Historical Hamiltonian Dynamics: symplectic and covariant

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Abstract

This paper presents a " historical " formalism for dynamical systems, in its Hamiltonian version (Lagrangian version was presented in a previous paper). It is universal, in the sense that it applies equally well to time dynamics and to field theories on space-time. It is based on the notion of (Hamiltonian) *histories*, which are sections of the (extended) phase space bundle. It is developed in the space of sections, in contradistinction with the usual formalism which works in the bundle manifold.

In field theories, the formalism remains covariant and does not require a spitting of space-time. It considers *space-time* exactly in the same manner than *time* in usual dynamics, both being particular cases of the *evolution domain*. It applies without modification when the histories (the fields) are forms rather than scalar functions, like in electromagnetism or in tetrad general relativity.

We develop a differential calculus in the infinite dimensional space of histories. It admits a (generalized) symplectic form which does not break the covariance. We develop a covariant symplectic formalism, with generalizations of usual notions like current conservation, Hamiltonian vector-fields, evolution vector-field, brackets, ... The usual multisymplectic approach derives form it, as well as the symplectic form introduced by Crnkovic and Witten in the space of *solutions*.

1 Introduction

Our historical Hamiltonian formalism is based on the notion of *history*. According to [24], histories " furnish the raw material from which reality is constructed".

This follows our previous work ([6], hereafter paper I) which presents a *Lagrangian* formalism on the same basis (see an outline in Appendix A). An history (or kinematical history) is a *possible* evolution of a dynamical system, also called a *configuration* [18]. An history which obeys the dynamical equations becomes a *physical* evolution, or particular solution.

Our approach applies equally well to usual time dynamics (tD) and to covariant field theories (FT), and allows further generalizations. These different contexts (tD and FT) only differ by their *evolution domain* (see below): the time line in tD; the space-time in FT. But all expressions or equations are identical in both cases. Thus space-time in FTs appears on the same footing than time in tD, with the only difference that it is 4 dimensional rather than monodimensional. In the case of FTs, our formalism remains entirely covariant and does not require any splitting of space-time.

In addition, it applies without modification to the case where the fields are not *functions*, but *forms* (*e.g.*, on space-time). This applies to electromagnetism or to general relativity in the tetrad formalism.

In paper I, we have presented its Lagrangian version. Here we present the Hamiltonian one. The most important result is the existence of a canonical (generalized) symplectic form which remains entirely covariant. In time dynamics, it is equivalent to the usual symplectic form. In field theories, it remains covariant and we show that the multisymplectic formalism may be seen as derived from it. To mention some general ideas underlying this approach,

- dynamics is defined, not versus time but versus an *evolution domain*. It reduces to the time line in tD (a particular case); to space-time for FT's. But it is treated exactly in the same manner in both cases.

- An history may be a function on the evolution domain (like a scalar field on space-time) but also, more generally, a differential form on it; in any case a section of a particular fiber bundle.

- A *particular solution* is an history which is an orbit of a [Hamiltonian] flow in the corresponding bundle. Such flows have the dimension of the evolution domain. They may be called *general solutions*.

- Our calculus does not hold in configuration space, or phase space, but in the space of histories which has infinite dimension It is inspired by *diffeology* [14] considerations. It may be seen as generalization of both [29] and the multisymplectic formalism, and as a synthesis between them.

- In the space of (Hamiltonian) histories, we define a canonical and covariant (generalized) symplectic form; equivalent to the usual symplectic form in tD; giving raise to both the multisymplectic form and the symplectic currents [29] in FTs.

- Covariant field theories (in space-time) and time dynamics appear as two particular cases of this formalism.

In some sense, our formalism appears as a generalized synthesis between the multisymplectic geometry (see, *e.g.*, [11]), the “ covariant phase space ”approaches (see, *e.g.*, [11]), the canonical approach and the geometry of the space of solutions. It remains entirely covariant.

The section 2 introduces the notion of history (2.1). It defines their prolongations to *velocity-histories* and *Hamiltonian histories*, involved in the Lagrangian and Hamiltonian formulations of dynamics, and introduces the *phase space bundle* (2.2). It also explicits our differential calculus in the space of Hamiltonian Histories (2.4). Section 3 expresses the Hamiltonian Dynamics in its historical formulation. It introduces the generalized symplectic form and the evolution H-vector field (3.1). It derives the universal historical evolution equations (3.2) and explicits the dynamical solution (3.3). Section 4 gives illustrations, applying the general formalism to time Dynamics (4.1), and to scalar field theories, where our formalism is compared to the multisymplectic one (4.2). Section 5 considers conservation (5.1) and symmetries (5.2) It discusses the notions of [generalized] Poisson brackets and observables (5.3). The last sections apply to electromagnetism (6) and to first order general relativity (7). An outline of the historical Lagrangian formalism presented in paper I is given in Appendix A.

2 General framework

Our framework applies equally well to (non relativistic) time dynamics and to relativistic (covariant) field theories. It is formulated in terms of *histories*, that we define below. Shortly, an history is associated to each degree of freedom of the dynamical system. We treat only the case of an unique degree of freedom; the generalization to multicomponent-systems is straightforward and is treated in illustrations below. The case of a form-field rather than a scalar field, like the electromagnetic potential in Maxwell theory, is treated as a single degree of freedom, to which correspond an unique (although not scalar-valued) history (a 1–history); or the tetrad general relativity, where the cotetrads forms e^I and and the spin connection forms ω^{IJ} are also 1–histories. We treat first the scalar field case and then extend to the form field.

2.1 Histories

A dynamical system is characterized by its configuration bundle $\mathbf{C} \to \mathcal{D}$.
Here, $\mathcal{D}$ is the *domain* of the theory. In usual time dynamics (tD), this is the time line $\mathbb{R}_t$, or an interval of it. In relativistic field theories (FT), this is space-time. We treat both cases equally, and more generally, $\mathcal{D}$ is a n-dimensional differentiable manifold, possibly with a given metric. We label $\mathcal{D}$ with arbitrary coordinates x^μ (the unique coordinate $t = x^0$ for the timeline in tD), which disappear in our final results which are covariant [1]. They generate adapted [local] coordinates in the various fiber bundles we will consider. Our philosophy is to treat $\mathcal{D}$ as some kind of " n-dimensional timeline " w.r.t. which the evolution is expressed.

An element of the fiber is a possible value of the dynamical variable. Most physical systems admit many degrees of freedom (or components). We treat the case of an unique component. The generalization to composite fields is straightforward as it will appear in the examples below. Thus for the particle in space, an history corresponds to each coordinate as $C: \ t \to q^i(t)$; for a *scalar* field in space-time, $C: \ (x^\mu) \to C(x^\mu)$ generally written $\varphi(x^\mu)$; for a composite field, one history for each component ϕ^A.

An history (or field-history, or configuration), that we always write C, is a section [2] of the configuration bundle $\mathbf{C}$: a function on $\mathcal{D}$ for the particle or for the scalar field; but, more generally, a differential form on $\mathcal{D}$ like in

[1] We define the non covariant forms $\mathrm{Vol} \stackrel{\mathrm{def}}{=} \mathrm{d}^n x$, $\mathrm{Vol}_\mu \stackrel{\mathrm{def}}{=} \partial_\mu \lrcorner \mathrm{Vol}$ and $\mathrm{Vol}_{\mu\alpha} \stackrel{\mathrm{def}}{=} \partial_\alpha, \partial_\mu \lrcorner$ Vol in $\mathcal{D}$, as usual.

