

Two-Dimensional, Compressible Time-Dependent Nozzle Flow

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Rocket exhaust nozzles utilizing steep inlet cone angles and tight throat contours to produce high acceleration of the gas flow have several advantages over conventional nozzles. Nozzles of this type are shorter and lighter, have smaller surface area, and have fewer heat transfer problems. Detailed design and performance analyses on steep inlet ($\phi_0 \geq 30^\circ$) rocket nozzles have been virtually impossible in the past. Severe two-dimensional effects in the inlet cone and throat regions of these nozzles have invalidated classical one-dimensional analysis. A general two-dimensional solution of the entire flow field is required before detailed analysis can be initiated.

This work develops a computational technique and a computer program for fast and accurate solution of flow fields in severely contoured axisymmetric nozzles. An asymptotic time-dependent finite-difference method developed by Moretti and Abbett is used in the solution of the governing fluid flow equations. The method is not restricted to a simplified thermodynamic model, and the technique presented can be extended for solution of the complete Navier-Stokes equations. The importance of boundary condition analysis is discussed. Computational techniques consistent with the goals of this report are used in the development of the boundary regions.

The program developed can construct a flow field for isentropic

axisymmetric nozzles with severe wall curvature. It is demonstrated that the solution constructs flow fields for not only transonic but also subsonic and supersonic conditions. Solutions are compared with experimental data for several axisymmetric nozzles.

TWO-DIMENSIONAL, COMPRESSIBLE
TIME-DEPENDENT NOZZLE FLOW

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TWO-DIMENSIONAL, COMPRESSIBLE
TIME-DEPENDENT NOZZLE FLOW

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NOMENCLATURE

English Symbols:

A	= UC + VD
a	speed of sound
B	= UK
C	= $\partial Y / \partial X$
D	= $\partial Y / \partial r$
E	= DG
F	= CG
G	= a^2
g	dummy variable
H	= D/Y
h	specific enthalpy
I	grid point index on Z
J	grid point index on Y
K	= $\partial Z / \partial X$
L	= KG
M	mach number
N	number of time steps
P	pressure ratio, p/p_0
p	pressure
R	= $\ln \rho$
\bar{R}	gas constant

R_t	$= r_c/r_t$
r	cylindrical radial coordinate
r_n	radial coordinate of nozzle wall
r_t	radius at nozzle throat
r_c	throat radius of curvature
t	time
U	nondimensional axial velocity
V	nondimensional radial velocity
W	total velocity
x	cylindrical axial coordinate
x_η	axial coordinate of nozzle wall point
x_o	coordinate stretching parameter
Y	transformed radial coordinate
Z	transformed axial coordinate

Greek Symbols:

τ	wall tangential coordinate
η	wall normal coordinate
ϕ	wall inclination angle
γ	density
ν	ratio of specific heats
σ	nondimensional normal velocity
ν	nondimensional tangential velocity

Superscripts:

'	dimensional quantity
\rightarrow	vector quantity

Subscripts:

- o nozzle entrance conditions
- n indicates conditions at nozzle wall
- t indicates conditions at nozzle throat
- x nozzle exit conditions

SUMMARY

Rocket exhaust nozzles utilizing steep inlet cone angles and tight throat contours to produce high acceleration of the gas flow have several advantages over conventional nozzles. Nozzles of this type are shorter and lighter, have smaller surface area, and have fewer heat transfer problems. Detailed design and performance analyses on steep inlet ($\phi_0 \geq 30^\circ$) rocket nozzles have been virtually impossible in the past. Severe two-dimensional effects in the inlet cone and throat regions of these nozzles have invalidated classical one-dimensional analysis. A general two-dimensional solution of the entire flow field is required before detailed analysis can be initiated.

This work develops a computational technique and a computer program for fast and accurate solution of flow fields in severely contoured axisymmetric nozzles. An asymptotic time-dependent finite-difference method developed by Moretti and Abbett is used in the solution of the governing fluid flow equations. The method is not restricted to a simplified thermodynamic model, and the technique presented can be extended for solution of the complete Navier-Stokes equations. The importance of boundary condition analysis is discussed. Computational techniques consistent with the goals of this report are used in the development of the boundary regions.

The program developed can construct a flow field for isentropic axisymmetric nozzles with severe wall curvature. It is demonstrated that the solution constructs flow fields for not only transonic but also

subsonic and supersonic conditions. Solutions are compared with experimental data for several axisymmetric nozzles.

CHAPTER I

INTRODUCTION

Nature and Purpose of the Problem

Flows through supersonic nozzles are of interest in design and development and in basic research. In application, nozzles are used in jet and rocket engines and in measuring flow rates. In research they are used in wind tunnels and in the study of non-equilibrium effects (1). This study was initiated to develop a computational technique and a computer program for the solution of two-dimensional flows through nozzles by applying new techniques to the solution of the two-dimensional flow equations. Prediction of the flow is basic to the study of other effects such as chemical reactions and heat transfer.

During the last three decades, a major emphasis has been placed on understanding the aerodynamic design and performance of converging-diverging exhaust nozzles. Investigators have been hindered, however, by intrinsic difficulties associated with the flow solution. One of the major problem areas has been the solution of the transonic region. Emphasis in this paper is therefore placed on the throat region of the nozzle where the flow is transonic. Several interesting phenomena associated with the transonic region of the flow field have been noticed which have important ramifications for the design of rocket exhaust nozzles.

One of the critical factors in the design of exhaust nozzles is the containment of high-temperature gases. The cooling requirements may

be the limiting factor in the design of rocket exhaust nozzles. A method for reducing the heat transfer, then, could have far-reaching benefits. Numerous investigators (see References 1-4) have noticed that the heat transfer from the gas to the wall in a nozzle throat is appreciably less than a standard heat transfer correlation predicts. This phenomenon has been found to be dependent on the Reynolds number and the convergent half-angle, i.e., the acceleration of the flow. It was found that in the region of the throat of a supersonic nozzle, a reduction of as much as 50 per cent in heat transfer below that typical for turbulent boundary layer could be obtained by increasing the convergent half-angle (1,2). This suggests advantages in utilizing nozzle designs creating high acceleration of the flow. There are other factors, however, which must be considered before this can be done.

To achieve high accelerations in nozzles, steep ($\phi_0 \geq 30^\circ$) inlet cone angles and tight throat contours ($R_c \geq 1.5$) are required. Considerable deviations in pressure measurements from classical one-dimensional isentropic flow behavior have been observed in the transonic region of such nozzles. Pressure measurements have shown that deviations of as much as 30 to 45 per cent from that for one-dimensional flow occur just down stream of the throat (1,5). These deviations result from radial velocity components caused by the taper and curvature of the nozzle (6). Similar deviations have been observed where measurements were made in the divergent region of conical nozzles (1). Other investigators have observed this phenomena in the convergent (7) and throat (8) regions of converging-diverging nozzles.

Thus a nozzle using a steep inlet angle and a tight throat contour

to produce high acceleration of the gas flow has several advantages. Also, nozzles of this type are shorter, weight less, have smaller surface area and fewer of the problems associated with heat transfer from hot exhaust gases. To design and build a nozzle of this type, however, a heat transfer correlation at the wall is required. To theoretically evaluate the heat transfer, the boundary layer acceleration must be known. This, in turn, is dependent upon the velocity of the flow in the neighborhood of the nozzle wall. A solution for the free stream flow conditions, then, is required for use in solving the boundary layer flow. It has been shown, however, that the classical one-dimensional analysis is no longer valid for nozzles with high entrance angles and tight throat contours. A general two-dimensional solution of the entire flow field is therefore necessary before a boundary layer investigation can be initiated.

A solution is needed for two-dimensional isentropic flow which is valid throughout a supersonic nozzle. The literature reveals that such a solution is virtually non-existent (1). The reason lies in the varying mathematical character of the equations describing the flow through the nozzle. The equations for subsonic, sonic and supersonic flow are elliptic, parabolic, and hyperbolic respectively. Existing studies, therefore, usually entail analysis in three different flow regions; namely, the subsonic or convergent region, the transonic or throat region, and the supersonic or divergent region. The solutions are then coupled to describe the entire flow field.

The solutions of the three regions, however, are not independent. There is a definite order in which the regions should be solved. The

subsonic and transonic solutions are interdependent and should be solved simultaneously. The subsonic-transonic solution then provides the boundary conditions for the supersonic solution (9). In the following paragraphs the character of each of the three nozzle flow regimes is briefly discussed.

In the subsonic region the governing equations are of elliptic type. The solution is classified as a boundary value problem of potential theory. When either the stream function or the velocity potential is considered as the dependent variable for incompressible inviscid subsonic flow, the governing equations reduce to Laplace's equation. The solution must satisfy Laplace's equation everywhere within the interior of the flow. At the boundaries either the Dirichlet condition (the dependent variable specified), the von Neuman condition (the normal derivative of the dependent variable specified), or a combination of these conditions must be satisfied by the solution (10). This simplicity does not carry over to the transonic region.

The equations controlling transonic flow must describe the transition from subsonic to supersonic conditions. The resulting equations are a set of non-linear partial differential equations with variable coefficients which cannot be solved in closed form. This region therefore, is the most difficult of the three flow regimes to solve. In the absence of an exact solution, investigators have been forced to make simplifying assumptions and solve the resultant equations numerically. Numerical methods capable of solving the steady state transonic equations have been developed for irrotational flow in nozzles. These methods have been successful only for the solution of flow fields in nozzles with moderate

wall curvature. They have severe limitations when applied to the analysis of rocket nozzles. The presence of extremely large velocity gradients in the nozzle throat give rise to numerical instabilities in the calculations. This causes the accuracy of the solutions to rapidly deteriorate. The solution is also erroneously uncoupled from the subsonic region.

In the supersonic region, the system of equations is hyperbolic. The solution is generally obtained by the method of characteristics, which uses a set of given data along an initial starting line to solve the equations of motion at a discrete set of points on an adjacent line. This is accomplished by transforming the governing partial differential equations into a characteristic coordinate system and numerically integrating the resulting system of ordinary differential equations along predetermined characteristic lines. This procedure is then repeated until the desired portion of the supersonic flow field is constructed (11). Solution of the two-dimensional supersonic equations by the method of characteristics is well developed but the flow conditions must be specified on a line upstream of the region to be solved. This boundary condition can be obtained only by solving the transonic region.

The problem then, is to develop a two-dimensional solution technique for the subsonic-transonic region of a conical converging-diverging nozzle with a steeply inclined entrance cone and a tight throat contour. This solution can then provide the boundary conditions necessary for the solution of the supersonic region by the method of characteristics.

In the following sections the techniques available for the solution of the combined subsonic-transonic region are presented. Their relative merits and disadvantages are discussed and the method of solution is

selected. The problems associated with the chosen technique are then analyzed to determine the most promising technique for solution of the entire flow field.

Previous Related Studies

The early studies of transonic two-dimensional and axisymmetric flow involve velocity perturbations about the sonic velocity. The continuity equation can be re-written in terms of a velocity perturbation potential and its partial derivatives. Meyer (12) first obtained a solution to this equation by expanding the perturbation potential in a power series and assuming a linear velocity distribution along the nozzle axis. Lighthill (13) made use of the series solution to make a qualitative analysis of the behavior of the flow near the sonic line. The method can be applied conveniently only to the indirect (or design) problem. That is, the flow field is developed dependent upon the assumed centerline velocity distribution. Any streamline may be a wall and, therefore, once the solution is obtained, the flow field for the streamline contour produced is known. Application of the method to the direct (or performance) problem is cumbersome and time-consuming because the centerline velocity distribution which will produce a given wall contour is not known.

The direct problem for symmetric two-dimensional and axisymmetric flow was first solved by Taylor (14) and Hooker (15) respectively. Using a double power series, Taylor evaluated the velocity perturbation potential up to and including fourth order terms. This involved the simultaneous solution of eight equations for the eight unknown series coefficients.

The perturbation solutions are fundamental in their approach. The

evaluation, however, of a double power series expansion for the general equations of motion is a major effort even for the simple case of a linear axial velocity distribution. The complexity cannot be justified, especially when the method cannot be conveniently utilized for performance analysis of nozzles. To overcome this drawback various authors have simplified the equations of motion and obtained approximate solutions for transonic flow in a nozzle.

Sauer (68) was the first to make a major simplification to the equations of motion. He wrote the governing equations in terms of the velocity perturbation potential. Then noting that several of the terms approached zero in the vicinity of the throat, he retained only the first order factors in these terms. This produced a series solution which was the first three terms of Meyer's solution. The technique was found to be applicable for nozzles with low inlet cone angles only. Several attempts have, therefore, been made to improve Sauer's original solution. Yur'ev (16) obtained a solution by including an extra term and Sims (17) expanded the power series solution to five terms. Mendelson (18) extended Meyer's power series solution by formulating recurrence relationships for the general series coefficients in terms of the velocity distribution specified along the nozzle axis. In all these cases no substantial improvement was made in accuracy over Sauer's original solution.

Oswatitsch and Rothstein (19) in an effort to eliminate the need to specify the axial velocity distribution, developed an iterative solution based on successive approximations to the flow field. Although Oswatitsch (20) later showed that the numerical technique was unstable when applied to nozzles with steep inlet cone angles, their work became

the basis for many investigators.

The most significant factor that influences the transonic flow pattern is the wall radius curvature in the throat region. Realizing this, Hall (21) produced a technique for symmetric nozzles. Using a perturbation technique, he wrote the series expansion in inverse powers of R_t , the ratio of throat radius of curvature to throat radius. Subsequent studies typified by the works of Moore and Hall (22) and Quan and Kliegel (23) have extended Hall's original solution to two-dimensional and annular nozzles with arbitrary profiles and dual gas flows. The solutions have shown favorable results, however, only for slender nozzles ($\phi_0 \geq 30^\circ$, $R_t \geq 1.5$). Increase the accuracy of the method for R_t less than one, Kliegel and Levine (24) reformulated the series expansion to inverse powers of $(R_t + 1)$. The method solved only the transonic flow region. The interdependency of subsonic and transonic solutions was not taken into account.

The streamline procedure developed first by Friedrichs (25, 26) is an attempt to improve on the perturbation methods. The procedure utilizes the full nonlinear partial differential equations of motion for inviscid, irrotational, isentropic transonic nozzle flow. The equation of continuity for steady, axisymmetric flow is expressed in terms of the stream function and the velocity potential. A transformation is then made using the velocity distribution along the nozzle axis. The resulting system of partial differential equations is then solved by a series expansion of the stream function. The method determines the flow field in both the subsonic and supersonic regions.

The streamline procedure has been adapted to the two-dimensional

problem by Liepman (27). Gray (28) generalized the technique to allow any curve in axisymmetric or two-dimensional flow to be selected as the reference line along which the velocity distribution is specified.

Hopkins and Hill (29, 30) and Thompson (31) have utilized the method in the study of asymmetric, two-dimensional annular plug, expansion-deflection type nozzles and two-dimensional curved channels.

Other procedures using a streamline technique have been formulated which numerically iterate the equations of motion across the flow field. The results of an iteration are used to approximate the partial derivatives in the axial direction for the next iteration. Utilizing a given velocity distribution along the axis Pirumov (32) constructed the transonic solution in a converging-diverging nozzle. Zupnik and Nilson (33) generalized the approach to solve the direct problem in two-dimensional and axisymmetric nozzles.

A variety of flows was analyzed by Emmons (34, 35) using a modified version of the classical relaxation technique discussed by Southwell (36). Hyperbolic nozzles were examined for a range of flows extending from the fully subsonic flow case to the shock free subsonic-supersonic flow case. The relaxation method was not as formalized as other methods and success was often dependent on the skill, intuition and problem knowledge of the practitioner.

A much-used procedure in the Soviet Union is the method of integral relations (39, 40). The method applies to problems in two-dimensional isentropic mixed flow. The computational region is first divided into a number of axial strips. The governing differential equations are then numerically integrated across these strips; while

the method of characteristics is used to find disturbance movements between strips.

A geometrical technique was developed by Ringleb (41, 42) which uses piecewise circular arcs to approximate the streamlines and equipotential lines. Using the infinite series expansion of Oswatitsch and Rothstein (19), Ringleb constructed the flow field in two-dimensional and axisymmetric nozzles. His procedure was later extended by Chou and Mortimer (41) by including an iterative boundary point computational technique which reduced the amount of specified information required to solve the flow. The method has been shown by Holt (42) to be numerically limited to nozzles with small inlet cone half-angles.

Selection of the Method of Solution

As is shown in the previous section, many methods have been developed which are capable of solving the steady-state equations for near-sonic conditions. These methods do not, however, completely couple the subsonic flow to the sonic flow and cannot solve the subsonic flow field. They also cannot handle rotational non-isentropic flows and are not considered in this paper. An alternate approach which does not have these restrictions and which may be used to solve the complete Navier-Stokes equations considers the mixed flow problem as an initial value problem in time. Presently only two solution methods for this approach are feasible: (1) the method of characteristics and (2) direct substitution of finite-difference approximations for the partial derivatives in the equations.

The inclusion of time in the basic flow equations as a third

independent variable alters the nature of the equations such that they are of the hyperbolic type throughout the flow field. The method of characteristics can therefore be used to solve the time dependent equations of motion for the entire nozzle. This is accomplished by deriving the compatibility equations from the basic equations and numerically integrating these equations along characteristic curves. This is a three-dimensional characteristic problem and results in numerous complex computer programs for the solution of the flow. Although the method of characteristics appears to be the most accurate method available, the time to write and execute these massive programs severely limits the utilization of the method for nozzle design or performance analyses.

The direct substitution of finite differences for the partial derivatives in the flow equations also has several problem areas which impede development of a general working technique for application to nozzle mixed flow solutions. The major problems associated with the approach are the proper treatment of boundary conditions and numerical stability of the difference equations. These problems are of a technical nature, however, and a well developed computational technique should be able to overcome these obstacles. The approach contains none of the fundamental errors associated with the methods discussed in the preceding section. Much work has therefore been done in this area in an effort to iron out the problems associated with the computation of mixed flows by the time-dependent finite-difference method.

The approach was originally suggested by von Neumann and Richtmyer (43). Lax (44) implemented their technique by writing the

equations in divergence-free form (conservation form) and replacing space and time derivatives with center and forward differences, respectively. He was successful in obtaining solutions for unsteady one-dimensional flows with shocks. Lax and Wendroff (45) extended the technique to include systems of equations in three independent variables. Their technique is referred to in the literature as the Lax-Wendroff one step method. They also developed the Lax-Wendroff two-step method to reduce computer storage and execution time requirements. Bernstein (46, 47) applied the technique to the solution of several multi-dimensional flow problems. Crocco (48), Fromm (49) and Thommen (50) have devised time-dependent finite-difference methods for solving the Navier-Stokes equations in various forms. These methods are formidable, however, due to the complex nature of these equations.

