BRANCHING RULES FOR QUANTUM TOROIDAL $\mathfrak{gl}_n$

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Abstract. We construct an analog of the subalgebra $U\mathfrak{gl}(n) \otimes U\mathfrak{gl}(m) \subset U\mathfrak{gl}(m+n)$ in the setting of quantum toroidal algebras and study the restrictions of various representations to this subalgebra.

1. Introduction

1.1. **Motivation: the AGT conjecture.** The quantum toroidal algebra, [GKV], associated with a semi-simple Lie algebra $\mathfrak{g}$ is the quantum version of the universal enveloping algebra of the Lie algebra of currents $\mathbb{C}^* \times \mathbb{C}^* \to \mathfrak{g}$.

In this paper we consider only the case $\mathfrak{g} = \mathfrak{gl}_n$, $n \geq 1$. The corresponding toroidal algebra $\mathcal{E}_n = \mathcal{E}_n(q_1, q_2, q_3)$, see Section 2.1, depends on three deformation parameters q_1, q_2, q_3 such that $q_1q_2q_3 = 1$. The algebra $\mathcal{E}_n(q_1, q_2, q_3)$ has two central elements which we denote by q^c and κ. In all representations appearing in this paper, one of the central elements, q^c, always acts by 1. In the limit $q_2 \to 1$, the algebra $\mathcal{E}_n$ becomes the universal central extension of the universal enveloping algebra of the Lie algebra $\mathbb{M}_n \otimes \mathbb{C}[Z^{\pm 1}, D^{\pm 1}]$, see Section 3.7. Here $\mathbb{M}_n$ is the algebra of $n \times n$ matrices, and $\mathbb{C}[Z^{\pm 1}, D^{\pm 1}]$ is the algebra of functions on the one-dimensional quantum torus: $ZD = q_1^n DZ$. The Lie algebra structure is given by the standard formula $[a, b] = ab - ba$.

The algebra $\mathcal{E}_n$ has another important so-called conformal limit. This limit is more subtle and it is obtained by setting $q_1 = \varepsilon^{\sigma_1}$, $q_2 = \varepsilon^{\sigma_2}$, $q_3 = \varepsilon^{\sigma_3}$ with $\sigma_1 + \sigma_2 + \sigma_3 = 0$, $\kappa = \varepsilon^k$, and sending $\varepsilon \to 1$. This limit is called conformal, because the limiting algebra has a vertex operator algebra (conformal algebra) structure. The limiting algebra depends on σ_1/σ_2 and k. Note that the algebra obtained via the conformal limit for special values of parameters is smaller than $\mathcal{E}_n$.

The conformal limit is important for the study of the AGT conjecture. The AGT conjecture, [AGT], claims that when parameters of the 4-dimensional topological super Yang-Mills field theory go to an appropriate limit, the theory becomes deeply connected to a 2-dimensional conformal field theory. At the same time the algebra $\mathcal{E}_n(q_1, q_2, q_3)$ acts by correspondences in the space of the K-theory of the moduli spaces of instantons related to the 4-dimensional topological super Yang-Mills field theory and the conformal limit of $\mathcal{E}_n(q_1, q_2, q_3)$ describes the relevant conformal field theory.

1.2. **Motivation: the coset constructions.** Consider a pair of affine Lie algebras: $\widehat{\mathfrak{gl}}_N$ and its subalgebra $\widehat{\mathfrak{gl}}_{N-n} \subset \widehat{\mathfrak{gl}}_N$ both with level k. The well-known coset construction of conformal field theory gives a new vertex operator algebra for this pair, which we denote $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-n})$. The coset algebras naturally appear in the problem of decompositions of representations. Consider a restriction of an integrable representation π of $\widehat{\mathfrak{gl}}_N$ with level k to the subalgebra $\widehat{\mathfrak{gl}}_{N-n}$.

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Then we have the decomposition $\pi = \oplus_\alpha W_\alpha \otimes \mathcal{R}_\alpha$, where $\mathcal{R}_\alpha$ are irreducible representations of $\widehat{\mathfrak{gl}}_{N-n}$, and spaces of multiplicities W_α are irreducible representations of the algebra $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-n})$.

The problem of decomposition of $U\mathfrak{gl}_N$ module after restriction to $U\mathfrak{gl}_{N-n}$ is closely related to the problem of finding the commutant of the subalgebra $U\mathfrak{gl}_{N-n}$ in $U\mathfrak{gl}_N$. This commutant can be described explicitly, and it is closely related to the Yangian of $\mathfrak{gl}_n$, see [O1], [O2]. Namely, the commutant is a factor of the Yangian and the Yangian can be viewed as the analytic continuation of the commutant with respect to the variable N. To get generators and relations of the coset algebra $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-n})$ one has to study the commutant in the affine setting.

Clearly, $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-n})$ contains the subalgebra $\widehat{\mathfrak{gl}}_n$ with level k generated by $E_{ij}(z) = \sum_{s\in\mathbb{Z}}(E_{ij}\otimes t^s)z^{-s}$, where $i,j = N-n+1,\dots,N$. It also contains the quadratic currents $E^{(2)}_{ij}(z) = \sum_{\alpha=1}^{N-n} : E_{i\alpha}(z)E_{\alpha j}(z) :$ with $i,j = N-n+1,\dots,N$. In fact, the algebra $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-n})$ is generated by $E_{ij}(z)$ and $E^{(2)}_{ij}(z)$. But in the operator product of the currents $E_{ij}(z)$ with $E_{ij}(w)$ one can find cubic currents $E^{(3)}_{ij}(z)$, then quartic currents $E^{(4)}_{i,j}(z)$ and so on. The coset algebra $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-n})$ is expected to be a factor of a quantization of the universal enveloping algebra of the double current Lie algebra $\mathfrak{gl}_n \otimes \mathbb{C}[z_1^{\pm1}, z_2]$, and then the currents $E^{(m)}_{ij}(z)$ should correspond to the currents $E_{ij}(z_1)z_2^m$.

The case of $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-1})$ is the best studied and is known as the W-algebra associated to $\mathfrak{gl}_k$. There exists a number of alternative constructions which produce the W-algebra, though in almost all cases, a rigorous proof of the identification is missing.

For example, consider the algebra obtained by the quantum Drinfeld-Sokolov reduction of $\widehat{\mathfrak{gl}}_M$ with level s, followed by the analytic continuation with respect to M [FF]. We follow the standard notation and denote the result by $W_{M,\frac{1}{s+M}}$. Then with this notation, we have $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-1}) \simeq W_{k,\frac{N+k+1}{N+k+2}}$. This statement is non-trivial, the direct check is tedious and has not been done yet.

There exists a dual coset construction of the algebra $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-1})$, where one takes $\widehat{\mathfrak{gl}}_k$ of level 1 times $\widehat{\mathfrak{gl}}_k$ of arbitrary level and considers the coset with respect to the diagonal embedding of $\widehat{\mathfrak{gl}}_k$.

There is an additional puzzling observation that the algebra $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-1})$ is isomorphic to the W-algebra constructed by the Drinfeld-Sokolov reduction from the Lie superalgebra $\widehat{\mathfrak{gl}}(N|N-1)$.

The algebra $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-n})$ depends on two parameters N, k, where k is the level of $\widehat{\mathfrak{gl}}_N$. The parameter k is a complex number, while N is natural. However, the structure constant in $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-n})$ depends on N algebraically. Therefore, we can make the analytic continuation with respect to N, then N becomes an arbitrary complex number. The quantum toroidal algebra $\mathcal{E}_n(q_1,q_2,q_3)$ is a quantization of the resulting algebra. Moreover, the conformal limit of algebra $\mathcal{E}_n(q_1,q_2,q_3)$ coincides with $\mathcal{C}_k(\widehat{\mathfrak{gl}}_N, \widehat{\mathfrak{gl}}_{N-n})$.

In particular, the algebra corresponding to the W-algebra, $\mathcal{E}_1(q_1,q_2,q_3)$, is the quantum toroidal algebra which has been most extensively studied. It is known as elliptic Hall algebra [BS], [S], [SV2], (q,γ) analog of $\mathcal{W}_{1+\infty}$, [M07], an elliptic deformation of the W algebra of type

$\mathfrak{gl}$, Ding-Iohara algebra, [FHHSY], spherical Cherednik DAHA [SV1], quantum continuous $\mathfrak{gl}_\infty$, [FFJMM1], [FFJMM2].

The coset construction has a quantum group version. Consider the quantum affine algebra $U_q(\widehat{\mathfrak{gl}}_N)$ with the subalgebra $U_q(\widehat{\mathfrak{gl}}_{N-1})$. Then the problem is to find the commutant of $U_q(\widehat{\mathfrak{gl}}_{N-1})$ in $U_q(\widehat{\mathfrak{gl}}_N)$. This is a non-trivial question which we suggest to solve using the quantum toroidal algebras.

Namely, one expects that there is an evaluation map $\mathcal{E}_N(q_1, q_2, q_3) \to U_q(\widehat{\mathfrak{gl}}_N)$ where $q^2 = q_2$ and the level of $\widehat{\mathfrak{gl}}_N$ depends on q_1 and κ. Then on the level of quantum toroidal algebras, we find a homomorphism of algebras $\varphi : \mathcal{E}_1 \otimes \mathcal{E}_{N-1} \to \tilde{\mathcal{E}}_N$ where $\tilde{\mathcal{E}}_N$ is a suitable completion of $\mathcal{E}_N$. In the Lie algebra limit $q_2 \to 1$ the map φ becomes very simple: it is just the map coming from the embedding

$$\mathbb{M}_1 \otimes \mathbb{C}[Z^{\pm 1}, D^{\pm 1}] \oplus \mathbb{M}_{N-1} \otimes \mathbb{C}[Z^{\pm 1}, D^{\pm 1}] \to \mathbb{M}_N \otimes \mathbb{C}[Z^{\pm 1}, D^{\pm 1}].$$

Note that on the other hand the conformal limit of φ is rather non-trivial.

Combining φ with the evaluation map, we obtain

$$\mathcal{E}_1 \otimes \mathcal{E}_{N-1} \to U_q(\widehat{\mathfrak{gl}}_N).$$

The image of the subalgebra $1 \otimes \mathcal{E}_{N-1}$ is $U_q(\widehat{\mathfrak{gl}}_{N-1})$ and $\mathcal{E}_1 \otimes 1$ is mapped to the commutant of $U_q(\widehat{\mathfrak{gl}}_{N-1})$ in $U_q(\widehat{\mathfrak{gl}}_N)$. Actually, we believe that the map of algebra $\mathcal{E}_1 \otimes 1$ to the commutant is surjective, but we do not discuss this fact in the present paper. Instead we concentrate on a family of irreducible representations of the algebra $\mathcal{E}_N$ and study the restriction on the product $\mathcal{E}_1 \otimes \mathcal{E}_{N-1}$. In all cases we consider, the multiplicities of irreducible representations of $\mathcal{E}_1 \otimes \mathcal{E}_{N-1}$ appearing in irreducible representations of $\mathcal{E}_N$ are one.

1.3. **Motivation: geometry.** The simplest integrable representation of $\mathcal{E}_n$ is called the Fock module, [VV2], [STU], [FJMM1], [FJMM2], [S]. The Fock module appears in geometry in the following way. Consider the Hilbert scheme H_d of ideals of codimension d in $\mathbb{C}[z_1, z_2]$. The plane $\mathbb{C}^2$ is equipped with an action of the torus $\mathbb{C}^* \times \mathbb{C}^*$ via $\alpha \times \beta :\ (z_1, z_2) \mapsto (\alpha z_1, \beta z_2)$ and of the cyclic group $\mathbb{Z}_p$ of order p via $\zeta(z_1, z_2) = (\zeta z_1, \zeta^{-1} z_2)$, where $\zeta \in \mathbb{C}^*$ is a root of unity of order p. These actions induce the corresponding actions in H_d.

Let $H_d^{(p)}$ be the manifold of the fixed points of $\mathbb{Z}_p$ in H_d. The manifold $H_d^{(p)}$ is smooth but not connected. It is known that the quantum toroidal algebra $\mathcal{E}_p(q_1, q_2, q_3)$ acts in the equivariant K-theory space $\mathcal{F} = \oplus_{d=0}^{\infty} K(H_d^{(p)})$, where q_1, q_2 are the equivariant parameters, by correspondences, see [N], [FT]. This representation of $\mathcal{E}_p$ is isomorphic to the Fock module. Moreover, geometrically one observes the following remarkable phenomenon.

A basis in $\mathcal{F}$ is given by fixed points of $\mathbb{C}^* \times \mathbb{C}^*$ action. This basis consists of eigenvectors of the Cartan subalgebra of $\mathcal{E}_p$. If $J \in H_d^{(p)}$ then $J \subset \mathbb{C}[z_1, z_2]$ is a homogeneous ideal such that the quotient $\mathbb{C}[z_1, z_2]/J$ is a d-dimensional representation of $\mathbb{Z}_p$. Irreducible representations of $\mathbb{Z}_p$ are all one dimensional, denote them $\nu_0, \nu_1, \ldots, \nu_{p-1}$. We call $J \in H_d^{(p)}$ of type $(a_0, \ldots, a_{p-1})$ if $\mathbb{C}[z_1, z_2]/J = \oplus_{i=0}^{p-1} a_i \nu_i$. Note that $a_0 + \cdots + a_{p-1} = d$. Denote $H_{a_0, \ldots, a_{p-1}} \subset H_d^{(p)}$ the set of ideals of type $(a_0, \ldots, a_{p-1})$.

Then $H_{a_0, \ldots, a_{p-1}}$ are exactly the connected components of $H_d^{(p)}$, and we have the geometric description of the weight decomposition of the Fock module: $\mathcal{F} = \oplus K(H_{a_1, \ldots, a_{p-1}})$.

The algebra $\mathcal{E}_p$ has a large group of automorphisms which is a toroidal version of Lusztig braid group, see [M99]. In particular, this group contains the root lattice of $\mathfrak{sl}_p$, which consists of the extensions of the affine translations to $\mathcal{E}_p$. This lattice is isomorphic to $\mathbb{Z}^{p-1}$ and it also acts in the Fock module $\mathcal{F}$. Geometric description of the action of the braid group is non-trivial, but one can observe the following stabilization of manifolds.

The group $\mathbb{Z}^{p-1}$ acts in the set of weights $\{(a_0, \dots, a_{p-1})\}$. Fix some $A = (a_0, \dots, a_{p-1})$ and let $T \in \mathbb{Z}^{p-1}$ be a generic element. Consider the sequence of manifolds $\mathcal{M}_s = H^{(p)}_{T^s A}$, $s = 0, 1, 2, \dots$. According to [N], for s large enough the manifolds $\mathcal{M}_s$ are all isomorphic and have a simple geometric description which can be described as follows. Consider the quotient $(\mathbb{C} \times \mathbb{C})/\mathbb{Z}_p$. It has the Kleinian singularity at the origin. Resolve this singularity and call the result X_p. Then X_p is a 2-dimensional smooth manifold with a natural action of the torus $\mathbb{C}^* \times \mathbb{C}^*$.

For s large enough the manifolds $\mathcal{M}_s$ is isomorphic to a connected component of the Hilbert scheme of torsion free sheaves on X_p. The choice of the connected component corresponds to the choice of $(a_0, \dots, a_{p-1})$.

On the other hand, the manifold X_p has p fixed points with respect to $\mathbb{C}^* \times \mathbb{C}^*$. Therefore, on the Hilbert scheme of X_p we have p commuting actions of $\mathcal{E}_1$. The i-th action is given by correspondences with support in the i-th point. Thus, in this limit of the Fock module, we observe an action of $\mathcal{E}_1^{\otimes p}$. One of the goals of this paper is to give a representation-theoretic explanation of this phenomenon.

Namely, the Cartan subalgebra of $\mathcal{E}_p$ is a commutative algebra generated by $\{K_i^{\pm}(z)\}$. The fixed points are eigenvectors with respect to the operators $K_i^{\pm}(z)$. Consider the operators $T^s K_i^{\pm}(z)$, $s = 0, 1, 2, \dots$, acting in the Fock module. For $v \in \mathcal{F}$, we have $T^s K_i^{\pm}(z) \cdot v = T^s \circ K_i^{\pm}(z) \circ T^{-s} v$. For each $v \in \mathcal{F}$, for large enough s, the vector $T^s K_i^{\pm}(z) \cdot v$ does not depend on s. The joint spectrum of the Cartan subalgebra is simple, so in the limit $s \to \infty$, we obtain a basis of the Fock module.