[2] For mathematical conditions imposed on them, see, *e.g.*, [1].

electromagnetism or in tetrad general relativity (see below). Thus, the *space of histories* $\mathcal{C} = \mathrm{Sect}(\mathbf{C})$, or possibly a subset of it. The histories which obey the dynamical equations are the particular solutions.

2.2 The Phase Space Bundle and Hamiltonian Histories

Given an history C, the corresponding *velocity-history* is its first jet extension (or prolongation), the pair $C_V \stackrel{\text{def}}{=} jC \stackrel{\text{def}}{=} (C, \mathrm{d}C)$ (with d the exterior derivative in $\mathcal{D}$) or (C, C_μ) in components. This is a section of the first jet bundle $\mathcal{J}\mathbf{C}$. In paper I, we have developed the Lagrangian historical formalism in this jet bundle (see A). [3]

Its affine dual $\mathcal{J}^*\mathbf{C} \to \mathbf{C}$. Its bundle manifold, the phase space [4], admits the adapted [Darboux] coordinates [5] x^μ, ϕ, p^μ, π. They act by duality [18] as

$$\langle (x^\mu, \phi, p^\mu, \pi), (x^\mu, \phi, v_\mu) \rangle = (p^\mu \ v_\mu + \pi) \ \mathrm{Vol}.$$

We see the polymomenta p^μ as the dual components of the (n-1)–form over $\mathcal{D}$, $p \stackrel{\text{def}}{=} p^\mu \ \mathrm{Vol}_\mu$, that we call the polymomentum [6].

The (extended) phase space bundle is the bundle $\mathbf{Y} = \mathcal{J}^*\mathcal{C} \to \mathcal{D}$. A section is a map

$$Y = (X^\mu, C, P, \Pi) : \ x^\mu \to X^\mu(x^\mu) = x^\mu, C(x^\mu), P(x^\mu), \Pi(x^\mu),$$

that we call an Hamiltonian history (hereafter H-history) [7]. The components are expressed in the table 1, where $\Omega^k_D = \Omega^k(\mathcal{D})$ is the space of k-forms on $\mathcal{D}$. We call P the *historical momentum.* The trivial maps X^μ, defined for convenience, will appear as the conjugate variables to the Π_μ.

Any H-history Y defines a n-dimensional hypersurface in the phase space, which is simply its image $\mathrm{Im}(Y)$. And Y is a diffeomorphism $\mathcal{D} \to \mathrm{Im}(Y) = Y(\mathcal{D})$. When the history is a solution, $\mathrm{Im}(Y)$ is an orbit of the evolution flux (see below).

We will work in the space of Hamiltonian histories rather than in the phase space bundle. We first define differential calculus in it.

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[3] An interesting different point of view [23] considers a field configuration as a section of the infinite jet bundle $\mathcal{J}^\infty\mathbf{C}$.

[4] Different authors use various appellations for this bundle or for its associated manifold: the covariant phase space bundle, the doubly extended phase space [7], the extended dual bundle [22], the extended multimomentum bundle [19], the *De Donder-Weyl multisymplectic manifold ...*

[5] For time dynamiccs, replace x^μ by t, ϕ by q, p^μ by p.

[6] Equivalently, the p^μ are the components of the dual polymomentum $\star p = p^\mu \ \mathrm{d}x^\mu$ (sum over indices).

[7] It is known that $\mathbf{Y}$ may also be seen as the bundle $\bigwedge^n_2 \mathrm{T}^*\mathbf{Q}$ of n-forms over $\mathbf{Q}$ which annihilates two arbitrary vertical vector-fields, see, *e.g.*, [11], [17]. In this case, p^μ and π appear as the coefficients in the expansion of such an n-form.

There is a canonical projection which projects it out to the *linear* dual [7] $\widetilde{\mathbf{Y}}$, forming the line bundle [22]

$$\rho : \ \mathbf{Y} \to \widetilde{\mathbf{Y}} : \ (x^\mu, \varphi, p^\mu, \pi) \to (x^\mu, \varphi, p^\mu).$$

Interestingly [22, 3, 8], the (scalar) Hamiltonian may be seen as a section $\widetilde{h}$ of that bundle, which defines the function H on $\widetilde{\mathbf{Y}}$ through

$$\widetilde{h}(x^\mu, \varphi, p^\mu) = \ (x^\mu, \varphi, p^\mu, H \ (x^\mu, \varphi, p^\mu)) \, .$$

It is equivalent to work in $\mathbf{Y}$ or in $\widetilde{\mathbf{Y}}$. Both are polysymplectic. For the relation between both approaches, see also [3, 22, 8].

Table 1: The components of an hamiltonian history

C	$\mathcal{D} \to \Omega^0_D$	$(x^\mu) \to C(x^\mu)$
$P = P^\nu \ \mathrm{Vol}_\nu$ P^ν	$\mathcal{D} \to \Omega^{n-1}_D$ $\mathcal{D} \to \Omega^0_D$	$(x^\mu) \to P(x^\mu)$ $(x^\mu) \to P^\nu(x^\mu)$
Π_ν $\Pi = \Pi_\nu \ \mathrm{d}x^\nu$	$\mathcal{D} \to \Omega^{n-1}_D$ $\mathcal{D} \to \Omega^n_D$	$(x^\mu) \to \Pi_\nu(x^\mu)$ $(x^\mu) \to \Pi(x^\mu)$
$X^\nu \ \mathrm{d}x^\nu$	$\mathcal{D} \to \Omega^0_D$	$(x^\mu) \to X^\nu(x^\mu) = x^\nu$

2.3 Extension to form-fields

Our formalism applies equally well in the case where a field-history C is a r-form, rather than a function (0-form), on $\mathcal{D}$. We treat explicitly the case $r = 1$. This applies to electromagnetism, where C corresponds to the Maxwell potential A; or to general relativity in tetrad formalism, where histories correspond to the cotetrad fields e^I and to the connection forms ω^{IJ}, see below. We do not consider separately the components of a form-field, but we treat it globally as an history C as in the table 2.

The scalar case corresponds to $r = 0$. When $r > 1$, the treatment is similar, with indices replaced by multi-indices (see paper I, and appendix C). The table 2 presents the components of an Hamiltonian history in the case $r = 1$. **In all formula, juxtaposition implies wedge product in $\mathcal{D}$.** We calculate now in the infinite dimensional space $\mathcal{Y}$ of Hamilton–histories.

Table 2: The components of an hamiltonian history

$C = C_\alpha \mathrm{d}x^\alpha$	$\mathcal{D} \to \Omega^r_D$	$(x^\mu) \to C(x^\mu) = C_\alpha(x^\mu) \ \mathrm{d}x^\alpha$
$P = P^{\mu\alpha} \ \mathrm{Vol}_{\mu\alpha}$	$\mathcal{D} \to \Omega^{n-1-r}_D$	$(x^\mu) \to P(x^\mu) = P^{\mu\alpha}(x^\mu) \ \mathrm{Vol}_{\mu\alpha}$

2.4 Differential Calculus with Hamiltonian histories

An Hamiltonian history (H-history) Y is a section of the bundle $\mathbf{Y}$. We call $\mathcal{Y}$ the infinite dimensional space of H-histories, and we construct differential calculus on it [8]. We represent such a section (a H-history, a " point " of $\mathcal{Y}$) as

$$Y = (X, C, P, \Pi) = (Y^A),$$

where we treat the $Y^A = X, C, P, \Pi$ (with $A = 1, 2, 3, 4$) like four coordinates in $\mathcal{Y}$[9].

