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Abstract

In this paper, we consider rods whose thickness vary linearly between ϵ and ϵ^2. Our aim is to study the asymptotic behavior of these rods in the framework of the linear elasticity. We use a decomposition method of the displacement fields of the form $u = U_e + \bar{u}$, where U_e stands for the translation-rotations of the cross-sections and $\bar{u}$ is related to their deformations. We establish a priori estimates. Passing to the limit in a fixed domain gives the problems satisfied by the bending, the stretching and the torsion limit fields which are ordinary differential equations depending on weights.

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1. Introduction

In this paper we are interested in analyzing the asymptotic behavior of a thin rod with different order of thickness in the framework of the linear elasticity. We consider a straight rod of fixed length where the cross-sections are bounded Lipschitz domains with small diameter of order varying between ϵ and ϵ^2. To be more precise, the order of the thickness of the rod is given by $\epsilon\rho_\epsilon(\cdot)$ where $\rho_\epsilon(\cdot)$ is a linear function depending on the cross-section of the rod such that it is 1 at the bottom and ϵ at the top of the rod. We investigate how the variable thickness of the rod affects to the a priori estimates and the limit problems.

Since the diameter of the rod tends to zero, this work belongs to the field of elliptic problems posed on thin domains. Many fields of science involve the study of thin domains, for example in solid mechanics (thin rods, plates, shells), fluid dynamics (lubrication, meteorology problems, ocean dynamics), physiology (blood circulation), etc. There are many papers dedicated to the study of the thin structures from the point of view of the elasticity, see e.g. [22, 21] for models of rods and [3, 4] for plates and shells.

Our work is based on the decomposition of a displacement of the rod according to [19]. Every displacement of the rod is the sum of an elementary displacement, it characterizes the translation and the rotation of the cross-sections, and a warping which is the residual displacement related to the deformation of the cross-section. This decomposition of the rod was introduced in [15] and [16] and it allows to obtain the Korn inequality as well as the asymptotic behavior of the strain tensor of a sequence of displacements in a simple and effective way.

The notion of the elementary displacement together with the unfolding method (see [8, 9]) has led to a new method in elasticity which has been successfully applied to many problems, see e.g [5, 6, 7] and

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[13, 14, 15, 16, 17, 18, 19]. References and other applications of the unfolding operator technique can be found in [10, 11, 12, 1].

Our paper is organized as follows. In Section 2 we describe the geometry of the rod, introduce the decomposition of a displacement field of the rod and we give some estimates of the decomposition fields in terms of the strain energy (Theorem 2.3). The proof of the Theorem 2.3 is based on the approximation of the displacement of the rod by a rigid body displacement. Of course, the estimates may depend on the function $\rho_\epsilon(\cdot)$.

Section 3 is dedicated to get a priori estimates for the different fields assuming that the rod is clamped at the bottom. These estimates have an essential importance in our study to pass to the limit. Moreover, we introduce the rescaling operator Π_ϵ which allows to work in a fixed domain. One particular feature of this transformation is that the ratio of the dilation of the fixed rod depends on the third variable, it is given by the function $\epsilon\rho_\epsilon(\cdot)$. Then a special care is dedicated to the estimate of the derivatives with respect to the third variable.

In Section 4 we give the limit of the displacements and we show a few relations between some of them. Since some of the a priori estimates established depend on the variable thickness $\rho_\epsilon(\cdot)$ we introduce some weighted Sobolev spaces which allow to obtain the limit fields in a natural way. In Section 5 we pose the problem of elasticity and we specify the assumptions on the applied forces. We show that the choice of the applied forces is reasonable to get the suitable estimate of the total elastic energy, so that the convergence results of the previous sections can be used. In Section 6 we derive the equations satisfied by the limit fields and we prove the strong convergence of the energy. Moreover, we deduce some strong convergences of the fields of the displacement's decomposition. Finally, in Section 7 we summarize the main results.

2. Decomposition of the displacement of a straight rod with different order of thickness

Let ω be a bounded domain in $\mathbb{R}^2$ with Lipschitzian boundary, diameter equal to R and star-shaped with respect to a disc of radius R_1. We choose the origin O of coordinates at the center of gravity of ω and we choose as coordinates axes $(O;\mathbf{e}_1)$ and $(O;\mathbf{e}_2)$ the principal axes of inertia of ω. Notice that, with this reference frame we have

$$\int_\omega x_1\,dx_1dx_2 = \int_\omega x_2\,dx_1dx_2 = \int_\omega x_1x_2\,dx_1dx_2 = 0. \tag{2.1}$$

The cross-section ω_{ϵ,x_3} of the rod is obtained by transforming ω with a dilatation of center O and ratio $\epsilon\rho_\epsilon(x_3)$, where

$$\rho_\epsilon(x_3) = 1 - \frac{x_3}{L}\big(1-\frac{\epsilon}{L}\big), \qquad x_3\in[0,L].$$

We assume $0<\epsilon<L/2$ and $0<R_1<1/2$ without loss of generality.

Definition 2.1. *The straight rod is defined as follows:*

$$\Omega_\epsilon = \{x=(x_1,x_2,x_3)\in\mathbb{R}^3 \mid x_3\in(0,L), (x_1,x_2)\in\omega_{\epsilon,x_3}\},$$

where $\omega_{\epsilon,x_3} = \Big\{(x_1,x_2)\in\mathbb{R}^2 \ \mid\ \Big(\dfrac{x_1}{\epsilon\rho_\epsilon(x_3)},\dfrac{x_2}{\epsilon\rho_\epsilon(x_3)}\Big)\in\omega\Big\} = \epsilon\rho_\epsilon(x_3)\omega.$

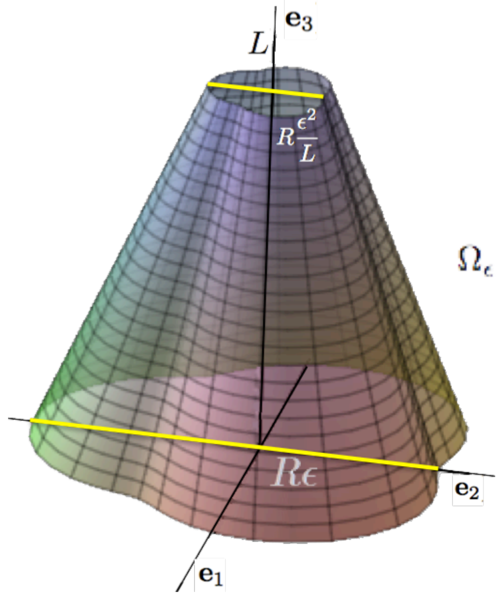

Figure 1: Straight rod Ω_ϵ.

Notice that the center line of the straight rod is the coordinate axis $(O;\mathbf{e}_3)$. Moreover, the thickness of the thin rod depends on x_3, it is given by the function $\epsilon\rho_\epsilon(x_3) = \epsilon - \frac{x_3}{L}\epsilon\big(1-\frac{\epsilon}{L}\big)$. Observe that the diameter of the lower boundary is order ϵ while the diameter of the upper boundary is order ϵ^2/L. (See Figure 1.)

Now, we define an elementary displacement associated to a displacement of the rod.

Definition 2.2. *The elementary displacement U_e, associated to $u \in L^1(\Omega_\epsilon;\mathbb{R}^3)$, is given by:*

$$U_e(x) = \mathcal{U}(x_3) + \mathcal{R}(x_3) \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2), \quad x \in \Omega_\epsilon,$$

where for a.e. $x_3 \in (0, L)$

$$\begin{cases} \mathcal{U}(x_3) = \dfrac{1}{|\omega|\rho_\epsilon(x_3)^2\epsilon^2} \displaystyle\int_{\omega_{\epsilon,x_3}} u(x_1,x_2,x_3)\, dx_1dx_2, \\ \mathcal{R}_3(x_3) = \dfrac{1}{(I_1+I_2)\rho_\epsilon(x_3)^4\epsilon^4} \displaystyle\int_{\omega_{\epsilon,x_3}} \big[(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \wedge u(x_1,x_2,x_3)\big] \cdot \mathbf{e}_3\, dx_1dx_2, \\ \mathcal{R}_\alpha(x_3) = \dfrac{1}{(I_{3-\alpha})\rho_\epsilon(x_3)^4\epsilon^4} \displaystyle\int_{\omega_{\epsilon,x_3}} \big[(x_1\mathbf{e}_1 + x_2\mathbf{e}_2) \wedge u(x_1,x_2,x_3)\big] \cdot \mathbf{e}_\alpha\, dx_1dx_2, \\ I_\alpha = \displaystyle\int_\omega x_\alpha^2\, dx_1dx_2, \ \ \textit{for } \alpha \in \{1,2\}. \end{cases} \tag{2.2}$$

The first component $\mathcal{U}$ of U_e is the displacement of the center line. The second component $\mathcal{R}$ represents the rotation of the cross-section. Under the action of an elementary displacement the cross-section ω_{ϵ,x_3} is translated by $\mathcal{U}(x_3)$ and it is rotated around the vector $\mathcal{R}(x_3)$ with an angle $\|\mathcal{R}(x_3)\|_2$, where $\|\cdot\|_2$ is the Euclidean norm in $\mathbb{R}^3$. Observe that, the torsion of the rod is given by the displacement $\mathcal{R}_3(x_3)\mathbf{e}_3 \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2)$.

Any displacement u of the rod can be decomposed as

$$u = U_e + \bar{u}. \tag{2.3}$$

The displacement $\bar{u}$ is the warping.

Next theorem gives estimates of the components of the elementary displacement U_e and of the warping $\bar{u}$ in terms of ϵ, ρ_ϵ and of the strain energy of the displacement u. Notice that if u belongs to $H^1(\Omega_\epsilon)$ the functions $\mathcal{U}$ and $\mathcal{R}$ belong to $H^1((0,L);\mathbb{R}^3)$.

Theorem 2.3. *Let $u \in H^1(\Omega_\epsilon;\mathbb{R}^3)$ and $u = U_e + \overline{u}$ the decomposition given by (2.2)-(2.3). Then the following estimates hold:*

$$\begin{cases} \displaystyle\Big\|\frac{\overline{u}}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} \le C\epsilon\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}, \\ \displaystyle\Big\|\rho_\epsilon\Big(\frac{d\mathcal{U}}{dx_3} - \mathcal{R}\wedge\mathbf{e}_3\Big)\Big\|_{L^2((0,L);\mathbb{R}^3)} \le \frac{C}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}, \\ \displaystyle\Big\|\rho_\epsilon^2\frac{d\mathcal{R}}{dx_3}\Big\|_{L^2((0,L);\mathbb{R}^3)} \le \frac{C}{\epsilon^2}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}, \\ \|\nabla\bar{u}\|_{[L^2(\Omega_\epsilon;\mathbb{R}^3)]^9} \le C\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}. \end{cases} \tag{2.4}$$

The constants are independent of ϵ and L.

Proof. To prove the above estimates we are going to introduce a partition of the rod Ω_ϵ in several small portions where every of these small rods are star-shaped with respect to suitable balls which verify that the ratio between their radius and the diameters of the portions remains uniformly bounded. Then we use the approximation of the displacement u by a rigid body displacement in each portion, (see Theorem 2.3 in [19]).

Step 1. Construction of the partition.

We start by considering the first portion of the rod

$$\Omega_\epsilon^0 = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_3 \in (0,\epsilon), (x_1, x_2) \in \omega_{\epsilon,x_3}\}.$$

First, notice that Ω_ϵ^0 has a diameter less than $(R+1)\epsilon$ and all the cross-sections of Ω_ϵ^0 are star-shaped with respect to a disc of radius $R_1\epsilon\rho_\epsilon(\epsilon)$. Therefore, by a simple geometrical argument, it is easy to check that this portion is star-shaped with respect to a ball of radius $R_1\epsilon\rho_\epsilon(\epsilon)$.

We consider now a partition of the interval [0,L] defined as

$$s_\epsilon^0 = 0 < s_\epsilon^1 = \epsilon < s_\epsilon^2 = s_\epsilon^1 + \epsilon\rho_\epsilon(s_\epsilon^1) < \cdots < s_\epsilon^{N_\epsilon} = s_\epsilon^{N_\epsilon-1} + \epsilon\rho_\epsilon(s_\epsilon^{N_\epsilon-1}) \le L \le s_\epsilon^{N_\epsilon+1} = s_\epsilon^{N_\epsilon} + \epsilon\rho_\epsilon(s_\epsilon^{N_\epsilon}).$$

Hence, the points of the partition $\{s_\epsilon^k\}$ are the elements of an arithmetico-geometric sequence

$$s_\epsilon^k = \epsilon\frac{1-\rho_\epsilon(\epsilon)^k}{1-\rho_\epsilon(\epsilon)} \quad\Longrightarrow\quad \lim_{k\to\infty} s_\epsilon^k = \frac{\epsilon}{1-\rho_\epsilon(\epsilon)} = \frac{L}{1-\dfrac{\epsilon}{L}} > L.$$

It makes sense to define N_ϵ as the largest integer such that $s_\epsilon^{N_\epsilon} \le L$.

The $(k+1)$-portion of the rod is defined as

$$\Omega_\epsilon^k = \{x \in \mathbb{R}^3 \mid x_3 \in \big(s_\epsilon^k, s_\epsilon^k + \epsilon\rho_\epsilon(s_\epsilon^k)\big), (x_1,x_2) \in \omega_{\epsilon,x_3}\}, \quad 0 \le k \le N_\epsilon - 2,$$

and

$$\Omega_\epsilon^{N_\epsilon-1} = \{x \in \mathbb{R}^3 \mid x_3 \in \big(s^{N_\epsilon-1}, L\big), (x_1,x_2) \in \omega_{\epsilon,x_3}\}.$$

Therefore, we obtain

$$\Omega_\epsilon = \mathrm{Int}\Big\{\bigcup_{k=0}^{N_\epsilon-1}\overline{\Omega_\epsilon^k}\Big\}.$$

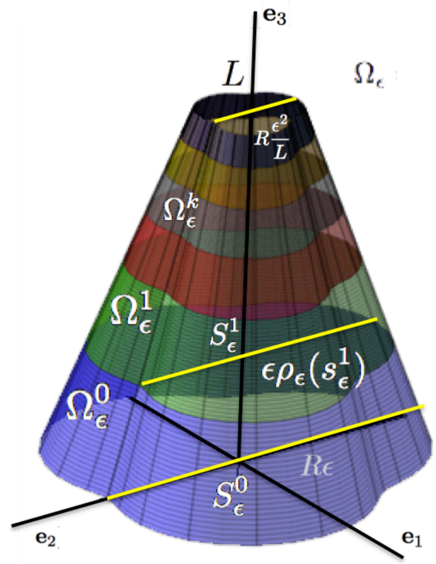

Figure 2: Partition of straight rod Ω_ϵ.

Step 2. Rigid body approximation of u in the portions.

Since Ω_ϵ^k $(0 \le k \le N_\epsilon - 2)$ is obtained by transforming Ω_ϵ^0 by a dilation of ratio $\rho_\epsilon(s_\epsilon^k)$ we can conclude that Ω_ϵ^k $(0 \le k \le N_\epsilon - 2)$ is star-shaped with respect to a ball of radius $R_1\epsilon\rho_\epsilon(s_\epsilon^{k+1})$ and its diameter is less than $(R+1)\epsilon\rho_\epsilon(s_\epsilon^k)$. Moreover, the last portion $\Omega_\epsilon^{N_\epsilon-1}$ is star-shaped with respect to a ball of radius $R_1\epsilon\rho_\epsilon(s_\epsilon^{N_\epsilon+1})$ and its diameter is less than $2(R+1)\epsilon\rho_\epsilon(s_\epsilon^{N_\epsilon-1})$.

From Theorem 2.3 in [19] there exists a rigid body displacement r_k $(0 \le k \le N_\epsilon - 1)$ such that

$$\begin{aligned}
\|u - r_k\|^2_{L^2(\Omega_\epsilon^k;\mathbb{R}^3)} &\le C(R+1)^2\epsilon^2\rho_\epsilon(s_\epsilon^k)^2\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega_\epsilon^k)]^9},\\
\|\nabla(u - r_k)\|^2_{[L^2(\Omega_\epsilon^k;\mathbb{R}^3)]^9} &\le C(R+1)^2\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega_\epsilon^k)]^9}.
\end{aligned} \tag{2.5}$$

The constants depend only on the reference cross-section ω and on the ratio between the diameter of the portion and the radius of the ball inside (see Theorem 2.3 in [19])

$$\frac{(R+1)\epsilon\rho_\epsilon(s_\epsilon^k)}{R_1\epsilon\rho_\epsilon(s_\epsilon^{k+1})} = \frac{R+1}{R_1}\frac{\rho_\epsilon(s_\epsilon^k)}{\rho_\epsilon(s_\epsilon^k) - \dfrac{\epsilon\rho_\epsilon(s_\epsilon^k)}{L}\Big(1 - \dfrac{\epsilon}{L}\Big)} = \frac{R+1}{R_1}\frac{1}{\rho_\epsilon(\epsilon)} \le \frac{4}{3}\frac{R+1}{R_1},\ 0 \le k \le N_\epsilon - 2. \tag{2.6}$$

Observe that for the last portion the ratio is less than $4\dfrac{R+1}{R_1}$.

Step 3. First estimate in (2.4).

Recall that the rigid body displacements r_k are of the form

$$r_k(x) = A_k + B_k \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + (x_3 - s_\epsilon^k)\mathbf{e}_3), \quad x = (x_1, x_2, x_3) \in \Omega_\epsilon^k \text{ and } A_k, B_k \in \mathbb{R}^3.$$

Now, we are going to prove $(0 \le k \le N_\epsilon - 2)$[1]

$$\|\mathcal{U} - A_k - B_k \wedge (x_3 - s_{\epsilon,k})\mathbf{e}_3\|^2_{L^2((s_\epsilon^k, s_\epsilon^{k+1});\mathbb{R}^3)} \le C\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega_\epsilon^k)]^9}, \tag{2.7}$$

[1] If $k = N_\epsilon - 1$ we have to replace s_ϵ^{k+1} by L.

$$\|\mathcal{R}-B_k\|^2_{L^2((s^k_\epsilon,s^{k+1}_\epsilon);\mathbb{R}^3)} \le \frac{C}{\epsilon^2\rho_\epsilon(s^{k+1}_\epsilon)^2}\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega^k_\epsilon)]^9}. \tag{2.8}$$

The constants do not depend on k and ϵ.

