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Shape Holomorphy of the stationary Navier-Stokes Equations*

Albert Cohen, Christoph Schwab and Jakob Zech

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Abstract

We consider the stationary Stokes and Navier-Stokes Equations for viscous, incompressible flow in parameter dependent bounded domains D_T, subject to homogeneous Dirichlet ("no-slip") boundary conditions on $\partial\mathrm{D}_T$. Here, D_T is the image of a given fixed nominal Lipschitz domain $\hat{\mathrm{D}} \subseteq \mathbb{R}^d$, $d \in \{2,3\}$, under a map $T : \mathbb{R}^d \to \mathbb{R}^d$. We establish shape holomorphy of Leray solutions which is to say, holomorphy of the map $T \mapsto (\hat{u}_T, \hat{p}_T)$ where $(\hat{u}_T, \hat{p}_T) \in H_0^1(\hat{\mathrm{D}})^d \times L^2(\hat{\mathrm{D}})$ denotes the pullback of the corresponding weak solutions and T varies in $W^{k,\infty}$ with $k \in \{1,2\}$, depending on the type of pullback. We consider in particular parametrized families $\{T_{\boldsymbol{y}} : \boldsymbol{y} \in U\} \subseteq W^{1,\infty}$ of domain mappings, with parameter domain $U = [-1,1]^{\mathbb{N}}$ and with affine dependence of $T_{\boldsymbol{y}}$ on $\boldsymbol{y}$. The presently obtained shape holomorphy implies summability results and n-term approximation rate bounds for gpc ("generalized polynomial chaos") expansions for the corresponding parametric solution map $\boldsymbol{y} \mapsto (\hat{u}_{\boldsymbol{y}}, \hat{p}_{\boldsymbol{y}}) \in H_0^1(\hat{\mathrm{D}})^d \times L^2(\hat{\mathrm{D}})$.

1 Introduction

In recent years, the significance of sparsity in representations of solution manifolds of parametric PDE models in the sciences for several classes of algorithms has been recognized. We mention only high-dimensional, sparse tensor approximation and sparse grid interpolation (see [6, 5] and the references there), Model Order Reduction (MOR for short) for PDEs in parametric geometries (see for example [11, 15] and the references in these articles), and Reduced Basis (RB for short) methods [12, 19].

A general principle to establish sparsity in high-dimensional, parametric solution families of PDEs has been identified in [8]: the holomorphic dependence of the solution on the PDE parameters with quantitative control on the size of the domain of analytic continuation, independent of the number of parameters in the model. One model class which is of particular interest in biomedical applications is viscous, incompressible flow at low or moderate Reynolds number. Low-parametric, sparse representations of viscous flows in the context of physiological fluid flow are of particular relevance for efficient, patient specific simulations. Accordingly, recent years have seen significant activity in MOR and RB algorithms for this problem class; we refer only to [15, 20, 14, 3, 4] and the references there.

Previous mathematical results on parameter sparsity of solution manifolds comprise scalar, elliptic PDEs with holomorphic dependence of solutions with respect to parameters in parametric representations of their domain of definition. We refer to [8, Section 4], where the holomorphic

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dependence of solutions on parameters in Fourier representations of scalar, linear elliptic coercive PDEs was shown. In [13], an analogous result was given for linear, scalar and possibly indefinite elliptic problems in families of parametric domains which are homeomorphic to the unit disk.

In the present paper, we establish shape holomorphy, i.e., the holomorphic dependence of solutions on transformations of the shape of the domain, for the Stokes and the nonlinear Navier-Stokes equations governing stationary, viscous and incompressible flow. Despite mathematical shape calculus having reached a certain maturity during the past decades due to its importance in shape optimization problems in science and engineering (see, e.g., [10] and the references there, and [24] and its references for the particular case of Navier-Stokes equations), we are not aware of results on the holomorphic dependence of solutions on the shape of the domain. As we explain in some detail in Section 5, the shape-holomorphy results obtained in the present paper imply, in particular, the possibility of low-parametric approximation of solutions on large families of domains represented by suitable holomorphic parametrizations, on high-dimensional parameter domains. This implies for a wide range of concrete domain representations used in computational engineering the existence of sparse surrogates of the parametric solution families. This, in turn, is crucial in a number of applications, and we indicate some of these in Section 5.

The presently obtained results and their proofs differ from the argument in [8, 13]: a first result which we establish holds under the usual small data hypothesis to ensure uniqueness of solutions of the stationary NSE and the novel approach to its proof is also applicable to all mentioned, linear elliptic problems. For a family of domains $\mathrm{D}_T = T(\hat{\mathrm{D}})$ where $\hat{\mathrm{D}}$ is a reference domain and T a bi-Lipschitz transformation, we show existence, uniqueness and holomorphic dependence of the pullback solution $(\hat{u}_T, \hat{p}_T)$ defined on $\hat{\mathrm{D}}$ with respect to the domain transformation T itself, rather than on real valued parameters which arise in a particular parametrization of T, which is the perspective taken in [8]. As an immediate consequence, we obtain parametric holomorphy in concrete, parametric families $\{T_{\boldsymbol{y}} : \boldsymbol{y} \in U\}$ of domain transformations. For example, if the transformation $T_{\boldsymbol{y}}$, with $\boldsymbol{y} = (y_j)_{j\in\mathbb{N}} \in [-1, 1]^{\mathbb{N}}$ depends holomorphically on each y_j. This more abstract approach has the advantage, that the assumption of a concrete parametrization of the transformation (typically with affine dependence on each y_j) is not necessary. For specific parametric families of mappings, such as the affine-parametric dependencies considered in [5, 6, 8], the present results imply in particular the results on analytic dependence of parametric solutions which were obtained in these references. We note that existence of domain differentials is classically invoked in numerical shape optimization, see, e.g., [17, 18, 24] and the references there. However, we are not aware of earlier results discussing the holomorphic dependence of solutions on the shape of the domain.

The structure of the paper is as follows. In Section 2, we introduce the formulation of the problem and set up terminology and notation. Sections 3 and 4 contain the main results: we transform the equations to a reference domain and prove holomorphic dependence of the weak pullback solutions to the stationary Stokes and Navier-Stokes problem on the transformation. To this end we separately treat two differing transformation models, the "plain pullback" and the Piola transformation. The space in which the transformation T varies depends on the type of pullback, namely $W^{1,\infty}$ in the case of the plain pullback and a weighted version of $W^{2,\infty}$ in the case of the Piola transformation. Finally, Section 5 is intended to give the practically relevant implications of our results in the context of uniform approximation: in Section 5.1 we first recall a framework for holomorphic dependence on a countable number of parameters. Following this, we establish holomorphic parameter dependence under sufficiently smooth transformations of the domain, and provide sparsity results in Section 5.2. Dimension-independent convergence rate bounds of gpc ("generalized polynomial chaos") approximations which are implied by the presently established holomorphy results are briefly indicated.

2 Problem Formulation

In the bounded Lipschitz domain $\mathrm{D} \subseteq \mathbb{R}^d$, $d \in \{2,3\}$, and for a given volume force $f \in L^2(\mathrm{D})^d$, we consider the *Stokes problem* on D, i.e. we seek a *velocity field* $u : \mathrm{D} \to \mathbb{R}^d$ and a *pressure* $p : \mathrm{D} \to \mathbb{R}$ solving

$$-\Delta u + \nabla p = f \qquad \text{in D,} \tag{2.1a}$$

$$\operatorname{div} u = 0 \qquad \text{in D,} \tag{2.1b}$$

$$u = 0 \qquad \text{on } \partial\mathrm{D}. \tag{2.1c}$$

We consider the variational formulation in $H_0^1(\mathrm{D})^d \times L^2_{\#}(\mathrm{D})$ where $L^2_{\#}(\mathrm{D}) := L^2(\mathrm{D})/\mathbb{R}$: find $(u,p) \in H_0^1(\mathrm{D})^d \times L^2_{\#}(\mathrm{D})$ such that

$$a(u,v) - b(v,p) = F(v), \qquad v \in H_0^1(\mathrm{D}), \tag{2.2a}$$

$$b(u,q) = 0, \qquad q \in L^2_{\#}(\mathrm{D}), \tag{2.2b}$$

where $F(v) := \int_{\mathrm{D}} f \cdot v$ and the bilinear forms are given by

$$a(u,v) := \int_{\mathrm{D}} \nabla u \cdot \nabla v \quad \text{and} \quad b(u,v) = \int_{\mathrm{D}} \operatorname{div}(v) p, \tag{2.3}$$

with $\nabla u \cdot \nabla v = \operatorname{tr}(\nabla u \nabla v^\top)$, where $\nabla u := \left(\frac{\partial u_i}{\partial x_j}\right)_{i,j=1,\dots,d}$. Note that the quotient space $L^2_{\#}(\mathrm{D})$ is equipped with the norm

$$\|p\|_{L^2_{\#}} := \min_{c \in \mathbb{R}} \|p - c\|_{L^2}.$$

This problem is known to be well-posed (see e.g. [1, Thm. 4.2.3]) due to the continuity of both bilinear forms on the appropriate spaces, the ellipticity of a in $H_0^1(\mathrm{D})^d$ and the inf-sup condition [2]

$$\inf_{p \in L^2_{\#}(\mathrm{D})} \sup_{u \in H_0^1(\mathrm{D})^d} \frac{b(u,p)}{\|u\|_{H_0^1} \|p\|_{L^2_{\#}}} \geq \beta = \beta(\mathrm{D}) > 0\ .$$

Closely connected to the linear Stokes problem is the nonlinear *stationary Navier-Stokes problem*, which reads

$$-\Delta u + (u \cdot \nabla) u + \nabla p = f \qquad \text{in D,} \tag{2.4a}$$

$$\operatorname{div} u = 0 \qquad \text{in D,} \tag{2.4b}$$

$$u = 0 \qquad \text{on } \partial\mathrm{D}. \tag{2.4c}$$

Recall that the vector $(u \cdot \nabla) v$ has components $\sum_{j=1}^d u_j \frac{\partial v_i}{\partial x_j}$ and may also be viewed as the matrix-vector product $\nabla v\, u$. As a weak formulation we derive similarly as above

$$a(u,v) - b(v,p) + t(u,u,v) = F(v), \qquad v \in (H_0^1(\mathrm{D}))^d, \tag{2.5a}$$

$$b(u,q) = 0, \qquad q \in L^2_{\#}(\mathrm{D}), \tag{2.5b}$$

where (a, b, F) are as before and the trilinear form is given by

$$t(u, v, w) = \int_{\mathrm{D}} (u \cdot \nabla) v \cdot w, \qquad u, v, w \in (H_0^1(\mathrm{D}))^d \,. \tag{2.6}$$

This trilinear form is continuous on $H_0^1(\mathrm{D})^d \times H_0^1(\mathrm{D})^d \times H_0^1(\mathrm{D})^d$ and antisymmetric in the last two arguments provided that $\mathrm{div}(u) = 0$. Classical arguments based on the Faedo-Galerkin method, Brouwer fixed point theorem, and limiting techniques ensure the existence of a solution (see e.g. [23, Chap. II, §1, Thm. 1.2]). By inserting u as test function, one obtains the standard energy estimate

$$\|u\|_{H_0^1(\mathrm{D})} \leq \|f\|_{H^{-1}(\mathrm{D})} \,. \tag{2.7}$$

The solution is ensured to be unique for sufficiently small data f [23, Chap. II, §1, Thm. 1.3].

We are interested in the situation where the domain may vary within a parametrized family, such that all domains in this family are contained in a "hold-all" domain

$$\mathrm{D} \subset \mathrm{D}_H \,. \tag{2.8}$$

Eventually we are interested in numerical reduced modeling methods that allow to approximate in some sense the mapping between the parameters and the solution. This requires to understand the dependence of the solution with respect to the parameter, and thus with respect to the domain. In order to be able to compare the different solutions, we will pull them back to some *nominal domain* $\hat{\mathrm{D}} \subseteq \mathbb{R}^d$, based on transformations $T : \hat{\mathrm{D}} \to \mathrm{D}$. The goal is to establish appropriate conditions in order for the pullbacks to depend smoothly on the transformation, which will be made more precise in the subsequent sections. The course of action is as follows: we first derive equivalent PDEs on the nominal domain $\hat{\mathrm{D}}$, via the plain pullback. Following this, we prove holomorphy of the mapping which associates a domain transformation T to the solution $(\hat{u}_T, \hat{p}_T)$ of the corresponding problem on the reference domain. We then proceed in the same manner for a second transformation type, the Piola pullback. Finally, concrete examples of transformation families depending on a parameter vector $\boldsymbol{y}$ will be given in section 5. Our results then imply the desired holomorphic dependence of the solution on parameters in parametric representations of the domain transformations.

3 The plain pullback

3.1 Domain transformation

We now derive equations in the reference domain $\hat{\mathrm{D}}$ which are satisfied by the pullback of the solutions in the physical domain D. Throughout section 3.1, $T : \hat{\mathrm{D}} \to \mathrm{D}$ denotes a bijective bi-Lipschitz map between these Lipschitz domains, that is, $T \in W^{1,\infty}(\hat{\mathrm{D}}, \mathrm{D})$ and $T^{-1} \in W^{1,\infty}(\mathrm{D}, \hat{\mathrm{D}})$. Up to multiplying by -1 in further expressions, we assume its Jacobian determinant to be positive a.e. in $\hat{\mathrm{D}}$.

Remark 3.1. *Note that if $\hat{\mathrm{D}}$ is a Lipschitz domain, the fact that T is bi-Lipschitz readily implies that $D = T(\hat{\mathrm{D}})$ is also a Lipschitz domain.*

We use the standard notation $x = T(\hat{x})$ with $\hat{x} \in \hat{\mathrm{D}}$ and $x \in \mathrm{D}$. We denote by dT the differential of T and by

$$J(\hat{x}) := \det(dT(\hat{x})), \tag{3.1}$$

the Jacobian. For all involved quantities, velocities, pressures and the right-hand side, we use the plain pullback transformation

$$\hat{\varphi}(\hat{x}) = \varphi(x) \quad \text{i.e.} \quad \hat{\varphi} := \varphi \circ T \,. \tag{3.2}$$

With such choices, the maps

$$L^2_{\#}(\mathrm{D}) \ni p \mapsto \hat{p} \in L^2_{\#}(\hat{\mathrm{D}}) \qquad \text{and} \qquad H^1_0(\mathrm{D})^d \ni u \mapsto \hat{u} \in H^1_0(\hat{\mathrm{D}})^d, \tag{3.3}$$

are isomorphisms. For the first map, this is due to the fact that for any constant c, we have $\hat{c} = c$ and thus

$$\|p - c\|_{L^2(\mathrm{D})} = \|(\hat{p} - c)J^{1/2}\|_{L^2(\hat{\mathrm{D}})}, \tag{3.4}$$

so that

$$m_0\|\hat{p}\|_{L^2_{\#}(\hat{\mathrm{D}})} \leq \|p\|_{L^2_{\#}(\mathrm{D})} \leq M_0\|\hat{p}\|_{L^2_{\#}(\hat{\mathrm{D}})}, \tag{3.5}$$

where $m_0 = m_0(T) := \min_{\hat{x}\in\hat{\mathrm{D}}} J^{1/2}(\hat{x}) > 0$ and $M_0 = M_0(T) := \max_{\hat{x}\in\hat{\mathrm{D}}} J^{1/2}(\hat{x}) < \infty$. For the second map, we have

$$\|u\|^2_{H^1_0(\mathrm{D})} := \int_{\mathrm{D}} \nabla u \cdot \nabla u = \int_{\mathrm{D}} \mathrm{tr}\,(\nabla u \nabla u^\top) = \int_{\hat{\mathrm{D}}} \mathrm{tr}(\nabla \hat{u}\, dT^{-1} dT^{-\top} \nabla \hat{u}^\top) J. \tag{3.6}$$

If A and B are $d \times d$ matrices, it is easily seen that

$$\|AB\|_{HS} \leq \min\{\|A\|\,\|B\|_{HS}, \|A\|_{HS}\,\|B\|\} \tag{3.7}$$

where $\|M\|_{HS} = \mathrm{tr}(MM^\top)$ is the Hilbert-Schmidt norm (also referred to as Frobenius norm, and denoted by $\|M\|_F$) and $\|M\|$ is the spectral norm. Indeed, it suffices to note that

$$\|AB\|^2_{HS} = \sum_{j=1}^{d} \|AB_j\|^2 \leq \|A\|^2 \sum_{j=1}^{d} \|B_j\|^2 = \|A\|^2 \|B\|^2_{HS}, \tag{3.8}$$

where B_j denotes the jth column of B, and thus (3.7) follows by exchanging A and B.