Steger and Lomax (51) suggested using a time-dependent relaxation technique. In the implementation of their suggestion, however, it was found that the relaxation technique required much refinement before it could effectively be used to solve transonic flow problems. Prozan (37, 38) developed the error minimization technique to improve on the existing relaxation methods. In this method the governing equations are rewritten in terms of a residual error. The set of differential equations are solved simultaneously and the residual reduced until the desired flow field is developed.

Many other techniques have been devised to obtain solutions by the time-dependent method. Several authors (52-56) have compared these numerical techniques on the basis of ease of coding, spatial and temporal resolution and execution time. The results of these compari-

sons indicate that a divergence form of the Lax-Wendroff one-step method originated by Moretti and Abbett (54) shows the most promise for the solution of the problem at hand. Some of the desirable characteristics of this method are:

(1) It is a direct method in the sense that the nozzle geometry is prescribed and controls the subsequent computation.

(2) The desired accuracy of the solution is set by the spatial grid size and not by a reformation of the analysis.

(3) The method is not restricted to a simplified thermodynamical model.

(4) It requires a relatively short execution time on a high speed computer.

(5) The computer storage required is relatively small.

For these reasons the time-dependent finite-difference method of Moretti and Abbett appears to be the best technique for obtaining a rapid, accurate solution to the transonic flow problem in an axisymmetric, rapidly converging-diverging nozzle. In the development of this method for solution on a high-speed computer, the following guidelines were used:

(1) The computer program should not require excessive execution time.

(2) The computational methods should be adaptable to the most general flow problem.

(3) Boundary point computational techniques should not be strictly mathematical in nature but should be chosen on the basis of physical considerations of the flow.

Experimental Studies

Early experimental investigations of gas flows through conical nozzles show the two-dimensionality of the flow but do not provide data for nozzles with high entrance cone half-angles or tight throat contours. Scheller and Bierlein (73) and Campbell and Farley (74) made measurements in the divergent region of conical nozzles. Fortini and Ehlers (7) recorded wall static pressure measurements in the convergent region and Stanton (8) measured velocity distributions in the throat region.

Nozzles of the type considered in this paper were studied by Back et al. (1, 4, 6, 69-71). The experimental measurements were made primarily on two severely contoured nozzles. The nozzles have a 15 degree divergent cone half-angle and 45 and 30 degree convergent half-angles with R_t equal to 0.625 and 2.0 respectively. They present static pressure measurements at the centerline, wall and various radii for axial stations along the axis of the nozzles. Mach number distributions are detailed and heat transfer and boundary layer data are presented. Shelton (72) made static pressure measurements in a conical nozzle with 30 degree convergent and 15 degree divergent half-angles for R_t ranging from 0.35 to 1.0. These reports represent the extent of experimental research on severely contoured conical nozzles.

CHAPTER II

TECHNICAL DISCUSSION

Treatment of Boundaries

The equations governing fluid flow are called the indefinite equations of motion. That is, the equations apply to any fluid flow problem in general, but do not define a specific problem. A proper set of boundary and/or initial conditions is required before a specific problem can be solved. For each problem there are a number of necessary and sufficient boundary conditions. In treatment of the equations by numerical techniques it is all too easy to overspecify boundary conditions. Equally disastrous, but not as common, is the under-specification of these conditions. But this difficulty is not reserved to the numerical investigator. Proper treatment of boundary conditions is the outstanding problem area for all the transonic flow solution techniques discussed in the introduction. In numerical techniques, however, the difficulties are compounded by the absence of a mathematical analysis of stability at the boundaries. Moretti (57) has indicated that the oscillations associated with numerical methods are generated at the boundaries and are not a fault of the numerical technique. Prozan (37) makes a special effort to point out that treatment of boundary conditions is the foremost problem associated with numerical solution of the flow field in a converging-diverging nozzle. This area must be studied extensively before any rewarding results can be obtained by

numerical techniques.

Subsonic Entrance

The entrance region offers a perplexing problem for the analytical investigator of Laval nozzle flow. In numerical solutions, the flow field is overlaid with a grid where intersection points on the grid are assumed to represent the area surrounding the point. The numerical approximation approaches the exact solution as the grid point spacing is reduced. The use of the grid requires that an entrance line be defined somewhere upstream of the area of interest where flow properties are known. This entrance must accurately represent the characteristics of the flow entering the nozzle. But the location of the line must not be so far upstream as to make the computational region excessively large. This increases computer execution times. Since the entrance flow is constant, one might choose an entrance line based on grid size considerations and assume constant values on it. In subsonic flow, however, any point is affected by all the other points in the flow. Changes in the downstream subsonic computational region create disturbances which propagate upstream. These disturbances must pass through the entrance plane (57). On an arbitrarily set computational entrance line, then, values must be up-dated in accord with the propagating wave. An arbitrary truncation of the subsonic flow field with constant flow properties assumed at the entrance cannot be used to model flow conditions at the entrance to a nozzle. Several authors have used this technique (37, 59) presumably under the assumption that the effect of wave propagation is negligible. Their results, however, are poor. The authors who have devised methods for computing the entrance flow with

some physical insight are briefly discussed in the following paragraphs.

Laval (60) utilizing the Lax-Wendroff two-step method for numerical solution of transonic flow, uses a parabolic extrapolation from the downstream flow on an arbitrarily assigned entrance plane. The value of the axial component of velocity is then corrected by assuming that the mass flow rate through the entrance section at time $t + \Delta t$ is equal to the mass flow rate through the throat section at time t .

Migdal, et al. (61) use a stretching of the axial coordinate which places the entrance at a station an infinite distance upstream from the throat. By using a constant angle inlet cone, the area at this point is infinite. The flow variables at the entrance, therefore, remain constant with time and equal to their stagnation values. Disturbances generated at the throat cannot reflect from an entrance plane of this type. This is also the procedure used by Prozan and Kooker (38) with their error minimization technique.

Serra (62), using the Lax-Wendroff one-step method, developed a technique utilizing a two-dimensional method of characteristics analysis to evaluate the flow variables on an arbitrarily assigned subsonic entrance plane. Three of the four dependent variables are specified a priori at each entrance grid point and the remaining parameters are determined by reducing the inviscid flow equations to normal form and employing a modified characteristics construction. The value of the fourth parameter is used to calculate values for remaining variables and the procedure is repeated until convergence is obtained to the values at advanced time.

The coordinate stretching was chosen as the best method for

treatment of the entrance region for the present problem. Although the method of Serra appears to be a valid technique, the execution time required for an iterative characteristics construction is not justifiable. The coordinate stretching method has the accuracy of the characteristics technique and much smaller execution times. Care must be taken, however, to supply enough upstream, 'buffer', grid stations to prevent the flow from abruptly changing from an infinite reservoir to a finite cross-section. If the buffer region for a particular problem is found to require an excessive number of grid stations, the execution time may also be excessive. Under these conditions the Serra technique may become the more acceptable method for computation of the values on the entrance plane.

Wall Points

In numerical solution of a flow field, information is transmitted from point to point via computation of finite-differences. Boundary grid points influence their neighbors and boundary condition information is transmitted into the flow field. At each computational step, therefore, the values of normal velocity, tangential velocity, and pressure must be calculated at the nozzle wall. The only proper boundary condition on the wall is the vanishing of the normal component of velocity. A wall point computational method must evaluate the boundary values using the boundary condition and information from interior points. Authors vary considerably in their treatment of this problem.

Laval (60) and Prozan et al. (37, 38) use a simple parabolic extrapolation from interior points to obtain values for wall points. These extrapolated values are used for the next time step and extrapo-

lation repeated. Errors are therefore compounded as the computation progresses. Such a procedure does not model the physical behavior of the flow. The wall partial derivatives obtained are dependent on the geometrical nature of the extrapolation curve rather than the physical properties of the flow.

Lapidus (59), using the Lax-Wendroff two-step method, devised an interesting technique for treatment of wall points. Property values at several points surrounding a particular wall point are averaged and the change in properties through the region are calculated. The calculated change is then used to up-date the wall point in such a manner as to make the momentum vector parallel to the wall. Although the physical characteristics of the flow were considered in the development of this technique, it has not yielded acceptable results.

Several authors (10, 46, 64) use the same difference equation at the wall as is used in the interior of the flow. This is accomplished by use of the reflection technique. That is, a virtual grid line is assumed beyond the wall. The values along this line are assumed to be a mirror image of the internal grid line immediately adjacent and parallel to the wall. This causes the normal derivatives of all dependent variables to vanish at the wall. This is legitimate only for the normal velocity. Forcing the remaining partials to zero is physically wrong.

Moretti and Abbett (63), recognizing that two-dimensional time-dependent equations are hyperbolic, utilize a quasi-one-dimensional method of characteristics to evaluate the flow parameters on the nozzle wall. Initially, first order Taylor series are used to obtain provi-

sional values at the wall. A point on a characteristic line which lies in the interior at the old time step is located by an iterative process. The values of the flow parameters are interpolated from the surrounding grid points. The compatibility equation is then integrated along the characteristic line to obtain the value of the pressure at the new time. This value is then compared with the value generated by the Taylor series and a correction made dependent on the difference in the two values. The procedure is repeated until convergence is obtained to the new time values. This method has been used by several authors (64, 65) with favorable results. The technique was programmed for evaluation of wall points for the axisymmetric nozzle problem of the present study. The iterative computation was found to double program execution times. The severe two dimensional effects in the nozzles of this study make the one-dimensional nature of the characteristics construction unjustifiable. The method was therefore discarded in favor of a simpler technique.

In this work, wall points are computed by evaluation of the governing equations in their reduced form at the wall. The equations are transformed to a normal-tangential-coordinate system and the normal component of velocity is set equal to zero. Central and backward differences are substituted for the derivatives in the axial and radial directions respectively. This method is consistent with physical behavior at the wall and reduces execution time substantially.

Centerline

Due to symmetry at the nozzle centerline, the behavior of the flow is characterized by the vanishing of all radial partial derivatives.

In numerical methods, several techniques are available for imposing this condition. In terms of computational accuracy, the centerline techniques are approximately equivalent. The choice of a method for the present problem therefore depends on the execution times required by each method.

The use of the reflection technique, discussed earlier, is completely valid for this case. The technique automatically sets all derivatives in the direction of the reflection equal to zero. The same difference equations applied in the interior can be utilized at the centerline. However, special consideration must be given the term V/r [see equation (16)]. By l' Hopital's Rule this term is zero and can be simply deleted from the governing equations at the centerline. Otherwise the equations remain unaltered. This approach requires the largest execution time of the techniques considered.

Alternately, the governing equations may be reduced by setting radial partial derivatives and V/r to zero. If the computer program is coded using the reduced equations, execution times are reduced since the routine does not evaluate radial derivatives at the centerline.

The method used in this paper for computation of property values at the centerline, reduced execution times still farther. By utilizing a series expansion approximation for each variable on the centerline in powers of the radial coordinate the condition of symmetry is imposed with minimal execution time.

Supersonic Exit

Since the flow equations for supersonic flow are hyperbolic, disturbances can only travel in a downstream direction. The upstream

flow solution is therefore insensitive to the method used for computation of the flow properties at the exit boundary. The general procedure for computation of this boundary is to use a simple linear extrapolation from upstream points.

Stability

It is an unfortunate fact that many times an attempt to solve a partial differential equation by a finite-difference technique leads only to a result which is completely unacceptable. The difference equation may have a rapidly growing and oscillating solution which bears no resemblance to the solution expected from the original differential equation. This results from computational instability (66).

Von Neumann (43) proposed a method utilizing Fourier components which could define the computational stability limits for a linear difference equation with constant coefficients. Courant, Friedrichs, and Levy (74), recognizing that a "domain of dependence" exists in hyperbolic equations, derived the familiar restriction

$$C \frac{\Delta t}{\Delta x} \leq 1 \quad (1)$$

Where:

c = constant
 Δt = time increment
 Δx = space increment

Their computational stability condition, often called the "Courant condition," restricts the distance a wave travels in one time increment to less than one space increment. Lax and Wendroff (45) analyzed their numerical method using a similar Fourier technique. The resulting

linearized stability condition is used consistently by the numerical investigators cited. For the present case this condition states that:

$$\Delta t \leq \frac{\Delta x}{\sqrt{8(U+a)}} \quad \text{and} \quad \Delta t \leq \frac{\Delta y}{\sqrt{8(V+a)}} \quad (2)$$

The equations describing motion of a fluid, however, are a set of coupled, nonlinear, partial differential equations with variable coefficients. Hirt (66) points out that Fourier stability analysis cannot predict instabilities in this type equation. Fourier analysis neglects several terms which contribute to instabilities. As reported by Lax and Wendroff (45), however, Burstein (46) found stability for values of t larger than those permitted by equation (2). The linearized stability analysis would then appear to be conservative for the variable coefficient problem. Equation (2) is therefore used as the stability condition in this paper.

Initial Conditions

Time-dependent methods require initial values of flow quantities at all points. The choice of initial conditions is somewhat arbitrary. The solution is asymptotic and apparently the flow will eventually approach steady-state conditions no matter what initial data is assumed. If the initial conditions come close to representing the steady state flow, however, a faster convergence to the final solution will be obtained. An initial guess which is substantially distant from the steady state values may produce an initial flow which is too violent. This may cause the computations to become unstable. This is an example of non-linear instability. For linear systems, stability is not in-

fluenced by the size of the initial data (59). The Lax-Wendroff schemes and the Moretti and Abbett technique are non-linear systems and therefore could become unstable given initial data which is far from the steady state solution. The general procedure for the two-dimensional and axisymmetric solution is to use the one-dimensional solution as initial conditions for the time-dependent difference equations.

In this study the one-dimensional solution used for initial conditions is altered to make the velocity vector parallel to the nozzle wall. The radial velocity is caused to decrease linearly across the nozzle to zero at the centerline. This places the initial conditions somewhat closer to the expected steady state solution; execution time is reduced and instabilities are avoided.

CHAPTER III

ANALYTICAL DEVELOPMENT

Coordinate SystemsInterior Points

A cylindrical coordinate system, fixed in the nozzle, is used in formulation of the governing equations in the physical plane (Figure 1). The axial coordinate is the centerline of the nozzle. The origin is situated at the nozzle throat with r representing the radial coordinate.

Because a time-dependent finite-difference technique is used to compute properties at interior points, a uniform mesh grid is desired for simplicity in formulating expressions for partial derivatives. By means of a coordinate transformation the physical plane of Figure 1 can be mapped into a rectangular region as shown in Figure 2. The region can then be divided into various constant Y and Z intervals. In the transformed plane (Figure 2) the radial coordinate varies between zero and one. That is

$$Y = \frac{r}{r_{\eta}} \quad (3)$$

In order to prescribe subsonic boundary conditions at the nozzle entrance (Line AB, Figure 1), such that disturbances are not reflected, it is necessary to consider the entrance plane as infinitely far from any