The proof is similar for both inequalities, we will show only the first one. Taking the mean value of $u-r_k$ over the cross-sections of the portion Ω^k_ϵ and by the definition of the elementary displacement and (2.1) we have

$$\begin{aligned}
&\|\mathcal{U}-A_k-B_k\wedge(x_3-s_{\epsilon,k})\mathbf{e}_3\|^2_{L^2((s^k_\epsilon,s^{k+1}_\epsilon);\mathbb{R}^3)} = \int_{s^k_\epsilon}^{s^{k+1}_\epsilon} |\mathcal{U}(x_3)-A_k-B_k\wedge(x_3-s_{\epsilon,k})\mathbf{e}_3|^2\,dx_3\\
&= \int_{s^k_\epsilon}^{s^{k+1}_\epsilon} \Big|\frac{1}{|\omega|\rho_\epsilon(x_3)^2\epsilon^2}\int_{\omega_{\epsilon,x_3}}[u(x_1,x_2,x_3)-r_k(x_1,x_2,x_3)]\,dx_1dx_2\Big|^2\,dx_3\\
&\le \int_{s^k_\epsilon}^{s^{k+1}_\epsilon} \frac{1}{|\omega|\rho_\epsilon(x_3)^2\epsilon^2}\int_{\omega_{\epsilon,x_3}}|u(x)-r_k(x)|^2\,dx\\
&\le \frac{1}{|\omega|\rho_\epsilon(s^{k+1}_\epsilon)^2\epsilon^2}\int_{\Omega^k_\epsilon}|u(x)-r_k(x)|^2\,dx.
\end{aligned}$$

Then using $(2.5)_1$ and taking into account (2.6) we obtain the expected estimate

$$\|\mathcal{U}-A_k-B_k\wedge(x_3-s_{\epsilon,k})\mathbf{e}_3\|^2_{L^2((s^k_\epsilon,s^{k+1}_\epsilon);\mathbb{R}^3)} \le \frac{C(R+1)^2\epsilon^2\rho_\epsilon(s^k_\epsilon)^2}{|\omega|\epsilon^2\rho_\epsilon(s^{k+1}_\epsilon)^2}\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega^k_\epsilon)]^9} \le C\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega^k_\epsilon)]^9},$$

where the constant does not depend on ϵ and k.

Consequently, from (2.7) and (2.8), taking into account the definition of the elementary displacement and $\displaystyle\int_{\omega_{\epsilon,x_3}} x_\alpha^2\,dx_1dx_2=\epsilon^4\rho_\epsilon(x_3)^4I_\alpha$ we have

$$\begin{aligned}
&\|U_e-r_k\|_{L^2(\Omega^k_\epsilon)} \le \|\mathcal{U}-A_k-B_k\wedge(x_3-s_{\epsilon,k})\mathbf{e}_3\|_{L^2(\Omega^k_\epsilon)}+\|(\mathcal{R}-B_k)\wedge(x_1\mathbf{e}_1+x_2\mathbf{e}_2)\|_{L^2(\Omega^k_\epsilon)}\\
&\le \int_{s^k_\epsilon}^{s^{k+1}_\epsilon}|\omega|\rho_\epsilon(x_3)^2\epsilon^2|\mathcal{U}(x_3)-A_k-B_k\wedge(x_3-s_{\epsilon,k})\mathbf{e}_3|^2\,dx_3+C\epsilon^4\rho_\epsilon(s^k_\epsilon)^4\|\mathcal{R}-B_k\|^2_{L^2(s^k_\epsilon,s^{k+1}_\epsilon)}\\
&\le C\epsilon^2\rho_\epsilon(s^k_\epsilon)^2\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega^k_\epsilon)]^9}.
\end{aligned}$$

Thus, we can replace r_k by U_e in $(2.5)_1$

$$\|u-U_e\|^2_{L^2(\Omega^k_\epsilon;\mathbb{R}^3)} \le C\epsilon^2\rho_\epsilon(s^k_\epsilon)^2\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega^k_\epsilon)]^9}.$$

Moreover, since $1\le\dfrac{\rho_\epsilon(s^k_\epsilon)}{\rho_\epsilon(x_3)}\le 2$ for $x_3\in(s^k_\epsilon,s^{k+1}_\epsilon)$, we get

$$\Big\|\frac{u-U_e}{\rho_\epsilon}\Big\|^2_{L^2(\Omega^k_\epsilon;\mathbb{R}^3)} \le C\epsilon^2\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega^k_\epsilon)]^9}.$$

Adding all these inequalities lead to the first estimate involving the warping

$$\Big\|\frac{u-U_e}{\rho_\epsilon}\Big\|^2_{L^2(\Omega_\epsilon;\mathbb{R}^3)} \le C\epsilon^2\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega_\epsilon)]^9}. \tag{2.9}$$

Step 4. Second estimate in (2.4).

First of all, we compute the derivative of $\mathcal{U}$ with respect to x_3. Since the diameter of the cross-section depends on x_3 we rewrite $\mathcal{U}$ performing a change of variables

$$\mathcal{U}(x_3) = \frac{1}{|\omega|\epsilon^2} \int_{\omega_\epsilon} u(\rho_\epsilon(x_3)s_1, \rho_\epsilon(x_3)s_2, x_3)\, ds_1 ds_2,$$

where $\omega_\epsilon = \epsilon\omega$. The derivative is given by

$$\frac{d\mathcal{U}}{dx_3}(x_3) = \frac{1}{|\omega|\epsilon^2} \int_{\omega_\epsilon} \Big[\frac{\partial u}{\partial x_1}\rho'_\epsilon(x_3)s_1 + \frac{\partial u}{\partial x_2}\rho'_\epsilon(x_3)s_2 + \frac{\partial u}{\partial x_3}\Big]\, ds_1 ds_2, \qquad \text{for a.e } x_3 \in (0, L).$$

Undoing the change of variables we get

$$\frac{d\mathcal{U}}{dx_3}(x_3) = \frac{1}{|\omega|\epsilon^2\rho_\epsilon(x_3)^2} \int_{\omega_{\epsilon,x_3}} \Big[\frac{\partial u}{\partial x_1}x_1\frac{\rho'_\epsilon(x_3)}{\rho_\epsilon(x_3)} + \frac{\partial u}{\partial x_2}x_2\frac{\rho'_\epsilon(x_3)}{\rho_\epsilon(x_3)} + \frac{\partial u}{\partial x_3}\Big]\, dx_1 dx_2, \qquad \text{for a.e } x_3 \in (0, L).$$

From (2.5) we have

$$\Big\|\frac{\partial u}{\partial x_i} - B_k \wedge \mathbf{e}_i\Big\|^2_{L^2(\Omega^k_\epsilon;\mathbb{R}^3)} \leq C\|(\nabla u)_\mathcal{S}\|^2_{[L^2(\Omega^k_\epsilon)]^9}, \quad i \in \{1,2,3\}.$$

Moreover, from (2.8) we may replace B_k by $\mathcal{R}$

$$\Big\|\frac{\partial u}{\partial x_i} - \mathcal{R} \wedge \mathbf{e}_i\Big\|^2_{L^2(\Omega^k_\epsilon;\mathbb{R}^3)} \leq C\|(\nabla u)_\mathcal{S}\|^2_{[L^2(\Omega^k_\epsilon)]^9}, \quad i \in \{1,2,3\}.$$

Adding all these inequalities we obtain

$$\Big\|\frac{\partial u}{\partial x_i} - \mathcal{R} \wedge \mathbf{e}_i\Big\|^2_{L^2(\Omega_\epsilon;\mathbb{R}^3)} \leq C\|(\nabla u)_\mathcal{S}\|^2_{[L^2(\Omega_\epsilon)]^9}, \quad i \in \{1,2,3\}. \tag{2.10}$$

Taking into account (2.1) we have for a.e. $x_3 \in (0, L)$

$$\begin{aligned}\frac{d\mathcal{U}}{dx_3}(x_3) - \mathcal{R}(x_3) \wedge \mathbf{e}_3 = \frac{1}{|\omega|\epsilon^2\rho_\epsilon(x_3)^2} \int_{\omega_{\epsilon,x_3}} &\Big[\Big(\frac{\partial u}{\partial x_1}(x) - \mathcal{R}(x_3) \wedge \mathbf{e}_1\Big)x_1\frac{\rho'_\epsilon(x_3)}{\rho_\epsilon(x_3)} \\ &+ \Big(\frac{\partial u}{\partial x_2}(x) - \mathcal{R}(x_3) \wedge \mathbf{e}_2\Big)x_2\frac{\rho'_\epsilon(x_3)}{\rho_\epsilon(x_3)} + \Big(\frac{\partial u}{\partial x_3}(x) - \mathcal{R}(x_3) \wedge \mathbf{e}_3\Big)\Big]\, dx_1 dx_2.\end{aligned}$$

Using (2.10) leads to $(0 \leq k \leq N_\epsilon - 2)$[2]

$$\begin{aligned}\Big\|\frac{d\mathcal{U}}{dx_3} - \mathcal{R} \wedge \mathbf{e}_3\Big\|^2_{L^2(s^k_\epsilon, s^{k+1}_\epsilon)} &\leq \frac{C}{\rho_\epsilon(s^{k+1}_\epsilon)^2}\|(\nabla u)_\mathcal{S}\|^2_{[L^2(\Omega^k_\epsilon)]^9} + \frac{C}{\epsilon^2\rho_\epsilon(s^{k+1}_\epsilon)^2}\|(\nabla u)_\mathcal{S}\|^2_{[L^2(\Omega^k_\epsilon)]^9} \\ &\leq \frac{C}{\epsilon^2\rho_\epsilon(s^{k+1}_\epsilon)^2}\|(\nabla u)_\mathcal{S}\|^2_{[L^2(\Omega^k_\epsilon)]^9}.\end{aligned}$$

Hence, since $1 \leq \dfrac{\rho_\epsilon(x_3)}{\rho_\epsilon(s^{k+1}_\epsilon)} \leq 2$ for $x_3 \in (s^k_\epsilon, s^{k+1}_\epsilon)$ we obtain

$$\Big\|\rho_\epsilon\Big(\frac{d\mathcal{U}}{dx_3} - \mathcal{R} \wedge \mathbf{e}_3\Big)\Big\|^2_{L^2(s^k_\epsilon, s^{k+1}_\epsilon)} \leq \frac{C}{\epsilon^2}\|(\nabla u)_\mathcal{S}\|^2_{[L^2(\Omega_\epsilon)]^9}. \tag{2.11}$$

[2] If $k = N_\epsilon - 1$ we have to replace s^{k+1}_ϵ by L.

Adding all these inequalities we get the desired estimate

$$\Big\|\rho_\epsilon\Big(\frac{d\mathcal{U}}{dx_3}-\mathcal{R}\wedge\mathbf{e}_3\Big)\Big\|^2_{L^2((0,L);\mathbb{R}^3)}\le\frac{C}{\epsilon^2}\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega_\epsilon)]^9}. \tag{2.12}$$

Step 5. Third estimate in (2.4).

First of all, we introduce the function:

$$V(x_3)=\frac{1}{\epsilon^4\rho_\epsilon(x_3)^4}\int_{\omega_{\epsilon,x_3}}[(x_1\mathbf{e}_1+x_2\mathbf{e}_2)\wedge u(x_1,x_2,x_3)]\,dx_1dx_2.$$

To calculate the derivative with respect to x_3 we perform a change of variables which allows us to write the function V as follows:

$$\begin{aligned}V(x_3)&=\frac{1}{\epsilon^4\rho_\epsilon(x_3)^2}\int_{\omega_\epsilon}(\rho_\epsilon(x_3)s_1\mathbf{e}_1+\rho_\epsilon(x_3)s_2\mathbf{e}_2)\wedge u(\rho_\epsilon(x_3)s_1,\rho_\epsilon(x_3)s_2,x_3)]\,ds_1ds_2\\&=\frac{1}{\epsilon^4\rho_\epsilon(x_3)}\int_{\omega_\epsilon}(s_1\mathbf{e}_1+s_2\mathbf{e}_2)\wedge u(\rho_\epsilon(x_3)s_1,\rho_\epsilon(x_3)s_2,x_3)]\,ds_1ds_2.\end{aligned}$$

Then deriving with respect to x_3 gives (for a.e. $x_3\in(0,L)$)

$$\begin{aligned}\frac{dV}{dx_3}(x_3)=&\frac{-2\rho'_\epsilon(x_3)}{\epsilon^4\rho_\epsilon(x_3)^2}\int_{\omega_\epsilon}[(s_1\mathbf{e}_1+s_2\mathbf{e}_2)\wedge u(\rho_\epsilon(x_3)s_1,\rho_\epsilon(x_3)s_2,x_3)]\,ds_1ds_2\\&+\frac{1}{\epsilon^4\rho_\epsilon(x_3)}\int_{\omega_\epsilon}\Big[(s_1\mathbf{e}_1+s_2\mathbf{e}_2)\wedge\Big(\frac{\partial u}{\partial x_1}\rho'_\epsilon(x_3)s_1+\frac{\partial u}{\partial x_2}\rho'_\epsilon(x_3)s_2+\frac{\partial u}{\partial x_3}\Big)\Big]\,ds_1ds_2\end{aligned}$$

Undoing the change of variables we have (for a.e. $x_3\in(0,L)$)

$$\begin{aligned}\frac{dV}{dx_3}(x_3)=&\frac{-2\rho'_\epsilon(x_3)}{\epsilon^4\rho_\epsilon(x_3)^5}\int_{\omega_{\epsilon,x_3}}[(x_1\mathbf{e}_1+x_2\mathbf{e}_2)\wedge u(x_1,x_2,x_3)]\,dx_1dx_2\\&+\frac{1}{\epsilon^4\rho_\epsilon(x_3)^4}\int_{\omega_{\epsilon,x_3}}\Big[(x_1\mathbf{e}_1+x_2\mathbf{e}_2)\wedge\Big(\frac{\partial u}{\partial x_1}\frac{\rho'_\epsilon(x_3)}{\rho_\epsilon(x_3)}x_1+\frac{\partial u}{\partial x_2}\frac{\rho'_\epsilon(x_3)}{\rho_\epsilon(x_3)}x_2+\frac{\partial u}{\partial x_3}\Big)\Big]\,dx_1dx_2.\end{aligned}$$

In view of (2.1) we can write (for a.e. $x_3\in(0,L)$)

$$\begin{aligned}\frac{dV}{dx_3}(x_3)=&\frac{-2\rho'_\epsilon(x_3)}{\epsilon^4\rho_\epsilon(x_3)^5}\int_{\omega_{\epsilon,x_3}}\Big[(x_1\mathbf{e}_1+x_2\mathbf{e}_2)\wedge\Big(u(x)-\mathcal{U}(x_3)-\mathcal{R}(x_3)\wedge(x_1\mathbf{e}_1+x_2\mathbf{e}_2)\Big)\Big]\,dx_1dx_2\\&+\frac{\rho'_\epsilon(x_3)}{\epsilon^4\rho_\epsilon(x_3)^5}\int_{\omega_{\epsilon,x_3}}\Big\{(x_1\mathbf{e}_1+x_2\mathbf{e}_2)\wedge\Big[x_1\Big(\frac{\partial u}{\partial x_1}(x)-\mathcal{R}(x_3)\wedge\mathbf{e}_1\Big)+x_2\Big(\frac{\partial u}{\partial x_2}(x)-\mathcal{R}(x_3)\wedge\mathbf{e}_2\Big)\Big\}\,dx_1dx_2\\&+\frac{1}{\epsilon^4\rho_\epsilon(x_3)^4}\int_{\omega_{\epsilon,x_3}}(x_1\mathbf{e}_1+x_2\mathbf{e}_2)\wedge\Big(\frac{\partial u}{\partial x_3}(x)-\mathcal{R}(x_3)\wedge\mathbf{e}_3\Big)\,dx_1dx_2.\end{aligned}$$

Using (2.9) and (2.11) leads to ($0\le k\le N_\epsilon-2$)[3]

$$\begin{aligned}\Big\|\frac{dV}{dx_3}\Big\|^2_{L^2((s^k_\epsilon,s^{k+1}_\epsilon);\mathbb{R}^3)}&\le\frac{C}{\epsilon^2\rho_\epsilon(s^{k+1}_\epsilon)^2}\Big\|\frac{u-U_e}{\rho_\epsilon}\Big\|^2_{L^2(\Omega^k_\epsilon;\mathbb{R}^3)}+\frac{C}{\epsilon^2\rho_\epsilon(s^{k+1}_\epsilon)^2}\sum_{i=1}^3\Big\|\frac{\partial u}{dx_3}-\mathcal{R}\wedge\mathbf{e}_i\Big\|^2_{L^2(\Omega^k_\epsilon;\mathbb{R}^3)}\\&\le\frac{C}{\rho_\epsilon(s^{k+1}_\epsilon)^4}\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega^k_\epsilon)]^9}+\frac{C}{\epsilon^4\rho_\epsilon(s^{k+1}_\epsilon)^4}\|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega^k_\epsilon)]^9}\end{aligned}$$

[3] If $k=N_\epsilon-1$ we have to replace s^{k+1}_ϵ by L.

$$\leq \frac{C}{\epsilon^4 \rho_\epsilon(s_\epsilon^{k+1})^4} \|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega_\epsilon^k)]^9}.$$

Thus, since $1 \leq \dfrac{\rho_\epsilon(x_3)}{\rho_\epsilon(s_\epsilon^{k+1})} \leq 2$ for $x_3 \in (s_\epsilon^k, s_\epsilon^{k+1})$ and adding all these inequalities we have

$$\Big\|\rho_\epsilon^2 \frac{dV}{dx_3}\Big\|^2_{L^2((0,L);\mathbb{R}^3)} \leq \frac{C}{\epsilon^4} \|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega_\epsilon)]^9}.$$

Since $(I_1 + I_2)\mathcal{R} = V + \dfrac{I_1}{I_2}(V \cdot \mathbf{e}_1)\mathbf{e}_1 + \dfrac{I_2}{I_1}(V \cdot \mathbf{e}_2)\mathbf{e}_2$ we get the required estimate

$$\Big\|\rho_\epsilon^2 \frac{d\mathcal{R}}{dx_3}\Big\|^2_{L^2((0,L);\mathbb{R}^3)} \leq \frac{C}{\epsilon^4} \|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega_\epsilon)]^9}. \tag{2.13}$$

Step 6. Fourth estimate.