Therefore,

$$m_1\|\hat{u}\|_{H^1_0(\hat{\mathrm{D}})} \leq \|u\|_{H^1_0(\mathrm{D})} \leq M_1\|\hat{u}\|_{H^1_0(\hat{\mathrm{D}})}, \tag{3.9}$$

where $m_1 = m_1(T) := \min_{\hat{x}\in\hat{\mathrm{D}}} J^{1/2}(\hat{x})\|dT(\hat{x})\|^{-1}$ and $M_1 = M_1(T) := \max_{\hat{x}\in\hat{\mathrm{D}}} J^{1/2}(\hat{x})\|dT^{-1}(\hat{x})\|$. By duality we also find that

$$H^{-1}(\mathrm{D})^d \ni f \mapsto \hat{f} \in H^{-1}(\hat{\mathrm{D}})^d, \tag{3.10}$$

is an isomorphism with constants M_1^{-1} and m_1^{-1}.

We first describe the equations satisfied by the pullback solutions of the Stokes system (2.1). By transformation, this system becomes

$$\begin{cases} a_T(\hat{u}, \hat{v}) - b_T(\hat{v}, \hat{p}) = F_T(\hat{v}), & \hat{v} \in (H^1_0(\hat{\mathrm{D}}))^d, \\ b_T(\hat{u}, \hat{q}) = 0, & \hat{q} \in L^2_{\#}(\hat{\mathrm{D}}), \end{cases} \tag{3.11}$$

where the bilinear forms and linear forms are defined as follows:

$$a_T(\hat{u}, \hat{v}) := \int_{\hat{\mathrm{D}}} \mathrm{tr}(\nabla \hat{u}\, dT^{-1} dT^{-\top} \nabla \hat{v}^\top) J, \tag{3.12a}$$

and

$$b_T(\hat{v},\hat{p}) := \int_{\hat{\mathrm{D}}} \mathrm{tr}(\nabla\hat{v}\, dT^{-1})\hat{p}J, \tag{3.12b}$$

and

$$F_T(\hat{v}) := \int_{\hat{\mathrm{D}}} \hat{f}\cdot\hat{v}J. \tag{3.12c}$$

The isomorphism properties ensure that these forms satisfy the continuity, coercivity and inf-sup properties on the spaces $(H_0^1(\hat{\mathrm{D}}))^d$ and $L^2_\#(\hat{\mathrm{D}})$ and therefore, with the notation

$$X = (H_0^1(\hat{\mathrm{D}}))^d \times L^2_\#(\hat{\mathrm{D}}), \tag{3.12d}$$

the system has a unique solution $(\hat{u}_T, \hat{p}_T) \in X$.

Likewise, with

$$t_T(\hat{u},\hat{v},\hat{w}) = \int_{\hat{\mathrm{D}}} (dT^{-1}\hat{u}\cdot\nabla)\hat{v}\cdot\hat{w}J, \tag{3.12e}$$

the pullback equations for Navier-Stokes have the form

$$\begin{cases} a_T(\hat{u},\hat{v}) - b_T(\hat{v},\hat{p}) + t_T(\hat{u},\hat{u},\hat{v}) = F_T(\hat{v}), & \hat{v}\in (H_0^1(\hat{\mathrm{D}}))^d, \\ b_T(\hat{u},\hat{q}) = 0, & \hat{q}\in L^2_\#(\hat{\mathrm{D}}). \end{cases} \tag{3.13}$$

They also admit a solution $(\hat{u}_T, \hat{p}_T) \in X$. We obtain an explicit dependence on T in the a-priori estimate by writing

$$m_1^2\|\hat{u}_T\|^2_{H_0^1(\hat{\mathrm{D}})} \leq a_T(\hat{u}_T,\hat{u}_T) = \|u_T\|^2_{H_0^1(\mathrm{D})} \leq \|f\|^2_{H^{-1}(\mathrm{D})} \leq \|f\|^2_{H^{-1}(\mathrm{D}_H)}, \tag{3.14}$$

and thus

$$\|\hat{u}_T\|_{H_0^1(\hat{\mathrm{D}})} \leq m_1(T)^{-1}\|f\|_{H^{-1}(\mathrm{D}_H)}. \tag{3.15}$$

We also obtain an explicit dependence on T in the small data assumption for uniqueness by using the continuity property of the trilinear form

$$|t_T(\hat{u},\hat{v},\hat{w})| \leq N_1\|\hat{u}\|_{H_0^1(\mathrm{D})}\|\hat{v}\|_{H_0^1(\mathrm{D})}\|\hat{w}\|_{H_0^1(\mathrm{D})}, \tag{3.16}$$

where $N_1 = N_1(T) = C_E^2 \max_{\hat{x}\in\hat{\mathrm{D}}} J(\hat{x})\|dT^{-1}(\hat{x})\|$ with C_E^2 the Sobolev embedding constant between $H_0^1(\hat{\mathrm{D}})$ and $L^4(\hat{\mathrm{D}})$. Then, if $\hat{u}_1$ and $\hat{u}_2$ are two different solutions, we may write

$$m_1^2\|\hat{u}_1-\hat{u}_2\|^2_{H_0^1(\hat{\mathrm{D}})} \leq a_T(\hat{u}_1-\hat{u}_2,\hat{u}_1-\hat{u}_2) = t_T(\hat{u}_1-\hat{u}_2,\hat{u}_2,\hat{u}_1-\hat{u}_2),$$

and

$$|t_T(\hat{u}_1-\hat{u}_2,\hat{u}_2,\hat{u}_1-\hat{u}_2)| \leq N_1\|\hat{u}_1-\hat{u}_2\|^2_{H_0^1}\|\hat{u}_2\|_{H_0^1} \leq N_1 m_1^{-1}\|\hat{u}_1-\hat{u}_2\|^2_{H_0^1}\|f\|_{H^{-1}(\mathrm{D}_H)}.$$

Therefore, uniqueness of $\hat{u}_T$ is ensured under the condition

$$\|f\|_{H^{-1}(\mathrm{D}_H)} < C_1, \quad C_1 = C_1(T) = \frac{m_1(T)^3}{N_1(T)}, \tag{3.17}$$

and uniqueness of $\hat{p}_T$ follows from the inf-sup condition.

Remark 3.2. *Note that* (3.11) *can also be expressed as*

$$\begin{cases} A_T\hat{u} - B_T^*\hat{p} = F_T, \\ B_T\hat{u} = 0, \end{cases} \tag{3.18}$$

where $A_T \in \mathcal{L}((H_0^1(\hat{\mathrm{D}}))^d, (H^{-1}(\hat{\mathrm{D}}))^m)$ *and* $B_T \in \mathcal{L}((H_0^1(\hat{\mathrm{D}}))^d, L^2_{\#}(\hat{\mathrm{D}}))$ *are the operators associated to the bilinear forms* a_T, b_T, *and likewise* (3.13) *is equivalently stated as*

$$\begin{cases} A_T\hat{u} - B_T^*\hat{p} + N_T(\hat{u}) = F_T, \\ B_T\hat{u} = 0, \end{cases} \tag{3.19}$$

where $N_T : H_0^1(\hat{\mathrm{D}})^d \to H^{-1}(\hat{\mathrm{D}})^d$ *is the nonlinear operator that maps* $\hat{u}$ *to the linear form* $\hat{v} \mapsto t_T(\hat{u}, \hat{u}, \hat{v})$.

3.2 Holomorphy of the plain domain-to-solution map

We are now interested in analyzing the smoothness properties of the domain-to-solution map

$$T \mapsto (\hat{u}_T, \hat{p}_T), \tag{3.20}$$

as T varies in a certain set $\mathfrak{T}$.

The differential of such domain to pullback solution maps is related to the notion of material derivative which is of common use in the context of shape optimization and which we next recall, see [22] for more details. If $V = V(x,t)$ is a vector field and T_t is the corresponding flow, then if u_Ω is the solution to some given PDE for the domain Ω, then we define

$$\dot{u}(\Omega, V) = \lim_{t\to 0} \frac{1}{t}\big(u_{T_t(\Omega)} \circ T_t - u_\Omega\big), \tag{3.21}$$

where the limit needs to be defined in a given topology, for example in the strong sense for a Sobolev space $W^{m,p}(\Omega)$. Then, a simple computation shows that if $\mathcal{F}_T$ is the Fréchet derivative at T of the map $T \mapsto \hat{u}_T$ for this topology, we have

$$\dot{u}(D_T, V) = \mathcal{F}_T(V_0 \circ T) \circ T^{-1}, \quad V_0 = V(\cdot, 0). \tag{3.22}$$

Our objective is to establish the holomorphy of the map (3.20) for problems (3.11) and (3.13) under certain assumptions, which are stated below.

Assumption 3.3. *The set* $\mathfrak{T}$ *is compact in* $W^{1,\infty}(\hat{\mathrm{D}}, \mathbb{R}^d)$ *and such that* $T^{-1} \in W^{1,\infty}(\mathrm{D}, \mathbb{R}^d)$ *for every* $T \in \mathfrak{T}$, *where* $\mathrm{D} = T(\hat{\mathrm{D}})$.

Note that, under Assumption 3.3, the quantities $\|dT(\hat{x})\|$, $\|dT^{-1}(\hat{x})\|$, $J(\hat{x})$ and $J^{-1}(\hat{x})$ are uniformly bounded over $\hat{x} \in \hat{\mathrm{D}}$ and $T \in \mathfrak{T}$. In particular the constant $C_1(T)$ from the previous small data assumption is bounded by below away from 0 uniformly for $T \in \mathfrak{T}$. Also note that the "hold-all" domain $\mathrm{D}_H = \bigcup_{T\in\mathfrak{T}} T(\hat{\mathrm{D}}) \subseteq \mathbb{R}^d$ is a bounded set.

Assumption 3.4. *The function* f *is analytic in an open neighborhood of* $\overline{\mathrm{D}}_H$.

For $\varepsilon > 0$, we define the complex valued ε-neighborhood of $\mathfrak{T}$,

$$\mathfrak{T}_\varepsilon := \{\tilde{T} \in W^{1,\infty}(\hat{\mathrm{D}}, \mathbb{C}^d) \,:\, \exists T \in \mathfrak{T},\ \|\tilde{T} - T\|_{W^{1,\infty}(\hat{\mathrm{D}})} < \varepsilon\}\,. \tag{3.23}$$

Remark 3.5. *Assumption 3.3 implies, by compactness of $\mathfrak{T}$, that there exists ε_1 such that the quantities $\|dT^{-1}(\hat{x})\|$ and $J^{-1}(\hat{x})$ are uniformly bounded over $\hat{x} \in \hat{\mathrm{D}}$ and $T \in \mathfrak{T}_{\varepsilon_1}$.*

Remark 3.6. *Assumption 3.4 implies that there exists $\varepsilon_2 > 0$ and an open set $\mathrm{D}_f \subseteq \mathbb{C}^d$ such that $\overline{T(\hat{\mathrm{D}})} \subseteq \mathrm{D}_f$ for all $T \in \mathfrak{T}_{\varepsilon_2}$ and such that f has an holomorphic extension $f : \mathrm{D}_f \to \mathbb{C}$.*

We first establish the holomorphic dependence of the domain-to-solution map (3.20) for the Stokes problem under the above assumptions. In what follows we use the notation

$$X_{\mathbb{C}} := H_0^1(\hat{\mathrm{D}}, \mathbb{C})^d \times L^2_{\#}(\hat{\mathrm{D}}, \mathbb{C}). \tag{3.24}$$

for the complex valued version of X.

Theorem 3.7. *Let Assumptions 3.3 and 3.4 be satisfied. Then there exists $\varepsilon = \varepsilon(\hat{\mathrm{D}}, \mathfrak{T}) > 0$ such that the domain-to-solution map $T \mapsto (\hat{u}_T, \hat{p}_T)$, with $(\hat{u}_T, \hat{p}_T)$ solving* (3.18)*, admits an extension on $\mathfrak{T}_\varepsilon$ as in* (3.23)*, which is holomorphic and uniformly bounded as a mapping from $W^{1,\infty}(\hat{\mathrm{D}}, \mathbb{C}^d)$ to $X_{\mathbb{C}}$.*

Proof. Let $T \in \mathfrak{T}$ and define $\mathrm{D} = T(\hat{\mathrm{D}})$. The bilinear form $a_T(\cdot, \cdot)$ is continuous and elliptic on $H_0^1(\hat{\mathrm{D}})^d \times H_0^1(\hat{\mathrm{D}})^d$, and the bilinear form b_T is continuous on $H_0^1(\hat{\mathrm{D}})^d \times L^2_{\#}(\hat{\mathrm{D}})$ and satisfies an inf-sup condition

$$\inf_{\hat{p} \in L^2_J(\hat{\mathrm{D}})} \sup_{\hat{u} \in H_0^1(\hat{\mathrm{D}})^d} \frac{b_T(\hat{u}, \hat{p})}{\|\hat{u}\|_{H_0^1(\hat{\mathrm{D}})} \|\hat{p}\|_{L^2_{\#}(\hat{\mathrm{D}})}} \geq \beta = \beta(T) > 0. \tag{3.25}$$