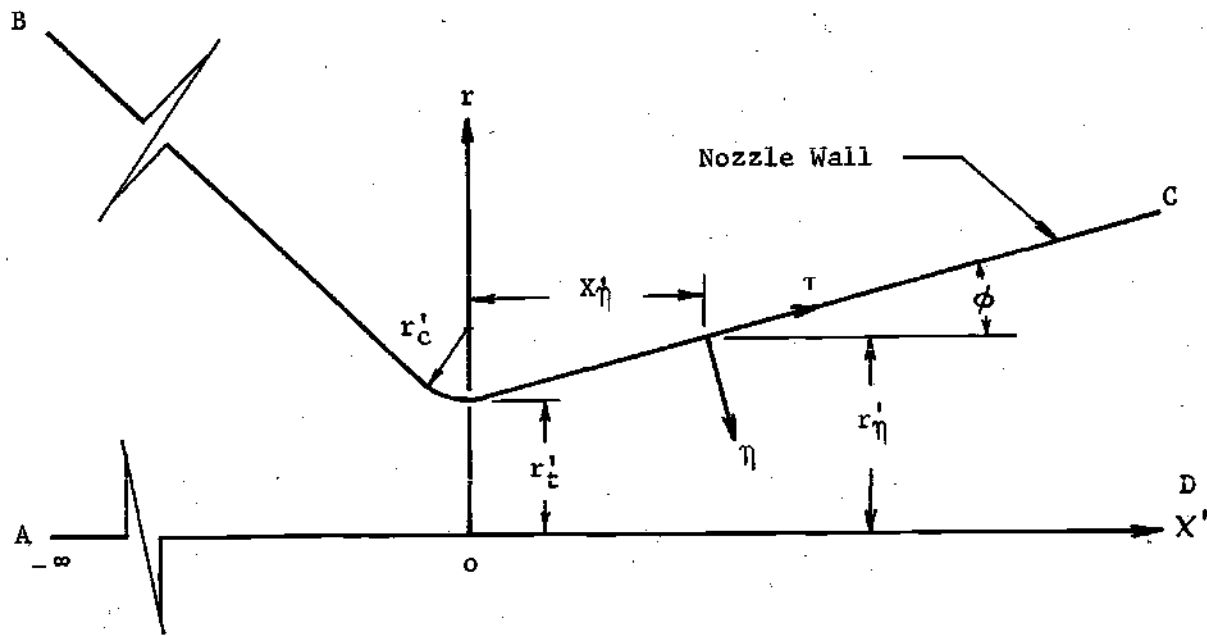


Figure 1. Physical Plane Axial-Radial and Normal-Tangential Coordinate Systems

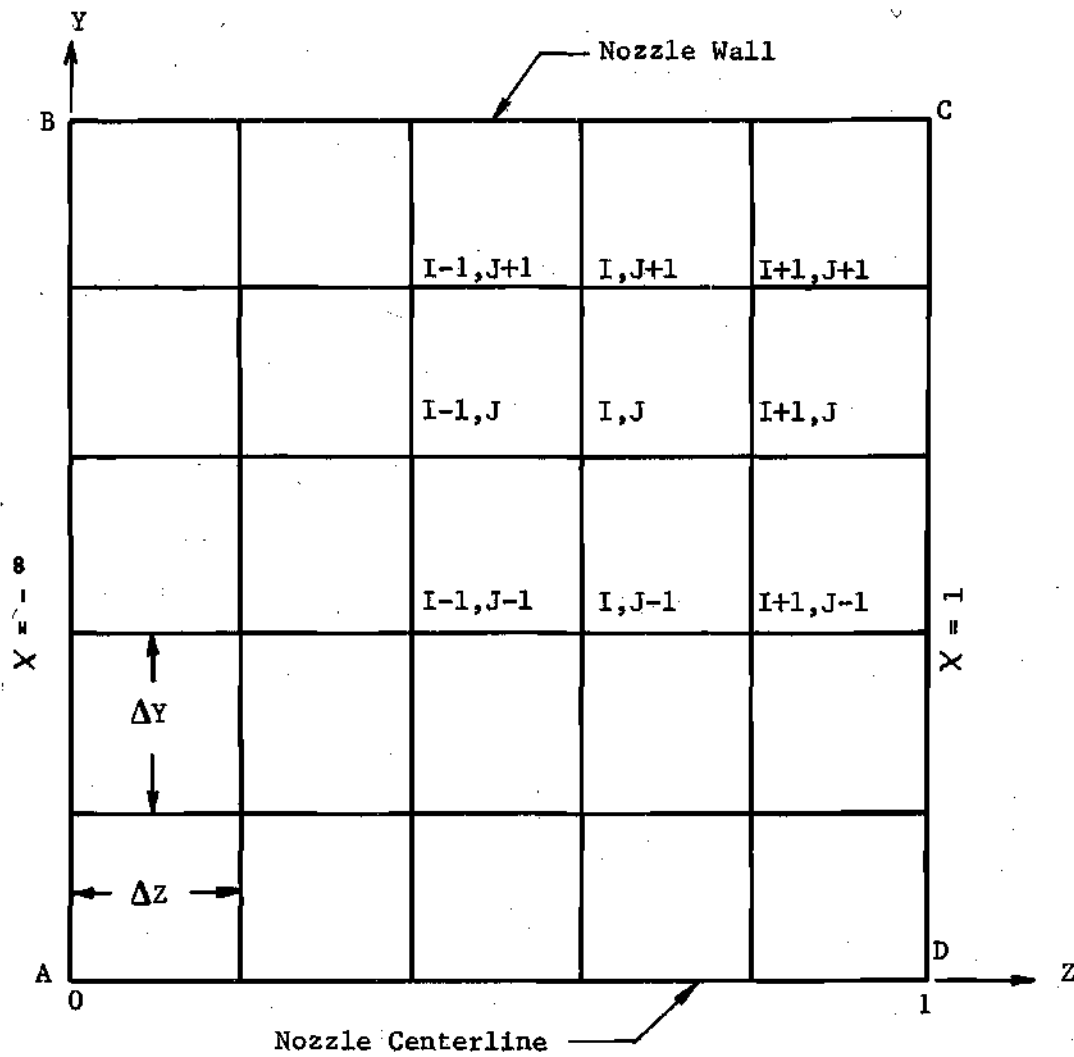


Figure 2. Finite Differencing Grid in the Transformed Plane

source of disturbances. The transformation

$$Z = \frac{1 + \exp(-2/X_0)}{1 + \exp(-2x/X_0)} \quad (4)$$

places the nozzle entrance (Line AB, Figures 1, 2 and 3) at $X = 1$. A rectangular grid on the transformed plane (see Figure 2) appears as an exponentially spaced grid on the physical plane (Figure 3). This allows the computational plane to extend to upstream infinity where flow parameters take on their stagnation values while permitting a closely spaced grid in the throat region ($-1 \leq X \leq 1$) where the greatest accuracy is required. The relative number of axial grid divisions falling within the throat region is controlled by the value of the stretching parameter, X_0 . Notice in Figure 4 that for equal divisions on Z , the corresponding number of divisions or X falling within the throat region decreases as X_0 is increased.

Wall Points

Since the normal component of velocity at the nozzle wall vanishes, a coordinate transformation from the cylindrical, x - y , coordinate system to a system in which the coordinate directions lie normal and tangential to the wall is desirable. A coordinate system is chosen in which the tangential coordinate is positive in the direction of the flow and the normal coordinate is positive inward (Figure 1). The origin resides at the particular wall point at which the properties are being computed. The new coordinate system, therefore, moves from wall point to wall

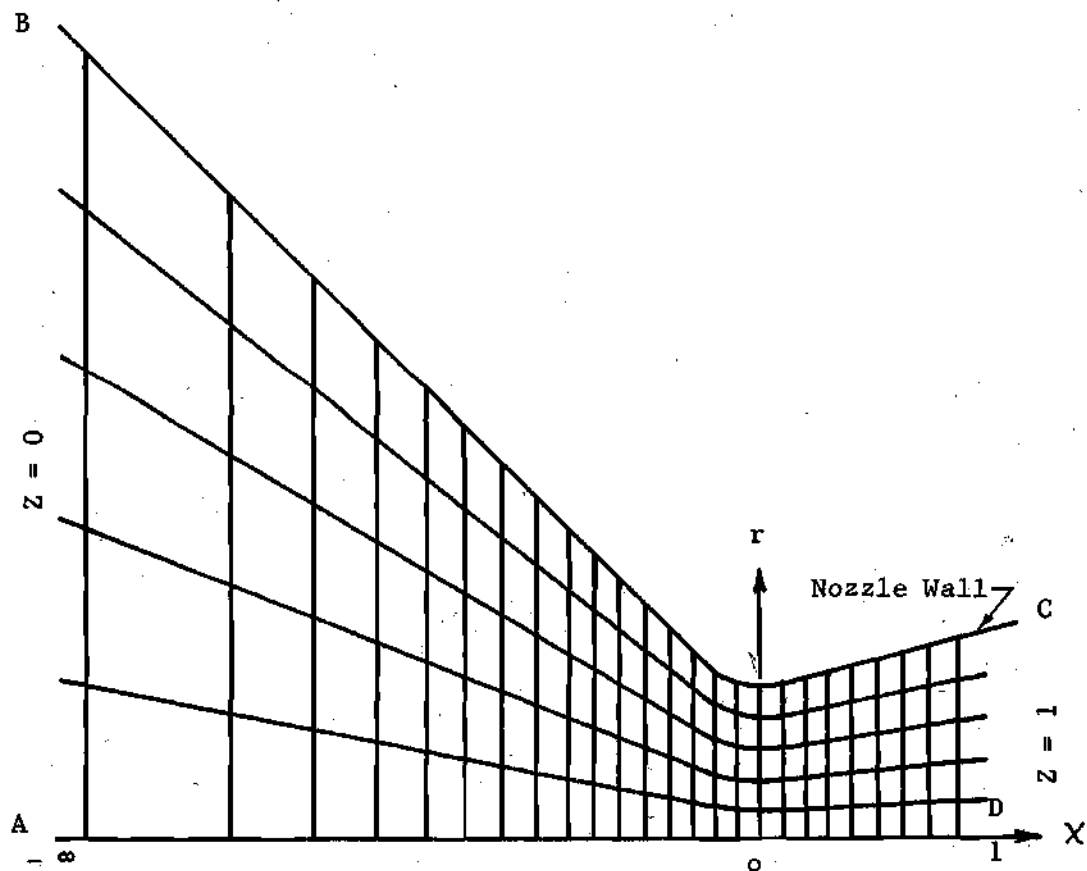


Figure 3. Finite Differencing Grid in the Physical Plane

$$z = \frac{1 + c \frac{-2}{x_0}}{1 + c \frac{-2x}{x_0}}$$

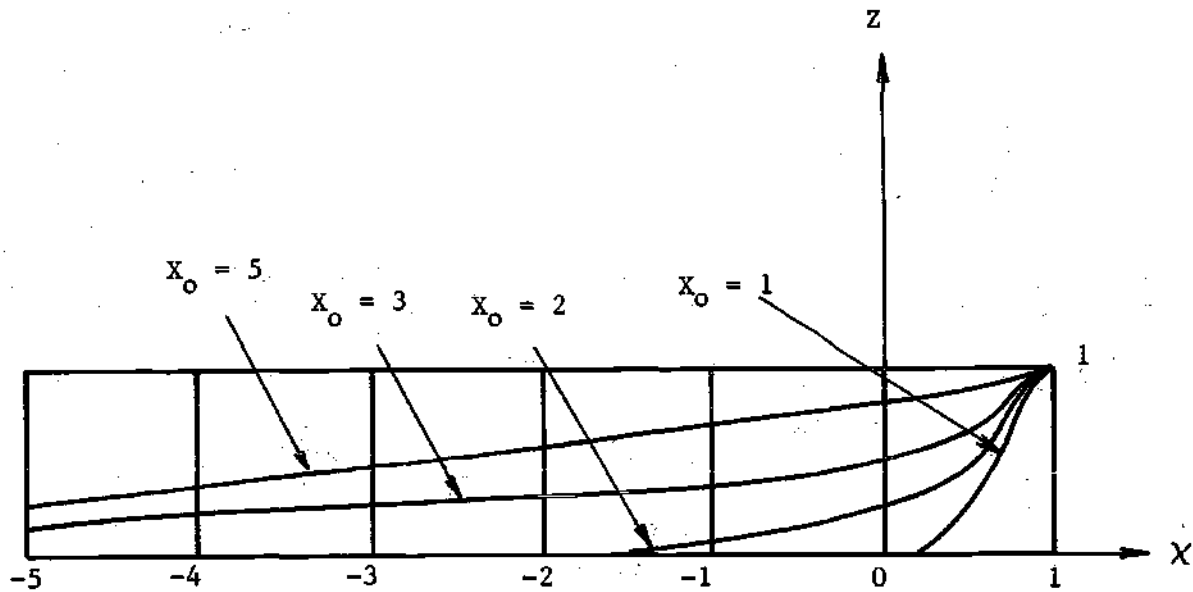


Figure 4. Effect of Stretching

point as the flow calculation develops. Therefore,

$$\tau = (X - X_\eta) \cos \phi + (r - r_\eta) \sin \phi \quad (5)$$

$$\eta = (X - X_\eta) \sin \phi - (r - r_\eta) \cos \phi \quad (6)$$

and

$$X = X_\eta + \tau \cos \phi + \eta \sin \phi \quad (7)$$

$$r = r_\eta + \tau \sin \phi - \eta \cos \phi \quad (8)$$

where x and y are the coordinates of the wall point under consideration. The angle ϕ is the inclination of the η - τ coordinate system with respect to the axial direction which is also equal to the wall inclination angle.

Non-dimensionalization Procedure

All coordinates and flow properties must be non-dimensionalized so that results are applicable to more than one particular nozzle-flow situation. This process should be carried out in such a manner as to leave the governing equations essentially unaffected with regard to their form. The area of particular interest lies in the transonic flow region. The significant length parameter is therefore taken to be the radius of the nozzle at the throat, r_t' . All length parameters are non-dimensionalized with respect to this significant radius. Pressure and density are non-dimensionalized with respect to their values at the nozzle entrance, (ρ_0', p_0') . The nozzle entrance is situated at an infinite distance upstream where the values of pressure and density remain constant at their stagnation values. The square root of the ratio of ρ_0 to ρ_0' is a measure of the speed of sound at the inlet and is used in non-dimensionalization of all velocity parameters. The

actual speed of sound at the inlet, $(\tau p'_0 / \rho'_0)^{\frac{1}{2}}$ was not used because it complicates the form of the non-dimensionalized flow equations. Time, which has the units of a length divided by a velocity, is therefore non-dimensionalized with respect to $r'_t / (\rho'_0 / p'_0)^{\frac{1}{2}}$. If any of the English Engineering systems of units is used the gravitational constant must be included in the expressions for velocity and time.

The non-dimensional parameters are defined as follows:

$$\begin{aligned}
 x &= x'/r'_t \\
 y &= y'/r'_t \\
 \eta &= \eta'/r'_t \\
 \tau &= \tau'/r'_t \\
 p &= p'/p'_0 \\
 \rho &= \rho'/\rho'_0 \\
 u &= u' / (p'_0 / \rho'_0)^{\frac{1}{2}} \\
 v &= v' / (p'_0 / \rho'_0)^{\frac{1}{2}} \\
 \sigma &= \sigma' / (p'_0 / \rho'_0)^{\frac{1}{2}} \\
 \gamma &= \gamma' / (p'_0 / \rho'_0)^{\frac{1}{2}} \\
 a &= a' / (p'_0 / \rho'_0)^{\frac{1}{2}} \\
 t &= t' r' / (p'_0 / \rho'_0)^{\frac{1}{2}} \quad (9)
 \end{aligned}$$

Development of Governing Equations

The working fluid is referred to as a perfect gas. By this is meant that surface effects, magnetic effects, electrical effects and chemical effects are not significant. The gas can be considered a pure substance which remains in a single phase. The specific heats are constant. The gas is non-viscous and obeys the ideal gas equation of

state:

$$p' = \rho' \bar{RT}' \quad (10)$$

The thermodynamic state of the system is, therefore, defined by any two independent properties. The remaining properties are related by the equation of state. With the inclusion of velocity as a third dependent variable the flow field is completely defined. Since the flow is defined by three dependent variables, three independent equations must be utilized to solve the flow. Assuming the flow to be reversible and adiabatic with negligible body forces, these equations are

conservation of mass

$$\frac{D\rho}{Dt} + \rho \Delta \cdot \vec{V} = 0 \quad (11)$$

conservation of momentum

$$\rho \frac{D\vec{V}}{Dt} + \Delta p = 0 \quad (12)$$

conservation of energy

$$\rho \frac{Dh}{Dt} = \frac{Dp}{Dt} \quad (13)$$

The equations of conservation of mass and momentum are expanded into cylindrical coordinates, assuming axisymmetric flow. The equations are non-dimensionalized as indicated previously. The expression relating pressure and density from the conservation of energy equation is utilized to eliminate pressure from the equations. For convenience

$$R = \ln \rho \quad (14)$$

is defined. As a result

$$a^2 = G = \tau \exp [(\tau - 1) R] \quad (15)$$

Using the newly defined parameter R and the resulting expression for the governing equations in non-dimensional form are

conservation of mass

$$\frac{\partial R}{\partial t} + U \frac{\partial R}{\partial x} + V \frac{\partial R}{\partial r} + \frac{\partial U}{\partial x} + \frac{\partial V}{\partial r} + \frac{V}{r} = 0 \quad (16)$$

conservation of axial momentum

$$\frac{\partial V}{\partial t} + V \frac{\partial V}{\partial r} + U \frac{\partial V}{\partial x} + G \frac{\partial R}{\partial x} = 0 \quad (17)$$

conservation of radial momentum

$$\frac{\partial V}{\partial t} + U \frac{\partial V}{\partial x} + V \frac{\partial V}{\partial r} + G \frac{\partial R}{\partial r} = 0 \quad (18)$$

Interior Points Coordinate Transformation

Using the coordinate transformations

$$Y = \frac{r}{r_t} \quad (19)$$

and

$$Z = \frac{1 + \exp(-2/X_0)}{1 + \exp(-2x/X_0)} \quad (20)$$

the governing equations are rewritten in the transformed plane. It is necessary to relate the fluid properties in the physical plane to those of the transformed plane. Any fluid property $g(x, r, t)$ in the physical plane is related to a fluid property $g(Z, Y, t)$ in the transformed plane

through the following equations:

$$\frac{\partial g}{\partial x} = K \frac{\partial g}{\partial Z} + C \frac{\partial g}{\partial Y} \quad (21)$$

$$\frac{\partial g}{\partial t} = D \frac{\partial g}{\partial x} \quad (22)$$

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial t} \quad (23)$$

Where

$$K = \frac{\partial Z}{\partial x} = \frac{2Z \exp\left(-\frac{2x}{X_0}\right)}{X_0 \left[1 + \exp\left(-\frac{2x}{X_0}\right)\right]}$$

$$C = \frac{\partial Y}{\partial x} = -\frac{Y}{r_n} \frac{d}{dx} (r_n) \quad (24)$$

$$D = \frac{\partial Y}{\partial r} = \frac{1}{r_n} \quad (25)$$

The transformation equations [equations (21), (23) and (23)] are used in conjunction with the governing equations for the physical plane [equations (16), (17) and (18)] to produce the governing equations for the transformed plane. These equations are

conservation of mass

$$\frac{\partial R}{\partial t} + B \frac{\partial R}{\partial Z} + A \frac{\partial R}{\partial Y} + K \frac{\partial U}{\partial Z} + C \frac{\partial U}{\partial Y} + D \frac{\partial V}{\partial Y} + HV = 0 \quad (26)$$

conservation of axial momentum

$$\frac{\partial U}{\partial t} + B \frac{\partial U}{\partial Z} + A \frac{\partial U}{\partial Y} + L \frac{\partial U}{\partial Z} + F \frac{\partial R}{\partial Y} = 0 \quad (27)$$

conservation of radial momentum

$$\frac{\partial V}{\partial t} + B \frac{\partial V}{\partial Z} + A \frac{\partial V}{\partial Y} + E \frac{\partial R}{\partial Y} = 0 \quad (28)$$

Where

$$A = UC + VD \quad (29)$$

$$B = UK \quad (30)$$

$$E = DG \quad (31)$$

$$F = CG \quad (32)$$

$$L = GK \quad (33)$$

$$G = a^2 \quad (34)$$

$$H = D/\gamma \quad (35)$$

Nozzle Wall Points Transformation

As discussed previously, a normal-tangential coordinate system is a convenient choice for calculation of the wall points. Any flow property $g(\tau, \eta, t)$ in the mobile wall fixed coordinate system is related to the flow property $g(x, r, t)$ in the stationary nozzle fixed coordinate system by the relations

$$\frac{\partial g}{\partial x} = \frac{\partial g}{\partial \tau} \cos \phi + \frac{\partial g}{\partial \eta} \sin \phi \quad (36)$$

$$\frac{\partial g}{\partial r} = \frac{\partial g}{\partial \tau} \sin \phi - \frac{\partial g}{\partial \eta} \cos \phi \quad (37)$$

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial t} \quad (38)$$

The components of velocity in the respective coordinate systems are related by

$$U = v \cos \phi + \sigma \sin \phi \quad (39)$$

$$V = v \sin \phi - \sigma \cos \phi \quad (40)$$

Using equations (26), (27) and (28) in conjunction with equations (36), (37), (38), (39), and (40) yields the governing equations for the mobile wall fixed coordinate system;

conservation of mass

$$\frac{\partial R}{\partial t} + v \frac{\partial R}{\partial \tau} + \sigma \frac{\partial R}{\partial \eta} + \frac{\partial \sigma}{\partial \eta} + \frac{\partial v}{\partial \tau} + \frac{v}{r} = 0 \quad (41)$$

conservation of tangential momentum

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial \tau} + \sigma \frac{\partial v}{\partial \eta} + G \frac{\partial R}{\partial \tau} = 0 \quad (42)$$

conservation of normal momentum

$$\frac{\partial \sigma}{\partial t} + v \frac{\partial \sigma}{\partial \tau} + \sigma \frac{\partial \sigma}{\partial \eta} + G \frac{\partial R}{\partial \eta} = 0 \quad (43)$$

For points on the nozzle wall the normal coordinate of velocity, σ , is zero. The governing equations for the wall points can therefore be simplified further. Setting $\sigma = 0$ in equations (41), (42), and (43), yields

conservation of mass

$$\frac{\partial R}{\partial t} + \frac{\partial \sigma}{\partial \eta} + v \frac{\partial R}{\partial \tau} + \frac{\partial v}{\partial \tau} + \frac{1}{r} v \sin \phi = 0 \quad (44)$$

conservation of tangential momentum

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial \tau} + G \frac{\partial R}{\partial \tau} = 0 \quad (45)$$

conservation of normal momentum

$$\frac{\partial R}{\partial \eta} = 0 \quad (46)$$

For computational convenience equations (44), (45) and (46) are transformed into the rectangular Y - Z coordinate system. This transformation simplifies the merging of the interior and wall points solutions.