In addition, we construct an embedding $\mathcal{E}_1^{\otimes p} \to \widetilde{\mathcal{E}}_p$. Then the action of this subalgebra in the above basis recovers the geometric action.

1.4. **The plan of the paper and the main results.** Here is the outline of the paper.

We denote $\mathcal{E}_n$ the quantum toroidal algebra of type $\mathfrak{gl}_n$.

Section 2 collects notation and basic facts about $\mathcal{E}_n$. We discuss the defining relations in Section 2.1, automorphisms in Section 2.4, representation theory in Section 2.5. In the literature, the cases $n \geq 3$ and $n = 1$ usually appear separately, while $n = 2$ is often omitted. We manage to write all formulas in a uniform way.

Section 3 contains the construction and the properties of $\mathcal{E}_m$ inside a suitable completion of $\mathcal{E}_n$, $m < n$. The main construction is described in Section 3.1, it defines fused currents via a quantum version of the operator product expansion. Then we prove our first main results, Theorem 3.1, see Section 3.4 and Theorem 3.4, see Section 3.6. Theorem 3.1 establishes that the fused currents do satisfy the relations of the quantum toroidal $\mathfrak{gl}_m$, and Theorem 3.4 proves that the upper left corner and the bottom right corner subalgebras $\mathcal{E}_n$ and $\mathcal{E}_m$ commute within $\mathcal{E}_{m+n}$. Our main method is the study of correlation functions, we develop the techniques in Section 3.3. In Section 3.7 we describe the Lie algebra limit of $\mathcal{E}_n$ and the meaning of our construction in this limit.

Section 4 is devoted to the study of the modules over $\mathcal{E}_{m+n}$ after restriction to $\mathcal{E}_m \otimes \mathcal{E}_n$. We write the formulas mainly in the case of $n = 1$. We give all details in the case of the Fock module and $n = 1$ to explain the approach and the logic of the proofs, see Section 4.1. Then we proceed to tensor products of Fock modules and their irreducible submodules. The main results are Theorem 4.9, Theorem 4.11 and Theorem 4.12. These theorems explicitly describe decompositions of various modules. We conclude with a conjectural formula for the decomposition of the so called Macmahon module.

2. Quantum toroidal algebras

In this section we introduce our notation concerning the quantum toroidal algebra of type $\mathfrak{gl}_n$. We also recall its basic features relevant to the present text.

2.1. **Generators and relations.** Let n be a natural number. We shall write $a \equiv b$ for $a \equiv b \bmod n$. Let $(a_{i,j})_{i,j=0}^{n-1}$ be the Cartan matrix of type $A_{n-1}^{(1)}$, and let $(m_{i,j})_{i,j=0}^{n-1}$ be a skew-symmetric matrix defined by $m_{i+1,i} = 1$ and $m_{i,j} = 0$ if $i \not\equiv j \pm 1$, where the suffix is to be read modulo n.

Fix non-zero complex numbers d, q. Throughout the text we shall use the parameters

$$q_1 = dq^{-1},\ q_2 = q^2,\ q_3 = d^{-1}q^{-1}\,,$$

so that $q_1q_2q_3 = 1$. We assume further that for $n_1, n_2, n_3 \in \mathbb{Z}$

$$q_1^{n_1}q_2^{n_2}q_3^{n_3} = 1 \text{ holds only if } n_1 = n_2 = n_3.$$

In particular, none of the q_i is a root of unity.

The *quantum toroidal algebra* of type $\mathfrak{gl}_n$, which we denote $\mathcal{E}_n$, is an associative unital $\mathbb{C}$-algebra defined by generators and relations to be given below.

The algebra $\mathcal{E}_n$ has generators

$$E_{i,k},\ F_{i,k},\ H_{i,r},\ K_i^{\pm 1},\ q^{\pm c} \quad (i \in \mathbb{Z}/n\mathbb{Z},\ k \in \mathbb{Z},\ r \in \mathbb{Z}/\{0\}).$$

In order to write down the defining relations, introduce the generating series

$$E_i(z) = \sum_{k\in\mathbb{Z}} E_{i,k}z^{-k}, \quad F_i(z) = \sum_{k\in\mathbb{Z}} F_{i,k}z^{-k}, \quad K_i^{\pm}(z) = K_i^{\pm 1}\exp(\pm(q-q^{-1})\sum_{r=1}^{\infty} H_{i,\pm r}z^{\mp r})\,.$$

Define further $g_{i,j}(z,w)$ by

$$n \geq 3: \quad g_{i,j}(z,w) = \begin{cases} z - q_2w & (i \equiv j), \\ z - q_1w & (i \equiv j-1), \\ z - q_3w & (i \equiv j+1), \\ z - w & (i \not\equiv j, j\pm 1). \end{cases}$$

$$n = 2: \quad g_{i,j}(z,w) = \begin{cases} z - q_2w & (i \equiv j), \\ (z - q_1w)(z - q_3w) & (i \not\equiv j). \end{cases}$$

$$n = 1: \quad g_{0,0}(z,w) = (z - q_1w)(z - q_2w)(z - q_3w).$$

and

$$d_{i,j}=\begin{cases}d^{\mp1} & (i\equiv j\mp1, n\geq 3),\\ -1 & (i\not\equiv j, n=2),\\ 1 & (\text{otherwise}).\end{cases}$$

Notation being as above, the defining relations for $\mathcal{E}_n$ [1] are as follows:

$$\begin{gathered}
K_iK_i^{-1}=K_i^{-1}K_i=1,\\
q^{\pm c}\text{ are central},\quad q^cq^{-c}=q^{-c}q^c=1\,,\\
K_i^{\pm}(z)K_j^{\pm}(w)=K_j^{\pm}(w)K_i^{\pm}(z),\\
\frac{g_{i,j}(q^{-c}z,w)}{g_{i,j}(q^cz,w)}K_i^-(z)K_j^+(w)=\frac{g_{j,i}(w,q^{-c}z)}{g_{j,i}(w,q^cz)}K_j^+(w)K_i^-(z),\\
d_{i,j}g_{i,j}(z,w)K_i^{\pm}(q^{(1\mp1)c/2}z)E_j(w)+g_{j,i}(w,z)E_j(w)K_i^{\pm}(q^{(1\mp1)c/2}z)=0,\\
d_{j,i}g_{j,i}(w,z)K_i^{\pm}(q^{(1\pm1)c/2}z)F_j(w)+g_{i,j}(z,w)F_j(w)K_i^{\pm}(q^{(1\pm1)c/2}z)=0\,,
\end{gathered}$$

$$\begin{gathered}
[E_i(z),F_j(w)]=\frac{\delta_{i,j}}{q-q^{-1}}\big(\delta\big(q^c\frac{w}{z}\big)K_i^+(z)-\delta\big(q^c\frac{z}{w}\big)K_i^-(w)\big),\\
d_{i,j}g_{i,j}(z,w)E_i(z)E_j(w)+g_{j,i}(w,z)E_j(w)E_i(z)=0,\\
d_{j,i}g_{j,i}(w,z)F_i(z)F_j(w)+g_{i,j}(z,w)F_j(w)F_i(z)=0.
\end{gathered}$$

In addition we impose the Serre relations as follows. We use the notation $[A,B]_p=AB-pBA$.

For $n\geq 3$,

$$\begin{gathered}
[E_i(z),E_j(w)]=0,\quad [F_i(z),F_j(w)]=0\quad (i\neq j,j\pm1),\\
\mathrm{Sym}_{z_1,z_2}[E_i(z_1),[E_i(z_2),E_{i\pm1}(w)]_q]_{q^{-1}}=0\,,\\
\mathrm{Sym}_{z_1,z_2}[F_i(z_1),[F_i(z_2),F_{i\pm1}(w)]_q]_{q^{-1}}=0\,.
\end{gathered}$$

For $n=2$, $i\not\equiv j$,

$$\mathop{\mathrm{Sym}}_{z_1,z_2,z_3}\big[E_i(z_1),\big[E_i(z_2),\big[E_i(z_3),E_j(w)\big]_{q^2}\big]\big]_{q^{-2}}=0\,,\tag{2.1}$$

$$\mathop{\mathrm{Sym}}_{z_1,z_2,z_3}\big[F_i(z_1),\big[F_i(z_2),\big[F_i(z_3),F_j(w)\big]_{q^2}\big]\big]_{q^{-2}}=0\,.\tag{2.2}$$

For $n=1$,

$$\begin{gathered}
\mathop{\mathrm{Sym}}_{z_1,z_2,z_3}\ z_2z_3^{-1}[E_0(z_1),[E_0(z_2),E_0(z_3)]]=0\,,\\
\mathop{\mathrm{Sym}}_{z_1,z_2,z_3}\ z_2z_3^{-1}[F_0(z_1),[F_0(z_2),F_0(z_3)]]=0\,.
\end{gathered}$$

[1] We have slightly changed the notation from [FJMM2]. The generators $K_i^{\pm}(z)$, $H_{i,r}$ here correspond to $K_i^{\pm}(q^{-c/2}z)$, $q^{rc/2}H_{i,r}$ there respectively. For $n=1$, see Remark 2 in Section 2.2

In the above, $\mathrm{Sym}_{z_1,\cdots,z_s}$ stands for the symmetrization in $z_1,\cdots,z_s$.

2.2. **Some technical points.** In this subsection we give a few remarks about the relations in $\mathcal{E}_n$, which are important for this work.

It is convenient to rewrite the relations involving $K_i^{\pm}(z)$ in terms of the generators $\{H_{i,r}\}$. Let $[x]=(q^x-q^{-x})/(q-q^{-1})$.

First of all, we have

$$K_iE_j(z)K_i^{-1}=q^{a_{i,j}}E_j(z),\ \ K_iF_j(z)K_i^{-1}=q^{-a_{i,j}}F_j(z).$$

The other relations are as follows.

For $n\geq 3$,

$$\begin{aligned}
&[H_{i,r},E_j(z)]=\frac{[ra_{i,j}]}{r}d^{-rm_{i,j}}q^{(r-|r|)c/2}\,z^rE_j(z)\,,\\
&[H_{i,r},F_j(z)]=-\frac{[ra_{i,j}]}{r}d^{-rm_{i,j}}q^{(r+|r|)c/2}\,z^rF_j(z)\,,\\
&[H_{i,r},H_{j,s}]=\delta_{r+s,0}\frac{[ra_{i,j}]}{r}\frac{q^{rc}-q^{-rc}}{q-q^{-1}}d^{-rm_{i,j}}\,.
\end{aligned}$$

For $n=2$,

$$\begin{aligned}
&[H_{i,r},E_j(z)]=a_{i,j}(r)z^rE_j(z)q^{(r-|r|)c/2}\,,\\
&[H_{i,r},F_j(z)]=-a_{i,j}(r)z^rF_j(z)q^{(r+|r|)c/2}\,,\\
&[H_{i,r},H_{j,s}]=\delta_{r+s,0}\,a_{i,j}(r)\frac{q^{rc}-q^{-rc}}{q-q^{-1}},
\end{aligned}$$

where $a_{i,i}(r)=[r](q^r+q^{-r})/r$, $a_{i,j}(r)=-[r](d^r+d^{-r})/r\ (i\neq j)$.

For $n=1$,

$$\begin{aligned}
&[H_{0,r},E_0(z)]=z^rb(r)E_0(z)q^{(r-|r|)c/2},\\
&[H_{0,r},F_0(z)]=-z^rb(r)F_0(z)q^{(r+|r|)c/2},\\
&[H_{0,r},H_{0,s}]=\delta_{r+s,0}\,b(r)\frac{q^{rc}-q^{-rc}}{q-q^{-1}},
\end{aligned}$$

where $b(r)=[r](q^r+q^{-r}-d^r-d^{-r})/r$.

The following elements of $\mathcal{E}_n$ are central,

$$\kappa=K_0\cdots K_{n-1}\,,\quad q^c\,.$$

The algebra $\mathcal{E}_n$ is $\mathbb{Z}^n\times\mathbb{Z}$-graded by the degree assignment

$$\begin{aligned}
&\deg E_{i,k}=(1_i,k)\,,\quad \deg F_{i,k}=(-1_i,k)\,,\quad \deg H_{i,r}=(0,r)\,,\\
&\deg K_i=\deg q^c=(0,0),
\end{aligned}\tag{2.3}$$

where $1_i=(0,\cdots,\overset{i-th}{1},\cdots,0)\in\mathbb{Z}^n$. For a homogeneous element $x\in\mathcal{E}_n$ with $\deg x=(d_0,\cdots,d_{n-1},k)$, we set $\mathrm{pdeg}\,x=\sum_{i=0}^{n-1}d_i$ and call it the *principal degree*. We have

$$\mathrm{pdeg}\,E_{i,k}=1\,,\quad \mathrm{pdeg}\,F_{i,k}=-1\,,\quad \mathrm{pdeg}\,H_{i,r}=0\,.\tag{2.4}$$

In Section 4 we use the classical weight of a homogeneous element

$$\text{cweight}\, x = \sum_{i=1}^{n-1}(d_i - d_0)\alpha_i \tag{2.5}$$

where α_i $(i = 1, \ldots, n-1)$ are the $\mathfrak{sl}_n$ roots.

The algebra $\mathcal{E}_n$ has also a formal coproduct

$$\begin{aligned}
&\Delta E_i(z) = E_i(z) \otimes 1 + K_i^-(C_1 z) \otimes E_i(C_1 z)\,,\\
&\Delta F_i(z) = F_i(C_2 z) \otimes K_i^+(C_2 z) + 1 \otimes F_i(z)\,,\\
&\Delta K_i^+(z) = K_i^+(z) \otimes K_i^+(C_1^{-1} z)\,,\\
&\Delta K_i^-(z) = K_i^-(C_2^{-1} z) \otimes K_i^-(z)\,,\\
&\Delta\, q^c = q^c \otimes q^c\,,
\end{aligned}$$

where we have set $C_1 = q^c \otimes 1$ and $C_2 = 1 \otimes q^c$. Since the right hand side contains an infinite sum of generators, these formulas are not a coproduct in the usual sense. Nevertheless for a certain class of modules they can be used to define a module structure on tensor products. For the details see [FJMM1],[FJMM2].

In the sequel, when necessary we shall exhibit the dependence on q_i explicitly and write $\mathcal{E}_n$ as $\mathcal{E}_n(q_1, q_2, q_3)$.

Remark 1. The definition of the quantum toroidal algebra with $n \geq 3$ is due to [GKV]. Our presentation of $\mathcal{E}_n$ $(n \geq 3)$ follows closely the one given in [TU].

To the authors' knowledge, the algebra $\mathcal{E}_1$ has been introduced for the first time in [BS], where it was termed the elliptic Hall algebra. Subsequently the same algebra has been rediscovered by other authors. In [M07] it was called a (q, γ) analog of $\mathcal{W}_{1+\infty}$, and in [FHHSY] it was called Ding-Iohara algebra. In [FFJMM1], [FFJMM2] we called it "quantum continuous $\mathfrak{gl}_\infty$".

Remark 2. In our previous paper [FJMM1] we have used an algebra which is an extension of $\mathcal{E}_1$ by an additional central element. The correspondence of the notation in [FJMM1] and the present paper is $e(z) = (1/(1-q_1))E_0(z)$, $f(z) = -(q^{-1}/(1-q_3))\mathfrak{c}F_0(z)$, $\psi^{\pm}(z) = \mathfrak{c}K_0^{\pm}(q^c z)$, where $\mathfrak{c}$ is an extra central element. In the generators $e(z), f(z), \psi^{\pm}(z)$, the defining relations are completely symmetric in the parameters q_1, q_2, q_3. Hence $\mathcal{E}_1(q_{\pi(1)}, q_{\pi(2)}, q_{\pi(3)}) = \mathcal{E}_1(q_1, q_2, q_3)$ for any permutation π of $\{1, 2, 3\}$. In contrast, in the case $n \geq 2$ the $q_1 \leftrightarrow q_3$ symmetry holds true, the map is given by (2.9) below, but q_2 plays a distinguished role.

Remark 3. We have not been able to find the Serre relations for $\mathcal{E}_2$ in the literature, except [M01] where the special case $d = q$ is treated. Our quartic relations are similar to that of [M01].

For $\mathcal{E}_2$ we also have cubic relations inspired by the consideration of 'fused currents' which will be discussed in Section 3.1 and Theorem 3.1. These cubic relations are not discussed in [M01]. As we show these cubic relations are equivalent to the quartic Serre relations (2.1), (2.2) in the presence of quadratic relations.