This implies that the pullback Stokes operator

$$S_T : (\hat{u}, \hat{p}) \mapsto S_T(\hat{u}, \hat{p}) = (A_T \hat{u} - B_T^* \hat{p}, B_T \hat{u}), \tag{3.26}$$

is an isomorphism from X to its dual $X^* = H^{-1}(\hat{\mathrm{D}})^d \times L^2_{\#}(\hat{\mathrm{D}})$. Obviously, the solution $(\hat{u}_T, \hat{p}_T)$ to (3.11) is defined as

$$(\hat{u}_T, \hat{p}_T) = S_T^{-1} H_T, \quad H_T := (F_T, 0).$$

We next extend S_T as an operator from the complex space

$X_{\mathbb{C}}$ to its dual $X_{\mathbb{C}}^* = H^{-1}(\hat{\mathrm{D}}, \mathbb{C})^d \times L^2_{\#}(\hat{\mathrm{D}}, \mathbb{C})$, and H_T as an element of $X_{\mathbb{C}}^*$. This is done by writing

$$(A_T \hat{u})(\hat{v}) := a_T(\hat{u}, \hat{v}) := \int_{\hat{\mathrm{D}}} \mathrm{tr}(\nabla \hat{u}\, dT^{-1} dT^{-\top} \nabla \hat{v}^\top) J, \tag{3.27}$$

as well as

$$(B_T \hat{v})(\hat{p}) := b_T(\hat{v}, \hat{p}) := \int_{\hat{\mathrm{D}}} \mathrm{tr}(\nabla \hat{v}\, dT^{-1}) \hat{p} J. \tag{3.28}$$

and

$$F_T(\hat{v}) := \int_{\hat{\mathrm{D}}} \hat{f}_T \cdot \hat{v} J, \quad \hat{f}_T \cdot \hat{v} := (\hat{f}_T)_1 \hat{v}_1 + \cdots + (\hat{f}_T)_d \hat{v}_d,$$

where $\hat{f}_T := f \circ T$. Note that, since we are not using complex conjugates in the above definitions, $X_{\mathbb{C}}^*$ is not the antidual of $X_{\mathbb{C}}$ but its regular dual. The operator S_T is an isomorphism from $X_{\mathbb{C}}$ onto $X_{\mathbb{C}}^*$: we may write for $(\hat{u}, \hat{p}) = (\hat{u}_1, \hat{p}_1) + i(\hat{u}_2, \hat{p}_2)$,

$$S_T(\hat{u}, \hat{p}) = S_T(\hat{u}_1, \hat{p}_1) + i S_T(\hat{u}_2, \hat{p}_2),$$

and for $(F,G) = (F_1,G_1) + i(F_2,G_2)$,

$$S_T^{-1}(F,G) = S_T^{-1}(F_1,G_1) + iS_T^{-1}(F_2,G_2).$$

We finally observe that $T \mapsto S_T$ has a well defined and holomorphic, extension, as a mapping from $W^{1,\infty}(\hat{\mathrm{D}},\mathbb{C})$ to $\mathcal{L}(X_{\mathbb{C}}, X_{\mathbb{C}}^*)$, at any $T \in \mathfrak{T}_{\varepsilon_1}$ where ε_1 is as in Remark 3.5, and in particular at any $T \in \mathfrak{T}$. This extension is defined by (3.27) and (3.28), and the holomorphy follows from that of the maps $T \mapsto dT^{-1}$ when T is invertible and $T \mapsto J$. It follows that for each $T \in \mathfrak{T}$, there is an $\varepsilon_T > 0$ such that $S_{\tilde{T}}$ is invertible for all $\tilde{T}$ in the ball

$$B(T,\varepsilon_T) := \{\tilde{T} \in W^{1,\infty}(\hat{D},\mathbb{C}^d) \, : \, \|\tilde{T} - T\|_{W^{1,\infty}} \leq \varepsilon_T\},$$

and such that

$$T \mapsto S_T^{-1}$$

is holomorphic, as a mapping from $W^{1,\infty}(\hat{\mathrm{D}},\mathbb{C}^d)$ to $\mathcal{L}(X_{\mathbb{C}}^*, X_{\mathbb{C}})$, over this ball. By a finite covering argument, using the compactness of $\mathfrak{T}$, we obtain that there exists $\varepsilon_3 > 0$ such that this mapping is bounded and holomorphic on $\mathfrak{T}_{\varepsilon_3}$. On the other hand, using Remark 3.6, we also find that the map

$$T \mapsto H_T := (F_T, 0),$$

has a bounded and holomorphic extension, as a mapping from $W^{1,\infty}(\hat{\mathrm{D}},\mathbb{C}^d)$ to $X_{\mathbb{C}}^*$, on $\mathfrak{T}_{\varepsilon_2}$. This proves that the domain-to-solution map

$$T \mapsto (\hat{u}_T, \hat{p}_T),$$

is bounded and holomorphic on $\mathfrak{T}_\varepsilon$, with $\varepsilon := \min\{\varepsilon_2, \varepsilon_3\}$. □

For the sake of completeness, we next provide with the explicit expression of the Fréchet derivative of the solution map which we denote here by $\mathcal{S} : T \mapsto (\hat{u}_T, \hat{p}_T)$.

Proposition 3.8. *Let the assumptions of Theorem 3.7 be satisfied. Then, the Fréchet derivative* $d\mathcal{S}(T)(H)$ *of* $\mathcal{S}$ *at* $T \in \mathfrak{T}$ *in direction* $H \in W^{1,\infty}(\hat{\mathrm{D}},\mathbb{C}^d)$ *is given by the unique solution* $(\hat{w},\hat{r}) \in X_{\mathbb{C}}$ *of*

$$\begin{aligned}
&a_T(\hat{w},\hat{v}) - b_T(\hat{v},\hat{r}) = \\
&\quad \int_{\hat{\mathrm{D}}} \operatorname{tr}\left(\nabla \hat{u}_T \left[J(dT^{-1}dH\,dT^{-1}dT^{-\top} + dT^{-1}dT^{-\top}dH^\top dT^{-\top}) - \operatorname{tr}(\operatorname{Cof}(dT)^\top dH)dT^{-1}dT^{-\top}\right] \nabla \hat{v}^\top\right) \\
&\quad - \int_{\hat{\mathrm{D}}} \operatorname{tr}\left(\nabla \hat{v} \left[J dT^{-1} dH\, dT^{-1} - \operatorname{tr}(\operatorname{Cof}(dT)^\top dH)dT^{-1}\right]\right) \hat{p}_T + \int_{\hat{\mathrm{D}}} (df \circ T)H \cdot \hat{v} J \\
&\quad + \int_{\hat{\mathrm{D}}} \hat{f}_T \cdot \hat{v} \operatorname{tr}(\operatorname{Cof}(dT)^\top dH),
\end{aligned} \tag{3.29a}$$

$$b_T(\hat{w},\hat{q}) = \int_{\hat{\mathrm{D}}} \operatorname{tr}\left(\nabla \hat{u}_T \left[J dT^{-1} dH\, dT^{-1} - \operatorname{tr}(\operatorname{Cof}(dT)^\top dH)dT^{-1}\right]\right) \hat{q}, \tag{3.29b}$$

for all $(\hat{v},\hat{q}) \in X_{\mathbb{C}}$*, where* $\operatorname{Cof}(M)$ *denotes the cofactor matrix of* M*.*

Proof. Let S_T as in (3.26) and denote by $\hat{w}(H)$, $\hat{r}(H)$ the solution of (3.29). Both $\hat{w}$ and $\hat{r}$ are bounded linear mappings in H. We will show that

$$\|S_T((\hat{u}_{T+H}, \hat{p}_{T+H}) - (\hat{u}_T, \hat{p}_T) - (\hat{w}(H), \hat{r}(H)))\|_{\mathcal{L}(X_\mathbb{C}, X_\mathbb{C}^*)} = o(\|H\|_{W^{1,\infty}}) \quad \text{as} \quad \|H\|_{W^{1,\infty}(\hat{\mathrm{D}})} \to 0. \tag{3.30}$$

Since S_T is boundedly invertible, there exists $C > 0$ with $\|S_T(\hat{u}, \hat{p})\| \geq C\|(\hat{u}, \hat{p})\|$ for all $(\hat{u}, \hat{p}) \in X_\mathbb{C}$, hence (3.30) implies $\|\hat{u}_{T+H} - \hat{u}_T - \hat{w}(H)\|_{H_0^1} + \|\hat{p}_{T+H} - \hat{p}_T - \hat{r}(H)\|_{L^2_\#} = o(\|H\|_{W^{1,\infty}})$, so that our claim is an immediate consequence of (3.30).

Identifying $F_T \in (H_0^1(\hat{\mathrm{D}}, \mathbb{C})^d)'$ with $(F_T, 0) \in X'_\mathbb{C}$, there holds

$$S_T((\hat{u}_{T+H}, \hat{p}_{T+H}) - (\hat{u}_T, \hat{p}_T)) = S_T(\hat{u}_{T+H}, \hat{p}_{T+H}) - S_{T+H}(\hat{u}_{T+H}, \hat{p}_{T+H}) + F_{T+H} - F_T \tag{3.31}$$

and explicitly writing down these terms evaluated at $(\hat{v}, \hat{q})$ we get

$$\int_{\hat{\mathrm{D}}} \mathrm{tr}\left(\nabla \hat{u}_{T+H}\left[dT^{-1}dT^{-\top}J_T - d(T+H)^{-1}d(T+H)^{-\top}J_{T+H}\right]\nabla \hat{v}^\top\right)$$
$$- \int_{\hat{\mathrm{D}}} \mathrm{tr}\left(\nabla \hat{v}\left[dT^{-1}J_T - d(T+H)^{-1}J_{T+H}\right]\right)\hat{p}_{T+H} + \int_{\hat{\mathrm{D}}}\left(\hat{f}_{T+H}J_{T+H} - \hat{f}_T J_T\right)\cdot \hat{v}, \tag{3.32a}$$

$$\int_{\hat{\mathrm{D}}} \mathrm{tr}\left(\nabla \hat{u}_{T+H}\left[dT^{-1}J_T - d(T+H)^{-1}J_{T+H}\right]\right)\hat{q} \tag{3.32b}$$

where we used the notation $J_T := \det dT$ to indicate the T dependence. Now, w.r.t. the L^∞-norm,

$$f \circ (T+H) = f \circ T + (df \circ T)H + O(\|H\|^2_{W^{1,\infty}}), \tag{3.33a}$$
$$d(T+H)^{-1} = dT^{-1} - dT^{-1}dH dT^{-1} + O(\|H\|^2_{W^{1,\infty}}), \tag{3.33b}$$
$$\det d(T+H) = \det dT + \mathrm{tr}(\mathrm{Cof}(dT)^\top dH) + O(\|H\|^2_{W^{1,\infty}}), \tag{3.33c}$$

as $\|H\|_{W^{1,\infty}} \to 0$, where we mean by this notation that e.g. $\det d(T+H) = \det dT + g$ and $\|g\|_{L^\infty(\hat{\mathrm{D}})} = O(\|H\|_{W^{1,\infty}(\hat{\mathrm{D}})})$ as $\|H\|_{W^{1,\infty}(\hat{\mathrm{D}})} \to 0$. Denote now by $l_H((\hat{u}_T, \hat{p}_T), (\hat{v}, \hat{q}))$ the right-hand side of (3.29), and we'll consider l_H as an operator of its first argument. Inserting (3.33) into (3.32) we arrive at

$$\begin{aligned} S_T((\hat{u}_{T+H}, \hat{p}_{T+H}) - (\hat{u}_T, \hat{p}_T)) &= l_H((\hat{u}_T, \hat{p}_T)) + l_H((\hat{u}_{T+H}, \hat{p}_{T+H}) - (\hat{u}_T, \hat{p}_T)) + O(\|H\|^2_{W^{1,\infty}}) \\ &= S_T(\hat{w}(H), \hat{r}(H)) + l_H((\hat{u}_{T+H}, \hat{p}_{T+H}) - (\hat{u}_T, \hat{p}_T)) + O(\|H\|^2_{W^{1,\infty}}). \end{aligned} \tag{3.34}$$

It is easily checked that $|l_H((\hat{u}, \hat{p}), (\hat{v}, \hat{q}))| \leq \|H\|_{W^{1,\infty}}\|(\hat{u}, \hat{p})\|\|(\hat{v}, \hat{q})\|$ for all $(\hat{u}, \hat{p})$, $(\hat{v}, \hat{q}) \in X_\mathbb{C}$ (note that l_H contains all first order H terms when expanding $S_{T+H} + (F_{T+H}, 0)$). Since $(\hat{u}_T, \hat{p}_T)$ depends continuously on T, we conclude $S_T(\hat{u}_{T+H} - \hat{u}_T, \hat{p}_{T+H} - \hat{p}_T) = S_T(\hat{w}(H), \hat{r}(H)) + o(\|H\|_{W^{1,\infty}})$ which implies (3.30). □

We next turn to the holomorphic dependence of the domain-to-solution map for the Navier-Stokes problem under the same Assumptions 3.3 and 3.4. In addition, we shall work under the following small data assumption.

Assumption 3.9. *The function f satisfies*

$$\|f\|_{H^{-1}(\mathrm{D}_H)} < C_{1,\mathfrak{T}}, \quad C_{1,\mathfrak{T}} := \min_{T\in\mathfrak{T}} C_1(T) > 0,$$

where $C_1(T)$ is given by (3.17).

This assumption thus ensures the existence and uniqueness of the solution $(\hat{u}_T, \hat{p}_T)$ of the Navier-Stokes problem (3.19) for all $T \in \mathfrak{T}$. We now establish the holomorphic dependence of the domain-to-solution map (3.20) for the Navier-Stokes problem under the above assumptions.

Theorem 3.10. *Let Assumptions 3.3, 3.4 and 3.9 be satisfied. Then there exists $\varepsilon = \varepsilon(\hat{\mathrm{D}}, \mathfrak{T}) > 0$ such that the domain-to-solution map $T \mapsto (\hat{u}_T, \hat{p}_T)$, with $(\hat{u}_T, \hat{p}_T)$ solving* (3.19)*, admits an extension on $\mathfrak{T}_\varepsilon$ as in* (3.23)*, which is holomorphic and uniformly bounded as a mapping from $W^{1,\infty}(\hat{\mathrm{D}}, \mathbb{C}^d)$ to $X_{\mathbb{C}}$.*

Proof. By the same treatment as in the proof of Theorem 3.7, we reach a formulation of the problem in the real valued space X of the form

$$S_T(\hat{u}, \hat{p}) - (N_T(\hat{u}), 0) - H_T = 0, \tag{3.35}$$

where S_T and H_T are defined as in the proof of the Theorem 3.7, and where N_T is the nonlinear operator defined from the trilinear form t_T which appears in (3.19). This equation has a unique solution $(\hat{u}_T, \hat{p}_T) \in X$ for any T in $\mathfrak{T}$. Using the shorthand $\hat{U} = (\hat{u}, \hat{p})$ and $\hat{U}_T = (\hat{u}_T, \hat{p}_T)$ we rewrite this equation

$$\mathcal{P}(\hat{U}, T) = 0. \tag{3.36}$$

We then follow the approach developed in [8] and summarized by Theorem 2.5 of [9]. In our application of this theorem $(T, \mathfrak{T}, X_{\mathbb{C}}, W^{1,\infty}, X_{\mathbb{C}}^*)$ play the role of $(a, \mathcal{A}, V, X, W)$ in the formulation of [9]. We first observe that $\mathcal{P}$ has a holomorphic extension from $X_{\mathbb{C}} \times \mathfrak{T}_{\varepsilon_{\mathcal{P}}}$ to $X_{\mathbb{C}}^*$, where $\varepsilon_{\mathcal{P}} := \min\{\varepsilon_1, \varepsilon_2\}$. This extension is defined by proceeding as in the proof of Theorem 3.7 for A_T, B_T and F_T, and by defining

$$N_T(\hat{u})(\hat{v}) := t_T(\hat{u}, \hat{u}, \hat{v}) = \int_{\hat{\mathrm{D}}} (dT^{-1}\hat{u} \cdot \nabla)\hat{u} \cdot \hat{v} J, \tag{3.37}$$

with again the convention that $\hat{f} \cdot \hat{v} := \hat{f}_1 \hat{v}_1 + \cdots + \hat{f}_d \hat{v}_d$. We then consider, for any given $T \in \mathfrak{T}$, the partial differential

$$L_T := \partial \mathcal{P}_{\hat{U}}(\hat{U}_T, T) \in \mathcal{L}(X_{\mathbb{C}}, X_{\mathbb{C}}^*), \tag{3.38}$$

and establish that this operator is an isomorphism. Similar to S_T in the proof of Theorem 3.7, it is sufficient to establish the isomorphism property between the real valued spaces X and X^*. The equation

$$L_T(\hat{u}, \hat{p}) = (F, G) \tag{3.39}$$

is equivalent to

$$\begin{cases} s_T(\hat{u}, \hat{v}) - b_T(\hat{v}, \hat{p}) = F(\hat{v}), & \hat{v} \in (H_0^1(\hat{\mathrm{D}}))^d, \\ b_T(\hat{u}, \hat{q}) = 0, & \hat{q} \in L_\#^2(\hat{\mathrm{D}}), \end{cases} \tag{3.40}$$

where

$$s_T(\hat{u}, \hat{v}) = a_T(\hat{u}, \hat{v}) + \int_{\hat{\mathrm{D}}} (dT^{-1}\hat{u}_T \cdot \nabla)\hat{u} \cdot \hat{v} J + \int_{\hat{\mathrm{D}}} (dT^{-1}\hat{u} \cdot \nabla)\hat{u}_T \cdot \hat{v} J \,. \tag{3.41}$$

Existence and uniqueness of the solution is ensured due to the inf-sup condition satisfied by b_T and the fact that the non-symmetric form s_T is still elliptic on $H_0^1(\hat{\mathrm{D}})^d$ due to the small data assumption. Indeed, we may write

$$s_T(\hat{u},\hat{u}) = a_T(\hat{u},\hat{u}) + \int_{\hat{\mathrm{D}}} (dT^{-1}\hat{u}\cdot\nabla)\hat{u}_T\cdot\hat{u}J, \tag{3.42}$$

where we have used the fact that $t_T(\hat{u}_T,\hat{u},\hat{u}) = 0$. Employing (3.15), we find

$$\left|\int_{\hat{\mathrm{D}}} (dT^{-1}\hat{u}\cdot\nabla)\hat{u}_T\cdot\hat{u}J\right| \leq N_1\|\hat{u}\|^2_{H_0^1(\mathrm{D})}\|\hat{u}_T\|_{H_0^1(\mathrm{D})} \leq N_1 m_1^{-1}\|f\|_{H^{-1}(\mathrm{D}_H)}\|\hat{u}\|^2_{H_0^1(\mathrm{D})}, \tag{3.43}$$

where N_1 and m_1 are the same as in the discussion on the uniqueness of the Navier-Stokes solution. Therefore, Assumption 3.9 implies the existence of a $\rho < 1$ such that

$$\left|\int_{\hat{\mathrm{D}}} (dT^{-1}\hat{u}_T\cdot\nabla)\hat{u}\cdot\hat{u}J\right| < \rho a_T(\hat{u},\hat{u}), \quad T\in\mathfrak{T}, \hat{u}\in H_0^1(\hat{\mathrm{D}})^d, \tag{3.44}$$

which shows that s_T is elliptic.