Keeping in mind the facts that

$$\frac{d}{dx} (r_\eta) = \tan \phi \quad (47)$$

and

$$r \text{ at wall} = r_\eta \quad (48)$$

the equations can be derived relating any property $g(\tau, \eta, t)$ in the mobile wall fixed coordinate system to the property $g(Z, Y, t)$ in the rectangular transformed plane. These equations are

$$\frac{\partial g}{\partial \eta} = K \sin \phi \frac{\partial g}{\partial Z} - \frac{D}{\cos \phi} \frac{\partial g}{\partial Y} \quad (49)$$

$$\frac{\partial g}{\partial \tau} = K \cos \phi \frac{\partial g}{\partial Z} \quad (50)$$

$$\frac{\partial g}{\partial t} = \frac{\partial g}{\partial t} \quad (51)$$

Using equations (49), (50), and (51) with equations (44), (45), and (46) yields the governing equations for points on the nozzle wall in the rectangular Y - Z coordinate system. These equations are:

conservation of mass

$$\frac{\partial R}{\partial t} + K \sin \phi \frac{\partial \sigma}{\partial Z} - \frac{D}{\cos \phi} \frac{\partial \sigma}{\partial Y} + K \cos \phi \frac{\partial R}{\partial Z} +$$

$$K \cos \phi \frac{\partial R}{\partial Z} + - \frac{v}{r} \sin \phi = 0$$

conservation of tangential momentum

$$\frac{\partial v}{\partial t} + v K \cos \phi \frac{\partial v}{\partial Z} + GK \cos \phi \frac{\partial R}{\partial Z} = 0 \quad (52)$$

conservation of normal momentum

$$K \sin \phi \frac{\partial R}{\partial Z} - \frac{D}{\cos \phi} \frac{\partial R}{\partial Y} = 0 \quad (53)$$

by substitution of equations (39), (40) and equation (36-38), equations (51), (52) and (53) could be written in the alternate forms:

$$\frac{\partial R}{\partial t} + B \frac{\partial R}{\partial Z} + K \frac{\partial U}{\partial Z} + C \frac{\partial U}{\partial Y} + D \frac{\partial V}{\partial Y} + HV = 0 \quad (54)$$

$$\frac{\partial U}{\partial t} + B \frac{\partial U}{\partial Z} + L \frac{\partial R}{\partial Z} + F \frac{\partial R}{\partial Y} = 0 \quad (55)$$

$$\frac{\partial V}{\partial t} + B \frac{\partial V}{\partial Z} + E \frac{\partial R}{\partial Y} \quad (56)$$

Numerical Technique

The interior region of the flow field is calculated using the method of Moretti and Abbett (63). The technique consists of expanding the fluid properties in a Taylor series in time. Lax and Wendroff (45), the principal investigators of this method, found that the term containing the second derivative was a necessary condition to insure convergence of the series. The method therefore uses the variable (a fluid property) and the first and second derivatives of the variable at time t_0 to compute the value of the variable at time $t_0 + \Delta t$. Written mathematically this statement is

$$g(t_0 + \Delta t) = g(t_0) + \frac{\partial g}{\partial t} \Delta t + \frac{\partial^2 g}{\partial t^2} \frac{(\Delta t)^2}{2}$$

where g represents a fluid property (R , U or V). The first time derivative of g is obtained from equations (26) through (28). Differentiation of equations (26) through (28) with respect to time produces the second time derivatives of g as follows:

$$\begin{aligned} \frac{\partial^2 R}{\partial t^2} = & -B \frac{\partial^2 R}{\partial z \partial t} - \frac{\partial B}{\partial t} \frac{\partial R}{\partial z} - A \frac{\partial^2 R}{\partial Y \partial t} - \frac{\partial A}{\partial t} \frac{\partial R}{\partial Y} - K \frac{\partial^2 U}{\partial Z \partial t} - \\ & C \frac{\partial^2 U}{\partial Y \partial t} - D \frac{\partial^2 V}{\partial Y \partial t} - H \frac{\partial V}{\partial t} \end{aligned} \quad (57)$$

$$\begin{aligned} \frac{\partial^2 U}{\partial t^2} = & -B \frac{\partial^2 U}{\partial z \partial t} - \frac{\partial U}{\partial z} \frac{\partial B}{\partial t} - A \frac{\partial^2 U}{\partial Y \partial t} - \frac{\partial A}{\partial t} \frac{\partial U}{\partial Y} - I \frac{\partial^2 R}{\partial Z \partial t} - \\ & \frac{\partial L}{\partial t} \frac{\partial R}{\partial t} - F \frac{\partial^2 R}{\partial Y \partial t} - \frac{\partial F}{\partial t} \frac{\partial R}{\partial Y} \end{aligned} \quad (58)$$

$$\frac{\partial^2 V}{\partial t^2} = B \frac{\partial^2 V}{\partial Z \partial t} - \frac{\partial B}{\partial t} \frac{\partial V}{\partial Z} - A \frac{\partial^2 V}{\partial Y \partial t} - \frac{\partial A}{\partial t} \frac{\partial V}{\partial Y} -$$

$$E \frac{\partial^2 R}{\partial Y \partial t} - \frac{\partial E}{\partial t} \frac{\partial R}{\partial Y} \quad (59)$$

Equations (57) through (59) contain crossed time and space derivatives. Interchange of the order of differentiation is valid for these functions. Equations (26) through (28) are therefore differentiated with respect to Y and Z to express the crossed time - and space - derivatives in terms of space-derivatives only. These terms are:

$$\frac{\partial^2 R}{\partial Z \partial t} = -B \frac{\partial^2 R}{\partial Z^2} - \frac{\partial B}{\partial Z} \frac{\partial R}{\partial Z} - A \frac{\partial^2 R}{\partial Y \partial Z} - \frac{\partial A}{\partial Z} \frac{\partial R}{\partial Y} -$$

$$K \frac{\partial^2 U}{\partial Z^2} - \frac{\partial K}{\partial Z} \frac{\partial U}{\partial Z} - C \frac{\partial^2 U}{\partial Y \partial Z} - \frac{\partial C}{\partial Z} \frac{\partial U}{\partial Y} - D \frac{\partial^2 V}{\partial Y \partial Z} -$$

$$\frac{\partial D}{\partial Z} \frac{\partial V}{\partial Y} - H \frac{\partial V}{\partial Z} - V \frac{\partial H}{\partial Z} \quad (60)$$

$$\frac{\partial^2 R}{\partial Y \partial t} = -B \frac{\partial^2 R}{\partial Z \partial Y} - \frac{\partial B}{\partial Y} \frac{\partial R}{\partial Z} - A \frac{\partial^2 R}{\partial Y^2} - \frac{\partial A}{\partial Y} \frac{\partial R}{\partial Y} -$$

$$K \frac{\partial^2 U}{\partial Z \partial Y} - C \frac{\partial^2 U}{\partial Y^2} - \frac{\partial C}{\partial Y} \frac{\partial U}{\partial Y} - D \frac{\partial^2 V}{\partial Y^2} - H \frac{\partial V}{\partial Y} \quad (61)$$

$$\frac{\partial^2 U}{\partial z \partial t} = -B \frac{\partial U}{\partial Z^2} - \frac{\partial B}{\partial Z} \frac{\partial U}{\partial Z} - A \frac{\partial^2 U}{\partial Y \partial Z} - \frac{\partial A}{\partial Z} \frac{\partial U}{\partial Y} -$$

$$L \frac{\partial^2 R}{\partial Z^2} - \frac{\partial L}{\partial Z} \frac{\partial R}{\partial Z} - F \frac{\partial^2 R}{\partial Y \partial Z} - \frac{\partial F}{\partial Z} \frac{\partial R}{\partial Y} \quad (62)$$

$$\begin{aligned} \frac{\partial^2 U}{\partial Y \partial t} &= -B \frac{\partial^2 U}{\partial Z \partial Y} - \frac{\partial B}{\partial Y} \frac{\partial U}{\partial Z} - A \frac{\partial^2 U}{\partial Y^2} - \frac{\partial A}{\partial Y} \frac{\partial U}{\partial Y} - \\ L \frac{\partial^2 R}{\partial Z \partial Y} &- \frac{\partial L}{\partial Y} \frac{\partial R}{\partial Z} - F \frac{\partial^2 R}{\partial Y^2} - \frac{\partial F}{\partial Y} \frac{\partial R}{\partial Y} \end{aligned} \quad (63)$$

$$\begin{aligned} \frac{\partial^2 V}{\partial Z \partial t} &= -B \frac{\partial^2 V}{\partial Z^2} - \frac{\partial B}{\partial Z} \frac{\partial V}{\partial Z} - A \frac{\partial^2 V}{\partial Y \partial Z} - \frac{\partial A}{\partial Z} \frac{\partial V}{\partial Y} - \\ E \frac{\partial^2 R}{\partial Y \partial Z} &- \frac{\partial E}{\partial Z} \frac{\partial R}{\partial Y} \end{aligned} \quad (64)$$

$$\begin{aligned} \frac{\partial^2 V}{\partial Y \partial t} &= -B \frac{\partial^2 V}{\partial Y \partial Z} - \frac{\partial B}{\partial Y} \frac{\partial V}{\partial Z} - A \frac{\partial^2 V}{\partial Y^2} - \frac{\partial A}{\partial Y} \frac{\partial V}{\partial Y} - \\ E \frac{\partial^2 R}{\partial Y^2} &- \frac{\partial E}{\partial Y} \frac{\partial R}{\partial Y} \end{aligned} \quad (65)$$

The first and second space derivatives occurring in the equations for the second time derivatives [see equations (57) through (65)] can be closely approximated by finite differences. A standard central finite-difference scheme has been used for evaluating the partial derivatives. The first and second space-derivatives of g in finite difference form are

$$\frac{\partial g}{\partial Z}(I, J) = [g(I + 1, J) - g(I - 1, J)] / 2\Delta Z \quad (66)$$

$$\frac{\partial g}{\partial Y}(I, J) = [g(I, J + 1) - g(I, J - 1)] / 2\Delta Y \quad (67)$$

$$\frac{\partial^2 g}{\partial Z^2}(I,J) = [g(I+1, J) - 2g(I,J) + g(I-1, J)] / (\Delta Z)^2 \quad (68)$$

$$\frac{\partial^2 g}{\partial Y^2}(I,J) = [g(I, J+1) - 2g(I,J) + g(I, J-1)] / (\Delta Y)^2 \quad (69)$$

$$\frac{\partial^2 g}{\partial Y \partial Z}(I,J) = [g(I+1, J+1) - g(I+1, J-1) - g(I-1, J+1) + g(I-1, J-1)] / 4\Delta Z \Delta Y \quad (70)$$

Where I and J refer to the grid point under consideration as shown in Figure 2. The remaining terms are evaluated as follows:

$$\frac{dD}{dZ} = -\frac{D^2}{K} \frac{dr_\eta}{dx} \quad (71)$$

$$\frac{\partial H}{\partial Z} = \frac{\partial D}{\partial Z} / Y \quad (72)$$

$$\frac{dK}{dZ} = 2\left(\frac{K}{Z} - \frac{1}{X_0}\right) \quad (73)$$

$$\frac{\partial C}{\partial Z} = Y \left(\frac{dC}{dZ} \frac{dr_\eta}{dx} + \frac{D}{K} \frac{d^2 r_\eta}{dx^2} \right) \quad (74)$$

$$\frac{\partial A}{\partial Z} = U \frac{\partial C}{\partial Z} + C \frac{\partial U}{\partial Z} + V \frac{dD}{dZ} + D \frac{\partial V}{\partial Z} \quad (75)$$

$$\frac{\partial B}{\partial Z} = U \frac{\partial K}{\partial Z} + K \frac{\partial U}{\partial Z} \quad (76)$$

$$\frac{\partial G}{\partial Z} = G(\gamma - 1) \frac{\partial R}{\partial Z} \quad (77)$$

$$\frac{\partial E}{\partial Z} = G \frac{dD}{dZ} + D \frac{\partial G}{\partial Z} \quad (78)$$

$$\frac{\partial F}{\partial Z} = G \frac{\partial C}{\partial Z} + C \frac{\partial G}{\partial Z} \quad (79)$$

$$\frac{\partial L}{\partial Z} = G \frac{\partial K}{\partial Z} + K \frac{\partial G}{\partial Z} \quad (80)$$

$$\frac{\partial C}{\partial Y} = -D \frac{dr_n}{dx} \quad (81)$$

$$\frac{\partial H}{\partial Y} = -\frac{D}{Y^2} \quad (82)$$

$$\frac{\partial A}{\partial Y} = U \frac{\partial C}{\partial Y} + C \frac{\partial U}{\partial Y} \quad (83)$$

$$\frac{\partial B}{\partial Y} = K \frac{\partial U}{\partial Y} \quad (84)$$

$$\frac{\partial G}{\partial Y} = G(\gamma - 1) \frac{\partial R}{\partial Y} \quad (85)$$

$$\frac{\partial E}{\partial Y} = D \frac{\partial G}{\partial Y} \quad (86)$$

$$\frac{\partial F}{\partial Y} = G \frac{\partial C}{\partial Y} + C \frac{\partial G}{\partial Y} \quad (87)$$

$$\frac{\partial L}{\partial Y} = K \frac{\partial G}{\partial Y} \quad (88)$$

$$\frac{\partial A}{\partial t} = C \frac{\partial U}{\partial t} + D \frac{\partial V}{\partial t} \quad (89)$$

$$\frac{\partial B}{\partial t} = K \frac{\partial U}{\partial t} \quad (90)$$

$$\frac{\partial G}{\partial t} = G(\gamma - 1) \frac{\partial R}{\partial t} \quad (91)$$

$$\frac{\partial E}{\partial t} = D \frac{\partial G}{\partial t} \quad (92)$$

$$\frac{\partial F}{\partial t} = C \frac{\partial G}{\partial t} \quad (93)$$

$$\frac{\partial L}{\partial t} = K \frac{\partial G}{\partial t} \quad (94)$$

The first and second derivatives of r_n with respect to x are calculated from the nozzle wall function $r = f(x)$.

As discussed in Chapter II, the initial conditions for two-dimensional numerical methods are usually taken to be the one-dimensional solution for the geometry under consideration. In this study the one-dimensional flow solution is altered in order to decrease execution time by bringing the initial conditions nearer to the final two-dimensional solution. The total velocity used for initial conditions is assumed equal to the one-dimensional value, that is, constant across the nozzle. Radial and axial components are calculated to make the total velocity parallel to the wall. The radial velocity is then assumed to vary linearly from this wall value to zero at the centerline. The axial component is then calculated from the known total and radial velocities.

CHAPTER IV

RESULTS

The equations developed in Chapter III are coded in the Fortran V language for implementation on a Univac 1108 digital computer. A listing of this program appears in the Appendix.

The program solves the flow field for a convergent-divergent conical nozzle. The entrance and exit cones are connected by a circular arc of radius r_t (see Figure 1). The nozzle geometry is described to the computer program by the input of the entrance and exit half-angles (ϕ_0, ϕ_x) and R_t , the ratio of throat radius of curvature r_c , to the nozzle radius of the throat, r_t .

The computational grid (see Figure 4) is described by the input of the number of axial and radial grid lines and the stretching parameter, X_0 (see equation 4).

Three nozzles are analyzed in the results. Two nozzles with entrance half-angles of 30° have exit half-angles of 15° . They differ in that one has an R_t ratio of 0.35, the other, $R_t = 0.55$. The third nozzle has an entrance angle of 45° and an exit of 15° with $R_t = 0.625$. The results are discussed in terms of computer runs involving these three nozzles.

The grid spacing used for these runs consists of 51 axial and 11 radial grid lines. A fine grid spacing produces very accurate approximation of derivatives by finite differences. This at the expense of

increased computation times. Fifty-one divisions axially was found to be necessary to damp out oscillations initiated by the change from an infinite to a finite nozzle crosssection at the entrance. Eleven grid lines in the radial direction was found to be minimal for accurate representation of radial derivatives. The values of the stretching parameter, X_0 , used are those which, after experimentation, yielded the best results as compared with experimental data.

Run 1 : Initial Check-out

To check out the computer program an essentially one-dimensional flow situation is solved. The entrance and exit cones for this nozzle have half-angles of three degrees. The cones are connected with a circular arc throat section with R_t equal to 100. The axial coordinate is stretched using X_0 equal to 10.00. The initial conditions for this nozzle should be very close to the steady state solution. Table 1 compares the axial Mach Number distribution at the nozzle wall for the initial conditions ($N=0$) and after 12 time steps ($N=12$). The steady state solution is seen to vary a maximum of 0.06% from the initial conditions. The program is thus considered operative.

