EFFECTIVENESS OF AN ARTIFICIAL FRESH-WATER BARRIER IN THE ALLEVIATION OF THE EFFECTS OF SALT-WATER INTRUSION

A THESIS

Presented to

The Faculty of the Graduate Division

by

Srisakdi Charmonman

In Partial Fulfillment

of the Requirements for the Degree

Doctor of Philosophy in the

School of Civil Engineering

Georgia Institute of Technology

June, 1964

EFFECTIVENESS OF AN ARTIFICIAL FRESH-WATER BARRIER

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Approved:

Chairman

Date approved by Chairman: 6/4/64

ACKNOWLEDGMENTS

The author wishes to express his sincere appreciation to his advisor, Dr. M. R. Carstens, for his suggestion of the thesis topic and his advice and guidance throughout the course of this research. Dr. William F. Atchison, Chief of the Rich Electronic Computer Center, kindly granted free computer time for this research. The interest and remarks of the other members of the reading committee, Regents' Professor C. E. Kindsvater and Professor A. B. Vesic, are greatly appreciated.

The author is grateful to U. S. AID and the SEATO Graduate School of Engineering for financial support of this program.

Assistance and cooperation in typing and proofreading from Mrs. Jacqueline M. Van Hook, Mrs. Gloria M. Herzog, and Mr. Larry Lyons are appreciated.

Sincere thanks are due to his wife, Mrs. Parasubhasri Charmonman, for her patience, valuable assistance, and encouragement.

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SUMMARY

Due to the gentle slope of the alluvial plain at the head of the Gulf of Siam, an appreciable area of land is subjected to salt-water intrusion under the land surface. In an area extending several miles from the Gulf shore, only salt-tolerant vegetation will thrive. A slight lowering of the interface between the intruded salt water and the overlying fresh water would result in a significant increase in agricultural land. One method of depressing the interface is to construct fresh-water canals near the Gulf shore and to maintain the fresh-water level in the canals above mean sea level.

The existence of the extensive mud flats at the head of the Gulf is indicative that land reclamation from the Gulf may be feasible. One method of reclamation would be to construct a series of parallel fresh-water canals with intermediate drains. Again the fresh-water level would be maintained above mean sea level. The elevated fresh-water canals might be constructed from hydraulically conveyed slurries dredged from the navigation channel of the Port of Bangkok. These materials are currently being wasted by dumping in deeper water from hopper dredges.

The object of this study is to formulate the fundamental seepage analysis necessary for a subsequent evaluation of the effectiveness and water-loss from various canal arrangements. The technical feasibility of these projects depends to a great extent upon the required discharge of fresh water into the fresh-water canal system. If the required discharge is slight, the necessary water might be obtained from an upstream diversion of the river.

Three specific problems were investigated. The first problem is to find the configuration of the interface between the fresh and the salt water under a single canal. This problem can be considered as representative of a canal along the middle of a long narrow mass of land piled up above the mean sea level of the Gulf. The second problem, to find the configuration of the interface under a single canal with natural ground-water flow, represents a canal parallel to the shore line along the head of the Gulf. The third problem is directed toward the reclamation of the mud flat by using a system of parallel canals with intermediate drains.

Some simplifying assumptions are necessary to make the two-phase seepage problems tractable. The aquifer is assumed to be two-dimensional, homogeneous and isotropic. Solutions for an isotropic aquifer can be applied to an anisotropic aquifer by a simple transformation. The fresh-water flow is assumed to be steady. The underlying salt water is assumed to be stationary. The interface between the fresh and salt water is assumed to be a line. The aquifer is assumed to be confined. Solutions for a confined aquifer are approximate solutions for an unconfined aquifer if the level of the ground-water table is assumed to coincide with the piezometric head along the confining line. Finally, since the slope of the seashore in the upper part of the Gulf of Siam is very slight, the outflow face is assumed to be horizontal.

The fundamental seepage analysis, which is formulated in Chapter II, consists of the solution of Laplace's equation with appropriate boundary conditions. Different methods of solutions are discussed in Chapter III. The three specific problems cannot be solved analytically at the present time. Barriers to analytical solutions are the boundary conditions. In the physical plane, the location of the interface is unknown and the interface is curvilinear. In the complex potential plane, not all of the boundary conditions can be satisfied by using conformal-mapping or separation-of-variables techniques. In Problems 1 and 3 a constant of integration along part of the boundary cannot be evaluated. In Problem 2 the domain consists of two adjoining rectangles. The boundary condition along the joining line is initially unknown. Therefore, numerical methods have to be employed.

The method of successive over-relaxation was utilized in the complex potential plane for the numerical solution of Laplace's equation. In the complex potential plane the domain consists of a rectangular region or of two adjoining rectangular regions. Even though Young's over-relaxation factor was derived for a rectangular domain with all boundary conditions specified on the function, the author found empirically that Young's over-relaxation factor reduced the number of iterations for the rectangular domain in which the boundary condition on one side is specified on the derivative of the function. Young's over-relaxation factor also reduced the number of iterations for a domain consisting of two adjoining rectangular regions. After vertical coordinates have been solved in the complex potential plane, horizontal coordinates on any streamline or piezometric-head line can be obtained by numerical integration of the inverse Cauchy-Riemann equations. The principal disadvantage of obtaining solutions in this manner is that the geometric characteristics in the physical plane are dependent variables.

A Burroughs B-5000 electronic computer (90,000-word storage capacity) was employed. The longest time required for a solution of a set of dimensionless boundary conditions was 30 minutes. The shortest time required for a solution of a set of dimensionless boundary condition was 3 minutes. The program written by the author in Extended Algol 60, is rather general and, with minor modification of the block entitled "Boundary Conditions", can be employed to solve any problem satisfying Laplace's equation in a rectangular domain or a domain consisting of two adjoining rectangular regions. Numerical differentiation of the function and numerical integration of the inverse Cauchy-Riemann equations are grouped in the block entitled "Find x".

All the three specific problems were solved. Ten sets of dimensionless boundary conditions were used for Problem 1. Solutions to thirty-four sets of dimensionless boundary conditions for Problem 2 were presented. Problem 3 was solved for twenty-one sets of dimensionless boundary conditions.

In Chapter V, engineering analyses were presented in order to estimate the water-loss from canals which might be installed to reclaim land at the head of the Gulf of Siam. Several assumptions were used in all of the analyses. The first assumption is that salt-intruded land is, by definition, land under which the salt water is within 2 meters of the land surface. In other words, in order for land to be suitable for agriculture, the interface between the salt water and fresh water must be at least two meters beneath the land surface. The second assumption is that the delta soil is a clay-silt soil. The third assumption is that the ratio of the specific weight of salt water to that of fresh water is 1.025. The specific-weight ratio at the head of the Gulf of Siam is probably less than the assumed mean value. The Gulf is elongated with appreciable fresh-water inflows from the Chao Phraya and Mae Klong rivers at the head. Therefore, the salt concentration is undoubtedly less than in the open sea. However, the assumption is conservative in regard to the amount of depression of the interface by means of seepage from fresh-water canals. The fourth assumption is that the water loss from the canal consists entirely of seepage losses. Obviously, evapo-transpiration losses would have to be included in a more comprehensive analysis.

Conclusions were drawn that: (1) the concept to employ on artificial fresh-water canal in the suppression of salt water and to employ parallel canals with intermediate drains in reclamation of the mud flats is promising and should be investigated further; (2) if the soil is in the clay- or silt-size range, the amount of water loss due to seepage is negligible; and (3) the feasibility of the mud-flat reclamation project cannot be determined from the equilibrium seepage condition alone but field experiments will be necessary in order to find a feasible scheme of leaching.

NOTATION

The symbols used frequently herein are defined as follows:

a = width of a rectangle

A, A', B, B', C, C', D, D' = points on the boundaries or subscripts pertaining to the points

A_1, A_2, B_1, B_2, C_1, $C_2 \cdots C_5$ = constants

aw = subscript standing for air-fresh-water interface

b = grid size

d = total derivative operator or mean diameter of soil particles, equation (2)

e = allowable error according to equation (109)

f = subscript for fresh water

f,F = function

g = gravitational acceleration

G = function

h = piezometric head = $p/\gamma + y$

i = $\sqrt{-1}$ or subscript denoting interface of fresh and salt water

I,J = numbers of grid points

IM, JN = number of grids on each side of a rectangle

k = coefficient of permeability of soil (real) or superscript denoting kth iteration (integer)

ℓ = length of a rectangle

L = a dummy variable defined by equation (62b)

m = subscript for model

M = a reference piezometric-head function = $-\gamma_s k h_s/\gamma_f$

n = subscript showing that the variable has different values corresponding to n = 1, 2, 3 $\cdots$

N = $k(\gamma_s/\gamma_f - 1)$

N, E, S, W = grid points north, east, south and west of the point O, or subscripts pertaining to the points

O = origin of the physical plane, or any point on the grid in N-E-S-W notation or subscript for the point O

p = pressure or subscript for prototype

P = a dummy variable used as real part of a complex variable, w.

q = discharge between any two streamlines

Q = net fresh-water discharge per unit width of aquifer

Q' = dimensionless discharge

r = subscript for ratio

R = Reynolds number = vd/ν

s = subscript for salt water or distance along any curve

S = a dummy variable used as imaginary part of a complex variable, w

T = subscript for total

u = x-component of velocity

u' = dimensionless x-component of velocity = u/N

v = y-component of velocity

v' = dimensionless y-component of velocity = v/N

V = total velocity = $\sqrt{u^2 + v^2}$

V' = dimensionless total velocity = V/N

w = complex potential = $\varphi + i\psi$

w' = dimensionless complex potential = $\varphi' + i\psi'$

x,y = coordinates of the physical plane

x',y' = dimensionless x and y respectively ($x' = Nx/Q$)

X,Y = function of x and y respectively

z = a complex variable = $x + iy$

z' = dimensionless complex variable = $x' + iy'$

α = counter-clockwise angle

β = a real number

γ = specific weight of liquid

δ = central difference operator

ζ = a complex variable = dz/dw

ζ' = a dimensionless complex variable = dz'/dw'

θ = clockwise angle

μ = dynamic viscosity

ν = kinematic viscosity

φ = piezometric-head function = $-kh$

φ' = dimensionless piezometric-head function = φ/Q

ψ = stream function

ψ' = dimensionless stream function = ψ/Q

ω = over-relaxation factor defined by equation (62a)

Ω = a dimensionless complex variable = ζ' = $(\cos\alpha + i \sin\alpha)/|V'|$

CHAPTER I

INTRODUCTION

General

The central portion of Thailand is occupied by the great, low alluvial plain of the Chao Phraya River (Fig. 1). Major portions of this alluvial plain are flooded annually as a result of the monsoon rains which occur from May to October. The alluvial plain is formed by sediment deposition during these periodic flood inundations. The mean slope of the alluvial plain is only approximately 1.6×10^{-4}, which may be attributed to the fine particle size of the silt and clay sediment deposits. This gentle slope prevails into the Gulf of Siam, where extensive mud flats are exposed during low tide in the area west of the mouth of the river.

Due to the gentle slope of the alluvial plain, an appreciable area of land is subjected to salt-water intrusion under the land surface at the head of the Gulf of Siam, particularly west of the mouth of the Chao Phraya River. In an area extending several miles from the Gulf shore only salt-tolerant vegetation will thrive. A slight lowering of the interface between the intruded salt water and the overlying fresh water would result in a significant increase in agricultural land suitable for rice production. One method of depressing the interface is to construct fresh-water canals near the Gulf shore and to maintain the fresh-water level in the canals above mean sea level.

The existence of the extensive mud flats at the head of the Gulf is indicative that land reclamation from the Gulf may be feasible. One method of reclamation would be to construct a series of parallel fresh-water canals with intermediate drains. Again the fresh-water level would be maintained above mean sea level. The drainage water would either be wasted over the peripheral canal into the sea or recirculated into the fresh-water canal system. The elevated fresh-water canals might be constructed from hydraulically conveyed slurries dredged from the navigation channel of the Port of Bangkok. These materials are currently being wasted by dumping in deeper waters from hopper dredges.

The technical feasibility of these projects depends primarily upon the required discharge of fresh water into the fresh-water canal system. Obviously a portion of the fresh water is lost by seepage into the salt-water body. If these losses are small, the necessary water might be obtained from an upstream diversion of the river. The object of this study is to formulate the fundamental seepage analysis necessary for a subsequent evaluation of the effectiveness and water-loss from various canal arrangements.

The idea of using a fresh-water barrier to suppress salt water is suggested by observing the results of an existing canal. In constructing a road from Bangkok to Cholburi, a city on the upper right of the Gulf of Siam as shown in Fig. 1, a fresh-water canal was developed from the borrow pit. This canal is parallel to and on the inland side of a portion of the road which is nearly parallel to the coastal line. On the inland side of the canal, rice, which cannot tolerate much salinity, has been growing satisfactorily, while on the seaward side only salt tolerant vegetation of little market value is growing. This existing condition indicates that the fresh-water canal has been responsible for the suppression of salt water.

If there were no natural groundwater flowing toward the sea prior to the excavation of the canal, the aquifer would be saturated with salt water as shown in Fig. 2(a). This aquifer is estimated to be clay or silty clay extending from the surface down to about 60 feet[1]. For this condition, the presence of a fresh-water canal, in which the fresh-water surface elevation is higher than the salt-water would cause fresh water flow out into the sea. This flow would replace part of the salt water, pushing it downward and seaward and creating an interface as shown in Fig. 2(b). If there was natural groundwater flowing prior to the presence of the fresh-water canal, the system would be under dynamic equilibrium, with an interface between the fresh and the salt water as shown in Fig. 3. In this case, the fresh-water canal would increase both the existing piezometric head and the fresh-water discharge, causing the interface to move further seaward and downward until the system again reached a state of dynamic equilibrium as shown in Fig. 4.

Specific Problems

The first problem is to find the configuration of the interface between the fresh and the salt water under a single canal. This problem can be considered as representative of a canal along the middle of a small cape or long, narrow island as shown in Fig. 5.

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1. According to B. Asanachinda and M. Dhamkrongartama, Geographical Maps of Thailand (Thai), Kuruspa, Bangkok, 1957, p. 21, a detailed investigation has not been executed.

The second problem, to find the configuration of the interface under a single canal with natural ground water flow from infinity, represents the previously mentioned case of the canal parallel to the shore line along a part of the road from Bangkok to Cholburi. A sketch of this condition is shown in Fig. 4, Fig. 6(a) and Fig. 7(a).

The third problem is directed toward the reclamation of the mud flats by using a system of parallel canals with intermediate drains as shown in Fig. 8(a).

Survey of Literature

There have been many analytical and experimental investigations of salt-water intrusion. Todd[2] has prepared an extensive abstract of literature pertaining to this subject. An extensive bibliography is included in his later book[3] on ground water. Another source of reference on salt-water encroachment was published by Winslow[4]. A review of previous works may be found in the dissertations of Henry[5] and Bear[6] and in a report by Harleman and Rumer[7].

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2. D. K. Todd, "An Abstract of Literature Pertaining to Sea-Water Intrusion and Its Control," Technical Bulletin 10, Sanitary Engineering Research Project, University of California, Berkeley, 1952.

3. D. K. Todd, Ground Water Hydrology, John Wiley, New York, 1959, pp. 294-296.

4. A. G. Winslow, Bibliography of Salt-Water Encroachment, U. S. Geological Survey, Washington, D.C., 1953.

5. H. R. Henry, Salt Intrusion into Coastal Aquifers, Ph.D. Dissertation, Faculty of Pure Science, Columbia University, 1960.

6. Jacob Bear, The Transition Zone Between Fresh and Salt Waters in Coastal Aquifers, Ph.D. Dissertation, Civil Engineering, University of California, 1961, pp. 9-15.

7. D. R. F. Harleman and R. R. Rumer, Jr., "The Dynamics of Salt-Water Intrusion in Porous Media," MIT Hydrodynamics Laboratory, Civil Engineering, Report No. 55, August 1962, pp. 2-8.

The earliest analysis of two-phase seepage flows appears to have been carried out in Europe, principally by Badon-Ghyben[8] and Herzberg[9], who worked independently. Both men observed that in some wells near the seacoast, the salt water was not encountered at sea-level, but was found below sea-level at a depth of the order of forty times the height of the fresh water above sea-level. They deduced that the fresh and salt water were at static equilibrium. This would mean that the mass of a unit vertical column of fresh water extending from the water table to the interface must be the same as that of the displaced salt water. Using the ratio of density of salt water to that of fresh water as 1.025, the analysis agreed with the natural phenomenon. However, it should be noted that the Ghyben and Herzberg "law" is valid only when both fluids are in static equilibrium and not when fresh water is moving. Kitagawa and Todd[10] used Ghyben and Herzberg's assumption with a one-dimensional form of Darcy's law to solve for the position of the interface as a function of the steady fresh-water discharge. The interface is parabolic. This is essentially equivalent to the method used in the Dupuit-Forchheimer[11] theory of gravity-flow systems.

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8. For example, see M. K. Hubbert, "The Theory of Groundwater Motion," The Journal of Geology 48, No. 8, Part 1, November - December 1940, p. 924.

9. Baurat Herzberg, "Die Wasserversorgung einiger Nordseebäder," Journal für Gasbelechtung und Wasserversorgung 44, Munich, 1901, pp. 815-819 and pp. 842-844.

10. For example, see Henry, op. cit., p. 2 and p. 28.

11. For example, see M. Muskat, The Flow of Homogeneous Fluids through Porous Media, J. W. Edwards, Ann Arbor, 1946, p. 359.

Hubbert[12] determined a more general relationship between potentials in two fluids for the case of one flowing and the other at rest. Ghyben and Herzberg's static solution can be deduced from Hubbert's solution. He also derived an equation for the slope of the interface and confirmed this equation experimentally[13]. The experimental apparatus consisted of a sand-filled box, a dense sugar solution at rest, and fresh-water flow above from each of the two ends to an outlet in the center. Meyer and Garder[14] applied Hubbert's theory in solving for the maximum rate of flow of oil with an underlying stationary water stratum. Perlmutter, Geraghty and Upson[15] found that field measurements did not agree with the Ghyben and Herzberg principle. The disagreement would be expected since the field condition is dynamic while Ghyben and Herzberg's assumption was a static condition. Hubbert's formula was found to give better agreement with the field measurements.

Jacob[16] employed the Dupuit assumption to develop two partial differential equations for the salt-water flow and fresh-water flow in an

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12. M. K. Hubbert, op. cit., pp. 785-944.

13. M. K. Hubbert, "Entrapment of Petroleum under Hydrodynamic Conditions," American Association of Petroleum Geologist, Bulletin 37, No. 8, August 1953, pp. 1954-2026.

14. H. I. Meyer, and A. O. Garder, "Mechanics of Two Immiscible Fluids in Porous Media," Journal of Applied Physics 25, No. 11, November 1954, pp. 1400-1406.

15. N. M. Perlmutter, J. J. Geraghty and J. E. Upson, "Relation between Fresh and Salt Ground Water in Southern Nassau and Southeastern Queens Counties, Long Island, New York," Economic Geology 54, No. 3, May 1959, pp. 416-435.

16. C. E. Jacob, "Salt-Water Encroachment in Inclined Confined Aquifers of Non-Uniform Thickness," Paper presented at 39th Meeting of American Geophysical Union, Washington, D.C., May 1958.

inclined confined aquifer of non-uniform thickness. Integration was possible in certain cases of steady flow, both opposed and parallel. Nonsteady states were treated by graphical integration between successive steady states.

Glover[17] utilized an exact solution, previously obtained by Kozeny[18], for the flow of ground-water under gravity forces for fresh-water seepage in a confined aquifer of infinite depth. The fresh water flows over stagnant salt water. The interface is parabolic.

Henry[19] employed the hodograph method and conformal mapping in solving problems of fresh water flowing from a recharge reservoir. The boundary of the recharge reservoir is a vertical bank extending down to the bottom of the constant-depth aquifer. The outflow face is either horizontal or vertical. Several numerical solutions with different values of distance from reservoir to the sea and depth of aquifer are presented.

Rumer and Harleman[20] derived an approximate solution for the interface within a confined aquifer of finite depth and length and with a vertical outflow face. The approximation is that the interface configuration and position is practically invariant whenever the length of intrusion is greater than the aquifer depth. Experiments were performed

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17. R. E. Glover, "The Pattern of Fresh-Water Flow in Coastal Aquifer," Journal of Geophysical Research 64, No. 4, April 1959, pp. 457-459.

18. See, for example, Muskat, op. cit., p. 326.

19. Henry, op. cit., pp. 9-29.

20. R. R. Rumer, Jr., and D. R. F. Harleman, "Intruded Salt-Water Wedge in Porous Media," Proceedings of the American Society of Civil Engineers 89, No. HY6, November 1963, pp. 193-220.

using a glass box filled with glass beads, sand and plastic spheres to confirm the approximate equation. Dispersion was also taken into account.

The solution of the first problem as in Fig. 5 has been attempted by Sutabutra[21], who employed the same transformation as Henry and found that only a very special case of a canal at an infinite distance from the sea can be solved. He also solved the problem of ground-water flow from infinity in a confined aquifer (see Fig. 3b) by using flow through a 360° bend. The result was a parabolic interface as previously obtained by others[22].

Bear[23] obtained a solution, by conformal mapping, of the problem on a very long cape or island (see Fig. 5), but his source of fresh water was uniform rainfall instead of seepage from a centrally located canal.

Shea[24] proposed using fresh water to prevent salt-water intrusion in Southern Dade County, Florida. In building a levee for flood protection in this area, a continuous borrow pit was excavated parallel to and on the land side of the embankment. Another levee on the land side of the borrow pit will form a canal to be filled with fresh water. Model studies in a sand box were performed. The conditions were different from the second problem (see Fig. 4) since the direction of fresh-water flow was

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21. Prathet Sutabutra, Two Problems in the Hydrodynamics of Salt-Water Intrusion, Master's Thesis, SEATO Graduate School of Engineering, Thailand, 1963.

22. See, for example, Henry, op. cit., p. 28 and p. 48; or Glover, op. cit., p. 458.

23. Jacob Bear, "Water-Table Aquifers Receiving Vertical Recharge," to be published in a work edited by Cooper in 1964.

24. Paul H. Shea, Model Study of a Means of Preventing Salt-Water Intrusion, unpublished paper, U.S. Army Engineer District, Jacksonville, 1961.

landward rather than seaward and natural ground-water flow was omitted in the model studies.

For the Sacramento-San Joaquin Delta of California, Marcus, Evenson and Todd[25] employed both Hele-Shaw and electric analog models to study the effects of fresh-water irrigation. In one case the irrigation water was applied uniformly over the surface. A second case involved the application of water by means of irrigation ditches. In both cases both fresh and salt water percolated into parallel drainage ditches.

Approaches to the Problems

Some simplifying assumptions are necessary to make the two-phase seepage problems tractable. The aquifer is assumed to be two-dimensional, isotropic, and homogeneous. Generally in homogeneous natural deposits the coefficient of permeability is greater in the horizontal direction than in the vertical. By a simple expansion or contraction of spatial coordinates, a given homogeneous, anisotropic flow region can be transformed into a fictitious isotropic region[26]. The fresh-water flow is assumed to be steady. The underlying salt water is assumed to be stationary. The interface between the fresh and salt water is assumed to be a line. Even though the interface in field conditions will be dispersed, the dispersion zone will tend to be narrow[27], so that a line can be used

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25. Hendrikus Marcus, D. E. Evenson and D. K. Todd, Seepage of Saline Water in Delta Lowlands, Water Resources Center Contribution No. 53, U. of California, Berkeley, 1962.

26. See, for example, M. E. Harr, Groundwater and Seepage, McGraw-Hill, New York, 1962, p. 29.

27. See, for example, J. S. Brown, "A Study of Coastal Ground Water with Specific Reference to Connecticut," Water Supply Paper 537, U.S. Geological Survey, Washington, D.C., 1925.

without serious error. In addition the assumption is made that the aquifer is confined as shown in Fig. 3(b), Fig. 5(d), Fig. 6(a), Fig. 7(a), and Fig. 8(b). This assumption would be a reasonable representation for canals in which the exposed slopes are small enough to preclude seepage on the slope as shown in Fig. 5(a). The discharge through the cross-hatched areas is neglected. The actual configuration of the ground-water table can be approximated by the piezometric-head level along the upper surface of the confined aquifer. This assumption has been employed by Kirkham[28] with good agreement between field data and analytical results. Finally, since the slope of the seashore in the upper part of the Gulf of Siam is very slight, the outflow face is assumed to be horizontal.

Solution by Means of an Electric Analog

Solution to the first problem of this investigation (see Fig. 5) has been attempted by means of an electric analog model at the SEATO Graduate School of Engineering in Thailand in 1960, under the direction of the author's present advisor. Since the position of the interface, which is one of the boundaries of the configuration, is initially unknown, successive estimates and adjustments were required. About 18 trials were needed for one specific set of dimensions. Since a number of sets of boundary conditions are required for each of the three problems in this investigation, an electric analog model was not chosen.

Analytical Solution

Analytical solution in the closed form is always the most desirable. As will be shown in the section of theory, the seepage flow satisfies

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28. Don Kirkham, "Seepage of Steady Rainfall through Soil into Drains," Transactions, American Geophysical Union 39, No. 5, October 1958, pp. 892-908.

Laplace's equation. Since the interface is curved and is initially unknown, the method of separation of variables is obviously not applicable. The method of separation of variables has been employed in the determination of steady-state temperature distribution[29] in which the physical boundaries were known. The solution of Laplace's equation for the first problem (Fig. 5) when the canal is at an infinite distance from the sea can be obtained by conformal transformation. In the second problem of this investigation, incomplete elliptic integrals of the third kind are encountered in the transformed plane. After numerical integration in that plane, numerical transformation would be required in going back to the physical plane. This method appears to be more difficult than to use numerical analysis throughout.

Solution by Means of Relaxation Technique

McNown, Hsu and Yih[30] reminded engineers of the applicability of relaxation technique. They presented both the principle and various examples. Shaw and Southwell[31] introduced a trial relaxation method for solving free-surface flow through a dam. However, this method requires excessive computer programming because the interface and consequently the distance from every adjacent point require repeated adjustment.

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29. K. S. Miller, Partial Differential Equations in Engineering Problems, Prentice Hall, New Jersey, 1961, pp. 101-105.

30. John S. McNown, En-Yun Hsu and Chia-Shun Yih, "Applications of the Relaxation Technique in Fluid Mechanics with Discussion by Others," Trans., ASCE 120, 1955, pp. 650-686.

31. F. S. Shaw and R. V. Southwell, "Relaxation Methods Applied to Engineering Problems VII. Problems Relating to the Percolation of Fluids through Porous Media," Proceedings of the Royal Society of London, Series A, Mathematical and Physical Sciences, 178, May 1941, pp. 1-17.

Furthermore, for every adjustment of the interface, relaxation has to be carried out until desirable accuracy is achieved.

Thom and Apelt[32] noted that relaxation could be done on the complex potential plane instead of the physical plane. The interface is a line of constant stream function. Therefore, if each of the other physical boundaries is either a line of constant stream function or a line of constant potential, the pattern in the complex potential plane will be rectangular. The problem of relaxation in the domain with a curved boundary can thus be reduced to relaxation in a rectangle.

Young[33] introduced a near-optimum over-relaxation factor for a rectangular domain. The factor was derived analytically and checked experimentally. With mesh size b, the required number of iterations is of the order of b^{-2} using no over-relaxation factor and only of the order of b^{-1} using Young's over-relaxation factor[34]. In one of the experiments[35], twenty grids were used for each side of a unit square. The number of iterations was reduced from 279 to 35.

The method utilized in this investigation is over-relaxation for y in the complex potential plane. The derivatives of y with respect to velocity potential or stream function are evaluated numerically.

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32. A. Thom and C. J. Apelt, Field Computation in Engineering and Physics, D. Van Nostrand Co., New York, 1961.

33. D. M. Young, Jr., Iterative Methods for Solving Partial Differential Equations of the Elliptic Type, Ph.D. Dissertation in Mathematics, Harvard, Boston, 1950.

34. D. M. Young, Jr., "Iterative Methods for Solving Partial Difference Equations of Elliptic Type," Transactions of the American Mathematical Society 76, 1954, p. 95.

35. Ibid., p. 110.

Corresponding values of x can be obtained by numerical integration of the inverse Cauchy-Riemann equation. Both streamlines and equipotential lines are then plotted from the obtained coordinates in the physical plane. The electronic computer employed was a Burroughs B-5000 (90,000-word storage capacity). The longest time required for a solution for a set of boundary conditions was 30 minutes. The program, written by the author in Extended Algol 60, is rather general and, with minor modification of the boundary conditions, canbe used to solve any problem satisfying Laplace's equation in a rectangle or two adjoining rectangles.

CHAPTER II

THEORY

Equations

The physical principles governing the behavior of fluids flowing through porous media are fundamentally the same as those for the motion of viscous fluids in any other flow systems. In this investigation the fluid will be considered incompressible. The law of conservation of matter (equation of continuity) for steady, incompressible, two-dimensional flow can be written as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1)$$

in which u and v are the components of the discharge velocity in the x and y directions respectively. Newton's second law of motion is utilized in deriving the Navier-Stokes equations[36]. In porous media flow, the Reynolds number which relates inertial to viscous forces is defined as

$$R = \frac{vd}{\nu} \qquad (2)$$

in which v = discharge velocity

d = mean diameter of soil particles, and

ν = kinematic viscosity

The inertia terms in the Navier-Stokes equations of motion can be omitted

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36. Hermann Schlichting, Boundary Layer Theory, McGraw-Hill, New York, 1960, pp. 42-54.

when the Reynolds number is less than unity[37]. Defining the piezometric head,

$$h \equiv \frac{p}{\gamma} + y \tag{3}$$

the simplified Navier-Stokes equations for porous media flow are

$$\frac{\partial h}{\partial x} = \frac{\mu}{\gamma}\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) \tag{4a}$$

$$\frac{\partial h}{\partial y} = \frac{\mu}{\gamma}\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2}\right) \tag{4b}$$

Differentiating equation (4a) with respect to x and (4b) with respect to y and combining, one obtains

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = \frac{\mu}{\gamma}\left[\frac{\partial^3 u}{\partial x^3} + \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} + \frac{\partial^3 v}{\partial y^3}\right]$$

Replacing u by v from the relation in the equation of continuity, equation (1), piezometric head will be found to satisfy the potential equation

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0 \tag{5}$$

and $h(x, y)$ is, therefore, a potential function.

Defining a stream function $\psi(x, y)$ by

$$u = \partial\psi/\partial y \tag{6a}$$

$$v = -\partial\psi/\partial x \tag{6b}$$

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37. *Ibid.*, p. 94.

eliminating h from equations (4a) and (4b), and replacing u and v by those in equations (6a) and (6b) one obtains

$$\frac{\partial^4 \psi}{\partial x^4} + \frac{2\partial^4 \psi}{\partial x^2 \partial y^2} + \frac{\partial^4 \psi}{\partial y^4} = 0 \tag{7a}$$

or, with vector notation,

$$\nabla^2(\nabla^2 \psi) = \nabla^4 \psi = 0 \tag{7b}$$

It is thus seen that the Navier-Stokes equations imply that the stream function of plane creeping motion or flow through porous media is a biharmonic or bipotential function. It is easier to solve Laplace's equation as equation (5) than to solve the biharmonic equation. By defining a piezometric head function,

$$\varphi \equiv -kh \tag{8}$$

where k is coefficient of permeability, equation (5) may be rewritten as

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0 \tag{9}$$

So the piezometric head function satisfies the potential equation and is thus a potential function.

Equation (9) can also be obtained by combining the equation of continuity, equation (1), and Darcy's law,

$$u = -k \frac{\partial h}{\partial x}; \; v = -k \frac{\partial h}{\partial y} \tag{10}$$

However, it should be noted that Darcy's law does not describe the conditions within an individual pore. It was not derived from the Navier-Stokes equations but was determined experimentally. Therefore, Darcy's law represents the statistical macroscopic equivalent of the Navier-Stokes equations. Since the introduction of this law in 1856, it has been the basis of theoretical development in the field of ground-water flow. It has been stated in various published investigations that if the Reynolds number, equation (2), is less than or equal to unity, valid results will be obtained from an application of Darcy's law[38]. Experiments showing the validity of Darcy's law for Reynolds number up to 12 have been recorded[39].

Important properties of the scalar potential functions, φ and ψ, are demonstrated in the following. Let a complex variable be defined as

$$w \equiv P + iS \tag{11}$$

where both P and S are functions of x and y. If another complex variable is defined as

$$z \equiv x + iy \tag{12}$$

equation (11) can be rewritten as

$$w = P + iS = F(z) \tag{13}$$

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38. Muskat, op. cit., p. 67.

39. Ibid.

If w and dw/dz are both single valued and finite, w is analytical. Differentiating equation (13) with respect to x and y respectively, one obtains

$$\frac{\partial P}{\partial x} + i\,\frac{\partial S}{\partial x} = \frac{dF(z)}{dz}\,\frac{\partial z}{\partial x} = F'(z)$$

$$\frac{\partial P}{\partial y} + i\,\frac{\partial S}{\partial y} = \frac{dF(z)}{dz}\,\frac{\partial z}{\partial y} = iF'(z)$$

Eliminating $F'(z)$ from the above two equations, and equating real and imaginary parts in the resulting equation one obtains

$$\frac{\partial P}{\partial x} = \frac{\partial S}{\partial y} \tag{14a}$$

$$\frac{\partial P}{\partial y} = \frac{-\partial S}{\partial x} \tag{14b}$$

Differentiating equations (14a) and (14b) with respect to x and y respectively, the resulting combination will be

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = \frac{\partial^2 S}{\partial x \partial y} - \frac{\partial^2 S}{\partial y \partial x}$$

If the functions are continuous and derivatives exist, $\frac{\partial^2 S}{\partial x \partial y}$ has the same value as $\frac{\partial^2 S}{\partial y \partial x}$, from which

$$\frac{\partial^2 P}{\partial x^2} + \frac{\partial^2 P}{\partial y^2} = 0$$

In similar manner, since $\frac{\partial^2 P}{\partial x \partial y}$ has the same value as $\frac{\partial^2 P}{\partial y \partial x}$, one obtains

$$\frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} = 0 \tag{15}$$

Therefore, P and S are potential or harmonic functions and w is a complex potential function. The families of curves $P(x,y) = C_1$ and $S(x,y) = C_2$, where C_1 and C_2 are constants, will form a mutually orthogonal network, since the slope of each family of curves are, respectively,

$$\left.\frac{\partial y}{\partial x}\right|_P = -\frac{\partial P/\partial x}{\partial P/\partial y}$$

and

$$\left.\frac{\partial y}{\partial x}\right|_S = -\frac{\partial S/\partial x}{\partial S/\partial y}$$

Making use of equations (14a) and (14b), one gets

$$\left.\frac{\partial y}{\partial x}\right|_P = \frac{\partial S/\partial y}{\partial S/\partial x}$$

which is negative reciprocal of $\left.\frac{\partial y}{\partial x}\right|_S$. Letting P be a velocity potential φ, as defined by equation (8), equation (9) is again obtained. Darcy's law can be rewritten as

$$u = \frac{\partial \varphi}{\partial x};\ v = \frac{\partial \varphi}{\partial y} \tag{16}$$

Utilizing equations (14a), (14b) and (16), keeping in mind that $P = \varphi$, one obtains

$$\frac{v}{u} = \frac{\partial \varphi/\partial y}{\partial \varphi/\partial x} = -\frac{\partial S/\partial x}{\partial S/\partial y}$$

Thus, the direction of the fluid at any point coincides with the tangent at that point to the curve $S(x,y)$ = constant. Therefore, these curves have to be streamlines and S is the stream function ψ. Replacing S by ψ, equation (15) becomes

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0 \tag{17}$$

The velocity components may be written as

$$u = -k\,\frac{\partial h}{\partial x} = \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y} \tag{18a}$$

$$v = -k\,\frac{\partial h}{\partial y} = \frac{\partial \varphi}{\partial y} = -\,\frac{\partial \psi}{\partial x} \tag{18b}$$

Consider the flow between the two streamlines ψ_1 and ψ_2 of Fig. 9. If q is the quantity of discharge between the two streamlines, then

$$q = \int_{\psi_1}^{\psi_2} (u\,dy - v dx) = \int_{\psi_1}^{\psi_2} d\psi = \psi_2 - \psi_1 \tag{19}$$

Equation (19) shows that the quantity of flow between two streamlines is a constant.

In a domain with a given set of boundary conditions, if either equipotential function or stream function is known the other can be obtained readily. Considering the total differential

$$d\psi = \frac{\partial \psi}{\partial x}\,dx + \frac{\partial \psi}{\partial y}\,dy$$

with the Cauchy-Riemann equations as shown in the last part of equation (18)

$$\psi = \int \left(\frac{\partial \varphi}{\partial x}\,dy - \frac{\partial \varphi}{\partial y}\,dx\right) \tag{20}$$

and, similarly for φ, one obtains

$$\varphi = \int \left(\frac{\partial \psi}{\partial y} dx - \frac{\partial \psi}{\partial x} dy \right) \tag{21}$$

Therefore, once one function is found, the other can be obtained from equation (20) or (21).

Both x and y also satisfy Laplace's equation in the complex potential plane provided w can be expressed in terms of z and dw/dz exists and differs from zero throughout the region. The specified conditions imply that z can also be expressed as a function of w.

$$z = G(w)$$

or,

$$x + iy = G(\varphi + i\psi)$$

Differentiating with respect to φ

$$\frac{\partial x}{\partial \varphi} + i \frac{\partial y}{\partial \varphi} = \frac{\partial G}{\partial \varphi} = \frac{dG}{dw} \frac{\partial w}{\partial \varphi} = \frac{dG}{dw} \tag{22}$$

and again,

$$\frac{\partial^2 x}{\partial \varphi^2} + i \frac{\partial^2 y}{\partial \varphi^2} = \frac{\partial}{\partial \varphi} \left(\frac{dG}{dw} \right) = \frac{d^2 G}{dw^2} \tag{23}$$

Similarly,

$$\frac{\partial x}{\partial \psi} + i \frac{\partial y}{\partial \psi} = i \frac{dG}{dw} \tag{24}$$

and

$$\frac{\partial^2 x}{\partial \psi^2} + i \frac{\partial^2 y}{\partial \psi^2} = - \frac{d^2 G}{dw^2} \tag{25}$$

Eliminating dG/dw from equations (22) and (24)

$$i \frac{\partial x}{\partial \varphi} - \frac{\partial y}{\partial \varphi} = \frac{\partial x}{\partial \psi} + i \frac{\partial y}{\partial \psi}$$

Equating real and imaginary parts

$$\frac{\partial x}{\partial \varphi} = \frac{\partial y}{\partial \psi} \tag{26a}$$

and

$$\frac{\partial x}{\partial \psi} = - \frac{\partial y}{\partial \varphi} \tag{26b}$$

This means that the inverse Cauchy-Riemann equations exist. Similar operation of equations (23) and (25) yields

$$\frac{\partial^2 x}{\partial \varphi^2} + \frac{\partial^2 x}{\partial \psi^2} = 0 \tag{27a}$$

and

$$\frac{\partial^2 y}{\partial \varphi^2} + \frac{\partial^2 y}{\partial \psi^2} = 0 \tag{27b}$$

Thus both x and y satisfy Laplace's equation in the complex potential plane.

