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Abstract. We expose different methods of regularizations of subsolutions in the context of discrete weak KAM theory that allow us to prove the existence and the density of $C^{1,1}$ subsolutions. Moreover, these subsolutions can be made strict and smooth outside of the Aubry set.

1 Introduction

We consider a smooth connected Riemannian manifold M endowed with the distance $\mathrm{d}(\,\cdot\,,\,\cdot\,)$ coming from the Riemannian metric. Fixing a cost function $c\colon M \times M \to \mathbb{R}$ we study the functions $u\colon M \to \mathbb{R}$ that satisfy

$$\forall (x, y) \in M \times M, \quad u(y) - u(x) \leqslant c(x, y).$$

We call them subsolutions, by analogy with those appearing in Weak KAM theory (see [3, 14] for example). We will denote by $\mathcal{SS}$ the set of subsolutions and by $\mathcal{SS}_C = \mathcal{SS} \cap C^0(M, \mathbb{R})$ the set of continuous subsolutions. These subsolutions are one of the important objects in discrete (in time) weak KAM theory. Some other aspects of this discrete theory have been discussed in [6, 16, 19]. This theory is also closely related to the time-periodic weak KAM theory discussed, for example, in [4, 10]. In many aspects, these various settings (discrete, time-periodic, autonomous) are similar, but differences appear for some specific questions. For example, the convergence of the Lax–Oleinik semi-group holds only in the autonomous setting; see [2,7,12,13]. The Hamilton–Jacobi equation does not have such a nice form in the discrete setting as in the autonomous setting; see [16]. Some other specific aspects of the discrete case are discussed in [19]. Concerning the regularity of subsolutions, the existence of C^1 subsolutions was obtained in [19] in the discrete setting by an adaptation of the original proof of Fathi and Siconolfi [14]. On the other hand, the proof of the existence of $C^{1,1}$ subsolutions given in [3] for the autonomous setting does not extend to the discrete setting. The existence of $C^{1,1}$ subsolutions was, however, obtained in [20] by a different method. Our goal here is to extend and simplify the results of this paper.

Defining, as usual, the discrete Lax–Oleinik operators

$$T_c^- u(x) = \inf_{y \in M} u(y) + c(y, x), \qquad T_c^+ u(x) = \sup_{y \in M} u(y) - c(x, y),$$

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we see that a function u is a subsolution if and only if one of the equivalent relations is verified:

$$u \leqslant T_c^- u \quad \text{or} \quad T_c^+ u \leqslant u.$$

Note that as a consequence the functions $T_c^- u$ and $T_c^+ u$ are themselves subsolutions whenever u is a subsolution. We will use the following hypothesis on c. More concrete hypotheses implying this one are given below.

Hypothesis 1 For each subsolution u, the functions $T_c^- u$ and $-T_c^+ u$ are locally semiconcave.[1]

Subsolutions do not necessarily exist, and when they exist they are not necessarily continuous (the continuity of subsolutions is discussed in [19]). Under Hypothesis 1, the existence of a continuous subsolution is implied by the existence of a (possibly discontinuous) subsolution u. Just consider the subsolution $T_c^- u$, which is locally semiconcave hence locally Lipschitz. See also Lemma 2.2.

Theorem 1.1 *If Hypothesis* 1 *holds, then the set of locally $C^{1,1}$ subsolutions is dense in the set of continuous subsolutions for the strong topology.*

We recall that the strong (or Whitney) topology on $C^0(M, \mathbb{R})$ is induced by the basis of open sets

$$O_{\epsilon,f} = \left\{ g \in C^0(M, \mathbb{R}), \forall x \in M, |f(x) - g(x)| < \epsilon(x) \right\},$$

where $f \in C^0(M, \mathbb{R})$ and ϵ is a continuous positive valued function on M. For further precisions on this topology, see [17, Chapter 2]. The existence of $C^{1,1}$ subsolutions was proved in [20], but the density is new. In [20], the existence of $C^{1,1}$ subsolutions is deduced from the following result of Ilmanen (see [5, 8, 15, 18]).

Theorem 1.2 *Let f and g be locally semiconcave functions on M such that $f + g \geqslant 0$. Then there exists a locally $C^{1,1}$ function u such that $-g \leqslant u \leqslant f$.*

We will offer a direct proof of Theorem 1.1, which is inspired from the proof of Ilmanen's Lemma given in [5]. Note that Theorem 1.1 implies Theorem 1.2. This follows immediately from the equivalence, for a given function u, between the following two properties:

- the function $g + u$ is bounded from below and $-g \leqslant u - \inf(g + u) \leqslant f$;
- the function u is a subsolution for the cost $c(x, y) = g(x) + f(y)$.

We need to introduce more definitions before we state our other results. The subsolution u is called free at x if

$$T_c^+ u(x) < u(x) < T_c^- u(x).$$

We define the set $\mathcal{A}_u$ as

$$\mathcal{A}_u := \left\{ x \in M, T_c^+ u(x) = u(x) = T_c^- u(x) \right\} \subset M$$

[1] Throughout the paper, what we call semiconcave is sometimes called semiconcave with a linear modulus.

and the Aubry set $\mathcal{A}$ as

$$\mathcal{A} := \bigcap_{u \in \mathcal{SS}} \mathcal{A}_u \subset M,$$

where the intersection is taken on all subsolutions. Under Hypothesis 1, the sets $\mathcal{A}_u$ are closed, since they are defined by the equality $T_c^+ u = T_c^- u$. The set $\mathcal{A}$ is then also closed. Moreover, it makes no difference to restrict the intersection to continuous subsolutions in the definition of $\mathcal{A}$ by Lemma 2.2. We say that the subsolution u is strict at (x, y) if

$$u(y) - u(x) < c(x, y).$$

Obviously, the subsolution u is strict at (x, y) and at (y, x) for each y if it is free at x. We define the set

$$\widehat{\mathcal{A}}_u := \left\{ (x, y) \in M^2 : u(y) - u(x) = c(x, y) \right\}.$$

We also define

$$\widehat{\mathcal{A}} := \bigcap_{u \in \mathcal{SS}} \widehat{\mathcal{A}}_u$$

where the intersection is taken on all subsolutions. Equivalently, if Hypothesis 1 holds, the intersection can be taken on continuous subsolutions by Lemma 2.2. This yields that $\widehat{\mathcal{A}}$ is also closed.

Theorem 1.3 *Assume that c satisfies Hypothesis* 1*. Given a subsolution u, there exists a subsolution v such that*

- *$v = u$ on $\mathcal{A}_u$;*
- *v is smooth and free on the complement of $\mathcal{A}_u$;*
- *v is locally $C^{1,1}$;*
- *v is strict at each pair (x, y) where u is strict.*

We can then obtain a subsolution that is as smooth, free, and strict as possible.

