On level-transitivity and exponential growth

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Ines Klimann
Univ Paris Diderot, Sorbonne Paris Cité, IRIF,
UMR 8243 CNRS, F-75013 Paris, France
klimann@liafa.univ-paris-diderot.fr

Abstract

We prove that if the group generated by an invertible and reversible Mealy automaton acts level-transitively on a regular rooted tree, then the semigroup generated by the dual automaton has exponential growth, hence giving a decision procedure of exponential growth for a restricted family of automaton (semi)groups.

The purpose of this note is to link up two classes of groups and semigroups highly studied for themselves: level-transitive (semi)groups and (semi)groups of exponential growth, through automaton (semi)groups.
On the one hand, level-transitive groups (or equivalently spherically transitive groups, depending on the authors) — *i.e.* groups acting transitively on every level of a regular rooted tree — have received special focus these last years because of branch groups, which form a particular class of level-transitive groups, one of the three classes into which the class of just infinite groups is naturally decomposed [8, 3].
On the other hand, the study on how (semi)groups grow has been highlighted since Milnor's question on the existence of groups of intermediate growth in 1968 [13] and the very first example of such a group given by Grigorchuk [6].
In this note, we prove that no semigroup of polynomial or intermediate growth can be generated by an invertible and reversible Mealy automaton whose dual generates a level-transitive group. Even if the problem of deciding the level-transitivity of an automaton group is still open, there exist some families of Mealy automata for which the level-transitivity of an element in the generated semigroup is decidable [15].

1 Basic notions

1.1 Semigroups and groups generated by Mealy automata

We first recall the formal definition of an automaton. A *(finite, deterministic, and complete) automaton* is a triple $\big(Q, \Sigma, \delta = (\delta_i \colon Q \to Q)_{i\in\Sigma}\big)$, where the *state set* Q and the *alphabet* Σ are non-empty finite sets, and the δ_i are functions.

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A *Mealy automaton* is a quadruple $(Q, \Sigma, \delta, \rho)$, such that (Q, Σ, δ) and (Σ, Q, ρ) are both automata. In other terms, a Mealy automaton is a complete, deterministic, letter-to-letter transducer with the same input and output alphabet. Its *size* is the cardinality of its state set.
The graphical representation of a Mealy automaton is standard, see Figure 1.

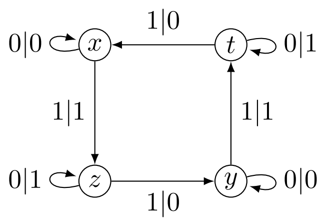

Figure 1: An example of a Mealy automaton which does not generate a free semigroup on its state set, but whose dual generates a level-transitive group.

Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ be a Mealy automaton. For each state $x \in Q$, the map $\rho_x : \Sigma \to \Sigma$ can be extended to a map $\rho_x : \Sigma^* \to \Sigma^*$ recursively defined by:

$$\forall i \in \Sigma,\ \forall \mathbf{s} \in \Sigma^*, \qquad \rho_x(i\mathbf{s}) = \rho_x(i)\rho_{\delta_i(x)}(\mathbf{s}) \ .$$

The image of the empty word is itself. The mapping ρ_x for each $x \in Q$ is length-preserving and prefix-preserving. We say that ρ_x is the function *induced* by x. For $\mathbf{x} = x_1 \cdots x_n \in Q^n$ with $n > 0$, set $\rho_{\mathbf{x}} \colon \Sigma^* \to \Sigma^*, \rho_{\mathbf{x}} = \rho_{x_n} \circ \cdots \circ \rho_{x_1}$. The semigroup of mappings from Σ^* to Σ^* generated by $\{\rho_x, x \in Q\}$ is called the *semigroup generated by* $\mathcal{A}$ and is denoted by $\langle \mathcal{A} \rangle_+$.

A Mealy automaton $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ is *invertible* if the functions ρ_x are permutations of Σ. In this case, the functions induced by the states are permutations on words of the same length and thus we may consider the group of mappings from Σ^* to Σ^* generated by $\{\rho_x, x \in Q\}$: it is called the *group generated by* $\mathcal{A}$ and is denoted by $\langle \mathcal{A} \rangle$.

