, \tag{6.22}$$

(see Figure 2).

Since $a_0 \neq 0$ (see (2.6)), each function Φ_δ looks like the restriction to $\mathbb{R}^+$ of a convex parabola. When $0 \leq \delta \leq \delta_1$, one has $L_\delta \geq 0$, and this convex parabola has a unique minimizer Z_δ on $\mathbb{R}^+$ which is also the minimizer of the function Φ_δ. A simple computation shows that Z_δ is given by

$$Z_\delta = \left(\frac{L_\delta}{(1+\theta)GC_N^{2+\theta}\|a_0\|_q}\right)^{1/\theta} = \left(\frac{\alpha - C_N^2\|a_0\|_{N/2} - \delta C_N^2\|f\|_{N/2}}{(1+\theta)GC_N^{2+\theta}\|a_0\|_q}\right)^{1/\theta}, \tag{6.23}$$

i.e (6.15), and that the minimum of Φ_δ, namely $\Phi_\delta(Z_\delta)$, is given by

$$\begin{cases} \Phi_\delta(Z_\delta) = \|f\|_{H^{-1}(\Omega)} - \dfrac{\theta}{1+\theta}\,\dfrac{L_\delta^{(1+\theta)/\theta}}{((1+\theta)GC_N^{2+\theta}\|a_0\|_q)^{1/\theta}} = \\[2ex] = \|f\|_{H^{-1}(\Omega)} - \dfrac{\theta}{1+\theta}\dfrac{(\alpha - C_N^2\|a_0\|_{N/2} - \delta C_N^2\|f\|_{N/2})^{(1+\theta)/\theta}}{((1+\theta)GC_N^{2+\theta}\|a_0\|_q)^{1/\theta}}. \end{cases} \tag{6.24}$$

When $0 \leq \delta \leq \delta_1$, the function L_δ is nonnegative, continuous and decreasing with respect to δ. Therefore Z_δ is continuous and decreasing with respect to δ, while $\Phi_\delta(Z_\delta)$ is continuous and increasing with respect to δ.

When $\delta = \delta_1$, one has $L_{\delta_1} = 0$, the function Φ_{δ_1} attains its minimum in $Z_{\delta_1} = 0$, and $\Phi_{\delta_1}(Z_{\delta_1}) = = \|f\|_{H^{-1}(\Omega)} > 0$, while the second smallness condition (2.15) is nothing but $\Phi_\gamma(Z_\gamma) \leq 0$. Therefore there exists a unique δ_0 with $\gamma \leq \delta_0 < \delta_1$ such that $\Phi_{\delta_0}(Z_{\delta_0}) = 0$. This is the definition of δ_0 given by (6.16) and (6.17) in Lemma 6.2.

Lemma 6.2 is proved. □

Remark 6.5 The case where equality takes places in inequality (2.15) corresponds to the case where $\delta_0 = \gamma$.

On the other hand, when (2.15) is a strict inequality, one has $\gamma < \delta_0$, and for δ with $\gamma \leq \delta < \delta_0$, the function Φ_δ has two distinct zeros Y_δ^- and Y_δ^+ with $0 < Y_\delta^- < Y_\delta^+$. Since

$$\begin{cases} \Phi_\delta(X) = GC_N^{2+\theta}\|a_0\|_q X^{1+\theta} - (\alpha - C_N^2\|a_0\|_{N/2} - \delta C_N^2\|f\|_{N/2})X + \|f\|_{H^{-1}(\Omega)} = \\[1ex] = \Phi_0(X) + \delta C_N^2\|f\|_{N/2}X, \end{cases}$$

the family of functions Φ_δ is an increasing family of functions on $\mathbb{R}^+$, and one has

$$0 < Y_\delta^- < Z_{\delta_0} < Y_\delta^+ \quad \text{if} \quad \gamma \leq \delta < \delta_0. \tag{6.25}$$

□

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