We generalize the notions of functions, vector-fields, differential forms... to H-maps, H-vector-fields, H-forms. This appears necessary to define a correct calculus. A H–map is an application

$$F : \mathcal{Y} \to \Omega(M) : Y = (Y^A) \to F(Y) = F(Y^A).$$

When $F(Y) \in \Omega^R(M)$, we call F a [0;R]-map. The Hamiltonian functional $\mathcal{H}$ will appear as a particular [0,n]–map. We write $\mathcal{C}(\mathcal{Y}) = \Omega^0(\mathcal{Y})$ the space of H-maps.

Hereafter, juxtaposition of H-maps will mean their wedge product on $\mathcal{D}$, always implicit. This gives to $\mathcal{C}(\mathcal{Y})$ an algebra structure. Also, the differential calculus on $\mathcal{D}$ is easily lifted to $\mathcal{Y}$ through the formula

$$(\mathrm{d}F)(Y) = \mathrm{d}(F(Y)).$$

We call occasionally d the *horizontal* derivative, but we do not consider it as part of the proper differential calculus on $\mathcal{Y}$. We introduce below a genuine external derivative D in $\mathcal{Y}$, different from d and commuting with it. This is analog to the double complex structure introduced by [4].

2.4.1 Derivations are vector-fields

We first define derivations of H-maps w.r.t. their arguments Y^A, under the form of basic partial derivative operators $\partial_A = \frac{\partial}{\partial Y^A}$ acting on $\mathcal{Y}$. This is accomplished through the variation formula (wedge product in $\mathcal{D}$ assumed)

$$\delta F = \frac{\partial F}{\partial Y^A}\ \delta Y^A = \frac{\partial F}{\partial X}\ \delta X + \frac{\partial C}{\partial X}\ \delta C + \frac{\partial F}{\partial P}\ \delta P + \frac{\partial F}{\partial \Pi}\ \delta \Pi \tag{1}$$

$$= \frac{\partial F}{\partial X^\mu}\ \delta X^\mu + \frac{\partial C}{\partial X}\ \delta C + \frac{\partial F}{\partial P}\ \delta P + \frac{\partial F}{\partial \Pi_\mu}\ \delta \Pi_\mu,$$

corresponding to the general variation of a H-history

$$\delta Y = (\delta X, \delta C, \delta P, \delta \Pi) = (\delta Y^A).$$

We call the operators ∂_A the basic H-vector-fields in $\mathcal{Y}$. The general H-vector-field on $\mathcal{Y}$ is $V = V^A\ \partial_A$, whose components $V^A \in \mathcal{C}(\mathcal{Y})$ are arbitrary H-maps. It acts on an H-map F, as $V(F) = V^A\ \frac{\partial F}{\partial Y^A}$ (wedge product in $\mathcal{D}$ still implicit) [10].

[8] Note that similar approaches ([30], [4]) consider elements of $\Omega(\mathrm{Sect}(\Omega_D \times \mathcal{D})$.

[9] X holds for the four X^μ; C holds for the infinite set of values $C(x)$ (or $C_\mu(x)$ if $r \neq 0$). Our notation allows us to manage this infinite set like one unique coordinate; similarly with P.

[10] This requires some conditions on the grade of F that we do not detail here.

2.4.2 H-forms

We define [differential] H-forms in $\mathcal{Y}$ through duality. First the *basis one-forms* $\mathrm{D}Y^A$ – which mean the collection $\mathrm{D}X^\mu, \mathrm{D}C, \mathrm{D}P, \mathrm{D}\Pi_\mu$ – through their actions on an arbitrary H-vector-field,

$$\langle \mathrm{D}Y^A, V\rangle = V^A.$$

The *general one-H-form* expands as

$$\alpha = \alpha_A \ \mathrm{D}Y^A,$$

whose components $\alpha_A \in \mathcal{C}(\mathcal{Y})$ are arbitrary H-maps (sum over repeated indices is always assumed). We have

$$\langle \alpha, V\rangle = \alpha_A \ V^A;$$

and the exterior derivative of a H–map F

$$\mathrm{D}F = \frac{\partial F}{\partial Y^A} \ \mathrm{D}Y^A.$$

This is just an other way to write equ.(1), after realizing that a variation of a H-history is simply the action of a H-vector-field $\delta = \delta^A \ \partial_A$ on it, namely

$$\delta Y^A = \delta(Y^A) = \langle \mathrm{D}Y^A, \delta\rangle = \delta^A$$

(this requires δ^A to be of the same grade than Y^A). When F is a [0,R]-map, we call $\mathrm{D}F$ a [1,R]-H-form; [0,R]-maps are [0,R]-H-forms.

The wedge product of H-forms, $\wedge$ (not to be confused with the wedge product on $\mathcal{D}$ which is always implicit), is defined as antisymmetrized tensor product, as usual. It generates [2,R]-H-forms, *etc.* The external derivative D also applies to [k,R]-forms and generates [k+1,R]-forms. Thus we have the rules expressed in table 3. Contraction of H-vector-fields with H-form is as usual.

Table 3: Differentials of H-forms

([k;R]-form) $\wedge$ ([k';R']-form) = [k+k';R+R']-form
d ([k;R]-form) = [k;R+1]-form ;
D ([k;R]-form) = [k+1;R]-form .

We have for instance

$$\mathrm{D}\mathcal{H} = \frac{\partial \mathcal{H}}{\partial X^\mu} \ \mathrm{D}X^\mu + \frac{\partial \mathcal{H}}{\partial C} \ \mathrm{D}Y^C + \frac{\partial \mathcal{H}}{\partial P} \ \mathrm{D}Y^P + \frac{\partial \mathcal{H}}{\partial \Pi_\mu} \ \mathrm{D}Y^{\Pi_\mu}$$

$$= \frac{\partial \mathcal{H}_0}{\partial C} \ \mathrm{D}Y^C + \frac{\partial \mathcal{H}_0}{\partial P} \ \mathrm{D}Y^P + \mathrm{d}x^\mu \ \mathrm{D}Y^{\Pi_\mu}$$

where we used equ.(2) in the last term.

These formulas also apply equally well in the case where the histories are not scalar, *i.e.*, [0,r]-histories rather than [0,0]-histories. We give in appendix B their explicit development for one-form valued histories, *i.e.*, [0,1]-histories rather than [0,0]-histories. They generalize easily to the general case of [0,r]-histories. We give in table 4 the grades of the different H-maps and H-forms involved (scalar case corresponds to $r = 0$).