We have thus established the needed properties for applying Theorem 2.5 of [9]. By an argument based on the implicit function theorem, the domain-to-solution map has a holomorphic and bounded extension on a domain $\mathfrak{T}_\varepsilon$, for sufficiently small $\varepsilon > 0$. □

Similar to the Stokes equations, we next provide with the explicit expression of the Fréchet derivative of the solution map, again denoted by $\mathcal{S} : T \mapsto (\hat{u}_T, \hat{p}_T)$.

Proposition 3.11. *Let the assumptions of Theorem 3.10 be satisfied. Then, the Fréchet derivative $d\mathcal{S}(T)(H)$ of $\mathcal{S}$ at $T\in\mathfrak{T}$ in direction $H\in W^{1,\infty}(\hat{\mathrm{D}},\mathbb{C}^d)$ is given by the unique solution $(\hat{w},\hat{r})\in X_{\mathbb{C}}$ of*

$$\begin{aligned}
&a_T(\hat{w},\hat{v}) - b_T(\hat{v},\hat{r}) + t_T(\hat{u}_T,\hat{w},\hat{v}) + t_T(\hat{w},\hat{u}_T,\hat{v}) = \\
&\quad\int_{\hat{\mathrm{D}}} \mathrm{tr}\left(\nabla\hat{u}_T\left[J(dT^{-1}dH\,dT^{-1}dT^{-\top} + dT^{-1}dT^{-\top}dH^\top dT^{-\top}) - \mathrm{tr}(\mathrm{Cof}(dT)^\top dH)dT^{-1}dT^{-\top}\right]\nabla\hat{v}^\top\right) \\
&\quad-\int_{\hat{\mathrm{D}}} \mathrm{tr}\left(\nabla\hat{v}\left[J dT^{-1}dH\,dT^{-1} - \mathrm{tr}(\mathrm{Cof}(dT)^\top dH)dT^{-1}\right]\right)\hat{p}_T \\
&\quad+\int_{\hat{\mathrm{D}}} (dT^{-1}dH\,dT^{-1}J - dT^{-1}\mathrm{tr}(\mathrm{Cof}(dT)^\top dH)(\hat{u}_T\cdot\nabla)\hat{u}_T\cdot\hat{v} \\
&\quad+\int_{\hat{\mathrm{D}}} (df\circ T)H\cdot\hat{v}J + \int_{\hat{\mathrm{D}}} \hat{f}_T\cdot\hat{v}\,\mathrm{tr}(\mathrm{Cof}(dT)^\top dH),
\end{aligned} \tag{3.45a}$$

$$b_T(\hat{w},\hat{q}) = \int_{\hat{\mathrm{D}}} \mathrm{tr}\left(\nabla\hat{u}_T\left[J dT^{-1}dH\,dT^{-1} - \mathrm{tr}(\mathrm{Cof}(dT)^\top dH)dT^{-1}\right]\right)\hat{q}, \tag{3.45b}$$

for all $(\hat{v},\hat{q})\in X_{\mathbb{C}}$.

Proof. We proceed in the same way as in the proof of Prop. 3.8. Let S_T as in (3.26), and denote by $\tilde{l}_H(\hat{u}_T,\hat{p}_T)(\hat{v},\hat{q})$ the right-hand side of (3.45). According to Theorem 3.10, the Fréchet derivative $d\mathcal{S}(\hat{u}_T,\hat{p}_T) : W^{1,\infty}(\hat{\mathrm{D}},\mathbb{C}^d)\to X_{\mathbb{C}}$ exists, and we shall denote it in the following by

$(\hat u_T', \hat p_T')$. Furthermore, instead of letting $(\hat w, \hat r) = (\hat w(H),\ \hat r(H))$ be the solution to (3.45), assume for now that for given $H \in W^{1,\infty}(\hat{\mathrm{D}}, \mathbb{C}^d)$, $(\hat w(H), \hat r(H)) \in X_{\mathbb{C}}$ solves

$$S_T(\hat w(H), \hat r(H)) = \tilde l_H(\hat u_T, \hat p_T) - N_T(\hat u_T, \hat u_T'(H)) - N_T(\hat u_T'(H), \hat u_T), \tag{3.46}$$

where as before $N_T(u) := t_T(u, u, \cdot)$, and now additionally $N_T(u, v) := t_T(u, v, \cdot)$; by a further abuse of notation, we also identify $N_T(u) \in (H_0^1(\hat{\mathrm{D}}, \mathbb{C})^d)'$ with $(N_T(u), 0) \in X_{\mathbb{C}}'$ (see (3.24)), and similarly for $N_T(u, v)$. Then the map $H \mapsto (\hat w(H), \hat r(H))$ is bounded and linear. We will show that there holds

$$\|S_T((\hat u_{T+H}, \hat p_{T+H}) - (\hat u_T, \hat p_T) - (\hat w(H), \hat r(H)))\|_{\mathcal{L}(X_{\mathbb{C}}, X_{\mathbb{C}}^*)} = o(\|H\|_{W^{1,\infty}}) \quad \text{as} \quad \|H\|_{W^{1,\infty}(\hat{\mathrm{D}})} \to 0\ . \tag{3.47}$$

This implies $(\hat w(H), \hat r(H)) = (\hat u_T'(H), \hat p_T'(H))$, and therefore $(\hat w(H), \hat r(H))$ solves (3.45). Next, (3.45) has a unique solution, since the bilinear form on the left-hand side (formerly denoted by s_T) is coercive. Finally, as it was argued in the proof of Thm. 3.10, the bounded invertibility of S_T then concludes the proof.

It remains to verify (3.47). Since $(\hat u_T, \hat p_T)$, $(\hat u_{T+H}, \hat p_{T+H})$ solve (3.19), we have

$$\begin{aligned} &S_T((\hat u_{T+H}, \hat p_{T+H}) - (\hat u_T, \hat p_T)) = \\ &\qquad S_T(\hat u_{T+H}, \hat p_{T+H}) - S_{T+H}(\hat u_{T+H}, \hat p_{T+H}) + F_{T+H} - N_{T+H}(u_{T+H}) - F_T + N_T(u_T), \end{aligned} \tag{3.48}$$

where F_T here and in the following also stands for $(F_T, 0)$, for any T. As in the proof of Theorem 3.7 (cp. (3.31), (3.34)), we get

$$\|S_T(\hat u_{T+H}, \hat p_{T+H}) - S_{T+H}(\hat u_{T+H}, \hat p_{T+H}) + F_{T+H} - F_T - l_H(\hat u_T, \hat p_T)\|_{\mathcal{L}(X_{\mathbb{C}}, X_{\mathbb{C}}^*)} = o(\|H\|_{W^{1,\infty}}) \tag{3.49}$$

as $\|H\|_{W^{1,\infty}} \to 0$ (note that in order to show (3.49) in the proof of Theorem 3.7, the only requirement was that $(\hat u_T, \hat p_T)$ depends continuously on T, which is also the case here, as the dependence is even holomorphic). In view of (3.46), (3.47) and (3.48) we are thus left with checking

$$\begin{aligned} &\|N_{T+H}(\hat u_{T+H}) - N_T(\hat u_T) - l_H(\hat u_T, \hat p_T) + \tilde l_H(\hat u_T, \hat p_T) - N_T(\hat u_T, \hat u_T'(H)) - N_T(\hat u_T'(H), \hat u_T)\|_{\mathcal{L}(X_{\mathbb{C}}, X_{\mathbb{C}}^*)} \\ &\qquad = o(\|H\|_{W^{1,\infty}}). \end{aligned} \tag{3.50}$$

This follows by employing (3.33) as well as $\|(\hat u_{T+H}, \hat p_{T+H}) - (\hat u_T, \hat p_T) - (\hat u_T'(H), \hat p_T'(H))\|_{X_{\mathbb{C}}} = o(\|H\|_{W^{1,\infty}})$ to gather all first order terms in (3.50). □

4 Piola Transform

4.1 Domain Transformation

We next study a second pullback where the Piola transform is applied to the velocity. This transform is defined as

$$dT(\hat x)\hat u(\hat x) = J(\hat x) u(x). \tag{4.1}$$

While we use the same notation $\hat u$, note that it now differs from the plain pullback studied in the previous sections. In addition to the assumption that $T : \hat{\mathrm{D}} \to \mathrm{D}$ is bi-Lipschitz with a Jacobian determinant that is positive a.e. in $\hat{\mathrm{D}}$, we impose the further assumption that T belongs to the weighted space

$$W^{2,\infty}_\omega(\hat{\mathrm{D}}) := \{T \in W^{1,\infty}(\hat{\mathrm{D}}) \;:\; \sup_{\hat{x}\in\hat{\mathrm{D}}} \omega(\hat{x})\|d^2T(\hat{x})\| < \infty\}, \tag{4.2}$$

where $\|d^2T(\hat{x})\| = \max_{i,j,k} |\frac{\partial T_i}{\partial x_j \partial x_k}(\hat{x})|$ stands for the maximum norm of all partial second derivatives of T at $\hat{x}$, and where

$$\omega(\hat{x}) = \mathrm{dist}(\hat{x}, \partial\hat{\mathrm{D}}). \tag{4.3}$$

The space $W^{2,\infty}_\omega$ is larger than $W^{2,\infty}(\hat{\mathrm{D}})$. It is a Banach space when equipped with the norm

$$\|T\|_{W^{2,\infty}_\omega} := \|T\|_{W^{1,\infty}(\hat{\mathrm{D}})} + |T|_{W^{2,\infty}_\omega}, \quad |T|_{W^{2,\infty}_\omega} := \sup_{\hat{x}\in\hat{\mathrm{D}}} \omega(\hat{x})\|d^2T(\hat{x})\| \,. \tag{4.4}$$

Remark 4.1. *The motivation for considering $W^{2,\infty}_\omega$ is that, unlike $W^{2,\infty}$ transformations, it allows to map smooth-domains onto domains with Lipschitz boundaries, which can therefore possess corners. A concrete construction for such transformations of star shaped domains is given in [2], and we shortly sketch (a slight modification of) it: Let $r : S^{d-1} \to \mathbb{R}^d_+$ be a strictly positive Lipschitz continuous function on the unit sphere, describing the Lipschitz boundary $\mathrm{D} := \{x \in \mathbb{R}^d \,:\, |x| < r(x/|x|)\}$. Denote by B_1 the (open) unit ball in $\mathbb{R}^d$. Additionally let $\varphi \in C^\infty_0(B_1)$, $\varphi \geq 0$ and $\int \varphi = 1$, as well as $\chi \in C^\infty([0,1])$, $\mathrm{supp}(\chi) \subseteq (0,1]$, $\chi \geq 0$, $\chi(1) = 1$ and χ is monotonically increasing. Set for $h \in (0,1]$ and $x \in B_1$*

$$g(h,x) := \frac{1}{h^d}\int_{\mathbb{R}^d} r\left(\frac{\xi}{|\xi|}\right)\varphi\left(\frac{x-\xi}{h}\right)\mathrm{d}\xi, \tag{4.5}$$

and furthermore with $0 < r_{\min} \leq \mathrm{ess\,inf}_{x\in S^{d-1}}\, r(x)$ and $\varepsilon > 0$ to be chosen subsequently

$$T(x) := x\left[r_{\min} + \chi(|x|)\left(g\left(\varepsilon(1-|x|), x/|x|\right) - r_{\min}\right)\right] \,. \tag{4.6}$$

We observe that $g \in C^\infty((0,1] \times S^{d-1})$ and thus $T \in C^\infty(B_1)$ (since $\chi(|x|) \in C^\infty(B_1)$ vanishes around 0). Furthermore, for $|x| = 1$ and $0 < h \leq \frac{1}{2}$ we have $|\partial^\alpha g(h,x)| \leq C\|r\|_{W^{1,\infty}}$ if $|\alpha| \leq 1$ and $|\partial^\alpha g(h,x)| \leq \frac{C}{h}\|r\|_{W^{1,\infty}}$ if $|\alpha| \leq 2$: For example, set $l(\xi) := r(\frac{\xi}{|\xi|})$ and observe that for $\frac{1}{2} \leq |\xi| \leq \frac{3}{2}$ it holds $\|\nabla l(\xi)\| \leq C\|r\|_{W^{1,\infty}}$, and therefore

$$\left|\frac{\partial}{\partial h}g(h,x)\right| = \left|\frac{\partial}{\partial h}\int_{\mathbb{R}^d} l(x-h\xi)\varphi(\xi)\mathrm{d}\xi\right| = \left|\int_{\mathbb{R}^d} \nabla l(x-h\xi)\cdot(-\xi)\varphi(\xi)\mathrm{d}\xi\right| \leq C\|r\|_{W^{1,\infty}}, \tag{4.7}$$

and

$$\left|\frac{\partial^2}{\partial h^2}g(h,x)\right| = \left|\frac{\partial}{\partial h}\frac{1}{h^d}\int_{\mathbb{R}^d} \nabla l(\xi)\cdot\frac{\xi-x}{h}\varphi\left(\frac{x-\xi}{h}\right)\right| \leq \frac{C}{h}\|r\|_{W^{1,\infty}} \tag{4.8}$$

by applying the chain rule and since $|\frac{\xi-x}{h}| \leq 1$ whenever $\varphi(\frac{\xi-x}{h}) \neq 0$. The remaining cases can be treated similarly.