Lemma 2.1. *In $\mathcal{E}_2$ we have the following cubic relations:*

$$\operatorname*{Sym}_{z_1,z_2}\Big[q_1(z_1-q_3w)(z_2-q_3w)E_i(z_1)E_i(z_2)E_j(w)-(1+q_2^{-1})(z_1-q_3w)(q_1z_2-w)E_i(z_1)E_j(w)E_i(z_2)$$

$$+q_3(q_1z_1-w)(q_1z_2-w)E_j(w)E_i(z_1)E_i(z_2)\Big]=0\,,$$

$$\operatorname*{Sym}_{z_1,z_2}\Big[q_3(q_1z_1-w)(q_1z_2-w)F_i(z_1)F_i(z_2)F_j(w)-(1+q_2^{-1})(q_1z_1-w)(z_2-q_3w)F_i(z_1)F_j(w)F_i(z_2)$$

$$+q_1(z_1-q_3w)(z_2-q_3w)F_j(w)F_i(z_1)F_i(z_2)\Big]=0\,,$$

and the relations obtained by interchanging q_1 with q_3.

Proof. Starting from the special case of the quartic relation

$$[E_{i,k},[E_{i,k},[E_{i,k},E_{j,l}]_{q^2}]]_{q^{-2}}=0\,,$$

we compute the commutator with $F_{i,\pm1-k}$. The result is

$$(1+q_2^{-1})[E_{i,k},[E_{i,k\pm1},E_{j,l}]_{q^{-2}}]=(q_1+q_3)[[E_{j,l\pm1},E_{i,k}]_{q^{-2}},E_{i,k}]\,.$$

Taking commutators with $H_{i,r}$ we obtain

$$\mathrm{Sym}_{k_1,k_2}\big((1+q_2^{-1})[E_{i,k_1},[E_{i,k_2\pm1},E_{j,l}]_{q^{-2}}]-(q_1+q_3)[[E_{j,l\pm1},E_{i,k_1}]_{q^{-2}},E_{i,k_2}]\big)=0\,,$$

or in current form

$$\mathrm{Sym}_{z_1,z_2}\big((1+q_2^{-1})z_2^{\pm1}[E_i(z_1),[E_i(z_2),E_j(w)]_{q^{-2}}]-(q_1+q_3)w^{\pm1}[[E_j(w),E_i(z_1)]_{q^{-2}},E_i(z_2)]\big)=0\,.$$

Modulo the quadratic relation $\mathrm{Sym}_{z,w}(z-q^2w)E_i(z)E_i(w)=0$, these equations are equivalent to the first identity in the lemma. □

In fact the quadratic relations

$$(z-q_1w)(z-q_3w)E_i(z)E_j(w)=(w-q_1z)(w-q_3z)E_j(w)E_i(z).$$

with $i\neq j$ also follow from the quartic relations.

On the other hand, the quartic Serre relations , are consequences of the quadratic and cubic relations, see the part of Section 3.3 concerning the Serre relations.

2.3. **Horizontal and vertical subalgebras.** In this subsection we describe subalgebras of $\mathcal{E}_n$ isomorphic to the quantum affine algebras $U_q(\widehat{\mathfrak{sl}}_n)$ and $U_q(\widehat{\mathfrak{gl}}_n)$ for $n\geq 2$.

The algebra $U_q(\widehat{\mathfrak{sl}}_n)$ has a presentation in terms of the Chevalley generators $\{e_i,f_i,t_i^{\pm1}\}$, $0\leq i\leq n-1$, as follows.

$$\begin{aligned}
&t_it_j=t_jt_i,\quad t_it_i^{-1}=t_i^{-1}t_i=1,\\
&t_ie_jt_i^{-1}=q^{a_{i,j}}e_j,\quad t_if_jt_i^{-1}=q^{-a_{i,j}}f_j,\\
&[e_i,f_j]=\delta_{i,j}\frac{t_i-t_i^{-1}}{q-q^{-1}},\\
&[e_i,e_j]=0,\quad [f_i,f_j]=0\quad (\text{if } a_{i,j}=0),\\
&[e_i,[e_i,e_j]_{q^{-1}}]_q=0,\quad [f_i,[f_i,f_j]_q]_{q^{-1}}=0\quad (\text{if } a_{i,j}=-1)\,.
\end{aligned}$$

When $n=2$, the last line is to be replaced by

$$[e_i,[e_i,[e_i,e_j]_{q^{-2}}]_1]_{q^2}=0, \quad [f_i,[f_i,[f_i,f_j]_{q^2}]_1]_{q^{-2}}=0 \quad (i\neq j).$$

Alternatively, $U_q(\widehat{\mathfrak{sl}}_n)$ has a presentation in terms of the Drinfeld generators $\{x^{\pm}_{i,l}, h_{i,r}, k_i^{\pm 1}, q^{\pm c}\}$, $1\le i\le n-1$, $l\in\mathbb{Z}$, $r\in\mathbb{Z}\backslash\{0\}$ with the relations

$$\begin{aligned}
&k_ik_i^{-1}=k_i^{-1}k_i=1, \quad q^cq^{-c}=q^{-c}q^c=1,\\
&q^{\pm c} \text{ are central}, \quad [k_i,k_j]=[k_i,h_{j,r}]=0\,,\\
&[h_{i,r},h_{j,s}]=\delta_{r+s,0}\frac{[ra_{i,j}]}{r}\frac{q^{rc}-q^{-rc}}{q-q^{-1}}\,,\\
&k_ix^{\pm}_{j,l}k_i^{-1}=q^{\pm a_{i,j}}x^{\pm}_{j,l}, \quad [h_{i,r},x^{\pm}_{j,l}]=\pm\frac{[ra_{i,j}]}{r}q^{(r\mp|r|)c/2}x^{\pm}_{j,l+r},\\
&[x^+_{i,k},x^-_{j,l}]=\frac{\delta_{i,j}}{q-q^{-1}}\left(q^{-lc}\phi^+_{i,k+l}-q^{-kc}\phi^-_{i,k+l}\right)\,,\\
&[x^{\pm}_{i,k+1},x^{\pm}_{i,l}]_{q^{\pm 2}}+[x^{\pm}_{i,l+1},x^{\pm}_{i,k}]_{q^{\pm 2}}=0,\\
&[x^{\pm}_{i,k},x^{\pm}_{j,l}]=0 \quad (\text{if } a_{i,j}=0),\\
&[x^{\pm}_{i,k+1},x^{\pm}_{j,l}]_{q^{\mp 1}}+[x^{\pm}_{j,l+1},x^{\pm}_{i,k}]_{q^{\mp 1}}=0 \quad (\text{if } a_{i,j}=-1),\\
&\mathrm{Sym}_{k_1,k_2}[x^{\pm}_{i,k_1},[x^{\pm}_{i,k_2},x^{\pm}_{j,l}]_{q^{-1}}]_q=0 \quad (\text{if } a_{i,j}=-1)\,.
\end{aligned}$$

In the above we have set

$$\sum_{\pm k\ge 0}\phi^{\pm}_{i,k}z^{-k}=k_i^{\pm 1}\exp\Bigl(\pm(q-q^{-1})\sum_{\pm r>0}h_{i,r}z^{-r}\Bigr)\,.$$

We choose the correspondence of these two generators as follows.

$$\begin{aligned}
&e_i=x^+_{i,0}, \quad f_i=x^-_{i,0}, \quad t_i=k_i \quad (1\le i\le n-1), \quad t_0t_1\cdots t_{n-1}=q^c,\\
&e_0=q^c(k_1\cdots k_{n-1})^{-1}[\cdots[x^-_{1,1},x^-_{2,0}]_q,\cdots,x^-_{n-1,0}]_q\,,\\
&f_0=[x^+_{n-1,0},\cdots[x^+_{2,0},x^+_{1,-1}]_{q^{-1}},\cdots]_{q^{-1}}k_1\cdots k_{n-1}q^{-c}\,.
\end{aligned}$$

In order to express the Drinfeld generators in terms of the Chevalley generators, it is useful to have the formulas:

$$\begin{aligned}
&x^-_{i,1}=(-1)^{i-1}(t_0\cdots t_{i-1}t_{i+1},\cdots t_{n-1})^{-1}[\cdots[e_0,e_{n-1}]_{q^{-1}}\cdots,e_{i+1}]_{q^{-1}},e_1]_{q^{-1}}\cdots,e_{i-1}]_{q^{-1}},\\
&h_{i,1}=(-1)^i[\cdots[e_0,e_{n-1}]_{q^{-1}}\cdots,e_{i+1}]_{q^{-1}},e_1]_{q^{-1}}\cdots,e_{i-1}]_{q^{-1}},e_i]_{q^{-2}},\\
&x^+_{i,-1}=(-1)^{i-1}[f_{i-1},\cdots[f_1,[f_{i+1},\cdots[f_{n-1},f_0]_q\cdots]_qt_0\cdots t_{i-1}t_{i+1},\cdots t_{n-1},\\
&h_{i,-1}=(-1)^i[f_i,[f_{i-1},\cdots[f_1,[f_{i+1},\cdots[f_{n-1},f_0]_q\cdots]_q]_{q^2}.
\end{aligned}$$

A characteristic feature of the algebra $\mathcal{E}_n$ is that for $n\ge 2$ it admits two different embeddings of $U_q(\widehat{\mathfrak{sl}}_n)$,

$$h,v:U_q(\widehat{\mathfrak{sl}}_n)\longrightarrow\mathcal{E}_n.$$

The embedding h is defined in terms of the Chevalley generators,

$$h:\ e_i\mapsto E_{i,0}, \quad f_i\mapsto F_{i,0}, \quad t_i\mapsto K_i \quad (0\le i\le n-1).$$

The embedding v is defined in terms of the Drinfeld generators,

$$v:\ x^+_{i,k}\mapsto d^{ik}E_{i,k},\quad x^-_{i,k}\mapsto d^{ik}F_{i,k},\quad k_i\mapsto K_i,\quad h_{i,r}\mapsto d^{ir}H_{i,r},\quad q^c\mapsto q^c$$
$$(1\le i\le n-1, k\in\mathbb{Z}, r\in\mathbb{Z}\backslash\{0\}).$$

We call h the *horizontal embedding*, and its image $h\big(U_q\widehat{\mathfrak{sl}}_n\big)$ the *horizontal subalgebra* of $\mathcal{E}_n$. Similarly we call v the *vertical embedding*, and its image $v\big(U_q\widehat{\mathfrak{sl}}_n\big)$ the *vertical subalgebra* of $\mathcal{E}_n$. We denote the horizontal and vertical subalgebras by $U_q^{hor}(\widehat{\mathfrak{sl}}_n)$ and $U_q^{ver}(\widehat{\mathfrak{sl}}_n)$ respectively. Note that if $x\in U_q^{hor}(\widehat{\mathfrak{gl}}_n)$ then $\deg x\in\mathbb{Z}^n\times\{0\}$, and if $x\in U_q^{ver}(\widehat{\mathfrak{gl}}_n)$ then $\deg x\in\{0\}\times\mathbb{Z}^{n-1}\times\mathbb{Z}$. Note also that $\mathcal{E}_n$ is generated by the union of $U_q^{hor}(\widehat{\mathfrak{sl}}_n)$ and $U_q^{ver}(\widehat{\mathfrak{sl}}_n)$.

As it is pointed out in [FJMM2], there are also Heisenberg subalgebras commuting with these subalgebras. For each $r\neq 0$, let $\{c_{i,r}\}_{i=0}^{n-1}$ be a non-trivial solution of the equation

$$\sum_{i=0}^{n-1} c_{i,r}[ra_{i,j}]d^{-rm_{i,j}}=0\quad (j=1,\dots,n-1).$$

Let $\mathfrak{a}^{ver}$ be the subalgebra of $\mathcal{E}_n$ generated by $H_r^{ver}=\sum_{i=0}^{n-1}c_{i,r}H_{i,r}$, $r\in\mathbb{Z}_{\neq0}$. Clearly $\mathfrak{a}^{ver}$ is a Heisenberg subalgebra with central element q^c which commutes with $U_q^{ver}\widehat{\mathfrak{sl}}_n$. We call the subalgebra generated by these two the *vertical quantum affine* $\mathfrak{gl}_n$ and denote it by $U_q^{ver}(\widehat{\mathfrak{gl}}_n)$.

Similarly there exists a Heisenberg subalgebra $\mathfrak{a}^{hor}$ which commutes with $U_q^{hor}\widehat{\mathfrak{sl}}_n$. In terms of the automorphism θ to be given in Theorem 2.2 below, we have $\mathfrak{a}^{hor}=\theta^{-1}\big(\mathfrak{a}^{ver}\big)$.

We call the subalgebra generated by these two the *horizontal quantum affine* $\mathfrak{gl}_n$ and denote it by $U_q^{hor}(\widehat{\mathfrak{gl}}_n)$. The central element $\kappa=h(q^c)$ belongs to the horizontal subalgebra, while $q^c=v(q^c)$ belongs to the vertical subalgebra.

2.4. **Automorphisms.** The algebra $\mathcal{E}_n(q_1,q_2,q_3)$ allows for various symmetries.

First, there exist automorphisms of algebras

$$\tau, s_a,\chi_j:\mathcal{E}_n(q_1,q_2,q_3)\to\mathcal{E}_n(q_1,q_2,q_3)\,,$$

where $a\in\mathbb{C}^\times$ and $0\le j\le n-1$, such that

$$\tau: E_i(z)\mapsto E_{i+1}(z),\quad F_i(z)\mapsto F_{i+1}(z),\quad K_i^\pm(z)\mapsto K_{i+1}^\pm(z), \tag{2.6}$$

$$s_a: E_i(z)\mapsto E_i(az),\quad F_i(z)\mapsto F_i(az),\quad K_i^\pm(z)\mapsto K_i^\pm(az)\,, \tag{2.7}$$

$$\chi_j: E_i(z)\mapsto E_i(z)z^{-\delta_{i,j}},\quad F_i(z)\mapsto F_i(z)z^{\delta_{i,j}},\quad K_i^\pm(z)\mapsto q^{\mp\delta_{i,j}c}K_i^\pm(z)\,, \tag{2.8}$$

and such that all these maps send q^c to itself.

We have $\tau^n=id$.

In addition, there exists an isomorphism of algebras

$$\iota:\mathcal{E}_n(q_1,q_2,q_3)\to\mathcal{E}_n(q_3,q_2,q_1),$$

given by

$$\iota: E_i(z)\mapsto E_{n-i}(z),\quad F_i(z)\mapsto F_{n-i}(z),\quad K_i^\pm(z)\mapsto K_{n-i}^\pm(z)\,, \tag{2.9}$$

and $\iota(q^c)=q^c$.

Of particular importance is the existence of an automorphism which exchanges the horizontal subalgebra $U_q^{hor}(\widehat{\mathfrak{gl}}_n)$ and the vertical subalgebra $U_q^{ver}(\widehat{\mathfrak{gl}}_n)$.

Let σ, η' be anti-automorphisms of $U_q(\widehat{\mathfrak{sl}}_n)$ given by

$$\sigma: \ e_i \mapsto e_i, \quad f_i \mapsto f_i, \quad t_i \mapsto t_i^{-1},$$
$$\eta': \ x^{\pm}_{i,k} \mapsto x^{\pm}_{i,-k}, \quad h_{i,r} \mapsto -q^{rc} h_{i,-r}, \quad k_i \mapsto k_i^{-1}, \quad q^c \mapsto q^c.$$

Theorem 2.2. [M99],[M01] *Let $n \geq 2$. There exists a unique automorphism θ of $\mathcal{E}_n$ such that*[2]

$$\theta \circ v = h, \quad \theta \circ h = v \circ \eta' \circ \sigma.$$

We have $\theta(q^c) = \kappa$ and $\theta(\kappa) = q^{-c}$.

Theorem 2.3. [BS],[M07] *There exists a unique automorphism θ of $\mathcal{E}_1$ such that* [3]

$$E_{0,0} \mapsto -q^c H_{0,-1}, \quad F_{0,0} \mapsto a^{-1} q^{-c} H_{0,1},$$
$$H_{0,1} \mapsto E_{0,0}, \quad H_{0,-1} \mapsto -a F_{0,0},$$
$$q^c \mapsto K_0, \quad K_0 \mapsto q^{-c},$$

where $a = q(1-q_1)(1-q_3)$.