Table 4: The grades of the H-maps and H-forms

H-form	$\mathcal{H}$	$\frac{\partial \mathcal{H}}{\partial C}$	$\frac{\partial \mathcal{H}}{\partial P}$	$\mathrm{D}\mathcal{H}$	Ω
grade	[0;n]	[0,n-r]	[0;r+1]	[1;n]	[2;n-1]

3 Dynamics and evolution

3.1 The symplectic H-form

The space of histories $\mathcal{Y}$ admits the canonical [1;n-1]-Hform

$$\Theta \stackrel{\mathrm{def}}{=} P\ \mathrm{D}C + \Pi_\mu\ \mathrm{D}X^\mu = P^\mu\ \mathrm{D}C\ \mathrm{Vol}_\mu + \pi\ \mathrm{D}X^\mu\ \mathrm{Vol}_\mu,$$

that we call the Poincaré-Cartan H-form [18]. Its [vertical] exterior derivative

$$\Omega \stackrel{\mathrm{def}}{=} \mathrm{D}\Theta = \mathrm{D}P \wedge \mathrm{D}C + \mathrm{D}\Pi_\mu \wedge\ \mathrm{D}X^\mu,$$

is a closed and non degenerate [2;n-1]–form on $\mathcal{Y}$, that we call the *symplectic H–form.* For a 0-history, $\Omega = (\mathrm{D}P^\mu \wedge \mathrm{D}C)\ \mathrm{Vol}_\mu$; for a 1-history, $\Omega = (\mathrm{D}P^{\alpha\mu} \wedge \mathrm{D}C_\alpha)\ \mathrm{Vol}_\mu$. For $r > 1$, the same formulas hold, with indices replaced by multi–indices. We will see below that it allows us to construct a genuine (scalar-valued) symplectic form *in the space of solutions*, which identifies to that of [29].

Under some conditions, a H–map F admits a *symplectic gradient* $\nabla_\Omega F$, a H-vector-field defined through

$$\nabla_\Omega F \lrcorner\, \Omega = \mathrm{D}F.$$

3.2 Evolution

The Dynamics is described by the *historical Hamiltonian*[11]

$$\mathcal{H} = h\ \mathrm{Vol} = H_0(C,P) + \Pi_\mu\ \mathrm{d}x^\mu. \tag{2}$$

This is a [0;n]-H-map $\mathcal{H}:\ \mathcal{Y} \to \Omega^n_D$.

[11] see appendix A for obtaining it as a result of a Legendre transform.

The *evolution vector-field* is defined as its symplectic gradient $Z = \nabla_\Omega \mathcal{H}$:

$$Z \lrcorner \Omega = \mathrm{D}\mathcal{H}.$$

We emphasize that there is no analog in the multisymplectic formalism. It expands as $Z = Z^A\ \partial_A$, and the equation above gives its components through $Z^A\ \Omega_{AB} = \frac{\partial \mathcal{H}}{\partial Y^B}$, with explicit solution

$$Z^P = -\frac{\partial \mathcal{H}}{\partial C}, \quad Z^C = \frac{\partial \mathcal{H}}{\partial P}, \qquad Z^{X^\mu} = \frac{\partial \mathcal{H}}{\partial \Pi_\mu} = \mathrm{d}x^\mu, \qquad Z^{\Pi_\mu} = -\frac{\partial \mathcal{H}}{\partial X^\mu} = 0 \quad (3)$$

(note the difference between x^μ and X^μ), where Z^P and Z^C are [0;n-r]- and [0,r+1]–Hmaps respectively. This evolution vector-field acts as a derivation operator on any H–map F, giving the H–map

$$Z(F) = Z^A\ \frac{\partial F}{\partial Y^A},$$

with the components given in equ.(3); in particular the derivatives of the “ coordinates ”, $Z(Y^A) = Z^A$. In particular $Z(X^\mu) = \mathrm{d}x^\mu$, $Z(\Pi^\mu) = 0$.

3.3 The dynamical solution

An H–history $Y = (Y^A)$ is a real motion (solution) when the evolution vector-field is tangent to it. This means $\mathrm{d}Y^A = Z(Y^A) = Z^A$, *i.e.*, using (3)

$$\boxed{\mathrm{d}C = \frac{\partial \mathcal{H}}{\partial P}; \quad \mathrm{d}P = -\frac{\partial \mathcal{H}}{\partial C}, \qquad \mathrm{d}X^\mu = \mathrm{d}x^\mu, \qquad \mathrm{d}\Pi_\mu = 0.} \qquad (4)$$

The two last are identities. We recall that d is the (horizontal) exterior derivative in $\mathcal{D}$, not be confused with exterior derivative D in $\mathcal{Y}$.

This “ historical ” version of the Hamilton–De Donder–Weyl equations applies to tD as well to FT. We show below that it leads to the usual dynamical equations. It includes the case where the field is a form rather than a map (*e.g.*, electromagnetism or general relativity), as we show in applications below. For a multi–component history (field) it holds for each component.

It is easy to check that the previous equations insures stationarity of the action $\int_\mathcal{D}\ \mathcal{L}$, with the Lagrangian H–map (see paper I)

$$\mathcal{L} = P\ \mathrm{d}C - \mathcal{H}.$$

Namely, using the commutativity between d and D,

$$\mathrm{D}\mathcal{L} = \mathrm{D}(P\ \mathrm{d}C - \mathcal{H}) = \mathrm{D}P\ \mathrm{d}C - P\ \mathrm{Dd}C - \mathrm{D}\mathcal{H} =$$

$$= \mathrm{D}P\ \mathrm{d}C - \epsilon(\mathrm{d}(P\ \mathrm{D}C) - \mathrm{d}P\ \mathrm{D}C) - (\frac{\partial \mathcal{H}}{\partial C}\ \mathrm{D}C + \frac{\partial \mathcal{H}}{\partial P}\ \mathrm{D}P).$$

Inserting the motion equations above, this reduces to $\mathrm{D}\mathcal{L} = -\mathrm{d}(P\ \mathrm{D}c)$, an exact form in $\mathcal{D}$ which gives zero contribution to the integral, QED.

4 Illustrations

4.1 Application to time Dynamics

In usual dynamics, $\mathcal{D}$ is the time line, Vol = dt, $\Pi = \pi$ dt and $\Pi_\mu = \Pi_t = \pi$. Then

$$\mathcal{H} = \frac{1}{2} \star P \ P + U(C) \ \mathrm{d}t + \pi \ \mathrm{d}t = h \ \mathrm{d}t,$$

with $h = \frac{1}{2} \ P \ P \ + U(C) \ + \pi$ the usual Hamiltonian *function*; $C = q$ and $P = p$ are zero-forms ($r = 0$). Then $\frac{\partial \mathcal{H}}{\partial C} = \frac{\partial h}{\partial C}$ d$t = U'(C)$ dt and $\frac{\partial \mathcal{H}}{\partial P} = \frac{\partial h}{\partial P}$ d$t = P$ dt are both [0;1]-Hmaps. $\Omega = \mathrm{D}P \wedge \mathrm{D}C + \mathrm{D}\Pi \wedge \mathrm{D}T$ is a [2,0]-form (a genuine scalar valued symplectic form).

Then, (4) immediately gives the usual Hamilton equations (we reintroduce the familiar notations):

$$\mathrm{d}C = \dot{C} \ \mathrm{d}t = \frac{\partial \mathcal{H}}{\partial P} = \frac{\partial h}{\partial P} \ \mathrm{d}t \implies \dot{C} = \dot{q} = \frac{\partial h}{\partial p}.$$

$$\mathrm{d}P = \dot{P} \ \mathrm{d}t = -\frac{\partial \mathcal{H}}{\partial c} = -\frac{\partial h}{\partial c} \ \mathrm{d}t \implies \dot{p} = -\frac{\partial h}{\partial c};$$

with

$$\mathrm{d}T = \frac{\partial \mathcal{H}}{\partial \Pi_t} = \mathrm{d}t; \quad \mathrm{d}\pi = \frac{\partial \mathcal{H}}{\partial T} = 0. \tag{5}$$