Boundary Conditions

Interface

On the interface of moving fresh water and stationary salt water the boundary condition can be found as follows. The pressure of fresh water at any point on the interface, Fig. 10, is the same as that of salt water at that point

$$p_f = p_s$$

where subscript f and s stand for fresh and salt water respectively. Let y along the interface be y_i. From the definition of piezometric head, equation (3),

$$h_s = \frac{p_s}{\gamma_s} + y_i$$

and

$$h_f = \frac{p_f}{\gamma_f} + y_i$$

Equating p_s and p_f and solving for h_f

$$h_f = \frac{\gamma_s}{\gamma_f} h_s - y_i \left(\frac{\gamma_s}{\gamma_f} - 1\right)$$

From the definition of piezometric-head function, equation (8),

$$\varphi_i = M + N y_i \tag{28a}$$

in which

$$M = - \frac{\gamma_s}{\gamma_f} k h_s \tag{28b}$$

and

$$N = k\left(\frac{\gamma_s}{\gamma_f} - 1\right) \tag{28c}$$

With a stationary salt-water phase, h_s and M are constant. Differentiating equation (28a) with respect to s, the distance along the interface,

$$\frac{\partial \varphi_i}{\partial s} = N \frac{\partial y_i}{\partial s} \tag{29}$$

Equation (10) can be rewritten for the total velocity V as

$$V = \frac{\partial \varphi}{\partial s} \tag{30}$$

Replacing $\frac{\partial \varphi}{\partial s}$ in equation (29) by V_i, one obtains

$$V_i = N \frac{\partial y_i}{\partial s}$$

The sine of the angle α, Fig. 10, between the streamline and the horizontal can be evaluated either from the velocity components or from the spatial derivative.

$$\frac{v}{V} = \frac{\Delta y}{\Delta s} = \lim_{\Delta s \to 0} \frac{\Delta y}{\Delta s} = \frac{\partial y}{\partial s}$$

Thus

$$V_i^2 = Nv_i$$

or

$$V_i^2 - Nv_i = 0 \tag{31}$$

<u>Seepage Face</u>

Even though the seepage face is not a streamline it may be regarded as an interface between fresh and salt water. Equations (28), (29), and (30) are valid if s is the distance along the seepage face and V is replaced by V_s as shown in Fig. 11. Letting θ be the clockwise angle from the x-axis to the seepage face

$$V_s = u \cos \theta - v \sin \theta$$

Replacing V_s from equation (29) and equation (30)

$$N \frac{\partial y_i}{\partial s} = u \cos \theta - v \sin \theta$$

Since $\sin \theta = \frac{\partial y}{\partial s}$;

$$N \sin \theta - u \cos \theta + v \sin \theta = 0$$

Measuring the angle α in a counterclockwise direction,

$$\alpha = 2\pi - \theta$$

from which

$$N \sin \alpha + u \cos \alpha + v \sin \alpha = 0 \tag{32}$$

In the fresh-water side on OA, Fig. 11,

$$\varphi = -kh_f = -k\left(\frac{p_f}{\gamma_f} + y\right)$$

On the seepage face, OA,

$$p_f = p_s = -\gamma_s y$$

and

$$\varphi_{OA} = -k\left(-\frac{\gamma_s}{\gamma_f} + 1\right) y$$

The boundary condition along the seepage face is therefore

$$\partial y/\partial\varphi|_{OA} = \frac{1}{N} \tag{33}$$

Dimensionless Representation

In order to obtain a solution for a given set of boundary conditions which is independent of dimensions in the physical plane, both the governing equations and the boundary conditions shall be made dimensionless. Letting primes denote dimensionless values,

$$y' = \frac{N}{Q}\, y; \quad x' = \frac{N}{Q}\, x \tag{34a}$$

$$\varphi' = \varphi/Q; \quad \psi' = \psi/Q \tag{34b}$$

Hence

$$\frac{\partial y}{\partial \varphi} = \frac{\partial Q y'/N}{\partial \varphi' Q} = \frac{1}{N}\frac{\partial y'}{\partial \varphi'}$$

and

$$\frac{\partial^2 y}{\partial \varphi^2} = \frac{\partial}{\partial \varphi}\left[\frac{1}{N}\frac{\partial y'}{\partial \varphi'}\right] = \frac{1}{NQ}\frac{\partial^2 y'}{\partial \varphi'^2}$$

Similarly,

$$\frac{\partial^2 y}{\partial \psi^2} = \frac{1}{NQ}\frac{\partial^2 y'}{\partial \psi'^2}$$

Since $\nabla^2 y = 0$, equation (27b), it follows that

$$\nabla^2 y' = 0 \tag{35}$$

The boundary conditions for y' can be summarized as follows:

1. Along the interfacial streamline (AB in Fig. 10),

$$\varphi_i' = \frac{M}{Q} + \frac{N}{Q}\; y_i$$

For convenience $M = -\frac{\gamma_s}{\gamma_f} h_s$ shall be taken as zero. In other words, the piezometric head of stationary salt water shall be used as the datum. Therefore,

$$\varphi_i' = y_i' \tag{36}$$

2. Along the line of zero y, y' remains zero. Along any line of constant y, y' will be N/Q times that constant.

3. Along the streamline of constant x, as CB in Fig. 5,

$$\left.\frac{\partial y'}{\partial \psi'}\right|_{BC} = 0 \tag{37}$$

4. Along the sloping seepage face as OA in Fig. 11, from equation (33)

$$\left.\frac{\partial y'}{\partial \varphi'}\right|_{OA} = 1 \tag{38}$$

5. Along the air-fresh water interface, for an unconfined aquifer, equation (3) may be rewritten as

$$h_f = \frac{p_f}{\gamma_f} + y \tag{39}$$

The pressure of fresh water along the air-water interface is the same as the pressure of the air. Letting the air pressure be zero, equation (39) becomes

$$h_f = y_{aw} \tag{40}$$

where aw denotes air-water. Combining equation (8) and equation (40),

$$\varphi_{aw} = -k\ h_f = -k\ y_{aw} \tag{41}$$

with $\varphi' = \varphi/Q$, $y' = Ny/Q$ and $N = k(\gamma_s/\gamma_f - 1)$ as defined in equations (34a), (34b), and (28c), equation (41) becomes

$$y'_{aw} = \left(1 - \frac{\gamma_s}{\gamma_f}\right)\varphi'_{aw} \tag{42}$$

CHAPTER III

METHOD OF SOLUTION OF LAPLACE'S EQUATION

The method of solution of Laplace's equation for two-dimensional steady flow of ground water in an isotropic homogeneous aquifer will be presented in this chapter. As stated in the introduction, the particular numerical method employed in this investigation will be explained in detail while other methods will be cursorily examined. The simplest case of natural ground-water flow in a confined aquifer as shown in Fig. 3(b) shall be taken to illustrate the application of some of the methods.

Numerical Method

Numerical methods of solving partial differential equations are based upon the theory of finite differences. The differential equation is first approximated by a difference equation, which is an algebraic equation showing the approximate relation of values of the dependent variable corresponding to a number of values of the independent variable. The task of solving Laplace's equation is thus reduced to solving a set of linear algebraic equations. Various kinds[40] of differences such as forward, central, backward, and divided have been defined. Using one kind of difference to approximate a particular partial differential equation may give a convergent solution while using another kind may not.

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40. See, for example, L. M. Milne-Thomson, The Calculus of Finite Differences, MacMillan, London, 1933.

The central difference is best for the solution of Laplace's equation.

Regardless of the kind of difference chosen, the domain must be superimposed with a network. Any kind of network such as triangular, square, rectangular or irregular polygon can be used. A square network as shown in Fig. 12(a) is obviously the most suitable for a rectangular domain. Numerical values of the function at all of the net points satisfying the difference equations and the boundary conditions constitutes a numerical solution of the differential equation. This solution is approximate. Greater accuracy can be attained by increasing the number of net points.

The mathematical definition of the kth central difference[41] of $y(\varphi)$ is

$$\delta^k y(\varphi) = \delta^{k-1}\, y(\varphi + \tfrac{b}{2}) - \delta^{k-1}\, y(\varphi - \tfrac{b}{2}) \tag{43}$$

in which δ is the central difference operator, y the dependent variable, φ the independent variable, and b is the change in φ between two adjacent net points. Replacing k by unity, equation (43) gives the first central difference as

$$\delta\, y(\varphi) = y(\varphi + \tfrac{b}{2}) - y(\varphi - \tfrac{b}{2}) \tag{44}$$

The first derivative of y with respect to φ can be approximated by dividing equation (44) by b, which implies that the slope of the curve $y(\varphi)$ at the point O_C (Fig. 12b) is obtained by linear approximation. Thus, the first derivative is approximated by the slope of the straight line

41. K. S. Kunz, Numerical Analysis, McGraw-Hill, New York, 1957, p. 66.

joining the points O_L and O_R. The smaller b is taken, the more accurate will be the approximation. In an area where y varies slowly with respect to φ, a relatively large b will give the desired accuracy, whereas if y varies rapidly with respect to φ, a relatively small b must be used to obtain equal accuracy.

The second central difference can be obtained from equation (43) by putting k = 2

$$\delta^2 y(\varphi) = \delta y(\varphi + \frac{b}{2}) - \delta y(\varphi - \frac{b}{2}) \tag{45}$$

which, by application of equation (44) becomes

$$\delta^2 y(\varphi) = y(\varphi + b) - 2y(\varphi) + y(\varphi - b) \tag{46}$$

The second derivative is obtained by dividing the second difference by b^2.

For the two-dimensional Laplace's equation, y is a function of two variables, $y(\varphi, \psi)$. Equation (46), which was derived with one variable, can be applied by alternately holding each variable constant. Thus

$$\frac{\partial^2 y}{\partial \varphi^2}(\varphi, \psi) \approx \frac{1}{b^2}\Big[y(\varphi + b, \psi) - 2y(\varphi, \psi) + y(\varphi - b, \psi)\Big] \tag{47}$$

and

$$\frac{\partial^2 y}{\partial \psi^2}(\varphi, \psi) \approx \frac{1}{b^2}\Big[y(\varphi, \psi + b) - 2y(\varphi, \psi) + y(\varphi, \psi - b)\Big] \tag{48}$$

The left hand side of Laplace's equation is obtained if equation (47) is added to equation (48). Thus the difference equation approximating Laplace's equation is

$$y(\varphi + b, \psi) + y(\varphi, \psi + b) + y(\varphi - b, \psi) + y(\varphi, \psi - b) - 4y(\varphi, \psi) = 0 \tag{49}$$

or

$$y(\varphi, \psi) = \frac{1}{4}\left[y(\varphi + b, \psi) + y(\varphi, \psi + b) + y(\varphi - b, \psi) + y(\varphi, \psi - b)\right] \quad (50a)$$

With I-J notation (Fig. 13), which is used in computer coding, equation (50a) can be written as

$$y(I, J) = \frac{1}{4}\left[y(I, J - 1) + y(I + 1, J) + y(I, J + 1) + y(I - 1, J)\right] \quad (50b)$$

or, with E-N-W-S notation as in Fig. 12(a), as

$$y_O = \frac{1}{4}(y_E + y_N + y_W + y_S) \quad (50c)$$

From equation (50), the value of the function at any point is simply the mean value of the function at the four adjacent points.

Equation (50) can also be obtained by Taylor's series expansion. Considering Fig. 12(a), if y is the function of φ only, the expansion about the point $\varphi = \varphi_O$ is

$$y = y_O + \left.\frac{dy}{d\varphi}\right|_O (\varphi - \varphi_O) + \frac{1}{2!}\left.\frac{d^2y}{d\varphi^2}\right|_O (\varphi - \varphi_O)^2 + \frac{1}{3!}\left.\frac{d^3y}{d\varphi^3}\right|_O (\varphi - \varphi_O)^3$$

$$+ \frac{1}{4!}\left.\frac{d^4y}{d\varphi^4}\right|_O (\varphi - \varphi_O)^4 + \ldots$$

If φ is replaced by $(\varphi + b)$ and by $(\varphi - b)$, and the two resulting equations are added, one obtains

$$y_E + y_W = 2y_O + \frac{2b^2}{2!}\left.\frac{d^2y}{d\varphi^2}\right|_O + \frac{2b^4}{4!}\left.\frac{d^4y}{d\varphi^4}\right|_O + \ldots$$

or

$$\frac{d^2y}{d\varphi^2} = \frac{1}{b^2}(y_E + y_W - 2y_O) + \frac{b^4}{12}\frac{d^4y}{d\varphi^4}\bigg|_O$$

Similarly, if y is assumed to be the function of ψ alone, one obtains

$$\frac{d^2y}{d\psi^2} = \frac{1}{b^2}\left(y_N + y_S - 2y_O\right) + \frac{b^4}{12}\frac{d^4y}{d\psi^4}\bigg|_O$$

Thus, if y is a function of two variables, φ and ψ,

$$\frac{\partial^2 y}{\partial\varphi^2} + \frac{\partial^2 y}{\partial\psi^2} \approx \frac{1}{b^2}\left(y_E + y_N + y_W + y_S - 4y_O\right) + O(b^4) \tag{51}$$

If b is small, the term containing b^4 is negligible and equation (50) is again obtained.

The following illustration shows that the task of solving Laplace's equation is reduced to solving a system of linear algebraic equations. Consider a function which satisfies Laplace's equation in a 2×3 rectangular domain. For purposes of illustration, the rectangle shall be superimposed by a network of six nets with the boundary conditions as shown in Fig. 14. Applying equation (50c) to point A, one obtains

$$y_A = \frac{1}{4}\left[y_B + 3 + 1 + 0\right]$$

or

$$4y_A - y_B = 4 \tag{52}$$

Similarly, for the point B,

$$y_A - 4y_B = 0 \tag{53}$$

Equations (52) and (53) show that, in this particular case, Laplace's equation is approximated by a system of two equations with two unknowns. In this example, there are two net points, excluding those on the boundary. In general, Laplace's equation can be approximated by a system of linear algebraic equations in which the number of equations and the number of unknowns are both equal to the total number of interior net points. A unique solution exists for this system. The solution is shown graphically in Fig. 15. Suppose, in constructing the network, one unit length of the side of the rectangle is divided into 20 equal spaces, and the rectangle is 2 by 9. The total number of interior points is then 39 × 179 = 6,981. This system of 6,981 equations with 6,981 unknowns can be solved in about 30 minutes using the Burroughs B-5000 electronic computer.

Numerical values have to be initially assigned to all of the interior grid points. Although an initial estimate of all zeroes is satisfactory, a better estimate can be made by observing the boundary conditions and noting that maximum or minimum values of the function must be on the boundary. This fact can be easily proved. Suppose the maximum value of the function is located at an interior point, say the point O of Fig. 12(a). Since y_O is the maximum, y_E, y_N, y_W and y_S have to be less than or equal to y_O. Another condition, from equation (50c), is that y_O is the average of the sum of the four adjacent points. This implies that y_E, y_N, y_W and y_S cannot be less than y_O. Therefore, all of the five points have maximum y. If this argument is carried on, every point on the domain will have maximum y, which is impossible except for a special trivial case of y equal to that maximum also on the boundary.

The case of natural ground-water flow within a confined aquifer with a horizontal outflow face, as shown in Fig. 16(a), is taken as an illustration of the method. On the non-dimensional complex potential w'-plane, Fig. 16(b), y' varies linearly with φ' along the boundary AB with a constant of proportionality of unity. Along OC, y' is zero. Along BC, y' varies linearly with ψ', as can be seen in Fig. 16(a). The linearity of the boundary conditions on all boundaries suggests that y' should vary linearly along either the line of constant φ' or ψ'. If this assumption of linear variation is made, at any point $y'(\varphi_1', \psi_1')$,

$$y_1'(\varphi_1', \psi_1') = -y'(\varphi_1', -1)\ \psi_1' \qquad (54)$$

Substitution of φ_1' for $y'(\varphi_1', -1)$ from the boundary condition on AB yields

$$y_1'(\varphi_1', \psi_1') = -\ \varphi_1'\ \psi_1'$$

or, in general,

$$y'(\varphi', \psi') = -\ \varphi'\ \psi' \qquad (55)$$

Differentiating equation (55) with respect to φ', one obtains

$$\frac{\partial y'}{\partial \varphi'}(\varphi', \psi') = -\ \psi' \qquad (56)$$

$\frac{\partial y'}{\partial \varphi'}$ in equation (56) can be replaced by $\frac{-\partial x'}{\partial \psi'}$, from the inverse Cauchy-Riemannequation, equation 26(b),

$$\frac{-\partial x'}{\partial \psi'}(\varphi', \psi') = -\ \psi' \qquad (57)$$

Partial integration of equation (57) gives

$$x' \left(\varphi', \psi'\right) = \frac{\psi'^2}{2} + F_1\left(\varphi'\right) + C_1 \tag{58}$$

Similarly, if equation (55) is differentiated with respect to ψ' and combined with the Cauchy-Riemannequation, partial integration yields

$$x' = \frac{-\varphi'^2}{2} + F_2\left(\psi'\right) + C_2 \tag{59}$$

Both equations (58) and (59) will be satisfied if

$$x' = \frac{-\varphi'^2}{2} + \frac{\psi'^2}{2} + C$$

At the point O, $x' = 0$, $\varphi' = 0$ and $\psi' = 0$. Hence, $C = 0$ and

$$x' = \frac{1}{2}\left(\psi'^2 - \varphi'^2\right) \tag{60}$$

Equations (55) and (60) are the solution of the problem of natural groundwater flow in a confined aquifer with a horizontal outflow face. This solution is identical with the solution obtained by other methods, such as conformal transformation. Therefore, the initial estimate that y' varies linearly with both φ' and ψ', suggested by observation of the boundary conditions, is the exact solution.

In general, when the initial estimate is not the exact solution, the difference equation approximating Laplace's equation, equation (50), has to be applied to all the interior net points. One application of equation (50) to all of the interior points is called one iteration regardless of the order of application, either from left to right, top to bottom, diagonally, or any other system. After the first iteration, the initial estimate can be compared with the value of the function after application of equation (50) to each point. The total number of interations required to achieve a desired accuracy varies with the judiciousness of the initial estimate and with the rate of convergence of the solution. If the difference at every interior point is negligible, say less than 0.0001, the last numerical value at all points can be taken as an approximate solution of Laplace's equation.

Application of equation (50) for a system of two equations with two unknowns can be illustrated graphically. This is done in Fig. 15 for the case of Laplace's equation in a rectangular domain with boundary conditions as shown in Fig. 14. Suppose an initial estimate is made that the solution is $y_A = 4$ and $y_B = 3.5$, shown as position 0 in Fig. 15. Solving equation (52) for y_A with $y_B = 3.5$, one obtains $y_A = 1.87$, which is graphically equivalent to changing the value from position 0 parallel to the y_A axis to meet the line $4y_A - y_B = 4$ at the position 1. Now, holding $y_A = 1.87$, equation (53) is solved for y_B, obtaining $y_B = 0.47$, thus moving from the position 1 parallel to the axis y_B to meet the line $y_A - 4y_B = 0$ at the position 2. At this stage, one iteration has been performed, consisting of one application of equation (50c) to the two interior points. The next iteration consists of moving two more steps, one parallel to y_A and the next parallel to y_B. This is continued until the exact solution is met at the intersection of the two lines of equations (52) and (53). Re-examination of Fig. 15 suggests that the number of iterations can be reduced if each step is made a little further from the line where the preceding iteration ended. For example, from the position 0, suppose the horizontal step is made to the point x = 1 instead of x = 1.87. The next vertical step, completing one iteration, will bring the approximate solution to the point which would otherwise require several iterations if the "over-step" were not made. This over-step method is formally known as "over-relaxation".

Fig. 15 again suggests that excessive over-relaxation will require more iterations to reach desirable accuracy. For example, if the horizontal step is made too long from the position 0, say to the point x = -5, many iterations will be needed to converge to the exact solution. Therefore, there is a limit for over-relaxation. Let ω be an over-relaxation factor defined by modification of equation (50c) as

$$y_O^{k+1} = \frac{\omega}{4}\left(y_E^k + y_N^k + y_W^k + y_S^k\right) - (\omega - 1)\ y_O^k \tag{61}$$

where superscript k indicates the kth iteration. It has been found theoretically[42] that $0 < \omega < 2$ will make the method of successive over-relaxation converge and that there is an optimum over-relaxation factor, $1 < \omega_{opt} < 2$, which will give the most rapid convergence. Also, the use of ω slightly larger than ω_{opt} is less costly in computation time than the use of ω slightly smaller than ω_{opt}. At the present, no theory exists for relating the optimum over-relaxation factor to the configuration of the domain and boundary conditions. However, for a rectangle with boundary conditions specified on the function itself, not its derivative, a near-optimum over-relaxation factor has been suggested by Young as

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42. See, for example, G. E. Forsythe, and W. R. Wasow, Finite Difference Methods for Partial Differential Equations, John Wiley, New York, 1960, pp. 242-283.

$$\omega = 1 + \frac{L}{(1 + \sqrt{1 - L})^2} \tag{62a}$$

in which

$$L = \left[\frac{(\cos \pi/IM + \cos \pi/JN)}{2}\right]^2 \tag{62b}$$

In equation (62b), IM and JN are the number of grids on the two sides of the rectangle.

After the values of y at all the grid points have been found, x will be evaluated by means of the inverse Cauchy-Riemannequation. As derivatives of y with respect to φ and ψ are required along the boundary of the complex potential plane, especially along the interface AB, central difference cannot be used. An approximate formula for differentiation including any number of grid points can be developed by the method of undetermined coefficients. Suppose four points are chosen, as shown in Fig. 17. The procedure is to assume that

$$\left.\frac{\partial y}{\partial \varphi}\right|_J = y'_J = C_1 y_J + C_2 y_{J-1} + C_3 y_{J-2} + C_4 y_{J-3} \tag{63}$$

Then, apply equation (63) to all polynomials from zero through third degree. The zero-degree polynomial, taking the origin at φ_J for simplicity, is

$$y = \varphi^0 = 1$$

and

$$y' = 0 = C_1 + C_2 + C_3 + C_4 \tag{64}$$

For the first-degree polynomial

$$y = \varphi$$

and

$$y' = 1 = C_1\varphi_J + C_2\varphi_{J-1} + C_3\varphi_{J-2} + C_4\varphi_{J-3}$$

Since the origin is assumed to be at φ_J, $\varphi_J = 0$, $\varphi_{J-1} = -b$, $\varphi_{J-2} = -2b$ and so on. The above equation becomes

$$1 = C_2 + 2C_3 + 3C_4 \tag{65}$$

Similarly, for the second-degree polynomial,

$$0 = C_2 + 4C_3 + 9C_4 \tag{66}$$

and, for the third-degree polynomial,

$$0 = C_2 + 8C_3 + 27C_4 \tag{67}$$

Equations (64) to (67) are four equations with four unknowns and therefore can be solved to give C_1, C_2, C_3, and C_4. Solving for C_1, C_2, C_3, and C_4 and substituting in equation (63)

$$\left.\frac{\partial y}{\partial \varphi}\right|_J = \frac{1}{b}\left(\frac{11}{16}\, y_J - 3y_{J-1} + \frac{3}{2}y_{J-2} - \frac{1}{3}y_{J-3}\right) \tag{68}$$

Equation (68) is exact for a third-degree polynomial. In a similar manner, the derivative at the point φ_{J-3} can be expressed as

$$\left.\frac{\partial y}{\partial \varphi}\right|_{J-3} = -\frac{1}{b}\left(\frac{11}{16}\, y_{J-3} - 3y_{J-2} + \frac{3}{2}y_{J-1} - \frac{1}{3}y_J\right) \tag{69}$$

The error term in terms of the order of $\frac{\partial^4 y}{\partial \varphi^4}$ can be obtained, say for equation (68), by setting

$$\left.\frac{\partial y}{\partial \varphi}\right|_J = \frac{1}{b}\left(\frac{11}{16}\, y_J - 3y_{J-1} + \frac{3}{2}y_{J-2} - \frac{1}{3}y_{J-3}\right) + C_5 y^{IV}$$

which, with the fourth-degree polynomial

$$y = \varphi^4$$

gives

$$4\varphi_J^3 = 0 = \frac{1}{b}\left[0 - 3(-b)^4 + \frac{3}{2}(-\ 2b)^4 - \frac{1}{3}(-\ 3b)^4\right] + 4!\ C_5$$

from which $C_5 = \frac{1}{4}\, b^3$. Thus

$$\left.\frac{\partial y}{\partial \varphi}\right|_J = \frac{1}{b}\left(\frac{11}{16}\, y_J - 3y_{J-1} + \frac{3}{2}y_{J-2} - \frac{1}{3}y_{J-3}\right) + \frac{1}{4}\, b^3 y^{IV}(\zeta) \tag{70}$$

where $\varphi_{J-3} < \zeta < \varphi_J$.

Simpson's one-third rule[43] is an integration formula which is exact for a third-degree polynomial.

$$\int_{x_0}^{x_2} f(x)\ dx = \frac{b}{3}\left[f_0 + 4f_1 + f_2\right] - \frac{1}{90}\, b^5 f^4(\zeta) \tag{71}$$

where $x_0 < \zeta < x_2$. Other differentiation or integration formulae including more points can be obtained by means of the method of undertermined coefficients.

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43. See, for example, Kunz, op. cit., p. 146.

Other Methods

Separation of Variables

The method of separation of variables seems to be the most elementary one. This method can be applied to partial differential equations whose order is higher than the second[44] and to equations with more than two independent variables[45]. However, difficulty is encountered in satisfying certain types of boundary conditions as, for example, on the interface of salt and fresh water in the physical plane where the position is initially unknown. Take, for example, the problem of natural ground-water flow from infinity in a confined aquifer as in Fig. 3(b). One of the governing equations is equation (9),

$$\frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial y^2} = 0$$

The basic principle of the method of separation of variables is to assume that the solution $\varphi(x,y)$ can be separated as the product of a function of x and a function of y,

$$\varphi(x, y) = X(x)\ Y(y) \tag{72}$$

Substituting equation (72) into equation (9) and rearranging, one obtains

$$X''/X = - Y''/Y \tag{73}$$

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44. Miller, <u>op. cit</u>., pp. 117-120.

45. <u>Ibid</u>., pp. 120-124.

where the primes on the functions X and Y represent differentiation with respect to its independent variable. As the right hand side is independent of x while the left hand side is independent of y, they can be equal for all x and y only when both are equal to a constant. This constant can be either greater than, equal to, or less than zero. The boundary conditions will dictate which of the possibilities is valid for each specific problem. For the problem in Fig. 3(b), the interfacial boundary is initially unknown and thus this method cannot be used. However, suppose the domain of interest is a rectangle with all but one of the four boundary conditions being zero, as shown in Fig. 18. The constant in this case was found to be less than zero, say $-\beta^2$ where β is a real number. Thus the partial differential equation, equation (9), is now reduced to two ordinary differential equations,

$$X'' + \beta^2 X = 0 \tag{74a}$$

$$Y'' - \beta^2 Y = 0 \tag{74b}$$

The solution of equation (74a) is

$$X(x) = A_1 \cos \beta x + B_1 \sin \beta x$$

from which the boundary condition $\varphi(0, y) = 0$ yields $A_1 = 0$ and the condition $\varphi(a, y) = 0$ yields $\beta = \frac{n\pi}{a}$, where n is positive integer. Thus

$$X(x) = B_n \sin \frac{n\pi x}{a} \tag{75}$$

The solution of equation (74b), with $\beta = \frac{n\pi}{a}$, is

$$Y(y) = A_n \sinh \frac{n\pi}{a} (y + B_2)$$

from which the boundary condition $\varphi(x, \ell) = 0$ gives $B_2 = -\ell$, and

$$Y(y) = A_n \sinh \frac{n\pi}{a} (y - \ell) \tag{76}$$

According to equation (72), the product of equation (75) and equation (76) is a solution of equation (9).

$$\varphi_n(x, y) = C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a} (y - \ell)$$

Since Laplace's equation is linear and homogeneous, any finite linear combination of solutions is a solution; and an infinite linear combination of solutions having suitable convergence and differentiability properties is also a solution. In particular, it can be shown that

$$\varphi(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{a} \sinh \frac{n\pi}{a} (y - \ell) \tag{77}$$

where

$$C_n = \frac{2}{a \sinh \frac{n\pi}{a} \ell} \int_0^a f(x) \sin \frac{n\pi}{a} x \, dx \tag{78}$$

is a solution and that it is unique. Theoretically, the problem in Fig. 18 is now solved. However, in practice, one may find some difficulties in integrating equation (78) and in finding the suitable form of the terms in equation (77) that will give convergent series at all points of interest.

If the boundary conditions along two or more edges of the rectangle are non-homogeneous[46], one may treat the problem by superposition of the solutions of two or more problems (similar to the one just discussed)

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46. A boundary condition is homogeneous if $C\varphi$ satisfies it whenever φ does, where C is any constant.

in which the boundary conditions are non-homogeneous on one edge only. For example, see Kirkham[47].

Conformal Transformation

The method of conformal transformation or conformal mapping is based on the theory of complex variables[48]. This method can be used to solve many kinds of flow problems[49]. For the first problem in this investigation (Fig. 5), only the special case of a canal located at an infinite distance from the shore can be solved by conformal tranformation. The problem of natural ground-water flow from an infinite distance, as in Fig. 3(b), shall be again taken to illustrate the method. After having been non-dimensionalized, the boundary conditions become as in Fig. 16(a). If non-dimensionalized by $N = k\left(\frac{\gamma_s}{\gamma_f} - 1\right)$, the inverse transformation[50] will become

$$\Omega = \zeta' = \frac{N}{\frac{dw}{dz}} = \frac{1}{\frac{dw'}{dz'}} = \frac{1}{|V'|}\, \ell^{i\alpha} = \frac{1}{|V'|}\,(\cos\alpha + i\,\sin\alpha) \tag{79}$$

with the prime denoting dimensionless quantities.

The physical plane shall be mapped onto the Ω-plane. The interfacial condition is

$$V'^2 - v' = 0 \tag{80}$$

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47. Kirkham, loc. cit.

48. See, for details, R. V. Churchill, Complex Variables and Applications, McGraw-Hill, New York, 1960.

49. See, for example, H. R. Vallentine, Applied Hydrodynamics, Butterworths, London, 1959, pp. 131-224.

50. Ibid., p. 147.

At the origin O, since $u = 0$, $v = \infty$ and $\alpha = \frac{\pi}{2}$, both the real and imaginary parts of Ω are zero. Along the seepage face OA, $\alpha = \frac{\pi}{2}$. In other words, the real part of Ω is zero along OA. At the point A which is on the interface, with $u = 0$, from equation (80), $v' = 1$ or $\Omega = 0 + i$. Along the interface AB, replacing V'^2 in equation (80) by $u'^2 + v'^2$, replacing u' by $v' \cos\alpha/\sin\alpha$, and multiplying through by $\sin^2\alpha$,

$$v'^2 - v' \sin^2\alpha = 0$$

From A to a point near B, $v' \neq 0$. Thus

$$v' = \sin^2\alpha$$

Substituting v' by V'^2 as shown in equation (80),

$$1 = \frac{\sin^2\alpha}{V'^2}$$

or

$$\frac{\sin\alpha}{|V'|} = 1 \qquad (81)$$

At B, which is at an infinite distance from the origin, both u' and v' may be taken as zero. Thus the real part of Ω will be infinite while the imaginary part is undefined. Nevertheless, at a point near B, equation (81) is applicable. At C, again both u' and v' can be taken as zero. Thus the real part of Ω is infinite and the imaginary part is undefined. Since Ω for both B and C have infinite real parts, they can be assumed to meet at infinity. Since $\alpha = 0$ along CO, the imaginary part of Ω is zero. Values of all variables at various points and parts are grouped as follows:

Table 1. Coordinates in Various Planes for Natural Ground-Water Flow.

Location	x'	y'	u'	v'	α	$\frac{\cos\alpha}{\lvert V'\rvert}$	$\frac{\sin\alpha}{\lvert V'\rvert}$
O	0	0	0	∞	$\pi/2$	0	0
OA	$x'>0$	0	0	$v'>0$	$\pi/2$	0	0-1
A	$x'>0$	0	0	1	$\pi/2$	0	1
AB	$-\infty\leq x'\leq x'_A$	$y'<0$	$u'>0$	$v'>0$	$0<\alpha\leq\pi/2$	$0-\infty$	1
B	$-\infty$	$-\infty$	0	0	0	∞	1
C	$-\infty$	0	0	0	0	∞	0-1
CO	$x'\leq 0$	0	$u'>0$	0	0	$0-\infty$	0

The configuration on the dimensionless complex potential plane, Fig. 16(b), can be obtained readily by values of φ and ψ chosen as shown in Fig. 16(a). The configuration on the Ω-plane is shown in Fig. 16(c). Both the Ω-plane and the w'-plane can be transformed onto the upper part of a plane by the Schwarz-Christoffel theorem[51] as they are simple closed polygons[52]. However, by observation, the transformation from the w'-plane to the Ω-plane is

$$\Omega = -w'$$

51. See, for example, Ibid., pp. 183-189.

52. Ibid., pp. 184-185.

Replacing Ω by $1/dw'/dz'$ from equation (79), with manipulation one obtains

$$dz' = -w' \; dw'$$

$$z' = \frac{-w'^2}{2} + C$$

At the origin O, $z' = w' = 0$, thus $C = 0$.

$$z' = \frac{w'^2}{2} \tag{82}$$

Replacing z' by $x' + iy'$ and w' by $\varphi' + i\psi'$ and equating real and imaginary parts, one obtains the previously obtained equations (60) and (55).

$$x' = \frac{1}{2}(\psi'^2 - \varphi'^2)$$

and

$$y' = -\varphi'\psi'$$

Replacing $\psi' = -1$ for the interface AB in equations (60) and (55) and eliminating φ', one obtains

$$x' = \frac{1}{2}(1 - y'^2) \tag{83}$$

Graphical Method

As demonstrated in Chapter II, Theory, both the stream function and the piezometric-head function satisfy Laplace's equation, equations (9) and (17). The families of curves $\psi(x,y) = C_1$ and $\varphi(x,y) = C_2$, where C_1 and C_2 are constants, form a mutually orthogonal network called a flow net. A flow net represents the only possible flow pattern for a given set of boundary conditions. The net can be constructed graphically by observing that (1) piezometric-head lines intersect streamlines, including fixed boundaries, at right angles except at stagnation points and points of theoretically infinite velocity; (2) each unit of the net is approximately square with equal median lines; and (3) the spacing of both the piezometric-head lines and the streamlines is inversely proportional to the velocity at any point. The principles of construction of flow nets are simple. However, considerable skill is required to attain correct flow nets at all points in the flow. Nevertheless, even without experience, flow nets can be roughly sketched for use as a guide in making initial estimates which can be subsequently refined by numerical methods. Flow nets also can be satisfactorily constructed when either streamlines or piezometric-head lines have been obtained by other methods, such as electrical analogy.

Electrical Analogy

Since the voltage in the steady flow of electric current in a conductor satisfies Laplace's equation, the electric-field pattern in a sheet of conducting material or a shallow bath of electrolyte is analogous to the flow pattern through porous media. If the configuration of the aquifer can be geometrically represented by a sheet of conducting material (for example, graphite paper), this method can be employed satisfactorily. The inflow and outflow water faces are represented by lines of high and low constant voltage, respectively. The voltage difference is analogous to the head difference. A voltmeter is used in locating lines of equal voltage, which correspond to equipotential or piezometric-head lines. Streamlines can be then drawn by utilizing the principles of flow net construction. Streamlines may also be obtained by using electric flow again, since the stream function also satisfies Laplace's equation. However, with the interface of the fresh and salt water being initially unknown, successive trail solutions are necessary. After each trial the boundary conditions on the interface are checked and the interfacial boundary is adjusted. This process must be repeated until the interfacial boundary satisfies the dynamic boundary condition.

Sand Boxes

The familiar sand boxes or flow tanks can be employed in the investigation of porous media flow. At least one side of the box is constructed of glass in order to observe the streamlines followed by injected dye. The box may be filled with sand or glass beads or plastic balls to represent the porous material. Wall piezometers are frequently installed in order to measure piezometric head.

The actual flow system is represented in the model. Equation (5) is used to determine the model scale ratio.

$$\frac{\partial^2 h}{\partial x^2} + \frac{\partial^2 h}{\partial y^2} = 0$$

Let subscripts r, m and p denote ratio, model and prototype, respectively.

$$h_r = \frac{h_m}{h_p}, \quad x_r = \frac{x_m}{x_p}, \quad y_r = \frac{y_m}{y_p} \tag{84}$$

Equation (5), which is for the prototype, becomes

$$\frac{\partial^2 h_p}{\partial x_p^2} + \frac{\partial^2 h_p}{\partial y_p^2} = 0 \tag{85}$$

Replacing all variables in equation (85) by those in equation (84), one obtains

$$\frac{\partial^2(h_m/h_r)}{\partial(x_m/x_r)^2} + \frac{\partial^2(h_m/h_r)}{\partial(y_m/y_r)^2} = 0$$

$$\frac{x_r^2}{h_r}\frac{\partial^2 h_m}{\partial x_m^2} + \frac{y_r^2}{h_r}\frac{\partial^2 h_m}{\partial x_m^2} = 0 \tag{86}$$

Equations (85) and (86) can be made identical except for the subscripts if

$$x_r = y_r \tag{87}$$

Thus an undistorted model is required for flow in an isotropic, homogeneous medium.

On the interface the boundary condition is, from equations (28a), (28b) and (28c),

$$h_{fp} = (\gamma_s/\gamma_f)_p\ h_{sp} - y_p(\gamma_s/\gamma_f - 1)_p \tag{88}$$

which, by means of equation (84), becomes

$$h_{fm}/h_r = (\gamma_s/\gamma_f)_p\ h_{sm}/h_r - y_m(\gamma_s/\gamma_f - 1)_p/y_r$$

or

$$h_{fm} = (\gamma_s/\gamma_f)_p\ h_{sm} - \left[y_m(\gamma_s/\gamma_f - 1)_p\right]h_r/y_r \tag{89}$$

Replacing h_r by $(\gamma_s/\gamma_f - 1)_r\ y_r$, equation (89) yields

$$h_{fm} = (\gamma_s/\gamma_f)_p\ h_{sm} - y_m\ (\gamma_s/\gamma_f - 1)_p\ (\gamma_s/\gamma_f - 1)_r \tag{90}$$

which will be identical to equation (88) except for subscript if

$$(\gamma_s/\gamma_f - 1)_r = (\gamma_s/\gamma_f - 1)_m/(\gamma_s/\gamma_f - 1)_p \tag{91}$$

$$h_{sm} = h_{sp} = 0 \tag{92}$$

and

$$h_r/(\gamma_s/\gamma_f - 1)_r\ y_r = 1 \tag{93}$$

Equation (91) is similar to equation (84). Equation (92) implies that both in the prototype and model, the head of stationary salt water has to be used as the datum. Equation (93) is the only one to impose an additional restriction on the model studies. If the prototype has large linear dimensions, the linear scale ratio ($y_r = y_m/y_p$) is necessarily small. The piezometric head ratio ($h_r = h_m/h_p$) should be as large as possible in order to maintain accuracy in the measurement of piezometric head. The ratio $(\gamma_s/\gamma_f - 1)_r$ can be selected to obtain the necessary accuracy in the measurement of piezometric head.