Hence $\|dT(x)\| \leq C\|r\|_{W^{1,\infty}}$, and $\|d^2T(x)\| \leq \frac{C}{1-|x|}\|r\|_{W^{1,\infty}}$ for $x \in B_1$, which shows $T \in W^{2,\infty}_\omega(B_1)$ and $\|T\|_{W^{2,\infty}_\omega(B_1)} \leq C\|r\|_{W^{1,\infty}}$. Finally, for sufficiently small $0 \leq \varepsilon \leq \frac{1}{2}$, the map $T : B_1 \to \mathrm{D}$ is a bijection: we have $T(0) = 0$, $T(x) = r(x)$ for $x \in S^{d-1}$ (by continuous extension) and

$$\begin{aligned}\frac{\partial|T(x)|}{\partial|x|} =& r_{\min} + \chi(|x|)\left(g\left(\varepsilon(1-|x|), x/|x|\right) - r_{\min}\right) + |x|\Big[\chi'(|x|)\left(g\left(\varepsilon(1-|x|), x/|x|\right) - r_{\min}\right) \\ &- \varepsilon\chi(|x|)g_h\left(\varepsilon(1-|x|), x/|x|\right)\Big],\end{aligned} \tag{4.9}$$

where g_h denotes the partial derivative of g w.r.t. its first argument. Now, $\chi' \geq 0$ and $g - r_{\min} \geq 0$ by definition of g and $r_{\min}$. Thus $|\frac{\partial |T(x)|}{\partial |x|}| \geq \frac{r_{\min}}{2}$ for all $x \in B_1$ if $\varepsilon > 0$ is small enough (where ε depends on $\|r\|_{W^{1,\infty}}$ through the above constants).

The map taking the polar coordinates $(\rho, \varphi, \theta_1, \ldots, \theta_{d-2})$ to x has the Jacobian determinant $\rho^{d-1} \prod_{j=1}^{d-2} \sin(\theta_j)^j$. Using this, a short computation reveals that the Jacobian determinant of T is given by the right hand side of (4.9) multiplied with $[r_{\min} + \chi(|x|)\,(g\,(\varepsilon(1-|x|), x/|x|) - r_{\min})]^{d-1}$, and consequently is bounded uniformly from above and below.

We finally note that T^{-1} also belong to $W^{2,\infty}_\omega(D)$. Indeed, we can express dT^{-1} via the cofactor matrix $\mathrm{Cof}(dT)$ according to

$$(dT(x))^{-1} = \frac{1}{J(x)} \mathrm{Cof}(dT(x)), \tag{4.10}$$

The term in $\nabla((dT)^{-1})$ thus consists of second order derivatives of T multiplied with (possibly several) lower order derivatives of T times $1/J^2$. The ratio between $\mathrm{dist}(x, \partial B_1)$ and $\mathrm{dist}(T(x), \partial \mathrm{D})$ being bounded from above and below, uniformly with respect to x since T is bi-Lipschitz, we conclude that $T^{-1} \in W^{2,\infty}_\omega(D)$.

The Piola transform

$$u \mapsto \mathcal{P}_T u = \hat{u}, \tag{4.11}$$

induces an isomorphism from $H^1_0(\mathrm{D})$ to $H^1_0(\hat{\mathrm{D}})$. This follows by writing

$$\begin{aligned}
\|u\|^2_{H^1_0(\mathrm{D})} &= \int_{\mathrm{D}} \mathrm{tr}\,(\nabla u \nabla u^\top) \\
&= \int_{\hat{\mathrm{D}}} \mathrm{tr}\Big(\nabla(J^{-1} dT \hat{u}) dT^{-1} dT^{-\top} (\nabla(J^{-1} dT \hat{u}))^\top\Big) J \\
&\leq Q_1(u) + Q_2(u),
\end{aligned}$$

where

$$Q_1(u) = 2 \int_{\hat{\mathrm{D}}} \mathrm{tr}\Big(dT \nabla \hat{u} dT^{-1} dT^{-\top} \nabla \hat{u}^\top dT^\top\Big) J^{-1}, \tag{4.12}$$

and

$$Q_2(u) = 2 \int_{\hat{\mathrm{D}}} \mathrm{tr}\Big(\nabla(J^{-1} dT)) \hat{u} dT^{-1} dT^{-\top} \hat{u}^\top \nabla(J^{-1} dT)^\top\Big) J\ . \tag{4.13}$$

The first term is treated similarly as when proving the norm equivalence for the plain pullback, and we obtain

$$Q_1(u) \leq C \|\hat{u}\|^2_{H^1_0(\hat{\mathrm{D}})}, \quad C = 2 \max_{\hat{x} \in \hat{\mathrm{D}}} \|dT(\hat{x})\|^2 \|dT^{-1}(\hat{x})\|^2 J^{-1}(\hat{x}).$$

The second term involves second derivatives of the transformation T and can be bounded by

$$Q_2(u) \leq C \int_{\hat{\mathrm{D}}} \|d^2 T\|^2 |\hat{u}|^2,$$

where $C = C(\|J\|_{L^\infty}, \|J^{-1}\|_{L^\infty}, \|dT\|_{L^\infty}, \|dT^{-1}\|_{L^\infty})$. Therefore

$$Q_2(u) \leq C |T|^2_{W^{2,\infty}_\omega} \int_{\hat{\mathrm{D}}} \omega^{-2} |\hat{u}|^2 \leq C |T|^2_{W^{2,\infty}_\omega} C_H \|\hat{u}\|^2_{H^1_0(\hat{\mathrm{D}})},$$

where we have used the Hardy inequality

$$\int_{\hat{\mathrm{D}}} \omega^{-2}|\varphi|^2 \le C_H \int_{\hat{\mathrm{D}}} |\nabla\varphi|^2, \quad \varphi \in H_0^1(\hat{\mathrm{D}}) \ . \tag{4.14}$$

This gives the upper bound in

$$m_2\|\hat{u}\|_{H_0^1(\hat{\mathrm{D}})} \le \|u\|_{H_0^1(\mathrm{D})} \le M_2\|\hat{u}\|_{H_0^1(\hat{\mathrm{D}})},$$

where $M_2 = M_2(T) = M_2(\|J\|_{L^\infty}, \|J^{-1}\|_{L^\infty}, \|dT\|_{L^\infty}, \|dT^{-1}\|_{L^\infty}, |T|_{W_\omega^{2,\infty}}, C_H)$, and the lower bound is proved in a similar way with $m_2(T) = M_2(T^{-1})^{-1}$ by remarking that $\mathcal{P}_{T^{-1}}$ is the inverse of $\mathcal{P}_T$, and therefore $u = \mathcal{P}_{T^{-1}}\hat{u}$.

One interesting property of the Piola transform is its behaviour with respect to the divergence, namely [1, (2.1.71)]

$$\mathrm{div} v(x) = J^{-1}(\hat{x})\mathrm{div}\hat{v}(\hat{x}) \ .$$

In particular, it preserves the divergence free property. We maintain the same plain pullback transformations for the pressure and right-hand side data as in the previous section.

We then obtain the transformed system (3.11), where the bilinear forms are now given by

$$a_T(\hat{u}, \hat{v}) := \int_{\hat{\mathrm{D}}} \mathrm{tr}\Big(\nabla(J^{-1}dT\hat{u})dT^{-1}dT^{-\top}\nabla(J^{-1}dT\hat{v})^\top\Big)J, \tag{4.15a}$$

and (see [1, (2.1.73)])

$$b_T(\hat{v}, \hat{p}) := \int_{\hat{\mathrm{D}}} \mathrm{div}(\hat{v})\hat{p}, \tag{4.15b}$$

and the linear form by

$$F_T(\hat{v}) := \int_{\hat{\mathrm{D}}} dT\hat{f}_T \cdot \hat{v}, \tag{4.15c}$$

where again $\hat{f}_T := f \circ T$.

The interest of the Piola transformation lies in that b_T has the same form as the original b, which allows us to use standard LBB compatible finite elements for the discretization, independently of T. The right hand side F_T has the same form as in (3.12c). The isomorphism property ensures that these forms satisfy the continuity, coercivity and inf-sup properties on the spaces $(H_0^1(\hat{\mathrm{D}}))^d$ and $L_\#^2(\hat{\mathrm{D}})$. Thus, the system has a unique solution $(\hat{u}_T, \hat{p}_T) \in X$.

Likewise, the Navier-Stokes equations lead to the transformed system (3.13), where the trilinear form is now given by

$$t_T(\hat{u}, \hat{v}, \hat{w}) = \int_{\hat{\mathrm{D}}} (\hat{u} \cdot \nabla)(J^{-1}dT\hat{v}) \cdot (dT\hat{w})J^{-1}, \tag{4.15d}$$

and also admits a solution $(\hat{u}_T, \hat{p}_T) \in X$. Similar to the plain pullback, we obtain $m_2^2\|\hat{u}\|_{H_0^1}^2 \le \|u\|_{H_0^1}^2 = a_T(\hat{u}, \hat{u})$, an a-priori estimate of the type

$$\|\hat{u}_T\|_{H_0^1(\hat{\mathrm{D}})} \le m_2(T)^{-1}\|f\|_{H^{-1}(\mathrm{D}_H)}, \tag{4.16}$$

and a continuity bound for the trilinear form, now with the constant

$$N_2 = N_2(\|J\|_{L^\infty}, \|J^{-1}\|_{L^\infty}, \|dT\|_{L^\infty}, \|dT^{-1}\|_{L^\infty}, |T|_{W_\omega^{2,\infty}}, C_H, C_E),$$

with C_E the embedding constant of $H_0^1(\hat{\mathrm{D}})$ into $L^4(\hat{\mathrm{D}})$ and with C_H denoting the constant in the Hardy inequality (4.14).

The same reasoning shows that uniqueness of $\hat{u}_T$ is ensured under the small data condition

$$\|f\|_{H^{-1}(\mathrm{D}_H)} \leq C_2, \quad C_2 = C_2(T) = \frac{m_2(T)^3}{N_2(T)}, \tag{4.17}$$

and uniqueness of $\hat{p}_T$ follows from the inf-sup condition. The operators A_T, B_T, and the nonlinear operator N_T are defined accordingly, similar to Remark 3.2. The linear form F_T is left unchanged.

4.2 Holomorphy of the Piola domain-to-solution map

We now establish similar holomorphy result as those of §3.2 for the domain-to-solution map (3.20) in the case of the Piola transform as T varies in a certain set $\mathfrak{T}$. We prove holomorphy of this map for problems (3.11) and (3.13). Since the arguments are quite similar to those used for the plain pullback, we only briefly sketch them. We strengthen Assumption 3.3 into the following.

Assumption 4.2. *The set $\mathfrak{T}$ is compact in $W^{2,\infty}_\omega(\hat{\mathrm{D}}, \mathbb{R}^d)$ and such that $T^{-1} \in W^{2,\infty}_\omega(\mathrm{D}, \mathbb{R}^d)$ for every $T \in \mathfrak{T}$.*

Likewise, we now define for $\varepsilon > 0$ the complex valued ε-neighborhood of $\mathfrak{T}$,

$$\mathfrak{T}_\varepsilon := \{\tilde{T} \in W^{2,\infty}_\omega(\hat{\mathrm{D}}, \mathbb{C}^d) \,:\, \exists T \in \mathfrak{T},\ \|\tilde{T} - T\|_{W^{2,\infty}_\omega(\hat{\mathrm{D}})} < \varepsilon\}\ . \tag{4.18}$$

We again introduce ε_1 and ε_2 as in Remarks 3.5 and 3.6. The following result holds for the Stokes problem.

Theorem 4.3. *Let Assumptions 4.2 and 3.4 be satisfied. Then there exists $\varepsilon = \varepsilon(\hat{\mathrm{D}}, \mathfrak{T}) > 0$ such that the domain-to-solution map $T \mapsto (\hat{u}_T, \hat{p}_T)$, with $(\hat{u}_T, \hat{p}_T)$ solving* (3.18) *where the operators are defined by* (4.15)*, admits an extension on $\mathfrak{T}_\varepsilon$ as in* (4.18)*, which is holomorphic and uniformly bounded as a mapping from $W^{2,\infty}_\omega(\hat{\mathrm{D}}, \mathbb{C}^d)$ to $X_\mathbb{C}$.*

Proof. The proof is similar to that of Theorem 3.7, with $W^{1,\infty}$ replaced by $W^{2,\infty}_\omega$. We introduce the isomorphism S_T analogous to (3.26), but now using (4.15a) and (4.15b) to define A_T, B_T, and write the solution as $(\hat{u}_T, \hat{p}_T) = S_T^{-1} H_T$ where $H_T := (F_T, 0)$ via (4.15c). We then extend S_T as an isomorphism from the complex space $X_\mathbb{C}$ to its dual $X^*_\mathbb{C}$, and H_T as an element of $X^*_\mathbb{C}$. We next observe that $T \mapsto S_T$ has a well defined and holomorphic, extension, as a mapping from $W^{2,\infty}_\omega(\hat{\mathrm{D}}, \mathbb{C})$ to $\mathcal{L}(X_\mathbb{C}, X^*_\mathbb{C})$, at any $T \in \mathfrak{T}_{\varepsilon_1}$ where ε_1 is as in Remark 3.5, and in particular at any $T \in \mathfrak{T}$. By the same covering argument, we obtain that there exists $\varepsilon_3 > 0$ such that the mapping $T \mapsto S_T^{-1}$ is bounded and holomorphic on $\mathfrak{T}_{\varepsilon_3}$. By Remark 3.6, we also find that the map $T \mapsto H_T := (F_T, 0)$ has a bounded and holomorphic extension, as a mapping from $W^{1,\infty}(\hat{\mathrm{D}}, \mathbb{C})$ (and thus also from $W^{2,\infty}_\omega(\hat{\mathrm{D}}, \mathbb{C})$) to $X^*_\mathbb{C}$, on $\mathfrak{T}_{\varepsilon_2}$. This proves that the domain-to-solution map is bounded and holomorphic on $\mathfrak{T}_\varepsilon$, with $\varepsilon := \min\{\varepsilon_2, \varepsilon_3\}$. □

For the Navier-Stokes problem, we modify the small data hypothesis, Assumption 3.9, as follows.

Assumption 4.4. *The function f satisfies*

$$\|f\|_{H^{-1}(\mathrm{D}_H)} < C_{2,\mathfrak{T}}, \quad C_{2,\mathfrak{T}} := \min_{T \in \mathfrak{T}} C_2(T) > 0,$$

where $C_2(T)$ is the constant in (4.17).