Observe that

$$\begin{aligned}
\frac{\partial}{\partial x_\alpha}(u - U_e) &= \frac{\partial u}{\partial x_\alpha} - \mathcal{R} \wedge \mathbf{e}_\alpha, \qquad \text{for } \alpha \in \{1, 2\},\\
\frac{\partial}{\partial x_3}(u - U_e) &= \frac{\partial u}{\partial x_3} - \frac{\partial \mathcal{U}}{\partial x_3} - \frac{\partial \mathcal{R}}{\partial x_3} \wedge (x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2)\\
&= \frac{\partial u}{\partial x_3} - \mathcal{R} \wedge \mathbf{e}_3 + \mathcal{R} \wedge \mathbf{e}_3 - \frac{\partial \mathcal{U}}{\partial x_3} - \frac{\partial \mathcal{R}}{\partial x_3} \wedge (x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2).
\end{aligned}$$

From these expressions and taking into account (2.10), (2.12) and (2.13) we can conclude

$$\|\nabla \bar{u}\|^2_{[L^2(\Omega_\epsilon;\mathbb{R}^3)]^9} \leq C \|(\nabla u)_{\mathcal{S}}\|^2_{[L^2(\Omega_\epsilon)]^9},$$

which ends the proof. □

3. Estimates for the clamped rod at the bottom.

From now on, we will assume that the rod Ω_ϵ is clamped at the bottom, $\Gamma_{\epsilon,0} = \omega_{\epsilon,0} \times \{0\}$. Then the space of admissible displacements of the rod is

$$H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon; \mathbb{R}^3) = \{u \in H^1(\Omega_\epsilon; \mathbb{R}^3) \mid u = 0 \text{ on } \Gamma_{\epsilon,0}\}.$$

Observe that the elementary displacement U_e associated to any $u \in H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon; \mathbb{R}^3)$ is equal to zero in the fixed part of the rod, $\mathcal{U}(0) = \mathcal{R}(0) = 0$.

Using estimates (2.4) and the boundary condition we deduce estimates on $\mathcal{R}$, $\dfrac{d\mathcal{U}}{dx_3}$ and $\mathcal{U}$.

Lemma 3.1. *Assuming the rod clamped at the bottom, then we have*

$$\left\{\begin{aligned}
\|\rho_\epsilon \mathcal{R}\|_{L^2((0,L);\mathbb{R}^3)} &\leq \frac{CL}{\epsilon^2} \|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9},\\
\Big\|\rho_\epsilon \frac{d\mathcal{U}_\alpha}{dx_3}\Big\|_{L^2(0,L)} &\leq \frac{CL}{\epsilon^2} \|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}, \quad \textit{for } \alpha \in \{1, 2\},\\
\Big\|\rho_\epsilon \frac{d\mathcal{U}_3}{dx_3}\Big\|_{L^2(0,L)} &\leq \frac{C}{\epsilon} \|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9},\\
\|\mathcal{U}_\alpha\|_{L^2(0,L)} &\leq \frac{CL^2}{\epsilon^2} \|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}, \quad \textit{for } \alpha \in \{1, 2\},\\
\|\mathcal{U}_3\|_{L^2(0,L)} &\leq \frac{CL}{\epsilon} \|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9},
\end{aligned}\right. \tag{3.1}$$

The constants are independent of ϵ and L.

Proof. We begin with the proof of the first estimate in (3.1). Since $\mathcal{R}(0)=0$ by integration by parts we have

$$\int_0^L 2\rho_\epsilon^3(x_3)\mathcal{R}(x_3)\frac{d\mathcal{R}}{dx_3}(x_3)\,dx_3 = -\int_0^L 3\rho_\epsilon^2(x_3)\rho_\epsilon'(x_3)\mathcal{R}^2(x_3)\,dx_3 + \rho_\epsilon^3(L)\mathcal{R}^2(L).$$

Then taking into account the facts that $\dfrac{1}{2L}\le -\rho_\epsilon'(x_3)=\dfrac{1}{L}\big(1-\dfrac{\epsilon}{L}\big)\le\dfrac{1}{L}$ $(0<\epsilon<\dfrac{L}{2})$ and $0\le\rho_\epsilon^3(L)\mathcal{R}^2(L)$ we get

$$\int_0^L \rho_\epsilon^2(x_3)\mathcal{R}^2(x_3)\,dx_3 \le \frac{2L}{3}\int_0^L \rho_\epsilon^3(x_3)\mathcal{R}(x_3)\frac{d\mathcal{R}}{dx_3}(x_3)\,dx_3.$$

Hence, by the Cauchy's inequality it follows that:

$$\int_0^L \rho_\epsilon^3(x_3)\mathcal{R}(x_3)\frac{d\mathcal{R}}{dx_3}(x_3)\,dx_3 \le \|\rho_\epsilon\mathcal{R}\|_{L^2((0,L);\mathbb{R}^3)}\Big\|\rho_\epsilon^2\frac{d\mathcal{R}}{dx_3}\Big\|_{L^2((0,L);\mathbb{R}^3)}$$

Finally, the above inequalities together with the third estimate in (2.4) allow us to obtain the required estimate

$$\|\rho_\epsilon\mathcal{R}\|_{L^2((0,L);\mathbb{R}^3)} \le \frac{2L}{3}\Big\|\rho_\epsilon^2\frac{d\mathcal{R}}{dx_3}\Big\|_{L^2((0,L);\mathbb{R}^3)} \le \frac{CL}{\epsilon^2}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}. \tag{3.2}$$

The constant is independent of ϵ and L.

The second estimate follows from $(2.4)_2$ and (3.2):

$$\Big\|\rho_\epsilon\frac{d\mathcal{U}}{dx_3}\Big\|_{L^2((0,L);\mathbb{R}^3)} \le \Big\|\rho_\epsilon\Big(\frac{d\mathcal{U}}{dx_3}-\mathcal{R}\wedge e_3\Big)\Big\|_{L^2((0,L))} + \|\rho_\epsilon(\mathcal{R}\wedge e_3)\|_{L^2((0,L);\mathbb{R}^3)} \le \frac{CL}{\epsilon^2}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}. \tag{3.3}$$

From the second estimate in (2.4) we obtain a better estimate for $\Big\|\rho_\epsilon\dfrac{d\mathcal{U}_3}{dx_3}\Big\|_{L^2(0,L)}$

$$\Big\|\rho_\epsilon\frac{d\mathcal{U}_3}{dx_3}\Big\|_{L^2(0,L)} = \Big\|\rho_\epsilon\Big(\frac{d\mathcal{U}}{dx_3}-\mathcal{R}\wedge\mathbf{e}_3\Big)\cdot\mathbf{e}_3\Big\|_{L^2((0,L);\mathbb{R}^3)} \le \frac{C}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}. \tag{3.4}$$

Finally, the estimates for $\mathcal{U}$ follows by a similar computation to $\mathcal{R}$. We first prove

$$\|\mathcal{U}_i^\epsilon\|_{L^2(0,L)} \le 2L\Big\|\rho_\epsilon\frac{d\mathcal{U}_i^\epsilon}{dx_3}\Big\|_{L^2(0,L)} \qquad \text{for } i=1,2,3$$

then due to $(3.1)_2$-$(3.1)_3$ we get $(3.1)_4$-$(3.1)_5$. □

In view of the definition of the elementary displacement (2.2) we can write explicitly the components of the displacement, the gradient and the symmetric gradient of the displacement

$$\begin{cases} u_1(x)=\mathcal{U}_1(x_3)-x_2\mathcal{R}_3(x_3)+\bar{u}_1(x),\\ u_2(x)=\mathcal{U}_2(x_3)+x_1\mathcal{R}_3(x_3)+\bar{u}_2(x),\\ u_3(x)=\mathcal{U}_3(x_3)-x_1\mathcal{R}_2(x_3)+x_2\mathcal{R}_1(x_3)+\bar{u}_3(x).\end{cases} \tag{3.5}$$

Remark 3.2. *Notice that, due to the definition of $\mathcal{U}$, $\mathcal{R}$ and (2.1) we know that the warping satisfies*

$$\int_{\omega_{\epsilon,x_3}} \bar{u}_i\,dx_1dx_2 = \int_{\omega_{\epsilon,x_3}}(x_1\bar{u}_2-x_2\bar{u}_1)\,dx_1dx_2 = \int_{\omega_{\epsilon,x_3}} x_\alpha\bar{u}_3\,dx_1dx_2 = 0, \quad i\in\{1,2,3\},\ \ \alpha\in\{1,2\}. \tag{3.6}$$

$$
\nabla u = \begin{pmatrix} \dfrac{\partial \bar u_1}{\partial x_1} & -\mathcal{R}_3 + \dfrac{\partial \bar u_1}{dx_2} & \dfrac{d\mathcal{U}_1}{dx_3} - x_2\dfrac{d\mathcal{R}_3}{dx_3} + \dfrac{\partial \bar u_1}{\partial x_3} \\ \mathcal{R}_3 + \dfrac{\partial \bar u_2}{\partial x_1} & \dfrac{\partial \bar u_2}{\partial x_2} & \dfrac{d\mathcal{U}_2}{dx_3} + x_1\dfrac{d\mathcal{R}_3}{dx_3} + \dfrac{\partial \bar u_2}{\partial x_3} \\ -\mathcal{R}_2 + \dfrac{\partial \bar u_3}{\partial x_1} & \mathcal{R}_1 + \dfrac{\partial \bar u_3}{\partial x_2} & \dfrac{d\mathcal{U}_3}{dx_3} - x_1\dfrac{d\mathcal{R}_2}{dx_3} + x_2\dfrac{d\mathcal{R}_1}{dx_3} + \dfrac{\partial \bar u_3}{\partial x_3} \end{pmatrix} \tag{3.7}
$$

$$
(\nabla u)_{\mathcal{S}} = \begin{pmatrix} \dfrac{\partial \bar u_1}{\partial x_1} & \dfrac{1}{2}\Big(\dfrac{\partial \bar u_1}{\partial x_2} + \dfrac{\partial \bar u_2}{\partial x_1}\Big) & \dfrac{1}{2}\Big(\dfrac{d\mathcal{U}_1}{dx_3} - x_2\dfrac{d\mathcal{R}_3}{dx_3} - \mathcal{R}_2 + \dfrac{\partial \bar u_3}{\partial x_1} + \dfrac{\partial \bar u_1}{\partial x_3}\Big) \\ * & \dfrac{\partial \bar u_2}{\partial x_2} & \dfrac{1}{2}\Big(\dfrac{d\mathcal{U}_2}{dx_3} + x_1\dfrac{d\mathcal{R}_3}{dx_3} + \mathcal{R}_1 + \dfrac{\partial \bar u_3}{\partial x_2} + \dfrac{\partial \bar u_2}{\partial x_3}\Big) \\ * & * & \dfrac{d\mathcal{U}_3}{dx_3} - x_1\dfrac{d\mathcal{R}_2}{dx_3} + x_2\dfrac{d\mathcal{R}_1}{dx_3} + \dfrac{\partial \bar u_3}{\partial x_3} \end{pmatrix} \tag{3.8}
$$

The previous Lemma 3.1 allows us to established the Korn's inequality for any displacement $u \in H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon;\mathbb{R}^3)$.

Lemma 3.3. *Assuming the rod clamped at the bottom boundary, then we have*

$$
\begin{cases} \|\nabla u\|_{[L^2(\Omega_\epsilon;\mathbb{R}^3)]^9} \le C\dfrac{L}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}, \\ \Big\|\dfrac{u_\alpha}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon)} \le C\dfrac{L^2}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}, \quad \textit{for } \alpha\in\{1,2\}, \\ \Big\|\dfrac{u_3}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon)} \le CL\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}, \end{cases} \tag{3.9}
$$

The constant does not depend on ϵ and L.

Proof. Recall that any displacement $u \in H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon;\mathbb{R}^3)$ can be written as $u = U_e + \bar u$. Then we get

$$
\|\nabla u\|_{[L^2(\Omega_\epsilon;\mathbb{R}^3)]^9} \le \|\nabla U_e\|_{[L^2(\Omega_\epsilon;\mathbb{R}^3)]^9} + \|\nabla \bar u\|_{[L^2(\Omega_\epsilon;\mathbb{R}^3)]^9}.
$$

Using (3.7), (2.4) and (3.1) one has the following estimate:

$$
\begin{aligned}
&\|\nabla U_e\|_{[L^2(\Omega_\epsilon;\mathbb{R}^3)]^9} \le \Big\|\epsilon\rho_\epsilon\frac{d\mathcal{U}}{dx_3}\Big\|_{L^2((0,L);\mathbb{R}^3)} + \big\|\epsilon\rho_\epsilon\mathcal{R}\wedge(\mathbf{e}_1+\mathbf{e}_2)\big\|_{L^2((0,L);\mathbb{R}^3)} \\
&+ \Big\|\frac{d\mathcal{R}}{dx_3}\wedge(x_1\mathbf{e}_1 + x_2\mathbf{e}_2)\Big\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} \le \frac{CL}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9} + \frac{CL}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9} + C\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9} \\
&\le \frac{CL}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}.
\end{aligned}
$$

Recall that $\|\nabla \bar u\|_{[L^2(\Omega_\epsilon;\mathbb{R}^3)]^9} \le C\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon]^9}$. Consequently, we obtain the first estimate in (3.9).

In view of (3.5) and taking into account estimates (2.4) and (3.1) we obtain

$$
\begin{aligned}
\Big\|\frac{u_\alpha}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} &\le \|\epsilon\,\mathcal{U}_\alpha\|_{L^2(0,L)} + \Big\|\frac{x_{3-\alpha}\mathcal{R}_3}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon)} + \Big\|\frac{\bar u_\alpha}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon)} \\
&\le \frac{CL^2}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9} + CL\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9} + C\epsilon\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9} \\
&\le \frac{CL^2}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}, \qquad \text{for } \alpha = 1,2.
\end{aligned}
$$

$$
\begin{aligned}
\Big\|\frac{u_3}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} &\le \|\epsilon\mathcal{U}_3\|_{L^2(0,L)} + \Big\|\frac{x_1\mathcal{R}_2}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon)} + \Big\|\frac{x_2\mathcal{R}_1}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon)} + \Big\|\frac{\bar{u}_3}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon)}\\
&\le CL\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon]^9},
\end{aligned}
$$

which ends the proof. □

3.1. Rescaling of the rod

In this paragraph we define an operator which changes the scale. It allows us to transform the rod Ω_ϵ into a domain independent of ϵ.

Set $\Omega = \omega \times (0, L)$, the reference beam. We rescale Ω_ϵ using the following operator:

$$
(\Pi_\epsilon\phi)(X_1, X_2, x_3) = \phi(\epsilon\rho_\epsilon(x_3)X_1, \epsilon\rho_\epsilon(x_3)X_2, x_3), \text{ for a.e. } (X_1, X_2, x_3) \in \Omega,
$$

defined for any function ϕ measurable on Ω_ϵ.

Observe that, if $\phi \in L^2(\Omega_\epsilon)$ then $(\Pi_\epsilon\phi) \in L^2(\Omega)$ and we have

$$
\|\Pi_\epsilon\phi\|_{L^2(\Omega)} = \frac{1}{\epsilon}\Big\|\frac{\phi}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon)}. \tag{3.10}
$$

Therefore, taking into account this above relation, we get the estimate for the rescaled warping $\Pi_\epsilon\bar{u}$

$$
\|\Pi_\epsilon\bar{u}\|_{L^2(\Omega;\mathbb{R}^3)} = \frac{1}{\epsilon}\Big\|\frac{\bar{u}}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} \le C\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}. \tag{3.11}
$$

In order to obtain the estimates for the derivatives of the warping observe that for any $\phi \in H^1(\Omega_\epsilon)$

$$
\begin{aligned}
&\frac{\partial(\Pi_\epsilon\phi)}{\partial X_\alpha} = \epsilon\rho_\epsilon\Pi_\epsilon\Big(\frac{\partial\phi}{\partial x_\alpha}\Big), \text{ for } \alpha = 1, 2,\\
&\frac{\partial(\Pi_\epsilon\phi)}{\partial x_3} = \epsilon\rho'_\epsilon X_1\Pi_\epsilon\Big(\frac{\partial\phi}{\partial x_1}\Big) + \epsilon\rho'_\epsilon X_2\Pi_\epsilon\Big(\frac{\partial\phi}{\partial x_2}\Big) + \Pi_\epsilon\Big(\frac{\partial\phi}{\partial x_3}\Big).
\end{aligned} \tag{3.12}
$$

We recall that $||\rho'_\epsilon||_{L^\infty(0,L)} \le \dfrac{1}{L}$, then from (2.4) and (3.10) we get

$$
\Big\|\frac{\partial(\Pi_\epsilon\bar{u})}{\partial X_\alpha}\Big\|_{L^2(\Omega;\mathbb{R}^3)} = \Big\|\frac{\partial\bar{u}}{\partial x_\alpha}\Big\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} \le C\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}, \text{ for } \alpha = 1, 2, \tag{3.13}
$$

$$
\begin{aligned}
\Big\|\rho_\epsilon\frac{\partial(\Pi_\epsilon\bar{u})}{\partial x_3}\Big\|_{L^2(\Omega;\mathbb{R}^3)} &\le \frac{C}{L}\Big(\Big\|\frac{\partial\bar{u}}{\partial x_1}\Big\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} + \Big\|\frac{\partial\bar{u}}{\partial x_2}\Big\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)}\Big) + \frac{1}{\epsilon}\Big\|\frac{\partial\bar{u}}{\partial x_3}\Big\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)}\\
&\le \frac{C}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}.
\end{aligned} \tag{3.14}
$$

In the same way, all the estimates in the previous sections over Ω_ϵ can be easily transposed over Ω.