The discharge scale ratio is obtained from the implication of similarity of velocity distributions. In the prototype, Darcy's law, equation (10), can be rewritten as

$$v_p = -k_p \frac{\partial h_p}{\partial y_p} \tag{94}$$

where y here stands for any direction. If, similar to equations (84) and (91), velocity ratio and coefficient of permeability ratio are defined, respectively, as

$$v_r = v_m/v_p,\ k_r = k_m/h_p \tag{95}$$

equation (94) becomes

$$v_m = -k_m \frac{\partial h_m}{\partial y_m} \frac{v_r y_r}{k_r h_r} \tag{96}$$

Equations (96) and (94) imply that

$$v_r \; y_r / k_r \; h_r = 1 \tag{97}$$

Consider the rate of flow of fresh water through an element of area

$$dQ_p = v_{np} \; dA_p = v_{np} \; d(x_p \; y_p) \tag{98}$$

where subscript n specifies that the velocity is normal to the incremental area. Let the discharge scale ratio be defined as

$$Q_r = Q_m / Q_p \tag{99}$$

Equation (98) becomes

$$dQ_m = v_{nm} \; d(x_m \; y_m) \cdot \frac{Q_r}{v_r \; x_r \; y_r}$$

For similarity,

$$Q_r / v_r \; x_r \; y_r = 1 \tag{100a}$$

or, together with conditions in equations (87) and (97),

$$Q_r / k_r \; h_r \; y_r = 1 \tag{100b}$$

Hele-Shaw Model

The Hele-Shaw models consist of viscous flow between two parallel plates a small distance apart. These models are based on the theory that the mean velocity, if it is very small, can be deduced[53] from the Navier-Stokes equations to be

$$\bar{u} = -\frac{a^2 g}{3\nu}\frac{\partial h}{\partial x} = -k_m \frac{\partial h}{\partial x} \tag{101a}$$

$$\bar{v} = -\frac{a^2 g}{3\nu}\frac{\partial h}{\partial y} = -k_m \frac{\partial h}{\partial y} \tag{101b}$$

in which a is half the channel width. The quantity $a^2g/3\nu$ can be considered to be the coefficient of permeability, k_m, of the channel. Equations (101a) and (101b) are equivalent to Darcy's law and, when combined with the equation of continuity, will yield Laplace's equation. If the piezometric head gradient is chosen to be the same in the prototype and the model, the scale ratio for velocity can be readily obtained from Darcy's law and equations (101a) and (101b),

$$v_r = v_m/v_p = k_m/k_p \tag{102}$$

Similar to the case of sand boxes, the wall is usually made of a transparent material for observing the injected dye which follows streamlines.

53. Harr, op. cit., pp. 144-147.

CHAPTER IV

SOLUTIONS OF THE THREE PROBLEMS

Flow from a Single Canal

For seepage flow from a single canal, only half of the configuration on the physical plane, Fig. 5(d), needs to be analyzed due to symmetry. The origin of the x-y axes is chosen to be at the shore, as shown in Fig. 19(a). The dimensionless piezometric-head function on the outflow face OA is taken as the datum of φ'; therefore, $\varphi'_{OA} = 0$. On the canal bed or inflow face CD, the value of the piezometric-head function is designated φ'_C, where subscript C denotes canal. The interface AB is chosen to be the line of $\psi' = 0$. Obviously, the continuation of this streamline, BC, also has the value $\psi'_{BC} = 0$. Referring to equation (19) the value of the stream function on the line DO must be half the discharge from the canal, $(Q_C/2)$. In order to convert both potential functions into dimensionless quantities, $Q_C/2$ is taken as the reference discharge. Thus, the value of ψ'_{DO} is unity.

The configuration in the physical plane, Fig. 19(a), can be mapped onto the complex potential plane as a rectangle as shown in Fig. 19(b). The line OA is a line of zero piezometric head and must lie on the ψ'-axis. The line CD is a constant piezometric-head function and thus is parallel to OA. The line ABC, along which $\psi' = 0$, is on the φ'-axis. The line OD, along which $\psi' = 1$, is parallel to ABC. The value of the piezometric-head function at the point B is designated as φ'_B.

All boundary conditions for y' are known. From Fig. 19(a), with the y'-axis as shown, the value of y' is zero along the line OA, OD and CD.

In accordance with equation (36), along the interface AB

$$\varphi_i' = y_i'$$

Since the streamline BC is parallel to y'-axis in the physical plane and all equipotential lines cross it at right angles,

$$\left.\frac{\partial y'}{\partial \psi'}\right|_{BC} = 0$$

Since all boundary conditions for y' are known and y', according to equation (35), satisfies Laplace's equation

$$\frac{\partial^2 y'}{\partial \varphi'^2} + \frac{\partial^2 y'}{\partial \psi'^2} = 0$$

in the rectangle OABCD on the complex potential plane, y' can be obtained by the over-relaxation method. Equation (35) is first approximated by equation (61) which in I-J notation becomes

$$y'\left(I,\ J\right) = \frac{\omega}{4}\left[y'(I,\ J-1) + y'(I+1,\ J) + y'(I,\ J+1) + y'(I-1,\ J)\right] - (1-\omega)\ y'\ (I,\ J) \qquad (103)$$

in which ω is the near-optimum over-relaxation factor

$$\omega = 1 + \frac{L}{(1+\sqrt{1-L})^2}$$

$$L = \left[(\cos \pi/IM + \cos \pi/JN)/2\right]^2$$

IM and JN are the number of grids on each of the two sides of the rectangle.

In order to apply the numerical method, the rectangle on the complex potential plane must have fixed dimensions. Re-examination of Fig. 19(b) reveals that the dimensions of the rectangle will be fixed if φ'_B and φ'_C are given numerical values. This implies that φ'_B and φ'_C must be taken as independent variables while x' and y' become dependent. Unfortunately, the physical dimensions cannot be specified initially but must be obtained in the last step of the solution. However, once solutions are found for a set of the dimensions, a chart showing the effect of either φ'_B or φ'_C on the physical variables can be prepared and utilized in finding the values of φ'_B and φ'_C which correspond to any desired physical dimensions. Values of φ'_B and φ'_C for which solutions are obtained in this investigation are given in Table 2.

Table 2. Chosen Values of Independent Variables for Problem 1.

Run No.	φ'_B	φ'_C
S1	-1.8	-2
S2	-1.4	-2
S3	-1.0	-2
S4	-0.6	-2
S5	-0.2	-2
S6	-3.8	-4
S7	-3.4	-4
S8	-3.0	-4
S9	-2.6	-4
S10	-2.2	-4

After the rectangle has been given fixed dimensions, a square network is superimposed on it. Five grids per one unit of φ' was first chosen, but it was found that unless the grid size is sufficiently fine, accuracy will be lost in the refinement. Therefore, the initial number of grids per one unit of φ' was changed to be ten. For $\varphi'_C = 4$, this results in 711 grid points. If the values of y' from the initial estimate are punched for all 711 points, it will be time-consuming. Thus the initial estimate was not made from the rough sketch of the flow net in the physical plane but was made in functional form by observing the boundary conditions on the complex potential plane. With the grid points counted as shown in Fig 19(c), the boundary conditions specified on the function become

$$y'(I, 0) = 0; \qquad I = 0, 1 \cdots MO \tag{104a}$$

$$y'(I, NC) = 0; \qquad I = 0, 1 \cdots MO \tag{104b}$$

$$y'(MO, J) = 0; \qquad J = 0, 1 \cdots NC \tag{104c}$$

$$y'(0, J) = \varphi'_B \; J/NB; \quad J = 0, 1 \cdots NB \tag{104d}$$

and the derivative can be approximated by the first central difference

$$\left.\frac{\partial y'}{\partial \psi'}\right|_{BC} = \left[y(1, J) - y(-1, J)\right]/2h$$

which, when equated to zero according to equation (37), yields

$$y'(1, J) = y'(-1, J); \qquad J = NB + 1, NB + 2 \cdots NC - 1 \tag{105}$$

Since all the boundary conditions on y' are linear, y' is initially estimated to be linear, that is, linear along the line BC;

$$y'(0, J) = \varphi'_B\left[1 - (J - NB)/NBC\right]; \; J = NB + 1, NB + 2, \cdots NC - 1 \tag{106a}$$

where

$$NBC = NC - NB \tag{106b}$$

and, linear along the line of constant φ';

$$y'(I, J) = y'(0, J)\,(1 - I/MO); \quad \begin{matrix} I = 1,2\cdots MO - 1 \\ J = 1,2\cdots NC - 1 \end{matrix} \tag{107}$$

After the initial estimate has been made, ω and L are evaluated according to equations (62a) and (62b). In the case of $\varphi'_C = -4$ and MO = 10, IM and JN are 10 and 40, respectively. For each iteration, equation (103) is applied to all the interior points. 711 points or 711 equations with 711 unknowns are solved in this case. The over-relaxation factor was derived for these interior points, not for the points on the boundary. Therefore, it should not be applied to the exterior points on the line BC where equation (50b),

$$y'(I, J) = \frac{1}{4}\left[y'(I, J - 1) + y'(I + 1, J) + y'(I, J + 1) + y'(I - 1, J)\right]$$

has to be used. Replacing the value of I by zero, as it is on BC, and then utilizing equation (105),

$$y'(0, J) = \frac{1}{4}\left[y'(0, J - 1) + 2y'(1, J) + y'(0, J + 1)\right];$$

$$J = NB + 1,\ NB + 2\cdots NC - 1 \tag{108}$$

Furthermore, the existence of the boundary conditions specified on the derivative of the function gives more linear algebraic equations, as in equation (108), to the system effecting the efficiency of the over-relaxation factor. Yet, as these additional equations constitute only a small percentage of the whole system, the number of iterations needed, for any desired accuracy, by using the over-relaxation factor should still be less than the number needed without using the factor.

After a specified accuracy is attained, the network is refined. As accuracy is lost in the refinement, a low order of accuracy should be specified for the coarse net. In this investigation, the criteria for refinement is chosen to be

$$\mathrm{Max}\left[(y^{k+1} - y^{k})/y^{k+1}\right] \leq e \tag{109}$$

where y^k denotes the function obtained from the kth iteration. Equation (109) shall be called "accuracy-check equation". It implies that the net will be refined when the maximum change of the function at any point is less than one percent, if e is taken to be 0.01. For simplicity, the size of the refined grid is chosen to be half of that of the coarse one as shown in Fig. 20. The value of the function at the point of intersection of two new grid lines, as the point O of Fig. 20, is obtained from the 90^o rotation of equation (50b) which is

$$y'(I, J) = \frac{1}{4}\Big[y'(I+1, J-1) + y'(I+1, J+1) + y'(I-1, J+1) + y'(I-1, J-1)\Big] \tag{110a}$$

or, with N-E-S-W notation,

$$y'_O = \frac{1}{4}\left(y'_{NE} + y'_{NW} + y'_{SW} + y'_{SE}\right) \tag{110b}$$

The value of y' at the remaining new points, such as N and W in Fig. 20, can be obtained by using equation (50b).

Relaxation is repeated on the fine grid with the new over-relaxation factor until the condition in equation (109) with e = 0.0001 is satisfied. Derivatives of y' with respect to φ' are evaluated for every point along the lines OA and DC by using equations (68) and (69) which in I-J notation become

$$\frac{\partial y'(I, 0)}{\partial \varphi'} = \frac{1}{b}\left[\frac{11}{6}\, y'(I, 0) - 3y'(I, 1) + \frac{3}{2}y'(I, 2) - \frac{1}{3}y'(I, 3)\right] \qquad (111)$$

and

$$\frac{\partial y'(I, NC)}{\partial \varphi'} = -\frac{1}{b}\left[\frac{11}{6}\, y'(I, NC) - 3y'(I, NC - 1) + \frac{3}{2}y'(I, NC - 2) - \frac{1}{3}y'(I, NC - 3)\right] \qquad (112)$$

where b = 1/20. Application of equation (111) or (112) implies that a third-order polynomial is first fitted to the four points and the slope of this polynomial approaching the end point is taken to be the approximation of the first derivative of the function at that point. Incremental values of x' are then evaluated by using the inverse Cauchy-Riemann equations, equations (26a) and (26b) and equation (71), combined for OA as

$$x'(I - 2, 0) - x'(I, 0) = \frac{b}{3}\left[\frac{\partial y'(I - 2, 0)}{\partial \varphi'} + 4\,\frac{\partial y'(I - 1, 0)}{\partial \varphi'} + \frac{\partial y'(I, 0)}{\partial \varphi'}\right] \qquad (113a)$$

and for DC as

$$x'(I - 2, NC) - x'(I, NC) = \frac{b}{3}\left[\frac{\partial y'(I - 2, NC)}{\partial \varphi'} + 4\,\frac{\partial y'(I - 1, NC)}{\partial \varphi'} + \frac{\partial y'(I, NC)}{\partial \varphi'}\right] \qquad (113b)$$

Similarly, x' along the interfacial streamline ABC and other streamlines can be evaluated. Therefore, the coordinates of points at constant interval of piezometric-head function along streamlines are obtained and both the streamlines and piezometric-head are plotted in the physical plane.

A typical flow net is shown in Fig. 21. Various dimensions are tabulated in Table 3. The effect of h_B/h_C on various dimensions and on the land-water ratio are shown for $\varphi'_C = -2.0$ in Fig. 22(a) and for $\varphi'_C = -4.0$ in Fig. 22(b).

A Single Canal With Natural Ground-Water Flow

Seepage flow from a single canal with natural ground-water flow is analyzed as two separate cases. The first case is shown in Fig. 4(a) and the second case in Fig. 4(b). The difference in the two cases is the direction of flow at C. In the first case, the head in the canal is the same as that of the approaching ground water resulting in the piezometric head gradient being vertical at C. In other words, the discharge is vertically downward at C. In the second case, the head in the canal is higher than that of the approaching ground water, resulting in a landward flow from the canal.

Condition A

As shown in Fig. 6(a), φ' on the outflow face OA and ψ' on the interfacial streamline AB are chosen to be zero. The piezometric-head function at the inflow face CD is designated as φ'_C. From equation (19), ψ on C'CFG is the natural discharge from infinity, Q_I. Similarly, ψ on DO is the total discharge from infinity and from the canal, Q_T. For convenience in comparing the solution of this problem with the problem

Table 3. Dimensions (Problem 1).

Run No.	φ' at Canal φ'_C	φ' Under ℄ of Canal $\varphi'_B = y'_B$	Seepage Face x'_A	Depth Under the Shore $y'_{B'}$	Land Between Canal and Shore x'_D	Half Width of Canal $x'_D - x'_C$	Land-Water Ratio $-x'_D(x'_D-x'_C)$	$h_B/h_C = \varphi'_B/\varphi'_C$	$x'_A+x'_D-x'_C$	$-x'_D/(x'_A+x'_D-x'_C)$
S1	-2	-1.8	0.489	-0.939	-1.082	1.691	0.64	0.90	2.180	0.50
S2	-2	-1.4	0.479	-0.889	-0.848	0.576	1.47	0.70	1.055	0.80
S3	-2	-1.0	0.446	-0.754	-0.510	0.198	2.57	0.50	0.644	0.79
S4	-2	-0.6	0.358	-0.516	-0.200	0.052	3.85	0.30	0.410	0.49
S5	-2	-0.2	0.148	-0.180	-0.021	0.005	4.20	0.10	0.153	0.14
S6	-4	-3.8	0.499	-1.000	-6.151	3.870	1.59	0.95	4.369	1.41
S7	-4	-3.4	0.498	-0.999	-5.652	1.566	3.61	0.85	2.064	2.74
S8	-4	-3.0	0.500	-0.998	-4.843	0.706	6.85	0.75	1.206	4.01
S9	-4	-2.6	0.499	-0.996	-3.893	0.321	12.11	0.65	0.820	4.75
S10	-4	-2.2	0.498	-0.988	-2.937	0.142	20.65	0.55	0.640	4.58

of natural ground-water flow of Fig. 3(b), Q_I is taken as a reference discharge. Therefore, ψ' on C'CFG becomes unity and that on DO, designated as Q_T', is equal to or greater than unity. Landward from the canal the spacing of streamlines in the vertical direction tends to become equal, similar to the case of natural ground-water flow alone. Therefore, it appears reasonable to assume that at a point at some distance inland from the canal, ψ' varies linearly with y'. Let φ' at this point be denoted by φ_B'.

Mapping of the physical plane onto the complex potential plane results in two adjoining rectangles as shown in Fig. 6(b). The upper rectangle OGCD represents the flow from the canal and ABC'G represents the natural ground-water flow. Since all boundary conditions for y' are known, relaxation is performed on this variable.

In order to fix the dimensions of the two rectangles, three variables, φ_C', φ_B' and Q_T', must be given numerical values. Therefore, in this problem, φ_C', φ_B' and Q_T' are taken as independent variables, leaving x' and y' as the independent variables.

There is probably an optimum over-relaxation factor for the relaxation of a configuration of two adjoining rectangles, but, up to the present time, this optimum value has not been found theoretically. If one set of dimensions were used several times, this factor could be found by trial and error. However, in this investigation, various sets of dimensions, as shown in Table 4, are used, with each set to be solved only once.

Table 4. Chosen Values of Independent Variables for Condition A of Problem 2.

Run No.	φ'_C	φ'_B	Q'_T
A1	-2	-6	2.2
A2	-2	-4	2.0
A3	-2	-4	1.8
A4	-2	-4	1.6
A5	-2	-4	1.4
A6	-2	-4	1.2
A7	-3	-6	2.2
A8	-3	-6	2.0
A9	-3	-6	1.8
A10	-3	-6	1.6
A11	-3	-6	1.4
A12	-3	-6	1.2
A13	-4	-8	2.2
A14	-4	-8	2.0
A15	-4	-8	1.8
A16	-4	-8	1.6
A17	-4	-8	1.4
A18	-4	-8	1.2
A19	-5	-10	2.2
A20	-5	-10	2.0
A21	-5	-10	1.8
A22	-5	-10	1.6
A23	-5	-10	1.2

In the absence of a theoretical value, the over-relaxation factor as defined in equations (62a) and (62b) shall be employed. The two adjoining rectangles in the dimensionless complex potential plane are divided into two overlapping rectangles by extending the vertical line DC to the point S on the φ' - axis, as shown in Fig. 6(c). One over-relaxation factor, ω_1, is used for the rectangle OASD and another, ω_2, for the rectangle CSBC'. No over-relaxation factor is used on the line CS. For every iteration, relaxation is performed from right to left and from bottom to top. These operations are performed first in the rectangle OASD using equation (103) with $\omega = \omega_1$; second, on the line CS using equation (50b) which is identical to equation (103) with ω as unity; and third, in the rectangle CSBC' using equation (103) with $\omega = \omega_2$. Ten nets per unit of φ' or unit of ψ' were taken initially, that is, MC in Fig. 6(a) was given the numerical value of ten. The nets were subsequently refined by choosing 20 per unit of φ' or ψ'.

The initial estimate of y' is made by observing the boundary conditions on the complex potential plane, Fig. 6(b), which may be rewritten in terms of I-J notation as,

on AO,	$y'(I, 0) = 0;$	$I = 0, 1, 2 \ldots MD$	(114a)
on CD,	$y'(I, ND) = 0;$	$I = MC, MC + 1, \ldots MD$	(114b)
on BC',	$y'(I, NB) = \varphi_B'(1 - I/MC);$	$I = 1, 2 \ldots MC$	(114c)
on AB,	$y'(0, J) = \varphi_B' J/NB;$	$J = 1, 2 \ldots NB$	(115a)
on C'C,	$y'(MC, J) = 0;$	$J = ND + 1, ND + 2 \ldots NB - 1$	(115b)
and on OD,	$y'(MD, J) = 0;$	$J = 1, 2 \ldots ND - 1$	(115c)

in which MC is the number of grids per one unit of φ',

$$MD = MC \times Q'_T \tag{116a}$$

$$ND = -MC \times \varphi'_C \tag{116b}$$

NB is assumed to be twice the value of ND. The assumption that NB = 2ND or that $\varphi'_B = 2\varphi'_C$ is simply a means of insuring that x_B is sufficiently landward from the canal. The value of y' is zero on all the boundaries except on AB, where y' varies linearly with φ', and on BC, where y' varies linearly with ψ'. The initial estimate of y' is based upon a linear variation with ψ'. In terms of I-J notation,

$$y'(I, J) = y'(0, J)\ (1 - I/MD); \quad \begin{array}{l} I = 1, 2, \ldots MD-1 \\ J = 1, 2, \ldots ND-1 \end{array} \tag{117a}$$

and,

$$y'(I, J) = y'(0, J)\ (1 - I/MC); \quad \begin{array}{l} I = 1, 2, \ldots MC-1 \\ J = ND, ND + 1, \ldots NB-1 \end{array} \tag{117b}$$

The number of iterations required is again determined by checking the desirable accuracy according to equation (109). The value of e is taken to be 0.01 for the coarse net and 0.0001 for the fine net. For some trial runs of the computer program, equation (109) was applied to all the interior net points. The maximum values of $(y^{k+1} - y^k)/y^{k+1}$ were obtained mostly in the region of flow from the fresh water canal, OGCD, not in the region of flow from infinity, GABC' (Fig. 6b). For expediency in all of the later runs, equation (109) was used as an accuracy check only in the region OGCD, which has about one-third as many interior points as the whole region. Additional computer time is saved by applying equation (109) to every fifth iteration for the coarse net and every second iteration for the fine net, instead of to every iteration.

After equation (109) has been satisfied, y' at all grid points are taken as a solution of equation (35). The values of x' for equipotential line OA and streamlines $\psi' = 0$, $\psi' = 0.2$, ..., $\psi' = Q_T' - 0.2$, and $\psi' = Q_T'$ were obtained by application of the inverse Cauchy-Riemann equation, equations (111), (112), (113a) and (113b). A typical flow net for a set of boundary conditions is shown in Fig. 23. Pertinent dimensions are presented in Table 5.

Condition B

Similar to the problem of Condition A, φ' on the outflow face, OA in Fig. 7(a), and ψ' on the interfacial streamline AB, are chosen to be zero. From equation (19), ψ on C'D' and on CD'FG is the natural discharge from infinity, Q_I. On DO, ψ is equal to Q_T, the total discharge. The reference discharge is again Q_I.

The dimensionless complex potential plane of this problem is shown in Fig. 7(b). The configuration of Fig. 7(b) is two adjoining rectangles with a slot along the line CD'. The rectangle OGCD is joined to the rectangle GABC' only on a portion, GD', of the line GD'C.

Application of the numerical method requires that the domain has numerical values for the dimensions. Therefore, for Fig. 7(b), φ_C', φ_B', $\varphi_{D'}$, and Q_T' must be given numerical values. Thus, φ_C', φ_B', $\varphi_{D'}$, and Q_T' are taken as independent variables, and the dependent variables are x' and y' for various points. If $\varphi_{D'}$ is the same as φ_C', this problem reduces to Problem 2A. Numerical values of φ_C', Q_T' and $\varphi_{D'}$ are shown in columns 2, 3, 4 of Table 6. The value of φ_B' was taken to be equal to $2\varphi_C'$.

The over-relaxation factor as defined in equations (62a) and (62b) shall again be employed. The two adjoining rectangles in the dimensionless

Table 5. Dimensions (Problem 2A).

Run No.	φ' at the Canal φ'_C	Total Discharge Q'_T	Seepage Face x'_A	Depth Under the Shore $y'_{A'}$	Distance Between Shore and Canal x'_D	x'_C	Canal Width $x'_D - x'_C$	x' at B x'_B	x'_B/x'_C	x' at ℄ of Canal $x'_{℄}$	y' at ℄ of Canal $x'_{℄}$
A1	-2	2.2	0.905	-1.55	-0.245	-0.807	0.562	-18.618	23.1	-0.526	-1.910
A2	-2	2.0	0.867	-1.53	-0.337	-0.856	0.519	- 6.668	7.8	-0.597	-1.923
A3	-2	1.8	0.819	-1.48	-0.464	-0.926	0.462	- 6.731	7.3	-0.695	-1.945
A4	-2	1.6	0.757	-1.41	-0.642	-1.028	0.386	- 6.823	6.6	-0.835	-1.972
A5	-2	1.4	0.683	-1.31	-0.898	-1.182	0.284	- 6.956	6.4	-1.040	-2.017
A6	-2	1.2	0.596	-1.17	-1.285	-1.432	0.147	- 7.159	5.0	-1.359	-2.080
A7	-3	2.2	1.031	-1.88	-0.950	-1.962	1.012	-15.50	8.9	-1.456	-2.736
A8	-3	2.0	0.959	-1.80	-1.184	-2.097	0.913	-15.63	7.5	-1.641	-2.763
A9	-3	1.8	0.879	-1.68	-1.485	-2.277	0.792	-15.78	7.1	-1.881	-2.799
A10	-3	1.6	0.791	-1.55	-1.879	-2.523	0.644	-16.00	6.4	-2.201	-2.845
A11	-3	1.4	0.697	-1.39	-2.413	-2.873	0.460	-16.31	5.7	-2.643	-2.904
A12	-3	1.2	0.600	-1.20	-3.180	-3.411	0.231	-16.77	4.9	-3.296	-2.986

(continued)

Table 5. Dimensions (Problem 2A) (Continued).

Run No.	φ' at the Canal φ'_C	Total Discharge Q'_T	Seepage Face x'_A	Depth Under the Shore y'_A	Distance Between Shore and Canal x'_D	x'_C	Canal Width $x'_D - x'_C$	x' at B x'_B	x'_B/x'_C	x' at ℄ of Canal $x'_{℄}$	y' at ℄ of Canal $x'_{℄}$
A13	-4	2.2	1.078	-2.06	-2.143	-3.611	1.468	-27.88	7.7	-2.877	-3.639
A14	-4	2.0	0.989	-1.93	-2.560	-3.871	1.311	-28.13	7.3	-3.216	-3.671
A15	-4	1.8	0.895	-1.77	-3.083	-4.205	1.122	-28.41	6.8	-3.644	-3.722
A16	-4	1.6	0.798	-1.59	-3.754	-4.656	0.902	-28.82	6.2	-4.205	-3.778
A17	-4	1.4	0.700	-1.40	-4.649	-5.285	0.636	-29.39	5.5	-4.967	-3.841
A18	-4	1.2	0.600	-1.20	-5.909	-6.223	0.314	-30.23	4.9	-6.066	-3.932
A19	-5	2.2	1.093	-2.16	-3.807	-5.733	1.926	-43.74	7.6	-4.770	-4.577
A20	-5	2.0	0.997	-1.98	-4.447	-6.152	1.705	-44.11	7.2	-5.300	-4.621
A21	-5	1.8	0.899	-1.80	-5.243	-6.696	1.453	-44.59	6.7	-5.970	-4.678
A22	-5	1.6	0.800	-1.60	-6.257	-7.417	1.160	-45.25	6.1	-6.837	-4.738
A23	-5	1.2	0.600	-1.20	-9.471	-9.868	0.397	-47.71	4.8	-9.670	-4.902

Note: φ'_B is taken to be $3\varphi'_C$ for Run 1 and to be $2\varphi'_C$ for Runs 2 - 23.

complex potential plane are divided into two overlapping rectangles, OGCD and GABC'. The overlapping portion is the line GD'. The over-relaxation factors are ω_1 and ω_2 for the rectangles GABC' and OGCD, respectively. No over-relaxation factor is used on the line GD'. Twenty nets per unit of φ' or ψ' were used and no refinement was made.

The boundary conditions are denoted by equations (114) and (115), with one addition, namely

$$y'(MC, J) = 0; \qquad J = NDP, NDP + 1...ND \tag{118}$$

on CD' as shown in Figs. 7(b) and 7(c). The initial estimate of y' is taken to be linear.

$$y'(I, J) = y'(0, J)(1 - I/MD); \qquad \begin{matrix} I = 1, 2...MD - 1 \\ J = 1, 2...NDP - 1 \end{matrix} \tag{119a}$$

$$y'(I, J) = y'(0, J)(1 - I/MC); \qquad \begin{matrix} I = 1, 2...MC - 1 \\ J = NDP, NDP + 1...NB - 1 \end{matrix} \tag{119b}$$

and,

$$y'(I, J) = y'(I, NDP - 1)\left[1 - (J - NDP + 1)/(ND - NDP + 1)\right]; \tag{119c}$$
$$I = MC + 1, MC + 2...MD - 1$$
$$J = NDP, NDP + 1...ND - 1$$

The number of iterations required is determined by the application of the accuracy-check equation, equation (109), to all the net points in the region OASCD of Fig. 7(c). The value of e of equation (109) is taken to be 0.0001. Application of equation (109) is made every fifth iteration when the number of iterations is less than sixty, and every second iteration otherwise.

The values of x' for equipotential lines OA and DC, and for streamlines $\psi' = 0, 0.2 \ldots Q_T'$, are again obtained by means of the inverse Cauchy-Riemann equation, equations (111), (112), (113a) and (113b). A typical flow net for a set of boundary conditions is shown in Fig. 25. The effect of $h_{D'}/h_C$ on various dimensions is shown in Fig. 26. Key dimensions are tabulated in Table 6.

Parallel Canals with Intermediate Drains

Seepage flow from parallel canals to intermediate drains, Fig. 8(a), is simplified as shown in Fig. 8(b). Mapping of the simplified physical plane into the complex potential plane yields a rectangle with boundary conditions on y' as shown in Fig. 8(c). The boundary conditions for this problem in the complex potential plane appear to be the same as those for the problem of a single canal, Fig. 19(b), except on the line of zero ψ'. In Fig. 8(c), if the point A' is made to coincide with the point A, the configuration and boundary conditions of Fig. 8(c) are identical to those of Fig. 19(b). Therefore, the solution of the problem of a single canal is a special case of the solution of the problem of parallel canals with intermediate drains. An additional independent variable, $\varphi_{A'}'$, is required in order to fix the configuration on the w-plane. For the single canal, $\varphi_{A'}'$ is zero.

The boundary conditions of the problem of a single canal, equations (104a), (104b), and (104c), are valid for this problem, but equation (104d) has to be modified as follows,

$$y'(0, J) = \varphi_B' \, J/NB; \qquad J = NA' + 1, NA' + 2 \ldots NB \tag{120}$$

Table 6. Dimensions (Problem 2B).

Run No.	φ' at the Canal φ'_C	Total Discharge Q'_T	φ' at D' $\varphi'_{D'}$	Seepage Face x'_A	Depth Under the Shore $y'_{A'}$	Distance Between Shore and Canal x'_D	x'_C	Canal Width $x'_D - x'_C$	x' at D' $x'_{D'}$	x' at B x'_B	$\frac{x'_B}{x'_{D'}}$	$h_{D'}/h_C = \varphi'_{D'}/\varphi'_C$	x' at ℄ of Canal $x'_{℄}$	y' at ℄ of Canal $y'_{℄}$
A1	-2	2.2	-2	0.905	-1.550	-0.245	-0.807	0.562	-0.807	18.618	23.1	1.0	-0.5260	-1.910
B1	-2	2.2	-1.5	0.815	-1.396	-0.133	-0.196	0.063	-0.328	6.978	21.3	0.75	-0.1645	-1.515
B2	-2	2.2	-1.0	0.681	-1.202	-0.036	-0.043	0.007	-0.119	- 7.265	61.1	0.50	-0.0395	-1.235
B3	-2	2.2	-0.5	0.554	-1.055	-0.0026	-0.0028	0.0002	-0.022	- 7.442	338.0	0.25	-0.0027	-1.058
N	-2	---	0	0.500	-1.000	0	0	0	0	---	---	0	0	-1.000
A5	-2	1.6	-2	0.757	-1.41	-0.642	-1.028	0.386	-1.028	- 6.823	6.6	1.0	-0.8350	-1.972
B4	-2	1.6	-1.5	0.717	-1.301	-0.344	-0.358	0.014	-0.465	- 7.096	15.3	0.75	-0.3510	-1.562
B5	-2	1.6	-1.0	0.644	-1.170	-0.103	-0.104	0.001	-0.169	- 7.306	43.2	0.50	-0.1037	-1.255
B6	-2	1.6	-0.5	0.550	-1.050	-0.009	-0.007	0	-0.027	- 7.447	127.6	0.25	-0.0090	-1.060
N	-2	---	0	0.500	-1.000	0	0	0	0	---	---	0	0	-1.000
B7	-3	2.2	-1.5	0.816	-1.395	-0.160	-0.163	0.003	-0.325	-16.976	52.2	0.50	-0.1615	-1.512
B8	-4	2.2	-2.0	0.922	-1.589	-0.398	-0.400	0.002	-0.657	-30.574	46.5	0.50	-0.3990	-1.859
B9	-4	2.2	-3.6	1.073	-2.040	-1.855	-2.189	0.334	-2.598	-28.486	11.0	0.90	-2.022	-3.210
B10	-4	2.2	-3.8	1.079	-2.070	-2.034	-2.665	0.631	-2.969	-28.148	9.5	0.95	-2.335	-3.399
B11	-5	2.2	-2.5	0.993	-1.768	-0.7636	-0.7644	0.001	-1.120	-48.056	42.9	0.50	-0.7640	-2.251

Note: $\varphi'_B = 2\varphi'_C$

Equation (105) must also be modified as follows,

$$y'(1, J) = y'(-1, J); \; J = 1, 2, \ldots NA', NB + 1, NB + 2 \ldots NC - 1 \qquad (121)$$

If y' is initially estimated to be linear, equations (106a), (106b) and (107) are applicable. For each iteration, equation (103) is applied to all interior grid points and equation (108) is applied to the portion of the line of zero ψ' where J = 1, 2...NA' and J = NB + 1, NB + 2...NC - 1. The remaining procedure is the same as for the problem of a single canal. A typical flow net is shown in Fig. 27. Key dimensions are tabulated in Table 7. The variations of physical dimensions with the value $h_{A'}/h_C$ for various values of the piezometric head in the canal (related to φ_C') are shown in Fig. 28. The variations of the ratio of land surface to water surface, and the variation of the ratio of land surface to half the sum of the depth of the interface at the centerline of the canal and at the centerline of the drain with h_B/h_C are shown in Fig. 29.

Accuracy of Numerical Solutions

In the numerical computations involved in either relaxation, integration or differentiation, there are two major sources of error, namely, "truncation errors" and "round-off errors". The truncation errors are caused by truncating all but the first few terms of an infinite series, such as Taylor's series, as illustrated after equation (51). The round-off errors arise from the necessity of using finite decimal numbers in the computations.

Truncation Errors

The truncation error is first made in approximating Laplace's equation by a finite difference equation. From equation (51), the

Table 7. Dimensions (Problem 3).

Run No.	φ' at Canal φ'_C	φ' Under ℄ of Canal $\varphi'_B=y'_B$	φ' Under ℄ of Drain $\varphi'_{A'}$	Half Width of Drain x'_A	Depth Under ℄ of Drain $y'_{A'}$	Land Surface x'_D	Half Width of Canal $x'_D-x'_C$	Total Water Surface x'_w	Land-Water Ratio $-x'_D/x'_w$	$h_B/h_C=\varphi'_B/\varphi'_C$	$h_{A'}/h_C=\varphi'_{A'}/\varphi'_C$	$\frac{x'_D}{(y'_A+y'_B)/2}$
S1	-4	-3.8	0	0.499	0	-6.151	3.870	4.369	1.41	0.95	0	3.24
P1	-4	-3.8	-0.4	0.465	-0.27	-6.148	3.870	4.335	1.42	0.95	0.10	3.06
P2	-4	-3.8	-0.8	0.366	-0.55	-6.094	3.870	4.236	1.44	0.95	0.20	2.86
P3	-4	-3.8	-1.2	0.266	-0.81	-5.928	3.870	4.136	1.43	0.95	0.30	2.57
P4	-4	-3.8	-1.6	0.182	-1.06	-5.605	3.864	4.051	1.45	0.95	0.40	2.31
P5	-4	-3.8	-2.0	0.119	-1.30	-5.108	3.867	3.986	1.28	0.95	0.50	2.01
S2	-4	-3.4	0	0.498	0	-5.652	1.566	2.064	2.74	0.85	0	3.32
P6	-4	-3.4	-0.4	0.464	-0.27	-5.649	1.566	2.030	2.78	0.85	0.10	3.07
P7	-4	-3.4	-1.6	0.182	-1.06	-5.105	1.565	1.747	2.92	0.85	0.40	2.29
P8	-4	-3.4	-2.0	0.118	-1.30	-4.605	1.563	1.681	2.74	0.85	0.50	1.96
S3	-4	-3.0	0	0.500	0	-4.843	0.706	1.206	4.01	0.75	0	3.23
P9	-4	-3.0	-0.4	0.456	-0.25	-4.838	0.706	1.162	4.16	0.75	0.10	2.97
P10	-4	-3.0	-1.6	0.181	-1.06	-4.291	0.705	0.886	4.85	0.75	0.40	2.12
P11	-4	-3.0	-2.0	0.116	-1.29	-3.782	0.702	0.818	4.63	0.75	0.50	1.76

(Continued)

Table 7. Dimensions (Problem 3) (Continued).

Run No.	φ' at Canal φ'_C	φ' Under L of Canal $\varphi'_B = y'_B$	φ' Under L of Drain $\varphi'_{A'}$	Half Width of Drain x'_A	Depth Under L of Drain $y'_{A'}$	Land Surface x'_D	Half Width of Canal $x'_D - x'_C$	Total Water Surface x'_W	Land-Water Ratio $-x'_D/x'_W$	$h_B/h_C = \varphi'_B/\varphi'_C$	$h_{A'}/h_C = \varphi'_{A'}/\varphi'_C$	$\frac{x'_D}{(y'_A+y'_B)/2}$
S4	-4	-2.6	0	0.494	0	-3.893	0.321	0.820	4.75	0.65	0	2.99
P12	-4	-2.6	-0.4	0.465	-0.27	-3.890	0.321	0.786	4.95	0.65	0.10	2.70
P13	-4	-2.6	-0.8	0.376	-0.58	-3.847	0.320	0.696	5.53	0.65	0.20	2.42
P14	-4	-2.6	-1.2	0.264	-0.81	-3.664	0.320	0.584	6.28	0.65	0.30	2.14
P15	-4	-2.6	-1.6	0.178	-1.05	-3.333	0.318	0.496	6.73	0.65	0.40	1.82
P16	-4	-2.6	-2.0	0.111	-1.25	-2.791	0.310	0.421	6.63	0.65	0.50	1.50
S5	-4	-2.2	0	0.498	0	-2.937	0.142	0.640	4.58	0.55	0	2.67
P17	-4	-2.2	-0.4	0.454	-0.25	-2.932	0.142	0.596	4.92	0.55	0.10	2.38
P18	-4	-2.2	-0.8	0.361	-0.54	-2.878	0.142	0.503	5.71	0.55	0.20	2.11
P19	-4	-2.2	-1.2	0.258	-0.80	-2.704	0.142	0.400	6.77	0.55	0.30	1.80
P20	-4	-2.2	-1.6	0.168	-1.02	-2.345	0.139	0.307	7.65	0.55	0.40	1.46
P21	-4	-2.2	-2.0	0.088	-1.30	-1.628	0.118	0.206	7.90	0.55	0.50	0.93

Note: As AA' is read from the flow net, only two decimal points are obtained.

truncation error is of the order of b^4. In this investigation b is 1/20 and therefore, b^4 is 6.25×10^{-6}, which is negligible. The numerical differentiation formula employed, equation (70), appears to give the highest truncation error, which is in the order of b^3 or 1.25×10^{-4} for b of 1/20. Simpson's one-third rule for integration, equation (71), has a truncation error in the order of b^5. Therefore, considering the truncation errors alone, the maximum error is in the order of 0.000125, which is not serious. If higher accuracy is desired, equation (70) should be replaced by one containing more points, or b should be made sufficiently small so that the truncation error is less than the maximum permissible round-off error.