Remark. Actually, in the case of $\mathcal{E}_2$, the existence of θ has been proved only in the case $q_1 = 1$ [M01]. It can be shown that with minor modifications the method of [M01] carries over to the general case.

We shall write

$$x^{\perp} = \theta^{-1}(x) \quad (x \in \mathcal{E}_n). \tag{2.10}$$

Then we have

$$E^{\perp}_{i,0} = E_{i,0}, \quad F^{\perp}_{i,0} = F_{i,0}, \quad K^{\perp}_i = K_i \quad (1 \leq i \leq n-1),$$
$$(q^c)^{\perp} = \kappa^{-1}, \quad \kappa^{\perp} = q^c, \tag{2.11}$$

and for $n \geq 2$

$$E^{\perp}_{0,0} = d\kappa^{-1} q^c K_0 [\cdots [F_{1,1}, F_{2,0}]_q, \cdots, F_{n-1,0}]_q,$$
$$F^{\perp}_{0,0} = d^{-1} \kappa q^{-c} [E_{n-1,0}, \cdots, [E_{2,0}, E_{1,-1}]_{q^{-1}}, \cdots]_{q^{-1}} K_0^{-1}.$$

We also have

$$H^{\perp}_{i,1} = -(-d)^{-i} \kappa^{-1} [\cdots [F_{0,0}, F_{n-1,0}]_q \cdots, F_{i+1,0}]_q, F_{1,0}]_q \cdots F_{i-1,0}]_q, F_{i,0}]_{q^2},$$
$$H^{\perp}_{i,-1} = -(-d)^i \kappa [E_{i,0}, [E_{i-1,0}, \cdots [E_{1,0}, [E_{i+1,0}, \cdots [E_{n-1,0}, E_{0,0}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-2}},$$

for $i = 1, \dots, n-1$,

[2]Our θ here is ψ of [M99].

[3]Our θ is ψ of [M07] followed by the automorphism $E_0(z) \mapsto q^{-c} E_0(z)$, $F_0(z) \mapsto q^c F_0(z)$, $K_0^{\pm}(z) \mapsto K_0^{\pm}(z)$, $q^c \mapsto q^c$. Unlike ψ, $\theta^4 \neq \mathrm{id}$.

and for $n \geq 2$ we have

$$\begin{aligned}
&F^{\perp}_{0,1} = (-d)^{-n} F_{0,-1}, \\
&H^{\perp}_{0,1} = -(-d)^{-n+1}\kappa^{-1}[\cdots[F_{1,1}, F_{2,0}]_q \cdots F_{n-1,0}]_q, F_{0,-1}]_{q^2}, \\
&E^{\perp}_{0,-1} = (-d)^{n} E_{0,1}, \\
&H^{\perp}_{0,-1} = -(-d)^{n-1}\kappa[E_{0,1}, [E_{n-1,0} \cdots [E_{2,0}, E_{1,-1}]_{q^{-1}} \cdots]_{q^{-1}}]_{q^{-2}}.
\end{aligned}$$

The following Lemma can be extracted from [M00]:

Lemma 2.4. *If $x \in \mathcal{E}_n$ has degree $(l, d_1 + l, \ldots, d_{n-1} + l, k)$ then $x^{\perp} = \theta^{-1}(x)$ has degree $(-k, d_1 - k, \ldots, d_{n-1} - k, l)$.*

In particular, the principal degrees of the 'perpendicular generators' are given by

$$\operatorname{pdeg} E^{\perp}_{i,k} = -nk - n\delta_{i,0} + 1\,, \quad \operatorname{pdeg} F^{\perp}_{i,k} = -nk + n\delta_{i,0} - 1\,, \quad \operatorname{pdeg} H^{\perp}_{i,k} = -nk\,. \tag{2.12}$$

Later on we shall use the formulas

(2.13)

$$\begin{aligned}
&\theta(H_{i,1}) = (-d)^{-i}[[\cdots[[\cdots[E_{0,0}, E_{n-1,0}]_{q^{-1}}, \cdots, E_{i+1,0}]_{q^{-1}}, E_{1,0}]_{q^{-1}}, \cdots, E_{i-1,0}]_{q^{-1}}, E_{i,0}]_{q^{-2}}\,, \\
&\theta(H_{i,-1}) = (-d)^{i}[F_{i,0}, [F_{i-1,0}, \cdots[F_{1,0}, [F_{i+1,0}, \cdots[F_{n-1,0}, F_{0,0}]_q \cdots]_q]_q \cdots]_q]_{q^2}\,,
\end{aligned}$$

where $1 \leq i \leq n-1$;

$$\begin{aligned}
\theta(H_{0,1}) &= (-d)^{-n+1}[[\cdots[E_{1,1}, E_{2,0}]_{q^{-1}}, \cdots, E_{n-1,0}]_{q^{-1}}, E_{0,-1}]_{q^{-2}}\,, \\
\theta(H_{0,-1}) &= (-d)^{n-1}[F_{0,1}, [F_{n-1,0}, \cdots, [F_{2,0}, F_{1,-1}]_q \cdots]_q]_{q^2}
\end{aligned} \tag{2.14}$$

We recall that for $n = 1$

$$\theta(H_{0,1}) = E_{0,0}, \quad \theta(H_{0,-1}) = -aF_{0,0}\,.$$

Let s_i, $i = 0, \ldots, n-1$, denote the Lusztig braid group automorphism of $U_q(\widehat{\mathfrak{sl}}_n)$,

$$\begin{aligned}
&s_i(e_i) = -f_i t_i, \quad s_i(f_i) = -t_i^{-1} e_i, \\
&s_i(e_j) = \begin{cases} [e_i, e_j]_{q^{-1}} & \text{if } n \geq 3; \\ \frac{1}{[2]}[e_i, [e_i, e_j]_{q^{-2}}] & \text{if } n = 2, \end{cases} \quad s_i(f_j) = \begin{cases} [f_j, f_i]_q, & \text{if } n \geq 3; \\ \frac{1}{[2]}[[f_j, f_i]_{q^2}, f_i] & \text{if } n = 2, \end{cases} \quad (j \equiv i \pm 1), \\
&s_i(e_j) = e_j, \quad s_i(f_j) = f_j \quad (j \not\equiv i, i \pm 1), \\
&s_i(t_j) = t_i^{-a_{i,j}} t_j\,.
\end{aligned}$$

Consider the automorphisms

$$T_{n-1|0} = \theta^{-1} \circ \chi_0 \chi_{n-1}^{-1} \circ \theta\,, \tag{2.15}$$

$$T = T^{n}_{n-1|0}\,. \tag{2.16}$$

Since each χ_j (see (2.8)) preserves the vertical subalgebra, $T_{n-1|0}$, T preserve the horizontal subalgebra. Note also that $T_{n-1|0}$, T restricted to $\mathfrak{a}^{hor}$ are identity operators.

We shall need the following result.

Lemma 2.5. *We have*

$$T^{-1} \circ h = h \circ (s_{n-1} \cdots s_1 s_0)^{n-1}.$$

Proof. Set $\mathcal{Y}_n = \zeta_0\chi_0\chi_{n-1}^{-1}$, where ζ_0 is the automorphism of $\mathcal{E}_n$ given by $E_0(z) \mapsto (-d)^{-n}E_0(z)$, $F_0(z) \mapsto (-d)^n F_0(z)$, leaving unchanged the rest of the generators. The lemma follows from Proposition 2 of [M99] by choosing $x = \varphi^{-1}(\mathcal{Y}_n^{-n})$, and noting that $\zeta_0 \circ v = v$. $\square$

The exists an action of the braid group on any integrable $U_q(\widehat{\mathfrak{sl}}_n)$ module. Therefore, there exists an action of T on any integrable $U_q^{hor}(\widehat{\mathfrak{gl}}_n)$ module.

2.5. **Representations.** In this subsection we present a family of $\mathcal{E}_n$-modules studied in our previous works [FJMM1],[FJMM2]. These are

- the vector representation $V^{(k)}(u)$,
- the Fock representation $\mathcal{F}^{(k)}(u)$,
- the representation $\mathcal{N}^{(k)}_{\alpha,\beta}(u)$.

In all cases the central element q^c acts as identity. These modules carry a discrete parameter $k \in \mathbb{Z}/n\mathbb{Z}$ which we call *color*, and a continuous parameter $u \in \mathbb{C}^\times$ which we call the *evaluation parameter*. In fact the general case can be obtained as a twist of the one for $k = 0$ and $u = 1$ by the automorphisms τ and s_a, given by (2.6) and (2.7) respectively.

First, we recall some terminology about representations.

An $\mathcal{E}_n$-module V is said to have *level* ℓ if the central element κ^{-1} acts as the scalar ℓ.

Let $\boldsymbol{\phi}(z) = (\phi_i^+(z), \phi_i^-(z))_{i\in\mathbb{Z}/n\mathbb{Z}}$ be a collection of formal series $\phi_i^\pm(z) \in \mathbb{C}[[z^{\mp1}]]$. A vector $v \in V$ is said to have *weight* $\boldsymbol{\phi}(z)$ if $K_i^\pm(z)v = \phi_i^\pm(z)v$ holds for all $i \in \mathbb{Z}/n\mathbb{Z}$. The module V is *weighted* if the action of the commuting family of operators $\{K_i^\pm(z)\}_{i\in\mathbb{Z}/n\mathbb{Z}}$ is diagonalizable in V. It is said to be *tame* if the joint spectrum of this action is simple.

The module V is *lowest weight* if it is generated by a weight vector v such that $F_i(z)v = 0$ for all $i \in \mathbb{Z}/n\mathbb{Z}$. Such a v is called a *lowest weight vector*, and its weight the lowest weight of V. Given $\boldsymbol{\phi}(z) = (\phi_i^+(z), \phi_i^-(z))_{i\in\mathbb{Z}/n\mathbb{Z}}$ with $\phi_i^+(\infty)\phi_i^-(0) = 1$, there exists a unique irreducible lowest weight module $L_{\boldsymbol{\phi}(z)}$ with lowest weight $\boldsymbol{\phi}(z)$.

Let $V = \oplus_{s\in\mathbb{Z}} V_s$ be a module $\mathbb{Z}$-graded by the principal degree. (This is the case for all modules we consider in this paper.) We say that V is *quasi-finite* if $\dim V_s < \infty$ for all s. It is known [M07], [FJMM2], that an irreducible lowest weight module $L_{\boldsymbol{\phi}(z)}$ is quasi-finite if and only if, for each i, $\phi_i^\pm(z)$ are expansions of a rational function $\phi_i(z)$, such that it is regular at $z = 0, \infty$ and $\phi_i(0)\phi_i(\infty) = 1$. If it is the case we say simply that the lowest weight is $\boldsymbol{\phi}(z) = \big(\phi_i(z)\big)_{i\in\mathbb{Z}/n\mathbb{Z}}$.

Vector representation. The vector representation $V^{(k)}(u)$ has a basis $\{[u]_j^{(k)}\}_{j\in\mathbb{Z}}$. For $n \geq 2$, the action of the generators is explicitly given as follows.

$$E_i(z)[u]_j^{(k)} = \begin{cases} \delta(q_1^{j+1}u/z)[u]_{j+1}^{(k)}\,, & i+j+1 \equiv k; \\ 0\,, & i+j+1 \not\equiv k; \end{cases}$$

$$F_i(z)[u]_{j+1}^{(k)} = \begin{cases} \delta(q_1^{j+1}u/z)[u]_j^{(k)}\,, & i+j+1 \equiv k; \\ 0\,, & i+j+1 \not\equiv k; \end{cases}$$

$$K_i^\pm(z)[u]_j^{(k)} = \begin{cases} \psi(q_1^j u/z)[u]_j^{(k)}\,, & j+i \equiv k; \\ \psi(q_1^j q_3^{-1} u/z)^{-1}[u]_j^{(k)}\,, & j+i+1 \equiv k; \\ [u]_j^{(k)}, & \text{otherwise.} \end{cases}$$

Here and after we set

$$\psi(z)=\frac{q-q^{-1}z}{1-z}\,.$$

For $n=1$ the formulas read

$$\begin{aligned}
E_0(z)[u]^{(0)}_j &= \delta(q_1^{j+1}u/z)[u]^{(0)}_{j+1}\,,\\
F_0(z)[u]^{(0)}_{j+1} &= q\frac{1-q_3}{1-q_1^{-1}}\delta(q_1^{j+1}u/z)[u]^{(0)}_j\,,\\
K^{\pm}_0(z)[u]^{(0)}_j &= \psi(q_1^ju/z)\psi(q_1^jq_3^{-1}u/z)^{-1}[u]^{(0)}_j\,.
\end{aligned}$$

The vector representation $V^{(k)}(u)$ is an irreducible, tame representation of level 1.

Fock representation. We use the following notation concerning partitions. A partition $\lambda=(\lambda_1,\lambda_2,\cdots)$ is a sequence of non-negative integers λ_i such that only finitely many are nonzero and $\lambda_j\ge\lambda_{j+1}$ for all j. In particular, we denote $\emptyset=(0,0,\dots)$. The dual partition λ' is given by $\lambda'_i=|\{j\mid\lambda_j\ge i\}|$. We identify a partition λ with the set of integer points (x,y) on the plane satisfying $1\le x\le\ell(\lambda)$ and $1\le y\le\lambda_x$, where $\ell(\lambda)=\lambda'_1$ is the length of λ. A pair of natural numbers (x,y) is a *convex corner of* λ if $\lambda'_{y+1}<\lambda'_y=x$. A pair of natural numbers (x,y) is a *concave corner of* λ if $\lambda'_y=x-1$ and in addition $y=1$ or $\lambda'_{y-1}>x-1$. Let $CC(\lambda)$ and $CV(\lambda)$ be the set of concave and convex corners of λ respectively.

Fixing $k\in\mathbb{Z}/n\mathbb{Z}$, to each point $(x,y)\in\mathbb{Z}^2$ we assign a color $k+x-y\in\mathbb{Z}/n\mathbb{Z}$. For $i\in\mathbb{Z}/n\mathbb{Z}$, introduce the set of concave (resp. convex) corners of λ of color i as follows.

$$\begin{aligned}
CC^{(k)}_i(\lambda)&=\{(x,y)\in CC(\lambda)\mid k+x-y\equiv i\}\,,\\
CV^{(k)}_i(\lambda)&=\{(x,y)\in CV(\lambda)\mid k+x-y\equiv i\}\,.
\end{aligned}$$

Finally, for a partition $\lambda=(\lambda_1,\lambda_2,\dots)$ and $j\in\mathbb{Z}_{\ge1}$ we write $\lambda\pm\mathbf{1}_j=(\lambda_1,\lambda_2,\dots,\lambda_j\pm1,\dots)$.

The Fock representation $\mathcal{F}^{(k)}(u)$ has a basis $\{|\lambda\rangle\}$ indexed by all partitions λ. It is realized as a linear subspace of the infinite tensor product of vector representations

$$\mathcal{F}^{(k)}(u)\subset V^{(k)}(u)\otimes V^{(k)}(uq_2^{-1})\otimes V^{(k)}(uq_2^{-2})\otimes\dots,$$

where

$$|\lambda\rangle=[u]^{(k)}_{\lambda_1-1}\otimes[uq_2^{-1}]^{(k)}_{\lambda_2-2}\otimes[uq_2^{-2}]^{(k)}_{\lambda_3-3}\otimes\dots\,.$$

Notation being as above, the action of $\mathcal{E}_n$ is given as follows.

For $i\in\mathbb{Z}/n\mathbb{Z}$, $j\in\mathbb{Z}_{\ge1}$ such that $k+j-\lambda_j\equiv i+1$, set

$$\langle\lambda+\mathbf{1}_j|E_i(z)|\lambda\rangle=\prod_{\substack{s=1,\\ k+s-\lambda_s\equiv i}}^{j-1}\psi(q_1^{\lambda_s-\lambda_j-1}q_3^{s-j})\prod_{\substack{s=1,\\ k+s-\lambda_s\equiv i+1}}^{j-1}\psi(q_1^{\lambda_j-\lambda_s}q_3^{j-s})\;\delta(q_1^{\lambda_j}q_3^{j-1}u/z),$$

$$\langle\lambda|F_i(z)|\lambda+\mathbf{1}_j\rangle=\prod_{\substack{s=j+1,\\ k+s-\lambda_s\equiv i}}^{\ell(\lambda)}\psi(q_1^{\lambda_s-\lambda_j-1}q_3^{s-j})\prod_{\substack{s=j+1,\\ k+s-\lambda_s\equiv i+1}}^{\ell(\lambda)+1}\psi(q_1^{\lambda_j-\lambda_s}q_3^{j-s})\;\delta(q_1^{\lambda_j}q_3^{j-1}u/z).$$

Further, for $i \in \mathbb{Z}/n\mathbb{Z}$, set

$$\langle\lambda|K_i^{\pm}(z)|\lambda\rangle = \prod_{(x,y)\in\ CV_i^{(k)}(\lambda)} \psi(q_3^x q_1^y q_2 u/z) \prod_{(x,y)\in\ CC_i^{(k)}(\lambda)} \psi(q_3^x q_1^y q_2^2 u/z)^{-1}.$$

We set all other matrix coefficients to be zero. In particular, we see that $E_i(z)$ adds, and $F_i(z)$ removes, a box of color i.