4. Asymptotic behavior of a sequence of displacements

Now we consider a sequence of admissible displacements $\{u^\epsilon\}_\epsilon$, where $u^\epsilon \in H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon; \mathbb{R}^3)$, satisfying

$$
\|(\nabla u^\epsilon)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9} \le C\epsilon^2,
$$

the constant does not depend on ϵ.

We are interested to describe the behaviour of the sequence $\{u^\epsilon\}_\epsilon$ as $\epsilon \to 0$. In the following proposition we introduce the weak limits of the fields of the displacement's decomposition in the rod. We denote by $\rho(x_3) = 1 - \dfrac{x_3}{L}$, $x_3 \in [0, L]$, the strong limit in $L^\infty(0, L)$ of ρ_ϵ. Observe that

$$
0 \le \rho(t) \le \rho_\epsilon(t) \quad \text{ for } t \in [0, L]. \tag{4.1}
$$

First of all, we introduce certain weighted Lebesgue and Sobolev spaces defined in the interval $(0, L)$.

- $L^2_{\rho^k}(0,L)$, $k \in \mathbb{N}$, consists of locally summable functions $\varphi : (0,L) \to \mathbb{R}$ equipped with the following norm:

$$\|\varphi\|_{L^2_{\rho^k}(0,L)} = \Big(\int_0^L [\rho^k(t)\varphi(t)]^2 \, dt \Big)^{1/2}.$$

  Obseve that, there exists a linear homeomorphism of $L^2(0,L)$ onto $L^2_{\rho^k}(0,L)$

$$T(\psi) = \frac{\psi}{\rho^k}, \qquad \text{for } \psi \in L^2(0,L).$$

  Then $L^2_{\rho^k}(0,L) = \Big\{ \varphi \in L^2_{loc}(0,L) \;\mid\; \rho^k\varphi \in L^2(0,L) \Big\}$ endowed with the norm above is a Banach space.

  Remark 4.1. *Observe that if $\{\Phi_\epsilon\}_\epsilon$ is a sequence of functions belonging to $L^2(\Omega)$ and satisfying $\rho^k_\epsilon \Phi_\epsilon \rightharpoonup \Psi$ weakly in $L^2(\Omega)$ then $\Phi_\epsilon \rightharpoonup \Phi = \dfrac{\Psi}{\rho^k}$ weakly in $L^2_{\rho^k}(\Omega)$. Conversely, if $\{\Phi_\epsilon\}_\epsilon$ is a sequence of functions such that $\rho^k_\epsilon \Phi_\epsilon$ is uniformly bounded in $L^2(\Omega)$ and satisfies $\Phi_\epsilon \rightharpoonup \Phi$ weakly in $L^2_{\rho^k}(\Omega)$ then $\rho^k_\epsilon \Phi_\epsilon \rightharpoonup \rho^k \Phi$ weakly in $L^2(\Omega)$. Here k belongs to $\mathbb{N}$.*

- We define the space $H^1_\rho(0,L)$ as follows:

$$H^1_\rho(0,L) = \Big\{ \varphi \in H^1_{loc}(0,L) \;\mid\; \rho\varphi' \in L^2(0,L), \varphi \in L^2(0,L) \text{ and } \varphi(0) = 0 \Big\},$$

  endowed with the following norm:

$$\|\varphi\|_{H^1_\rho(0,L)} = \Big(\int_0^L [\rho(t)\varphi'(t)]^2 \, dt \Big)^{1/2}.$$

  We use this norm since as in the proof of Lemma 3.1 we can easily obtain

$$\|\varphi\|_{L^2(0,L)} \leq 2L \|\rho\varphi'\|_{L^2(0,L)}, \text{ for } \varphi \in H^1_\rho(0,L). \tag{4.2}$$

  Since ρ^{-k}, $k \in \mathbb{N}$, is locally integrable we can conclude that $H^1_\rho(0,L)$ is a Banach space, see [20].

- Analogously, $H^1_{\rho^2}(0,L)$ and $H^2_{\rho^2}(0,L)$ are the Banach spaces which contain the functions $\varphi : (0,L) \to \mathbb{R}$ such that

$$H^1_{\rho^2}(0,L) = \Big\{ \varphi \in H^1_{loc}(0,L) \;\mid\; \rho^2\varphi' \in L^2(0,L), \rho\varphi \in L^2(0,L) \text{ and } \varphi(0) = 0 \Big\},$$

$$H^2_{\rho^2}(0,L) = \Big\{ \varphi \in H^2_{loc}(0,L) \;\mid\; \rho^2\varphi'' \in L^2(0,L), \rho\varphi' \in L^2(0,L), \varphi \in L^2(0,L) \text{ and } \varphi(0) = 0 \Big\}.$$

  We define their norms to be

$$\|\varphi\|_{H^1_{\rho^2}(0,L)} = \Big(\int_0^L [\rho^2(t)\varphi'(t)]^2 \, dt \Big)^{1/2}.$$

$$\|\varphi\|_{H^2_{\rho^2}(0,L)} = \Big(\int_0^L [\rho^2(t)\varphi''(t)]^2 \, dt \Big)^{1/2}.$$

  We can easily prove that

$$\begin{aligned} \|\rho\varphi\|_{L^2(0,L)} &\leq \frac{2L}{3} \|\varphi\|_{H^1_{\rho^2}(0,L)} \qquad \text{for any } \varphi \in H^1_{\rho^2}(0,L), \\ \|\rho\varphi'\|_{L^2(0,L)} &\leq \frac{2L}{3} \|\varphi\|_{H^2_{\rho^2}(0,L)} \qquad \text{for any } \varphi \in H^2_{\rho^2}(0,L), \end{aligned} \tag{4.3}$$

  then (4.2) yields $\|\varphi\|_{L^2(0,L)} \leq \dfrac{4L^2}{3} \|\varphi\|_{H^2_{\rho^2}(0,L)}$ for any $\varphi \in H^2_{\rho^2}(0,L)$.

Similarly we define some weighted spaces in the fixed domain Ω

$$L^2_\rho(\Omega) = \Big\{\phi \in L^2_{loc}(\Omega) \mid \rho\phi \in L^2(\Omega)\Big\},$$

$$H^1_\rho(\Omega) = \Big\{\phi \in H^1_{loc}(\Omega) \mid \rho\frac{\partial\phi}{\partial x_3} \in L^2(\Omega) \text{ and } \frac{\partial\phi}{\partial X_1}, \frac{\partial\phi}{\partial X_2}, \phi \in L^2(\Omega)\Big\}.$$

They are Banach spaces endowed with their respective norms

$$\|\phi\|_{L^2_\rho(\Omega)} = \Big(\int_\Omega [\rho(x_3)\phi(x)]^2\, dX_1 dX_2 dx_3\Big)^{1/2}.$$

$$\|\phi\|_{H^1_\rho(\Omega)} = \Big(\int_\Omega \Big\{\Big(\rho\frac{\partial\phi}{\partial x_3}\Big)^2 + \Big(\frac{\partial\phi}{\partial X_2}\Big)^2 + \Big(\frac{\partial\phi}{\partial X_1}\Big)^2 + \phi^2\Big\}\, dX_1 dX_2 dx_3\Big)^{1/2}.$$

Proposition 4.2. *Let $\{u^\epsilon\}_\epsilon$ be a sequence of displacements such that $u^\epsilon \in H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon;\mathbb{R}^3)$ and*

$$\|(\nabla u^\epsilon)_S\|_{[L^2(\Omega_\epsilon)]^9} \le C\epsilon^2, \tag{4.4}$$

where the constant C is independent of ϵ. Then for a subsequence, still denoted by $\{\epsilon\}$,

- *there exist $\mathcal{U} \in [H^1_\rho(0,L)]^3$, $\mathcal{R} \in [H^1_{\rho^2}(0,L)]^3$ and $\mathcal{Z} \in L^2_\rho((0,L);\mathbb{R}^3)$ such that,*

$$\mathcal{U}^\epsilon_\alpha \rightharpoonup \mathcal{U}_\alpha \text{ weakly in } H^1_\rho(0,L), \text{ for } \alpha = 1,2, \tag{4.5}$$

$$\frac{1}{\epsilon}\mathcal{U}^\epsilon_3 \rightharpoonup \mathcal{U}_3 \text{ weakly in } H^1_\rho(0,L), \tag{4.6}$$

$$\mathcal{R}^\epsilon \rightharpoonup \mathcal{R} \text{ weakly in } [H^1_{\rho^2}(0,L)]^3, \tag{4.7}$$

$$\frac{1}{\epsilon}\Big(\frac{d\mathcal{U}^\epsilon}{dx_3} - \mathcal{R}^\epsilon \wedge \mathbf{e}_3\Big) \rightharpoonup \mathcal{Z} \text{ weakly in } L^2_\rho((0,L);\mathbb{R}^3), \tag{4.8}$$

$$\mathcal{R}(0) = 0,\ \ \mathcal{U}_\alpha(0) = 0,\ \ \mathcal{U}_3(0) = 0. \tag{4.9}$$

- *there exist $\bar{u} \in L^2\big((0,L); H^1(\omega;\mathbb{R}^3)\big)$, $u \in [H^1_\rho(\Omega)]^3$ and $\mathcal{K} \in H^1_\rho(\Omega;\mathbb{R}^3)$ such that*

$$\frac{1}{\epsilon^2}\Pi_\epsilon(\bar{u}^\epsilon) \rightharpoonup \bar{u} \text{ weakly in } L^2\big((0,L); H^1(\omega;\mathbb{R}^3)\big) \tag{4.10}$$

$$\Pi_\epsilon(u^\epsilon_\alpha) \rightharpoonup u_\alpha \text{ weakly in } H^1_\rho(\Omega), \tag{4.11}$$

$$\frac{1}{\epsilon}\Pi_\epsilon(u^\epsilon_3) \rightharpoonup u_3 \text{ weakly in } H^1_\rho(\Omega), \tag{4.12}$$

$$\frac{1}{\epsilon}\Pi_\epsilon(u^\epsilon - \mathcal{U}^\epsilon) \rightharpoonup \mathcal{K} \text{ weakly in } H^1_\rho(\Omega;\mathbb{R}^3). \tag{4.13}$$

Moreover, we have the following relations between the limit fields:

$$\frac{d\mathcal{U}_1}{dx_3} = \mathcal{R}_2, \quad \frac{d\mathcal{U}_2}{dx_3} = -\mathcal{R}_1, \tag{4.14}$$

$$\begin{aligned}
u_1(X_1,X_2,x_3) &= \mathcal{U}_1(x_3), \text{ for a. e. } (X_1,X_2,x_3) \in \Omega,\\
u_2(X_1,X_2,x_3) &= \mathcal{U}_2(x_3), \text{ for a. e. } (X_1,X_2,x_3) \in \Omega,\\
u_3(X_1,X_2,x_3) &= \mathcal{U}_3(x_3) - \rho(x_3)X_1\frac{d\mathcal{U}_1}{dx_3}(x_3) - \rho(x_3)X_2\frac{d\mathcal{U}_2}{dx_3}(x_3), \text{ for a. e. } (X_1,X_2,x_3) \in \Omega.
\end{aligned} \tag{4.15}$$

$$\mathcal{K}_1(X_1,X_2,x_3) = -\rho(x_3)X_2\mathcal{R}_3(x_3) \text{ for a.e. } (X_1,X_2,x_3) \in \Omega.$$

$$\mathcal{K}_2(X_1,X_2,x_3) = \rho(x_3)X_1\mathcal{R}_3(x_3) \text{ for a.e. } (X_1,X_2,x_3) \in \Omega.$$

$$\mathcal{K}_3(X_1,X_2,x_3) = -\rho(x_3)X_1\frac{d\mathcal{U}_1}{dx_3}(x_3) - \rho(x_3)X_2\frac{d\mathcal{U}_2}{dx_3}(x_3) \text{ for a.e. } (X_1,X_2,x_3) \in \Omega.$$

Proof. First we get the weak limits, up to a subsequence still denoted by ϵ, of the different fields. Then we derive a few relations between some of them.

Step 1. The convergences.

Taking into account (4.1)-(4.2) and $(3.1)_2$-$(3.1)_3$ we have

$$\|\mathcal{U}^\epsilon_\alpha\|_{H^1_\rho(0,L)} \le C, \qquad \|\mathcal{U}^\epsilon_3\|_{H^1_\rho(0,L)} \le C\epsilon, \qquad \text{for } \alpha = 1, 2.$$

Then we obtain the following convergences:

$$\mathcal{U}^\epsilon_\alpha \rightharpoonup \mathcal{U}_\alpha \text{ weakly in } H^1_\rho(0,L), \text{ for } \alpha = 1, 2.$$

$$\frac{1}{\epsilon}\mathcal{U}^\epsilon_3 \rightharpoonup \mathcal{U}_3 \text{ weakly in } H^1_\rho(0,L).$$

According to (4.1) we get

$$\|\mathcal{R}^\epsilon_i\|_{H^1_{\rho^2}(0,L)} \le \Big\|\rho^2_\epsilon \frac{d\mathcal{R}_i}{dx_3}\Big\|_{L^2(0,L)} \text{ for } i = 1, 2, 3.$$

Due to estimates $(2.4)_3$ and (4.3) we obtain

$$\mathcal{R}^\epsilon \rightharpoonup \mathcal{R} \text{ weakly in } [H^1_{\rho^2}(0,L)]^3.$$

Again due to (4.1) we have

$$\Big\|\frac{d\mathcal{U}^\epsilon}{dx_3} - \mathcal{R}^\epsilon \wedge \mathbf{e}_3\Big\|_{L^2_\rho((0,L);\mathbb{R}^3)} \le \Big\|\rho_\epsilon\Big(\frac{d\mathcal{U}}{dx_3} - \mathcal{R} \wedge e_3\Big)\Big\|_{L^2((0,L);\mathbb{R}^3)}.$$

In view of estimate $(2.4)_2$ we get

$$\frac{1}{\epsilon}\Big(\frac{d\mathcal{U}^\epsilon}{dx_3} - \mathcal{R}^\epsilon \wedge \mathbf{e}_3\Big) \rightharpoonup \mathcal{Z} \text{ weakly in } L^2_\rho((0,L);\mathbb{R}^3).$$

Thanks to the estimates (3.11) and (3.13) the sequence $\frac{1}{\epsilon^2}\Pi_\epsilon(\bar{u}^\epsilon)$ is bounded in $L^2\big((0,L);H^1(\omega;\mathbb{R}^3)\big)$. Then we obtain

$$\frac{1}{\epsilon^2}\Pi_\epsilon(\bar{u}^\epsilon) \rightharpoonup \bar{u} \text{ weakly in } L^2\big((0,L);H^1(\omega;\mathbb{R}^3)\big).$$

From property (3.10) of the rescaling operator and the estimate $(3.9)_2$ we have

$$\big\|\Pi_\epsilon u^\epsilon_\alpha\big\|_{L^2(\Omega)} = \frac{1}{\epsilon}\Big\|\frac{u^\epsilon_\alpha}{\rho_\epsilon}\Big\|_{L^2(\Omega_\epsilon)} \le C\frac{L}{\epsilon^2}\|(\nabla u)_\mathcal{S}\|_{[L^2(\Omega_\epsilon)]^9}, \quad \text{for } \alpha = 1, 2. \tag{4.16}$$

Moreover, taking into account the derivation rule (3.12) and the estimates $(3.9)_1$ we have

$$\Big\|\frac{\partial(\Pi_\epsilon u^\epsilon_\alpha)}{\partial X_\beta}\Big\|_{L^2(\Omega)} = \Big\|\frac{\partial u^\epsilon_\alpha}{\partial x_\beta}\Big\|_{L^2(\Omega_\epsilon)} \le \frac{C}{\epsilon}\|(\nabla u^\epsilon)_\mathcal{S}\|_{[L^2(\Omega_\epsilon)]^9}, \text{ for } \alpha, \beta = 1, 2, \tag{4.17}$$

$$\begin{aligned}\Big\|\rho\frac{\partial(\Pi_\epsilon u^\epsilon_\alpha)}{\partial x_3}\Big\|_{L^2(\Omega)} \le \Big\|\rho_\epsilon\frac{\partial(\Pi_\epsilon u^\epsilon_\alpha)}{\partial x_3}\Big\|_{L^2(\Omega)} &\le \frac{C}{L}\Big(\Big\|\frac{\partial u^\epsilon_\alpha}{\partial x_1}\Big\|_{L^2(\Omega_\epsilon)} + \Big\|\frac{\partial u^\epsilon_\alpha}{\partial x_2}\Big\|_{L^2(\Omega_\epsilon)}\Big) + \frac{1}{\epsilon}\Big\|\frac{\partial u^\epsilon_\alpha}{\partial x_3}\Big\|_{L^2(\Omega_\epsilon)}\\ &\le \frac{C}{\epsilon^2}\|(\nabla u^\epsilon)_\mathcal{S}\|_{[L^2(\Omega_\epsilon)]^9}, \quad \text{for } \alpha = 1, 2.\end{aligned} \tag{4.18}$$

Therefore, from (4.16), (4.17) and (4.18) we get

$$\Pi_\epsilon(u^\epsilon_\alpha) \rightharpoonup u_\alpha \text{ weakly in } H^1_\rho(\Omega), \quad \text{for } \alpha = 1, 2.$$

In the same way, from $(3.9)_1$, $(3.9)_3$ and (3.12) we obtain

$$\left\|\Pi_\epsilon u_3^\epsilon\right\|_{L^2(\Omega)} \leq C\frac{L}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}, \tag{4.19}$$

$$\left\|\frac{\partial(\Pi_\epsilon u_3^\epsilon)}{\partial X_\beta}\right\|_{L^2(\Omega)} \leq \frac{C}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}, \text{ for } \beta = 1, 2, \tag{4.20}$$

$$\left\|\rho\frac{\partial(\Pi_\epsilon u_3^\epsilon)}{\partial x_3}\right\|_{L^2(\Omega)} \leq \frac{C}{\epsilon}\|(\nabla u)_{\mathcal{S}}\|_{[L^2(\Omega_\epsilon)]^9}. \tag{4.21}$$