Round-Off Errors

As the maximum truncation error is in the order of 1.25×10^{-4} for $b = 1/20$, increased accuracy of the final results cannot be obtained by reducing the maximum permissible round-off error below this value. However, the maximum round-off error for relaxation is set at 1.00×10^{-4}, slightly less than 1.25×10^{-4}, because round-off in the intermediate steps decreases the accuracy of the final result through cumulative errors. Although the Burroughs B-220 and B-5000 computers can maintain eight and twelve significant digits, respectively, increased accuracy significantly increases the computer time required.

CHAPTER V

ENGINEERING ASPECTS

The objective of this study was to formulate the fundamental seepage analysis necessary for a subsequent evaluation of the effectiveness and water-loss from various canal arrangements. The fundamental seepage analysis, which was formulated in Chapter II, consisted of the solution of Laplace's equation with appropriate boundary conditions. Different methods of solution were discussed in Chapter III, including the method used in this study. A limited number of solutions were presented in Chapter IV. The choice of the problems and boundary conditions investigated was based upon the physical conditions at the head of the Gulf of Siam and the possibility of reclaiming salt-intruded land.

In this chapter, engineering analyses are presented in order to estimate the water-loss from canals which might be installed to reclaim land at the head of the Gulf of Siam. Several assumptions are used in all of the following analyses. The first assumption is that salt-intruded land is land under which the salt water is within 2m of the land surface. The second assumption is that the coefficient of permeability of the delta soil is $1(10^{-5})$ cm/sec, which is a reasonable value for a clay-silt soil. The third assumption is that γ_s/γ is 1.025. It is probable that the specific-weight ratio is less at the head of the Gulf of Siam than the assumed mean value of 1.025. The Gulf is elongated with appreciable fresh water inflows from the Chao Phraya and Mae Klong rivers at the head. Under these conditions, the salt concentration is undoubtedly less than in the open sea. However, the assumption is conservative in regard to the amount of depression of the interface by means of seepage from fresh-water canals. The fourth assumption is that the water-loss from the canal consists entirely of seepage losses. Obviously evapo-transpiration losses would have to be included in a more comprehensive analysis.

The first analysis concerns the water-loss from a single canal. The physical condition corresponds to constructing a straight fill in the mud-flat area. The fill is 21.0 meters wide at mean Gulf level. The top of the fill is above high-tide level plus wave height, say, 1.5 meters above mean Gulf level. The fill is one km in length. A fresh-water canal is located on the axis of the fill, and the bottom of the canal is at mean Gulf level. The sides of the fill and the canal are placed on a 1 to 1 slope. These geometric conditions, which are shown in Fig. 30, were obtained from Run S10, Table 3, in the following manner. Equations (8), (34a), (34b) and (28c) shall be utilized.

$$\varphi = -kh \tag{8}$$

$$x' = Nx/Q;\ y' = Ny/Q \tag{34a}$$

$$\psi' = \psi/Q;\ \varphi' = \varphi/Q \tag{34b}$$

$$N = k\left(\frac{\gamma_s}{\gamma_f} - 1\right) \tag{28c}$$

For the problem of a single canal, the reference discharge, Q, is equal to one-half the discharge from the canal, $Q_C/2$. In order to demonstrate the effect of the coefficient of permeability on fresh-water discharge, all other variables shall be kept constant. The ratio of the specific weight of salt water to that of fresh water is taken as 1.025. The length of the land surface between the canal and the shore is assumed to be 10 meters. From the value of $x'_D = -2.937$ in Table 3 and $x_D = -1{,}000$ centimeters, equation (34a) yields

$$Q/N = Q_C/2N = 340.5 \text{ cm} \tag{122a}$$

Utilizing equation (28c) and the assumed value of $\gamma_s/\gamma_f = 1.025$, equation (122a) becomes

$$Q/k = Q_C/2k = 8.51 \text{ cm} \tag{122b}$$

Replacing Q/N in equation (34a) by 340.5 from equation (122a), one obtains

$$x(\text{cm}) = 340.5\ x'; \quad y(\text{cm}) = 340.5\ y' \tag{123}$$

Equation (123) is utilized in converting dimensionless distances in Table 3 to dimensional values. The length of the seepage face is 0.498 x 340.5 = 169.7 cm. The depth of the interface under the shore is -0.988 x 340.5 = 336.5 cm. Half of the width of the canal is 0.142 x 340.5 = 48.4 cm. The piezometric head of the canal can be found from equations (8) and (34b). Substituting the expression for φ from equation (8) into equation (34b), one obtains

$$h_C = -\frac{Q\varphi'_C}{k} \tag{124}$$

By means of equation (122b), equation (124) can be rewritten as

$$h_C(\text{cm}) = -8.51\ \varphi'_C \tag{125}$$

From Table 3, $\varphi'_C = -4$. Hence, equation (125) yields $h_C = 34.04$ cm.

With all other variables being held constant, equation (122b) shows the relationship between fresh-water discharge from the canal Q_C, and the coefficient of permeability, k. Some typical values which illustrate this influence are shown in Table 8.

Table 8. A Typical Variation of Q_C with respect to k. (Problem 1).

Type of Soil	Coefficient of Permeability[a] k, cm/sec	Fresh-Water Discharge Q_C cm^2/sec
Clay	1×10^{-6} and smaller	1.7×10^{-5} and less
Silt	1×10^{-5} — 5×10^{-4}	1.7×10^{-4} — 8.5×10^{-3}
Silty Sand	1×10^{-4} — 2×10^{-3}	1.7×10^{-3} — 3.4×10^{-2}
Fine Sand	1×10^{-3} — 5×10^{-2}	1.7×10^{-2} — 0.85
Coarse Sand	1×10^{-2} — 1	0.17 — 17

[a]From Harr, op. cit., p. 8.

From Table 8, for clay-silt size with k of $1(10^{-5})$ cm/sec, Q is $1.7(10^{-4})$ cm^2/sec per cm of canal or 17 cm^3/sec/km. Thus, the water-loss from seepage is negligible if the fill is constructed of soil of the clay-silt size. On the other hand, if the fill was composed of coarse sand, k of $1(10^{-1})$ cm/sec, the seepage from the canal required to maintain the interface shown in Fig. 30 would be 1.7 cm^3/sec/cm, or $1.7(10^5)$ cm/sec/km, or 0.17 m^3/sec/km. This is an excessive amount of discharge to be wasted in the canal in order to obtain only 1.4 hectares of agricultural land.

This particular example is rather unrealistic inasmuch as the sides of the fill would be subject to severe wave erosion. Nevertheless, the example is indicative that the water-loss from seepage is negligible with the fill constructed of soil dredged from Bangkok harbor, which is of the clay-silt size.

A more realistic reclamation scheme would be to place the dredged materials over a large area to raise the land surface slightly above mean Gulf level. The reclaimed area would be traversed with parallel alternating canals and drains. In this case, only the periphery of the reclaimed area would have to be leveed and revetted against high tide plus storm-waves. Inasmuch as extensive mud flats are exposed at low tide and since the tide varies about one meter, the mud-flat area must be nearly at mean Gulf level. In fact, some areas probably exist where the mud-flat elevation is already higher than mean Gulf level. In such areas the reclamation construction would be limited to construction of a peripheral revetted levee; excavation of canals and drains; land leveling; construction of a fresh-water diversion canal from the river to the reclaimed area; and, construction of gate structures in the peripheral levee to release drainage canal water during low tide.

As a design example, consider the results of run number P21 in Table 7. Again, γ_s/γ_f shall be assumed to be 1.025. Let the depth of the interface under the centerline of the drain be 2.6 meters. With y'_A, equal to -1.30 from Table 7, and replacing Q by $Q_C/2$, equation (34a) gives

$$Q/N = Q_C/2N = 200 \text{ cm} \qquad (126)$$

Replacing the value of Q/N from equation (126) into equation (34a), one obtains

$$x(cm) = 200\ x';\ y(cm) = 200y' \tag{127}$$

Dimensionless linear quantities in Table 7 are made dimensional by means of equation (127). The width of the drain is 2 x 0.088 x 2 = 0.35 meters. The land surface between the canal and the drain is 2 x 1.63 = 3.36 meters. The width of the canal is 4 x 0.118 = 0.47 meters as shown in Fig. 31.

A relationship between Q_C and k can be obtained from equation (126). With $\gamma_s/\gamma_f = 1.025$, equation (28c) yields $N = k/40$. Therefore, from equation (126),

$$Q_C = 10\ k \tag{128}$$

Replacing φ in equation (34b) by the expression from equation (8), with $Q = Q_C/2$, one obtains

$$\varphi_C' = -2kh_C/Q_C$$

from which equation (128) and $\varphi_C' = -4$ result in $h_C = 0.20$ meters.

If the project were rectangular in plan, such as 7.6 km wide by 10 km in length, there would be 1000 canals and drains, each 10 km in length. The total length of canals would be 10,000 km. The total discharge from canals to drains is calculated from equation (128) as follows:

$$Q(\text{total project}) = (10)\ (1)\ (10^{-5})\ (10^{5})\ (10^{2}) = 1000\ cm^3/sec$$

The discharge wasted to drainage by seepage is negligible. The total water requirement of the project would be that required for evapotranspiration.

From the standpoint of water-loss, the negligible discharge is favorable; but the negligible discharge may not be sufficient from the standpoint of leaching the salts from the soil in the reclaimed area. For example, the water seeping from the canal would be in transit about 20 years before reaching the drain under the assumed conditions. With this slow movement, the leaching would also be negligible over a reasonable time period. Since the assumptions are reasonable, the conclusion is that the feasibility of mud-flat reclamation for agriculture depends primarily upon finding a feasible method of initially leaching the soil. The reclaimed area could be inundated during the monsoon season until the fresh-water depth was a meter or one and one half meters above mean Gulf level. In view of the extremely low seepage velocity, the author feels that a mechanical loosening of the soil would also be required. Perhaps a combination of inundation, deep scarifying, and draining would remove sufficient salts to allow some type of vegetation to grow. Vegetation which is salt-tolerant and which transports salts to the stem and leaf system would be desirable, since the vegetable matter could be easily removed from the reclamation area. From the above discussion, it is apparent that the feasibility of such a project cannot be determined from the equilibrium seepage condition alone, but that field experiments will be necessary in order to find a feasible scheme of leaching.

As another example, the salt-intruded land (with salt water less than 2 meters from the land surface) is assumed to extend one kilometer inland from the high tide level. During and following the monsoon season, which ends approximately in October in the region around the head of the Gulf of Siam[54], fresh water covers the entire land surface down to the Gulf and the salt water is depressed to a greater depth under the land. By early December the surface water has drained from the flood plain. From December until the start of the monsoon season, in late May, the interface between the salt water and fresh water moves upward. Thus, the critical period is just prior to the monsoon season, when the interface has reached a maximum elevation. If the salt water moves upward into the root zone of the vegetation, undoubtedly, some salt water will remain in the soil pores during the subsequent lowering of the interface. The portion of the land where salt water has reached the root zone of the vegetation would be classed as salt-intruded even after the interface has been lowered. Salt water remaining in the soil pores would damage the crops or retard crop growth. Therefore, if the interface can be maintained below the root zone, a portion of land which would otherwise be salt-intruded can be preserved for cultivation. One method to maintain the interface below the root zone is to construct an artificial fresh-water canal parallel to the shore line, similar to the canal paralleling the road from Bangkok to Cholburi (Chapter I).

In order to apply the numerical solutions for the design of a parallel canal, the initial location of the interface and ground-water table must be determined. The initial conditions, Fig. 3(a), can be approximated by a confined aquifer as shown in Fig. 3(b). For this case,

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54. The Royal Irrigation Department, The Greater Chao Phya Project, Ministry of Agriculture, Bangkok, Thailand, 1957, p. 6.

equation (83) defines the interface geometry in terms of the physical variables,

$$x' = \frac{1}{2}\left(1 - y'^2\right) \tag{83}$$

The physical variables are as follows: (1) The land slope is $1.6(10^{-4})$, (2) the interface is 1.84 m below mean Gulf level at a distance of one km from the Gulf, (3) the soil mass is assumed to be anisotropic in which k_x is $4(10^{-5})$ cm/sec and k_y is $1(10^{-5})$ cm/sec, (4) the specific-weight ratio, γ_s/γ, is 1.025 and (5) the parallel canal is to be located where the interface is one meter below mean Gulf level.

This example differs from the previous two in that the soil mass is taken to be anisotropic. Sedimentary deposits containing clay are generally anisotropic. By means of a coordinate transformation[55], seepage flow through homogeneous anistropic media can be transformed into flows through homogeneous isotropic media. The transformation is accomplished by multiplying the horizontal distances by $\sqrt{k_y/k_x}$ and by using $k = \sqrt{k_x k_y}$ as the coefficient of permeability in the transformed plane.

Using the above transformation and equation (83), the discharge through the aquifer, Q_I, and the location of the canal can be determined as follows

$$x = \frac{1}{2}\sqrt{\frac{k_x}{k_y}}\left[\frac{Q_I}{\sqrt{k_x k_y}(\gamma_s/\gamma_f - 1)} - \frac{\sqrt{k_x k_y}(\gamma_s/\gamma_f - 1)y^2}{Q_I}\right] \tag{129}$$

55. Harr, op. cit., pp. 29-31.

Q_I is determined from equation (129) where x is -1000m and y is -1.84m. From this calculation, Q_I is 1.69(10^{-7}) cm^3/sec/cm. The location of the canal is also determined from equation (129), where y is -1.0m and Q_I is 1.69(10^{-7}) cm^3/sec/cm. From this calculation, x = -295m, or the canal is located about 300m from the mean Gulf level shoreline.

The level of the existing ground-water table is approximated by the piezometric head of the confining surface of the aquifer, Fig. 3(b). Replacing ψ' in equation (60), which is the solution of the problem of Fig. 3(b), by zero, one obtains

$$h = \left[-4(\gamma_s/\gamma_f - 1)\ x\sqrt{\frac{k_y}{k_x}}\ Q_I/\sqrt{k_x k_y}\ \right]^{1/2} \tag{130}$$

Equation (130) is employed in evaluating the level of the ground-water table. At x = 1 km, the level of ground-water table is 6.59 cm. At x = 295 m, the level of the ground-water table is 3.52 cm, as shown in Fig. 32.

Since the purpose of constructing an artificial fresh-water canal at x = 295 m is to depress the existing interface to 2 m below the ground surface, the solution of either Problem 2A or Problem 2B should be utilized.

$$Q_I/N = \frac{1.6925\times10^{-7}}{2\times10^{-5}\times0.025} = 0.3385$$

According to equation (34a), y' at the centerline of the proposed canal must be

$$y'_{℄} = -2/0.3385 = -5.9$$

From Tables 5 and 6, the minimum value of the numerically obtained solutions for y_L is only -4.9. Obviously the range of the solutions shown in Table 5 and 6 is insufficient for this physical problem.

In the absence of solutions of Problem 2 with $y'_{℄} \leq -5.9$, the solution for Problem 1 as shown in Table 3 shall be employed. The head in the single canal shall be maintained at the same level as that of the approaching ground-water table, that is, 3.5 cm. The discharge from the fresh-water canal shall be made large enough to depress the salt water to 2 m below the mean Gulf level. The interface of fresh water and salt water resulting from a single canal is shown by the dotted line in Fig. 32. Without the presence of natural ground water, the single canal can depress the salt water down to 2 m below the high-tide level. Therefore, with natural ground water present, the combined fresh water would depress the salt water more than 2 m below mean Gulf level.

From the solution of run number S7 in Table 3, $\varphi'_C = -4$, $\varphi'_B = y'_B = -3.4$, $x'_D = -5.652$, $x'_D - x'_C = 1.566$, and $x'_A = 0.498$. From equation (34a),

$$Q/N = y/y' = -200/-3.4 = 58.8$$

The value of Q/N = 58.8 is employed in converting the dimensionless quantities to dimensional quantities. The width of the canal is found to be 1.84 m. Other dimensions are shown in Fig. 32. The fresh-water discharge from the canal is evaluated by using equations (34b) and (8).

$$Q_C = -2h_C\sqrt{k_x k_y}/\varphi'_C = 2 \times 3.52 \times 2 \times 10^{-5}/4 = 3.52 \times 10^{-5}\ \text{cm}^3/\text{sec/cm}$$

If the canal were 100 km long, the total discharge would be $3.52 \times 10^{-5} \times 10^{-6} \times 10^{7} = 3.52 \times 10^{-4}$ m^3/sec/100 km. The linear distance of land surface gained is about 704 m, as shown in Fig. 32. The total area of land gained is therefore 7040 hectares.

Again, the amount of discharge wasted by seepage is negligible. As mentioned previously, negligible discharge is good from the standpoint of water-loss but bad in the standpoint of leaching. However, existing vegetation on the Gulf shore would serve as a mechanism for leaching after the canal was installed.

CHAPTER VI

CONCLUSIONS

The effectiveness of fresh-water canals for alleviation of salt-water intrusion was investigated. The lowering of the interface between the salt-water-saturated soil and the fresh-water-saturated soil was determined by numerical solution of the applicable mathematical equations.

The following conclusions can be grouped into two categories. Conclusions 1-4, inclusive, pertain to the method of solution. Conclusions 5-7, inclusive, pertain to the results of the solution.

1. Seepage-flow solutions were obtained in the complex potential plane from the equation, $\nabla^2 y = 0$. Boundary conditions along free surfaces or along fluid interfaces are readily formulated. The physical plane was reconstructed by integration of the inverse Cauchy-Riemann equations. The author could find no reference in which this technique had been used in the analysis of seepage flow.

2. The problems solved <u>in this study</u> cannot be solved analytically at the present time. Barriers to analytical solution are the boundary conditions. In the physical plane the location of the interface is unknown and the interface is curvilinear. In the complex potential plane, along one boundary the boundary condition is mixed. In Problems 1 and 3 the boundary condition is a derivative function along a part of one boundary and is a linear algebraic function along the remainder of that boundary. In Problem 2 the boundary condition is a linear algebraic

function along part of the boundary and must be mated to an adjoining region along the remainder of that boundary.

3. The over-relaxation factor derived by Young was found empirically to reduce the number of iterations.

4. The computer programs are general for the solution of Laplace's equation. One program is for the solution within a rectangular domain. The second program is for the solution in a domain consisting of two rectangular regions which are joined together on one side. The third is for the solution in a domain consisting of two rectangular regions which are joined over a part of one side. The boundary conditions can be mixed on any boundary of the rectangular region by instructions in the block of the program entitled "Boundary Conditions".

5. The concept to employ an artificial fresh-water canal in the suppression of salt water and to employ parallel canals with intermediate drains for reclamation of mud flats is promising and should be investigated further.

6. If the soil is in the clay- or silt-size range, the amount of water-loss due to seepage is negligible.

7. The feasibility of mud-flat reclamation cannot be determined from the equilibrium seepage condition alone but field experiments will be necessary in order to find a feasible scheme of leaching.

APPENDICES

APPENDIX A

ILLUSTRATIONS

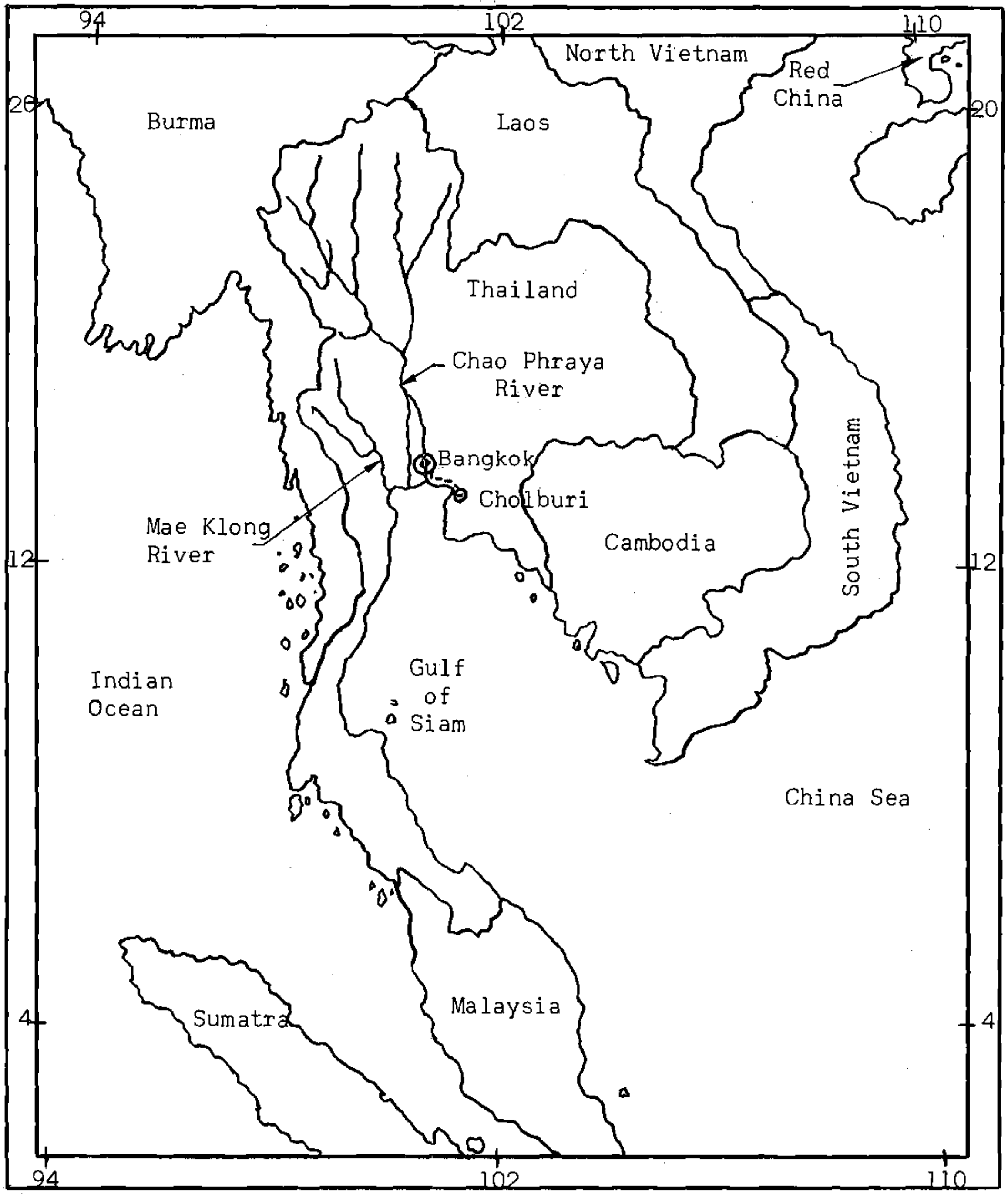
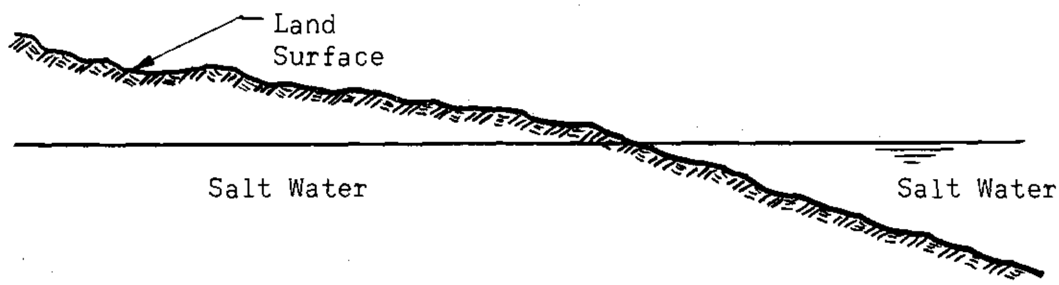
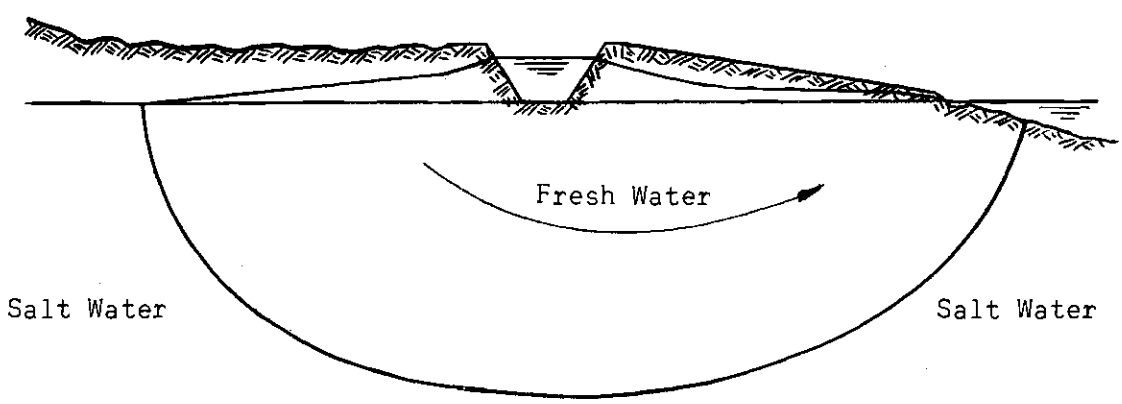


Figure 1. Map of Thailand.

Taken from B. Asanachinda and M. Dhamkrongartama, Geographical Maps of of Thailand (Thai), Kuruspa, Bangkok, 1957, p. viii.

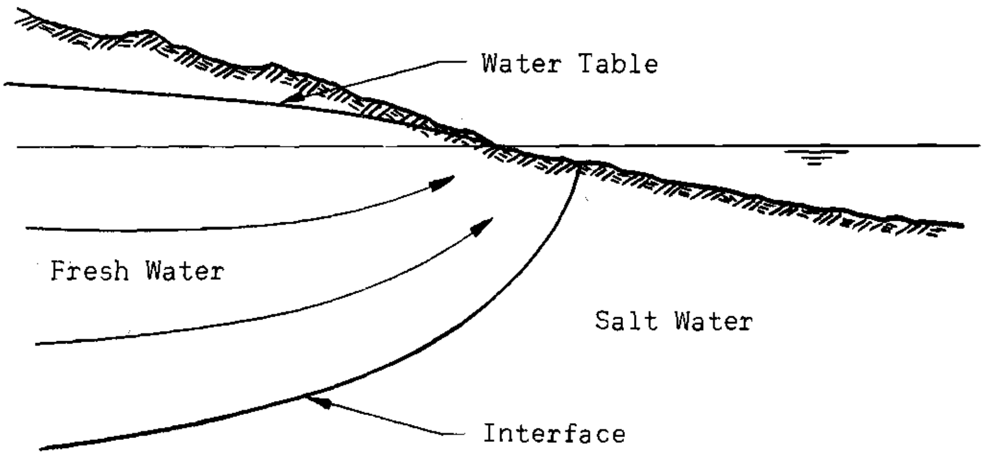


a.) Aquifer Saturated With Salt Water.

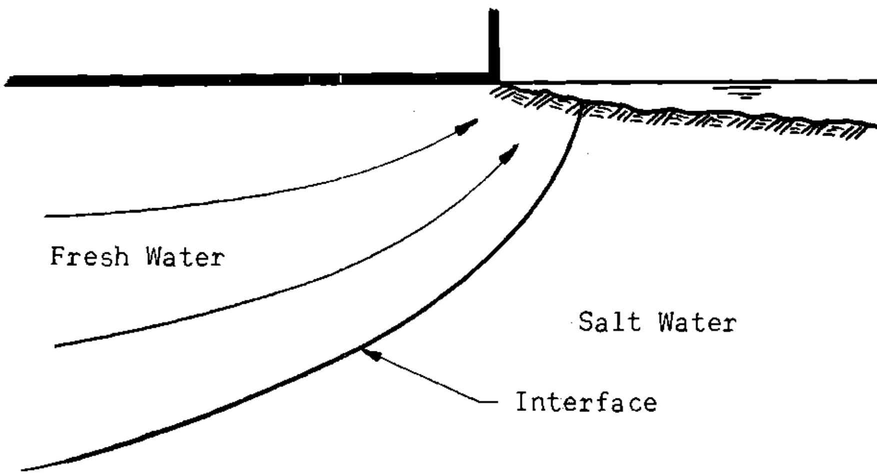


b.) A Fresh-Water Canal Near the Shore.

Figure 2. Salt-Water Intrusion Without Natural Ground Water.



a.) Unconfined Aquifer.



b.) Confined Aquifer.

Figure 3. Natural Ground-Water Flow in Coastal Aquifer.

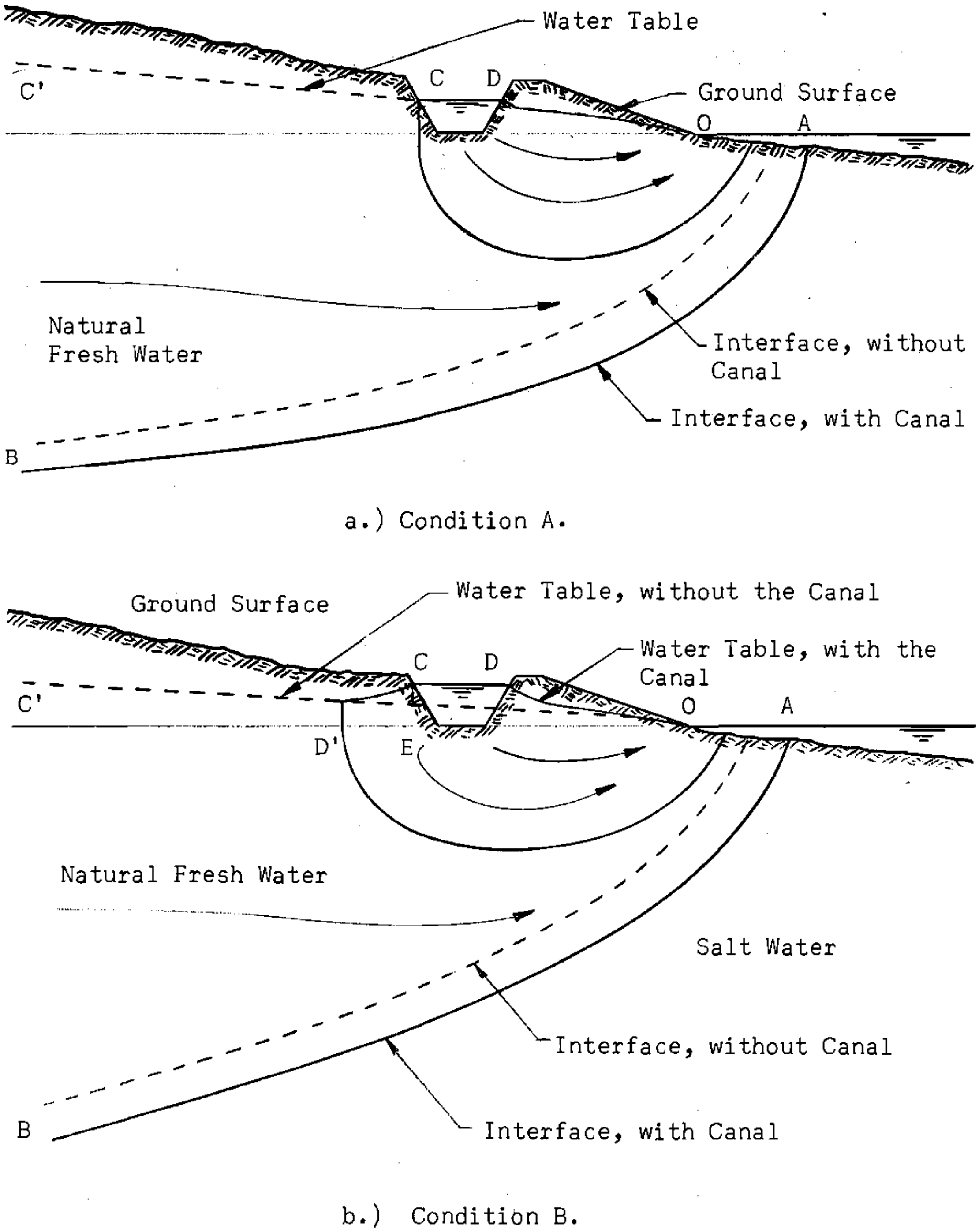
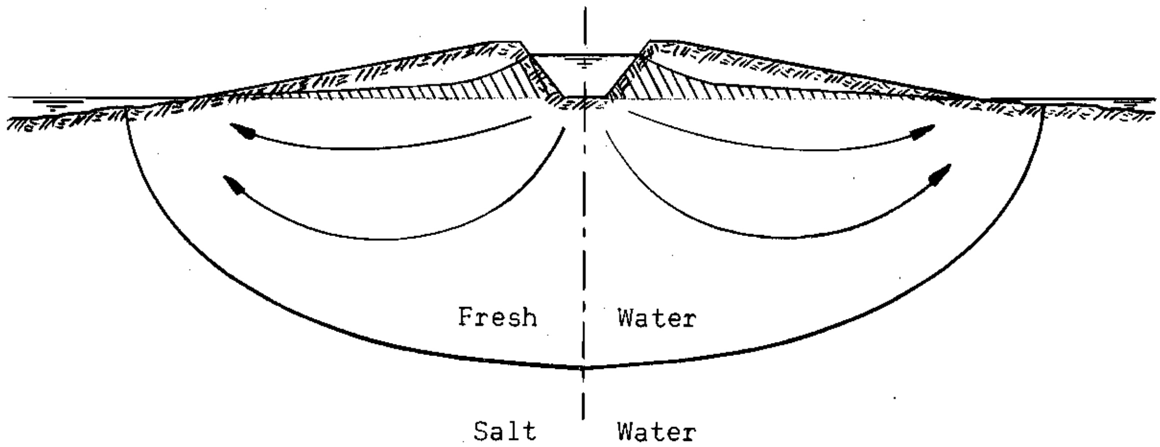
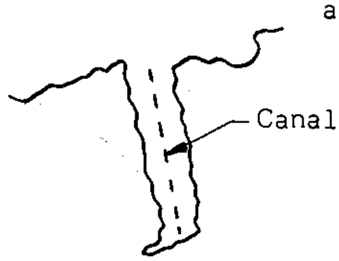


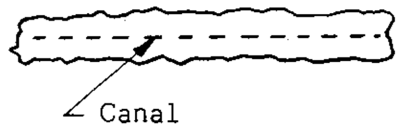
Figure 4. A Fresh-Water Canal with Natural Ground-Water Flow.



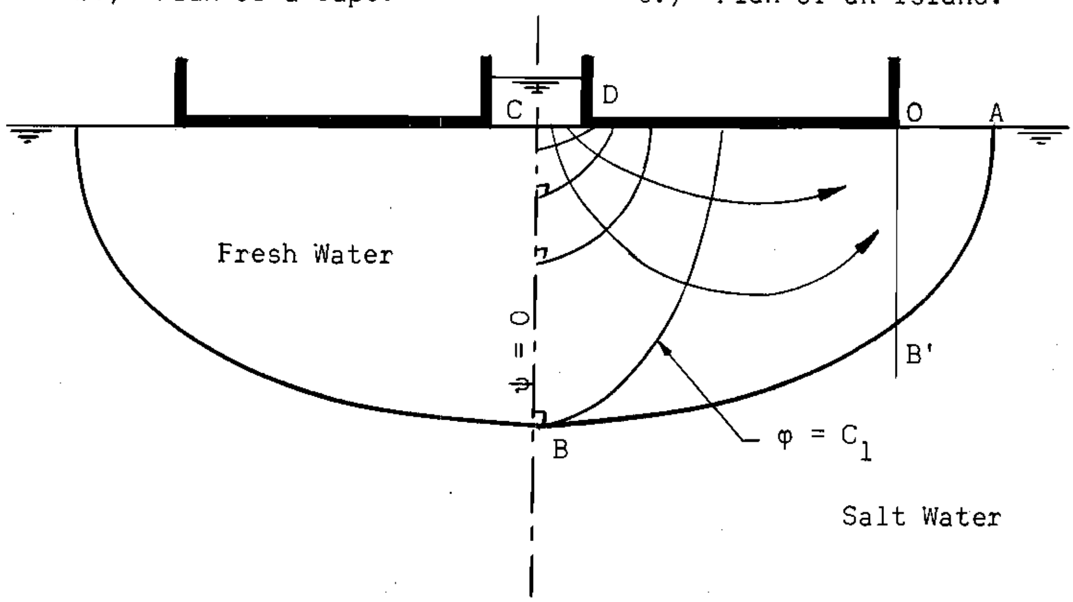
a.) Cross-Section.



b.) Plan of a Cape.

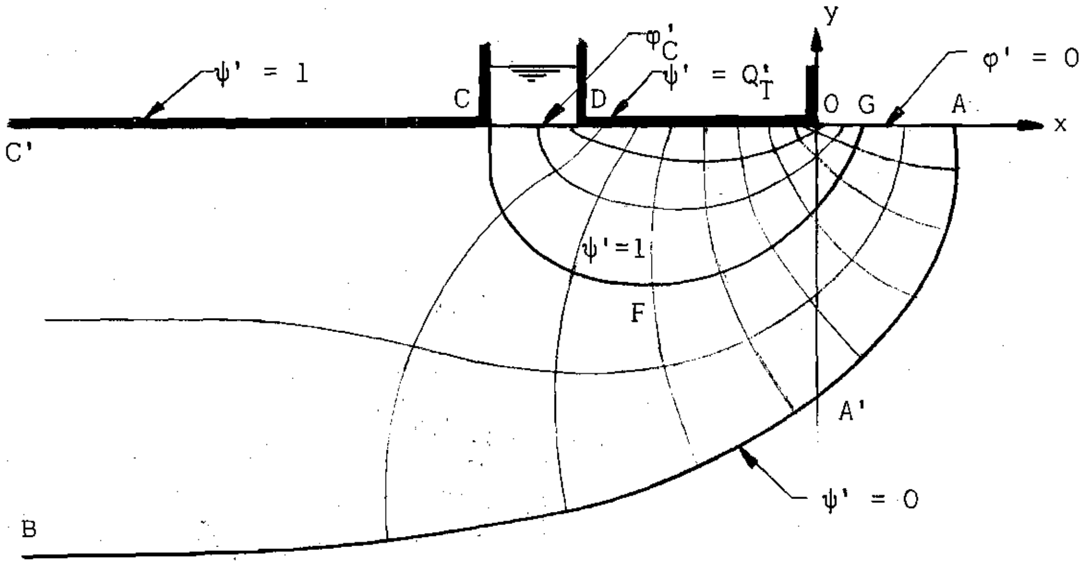


c.) Plan of an Island.

