On SL(3,C)-representations of the Whitehead link group

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Abstract

We describe a family of representations in SL(3,ℂ) of the fundamental group π of the Whitehead link complement. These representations are obtained by considering pairs of regular order three elements in SL(3,ℂ) and can be seen as factorising through a quotient of π defined by a certain exceptional Dehn surgery on the Whitehead link. Our main result is that these representations form an algebraic component of the SL(3,ℂ)-character variety of π.

1 Introduction

Let M be a manifold. The description of the character variety of $\pi_1(M)$ in a Lie group G is closely related to the study of geometric structures on M modelled on a G-space X. In this setting, representations of $\pi_1(M)$ into G appear as holonomies of (X, G)-structures. In the case of a hyperbolic 3-manifold M, a natural target group is PSL(2,ℂ) (or SL(2,ℂ)), as the holonomy of a hyperbolic structure on M has image contained in PSL(2,ℂ). The study of these character varieties was initiated by Thurston, in the non-compact case, who described a natural way of constructing explicit representations of $\pi_1(M)$ in PSL(2,ℂ) using ideal triangulations of M (see [Thu]). The rough idea is to parametrise hyperbolic ideal tetrahedra using cross-ratios, and to analyse the possible ways of constructing the hyperbolic structure on M by gluing together these ideal tetrahedra. This method gives rise to a familly of polynomial equations expressed in terms of a family of cross-ratios, which are often referred to as *Thurston's gluing equations* (see Chapter 4 of [Thu]). The output of this method is a subvariety of $\mathbb{C}^n$ consisting of those tuples of parameters that satisfy Thurston's equations, which is called the *deformation variety*. Representations can be expressed in terms of the cross-ratios, and one of the main interests of the deformation variety is that it allows explicit computations, which are very precious for experiments.

Thurston's approach has been generalized for the target groups $\mathrm{SL}(n, \mathbb{C})$ in [BFG14, GTZ15, DGG13, GGZ15]. This generalisation is geometrically meaningful. Indeed, the subgroups $\mathrm{SU}(2, 1)$ and $\mathrm{SL}(3, \mathbb{R})$ of $\mathrm{SL}(3, \mathbb{C})$ correspond respectively to spherical CR structures (see below) and real projective flag structures (see [FST15]), whereas $\mathrm{SL}(4, \mathbb{R})$ corresponds to projective structures on 3-manifolds, well-studied in the convex case [Ben08]. In the case of SU(2,1), the first examples following this point of view were produced by Falbel in [Fal08], who constructed and studied examples of representations of the figure 8 knot group to SU(2,1) (see also [DF15, FW14]). A parallel is also to be drawn with higher Teichmüller theory in case of surfaces. In this note, we focus on the target group $\mathrm{SL}(3, \mathbb{C})$.

Though simple in spirit, this method of describing representation varieties becomes very involved when the number of tetrahedra grows. In fact, the only $\mathrm{SL}(3, \mathbb{C})$-character variety of a hyperbolic

3-manifold that has been completely described so far with this approach is the one of the figure 8 knot complement [FGK+16], which admits an ideal triangulation by two tetrahedra (see Section 3.1 of [Thu]). Different methods have been used to describe examples of character varieties. In [HMP15], Heusener, Muñoz and Porti gave another description the character variety of the figure 8 group starting directly from the group presentation. In [MP16], Munoz and Porti described character varieties for torus knots. We will consider here the example of the Whitehead link complement (which can be triangulated by 4 ideal simplices). Denote by π its fundamental group. A possible presentation for π is:

$$\langle x, y \,|\, [x,y][x,y^{-1}][x^{-1},y^{-1}][x^{-1},y] \rangle.$$

Let $\chi_3(\pi)$ be the corresponding $\mathrm{SL}(3,\mathbb{C})$-character variety, that is the GIT quotient

$$\chi_3(\pi) = \mathrm{Hom}(\pi, \mathrm{SL}(3,\mathbb{C}))//\mathrm{SL}(3,\mathbb{C}).$$

The full computation of $\chi_3(\pi)$ is not achieved as for today and seems a difficult task. Our goal here is to describe an algebraic component of $\chi_3(\pi)$ that contains many examples of geometrically meaningful representations.

Our motivation comes from the study of the so-called *spherical CR structures* on hyperbolic 3-manifolds. These structures are examples of (G,X)-structures where X is $\mathbf{S}^3$ and G is PU(2,1). The holonomy of such a structure is thus a representation of $\pi_1(M)$ in PU(2,1). This motivates the study of representations of π to $\mathrm{SU}(2,1)$ which is a real form of SL(3,$\mathbb{C}$). Recall that $\mathrm{PU}(2,1)$ is the group of holomorphic isometries of the complex hyperbolic plane $\mathbf{H}^2_{\mathbb{C}}$ and $\mathrm{SU}(2,1)$ is a triple cover of it. The sphere $\mathbf{S}^3$ is the boundary at infinity of $\mathbf{H}^2_{\mathbb{C}}$. In particular, spherical CR structures arise naturally on the boundary at infinity of quotients of $\mathbf{H}^2_{\mathbb{C}}$ with non-empty discontinuity region. Spherical CR structures can also be thought of as examples of *projective flag structures* on 3-manifolds on M, of which holonomies are representations of $\pi_1(M)$ to PSL(3,$\mathbb{C}$).

Striking examples of spherical CR structures have been produced by R. Schwartz in [Sch01, Sch07] about fifteen years ago. There, Schwartz described what is now called a *spherical CR uniformisation* of the Whitehead link complement, that is a spherical CR structure with the additional property that the holonomy representation has non-empty discontinuity region with quotient homeomorphic to the Whitehead link complement (see [Der15] for a precise definition). Since then, Deraux and Falbel [DF15] produced a spherical CR uniformisation of the complement of the figure eight knot, Deraux [Der] and Acosta [Aco15] deformed this uniformisation, Deraux [Der15] described a uniformisation of the manifold m_{009}, and Parker-Will [PW15] described another uniformisation of the Whitehead link complement, different from Schwartz's one. All these examples have the common property that the image of their holonomy representation is a subgroup of $\mathrm{PU}(2,1)$ generated by a pair of regular order three elements (see the introdution of [PW15] for a list).

Our goal here is to provide a common frame for all these examples. We will show that all these representations belong to a common algebraic component of the character variety of π in SL(3,$\mathbb{C}$). This component is formed by representations $\rho : \pi \longrightarrow \mathrm{SL}(3,\mathbb{C})$ that factor through the group $\pi' = \mathbb{Z}_3 * \mathbb{Z}_3$. The latter group is a quotient of π, and more precisely is the fundamental group of a compact exceptional Dehn filling of the Whitehead link complement, as we will see later on. Define X_0 as the subset of the character variety of π' corresponding to representations generated by two regular order three elements of $\mathrm{SL}(3,\mathbb{C})$ (recall that an order three element in $\mathrm{SL}(3,\mathbb{C})$ is regular if and only if its trace is 0).

Theorem 1. *The character variety $\chi_3(\pi)$ contains X_0 as an algebraic component of dimension 4. All representation classes in this component factor through π'.*

We observe here a very similar situation to what happens in the case of the 8-knot complement. There, a 2-dimensional component of the character variety is formed by representations factorising through a quotient. This quotient is the fundamental group of a non-hyperbolic Dehn surgery on the 8-knot complement (see [HMP15, Proposition 10.3] or [FGK$^+$16, Section 5.3]). Moreover this quotient is isomorphic to the $(3, 3, 4)$-triangle group, which is a quotient of π'. As such (see Proposition 3), the aforementioned 2-dimensional component for the 8-knot complement is a slice of the 4-dimensional component for the Whitehead link complement described in Theorem 1.

The proof of Theorem 1 has two steps.