Table 1. Comparison of Initial Conditions and Final Solution for an Essentially One-dimensional Nozzle ($\phi_0 = -3$, $\phi_x = 3$, $N = 51$, $R_c = 100$, $X_0 = 10$, $N = 11$)

1	x/r	$N = 0$	$N = 12$	% Deviation
6	-14.221	0.2310	0.22310	0.00
11	-10.455	0.3076	0.3076	0.00
16	- 8.109	0.3801	0.3801	0.00
21	- 6.330	0.4590	0.4590	0.00
26	- 4.849	0.5539	0.5541	0.03
31	- 3.543	0.6567	0.6568	0.01
36	- 2.344	0.7624	0.7627	0.04
41	- 1.209	0.8726	0.8730	0.05
46	- 0.103	0.9888	0.9894	0.06
51	- 1.00	1.1130	1.1130	0.00

Run 2 : 30 - 15 Nozzle, $R_r = 0.35$

Static pressure ratio distributions calculated using the computer program developed for this paper are presented for a conical nozzle having convergent and divergent half-angles of 30 and 15 degrees respectively. The nozzle has a ratio of throat radius of curvature to throat radius of 0.35. Distributions are calculated along the axis and wall of the nozzle. The wall pressure distribution is compared with the experimental measurements of Shelton (72). The coordinate stretching parameter for this run was set at 6.0. The results appear in Figure 5 for the 180th time step ($N=180$). The execution time for the run was 3.66 minutes on an Univac 1108 digital computer.

Good agreement is observed between the numerical and experimental pressure ratio distributions along the nozzle wall, except in the supersonic region. This region can more accurately be described by two-dimensional steady state characteristics method using the present transonic boundary conditions.

The considerable two-dimensionality of the flow in the vicinity of the throat is clearly demonstrated by the extreme divergence between the wall and centerline pressure ratios in Figure 5.

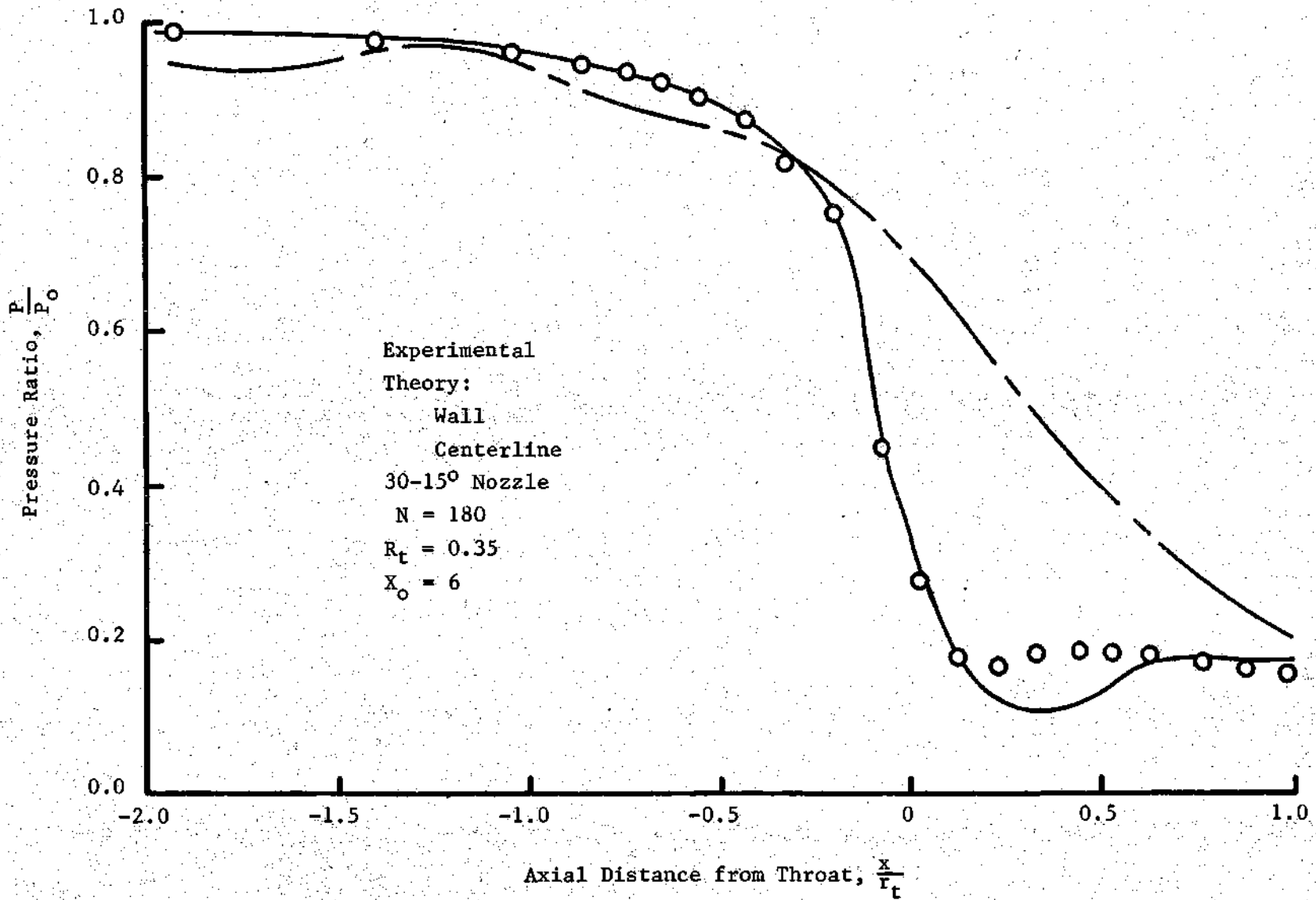


Figure 5. Static to Stagnation Pressure Ratio Distribution in a 30-15° Conical Nozzle

Run 3 : 30 - 15 Nozzle, $R_t = 0.55$

Figure 6 presents the results of Run 3. The computer program was set up, as on Run 2, for a 30 - 15 degree conical nozzle. On this run a curvature ratio of 0.55 was used ($R_t = 0.55$). A slightly less severe stretching was used ($X_0 = 3.0$). The wall and centerline pressure ratio distributions appear in Figure 6 compared with the wall pressure measurements of Shelton (72). The execution time for this run was 3.87 minutes for 200 time steps.

Very good agreement is observed within the transonic region of the flow field. As on Run 2, the theoretical curve dips well below the experimental in the supersonic region. Close examination reveals a deviation within the subsonic region. This deviation is oscillatory in nature and increases near the centerline.

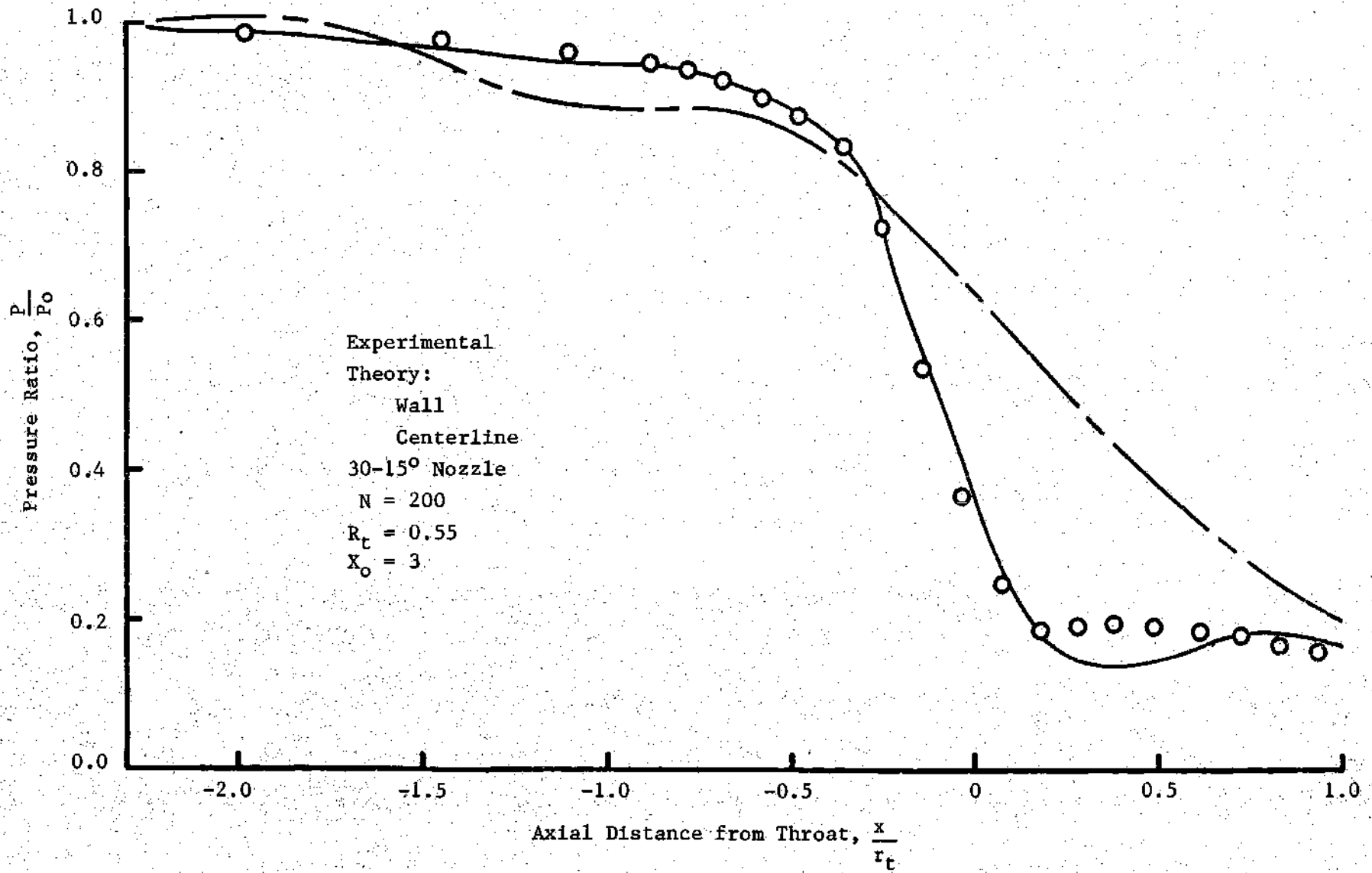


Figure 6. Static to Stagnation Pressure Distribution in a 30-15° Nozzle

Run 4 : 45 - 15 Nozzle, $R_t = 0.625$

Run 4 investigates a more severely contoured nozzle. A 45 degree inlet half-angle is used with a 15 degree exit angle. A stretching parameter value of 3.0 is used. The computation is allowed to march for 180 time steps. Figure 7 presents the numerical results for the Mach Number distributions along the wall and centerline of the nozzle. These are compared with the experimental observations of Back et al. (75). This run required a 3.76 minutes of computer time. As in the 30 - 15 nozzles, deviations from the experimental are observed in the subsonic and supersonic regions. Very good results are evident within the transonic region.

Mach line distributions for Run 4 appear in Figure 8. Experimental data is that of Back et al. (75).

The progression of the solution to steady state is indicated in Figure 9. Notice that within the transonic region ($\frac{x}{r_t} = 0.292$) the time marching routine has reached steady state by the 150th time step. Just upstream of the throat ($\frac{x}{r_t} = 1.119$) the routine requires 300 time steps to obtain steady state conditions. Near the nozzle entrance ($\frac{x}{r_t} = -4.906$) within the subsonic region, the routine does not reach steady state. This indicates an error generating source within the subsonic region. If left to accumulate, this error could destroy the accuracy of the time marching routine, throughout the flow field.