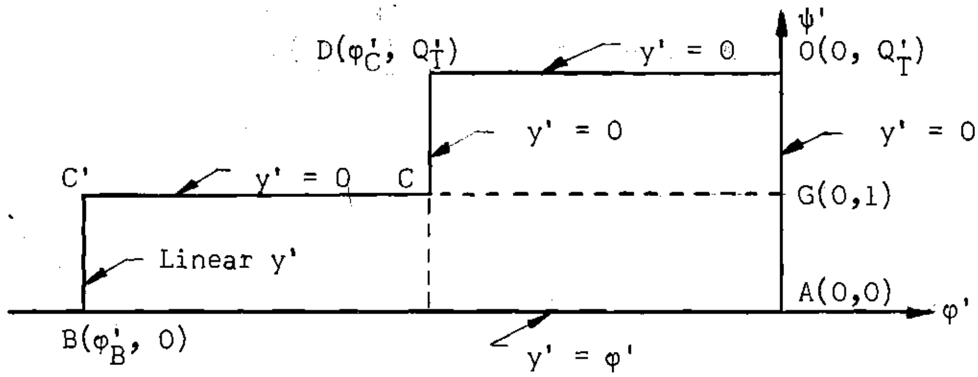


d.) Simplified Single Canal.

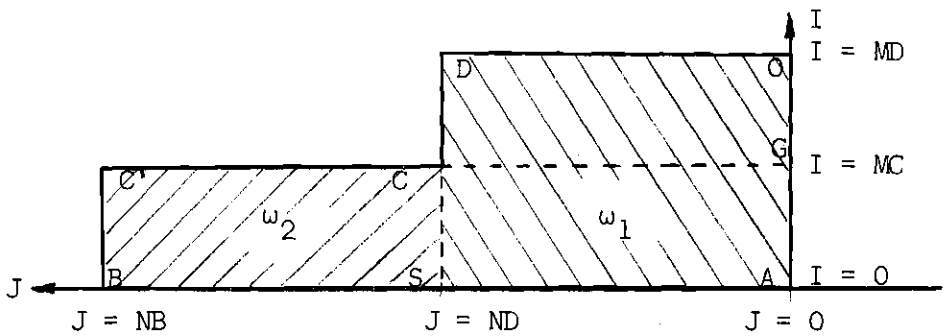
Figure 5. Problem 1 — Single Canal.



a.) Simplified Physical Plane, $z = x + iy$

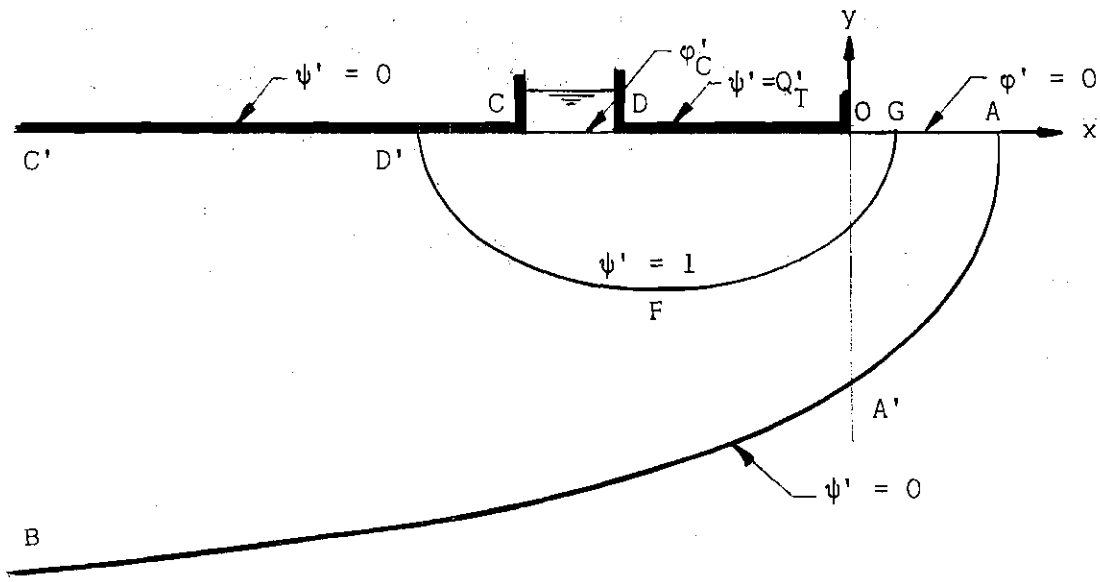


b.) Dimensionless Complex Potential Plane, $w' = \varphi' + i\psi'$.

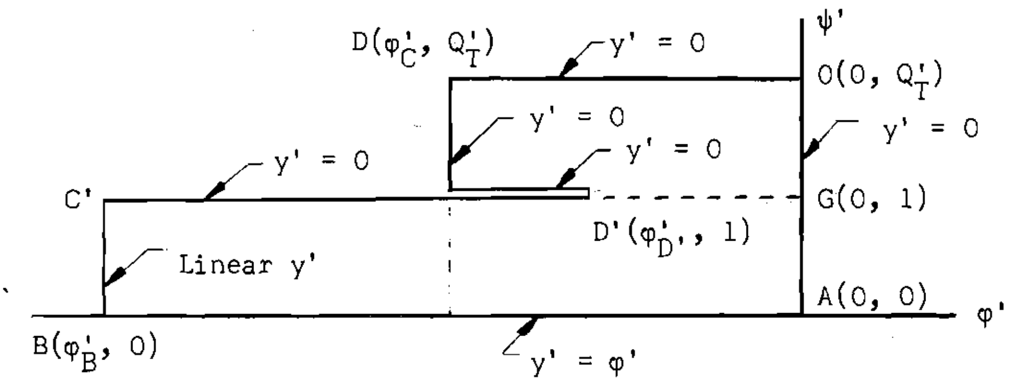


c.) I-J Notation.

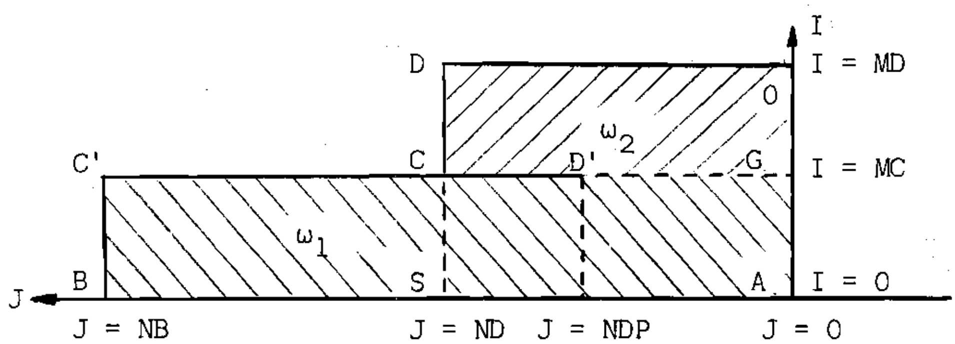
Figure 6. Condition A of Problem 2.



a.) Simplified Physical Plane, $z = x + iy$



b.) Dimensionless Complex Potential Plane, $w' = \varphi' + i\psi'$



c.) I - J Notation

Figure 7. Condition B of Problem 2.

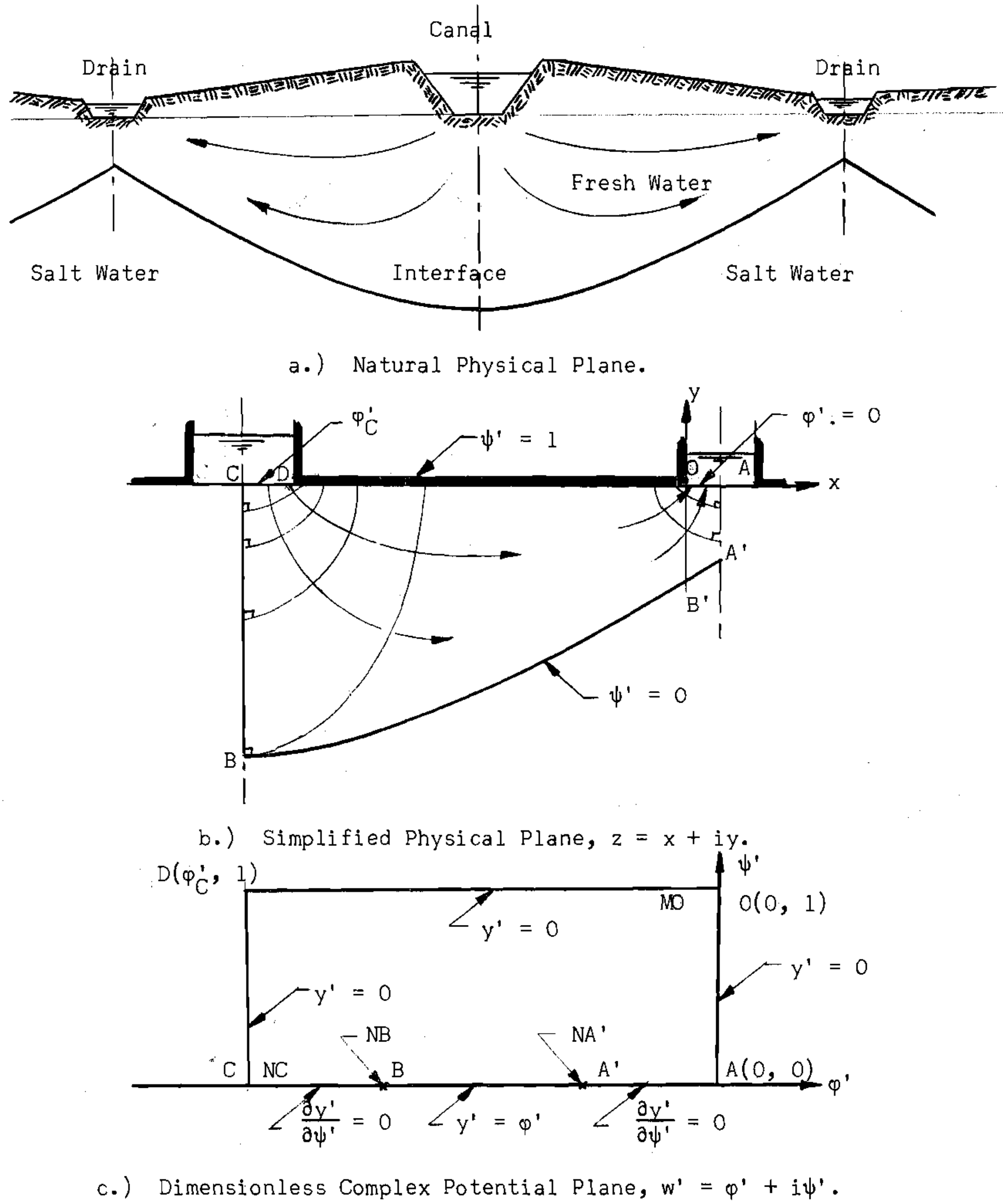


Figure 8. Problem 3 — Parallel Canals with Intermediate Drains.

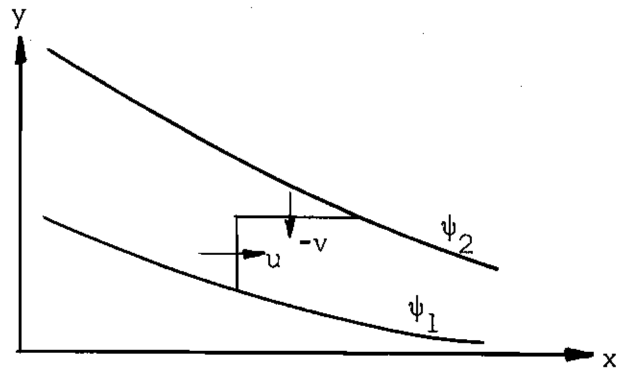


Figure 9. Flow between Two Streamlines.

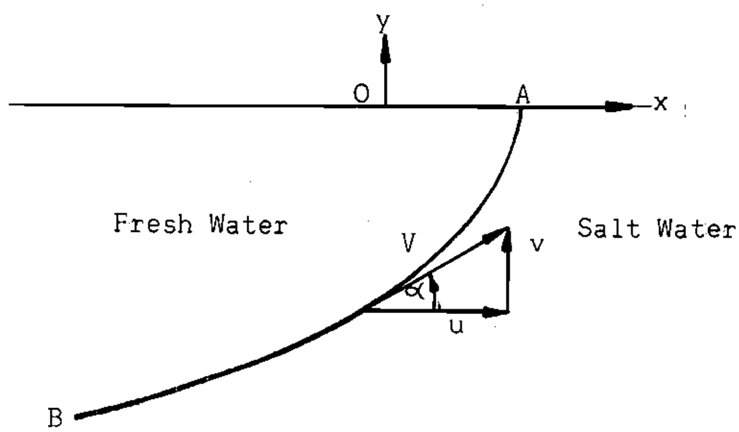


Figure 10. Interface.

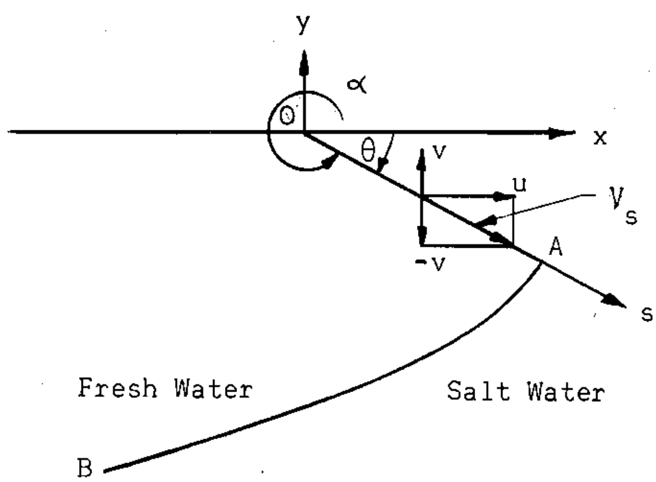
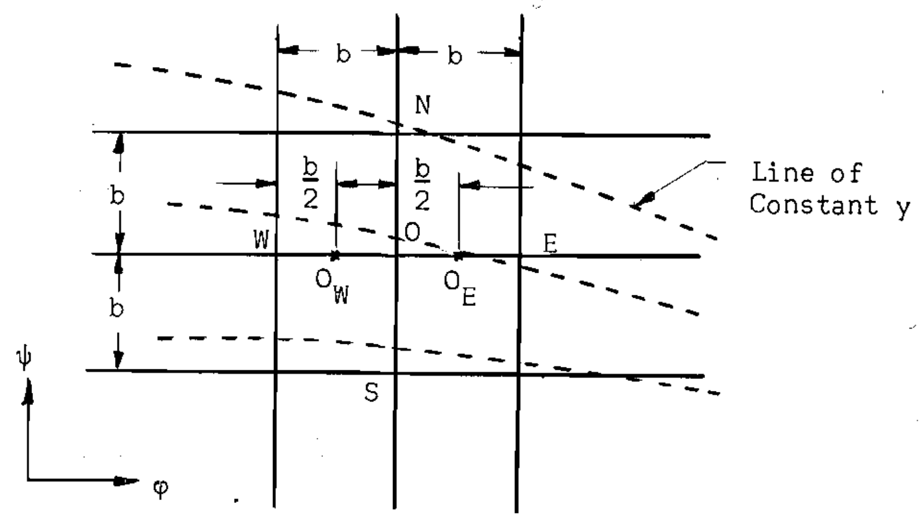
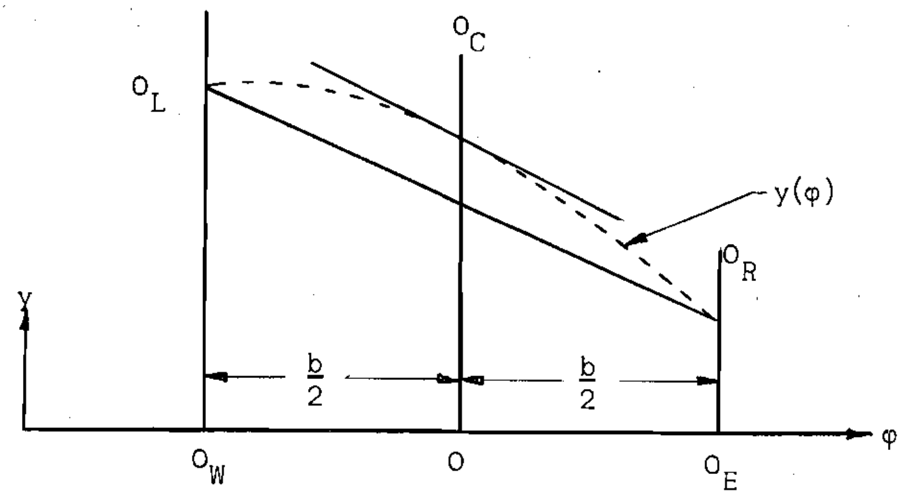


Figure 11. Seepage Face.



a.) Square Network with E-N-W-S Notation.



b.) First Central Difference at O.

Figure 12. Square Net with Linear Approximation.

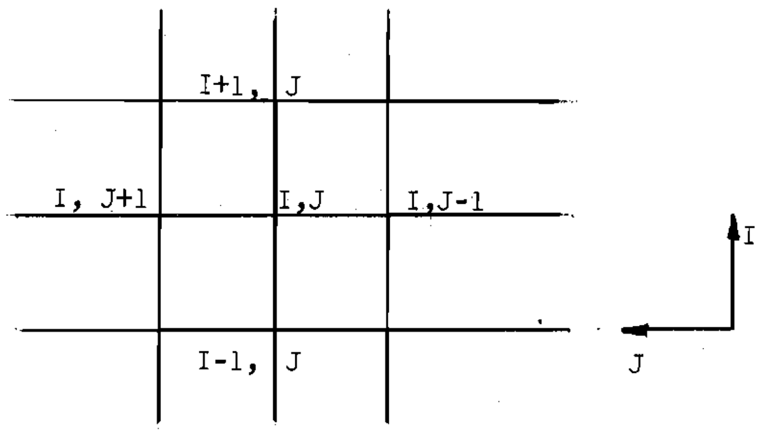


Figure 13. I - J Notation on Square Net.

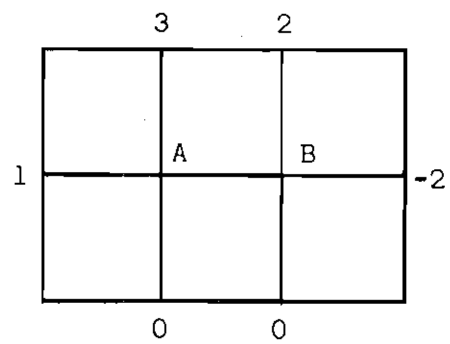


Figure 14. A Simple Case of Laplace's Equation in a Rectangular Domain.

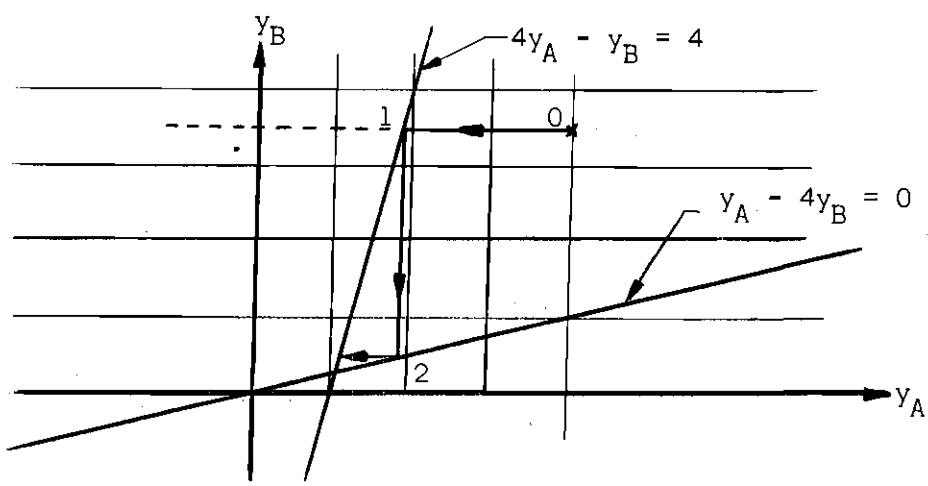
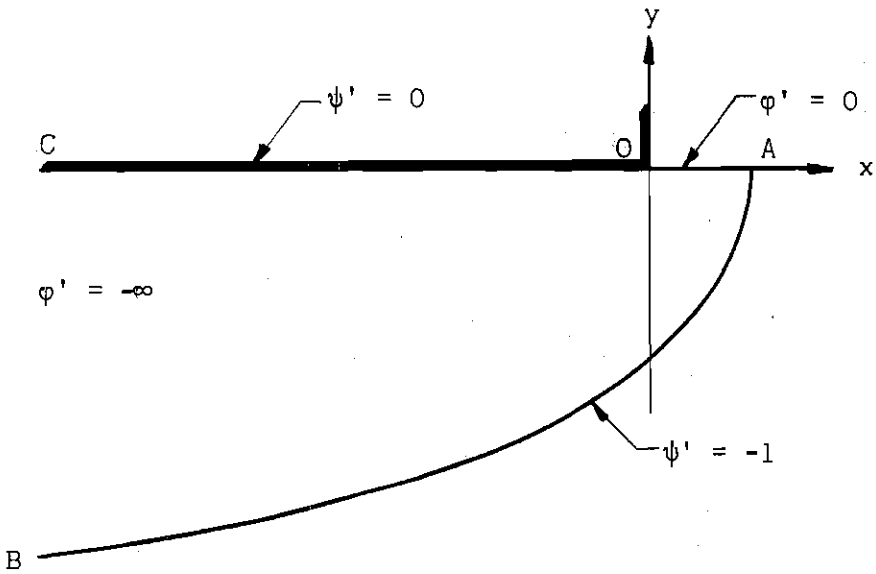
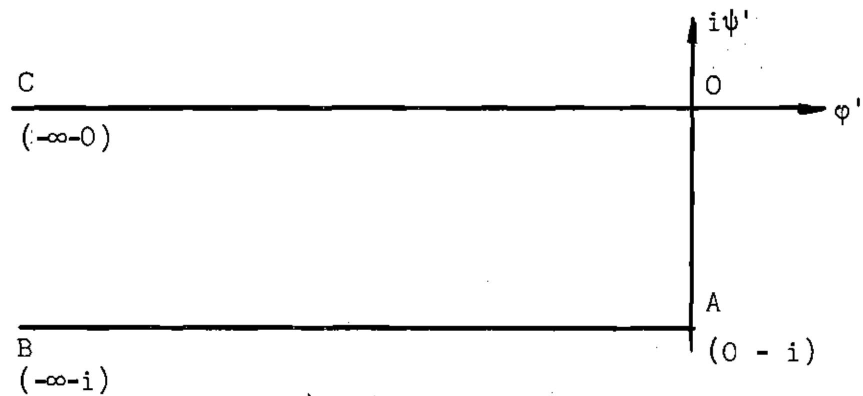


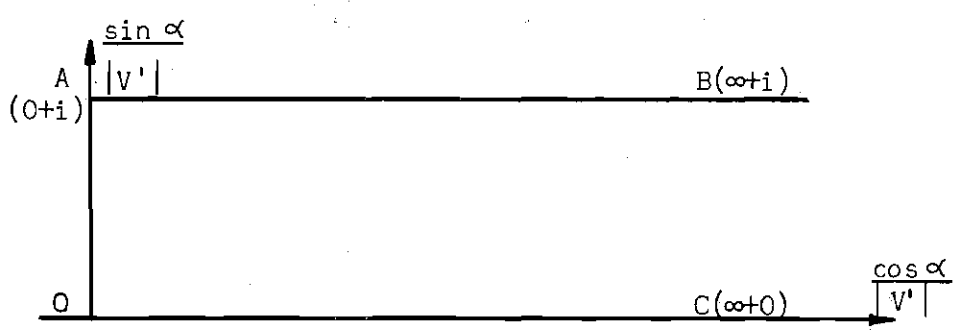
Figure 15. Two Equations with Two Unknowns.



a.) Nondimensionalized Physical Plane.



b.) w'-plane.



c.) Ω- or ζ'-plane.

Figure 16. Natural Ground-Water Flow From Infinity.

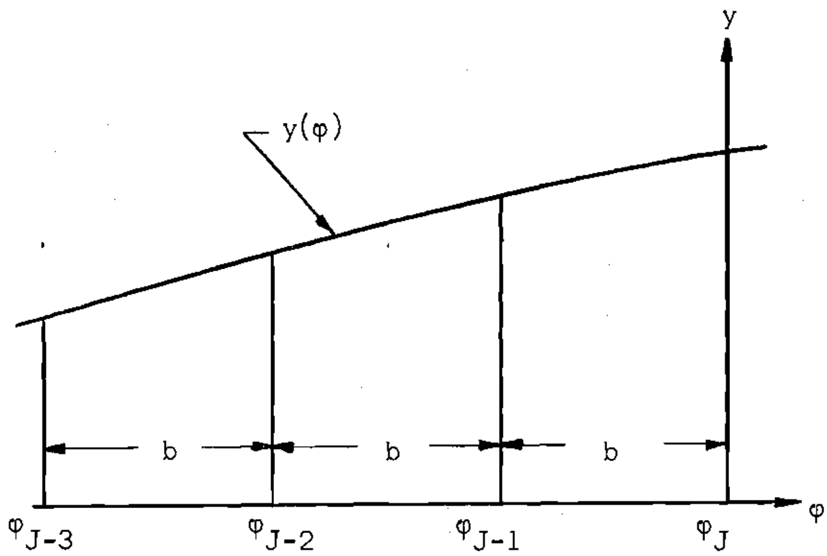


Figure 17. Fitting a Line to Four Points.

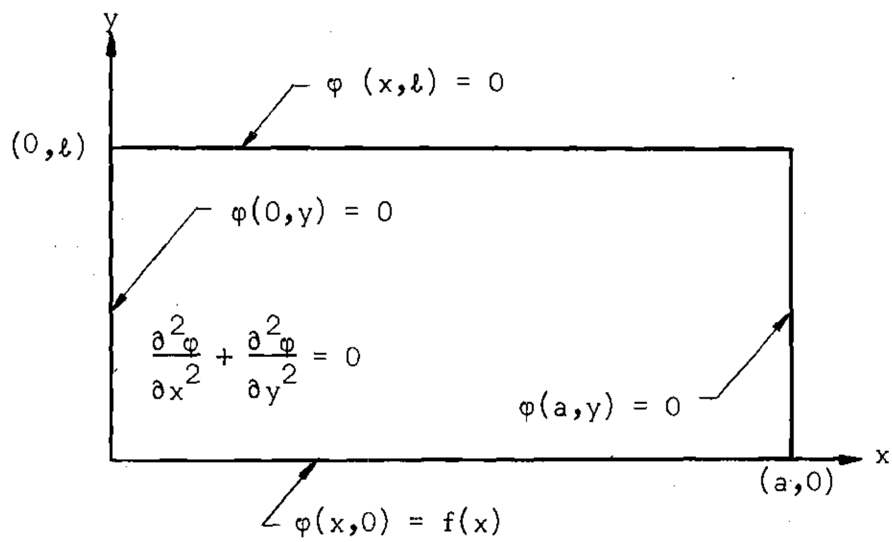
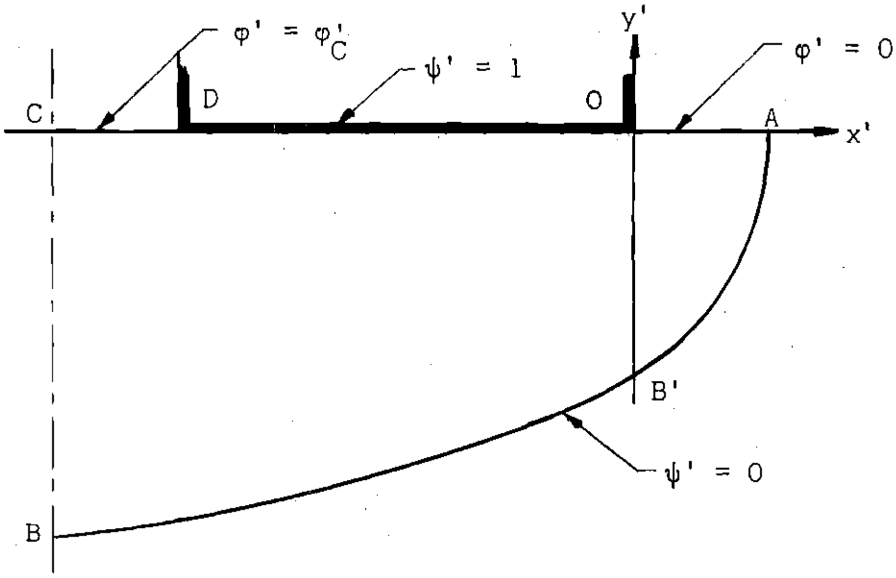
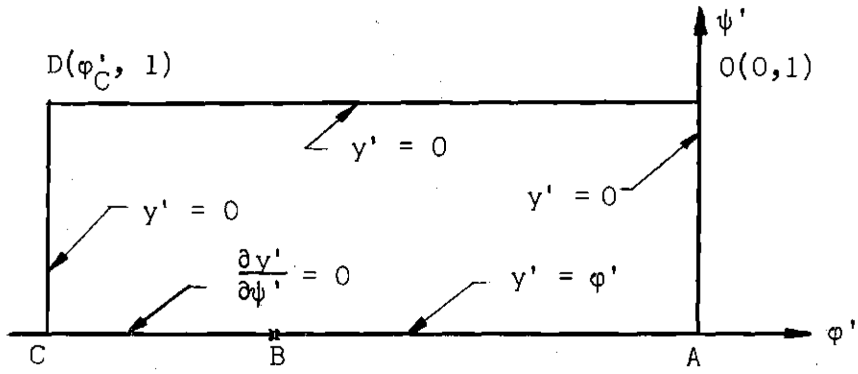


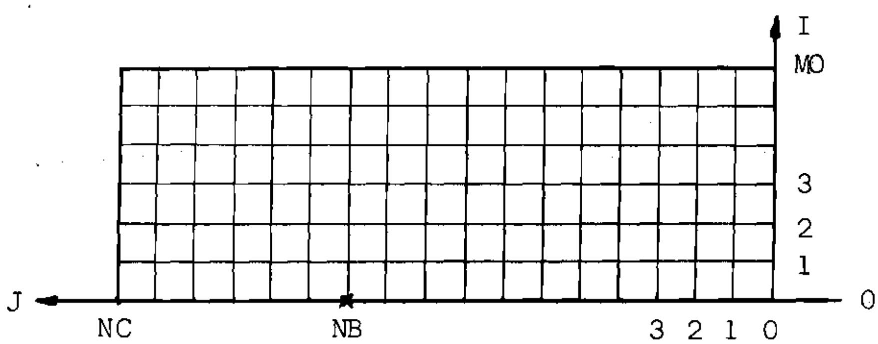
Figure 18. A Typical Rectangle with Homogeneous Boundary Conditions.



a.) Dimensionless Physical Plane, $z' = x' + iy'$.



b.) w'-plane.



c.) w'-plane with Network.

Figure 19. Single Canal.

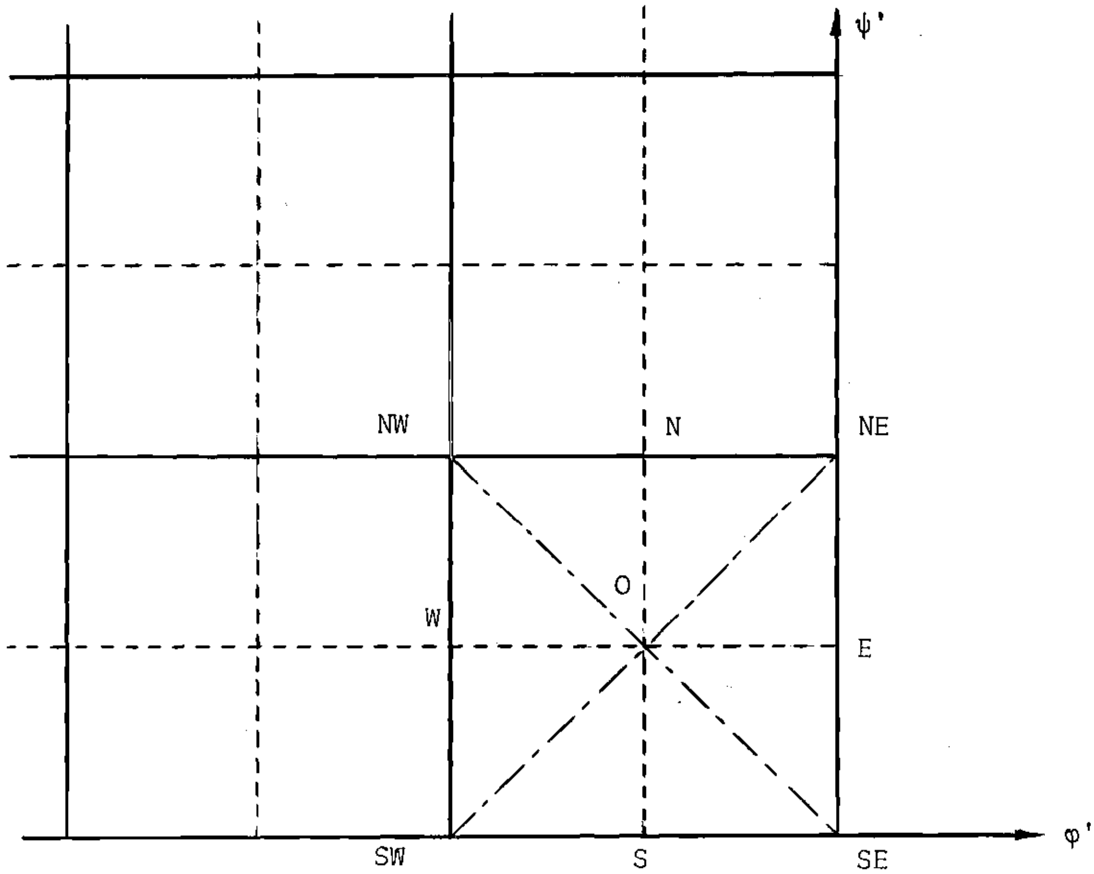


Figure 20. Refinement.

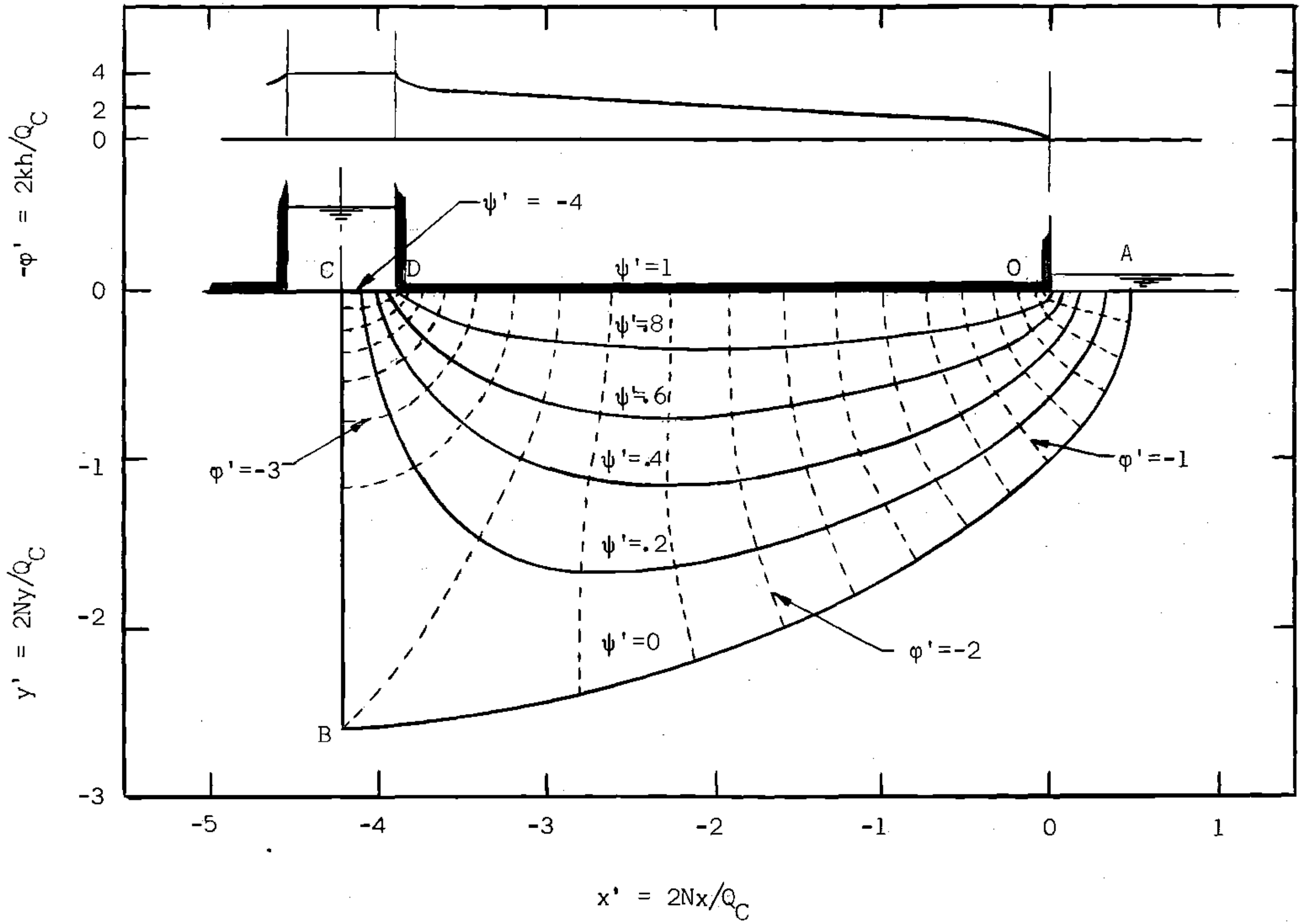


Figure 21. A Typical Flow Net (Problem 1, $\varphi_C' = -4$, $\varphi_B' = -2.6$).