Here we used the bra-ket notation for the matrix elements of the linear operators acting in $\mathcal{F}^{(k)}(u)$ in the basis $\{|\lambda\rangle\}$.

The Fock representation $\mathcal{F}^{(k)}(u)$ is an irreducible, tame, lowest weight representation of level q with lowest weight $(\phi_i(z))$ where

$$\phi_i(z) = \begin{cases} \dfrac{q^{-1} - qu/z}{1-u/z} & (i \equiv k) \\ 1 & (i \not\equiv k) \end{cases}.$$

We remark that the Fock representation was given in [Sa] using vertex operators (for the perpendicular generators), and in [VV2],[STU] using the q-wedge spaces. The explicit formula for the action of $\mathcal{E}_1$ in the Fock space was found in [FT].

Representation $\mathcal{N}^{(p)}_{\alpha,\beta}(u)$. The representation $\mathcal{N}^{(p)}_{\alpha,\beta}(u)$ is defined as a submodule of a finite tensor product of Fock representations. Let α, β be partitions with m parts, such that $\alpha_m = \beta_m = 0$. Given a color $p \in \mathbb{Z}/n\mathbb{Z}$ and an evaluation parameter $u \in \mathbb{C}^\times$, set

$$p_i = p - \alpha_i + \beta_i\,, \quad u_i = q_1^{\alpha_i} q_2^{i-1} q_3^{\beta_i} u\,, \quad i = 1, \cdots, m\,. \tag{2.17}$$

Consider the linear subspace

$$\mathcal{N}^{(p)}_{\alpha,\beta}(u) \subset \mathcal{F}^{(p_1)}(u_1) \otimes \cdots \otimes \mathcal{F}^{(p_m)}(u_m) \tag{2.18}$$

spanned by vectors $|\lambda^{(1)}\rangle \otimes \cdots \otimes |\lambda^{(m)}\rangle$, where $\lambda^{(i)}$ are partitions satisfying the conditions

$$\lambda^{(i)}_j \geq \lambda^{(i+1)}_{j+b_i} - a_i, \quad i = 1, \cdots, m-1, \tag{2.19}$$

where

$$a_i = \alpha_i - \alpha_{i+1}, \qquad b_i = \beta_i - \beta_{i+1}. \tag{2.20}$$

Then $\mathcal{N}^{(p)}_{\alpha,\beta}(u)$ is a well-defined $\mathcal{E}_n$-submodule of the tensor product module $\mathcal{F}^{(p_1)}(u_1) \otimes \cdots \otimes \mathcal{F}^{(p_m)}(u_m)$. Moreover it is an irreducible, tame, quasi-finite lowest weight module of level q^m and lowest weight $(\phi_i(z))$, where

$$\phi_i(z) = \prod_{j: p_j \equiv i} \frac{q^{-1} - qu_j/z}{1 - u_j/z}\,.$$

3. Construction of subalgebras

In this section we describe a family of subalgebras of a completion of $\mathcal{E}_n$. These subalgebras satisfy the relations of $\mathcal{E}_m$ with $m < n$ and act in all lowest weight representations of $\mathcal{E}_n$. In this section, except in Section 3.5, we always work in perpendicular generators, see (2.10) and (2.11). We use similar notation for the generating series, e.g. $E_i^{\perp}(z)$, $F_i^{\perp}(z)$, etc.

3.1. **Definition of current $E^{\perp}_{n-1|0}(z)$.** We give the definition of the "fused" current $E^{\perp}_{n-1|0}(z)$. The construction mimics the extraction of the polar term in the operator product of $E^{\perp}_0(z)$ and of $E^{\perp}_{n-1}(z)$.

Let first $n \geq 3$. We have the relation

$$(d^{-1}z - q^{-1}w)E^{\perp}_{n-1}(z)E^{\perp}_0(w) = (d^{-1}q^{-1}z - w)E^{\perp}_0(w)E^{\perp}_{n-1}(z),$$

which in components is

$$E^{\perp}_{n-1,k+1}E^{\perp}_{0,r} - q_1 E^{\perp}_{n-1,k}E^{\perp}_{0,r+1} = q^{-1}E^{\perp}_{0,r}E^{\perp}_{n-1,k+1} - dE^{\perp}_{0,r+1}E^{\perp}_{n-1,k}. \tag{3.1}$$

We start with the representation-theoretical version.

We define another grading $\deg^{\perp}$ on $\mathcal{E}_n$ such that:

$$\deg^{\perp} E^{\perp}_{i,k} = \deg^{\perp} F^{\perp}_{i,k} = \deg^{\perp} H^{\perp}_{i,k} = k. \tag{3.2}$$

From Lemma 2.4, it follows that $-\deg^{\perp} x$ is equal to the 0-th component of $\deg x$.

Call a graded $\mathcal{E}_n$ module V *admissible* if for every vector $v \in V$ there exists $N(v)$ such that $gv = 0$ for all $g \in \mathcal{E}_n$ with $\deg^{\perp} g > N(v)$. If a representation is admissible then the formal series $E^{\perp}_i(z)v$, $F^{\perp}_i(z)v$, $K^{-,\perp}_i(z)v$ are actually Laurent series in z and the series $K^{+,\perp}_i(z)v$ is a polynomial in z^{-1}. Note that in terms of the standard generators, the condition of admissibility is written in terms of the principal grading, (see (2.4), (2.12)). In particular, all lowest weight modules defined in Section 2.5 are admissible.

Let V be an admissible representation. Then from (3.1), we obtain that given $k \in \mathbb{Z}$ and $v \in V$, we have

$$E^{\perp}_{n-1,s+k}E^{\perp}_{0,-s}v = q_1^{-1}E^{\perp}_{n-1,s+1+k}E^{\perp}_{0,-s-1}v$$

for large enough s.

Define

$$E^{\perp}_{n-1|0,k}v = q_1^{-s-k}E^{\perp}_{n-1,s+k}E^{\perp}_{0,-s}v, \tag{3.3}$$

where s is sufficiently large. Clearly, $E^{\perp}_{n-1|0,k}$ is a well-defined operator acting in V. We set $E^{\perp}_{n-1|0}(z) = \sum_{k\in\mathbb{Z}} E^{\perp}_{n-1|0,k}z^{-k}$.

Equivalently, we can define:

$$E^{\perp}_{n-1|0}(z) = \lim_{z'\to z}(1 - \frac{z}{z'})E^{\perp}_{n-1}(q_1 z')E^{\perp}_0(z). \tag{3.4}$$

Using (3.1) repeatedly, we can also write

$$q_1^{-s-k}E^{\perp}_{n-1,s+k}E^{\perp}_{0,-s} = q_1^{-k}E^{\perp}_{n-1,k}E^{\perp}_{0,0} + \sum_{i=0}^{s-1} q_1^{-1-i-k}(q^{-1}E^{\perp}_{0,-1-i}E^{\perp}_{n-1,k+1+i} - dE^{\perp}_{0,-i}E^{\perp}_{n-1,k+i}).$$

Therefore we can equivalently define:

$$E^{\perp}_{n-1|0,k} = q_1^{-k}E^{\perp}_{n-1,k}E^{\perp}_{0,0} + \sum_{i=0}^{\infty} q_1^{-1-i-k}(q^{-1}E^{\perp}_{0,-1-i}E^{\perp}_{n-1,k+1+i} - dE^{\perp}_{0,-i}E^{\perp}_{n-1,k+i}).$$

Note that the sum evaluated on any vector v in an admissible representation becomes finite. This formula shows that the operator $E^{\perp}_{n-1|0,k}$ belongs to the completion of $\mathcal{E}_n$ with respect to grading (3.2).

There is one more useful way to write the operators $E^{\perp}_{n-1|0,k}$, which we use in Section 4. Let $T_{n-1|0}$ be the automorphism of $\mathcal{E}_n$ given by (2.15).

We have

$$T_{n-1|0}E^{\perp}_0(z)=z^{-1}E^{\perp}_0(z),\quad T_{n-1|0}E^{\perp}_{n-1}(z)=zE^{\perp}_{n-1}(z),$$
$$T_{n-1|0}F^{\perp}_0(z)=zF^{\perp}_0(z),\quad T_{n-1|0}F^{\perp}_{n-1}(z)=z^{-1}F^{\perp}_{n-1}(z),$$

and $T_{n-1|0}$ preserves currents with indexes different from 0 and $n-1$ as well as q^c and κ. In particular,

$$T_{n-1|0}K^{\pm,\perp}_i(z)=\kappa^{\mp\delta_{i,n-1}\pm\delta_{i,0}}K^{\pm,\perp}_i(z).$$

Then we clearly have

$$E^{\perp}_{n-1|0,k}=\lim_{s\to\infty}q_1^{-s-k}T^s_{n-1|0}(E^{\perp}_{n-1,k}E^{\perp}_{0,0}).$$

Finally, let us consider the case $n=2$. In this case, we replace the product (3.3) by the following stable combination. We set

$$E^{(1),\perp}_{1|0,k}=q_1^{-s-k}(E^{\perp}_{1,s+k}E^{\perp}_{0,-s}-q_3E^{\perp}_{1,s+k-1}E^{\perp}_{0,-s+1}),$$

where s is sufficiently large. Equivalently we have

$$E^{(1),\perp}_{1|0}(z)=\lim_{z'\to z}(1-\frac{z}{z'})(1-\frac{q_3z}{q_1z'})E^{\perp}_1(q_1z')E^{\perp}_0(z)$$

and

$$E^{(1),\perp}_{1|0,k}=\lim_{s\to\infty}q_1^{-s-k}T^s_{1|0}(E^{\perp}_{1,k}E^{\perp}_{0,0}-q_3E^{\perp}_{1,k-1}E^{\perp}_{0,1}).$$

For $n=2$ we write an extra upper index for the reason explained in Section 3.2, see (3.11), (3.12) below.

3.2. **Other operators.** We collect operators obtained by the construction described in Section 3.1.

Similarly to Section 3.1, we define a number of other currents. We use formulas of type (3.4) keeping in mind that it is always justified by formulas of type (3.3).

For $n\geq 3$ and $i=0,1,\dots,n-1$, define

$$E^{\perp}_{i|i+1}(z)=\lim_{z'\to z}(1-\frac{z}{z'})E^{\perp}_i(q_1z')E^{\perp}_{i+1}(z),\tag{3.5}$$

$$F^{\perp}_{i|i+1}(z)=\lim_{z'\to z}(1-\frac{z'}{z})F^{\perp}_{i+1}(z)F^{\perp}_i(q_1z'),\tag{3.6}$$

$$K^{\pm,\perp}_{i|i+1}(z)=K^{\pm,\perp}_i(q_1z)K^{\pm,\perp}_{i+1}(z).\tag{3.7}$$

We also have another family of operators defined by

$$E^{\perp}_{i+1|i}(z)=\lim_{z'\to z}(1-\frac{z}{z'})E^{\perp}_{i+1}(q_3z')E^{\perp}_i(z),\tag{3.8}$$

$$F^{\perp}_{i+1|i}(z)=\lim_{z'\to z}(1-\frac{z'}{z})F^{\perp}_i(z)F^{\perp}_{i+1}(q_3z'),\tag{3.9}$$

$$K^{\pm,\perp}_{i+1|i}(z)=K^{\pm,\perp}_{i+1}(q_3z)K^{\pm,\perp}_i(z).\tag{3.10}$$

All such currents of the same type (e.g. of type E) are related to each other by $\mathcal{E}_n$ automorphisms τ and ι, see (2.6) and (2.9), for example, $E^{\perp}_{i|i+1}(z) = \theta^{-1}\circ\tau^{i+1}\circ\theta\big(E^{\perp}_{n-1|0}(z)\big)$ and $E^{\perp}_{i+1|i}(z) = \theta^{-1}\circ\iota\circ\theta\big(E^{\perp}_{n-i-1|n-i}(z)\big)$.

Moreover, we use our construction recursively to obtain new currents. For example, we set

$$E^{\perp}_{i|i+1|i+2}(z) \;=\; \lim_{z''\to z'}\lim_{z'\to z}(1-\frac{z'}{z''})(1-\frac{z}{z'})E^{\perp}_i(q_1^2z'')E^{\perp}_{i+1}(q_1z')E^{\perp}_{i+2}(z),$$

or

$$E^{\perp}_{i|i+1|i}(z) \;=\; \lim_{z''\to z'}\lim_{z'\to z}(1-\frac{z'}{z''})(1-\frac{z}{z'})E^{\perp}_i(q_1q_3z'')E^{\perp}_{i+1}(q_3z')E^{\perp}_i(z).$$

One can justify this recursive definition directly, but we defer our discussion to Section 3.4.

Note that our notation $E^{\perp}_{i|i+1|i}(z)$ contains complete information about the shifts of arguments participating in the corresponding definition. Namely, $i|i+1$ signifies the relative shift of q_3 while $i+1|i$ signifies the relative shift of q_1.

For $n=2$, the construction of these currents is quite parallel, but we write an additional upper index to distinguish the formulas in (3.5)–(3.7) and those in (3.8)–(3.10), e.g.,

$$E^{(1),\perp}_{1|0}(z) = \lim_{z'\to z}(1-\frac{z}{z'})(1-\frac{q_3z}{q_1z'})E^{\perp}_1(q_1z')E^{\perp}_0(z), \tag{3.11}$$

$$E^{(3),\perp}_{1|0}(z) = \lim_{z'\to z}(1-\frac{z}{z'})(1-\frac{q_1z}{q_3z'})E^{\perp}_1(q_3z')E^{\perp}_0(z). \tag{3.12}$$

In what follows we also use the notation

$$E^{\perp}_{n-1||0}(z) = E^{\perp}_{n-1|n-2|\dots|1|0}(z),\;\; F^{\perp}_{n-1||0}(z) = F^{\perp}_{n-1|n-2|\dots|1|0}(z),\;\; K^{\pm,\perp}_{n-1||0}(z) = K^{\pm,\perp}_{n-1|n-2|\dots|1|0}(z). \tag{3.13}$$

3.3. **Correlation functions.** We discuss properties of correlation functions in admissible representations.

Let V be an admissible representation. Choose an arbitrary graded basis. Choose arbitrary basis vectors v_1, v_2. We use the standard notation for the correlation functions. For example, we write $\langle E^{\perp}_1(z)E^{\perp}_2(w)\rangle$ for the matrix coefficient $\langle v_1|E^{\perp}_1(z)E^{\perp}_2(w)|v_2\rangle$ of the operator $E^{\perp}_1(z)E^{\perp}_2(w)$. We study properties common to all correlation functions, and therefore it is not important which admissible representation and which particular matrix element we consider, thus we omit this information from our notation. In all calculations V, v_1, v_2 are arbitrary but fixed.

By the word "current" we mean either $E^{\perp}_i(z)$, $F^{\perp}_i(z)$, $K^{-,\perp}_i(z)$ or $K^{+,\perp}_i(z)$. Later we will also use the fused currents.

Algebraic relations between currents translate into properties of correlation functions.

Moreover, if an element g of $\mathcal{E}_n$ acts by zero in all admissible representations, then $g=0$. Indeed, such a statement is known to be true in the setting of Lie algebras, so it holds for $U\mathcal{L}'_n(d)$, see Section 3.7. (The proof is analogous to ii) of Theorem 8.4.4 in [D].) But since $\mathcal{E}_n$ is a quantization of $U\mathcal{L}'_n(d)$ and all admissible representations of $U\mathcal{L}'_n(d)$ quantize to representations of $\mathcal{E}_n$, this fact is true for $\mathcal{E}_n$. Finally, note that admissible representations are graded and if an element $g=\sum_{i=1}^{\infty} g_i$ with $\deg^{\perp} g_i = i$ of the completion of $\mathcal{E}_n$ acts by zero in an admissible representation, then all g_i do. Therefore, we have the converse statement: if all correlation functions satisfy a given property then the currents satisfy an algebraic relation inside $\mathcal{E}_n$.