In a Mealy automaton $\mathcal{A} = (Q, \Sigma, \delta, \rho)$, the sets Q and Σ play dual roles. So we may consider the *dual (Mealy) automaton* defined by $\mathfrak{d}(\mathcal{A}) = (\Sigma, Q, \rho, \delta)$. We extend to δ the former notations on ρ, in a natural way. Hence $\delta_i \colon Q^* \to Q^*, i \in \Sigma$, are the functions induced by the states of $\mathfrak{d}(\mathcal{A})$, and for $\mathbf{s} = s_1 \cdots s_n \in \Sigma^n$ with $n > 0$, we set $\delta_{\mathbf{s}} \colon Q^* \to Q^*,\ \delta_{\mathbf{s}} = \delta_{s_n} \circ \cdots \circ \delta_{s_1}$.
A Mealy automaton $(Q, \Sigma, \delta, \rho)$ is *reversible* if its dual is invertible, that is if the functions δ_i are permutations of Q. Note that a connected component of a reversible Mealy automaton is always strongly connected.

An automaton group or semigroup can be seen as acting on a regular rooted tree representing the language of all words on its alphabet.

1.2 Growth of a semigroup or of a group

Let H be a semigroup generated by a finite set S. The *length* of an element g of the semigroup, denoted by $|g|$, is the length of its shortest decomposition:

$$|g| = \min\{n \mid \exists s_1, \ldots, s_n \in S,\ g = s_1 \cdots s_n\} \ .$$

The *growth function* γ_H^S of the semigroup H with respect to the generating set S enumerates the elements of H with respect to their length:

$$\gamma_H^S(n) = \#\{g \in H\,;\, |g| \leq n\}\ .$$

The *growth functions* of a group are defined similarly by taking symmetrical generating sets.

The growth functions corresponding to two generating sets are equivalent [12], so we may define the *growth* of a group or a semigroup as the equivalence class of its growth functions. Hence, for example, a finite (semi)group has a bounded growth, while an infinite abelian (semi)group has a polynomial growth, and a non-abelian free (semi)group has an exponential growth.

It is quite easy to obtain groups of polynomial or exponential growth. Answering a question of Milnor [13], Grigorchuk gave the very first example of an automaton group of intermediate growth [6]: faster than any polynomial, slower than any exponential, opening thus a new classification criterium for groups, that has been deeply studied since this seminal article (see [7] and references therein). This example is an automaton group. It is now known as *the Grigorchuk group.*

1.3 Level-Transitivity

The action of a (semi)group generated by an invertible Mealy automaton $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ is *level-transitive* if its restriction to Σ^n has a unique orbit, for any n (this notion is equivalently called *spherically transitive* [9]). From a dual point of view it means that the powers of the dual $\mathfrak{d}(\mathcal{A})$ are connected, the *n-th power* of the automaton $\mathfrak{d}(\mathcal{A})$ being the Mealy automaton

$$\mathfrak{d}(\mathcal{A})^n = \left(\ \Sigma^n, Q, (\rho_x \colon \Sigma^n \to \Sigma^n)_{x \in Q}, (\delta_{\mathbf{s}} \colon Q \to Q)_{\mathbf{s} \in \Sigma^n}\ \right)\ .$$

Note that all the powers of a reversible Mealy automaton are reversible.
The next theorem is proved in [10]:

Theorem 1. *Let $\mathcal{A}$ be a reversible automaton with a prime number of states. If the action of $\mathfrak{d}(\mathcal{A})$ is level-transitive, then the semigroup $\langle \mathcal{A} \rangle_+$ is free on the automaton state set.*

In [10] the hypothesis of the prime number of states was erroneously conjectured to be not mandatory. In fact, the Mealy automaton of Figure 1 given by Laurent Bartholdi (personal communication) does not generate a free semigroup on its state set, even though its dual generates a level-transitive group.
Although deciding the level transitivity of an automaton group or of an element of an automaton group are open problems [9, Problems 7.2.1(e+f)], this former problem has received a solution in some cases [15] and it is even implemented in the `GAP` packages `FR` and `automgrp` [4, 2, 14].

Note that there exists a previous result linking the level-transitivity of a group and the freeness of an automaton group on its sate set. In [5], Glasner and Mozes associate to a Mealy automaton a special graph called *VH-square complex*, based on a natural tiling set. They prove that if the action of the fundamental group of this graph acts level-transitively, then the group generated by the Mealy automaton is free.