Theorem 1.4 *If c satisfies Hypothesis* 1 *and admits a subsolution, then there exists a locally $C^{1,1}$ subsolution that is free and smooth in the complement of $\mathcal{A}$, and strict on the complement of $\widehat{\mathcal{A}}$.*

Observe as a consequence that the projections of $\widehat{\mathcal{A}}$ on both the first and the second factor are contained in $\mathcal{A}$ (and, under the additional Hypothesis 2, each of these projections is equal to $\mathcal{A}$). Strict $C^{1,1}$ subsolutions were obtained in [20] under an additional twist assumption. We will use a simple trick from [3] to obtain the general result directly from Theorem 1.1. That the subsolutions can be made smooth outside of $\mathcal{A}$ is well known. It will certainly not be a surprise to specialists that this can be done without destroying the global $C^{1,1}$ regularity, although we do not know any reference for this statement. We prove it using a regularization procedure due to De Rham [11]. This proof also applies to the “classical” (as opposed to discrete) weak KAM theory.

The abstract Hypothesis 1 holds in a more concrete setting introduced in [19].

Hypothesis 2 The function c satisfies the following properties:

- *uniform super-linearity*: for every $k \geqslant 0$, there exists $C(k) \in \mathbb{R}$ such that

$$\forall (x, y) \in M \times M, \quad c(x, y) \geqslant k\mathrm{d}(x, y) - C(k);$$

- *uniform boundedness*: for every $R \in \mathbb{R}$, there exists $A(R) \in \mathbb{R}$ such that

$$\forall (x, y) \in M \times M, \quad \mathrm{d}(x, y) \leqslant R \Rightarrow c(x, y) \leqslant A(R);$$

- *local semiconcavity*: for each point (x_0, y_0) there is a domain of chart containing (x_0, y_0) and a smooth function $f(x, y)$ such that $c - f$ is concave in the chart. (This holds for example if c is C^2 or locally $C^{1,1}$).

This hypothesis has two important consequences, as proved in [19]. First, it implies Hypothesis 1. Second, it implies that the extrema in the definitions of $T_c^{\pm} u(x)$ are reached for each continuous subsolution u and each $x \in M$. This in turn implies that the the projection of $\widehat{\mathcal{A}}$ on the first, as well as on the second, factor are equal to $\mathcal{A}$, which corresponds to the projected Aubry set introduced in [19].

Lemma 1.5 *Assume that c satisfies Hypothesis* 2*. Given $x \in \mathcal{A}$, there exist y and z such that (x, z) and (y, x) are in $\widehat{\mathcal{A}}$.*

Proof Let w be a continuous subsolution that is strict outside of $\widehat{\mathcal{A}}$ (such a solution exists by Theorem 1.4). Let y be such that $T_c^- w(x) = w(y) + c(y, x)$. Since $x \in \mathcal{A}$ we obtain that $w(x) - w(y) = c(y, x)$. Hence $(y, x) \in \widehat{\mathcal{A}}_w = \widehat{\mathcal{A}}$. The existence of z is proved in the same way, using T_c^+. ■

Finally, let us mention one last setting in which Hypothesis 1 holds :

Hypothesis 3 The function c is locally bi-semiconcave, *i.e.*, for all $(x, y) \in M \times M$ we can find the following:

- neighborhoods U and V of respectively x and y,
- diffeomorphisms φ_1 and φ_2 from B_n to respectively U and V (B_n is the unit ball in $\mathbb{R}^n$),
- smooth functions f and g from B_n to $\mathbb{R}$,

such that for each $x \in M$, the function $z \mapsto c(x, \varphi_2(z)) - g(z)$ is concave and for all $y \in M$, the function $z \mapsto c(\varphi_1(z), y) - f(z)$ is concave.

It is easy to prove, as in [20, Proposition 4.6], that Hypothesis 3 also implies Hypothesis 1 (using that an infimum of equi-semiconcave functions is itself semiconcave).

2 Preliminaries

Here we gather some useful facts obtained from elementary manipulations of the Lax–Oleinik operators. Let us first list, without proof, some properties of the operators $T_c^{\pm}$.

- Monotony: $u \leqslant v \Rightarrow T_c^{\pm} u \leqslant T_c^{\pm} v$.
- Convexity: Given a sequence u_n of functions and a sequence a_n of non-negative numbers such that $\sum_{n\in\mathbb{N}} a_n = 1$, and such that the series

$$\sum_{n\in\mathbb{N}} a_n T_c^- u_n, \quad \sum_{n\in\mathbb{N}} a_n u_n, \quad \text{and} \quad \sum_{n\in\mathbb{N}} a_n T_c^+ u_n$$

are converging point-wise, we have

$$T_c^-\Big(\sum_{n\in\mathbb{N}} a_n u_n\Big) \geqslant \sum_{n\in\mathbb{N}} a_n T_c^- u_n, \quad T_c^+\Big(\sum_{n\in\mathbb{N}} a_n u_n\Big) \leqslant \sum_{n\in\mathbb{N}} a_n T_c^+ u_n.$$

The set $\mathcal{SS}$ of subsolutions is convex, and it is closed under point-wise convergence. A convex combination $\sum_{n\in\mathbb{N}} a_n u_n$ of subsolutions, with a point-wise convergent sum, is a subsolution; it is free at x (resp. strict at (x, y)) provided there exists n such that $a_n > 0$ and such that u_n is free at x (resp. strict at (x, y)).
- We have the equalities $T_c^+ \circ T_c^- \circ T_c^+ = T_c^+$ and $T_c^- \circ T_c^+ \circ T_c^- = T_c^-$.
- We have the inequalities

$$T_c^+ \circ T_c^- u \leqslant u, \quad T_c^- \circ T_c^+ u \geqslant u$$

for each function u.
- If u is a subsolution, then

$$T_c^+ u \leqslant T_c^+ \circ T_c^- u \leqslant u \leqslant T_c^- \circ T_c^+ u \leqslant T_c^- u \tag{2.1}$$

The following criterion for subsolutions is taken from [19].

Lemma 2.1 *Let u be a subsolution and let us consider a function v such that*

$$u \leqslant v \leqslant T_c^- u.$$

Then v itself is a subsolution.

Proof The statement follows from the inequalities $u \leqslant v \leqslant T_c^- u \leqslant T_c^- v$. ∎

Playing with the Lax–Oleinik operators also leads to the following lemma.

Lemma 2.2 *Let u be a subsolution, then the subsolution*

$$v := \frac{T_c^+ u + T_c^+ \circ T_c^- u + T_c^- \circ T_c^+ u + T_c^- u}{4}$$

is free on the complement of $\mathcal{A}_u$, equal to u on $\mathcal{A}_u$, and strict on the complement of $\widehat{\mathcal{A}}_u$. If Hypothesis 1 holds, then v is locally Lipschitz.

We then have $\mathcal{A}_v \subset \mathcal{A}_u$, but this inclusion is not necessarily an equality.