We now discuss the dictionary between algebraic relations and properties of correlation functions. In the dictionary we consider correlation functions of two or three currents. The correlation functions of many currents satisfy the same properties for each subset of two or three currents.

Quadratic relations. Consider two currents satisfying a quadratic relation. For example, consider $\langle E_1^\perp(z)E_2^\perp(w)\rangle$, $n\geq 3$. Then the quadratic relation for the currents $E_1^\perp(z)$ and $E_2^\perp(w)$ is

$$(d^{-1}z-q^{-1}w)E_1^\perp(z)E_2^\perp(w)=(d^{-1}q^{-1}z-w)E_2^\perp(w)E_1^\perp(z).$$

This is equivalent to the following form of the correlation functions:

$$\langle E_1^\perp(z)E_2^\perp(w)\rangle=\frac{p(z,w)}{d^{-1}z-q^{-1}w},\qquad \langle E_2^\perp(w)E_1^\perp(z)\rangle=\frac{p(z,w)}{d^{-1}q^{-1}z-w},\tag{3.14}$$

where $p(z,w)$ is a Laurent polynomial.

Here the right hand side of the first equation is understood as an expansion in w/z, while the right hand side of the second equation as an expansion in z/w. Such a convention should be clear and we often do not mention it.

Apart from the poles in the correlation functions which are dictated by the quadratic relations we also have symmetries, when several of the currents are the same. For example, we have

$$\langle E_1^\perp(z_1)E_1^\perp(z_2)\rangle=\frac{p(z_1,z_2)}{z_1-q_2z_2},\tag{3.15}$$

where $p(z_1,z_2)$ is a Laurent polynomial (different from the one in (3.14)) such that $p(z_1,z_2)=-p(z_2,z_1)$. In particular, we have $p(z,z)=0$.

Commuting currents. This is an important special case of the quadratic relations. If two currents commute, their correlation function is a Laurent polynomial (no poles). Of course, the converse is not true in general. For example, since the currents $K^{\pm,\perp}(w)$ are power series in $w^{\mp 1}$, clearly, the correlation functions $\langle E_i^\perp(z)K_i^{+,\perp}(w)\rangle$ and $\langle K_i^{-,\perp}(w)E_i^\perp(z)\rangle$ do not have poles but $\langle E_i^\perp(z)K_i^{-,\perp}(w)\rangle$ and $\langle K_i^{+,\perp}(w)E_i^\perp(z)\rangle$ do. In particular, these currents do not commute.

However, in order to prove that two currents commute, often it is sufficient to check the absence of poles, since we have that the correlation functions in different orders of currents are equal as rational functions, see Section 3.6.

Serre relations. The Serre relations (see Section 2.1) are equivalent to the *wheel conditions* for the correlation functions. For example,

$$\langle E_1^\perp(z_1)E_1^\perp(z_2)E_2^\perp(w)\rangle=\frac{p(z_1,z_2,w)}{(z_1-q_2z_2)(d^{-1}z_1-q^{-1}w)(d^{-1}z_2-q^{-1}w)},$$

where $p(z_1,z_2,w)$ is a Laurent polynomial skew-symmetric in z_1,z_2 satisfying the following wheel condition:

$$p(z,q_2z,q_1q_2z)=p(z,q_2z,q_2q_3z)=0.$$

Note that due to the skew-symmetry, we also have $p(z,q_2^{-1}z,q_1z)=p(z,q_2^{-1}z,q_3z)=0$.

This fact is not completely trivial, therefore we sketch the computation for the most difficult case of $n=1$. Recall the function $g_{00}(z,w)=g(z,w)=(z-q_1w)(z-q_2w)(z-q_3w)$. Then the Serre relation

$$\operatorname*{Sym}_{z_1,z_2,z_3} \; z_2z_3^{-1}[E_0^\perp(z_1),[E_0^\perp(z_2),E_0^\perp(z_3)]]=0$$

is equivalent to:

$$p(z_1,z_2,z_3)\left(\operatorname*{Asym}_{z_1,z_2,z_3}(z_2z_3^{-1}(\frac{1}{g(z_1,z_2)g(z_1,z_3)g(z_2,z_3)}+\frac{1}{g(z_1,z_3)g(z_1,z_2)g(z_3,z_2)}\right.$$
$$\left.-\frac{1}{g(z_2,z_3)g(z_2,z_1)g(z_3,z_1)}-\frac{1}{g(z_3,z_2)g(z_3,z_1)g(z_2,z_1)}))\right)=0. \tag{3.16}$$

Let us study the result of the anti-symmetrization. First, one checks that as a rational function, it is zero. However, all the terms have to be expanded in their own region. We change all the expansions to the region $|z_1|\gg|z_2|\gg|z_3|$ by adding the delta functions.

The coefficient of a single delta function is obtained from the sum of twelve terms out of twenty four terms present in (3.16). One checks that this sum has a zero at the support of the corresponding delta function. Therefore single delta functions do not appear.

However, we do have products of two delta functions. The corresponding coefficient is computed from four terms (3.16) and it is non-trivial. For example, we have $\delta(z_1/(q_1z_2))\delta(q_3z_1/z_3)$ with a non-zero coefficient. Therefore, this product of delta functions is absent if and only if $p(z_1,q_1^{-1}z_1,q_3z_1)=0$.

Let us also comment on the Serre relations in the $n=2$ case. As discussed above, we impose the quartic relations, (2.1), (2.2), following [M01]. By Lemma 2.1 we also have the cubic relations. These relations are inspired by Theorem 3.1 and they are equivalent to the wheel condition for the correlation functions.

We show here that quartic relations follow from the cubic ones. Let

$$\langle E_0^\perp(z_1)E_0^\perp(z_2)E_0^\perp(z_3)E_1^\perp(w)\rangle=\frac{p(z_1,z_2,z_3,w)}{\prod_{i=1}^{3}(z_i-q_1w)(z_i-q_3w)\prod_{i<j}(z_i-q_2z_j)}.$$

Then $p(z_1,z_2,z_3,w)$ is a Laurent polynomial which is skew-symmetric in z_1,z_2,z_3 and vanishing if $z_i=z_j$, or if $z_2=q_2z_1$ and $w=q_1z_2$, or if $z_2=q_2z_1$ and $w=q_3z_2$. Relation (2.1) is equivalent to

$$p(z_1,z_2,z_3,w)\;\operatorname*{Asym}_{z_1,z_2,z_3}\frac{1}{\prod_{i<j}(z_i-q_2z_j)}\Big(\frac{1}{\prod_{i=1}^{3}g_{12}(z_i,w)}$$
$$-\frac{q_2+1+q_2^{-1}}{\prod_{i=1}^{2}g_{12}(z_i,w)g_{12}(w,z_3)}+\frac{q_2+1+q_2^{-1}}{g_{12}(z_1,w)\prod_{i=1}^{2}g_{12}(z_i,w)}-\frac{1}{\prod_{i=1}^{3}g_{12}(w,z_i)}\Big)=0.$$

Here as before in the case of $n=2$, $g_{12}(z,w)=(z-q_1w)(z-q_3w)$. This equality is established in the same way as (3.16) using the vanishing conditions of the Laurent polynomial $p(z_1,z_2,z_3,w)$. Hence the quartic relations are consequences of the quadratic and cubic relations.

Commutators of the E and F currents. The relation

$$[E_i^{\perp}(z), F_i^{\perp}(w)] = \frac{1}{q-q^{-1}}(\delta(\kappa^{-1}w/z)K_i^{+,\perp}(z) - \delta(\kappa^{-1}z/w)K_i^{-,\perp}(z))$$

holds if and only if the following formulas are satisfied for all correlation functions:

$$\langle E_i^{\perp}(z)F_i^{\perp}(w)\rangle = \frac{p(z,w)}{(z-\kappa w)(z-\kappa^{-1}w)}, \qquad \langle F_i^{\perp}(w)E_i^{\perp}(z)\rangle = \frac{p(z,w)}{(z-\kappa w)(z-\kappa^{-1}w)},$$

where $p(z,w)$ is a Laurent polynomial and

$$\langle K_i^{+,\perp}(z)\rangle = \kappa^{-1}z^{-2}\frac{q-q^{-1}}{\kappa^{-1}-\kappa}p(z,\kappa z) = -\kappa^{-1}z^{-1}(q-q^{-1})\operatorname{Res}_{w=\kappa z}\langle E_i^{\perp}(z)F_i^{\perp}(w)\rangle,$$

$$-\langle K_i^{-,\perp}(w)\rangle = \kappa^{-1}w^{-2}\frac{q-q^{-1}}{\kappa^{-1}-\kappa}p(\kappa w,w) = -\kappa^{-1}w^{-1}(q-q^{-1})\operatorname{Res}_{z=\kappa w}\langle E_i^{\perp}(z)F_i^{\perp}(w)\rangle.$$

3.4. **The relations for the fused currents.** In this section we describe the subalgebra generated by fused currents.

From now on we fix η_i, $i=1,2,3$ such that $q_i = \exp(\eta_i)$. For $x \in \mathbb{Q}$, we set $q_i^x = e^{x\eta_i}$.

Set

$$\tilde{E}_i^{\perp}(z) = E_i^{\perp}(q_1^{\frac{i}{n-1}}z), \qquad \tilde{F}_i^{\perp}(z) = F_i^{\perp}(q_1^{\frac{i}{n-1}}z), \qquad \tilde{K}_i^{\pm,\perp}(z) = K_i^{\pm,\perp}(q_1^{\frac{i}{n-1}}z),$$

$i = 1,\dots,n-2$, and set

$$\tilde{E}_0^{\perp}(z) = E_{n-1|0}^{\perp}(z), \qquad \tilde{F}_0^{\perp}(z) = F_{n-1|0}^{\perp}(z), \qquad \tilde{K}_0^{\pm,\perp}(z) = K_{n-1|0}^{\pm,\perp}(z).$$

Set

$$\tilde{q}_1 = q_1 \cdot q_1^{\frac{1}{n-1}}, \qquad \tilde{q}_2 = q_2, \qquad \tilde{q}_3 = q_3 \cdot q_1^{-\frac{1}{n-1}}.$$

Theorem 3.1. *The currents $\tilde{E}_i^{\perp}(z), \tilde{F}_i^{\perp}(z), \tilde{K}_i^{\pm,\perp}(z)$, $i = 0,1,\dots,n-2$, satisfy the relations of the toroidal algebra $\mathcal{E}_{n-1}(\tilde{q}_1,\tilde{q}_2,\tilde{q}_3)$.*

In the proof of this theorem we use the following simple lemma.

Lemma 3.2. *Let $f(z_1,z_2,w_1,w_2)$ be a Laurent polynomial, a,b,c,d complex numbers, such that $f(w,az,bz,cz) = f(dz,az,bz,w) = 0$. Then*

$$\frac{f(dz,az,bw,cw)}{z-w}\Big|_{z=w} = \frac{f(dz,aw,bz,cw)}{w-z}\Big|_{z=w}.$$

Proof. We have

$$\frac{f(dz,az,bw,cw)}{z-w}\Big|_{z=w} = \frac{\partial}{\partial z}f(dz,az,bw,cw)\Big|_{z=w} = \frac{\partial}{\partial y}f(dz,ay,bw,cw)\Big|_{y=w}.$$

On the other hand,

$$\frac{f(dz,aw,bz,cw)}{w-z}\Big|_{z=w} = \frac{\partial}{\partial w}f(dz,aw,bz,cw)\Big|_{z=w} = \frac{\partial}{\partial y}f(dz,ay,bw,cw)\Big|_{y=w}.$$

$\square$

Proof of Theorem 3.1. We check the relations for the correlation functions. By the construction, the correlation functions of the fused currents are extracted from those of the standard currents in the way similar to obtaining $\langle K_i^{\pm,\perp}(z)\rangle$ from $\langle E_i^\perp(z)F_i^\perp(w)\rangle$. For example, if

$$\langle E_{n-1}^\perp(q_1z_1)E_0^\perp(z_2)\rangle = \frac{p(z_1,z_2)}{z_1-z_2}$$

then

$$\langle E_{n-1|0}^\perp(z)\rangle = -z^{-1}p(z,z).$$

It enables us to check its properties.

All quadratic relations and Serre relations are checked by a straightforward computation. The cases of $n=3$ and $n=2$ are slightly different but not much more difficult. For example, let $n=2$, and let us check the relation $\tilde g(z,w)E_{1|0}^{(1),\perp}(z)E_{1|0}^{(1),\perp}(w) = \tilde g(w,z)E_{1|0}^{(1),\perp}(w)E_{1|0}^{(1),\perp}(z)$, where

$$\tilde g(z,w) = (z-\tilde q_1 w)(z-\tilde q_2 w)(z-\tilde q_3 w) = (z-q_1^2w)(z-q_2w)(z-q_3q_1^{-1}w).$$

We have

$$\langle E_1^\perp(q_1z')E_0^\perp(z)E_1^\perp(q_1w')E_0^\perp(w)\rangle = \frac{z'w'p(q_1z',z,q_1w',w)}{(z'-q_2w')(z-q_2w)}$$
$$\times\frac{1}{(z'-w)(z'-q_3q_1^{-1}w)(z-q_1^2w')(z-q_1q_3w')(z-z')(q_1z'-q_3z)(w'-w)(q_1w'-q_3w)}.$$

Then

$$\langle E_{1|0}^{(1),\perp}(z)E_{1|0}^{(1),\perp}(w)\rangle = \frac{p(q_1z,z,q_1w,w)}{(z-q_2w)^2(z-w)(z-q_3q_1^{-1}w)(z-q_1^2w)(z-q_1q_3w)}.$$

The factor $(z-q_1q_3w)$ and one factor of $(z-q_2w)$ cancel due to the wheel conditions for the Laurent polynomial $p(z',z,w',w)$. Finally the pole $z-w$ is absent due to the skew-symmetry property of $p(z',z,w',w)$.

The most difficult calculation is the EF relation for the fused current. Here are some details in the case $n\geq 3$. Consider

$$\begin{aligned} R(z_1,z_2,w_1,w_2) &:= \langle E_{n-1}^\perp(q_1z_2)E_0^\perp(z_1)F_0^\perp(w_1)F_{n-1}^\perp(q_1w_2)\rangle \\ &= \frac{p(z_1,z_2,w_1,w_2)}{(z_1-z_2)(w_1-w_2)(z_1-\kappa^{-1}w_1)(z_1-\kappa w_1)(z_2-\kappa^{-1}w_2)(z_2-\kappa w_2)}. \end{aligned}$$

Then we have

$$\langle E_{n-1}^\perp(q_1z_2)K_0^{+,\perp}(z_1)F_{n-1}^\perp(q_1w_2)\rangle = \frac{\kappa^{-1}z_1^{-2}(q-q^{-1})p(z_1,z_2,\kappa z_1,w_2)}{(z_1-z_2)(\kappa z_1-w_2)(\kappa^{-1}-\kappa)(z_2-\kappa^{-1}w_2)(z_2-\kappa w_2)}.$$

But this correlation function does not have a pole at $z_1=z_2$, therefore the Laurent polynomial $p(z_1,z_2,w_1,w_2)$ satisfies

$$p(z,z,\kappa z,w_2)=0. \tag{3.17}$$

Similarly, considering the correlation function $\langle E_0^\perp(z_1)F_0^\perp(w_1)K_{n-1}^{+,\perp}(q_1z_2)\rangle$, we obtain

$$p(z,\kappa^{-1}w,w,w)=0. \tag{3.18}$$

From Lemma 3.2, and the conditions (3.17), (3.18) we obtain that

$$\frac{p(z_1,z_1,w_1,w_1)}{z_1-\kappa^{-1}w_1}\Big|_{w_1=\kappa z_1}=\frac{p(z_1,z_2,\kappa z_1,\kappa z_2)}{z_2-z_1}\Big|_{z_2=z_1}.$$

Using this identity, it is straightforward to check that

$$\begin{aligned}&\frac{q-q^{-1}}{\kappa^2 z_1^3}\operatorname{Res}_{w_1=\kappa z_1}\operatorname{Res}_{z_2=z_1}\operatorname{Res}_{w_2=w_1}R(z_1,z_2,w_1,w_2)\\ &\quad=\Big\{\frac{q-q^{-1}}{\kappa z_2}\operatorname{Res}_{w_2=\kappa z_2}\frac{q-q^{-1}}{\kappa z_1}\operatorname{Res}_{w_1=\kappa z_1}\frac{q^{-1}(\kappa z_1-w_2)}{\kappa z_1-q_2^{-1}w_2}R(z_1,z_2,w_1,w_2)\Big\}\Big|_{z_2=z_1}.\end{aligned}$$

Similarly one obtains an equation involving residues at $z_1=\kappa w_1$ and $z_2=\kappa w_2$.