Hence, we get

$$\frac{1}{\epsilon}\Pi_\epsilon(u_3^\epsilon) \rightharpoonup u_3 \text{ weakly in } H^1_\rho(\Omega).$$

From the definition of the elementary displacement we have

$$u^\epsilon(x) - \mathcal{U}^\epsilon(x_3) = \mathcal{R}^\epsilon(x_3) \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2) + \bar{u}^\epsilon(x).$$

Hence, in view of $(3.1)_1$, (3.11), the property (3.10) of the rescaling operator and the assumption (4.4) we obtain the following estimate:

$$\frac{1}{\epsilon}\left\|\Pi_\epsilon(u^\epsilon - \mathcal{U}^\epsilon)\right\|_{L^2(\Omega;\mathbb{R}^3)} \leq \frac{1}{\epsilon^2}\left\|\frac{\mathcal{R}^\epsilon(x_3) \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2)}{\rho_\epsilon}\right\|_{L^2(\Omega_\epsilon,\mathbb{R}^3)} + \frac{1}{\epsilon}\left\|\Pi_\epsilon\bar{u}\right\|_{L^2(\Omega;\mathbb{R}^3)} \leq C. \tag{4.22}$$

Now using the rule of the derivation (3.12) and (3.10) we have for $\alpha = 1, 2$

$$\frac{1}{\epsilon}\left\|\frac{\partial\Pi_\epsilon(u^\epsilon - \mathcal{U}^\epsilon)}{\partial X_\alpha}\right\|_{L^2(\Omega;\mathbb{R}^3)} = \frac{1}{\epsilon}\left\|\frac{\partial(u^\epsilon - \mathcal{U}^\epsilon)}{\partial x_\alpha}\right\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} \leq \frac{1}{\epsilon}\left\|\mathcal{R}^\epsilon(x_3) \wedge \mathbf{e}_\alpha\right\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} + \frac{1}{\epsilon}\left\|\frac{\partial\bar{u}}{\partial x_\alpha}\right\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} \leq C,$$

$$\begin{aligned}\frac{1}{\epsilon}\left\|\rho\frac{\partial\Pi_\epsilon(u^\epsilon - \mathcal{U}^\epsilon)}{\partial x_3}\right\|_{L^2(\Omega;\mathbb{R}^3)} &\leq \frac{C}{\epsilon}\Big(\sum_{\alpha=1}^{2}\left\|\frac{\partial(u^\epsilon - \mathcal{U}^\epsilon)}{\partial x_\alpha}\right\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)}\Big) + \frac{1}{\epsilon^2}\left\|\frac{\partial(u^\epsilon - \mathcal{U}^\epsilon)}{\partial x_3}\right\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} \\ &\leq C + \frac{1}{\epsilon^2}\left\|\frac{d\mathcal{R}^\epsilon}{dx_3} \wedge (x_1\mathbf{e}_1 + x_2\mathbf{e}_2)\right\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} + \frac{1}{\epsilon^2}\left\|\frac{\partial\bar{u}^\epsilon}{\partial x_3}\right\|_{L^2(\Omega_\epsilon;\mathbb{R}^3)} \leq C.\end{aligned}$$

Consequently, from the two above estimates and (4.22) we get the last weak convergence

$$\frac{1}{\epsilon}\Pi_\epsilon(u^\epsilon - \mathcal{U}^\epsilon) \rightharpoonup \mathcal{K} \text{ weakly in } H^1_\rho(\Omega;\mathbb{R}^3).$$

Step 2. Relations between the limit fields.

Now we are going to establish the relations between the weak limits. First consider $(2.4)_2$ which implies

$$\Big(\frac{d\mathcal{U}^\epsilon}{dx_3} - \mathcal{R}^\epsilon \wedge \mathbf{e}_3\Big) \to 0 \text{ strongly in } L^2_\rho((0, L);\mathbb{R}^3),$$

as ϵ tends to 0. Then (4.5) and (4.7) give

$$\frac{d\mathcal{U}_1}{dx_3} = \mathcal{R}_2, \quad \frac{d\mathcal{U}_2}{dx_3} = -\mathcal{R}_1. \tag{4.23}$$

It follows that $\mathcal{U}_\alpha \in H^2_{\rho^2}(0, L)$, for $\alpha = 1, 2$.

Now, from (3.5) we can write

$$(\Pi_\epsilon u_1^\epsilon)(X_1, X_2, x_3) = \mathcal{U}_1^\epsilon(x_3) - \epsilon\rho_\epsilon X_2\mathcal{R}_3^\epsilon(x_3) + (\Pi_\epsilon\bar{u}_1^\epsilon)(X_1, X_2, x_3), \text{ for a.e. } (X_1, X_2, x_3) \in \Omega. \tag{4.24}$$

In view of (4.5), (4.7), (4.10) and (4.11) by passing to the limit in (4.24) we obtain

$$u_1(X_1, X_2, x_3) = \mathcal{U}_1(x_3), \text{ for a.e. } (X_1, X_2, x_3) \in \Omega.$$

Repeating the above arguments for $(\Pi_\epsilon u_2^\epsilon)$ we conclude that

$$u_2(X_1, X_2, x_3) = \mathcal{U}_2(x_3), \text{ for a.e. } (X_1, X_2, x_3) \in \Omega.$$

Notice that u_α does not depend on the variables (X_1, X_2), for $\alpha = 1, 2$.

From (3.5) we have for a.e. $(X_1, X_2, x_3) \in \Omega$

$$\frac{1}{\epsilon}(\Pi_\epsilon u_3^\epsilon)(X_1, X_2, x_3) = \frac{1}{\epsilon}\mathcal{U}_3^\epsilon(x_3) - \rho_\epsilon X_1 \mathcal{R}_2^\epsilon(x_3) + \rho_\epsilon X_2 \mathcal{R}_1^\epsilon(x_3) + \frac{1}{\epsilon}(\Pi_\epsilon \bar{u}_3^\epsilon)(X_1, X_2, x_3).$$

Now, using (4.6), (4.7), (4.10) and (4.12) we pass to the limit in the equality above and we get

$$u_3(X_1, X_2, x_3) = \mathcal{U}_3(x_3) - \rho X_1 \mathcal{R}_2(x_3) + \rho X_2 \mathcal{R}_1(x_3) \text{ for a.e. } (X_1, X_2, x_3) \in \Omega. \tag{4.25}$$

Observe that, due to (4.23), (4.25) can be written as

$$u_3(X_1, X_2, x_3) = \mathcal{U}_3(x_3) - \rho X_1 \frac{d\mathcal{U}_1}{dx_3}(x_3) - \rho X_2 \frac{d\mathcal{U}_2}{dx_3}(x_3), \text{ for a.e. } (X_1, X_2, x_3) \in \Omega. \tag{4.26}$$

Now we turn to the identification of $\mathcal{K}_i$. In view of (3.5) we have

$$\frac{1}{\epsilon}\Pi_\epsilon(u_1^\epsilon - \mathcal{U}_1^\epsilon) = -\rho_\epsilon X_2 \mathcal{R}_3^\epsilon(x_3) + \frac{1}{\epsilon}(\Pi_\epsilon \bar{u}_1^\epsilon)(X_1, X_2, x_3), \text{ for a.e. } (X_1, X_2, x_3) \in \Omega. \tag{4.27}$$

From (4.7), (4.10), (4.13) by passing to the limit in (4.27) we obtain

$$\mathcal{K}_1(X_1, X_2, x_3) = -\rho(x_3) X_2 \mathcal{R}_3(x_3) \text{ for a.e. } (X_1, X_2, x_3) \in \Omega.$$

Proceeding as above for $\frac{1}{\epsilon}\Pi_\epsilon(u_2^\epsilon - \mathcal{U}_2^\epsilon)$ we get

$$\mathcal{K}_2(X_1, X_2, x_3) = \rho(x_3) X_1 \mathcal{R}_3(x_3) \text{ for a.e. } (X_1, X_2, x_3) \in \Omega.$$

Finally we obtain the expression for $\mathcal{K}_3$. From (3.5) we have

$$\frac{1}{\epsilon}\Pi_\epsilon(u_3^\epsilon - \mathcal{U}_3^\epsilon) = -\rho_\epsilon X_1 \mathcal{R}_2^\epsilon(x_3) + \rho_\epsilon X_2 \mathcal{R}_1^\epsilon(x_3) + \frac{1}{\epsilon}(\Pi_\epsilon \bar{u}_3^\epsilon)(X_1, X_2, x_3).$$

Convergences (4.7), (4.10), (4.13) allow to pass to the limit and we get

$$\mathcal{K}_3(X_1, X_2, x_3) = -\rho X_1 \mathcal{R}_2(x_3) + \rho X_2 \mathcal{R}_1(x_3) \text{ for a. e. } (X_1, X_2, x_3) \in \Omega.$$

Equivalently, from (4.23) we have

$$\mathcal{K}_3(X_1, X_2, x_3) = -\rho X_1 \frac{d\mathcal{U}_1}{dx_3}(x_3) - \rho X_2 \frac{d\mathcal{U}_2}{dx_3}(x_3) \text{ for a.e. } (X_1, X_2, x_3) \in \Omega. \qquad \square$$

Remark 4.3. *It is worth to note that the limit displacement fields is a kind of Bernoulli-Navier displacement.*

Also observe that the limit warping $\bar{u}$ verifies the following conditions:

$$\int_\omega \bar{u}_i \, dX_1 dX_2 = \int_\omega (X_1 \bar{u}_2 - X_2 \bar{u}_1) \, dX_1 dX_2 = \int_\omega X_\alpha \bar{u}_3 \, dX_1 dX_2 = 0, \quad i \in \{1, 2, 3\}, \ \ \alpha \in \{1, 2\}. \tag{4.28}$$

To conclude this section, we give the asymptotic behavior of the gradient and the symmetric gradient. We define the field $\tilde u_3\in L^2((0,L);H^1(\omega))$ by setting

$$\tilde u_3(X_1,X_2,x_3)=\bar u_3(X_1,X_2,x_3)+\rho(x_3)\mathcal{Z}_1(x_3)X_1+\rho(x_3)\mathcal{Z}_2(x_3)X_2\quad\text{for a.e. } (X_1,X_2,x_3)\in\Omega.$$

Lemma 4.4. *In view of* (4.5)-(4.13) *we obtain*

$$\Pi_\epsilon(\nabla u^\epsilon)\rightharpoonup Z \text{ weakly in } [L^2_\rho(\Omega)]^9,\qquad \frac{1}{\epsilon}\Pi_\epsilon\big((\nabla u^\epsilon)_{\mathcal S}\big)\rightharpoonup T \text{ weakly in } [L^2_\rho(\Omega)]^9,\tag{4.29}$$

where

$$Z=\begin{pmatrix}0 & -\mathcal{R}_3 & \mathcal{R}_2\\ \mathcal{R}_3 & 0 & -\mathcal{R}_1\\ -\mathcal{R}_2 & \mathcal{R}_1 & 0\end{pmatrix},\quad T=\begin{pmatrix}\dfrac{\partial\bar u_1}{\partial X_1} & \dfrac12\Big(\dfrac{\partial\bar u_1}{\partial X_2}+\dfrac{\partial\bar u_2}{\partial X_1}\Big) & \dfrac12\Big(-\rho^2X_2\dfrac{d\mathcal{R}_3}{dx_3}+\dfrac{\partial\tilde u_3}{\partial X_1}\Big)\\ * & \dfrac{\partial\bar u_2}{dX_2} & \dfrac12\Big(\rho^2X_1\dfrac{d\mathcal{R}_3}{dx_3}+\dfrac{\partial\tilde u_3}{\partial X_2}\Big)\\ * & * & \rho\dfrac{d\mathcal{U}_3}{dx_3}-\rho^2X_1\dfrac{d\mathcal{R}_2}{dx_3}+\rho^2X_2\dfrac{d\mathcal{R}_1}{dx_3}\end{pmatrix}.$$

Proof. Step 1. Determination of the matrix Z.

In view of (3.7) to obtain the Z_{ij}'s we only need to take into account the following convergences:

- From (3.11), (3.13), (3.14), (4.1) and (4.4) we get

$$\frac{1}{\epsilon}\Pi_\epsilon\bar u_j^\epsilon\rightharpoonup 0 \text{ weakly in } H^1_\rho(\Omega),\ \text{for } j=1,2,3.\tag{4.30}$$

  Hence

$$\frac{1}{\epsilon}\Pi_\epsilon\Big(\frac{\partial\bar u_j^\epsilon}{\partial x_3}\Big)\rightharpoonup 0 \text{ weakly in } L^2_\rho(\Omega),\ \text{for } j=1,2,3.\tag{4.31}$$

- Since $\mathcal{U}^\epsilon$ and $\mathcal{R}^\epsilon$ are independent of x_1 and x_2 we have

$$\Pi_\epsilon(\mathcal{R}^\epsilon)=\mathcal{R}^\epsilon,\quad \Pi_\epsilon(x_\alpha\mathcal{R}^\epsilon)=\epsilon\rho_\epsilon X_\alpha\mathcal{R}^\epsilon,\ \text{for } \alpha=1,2,\quad \Pi_\epsilon\Big(\frac{d\mathcal{U}^\epsilon}{dx_3}\Big)=\frac{d\mathcal{U}^\epsilon}{dx_3}.$$

  Then in view of (4.5), (4.6), (4.7) and (4.14) we obtain

$$\begin{aligned}&\Pi_\epsilon(\mathcal{R}^\epsilon)\rightharpoonup\mathcal{R} \text{ weakly in } [L^2_\rho(\Omega)]^3,\\ &\Pi_\epsilon(x_\alpha\mathcal{R}^\epsilon)\to 0 \text{ strongly in } [L^2_\rho(\Omega)]^3,\ \text{for } \alpha=1,2,\\ &\Pi_\epsilon\Big(\frac{d\mathcal{U}^\epsilon_\alpha}{dx_3}\Big)\rightharpoonup\frac{d\mathcal{U}_\alpha}{dx_3}=(-1)^{3-\alpha}\mathcal{R}_{3-\alpha} \text{ weakly in } L^2_\rho(\Omega),\ \text{for } \alpha=1,2,\\ &\Pi_\epsilon\Big(\frac{d\mathcal{U}^\epsilon_3}{dx_3}\Big)\to 0 \text{ strongly in } L^2_\rho(\Omega).\end{aligned}$$

Step 2. Determination of the matrix T.

From (3.12) we have

$$\frac{1}{\epsilon}\Pi_\epsilon\Big(\frac{\partial\bar u^\epsilon}{\partial x_\alpha}\Big)=\frac{1}{\epsilon^2\rho_\epsilon}\frac{\partial(\Pi_\epsilon\bar u^\epsilon)}{\partial X_\alpha}\ \text{for } \alpha=1,2.$$

Then in view of (3.8) and using convergence (4.10) we obtain

$$T_{\alpha\beta}=\frac12\Big(\frac{\partial\bar u_\alpha}{\partial X_\beta}+\frac{\partial\bar u_\beta}{\partial X_\alpha}\Big)\ \text{for } \alpha,\beta=1,2.$$

Applying the rescaling operator to (3.8) we get

$$\frac{\rho}{\epsilon}\Pi_\epsilon\big((\nabla u^\epsilon)_{\mathcal{S}}\big)_{13}=\frac{1}{2}\Big[\frac{\rho}{\epsilon}\Big(\frac{d\mathcal{U}_1^\epsilon}{dx_3}-\mathcal{R}_2^\epsilon\Big)-\rho\rho_\epsilon X_2\frac{d\mathcal{R}_3^\epsilon}{dx_3}+\frac{\rho}{\rho_\epsilon}\frac{1}{\epsilon^2}\frac{\partial(\Pi_\epsilon\bar{u}_3^\epsilon)}{\partial X_1}+\frac{\rho}{\epsilon}\Pi_\epsilon\Big(\frac{\partial\bar{u}_1^\epsilon}{\partial x_3}\Big)\Big].$$

Convergences (4.7), (4.8), (4.10) and (4.31) allow us to pass to the limit and we obtain

$$T_{13}=\frac{1}{2}\Big(\rho\mathcal{Z}_1-\rho^2X_2\frac{d\mathcal{R}_3}{dx_3}+\frac{\partial\bar{u}_3}{\partial X_1}\Big)=\frac{1}{2}\Big(-\rho^2X_2\frac{d\mathcal{R}_3}{dx_3}+\frac{\partial\tilde{u}_3}{\partial X_1}\Big).$$

Similar calculations which are not repeated here allows us to get

$$T_{23}=\frac{1}{2}\Big(\rho\mathcal{Z}_2+\rho^2X_1\frac{d\mathcal{R}_3}{dx_3}+\frac{\partial\bar{u}_3}{\partial X_2}\Big)=\frac{1}{2}\Big(\rho^2X_1\frac{d\mathcal{R}_3}{dx_3}+\frac{\partial\tilde{u}_3}{\partial X_2}\Big).$$

To identify T_{33} observe that from (3.8) we have

$$\frac{\rho}{\epsilon}\Pi_\epsilon\big((\nabla u^\epsilon)_{\mathcal{S}}\big)_{33}=\frac{\rho}{\epsilon}\frac{d\mathcal{U}_3^\epsilon}{dx_3}-\rho\rho_\epsilon X_1\frac{d\mathcal{R}_2^\epsilon}{dx_3}+\rho\rho_\epsilon X_2\frac{d\mathcal{R}_1^\epsilon}{dx_3}+\frac{\rho}{\epsilon}\Pi_\epsilon\Big(\frac{\partial\bar{u}_3^\epsilon}{\partial x_3}\Big).$$

Convergences (4.6), (4.7), (4.10) and (4.31) allow us to pass to the limit and we obtain

$$T_{33}=\rho\frac{d\mathcal{U}_3}{dx_3}-\rho^2X_1\frac{d\mathcal{R}_2}{dx_3}+\rho^2X_2\frac{d\mathcal{R}_1}{dx_3}.$$

According to (4.14), T_{33} can be expressed as

$$T_{33}=\rho\frac{d\mathcal{U}_3}{dx_3}-\rho^2X_1\frac{d^2\mathcal{U}_1}{dx_3^2}-\rho^2X_2\frac{d^2\mathcal{U}_2}{dx_3^2}. \tag{4.32}$$

□

5. Position of the elastic problem

We consider the standard linear isotropic equations of elasticity in Ω_ϵ. The displacement field in Ω_ϵ is denoted by

$$u^\epsilon:\Omega_\epsilon\to\mathbb{R}^3.$$

The linearized deformation field in Ω_ϵ is defined by

$$\gamma_{ij}(u^\epsilon)=\frac{1}{2}\Big(\frac{\partial u_j^\epsilon}{\partial x_i}+\frac{\partial u_i^\epsilon}{\partial x_j}\Big),\quad i,j=1,2,3.$$

The Cauchy stress tensor in Ω_ϵ is linked to $\gamma(u^\epsilon)$ through the standard Hooke's law

$$\sigma_{ij}^\epsilon=\lambda\Big(\sum_{k=1}^3\gamma_{kk}(u^\epsilon)\Big)\delta_{ij}+2\mu\gamma_{ij}(u^\epsilon),\quad i,j=1,2,3,$$

where λ and μ denotes the Lame's coefficients of the elastic material and $\delta_{ij}=0$ if $i\neq j$ and $\delta_{ij}=1$ if $i=j$. The equation of equilibrium in Ω_ϵ is

$$-\sum_{j=1}^3\frac{\partial\sigma_{ij}^\epsilon}{\partial x_j}=f_i^\epsilon\ \text{ in }\Omega_\epsilon,\quad i=1,2,3, \tag{5.1}$$

where $f^\epsilon:\Omega_\epsilon\to\mathbb{R}^3$ denotes the applied force.