Proof To prove that v is free on the complement of $\mathcal{A}_u$, we consider a point x at which v is not free, and prove that $x \in \mathcal{A}_u$. We either have $T_c^+ v(x) = v(x)$ or $T_c^- v(x) = v(x)$. In the first case, we have

$$4v(x) = 4T_c^+ v(x) \leqslant T_c^+ \circ T_c^+ u(x) + T_c^+ \circ T_c^+ \circ T_c^- u(x) + T_c^+ \circ T_c^- \circ T_c^+ u(x) + T_c^+ \circ T_c^- u(x)$$

hence the inequalities

$$T_c^+ \circ T_c^+ u(x) \leqslant T_c^+ u(x), \quad T_c^+ \circ T_c^+ \circ T_c^- u(x) \leqslant T_c^+ \circ T_c^- u(x)$$

$$T_c^+ \circ T_c^- \circ T_c^+ u(x) = T_c^+ u(x) \leqslant T_c^- \circ T_c^+ u(x), \quad T_c^+ \circ T_c^- u(x) \leqslant T_c^- u(x)$$

sum to an equality, hence they are equalities. In view of (2.1) the two last equalities imply that $T_c^+ u(x) = u(x) = T_c^- u(x)$. The second case is similar. It then follows from Lemma 2.3 that v is strict outside of $\widehat{\mathcal{A}}_u$. ■

The following lemma allows us to reduce strictness questions to freedom questions and completes the proof of Lemma 2.2.

Lemma 2.3 *Let u, v be subsolutions such that v is free outside of $\mathcal{A}_u$ and equal to u on $\mathcal{A}_u$, then v is strict at each point (x, y) where u is strict.*

Proof Let (x, y) be a pair at which v is not strict. Then $v(y) - v(x) = c(x, y)$, hence $T_c^- v(y) = v(y)$ and $T_c^+ v(x) = v(x)$. Since v is free outside of $\mathcal{A}_u$, this implies that both x and y belong to $\mathcal{A}_u$. Since $u = v$ on $\mathcal{A}_u$, we conclude that

$$u(y) - u(x) = v(y) - v(x) = c(x, y),$$

hence u is not strict at (x, y). ■

It will also be useful to quantify the freedom of a subsolution u by its leverage function.

Definition 2.4 The leverage function $\lambda_u \colon M \to [0, \infty)$ of the subsolution u is defined by:

$$\lambda_u(x) := \tfrac{1}{3} \min\left(T_c^- u(x) - u(x),\ u(x) - T_c^+ u(x)\right).$$

Note that u is free at x if and only if $\lambda_u(x) > 0$.

Lemma 2.5 *Let u be a subsolution and let v be another function such that $|u - v| \leqslant \lambda_u$. Then v is itself a subsolution. Moreover, if u is free at x, then so is v, and if u is strict at (x, y), then so is v.*

Proof By definition, we have

$$3\max\{\lambda_u(x), \lambda_u(y)\} \leqslant \max\{u(x) - T_c^+ u(x), T_c^- u(y) - u(y)\} \leqslant c(x, y) - u(y) + u(x).$$

We conclude that

$$0 \leqslant \max\{\lambda_u(x), \lambda_u(y)\} \leqslant c(x, y) - v(y) + v(x),$$

hence that

$$T_c^+ v(x) + \lambda_u(x) \leqslant v(x) \leqslant T_c^- v(x) - \lambda_u(x)$$

for each x, which implies that v is a subsolution that is free at points where u is free. The last claim follows from Lemma 2.3. ■

3 The Uniform Case on $\mathbb{R}^n$ and the Jensen Transforms

In this section we work on $M = \mathbb{R}^n$. A function $u\colon \mathbb{R}^n \to \mathbb{R}$ is called k-semiconcave if $u(x) - k\|x\|^2$ is concave. We introduce the following more quantitative version of Hypothesis 1 on the cost c.

Hypothesis* 1-*K There exists a constant K such that for each subsolution u, the functions $T_c^- u$ and $-T_c^+ u$ are K-semiconcave.

One setting that implies this condition is the following version of Hypothesis 3.

Hypothesis* 3-*K There exists a constant K such that the function $x \mapsto c(x, y)$ is K-semiconcave for each y and the function $y \mapsto c(x, y)$ is K-semiconcave for each x.

We will use the Jensen transforms, which, for a function $u\colon \mathbb{R}^n \to \mathbb{R}$ and a positive real number t, yield the functions

$$J^{-t}u(x) = \inf_{y\in\mathbb{R}^n} \left(u(y) + \frac{1}{t}\|y - x\|^2 \right), \qquad J^{+t}u(x) = \sup_{y\in\mathbb{R}^n} \left(u(y) - \frac{1}{t}\|y - x\|^2 \right).$$

These are nothing but the Lax–Oleinik operators associated with the costs $c_t(x, y) = \frac{1}{t}\|y - x\|^2$.

Theorem 3.1 *Let u be a uniformly continuous subsolution. The function $J^{-t} \circ J^{+2t} \circ J^{-t}u$ is finite, and, for t small enough, it is a $C^{1,1}$ subsolution. Moreover, it converges uniformly to u as $t \to 0$. More precisely, if u is a uniformly continuous subsolution, then for $t, s < K^{-1}$ the functions $J^{-t} \circ J^{+(t+s)} \circ J^{-s}u$ and $J^{+t} \circ J^{-(t+s)} \circ J^{+s}u$ are $C^{1,1}$ subsolutions that converge uniformly to u as $t, s \to 0$. Moreover, we have*

$$T_c^+ \circ T_c^- u \leqslant J^{-t} \circ J^{+(t+s)} \circ J^{-s}u \leqslant T_c^- u, \quad T_c^+ u \leqslant J^{+t} \circ J^{-(t+s)} \circ J^{+s}u \leqslant T_c^- \circ T_c^+ u.$$

Note that the last inequalities imply that $J^{-t} \circ J^{+(t+s)} \circ J^{-s}u$ and $J^{+t} \circ J^{-(t+s)} \circ J^{+s}u$ are subsolutions by Lemma 2.1. We recall a few properties of the Jensen transforms, most of which are proved in [5] or [1]. Both families of operators J^- and J^+ are semi-groups. They are monotonous in the following ways:

$$\inf u \leqslant J^{-s}u \leqslant J^{-t}u \leqslant u \leqslant J^{+t}u \leqslant J^{+s}u \leqslant \sup u \quad \forall s > t > 0,$$

and

$$u \leqslant v \Longrightarrow \{\forall t \geqslant 0,\ J^{-t}u \leqslant J^{-t}v \text{ and } J^{+t}u \leqslant J^{+t}v\}.$$

A continuous function $\rho\colon [0, \infty) \to [0, \infty)$ such that $\rho(0) = 0$ is called a modulus of continuity. A function f is said to be ρ-continuous if $|f(y) - f(x)| \leqslant \rho(\|y - x\|)$ for all x and y. Given a modulus of continuity ρ, there exists a modulus of continuity ϵ such that, for each ρ-continuous function u, the following properties hold:

- the functions $J^{-t}u$ and $J^{+t}u$ are finite-valued and ρ-continuous for each $t \geqslant 0$;
- $J^{-t}u$ is t^{-1}-semiconcave and $J^{+t}u$ is t^{-1}-semiconvex;

- $\| J^{-t}u - u\|_\infty + \| J^{+t}u - u\|_\infty \leqslant \epsilon(t)$;
- $J^{-t} \circ J^{+t}u \geqslant u$ and $J^{+t} \circ J^{-t}u \leqslant u$;
- the equality $J^{-t} \circ J^{+t}u = u$ (resp. $J^{+t} \circ J^{-t}u = u$) holds if and only if u is t^{-1}-semiconcave (resp. t^{-1}-semiconvex);
- if u is semiconvex (resp. semiconcave) then $J^{-t} \circ J^{+t}u$ (resp. $J^{+t} \circ J^{-t}u$) is $C^{1,1}$ (and finite valued).