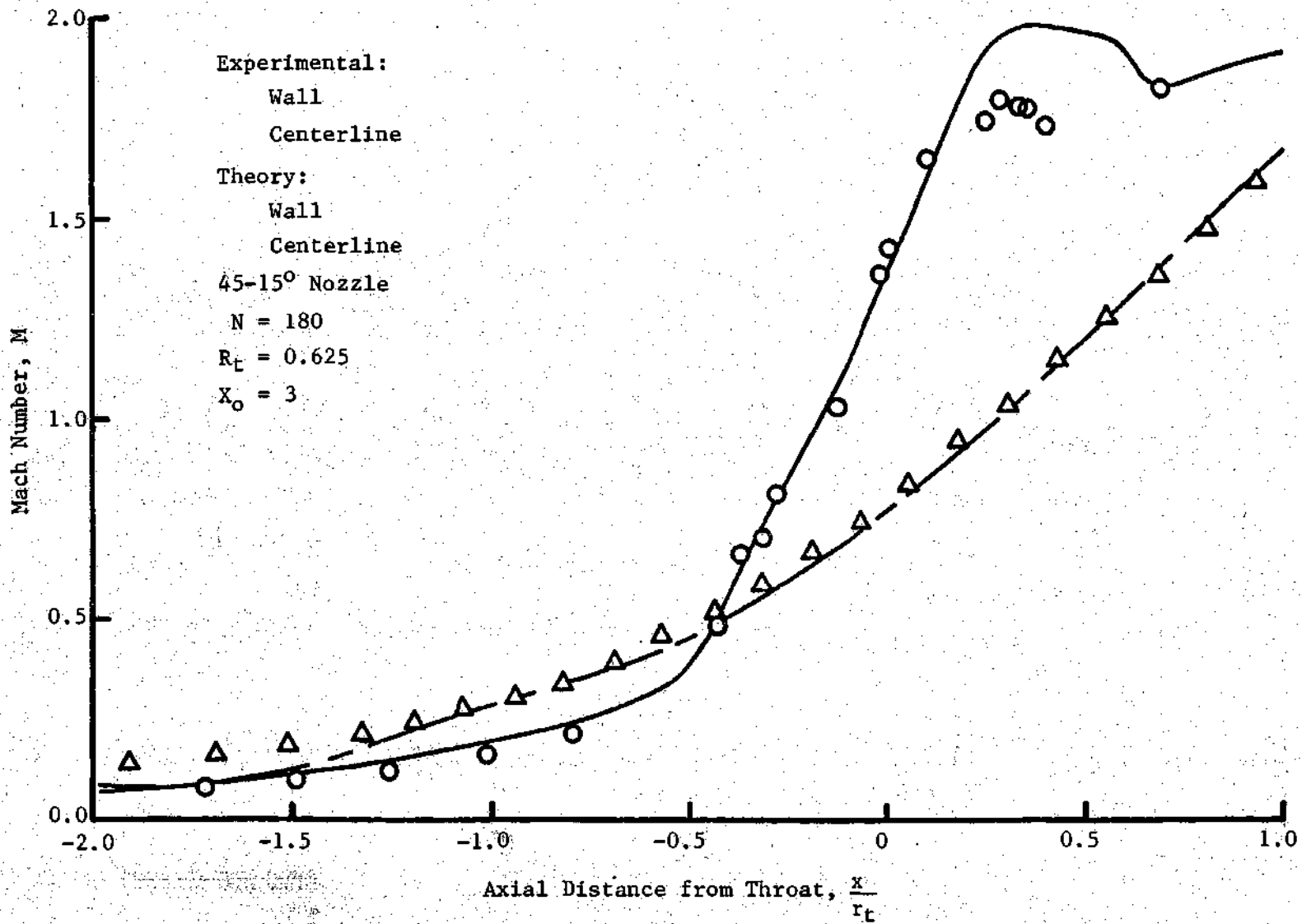


Figure 7. Mach Number Distribution in a 45-15° Conical Nozzle

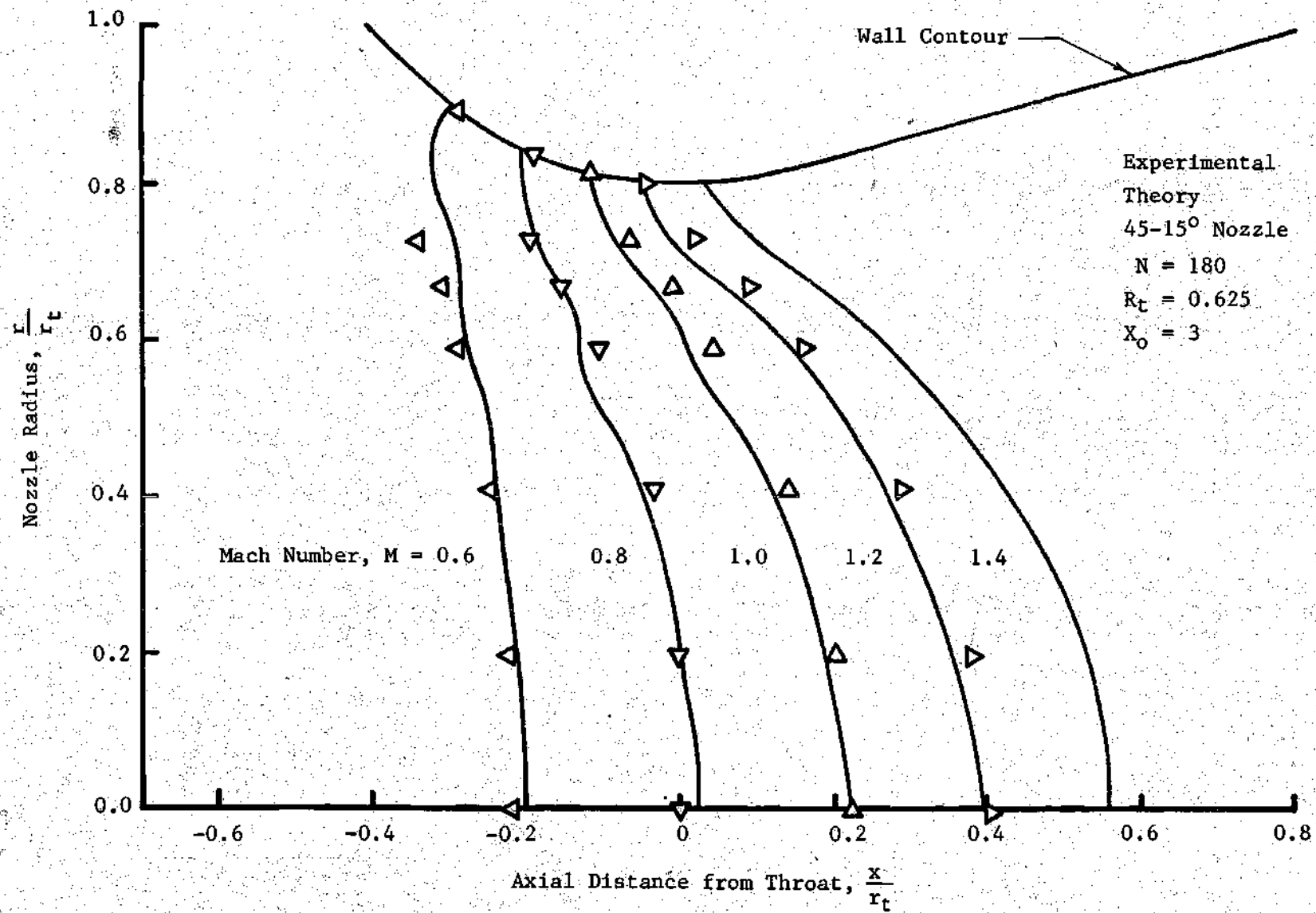


Figure 8. Mach Line Distribution in a 45-15° Nozzle

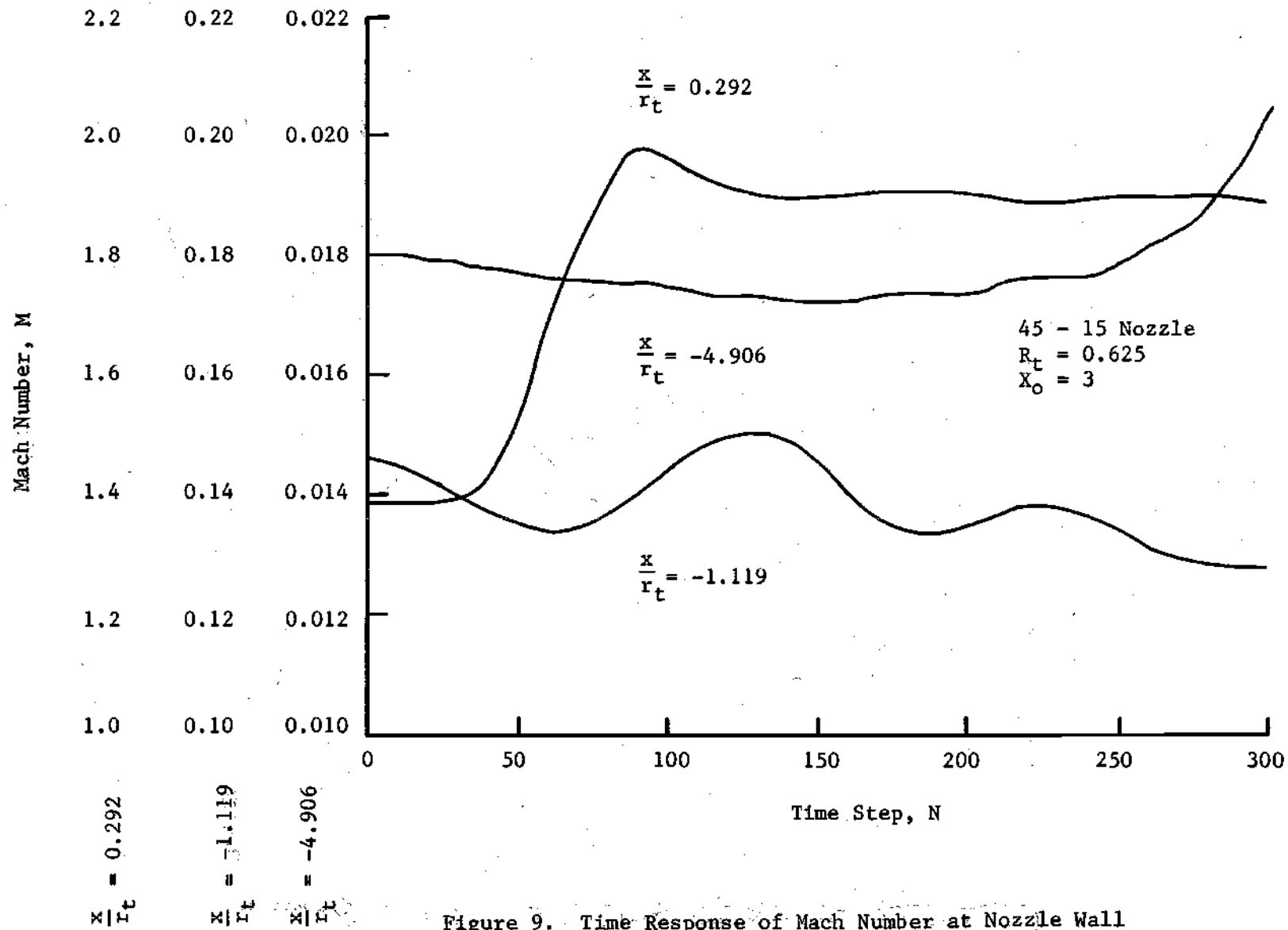


Figure 9. Time Response of Mach Number at Nozzle Wall

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

The computational method developed in this thesis permits the solution of flows over a regime which includes subsonic, transonic and supersonic flows. The computer program requires only the input of nozzle geometry parameters for solution of the flow field. The numerical technique is written in the Fortran V language. The program can be implemented on any large scale digital computer. Batch processing or demand facilities may be used with equal ease.

The results show excellent agreement with experimental measurements within the transonic region. Deviations from experimental values within the subsonic and supersonic regions indicate that an error generating source exists near the nozzle entrance region.

With sufficient feel for the progress of the time marching technique, effective values of the stretching parameter and grid spacing can be obtained. This does not insure that the chosen values are those that will yield the most accurate results or the most rapid convergence. The effect of the stretching parameter and grid spacing for varying inlet cone half-angles and throat contours on convergence and stability in the numerical technique has not yielded to mathematical correlation. The optimum parameter values for a given nozzle cannot be known before the initial execution. This is a disadvantage of the method as presented. Additional study is required to correlate these

parameters.

It is concluded that the numerical technique developed in this report must be restricted to flow field solutions in the throat region of converging-diverging nozzles. Accurate results can be obtained for transonic flows but not for flows in the subsonic or supersonic regions using this method. The transonic flow solution generated can be used as boundary conditions for a method of characteristics solution in the supersonic region. The errors generated within the subsonic region, however, have the capability of destroying the results for the entire flow field.

It is recommended that a study be implemented to determine the nature and extent of error generation within the finite-difference equations. Emphasis should be placed on determination of the effects at the nozzle boundaries and entrance region. Subsequently the choice of grid size and stretching parameter should be correlated for optimization of accuracy and execution time. Only after these studies are completed can the numerical technique developed herein be considered reliable for the solution of two-dimensional flow fields.

APPENDIX

A listing of the computer program written to utilize the numerical technique developed for this thesis follows. This program may be executed on any large scale digital computer. It contains logic for both batch and demand processing. Plotting routines written for the Typograph Demand Terminal are also included. Comments are included within the listing. This makes the program essentially self-explanatory.

THIS PROGRAM COMPUTES INITIAL CONDITIONS

INPUT DATA CARDS:

(1) NX (INTEGER)
 (2) NR (INTEGER)
 (3) X0 (REAL)
 (4) RC (REAL)
 (5) RT (REAL)
 (6) AGLIN (REAL, DEGREES)
 (7) AGLOUT (REAL, DEGREES)

OUTPUT FILE: 10

LOCATION OF
 VARIABLES:

D(1,I) = I
 D(2,I) = X
 D(3,I) = Z
 D(4,I) = YN
 D(5,I) = DYNDX
 D(6,I) = D
 D(7,I) = K
 D(8,I) = PHI (RADIANS)
 D(9,I) = PHI (DEGREES)
 D(10,I) = SIN (PHI)
 D(11,I) = COS (PHI)
 D(12,I) = DKZ
 D(13,I) = DDZ
 D(14,I) = DCZDY
 D(15,I) = DCY
 D(16,I) = D2YNDX

REAL K
 DOUBLE PRECISION X0,DELZ,DELY,YN1
 COMMON/BLK7/U(51,51),R(51,51)
 COMMON/BLK6/YN1(51)
 DIMENSION D(16,51),C(51,51),H(51,51),V(51,51),THA(51,51),HX(51,51)

1000 FORMAT()
 1001 FORMAT(' ENTER NUMBER OF AXIAL GRID POINTS')
 1002 FORMAT(' X0 = ?')
 1003 FORMAT(' ARC RADIUS = ?')
 1004 FORMAT(' THROAT RADIUS = ?')
 1005 FORMAT(' ENTRANCE ANGLE = ?')
 1006 FORMAT(' EXIT ANGLE = ?')
 1007 FORMAT(' ENTER NUMBER OF RADIAL GRID POINTS')

$$X(Z) = (X0/2.)*(ALOG(Z)-ALOG(1+EXP(-2./X0)-Z))$$

$$K(Z) = (((2.*Z)/X0) * EXP((-2.*X(Z))/X0)) / (1 + EXP((-2.*X(Z))/X0))$$

1
 WRITE(6,1001)
 READ (5,1000) NX
 WRITE (6,1007)
 READ (5,1000) NR
 WRITE (6,1002)
 READ(5,1000) X0
 WRITE(6,1003)

```

READ(5,1000) RC
WRITE(6,1004)
READ(5,1000) RT
WRITE(6,1005)
READ(5,1000) AGLIN
WRITE(6,1006)
READ(5,1000) AGLOUT
CONV = 57.2957795
AGLIN = AGLIN/CONV
AGLOUT = AGLOUT/CONV
PI = 3.1415927
RC = RC/RT
TI = RC*SIN(AGLIN)
TO = RC*SIN(AGLOUT)
TNI = TAN(AGLIN)
TNO = TAN(AGLOUT)
DZ = 1./(NX-1)
DY = 1./(NR-1)
DELZ = 1.000/(NX - 1.000)
X0 = DBLE(X0)
DELY = 1.000/(NR-1.000)
GAMMA = 1.4
D(1,1) = 1.0
D(2,1) = - 1.0E10
D(3,1) = 0.0
D(4,1) = 1.0E10
D(5,1) = -1.0
D(6,1) = 0.0
D(7,1) = 0.0
DO 10 I = 1,NX
10 C(I,1) = 0.0
DO 110 J = 1,NR
H(1,J) = 0.0
110 C(1,J) = 0.0
DO 20 I = 2, NX
Z = ( I - 1 ) * DZ
D(1,I) = I
D(2,I) = X(Z)
D(3,I) = Z
IF ( D(2,I) - TI ) 11, 11, 12
11 D(4,I) = D(2,I) * TNI + 1 + RC - RC/COS(AGLIN)
D(5,I) = TNI
D(16,I) = 0.0
GO TO 15
12 IF ( D(2,I) - TO ) 14, 13, 13
13 D(4,I) = D(2,I) * TNO + 1 + RC - RC/COS(AGLOUT)
D(5,I) = TNO
D(16,I) = 0.0
GO TO 15
14 D(4,I) = RC + 1 - SQRT(RC**2-D(2,I)**2)
D(5,I) = D(2,I) / SQRT ( RC**2 - D(2,I)**2 )
D(16,I) = ( SQRT ( RC ** 2 - D(2,I) ** 2 ) + D(5,I)*D(2,I) )
& / ( RC ** 2 - D(2,I) ** 2 )
15 YN1(I) = DBLE( D(4,I) )

```

```

D(6,I) = 1.0 / D(4,I)
D(7,I) = K(Z)
D(8,I) = ATAN( D(5,I) )
D(9,I) = D(8,I) * CONV
D(10,I) = SIN( D(8,I) )
D(11,I) = COS( D(8,I) )
D(12,I) = 2 * ( D(7,I) / D(3,I) - 1 / X0 )
D(13,I) = - D(6,I) ** 2 * D(5,I) / D(7,I)
D(14,I) = D(13,I) * D(5,I) + D(6,I) * D(16,I) / D(7,I)
D(15,I) = - D(5,I) * D(6,I)
DO 20 J = 2, NR
Y = ( J - 1 ) * DY
C(I,J) = -1 * D(5,I) * D(6,I) * Y
20 H(I,J) = D(6,I) / Y
CALL ONEDIM(X0,DELZ,DELY)
DO 30 I = 2,NX
HX(I,NR) = U(I,NR)
30 V(I,NR) = HX(I,NR) * D(10,I)
DO 40 I = 2,NX
DO 40 J = 1,NR
Y = (J-1) * DY
HX(I,J) = HX(I,NR)
V(I,J) = V(I,NR) * Y
U(I,J) = SQRT ( HX(I,J)**2 - V(I,J)**2 )
THA(I,J) = ATAN( V(I,J)/U(I,J) ) * CONV
40 R(I,J) = (1/(GAMMA-1)) * ALOG(1 - ((HX(I,J)**2) * (GAMMA-1))
1 / (2 * GAMMA))
AGLIN = AGLIN * CONV
AGLOUT = AGLOUT * CONV
WRITE (10) NX, NR, X0, RC, RT, AGLIN, AGLOUT, D, C, H, R, U, V, THA, HX
END FILE 10
REWIND 10
STOP
END

```

```

FUNCTION FDERIV(K)
COMMON/BLK5/OMACH(150)
DOUBLE PRECISION FDERIV,OMACH
FDERIV = (0.83333 + 0.16667*OMACH(K)**2)**2 -
1 (0.83333 + 0.16667*OMACH(K)**2)**3 / OMACH(K)**2
RETURN
END

```

```

SUBROUTINE ONEDIM(X0,DELZ,DELY)
DOUBLE PRECISION AREA,X,DELZ,DELY,X0,YN1,Z,AMACH
COMMON/BLK7/U(51,51),R(51,51)
COMMON/BLK6/YN1(51)
N = 1.0/DELZ
J = 1.0/DELY
N = N + 1
J = J + 1
DO 20 I = 2,N+1
Z = (I - 1)*DELZ
X = (-X0/2.)*DLOG(( 1. + DEXP(-2./X0) - Z) / Z)
AREA = YN1(I)**2
CALL XMACH(AREA,X,AMACH)
U(1,I) = (1.18322*AMACH) / DSQRT( 1. + 0.2*AMACH**2)
20 R(1,I) = -DLOG((1. + 0.2*AMACH**2)**2.5)
DO 40 L = 2,N+1
DO 40 K = 1,J+1
U(L,K) = U(1,L)
40 R(L,K) = R(1,L)
DO 50 K = 1,J
U(1,K) = 0.0
R(1,K) = 0.0
50 CONTINUE
RETURN
END

```

```

SUBROUTINE XMACH(AREA,X,AMACH)
COMMON/BLK5/OMACH(150)
DOUBLE PRECISION FDERIV,AMACH,OMACH,F,X,AREA
IF(X)100,100,105
100 OMACH(1) = 0.01
IF ( AREA .GE. 55) OMACH(1) = 1.0D-3
GO TO 110
105 OMACH(1) = 3.0
110 K = 1
120 IF(K = 74)130,130,140
130 OMACH(K+1) = OMACH(K) - F(K,AREA) / FDERIV(K)
IF(DABS(OMACH(K+1) - OMACH(K)) .LE. 0.0000001)GO TO 140
K = K + 1
GO TO 120
140 AMACH = OMACH(K + 1)
RETURN
END

```



```

1002 FORMAT(' DT = ',F8.4)
1005 FORMAT(' N = ',I5,' ; LOOK = 1')
1006 FORMAT()
1007 FORMAT(' HOW MANY TIME STEPS ?')
1008 FORMAT(' STOP = 1')
1009 FORMAT(5X,' IS THE PLOT SWITCH OFF ?')
1010 FORMAT(' ENTER 1 FOR A LOOK AT THE INITIAL CONDITIONS, N=',I5)
1011 FORMAT(1H1,10X,'1',8X,'15',11X,'16'//11X,'I',7X,
&'DCY',9X,'D2YNDX'//)
1012 FORMAT(9X,F3.0,5X,F6.4,7X,F6.4/)
1013 FORMAT(' FC = ')
1014 FORMAT(' FC = ? NO CHG. = NEG. ')
1016 FORMAT(' DT PRINT = 1')
1017 FORMAT(' TIME = ',1PE10.4)
1018 FORMAT(' SAVE IT ? ENTER 1')
1019 FORMAT(' N=',I4,' DT=',1PE9.3,' RTM=',1PE9.3,' DTSO2= ',1PE9.3)
1020 FORMAT(' FC=',1PE9.3,' NE=',I4,' NSK=',I4)
1021 FORMAT(1X,A3,9(3X,1PE9.3))
1024 FORMAT('// N=',I4,2X,I2,' ',I2,8(7X,I2,' ',I2),4X,'RTM=',1PE9.3)
1025 FORMAT(' NX= ',I2,' NR= ',I2,' X0= ',F4.1/
+ ' RT= ',F5.2,' RC= ',F5.2/
+ ' AGLIN= ',F6.2,' AGLOUT= ',F6.2)
1026 FORMAT(1H1//3X,'STRETCHING PARAMETER =',F7.2/3X,
+' ENTRANCE ANGLE =',F7.2/3X,' EXIT ANGLE =',F7.2/3X,
+' THROAT CONTOUR PARAMETER =',F6.3/3X,' GRID SIZE :',I3,
+' X',I3/3X,' TIME STEP H',I5,//////)
1027 FORMAT(//3X,' J =',I3//)
1028 FORMAT(4X,' I',10X,' R',15X,' U',15X,' V',14X,' THA',14X,' HX',
+14X,' M',15X,' P'//)
1029 FORMAT(3X,I2,1P7E16.5)

```

C
C

```

IF ( OP('K') ) ICK = 1
IF ( OP('I') ) II = 1
IF ( OP('B') ) IBR = 1
IF ( OP('S') ) ICS = 1
IF ( OP('J') ) IJN = 1
IF ( OP('P') ) ISP = 1
IF ( OP('C') ) BRIEF = .TRUE.
IF ( OP('L') ) LONG = .TRUE.
IF ( OP('F') ) FINAL = .TRUE.

```

C
C
C

CONSTANTS

```

GAMMA = 1.4
DEMNI = SQRT(8.)