These two equations are equivalent to the needed relation

$$\begin{aligned}&[E^{\perp}_{n-1|0}(z),F^{\perp}_{n-1|0}(w)]\\ &\quad=\frac{1}{q-q^{-1}}(\delta(\kappa^{-1}w/z)K^{+,\perp}_{n-1}(q_1z)K^{+,\perp}_0(z)-\delta(\kappa^{-1}z/w)K^{-,\perp}_{n-1}(q_1w)K^{-,\perp}_0(w)).\quad\square\end{aligned}$$

We denote the subalgebra of $\mathcal{E}_n$ described in the theorem by $\mathcal{E}^{n-1|0}_{n-1}$.

Note that Theorem 3.1 only shows that $\mathcal{E}^{n-1|0}_{n-1}$ is a factor of toroidal algebra $\mathcal{E}_{n-1}$ with parameters $\tilde q_1,\tilde q_2,\tilde q_3$. However, using the classical limit, see Section 3.7, we obtain that in fact the algebra $\mathcal{E}^{n-1|0}_{n-1}$ has the same size as $\mathcal{E}_{n-1}$ and therefore is isomorphic to $\mathcal{E}_{n-1}(\tilde q_1,\tilde q_2,\tilde q_3)$. Note that while $\mathcal{E}^{n-1|0}_{n-1}$ is a subalgebra of a completion of $\mathcal{E}_n$, its classical limit is a subalgebra of uncompleted classical limit of $\mathcal{E}_n$, see Section 3.7.

Note also that if V is an admissible representation for $\mathcal{E}_n$ then V is an admissible representation of $\mathcal{E}^{n-1|0}_{n-1}$. Therefore, Theorem 3.1 justifies the recursive use of the construction of the fused currents, see Section 3.1.

Let $k\in\{1,\dots,n-1\}$.

Using the theorem recursively, we obtain subalgebras $\mathcal{E}^{k|k+1|\dots|n-1|0}_k$ generated by currents $\tilde E^{\perp}_i(z)=E^{\perp}_i\big(q_1^{\frac{n-k}{k}i}z\big)$, $\tilde F^{\perp}_i(z)=F^{\perp}_i\big(q_1^{\frac{n-k}{k}i}z\big)$, $\tilde K^{\pm,\perp}_i(z)=K^{\pm,\perp}_i\big(q_1^{\frac{n-k}{k}i}z\big)$, $i=1,\dots,k-1$, and $\tilde E^{\perp}_0(z)=E^{\perp}_{k|k+1|\dots|n-1|0}(z)$, $\tilde F^{\perp}_0(z)=F^{\perp}_{k|k+1|\dots|n-1|0}(z)$, $\tilde K^{\pm,\perp}_0(z)=K^{\pm,\perp}_{k|k+1|\dots|n-1|0}(z)$.

The subalgebra $\mathcal{E}^{k|k+1|\dots|n-1|0}_k$ is isomorphic to $\mathcal{E}_k(\tilde q_1,\tilde q_2,\tilde q_3)$ with $\tilde q_i$ given by

$$\tilde q_1=q_1\cdot q_1^{\frac{n-k}{k}},\qquad \tilde q_2=q_2,\qquad \tilde q_3=q_3\cdot q_1^{-\frac{n-k}{k}}.$$

In the same way, we obtain subalgebras $\mathcal{E}^{n-k|\dots|1|0}_k$ which are generated by currents $\tilde E^{\perp}_i(z)=E^{\perp}_i\big(q_3^{\frac{n-k}{k}i}z\big)$, $\tilde F^{\perp}_i(z)=F^{\perp}_i\big(q_3^{\frac{n-k}{k}i}z\big)$, $\tilde K^{\pm,\perp}_i(z)=K^{\pm,\perp}_i\big(q_3^{\frac{n-k}{k}i}z\big)$, $i=n-k+1,\dots,n-1$, and $\tilde E^{\perp}_0(z)=E^{\perp}_{n-k|\dots|1|0}(z)$, $\tilde F^{\perp}_0(z)=F^{\perp}_{n-k|\dots|1|0}(z)$, $\tilde K^{\pm,\perp}_0(z)=K^{\pm,\perp}_{n-k|\dots|1|0}(z)$. The subalgebra $\mathcal{E}^{n-k|\dots|1|0}_k$ is isomorphic to $\mathcal{E}_k(\tilde q_1,\tilde q_2,\tilde q_3)$ with $\tilde q_i$ given by

$$\tilde q_1=q_1\cdot q_3^{-\frac{n-k}{k}},\qquad \tilde q_2=q_2,\qquad \tilde q_3=q_3\cdot q_3^{\frac{n-k}{k}}.$$

We abbreviate $\mathcal{E}^{n-1|\dots|1|0}_1$ to $\mathcal{E}^{n-1||0}_1$.

In Section 3.7 below we explain that, in the classical limit, the embedding of the subalgebra $\mathcal{E}^{k|k+1|\dots|n-1|0}_k$ into the completion of $\mathcal{E}_n$ corresponds to the embedding of submatrices into the upper-left corner. Similarly, the embedding $\mathcal{E}_k^{n-k|\dots|1|0}$ to the completion of $\mathcal{E}_n$ corresponds to the embedding of submatrices into the lower-right corner.

3.5. **Computation of $\tilde{H}_{i,1}$.** The algebra $\mathcal{E}_{n-1}^{n-1|0}$, see Theorem 3.1, is defined in terms of the perpendicular generators. It is not easy to write the standard (non-perpendicular) generators of $\mathcal{E}_{n-1}^{n-1|0}$ in terms of standard generators of $\mathcal{E}_n$ in general. In this section we compute such a formula for $\tilde{H}_{i,1}$. This is used in Section 4.

Lemma 3.3. *For $i = 0, \dots, n-2$ we have*

$$\tilde{H}_{i,1} = (-q) q_1^{-\frac{i}{n-1}} \lim_{s\to\infty} q_1^{-s} T_{n-1|0}^s \left(H_{i,1} + \delta_{i,0} q_1^{-1} H_{n-1,1}\right) .$$

Proof. By the definition of the symbol $\perp$ we have $\tilde{H}_{i,1} = \tilde{\theta}\big(\tilde{H}_{i,1}^\perp\big)$.

First assume $n \geq 3$. Set $\tilde{d} = d q_1^{\frac{1}{n-1}}$. Then, for $1 \leq i \leq n-2$, we have from (2.13)

$$\tilde{H}_{i,1} = (-\tilde{d})^{-i} [[\cdots[[\cdots[\tilde{E}_{0,0}^\perp, \tilde{E}_{n-2,0}^\perp]_{q^{-1}}, \cdots, \tilde{E}_{i+1,0}^\perp]_{q^{-1}}, \tilde{E}_{1,0}^\perp]_{q^{-1}}, \cdots, \tilde{E}_{i-1,0}^\perp]_{q^{-1}}, \tilde{E}_{i,0}^\perp]_{q^{-2}} .$$

Substituting $\tilde{E}_{j,0}^\perp = E_{j,0}^\perp$ $(1 \leq j \leq n-2)$ and

$$\tilde{E}_{0,0}^\perp = (-q) \lim_{s\to\infty} q_1^{-s} T_{n-1|0}^s [E_{0,0}^\perp, E_{n-1,0}^\perp]_{q^{-1}} ,$$

we find

$$\tilde{H}_{i,1} = (-\tilde{d})^{-i} (-q) \lim_{s\to\infty} q_1^{-s} T_{n-1|0}^s \big((-d)^i H_{i,1}\big)$$

which gives the desired result.

Consider the case $i = 0$. Using the quadratic relation $[E_{0,0}^\perp, E_{1,0}^\perp]_{q^{-1}} = -d[E_{1,1}^\perp, E_{0,-1}^\perp]_{q^{-1}}$ we rewrite $H_{n-1,1}$ as follows.

$$H_{n-1,1} = (-d)^{-n+2} [[[\cdots[E_{1,1}^\perp, E_{2,0}^\perp]_{q^{-1}}, \cdots, E_{n-2,0}^\perp]_{q^{-1}}, E_{0,-1}^\perp]_{q^{-1}}, E_{n-1,0}^\perp]_{q^{-2}} .$$

Setting $X = [[E_{1,1}^\perp, E_{2,0}^\perp]_{q^{-1}}, \cdots, E_{n-2,0}^\perp]_{q^{-1}}$ and using (2.14) we obtain

$$\begin{aligned}(-d)^{n-1}\big(H_{0,1} + q_1^{-1} H_{n-1,1}\big) &= [[X, E_{n-1,0}^\perp]_{q^{-1}}, E_{0,-1}^\perp]_{q^{-2}} - q[[X, E_{0,-1}^\perp]_{q^{-1}}, E_{n-1,0}^\perp]_{q^{-2}} \\ &= X[E_{n-1,0}^\perp, E_{0,-1}^\perp]_q - q^{-2}[E_{n-1,0}^\perp, E_{0,-1}^\perp]_{q^{-1}} X - (1-q^{-2}) E_{0,-1}^\perp X E_{n-1,0}^\perp .\end{aligned}$$

It follows that

$$\lim_{s\to\infty} q_1^{-s} T_{n-1|0}^s (H_{0,1} + q_1^{-1} H_{n-1,1}) = (-d)^{-n+1} [X, E_{n-1|0,-1}^\perp]_{q^{-2}} .$$

We obtain the statement by noting that $\tilde{E}_{1,1}^\perp = q_1^{-\frac{1}{n-1}} E_{1,1}^\perp$.

The case $n = 2$ can be checked directly, by noting that $\tilde{H}_{0,1} = \tilde{E}_{0,0}^\perp = E_{1|0,0}^\perp$. ∎

3.6. **Commuting subalgebras.** We show that the constructed "upper left corner" subalgebras commute with "lower right corner" subalgebras.

Theorem 3.4. *For each k, the $\mathcal{E}_n$ subalgebras $\mathcal{E}_{n-k}^{k|\dots|1|0}$ and $\mathcal{E}_k^{k|k+1|\dots|n-1|0}$ commute.*

Proof. The theorem is proved by the same techniques as Theorem 3.1.

For example, let us check the commutativity of $E_1^\perp(z) \in \mathcal{E}_k^{k|k+1|\dots|n-1|0}$ with $E_{k|\dots|1|0}^\perp(w) \in \mathcal{E}_{n-k}^{k|\dots|1|0}$. We consider the correlation function

$$\begin{aligned}&\langle E_1^\perp(z)E_k^\perp(q_3^k w_k)E_{k-1}^\perp(q_3^{k-1}w_{k-1})\dots E_0^\perp(w)\rangle\\&\quad=\frac{p(z,w_k,\dots,w_1,w)}{(z-q_2q_3w_1)(z-q_1q_3^2w_2)(z-q_1^{-1}w)(w_k-w_{k-1})\dots(w_1-w)}.\end{aligned}$$

We need to show that the poles at $z=q_2q_3w_1$, $z=q_1q_3^2w_2$ and $z=q_1^{-1}w$ disappear when we multiply by $(w_k-w_{k-1})\dots(w_1-w)$ and set $w_k=w_{k-1}=\dots=w_1=w$. But this follows from the wheel conditions for the Laurent polynomial $p(z,w_k,\dots,w_1,w)$.

Let us check the commutativity of $F_1^\perp(z) \in \mathcal{E}_k^{k|k+1|\dots|n-1|0}$ with $E_{k|\dots|1|0}^\perp(w) \in \mathcal{E}_{n-k}^{k|\dots|1|0}$. We consider the correlation function

$$\begin{aligned}&\langle F_1^\perp(z)E_k^\perp(q_3^k w_k)E_{k-1}^\perp(q_3^{k-1}w_1)\dots E_0^\perp(w)\rangle\\&\quad=\frac{p(z,w_k,\dots,w_1,w)}{(z-\kappa^{-1}q_3w_1)(z-\kappa q_3w_1)(w_k-w_{k-1})\dots(w_1-w)}.\end{aligned}$$

We need to show that the poles at $z=\kappa^{-1}q_3w_1$, $z=\kappa q_3w_1$ disappear when we multiply by $(w_k-w_{k-1})\dots(w_1-w)$ and set $w_k=w_{k-1}=\dots=w_1=w$. We have

$$\begin{aligned}&\mathrm{Res}_{z=q_3\kappa w_1}\langle F_1^\perp(z)E_k^\perp(q_3^k w_k)\dots E_2^\perp(q_3^2w_2)E_1^\perp(w_1q_3)E_0^\perp(w)\rangle\\&\quad=-\frac{q_3\kappa w_1^{-1}}{q-q^{-1}}\langle E_k^\perp(q_3^kw_k)\dots E_2^\perp(q_3^2w_2)K_1^{+,\perp}(q_3w_1)E_0^\perp(w)\rangle\\&\quad=-q_3^{-1}d^{-1}\frac{q_3\kappa w_1}{q-q^{-1}}\frac{q_1q_3w_1-w}{w_1-w}\langle E_k^\perp(q_3^kw_k)\dots E_2^\perp(q_3^2w_2)E_0^\perp(w)K_1^{+,\perp}(q_3w_1)\rangle.\end{aligned}$$

In the last expression there is no pole at $w_2=w_1$. It implies that we have the identity $p(\kappa q_3w_1,w_k,\dots,w_1,w_1,w)=0$ and the pole $z=\kappa q_3w_1$ disappears.

We omit further details. □

3.7. **Classical limit.** In this subsection we explain the meaning of the fused currents in the classical limit.

The quantum toroidal $\mathfrak{gl}_n$ algebra $\mathcal{E}_n=\mathcal{E}_n(q_1,q_2,q_3)$ contains two parameters q,d. By the classical limit we mean $q\to1$. The algebra $\mathcal{E}_n(q_1,q_2,q_3)$ in the limit is known to have the following description.

Consider the algebra $\mathcal{A}_n(d)=\mathbb{M}_n\otimes\mathbb{C}[Z^{\pm1},D^{\pm1}]$, where $\mathbb{M}_n$ stands for the algebra of $n\times n$ matrices, and $\mathbb{C}[Z^{\pm1},D^{\pm1}]$ is the algebra generated by symbols Z,D satisfying $DZ=d^{-n}ZD$. We regard $\mathcal{A}_n(d)$ as a Lie algebra by commutators. Let $\mathcal{L}_n(d)=\mathcal{A}_n(d)\oplus\mathbb{C}c_1\oplus\mathbb{C}c_2$ be its two-dimensional central extension, where the Lie bracket is given by

$$\begin{aligned}[M_1\otimes Z^{r_1}D^{s_1},M_2\otimes Z^{r_2}D^{s_2}]&=\bigl(d^{-nr_2s_1}M_1M_2-d^{-nr_1s_2}M_2M_1\bigr)\otimes Z^{r_1+r_2}D^{s_1+s_2}\\&\quad+\delta_{r_1+r_2,0}\delta_{s_1+s_2,0}\,d^{-nr_2s_1}\mathrm{tr}\bigl(M_1M_2\bigr)\cdot(r_1c_1+s_1c_2)\,,\end{aligned}$$

for $M_i\in\mathbb{M}_n$, $r_i,s_i\in\mathbb{Z}$, $i=1,2$. Let further $\mathcal{L}'_n(d)$ be the Lie subalgebra of $\mathcal{L}_n(d)$ spanned by c_1,c_2 and elements $\sum M_{r,s}Z^rD^s\in\mathcal{A}_n(d)$ such that $\mathrm{tr}(M_{0,0})=0$. The classical limit of $\mathcal{E}_n(q_1,q_2,q_3)$ is the universal enveloping algebra $U\mathcal{L}'_n(d)$.