- First we prove that X_0 is a closed Zariski subset in $\chi_3(\pi)$ and has dimension at least 4 (see Proposition 6).
- Secondly we consider the particular point of X_0 associated to a representation ρ_0 and show that the dimension of the Zariski tangent space to $\chi_3(\pi)$ at this point is also 4 (see Proposition 7). As a consequence, the dimension of the complex algebraic variety X_0 is at most 4. The representation ρ_0 has been analysed geometrically in [PW15] and corresponds to a spherical CR uniformisation of the Whitehead link complement.

The main technical part in our work is the proof of Proposition 7. We choose to prove this proposition using a method that is not specific to the Whitehead link complement, and we believe that it could be used to study further examples. It involves the so-called deformation variety as described in [FGK$^+$16]. The latter is an affine algebraic set, which is – at least around $[\rho_0]$ – a ramified covering of the character variety. The purpose of shifting to the deformation variety is that it allows effective computations via decorated representations and triangulations, as in [FGK$^+$16].

In the last section, we describe an explicit family of pairwise unconjugate representations of π', of which conjugacy classes form a Zariski open subset of X_0. This is called, in the works of Culler, Morgan and Shalen [CS83, MS84] a *tautological representation* of X_0 [CS83, MS84]. These representations are defined by pairs of regular order three matrices (A, B) with no common eigenvector that are parametrised by the traces of the four products AB, $A^{-1}B$, $A^{-1}B^{-1}$ and AB^{-1}. These parameters are natural coordinates on X_0, in view of Lawton's description of the character variety of the rank 2 free group given in Theorem 5.

This paper is organised as follows. In Section 2, we describe the Whitehead link complement and its fundamental group from the perspective of (real) hyperbolic geometry. In particular, we gather together classical information on presentations and parabolic subgroups of π that will be needed further. Section 3 is devoted to the character variety of π. We provide basic definitions and facts on these objects. We prove Proposition 6, state Proposition 7 and derive Theorem 1 from them. In Section 4, we present the deformation variety and prove Proposition 7. Eventually, in Section 5, we describe an explicit parametrisation of the representations in X_0. The interested reader may also want to use the companion Sage notebook [Gui] which combines the use of SnapPy [CDW] and SageMath [Dev16] to illustrate our method.

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2 Hyperbolic geometry of the Whitehead Link Complement

The Whitehead link is depicted on Figure 1. We denote by W its complement in $\mathbf{S}^3$ and by π the fundamental group of W. It is a well-known fact that W carries a (unique) complete hyperbolic structure.

2.1 The ideal octahedron

The hyperbolic structure of W can be explicitely described by considering an ideal regular octahedron (Figure 1) together with identifications of its faces by hyperbolic isometries. We refer the reader to Section 3.3 of [Thu], or to Example 1 in [Wie78] for details. We briefly recall here the description of this structure which is given in [Wie78]. We will keep the notation used there.

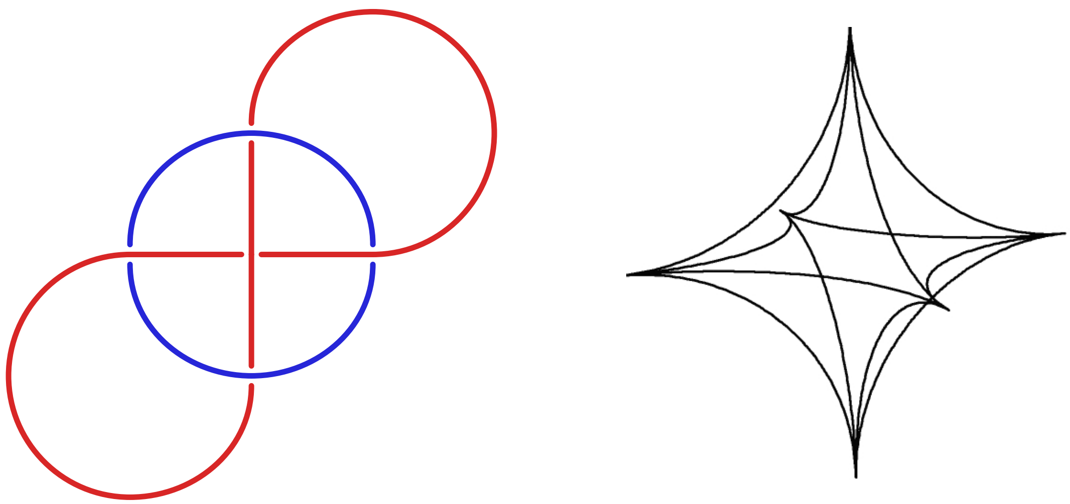


Figure 1: The Whitehead link, and a hyperbolic regular ideal octahedron.

We denote by $\mathcal{O}$ the ideal octahedron of which vertices are given, in the upper half-space model of $\mathbf{H}^3_{\mathbb{R}}$ by

$$\infty,\ 0,\ -1,\ -1+i,\ i,\ \frac{-1+i}{2}.$$

A flattened version of this octahedron is pictured on Figure 2. We denote by u, w_1 and t_2 the isometries of $\mathbf{H}^3_{\mathbb{R}}$ associated to the following elements of SL(2,$\mathbb{C}$); this choice of notation is the same as in [Wie78].

$$u = \begin{bmatrix} 1 & i \\ 0 & 1 \end{bmatrix},\ t_2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \text{ and } w_1 = \begin{bmatrix} 1 & 0 \\ -1-i & 1 \end{bmatrix}. \tag{1}$$

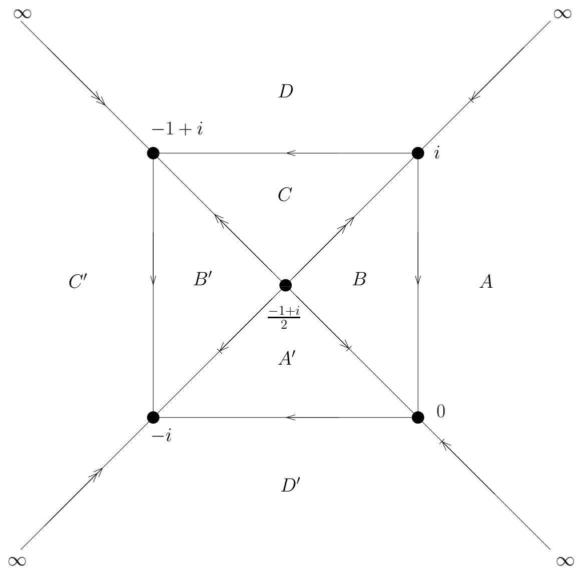
Note that t_2 belongs to the group $\langle u, w_1\rangle$, since $t_2 = (w_1^{-1}u)^2 w_1 u w_1^{-1} u$. We equip $\mathcal{O}$ with the face identifications described on Figure 2. This particular choice gives a holonomy representation with image a subgroup of the Bianchi group PSL(2,$\mathbb{Z}[i]$) isomorphic to the fundamental group of the Whitehead link complement. This subgroup can be seen to have index 12 in PSL(2,$\mathbb{Z}[i]$) (see [Wie78]). The octahedron from Figure 2 is a fundamental domain for the action on $\mathbf{H}^3_{\mathbb{R}}$ of the group generated by u and w_1. Applying Poincaré's polyhedron theorem, one obtains the following presentation for π.

$$\langle x, y \,|\, [x,y][x,y^{-1}][x^{-1},y^{-1}][x^{-1},y]\rangle. \tag{2}$$

The correspondance between (2) and the face identifications on Figure 2 is given by $x \leftrightarrow u$ and $y \leftrightarrow w_1$. Identifying projective transformations and matrices, we denote by Γ the subgroup of PSL(2,$\mathbb{C}$) generated by u and w_1.