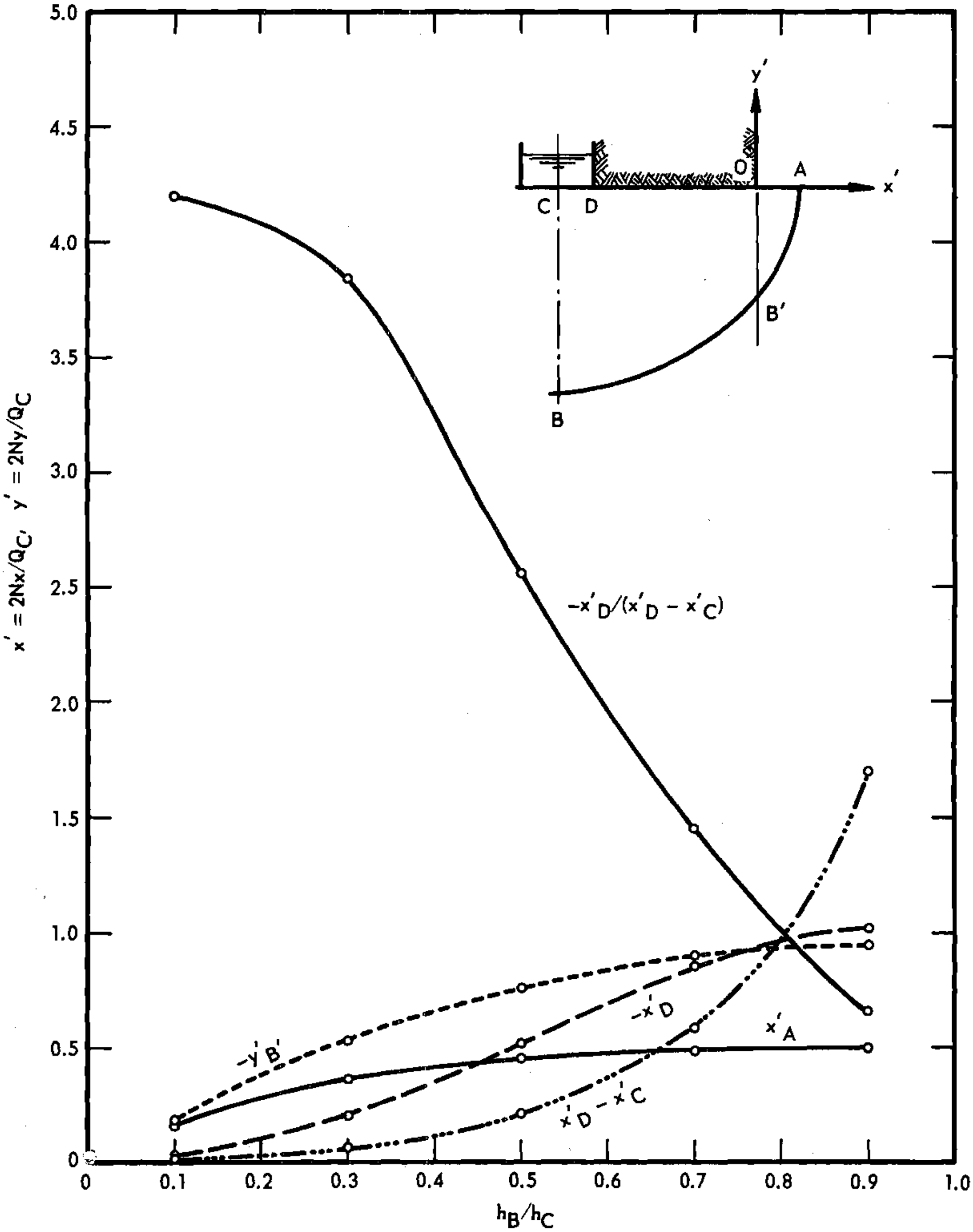


Figure 22(a). Effect of h_B/h_C Variation on Key Dimensions (Problem 1, $\emptyset'_C = -4.0$).

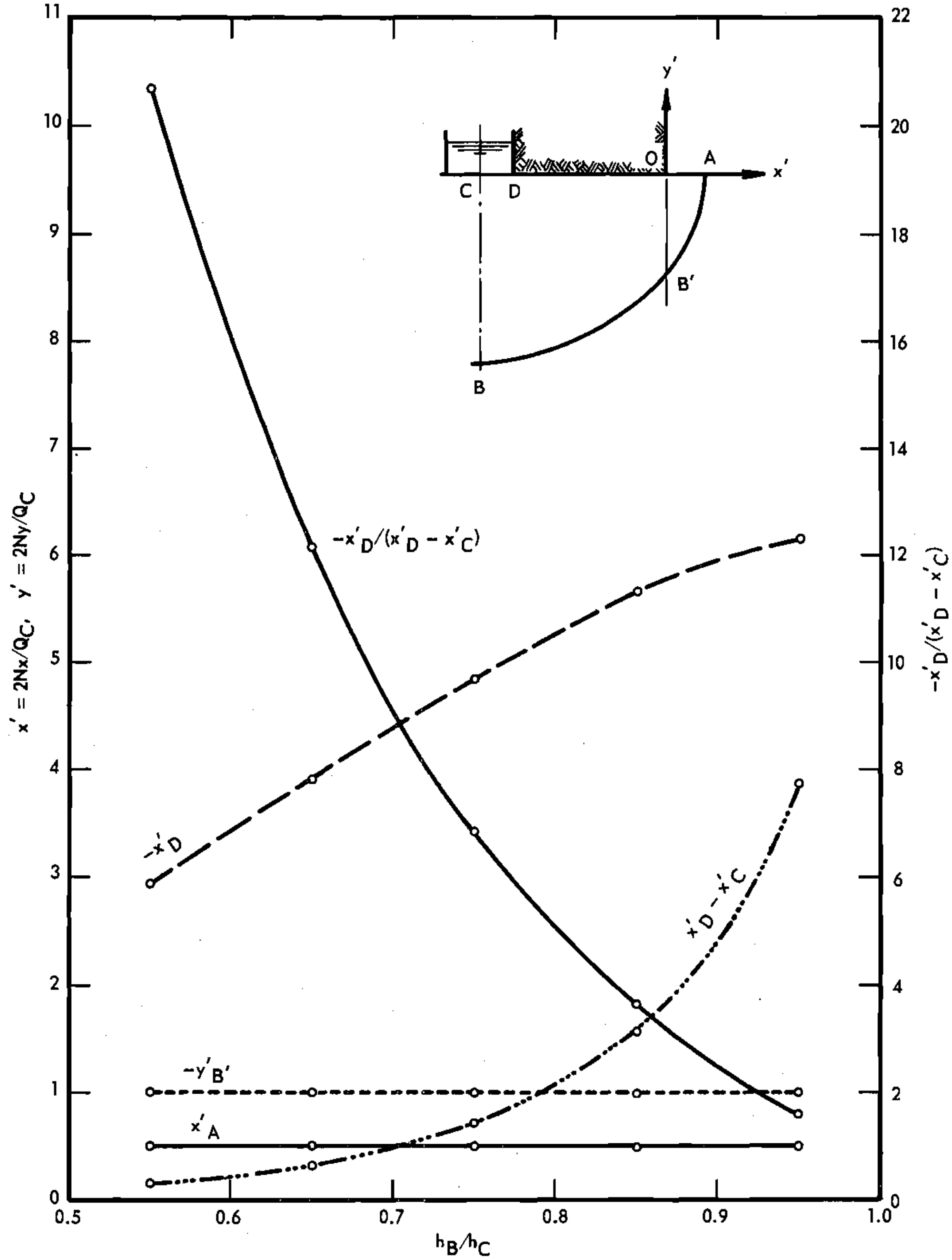


Figure 22(b). Effect of h_B/h_C Variation on Key Dimensions (Problem 1, $\phi'_C = -2.0$).

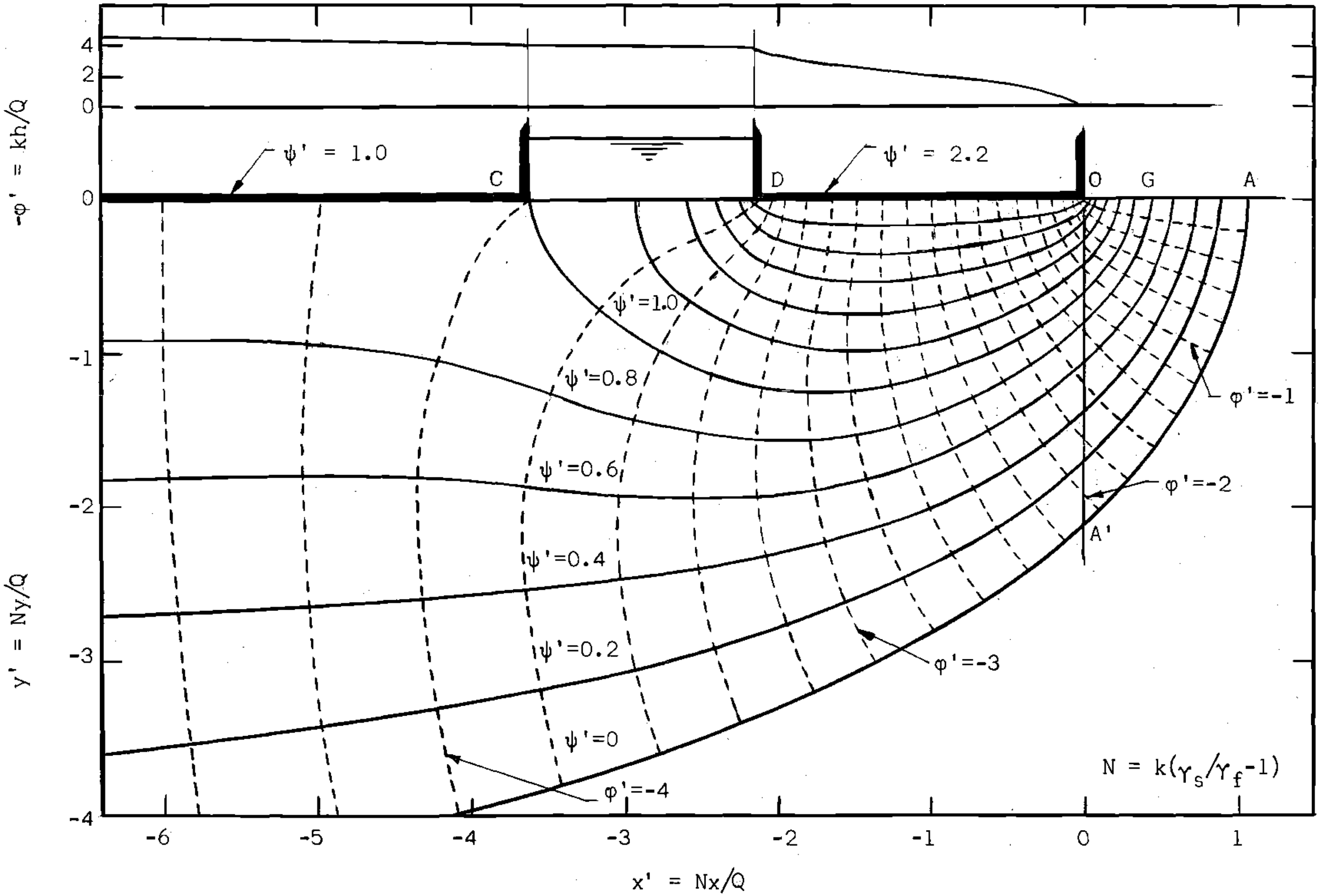


Figure 23. A Typical Flow Net (Problem 2A, $\varphi_C' = -4.0$, $Q_T' = 2.2$).

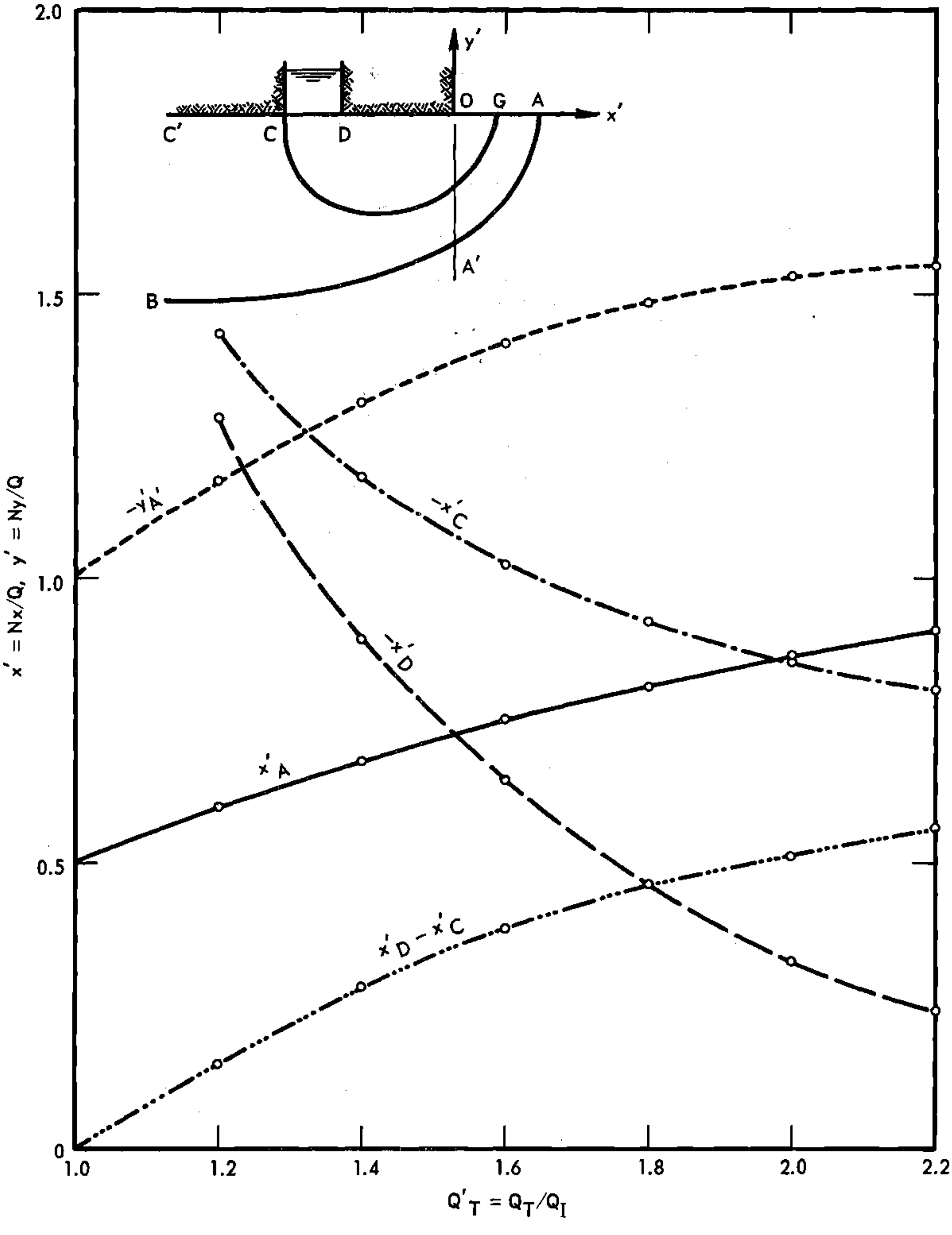


Figure 24(a). Effect of Q'_T Variation on Key Dimensions (Problem 2A, $\phi'_C = -2.0$).

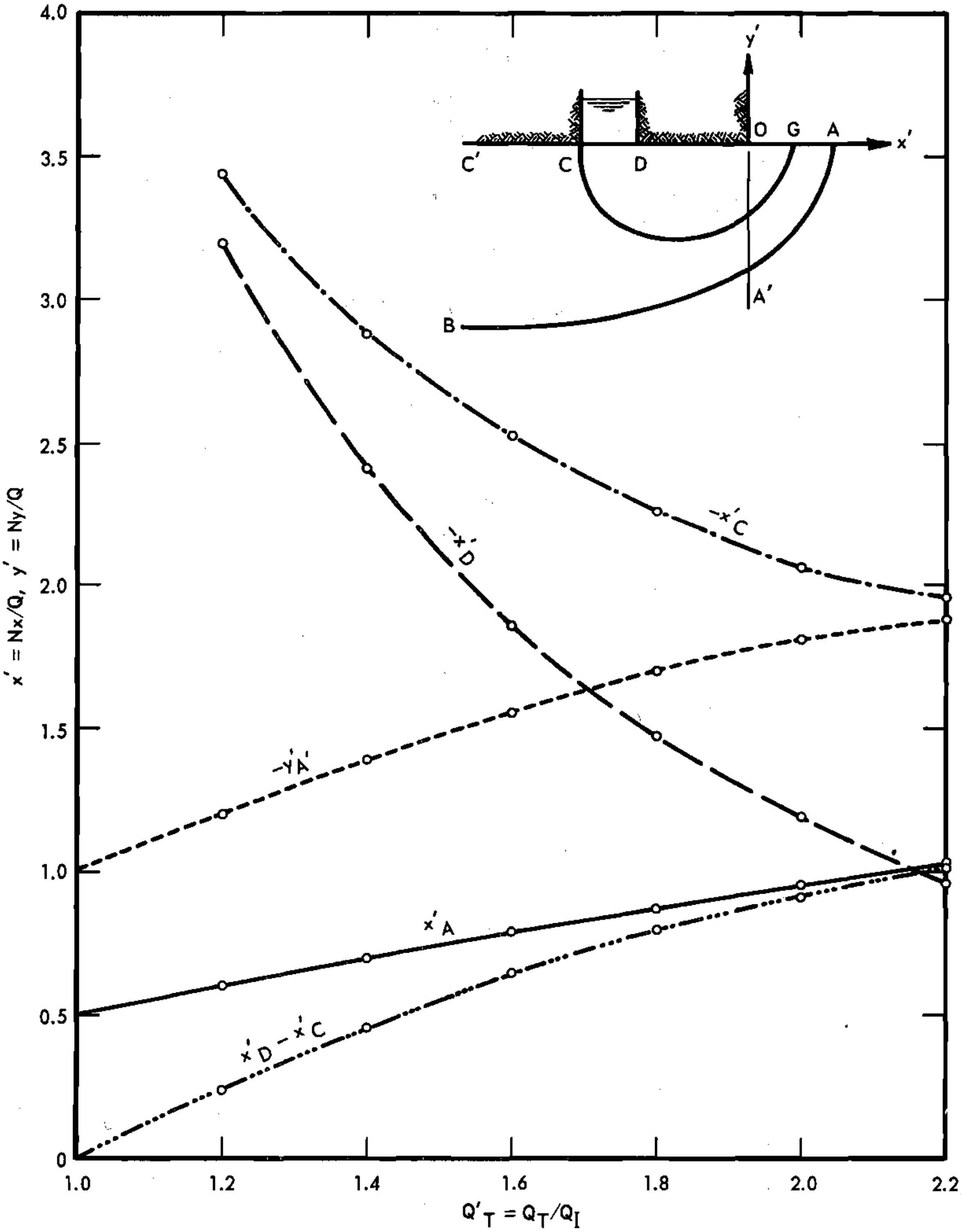


Figure 24(b). Effect of Q'_T Variation on Key Dimensions (Problem 2A, $\emptyset'_C = -3.0$).

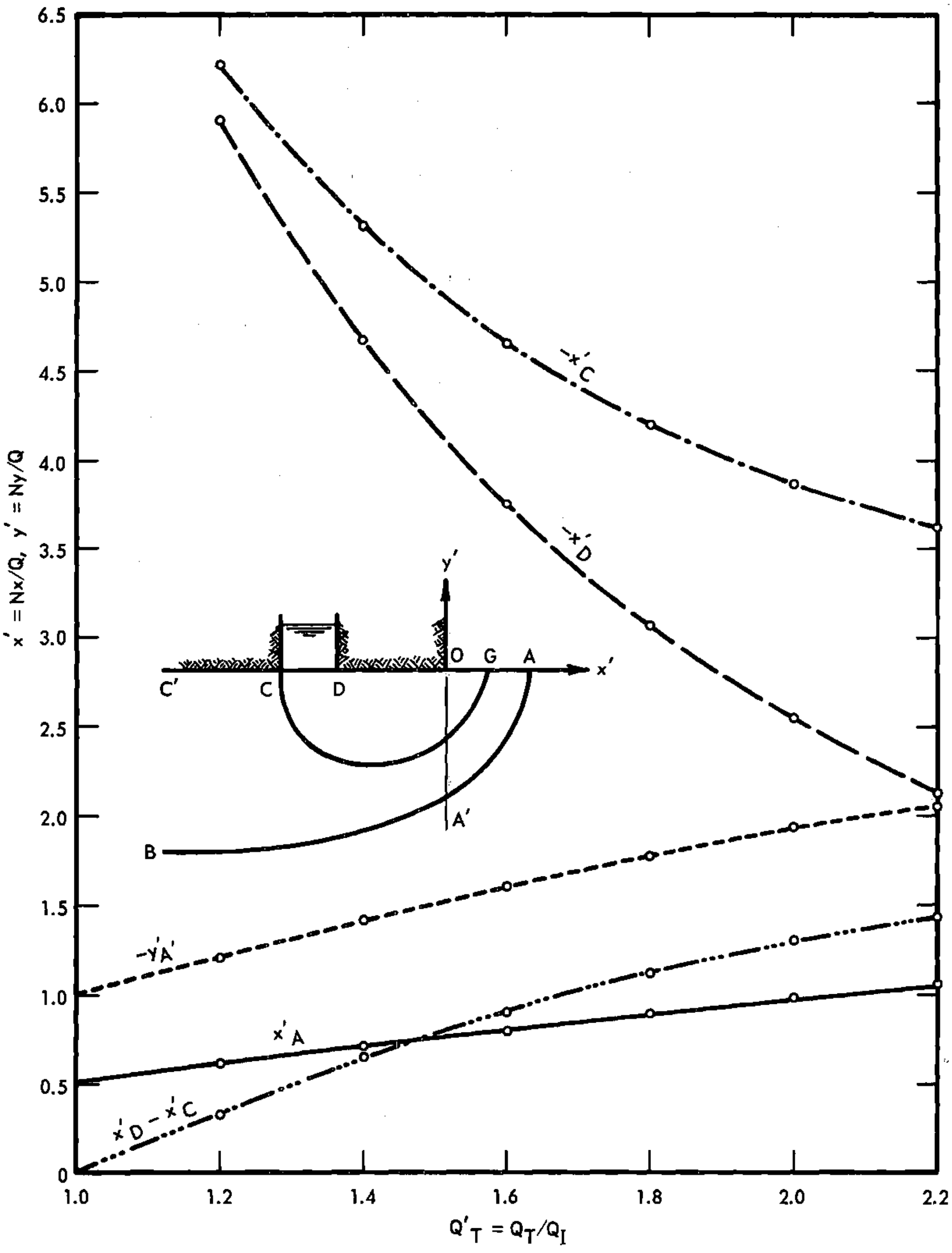


Figure 24(c). Effect of Q'_T Variation on Key Dimensions (Problem 2A, $\phi'_C = -4.0$).

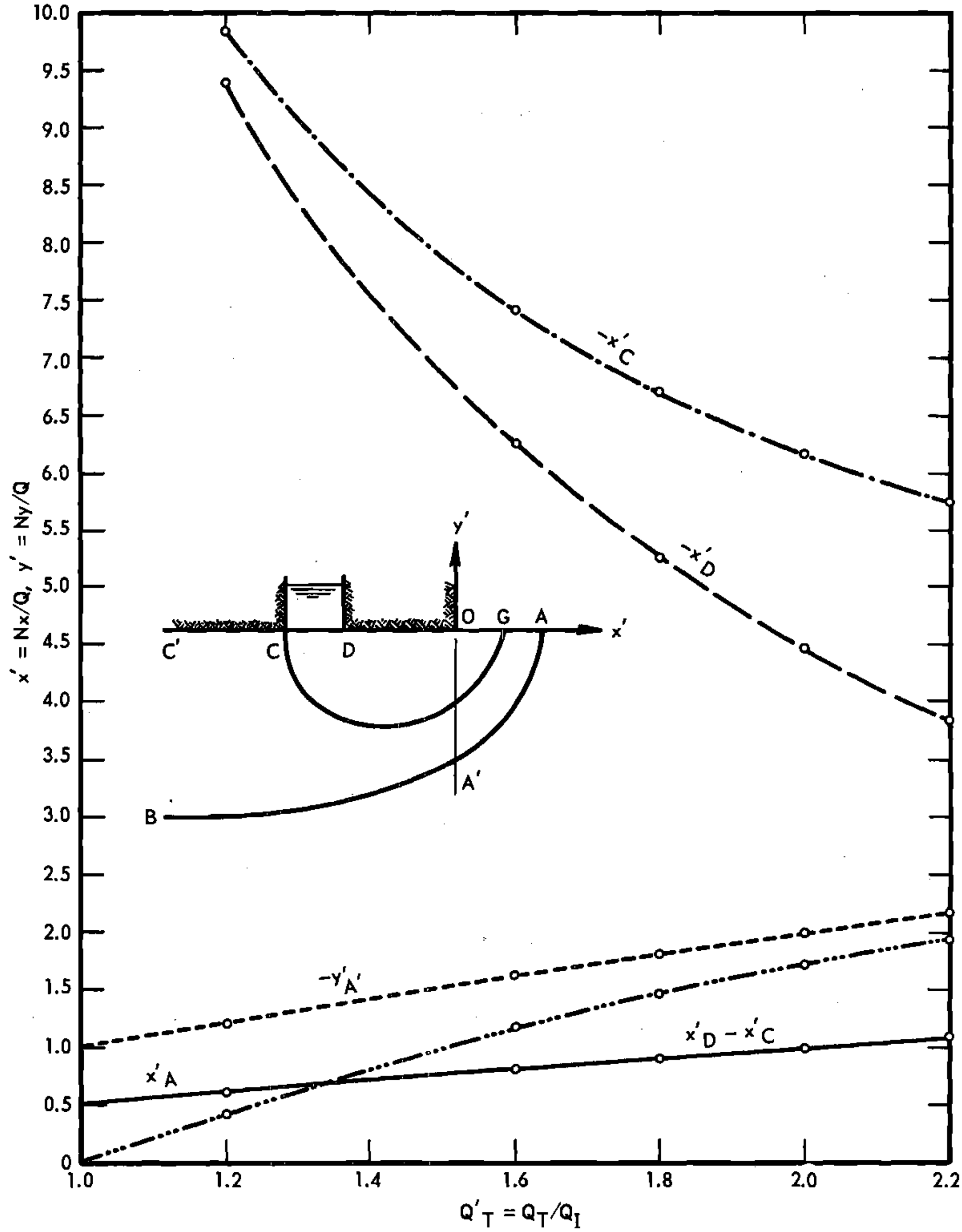


Figure 24(d). Effect of Q'_T Variation on Key Dimensions (Problem 2A, ϕ'_C = -5.0).

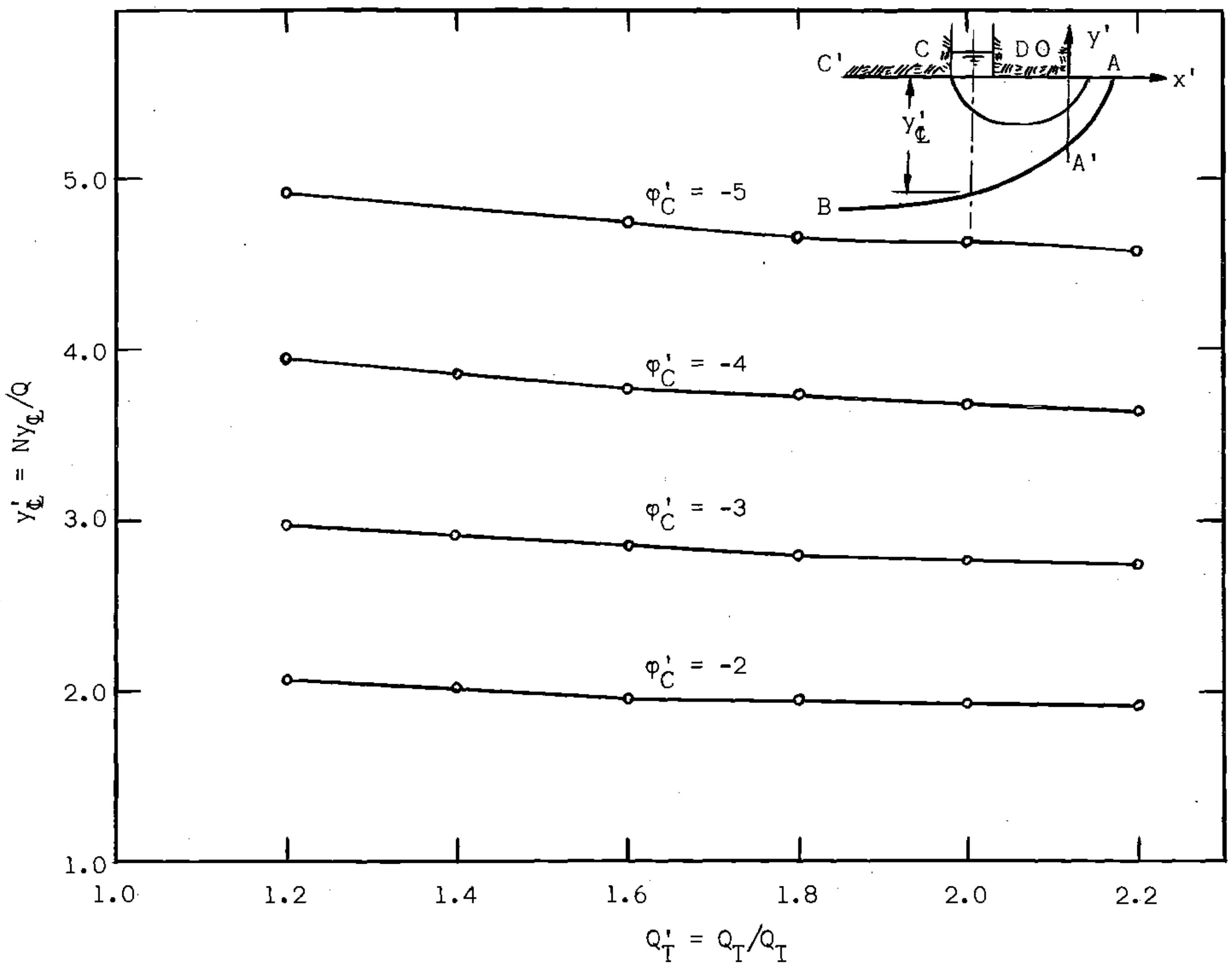


Figure 24(e). Effect of Q'_T Variation on y' at the Centerline of the Canal (Problem 2A).

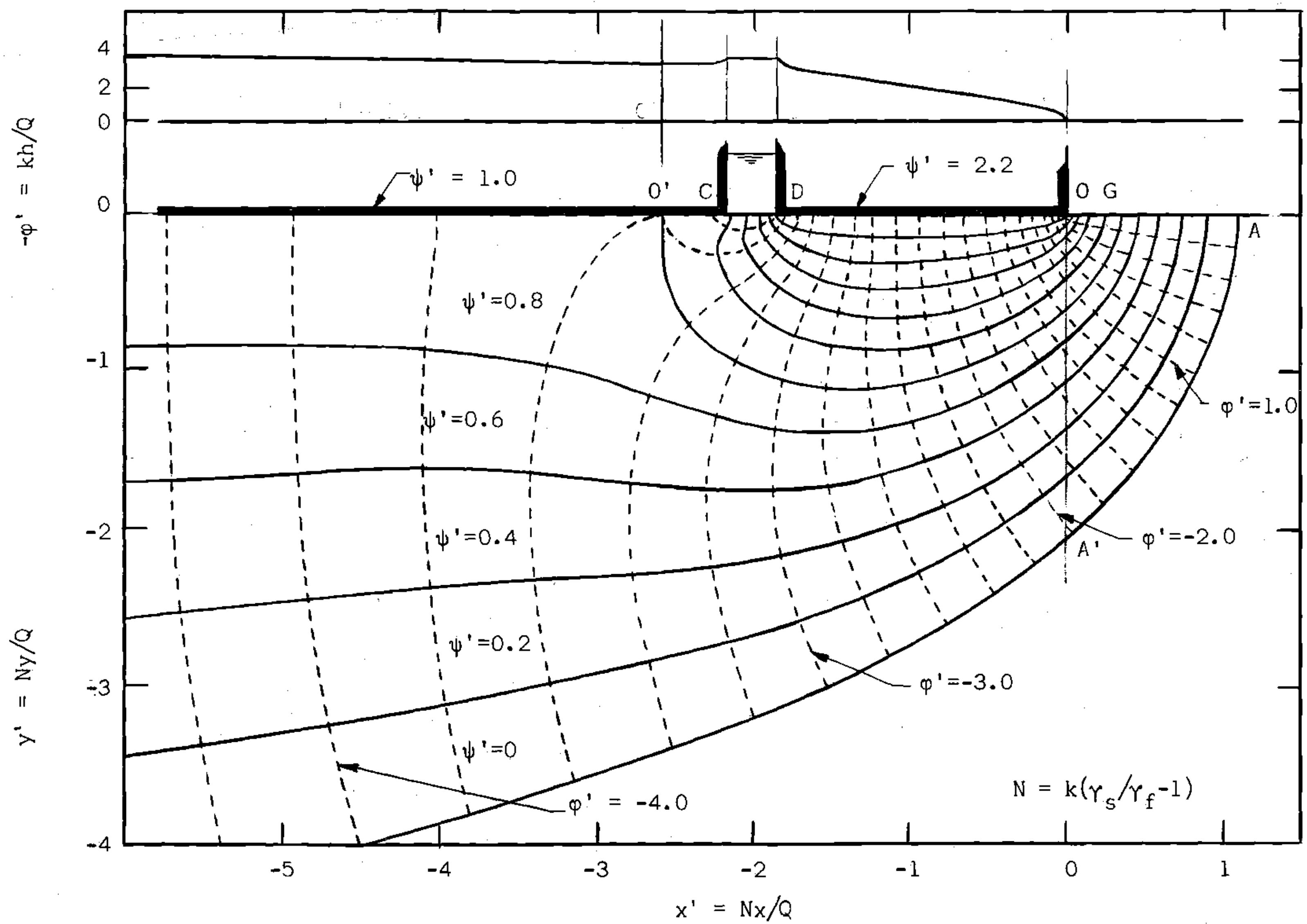


Figure 25. A Typical Flow Net (Problem 2B, $\varphi_C' = -4.0$, $\varphi_{D'}' = -3.6$, $Q_T' = 2.2$).

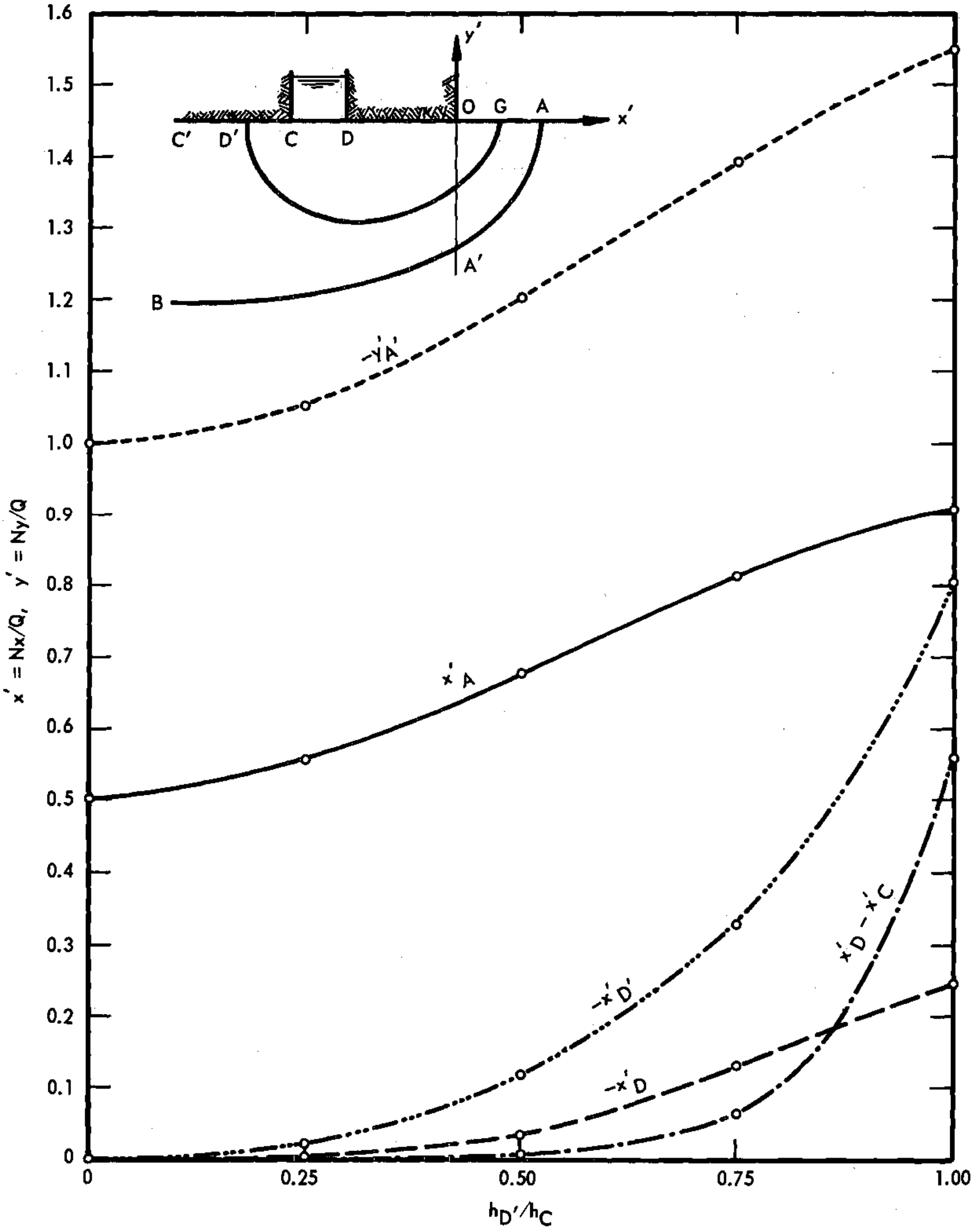


Figure 26(a). Effect of $h_{D'}/h_C$ Variation on Key Dimensions (Problem 2B, $\varnothing'_C = -2$, $Q'_T = 2.2$).

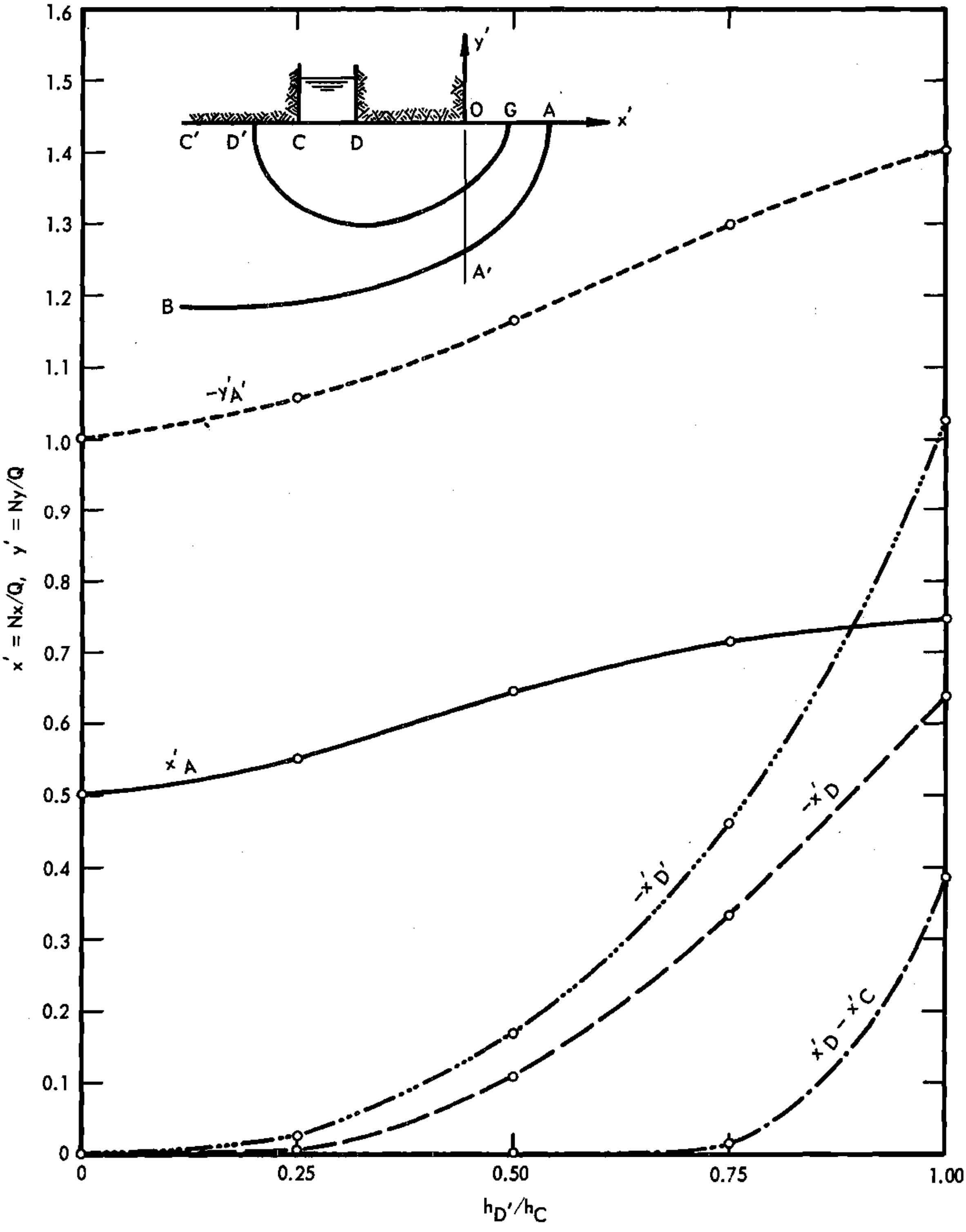


Figure 26(b). Effect of $h_{D'}/h_C$ Variation on Key Dimensions (Problem 2B, $\phi'_C = -2$, $Q'_T = 1.6$).