We assume that the rod is clamped at the bottom, $\Gamma_{\epsilon,0}=\omega_{\epsilon,0}\times\{0\}$

$$u^\epsilon = 0 \text{ on } \Gamma_{\epsilon,0},$$

and at the boundary $\partial\Omega_\epsilon \backslash \Gamma_{\epsilon,0}$ it is free

$$\sigma^\epsilon \nu_\epsilon = 0 \ \text{ on } \partial\Omega_\epsilon \backslash \Gamma_{\epsilon,0},$$

where ν_ϵ denotes the exterior unit normal to Ω_ϵ.

Taking into account that the space of admissible displacements of the rod is

$$H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon;\mathbb{R}^3) = \{u^\epsilon \in H^1(\Omega_\epsilon;\mathbb{R}^3) \mid u^\epsilon = 0 \text{ on } \Gamma_{\epsilon,0}\},$$

the variational formulation of (5.1) is

$$\begin{cases} \quad u^\epsilon \in H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon;\mathbb{R}^3), \\ \displaystyle\int_{\Omega_\epsilon} \sum_{i,j=1}^3 \sigma^\epsilon_{ij}\gamma_{ij}(v)\,dx = \int_{\Omega_\epsilon} \sum_{i=1}^3 f^\epsilon_i v_i\,dx, \quad \forall v \in H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon;\mathbb{R}^3). \end{cases} \tag{5.2}$$

For any $v \in H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon;\mathbb{R}^3)$, the total elastic energy is denoted by

$$\mathcal{E}(v) = \int_{\Omega_\epsilon} \Big[\lambda\Big(\sum_{k=1}^3 \gamma_{kk}(v)\Big)^2 + 2\mu \sum_{i,j=1}^3 \big(\gamma_{ij}(v)\big)^2\Big]\,dx.$$

Observe that there exists a constant which depends only on λ and μ such that for any $w \in H^1(\Omega_\epsilon;\mathbb{R}^3)$ we have

$$C\|(\nabla w)_S\|^2_{[L^2(\Omega_\epsilon)]^9} \le \mathcal{E}(w). \tag{5.3}$$

Taking $v = u^\epsilon$ in (5.2) leads to the usual energy relation

$$\mathcal{E}(u^\epsilon) = \int_{\Omega_\epsilon} \sum_{i=1}^3 f^\epsilon_i u^\epsilon_i\,dx. \tag{5.4}$$

5.1. Assumption on the forces

In view of the energy relation (5.4) and the estimates of the previous sections we assume throughout the paper

$$\begin{cases} F^\epsilon_1(x) = \epsilon^2 f^\epsilon_1(x_3) - x_2 g^\epsilon_3(x_3), \ \text{for } x \in \Omega_\epsilon \\ F^\epsilon_2(x) = \epsilon^2 f^\epsilon_2(x_3) + x_1 g^\epsilon_3(x_3), \ \text{for } x \in \Omega_\epsilon \\ F^\epsilon_3(x) = \epsilon f^\epsilon_3(x_3) + x_1 g^\epsilon_1(x_3) + x_2 g^\epsilon_2(x_3) \text{ for } x \in \Omega_\epsilon, \end{cases} \tag{5.5}$$

where $f^\epsilon, g^\epsilon \in L^2((0,L);\mathbb{R}^3)$ and they satisfy

$$\big\|\rho^2_\epsilon f^\epsilon\big\|_{L^2((0,L);\mathbb{R}^3)} + \big\|\rho^3_\epsilon g^\epsilon\big\|_{L^2((0,L);\mathbb{R}^3)} \le C \tag{5.6}$$

the constant does not depend on ϵ. Moreover we assume the following weak convergences:

$$\begin{cases} f^\epsilon \longrightarrow f \quad \text{strongly in } L^2_{\rho^2}((0,L);\mathbb{R}^3), \\ g^\epsilon \longrightarrow g \quad \text{strongly in } L^2_{\rho^3}((0,L);\mathbb{R}^3). \end{cases} \tag{5.7}$$

As a consequence, from (5.4) and the relations (3.6) we get an estimate of the total elastic energy

$$\mathcal{E}(u^\epsilon) = \int_0^L \Big[\epsilon^2 f^\epsilon_1(x_3)|\omega|\rho_\epsilon(x_3)^2\epsilon^2\mathcal{U}^\epsilon_1(x_3) + \epsilon^2 f^\epsilon_2(x_3)|\omega|\rho_\epsilon(x_3)^2\epsilon^2\mathcal{U}^\epsilon_2(x_3) + \epsilon f^\epsilon_3(x_3)|\omega|\rho_\epsilon(x_3)^2\epsilon^2\mathcal{U}^\epsilon_3(x_3)\Big]\,dx_3$$

$$+\int_0^L \Big[g_3^\epsilon(x_3)(I_1+I_2)\rho_\epsilon(x_3)^4\epsilon^4\mathcal{R}_3^\epsilon(x_3)+g_2^\epsilon(x_3)(I_2)\rho_\epsilon(x_3)^4\epsilon^4\mathcal{R}_1^\epsilon(x_3)-g_1^\epsilon(x_3)(I_1)\rho_\epsilon(x_3)^4\epsilon^4\mathcal{R}_2^\epsilon(x_3)\Big]\,dx_3$$

Due to $(3.1)_1$, $(3.1)_4$, $(3.1)_5$ and (5.3)-(5.6) we have

$$\mathcal{E}(u^\epsilon)\le C\epsilon^2\big(\|\rho_\epsilon^2 f^\epsilon\|_{L^2((0,L);\mathbb{R}^3)}+\|\rho_\epsilon^3 g^\epsilon\|_{L^2((0,L);\mathbb{R}^3)}\big)\|(\nabla u^\epsilon)_S\|_{[L^2(\Omega_\epsilon)]^9}\le C\epsilon^2\mathcal{E}(u^\epsilon)^{1/2}.$$

That leads to

$$\mathcal{E}(u^\epsilon)^{1/2}\le C\epsilon^2.$$

Hence

$$\|(\nabla u^\epsilon)_S\|_{[L^2(\Omega_\epsilon)]^9}\le C\epsilon^2.$$

Remark 5.1. *Observe that the assumptions on the applied forces were assumed in order to obtain the appropriate estimate on the energy naturally.*

6. The limit problems

In this section we obtain the equations satisfied by the limit fields $\mathcal{U}$, $\mathcal{R}$ and $\bar u$. To do this, we assume that the forces are given by (5.5) and satisfy (5.6)-(5.7). First, we apply the rescaling operator Π_ϵ to the original variational formulation of the problem (5.2)

$$\int_\Omega \rho_\epsilon^2\sum_{i,j=1}^3 \Pi_\epsilon(\sigma_{ij}^\epsilon)\Pi_\epsilon(\gamma_{ij}(v))\,dX_1dX_2dx_3=\int_\Omega \rho_\epsilon^2\sum_{i=1}^3 \Pi_\epsilon(F_i^\epsilon)\Pi_\epsilon(v_i)\,dX_1dX_2dx_3,\quad \forall v\in H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon;\mathbb{R}^3).\tag{6.1}$$

We will pass to the limit in (6.1) as ϵ tends to zero. In order to accomplish this we need specific choices of the test function v. We begin studying the behavior of the limit of the residual displacement $\bar u^\epsilon$.

6.1. Equations for $\bar u$

Let ϕ be in $H^1(\omega,\mathbb{R}^3)$ and φ be in $C^\infty[0,L]$ such that $\varphi(0)=0$, we define the test function $v^\epsilon\in H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon;\mathbb{R}^3)$ by

$$v^\epsilon(x_1,x_2,x_3)=\epsilon\varphi(x_3)\phi\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big),\quad (x_1,x_2,x_3)\in\Omega_\epsilon.\tag{6.2}$$

Then we have

$$\begin{aligned}
\gamma_{11}(v^\epsilon)&=\frac{1}{\rho_\epsilon}\varphi(x_3)\frac{\partial\phi_1}{\partial X_1}\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big),\\
\gamma_{12}(v^\epsilon)&=\frac{1}{2\rho_\epsilon}\varphi(x_3)\Big[\frac{\partial\phi_1}{\partial X_2}+\frac{\partial\phi_2}{\partial X_1}\Big]\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big),\\
\gamma_{13}(v^\epsilon)&=\frac12\Big[-\varphi(x_3)\frac{\partial\phi_1}{\partial X_1}\frac{x_1\rho_\epsilon'}{\rho_\epsilon^2}-\varphi(x_3)\frac{\partial\phi_1}{\partial X_2}\frac{x_2\rho_\epsilon'}{\rho_\epsilon^2}+\epsilon\varphi'(x_3)\phi_1+\frac{1}{\rho_\epsilon}\varphi(x_3)\frac{\partial\phi_3}{\partial X_1}\Big]\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big)\\
&=-\frac{\varphi(x_3)\rho_\epsilon'}{2\rho_\epsilon^2}\Big[\sum_{\alpha=1}^2\frac{\partial\phi_1}{\partial X_\alpha}\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big)x_\alpha\Big]+\frac{\epsilon}{2}\varphi'(x_3)\phi_1\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big)+\frac{1}{2\rho_\epsilon}\varphi(x_3)\frac{\partial\phi_3}{\partial X_1}\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big),\\
\gamma_{22}(v^\epsilon)&=\frac{1}{\rho_\epsilon}\varphi(x_3)\frac{\partial\phi_2}{\partial X_2}\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big),\\
\gamma_{23}(v^\epsilon)&=-\frac{\varphi(x_3)\rho_\epsilon'}{2\rho_\epsilon^2}\Big[\sum_{\alpha=1}^2\frac{\partial\phi_2}{\partial X_\alpha}\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big)x_\alpha\Big]+\frac{\epsilon}{2}\varphi'(x_3)\phi_2\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big)+\frac{1}{2\rho_\epsilon}\varphi(x_3)\frac{\partial\phi_3}{\partial X_2}\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big),\\
\gamma_{33}(v^\epsilon)&=-\frac{\varphi(x_3)\rho_\epsilon'}{\rho_\epsilon^2}\Big[\frac{\partial\phi_3}{\partial X_1}\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big)x_1+\frac{\partial\phi_3}{\partial X_2}\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big)x_2\Big]+\epsilon\varphi'(x_3)\phi_3\Big(\frac{x_1}{\epsilon\rho_\epsilon},\frac{x_2}{\epsilon\rho_\epsilon}\Big).
\end{aligned}$$

Hence, using the properties of the rescaling operator we get the following strong convergences in $L^2(\Omega)$:

$$
\begin{aligned}
&\rho_\epsilon \Pi_\epsilon(\gamma_{13}(v^\epsilon)) \to \frac{1}{2}\varphi(x_3)\frac{\partial \phi_3}{\partial X_1},\\
&\rho_\epsilon \Pi_\epsilon(\gamma_{23}(v^\epsilon)) \to \frac{1}{2}\varphi(x_3)\frac{\partial \phi_3}{\partial X_2},\\
&\rho_\epsilon \Pi_\epsilon(\gamma_{33}(v^\epsilon)) \to 0.
\end{aligned}
\tag{6.3}
$$

Moreover, $\rho_\epsilon\Pi_\epsilon(\gamma_{11}(v^\epsilon)), \rho_\epsilon\Pi_\epsilon(\gamma_{12}(v^\epsilon))$ and $\rho_\epsilon\Pi_\epsilon(\gamma_{22}(v^\epsilon))$ are independent of ϵ, since

$$
\begin{aligned}
&\rho_\epsilon\Pi_\epsilon(\gamma_{11}(v^\epsilon))(X_1,X_2,x_3) = \varphi(x_3)\frac{\partial \phi_1}{\partial X_1}(X_1,X_2),\\
&\rho_\epsilon\Pi_\epsilon(\gamma_{12}(v^\epsilon))(X_1,X_2,x_3) = \frac{1}{2}\varphi(x_3)\Big[\frac{\partial \phi_1}{\partial X_2}+\frac{\partial \phi_2}{\partial X_1}\Big](X_1,X_2),\\
&\rho_\epsilon\Pi_\epsilon(\gamma_{22}(v^\epsilon))(X_1,X_2,x_3) = \varphi(x_3)\frac{\partial \phi_2}{\partial X_2}\Big(X_1,X_2\Big).
\end{aligned}
\tag{6.4}
$$

Now, we take v^ϵ as test function in (6.1), we have

$$
\frac{1}{\epsilon}\Pi_\epsilon(v^\epsilon)(X_1,X_2,x_3) = \varphi(x_3)\phi(X_1,X_2), \qquad \text{for } (X_1,X_2,x_3)\in\Omega.
$$

Then we pass to the limit. As far as the right hand side of (6.1) is concerned, taking into account the assumptions (5.5)-(5.6) and (5.7) we have

$$
\begin{aligned}
&\Pi_\epsilon(F_1^\epsilon) = \epsilon^2 f_1^\epsilon(x_3) - \rho_\epsilon\epsilon X_2 g_3^\epsilon(x_3) \to 0 \quad \text{strongly in } L^2_{\rho^2}(\Omega),\\
&\Pi_\epsilon(F_2^\epsilon) = \epsilon^2 f_2^\epsilon(x_3) + \rho_\epsilon\epsilon X_1 g_3^\epsilon(x_3) \to 0 \quad \text{strongly in } L^2_{\rho^2}(\Omega),\\
&\Pi_\epsilon(F_3^\epsilon) = \epsilon f_3^\epsilon(x_3) + \rho_\epsilon\epsilon X_1 g_1(x_3) + \rho_\epsilon\epsilon X_2 g_2^\epsilon(x_3) \to 0 \quad \text{strongly in } L^2_{\rho^2}(\Omega).
\end{aligned}
$$

Hence, dividing by ϵ the right hand side of (6.1) and passing to the limit gives

$$
\int_\Omega \frac{1}{\epsilon}\rho_\epsilon^2\sum_{i=1}^{3}\Pi_\epsilon(F_i^\epsilon)\Pi_\epsilon(v_i^\epsilon)\,dX_1dX_2dx_3 \to 0.
\tag{6.5}
$$

On the other hand, using the convergences (6.3), (6.4) and (4.29) we obtain the convergence of the left hand side (divided by ϵ) when ϵ goes to 0

$$
\begin{aligned}
&\int_\Omega \frac{\rho_\epsilon^2}{\epsilon}\sum_{i,j=1}^{3}\Pi_\epsilon(\sigma_{ij}^\epsilon)\Pi_\epsilon(\gamma_{ij}(v))\,dX_1dX_2dx_3\\
&\to \int_\Omega \Big\{(\lambda+2\mu)\varphi\Big(\frac{\partial \bar u_1}{\partial X_1}\frac{\partial \phi_1}{\partial X_1}+\frac{\partial \bar u_2}{\partial X_2}\frac{\partial \phi_2}{\partial X_2}\Big)+\lambda\varphi\Big(\frac{\partial \bar u_2}{\partial X_2}\frac{\partial \phi_1}{\partial X_1}+\frac{\partial \bar u_1}{\partial X_1}\frac{\partial \phi_2}{\partial X_2}\Big)\Big\}\,dXdx_3\\
&+\int_\Omega \Big\{\mu\varphi\Big(\frac{\partial \bar u_1}{\partial X_2}+\frac{\partial \bar u_2}{\partial X_1}\Big)\Big(\frac{\partial \phi_1}{\partial X_2}+\frac{\partial \phi_2}{\partial X_1}\Big)+\lambda\varphi\Big(\rho\frac{d\mathcal{U}_3}{dx_3}-\rho^2X_1\frac{d^2\mathcal{U}_1}{dx_3^2}-\rho^2X_2\frac{d^2\mathcal{U}_2}{dx_3^2}\Big)\Big(\frac{\partial \phi_1}{\partial X_1}+\frac{\partial \phi_2}{\partial X_2}\Big)\Big\}\,dXdx_3\\
&+\int_\Omega \Big\{\mu\varphi\frac{\partial \phi_3}{\partial X_1}\Big(-\rho^2X_2\frac{d\mathcal{R}_3}{dx_3}+\frac{\partial \tilde u_3}{\partial X_1}\Big)+\mu\varphi\frac{\partial \phi_3}{\partial X_2}\Big(\rho^2X_1\frac{d\mathcal{R}_3}{dx_3}+\frac{\partial \tilde u_3}{\partial X_2}\Big)\Big\}\,dXdx_3
\end{aligned}
\tag{6.6}
$$

where ϕ be in $H^1(\omega,\mathbb{R}^3)$ and φ be in $C^\infty[0,L]$ such that $\varphi(0)=0$. Due to the convergence (6.5), the above limit is equal to zero. Since φ is arbitrary, we can localized with respect to x_3; that gives

$$
\begin{aligned}
&\int_\omega \Big\{(\lambda+2\mu)\Big(\frac{\partial \bar u_1}{\partial X_1}\frac{\partial \phi_1}{\partial X_1}+\frac{\partial \bar u_2}{\partial X_2}\frac{\partial \phi_2}{\partial X_2}\Big)+\lambda\Big(\frac{\partial \bar u_2}{\partial X_2}\frac{\partial \phi_1}{\partial X_1}+\frac{\partial \bar u_1}{\partial X_1}\frac{\partial \phi_2}{\partial X_2}\Big)\Big\}\,dX_1dX_2\\
&+\int_\omega \Big\{\mu\Big(\frac{\partial \bar u_1}{\partial X_2}+\frac{\partial \bar u_2}{\partial X_1}\Big)\Big(\frac{\partial \phi_1}{\partial X_2}+\frac{\partial \phi_2}{\partial X_1}\Big)+\lambda\Big(\rho\frac{d\mathcal{U}_3}{dx_3}-\rho^2X_1\frac{d^2\mathcal{U}_1}{dx_3^2}-\rho^2X_2\frac{d^2\mathcal{U}_2}{dx_3^2}\Big)\Big(\frac{\partial \phi_1}{\partial X_1}+\frac{\partial \phi_2}{\partial X_2}\Big)\Big\}\,dX_1dX_2\\
&+\int_\omega \Big\{\mu\frac{\partial \phi_3}{\partial X_1}\Big(-\rho^2X_2\frac{d\mathcal{R}_3}{dx_3}+\frac{\partial \tilde u_3}{\partial X_1}\Big)+\mu\frac{\partial \phi_3}{\partial X_2}\Big(\rho^2X_1\frac{d\mathcal{R}_3}{dx_3}+\frac{\partial \tilde u_3}{\partial X_2}\Big)\Big\}\,dX_1dX_2 = 0
\end{aligned}
\tag{6.7}
$$