Using these properties, we now prove Theorem 3.1. Let u be a uniformly continuous subsolution, with modulus ρ. Since the function u is a subsolution, we have $u \leqslant T_c^- u$, hence $T_c^- u$ is finite-valued. Our hypothesis is that the function $T_c^- u$ is K-semiconcave. For $s < K^{-1}$, we have

$$u \leqslant J^{-s} \circ J^{+s}u \leqslant J^{-s} \circ J^{+s}(T_c^- u) = T_c^- u,$$

where the last inequality follows from the K-semiconcavity of $T_c^- u$ and the properties of $J^- \circ J^+$ listed above. We conclude that the function $J^{-s} \circ J^{+s}u$ is a ρ-continuous, s^{-1}-semiconcave subsolution. Similarly, if u is ρ-continuous and $t < K^{-1}$, then the function $J^{+t} \circ J^{-t}u$ is a ρ-continuous, t^{-1}-semiconvex subsolution. Applying this observation to the function $J^{-s} \circ J^{+s}u$, we conclude that $J^{+t} \circ J^{-t} \circ J^{-s} \circ J^{+s}u$ is a ρ-continuous subsolution. This subsolution is $C^{1,1}$, since $J^{-s} \circ J^{+s}u$ is semiconcave. We have the inequality

$$\begin{aligned} T_c^+ \circ T_c^- u = J^{+t} \circ J^{-t}(T_c^+ \circ T_c^- u) &\leqslant J^{+t} \circ J^{-t}u \leqslant J^{+t} \circ J^{-t} \circ J^{-s} \circ J^{+s}u \\ &\leqslant J^{+t} \circ J^{-t}(T_c^- u) \leqslant T_c^- u. \end{aligned}$$

Finally, we have

$$u \leqslant J^{-s} \circ J^{+s}u \leqslant J^{-s}(u + \| J^{+s}u - u\|_\infty) \leqslant \| J^{+s}u - u\|_\infty + \| J^{-s}u - u\|_\infty + u \leqslant u + \epsilon(s)$$

and similarly $u - \epsilon(t) \leqslant J^{+t} \circ J^{-t}u \leqslant u$, hence

$$u - \epsilon(t) \leqslant J^{+t} \circ J^{-t}u \leqslant J^{+t} \circ J^{-(t+s)} \circ J^{+s}u \leqslant J^{-s} \circ J^{+s}u \leqslant u + \epsilon(s),$$

where ϵ is the modulus associated with ρ in the list of properties of J. ■

4 The General Case

In this section, we come back to the general setting and prove Theorem 1.1. We derive it from the uniform version using partitions of unity, as was done in [5] for Ilmanen's lemma. We fix a locally finite atlas $(\phi_i)_{i\in I}$ constituted of smooth maps $\phi_i \colon B_n \to M$, where B_n is the open unit ball. We assume that all the images $\phi_i(B_n)$, for $i \in I$, are relatively compact in M. Moreover, we consider a smooth partition of unity $(g_i)_{i\in I}$ subordinated to the locally finite open covering $\big(\phi_i(B_n)\big)_{i\in I}$. Given positive numbers $a_i, b_i, i \in I$, we define the operators

$$\forall x \in M, \quad Su(x) = \sum_{i\in I} \big[J^{-a_i} \circ J^{+a_i}(g_i u \circ \phi_i)\big] \circ \phi_i^{-1}(x),$$

$$\forall x \in M, \quad \check{S}u(x) = \sum_{i\in I} [J^{+b_i} \circ J^{-b_i}(g_i u \circ \phi_i)] \circ \phi_i^{-1}(x).$$

The functions in the sums are extended to the whole of M by the value zero outside of the domain $\phi_i(B_n)$. The sums are locally finite, hence well defined. Theorem 1.1 is a consequenct of the following theorem.

Theorem 4.1 *Assume that the cost c satisfies Hypothesis 1. Let u be a continuous subsolution and let $\epsilon\colon M \to]0,\infty)$ be a continuous function. For suitably chosen positive constants $(a_i)_{i\in I}$ and $(b_i)_{i\in I}$, the function $\check{S}\circ S(u)$ is a locally $C^{1,1}$ subsolution such that $|u-\check{S}\circ Su| \leqslant \epsilon$ and*

$$T_c^+ \circ T_c^- u \leqslant \check{S}\circ Su \leqslant T_c^- u.$$

Proof Since the image $\phi_i(B_n)$ is relatively compact and since the atlas is locally finite the set $A_i = \{ j \in I,\ \ \phi_j(B_n)\cap\phi_i(B_n) \neq \varnothing\}$ is finite. Let us denote by e_i its cardinal. Setting

$$\epsilon_i := \frac{\min_{j\in A_i}\inf_{x\in B_n}\epsilon\big(\phi_j(x)\big)}{2\max_{j\in A_i} e_j},$$

we observe that

$$\forall i\in I, \quad \sum_{j\in A_i}\epsilon_j \leqslant \frac{1}{2}\inf_{x\in B_n}\epsilon\big(\phi_i(x)\big).$$

Let us make the convention to extend all functions that are compactly supported inside B_n, like $(g_i u)\circ\phi_i$ by the value 0 to the whole of $\mathbb{R}^n$. For each i, we choose a positive constant a_i such that

$$\Big\|(g_i u)\circ\phi_i - J^{-a_i}\circ J^{+a_i}\big((g_i u)\circ\phi_i\big)\Big\|_\infty < \epsilon_i. \tag{4.1}$$