```

C
C

```

READ VISCOSITY PARAMETER
IF ( IBR .NE. 1 ) PRINT 1013
READ (5,1006) FC

```

C
C
C

START OPTION

```

IF ( ICS .NE. 1 ) GO TO 10

```



```

READ(10) NX,NR,X0,RC,RT,AGLIN,AGLOUT,D,C,H,R,U,V,THA,HX
REWIND 10
N = 0
DY = 1./(NR-1)
DTM = 10.0
DO 8 J = 2,NR
DO 8 I = 2,NX
DTX = DTT(D(2,I),D(2,I-1),U(I,J),R(I,J))
Y = (J-1) * DY * D(4,I)
YM = (J-2) * DY * D(4,I)
DTY = DTT(Y,YM,V(I,J),R(I,J))
DTS = AMIN1(DTX,DTY)
IF ( DTS .GT. DTM ) GO TO 8
DTM = DTS
IMX = I
JMX = J
8 CONTINUE
DT = 0.98 * DTM
TT = 'DTM'
WRITE (6,1001) TT,IMX,JMX,DTM
WRITE (6,1002) DT
9 CONTINUE
NSKCTR = 0
RTM = 0.0
GO TO 11

```

C
C
C

CONTINUATION OPTION

```

10 READ (10) NX,NR,X0,RC,RT,AGLIN,AGLOUT,D,C,H
READ (11) N,DT,RTM,R,U,V,THA,HX
REWIND 10
REWIND 11
NSKCTR = 0

```

C
C
C

CONSTANTS

```

11 DZ = 1./(NX - 1)
DY = 1./(NR-1)
NT = NX - 1
NTJ = NR - 1
GGMO = GAMMA - 1
TZ = 2 * DZ
TY = 2 * DY
DZ2 = DZ ** 2
DY2 = DY ** 2
FYZ = 4 * DY * DZ
TYZ = 2 * DY * DZ
DYZ2 = TY * DZ
RC = RC*RT
WRITE (6,1025) NX, NR, X0, RT, RC, AGLIN, AGLOUT
RC = RC/RT
IF ( ICK .EQ. 1 )
+CALL CHECK('GGMO',GGMO,'TZ ',TZ ,'TY ',TY ,'DZ2 ',DZ2 ,
+          'DY2 ',DY2 ,'FYZ ',FYZ , 1 )

```

```

I1 = ((NX-1)*3)/10
I2 = ((NX-1)*7)/10
I3 = NX-1
J1 = 1
J2 = (NR-1)/2
J3 = NR

C
C READ TIME STEPS PER RUN
  IF ( IJN .EQ. 1 ) READ(5,1006) NE

C
C READ OUTPUT SPACING
  IF ( ISP .EQ. 1 ) READ(5,1006) NSK
  IF ( IBR .NE. 1 ) GO TO 20
  WRITE (6,1020) FC,NE,NSK

C
C PRINT INITIAL CONDITIONS (BATCH OPTION)
  IF ( II .NE. 1 ) GO TO 20

C
  WRITE (6,1)
  DO 12 I = 2,NX
  WRITE (6,2) (D(J,I),J=1,14)
12 CONTINUE
  WRITE (6,1011)
  DO 13 I = 2,NX
  WRITE (6,1012) D(1,I),D(15,I),D(16,I)
13 CONTINUE
  CALL TABLES(C,N,1,'C')
  CALL TABLES(H,N,2,'H')
  CALL TABLES(R,N,3,'R')
  CALL TABLES(U,N,4,'U')
  CALL TABLES(V,N,5,'V')
  CALL TABLES(THA,N,6,'THA')
  CALL TABLES(HX,N,7,'HX')
  GO TO 1022

C
C PRINT INITIAL CONDITIONS (DEMAND)
  20 IF ( IBR .EQ. 1 ) GO TO 23
  PRINT 1010, N
  READ 1006, INITCN
  NSV = 1
  IF ( INITCN .NE. 1 ) GO TO 405
1022 DO 22 J = 1,NR
  DO 22 I = 1,NX
  P(I,J) = EXP(GAMMA*R(I,J))
  M(I,J) = HX(I,J)/SQRT(GAMMA*EXP(GGMO*R(I,J)))
  22 CONTINUE

C
C PRINT P AND M (BATCH)
  IF ( IBR .NE. 1 ) GO TO 1023
  CALL TABLES (M,N,8,'M')
  CALL TABLES (P,N,9,'P')

```

GO TO 23
1023 NPLTN = 1
GO TO 405

C ***** PROPERTY COMPUTATIONS *****
C

23 NS = N + 1
IF (IJN .EQ. 1) NE = N + NE
IF (IJN .NE. 1) NE = NS
IF (ICK .EQ. 1) CALL CHECK('RTM',RTM,'DT',DT,'NT',FLOAT(NT),
+'N',FLOAT(N),'NS',FLOAT(NS),'NE',FLOAT(NE),1)

24 NSV = 0
DTM = 1.0

25 DO 400 N = NS,NE
DTM = 1.0
RTM = RTM + DT
YTT = DY * DT
DTSO2 = DT ** 2 / 2
IF (ICK .EQ. 1) WRITE (6,1019) N,DT,RTM,DTSO2

C
C
C RETAIN NORMAL VELOCITY AT NR-1

J = NR
DO 29 I = 2,NX
S(1,I) = U(I,J-1) * D(10,I) - V(I,J-1) * D(11,I)
29 CONTINUE

C
C
C INTERIOR *****

DO 50 J = 2,NTJ
Y = (J - 1) * DY
YSO = Y ** 2
DO 40 I = 2,NT

A = U(I,J) * C(I,J) + V(I,J) * D(6,I)
B = U(1,J) * D(7,I)
G = GAMMA * EXP (GGMO * R(I,J))
E = D(6,I) * G
F = C(1,J) * G
L = D(7,I) * G

C
C IF (ICK .EQ. 1)
+CALL CHECK('A',A,'B',B,'G',G,'E',E,
+'F',F,'L',L,'3')

DRZ = (R(I+1,J) - R(I-1,J)) / TZ
DUZ = (U(I+1,J) - U(I-1,J)) / TZ
DVZ = (V(I+1,J) - V(I-1,J)) / TZ
DRY = (R(I,J+1) - R(I,J-1)) / TY
DUY = (U(I,J+1) - U(I,J-1)) / TY
DVY = (V(I,J+1) - V(I,J-1)) / TY

C
C IF (ICK .EQ. 1)

```

+CALL CHECK('DRZ ','DRZ ','DUZ ','DUZ ','DVZ ','DVZ ','DRY ','DRY ','
+          'DUY ','DUY ','DVY ','DVY ',' 4 )
C
GGMG = GGMO * G
C
DRT = - B * DRZ - A * DRY - D(7,I) * DUZ - C(I,J) * DUY
1     - D(6,I) * DVY - H(I,J) * V(I,J)
DUT = - B * DUZ - A * DUY - L * DRZ - F * DRY
DVT = - B * DVZ - A * DVY - E * DRY
C
DRZZ = ( R(I+1,J) - 2 * R(I,J) + R(I-1,J) ) / DZ2
DUZZ = ( U(I+1,J) - 2 * U(I,J) + U(I-1,J) ) / DZ2
C
IF ( ICK .EQ. 1 )
+CALL CHECK('GGMG','GGMG','DRT','DRT','DUT','DUT','DVT','DVT','
+          'DRZZ','DRZZ','DUZZ','DUZZ',' 5 )
DVZZ = ( V(I+1,J) - 2 * V(I,J) + V(I-1,J) ) / DZ2
DRYZ = ( R(I+1,J+1) - R(I+1,J-1) - R(I-1,J+1) + R(I-1,J-1) ) / FYZ
DUYZ = ( U(I+1,J+1) - U(I+1,J-1) - U(I-1,J+1) + U(I-1,J-1) ) / FYZ
DVYZ = ( V(I+1,J+1) - V(I+1,J-1) - V(I-1,J+1) + V(I-1,J-1) ) / FYZ
DRYY = ( R(I,J+1) - 2 * R(I,J) + R(I,J-1) ) / DY2
DUYY = ( U(I,J+1) - 2 * U(I,J) + U(I,J-1) ) / DY2
IF ( ICK .EQ. 1 )
+CALL CHECK('DVZZ','DVZZ','DRYZ','DRYZ','DUYZ','DUYZ','DVYZ','DVYZ','
+          'DRYY','DRYY','DUYY','DUYY',' 6 )
DVYY = ( V(I,J+1) - 2 * V(I,J) + V(I,J-1) ) / DY2
DHZ = D(13,I) / Y
DCZ = D(14,I) * Y
DAZ = U(I,J) * DCZ + C(I,J) * BUZ + V(I,J) * D(13,I)
+ D(6,I) * DVZ
DBZ = U(I,J) * D(12,I) + D(7,I) * DUZ
DGZ = GGMG * DRZ
IF ( ICK .EQ. 1 )
+CALL CHECK('DVYY','DVYY','DHZ','DHZ','DCZ','DCZ','DAZ','DAZ','
+          'DBZ','DBZ','DGZ','DGZ',' 7 )
DEZ = G * D(13,I) + D(6,I) * DGZ
DFZ = G * DCZ + C(I,J) * DGZ
DLZ = G * D(12,I) + D(7,I) * DGZ
DHY = - D(6,I) / YSQ
DAY = U(I,J) * D(15,I) + C(I,J) * DUY
DBY = D(7,I) * DUY
IF ( ICK .EQ. 1 )
+CALL CHECK('DEZ','DEZ','DFZ','DFZ','DLZ','DLZ','DHY','DHY','
+          'DAY','DAY','DBY','DBY',' 8 )
DGY = GGMG * DRY
DEY = D(6,I) * DGY
DFY = G * D(15,I) + C(I,J) * DGY
DLY = D(7,I) * DGY
DAT = C(I,J) * DUT + D(6,I) * DVT
DBT = D(7,I) * DUT
IF ( ICK .EQ. 1 )
+CALL CHECK('DGY','DGY','DEY','DEY','DFY','DFY','DLY','DLY','
+          'DAT','DAT','DBT','DBT',' 9 )
DGT = GGMG * DRT

```

```

DET = D(6,I) * DGT
DFT = C(I,J) * DGT
DLT = D(7,I) * DGT
DRTZ = - B * DRZZ - DBZ * DRZ - A * DRYZ - DAZ * DRY
.   - D(7,I) * DUZZ - D(12,I) * DUZ - C(I,J) * DUYZ
.   - DCZ * DUY - D(6,I) * DVYZ - D(13,I) * DVY - H(I,J) * DVZ
.   - V(I,J) * DHZ
DRTY = - B * DRYZ - DBY * DRZ - A * DRYY - DAY * DRY
.   - D(7,I) * DUYZ - C(I,J) * DUY - D(15,I) * DUY
.   - D(6,I) * DVYY - H(I,J) * DVY - V(I,J) * DHY
IF ( ICK .EQ. 1 )
+CALL CHECK('DGT','DGT','DET','DET','DFT','DFT','DLT','DLT',
+          'DRTZ','DRTZ','DRTY','DRTY', 10 )
DUTZ = - B * DUZZ - DBZ * DUZ - A * DUYZ - DAZ * DUY
.   - L * DRZZ - DLZ * DRZ - F * DRYZ - DFZ * DRY
DUTY = - B * DUYZ - DBY * DUZ - A * DUY - DAY * DUY
.   - L * DRYZ - DLY * DRZ - F * DRYY - DFY * DRY
DVTZ = - B * DVZZ - DBZ * DVZ - A * DVYZ - DAZ * DVY
.   - E * DRYZ - DEZ * DRY
DVTY = - B * DVYZ - DBY * DVZ - A * DVYY
.   - DAY * DVY - E * DRYY - DEY * DRY
DRTT = - B * DRTZ - DBT * DRZ - A * DRTY - DAT * DRY
.   - D(7,I) * DUTZ - C(I,J) * DUTY - D(6,I) * DVTY
.   - H(I,J) * DVT
DUTT = - B * DUTZ - DBT * DUZ - A * DUTY - DAT * DUY
.   - L * DRTZ - DLT * DRZ - F * DRTY - DFT * DRY
IF ( ICK .EQ. 1 )
+CALL CHECK('DUTZ','DUTZ','DUTY','DUTY','DVTZ','DVTZ','DVTY','DVTY',
+          'DRTT','DRTT','DUTT','DUTT', 11 )
DVTT = - B * DVTZ - DBT * DVZ - A * DVTY
.   - DAT * DVY - E * DRTY - DET * DRY
RI(I,J) = R(I,J) + DT * DRT + DTS2 * DRTT
UI(I,J) = U(I,J) + DT * DUT + DTS2 * DUTT
VI(I,J) = V(I,J) + DT * DVT + DTS2 * DVTT
IF ( ICK .EQ. 1 )
+CALL CHECK('DVTT','DVTT','R','R(I,J)','U','U(I,J)','V','V(I,J)',
+          'Y','Y','YSQ','YSQ', 12 )

```

```

40 CONTINUE
50 CONTINUE

```

C
C
C

```

CENTERLINE *****

```

```

J = 1
DO 51 I = 2,NT
RI(I,J) = ( 4 * RI(I,J+1) - RI(I,J+2) ) / 3
UI(I,J) = ( 4 * UI(I,J+1) - UI(I,J+2) ) / 3
VI(I,J) = 0.0
51 CONTINUE

```

C
C
C
C
C

```

WALL *****

```

```

J = NR

```

```

COMPUTE NORMAL VELOCITY FOR NEW TIME AT NR-1

```

```

C
DO 55 I = 2,NX
S(2,I) = U(I,J-1) * D(10,I) - V(I,J-1) * D(11,I)
55 CONTINUE
C
DO 60 I = 2,NT
C
DRZ = ( R(I+1,J) - R(I-1,J) ) / TZ
DNZ = ( HX(I+1,J) - HX(I-1,J) ) / TZ
DSY = - S(1,I) / DY
G = GAMMA * EXP ( GGMO * R(I,J) )
GGMG = GGMO * G
DRZZ = ( R(I+1,J) - 2 * R(I,J) + R(I-1,J) ) / DZ2
IF (ICK .EQ. 1) CALL CHECK('DRZ',DRZ,'DNZ',DNZ,'DSY',DSY,
+ 'G',G,'GGMG',GGMG,'DRZZ',DRZZ,13)
DNZZ = ( HX(I+1,J) - 2 * HX(I,J) + HX(I-1,J) ) / DZ2
DSYZ = ( S(1,I-1) - S(1,I+1) ) / TYZ
DGZ = GGMG * DRZ
DRT = ( D(6,I) / D(11,I) ) * DSY - D(10,I) * HX(I,J) * H(I,J)
+ D(7,I) * D(11,I) * ( HX(I,J) * DRZ + DNZ )
DGT = GGMG * DRT
DHZ = D(13,I)
IF ( ICK .EQ. 1 ) CALL CHECK ('DNZZ',DNZZ,'DSYZ',DSYZ,
+ 'DGZ',DGZ,'DRT',DRT,'DGT',DGT,'DHZ',DHZ,14)
DNT = - D(7,I) * D(11,I) * ( HX(I,J) * DNZ + G * DRZ )
DRTZ = ( 1 / D(11,I) ) * ( D(6,I) * DSY + D(13,I) * DSY )
+ D(10,I) * ( HX(I,J) * DHZ + H(I,J) * DNZ )
+ D(11,I) * ( ( HX(I,J) * DRZZ + DNZ * DRZ + DNZZ ) * D(7,I)
+ D(12,I) * ( HX(I,J) * DRZ + DNZ ) )
DNTZ = - D(11,I) * ( D(7,I) * ( HX(I,J) * DNZZ + DNZ ** 2
+ G * DRZZ + DGZ * DRZ ) + D(12,I) * ( HX(I,J)
* DNZ + G * DRZ ) )
DSTY = ( S(1,I) - S(2,I) ) / YTT
DRTT = D(6,I) * DSTY / D(11,I) - D(10,I) * H(I,J) * DNT - D(7,I) *
+ D(11,I) * ( HX(I,J) * DRTZ + DNT * DRZ + DNTZ )
DNNT = - D(7,I) * D(11,I) * ( HX(I,J) * DNTZ + DNT * DNZ
+ G * DRTZ + DGT * DRZ )
IF ( ICK .EQ. 1 ) CALL CHECK ('DNT',DNT,'DRTZ',DRTZ,
+ 'DNTZ',DNTZ,'DSTY',DSTY,'DRTT',DRTT,'DNNT',DNNT,15)
RI(I,J) = R(I,J) + DT * DRT + DTSD2 * DRTT
HXI(I) = HX(I,J) + DT * DNT + DTSD2 * DNNT
UI(I,J) = HXI(I) * D(11,I)
VI(I,J) = HXI(I) * D(10,I)
60 CONTINUE
C
C
C
SUPERSONIC BOUNDARY
C
I = NX
DO 70 J = 1,NR
RI(I,J) = 2 * R(NX-1,J) - R(NX-2,J)
UI(I,J) = 2 * U(NX-1,J) - U(NX-2,J)
VI(I,J) = 2 * V(NX-1,J) - V(NX-2,J)
70 CONTINUE
C

```

```

DO 75 I = 1,NX
DO 75 J = 1,NR
R(I,J) = RI(I,J)
U(I,J) = UI(I,J)
V(I,J) = VI(I,J)
75 CONTINUE

```

C
C
C
C

MINIMUM DT SEARCH

```

DO 90 J = 2,NR
DO 80 I = 2,NX
DTX = DTY ( D(2,I), D(2,I-1), U(I,J), R(I,J) )
Y = (J-1)*DY*D(4,I)
YM = (J-2)*DY*D(4,I)
DTY = DTT(Y,YM,V(I,J),R(I,J))
DTS = AMIN1(DTX,DTY)
IF ( DTS .GT. DTM ) GO TO 80
DTM = DTS
IMX = I
JMX = J
IF ( ICK .EQ. 1 ) CALL CHECK ( 'DTM',DTM,'DTY',DTY,'DTX',DTX,
1'DTS',DTS,'Y',Y,'YM',YM,92) Q*** CHECK 92 ****
80 CONTINUE
90 CONTINUE

```

C
C
C

COMPUTE THA,HX,M&P FOR FC=0.0

```

81 IF ( FC .GT. 1.0E-3 ) GO TO 335
DO 331 I = 1,NX
DO 331 J = 1,NR
331 CALL ANGLE
@ GO TO 551

```

C
C
C

ARTIFICIAL VISCOSITY

```

335 DO 500 J = 1,NR
DO 500 I = 2,NX
UTM = ABS( U(I,J) - U(I-1,J) )
RI(I,J) = UTM * ( R(I,J) - R(I-1,J) )
UI(I,J) = UTM * ( U(I,J) - U(I-1,J) )
VI(I,J) = UTM * ( V(I,J) - V(I-1,J) )
IF ( ICK .EQ. 1 ) CALL CHECK ( 'RI',RI(I,J),'UI',UI(I,J),
8'VI',VI(I,J),'R',R(I,J),'UTM',UTM,'V',V(I,J),13) Q*** CHECK 13 ***
500 CONTINUE
DO 515 J = 1,NR
DO 510 I = 2,NT
Y = (J-1)*DY
FF = FC*(1-D(3,I))*Y
R(I,J) = R(I,J) + FF * ( RI(I+1,J) - RI(I,J) )
U(I,J) = U(I,J) + FF * ( UI(I+1,J) - UI(I,J) )
V(I,J) = V(I,J) + FF * ( VI(I+1,J) - VI(I,J) )
IF ( ICK .EQ. 1 ) CALL CHECK ( 'R',R(I,J),'U',U(I,J),
8'V',V(I,J),'RI',RI(I,J),'UI',UI(I,J),'VI',VI(I,J),14) Q***CHECK 14