2.2 Triangulating the ideal octahedron

Connecting the vertices with coordinates ∞ and $\frac{-1+i}{2}$ by an edge, one obtains a decomposition of the octahedron $\mathcal{O}$ as a union of four ideal tetrahedra, which we label as follows (see Figure 2).



$$\alpha = w_1 u^{-1} \quad : A \longmapsto A'$$

$$\beta = w_1 u^{-1} w_1^{-1} \quad : B \longmapsto B'$$

$$\gamma = t_2^{-1} u w_1^{-1} \quad : C \longmapsto C'$$

$$\delta = u^{-1} \quad : D \longmapsto D',$$

Figure 2: An octahedron with face identifications

$$\Delta_0 = \left(i, 0, \frac{-1+i}{2}, \infty\right) \qquad \Delta_1 = \left(-1+i, i, \frac{-1+i}{2}, \infty\right)$$
$$\Delta_2 = \left(-i, -1+i, \frac{-1+i}{2}, \infty\right) \quad \Delta_3 = \left(0, -i, \frac{-1+i}{2}, \infty\right) \tag{3}$$

This gives an ideal triangulation of the manifold W which is the one used by the software SnapPy [CDW] to provide another presentation of π given by

$$\langle a, b | aba^{-3}b^2a^{-1}b^{-1}a^3b^{-2}\rangle. \tag{4}$$

Since the relator in (4) satisfies $aba^{-3}b^2a^{-1}b^{-1}a^3b^{-2} = a[ba^{-3}b^2, a^{-1}b]a^{-1}$, this second presentation is actually equivalent to

$$\langle a, b | [ba^{-3}b^2, a^{-1}b]\rangle. \tag{5}$$

An isomorphism between (2) and (4) is given by the changes of generators

$$(x, y) = (ab^{-1}, ab^{-1}a) \text{ and } (a, b) = (x^{-1}y, x^{-2}y). \tag{6}$$

Definition 1. The geometric representation of π is the morphism $\rho_{geom} : \pi \longrightarrow \mathrm{PSL}(2, \mathbb{C})$ defined by

$$\rho_{geom}(a) = u^{-1}w_1 = \begin{bmatrix} i & -i \\ -1-i & 1 \end{bmatrix} \text{ and } \rho_{geom}(b) = u^{-2}w_1 = \begin{bmatrix} -1+2i & -2i \\ -1-i & 1 \end{bmatrix}. \tag{7}$$

The geometric representation is thus a discrete and faithful representation of π in PSL(2,$\mathbb{C}$) with image Γ. The above matrices for a and b are obtained by identifying respectively x and y to u and w_1 in (6). Note that $\rho(b^{-1}a^3b^{-1}a^{-1}) = t_2$.

2.3 Stabilizers of the vertices and peripheral curves

There are two orbits of vertices modulo the identifications in the octahedron $\mathcal{O}$, the one of ∞ and the one of 0. It is a simple exercise using the face identifications to verify that the stabilisers of these two points are respectively $\Gamma_\infty = \langle u, t_2 \rangle$, and $\Gamma_0 = \langle w_1, uw_1^{-1}u^{-1}t_2^{-1}u^{-1}w_1^{-1}u \rangle$. The second generator for Γ_0 is the projective transformation associated to

$$\begin{bmatrix} 1 & 0 \\ 2-2i & 1 \end{bmatrix}.$$

We express now these stabilisers in terms of a and b.

Proposition 2. *The stabilizers of* 0 *and* ∞ *in* Γ *are the images by* ρ_{geom} *of the two subgroups of* π *respectively given by* $\langle ab^{-1}a, s_0 \rangle$ *and* $\langle ab^{-1}, s_\infty \rangle$, *where* $s_0 = [a, b^{-1}]a^{-1}b^2a^{-3}[b, a]$ *and* $s_\infty = b^{-1}a^3b^{-1}a^{-1}$.

Remark 1. SnapPy provides the following generators for the first homology of the boundary tori of the Whitehead link complement using the presentation (5) (m_i stands for meridian, and l_i for longitude):

$$m_1 = a^{-2}b,\ l_1 = a^{-2}bab^{-2}ab,\ m_2 = b^{-1}a,\ l_2 = b^{-1}ab^{-1}aba^{-3}ba. \tag{8}$$

By a direct computation, on verifies that

$$\begin{aligned} \rho_{geom}(am_1a^{-1}) &= w_1 \quad \rho_{geom}(al_1a^{-1}) = w_1^{-2}s_0 \\ \rho_{geom}(am_2a^{-1}) &= x \quad \rho_{geom}(al_2a^{-1}) = t_2^{-1}u^2 \end{aligned}$$

We see therefore that m_1 and l_1 correspond to the cusp of W associated to the (orbit of) 0, and that m_2 and l_2 correspond to the one associated to the (orbit of) ∞.

3 The SL(3,ℂ)-character variety

3.1 Generalities

Definition 2. Let G be a finitely generated group. The *representation variety* of G in SL(3,ℂ) is

$$\mathrm{Hom}(G, \mathrm{SL}(3, \mathbb{C}).$$

Its GIT quotient

$$\mathrm{Hom}(G, \mathrm{SL}(3, \mathbb{C}))//\mathrm{SL}(3, \mathbb{C})$$

is called the SL(3,ℂ)-*character variety*, denoted by $\chi_3(G)$.

We refer the reader to [Sik12, Heu16] for classical definitions about representation and character varieties and associated objects. A remark is important for our purposes: if G' is a quotient of G, then there is a natural map:

$$\mathrm{Hom}(G', \mathrm{SL}(3, \mathbb{C})) \hookrightarrow \mathrm{Hom}(G, \mathrm{SL}(3, \mathbb{C})).$$

Indeed, a representation $\rho' : G' \to \mathrm{SL}(3, \mathbb{C})$ is naturally promoted to a representation

$$\rho : G \twoheadrightarrow G' \xrightarrow{\rho'} \mathrm{SL}(3, \mathbb{C}).$$

As the projection $G \twoheadrightarrow G'$ is surjective, this map is injective and moreover two representation ρ' and $\bar{\rho}'$ in $\mathrm{Hom}(G', \mathrm{SL}(3, \mathbb{C}))$ are conjugate if and only if their associated ρ and $\bar{\rho}$ in $\mathrm{Hom}(G, \mathrm{SL}(3, \mathbb{C}))$ are conjugate.

As a consequence of this discussion, we obtain

Proposition 3. *If* G' *is a quotient of* G, *then* $\chi_3(G') \subset \chi_3(G)$.

3.2 A quotient of π

We denote by π' the quotient of π defined by the extra relations $a^3 = b^3 = 1$. More precisely, a presentation for π' is given by $\pi' = \langle a, b | a^3, b^3, [ba^{-3}b^2, a^{-1}b]\rangle$. Clearly, the group π' is isomorphic to $\mathbb{Z}_3 \star \mathbb{Z}_3$: the last relation is a consequence of the first two. Moreover, π' is the fundamental group of a double Dehn surgery on the Whitehead link:

Proposition 4. *The group $\pi' \simeq \mathbb{Z}_3 \star \mathbb{Z}_3$ is isomorphic to the quotient of π defined by the two relations $m_1^3 l_1^{-1}$ and $m_2^3 l_2^{-1}$.*

The proof of Proposition 4 is a direct verification from the definition of l_i and m_i given in Remark 1: the two conditions $m_1^3 l_1^{-1} = m_2^3 l_2^{-1} = 1$ imply that $a^3 = b^3 = 1$. In terms of Dehn surgery, π' is the fundamental group of the double Dehn surgery of slopes $(-3, -3)$ on the Whitehead link. This double Dehn surgery is not hyperbolic: this may be verified using SnapPy (see Section 1 of the companion Sage notebook to this paper [Gui]). More precisely, it can be seen to be the connected sum of two lens spaces (see [MP06, Table 2]).

3.3 A lower bound for $\dim(X_0)$

We are now going to describe the SL(3,$\mathbb{C}$)-character variety of π'. To this end, we use Lawton's theorem on the SL(3,$\mathbb{C}$)-character variety of the rank two free group F_2 [Law10].