Such a constant exists because the function $(g_i u)\circ\phi_i$ is uniformly continuous on $\mathbb{R}^n$. Since $T_c^- u$ is locally semiconcave, the function $(g_i T_c^- u)\circ\phi_i$, extended by zero outside of B_n, is semiconcave on $\mathbb{R}^n$ (see [5]). We can assume by taking $a_i > 0$ small enough that it is a_i^{-1}-semiconcave, so that

$$[g_i u]\circ\phi_i \leqslant J^{-a_i}\circ J^{+a_i}\big([g_i u]\circ\phi_i\big) \leqslant J^{-a_i}\circ J^{+a_i}\big([g_i T_c^- u]\circ\phi_i\big) = \big[g_i T_c^- u\big]\circ\phi_i$$

on $\mathbb{R}^n$. This implies in particular that the function $J^{-a_i}\circ J^{+a_i}(g_i u\circ\phi_i)$ is supported in B_n. As a consequence, the function $\big[J^{-a_i}\circ J^{+a_i}(g_i u\circ\phi_i)\big]\circ\phi_i^{-1}$, extended by zero outside of $\phi_i(B_n)$, is locally semiconcave on M, hence the function Su is locally semiconcave, being a locally finite sum of locally semiconcave functions. By summation we get

$$u = \sum_{i\in I}(g_i u)\circ\phi_i\circ\phi_i^{-1} \leqslant Su \leqslant \sum_{i\in I}\big[g_i T_c^- u\big]\circ\phi_i\circ\phi_i^{-1} = T_c^- u,$$

which, by Lemma 2.1, implies that Su is a subsolution. We have $|u - Su| < \epsilon/2$ by (4.1).

Next, we chose b_i such that $[g_i T_c^+ \circ T_c^- u] \circ \phi_i$ is b_i^{-1}-semiconvex, which implies that

$$[g_i T_c^+ \circ T_c^- u] \circ \phi_i = J^{+b_i} \circ J^{-b_i}\big([g_i T_c^+ \circ T_c^- u] \circ \phi_i\big) \leqslant J^{+b_i} \circ J^{-b_i}\big([g_i u] \circ \phi_i\big)$$
$$\leqslant J^{+b_i} \circ J^{-b_i}\big([g_i Su] \circ \phi_i\big) \leqslant [g_i Su] \circ \phi_i.$$

As above, this implies that $J^{+b_i} \circ J^{-b_i}\big([g_i Su] \circ \phi_i\big)$ is supported on B_n. Note that it is also $C^{1,1}$, hence the function $\big(J^{+b_i} \circ J^{-b_i}\big([g_i Su] \circ \phi_i\big)\big) \circ \phi_i^{-1}$, extended by zero outside of $\phi_i(B_n)$, is locally $C^{1,1}$ on M. By summation, we obtain that

$$T_c^+ \circ T_c^- u \leqslant \check{S}u \leqslant \check{S} \circ Su \leqslant Su \leqslant T_c^- u,$$

which implies that $\check{S} \circ Su$ is a subsolution. This function is locally $C^{1,1}$ as a locally finite sum of locally $C^{1,1}$ functions. Finally, we can assume by possibly reducing b_i that

$$\big\| (g_i Su) \circ \phi_i - J^{+b_i} \circ J^{-b_i}\big((g_i Su) \circ \phi_i\big) \big\|_\infty < \epsilon_i,$$

which implies that $|\check{S} \circ Su - Su| \leqslant \epsilon/2$ hence that $|\check{S} \circ Su - Su| \leqslant \epsilon$. ■

Theorem 4.2 *We assume Hypothesis* 1. *Let $\Omega \subset M$ be an open set and let u be a continuous subsolution that is free on Ω. Then the subsolution u belongs to the closure, for the strong topology, of the set of $C^{1,1}$ subsolutions that are free on Ω and equal to u on $\mathcal{A}_u$.*

Proof Let $\epsilon\colon M \to]0, \infty)$ be a continuous function. We can chose a_i and b_i in such a way that $\check{S}\circ Su$ is a subsolution that is equal to u on $\mathcal{A}_u$, and such that $|\check{S}\circ Su - u| \leqslant \epsilon$. However, $\check{S} \circ Su$ need not be free on Ω. To preserve the freedom of u, we work with the modified cost

$$\tilde{c}(x, y) = c(x, y) - \psi(y),$$

where ψ is a smooth bounded function such that $0 \leqslant \psi \leqslant \lambda_u$ (the leverage function of u), with strict inequalities on Ω. The associated Lax–Oleinik operator is

$$T_{\tilde{c}}^- v(x) = -\psi(x) + T_c^- v(x).$$

Each subsolution for the cost $\tilde{c}$ is thus a subsolution for the cost c, and $\tilde{c}$ satisfies Hypothesis 1. Moreover, the function u is a subsolution for the cost $\tilde{c}$. We apply Theorem 4.1 and get a locally $C^{1,1}$ subsolution w^- for the cost $\tilde{c}$, which satisfies $|w^- - u| \leqslant \epsilon$ and $w^- = u$ on $\mathcal{A}_u$. This function then satisfies

$$T_c^- w^- = \psi + T_{\tilde{c}}^- w^- \geqslant \psi + w^-,$$

hence it is a subsolution for the cost c. Similarly, by applying Theorem 4.1 with the modified cost $c(x, y) - \psi(x)$, we get a locally $C^{1,1}$ subsolution w^+ (for the cost c) such that $T_c^+ w^+ \leqslant w^+ - \psi$, $|w^+ - u| \leqslant \epsilon$, and $w^+ = u$ on $\mathcal{A}_u$. We then set $w := (w^+ + w^-)/2$ and claim that this locally $C^{1,1}$ subsolution is free on Ω. This follows from the inequalities

$$T_c^- w \geqslant \big(T_c^- w^- + T_c^- w^+\big)/2 \geqslant w + \psi/2,$$
$$T_c^+ w \leqslant \big(T_c^+ w^- + T_c^+ w^+\big)/2 \leqslant w - \psi/2,$$

since ψ is positive on Ω. We also obviously have $|w - u| \leqslant \epsilon$ and $w = u$ on $\mathcal{A}_u$. ■

5 Proof of Theorem 1.3

We will build, successively, subsolutions v_1, v_2, v_3 that are all equal to u on $\mathcal{A}_u$ and free on the complement Ω of $\mathcal{A}_u$. By Lemma 2.3, this also implies that the subsolutions v_i are strict where u is strict. We take

$$v_1 = \frac{T_c^+ u + T_c^+ \circ T_c^- u + T_c^- \circ T_c^+ u + T_c^- u}{4},$$

which is continuous, equal to u on $\mathcal{A}_u$, and free on the complement of $\mathcal{A}_u$ by Lemma 2.2.

We then build v_2 by applying Theorem 4.2 to v_1, with $\Omega = M \setminus \mathcal{A}_u$, and get a locally $C^{1,1}$ subsolution v_2 that is free on Ω and equal to u on $\mathcal{A}_u$.

The following mollification result, which will be proved in the Appendix using a procedure due to De Rham, allows to smooth our subsolution on Ω.