```

```

510 CONTINUE
R(NX,J) = 2 * R(NX-1,J) - R(NX-2,J)
U(NX,J) = 2 * U(NX-1,J) - U(NX-2,J)
V(NX,J) = 2 * V(NX-1,J) - V(NX-2,J)
IF ( ICK .EQ. 1 ) CALL CHECK ( 'R',R(I,J), 'U',U(I,J),
&'V',V(I,J), 'FF',FF, 'UTM',UTM, 'NX',FLOAT(NX),15) @*** CHECK 15 ***
515 CONTINUE
DO 525 I = 2,NX
DO 520 J = 2,NR
UTM = ABS( V(I,J) - V(I,J-1) )
RI(I,J) = UTM * ( R(I,J) - R(I,J-1) )
UI(I,J) = UTM * ( U(I,J) - U(I,J-1) )
VI(I,J) = UTM * ( V(I,J) - V(I,J-1) )
IF ( ICK .EQ. 1 ) CALL CHECK ( 'RI',RI(I,J), 'UI',UI(I,J),
&'VI',VI(I,J), 'UTM',UTM, 'R',R(I,J), 'V',V(I,J),16) @*** CHECK 16 **
520 CONTINUE
RI(I,1) = ( 4 * RI(I,2) - RI(I,3) ) / 3
UI(I,1) = ( 4 * UI(I,2) - UI(I,3) ) / 3
VI(I,1) = 0.0
525 CONTINUE
DO 535 I = 2,NT
DO 530 J = 2,NTJ
Y = (J-1)*DY
FF = FC*(1-D(3,I))*Y
R(I,J) = R(I,J) + FF * ( RI(I,J+1) - RI(I,J) )
U(I,J) = U(I,J) + FF * ( UI(I,J+1) - UI(I,J) )
V(I,J) = V(I,J) + FF * ( VI(I,J+1) - VI(I,J) )
IF ( ICK .EQ. 1 ) CALL CHECK ( 'R',R(I,J), 'U',U(I,J),
&'V',V(I,J), 'RI',RI(I,J), 'UI',UI(I,J), 'VI',VI(I,J),17) @** CHECK 17
526 CONTINUE
CALL ANGLE
530 CONTINUE
J = 1
R(I,J) = ( 4 * R(I,J+1) - R(I,J+2) ) / 3
U(I,J) = ( 4 * U(I,J+1) - U(I,J+2) ) / 3
V(I,J) = 0.0
IF ( ICK .EQ. 1 ) CALL CHECK ( 'R',R(I,J), 'U',U(I,J), 'V',
&V(I,J), 'DTS',DTS, 'IMX',FLOAT(IMX), 'JMX',FLOAT(JMX),19) @* CHECK 19
532 CONTINUE
CALL ANGLE
535 CONTINUE
I = NX
DO 540 J = 1,NTJ
R(I,J) = 2 * R(I-1,J) - R(I-2,J)
U(I,J) = 2 * U(I-1,J) - U(I-2,J)
V(I,J) = 2 * V(I-1,J) - V(I-2,J)
IF ( ICK .EQ. 1 ) CALL CHECK ( 'R',R(I,J), 'U',U(I,J), 'V',
&V(I,J), 'DTS',DTS, 'IMX',FLOAT(IMX), 'JMX',FLOAT(JMX),20) @* CHECK 20
536 CONTINUE
CALL ANGLE
540 CONTINUE
J = NR
DO 550 I = 2,NT
HX(I,J) = SQRT( U(I,J)**2 + V(I,J)**2 )

```



```

U(I,J) = HX(I,J) * D(11,I)
V(I,J) = HX(I,J) * D(10,I)
CALL ANGLE
550 CONTINUE
I = NX
HX(I,J) = 2 * HX(I-1,J) - HX(I-2,J)
U(I,J) = HX(I,J) * D(11,I)
V(I,J) = HX(I,J) * D(10,I)
CALL ANGLE
I = 1
DO 1550 J = 1, NR
CALL ANGLE
1550 CONTINUE
C
551 DT = 0.98*DTM
C
C PRINT DT (OPTION)
C
IF ( LONG ) GO TO 556
IF ( BRIEF ) GO TO 390
IF ( ISP .EQ. 1 ) GO TO 360
IF ( IPRT .NE. 1 ) GO TO 350
340 WRITE (6,1000) N
TT = 'DTM'
WRITE (6,1001) TT, IMX, JMX, DTM
WRITE (6,1002) DT
C
C OUTPUT
C
350 IF ( LONG ) GO TO 556
IF ( IBR .NE. 1 ) GO TO 390
IF ( NDPT ) GO TO 380
360 NSKCTR = NSKCTR + 1
IF ( ISP .NE. 1 ) GO TO 370
IF ( NSKCTR .NE. NSK ) GO TO 390
NSKCTR = 0
370 CONTINUE
IF ( ISP .NE. 1 ) GO TO 380
NDPT = .TRUE.
GO TO 340
380 NDPT = .FALSE.
CALL TABLES ( R, N, 16, 'R' )
CALL TABLES ( U, N, 17, 'U' )
CALL TABLES ( V, N, 18, 'V' )
CALL TABLES ( HX, N, 19, 'HX' )
CALL TABLES ( THA, N, 20, 'THA' )
CALL TABLES ( M, N, 21, 'M' )
CALL TABLES ( P, N, 22, 'P' )
C
C LONG LIST OPTION
C
556 IF ( .NOT. LONG ) GO TO 390
IF ( ISP .NE. 1 ) GO TO 557
NSKCTR = NSKCTR + 1

```

```

      IF ( NSKCTR .NE. NSK ) GO TO 390
      NSKCTR = 0
557 CONTINUE
      WRITE (6,1026) X0,AGLIN,AGLOUT,RC,NX,NR,N
      DO 554 J = 1,NR
      WRITE (6,1027) J
      WRITE (6,1028)
      DO 553 I = 1,NX
553 WRITE (6,1029) I,R(I,J),U(I,J),V(I,J),THA(I,J),HX(I,J),
      + M(I,J),P(I,J)
554 CONTINUE
      GO TO 400
C
C BRIEF LIST OPTION
C
390 IF ( .NOT. BRIEF ) GO TO 400
      WRITE (6,1024) N,I1,J1,I2,J1,I3,J1,I1,J2,I2,J2,I3,J2,
      +I1,J3,I2,J3,I3,J3,RTM
      WRITE (6,1021) QR, R(I1,J1), R(I2,J1), R(I3,J1),
      + R(I1,J2), R(I2,J2), R(I3,J2),
      + R(I1,J3), R(I2,J3), R(I3,J3)
      WRITE (6,1021) QU, U(I1,J1), U(I2,J1), U(I3,J1),
      + U(I1,J2), U(I2,J2), U(I3,J2),
      + U(I1,J3), U(I2,J3), U(I3,J3)
      WRITE (6,1021) QV, V(I1,J1), V(I2,J1), V(I3,J1),
      + V(I1,J2), V(I2,J2), V(I3,J2),
      + V(I1,J3), V(I2,J3), V(I3,J3)
      WRITE (6,1021) QTHA, THA(I1,J1), THA(I2,J1), THA(I3,J1),
      + THA(I1,J2), THA(I2,J2), THA(I3,J2),
      + THA(I1,J3), THA(I2,J3), THA(I3,J3)
      WRITE (6,1021) QHX, HX(I1,J1), HX(I2,J1), HX(I3,J1),
      + HX(I1,J2), HX(I2,J2), HX(I3,J2),
      + HX(I1,J3), HX(I2,J3), HX(I3,J3)
      WRITE (6,1021) QM, M(I1,J1), M(I2,J1), M(I3,J1),
      + M(I1,J2), M(I2,J2), M(I3,J2),
      + M(I1,J3), M(I2,J3), M(I3,J3)
      WRITE (6,1021) QP, P(I1,J1), P(I2,J1), P(I3,J1),
      + P(I1,J2), P(I2,J2), P(I3,J2),
      + P(I1,J3), P(I2,J3), P(I3,J3)
C
C 400 CONTINUE
      N = NE
C
C DEMAND MODE ONLY
C
      IF ( IBR .EQ. 1 ) GO TO 421
      PRINT 1017,RTM
      PRINT 1005,N
      READ 1006,NPLTN
405 IF ( NPLTN .EQ. 1 ) CALL PLOT
      PRINT 1008
      READ 1006, NSTG0
      IF ( NSTG0 .NE. 1 ) GO TO 410
      IF ( NPLOT .EQ. 1 ) PRINT 1009

```

```

GO TO 420
410 PRINT 1007
READ 1006, NOSTPS
NS = N + 1
NE = NOSTPS + N
PRINT 1014
READ 1006, CHGFF
IF ( CHGFF .GE. 0.0 ) FC = CHGFF
IF ( IBR .NE. 1 ) PRINT 1016
READ 1006, IPRT
IF ( NSV .EQ. 1 ) GO TO 24
GO TO 25
C
420 PRINT 1018
READ 1006, NSVE
IF ( NSVE .NE. 1 ) GO TO 422
C
C
LONG FINAL LIST OPTION
C
421 IF ( .NOT. FINAL ) GO TO 425
WRITE (6,1026) X0,AGLIN,AGLOUT,RC,NX,NR,N
DO 424 J = 1,NR
WRITE (6,1027) J
WRITE (6,1028)
DO 423 I = 1,NX
423 WRITE (6,1029) I,R(I,J),U(I,J),V(I,J),THA(I,J),HX(I,J),
+ M(I,J),P(I,J)
424 CONTINUE
C
C
425 WRITE (11) N,DT,RTM,R,U,V,THA,HX
END FILE 11
REWIND 11
422 CONTINUE
STOP
END

FUNCTION F(K,AREA)
COMMON/BLK5/OMACH(150)
DOUBLE PRECISION F,OMACH,AREA
F = (0.83333 + 0.16667*OMACH(K)**2)**3 / OMACH(K) - AREA
RETURN
END

```

```

SUBROUTINE PLOT
REAL M
DIMENSION X(51), Y(51)
COMMON /BLK1/ R(51,51), U(51,51), V(51,51), THA(51,51),
+ HX(51,51), M(51,51), P(51,51), GAMMA
COMMON /BLK2/ RT, D(16,51), NPLOT, DY
COMMON /BLK3/ NX, NR
COMMON /BLK4/ N, I, J
1 FORMAT(' CODE LIST = 1')
1001 FORMAT(/' PARAMETER CODES:')
+/' RADIAL: 1/R, 2/U, 3/V, 4/HX, 5/THA, 6/P, 7/M, /
+2X, ' AXIAL: 8/R, 9/U, 10/V, 11/HX, 12/THA, 13/P, 14/M')
2 FORMAT()
3 FORMAT(' WHICH COLUMN ? ')
4 FORMAT(///' N =', I4, 4X, 'I =', I3/' J', 7X, A3, 10X, 'Y')
5 FORMAT(1X, I2, 2(3X, 1PE9, 3))
7 FORMAT(' POINT LIST = 1')
8 FORMAT(//41X, A3, 'FOR N=', I4, ', I=', I2///)
9 FORMAT(//33X, 'TOTAL VELOCITY VS ', A1, ' FOR N=', I4, 1X, A1, '=', I2///)
10 FORMAT(//20X, 'VELOCITY ANGLE IN DEGREES VS ', A1, ' FOR N=', I4,
+1X, A1, '=', I2, ///)
11 FORMAT(' PLOT = 1')
12 FORMAT(' MORE LOOK = 1')
13 FORMAT(' WHICH ROW ? ')
14 FORMAT(//25X, 'PRESSURE RATIO VS X FOR N =', I4, ', J =', I3///)
15 FORMAT(/' N =', I4, 4X, 'J =', I3/' I', 7X, A3, 9X, 'X')
16 FORMAT(//28X, 'MACH NUMBER VS X FOR N=', I4, ', J=', I2, ///)
1016 FORMAT(//33X, A3, ' VS X FOR N=', I4, ' J=', I2, ///)
1017 FORMAT(' ENTER CODE')
1018 FORMAT(' COME AGAIN ?')
PRINT 1
READ 2, NPRCD
IF ( NPRCD .EQ. 1 ) PRINT 1001
217 PRINT 1017
218 READ 2, ITX
IF ( ITX .LE. 14 .AND. ITX .GE. 1 ) GO TO 219
PRINT 1018
GO TO 218
219 IF ( ITX .EQ. 1 .OR. ITX .EQ. 8 ) AR = 'R'
IF ( ITX .EQ. 2 .OR. ITX .EQ. 9 ) AR = 'U'
IF ( ITX .EQ. 3 .OR. ITX .EQ. 10 ) AR = 'V'
IF ( ITX .EQ. 4 .OR. ITX .EQ. 11 ) AR = 'HX'
IF ( ITX .EQ. 5 .OR. ITX .EQ. 12 ) AR = 'THA'
IF ( ITX .EQ. 6 .OR. ITX .EQ. 13 ) AR = 'P'
IF ( ITX .EQ. 7 .OR. ITX .EQ. 14 ) AR = 'M'
IF ( ITX .LE. 7 ) GO TO 18
PRINT 13
300 READ 2, IRW
IF ( IRW .LE. NR .AND. IRW .GE. 1 ) GO TO 19
PRINT 1018
GO TO 300
18 PRINT 3
220 READ 2, ICL
IF ( ICL .LE. NX .AND. ICL .GE. 1 ) GO TO 19

```

```
PRINT 1018
GO TO 220
19 PRINT 7
READ 2,IT
GO TO (20,40,60,80,100,110,1120,1117,1119,1210,1230,1250,
+114,116),ITX
20 DO 30 I = 1,NR
30 X(I) = R(ICL,I)
GO TO 120
40 DO 50 I = 1,NR
50 X(I) = U(ICL,I)
GO TO 120
60 DO 70 I = 1,NR
70 X(I) = V(ICL,I)
GO TO 120
80 DO 90 I = 1,NR
90 X(I) = HX(ICL,I)
GO TO 120
100 DO 110 I = 1,NR
110 X(I) = THA(ICL,I)
GO TO 120
1110 DO 1111 I = 1,NR
1111 X(I) = P(ICL,I)
GO TO 120
1120 DO 1121 I = 1,NR
1121 X(I) = M(ICL,I)
GO TO 120
112 DO 113 I = 2,NX
113 X(I) = D(2,I)*RT
X(1) = X(2)
GO TO 118
114 DO 115 I = 2,NX
115 Y(I) = P(I,IRW)
Y(1) = Y(2)
GO TO 112
116 DO 117 I = 2,NX
117 Y(I) = M(I,IRW)
Y(1) = Y(2)
GO TO 112
1117 DO 1118 I = 2,NX
1118 Y(I) = R(I,IRW)
Y(1) = Y(2)
GO TO 112
1119 DO 1200 I = 2,NX
1200 Y(I) = U(I,IRW)
Y(1) = Y(2)
GO TO 112
1210 DO 1220 I = 2,NX
1220 Y(I) = V(I,IRW)
Y(1) = Y(2)
GO TO 112
1230 DO 1240 I = 2,NX
1240 Y(I) = HX(I,IRW)
Y(1) = Y(2)
```

```

GO TO 112
1250 DO 1260 I = 2,NX
1260 Y(I) = THA(I,IRW)
      Y(1) = Y(2)
      GO TO 112
118 IF ( IT .NE. 1 ) GO TO 150
     PRINT 15,N,IRW,AR
     DO 119 I = 2,NX
119 PRINT 5,I,Y(I),X(I)
     GO TO 150
120 DO 130 I = 1,NR
130 Y(I) = (I-1) * DY * D(4,ICL) * RT
     IF ( IT .NE. 1 ) GO TO 150
     PRINT 4,N,ICL,AR
     DO 140 I = 1,NR
140 PRINT 5,I,X(I),Y(I)
150 PRINT 11
     READ 2,NPLOT
     IF ( NPLOT .NE. 1 ) GO TO 210
     IF ( ITX .LE. 7 ) NG = NR
     IF ( IIX .GE. 8 ) NG = NX
     CALL EZSUB(X,Y,NG)
     GO TO (160,160,160,170,180,160,160,183,183,183,184,185,
+181,182) ,ITX
160 PRINT 8,AR,N,ICL
     GO TO 200
170 AY = 'Y'
     AC = 'I'
     PRINT 9,AY,N,AC,ICL
     GO TO 200
180 AY = 'Y'
     AC = 'I'
     PRINT 10,AY,N,AC,ICL
     GO TO 200
181 PRINT 14,N,IRW
     GO TO 200
182 PRINT 16,N,IRW
     GO TO 200
183 PRINT 1016,AR,N,IRW
     GO TO 200
184 AX = 'X'
     AC = 'J'
     PRINT 9,AX,N,AC,IRW
     GO TO 200
185 AX = 'X'
     AC = 'J'
     PRINT 10,AX,N,AC,IRW
     GO TO 200
200 CONTINUE
210 PRINT 12
     READ 2,MORE
     IF ( MORE .EQ. 1 ) GO TO 217
     RETURN
     END

```

```

SUBROUTINE ANGLE
REAL M
COMMON /BLK1/ R(51,51), U(51,51), V(51,51), THA(51,51),
+ HX(51,51), M(51,51), P(51,51), GAMMA
COMMON /BLK3/ NX, NR
COMMON /BLK4/ N, I, J
THA(I,J)=0.0
IF(U(I,J) .EQ. 0.0) GO TO 1
IF(V(I,J) .EQ. 0.0) GO TO 2
THA(I,J)=ATAN(V(I,J)/U(I,J))
THA(I,J)=THA(I,J)*57.295779
IF (U(I,J) .LT. 0.0) THA(I,J)=THA(I,J)+180.0
GO TO 3
1 IF(V(I,J) .EQ. 0.0) GO TO 3
IF(V(I,J) .LT. 0.0) THA(I,J)=-90.0
IF(V(I,J) .GT. 0.0) THA(I,J)= 90.0
GO TO 3
2 IF(U(I,J) .LT. 0.0) THA(I,J)=180.0
3 HX(I,J)=SQRT(U(I,J)**2+V(I,J)**2)
M(I,J) = HX(I,J)/SQRT(GAMMA*EXP((GAMMA-1)*R(I,J)))
P(I,J) = EXP(GAMMA*R(I,J))
RETURN
END

```

```

SUBROUTINE OPTR(O,LET,TF)
LOGICAL TF,LOG
TF=.FALSE.
N=FLD(0,6,LET)-5
LOG=BDOL(FLD(9+N,1,0))
IF(LOG)TF=.TRUE.
RETURN
END

```

```

FUNCTION OP(LIT)
IF(L)2, 2
CALL DEMOPT(A)
L=1
CONTINUE
CALL OPTR(A,LIT,OP)
RETURN
END
2

```


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