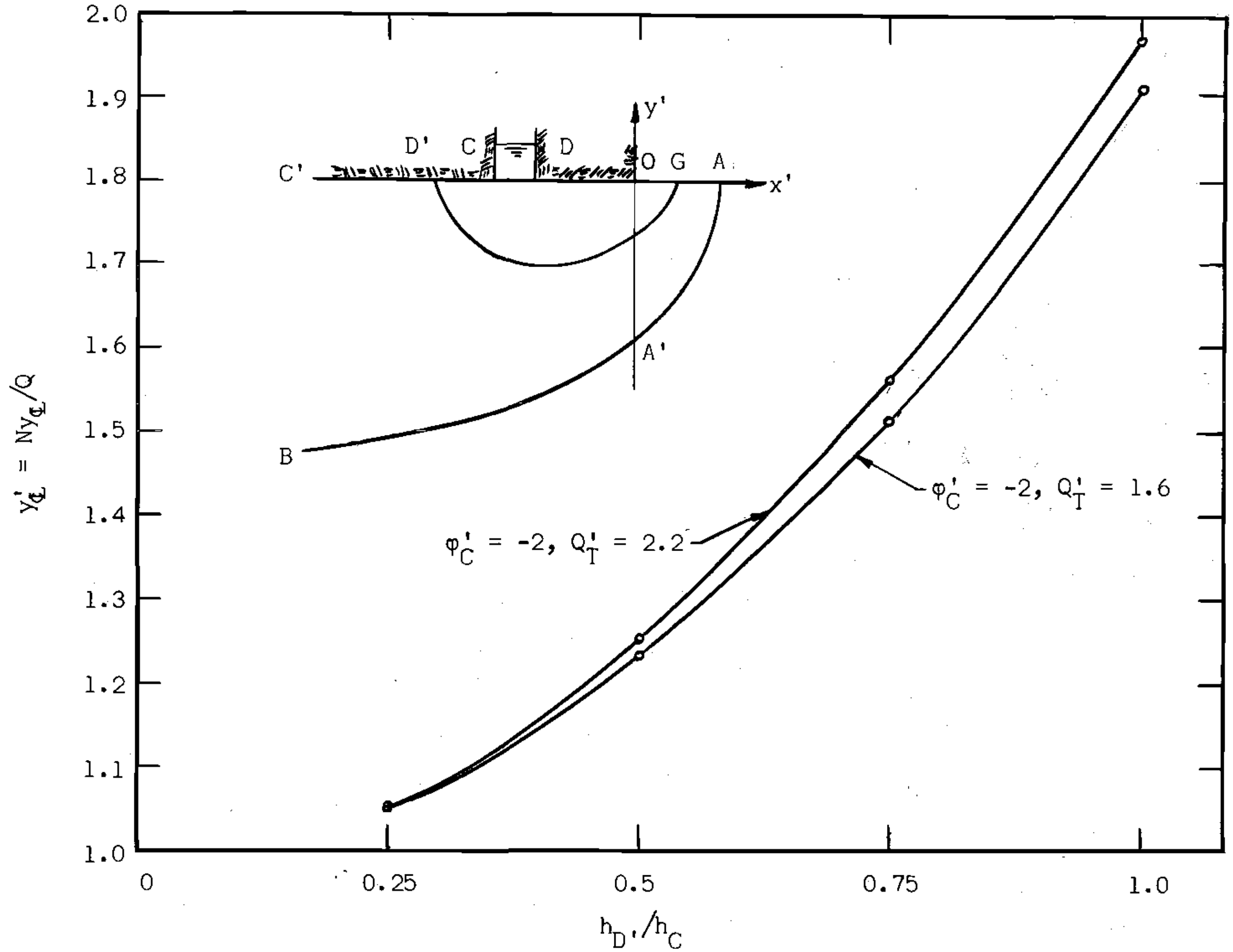


Figure 26(c). Effect of $h_{D'}/h_C$ Variation on y' at the Centerline of the Canal (Problem 2B).

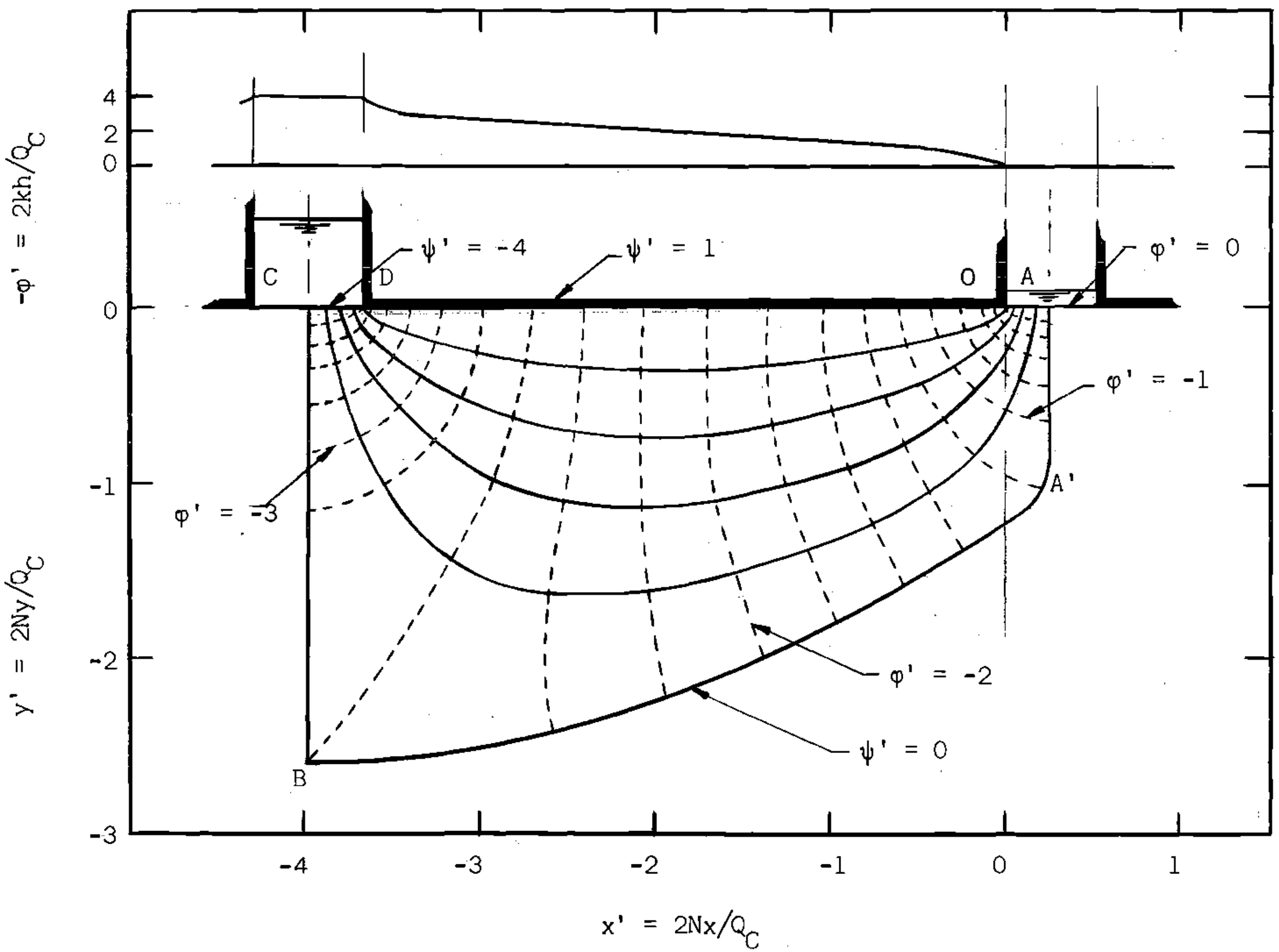


Figure 27. A Typical Flow Net (Problem 3, $\varphi'_C = -4.0$, $\varphi'_B = -2.6$, $\varphi'_{A'} = -1.2$).

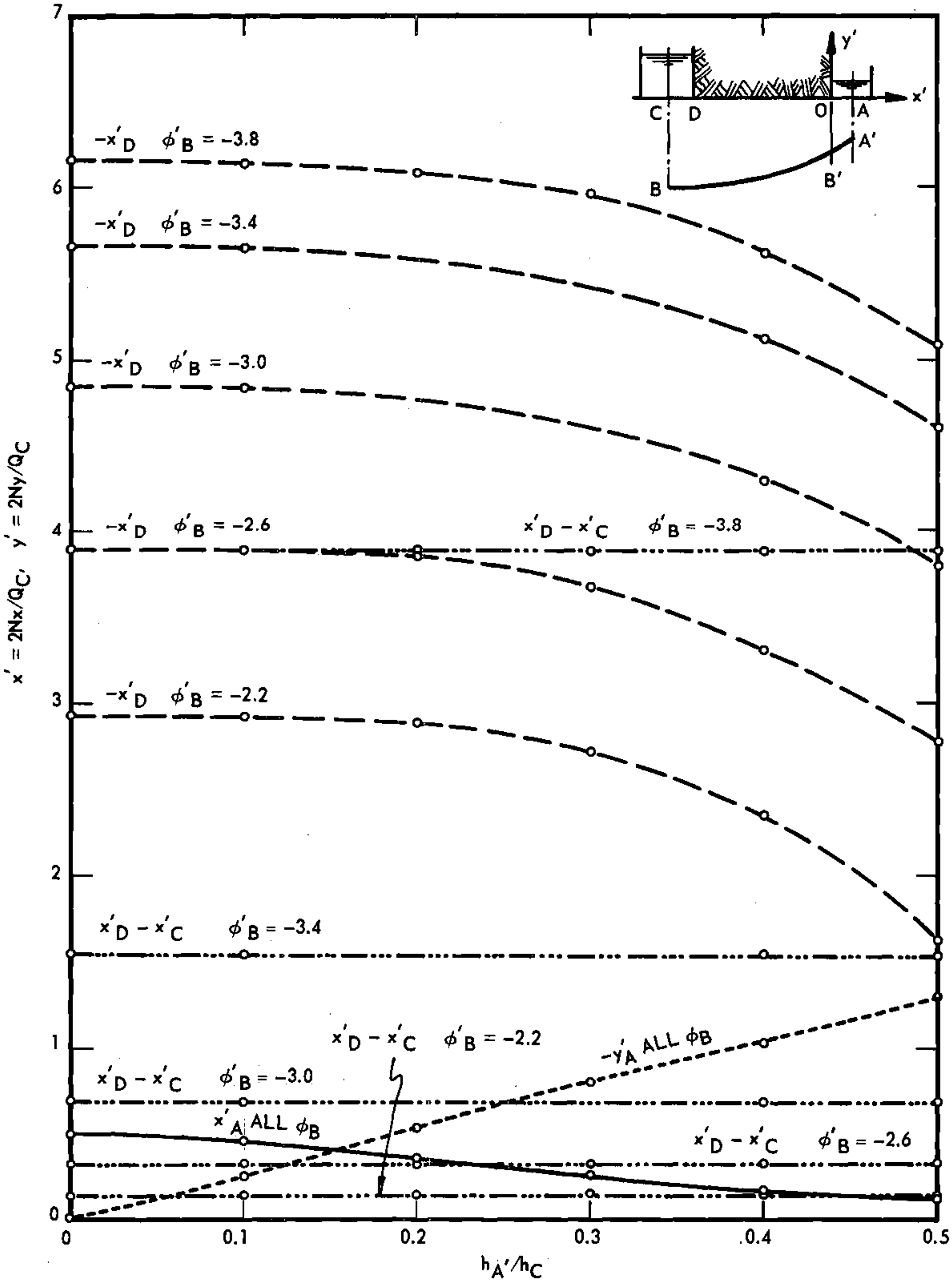


Figure 28. Effects of $h_{A'}/h_C$ Variation on Key Demensions (Problem 3).

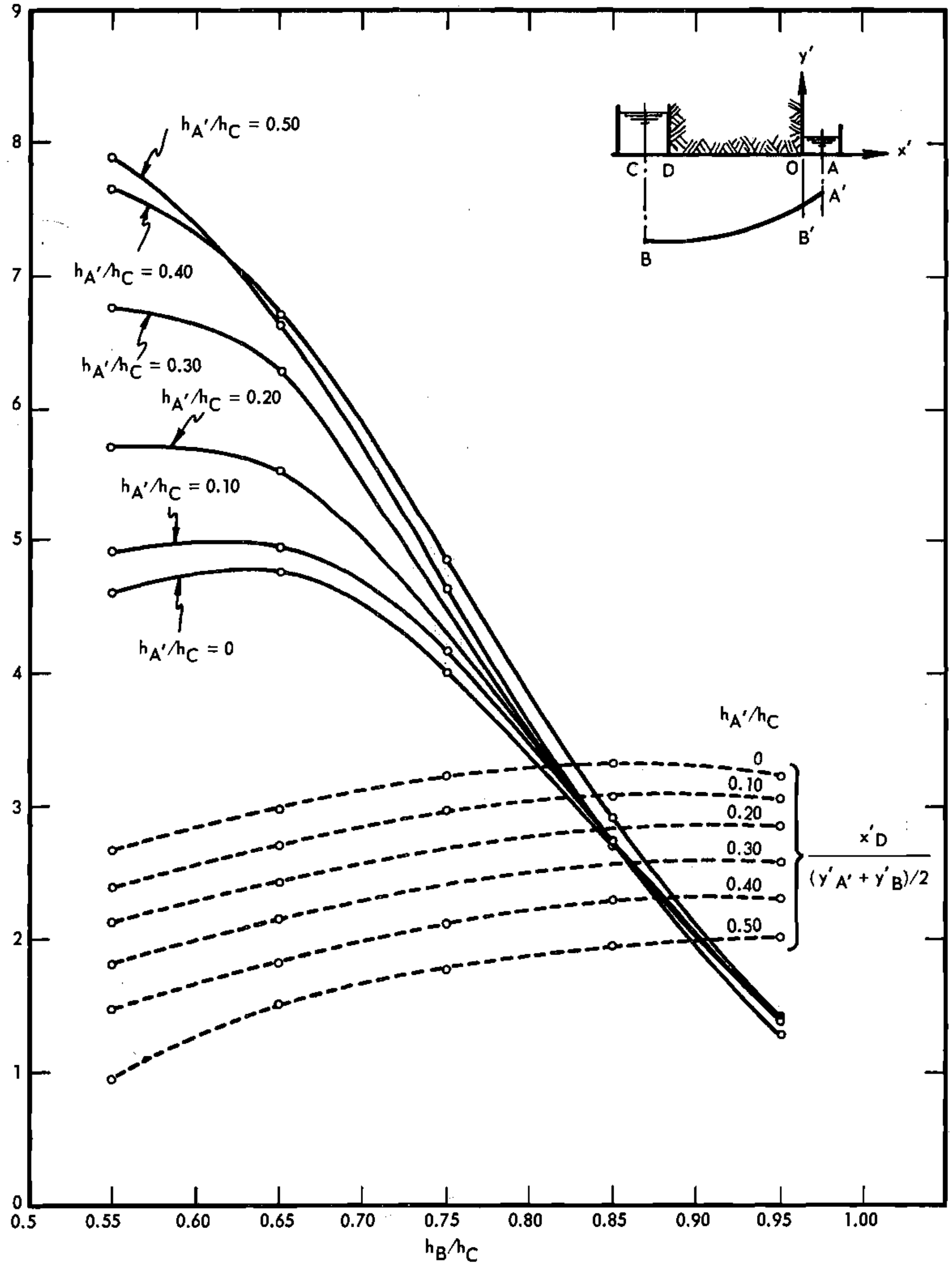


Figure 29. Effect of h_B/h_C Variation on Key Ratios (Problem 3).

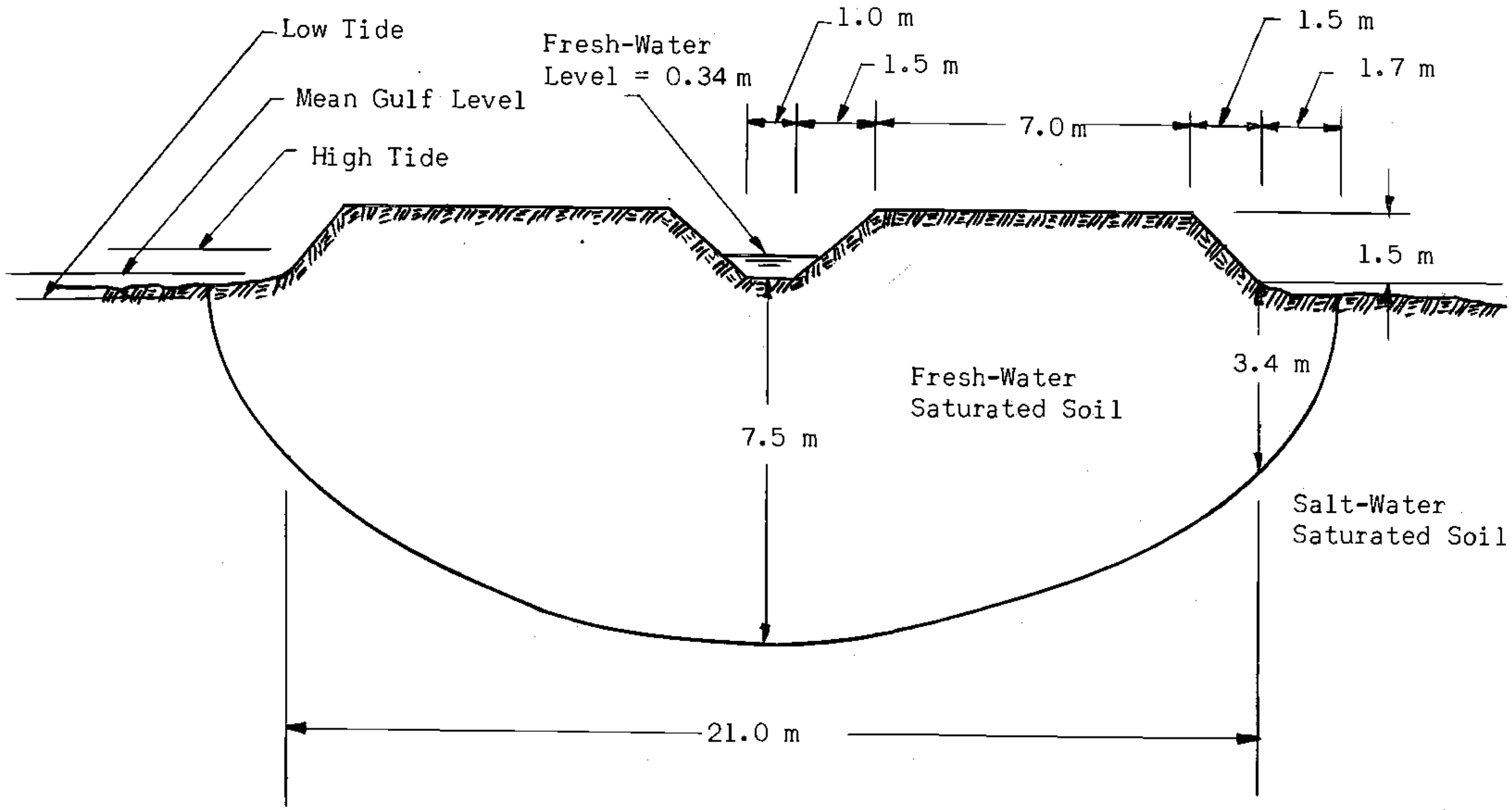


Figure 30. Single Canal on A Fill in the Mud-Flat Area.

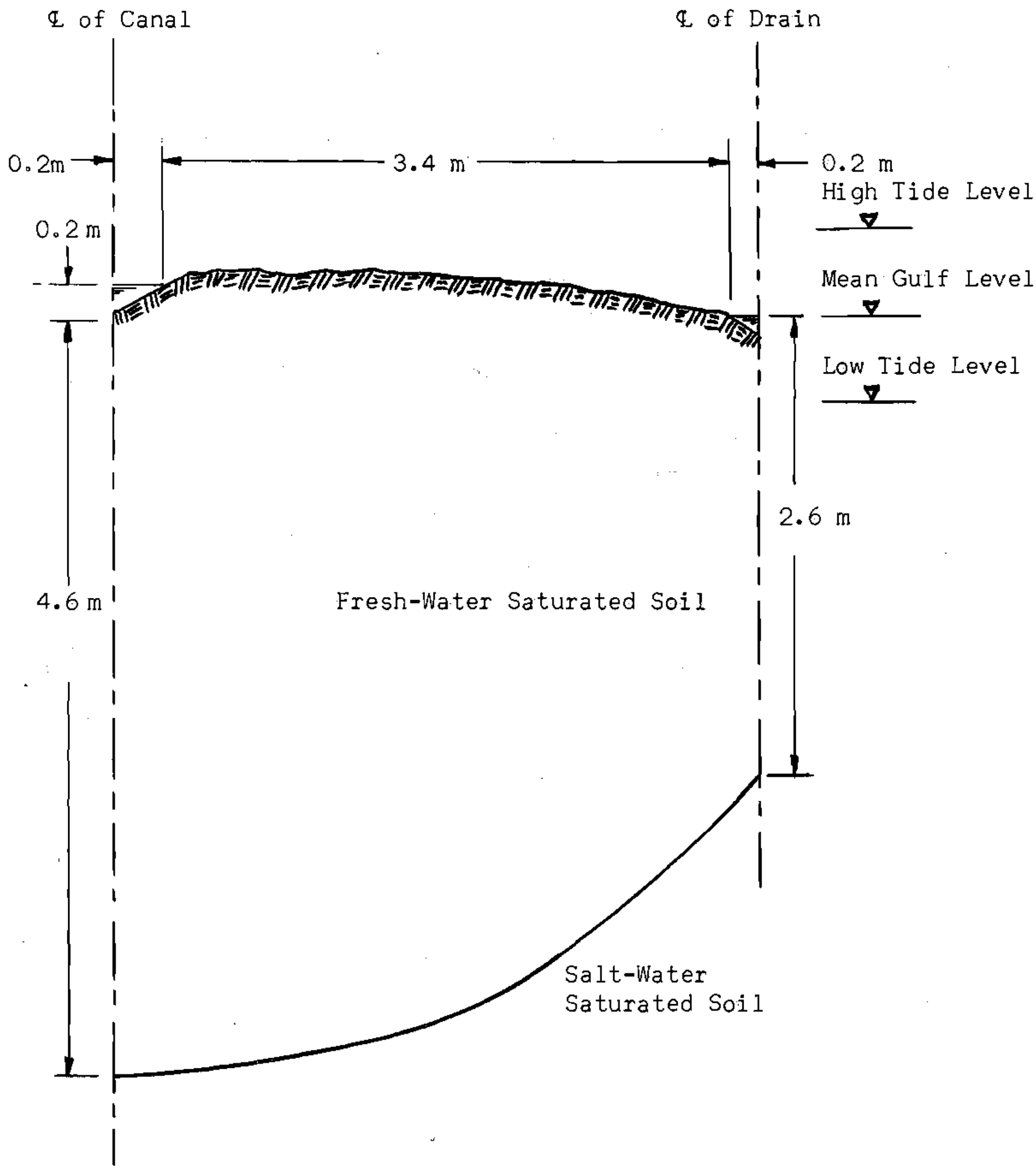


Figure 31. Cross-Section Elevation View Through a Canal and Drain

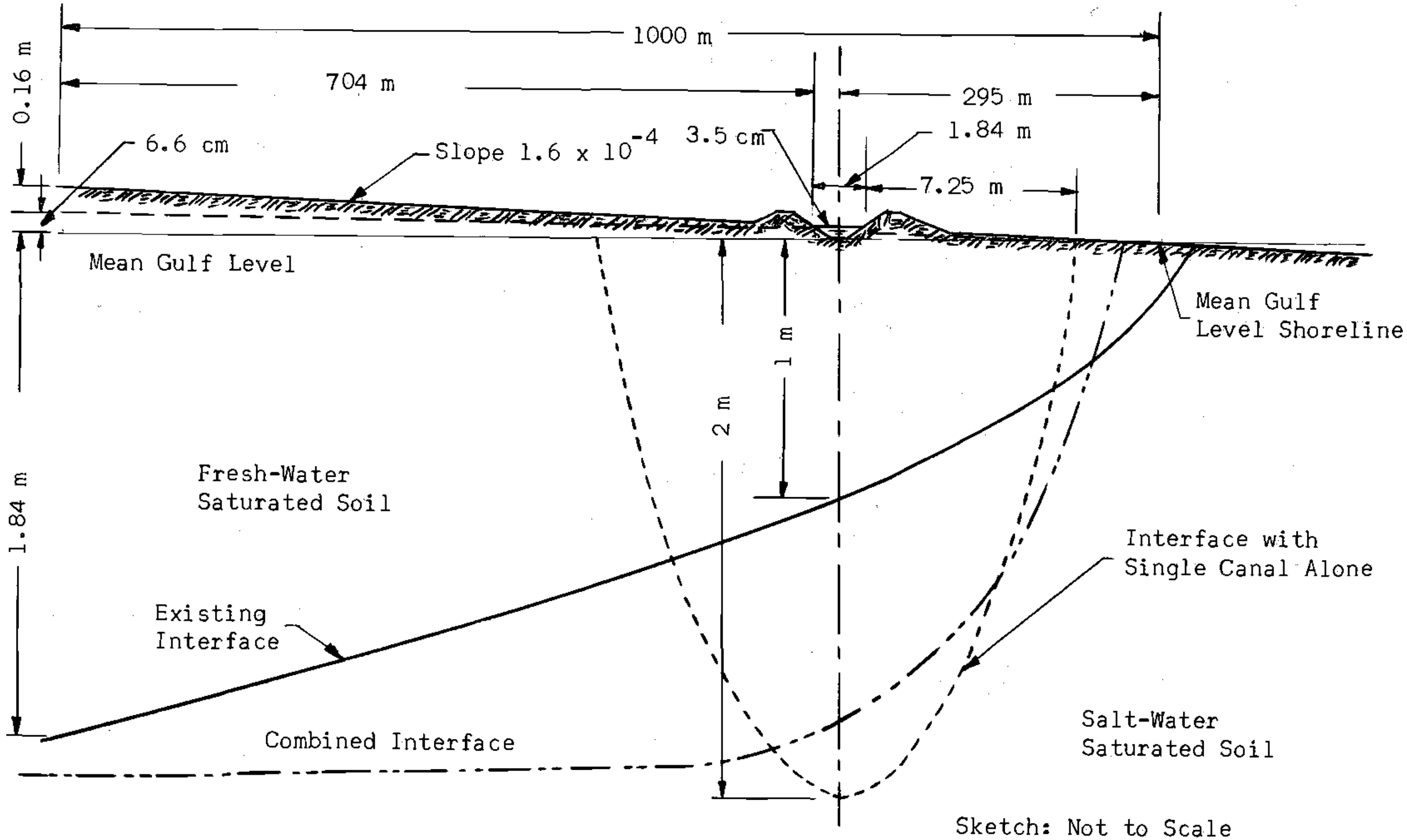


Figure 32. Single Canal Parallel to the Shore Line.

APPENDIX B

COMPUTER PROGRAM
(PROBLEMS 1 AND 3)

```
BEGIN     COMMENT FLOW FROM A SERIES OF PARALLEL CANALS WITH
          INTERMEDIATE DRAINS..
          NONDIMENSIONALIZED BY 1/2 OF Q FROM THE CANAL..
          SUCCESSIVE OVERRELAXATION METHOD ON W-PLANE..
          B-5000..        SRISAKDI CHARMONMAN..     MARCH 3, 1964
          INPUT:      MO = NUMBER OF GRID-POINTS FOR PHI = 1
                      SICPHI = COMMON PHI UNDER THE DRAIN
                      SOCPHI = COMMON PHI UNDER THE CANAL
                      PHISO = PHI AT THE CANAL
          PHI SHOULD BE SUCH THAT ITS PRODUCT WITH MO IS
          EVEN INTEGER  AND IT HAS BEEN CHOSEN TO BE NEGATIVE ;
INTEGER   I,J,K, MO,ND,NB,NC, D ;
REAL      L, W1, W11, SICPHI, SOCPHI, PHISO ;
          LABEL REREAD, EXIT ;
PROCEDURE FACTOR(K,KK) ;
          INTEGER K, KK ;
BEGIN     REAL L ;
          L ← ( ( COS( 3.1415927/K ) + COS(3.1415927/KK)) ) / 2) * 2 ;
          W1 ← 1 + L / ( ( 1 + SQRT(1-L) ) * 2 )
END FACTOR ;
FILE OUT LY12 1(1,15) ;
FORMAT TD1(// X8, "K=", I4,/ X8, "J=", I4,/
               X19,"I = 0,6..",X9,"I = 1,7..",X9,"I = 2,8..",X9,
          "I = 3,9..",X8,"I = 4,10..",X8,"I = 5,11..",/// ) ;
FORMAT TD2(// X8, "K=", I4,/ X8, "J=", I4, /
          X19, "I = 0,12",X10,"I = 2,14",X10,"I = 4,16",X10,
          "I = 6,18",X9, "I = 8,20", X10, "I = 10", ///) ;
FORMAT OUT Y12(X12, 6F18.9) ;
FILE IN FLMP(1,10) ;
FORMAT IN FTMP (X8,I3,3(X9,F6.3)    ) ;
LIST INMP (MO, PHISO, SICPHI, SOCPHI) ;
REREAD:   READ(FLMP, FTMP, INMP) [EXIT]    ;
          ND ←-PHISO × MO ;
          NC ←-SOCPHI × MO ;
          NB ←-SICPHI × MO ;
BEGIN
FORMAT OUT FTMN(X3,"MO       ND     NC      NB                ", /   4I7) ;
```

```
LIST LMN(MO,ND,NC,NB) ;
WRITE(LY12[DBL], FTMN, LMN)
END ;
BEGIN     COMMENT RELAXATION ON BOTH COURSE AND FINE GRID ;
ARRAY Y1[0:MO, 0:ND] ;
PROCEDURE YWRITE(D, FTD12) ;
          INTEGER D ;
          FORMAT FTD12 ;
BEGIN
          WRITE(LY12[DBL],FTD12, K,J) ;
          FOR J ← ND STEP -D UNTIL 0 DO
WRITE(LY12        , Y12, FOR I ← 0 STEP D UNTIL MO DO Y1[I,J] ) ;
END YWRITE ;
DEFINE    YY1 = Y1[I,J+D] + Y1[I,J-D] + Y1[I+D,J] #,
          YBC = ( YY1 + Y1[D,J] ) / 4 # ;
PROCEDURE ITERATE(D, UPM,UPN) ;
          INTEGER D, UPM, UPN ;
BEGIN     K ← K+1 ;
          I ← 0 ;
          FOR J ← D STEP D UNTIL NB DO
          Y1[0,J] ← YBC ;
          FOR J ← NC    STEP D UNTIL ND-D DO
          Y1[0,J] ← YBC ;
          FOR I ← D STEP D UNTIL UPM DO
          FOR J ← D STEP D UNTIL UPN DO
          Y1[I,J] ← W1×( YY1 + Y1[I-D,J] ) /4 - W11×Y1[I,J]
END ITERATE ;
BEGIN
COMMENT BOUNDARY CONDITIONS ;
          REAL DELPHI ;
          INTEGER NBC ;
          FOR J ← 0 STEP 1 UNTIL ND DO
          Y1[MO,J] ← 0 ;
          NBC ← NC -NB ;
          DELPHI ← SOCPHI - SICPHI ;
          FOR J ← NB  STEP 1 UNTIL NC DO
          Y1[0,J] ← SICPHI + DELPHI × (J-NB) / NBC ;
```

```
            FOR I ← 0 STEP 1 UNTIL MO-1 DO
BEGIN       Y1[I,0] ← 0 ;
            Y1[I,ND] ← 0
END ;
END         BOUNDARY CONDITIONS ;
BEGIN       COMMENT LINEAR INITIAL GUESS ;
            INTEGER NDC ;
            NDC ← ND - NC ;
            FOR J ← 2 STEP 2 UNTIL NB-2 DO
            Y1[O,J] ← SICPHI×J/NB ;
            FOR J ← NC+2 STEP 2 UNTIL ND-2 DO
            Y1[O,J] ← SOCPHI×(1-(J-NC)/NDC) ;
            FOR I ← 2 STEP 2 UNTIL MO-2 DO
            FOR J ← 2 STEP 2 UNTIL ND-2 DO
            Y1[I,J] ← Y1[O,J] × (1-I/MO) ;
END         INITIAL GUESS ;
BEGIN       COMMENT RELAXATION AND DEVIATE ;
            REAL DEV ;
PROCEDURE DEVIATE( D, UPM, UPN) ;
            INTEGER D, UPM, UPN ;
BEGIN
            INTEGER IM1, JM1 ;
FORMAT OUT FTDE(X3, " K      W1         DEV                   IM1  JM1  ", /
            I5, 2F10.6, X10, 2I5 ) ;
            LIST LDE(K, W1,      DEV, IM1, JM1) ;
            ARRAY YD[0:UPM, 0:UPN] ;
            FOR J ← D STEP D UNTIL UPN DO
            FOR I ← 0  STEP D UNTIL UPM DO
            YD[I,J] ← Y1[I,J] ;
            ITERATE(D, UPM, UPN) ;
            DEV ← 0 ;
            FOR J ← D STEP D UNTIL UPN DO
            FOR I ← 0  STEP D UNTIL UPM DO
BEGIN       YD[I,J] ← ABS(YD[I,J] - Y1[I,J]) ;
            IF DEV < YD[I,J] THEN
BEGIN       DEV ← YD[I,J] ;
            IM1 ← I ;
```

```
         JM1 ← J
END
ELSE
END ;
         WRITE(LY12[DBL], FTDE, LDE) ;
END DEVIATE ;
LABEL    REIT, REFINE ;
         REAL ALLOW ;        ALLOW ← 0.01  ;
         FACTOR(MO/2, ND/2) ;
         YWRITE(2, TD2) ;
         K ← 0 ;
         W11 ← W1-1 ;
         D ← 2 ;
REIT:    ITERATE(D, MO-D, ND-D) ;
         IF K MOD 5 = 0 THEN
BEGIN    DEVIATE(D, MO-D, ND-D) ;
         IF DEV > ALLOW THEN GO TO REIT
                        ELSE GO TO REFINE
END
         ELSE GO TO REIT ;
REFINE: J← NB ;      YWRITE(2, TD2) ;
BEGIN    DEFINE YF1 = Y1[I-1,J-1]+Y1[I+1,J-1]+Y1[I+1,J+1]+Y1[I-1,J+1]#,
         YYX = ( YY1 + Y1[I-D,J] ) / 4 # ,
         YBC = ( YY1 + Y1[D,J] ) / 4 # ;
         D ← 1 ;
         FOR J ← 1 STEP 2 UNTIL ND-1 DO
         FOR I ← 1 STEP 2 UNTIL MO-1 DO
         Y1[I,J] ← YF1 / 4 ;
         FOR I ← 1 STEP 2 UNTIL MO-1 DO
         FOR J ← 2 STEP 2 UNTIL ND-2 DO
         Y1[I,J] ← YYX ;
         FOR I ← 2 STEP 2 UNTIL MO-2 DO
         FOR J ← 1 STEP 2 UNTIL ND-1 DO
         Y1[I,J] ← YYX ;
         I ← 0 ;
         FOR J ← 1 STEP 2 UNTIL NB-1 DO
         Y1[0,J] ← YBC ;
```

```
             FOR J ← NC+1 STEP 2 UNTIL ND-1 DO
             Y1[0,J] ← YBC ;
             J ← ND ;
             YWRITE(1, TD1) ;
      END REFINEMENT ;
      BEGIN  COMMENT RELAXATION ON FINE GRID ;
             REAL ALLOW ;
             LABEL PARN, FINDX;
             ALLOW ← 0.0001 ;
             FACTOR(MD,ND) ;
             W11 ← W1-1 ;
      PARN:  ITERATE(D, MD-1, ND-1) ;
             IF K MOD 2 = 0  THEN
      BEGIN  DEVIATE(1, MD-1, ND-1) ;
             IF DEV > ALLOW THEN GO TO PARN
                             ELSE GO TO FINDX
      END
             ELSE GO TO PARN ;
      FINDX: YWRITE (1, TD1) ;
      END RELAXATION ON FINE GRID ;
      END RELAXATION AND DEVIATION ;
      BEGIN  COMMENT 4-POINT-DIFFERENTIATION FORMULA AND SIMPSON-RULE
             SHALL BE USED ;
             INTEGER IM, JM, KK ;
             REAL  C, H, HV3, X16, X12, X8, X4, XA, XE, XD ;
             ARRAY X[0:4, 0:ND] ;
      FORMAT HEADX("PARALLEL CANALS WITH INTERMEDIATE DRAINS", //,
             "PHISO=", F10.6, X5, "SICPHI=",F10.6,X5,"SOCPHI=",F10.6,/
             X16, "PHI=", F8.3, X2, "TO", F8.3, /
             X16, "PSI=", F8.3, X2, "TO", F8.3, //
                                    X18, "DERIVATIVE",X17,"DELTA X",X22,
             "X"X23,"COR. Y", / ) ;
      FORMAT XX(X4, 4F26.8    ) ;
      PROCEDURE XWRITE(H1,H2,S1,S2) ;
             REAL  H1,H2,S1,S2 ;
      BEGIN  WRITE(LY12[DBL], HEADX, PHISO, SICPHI, SOCPHI,H1,H2,S1,S2) ;
             FOR J ← JM STEP 1 UNTIL ND-1 DO
```

```
        WRITE(LY12      , XX, X[1,J], X[3,J], X[4,J], X[0,J],
                               X[2,J+1] ) ;
        WRITE(LY12      , XX, X[1,J], X[3,J], X[4,J], X[0,J] ) ;
END XWRITE ;
PROCEDURE DIFINT(XX ) ;
        REAL XX ;
BEGIN   DEFINE DIF = (C×X[0,IM] - 3×X[1,IM] + 1.5×X[2,IM]
                  - X[3,IM] / 3 ) / H # ,
        INT = (X[1,IM] + 4×X[2,J] + X[1,J] ) × HV3 # ;
        INTEGER NN, CC ;
        JM ← ND - IM / 2 ;
        KK ← JM+1 ;
        CC ← ( ND + NC ) / 2 ;
        IF XX = XA THEN
BEGIN   NN ← CC ;
        FOR J ← ND STEP -1 UNTIL CC+1 DO
BEGIN   X[0,J] ← X[0,2×J-ND] ;
        FOR I ← 1 STEP 1 UNTIL 3 DO
        X[I,J] ← 0 ;
        X[4,J] ← XD
END END
ELSE    NN ← ND ;
        FOR J ← NN STEP -1 UNTIL KK DO
BEGIN   IM ← 2×J - ND ;
        X[1,J] ← DIF ;
        X[0,J] ← X[0,IM] ;
        IM ← IM-1 ;
        X[2,J] ← DIF
END ;
        J← JM ;
BEGIN   IM ← 2×J - ND ;
        X[1,J] ← DIF ;
        X[0,J] ← X[0,IM]
END ;
        X[3,JM] ← 0 ;
        X[4,JM] ← XX ;
        FOR J ← KK STEP 1 UNTIL NN DO
```

```
BEGIN    IM ← J-1 ;
         X[3,J] ← INT ;
         X[4,J] ← X[4,IM] + X[3,J]
END ;
         IF XX = XA THEN
BEGIN
         X[0,CC] ← SDCPHI ;
         X[4,CC] ← XD ;
         X[3,CC] ← XD - X[4,CC-1]
END END DIFINT ;
         C ← 11 / 6 ;
         H ← 1 / MD ;          HV3 ← H / 3 ;
COMMENT  DA ;
         IM ← MD ;
         FOR I ← MD STEP -1 UNTIL 0 DO
         FOR J ← 0  STEP  1 UNTIL 3 DO
         X[J, ND-I] ← Y1[I,J] ;
         XA ← 10000 ;
         DIFINT(0) ;
         XWRITE(0,0,0,0,0,1,0) ;
         XA ← X[4,ND] ;
         X4 ← X[4,ND-2] ;
         X8 ← X[4,ND-4] ;
         X12 ← X[4,ND-6] ;
         X16 ← X[4,ND-8] ;
BEGIN
         COMMENT STREAMLINES FROM OE TO ABCD ;
PROCEDURE PSILINE(XOA, M) ;
         REAL XOA ;
         INTEGER M ;
BEGIN
         IM ← ND ;
         FOR J ← 0 STEP 1 UNTIL ND DO
         FOR I ← M  STEP 1 UNTIL M +3 DO
         X[I-M ,    J] ← Y1[I,J] ;
         DIFINT(XOA ) ;
         XWRITE(0,0,PHISO,M/20,M/20)
```

```
END PSILINE ;
COMMENT OE ;
        IM ← ND ;
        FOR I ← MO STEP -1 UNTIL MO-3 DO
        FOR J ← 0 STEP 1 UNTIL ND DO
        X[MO-I, J] ← Y1[I,J] ;
        DIFINT(0 ) ;
        FOR J ← ND STEP -1 UNTIL KK DO
        X[4,J] ← -X[4,J] ;
        XWRITE(0.0,PHISO,1.0,1.0) ;
COMMENT ED ;
        XE ← -X[4,ND] ;
        IM ← MO ;
        FOR I ← MO STEP -1 UNTIL 0 DO
        FOR J ← ND STEP -1 UNTIL ND-3 DO
        X[ND-J, ND-I] ← Y1[I,J] ;
        DIFINT(XE) ;
        FOR J ← ND STEP -1 UNTIL JM DO
        X[4,J] ← -X[4,J] ;
        XD ← X[4,ND] ;
        XWRITE(PHISO, PHISO, 0.0, 1.0) ;
        PSILINE(X16, 16) ;
        PSILINE(X12, 12) ;
        PSILINE(X8 , 8 ) ;
        PSILINE(X4 , 4 ) ;
        PSILINE(XA , 0 ) ;
END ALL STREAMLINES ;
END FINDX ;
END RELAXATION ON BOTH COURSE AND FINE GRIDS ;
        GO TO REREAD ;
EXIT: END.
```