Theorem 5.1 *Let f be a locally $C^{k,1}$ function on M and let $\epsilon\colon M \to [0, \infty)$ be a continuous function. Then, there exists a locally $C^{k,1}$ function $g\colon M \to \mathbb{R}$ that is smooth on the open set $\Omega := \epsilon^{-1}(0, +\infty)$ and satisfies, for all $x \in M$,*

$$|f(x) - g(x)| + \|\mathrm{d}_x f - \mathrm{d}_x g\| + \cdots + \|\mathrm{d}_x^k f - \mathrm{d}_x^k g\| \leqslant \epsilon(x).$$

More precisely, we apply Theorem 5.1 to the function $f = v_2$, with $k = 1$, and with a function $\epsilon(x)$ such that $\epsilon = 0$ on $\mathcal{A}_u$, $\epsilon > 0$ on Ω (the complement of $\mathcal{A}_u$), and $\epsilon \leqslant \lambda_{v_2}$ (the leverage function of v_2). We get a $C^{1,1}$ function v_3, which is smooth on Ω and is equal to u on $\mathcal{A}_u$. Since $|v_3 - v_2| \leqslant \lambda_{v_2}$, Lemma 2.5 implies that v_3 is a subsolution that is free on Ω. Lemma 2.3 then implies that v_3 is strict where u is strict. ■

6 Proof of Theorem 1.4

It is enough to prove the existence of a subsolution u that is free on the complement of $\mathcal{A}$ and strict on the complement of $\widehat{\mathcal{A}}$. Theorem 1.3 then implies the existence of a locally $C^{1,1}$ solution v that is free and smooth on the complement of $\mathcal{A}$, and that is strict on the complement of $\widehat{\mathcal{A}}$. We start with the following lemma.

Lemma 6.1 *If c satisfies Hypothesis* 1 *and admits a subsolution, then there exists a continuous subsolution w_1 that is free on the complement of $\mathcal{A}$.*

Proof Let us consider a point $x \notin \mathcal{A}$. By definition, there exists a subsolution v_x such that $x \notin \mathcal{A}_{v_x}$, hence, by Lemma 2.2, there exists a continuous subsolution $u_x \in \mathcal{SS}_C$ that is free at x. By continuity of u_x, $T_c^- u_x$, and $T_c^+ u_x$ we may consider a positive number ϵ_x and an open neighborhood of x, O_x, on which the following holds:

$$\forall y \in O_x, \quad T_c^- u_x(y) - \epsilon_x > u_x(y) > T_c^+ u_x(y) + \epsilon_x.$$

The set $M \setminus \mathcal{A}$ satisfies the Lindelöf property (it is a separable metric space). We can thus extract a countable covering O_n, $n \in \mathbb{N}$ of the covering O_x, $x \in M \setminus \mathcal{A}$. Denoting by u_n and ϵ_n the continuous subsolution and positive real number associated with O_n, we consider a convex combination

$$w_1 = \sum_{n\in\mathbb{N}} a_n u_n,$$

where a_n is a sequence of positive numbers such that $\sum_{\mathbb{N}} a_n = 1$ and such that the sum in the definition of w_1 is normally convergent on each compact set. The function w_1 is then a continuous subsolution. For each $x \notin \mathcal{A}$, there exists $n_0 \in \mathbb{N}$ such that $x \in O_{n_0}$, and we have

$$T_c^- w_1(x) = T_c^- \Big(\sum_{n\in\mathbb{N}} a_n u_n\Big)(x) \geqslant \sum_{n\in\mathbb{N}} a_n T_c^- u_n(x) \geqslant a_{n_0}\epsilon_{n_0} + \sum_{n\in\mathbb{N}} a_n u_n > w_1(x).$$

A similar computation shows that $T_c^+ w_1(x) < w_1(x)$. ■

Lemma 6.2 *If there exists a continuous subsolution, then there exists a continuous subsolution w_2 that is strict at each pair (x, y) where a strict continuous subsolution exists. Under Hypothesis 1, the subsolution w_2 is then strict outside of $\widehat{\mathcal{A}}$.*

Proof Since M is separable, the set $\mathcal{SS}_C$ of continuous subsolutions is also separable (for the compact-open topology), and we consider a dense subsequence $(u_n)_{n\in\mathbb{N}}$. Set

$$w_2 = \sum_{n\in\mathbb{N}} a_n u_n, \tag{6.1}$$

where the a_n are positive real numbers such that $\sum a_n = 1$ and the sum (6.1) is uniformly convergent on each compact subset. The function w_2 is a subsolution since it is a convex combination of subsolutions. If now $(x, y) \in \widehat{\mathcal{A}}_{w_2}$, summing the inequalities

$$\forall n \in \mathbb{N}, \quad a_n\big(u_n(y) - u_n(x)\big) \leqslant a_n c(x, y),$$

gives an equality; therefore, all inequalities are equalities and

$$\forall n \in \mathbb{N}, \quad (x, y) \in \widehat{\mathcal{A}}_{u_n}.$$

By density of the sequence u_n, we deduce that $(x, y) \in \widehat{\mathcal{A}}_u$ for each continuous solution u. Under Hypothesis 1, $\widehat{\mathcal{A}}$ is exactly the set of pairs at which no continuous subsolution is strict: $\widehat{\mathcal{A}} = \bigcap_{u\in\mathcal{SS}_C} \widehat{\mathcal{A}}_u$ hence, $(x, y) \in \widehat{\mathcal{A}}$. ■

To finish the proof of Theorem 1.4, we consider the subsolution $u = (w_1 + w_2)/2$. This subsolution is free on the complement of $\mathcal{A}$, because w_1 is, and it is strict on the complement of $\widehat{\mathcal{A}}$, because w_2 is. ■

A Proof of Theorem 5.1

We prove Theorem 5.1 using a regularization procedure due to De Rham; see [11]. The idea of De Rham is to construct an action $\mathfrak{t}$ of $\mathbb{R}^n$ on $\mathbb{R}^n$ by smooth diffeomorphisms supported on the unit sphere B_n, in such a way that the induced action on B_n is conjugated to the standard action of $\mathbb{R}^n$ on itself by translations. More precisely, there exists a diffeomorphism $\mathfrak{h}\colon B_n \to \mathbb{R}^n$ and diffeomorphisms $\mathfrak{t}_y$, $y \in \mathbb{R}^n$, of $\mathbb{R}^n$, equal to the identity outside of the open unit ball B_n, such that the map $(x, y) \mapsto \mathfrak{t}_y(x)$ is smooth and such that $\mathfrak{h} \circ \mathfrak{t}_y = y + \mathfrak{h}$ on B_n. This implies that $\mathfrak{t}$ is an action of the group $\mathbb{R}^n$ on $\mathbb{R}^n$, which means that $\mathfrak{t}_y \circ \mathfrak{t}_{y'} = \mathfrak{t}_{y+y'}$ for each y, y'. Since $\mathfrak{t}$ is smooth, $\mathfrak{t}_0 = \mathrm{Id}$, and $\mathfrak{t}_y = \mathrm{Id}$ outside of the unit ball, the maps $\mathfrak{t}_y$ converge uniformly to the identity as $y \longrightarrow 0$, and all their derivatives converge uniformly to the derivatives of the identity.