1.4 Minimization and Nerode classes

Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ be a Mealy automaton.
The *Nerode equivalence* $\equiv$ *on* Q is the limit of the sequence of increasingly finer equivalences $(\equiv_k)$ recursively defined by:

$$\forall x, y \in Q, \qquad x \equiv_0 y \iff \rho_x = \rho_y \,,$$
$$\forall k \geqslant 0,\; x \equiv_{k+1} y \iff \big(x \equiv_k y \quad \wedge \quad \forall i \in \Sigma,\; \delta_i(x) \equiv_k \delta_i(y)\big) \,.$$

Since the set Q is finite, this sequence is ultimately constant. For every element x in Q, we denote by $[x]$ the class of x w.r.t. the Nerode equivalence, called the *Nerode class* of x. Extending to the n-th power of $\mathcal{A}$, we denote by $[\mathbf{x}]$ the Nerode class in Q^n of $\mathbf{x} \in Q^n$.
Two states of a Mealy automaton belong to the same Nerode class if and only if they represent the same element in the generated semigroup, *i.e.* if and only if they induce the same action on Σ^*.

The *minimization* of $\mathcal{A}$ is the Mealy automaton $\mathfrak{m}(\mathcal{A}) = (Q/{\equiv}, \Sigma, \tilde{\delta}, \tilde{\rho})$, where for every (x, i) in $Q \times \Sigma$, $\tilde{\delta}_i([x]) = [\delta_i(x)]$ and $\tilde{\rho}_{[x]} = \rho_x$. This definition is consistent with the standard minimization of "deterministic finite automata" where instead of considering the mappings $(\rho_x : \Sigma \to \Sigma)_x$, the computation is initiated by the separation between terminal and non-terminal states.

The following remarks will be useful for the rest of the paper:

Remark 2. *If two words of Q^* are equivalent, so are their images under the action of any element of $\langle \mathfrak{d}(\mathcal{A}) \rangle_+$.*

Remark 3. *The Nerode classes of a connected reversible Mealy automaton (*i.e.* a Mealy automaton with exactly one connected component) have the same cardinality.*

Remark 4. *It is known from [11] that a reversible automaton generates a finite semigroup if and only if the sizes of the connected components of its powers are uniformely bounded. It is straightforward to adapt the proof to show that a reversible automaton generates a finite semigroup if and only if the sizes of the minimizations of the connected components of its powers are uniformely bounded.*

Remark 5. *Let $\mathcal{A}$ and $\mathcal{B}$ be two reversible connected Mealy automata on the same alphabet Σ, x some state of $\mathcal{A}$, and y some state of $\mathcal{B}$. If x and y have the same action on Σ^*, then $\mathfrak{m}(\mathcal{A})$ and $\mathfrak{m}(\mathcal{B})$ are isomorphic; in particular they have the same size. Indeed the image of x in $\mathcal{A}$ by some word $\mathbf{s} \in \Sigma^*$ and the image of y in $\mathcal{B}$ by this same word $\mathbf{s}$ have necessarily the same action on Σ^*, and $\mathcal{A}$ and $\mathcal{B}$ being strongly connected (because they are connected and reversible), for every state of $\mathcal{A}$ there is a state of $\mathcal{B}$ which acts similarly on Σ^*, and vice-versa.*

2 Main result

As already said, Theorem 1 cannot be generalized to any number of states. Our attempt here is to understand which property, weaker than freeness, can be deduced from the level-transitivity of the group generated by the dual automaton, without any hypothesis on the size of the state set. To obtain the following result, we need to add some invertibility hypothesis:

Theorem 6. *Let $\mathcal{A}$ be an invertible and reversible Mealy automaton. If the action of $\mathfrak{d}(\mathcal{A})$ is level-transitive, then the semigroup $\langle\mathcal{A}\rangle_+$ has exponential growth.*

Note that the exponential growth of the semigroup $\langle\mathcal{A}\rangle_+$ implies the exponential growth of the group $\langle\mathcal{A}\rangle$.

Let us first look at the structure of the Nerode classes of two consecutive powers of the state set, in the case of an invertible and reversible Mealy automaton whose dual generates a level-transitive group.