Theorem 5 (Lawton [Law10])**.** *The map ψ defined by*

$$\begin{array}{ccc} \mathrm{SL}(3,\mathbb{C}) \times \mathrm{SL}(3,\mathbb{C}) & \to & \mathbb{C}^8 \\ (A, B) & \mapsto & (\mathrm{tr}A, \mathrm{tr}B, \mathrm{tr}AB, \mathrm{tr}A^{-1}B, \mathrm{tr}A^{-1}, \mathrm{tr}B^{-1}, \mathrm{tr}A^{-1}B^{-1}, \mathrm{tr}AB^{-1}) \end{array}$$

is onto $\mathbb{C}^8$ and descends to a (double) branched cover $\underline{\psi} : \chi_3(F_2) \longrightarrow \mathbb{C}^8$.

The theorem in Lawton's work is more precise and gives an explicit polynomial in 9 variables defining $\chi_3(F_2)$ as a hypersurface in $\mathbb{C}^9$ covering $\mathbb{C}^8$. Namely, the above double cover corresponds to the fact that the traces of the nine words A, B, AB, $A^{-1}B$, A^{-1}, B^{-1}, $A^{-1}B^{-1}$, AB^{-1} and $[A, B]$ sastify a relation of the form

$$(\mathrm{tr}[A, B])^2 - S \cdot \mathrm{tr}[A, B] + P = 0, \tag{9}$$

where S and P are polynomials in the traces of the first eight above words. In other words, once the traces of A, B, AB, $A^{-1}B$, A^{-1}, B^{-1}, $A^{-1}B^{-1}$, AB^{-1} are fixed, the trace of $[A, B]$ is determined up to the choice of a root of (9). We provide the precise values of S and P in Section 7 of the Sage notebook [Gui], they may also be found in Lawton's [Law10], or in [Wil16].

We can now give an alternate definition of the set X_0 considered in the introduction:

Definition 3. Let $X_0 \subset \chi_3(F_2)$ be the inverse image by $\underline{\psi}$ of the subspace

$$V = \{(0, 0, z_1, z_2, 0, 0, z_3, z_4), z_i \in \mathbb{C}\}.$$

By Proposition 3, the sequence of quotients $F_2 \twoheadrightarrow \pi \twoheadrightarrow \pi'$ gives rise to a sequence of inclusions:

$$\chi_3(\pi') \subset \chi_3(\pi) \subset \chi_3(F_2).$$

With these inclusions in mind, we see that X_0 is actually included in $\chi_3(\pi')$. We can even be more specific:

Proposition 6. *The set X_0 is an irreducible Zariski closed subset of $\chi_3(\pi')$. Its dimension is at least 4.*

Proof. First, X_0 is included in $\chi_3(\pi')$. Indeed, the condition $\psi(A,B) \in V$ rewrites

$$\text{tr}A = \text{tr}A^{-1} = \text{tr}B = \text{tr}B^{-1} = .0$$

This implies that both A and B have order three. Indeed, the characteristic polynomial of a matrix $M \in$SL(3,$\mathbb{C}$) is equal to $X^3 - \text{tr}MX^2 + \text{tr}M^{-1}X - 1$ and thus if $\text{tr}M = \text{tr}M^{-1} = 0$ we have $M^3 = Id$.

By construction, X_0 is Zariski closed. Its irreducibility is not hard to verify using the explicit form of the branched 2-cover of Theorem 5 given in [Law10]. For example, using the parametrisation given in Section 5, it is easily seen that the double cover is indeed a branched one and not the union of two distinct sheets: the discriminant of the quadratic equation defining the double cover is not a square. For a direct and precise proof, see also [Aco16, Section 4.1]. Eventually the dimension follows from the fact it is the pull-back of $\mathbb{C}^4$ by $\underline{\psi}$. □

Remark 2. In fact X_0 is a component of $\chi_3(\pi')$ and the only one of positive dimension. Moreover it contains every irreducible representation of π'. The first part of this statement is a consequence of Theorem 1. The character variety $\chi_3(\pi')$, with a focus on real points, has been studied in details in [Aco16].

3.4 An upper bound for $\dim(X_0)$

We give in this section an upper bound on the dimension of the component of $\chi_3(\pi)$ containing X_0 by looking at a specific point to determine the Zariski tangent space. To this end, we consider the following elements of SL$(3,\mathbb{C})$:

$$S = \begin{pmatrix} 1 & \frac{\sqrt{3}-i\sqrt{5}}{2} & -1 \\ \frac{-\sqrt{3}-i\sqrt{5}}{2} & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad \text{et} \quad T = \begin{pmatrix} 1 & \frac{\sqrt{3}+i\sqrt{5}}{2} & -1 \\ \frac{\sqrt{3}-i\sqrt{5}}{2} & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

The matrices S and T have order three and satisfy $\text{tr}(S) = \text{tr}(S^{-1}) = \text{tr}(T) = \text{tr}(T^{-1}) = 0$. We define a point $[\rho_0]$ in the character variety of π by setting:

$$\rho_0(a) = S \quad \rho_0(b) = T. \tag{10}$$

By construction, $[\rho_0]$ belongs to X_0. The key step in the proof of Theorem 1 is the following

Proposition 7. *The Zariski tangent space to $\chi_3(\pi)$ at $[\rho_0]$ has dimension 4.*

We postpone the proof of Proposition 7 to the next section and proceed with the proof of Theorem 1. Recall that we need to prove that X_0 is an algebraic component of $\chi_3(\pi)$ containing $[\rho_0]$.

Proof of Theorem 1. Let X be an algebraic component of $\chi_3(\pi)$ containing X_0. The class $[\rho_0]$ belongs to X. By Proposition 7, the dimension of X is at most 4: it is bounded above by the dimension of any Zariski tangent space. But, by Proposition 6, $X_0 \subset X$ is a Zariski closed subset of dimension 4. Hence, $X = X_0$ and the theorem is proved. □

4 Decorated representations and the deformation variety

We are going to compute the dimension of the Zariski tangent space of $\chi_3(\pi)$ at $[\rho_0]$, in order to prove Proposition 7. To this end, we will use a variation of the character variety – called *decorated character variety* – and a specific set of coordinates on it – the *deformation variety*. The deformation variety is well-adapted to explicit computation. The equations defining this variety may be reconstructed using SnapPy's command `gluing_equations_pgl` [CDW].

The tools hereafter presented are suitable for character varieties with target group the quotient $\mathrm{PGL}(3,\mathbb{C})$ rather than $\mathrm{SL}(3,\mathbb{C})$. This will not be a problem, as we use these tools for computing local dimension around a point which belongs to both character varieties. Indeed, if ρ is a representation of π in $\mathrm{SL}(3,\mathbb{C})$, the local dimensions at $[\rho]$ of the character varieties for $\mathrm{SL}(3,\mathbb{C})$ and $\mathrm{PGL}(3,\mathbb{C})$ are the same.

4.1 Decorated representations

We first recall basic definitions. More details can be found in [BFG+13].

Definition 4. A *flag* of $\mathbb{P}(\mathbb{C}^3)$ is a pair $([x],[f])$ in $\mathbb{P}(\mathbb{C}^3)\times\mathbb{P}((\mathbb{C}^3)^\vee)$ such that $f(x)=0$. We denote by $\mathcal{F}l_3$ the set of flags of $\mathbb{P}(\mathbb{C}^3)$.

Geometrically a flag is a pair formed by a point in $\mathbb{P}(\mathbb{C}^3)$ and a projective line containing it.