4.2 Scalar field; Link with Multisymplectic

For classical field theories, $\mathcal{D} = M$ is space-time ($n = 4$). A scalar ($r = 0$) field C is usually written φ. Then $P = P^\mu$ Vol$_\mu$ is a [0,3]-Hmap, with *dual* components P^μ, $\mathcal{H} = h_0$ Vol $+ \Pi_\mu$ dx^μ is a [0,4]-Hmap. We have

$$\mathrm{d}C = C_{,\mu} \ \mathrm{d}x^\mu, \ P = P^\mu \ \mathrm{Vol}_\mu, \ \mathrm{d}P = P^\mu{}_{,\alpha} \ \mathrm{d}x^\alpha \ \mathrm{Vol}_\mu = P^\mu{}_{,\mu} \ \mathrm{Vol}.$$

Then $\frac{\partial \mathcal{H}}{\partial P} = \frac{\partial h}{\partial P^\mu}$ dx^μ is a [0,1]-Hmap; $\frac{\partial \mathcal{H}}{\partial C} = \frac{\partial h}{\partial C}$ Vol is a [0;4]-Hmap. The symplectic [2,3]-Hform

$$\Omega = \mathrm{D}P \wedge \mathrm{D}C + \mathrm{D}\Pi_\mu \wedge \mathrm{D}X^\mu = \mathrm{D}P^\mu \wedge \mathrm{D}C \ \mathrm{Vol}_\mu + \mathrm{D}\Pi_\mu \wedge \mathrm{D}X^\mu,$$

with Vol$_\mu$ a 3-form on space-time $\mathcal{D} = M$ (not on $\mathcal{Y}$). Then, equ.(4) implies the usual Hamilton equations

$$C_{,\mu} = \frac{\partial h}{\partial P^\mu}; \qquad (P^\alpha)_{,\alpha} = -\frac{\partial h}{\partial C}.$$

Assuming the standard Hamiltonian for scalar field theories,

$$H = \frac{1}{2} \star P \ P + U(C) \ \mathrm{Vol} + \frac{1}{2} \star \Pi_\mu \ \mathrm{d}x^\mu = (\frac{1}{2} \ P^\mu \ P_\mu \ + U(C) \ \mathrm{Vol} + \frac{1}{2} \ \pi^2) \ \mathrm{Vol},$$

we obtain

$$\mathrm{d}C = C_{,\mu} \ \mathrm{d}x^\mu = \star P = P^\mu \ \mathrm{d}x^\mu \Rightarrow C_{,\mu} = P^\mu;$$

$$\mathrm{d}P = -\frac{\partial \mathcal{H}}{\partial C} = -U'(C) \ \mathrm{Vol} \implies \mathrm{d}P^\mu \ \mathrm{Vol}_\mu = -U'(C) \ \mathrm{d}x^\mu \ \mathrm{Vol}_\mu \implies P^\mu_{,\mu} = C_{,\mu\mu} = -U'(C).$$

4.2.1 Link with Multisymplectic

The multisymplectic form appears as an emanation of our symplectic H-form, as the 5-form in the phase space bundle manifold $\mathbf{Y}$ (not on S^Y),

$$\Omega_M = \bar{\mathrm{d}}p^\mu \ \bar{\wedge}\ VOL_\mu \ \bar{\wedge}\ \bar{\mathrm{d}}\varphi + \ \bar{\mathrm{d}}\pi \ \bar{\wedge}\ VOL,$$

where all forms, exterior derivative $\bar{\mathrm{d}}$ and wedge product $\bar{\wedge}$ are in the bundle manifold $\mathbf{Y}$, $VOL \stackrel{\text{def}}{=} \epsilon_{\mu\alpha\beta\gamma}\ \bar{\mathrm{d}}x^\mu \ \bar{\wedge}\ \bar{\mathrm{d}}x^\alpha \ \bar{\wedge}\ \bar{\mathrm{d}}x^\beta \ \bar{\wedge}\ \bar{\mathrm{d}}x^\gamma$ and $VOL_\mu \stackrel{\text{def}}{=} \epsilon_{\mu\alpha\beta\gamma}\ \bar{\mathrm{d}}x^\alpha \ \bar{\wedge}\ \bar{\mathrm{d}}x^\beta \ \bar{\wedge}\bar{\mathrm{d}}x^\gamma$.

4.2.2 Application to r-histories

Exactly the same formalism applies when fields are forms rather than scalar functions, with indices replaced by multi-indices (see C):

$$c = c_{\underline{\alpha}}\ \mathrm{d}x^{\underline{\alpha}},\ \mathrm{d}c = c_{\underline{\alpha},\mu}\ \mathrm{d}x^{\underline{\alpha}\mu};$$

$$P = P^{\underline{\alpha}\mu}\ \mathrm{Vol}_{\underline{\alpha}\mu},\ \mathrm{d}P = P^{\underline{\alpha}\mu}{}_{,\beta}\ \mathrm{Vol}_{\underline{\alpha}\mu}\ \mathrm{d}^\beta,\ \mathcal{H} = h\ \mathrm{Vol},$$

$$\frac{\partial\mathcal{H}}{\partial P} = \frac{\partial h}{\partial P^{\underline{\alpha}\mu}}\ \mathrm{d}^{\underline{\alpha}\mu},\ \frac{\partial\mathcal{H}}{\partial c} = \frac{\partial h}{\partial c^{\underline{\alpha}}}\ \mathrm{Vol}_{\underline{\alpha}},$$

giving the Hamilton equations

$$c_{\underline{\alpha},\mu} = \frac{\partial h}{\partial P^{\underline{\alpha}\mu}}; \qquad (P^{\underline{\alpha}\mu})_{,\mu} = -\frac{\partial h}{\partial c_{\underline{\alpha}}},$$

where all multi-indexes are antisymmetrized.

5 Conservation and symmetries

5.1 On shell conservation

Interestingly, equ.(4) implies, *on shell*,

$$\mathrm{D}\mathcal{H} = \frac{\partial\mathcal{H}}{\partial c}\ \mathrm{D}c + \frac{\partial\mathcal{H}}{\partial P}\ \mathrm{D}P \simeq \mathrm{d}c\ \mathrm{D}P - \mathrm{d}P\ \mathrm{D}c$$

$$\implies \mathrm{D}\mathrm{D}\mathcal{H} = 0 = \mathrm{D}\mathrm{d}c\ \mathrm{D}P - \mathrm{D}\mathrm{d}P\ Dc = \mathrm{d}\Omega$$

after derivation: the generalized symplectic form is conserved on shell. This is the covariant version of the on shell conservation of the symplectic current in the multisymplectic formalism.

Since the value of Ω is a $(n-1)$-form on $\mathcal{D}$, it can be integrated along a 1-codimensional hypersurface of $\mathcal{D}$. This provides a canonical scalar-valued symplectic form on the space of solutions since the on–shell conservation of Ω implies that this symplectic form does not depend on the choice of the hypersurface (assumed Cauchy for FTs). Thus, this provides a canonical (scalar valued) symplectic form on the space of histories, which identifies with that introduced by [29], so that our result may be seen as a generalization of their work and its link with the multi–symplectic formalism.

5.2 Symmetries

We recall that a solution is a H-history Y verifying $Z(Y) = \mathrm{d}Y$ or, in coordinates, $Z^A = \mathrm{d}Y^A$. Any vector-field δ (of convenient grade) defines a variation $\delta(Y)$ of that history. One may check immediately that the variation of a solution remains a solution, *i.e.*, that

$$Z(Y) = \mathrm{d}Y \implies Z(\delta(Y)) = \mathrm{d}\delta(Y).$$

- A symmetry is a Hamiltonian vector-field δ that preserves H:

  $$0 = \delta(H) = \delta \lrcorner \mathrm{D}H = \delta \lrcorner (Z \lrcorner \mathrm{D}H) = -Z \lrcorner (\delta \lrcorner \omega) = \omega(\delta, Z).$$

  In coordinates, this implies $\delta^A \ \frac{\partial H}{\partial Y^A} = 0$.