Lemma 7. *Let $\mathcal{A} = (Q, \Sigma, \delta, \rho)$ be an invertible and reversible automaton whose dual generates a level-transitive group.*
Let $(C_i)_{1\leq i\leq k}$ be the Nerode classes of Q^n for some n, and D be a Nerode class of Q^{n+1}. We have

$$D = \bigcup_{q\in Q_D} C_{i_{q,D}} q \quad \textit{and} \quad D = \bigcup_{q\in Q'_D} q C_{i'_{q,D}} \ ,$$

where $Q_D \subseteq Q$ and $Q'_D \subseteq Q$ have the same cardinality, and the $(i_{q,D})_{q\in Q_D}$ on the one hand and the $(i'_{q,D})_{q\in Q_D}$ on the other are pairewise distinct.
The automata $\mathfrak{m}(\mathcal{A}^n)$ and $\mathfrak{m}(\mathcal{A}^{n+1})$ have the same size if and only if $Q_D = Q'_D = Q$.

Proof. We prove the first decomposition.
Let $\mathbf{u} \in Q^n$ and $\mathbf{v} \in [\mathbf{u}]$, then $\mathbf{u}q$ and $\mathbf{v}q$ are in the same Nerode class of Q^{n+1}. So $\mathbf{u}q \in D$ implies $[\mathbf{u}]q \subseteq D$.
Let C_i and C_j be two different Nerode classes of Q^n, then it is impossible that $C_iq \subseteq D$ and $C_jq \subseteq D$ because by hypothesis q induces a bijection on Σ^*.
As a consequence, the ratio between the cardinality of a Nerode class of Q^{n+1} and the cardinality of a Nerode class of Q^n (which does not depend on these classes by Remark 3) is the integer $\#Q_D = \#Q'_D$ which is less than or equal to $\#Q$. It is equal to $\#Q$ if and only if $\mathfrak{m}(\mathcal{A}^n)$ and $\mathfrak{m}(\mathcal{A}^{n+1})$ have the same size. □ □

Proposition 8. *Let $\mathcal{A}$ be an invertible and reversible Mealy automaton whose dual generates a level-transitive group, and n be an integer.*
If $\#\mathfrak{m}(\mathcal{A}^{n+1}) = \#\mathfrak{m}(\mathcal{A}^n)$, then $\#\mathfrak{m}(\mathcal{A}^{n+2}) = \#\mathfrak{m}(\mathcal{A}^{n+1})$. □

Proof. Let us denote by Q the state set of $\mathcal{A}$, by $C_1, \ldots, C_k$ the Nerode classes of Q^n, and by $D_1, \ldots, D_k$ the Nerode classes of Q^{n+1} (by hypothesis Q^{n+1} has as many Nerode classes as Q^n).
Let $r \in Q$: rD_1 is included in some Nerode class of Q^{n+2}. But by Lemma 7 we have

$$rD_1 = r\bigcup_{q\in Q} C_{i_q} q = \bigcup_{q\in Q} rC_{i_q} q \ ,$$

where C_{i_q} is written for $C_{i_{q,D_1}}$. Now, for a fixed $q \in Q$, rC_{i_q} is included in some Nerode class of Q^{n+1}, say D_{j_q}. Hence

$$\bigcup_{q\in Q} D_{j_q} q$$

is included in a Nerode class of Q^{n+2} and by Lemma 7 we obtain the result. □
□

We can now prove Theorem 6.

Proof. If $\mathfrak{d}(\mathcal{A})$ is level-transitive, then $\langle \mathfrak{d}(\mathcal{A}) \rangle$ is infinite and so is $\langle \mathcal{A} \rangle_+$ (see for example [1, 11]). But if there exists n such that $\mathfrak{m}(\mathcal{A}^n)$ and $\mathfrak{m}(\mathcal{A}^{n+1})$ have the same size, then for any $m \geq n$, $\mathfrak{m}(\mathcal{A}^m)$ has this same size, and the generated semigroup should be finite by Remark 4.
So for any n, the ratio of the size of $\mathfrak{m}(\mathcal{A}^{n+1})$ and the size of $\mathfrak{m}(\mathcal{A}^n)$, which is known to be an integer by Lemma 7, is at least 2. So the sequence $(\#\mathfrak{m}(\mathcal{A}^n))_n$ increases and is componentwise greater than or equal to $(2^n)_n$. By Remark 5, this implies that, for any integer n, the actions induced by the states of $\mathcal{A}^n$ cannot be induced by previous powers of $\mathcal{A}$ and hence $\langle \mathcal{A} \rangle_+$ has an exponential growth. □ □

Open problems For this result to be fully applicable, it remains of course to find a procedure to decide if an invertible and reversible Mealy automaton is level-transitive. Another interesting question would be: can an invertible and reversible Mealy automaton generate a semigroup of polynomial or of intermediate growth?

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