Definition 5. Let Γ be the fundamental group of a finite volume, cusped hyperbolic manifold M, let $\mathcal{P}\subset\mathbf{H}^3_{\mathbb{R}}$ be the set of parabolic fixed point of Γ and let ρ be a representation $\rho:\Gamma\longrightarrow\mathrm{PGL}(3,\mathbb{C})$. A *decoration* of ρ is a map $\phi:\mathcal{P}\longrightarrow\mathcal{F}l_3$ which is (Γ,ρ)-equivariant. A pair (ρ,ϕ) is called a *decorated representation.*

Definition 6. The *decorated representation variety* is

$$\mathrm{DecHom}(\Gamma)=\{(\rho,\phi),\rho\in\mathrm{Hom}(\Gamma,\mathrm{PGL}(3,\mathbb{C})),\phi\text{ is a decoration of }\rho\}.$$

The *decorated character variety* is the GIT quotient

$$\mathrm{Dec}\chi_3=\mathrm{DecHom}(\Gamma)//\mathrm{PGL}(3,\mathbb{C})$$

The precise links between the different versions of representation and character varieties are described in details in the introduction of [FGK+16]. It should be noted that for a given generic representation $\rho:\Gamma\to\mathrm{PGL}(3,\mathbb{C})$, there exists only a finite number of possible decorations. Indeed, if $p\in\mathcal{P}$ is the fixed point of a parabolic element $\gamma\in\Gamma$, it should be mapped by ϕ to a flag that is invariant for $\rho(\gamma)$. By equivariance, the map ϕ is completely determined by its values on a choice of representatives of the orbits of Γ on $\mathcal{P}$. As there is a finite number of cusps, and that each generic element in $\mathrm{SL}(3,\mathbb{C})$ preserves a finite number of flags, the number of possible ϕ for a given ρ is finite.

For our purposes, we chose the point $[\rho_0]$ in $\chi_3(\pi)$ (ρ_0 is described in Section 3.4).

Proposition 8. *The representation ρ_0 admits a unique decoration.*

Proof. The (hyperbolic) Whitehead link complement has two cusps, which are represented by the stabilisers of 0 and ∞ (see Proposition 2). Therefore, the equivariance property implies that a decoration ϕ of ρ_0 is completely determined by the images $\phi(0)$ and $\phi(\infty)$. The images by ρ_0 of the stabilisers of 0 and ∞ are respectively the cyclic groups $\langle ST^{-1}S\rangle$ and $\langle ST^{-1}\rangle$. Indeed, the images of the stabilisers of 0 and ∞ by ρ_0 are respectively $\langle\rho_0(ab^{-1}a),\rho_0(s_0)\rangle$ and $\langle\rho_0(ab^{-1}),\rho_0(s_\infty)\rangle$ (this follows directly from Proposition 2). The images by ρ_0 of s_0 and s_∞ are $ST^{-1}S$ and TS^{-1} : this is a direct verification using $S^3=T^3=1$.

Now, the two maps ST^{-1} and $ST^{-1}S$ are regular unipotent, and thus each of them has only one invariant flag. Therefore ρ_0 can only be decorated in one way : ϕ must map 0 to the invariant flag of $ST^{-1}S$ and ∞ to the one of ST^{-1}. □

The invariant flags of ST^{-1} and $ST^{-1}S$ (as well as those of various elements in the group) are made explicit in Table 2. A consequence of Proposition 8 is that around $[\rho_0]$, the decorated character variety is a finite ramified cover of the character variety (see also [Gui15]). As a consequence, the local dimension around $[\rho_0]$ can be equivalently computed at the level of $\chi_3(\pi)$ or of the decorated character variety.

4.2 Using a triangulation: the deformation variety

A configuration of ordered points in a projective space $\mathbb{P}(V)$ is said to be in general position when they are all distinct and no three points are contained in the same line. This notion applies to configurations of projective lines by duality. A configuration of n flags $(([x_1],[f_1],\cdots,([x_n],[f_n])))$ is in *general position* whenever the n points $([x_i])_{i=1}^n$ are in general position and the forms $[f_j]$ satisfy $f_j(x_i)\neq 0$ when $i\neq j$.

Definition 7. We call *tetrahedon of flags* in $\mathbb{P}(\mathbb{C}^3)$ any 4-tuple of flags in general position.

We briefly recall now the definitions of the main projective invariants we are going to use as well as the relations among them. We refer the reader to [BFG14] for more details. Let $T=(F_1,F_2,F_3,F_4)$ be a tetrahedron of flags.

1. **Triple ratio.** Let (ijk) be a face of T (oriented accordingly to the orientation of the tetrahedron) of flags in general position. Its *triple ratio* is the quantity

$$z_{ijk}=\frac{f_i(x_j)f_j(x_k)f_k(x_i)}{f_i(x_k)f_j(x_i)f_k(x_j)}. \tag{11}$$

2. **Cross-ratio.** For each oriented edge (ij), we define k and l in such a way that the permutation $(1,2,3,4)\longmapsto(i,j,k,l)$ is even. Viewing the set of lines through $[x_i]$ as a projective line, we associate to (ij) the cross-ratio

$$z_{ij}=\big[\ker(f_i),(x_ix_j),(x_ix_k),(x_ix_l)\big]. \tag{12}$$

These invariants can be thought of as decorating tetrahedra as shown in Figure 3 : to each face is associated a triple ratio, and to each edge are associated a pair of cross-ratios. Namely, the two cross ratios z_{ij} and z_{ji} are associated to the edge (ij).

The above projective invariants are tied by the following internal relations:

$$z_{ijk}=\frac{1}{z_{ikj}},\ z_{ijk}=-z_{il}z_{jl}z_{kl}$$

$$z_{ik}=\frac{1}{1-z_{ij}},\ z_{ij}z_{ik}z_{il}=-1. \tag{IR}$$

In particular, the triple ratio can be expressed purely in terms of cross ratios. These invariants can be used to parametrise the set of tetrahedra of flags :

Proposition 9 (Proposition 2.10 of [BFG14])**.** *A tetrahedron of flags is uniquely determined up to the action of* $\mathrm{PGL}(3,\mathbb{C})$ *by the 4-tuple* $(z_{12},z_{21},z_{34},z_{43})$ *in* $(\mathbb{C}\setminus\{0,1\})^4$.

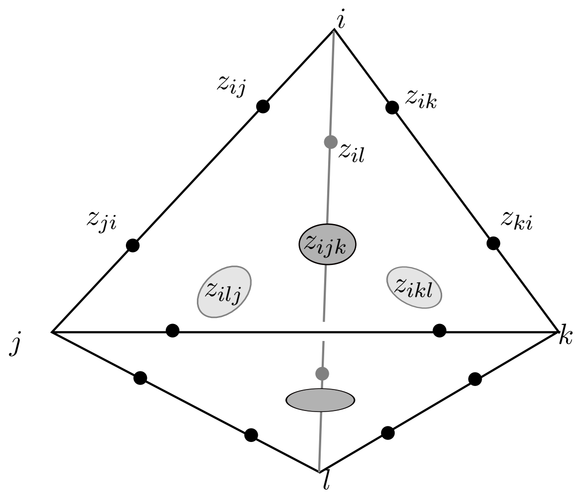


Figure 3: The z-coordinates for a tetrahedron.

Remark that 0 and 1 are forbidden values (as is ∞) because we assume the flags to be in general position: hence every cross-ratio is the cross-ratio of four different points.

Let now M be an ideally triangulated cusped hyperbolic 3-manifold. Denote by ν the number of tetrahedra and by $(\Delta_\mu)_{\mu=1}^{\nu}$ the family of tetrahedra triangulating M. We construct a decorated representation of $\Gamma = \pi_1(M)$ by turning each tetrahedron into a tetrahedron of flags and compute the (decorated) monodromy of their gluing. We only need to ensure that we may glue the tetrahedra together in a consistent way :

1. whenever two tetrahedra Δ and Δ' are glued together along faces $T \subset \Delta$ and $T' \subset \Delta'$, T and T' should have the same shape, that is the same triple ratio up to inversion,

2. around each edge of the triangulation, the monodromy should be the identity.

These gluing conditions are described in details in [BFG$^+$13, Section 2.3]. They give an equation for each face of the triangulation and two for each edge, which are respectively called the *face equations* and the *edge equations*.

Definition 8. The deformation variety of M, denoted $\mathrm{Defor}_3(M)$, is the subset of $\mathbb{C}^{12\nu}$ given by the $(z_{ij}(\Delta_\mu))_{0\leqslant i\neq j\leqslant 3,\, 1\leqslant\mu\leqslant\nu}$ verifying the internal relation (IR) for each tetrahedron T_μ together with the face and edge equations.