Let us give some details on the construction of $\mathfrak{h}$ and $\mathfrak{t}$. We set

$$\mathfrak{h}(x) = \frac{h(\|x\|)}{\|x\|}x,$$

where $h\colon [0, 1[\to \mathbb{R}_+$ is a smooth, strictly increasing ($h' > 0$) function such that

$$\begin{cases} h(r) = r, & 0 \leqslant r \leqslant 1/3, \\ h(r) = \exp((r-1)^{-2}), & 2/3 \leqslant r < 1. \end{cases}$$

We then define $\mathfrak{t}_y$, for each $y \in \mathbb{R}^n$ by

$$\begin{cases} \mathfrak{t}_y(x) = \mathfrak{h}^{-1}\big(\mathfrak{h}(x) + y\big) & \text{if } x \in B_n, \\ \mathfrak{t}_y(x) = x & \text{if } x \in \mathbb{R}^n \setminus B_n. \end{cases}$$

It is clear from these formulæ that $\mathfrak{t}_{y+y'} = \mathfrak{t}_y \circ \mathfrak{t}_{y'}$. The only issue is the smoothness of $\mathfrak{t}$. Differentiating the previous group property with respect to y' and taking $y' = 0$ yields the following relation:

$$\frac{\partial}{\partial y}\mathfrak{t}_y = \frac{\partial}{\partial y}\mathfrak{t}_0 \circ \mathfrak{t}_y.$$

This implies that

$$\mathfrak{t}_y(x) = x + \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{t}_{ty}(x)\mathrm{d}t = x + \int_0^1 \Big(\frac{\partial}{\partial y}\mathfrak{t}_{ty}(x)\Big)\, y\mathrm{d}t = x + \int_0^1 \Big(\frac{\partial}{\partial y}\mathfrak{t}_0\big(\mathfrak{t}_{ty}(x)\big)\Big)\, y\mathrm{d}t.$$

In other words, the map $\mathfrak{t}_y$ is the time-one flow of the vector field $X_y(x) := M(x)y$, where $M(x) = \partial_y \mathfrak{t}_y(x)_{|y=0}$. In order to prove that the map $\mathfrak{t}$ is smooth, it is enough to observe that the matrix $M(x)$ depends smoothly on x. This matrix can be computed, recalling that the gradient of the norm $x \mapsto \|x\|$ is $r_x := x/\|x\|$:

$$M(x) = \mathrm{d}_{\mathfrak{h}(x)}\mathfrak{h}^{-1} = \frac{1}{h'(\|x\|)}r_x^{\,t} r_x + \frac{\|x\|}{h(\|x\|)}(I_n - r_x^{\,t} r_x).$$

Since $1/h$, $1/h'$, as well as all their derivatives go to 0 when $\|x\| \to 1$, we conclude that $M(x)$ is smooth.

We have exposed the construction of $\mathfrak{h}$ and $\mathfrak{t}$. They allow us to define a local regularization procedure with the help of a smooth kernel $K_1\colon \mathbb{R}^n \to [0, \infty)$. We assume that K_1 is supported in the unit ball B_n and that $\int K_1 = 1$. For $\eta > 0$, we set $K_\eta(x) = \eta^{-n} K_1(\eta^{-1}x)$.

Lemma A.1 *Let $O \subset \mathbb{R}^n$ be an open set containing $\overline{B}_n$. Given a locally integrable function $f\colon O \to \mathbb{R}$ and $\eta \in]0,1[$, we define*

$$f_\eta(x) = \int_{\mathbb{R}^n} f\big(\mathfrak{t}_y(x)\big) K_\eta(-y)\mathrm{d}y.$$

The following assertions hold:

(i) *the function f_η is C^∞ in B_n, and equal to f outside of B_n;*
(ii) *if f is C^k on O, then so are the functions f_η, and $f_\eta \to f$ in C^k as $\eta \to 0$;*
(iii) *if f is $C^{k,1}$ on O, then so are the functions f_η, and*

$$\limsup_{\eta \to 0} \mathrm{Lip}(\mathrm{d}^k f_\eta) \leqslant \mathrm{Lip}(\mathrm{d}^k f);$$

(iv) *if, in some open set $O' \subset O$, f is C^l in O', then so is f_η.*

Proof On B_n we have

$$f_\eta \circ \mathfrak{h}^{-1} = (f \circ \mathfrak{h}^{-1}) \star K_\eta,$$

where $\star$ is the convolution. Since the functions K_η are smooth, this implies the first claim. Writing

$$f_\eta - f = \int_{B(0,\eta)} (f \circ \mathfrak{t}_y - f) K_\eta(-y)\mathrm{d}y$$

and observing that $f \circ \mathfrak{t}_y - f \to 0$ in $C^k(\mathbb{R}^n, \mathbb{R}^n)$ as $y \to 0$ (because $\mathfrak{t}_y \to \mathrm{Id}$ in $C^k(\mathbb{R}^n, \mathbb{R}^n)$) yields the second claim. We will now prove that

$$\limsup_{y \to 0} \mathrm{Lip}\left(\mathrm{d}^k(f \circ \mathfrak{t}_y)\right) \leqslant \mathrm{Lip}(\mathrm{d}^k f), \tag{A.1}$$

which yields the third claim in view of the relation

$$\mathrm{d}^k_x f_\eta = \int_{B(0,\eta)} \mathrm{d}^k_x(f \circ \mathfrak{t}_y) K_\eta(-y)\mathrm{d}y.$$

Let us consider a component $\partial^\alpha_x(f \circ \mathfrak{t}_y)$ of the differential $\mathrm{d}^k(f \circ \mathfrak{t}_y)$, where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index such that $|\alpha| = \sum \alpha_i = k$. By the Faà di Bruno formula, expressed in terms of partial differentials (see [9] for example), we have

$$\partial^\alpha_x(f \circ \mathfrak{t}_y) = \sum_{1 \leqslant |\lambda| \leqslant |\alpha|} \partial^\lambda_{\mathfrak{t}_y(x)} f \cdot B_{\alpha,\lambda}(\mathrm{d}_x \mathfrak{t}_y, \dots, \mathrm{d}^{|\alpha|}_x \mathfrak{t}_y),$$

where the $B_{\alpha,\lambda}$ are universal multi-variable polynomials with no constant terms. These polynomials satisfy the equalities

$$B_{\alpha,\alpha}(\mathrm{Id}, 0, \dots, 0) = 1 \quad \text{and} \quad B_{\alpha,\lambda}(\mathrm{Id}, 0, \dots, 0) = 0$$

for all $\lambda \neq \alpha$. Since $\mathrm{t}_y \to Id$ in C^∞, the first of these equalities implies that the function $x \mapsto B_{\alpha,\alpha}(\mathrm{d}_x \mathrm{t}_y, \dots, \mathrm{d}_x^{|\alpha|} \mathrm{t}_y)$ is converging to 1 in C^∞. Concerning the other factor in this term, we have

$$\mathrm{Lip}\left((\partial^\alpha f) \circ \mathrm{t}_y\right) \leqslant \mathrm{Lip}(\partial^\alpha f)\,\mathrm{Lip}(\mathrm{t}_y) \longrightarrow \mathrm{Lip}(\partial^\alpha f).$$

We deduce that the upper limit of the Lipschitz constants of the term corresponding to $\lambda = \alpha$ is not greater than $\mathrm{Lip}(\partial^\alpha f)$.