Theorem 4.5. *Let Assumptions 4.2, 3.4 and 4.4 be satisfied. Then there exists $\varepsilon = \varepsilon(\hat{\mathrm{D}}, \mathfrak{T}) > 0$ such that the domain-to-solution map*

$$T \mapsto (\hat{u}_T, \hat{p}_T), \tag{4.19}$$

with $(\hat{u}_T, \hat{p}_T)$ solving (3.19) *where the operators are defined by* (4.15)*, admits an extension to the set $\mathfrak{T}_\varepsilon$ as in* (3.23)*, which is holomorphic and uniformly bounded as a mapping from $W^{2,\infty}_\omega(\hat{\mathrm{D}}, \mathbb{C}^d)$ to $X_\mathbb{C}$.*

Proof. The proof is similar to that of Theorem 3.10, with $W^{1,\infty}$ replaced by $W^{2,\infty}_\omega$, based on the general formulation (3.36), and following the approach from Theorem 2.5 of [9] which is an extension of [8]. We find that $\mathcal{P}$ has a holomorphic extension from $X_\mathbb{C} \times \mathfrak{T}_{\varepsilon_\mathcal{P}}$ to $X_\mathbb{C}^*$, where $\varepsilon_\mathcal{P} := \min\{\varepsilon_1, \varepsilon_2\}$, and establish the isomorphism property of the operator $L_T := \partial \mathcal{P}_{\hat{U}}(\hat{U}_T, T) \in \mathcal{L}(X_\mathbb{C}, X_\mathbb{C}^*)$. In this case, the bilinear form s_T satisfies

$$s_T(\hat{u}, \hat{u}) = a_T(\hat{u}, \hat{u}) + \int_{\hat{\mathrm{D}}} (\hat{u} \cdot \nabla)(J^{-1} dT \hat{u}_T) \cdot (dT \hat{u}) J^{-1}, \tag{4.20}$$

and the small data assumption ensures ellipticity on the Hilbert space $H^1_0(\hat{\mathrm{D}}, \mathbb{R})^d$ of real-valued velocities by the same argument as in (3.42)-(3.44). The proof is concluded in a similar manner. □

In the same way as in Propositions 3.8 and 3.11 we obtain for the domain-to-solution maps $\mathcal{S} : T \mapsto (\hat{u}_T, \hat{p}_T)$ for Stokes and Navier-Stokes, respectively:

Proposition 4.6. *Let the assumptions of Theorem 4.3 be satisfied. Then, the Fréchet derivative $d\mathcal{S}(T)(H)$ of $\mathcal{S}$ at $T \in \mathfrak{T}$ in direction $H \in W^{2,\infty}_\omega(\hat{\mathrm{D}}, \mathbb{C}^d)$ is given by the unique solution $(\hat{w}, \hat{r}) \in X_\mathbb{C}$ of*

$$\begin{aligned}
&a_T(\hat{w}, \hat{v}) - b_T(\hat{v}, \hat{r}) = \\
&\quad \int_{\hat{\mathrm{D}}} \mathrm{tr}\left(\nabla(J^{-1} dT \hat{u}_T) \left[J(dT^{-1} dH dT^{-1} dT^{-\top} + dT^{-1} dT^{-\top} dH^\top dT^{-\top})\right] \nabla(J^{-1} dT \hat{v})^\top\right) \\
&\quad - \int_{\hat{\mathrm{D}}} \mathrm{tr}\left(\nabla\left(J^{-1} dT \hat{u}_T\right) dT^{-1} dT^{-\top} \nabla(J^{-1} dT \hat{v})^\top\right) \mathrm{tr}(\mathrm{Cof}(dT)^\top dH) \\
&\quad - \int_{\hat{\mathrm{D}}} \mathrm{tr}\left(\nabla\left(\left(\frac{-\mathrm{tr}(\mathrm{Cof}(dT)^\top dH)}{J^2} dT + J^{-1} dH\right) \hat{u}_T\right) dT^{-1} dT^{-\top} \nabla(J^{-1} dT \hat{v})^\top\right) J \\
&\quad - \int_{\hat{\mathrm{D}}} \mathrm{tr}\left(\nabla(J^{-1} dT \hat{u}_T) dT^{-1} dT^{-\top} \nabla\left(\left(\frac{-\mathrm{tr}(\mathrm{Cof}(dT)^\top dH)}{J^2} dT + J^{-1} dH\right) \hat{v}\right)^\top\right) J \\
&\quad + \int_{\hat{\mathrm{D}}} (df \circ T) H \cdot \hat{v} J + \int_{\hat{\mathrm{D}}} \hat{f}_T \cdot \hat{v}\, \mathrm{tr}(\mathrm{Cof}(dT)^\top dH),
\end{aligned} \tag{4.21a}$$

$$b_T(\hat{w}, \hat{q}) = 0, \tag{4.21b}$$

for all $(\hat{v}, \hat{q}) \in X_\mathbb{C}$.

Proposition 4.7. *Let the assumptions of Theorem 4.5 be satisfied. Then, the Fréchet derivative $d\mathcal{S}(T)(H)$ of $\mathcal{S}$ at $T \in \mathfrak{T}$ in direction $H \in W^{2,\infty}_\omega(\hat{\mathrm{D}}, \mathbb{C}^d)$ is given by the unique solution $(\hat{w}, \hat{r}) \in X_\mathbb{C}$*

of

$$
\begin{aligned}
&a_T(\hat{w},\hat{v}) - b_T(\hat{v},\hat{r}) + t_T(\hat{u}_T,\hat{w},\hat{v}) + t_T(\hat{w},\hat{u}_T,\hat{v}) = \\
&\quad \int_{\hat{\mathrm{D}}} \operatorname{tr}\left(\nabla(J^{-1}dT\hat{u}_T)\left[J(dT^{-1}dH dT^{-1}dT^{-\top} + dT^{-1}dT^{-\top}dH^\top dT^{-\top})\right]\nabla(J^{-1}dT\hat{v})^\top\right) \\
&\quad - \int_{\hat{\mathrm{D}}} \operatorname{tr}\left(\nabla\left(J^{-1}dT\hat{u}_T\right)dT^{-1}dT^{-\top}\nabla(J^{-1}dT\hat{v})^\top\right)\operatorname{tr}(\operatorname{Cof}(dT)^\top dH) \\
&\quad - \int_{\hat{\mathrm{D}}} \operatorname{tr}\left(\nabla\left(\left(\frac{-\operatorname{tr}(\operatorname{Cof}(dT)^\top dH)}{J^2}dT + J^{-1}dH\right)\hat{u}_T\right)dT^{-1}dT^{-\top}\nabla(J^{-1}dT\hat{v})^\top\right)J \\
&\quad - \int_{\hat{\mathrm{D}}} \operatorname{tr}\left(\nabla(J^{-1}dT\hat{u}_T)dT^{-1}dT^{-\top}\nabla\left(\left(\frac{-\operatorname{tr}(\operatorname{Cof}(dT)^\top dH)}{J^2}dT + J^{-1}dH\right)\hat{v}\right)^\top\right)J \\
&\quad - \int_{\hat{\mathrm{D}}} (\hat{u}_T\cdot\nabla)\left(\left(\frac{-\operatorname{tr}(\operatorname{Cof}(dT)^\top dH)}{J^2}dT + J^{-1}dH\right)\hat{u}_T\right)(dT\hat{v})J^{-1} \\
&\quad - \int_{\hat{\mathrm{D}}} (\hat{u}_T\cdot\nabla)(J^{-1}dT\hat{u}_T)\left(\left(J^{-1}dH - \frac{\operatorname{tr}(\operatorname{Cof}(dT)^\top dH)}{J^2}dT\right)\hat{v}\right) \\
&\quad + \int_{\hat{\mathrm{D}}} (df\circ T)H\cdot\hat{v}J + \int_{\hat{\mathrm{D}}} \hat{f}_T\cdot\hat{v}\operatorname{tr}(\operatorname{Cof}(dT)^\top dH), && (4.22a)\\
&b_T(\hat{w},\hat{q}) = 0, && (4.22b)
\end{aligned}
$$

for all $(\hat{v},\hat{q}) \in X_{\mathbb{C}}$.

5 Sparse Polynomial Approximation

Up to this point, we verified the holomorphic dependence of solutions to the Stokes and, under small data hypotheses, also of solutions to the Navier-Stokes problem on regular domain transformations T, with respect to the topology $W^{1,\infty}$ (plain pullback) or $W^{2,\infty}_{\omega}$ (Piola transformation), on the admissible family $\mathfrak{T}$ of transformations. For computational purposes, a parametrization of the domain transformations T is often used. For example, in Fourier series representations of T, $\boldsymbol{y}$ would denote the sequence of Fourier coefficients.

We consider parametrizations of the form

$$U \ni \boldsymbol{y} \mapsto T_{\boldsymbol{y}} \in \mathfrak{T}, \tag{5.1}$$

with

$$\boldsymbol{y} = (y_j)_{j=1,\dots,s} \in U := [-1,1]^s, \tag{5.2}$$

with $s \in \mathbb{N}$ or $s = \infty$ in the case of infinitely many parameters. This entails a parametric solution map

$$U \ni \boldsymbol{y} \mapsto (\hat{u}(\boldsymbol{y}), \hat{p}(\boldsymbol{y})) \in X, \tag{5.3}$$

with $(\hat{u}(\boldsymbol{y}), \hat{p}(\boldsymbol{y}))$ being the pullback solution corresponding to the transformation $T_{\boldsymbol{y}}$. The shape holomorphy results established in the previous sections imply results on holomorphic parameter dependence of the solutions, provided the parametrizations $\boldsymbol{y} \mapsto T_{\boldsymbol{y}}$ are themselves holomorphic. One key aspect is the description of the range of $\boldsymbol{y}$ on which the holomorphic extensions of the above maps are defined. Indeed, convergence rates of sparse polynomial approximations hold for parametric maps which admit holomorphic extensions in complex domains. For certain type of domains, these rates can be proved to be independent of the parameter dimension, in the sense that they remain valid in the infinite-dimensional case $s = \infty$. This is formalized by the notion of $(\boldsymbol{b}, \varepsilon)$-holomorphy introduced in [8], and recalled in Definition 5.1 below.

This analysis applies to numerous parametric geometry representations commonly used in engineering simulation. We mention only Fourier and wavelet representations (which lead to affine-parametric representations) and NURBS representations (where rational functions appear).

This section is structured as follows. We first review corresponding results from [8]. Thereafter, we illustrate the abstract results for certain transformation families $(T_{\boldsymbol{y}})_{\boldsymbol{y}\in U}$ and precise, for these families, the holomorphic parameter dependence in the sense given by Definition 5.1 below.

5.1 Holomorphy and sparse polynomial approximation

For $s > 1$, introduce the Bernstein ellipse in the complex plane:

$$\mathcal{E}_s := \left\{ \frac{w + w^{-1}}{2} : 1 \leq |w| \leq s \right\} \subset \mathbb{C}, \tag{5.4}$$

which has foci at $z = \pm 1$ and semi-axes of length $a := (s + s^{-1})/2 > 1$ and $b := (s - s^{-1})/2 > 0$, such that $a + b = s > 1$. We denote by

$$\mathcal{E}_{\boldsymbol{\rho}} := \bigotimes_{j \geq 1} \mathcal{E}_{\rho_j} \subset \mathbb{C}^{\mathbb{N}} , \tag{5.5}$$

the tensorized poly-ellipse when $\boldsymbol{\rho} := (\rho_j)_{j\geq 1}$ is a sequence of semi-axis sums $\rho_j > 1$. We denote $U := [-1, 1]^{\mathbb{N}}$. With the convention $\mathcal{E}_1 = [-1, 1]$, we also admit $\rho_j = 1$ in (5.5). Therefore $U \subseteq \mathcal{E}_{\boldsymbol{\rho}}$ with equality when $\rho_j = 1$ for all j.

In [8], convergence rates of sparse polynomial approximations to general parametric maps $u : \boldsymbol{y} \mapsto u(\boldsymbol{y})$ from U to X are established by holomorphic extensions of parametric solutions u to poly-ellipses $\mathcal{E}_{\boldsymbol{\rho}}$ of the above type. The size of these poly-ellipses is quantified according to the following definition.

Definition 5.1. *For a positive sequence $\boldsymbol{b} = (b_j)_{j\geq 1} \in \ell^p(\mathbb{N})$ for some $0 < p < 1$ and for some $\varepsilon > 0$, a parametric mapping $U \ni \boldsymbol{y} \mapsto u(\boldsymbol{y}) \in X$ satisfies the $(\boldsymbol{b}, \varepsilon)$-holomorphy assumption in the Banach space X for some $0 < p \leq 1$ if and only if the following holds: There exists a constant $C = C(\varepsilon, \boldsymbol{b})$ such that for any sequence $\boldsymbol{\rho} := (\rho_j)_{j\geq 1}$ of semi-axis sums $\rho_j > 1$ that is $(\boldsymbol{b}, \varepsilon)$-admissible, i.e.*

$$\sum_{j=1}^{\infty} (\rho_j - 1) b_j \leq \varepsilon, \tag{5.6}$$

the parametric map $\boldsymbol{y} \mapsto u(\boldsymbol{y}) \in X$ admits a complex extension $\boldsymbol{z} \mapsto u(\boldsymbol{z})$ (taking values in the complexification $X_{\mathbb{C}}$ of X) that is a holomorphic mapping with respect to each variable z_j on a set of the form $\mathcal{O}_{\boldsymbol{\rho}} := \bigotimes_{j\geq 1} \mathcal{O}_{\rho_j}$, where $\mathcal{O}_{\rho_j} \subset \mathbb{C}$ is an open set containing $\mathcal{E}_{\rho_j}$, and this extension is uniformly bounded on $\mathcal{E}_{\boldsymbol{\rho}}$ in (5.5) according to

$$\sup_{\boldsymbol{z} \in \mathcal{E}_{\boldsymbol{\rho}}} \|u(\boldsymbol{z})\|_X \leq C . \tag{5.7}$$

Remark 5.2. *The notion of $(\boldsymbol{b},\varepsilon)$-holomorphy does not automatically entail continuity in the infinite dimensional case, when the parameter set $U = [-1,1]^\infty$ is equipped with classical topologies. For this reason, we sometimes add continuity as an extra assumption in some of the subsequent statements.*

Consider for example

$$u(\boldsymbol{y}) := \begin{cases} 1 & \text{if } |\{j \,:\, y_j \neq 0\}| < \infty \\ 0 & \text{otherwise,} \end{cases} \tag{5.8}$$

for any $\boldsymbol{y} \in U$. Then (5.8) also defines a uniformly bounded extension onto $\boldsymbol{y} \in \mathbb{C}^{\mathbb{N}}$ that is holomorphic as a function of each y_j, but is discontinuous as a function of $\boldsymbol{y}$, for example when U is equipped with the product topology or any ℓ^p topology. In particular it is $(\boldsymbol{b},\varepsilon)$-holomorphic for the null sequence $\boldsymbol{b} = 0$, for any $\varepsilon > 0$. Note however that in this example, u is equal to the continuous null function almost everywhere in the sense of the uniform probability measure on U.

The significance of $(\boldsymbol{b},\varepsilon)$-holomorphy lies in that it yields approximation rates which are free from the curse of dimensionality. In particular,

(a) $(\boldsymbol{b},\varepsilon)$-holomorphic parametric solution maps $\boldsymbol{y} \mapsto u(\boldsymbol{y})$ allow for sparse polynomial approximations with dimension-independent n-term convergence rates which depend only on the summability exponent p of the sequence $\boldsymbol{b} \in \ell^p(\mathbb{N})$, see [8].

(b) Such polynomial approximations of $(\boldsymbol{b},\varepsilon)$-holomorphic parametric solution maps can also be constructively approximated by sparse, Smolyak type interpolation methods, see [6, 8].

In relation with such results, the concept of $(\boldsymbol{b},\varepsilon)$-holomorphy has been exploited in the context of Bayesian inverse problems [21] and for the construction of low-parametric, reduced basis surrogates [3, 4].

We next explain items (a) and (b), detailing in particular computational approximation strategies for the efficient computation of sparse approximations of countably-parametric solution families. We first recall the main polynomial approximation results from [8]. To state these results, for any coefficient bound sequence $c := (c_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}} \subset \mathbb{R}$, where $\mathcal{F} = \{\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} : |\boldsymbol{\nu}| < \infty\}$ denotes the set of all finitely supported multi-indices, we associate its downward closed envelope $\mathbf{c} := (\mathbf{c}_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\mathcal{F}}$ defined by

$$\mathbf{c}_{\boldsymbol{\nu}} := \sup_{\boldsymbol{\mu}\geq\boldsymbol{\nu}} |c_{\boldsymbol{\nu}}|, \quad \boldsymbol{\nu} \in \mathcal{F}, \tag{5.9}$$

where $\boldsymbol{\mu} \geq \boldsymbol{\nu}$ means that $\mu_j \geq \nu_j$ for all j. We also say that a set $\Lambda \subset \mathcal{F}$ is *downward closed* if and only if

$$\boldsymbol{\nu} \in \Lambda \quad \text{and} \quad \boldsymbol{\mu} \leq \boldsymbol{\nu} \Rightarrow \boldsymbol{\mu} \in \Lambda. \tag{5.10}$$

For $p > 0$, we introduce the space $\ell^p_m(\mathcal{F})$ of sequences that have their downward closed envelope in $\ell^p(\mathcal{F})$. We equip U with the uniform probability measure and consider the associated Bochner spaces $L^2(U,X)$ and $L^\infty(U,X)$.