6.1.1. Determination of $\tilde{u}_3$

First, we choose $\phi_1 = \phi_2 = 0$. In view of (6.7) we have

$$\int_\omega \Big\{ \frac{\partial \phi_3}{\partial X_1}\Big(-\rho^2 X_2 \frac{d\mathcal{R}_3}{dx_3} + \frac{\partial \tilde{u}_3}{\partial X_1}\Big) + \frac{\partial \phi_3}{\partial X_2}\Big(\rho^2 X_1 \frac{d\mathcal{R}_3}{dx_3} + \frac{\partial \tilde{u}_3}{\partial X_2}\Big)\Big\} dX_1 dX_2 = 0, \quad \text{a.e. in } [0,L].$$

Then the field $\tilde{u}_3 \in L^2((0,L); H^1(\omega))$ satisfies

$$\int_\omega \nabla_X \tilde{u}_3 \nabla_X \phi_3 \, dX = -\rho^2 \frac{d\mathcal{R}_3}{dx_3} \int_\omega \Big\{ -X_2 \frac{\partial \phi_3}{\partial X_1} + X_1 \frac{\partial \phi_3}{\partial X_2}\Big\} dX. \tag{6.8}$$

Now, we introduce the function χ as the unique solution of the following torsion problem:

$$\begin{cases} \chi \in H^1(\omega), \quad \displaystyle\int_\omega \chi \, dX = 0, \\ \displaystyle\int_\omega \nabla_X \chi \nabla_X \psi \, dX = -\int_\omega \Big\{ -X_2 \frac{\partial \psi}{\partial X_1} + X_1 \frac{\partial \psi}{\partial X_2}\Big\} dX, \quad \forall \psi \in H^1(\omega), \end{cases} \tag{6.9}$$

Taking χ as test function in (6.9) gives

$$\|\nabla \chi\|^2_{[L^2(\omega)]^2} \le I_1 + I_2.$$

By contradiction, we easily prove $\|\nabla \chi\|^2_{[L^2(\omega)]^2} < I_1 + I_2$. We set

$$K = I_1 + I_2 + \int_\omega \Big\{ -X_2 \frac{\partial \chi}{\partial X_1} + X_1 \frac{\partial \chi}{\partial X_2}\Big\} dX_1 dX_2 = I_1 + I_2 - \|\nabla \chi\|^2_{[L^2(\omega)]^2} > 0. \tag{6.10}$$

The above constant which depends on the geometry of the reference cross section ω, is the St Venant torsional stiffness.

Since $\tilde{u}_3$ verifies (6.8) and also $\int_\omega \tilde{u}_3(X_1, X_2, x_3)\, dX_1 dX_2 = 0$ for a.e. x_3 in $(0,L)$, we get

$$\tilde{u}_3(X_1, X_2, x_3) = \chi(X_1, X_2)\rho^2(x_3) \frac{d\mathcal{R}_3}{dx_3}(x_3) \qquad \text{for a.e.} (X_1, X_2, x_3) \in \Omega$$

which in turn gives

$$T_{13} = \Big(-X_2 + \frac{\partial \chi}{\partial X_1}\Big) \frac{\rho^2}{2} \frac{d\mathcal{R}_3}{dx_3}, \qquad T_{23} = \Big(X_1 + \frac{\partial \chi}{\partial X_2}\Big) \frac{\rho^2}{2} \frac{d\mathcal{R}_3}{dx_3}. \tag{6.11}$$

6.1.2. Determination of $\bar{u}_\alpha$, $\alpha = 1,2$

Now taking $\phi_3 = 0$ in (6.7) yields

$$\begin{aligned} &\int_\omega \Big\{ (\lambda + 2\mu)\Big(\frac{\partial \bar{u}_1}{\partial X_1}\frac{\partial \phi_1}{\partial X_1} + \frac{\partial \bar{u}_2}{\partial X_2}\frac{\partial \phi_2}{\partial X_2}\Big) + \lambda\Big(\frac{\partial \bar{u}_2}{\partial X_2}\frac{\partial \phi_1}{\partial X_1} + \frac{\partial \bar{u}_1}{\partial X_1}\frac{\partial \phi_2}{\partial X_2}\Big)\Big\} dX \\ &+ \int_\omega \Big\{ \mu\Big(\frac{\partial \bar{u}_1}{\partial X_2} + \frac{\partial \bar{u}_2}{\partial X_1}\Big)\Big(\frac{\partial \phi_1}{\partial X_2} + \frac{\partial \phi_2}{\partial X_1}\Big)\Big\} dX \\ &= -\int_\omega \Big\{ \lambda\Big(\rho \frac{d\mathcal{U}_3}{dx_3} - \rho^2 X_1 \frac{d^2\mathcal{U}_1}{dx_3^2} - \rho^2 X_2 \frac{d^2\mathcal{U}_2}{dx_3^2}\Big)\Big(\frac{\partial \phi_1}{\partial X_1} + \frac{\partial \phi_2}{\partial X_2}\Big)\Big\} dX \qquad \text{a.e. in } (0,L) \end{aligned} \tag{6.12}$$

for any $\phi_\alpha \in H^1(\omega)(\alpha = 1,2)$. Then the variational problem (6.12) corresponds to a classical 2d elastic problem for $(\bar{u}_1, \bar{u}_2)$. Taking into account the relations (4.28), the above variational problem admits a unique solution. Then we obtain

$$\frac{\partial \bar{u}_1}{\partial X_1}(X_1, X_2, \cdot) = -\nu \Big\{ \rho \frac{d\mathcal{U}_3}{dx_3} - \rho^2 X_1 \frac{d^2\mathcal{U}_1}{dx_3^2} - \rho^2 X_2 \frac{d^2\mathcal{U}_2}{dx_3^2}\Big\}, \tag{6.13}$$

$$\frac{\partial \bar{u}_2}{\partial X_2}(X_1,X_2,\cdot) = -\nu\Big\{\rho\frac{d\mathcal{U}_3}{dx_3} - \rho^2 X_1\frac{d^2\mathcal{U}_1}{dx_3^2} - \rho^2 X_2\frac{d^2\mathcal{U}_2}{dx_3^2}\Big\}, \tag{6.14}$$

$$\Big(\frac{\partial \bar{u}_1}{\partial X_2} - \frac{\partial \bar{u}_2}{\partial X_1}\Big)(X_1,X_2,\cdot) = 0 \qquad \text{a.e. in } \Omega \tag{6.15}$$

where $\nu = \dfrac{\lambda}{2(\lambda+\mu)}$ is the Poisson coefficient of the material.

As a consequence we get

$$T_{12} = 0, \qquad T_{11} = T_{22} = -\nu\Big\{\rho\frac{d\mathcal{U}_3}{dx_3} - \rho^2 X_1\frac{d^2\mathcal{U}_1}{dx_3^2} - \rho^2 X_2\frac{d^2\mathcal{U}_2}{dx_3^2}\Big\}. \tag{6.16}$$

6.2. Equations for $\mathcal{U}_1$,$\mathcal{U}_2$ and $\mathcal{R}_3$

Now we consider the functions φ_1, φ_2 and φ_3 in $C^\infty[0,L]$ such that $\varphi_1(0) = \varphi_1'(0) = \varphi_2(0) = \varphi_2'(0) = \varphi_3(0) = 0$. We construct a test field $\phi^\epsilon \in H^1_{\Gamma_{\epsilon,0}}(\Omega_\epsilon;\mathbb{R}^3)$ as follows:

$$\begin{aligned}
\phi_1^\epsilon(x) &= \varphi_1(x_3) - x_2\varphi_3(x_3),\\
\phi_2^\epsilon(x) &= \varphi_2(x_3) + x_1\varphi_3(x_3),\\
\phi_3^\epsilon(x) &= -x_1\varphi_1'(x_3) - x_2\varphi_2'(x_3).
\end{aligned}$$

Then we get

$$\begin{aligned}
&\gamma_{11}(\phi^\epsilon) = 0,\ \gamma_{22}(\phi^\epsilon) = 0,\ \gamma_{12}(\phi^\epsilon) = 0,\\
&\gamma_{13}(\phi^\epsilon) = -\frac{1}{2}x_2\varphi_3'(x_3),\\
&\gamma_{23}(\phi^\epsilon) = \frac{1}{2}x_1\varphi_3'(x_3),\\
&\gamma_{33}(\phi^\epsilon) = -x_1\varphi_1''(x_3) - x_2\varphi_2''(x_3).
\end{aligned}$$

Applying the rescaling operator Π_ϵ to the previous expressions gives

$$\begin{aligned}
&\Pi_\epsilon\big(\gamma_{11}(\phi^\epsilon)\big) = \Pi_\epsilon\big(\gamma_{22}(\phi^\epsilon)\big) = \Pi_\epsilon\big(\gamma_{12}(\phi^\epsilon)\big) = 0,\\
&\Pi_\epsilon\big(\gamma_{13}(\phi^\epsilon)\big) = -\frac{1}{2}\epsilon\rho_\epsilon X_2\varphi_3'(x_3),\\
&\Pi_\epsilon\big(\gamma_{23}(\phi^\epsilon)\big) = \frac{1}{2}\epsilon\rho_\epsilon X_1\varphi_3'(x_3),\\
&\Pi_\epsilon\big(\gamma_{33}(\phi^\epsilon)\big) = -\epsilon\rho_\epsilon X_1\varphi_1''(x_3) - \epsilon\rho_\epsilon X_2\varphi_2''(x_3).
\end{aligned}$$

In order to obtain the limit problem as ϵ tends to 0, we consider $v = \phi^\epsilon$ in (6.1), it leads to

$$\int_\Omega \rho_\epsilon^2 \sum_{i,j=1}^3 \Pi_\epsilon(\sigma_{ij}^\epsilon)\Pi_\epsilon(\gamma_{ij}(\phi^\epsilon))\,dX_1dX_2dx_3 = \int_\Omega \rho_\epsilon^2 \sum_{i=1}^3 \Pi_\epsilon(F_i^\epsilon)\Pi_\epsilon(\phi_i^\epsilon)\,dX_1dX_2dx_3. \tag{6.17}$$

We divide the above equality by ϵ^2. Then using the convergence (4.29) and the definition of the test function we can pass to the limit in the left-hand side to obtain

$$\lim_{\epsilon\to 0}\int_\Omega \frac{\rho_\epsilon^2}{\epsilon^2}\sum_{i,j=1}^3 \Pi_\epsilon(\sigma_{ij}^\epsilon)\Pi_\epsilon(\gamma_{ij}(\phi^\epsilon))\,dX_1dX_2dx_3$$

$$= \mu \int_\Omega \rho^2 \Big[- X_2\varphi_3'(x_3)T_{13} + X_1\varphi_3'(x_3)T_{23}\Big]\, dX_1dX_2dx_3$$

$$+ \int_\Omega \rho^2 \Big[\Big(- \sum_{\alpha=1}^2 X_\alpha\varphi_\alpha''(x_3)\Big)\Big((\lambda+2\mu)\Big(\rho\frac{d\mathcal{U}_3}{dx_3} - \rho^2 X_1\frac{d^2\mathcal{U}_1}{dx_3^2} - \rho^2 X_2\frac{d^2\mathcal{U}_2}{dx_3^2}\Big) + \lambda\Big(\frac{\partial\bar u_1}{\partial x_1} + \frac{\partial\bar u_2}{\partial x_2}\Big)\Big)\Big]\, dX_1dX_2dx_3.$$

Moreover, taking into account (6.11) and (6.16), the above limit is equal to

$$\begin{aligned}&\int_\Omega \rho^4\frac{d\mathcal{R}_3}{dx_3}\Big[- \frac{\mu}{2}X_2\varphi_3'(x_3)\Big(- X_2 + \frac{\partial\chi}{\partial X_1}\Big) + \frac{\mu}{2}X_1\varphi_3'(x_3)\Big(X_1 + \frac{\partial\chi}{\partial X_2}\Big)\Big]\, dX_1dX_2dx_3\\ &+ \int_\Omega \rho^2\Big[E\Big(- X_1\varphi_1''(x_3) - X_2\varphi_2''(x_3)\Big)\Big(\rho\frac{d\mathcal{U}_3}{dx_3} - \rho^2 X_1\frac{d^2\mathcal{U}_1}{dx_3^2} - \rho^2 X_2\frac{d^2\mathcal{U}_2}{dx_3^2}\Big)\Big]\, dX_1dX_2dx_3,\end{aligned} \tag{6.18}$$

where $E = \dfrac{\mu(3\lambda+2\mu)}{\lambda+\mu}$ is the Young's modulus of the elastic material.

On the other hand, in view of the assumptions (5.5), (5.7) and the definition of the test field we obtain the following limit for the right-hand side:

$$\begin{aligned}&\lim_{\epsilon\to0}\int_\Omega \frac{\rho_\epsilon^2}{\epsilon^2}\sum_{i=1}^3 \Pi_\epsilon(F_i^\epsilon)\Pi_\epsilon(\phi_i^\epsilon)\, dX_1dX_2dx_3\\ &= \int_\Omega \rho^2\Big\{f_1\varphi_1 + \rho^2X_2^2g_3\varphi_3 + f_2\varphi_2 + \rho^2X_1^2g_3\varphi_3 - \rho^2X_1^2g_1\varphi_1' - \rho^2X_2^2g_2\varphi_2'\Big\}\, dX_1dX_2dx_3.\end{aligned} \tag{6.19}$$

Hence, by (6.18) and (6.19) the limit equation of (6.17) is given by

$$\begin{aligned}&\int_\Omega \rho^4\frac{d\mathcal{R}_3}{dx_3}\Big[- \frac{\mu}{2}X_2\varphi_3'(x_3)\Big(- X_2 + \frac{\partial\chi}{\partial X_1}\Big) + \frac{\mu}{2}X_1\varphi_3'(x_3)\Big(X_1 + \frac{\partial\chi}{\partial X_2}\Big)\Big]\, dX_1dX_2dx_3\\ &+ \int_\Omega \rho^2\Big[E\Big(- X_1\varphi_1''(x_3) - X_2\varphi_2''(x_3)\Big)\Big(\rho\frac{d\mathcal{U}_3}{dx_3} - \rho^2 X_1\frac{d^2\mathcal{U}_1}{dx_3^2} - \rho^2 X_2\frac{d^2\mathcal{U}_2}{dx_3^2}\Big)\Big]\, dX_1dX_2dx_3\\ &= \int_\Omega \rho^2\Big\{f_1\varphi_1 + \rho^2X_2^2g_3\varphi_3 + f_2\varphi_2 + \rho^2X_1^2g_3\varphi_3 - \rho^2X_1^2g_1\varphi_1' - \rho^2X_2^2g_2\varphi_2'\Big\}\, dX_1dX_2dx_3,\end{aligned} \tag{6.20}$$

for any $\varphi_3 \in C^\infty[0,L]$ such that $\varphi_3(0) = 0$ and for $\varphi_1, \varphi_2 \in C^\infty[0,L]$ such that $\varphi_1(0) = \varphi_1'(0) = \varphi_2(0) = \varphi_2'(0) = 0$. We simplify (6.20)

$$\begin{aligned}&\frac{K\mu}{2}\int_{(0,L)}\rho^4\frac{d\mathcal{R}_3}{dx_3}\varphi_3'dx_3 + EI_1\int_{(0,L)}\rho^4\frac{d^2\mathcal{U}_1}{dx_3^2}\varphi_1''\,dx_3 + EI_2\int_{(0,L)}\rho^4\frac{d^2\mathcal{U}_2}{dx_3^2}\varphi_2''\,dx_3\\ =&(I_1+I_2)\int_{(0,L)}\rho^4g_3\varphi_3dx_3 + |\omega|\int_{(0,L)}\rho^2\{f_1\varphi_1 + f_2\varphi_2\}dx_3 - \sum_{\alpha=1}^2 I_\alpha\int_{(0,L)}\rho^4g_\alpha\varphi_\alpha'\,dx_3.\end{aligned} \tag{6.21}$$

First we choose $\varphi_1 = \varphi_2 = 0$ in (6.21). Taking into account the boundary condition $\mathcal{R}_3(0) = 0$, the function $\mathcal{R}_3$ is the unique solution of

$$\begin{cases} -\dfrac{K\mu}{2}\dfrac{d}{dx_3}\Big(\rho^4\dfrac{d\mathcal{R}_3}{dx_3}\Big) = (I_1+I_2)\rho^4g_3\\ \mathcal{R}_3(0) = 0,\end{cases} \tag{6.22}$$

where K is given by (6.10).