On the other hand, for each of the terms with $\lambda \neq \alpha$, the function

$$x \longmapsto B_{\alpha,\lambda}(\mathrm{d}_x \mathrm{t}_y, \dots, \mathrm{d}_x^{|\alpha|} \mathrm{t}_y)$$

is converging to 0 in C^∞, hence the Lipschitz constant of the function

$$x \longmapsto \partial^\lambda_{\mathrm{t}_y(x)} f \cdot B_{\alpha,\lambda}(\mathrm{d}_x \mathrm{t}_y, \dots, \mathrm{d}_x^{|\alpha|} \mathrm{t}_y)$$

is converging to 0. We conclude that

$$\limsup \mathrm{Lip}\left(\partial^\alpha (f \circ \mathrm{t}_y)\right) \leqslant \mathrm{Lip}(\partial^\alpha f),$$

which implies (A.1), hence the third point of the statement.

Regarding the last claim of the statement, we consider the set

$$\Omega := \bigcap_{y \in \overline{B}(0,\eta)} \mathrm{t}_y^{-1}(O'),$$

and claim that Ω is open. Assuming the claim, we observe that the function f_η is smooth in B_n and that it is C^l in Ω. Since the maps t_y are all the identity outside of B_n, the set Ω contains $O' - B_n$. We have covered O' by two open sets, B_n and Ω, such that the f_η is C^l on each of them. We conclude that this function is C^l on O'.

To prove that Ω is open, we fix $x_0 \in \Omega$. For each $y_0 \in \overline{B}(0, \eta)$, we have $\mathrm{t}_{y_0}(x_0) \in O'$, hence there exists an open set U_{y_0} containing y_0 and an open set Ω_{y_0} containing x_0 such that $\mathrm{t}_y(x) \in O'$ for all $(x, y) \in \Omega_{y_0} \times U_{y_0}$. By compactness, there exists finitely many points $y_i \in \overline{B}(0, \eta)$ such that the open sets U_{y_i} cover $\overline{B}(0, \eta)$. The open intersection $\bigcap_i \Omega_{y_i}$, which contains x_0, is then contained in Ω. Since this holds for each $x_0 \in \Omega$, we have proved that Ω is open. ∎

Lemma A.2 *Let O be open subsets of $\mathbb{R}^n$ and let $f\colon O \to \mathbb{R}$ be a $C^{k,1}$ function. Given a continuous function $\epsilon\colon O \to [0, \infty)$, there exists a function f_ϵ such that:*

(i) *the function f_ϵ is C^∞ in the open set $\{x \in O, \epsilon(x) > 0\} \subset O$;*
(ii) $|f_\epsilon(x) - f(x)| + \|\mathrm{d}_x f_\epsilon - \mathrm{d}_x f\| + \cdots + \|\mathrm{d}^k_x f_\epsilon - \mathrm{d}^k_x f\| \leqslant \epsilon(x)$ *for each $x \in O$;*
(iii) *the function f_ϵ is $C^{k,1}$ on O, and* $\mathrm{Lip}(\mathrm{d}^k f_\epsilon) \leqslant 1 + \mathrm{Lip}(\mathrm{d}^k f)$.

Proof Let us denote by F the closed set $\{\epsilon = 0\}$. The complement of F in O is open, and we consider a locally finite covering $(O_i)_{i\in\mathbb{N}^*}$ of $O \setminus F$ by open balls compactly included in $O\setminus F$. Since $\inf\{\epsilon(x), \quad x \in O_i\} > 0$. we can construct inductively, using Lemma A.1 a sequence of functions, $(f_i)_{i\in\mathbb{N}}$ such that

- $f_0 = f$;
- for each $i \in \mathbb{N}$, the function f_{i+1} is C^∞ in $O_1 \cup \cdots \cup O_{i+1}$;
- for each $i \in \mathbb{N}$, the functions f_i and f_{i+1} are equal in $O \setminus O_{i+1}$;
- for each $i \in \mathbb{N}$, the function f_{i+1} is $C^{k,1}$ in O, and $\mathrm{Lip}(\mathrm{d}^k f_{i+1}) \leqslant 2^{-i-1} + \mathrm{Lip}(\mathrm{d}^k f_i)$;
- $|f_{i+1}(x) - f_i(x)| + \|\mathrm{d}_x f_{i+1} - \mathrm{d}_x f_i\| + \cdots + \|\mathrm{d}^k_x f_{i+1} - \mathrm{d}^k_x f_i\| \leqslant 2^{-1-i}\epsilon(x)$ for each $x \in O$, $i \in \mathbb{N}$,

Each point of O has a neighborhood on which the sequence f_i is eventually constant, hence the limit $f_\epsilon := \lim f_i$ is well defined and smooth on $\bigcup_i O_i = O\setminus F$. The desired estimates on f_ϵ follow immediately from the inductive estimates by summation. ■

Proof of Theorem 5.1 We fix a locally finite atlas $(\phi_i)_{i\in\mathbb{N}^*}$ constituted of smooth maps $\phi_i\colon 2B_n \to M$, where B_n is the open unit ball. We assume that all the images $\phi_i(2B_n)$, $i \in \mathbb{N}^*$ are relatively compact in M and that the $\phi_i(B_n)$, $i \in \mathbb{N}^*$ still cover M. By Lemma A.2, it is possible to construct inductively a sequence of functions f_i, by iteratively modifying $f_i \circ \phi_{i+1}$ on B_n, such that

- $f_0 = f$;
- for each $i \in \mathbb{N}$, the function f_{i+1} is C^∞ in $\bigcup_{j\leqslant i+1} \phi_j(B_n) \cap \Omega$;
- for each $i \in \mathbb{N}$, in $M \setminus \phi_{i+1}(B_n)$, the functions f_i and f_{i+1} are equal;
- for each $i \in \mathbb{N}$, the function f_{i+1} is $C^{k,1}$ on M;
- for each $i \in \mathbb{N}$, $x \in M$, $|f_i(x) - f_{i+1}(x)| + \cdots + \|\mathrm{d}^k_x f_i - \mathrm{d}^k_x f_{i+1}\| \leqslant 2^{-i-1}\epsilon(x)$.

Each point $x \in M$ has a neighborhood on which the sequence f_i is eventually constant, hence the limit $g = \lim f_i$ is well defined, locally $C^{k,1}$, and smooth on Ω. The inequality on the differentials follows by summation from the iterative assumptions. ■

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