Assuming that the solution map $\boldsymbol{y} \mapsto u(\boldsymbol{y})$ belongs to $L^2(U,X)$, we consider generalized polynomial chaos approximations constructed by truncation of the tensorized Legendre series

$$u(\boldsymbol{y}) = \sum_{\boldsymbol{\nu}\in\mathcal{F}} u_{\boldsymbol{\nu}} P_{\boldsymbol{\nu}}(\boldsymbol{y}), \tag{5.11}$$

where $P_{\boldsymbol{\nu}}(\boldsymbol{y}) := \prod_{j\geq 1} P_{\nu_j}(y_j)$, with P_n denoting the univariate Legendre polynomial of degree n for the interval $[-1,1]$ with the classical normalization $\|P_n\|_{L^\infty([-1,1])} = |P_n(1)| = 1$. We obtain an $L^2(U,X)$-orthonormal basis by taking $L_{\boldsymbol{\nu}}(\boldsymbol{y}) := \prod_{j\geq 1} L_{\nu_j}(y_j)$, with L_n denoting P_n normalized in $L^2([-1,1],\frac{dt}{2})$. The series (5.11) may then be rewritten as

$$u(\boldsymbol{y}) = \sum_{\boldsymbol{\nu}\in\mathcal{F}} v_{\boldsymbol{\nu}} L_{\boldsymbol{\nu}}(\boldsymbol{y}) \ , \tag{5.12}$$

where

$$u_{\boldsymbol{\nu}} = \left(\prod_{j\geq 1}(1+2\nu_j)\right)^{1/2} v_{\boldsymbol{\nu}} \ . \tag{5.13}$$

Note that both series (5.11) and (5.12) converge unconditionally in $L^2(U,X)$.

Theorem 5.3 ([8])**.** *Assume that the solution map $\boldsymbol{y} \mapsto u(\boldsymbol{y})$ belongs to $L^2(U,X)$, and that it is $(\boldsymbol{b},\varepsilon)$-holomorphic for some $\boldsymbol{b} \in \ell^p$ with $0 < p < 1$ and $\varepsilon > 0$. Then, the sequences $(\|u_{\boldsymbol{\nu}}\|_X)_{\boldsymbol{\nu}\in\mathcal{F}}$ and $(\|v_{\boldsymbol{\nu}}\|_X)_{\boldsymbol{\nu}\in\mathcal{F}}$ of (norms of) the tensorized Legendre coefficients belong to $\ell^p_m(\mathcal{F})$, and the series* (5.11) *and* (5.12) *converge unconditionally in $L^\infty(U,X)$. There exist sequences $(\Lambda_n^2)_{n\geq 1}$ and $(\Lambda_n^\infty)_{n\geq 1}$, of nested downward closed subsets of $\mathcal{F}$ and a constant C such that, with $\#(\Lambda_n^2) = \#(\Lambda_n^\infty) = n$, there holds*

$$\|u - \sum_{\boldsymbol{\nu}\in\Lambda_n^\infty} v_{\boldsymbol{\nu}} L_{\boldsymbol{\nu}}\|_{L^\infty(U,X)} \leq C(n+1)^{-s}, \quad s = \frac{1}{p} - 1, \tag{5.14}$$

and

$$\|u - \sum_{\boldsymbol{\nu}\in\Lambda_n^2} v_{\boldsymbol{\nu}} L_{\boldsymbol{\nu}}\|_{L^2(U,X)} \leq C(n+1)^{-r}, \quad r = \frac{1}{p} - \frac{1}{2}. \tag{5.15}$$

Here, for a general downward closed set Λ we have defined the X-valued polynomial spaces

$$X_\Lambda := X \otimes \mathbb{P}_\Lambda = \left\{\sum_{\boldsymbol{\nu}\in\Lambda} v_{\boldsymbol{\nu}} \boldsymbol{y}^{\boldsymbol{\nu}} : v_{\boldsymbol{\nu}} \in X\right\} \ , \quad \mathbb{P}_\Lambda := \mathrm{span}\{\boldsymbol{y} \mapsto \boldsymbol{y}^{\boldsymbol{\nu}} \ : \ \nu \in \Lambda\} \ . \tag{5.16}$$

The preceding theorem gives approximation rate bounds in the spaces $X_{\Lambda_n^2}$ and $X_{\Lambda_n^\infty}$ which are free from the curse of dimensionality. The sets Λ_n^2 and Λ_n^∞ can be taken as the set of indices $\boldsymbol{\nu}$ corresponding to n largest values in the monotone envelopes of the sequences $(\|v_{\boldsymbol{\nu}}\|_X)_{\boldsymbol{\nu}\in\mathcal{F}}$ and $(\|u_{\boldsymbol{\nu}}\|_X)_{\boldsymbol{\nu}\in\mathcal{F}}$, respectively, or of certain computable surrogate quantities for these sequences.

Polynomial interpolation is a natural alternate strategy to the truncation of orthonormal series for the construction of polynomial approximation, which is based on interpolation. Interpolation processes on the spaces X_Λ for general downward closed sets Λ of multi-indices have been introduced and studied in [6]. Given $\boldsymbol{z} := (z_j)_{j\geq 1}$, a sequence of pairwise distinct points of $[-1,1]$, we associate with any finite subset $\Lambda \subset \mathcal{F}$ the following sparse grid in U:

$$\Gamma_\Lambda := \{\boldsymbol{z}_{\boldsymbol{\nu}} : \boldsymbol{\nu} \in \Lambda\} \quad \text{where} \quad \boldsymbol{z}_{\boldsymbol{\nu}} := (z_{\nu_j})_{j\geq 1} \ . \tag{5.17}$$

If $\Lambda \subset \mathcal{F}$ is downward closed, then the sparse grid Γ_Λ is unisolvent for X_Λ: for any function u defined in Γ_Λ and taking values in X, there exists a unique sparse grid interpolation polynomial $I_\Lambda u$ in X_Λ that coincides with u on Γ_Λ. This unique interpolation polynomial $I_\Lambda u \in X_\Lambda$ can be evaluated recursively as follows: if we write $\Lambda := \{\boldsymbol{\nu}^1, \ldots, \boldsymbol{\nu}^N\}$ such that for any $k = 1, \ldots, N$, $\Lambda_k := \{\boldsymbol{\nu}^1, \ldots, \boldsymbol{\nu}^k\}$ is downward closed, then

$$I_\Lambda u = \sum_{i=1}^{N} u_{\boldsymbol{\nu}^i} Q_{\boldsymbol{\nu}^i}, \tag{5.18}$$

where the polynomials $(Q_{\boldsymbol{\nu}})_{\boldsymbol{\nu}\in\Lambda}$ are a hierarchical basis of $\mathbb{P}_\Lambda$ given by

$$Q_{\boldsymbol{\nu}}(\boldsymbol{y}) := \prod_{j\geq 1} q_{\nu_j}(y_j) \quad \textit{where} \quad q_0(t) = 1 \text{ and } q_k(t) = \prod_{j=0}^{k-1} \frac{t - z_j}{z_k - z_j}, \quad k \geq 1, \tag{5.19}$$

and where the coefficients $u_{\boldsymbol{\nu}^k} \in X$ are recursively defined by

$$u_{\boldsymbol{\nu}^1} := u(z_0), \quad u_{\boldsymbol{\nu}^{k+1}} := u(z_{\boldsymbol{\nu}^{k+1}}) - I_{\Lambda_k} u(z_{\boldsymbol{\nu}^{k+1}}) = u(z_{\boldsymbol{\nu}^{k+1}}) - \sum_{i=1}^{k} u_{\boldsymbol{\nu}^i} Q_{\boldsymbol{\nu}^i}(z_{\boldsymbol{\nu}^{k+1}}) \,. \tag{5.20}$$

The following result recovers the same approximation rate $\mathcal{O}(n^{-s})$ in $L^\infty(U, V)$ as in (5.14) for the interpolation based on a different choice of downward closed sets.

Theorem 5.4 ([8]). *Assume that the series* (5.11) *and* (5.12) *converge unconditionally in* $L^\infty(U, X)$ *towards the map* $\boldsymbol{y} \mapsto u(\boldsymbol{y})$*, and assume that this map is* $(\boldsymbol{b}, \varepsilon)$*-holomorphic for some* $\boldsymbol{b} \in \ell^p$ *with* $0 < p < 1$ *and for some* $\varepsilon > 0$*. Assume in addition that* $\boldsymbol{y} \mapsto u(\boldsymbol{y})$ *is continuous from* U *equipped with the product topology to* X*. Assume, moreover, that the Lebesgue constants of the* n*-sections of the sequence* $\boldsymbol{z} := (z_j)_{j\geq 1}$ *of pairwise distinct points of* $[-1, 1]$ *constituting the sparse grid* (5.17) *are bounded polynomially as* $(1 + n)^\theta$ *for some* $\theta \geq 0$*. Then, there exists a constant* $C > 0$ *and a nested sequence of downward closed sets* $(\Lambda_n)_{n\geq 1}$ *with* $\#(\Lambda_n) = n$ *for which*

$$\|u - I_{\Lambda_n} u\|_{L^\infty(U,X)} \leq C(n + 1)^{-s}, \quad s = \frac{1}{p} - 1. \tag{5.21}$$

One example of a sequence $\boldsymbol{z} := (z_j)_{j\geq 1}$ such that the Lebesgue constants of the n-sections are bounded polynomially is the so-called R-Leja sequence, see [7]. The above results are a natural motivation for establishing holomorphy of the parametric mapping $U \ni \boldsymbol{y} \mapsto (\hat{u}(\boldsymbol{y}), \hat{p}(\boldsymbol{y})) = (\hat{u}_{T_{\boldsymbol{y}}}, \hat{p}_{T_{\boldsymbol{y}}})$ of the Stokes and the Navier-Stokes pullback solutions.

5.2 Parametric Domain Transformations

Assume that $\hat{\mathrm{D}} \subseteq \mathbb{R}^d$ is a bounded Lipschitz domain. We now consider bijective domain transformations $T_{\boldsymbol{y}} : \hat{\mathrm{D}} \to \mathrm{D}_{\boldsymbol{y}}$, from the fixed nominal domain $\hat{\mathrm{D}}$ to the physical domains $\mathrm{D}_{\boldsymbol{y}} := T_{\boldsymbol{y}}(\hat{\mathrm{D}})$. Here and in the following $\boldsymbol{y} \in U = [-1, 1]^{\mathbb{N}}$. Furthermore, the space S defined for $\mathrm{D} \subseteq \mathbb{R}^d$ by either

$$S(\mathrm{D}) = W^{1,\infty}(\mathrm{D}, \mathbb{R}^d) \qquad \text{or} \qquad S(\mathrm{D}) = W^{2,\infty}_{\omega}(\mathrm{D}, \mathbb{R}^d) \tag{5.22}$$

is fixed throughout this subsection, and will be specified subsequently. In order to derive $(\boldsymbol{b}, \varepsilon)$-holomorphy of the parametric solution map, we consider transformations $T_{\boldsymbol{y}}$ where the dependence of $T_{\boldsymbol{y}}$ on $\boldsymbol{y}$ is affine, as is customary for example in Fourier, spline or wavelet representations.

Let $\overline{T} \in S(\hat{\mathrm{D}})$ such that $\overline{T}^{-1} \in S(\overline{T}(\hat{\mathrm{D}}))$, and $\psi_j \in S(\hat{\mathrm{D}})$ for every $j \in \mathbb{N}$. This implies that $\overline{T}$ is bi-Lipschitz and there exists $0 < \kappa_1 \leq \kappa_2 < \infty$ s.t. for each a, b, satisfying that their convex envelope is in the open bounded Lipschitz domain $\hat{\mathrm{D}}$,

$$\kappa_1 \|a - b\| \leq \|\overline{T}(a) - \overline{T}(b)\| \leq \kappa_2 \|a - b\|. \tag{5.23}$$

Here, for $x \in \mathbb{R}^d$, the expression $\|x\|$ denotes the Euclidean norm and, for a matrix $M \in \mathbb{R}^{d\times d}$, $\|M\|$ stands for its spectral norm. Let again $U = [-1, 1]^{\mathbb{N}}$, and set for $\boldsymbol{y} = (y_j)_{j\geq 1} \in U$

$$T_{\boldsymbol{y}} := \overline{T} + \sum_{j\in\mathbb{N}} y_j \psi_j. \tag{5.24}$$

In order to establish $(\boldsymbol{b}, \varepsilon)$-holomorphy of the resulting parameter to solution map, we impose the following assumption.

Assumption 5.5. *For the sequence $\boldsymbol{b} = (b_j)_{j\geq 1}$ defined by $b_j := \|\psi_j\|_{S(\hat{\mathrm{D}})}$, $j \in \mathbb{N}$, there exists $p \in (0, 1)$ with $\boldsymbol{b} \in \ell^p$. In addition, $T_{\boldsymbol{y}}$ is invertible with $T_{\boldsymbol{y}}^{-1} \in S(T_{\boldsymbol{y}}(\hat{\mathrm{D}}))$ for all $\boldsymbol{y} \in U$.*

Remark 5.6. *In the case of Fourier, spline or wavelets representations, the summability exponent p of the sequence $\boldsymbol{b} \in \ell^p$ for which Assumption 5.5 holds is typically related to the amount of spatial Sobolev or Besov smoothness of the domain transformations $T_{\boldsymbol{y}}$.*

We further give concrete examples of domains with parametrized boundaries such that Assumption 5.5 holds either with $S = W^{1,\infty}$ or $S = W^{2,\infty}$. This assumption immediately implies absolute convergence of $T_{\boldsymbol{y}}$ in (5.24), uniformly with respect to $\boldsymbol{y} \in U$, in the sense that

$$\sup_{\boldsymbol{y}\in U} \sum_{j\in\mathbb{N}} \|y_j \psi_j\|_{S(\hat{\mathrm{D}})} = \sum_{j\in\mathbb{N}} \|\psi_j\|_{S(\hat{\mathrm{D}})} < \infty. \tag{5.25}$$

In turn, the domain $\mathrm{D}_{\boldsymbol{y}} = T_{\boldsymbol{y}}(\hat{\mathrm{D}})$ is well defined for all $\boldsymbol{y} \in U$. We next discuss other important implications of Assumption 5.5. The first one is the continuity of the map $\boldsymbol{y} \mapsto T_{\boldsymbol{y}}$.

Lemma 5.7. *Let Assumption 5.5 be satisfied. Then, the map $\boldsymbol{y} \mapsto T_{\boldsymbol{y}}$ is continuous from U equipped with the product topology to $S(\hat{\mathrm{D}})$, and the family*

$$\mathfrak{T} := \{T_{\boldsymbol{y}} \,:\, \boldsymbol{y} \in U\}, \tag{5.26}$$

is compact in $S(\hat{\mathrm{D}})$.

Proof. Since $(\|\psi_j\|_{S(\hat{\mathrm{D}})})_{j\in\mathbb{N}} \in \ell^1$, continuity follows by the following standard argument: if $\boldsymbol{y}_n = (y_{j,n})_{j\geq 1}$ converges pointwise to $\boldsymbol{y} = (y_j)_{j\geq 1}$, then we may write

$$\|T_{\boldsymbol{y}} - T_{\boldsymbol{y}_n}\|_{S(\hat{\mathrm{D}})} \leq \sum_{1\leq j\leq J} |y_{j,n} - y_j| \, \|\psi_j\|_{S(\hat{\mathrm{D}})} + 2 \sum_{j>J} \|\psi_j\|_{S(\hat{\mathrm{D}})}. \tag{5.27}$$

The second term is made arbitrarily small by taking J large enough and the first term goes to 0 by pointwise convergence. Compactness of $\mathfrak{T}$ follows from the compactness of U with respect to the product topology. □

Note that the compactness of $\mathfrak{T}$ established in Lemma 5.7 together with the invertibility of $T_{\boldsymbol{y}}$ validate Assumptions 3.3 and 4.2. The second implication of Assumption 5.5 concerns the $(\boldsymbol{b}, \varepsilon)$-holomorphy of the map $\boldsymbol{y} \mapsto T_{\boldsymbol{y}}$.