In the case of the Whitehead Link Complement, the gluing equations are the 16 monomial equations displayed in Table 1. Hence, the *deformation variety* of the Whitehead Link Complement is the affine algebraic subset of $\mathbb{C}^{48}$ defined by the 32 internal relations (IR) and 16 gluing equations of Table 1. The holonomy map, as defined in [BFG14, GGZ15], is a well-defined map from the deformation variety to the character variety $\chi_3(\pi)$.

4.3 Finding $[\rho_0]$ in the deformation variety

The specific representation ρ_0 we consider is defined by $\rho_0(a) = S$ and $\rho_0(b) = T$ where S and T are the order three elements in SL(3,$\mathbb{C}$) given in Section 3.4. We have seen in Proposition 8 that ρ_0 admits a unique decoration. In Table 2, we provide the flags associated by this decoration to the six vertices

Face equations	Edge equations
$z_{41}(\Delta_0)z_{31}(\Delta_0)z_{21}(\Delta_0)z_{41}(\Delta_1)z_{31}(\Delta_1)z_{21}(\Delta_1) = 1$	$z_{43}(\Delta_0)z_{34}(\Delta_1)z_{34}(\Delta_2)z_{34}(\Delta_3) = 1$
$z_{42}(\Delta_0)z_{32}(\Delta_0)z_{12}(\Delta_0)z_{42}(\Delta_2)z_{32}(\Delta_2)z_{12}(\Delta_2) = 1$	$z_{34}(\Delta_0)z_{43}(\Delta_1)z_{43}(\Delta_2)z_{43}(\Delta_3) = 1$
$z_{41}(\Delta_2)z_{31}(\Delta_2)z_{21}(\Delta_2)z_{42}(\Delta_3)z_{32}(\Delta_3)z_{12}(\Delta_3) = 1$	$z_{21}(\Delta_0)z_{12}(\Delta_1)z_{21}(\Delta_2)z_{21}(\Delta_3) = 1$
$z_{34}(\Delta_2)z_{24}(\Delta_2)z_{14}(\Delta_2)z_{43}(\Delta_3)z_{23}(\Delta_3)z_{13}(\Delta_3) = 1$	$z_{12}(\Delta_0)z_{21}(\Delta_1)z_{12}(\Delta_2)z_{12}(\Delta_3) = 1$
$z_{43}(\Delta_0)z_{23}(\Delta_0)z_{13}(\Delta_0)z_{34}(\Delta_3)z_{24}(\Delta_3)z_{14}(\Delta_3) = 1$	$z_{24}(\Delta_0)z_{23}(\Delta_0)z_{13}(\Delta_0)z_{24}(\Delta_1)z_{23}(\Delta_1)z_{13}(\Delta_1)z_{14}(\Delta_2)z_{23}(\Delta_3) = 1$
$z_{34}(\Delta_0)z_{24}(\Delta_0)z_{14}(\Delta_0)z_{34}(\Delta_1)z_{24}(\Delta_1)z_{14}(\Delta_1) = 1$	$z_{42}(\Delta_0)z_{31}(\Delta_0)z_{32}(\Delta_0)z_{42}(\Delta_1)z_{31}(\Delta_1)z_{32}(\Delta_1)z_{41}(\Delta_2)z_{32}(\Delta_3) = 1$
$z_{42}(\Delta_1)z_{32}(\Delta_1)z_{12}(\Delta_1)z_{41}(\Delta_3)z_{31}(\Delta_3)z_{21}(\Delta_3) = 1$	$z_{41}(\Delta_0)z_{41}(\Delta_1)z_{42}(\Delta_2)z_{31}(\Delta_2)z_{32}(\Delta_2)z_{42}(\Delta_3)z_{41}(\Delta_3)z_{31}(\Delta_3) = 1$
$z_{43}(\Delta_1)z_{23}(\Delta_1)z_{13}(\Delta_1)z_{43}(\Delta_2)z_{23}(\Delta_2)z_{13}(\Delta_2) = 1$	$z_{14}(\Delta_0)z_{14}(\Delta_1)z_{24}(\Delta_2)z_{23}(\Delta_2)z_{13}(\Delta_2)z_{24}(\Delta_3)z_{13}(\Delta_3)z_{14}(\Delta_3) = 1$

Table 1: Gluing equations for the Whitehead Link Complement

of the octahedron described in Section 2.1 (see also the companion notebook [Gui, Section 4]). Note that ρ_0 maps every stabiliser of a vertex of the octahedron to a cyclic group. The flag associated to this vertex is in fact invariant under the image by ρ_0 of the stabiliser.

As a result, each tetrahedron $(\Delta_\nu)_{1\leqslant\nu\leqslant 4}$ is decorated by four flags. For instance, the tetrahedron $\Delta_0 = (i, 0, \frac{-1+i}{2}, \infty)$ is decorated by $(F_i, F_0, F_{\frac{-1+i}{2}}, F_\infty)$ and the other three tetrahedra are decorated in a similar way (the tetrahedra are listed in Section 2.2). It is a simple calculation to compute the cross-ratios associated to these flags as explained in Section 4.2. Table 3 displays, for each tetrahedron, the values of coordinates z_{12}, z_{21}, z_{34}, z_{43} (see also the companion notebook [Gui, Section 5]). The values of the other coordinates can be deduced from them using the internal relations (IR).

Remark 3. Note the high degree of symmetry of the considered decorated representation: the tetrahedra are all the same up to the action of $\mathrm{SL}(3,\mathbb{C})$ and a renumbering of the vertices.

4.4 Computation of the Zariski tangent space: proof of Proposition 7.

The deformation variety is the (algebraic) set of all tuples of 48 complex numbers satisfying both the internal relations and the gluing equations (compare to [BFG+13, Section 3]). In other words it is the intersection of the inverse images

$$\mathrm{IR}^{-1}(1,\ldots,1) \cap \mathrm{GR}^{-1}(1,\ldots,1),$$

where the two maps IR and GR are defined by

- IR : $(\mathbb{C}\setminus\{0,1\})^{48} \to (\mathbb{C}^*)^{32}$ is the map representing the internal relations (IR): it sends a collection $(z_{ij}(\Delta_\nu))$ (for every half-edge ij and $1 \leqslant \nu \leqslant 4$) to the collection of 32 complex numbers given by:

$$(-z_{ij}(\Delta_\nu)z_{ik}(\Delta_\nu)z_{il}(\Delta_\nu), \quad z_{ik}(\Delta_\nu)(1 - z_{ij}(\Delta_\nu))).$$

- GE : $(\mathbb{C}\setminus\{0,1\})^{48} \to (\mathbb{C}^*)^{16}$ is the collection of left-hand sides in the gluing equations of table 1.

We denote by (IR, GE) the map $(\mathbb{C}\setminus\{0,1\})^{48} \to (\mathbb{C}^*)^{48}$ given by the previous two maps. The Zariski tangent space to the deformation variety is the kernel of the tangent map to (IR, GE). As those maps are mostly monomial, we choose to write their tangent maps in the following basis of tangent spaces

Vertex	Generator of its stabilizer in the image of ρ_0	Invariant flag
∞	ST^{-1}	$F_\infty : \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, [0,0,1]$
0	$ST^{-1}S$	$F_0 : \begin{bmatrix} 1 \\ -\frac{3\sqrt{3}+i\sqrt{5}}{4} \\ -1 \end{bmatrix}, [1, \frac{3\sqrt{3}-i\sqrt{5}}{4}, -1]$
i	$S^{-1}T^{-1}$	$F_i : \begin{bmatrix} 1 \\ -\frac{\sqrt{3}+i\sqrt{5}}{4} \\ \frac{-1+i\sqrt{15}}{4} \end{bmatrix}, [\frac{1+i\sqrt{15}}{4}, \frac{\sqrt{3}-i\sqrt{5}}{4}, -1]$
$-1+i$	$T^{-1}ST^{-1}$	$F_{-1+i} : \begin{bmatrix} 1 \\ \frac{3\sqrt{3}-i\sqrt{5}}{4} \\ -1 \end{bmatrix}, [1, -\frac{3\sqrt{3}+i\sqrt{5}}{4}, -1]$
$-i$	$T^{-1}S^{-1}$	$F_{-i} : \begin{bmatrix} 1 \\ \frac{\sqrt{3}-i\sqrt{5}}{4} \\ -\frac{1+i\sqrt{15}}{4} \end{bmatrix}, [\frac{1-i\sqrt{15}}{4}, -\frac{\sqrt{3}+i\sqrt{5}}{4}, -1]$
$\frac{-1+i}{2}$	TS	$F_{\frac{-1+i}{2}} : \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, [1,0,0]$