- Being Hamiltonian, δ is a symplectic gradient:

  $$\delta \lrcorner \omega = \mathrm{D}U.$$

  Then

  $$\delta(H) = -Z \lrcorner (\mathrm{D}U) = -Z(U) = 0 :$$

  the quantity U is conserved on shell.

5.3 Generalized Poisson bracket and observables

The main result here is the introduction of the historical symplectic H–form Ω. Is it possible to define a Poisson-like bracket from it ? The formula above suggests that the canonical “ variables ” are the forms C and P and that the bracket of two Hmaps could be defined as

$$\{f, g\} = \frac{\partial f}{\partial C} \ \frac{\partial g}{\partial P} - \frac{\partial g}{\partial C} \ \frac{\partial f}{\partial P} = X_f \lrcorner \mathrm{D}g,$$

involving the multisyplectic gradient X_f such that $X_f \lrcorner \Omega = \mathrm{D}f$.

Table 5: The types of the Hmaps and Hforms involved

c	P	$f, \mathrm{D}f$	$g, \mathrm{D}g$	Ω	X_f	$\{f, g\}$
[0; r]	[0;n-r-1]	[0;R],[1;R]	[0;S],[1;S]	[2,n-1]	[-1;R+1-n]	[0,S+R+1-n]

We give in table 5 the grades of the various quantities involved. The grade [-1;R+1-n] for the vector-field indicates that the inner product with a [1;K]–Hform gives a [0;R+1-n+K]–Hmap. This definition requires that the quantities involved are well defined and we restrict the validity of our bracket to such cases. This occurs when f and g have both degrees greater or equal to those of c and P, namely r and $n - r - 1$; or, alternatively, when f or g does not depend on the “ canonical variables ”. To illustrate, we have

$$\{P, c\} = 1;$$

$$\{\mathcal{H}, c\} = \frac{\partial \mathcal{H}}{\partial P} = \mathrm{d}c;$$
$$\{\mathcal{H}, P\} = -\frac{\partial \mathcal{H}}{\partial c} = \mathrm{d}P.$$

These formulas validate the definition of our bracket. It is a generalization of that proposed by [15].

It is defined for Hmaps, whose values are forms, rather than scalar functions. However, an observable is generally considered as scalar-valued, not form-valued. But any form provides a scalar by integration over a submanifold of adapted dimension. Thus, it seems a convenient point of view to consider generalized observables as form-valued, from which *non–local* scalar observables are extracted through integration over intermediary submanifolds. This corresponds indeed to what is done in Loop Quantum Gravity through the introduction of the Holonomy-Flux algebra.

The observables which commute with the Hamiltonian and with the constraints correspond to the *complete observables* in the sense of [20, 5] (see also [28]).

6 Application to electromagnetism

The usual treatment of electromagnetism considers the components A_μ of the electromagnetic form A as the dynamical variables, with the scalar Lagrangian $L = \frac{1}{2}\ F^{\mu\nu}\ F_{\mu\nu}$, where $F_{\mu\nu} \stackrel{\text{def}}{=} \partial_\mu A_\nu - \partial_\nu A_\mu$. Indices are lowered / raised with the fixed flat Minkowski metric.

1) The usual (non covariant) analysis proceeds by fixing one time coordinate $t = x^0$, so that

$$L = F^{0i}\ (\partial_0 A_i - \partial_i A_0) + \tfrac{1}{2}\ F^{ij}\ (\partial_i A_j - \partial_j A_i).$$

We obtain the conjugate momenta $P^0 \stackrel{\text{def}}{=} \frac{\partial \mathcal{L}}{\partial \dot{A}_0} = 0$ and $P^i = F^{0i} = \partial_0 A_i - \partial_i A_0$. The first relation appears as the primary constraint $P^0 = 0$ and the second inverts as $\partial_0 A_i = P^i + \partial_i A_0$. Applying a partial Legendre transform leads to the Hamiltonian

$$\begin{aligned} H &= \lambda\ P^0 + \dot{A}_i\ P^i - [P^i\ P^i + \tfrac{1}{2}\ F^{ij}\ (\partial_i A_j - \partial_j A_i)] \\ &= \lambda\ P^0 + (\partial_i A_0)\ P^i - \tfrac{1}{2}\ F^{ij}\ (\partial_i A_j - \partial_j A_i). \end{aligned}$$

The primary constraint is second class and generates the secondary constraint $(P^i)_{,i} = (F^{0i})_{,i} = 0$: the Gauss law. Finally, the motion equations give $\dot{A}_i = (A_0)_i$ and $\dot{P}^i = -(Fij)_j$. This may be synthetized in $F^{\mu\nu}_{,\nu} = 0$.

2) The (covariant) multisymplectic analysis starts from the same Lagrangian and, now, associates to each variable A_μ the four polymomentum components $p^{\mu\nu} = \frac{1}{2}\ (F_{\mu\nu} - F_{\nu\mu})$. They obey the constraints $C_{\mu\nu} = p_{\mu\nu} + p_{\nu\mu} = 0$. The Hamiltonian

$$\lambda^{\mu\nu}\ C_{\mu\nu} - \tfrac{1}{2}\ p^{\mu\nu}\ p_{\mu\nu},$$

leads to the usual equations, via a multisymplectic analysis (see, *e.g.*[26]).

3) Adopting our formalism, we write $\mathcal{L} = L\ \mathrm{Vol} = \frac{1}{2}\ \mathrm{d}A\ \star\mathrm{d}A$ so that $P = \star\mathrm{d}A$, which inverts as $\mathrm{d}A = \star P$: there is no constraint and our Hamiltonian takes the form $\mathcal{H} = \frac{1}{2}\ P\ \star P$.

This gives the motion equations

$$\mathrm{d}A = \frac{\partial \mathcal{H}}{\partial P} = \star P;$$
$$\mathrm{d}P = 0,$$

which condense into $\mathrm{d}\star\mathrm{d}A = 0$.

7 Application to canonical gravity

7.1 Dynamics in the first order formalism

The dynamical variables are the cotetrad components e^I and the Lorentz connection forms ω^{IJ}, with conjugated polymomenta P_I and Π_{IJ}. We calculated them in paper I, namely $P_I = 0$ and

$$\Pi_{IJ} = \mathcal{P}_{IJ} \stackrel{\text{def}}{=} \epsilon_{IJKL}\, e^{KL}. \tag{6}$$

They generate primary constraints and we write the Hamiltonian H-form

$$\begin{aligned}\mathcal{H} &= P_I\, V^I + (\Pi_{KL} - \mathcal{P}_{KL})\, W^{KL} + P_I\, \mathrm{d}e^I + \Pi_{KL}\, \mathrm{d}\omega^{KL} - \epsilon_{IJKL}\, e^I\, e^J\, (\mathrm{d}\,\omega^{KL} + (\omega\omega)^{KL}) \\ &= P_I\, V^I + (\Pi_{KL} - \mathcal{P}_{KL})\, W^{KL} + P_I\, \mathrm{d}e^I - \Pi_{KL}(\omega\omega)^{KL}\end{aligned}$$

with the Lagrange multipliers V^I and W^{KL}.