APPENDIX C

COMPUTER PROGRAM
(PROBLEM 2A)

```
BEGIN COMMENT FLOW FROM A CANAL AND FROM INFINITY..
          NONDIMENSIONALIZED BY Q AT INFINITY.. BPHI = R×DPHI..
          SUCCESSIVE OVER@RELAXATION METHOD ON W@PLANE..
          B@5000..  SRISAKDI CHARMONMAN..  JANUARY 1964..
          INPUT Q AND PHI SHOULD BE SUCH THAT THE PRODUCT OF EACH WITH
          MD IS EVEN INTEGER AND PHI HAS BEEN CHOSEN TO BE NEGATIVE;
INTEGER   I,IM1,IM2, J,JM1,JM2, K,KK,D,DA,DB,DC,
          MD2,MD1,MD, ME2,ME1,ME, ND2,ND1,ND,NDP,NDP2, NB2,NB1,NB,NBP ;
LABEL     REREAD, EXIT ;
REAL      L,W1,W11,W2,W21,DPHI,BPHI,QT, R ;
OWN       REAL DEV ;
PROCEDURE FACTOR(K,KK) ;
          INTEGER K, KK ;
BEGIN     REAL L ;
          L ← ( ( COS( 3.1415927/K ) + COS(3.1415927/KK) ) / 2) * 2 ;
          W2 ← 1 + L / ( ( 1 + SQRT(1-L) ) * 2 )
END FACTOR ;
FILE OUT LY12 1(1,15) ;
FORMAT TITLE  (X19,"I = 0,6..",X9,"I = 1,7..",X9,"I = 2,8..",X9,
          "I = 3,9..",X8,"I = 4,10..",X8,"I = 5,11..",/
          X8, "K=", I4,/ X8, "J=", I4, / ) ;
FORMAT OUT Y12(X12, 6F18.9) ;
FILE IN FLMP(1,10) ;
FORMAT IN FTMPQ(X8,I3,X8,F6.3,X6,F6.3, X4, F6.3 ) ;
LIST INMPQ(MD, DPHI, QT, R) ;
REREAD:   READ(FLMP, FTMPQ, INMPQ) [EXIT] ;
          MD1 ← MD- 1 ;  MD2 ← MD1 - 1 ;
          ME ← MD×QT ;
          ME1 ← ME - 1 ; ME2 ← ME1 - 1 ;
          ND ←-DPHI×MD ;
          ND1 ← ND - 1 ; ND2 ← ND1 - 1 ;
          NDP ← ND + 1 ; NDP2 ← NDP + 1 ;
          NB ← R×ND ;
          NB1 ← NB - 1 ; NB2 ← NB1 - 1 ;
          NBP ← NB + 1 ;
BEGIN
FORMAT OUT FTMN(X3,"ND1      ND     NDP      NB      MD       ME", /  6I7) ;
```

```
LIST LMN(    ND1,ND,NDP,NB,MD,ME) ;
WRITE(LY12[DBL], FTMN, LMN)
END ;
BEGIN    COMMENT RELAXATION ON BOTH COURSE AND FINE GRID ;
    ARRAY Y1[0:ME, 0:ND], Y2[0:MD, NDP:NB] ;
PROCEDURE  YWRITE(D) ;
         INTEGER D ;
BEGIN
         WRITE(LY12[DBL],TITLE, K,J) ;
         FOR J ← NB STEP -D UNTIL ND+D DO
WRITE(LY12      , Y12, FOR I ← 0 STEP D UNTIL MD DO Y2[I,J] ) ;
         WRITE(LY12[DBL], TITLE, K, J) ;
         FOR J ← ND STEP -D UNTIL 0 DO
WRITE(LY12      , Y12, FOR I ← 0 STEP D UNTIL ME DO Y1[I,J] )
END YWRITE ;
DEFINE   YY1 = Y1[I-D,J] + Y1[I,J-D] + Y1[I+D,J] #,
         YY2 = Y2[I-D,J] + Y2[I+D,J]+ Y2[I,J+D] # ;
PROCEDURE RELAX(JM1,JM2,IM1,IM2) ;
         VALUE JM2,IM1,IM2 ;
         INTEGER JM1, JM2, IM1, IM2 ;
BEGIN
         IF JM1 > 2 THEN
BEGIN    FOR J ← JM1 STEP D UNTIL JM2 DO
         FOR I ← IM1 STEP D UNTIL IM2 DO
         Y2[I,J] ← W2×(YY2+Y2[I,J-D])/4 - W21×Y2[I,J]
END
         ELSE
BEGIN    FOR J ← JM2 STEP -D UNTIL JM1 DO
         FOR I ← IM1 STEP D UNTIL MD-D DO
         Y1[I,J] ← W1×(YY1+Y1[I,J+D])/4- W11×Y1[I,J] ;
         FOR J ← JM2 STEP -D UNTIL JM1 DO
         FOR I ← MD STEP D UNTIL IM2 DO
         Y1[I,J] ← W1×(YY1+Y1[I,J+D])/4- W11×Y1[I,J]
END  END RELAX ;
PROCEDURE ITERATE(D, DA, DB, DC) ;
         INTEGER D, DA, DB, DC ;
BEGIN    K ← K+1 ;
```

```
        FOR I ← D STEP D UNTIL DA DO
BEGIN   J ← ND ;
        Y1[I,ND] ← (YY1 + Y2[I,DB])/4 ;     J ← DB ;
        Y2[I,DB] ← W2×(YY2+Y1[I,J-D])/4 - W21×Y2[I,J]
END ;
        RELAX(D,ND-D,D,DC) ;
        RELAX(DB+D,NB-D,D,MD-D)
END ITERATE ;
BEGIN
COMMENT BOUNDARY CONDITIONS ;
        FOR J ← 0 STEP 1 UNTIL ND DO
BEGIN   Y1[ME,J] ← 0 ;
        Y1[0,J] ← DPHI×J/ND
END ;
        BPHI ← R×DPHI ;
        FOR J ← NDP STEP 1 UNTIL NB DO
BEGIN   Y2[MD,J] ← 0 ;
        Y2[0,J] ← BPHI×J/NB
END ;
        FOR I ← 1 STEP 1 UNTIL MD1 DO
BEGIN   Y1[I,0] ← 0 ;
        Y2[I,NB] ← BPHI×(1-I/MD)
END ;
        FOR I ← MD STEP 1 UNTIL ME1 DO
BEGIN   Y1[I,0] ← 0 ;
        Y1[I,ND] ← 0
END  END BOUNDARY CONDITIONS ;
BEGIN   COMMENT LINEAR INITIAL GUESS ;
        FOR I ← 2 STEP 2 UNTIL ME2 DO
BEGIN   W1← 1-I/ME ;
        FOR J ← 2 STEP 2 UNTIL ND2 DO
        Y1[I,J] ← Y1[0,J]×W1
END ;
        FOR I ← 2 STEP 2 UNTIL MD2 DO
BEGIN   W1← 1-I/MD ;
        Y1[I,ND] ← Y1[0,ND]×W1;
        FOR J ← NDP2 STEP 2 UNTIL NB2 DO
```

```
         Y2[I,J] ← Y2[0,J]×W1
END  END INITIAL GUESS ;
BEGIN    COMMENT RELAXATION AND DEVIATE ;
PROCEDURE DEVIATE( D, DC) ;
         INTEGER D, DC ;
BEGIN    COMMENT SINCE THE FLOW NET IN THE REGION OF FLOW FROM THE
         SOURCE IS FINER THAN THAT ELSEWHERE, MAXIMUM DEVIATION OF Y IN
         THIS AREA CAN BE TAKEN AS THE REPRESENTATIVE OF THE WHOLE ;
         INTEGER IM1, JM1 ;
FORMAT OUT FTDE(X3, "K       W1        W2        DEV     IM1     JM1", /
         I5, 3F10.6, 2I7) ;
         LIST LDE(K, W1, W2, DEV, IM1, JM1) ;
         ARRAY YD[20:ME-D, D:ND-D] ;
         FOR J ← D STEP D UNTIL ND-D DO
         FOR I ← 20 STEP D UNTIL DC DO
         YD[I,J] ← Y1[I,J] ;
         ITERATE(D, MD-D, ND÷D, ME-D) ;
         DEV ← 0 ;
         FOR J ← D STEP D UNTIL ND-D DO
         FOR I ← 20 STEP D UNTIL DC DO
BEGIN    YD[I,J] ← ABS(YD[I,J] - Y1[I,J]) ;
         IF DEV < YD[I,J] THEN
BEGIN    DEV ← YD[I,J] ;
         IM1 ← I ;
         JM1 ← J
END
ELSE
END ;
         DEV ← DEV / ABS(Y1[IM1,JM1]) ;
         WRITE(LY12[DBL], FTDE, LDE) ;
END DEVIATE ;
LABEL    REIT, REFINE ;
         REAL ALLOW ;       ALLOW ← 0.01  ;
         FACTOR(ME/2, ND/2) ;
         W1 ← W2 ;
         FACTOR(MD/2, (NB-ND)/2) ;
         YWRITE(2) ;
```

```
            K ← 0 ;
            W11 ← W1-1 ;
            W21 ← W2-1 ;    D ← 2 ;
REIT:       ITERATE(2, MD2, ND+D, ME-D) ;
            IF K MOD 5 = 0 THEN
BEGIN       DEVIATE(2, ME-2 ) ;
            IF DEV > ALLOW THEN GO TO REIT
                           ELSE GO TO REFINE
END
            ELSE GO TO REIT ;
REFINE:  J← NB ;        YWRITE(2) ;
BEGIN       DEFINE YF1 = Y1[I-1,J-1]+Y1[I+1,J-1]+Y1[I+1,J+1]+Y1[Y-1,J+1]#,
                   YF2 = Y2[I+1,J+1]+Y2[I-1,J+1]# ;
            D ← 1 ;
            FOR J ← 1 STEP 2 UNTIL ND1 DO
            FOR I ← 1 STEP 2 UNTIL ME1 DO
            Y1[I,J] ← YF1 / 4 ;
            FOR I ← 1 STEP 2 UNTIL MD1 DO
BEGIN       J ← NDP ;
            Y2[I,NDP] ← (YF2 + Y1[I-1,J-1]+Y1[I+1,J-1]) / 4 ;
            FOR J ← NDP2+1 STEP 2 UNTIL NB1 DO
            Y2[I,J] ← (YF2 + Y2[I-1,J-1]+Y2[I+1,J-1]) / 4
END ;
            FOR I ← 1 STEP 2 UNTIL ME1 DO
            FOR J ← 2 STEP 2 UNTIL ND2 DO
            Y1[I,J] ← (YY1 + Y1[I,J+D]) / 4 ;
            FOR I ← 1 STEP 2 UNTIL MD1 DO
BEGIN       J ← ND ;
            Y1[I,ND] ← (YY1 + Y2[I,J+D]) / 4 ;
            FOR J ← NDP2 STEP 2 UNTIL NB2 DO
            Y2[I,J] ← (YY2 + Y2[I,J-D]) / 4
END ;
            FOR I ← 2 STEP 2 UNTIL ME2 DO
            FOR J ← 1 STEP 2 UNTIL ND1 DO
            Y1[I,J] ← (YY1 + Y1[I,J+D]) / 4 ;
            FOR I ← 2 STEP 2 UNTIL MD2 DO
BEGIN       J ← NDP ;
```

```
          Y2[I,NDP] ← (YY2 + Y1[I,J-D]) / 4 ;
          FOR J ← NDP2+1 STEP 2 UNTIL NB1 DO
          Y2[I,J] ← (YY2 + Y2[I,J-1]) / 4
END ;
          YWRITE(1)
END REFINEMENT ;
BEGIN     COMMENT RELAXATION ON FINE GRID ;
          REAL ALLOW ;
          LABEL PARN, FINDX;
          ALLOW ← 0.0001 ;
          FACTOR(ME,ND) ;
          W1 ← W2 ;
          W11 ← W1-1 ;
          FACTOR(MD, NB-ND) ;
          W21 ← W2-1 ;
PARN:     ITERATE(1, MD1, NDP, ME1) ;
          IF K MOD 2 = 0  THEN
BEGIN     DEVIATE(1, ME1) ;
          IF DEV > ALLOW THEN GO TO PARN
                         ELSE GO TO FINDX
END
          ELSE GO TO PARN ;
FINDX:    YWRITE (1)
END RELAXATION ON FINE GRID ;
END RELAXATION AND DEVIATION ;
BEGIN     COMMENT 4-POINT-DIFFERENTIATION FORMULA AND SIMPSON-RULE
          SHALL BE USED ;
          INTEGER IM, JM, M ;
          REAL C, H, HV3, DEV, XA,XF ;
          REAL  X4, X8, X12, X16, XD, XG ;
          ARRAY  X[0:4, 0:NB] ,   XOA[0:(ME-20)/4] ;
FORMAT HEADX("FLOW FROM A SOURCE AND FROM INFINITY WITH DPHI =",F10.6,
          "TOTAL Q =", F10.6, / X18, "DERIVATIVE",X17,"DELTA X",X22,
          "X"X23,"COR, Y", / ) ;
FORMAT XX(X4, 4F26.8     ) ;
PROCEDURE XWRITE ;
BEGIN     WRITE(LY12[DBL], HEADX, DPHI, QT) ;
```

```
            FOR J ← JM STEP 1 UNTIL NB1 DO
            WRITE(LY12      , XX, X[1,J], X[3,J], X[4,J], X[0,J],
                                  X[2,J+1] ) ;
            WRITE(LY12      , XX, X[1,J], X[3,J], X[4,J], X[0,J] ) ;
END XWRITE ;
PROCEDURE DIFINT(DEV) ;
            REAL DEV ;
BEGIN       DEFINE DIF = (C×X[0,IM] - 3×X[1,IM] + 1.5×X[2,IM]
                          - X[3,IM] / 3 ) / H # ,
            INT = (X[1,IM] + 4×X[2,J] + X[1,J] ) × HV3 # ;
            JM ← NB - IM / 2 ;
            KK ← JM+1 ;
            FOR J ← NB STEP -1 UNTIL KK DO
BEGIN       IM ← 2×J - NB ;
            X[1,J] ← DIF ;
            X[0,J] ← X[0,IM] ;
            IM ← IM-1 ;
            X[2,J] ← DIF
END ;
            J← JM ;
BEGIN       IM ← 2×J - NB ;
            X[1,J] ← DIF ;
            X[0,J] ← X[0,IM]
END ;
            X[3,JM] ← 0 ;
            X[4,JM] ← DEV ;
            FOR J ← KK STEP 1 UNTIL NB DO
BEGIN       IM ← J-1 ;
            X[3,J] ← INT ;
            X[4,J] ← X[4,IM] + X[3,J]
END END DIFINT ;
            C ← 11 / 6 ;
            H ← 1 / MD ;          HV3 ← H / 3 ;
COMMENT     DA ;
            IM ← ME ;
            FOR I ← ME STEP -1 UNTIL 0 DO
            FOR J ← 0  STEP  1 UNTIL 3 DO
```

```
        X[J, NB-I] ← Y1[I,J] ;
        DIFINT(0) ;
        XWRITE ;
        XA ← X[4,NB] ;
        X4  ← X[4,NB-2] ;  X8  ← X[4,NB-4] ;  X12 ← X[4,NB-6] ;
        X16 ← X[4,NB-8] ;
        FOR M ← 20 STEP 4 UNTIL ME-4 DO
        XDA[(M-20)/4] ← X[4,NB-M/2] ;
BEGIN
COMMENT FD ;
PROCEDURE XSOURCE(XF, MD) ;
        REAL XF ;       INTEGER MD ;
BEGIN
        IM ← ND ;       JM ← NB-ND ;
        FOR J ← 0 STEP 1 UNTIL ND DO
        FOR I ← MD STEP 1 UNTIL MD+3 DO
        X[I-MD, JM+J] ← Y1[I,J] ;
        DIFINT(XF) ;
        XWRITE ;
END XSOURCE ;
        IM ← ND ;       JM ← NB-ND ;
        FOR I ← ME STEP -1 UNTIL ME-3 DO
        FOR J ← 0  STEP  1 UNTIL ND DO
        X[ME-I,JM+J] ← Y1[I,J] ;
        DIFINT(0) ;
        FOR  J ← NB STEP -1 UNTIL KK DO X[4,J] ← -X[4,J]  ;
        XWRITE ;
        FOR M ← ME-4 STEP -4 UNTIL 20 DO
        XSOURCE(XDA[(M-20)/4], M) ;
        XD ← X[4,NB]
END ;
BEGIN
COMMENT AB ;
PROCEDURE XINFINITY(XA, Z) ;
        REAL XA ;       INTEGER Z ;
BEGIN
        IM ← NB ;
```

```
        FOR I ← Z STEP 1 UNTIL Z+3 DO
BEGIN   FOR J ← 0 STEP 1 UNTIL ND DO
        X[I-Z,J] ← Y1[I,J] ;
        FOR J ← ND+D STEP 1 UNTIL NB DO
        X[I-Z,J] ← Y2[I,J]
END ;
        DIFINT(XA) ;
        XWRITE ;
END XINFINITY ;
        XINFINITY(X16, 16) ;            XINFINITY(X12, 12) ;
        XINFINITY(X8, 8) ;              XINFINITY(X4, 4) ;
        XINFINITY(XA, 0)
END
END FINDX ;
END RELAXATION ON BOTH COURSE AND FINE GRIDS ;
        GO TO REREAD ;
EXIT:   END.
```

APPENDIX D

COMPUTER PROGRAM
(PROBLEM 2B)

```
BEGIN     COMMENT  FLOW FROM A CANAL AND FROM INFINITY..               1
          NONDIMENSIONALIZED BY Q AT INFINITY..  BPHI = R×DPHI..       2
          SUCCESSIVE OVERRELAXATION ON W-PLANE..                       3
          ONLY ONE GRID SIZE WHICH IS SMALL ENOUGH IS USED..           4
          B-5000..   SRISAKDI CHARMONMAN..   APRIL 1964..              5
          MD×QT MUST BE DIVISIBLE BY 4 ;                               6
FILE OUT  LY12 1(1,15) ;                                               7
LABEL     REREAD, EXIT ;                                               8
REAL      DPHI, PHIDP, QT, R ;                                         9
INTEGER   MD, ME, NDP, ND, NB ;                                        10-11
FILE IN FLMP(1,10) ;                                                   12
FORMAT IN FTMPQ(X6, I3, 4(X7, F6.3)) ;                                 13
LIST INMPQ(MD, DPHI, PHIDP, QT, R) ;                                   14
REREAD:   READ(FLMP, FTMPQ, INMPQ) [EXIT] ;                            15
          ME ← MD×QT ;                                                 16
          NDP ← -MD×PHIDP ;                                            17
          ND ← - MD×DPHI ;                                             18
          NB ← R×ND ;                                                  19-20
BEGIN                                                                  21
ARRAY     Y1[0:MD, 0:NB] ,  Y2[MD:ME, 0:ND] ;                          22
INTEGER   I, J, M1D, MD1, M1E, N1DP, N1D, N1B ;                        23
          M1D ← MD-1 ;                                                 24
          MD1 ← MD+1 ;                                                 25
          M1E ← ME-1 ;                                                 26
          N1DP ← NDP-1 ;                                               27
          N1D ← ND-1 ;                                                 28
          N1B ← NB-1 ;                                                 29
BEGIN     COMMENT BOUNDARY CONDITIONS ;                                30
REAL      DELPHI, DELPBC ;                                             31
          DELPHI ← -1/MD ;                                             32
          FOR J ← 1 STEP 1 UNTIL N1DP DO                               33
BEGIN     Y1[0,J] ← J×DELPHI ;                                         34
          Y2[ME,J] ← 0                                                 35
END ;                                                                  36
          FOR J ← NDP STEP 1 UNTIL N1D DO                              37
BEGIN     Y1[0,J] ← J×DELPHI ;                                         38
          Y1[MD,J] ← 0 ;                                               39
```

```
            Y2[MD,J] ← 0 ;
            Y2[ME,J] ← 0
END ;
            FOR J ← ND STEP 1 UNTIL N1B DO
BEGIN       Y1[0,J] ← J×DELPHI ;
            Y1[MD,J] ← 0
END ;
            DELPBC ← -R×DPHI×DELPHI ;
            FOR I ← 0 STEP 1 UNTIL MD DO
BEGIN
            Y1[I,0] ← 0 ;
            Y1[I,NB] ← DELPBC×(MD-I)
END ;
            FOR I ← MD STEP 1 UNTIL ME DO
BEGIN       Y2[I,0] ← 0 ;
            Y2[I,ND] ← 0
END END BOUNDARY CONDITIONS ;
BEGIN       COMMENT LINEAR INITIAL GUESS ;
REAL        DELY ;
INTEGER PDN ;
            FOR J ← 1 STEP 1 UNTIL N1DP DO
BEGIN       DELY ← Y1[0,J] / ME ;
            FOR I ← 1 STEP 1 UNTIL MD DO
            Y1[I,J] ← DELY×(ME-I) ;
            FOR I ← MD STEP 1 UNTIL M1E DO
            Y2[I,J] ← DELY×(ME-I)
END ;
            FOR J ← NDP STEP 1 UNTIL N1B DO
BEGIN       DELY ← Y1[0,J] / MD ;
            FOR I ← 1 STEP 1 UNTIL M1D DO
            Y1[I,J] ← DELY×(MD-I)
END ;
            PDN ← 1 / (ND-N1DP) ;
            FOR I ← MD1 STEP 1 UNTIL M1E DO
BEGIN       DELY ← Y2[I,N1DP] ×PDN ;
            FOR J ← NDP STEP 1 UNTIL N1D DO
            Y2[I,J] ← DELY×(ND-J)
```

```
END END INITIAL GUESS ;
BEGIN    COMMENT OVERRELAXATION ;
REAL     W1, W1M, W2, W2M ;
LABEL    REIT, YWRITE ;
INTEGER K ;
DEFINE   YY1 = Y1[I-1,J] + Y1[I,J-1] + Y1[I,J+1] #,
         YY2 = Y2[I,J-1] + Y2[I+1,J] + Y2[I,J+1] # ;
PROCEDURE ITERATE ;
BEGIN    K ← K+1 ;
         FOR I ← 1 STEP 1 UNTIL M1D DO
         FOR J ← 1 STEP 1 UNTIL N1B DO
         Y1[I,J] ← W1×(YY1+Y1[I+1,J])- W1M×Y1[I,J] ;
         I ← MD ;
         FOR J ← 1 STEP 1 UNTIL N1DP DO
BEGIN    Y1[I,J] ← 0.25 × (YY1 + Y2[MD1,J] ) ;
         Y2[I,J] ← Y1[I,J]
END ;
         FOR I ← MD1 STEP 1 UNTIL M1E DO
         FOR J ← 1 STEP 1 UNTIL N1D DO
         Y2[I,J] ← W2×(YY2+Y2[I-1,J]) - W2M×Y2[I,J]
END ITERATE ;
BEGIN    COMMENT OVERRELAXATION FACTOR ;
PROCEDURE FACTOR(K,KK) ;
         INTEGER K, KK ;
BEGIN    REAL L ;
         L ← ( ( COS( 3.1415927/K ) + COS(3.1415927/KK) ) / 2) * 2 ;
         W2 ← 1 + L / ( ( 1 + SQRT(1-L) ) * 2 )
END FACTOR ;
         FACTOR(MD,NB) ;
         W1M ← W2 - 1 ;
         W1 ← W2× 0.25 ;
         FACTOR(ME-MD, ND) ;
         W2M ← W2-1 ;
         W2 ← W2× 0.25
END OVERRELAXATION FACTOR ;
         K ← 0 ;
REIT:    ITERATE ;
```

```
BEGIN    COMMENT ACCURACY CHECK ;
INTEGER X ;
         IF K < 60 THEN X ← 5 ELSE X ← 2 ;
         IF K MOD X = 0 THEN
BEGIN
INTEGER  IM1, JM1 ;
REAL     DEV, YMAXD ;
ARRAY    YD[1:M1E, 1:N1D] ;
FORMAT OUT FTDE(X3, "K      W1          W2          DEV     IM1     JM1" /
         I5, 3F10.6, 2I7) ;
         LIST LDE(K, W1, W2, DEV, IM1, JM1) ;
         DEV ← 0 ;
         FOR J ← 1 STEP 1 UNTIL N1D DO
BEGIN    FOR I ← 1 STEP 1 UNTIL MD DO
         YD[I,J] ← Y1[I,J] ;
         FOR I ← MD1 STEP 1 UNTIL M1E DO
         YD[I,J] ← Y2[I,J]
END ;
         ITERATE ;
         FOR J ← 1 STEP 1 UNTIL N1D DO
BEGIN    FOR I ← 1 STEP 1 UNTIL M1D DO
         YD[I,J] ← ABS(Y1[I,J] - YD[I,J] ) ;
         YD[MD,J] ← ABS(Y1[MD,J] - YD[MD,J] ) ;
         FOR I ← MD1 STEP 1 UNTIL M1E DO
         YD[I,J] ← ABS(Y2[I,J] - YD[I,J] )
END ;
         FOR J ← 1 STEP 1 UNTIL N1D DO
         FOR I ← 1 STEP 1 UNTIL M1E DO
         IF YD[I,J] > DEV THEN
BEGIN    DEV ← YD[I,J] ;
         IM1 ← I ;
         JM1 ← J
END ;
         IF IM1 > MD THEN YMAXD ← Y2[IM1,JM1]
                     ELSE YMAXD ← Y1[IM1,JM1] ;
         DEV ← DEV / ABS(YMAXD) ;
         WRITE(LY12, FTDE, LDE) ;
```

```
          IF DEV > 0.0001 THEN GO TO REIT
                          ELSE GO TO YWRITE
END
ELSE      GO TO REIT
END ACCURACY CHECK ;
YWRITE:
BEGIN
FORMAT OUT TITL1(//  X19, "Y FOR FLOW FROM INFINITY" //
          X10,"I = 0,6..",X9,"I = 1,7..", X9,"I = 2,8..",X9,
          "I = 3,9..",X8,"I = 4,10..",X8,"I = 5,11..",//
          X1, "K=", I4,/ X8, "J=", I4, / ) ;
FORMAT OUT TITL2 ( //  X19, "Y FOR FLOW FROM THE CANAL" //
          X10,"I = MD,MD+6..",X5,"I = MD+1,MD+7..",X3,
          "I = MD+2,MD+8..",X3,"I = MD+3,MD+9..",X2,
          "I = MD+4,MD+10..",X2,"I = MD+5,MD+11..",//) ;
FORMAT OUT Y12(X3 , 6F18.9) ;
          WRITE(LY12, TITL1, K, J) ;
          FOR J ← NB STEP -1 UNTIL 0 DO
          WRITE(LY12, Y12, FOR I ← 0 STEP 1 UNTIL MD DO Y1[I,J] ) ;
          WRITE(LY12, TITL2) ;
          FOR J ← ND STEP -1 UNTIL 0 DO
          WRITE(LY12, Y12, FOR I ← MD STEP 1 UNTIL ME DO Y2[I,J])
END END OVERRELAXATION ;
BEGIN     COMMENT FOUR-POINT-DIFFERENTIATION FORMULA AND SIMPSON-RULE
          SHALL BE USED ;
INTEGER   IM, JM, M, NBD, MOA ;
REAL      C, H, HV3, XDP, BPHI , XE ;
          MOA ← 0.25 × ME ;
          BPHI ← R × DPHI ;
BEGIN
ARRAY     X[0:3, 0:NB],  XDA[0:MOA] ;
FORMAT OUT HEADX( ////  X2, "SINGLE CANAL PLUS NATURAL GW FLOW", X1,
         "WITH PHI AT THE CANAL =", F10.6, X5, "PHIDP =", F10.6,X5,
          "TOTAL Q =", F10.6 /
          X16, "PHI =", F8.3, X2, "TO", F8.3 /
          X16, "PSI =", F8.3, X2, "TO", F8.3 //
          X51, "X", X22, "COR. Y" //) ;
```

```
FORMAT OUT XX(X30, 2F26.8) ;
PROCEDURE XWRITE(H1,H2,S1,S2) ;
          REAL H1,H2,S1,S2 ;
BEGIN     WRITE(LY12, HEADX, DPHI, PHIDP, QT, H1,H2, S1,S2) ;
          FOR J ← JM STEP 2 UNTIL NB DO
          WRITE(LY12, XX, X[3,J], X[0,J] )
END XWRITE ;
PROCEDURE DIFINT(XJM) ;
          REAL XJM ;
BEGIN
DEFINE    DIF = (C×X[0,J]-3×X[1,J]+1.5×X[2,J]-X[3,J]/3)×MD #,
          INT = (X[1,J] + 4×X[1,J-1] + X[1,J-2]) × HV3 # ;
          JM ← NB - IM ;
          FOR J ← NB STEP -1 UNTIL JM DO
          X[1,J] ← DIF ;
          X[3,JM] ← XJM ;
          FOR J ← JM+2 STEP 2 UNTIL NB DO
BEGIN     X[2,J] ← INT ;
          X[3,J] ← X[3,J-2] + X[2,J]
END END DIFINT ;
          C ← 11 / 6 ;
          H ← 1 / MD ;
          HV3 ← H / 3 ;
BEGIN     COMMENT OA ;
          IM ← ME ;
          FOR J ← 0 STEP 1 UNTIL 3 DO
BEGIN     FOR I ← ME STEP -1 UNTIL MD1 DO
          X[J, NB-I] ← Y2[I,J] ;
          FOR I ← MD STEP -1 UNTIL 0 DO
          X[J, NB-I] ← Y1[I,J]
END ;
          DIFINT(0) ;
          XWRITE(0,0, QT,0) ;
          FOR M ← MOA STEP -1 UNTIL 0 DO
          XOA[MOA - M] ← X[3, JM+4×M]
END OA ;
BEGIN     COMMENT OE ;
```

```
            IM ← ND ;
            NBD ← NB - ND ;
            FOR I ← ME STEP -1 UNTIL ME-3 DO
            FOR J ← 0 STEP 1 UNTIL ND DO
            X[ME-I, NBD+J] ← Y2[I,J] ;
            DIFINT(0) ;
            FOR J ← JM STEP 2 UNTIL NB DO
            X[3,J] ← -X[3,J] ;
            XWRITE(0,DPHI, QT,QT) ;
            XE ← - X[3,NB]
END DE ;
BEGIN       COMMENT ED ;
INTEGER     MN ;
            MN ← NB+MD ;
            IM ← ME-MD ;
            FOR J ← ND STEP -1 UNTIL ND-3 DO
            FOR I ← ME STEP -1 UNTIL MD DO
            X[ND-J, MN-I] ← Y2[I,J] ;
            DIFINT(XE) ;
            FOR J ← JM STEP 2 UNTIL NB DO
            X[3,J] ← -X[3,J] ;
            XWRITE(DPHI, DPHI, 1, QT)
END ED ;
BEGIN
INTEGER M ;
PROCEDURE XCANAL(XJM, M) ;
            REAL XJM ;
            INTEGER M ;
BEGIN
REAL        PSI ;
            IM ← ND ;
            FOR J ← 0 STEP 1 UNTIL ND DO
            FOR I ← M STEP 1 UNTIL M+3 DO
            X[I-M, NBD+J] ← Y2[I,J] ;
            DIFINT(XJM) ;
            PSI ← M×H ;
           XWRITE(0,DPHI, PSI,PSI)
```

```
END XCANAL ;
        FOR M ← ME-4 STEP -4 UNTIL MD+4 DO
        XCANAL(XOA[0.25×M], M)
END ;
BEGIN   COMMENT  XFDP ;
INTEGER NDPB ;
        NDPB ← NB - NDP ;
        IM ← NDP ;
        FOR I ← MD STEP 1 UNTIL MD+3 DO
        FOR J ← 0 STEP 1 UNTIL IM DO
        X[I-MD, NDPB+J] ← Y2[I,J] ;
        DIFINT(XOA[0.25×MD]) ;
        XWRITE(0, PHIDP, 1, 1)
END XFDP ;
BEGIN
INTEGER M ;
PROCEDURE XINFINITY(XJM, M) ;
        REAL XJM ;
        INTEGER M ;
BEGIN   REAL PSI ;
        IM ← NB ;
        FOR J ← 0 STEP 1 UNTIL NB DO
        FOR I ← M STEP 1 UNTIL M+3 DO
        X[I-M, J] ← Y1[I,J] ;
        DIFINT(XJM) ;
        PSI ← M×H ;
       XWRITE(0,BPHI, PSI,PSI)
END XINFINITY ;
        FOR M ← MD-4 STEP -4 UNTIL 0 DO
        XINFINITY(XOA[0.25×M], M)
END
END
END FINDX ;
END RELAXATION AND FINDX ;
        GO TO REREAD ;
EXIT: END.
```

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VITA

Srisakdi Charmonman was born in Bangkok, Thailand on March 3, 1938. His father's former positions include Dean of the School of Law (Thammasart University), Minister of Justice, and acting Prime Minister of Thailand. He attended a private school through the fourth grade and a governmental secondary school until the tenth grade. In the first trisemester examination of a class of 120 in the ninth grade he was first with "extraordinary high percentage" and was promoted to the tenth grade without having to complete the ninth. In a governmental high school he was elected as cheer leader, selected as an adviser to the school year book and graduated as one of the first fifty out of about 20,000 students. In a general election in Thailand, he was appointed committee member of a voting poll and also chairman of that poll. At the Faculty of Engineering, Chulalongkorn University, he was appointed secretary of the engineering student body of about 1000 members in 1958. He graduated as a Bachelor of Engineering as the first of a class of about 90 in 1959.

After having graduated from Chulalongkorn University with four years of military courses, he was enlisted in the Royal Thai Air Forces for one-month training, appointed 2nd Lieutenant and released as a reserve officer.

The SEATO Graduate School of Engineering granted him a full scholarship to study for a master's degree for the period September 1959 to March 1961. He graduated as a Master of Engineering majoring in Hydraulics as the first of a class of fourteen in 1961. His thesis was entitled "Design, Construction and Performance Evaluation of an Axial-Flow Pump to Use With

Thai-Style Outboard Boat Motors". During the time he studied at the SEATO Graduate School he gained some teaching experiences by holding a part-time position as an instructor of mathematics at an evening school. He taught arithmetic, algebra, geometry and trigonometry in the twelfth grade eight hours per week.

From March 1961 to December 1961 he was a research associate at the SEATO School. His duties were to set up, calibrate and design minor parts of new equipment in the Hydraulic Laboratory under the direction of the Director of Research.

As the first of the first graduating class of the SEATO School he obtained a scholarship to study for a doctor's degree in the United States. He chose to study at Georgia Institute of Technology where he could work with Dr. Carstens with whom he had been associated at the SEATO Graduate School of Engineering.

He became engaged to Miss Parasubhasri Subhajalasaya in December 1961 and they were married in November 1962. She will receive a Master's in Librarianship from Emory University, Atlanta, Georgia in June, 1964.

Mr. Charmonman is an associate member of the American Society of Civil Engineers, a member of the Association for Computing Machinery and a member of the Society of the Sigma Xi.

His interests are in writing Thai poems, and in translating English into Thai. He enjoys table-tennis and swimming when time permits.