Table 2: The unique decoration of ρ_0

at the source and target. We take, for each coordinate z, the vector field $z\frac{\partial}{\partial z}$. In these basis, the tangent map to a function $\phi : (\mathbb{C}^*)^{48} \to \mathbb{C}^*$ has entries of the form

$$\frac{z}{\phi}\frac{\partial \phi}{\partial z}.$$

This follows from the following elementary lemma:

Lemma 10. *Let $f : \mathbb{C}^* \to \mathbb{C}^*$ be a differentiable function. In the basis $z\frac{\partial}{\partial z}$, the tangent map Tf at $a \in \mathbb{C}^*$ has coordinate $\frac{a}{f(a)}\frac{\partial f}{\partial z}$.*

Proof. In usual coordinates, by definition of partial derivative, the tangent map $T_a f$ at a point a maps

$$v \in T_a\mathbb{C}^* \quad \text{to} \quad w = \frac{\partial f}{\partial z} v \in T_{f(a)}\mathbb{C}^*.$$

The basis change from $\frac{\partial}{\partial z}$ to $z\frac{\partial}{\partial z}$ in both tangent spaces transforms v into $u = \frac{v}{a}$ and w into $\frac{w}{f(a)}$. Hence, in new basis, u is sent to

$$\frac{w}{f(a)} = \frac{1}{f(a)}\frac{\partial f}{\partial z} v = \frac{a}{f(a)}\frac{\partial f}{\partial z} u.$$

□

Tetrahedron	z_{12}	z_{21}	z_{34}	z_{43}
Δ_0	$\frac{7+i\sqrt{15}}{4}$	$\frac{-1-i\sqrt{15}}{8}$	$\frac{7-i\sqrt{15}}{4}$	$\frac{-1+i\sqrt{15}}{8}$
Δ_1	$\frac{-1+i\sqrt{15}}{8}$	$\frac{7-i\sqrt{15}}{4}$	$\frac{-1-i\sqrt{15}}{8}$	$\frac{7+i\sqrt{15}}{4}$
Δ_2	$\frac{7-i\sqrt{15}}{4}$	$\frac{-1+i\sqrt{15}}{8}$	$\frac{-1-i\sqrt{15}}{8}$	$\frac{7+i\sqrt{15}}{4}$
Δ_3	$\frac{-1-i\sqrt{15}}{8}$	$\frac{7+i\sqrt{15}}{4}$	$\frac{-1+i\sqrt{15}}{8}$	$\frac{7-i\sqrt{15}}{4}$

Table 3: Coordinates for $[\rho_0]$ in the deformation variety

We will apply this lemma to each coordinate of the map $(\mathrm{IR}, \mathrm{GE})$ to obtain a matrix for the tangent map to $(\mathrm{IR}, \mathrm{GE})$ at the point $[\rho_0]$. This matrix, denoted J, has size 48×48 and is depicted in Table 4 (see Remark 4 below). To construct J, we have to deal with two kinds of functions, depending on the equations that form the the maps IR and GE : monomial maps or maps of the form $z_{ik}(\Delta_\nu)(1 - z_{ij}(\Delta_\nu))$.

- If f is a monomial map, its tangent map has integer entries equal to the exponent of the relative variable. As an example, this formula applied to the map $-z_{12}(\Delta_0)z_{13}(\Delta_0)z_{14}(\Delta_0)$ gives all entries equal to 0 except for those associated to the variables $z_{12}(\Delta_0)$, $z_{13}(\Delta_0)$ and $z_{14}(\Delta_0)$ which give three entries equal to 1. The same phenomenon appears for each of the first sixteen rows of the matrix J. The gluing equations are also monomials (see Table 1), but involve more variables. These correspond to lines 33 to 48 of the matrix J, that have all their coefficients equal to 0 except for 4, 6 or 8 ones that are equal to 1.

- if f has the form[1] $z_{ik}(1 - z_{ij})$, its tangent map has every entry 0 except the ones corresponding to z_{ij} and z_{ik}. Those two are respectively $-\frac{z_{ij}}{1-z_{ij}}$ and 1. Note that, at a point verifying the internal relations, we have the additional relation $-\frac{z_{ij}}{1-z_{ij}} = z_{il}$ (see also the computation in [BFG+13, Section 5], especially Lemma 5.3). Hence the entries for such a map are 0, 1 or z_{il}. Those appear in rows 17 to 32 of the matrix J displayed in Table 4.

We see thus that J has entries either integer or of the form $z_{il}(\Delta_\nu)$. Note moreover that the last 16 rows, corresponding to the gluing equations, can be accessed directly by SnapPy [CDW] under SageMath [Dev16]: it is the Neumann-Zagier datum. This part of J is directly given by the commands:

```
import snappy;
Triangulation("5^2_1").gluing_equations_pgl(3,equation_type='non_peripheral').matrix
```

The next step is to compute the kernel of J. As all entries are in the number field $\mathbb{Q}[i, \sqrt{3}, \sqrt{5}]$, a computer algebra system such as Sage computes it exactly. As a result, the dimension of this kernel is 4 (see Section 6 of the Sage notebook [Gui]). We deduce that the dimension of the Zariski tangent space at the decoration of $[\rho_0]$ to the deformation variety is 4.

[1]We drop here the indication of the tetrahedron Δ_ν for $0 \leqslant \nu \leqslant 3$ in order to simplify the notations.

Remark 4. To write the matrix J, we choose the same order on the variables $z_{ij}(\Delta_\nu)$ as SnapPy does. As the recise order it is not very enlightning, we omit this discussion here. A change of order on the variables amounts to a permutation of the columns of J, which does not affect its kernel.

Note that at $[\rho_0]$, the two subgroups generated by the pairs $(\rho_0(l_i), \rho_0(m_i))$ for $i = 1, 2$ are regular unipotent: there is only one invariant flag for each one. As noted before, it implies that, locally the holonomy map between the deformation variety and the actual character variety is a finite ramified covering. This concludes the proof of Proposition 7: the Zariski tangent space to the character variety at $[\rho_0]$ also has dimension 4.

Row 1:	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0
Row 16:	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	1	1
Row 17:	1	0	x̄	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	1	0	ȳ	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	1	0	x	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	1	0	y	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	0	x̄	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	ȳ	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	x	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	y	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	x	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	y	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	x̄	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	ȳ	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	y	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	x	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	ȳ	0	0	0
Row 32:	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	x̄
Row 33:	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0
	0	1	0	0	1	1	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1	0	0	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0	0	0	1	0	0	1	1
	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	1	0	0
	0	0	1	0	1	0	1	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	1	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1
	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0	0	1	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	1	0	1	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	0	0
Row 48:	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0	1	1	0	0	0	0	0	0	0	1	0	1	0

Table 4: The matrix J, where $\mathrm{x} = \frac{9+i\sqrt{15}}{8}$ and $\mathrm{y} = -\frac{3+i\sqrt{15}}{4}$.

5 A parametrisation of X_0

We describe in this section a parametrisation of a Zariski open subset of X_0 by actual matrices. More precisely, given four generic complex numbers z_1, z_2, z_3 and z_4 we are going to provide two pairs (A, B) of regular order three elements in SL(3,$\mathbb{C}$) such that

- A and B have no common eigenvector in $\mathbb{C}^3$,
- A and B satisfy

$$\mathrm{tr}(AB) = z_1,\ \mathrm{tr}(A^{-1}B) = z_2,\ \mathrm{tr}(A^{-1}B^{-1}) = z_3,\ \mathrm{tr}(AB^{-1}) = z_4. \tag{13}$$

Recall that the traces of A, B and their inverses are zero as they are regular order three element. The genericity condition will be made explicit in Proposition 11.