The development of equ.(4) gives the motion equations:

- $$\mathrm{d}e^I = \frac{\partial \mathcal{H}}{\partial P_I} = V^I + \mathrm{d}e^I,$$
  giving $V^I = 0$;
- $$\mathrm{d}\omega^{IJ} = \frac{\partial \mathcal{H}}{\partial \Pi_{IJ}} = W^{IJ} - (\omega\omega)^{IJ}; \tag{7}$$
- $$\mathrm{d}P_I = -\frac{\partial \mathcal{H}}{\partial e^I} = 2\, \epsilon_{KLIJ}\, e^J\, W^{KL}. \tag{8}$$
  The two latter combine to give the secondary constraint
  $\mathrm{d}P^I = 0 = 2\, \epsilon_{KLIJ}\, e^J\, (\mathrm{d}\omega^{KL} + (\omega\omega)^{KL})$, leading to zero Ricci curvature;
- $$\mathrm{d}\Pi_{IJ} = -\frac{\partial \mathcal{H}}{\partial \omega^{IJ}} = 2\, \epsilon_{PQK[I}\, e^{PQ}\, \omega^K{}_{J]} = 2\, \epsilon_{NKIJ}\, e^{NM}\, \omega^K{}_M \tag{9}$$
  appears as a secondary constraint giving zero torsion, as can be checked using identities 6 and 16.

Appendices

A Outline of paper I

A.1 Velocity–Histories

Histories and velocity-histories are defined as in the text (2.2). We call $\mathcal{S}_V \subset \mathrm{Sect}(\mathbf{V})$ the space of velocity-histories (technically, an *exterior differential system* [3]). Since $\mathcal{J}$ is canonical, there is a one-to-one correspondence between histories and velocity-histories.

We express the Lagrangian dynamics in $\mathcal{S}_V$ rather than in the jet bundle itself. We treat $\mathcal{S}_V$ like an infinite dimensional manifold where C and $\mathrm{d}C$ play the role of coordinates. We define H-maps $\mathcal{S}_V \to \Omega_D$ as generalizations of functions. They form the algebra $\Omega^0(\mathcal{S}_V)$, and we have defined derivations w.r.t. their arguments C and $\mathrm{d}C$. We have also defined differential forms on $\mathcal{S}_V$, forming the spaces $\Omega^r(\mathcal{S}_V)$, and an exterior derivative $\mathrm{D}:\ \Omega^r(\mathcal{S}_V) \to \Omega^{r+1}(\mathcal{S}_V)$, which commutes with d (occasionally called the *horizontal* exterior derivative).

Dynamics is described through the *Lagrangian functional*

$$\mathcal{L}:\ \mathcal{S}_V \to \Omega_D^n:\ C_V \overset{\mathrm{def}}{=} (C, \mathrm{d}C) \to \mathcal{L}(C_V),$$

a H-map over $\mathcal{S}_V$, of type [0,n].

We define the *historical momentum*

$$P \overset{\mathrm{def}}{=} \frac{\partial \mathcal{L}}{\partial(\mathrm{d}C)} = P^{\underline{\mu}}\ \mathrm{Vol}_{\underline{\mu}} \tag{10}$$

as a [0;n-r-1]-Hmap admitting the dual components $P^{\underline{\mu}}$. This formula is written with multi-indexes (see paper I); they reduce to ordinary indices when C is a 0-history; to an antisymmetric pair of indices when C is a 1-history.

Then, applying our differential calculus, we have (wedge products between forms in $\mathcal{D}$ are implicitly assumed)

$$\mathrm{D}\mathcal{L} = \mathrm{D}C\ \frac{\partial \mathcal{L}}{\partial C} + \mathrm{D}(\mathrm{d}C)\ P = \mathrm{D}C\ \left(\frac{\delta^{EL}\mathcal{L}}{\delta C}\right) - \mathrm{d}\Theta. \tag{11}$$

We have defined the EL derivative

$$\frac{\delta^{EL}\mathcal{L}}{\delta C} \overset{\mathrm{def}}{=} \frac{\partial \mathcal{L}}{\partial C} - \epsilon_c\ \mathrm{d}\frac{\partial \mathcal{L}}{\partial(\mathrm{d}C)}, \tag{12}$$

with $\epsilon_c = (-1)^{\text{grade of } C}$; and also the *historical Lagrange form* (or Lagrange H-form) as the [1; n-1]-form

$$\Theta \overset{\mathrm{def}}{=} -\mathrm{D}C\ P = \mathrm{D}C\ \frac{\partial \mathcal{L}}{\partial(\mathrm{d}C)}.$$

The latter gives by derivation the [2; n-1]-form $\mathrm{D}\Theta = \mathrm{D}P \wedge \mathrm{D}C$ (implicit wedge product in $\mathcal{D}$) which we call the *symplectic H-form* (and Θ the generalized symplectic potential). This is the *historical version* of the symplectic structure on $\mathrm{T}M$ (see, *e.g.*[16, 2]). [12]

[12] or of the pre-sympletic structure of the evolution space.

An arbitrary variation of an history is seen as the result of the application of a vector-field δ in $\mathcal{S}_V$ as

$$\delta C = \langle \mathrm{D}C, \delta \rangle; \quad \delta(\mathrm{d}C) = \langle \mathrm{D}(\mathrm{d}C), \delta \rangle.$$

This leads to equ.(11). Since the last term in this equation does not contribute to the action, stationarity corresponds to the Euler-Lagrange equation $\frac{\delta^{EL}\mathcal{L}}{\delta C} = 0$.

These equations are explicitely covariant. They apply equally well to tD and FT's, and they include the case where the C is a r-history, *i.e.*, a form rather than a function.

A.2 Symmetries

A vector-field δ is a symmetry generator when it does not modifies the action. This means that it modifies $\mathcal{L}$ by an exact form (in $\mathcal{D}$) $\mathrm{d}X$ only. Hence, for a symmetry,

$$\delta C \ (\frac{\delta^{EL}\mathcal{L}}{\delta C}) - \mathrm{d}(\delta C \ P) = \mathrm{d}X.$$

Defining the *Noether current* ([n-1]-H-map) $j \stackrel{\text{def}}{=} X + \delta C \ P$, we have the conservation law

$$\mathrm{d}j = \delta C \ \frac{\delta^{EL}\mathcal{L}}{\delta C} \ \simeq \ 0 \text{ (on shell)}.$$

Locally, $j = \mathrm{d}Q$ which defines the *Noether charge density* $(n-2)$-H-map Q [27].

A diffeomorphism of $\mathcal{D}$ is obviously a symmetry since in that case $\delta\mathcal{L} = L_\zeta \mathcal{L} = \mathrm{d}(\zeta \lrcorner \mathcal{L})$, where ζ is the generator.

The historical Legendre transform (see below) will allow the change of variables $(C, \mathrm{d}C) \rightsquigarrow (C, P)$ at the basis of the Hamiltonian formalism.

A.3 Legendre transform

The (usual) Legendre transform transports the dynamics from $\mathbf{V}$ to $\mathbf{Y}$. It is defined as the fiber-preserving map [9]

$$T_L : \ \mathbf{V} \to \mathbf{Y} : \ (x^\mu, \varphi, v_\mu) \rightsquigarrow (x^\mu, \varphi, p^\mu, \pi),$$

here for a scalar field, in adapted coordinates. [13] It may be non invertible, what is expressed by primary constraints. We assume now a non degenerate Legendre transform, constraints are discussed in the examples.