In (6.21) we take $\varphi_3 = 0$. Since φ_1 and φ_2 are arbitrary in $C^\infty[0,L]$ such that $\varphi_1(0) = \varphi_1'(0) = \varphi_2(0) = \varphi_2'(0) = 0$, that gives the bending problems satisfied by $\mathcal{U}_1$ and $\mathcal{U}_2$

$$\begin{cases} EI_\alpha\dfrac{d^2}{dx_3^2}\Big(\rho^4\dfrac{d^2\mathcal{U}_\alpha}{dx_3^2}\Big) = |\omega|\rho^2f_\alpha + I_\alpha\dfrac{d}{dx_3}\big(\rho^4g_\alpha\big),\\ \mathcal{U}_\alpha(0) = \dfrac{d\mathcal{U}_\alpha}{dx_3}(0) = 0,\end{cases}\qquad \text{for } \alpha = 1,2. \tag{6.23}$$

Recall that in order to obtain (6.22)-(6.23), we have used the fact that $\rho(L) = 0$.

6.3. Equation for $\mathcal{U}_3$

In this step we derive the equation satisfied by $\mathcal{U}_3$. In order to get this, in (6.1) we consider as test field $v(x_1, x_2, x_3) = (0, 0, \varphi(x_3))$ in $H^1(\Omega_\epsilon; \mathbb{R}^3)$ such that $\varphi \in C^\infty[0, L]$ with $\varphi(0) = 0$. Due to the assumptions (5.5), (5.7), the definition of the test field v and taking into account (2.1) the limit of (6.1) devided by ϵ gives

$$\int_{(0,L)} E\rho^2 \frac{d\mathcal{U}_3}{dx_3}\varphi_3'\, dx_3 = \int_{(0,L)} \rho^2 f_3 \varphi_3\, dx_3.$$

Hence, since φ is any function in $C^\infty[0, L]$ such that $\varphi(0) = 0$ and $\rho(L) = 0$ we can conclude that $\mathcal{U}_3$ verifies the following compression-traction equation for elastic rods:

$$\begin{cases} -E\dfrac{d}{dx_3}\Big(\rho^2 \dfrac{d\mathcal{U}_3}{dx_3}\Big) = \rho^2 f_3, \\ \mathcal{U}_3(0) = 0. \end{cases} \tag{6.24}$$

6.4. Convergence of the total elastic energy

In the above subsections all the limit problems admit a unique solution. As a consequence the whole sequences $\Big\{\dfrac{1}{\epsilon^2}\overline{u}^\epsilon\Big\}_\epsilon$, $\{\mathcal{U}_\alpha^\epsilon\}_\epsilon$, $\Big\{\dfrac{1}{\epsilon}\mathcal{U}_3^\epsilon\Big\}_\epsilon$ and $\{\mathcal{R}_3^\epsilon\}_\epsilon$ converge weakly to their limit.

In this subsection we prove that the rescaled energy $\dfrac{\mathcal{E}(u^\epsilon)}{\epsilon^4}$ converges to the elastic limit energy as ϵ tends to zero and that some weak convergences are in fact strong convergences.

Lemma 6.1. *Under the assumptions* (5.5), (5.6) *and* (5.7) *on the applied forces, we obtain the following convergence for the total elastic energy*

$$\lim_{\epsilon\to 0} \frac{\mathcal{E}(u^\epsilon)}{\epsilon^4} = \int_\Omega \Big\{\lambda Tr(T)Tr(T) + \sum_{i,j=1}^3 2\mu T_{ij}T_{ij}\Big\} dX_1 dX_2 dx_3, \tag{6.25}$$

where T is the limit of the symmetric gradient defined in (4.29).

Proof. Taking $v = u^\epsilon$ in (5.2), dividing by ϵ^4, then using the properties of Π_ϵ and by standard weak lower-semi-continuity, we obtain

$$\int_\Omega \Big\{\lambda Tr(T)Tr(T) + \sum_{i,j=1}^3 2\mu T_{ij}T_{ij}\Big\} dX_1 dX_2 dx_3 \le \liminf_{\epsilon\to 0} \frac{\mathcal{E}(u^\epsilon)}{\epsilon^4}. \tag{6.26}$$

We have

$$\frac{\mathcal{E}(u^\epsilon)}{\epsilon^4} = \int_\Omega \frac{\rho_\epsilon^2}{\epsilon^2} \sum_{i,j=1}^3 \Pi_\epsilon(\sigma_{ij}^\epsilon)\Pi_\epsilon(\gamma_{ij}(u^\epsilon))\, dX_1 dX_2 dx_3 = \int_\Omega \frac{\rho_\epsilon^2}{\epsilon^2} \sum_{i=1}^3 \Pi_\epsilon(F_i^\epsilon)\Pi_\epsilon(u_i^\epsilon)\, dX_1 dX_2 dx_3.$$

The last term in the above equality is equal to

$$\int_\Omega \frac{\rho_\epsilon^2}{\epsilon^2} \sum_{i=1}^3 \Pi_\epsilon(F_i^\epsilon)\Pi_\epsilon(u_i^\epsilon)\, dX_1 dX_2 dx_3 = \sum_{\alpha=1}^2 \int_\Omega \rho_\epsilon^2 f_\alpha^\epsilon(x_3)\Pi_\epsilon(u_\alpha^\epsilon)\, dX_1 dX_2 dx_3 - \int_\Omega \frac{\rho_\epsilon^3}{\epsilon} X_2 g_3^\epsilon(x_3)\Pi_\epsilon(u_1^\epsilon)$$
$$+ \int_\Omega \frac{\rho_\epsilon^3}{\epsilon} X_1 g_3^\epsilon(x_3)\Pi_\epsilon(u_2^\epsilon) + \int_\Omega \frac{\rho_\epsilon^2}{\epsilon} f_3^\epsilon(x_3)\Pi_\epsilon(u_3^\epsilon)\, dX_1 dX_2 dx_3 + \sum_{\alpha=1}^2 \int_\Omega \frac{\rho_\epsilon^3}{\epsilon} X_\alpha g_\alpha^\epsilon(x_3)\Pi_\epsilon(u_3^\epsilon)\, dX_1 dX_2 dx_3.$$

Then (3.11), (4.7), (4.11), (4.12), (4.15) and (5.7) lead to

$$\limsup_{\epsilon\to 0}\frac{\mathcal{E}(u^\epsilon)}{\epsilon^4}=\int_\Omega\Big[\sum_{i=1}^3\rho^2 f_i u_i+\sum_{\alpha=1}^2\rho^4\big(X_\alpha^2 g_3\mathcal{R}_3-X_\alpha^2 g_\alpha\frac{d\mathcal{U}_\alpha}{dx_3}\big)\Big]\,dX_1dX_2dx_3$$
$$=|\omega|\int_{(0,L)}\rho^2 f\cdot\mathcal{U}\,dx_3+(I_1+I_2)\int_{(0,L)}\rho^4 g_3\mathcal{R}_3dx_3$$
$$-I_1\int_{(0,L)}\rho^4 g_1\frac{d\mathcal{U}_1}{dx_3}\,dx_3-I_2\int_{(0,L)}\rho^4 g_2\frac{d\mathcal{U}_2}{dx_3}\,dx_3. \qquad (6.27)$$

Besides, since T is a symmetric matrix we know that it verifies the following algebraic identity

$$\lambda Tr(T)Tr(T)+\sum_{i,j=1}^3 2\mu T_{ij}T_{ij}=ET_{33}^2+\frac{E}{(1+\nu)(1-2\nu)}(T_{11}+T_{22}+2\nu T_{33})^2$$
$$+\frac{E}{2(1+\nu)}[(T_{11}-T_{22})^2+4(T_{12}^2+T_{13}^2+T_{23}^2)].$$

Then, in view of (2.1), (4.32), (6.11), (6.16) and (6.9) we have

$$\int_\Omega\lambda Tr(T)Tr(T)+\sum_{i,j=1}^3 2\mu T_{ij}T_{ij}\,dX_1dX_2dx_3=E\int_\Omega\rho^2\Big(\frac{d\mathcal{U}_3}{dx_3}\Big)^2\,dX_1dX_2dx_3$$
$$+E\int_0^L\rho^4\Big(I_1\Big(\frac{d\mathcal{R}_2}{dx_3}\Big)^2+I_2\Big(\frac{d\mathcal{R}_1}{dx_3}\Big)^2\Big)\,dX_1dX_2dx_3+\frac{K\mu}{2}\int_0^L\rho^4\Big(\frac{d\mathcal{R}_3}{dx_3}\Big)^2\,dx_3$$
$$=E|\omega|\int_\Omega\rho^2\Big(\frac{d\mathcal{U}_3}{dx_3}\Big)^2\,dx_3+\sum_{\alpha=1}^2 EI_\alpha\int_{(0,L)}\rho^4\Big(\frac{d^2\mathcal{U}_\alpha}{dx_3^2}\Big)^2\,dx_3+\frac{K\mu}{2}\int_{(0,L)}\rho^4\Big(\frac{d\mathcal{R}_3}{dx_3}\Big)^2dx_3. \qquad (6.28)$$

We recall that

$$E|\omega|\int_\Omega\rho^2\Big(\frac{d\mathcal{U}_3}{dx_3}\Big)^2\,dx_3+\sum_{\alpha=1}^2 EI_\alpha\int_{(0,L)}\rho^4\Big(\frac{d^2\mathcal{U}_\alpha}{dx_3^2}\Big)^2\,dx_3+\frac{K\mu}{2}\int_{(0,L)}\rho^4\Big(\frac{d\mathcal{R}_3}{dx_3}\Big)^2dx_3$$
$$=|\omega|\int_{(0,L)}\rho^2 f\cdot\mathcal{U}\,dx_3-\sum_{\alpha=1}^2 I_\alpha\int_{(0,L)}\rho^4 g_\alpha\frac{d\mathcal{U}_\alpha}{dx_3}\,dx_3+(I_1+I_2)\int_{(0,L)}\rho^4 g_3\mathcal{R}_3dx_3.$$

Finally we obtain

$$\lim_{\epsilon\to 0}\frac{\mathcal{E}(u^\epsilon)}{\epsilon^4}=\int_\Omega\Big\{\lambda Tr(T)Tr(T)+\sum_{i,j=1}^3 2\mu T_{ij}T_{ij}\Big\}dX_1dX_2dx_3,$$

which gives us the convergence of the rescaled energy to the total energy of the problems (6.23), (6.24) and (6.22) as ϵ goes to zero. $\square$

Now we can deduce the strong convergences of the fields of the displacement decomposition using the strong convergence of the energy. In view of the weak convergence of the symmetric gradient (4.29), the strict convexity of the elastic energy implies that the convergence of the symmetric gradient is strong

$$\frac{1}{\epsilon}\Pi_\epsilon\big((\nabla u^\epsilon)_{\mathcal{S}}\big)\to T \text{ strongly in } [L^2_\rho(\Omega)]^9. \qquad (6.29)$$

As a consequence we get

$$\frac{\rho}{\epsilon}\Pi_\epsilon\big((\nabla u^\epsilon)_{\mathcal{S}}\big)_{33}=\frac{\rho}{\epsilon}\frac{d\mathcal{U}_3^\epsilon}{dx_3}-\rho\rho_\epsilon X_1\frac{d\mathcal{R}_2^\epsilon}{dx_3}+\rho\rho_\epsilon X_2\frac{d\mathcal{R}_1^\epsilon}{dx_3}+\frac{\rho}{\epsilon}\Pi_\epsilon\Big(\frac{\partial\bar{u}_3^\epsilon}{\partial x_3}\Big)$$

$$\to T_{33} = \rho \frac{d\mathcal{U}_3}{dx_3} - \rho^2 X_1 \frac{d\mathcal{R}_2}{dx_3} + \rho^2 X_2 \frac{d\mathcal{R}_1}{dx_3} \text{ strongly in } L^2(\Omega). \tag{6.30}$$

Moreover, using $\displaystyle\int_\omega \Pi_\epsilon(\bar{u}_3^\epsilon)\, dX_1 dX_2 = \int_\omega X_\alpha \Pi_\epsilon(\bar{u}_3^\epsilon)\, dX_1 dX_2 = 0$, for $\alpha \in \{1,2\}$, and taking into account convergence (4.30) we may deduce from (6.30) that

$$\frac{\rho}{\epsilon} \frac{d\mathcal{U}_3^\epsilon}{dx_3} \to \rho \frac{d\mathcal{U}_3}{dx_3}, \quad \rho^2 \frac{d\mathcal{R}_\alpha^\epsilon}{dx_3} \to \rho^2 \frac{d\mathcal{R}_\alpha}{dx_3}, \text{ strongly in } L^2(0,L),\ (\alpha = 1,2), \tag{6.31}$$

as ϵ tends to zero. Then, in view of the weak convergences (4.6) and (4.7), (6.31) implies that

$$\frac{1}{\epsilon} \mathcal{U}_3^\epsilon \to \mathcal{U}_3 \text{ strongly in } H_\rho^1(0,L), \tag{6.32}$$

$$\mathcal{R}_\alpha^\epsilon \to \mathcal{R}_\alpha \text{ strongly in } H_{\rho^2}^1(0,L), \quad \text{for } \alpha = 1,2. \tag{6.33}$$

Moreover, from (4.8) and (6.33) we have

$$\mathcal{U}_\alpha^\epsilon \to \mathcal{U}_\alpha \text{ strongly in } H_\rho^1(0,L), \text{ for } \alpha = 1,2.$$

Hence, due to the decomposition (3.5) and the previous strong convergences we deduce

$$\Pi_\epsilon(u_\alpha^\epsilon) \to \mathcal{U}_\alpha \text{ strongly in } H_\rho^1(\Omega), \text{ for } \alpha = 1,2.$$

$$\frac{1}{\epsilon} \Pi_\epsilon(u_3^\epsilon) \to \mathcal{U}_3 - \rho X_1 \frac{d\mathcal{U}_1}{dx_3} - \rho X_2 \frac{d\mathcal{U}_2}{dx_3} \text{ strongly in } H_\rho^1(\Omega).$$

We also have

$$\frac{1}{\epsilon^2} \gamma_{\alpha\beta}\big(\Pi_\epsilon \bar{u}^\epsilon\big) \to \gamma_{\alpha\beta}(\bar{u}) \text{ strongly in } L^2(\Omega), \text{ for } \alpha,\beta = 1,2.$$

We recall that the warping functions satisfy (4.28). Then from the 2d Korn inequality we derive

$$\sum_{\alpha=1}^{2} \Big\| \frac{1}{\epsilon^2} \Pi_\epsilon\big(\bar{u}_\alpha^\epsilon\big) - \bar{u}_\alpha \Big\|_{L^2(\Omega)} + \sum_{\alpha\beta=1}^{2} \Big\| \frac{1}{\epsilon^2} \frac{\partial \Pi_\epsilon\big(\bar{u}_\alpha^\epsilon\big)}{\partial X_\beta} - \frac{\partial \bar{u}_\alpha}{\partial X_\beta} \Big\|_{L^2(\Omega)} \le C \sum_{\alpha\beta=1}^{2} \Big\| \frac{1}{\epsilon^2} \gamma_{\alpha\beta}\big(\Pi_\epsilon \bar{u}^\epsilon\big) - \gamma_{\alpha\beta}(\bar{u}) \Big\|_{L^2(\Omega)}.$$

That leads to

$$\frac{1}{\epsilon^2} \Pi_\epsilon\big(\bar{u}_\alpha^\epsilon\big) \to \bar{u}_\alpha \text{ strongly in } L^2((0,L); H^1(\omega)), \text{ for } \alpha = 1,2.$$

7. Conclusion

In this last section we summarize the results obtained in the previous sections.

Theorem 7.1. *Let u^ϵ be the solution of the elasticity problem* (5.2). *Under the assumptions* (5.5)-(5.7) *on the applied forces, the sequence $\{u^\epsilon\}$ satisfies the following convergences*

$$\Pi_\epsilon(u_\alpha^\epsilon) \to \mathcal{U}_\alpha \text{ strongly in } H_\rho^1(\Omega), \text{ for } \alpha = 1,2,$$

$$\frac{1}{\epsilon} \Pi_\epsilon(u_3^\epsilon) \to \mathcal{U}_3 - \rho X_1 \frac{d\mathcal{U}_1}{dx_3} - \rho X_2 \frac{d\mathcal{U}_2}{dx_3} \text{ strongly in } H_\rho^1(\Omega),$$

where $\mathcal{U}_\alpha$ is the solution of the bending problem (6.23) *and $\mathcal{U}_3$ is the weak solution of the stretching problem* (6.24). *Moreover, we have*

$$\frac{1}{\epsilon} \Pi_\epsilon\big(\gamma_{ij}(u^\epsilon)\big) \to T_{ij} \text{ strongly in } L_\rho^2(\Omega), \text{ for } i,j = 1,2,3.$$

where

$$T_{11}=T_{22}=-\nu T_{33},\qquad T_{33}=\rho\frac{d\mathcal{U}_3}{dx_3}-\rho^2X_1\frac{d^2\mathcal{U}_1}{dx_3^2}-\rho^2X_2\frac{d^2\mathcal{U}_2}{dx_3^2},$$
$$T_{12}=0,\qquad T_{13}=\Big(-X_2+\frac{\partial\chi}{\partial X_1}\Big)\frac{\rho^2}{2}\frac{d\mathcal{R}_3}{dx_3},\qquad T_{23}=\Big(X_1+\frac{\partial\chi}{\partial X_2}\Big)\frac{\rho^2}{2}\frac{d\mathcal{R}_3}{dx_3},$$

with $\chi\in H^1(\omega)$ *is the solution of the torsion problem* (6.9) *and* $\mathcal{R}_3$ *the weak solution of* (6.22).

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