Lemma 5.8. *Let Assumption 5.5 be satisfied. Then the map $\boldsymbol{y} \mapsto T_{\boldsymbol{y}}$ is $(\boldsymbol{b}, \varepsilon)$-holomorphic for the same $\boldsymbol{b}$ used in Assumption 5.5, and for any $\varepsilon > 0$.*

Proof. Let $\boldsymbol{\rho}$ be any $(\boldsymbol{b}, \varepsilon)$-admissible sequence, that is, such that

$$\sum_{j\geq 1}(\rho_j - 1)b_j \leq \varepsilon. \tag{5.28}$$

Then we may define the domain $\mathcal{O}_{\boldsymbol{\rho}} = \prod_{j\geq 1} \mathcal{O}_{\rho_j}$, by taking for the univariate domain

$$\mathcal{O}_s := \{z \in \mathbb{C} \,:\, \operatorname{dist}(z, [-1, 1]) < s - 1\}. \tag{5.29}$$

It is readily checked that $\mathcal{O}_s$ is an open neighborhood of $\mathcal{E}_s$ for $s > 1$. It now suffices to remark that the extension

$$T_{\boldsymbol{z}} = \overline{T} + \sum_{j\geq 1} z_j \psi_j, \tag{5.30}$$

is well defined in $S(\hat{\mathrm{D}})$ for any $\boldsymbol{z}$ in $\mathcal{O}_{\boldsymbol{\rho}}$ since the above series converges absolutely in $S(\hat{\mathrm{D}})$. Holomorphy in each variable is trivial since it is an affine function. $\square$

Finally, recalling the ε-neighborhood of $\mathfrak{T}$ in $S(\hat{D}, \mathbb{C}^d)$ which is denoted by $\mathfrak{T}_\varepsilon$, we have the following result.

Lemma 5.9. *Let Assumption 5.5 be satisfied. Let $\boldsymbol{\rho}$ be any $(\boldsymbol{b}, \varepsilon)$-admissible sequence and $\mathcal{O}_{\boldsymbol{\rho}}$ be as in the proof of Lemma 5.8. Then, the holomorphic extension obtained in this Lemma satisfies that*

$$\boldsymbol{z} \in \mathcal{O}_{\boldsymbol{\rho}} \Rightarrow T_{\boldsymbol{z}} \in \mathfrak{T}_\varepsilon, \tag{5.31}$$

Proof. From the definition of $\mathcal{O}_{\boldsymbol{\rho}}$, for any $\boldsymbol{z} \in \mathcal{O}_{\boldsymbol{\rho}}$ there exists $\boldsymbol{y} \in U$ such that

$$|z_j - y_j| \leq \rho_j - 1, \quad j \geq 1. \tag{5.32}$$

Therefore

$$\|T_{\boldsymbol{z}} - T_{\boldsymbol{y}}\|_{S(\hat{\mathrm{D}})} \leq \sum_{j\geq 1}(\rho_j - 1)\|\psi_j\|_{S(\hat{\mathrm{D}})} = \sum_{j\geq 1}(\rho_j - 1)b_j \leq \varepsilon. \tag{5.33}$$

Since $T_{\boldsymbol{y}} \in \mathfrak{T}$, this shows that $T_{\boldsymbol{z}} \in \mathfrak{T}_\varepsilon$. $\square$

We next give two natural examples of parametrized domains for which Assumption 5.5 can be established with $S = W^{1,\infty}$ or $S = W^{2,\infty}_\omega$.

Example 5.10. *We consider star shaped domains for $d = 2$. It is then natural to use polar coordinates, and we write $\hat{x} = \sigma(\cos\varphi, \sin\varphi)^\top$. Let $\hat{r}$, $r \in W^{1,\infty}_{\mathrm{per}}(0, 2\pi)$ be two 2π periodic functions with Lipschitz constant λ s.t. $r_{\min} \leq \hat{r}, r \leq r_{\max}$ for some $0 < r_{\min} \leq r_{\max} < \infty$. Set*

$$\hat{\mathrm{D}} := \{\sigma(\cos\varphi, \sin\varphi)^\top \,:\, \sigma \leq \hat{r}(\varphi)\} \qquad \textit{and} \qquad \mathrm{D} := \{\sigma(\cos\varphi, \sin\varphi)^\top \,:\, \sigma \leq r(\varphi)\}, \tag{5.34}$$

and introduce the transformation $T(\hat{x}) := \frac{r(\varphi)}{\hat{r}(\varphi)}\hat{x} = x$*, mapping* $\hat{\mathrm{D}}$ *to* D*. Since* $\hat{r},\ r \in W^{1,\infty}(0,2\pi)$*, both domains are Lipschitz. With* $q(\varphi) := \frac{r(\varphi)}{\hat{r}(\varphi)}$ *one obtains*

$$dT(\hat{x}) = \begin{pmatrix} q(\varphi) - q'(\varphi)\sin(\varphi)\cos(\varphi) & q'(\varphi)\cos(\varphi)^2 \\ -q'(\varphi)\sin(\varphi)^2 & q(\varphi) + q'(\varphi)\sin(\varphi)\cos(\varphi) \end{pmatrix} \tag{5.35}$$

and $\det dT = q(\varphi)^2 \geq (\frac{r_{\min}}{r_{\max}})^2$*. Therefore* $T \in W^{1,\infty}(\hat{\mathrm{D}})$*. Switching the roles of* $\hat{r}$ *and* r *gives the inverse transformation* T^{-1}*, which is also in* $W^{1,\infty}(\mathrm{D})$*. The* $W^{1,\infty}$ *norms of* T *and* T^{-1} *solely depend on* λ*,* $r_{\min}$ *and* $r_{\max}$*.*

In order to parametrize the transformations, let now $(r_j)_{j\in\mathbb{N}}$ *be a sequence of functions in* $W^{1,\infty}_{\mathrm{per}}(0,2\pi)$*, s.t. for some* $p \in (0,1]$ *and for* $\varphi \in [0,2\pi)$

$$(\|r_j\|_{W^{1,\infty}_{\mathrm{per}}(0,2\pi)})_{j\in\mathbb{N}} \in \ell^p(\mathbb{N}) \quad \textit{and} \quad r_{\min} \leq \hat{r}(\varphi) - \sum_{j\in\mathbb{N}} |r_j(\varphi)| \leq \hat{r}(\varphi) + \sum_{j\in\mathbb{N}} |r_j(\varphi)| \leq r_{\max}. \tag{5.36}$$

Setting

$$\overline{T}(\hat{x}) := \mathrm{Id} \quad \text{and} \quad \psi_j(\hat{x}) := \sigma \frac{r_j(\varphi)}{\hat{r}(\varphi)}(\cos\varphi, \sin\varphi)^\top, \tag{5.37}$$

leads to the affinely parametrized transformation family as in (5.24)*, with* $T_{\boldsymbol{y}}$ *corresponding to the boundary described by*

$$\varphi \mapsto r_{\boldsymbol{y}}(\varphi) := \hat{r}(\varphi) + \sum_{j\in\mathbb{N}} y_j r_j(\varphi) \in W^{1,\infty}_{\mathrm{per}}(0,2\pi). \tag{5.38}$$

Note that

$$T_{\boldsymbol{y}}(\hat{x}) = \sigma \frac{r_{\boldsymbol{y}}(\varphi)}{\hat{r}(\varphi)}(\cos\varphi, \sin\varphi)^\top. \tag{5.39}$$

Let us check Assumption 5.5 for $S = W^{1,\infty}$*. By* (5.35) *and the definition of* ψ_j*, we get*

$$\|\psi_j\|_{W^{1,\infty}(\hat{\mathrm{D}})} \leq C\|r_j\|_{W^{1,\infty}_{\mathrm{per}}(0,2\pi)}, \tag{5.40}$$

for some $C = C(\hat{r})$*. Hence* $(\|\psi_j\|_{W^{1,\infty}(\hat{\mathrm{D}})})_{j\in\mathbb{N}} \in \ell^p$ *and in particular* $\boldsymbol{y} \mapsto T_{\boldsymbol{y}} \in W^{1,\infty}(\hat{\mathrm{D}})$ *is continuous. The inverse transformation* $T_{\boldsymbol{y}}^{-1}$ *is obviously given by*

$$T_{\boldsymbol{y}}^{-1}(x) = \sigma \frac{\hat{r}(\varphi)}{r_{\boldsymbol{y}}(\varphi)}(\cos\varphi, \sin\varphi)^\top, \quad x = \sigma(\cos\varphi, \sin\varphi)^\top. \tag{5.41}$$

Since $r_{\boldsymbol{y}}(\varphi) \geq r_{\min}$ *for all* $\boldsymbol{y} \in U$ *and* $\varphi \in [0,2\pi)$*, this transformation is well defined and in addition, by the same argument used for bounding* $\|\psi_j\|_{W^{1,\infty}(\hat{\mathrm{D}})}$*,*

$$\|T_{\boldsymbol{y}}^{-1}\|_{W^{1,\infty}(\mathrm{D}_{\boldsymbol{y}})} \lesssim \left\|\frac{\hat{r}}{r_{\boldsymbol{y}}}\right\|_{W^{1,\infty}} \lesssim \|r_{\boldsymbol{y}}\|_{W^{1,\infty}}. \tag{5.42}$$

where the multiplicative constant depend on $\hat{r}$ *and* $r_{\min}$*, and the right side is bounded independently of* $\boldsymbol{y}$*.*

Example 5.11. *We can use the construction in Remark 4.1, to adapt the above example for $S = W^{2,\infty}_{\omega}$ transformations, therefore allowing us to use the Piola pullback. Denote by B_1 the unit disc in $\mathbb{R}^2$ and let r, D, as in Example 5.10. Additionally set $\hat{r} \equiv 1$, hence $B_1 = \hat{\mathrm{D}}$. We have already seen in Remark 4.1 that for $\varepsilon > 0$ small enough (in dependence of $\|r\|_{W^{1,\infty}}$), T given in (4.6) is a bijection from B_1 to D, with $T \in W^{2,\infty}_{\omega}(B_1)$ and $T^{-1} \in W^{2,\infty}_{\omega}(\mathrm{D})$, and its Jacobian determinant is bounded uniformly from above and below.*

Let now $(r_j)_{j\in\mathbb{N}}$, $r_{\min}$ as in (5.36), χ as in Remark 4.1 and set

$$\overline{T}(x) := x\left[r_{\min} + \chi(|x|)(1 - r_{\min})\right], \tag{5.43}$$

$$g_j(x,h) := \frac{1}{h^2}\int_{\mathbb{R}^2} r_j(\xi/|\xi|)\varphi((x-\xi)/h)\mathrm{d}\xi, \tag{5.44}$$

$$\psi_j(x) := x\chi(|x|)\left(g_j\left(\varepsilon(1-|x|), x/|x|\right) - r_{\min}\right). \tag{5.45}$$

Then the transformation $T_{\boldsymbol{y}}$ in (5.24) corresponds to (4.6) for $r := 1 + \sum_{j\in\mathbb{N}} y_j r_j$. The validity of Assumption 5.5 for $S = W^{2,\infty}_{\omega}$ is now verified similarly as in Example 5.10.

We finally combine Assumption 5.5 with the results on holomorphic dependence of the domain to solution map $T \mapsto (\hat{u}_T, \hat{p}_T)$ established in the previous sections, in order to establish the $(\boldsymbol{b},\varepsilon)$-holomorphy of the parameter to solution map $\boldsymbol{y} \mapsto (\hat{u}(\boldsymbol{y}), \hat{p}(\boldsymbol{y}))$.

Corollary 5.12. *Denote by $(\hat{u}(\boldsymbol{y}), \hat{p}(\boldsymbol{y})) \in X = (H^1_0(\hat{\mathrm{D}}))^d \times L^2_{\#}(\hat{\mathrm{D}})$ the solution of (3.18) or (3.19) with respect to the transformation $T = T_{\boldsymbol{y}}$, such that either of the following holds:*

- *i) the operators in (3.18), (3.19) are defined via (3.12), Assumption 5.5 is satisfied with $S = W^{1,\infty}$, and in case of (3.19) additionally Assumption 3.9 holds.*
- *ii) the operators in (3.18), (3.19) are defined via (4.15), Assumption 5.5 is satisfied with $S = W^{2,\infty}_{\omega}$, and in case of (3.19) additionally Assumption 4.4 holds.*

Then there exists $\varepsilon > 0$ such that the parameter to solution map $U \ni \boldsymbol{z} \mapsto (\hat{u}(\boldsymbol{y}), \hat{p}(\boldsymbol{y})) \in X$ is $(\boldsymbol{b},\varepsilon)$-holomorphic, with the same sequence $\boldsymbol{b}$ as in Assumption 5.5. In addition this map is continuous from U equipped with the product topology to X.

Proof. As already noted in Lemma 5.7, Assumption 5.5 implies that the set $\mathfrak{T} := \{T_{\boldsymbol{y}} : \boldsymbol{y} \in U\}$ is compact, as it is the image of a compact set under a continuous map, and validates Assumptions 3.3 and 4.2. By Theorems 3.7, 3.10, 4.3 and 4.5, the domain to solution maps $T \mapsto (\hat{u}_T, \hat{p}_T)$ admit holomorphic extensions over some set $\mathfrak{T}_\varepsilon$ which is an ε neighborhood of $\mathfrak{T}$ in $S(\hat{D}, \mathbb{C}^d)$. In particular, this map is continuous on $\mathfrak{T}$. This together with Lemma 5.7 implies by composition the continuity of the parameter to solution map.

We now take any $(\boldsymbol{b},\varepsilon)$-admissible sequence $\boldsymbol{\rho}$ and define $\mathcal{O}_{\boldsymbol{\rho}}$ as in the proof of Lemma 5.8. Then according to Lemma 5.9, we know that $T_{\boldsymbol{z}} \in \mathfrak{T}_\varepsilon$ for all $\boldsymbol{z} \in O_{\boldsymbol{\rho}}$. The holomorphy in each variable of the extended parameter to solution map

$$\boldsymbol{z} \mapsto (\hat{u}(\boldsymbol{z}), \hat{p}(\boldsymbol{z})) = (\hat{u}_{T_{\boldsymbol{z}}}, \hat{p}_{T_{\boldsymbol{z}}}) \in X_{\mathbb{C}}, \tag{5.46}$$

now follows from composition of holomorphic maps □

6 Conclusions

For the stationary Stokes and Navier-Stokes equations in a bounded Lipschitz domain, with no-slip boundary conditions, and subject to analytic volume force which satisfies a small data assumption, we have shown the holomorphic dependence of the velocity and the pressure on the domain.

Two classes of domain parametrizations were admitted: bi-Lipschitz transformations as well as the divergence preserving Piola transformation. In either case, the domain-to-solution map of the solutions' pullback to a reference domain was shown to depend holomorphically on the domain parametrization T. For the Piola transformation, the appearance of second spacial derivatives of the domain parametrizations and the admission of Lipschitz domains mandated in particular the use of the weighted space $W^{2,\infty}_{\omega} \subset W^{1,\infty}$ and the no-slip boundary conditions in order to compensate the growth of the second derivatives near the boundary.

The presently obtained results imply holomorphic-parametric dependence of solutions of viscous, incompressible flow on the geometry, which is assumed to be a parametric image of a fixed, "nominal" domain under a sufficiently regular map.

We indicated several convergence rate results for sparse, polynomial approximation schemes of "generalized polynomial chaos" type of the parametric solution families which follow from the presently established holomorphy results. These, in turn, imply convergence rates of greedy approximations which have recently been found to perform successfully for such problems in [11, 12, 19, 16] and the references there. Concrete approximation algorithms, including also the spatial approximation of the solutions, with precise regularity requirements, which realize the rates will be addressed in a forthcoming publication.

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