We lift the Legendre transform to the *historical Legendre map* which applies to sections, the duality

$$\mathcal{T}_L : \ \mathcal{S}_V \to \mathcal{Y} : \ C = (C, \mathrm{d}C) \rightsquigarrow Y = (C, P)$$

between velocity-histories and Hamiltonian histories. This results from the simple remark that a fiber-preserving map between fiber bundles induces a map between their spaces of sections. Concretely, the velocity history C_V is transformed, by composition with T_L, as $Y = T_L \circ C_V$.

[13] It admits a *restricted* version

$$\mathbf{V} \rightsquigarrow \widetilde{\mathbf{Y}} : \ (x^\mu, \varphi, v_\mu) \rightsquigarrow (x^\mu, \varphi, p^\mu).$$

A.4 The historical Hamiltonian

We define the *historical Hamiltonian* on the historical phase space

$$\mathcal{H}: \mathcal{Y} \to \Omega^n(\mathcal{D}): Y \to \mathcal{H}(Y) = \Lambda^i \Gamma_i + \Pi \text{ Vol} + P \text{ d}C - \mathcal{L} \qquad (13)$$

(wedge product assumed). In this expression, $\text{d}C$ and $\mathcal{L}$ are expressed as functionals of C and P, as far as allowed by inversion of the Legendre map, so that $\mathcal{H}$ is a [0;n]–Hmap. The Λ^i and Γ_i are Lagrange multipliers and constraints, which are now defined as Hmaps also (see illustrations in examples). This definition holds for r–histories.

B Details of Calculations

- For a *scalar field*, a field-history is a zero-form C on $\mathcal{D}$. The momentum is a 3 form $P = P^\mu \text{ Vol}_\mu$ (its Hodge dual $\star P = P^\mu \text{ d}x^\mu$ is a 1-form).

  The Hamiltonian functional is a [0;4]-H-map $\mathcal{H} = h$ Vol. Its external derivative

  $$\text{D}\mathcal{H} = \frac{\partial \mathcal{H}}{\partial C} \text{ D}C + \frac{\partial \mathcal{H}}{\partial P} \text{ D}P... = \frac{\partial h}{\partial C} \text{ D}C \text{ Vol} + \frac{\partial h}{\partial P^\mu} \text{ D}P^\mu \text{ Vol}...,$$

  so that
  $$\frac{\partial \mathcal{H}}{\partial C} = \frac{\partial h}{\partial C} \text{ Vol}, \qquad \frac{\partial \mathcal{H}}{\partial P} = \frac{\partial h}{\partial P^\mu} \text{ d}x^\mu.$$

- For a *one-form field*, a field–history is a one form $C = C_\alpha \text{ d}x^\alpha$ on $\mathcal{D}$. The momentum is a 2 form $P = P^{\alpha\mu} \text{ Vol}_{\alpha\mu}$.

  The Hamiltonian is a [0;4]-H-map $\mathcal{H} = h$ Vol. Its external derivative

  $$\text{D}\mathcal{H} = \frac{\partial \mathcal{H}}{\partial C} \text{ D}C + \frac{\partial \mathcal{H}}{\partial P} \text{ D}P... = \frac{\partial h}{\partial C_\alpha} \text{ D}C_\alpha \text{ Vol} + \frac{\partial h}{\partial P^{\alpha\mu}} \text{ D}P^{\alpha\mu} \text{ Vol}...,$$

  so that
  $$\frac{\partial \mathcal{H}}{\partial C} = \frac{\partial h}{\partial C_\alpha} \text{ Vol}_\alpha \ , , \qquad \frac{\partial \mathcal{H}}{\partial P} = \frac{\partial h}{\partial P^{\alpha\mu}} \text{ d}x^\mu \text{ d}x^\alpha.$$

C Multi-index notations

For a r-history we write
$$C = C_{\underline{\mu}} \text{ d}^{\underline{\mu}},$$

where $\underline{\mu}$ means the (antisymmetrized) sequence $(\mu_1, ..., \mu_r)$ and $\text{d}^{\underline{\mu}}$ means $\text{d}x^{\mu_1}...\text{d}x^{\mu_r}$.

Similarly, the momentum,

$$P = P^{\underline{\nu}} \text{ Vol}_{\underline{\nu}} = P^{\underline{\nu}} \epsilon_{\underline{\nu},\underline{\rho}} \text{ d}^{\underline{\rho}}; \quad \underline{\nu} \overset{\text{def}}{=} \nu_1, ..., \nu_{r+1},$$

with $\epsilon_{\underline{\nu},\underline{\rho}} \overset{\text{def}}{=} \epsilon_{\nu_1,...,\nu_{r+1},\rho_1,...,\rho_{n-r-1}}$; $\quad \text{Vol}_{\underline{\nu}} \overset{\text{def}}{=} \partial_{\underline{\nu}} \lrcorner \text{ Vol} = \epsilon_{\underline{\nu},\underline{\rho}} \text{ d}^{\underline{\rho}}$; involving the multivector $\partial_{\underline{\nu}} = (\partial_{\nu_1}, ..., \partial_{\nu_{r+1}})$.

We expand similarly a [0, R]-Hmap as

$$F = F_{\underline{\alpha}} \text{ d}^{\underline{\alpha}}.$$

It results, *e.g.*,

$$\frac{\partial F}{\partial C} = \frac{\partial F_{\underline{\alpha}}}{\partial C_{\underline{\mu}}} \; (\partial_{\underline{\mu}} \lrcorner \mathrm{d}^{\underline{\alpha}}), \tag{14}$$

$$\frac{\partial F}{\partial P} = \epsilon^{\underline{\nu}\underline{\rho}} \;\; \frac{\partial F_{\underline{\alpha}}}{\partial P^{\underline{\nu}}} \; (\partial_{\underline{\rho}} \lrcorner \mathrm{d}^{\underline{\alpha}}). \tag{15}$$

Note that the validity of these formulas implies conditions for the grades, namely $R \geq r$ and $R \geq n-r-1$ respectively. We will restrict to such situations sufficient for our purpose, although generalizations are possible.

D An identity

To prove the identity :

$$\epsilon_{JKAB} \; e^{JI} \; \omega^{K}{}_{I} = -\epsilon_{J[A \; MN} \; e^{MN} \; \omega^{J}{}_{B]}, \tag{16}$$

we use

$$\star e^{IJ} = \tfrac{1}{2} \; \epsilon^{IJ}{}_{MN} \; e^{MN}; \;\; \tfrac{1}{2} \; e^{IJ} = -\epsilon^{IJ}{}_{MN} \; (\star e^{MN}).$$

Then,

$$\epsilon_{JKAB} \, e^{IJ} \; \omega^{K}{}_{I} = \tfrac{1}{2} \, \epsilon_{JKAB} \; \epsilon^{IJ}{}_{MN} \; (\star e^{MN}) \; \omega^{K}{}_{I} = \eta_{BC} \; (\star e^{KC}) \; \omega_{KA} - \eta_{AC} \; (\star e^{KC}) \; \omega_{KB}$$

$$= -\epsilon_{K[BMN} \; e^{MN} \; \omega^{K}{}_{A]}, \;\; QED$$

Similarly,

$$\epsilon_{IJKL} \; e^{IJ} \; \omega^{K}{}_{A} \; \omega^{AL} = -\epsilon_{JKMN} \; e^{IJ} \; \omega^{K}{}_{I} \; \omega^{MN}. \tag{17}$$

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