We know from Lawton's theorem that X_0 is a double cover over $\mathbb{C}^4$. Parameters on $\mathbb{C}^4$ are given by the four trace parameter z_1, z_2, z_3, z_4. To describe this double cover, define the quantity

$$\Delta = z_1^2 z_3^2 - 2z_1z_2z_3z_4 + z_2^2 z_4^2 - 4z_1^3 - 4z_2^3 - 4z_3^3 - 4z_4^3 + 18z_1z_3 + 18z_2z_4 - 27,$$

which is the discriminant of the trace equation (9) in Theorem 5, in the case where the traces of A, B and their inverses vanish. We denote by δ a square root of Δ. Let j be a non trivial cube root of 1 and $k = \mathbb{Q}[j](z_1, z_2, z_3, z_4)$.

Proposition 11. *Let (a, b, c, d) be the following elements in $k[\delta]$:*

$$\begin{aligned}
4a &= \frac{z_1z_3 - z_2z_4 + 6jz_1 + 6j^2z_3 + 9 + \delta}{jz_1 + z_2 + j^2z_3 + z_4 + 3} \\
4d &= \frac{z_1z_3 - z_2z_4 + 6j^2z_1 + 6jz_3 + 9 - \delta}{j^2z_1 + z_2 + jz_3 + z_4 + 3} \\
b &= \frac{z_1 - z_2 - j^2z_3 + j^2z_4 + 3(j^2 - 1)}{z_1 + z_2 + j^2z_3 + j^2z_4 + 3j} a + (j - 1)\frac{z_1 + jz_2 + jz_3 + z_4 + 3j^2}{z_1 + z_2 + j^2z_3 + j^2z_4 + 3j} \\
c &= \frac{z_1 + jz_2 - jz_3 - z_4 + 3(j - 1)}{z_1 + jz_2 + jz_3 + z_4 + 3j^2} d + (j^2 - 1)\frac{z_1 + z_2 + j^2z_3 + j^2z_4 + 3j}{z_1 + jz_2 + jz_3 + z_4 + 3j^2}.
\end{aligned}$$

Then for any 4-tuple of complex numbers (z_1, z_2, z_3, z_4) such that a, b c and d are well-defined, any pair (A, B) of order three regular elements of SL(3,$\mathbb{C}$), satisfying (13) *and such that the j^2-eigenline for A is different from the j-eigenline for B, is conjugate in SL(3,$\mathbb{C}$) to one of the two pairs defined by*

$$A = \begin{bmatrix} j & 0 & 0 \\ j^2 & 1 & 0 \\ b + a & 2ja & j^2 \end{bmatrix} \text{ and } B = \begin{bmatrix} j & 2j^2d & c + d \\ 0 & 1 & j \\ 0 & 0 & j^2 \end{bmatrix}. \tag{14}$$

Proof. A direct computation of the traces of AB, $A^{-1}B$, $A^{-1}B^{-1}$ and AB^{-1} with the given values leads to a verification of our parametrization [Gui, Section 7]. We now indicate how to obtain these values.

Recall first that regular order three elements in SL(3,$\mathbb{C}$) have eigenvalue spectrum $\{1, j, j^2\}$. First, one may conjugate the pair (A, B) so that A and B are respectively lower and upper triangle, with eigenvalues organised as in (14). This amounts to choosing a basis of $\mathbb{C}^3$ of the form (v_B, v, v_A) where v_A (resp. v_B) is a j^2-eigenvector for A (resp. a j-eigenvector for B), and v is a non zero vector in the intersection $V_A \cap V_B$, where V_A (resp. V_B) is spanned by v_A and a 1-eigenvector of A (resp. v_B and a 1 eigenvector for B). Conjugating by a diagonal matrix allows to bring off-diagonal coefficients equal to j^2 in A and to j in B as shown in (14).

We need now to determine $a, b, c,$ and d from (13). These four conditions correspond to the following system of equations.

$$(\Sigma) \quad \begin{cases} (a + b)(d + c) + 2aj^2 + 2jd = z_1 & (15) \\ (a - b)(d + c) - 2a - 2d + 3 = z_2 & (16) \\ (a - b)(d - c) + 2ja + 2j^2d = z_3 & (17) \\ (a + b)(d - c) - 2a - 2d + 3 = z_4 & (18) \end{cases}$$

This system is relatively easy to solve using a computer and, for instance, Gröbner bases. However, it is also solvable by hand, and we indicate now how to do it. Before going any further, let us observe that conjugating the pair (A, B) by the matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

amounts to do following exchanges in Σ:

$$a \longleftrightarrow d,\ b \longleftrightarrow c,\ j \longleftrightarrow j^2. \tag{19}$$

More precisely, the left-hand sides of (15) and (17) are preserved by these changes, whereas the left-hand sides of (16) and (18) are exchanged.

Next, we compute linear combinations of the above equations, and obtain the following equivalent system:

$$(\Sigma') \begin{cases} (15)+(16)+(17)+(18): & 4ad - 6a - 6d - z_1 - z_2 - z_3 - z_4 + 6 = 0 & (20) \\ (15)-(16)+(17)-(18): & 4bc + 2a + 2d - z_1 + z_2 - z_3 + z_4 - 6 = 0 & (21) \\ (15)-(17): & 2ac + 2bd - 4aj + 4dj - 2a + 2d - z_1 + z_3 = 0 & (22) \\ (16)-(18): & 2ac - 2bd - z_2 + z_4 = 0 & (23) \end{cases}$$

To obtain the value of a annonced in the statement, we proceed ad follows.

- The two equations (22) and (23) are linear in b and c. We solve them to obtain expressions of b and c in terms of a and d.
- Plugging these expressions of b and c in (21) and taking numerator, we obtain an equation that relates a and d and involves the monomials a^2d, ad^2, a^2, d^2, ad, a and d. This equation can by simplified by observing that the product ad can be expressed as an affine function of a and d using (20). Doing so, most of the monomials simplify and we obtain a linear relation between a and d. This yields an expression of d as a function of a, which can be inserted back in (20). We obtain this way a quadratic equation in a, which is:

$$\begin{aligned} 0 = & 4(jz_1 + z_2 + j^2z_3 + z_4 + 3)a^2 - 2(6jz_1 + 6j^2z_3 + z_1z_3 - z_2z_4 + 9)a \\ & + 9 + 6jz_1 - 3z_2 + 6j^2z_3 - 3z_4 + j^2z_1^2 + z_2^2 + jz_3^2 + z_4^2 \\ & - jz_1z_2 + 2z_1z_3 - jz_1z_4 - j^2z_2z_3 - z_2z_4 - j^2z_3z_4 \end{aligned} \tag{24}$$

The discriminant of this quadratic equation is Δ, and we obtain in turn two possible values for a, corresponding to the two square roots of Δ. We obtain in turn the value of d given in the statement. Note that d is obtained from a by the exchanges $j \longleftrightarrow j^2$ and $z_2 \longleftrightarrow z_4$. This correspond to the symmetry of the system (Σ) given in (19).

As observed above, knowing the values of a and d gives us the values of b and c. However, the expressions obtained by solving (22) and (23) are not exactly those given in the statement, and simplifying them is quite intricate. The following strategy gives a way to determine b and c more directly. First of all, we know from Lawton's theorem and our determination of a and d that b belongs to $k[\delta]$ (recall that $k = \mathbb{Q}[j](z_1, z_2, z_3, z_4)$). Hence, we may look for it under the form $b = aP + Q$ where P and Q belong to k. We use this form in the equation (22)−(23), and also plug the values of a and d. This leads to an equation, linear in P and Q, between two elements of $k[\delta]$. Isolating the coefficient of δ and the remaining part, we find 2 linear equations in P and Q. Solving those equations leads to the given value for b. The value for c can be obtained using the symmetries of Σ given in (19). □

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