To see this explicitly, set $K_i^{\perp} = q^{H_{i,0}^{\perp}}$, $\kappa = q^{-c_1}$, $c_2 = \sum_{i=0}^{n-1} H_{i,0}^{\perp}$. It is then straightforward to check that the limit $q \to 1$ of the defining relations for the generators $E_{i,k}^{\perp}, F_{i,k}^{\perp}, H_{i,k}^{\perp}$ of $\mathcal{E}_n(q_1, q_2, q_3)$ are satisfied by the following elements of $\mathcal{L}_n'(d)$:

$$\bar{E}_{i,k}^{\perp} = \begin{cases} E_{i,i+1} \otimes Z^k \, d^{-ik} & (1 \le i \le n-1)\,, \\ E_{n,1} \otimes DZ^k & (i=0)\,, \end{cases} \tag{3.19}$$

$$\bar{F}_{i,k}^{\perp} = \begin{cases} E_{i+1,i} \otimes Z^k \, d^{-ik} & (1 \le i \le n-1)\,, \\ E_{1,n} \otimes Z^k D^{-1} & (i=0)\,, \end{cases}$$

$$\bar{H}_{i,k}^{\perp} = \begin{cases} \big(E_{i,i} - E_{i+1,i+1}\big) \otimes Z^k \, d^{-ik} & (1 \le i \le n-1)\,, \\ \big(d^{-nk} E_{n,n} - E_{1,1}\big) \otimes Z^k \ + c_2 \delta_{k,0} & (i=0)\,. \end{cases}$$

Here $E_{i,j} \in \mathbb{M}_n$ are the matrix units. As it is noted in [M99], the automorphism $\theta \in \operatorname{Aut} \mathcal{E}_n$ reduces in the classical limit to the Lie algebra automorphism $\bar{\theta} \in \operatorname{Aut} \mathcal{L}_n'(d)$ given by the rule

$$Z \mapsto D, \quad D \mapsto Z^{-1}, \quad c_1 \mapsto c_2, \quad c_2 \mapsto -c_1,$$

and $M \mapsto M$ for $M \in \mathbb{M}_n$.

Let us examine the classical limit of the fused currents. For simplicity we consider the case $n \ge 3$. Recall that the Fourier components of the current $E_{n-1|0}^{\perp}(z)$ are defined to be

$$E_{n-1|0,r}^{\perp} = \lim_{s \to \infty} q_1^{-r-s} E_{n-1,r+s}^{\perp} E_{0,-s}^{\perp} = \lim_{s \to \infty} q_1^{-r-s} [E_{n-1,r+s}^{\perp}, E_{0,-s}^{\perp}]\,,$$

where the second equality is due to the meaning of the completion. In view of (3.19), the classical limit of this expression is

$$\bar{E}_{n-1|0,r}^{\perp} = d^{-r-s} [\bar{E}_{n-1,r+s}^{\perp}, \bar{E}_{0,-s}^{\perp}] = E_{n-1,1} \otimes DZ^r\,.$$

This holds true for all s, without taking the limit $s \to \infty$ nor introducing the completion. Similarly the classical limit of $F_{n-1|0,r}^{\perp}$ is

$$\bar{F}_{n-1|0,r}^{\perp} = d^{s} [\bar{F}_{0,r+s}^{\perp}, \bar{F}_{n-1,-s}^{\perp}] = E_{1,n-1} \otimes Z^r D^{-1}\,.$$

These elements along with the other generators $\bar{E}_{i,r}^{\perp} d^{-ir/(n-1)}$, $\bar{F}_{i,r}^{\perp} d^{-ir/(n-1)}$ for $1 \le i \le n-2$ generate a subalgebra of $\mathcal{L}_n'(d)$ isomorphic to $\mathcal{L}_{n-1}'(\tilde{d})$, where $\tilde{d} = d^{n/(n-1)}$ (note that $DZ = \tilde{d}^{-n+1} ZD$). This is nothing but the one induced from the upper left corner embedding of matrix algebras

$$\mathbb{M}_{n-1} \hookrightarrow \mathbb{M}_n, \quad M' \mapsto \begin{pmatrix} M' & 0 \\ 0 & 0 \end{pmatrix}.$$

In a similar manner the classical counterparts of $E_{n-1||0,r}^{\perp}$, $F_{n-1||0,r}^{\perp}$ generate a subalgebra $\mathcal{L}_1'(d^n)$ commuting with $\mathcal{L}_{n-1}'(\tilde{d})$. The former corresponds to the bottom right corner embedding

$$\mathbb{M}_1 \hookrightarrow \mathbb{M}_n, \quad M'' \mapsto \begin{pmatrix} 0 & 0 \\ 0 & M'' \end{pmatrix}.$$

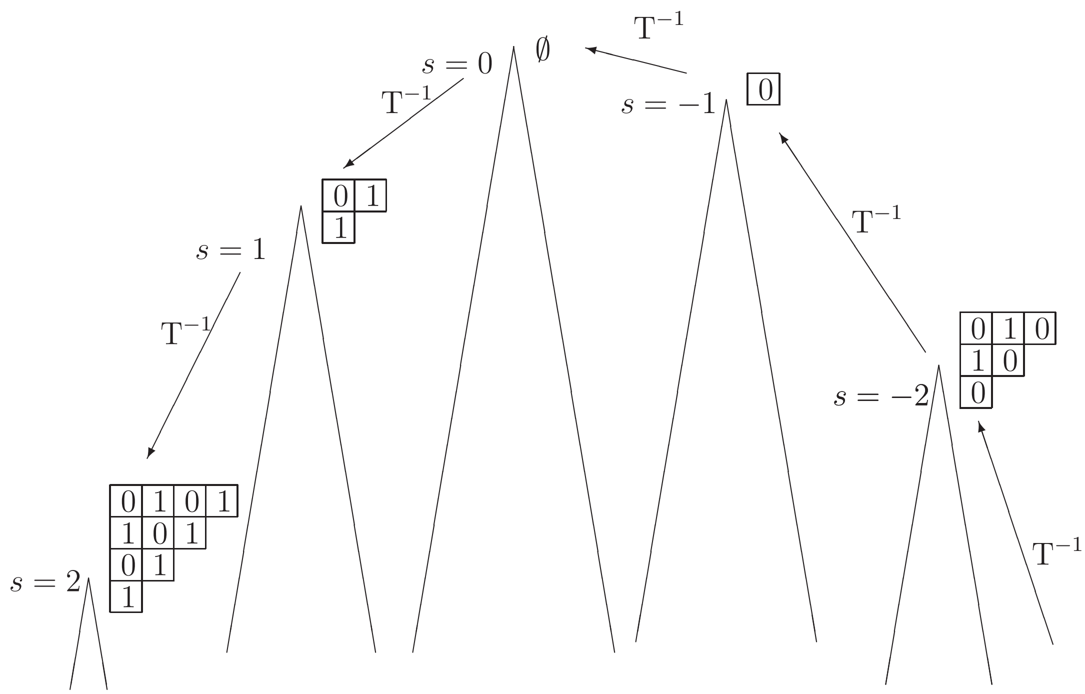

Figure 1. The $\mathcal{E}_2$ module $\mathcal{F}^{(0)}(u)$

4. Branching rules

In this section we study the restriction of various $\mathcal{E}_n$ modules to the subalgebra $\mathcal{E}_{n-1}^{n-1|0} \otimes \mathcal{E}_1^{n-1||0}$.

The logic of the computation is the same in all cases, but we start with Fock spaces, and specifically with $n = 2$ where the situation is the easiest to describe.

4.1. **Fock modules for $\mathcal{E}_2$.** In this section we study decompositions of the modules of level q for $\mathcal{E}_2$.

Consider the module $\mathcal{F}^{(0)}(u)$, see Section 2.5. This module has a basis labeled by partitions. In addition, it is convenient to represent this module by the following familiar picture, see Figure 1.

On this picture the module $\mathcal{F}^{(0)}(u)$ looks similar to that of the vacuum $\widehat{\mathfrak{sl}}_2$ integrable module of (additive) level one, but actually it is not the same. It is similar simply because the Fock module restricted to the horizontal algebra $U_q^{hor}\widehat{\mathfrak{sl}}_2$ is a level q module (in the sense $\kappa^{-1} = q$). However, the reader should be warned that our space is in fact the vacuum $U_q\widehat{\mathfrak{gl}}_2$ module. In other words, we have a Heisenberg current commuting with the $U_q^{hor}\widehat{\mathfrak{sl}}_2$, see Section 2.3, and our module is the tensor product of the Fock space of the Heisenberg algebra with the vacuum $U_q^{hor}\widehat{\mathfrak{sl}}_2$ integrable module of level q. Thus the module $\mathcal{F}^{(0)}(u)$ is the vacuum $U_q^{hor}\widehat{\mathfrak{gl}}_2$ module.

We have the usual $\mathfrak{sl}_2$ weight decomposition given by values of $K_1K_0^{-1}$. We called this weight "cweight" (see (2.5)).

The cweight of a partition is given by $\sharp\{$boxes of color 1$\} - \sharp\{$boxes of color 0$\}$ in the corresponding colored Young diagram. On Figure 1, the cweight increases from the right to the left and it is denoted by s. The cones which look downward picture vectors of the same cweight.

We also have the principal degree given by $\operatorname{pdeg} E_i(z) = 1$, $\operatorname{pdeg} F_i(z) = -1$. It counts the total number of boxes and in Figure 1 the principal degree increases from the top to the bottom.

The action of the i-th generator of the Heisenberg current increases the principal degree by $2i$ and does not change the cweight.

We have the action of the shift element T, see (2.16), on the Fock space as shown in Figure 1. Precisely, we have $T^{-1} = s_1 s_0$ where the s_i are the Lusztig simple reflections.

Our first observation is the following combinatorial "tensor product" decomposition of the sector with large cweight s. Let $\Lambda^s = (2s, 2s-1, \dots, 2, 1)$ and $\Lambda^{-s} = (2s-1, 2s-2, \dots, 2, 1)$ for $s > 0$ and let Λ^0 be the empty partition. Then $|\Lambda^s\rangle$ is the vector of the lowest degree of cweight s. The degree of $|\Lambda^s\rangle$ is $s(2s+1)$. Fix two partitions λ and μ with, say, k parts and let $|s|$ be larger than k. Let $\Lambda^s_{\lambda,\mu}$ be the unique partition of degree $s(2s+1) + 2|\lambda| + 2|\mu|$ and cweight s such that for $i = 1, \dots, k$ we have

$$(\Lambda^s_{\lambda,\mu})_i = \Lambda^s_i + 2\lambda_i,$$
$$(\Lambda^s_{\lambda,\mu})'_i = (\Lambda^s)'_i + 2\mu_i.$$

Informally speaking, $\Lambda^s_{\lambda,\mu}$ is obtained from Λ^s by attaching the partitions of λ and μ made out of dominoes to the top and the bottom respectively, see Figure 2.

Denote by S^s the subspace of $\mathcal{F}^{(0)}(u)$ of cweight s. Denote by $S^s_{\leq 2k}$ the subspace of S^s consisting of vectors which have degree at most $s(2s+1) + 2k$. We have the following purely combinatorial lemma.

Lemma 4.1. *If $2s > k$, the vectors $|\Lambda^s_{\lambda,\mu}\rangle$ with $|\mu| + |\lambda| \leq k$ form a basis of $S^s_{\leq 2k}$.*

Proof. Partitions of vectors in $S^s_{\leq 2k}$ have s more boxes of color 1 than color 0. Each odd row contains at least as many boxes of color 0 as color 1. Each even row contains at most one more box of color 1 than color 0. It follows that every such partition contains Λ^s. Hence any such partition is obtained from Λ^s by adding r boxes of color 1 and r boxes of color 0 where $r \leq k$. It is easy to see that the only way to do it is as described in the lemma. $\square$

Another important statement is the following lemma.

Lemma 4.2. *Let λ, μ be partitions and let $2s > |\lambda| + |\mu|$. Then $T^{-1}|\Lambda^s_{\lambda,\mu}\rangle = a_{s,\lambda,\mu}|\Lambda^{s+1}_{\lambda,\mu}\rangle$, where $a_{s,\lambda,\mu}$ is a non-zero constant.*

Proof. For $s \geq 0$, one cannot remove boxes of color 0 from $|\Lambda^s_{\lambda,\mu}\rangle$. It is therefore a lowest weight vector with respect to $U_q\mathfrak{sl}_2$ generated by $E_{0,0}, F_{0,0}$. Then $s_0|\Lambda^s_{\lambda,\mu}\rangle$ is a non-zero constant multiple of the corresponding highest weight vector, that is $|\Lambda^{-s-1}_{\lambda,\mu}\rangle$. Similarly $s_1|\Lambda^{-s-1}_{\lambda,\mu}\rangle$ is a non-zero constant multiple of $|\Lambda^{s+1}_{\lambda,\mu}\rangle$. The lemma follows. $\square$

For $s \in \mathbb{Z}$, and partitions λ, μ, we choose any integer $r = r(s, \lambda, \mu)$ such that $2(r+s) > |\lambda| + |\mu|$ and define vectors $v^s_{\lambda,\mu}$ by the formula

$$v^s_{\lambda,\mu} = T^r|\Lambda^{r+s}_{\lambda,\mu}\rangle.$$

Note that different choices of r change vectors $v^s_{\lambda,\mu}$ by non-zero scalars.

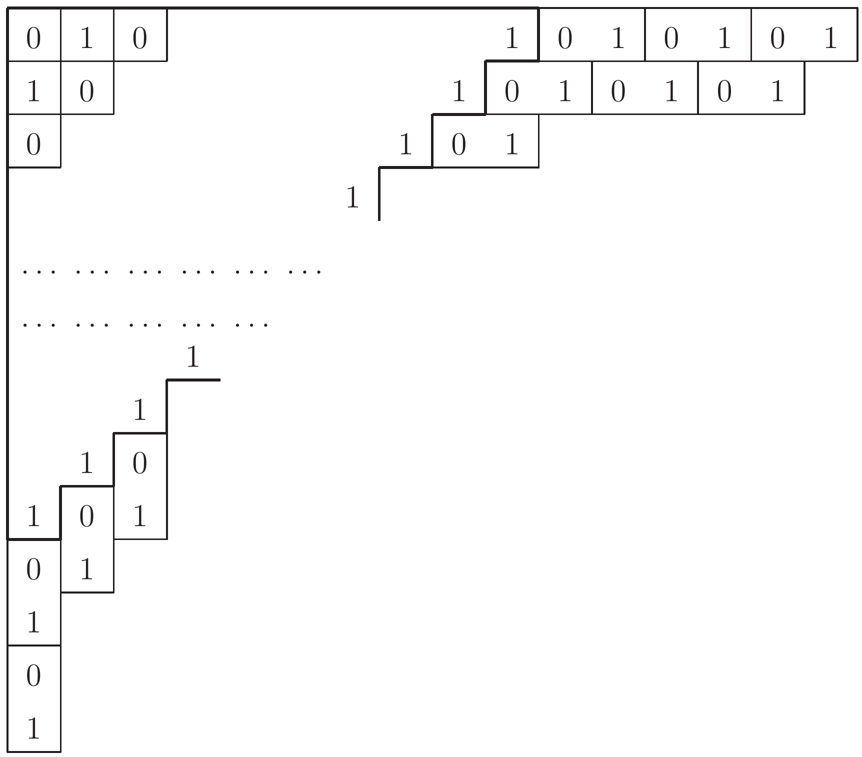

FIGURE 2. The partition $\Lambda^s_{\lambda,\mu}$ with $\lambda=(3,3,1)$ and $\mu=(2,1,1)$.

Corollary 4.3. *The vectors $v^s_{\lambda,\mu}$ form a basis of the space S^s.*

Proof. The corollary follows from Lemma 4.2 and Lemma 4.1. $\square$

Recall that we have the subalgebra $\mathcal{E}^{(1)}_1\otimes\mathcal{E}^{(3)}_1\subset\mathcal{E}_2$. The subalgebra $\mathcal{E}^{(1)}_1$ is generated by currents $E^{(1),\perp}_{1|0}(z)$, $F^{(1),\perp}_{1|0}(z)$, $K^{\pm(1),\perp}_{1|0}(z)$ and the subalgebra $\mathcal{E}^{(3)}_1$ is generated by currents $E^{(3),\perp}_{1|0}(z)$, $F^{(3),\perp}_{1|0}(z)$, $K^{\pm(3),\perp}_{1|0}(z)$. Now we are in a position to establish the decomposition of the $\mathcal{E}_2$-module $\mathcal{F}^{(0)}$.

We write $\mathcal{F}^{(2;0)}(u)$ for the $\mathcal{E}_2$ Fock module $\mathcal{F}^{(0)}(u)$, we also write $\mathcal{F}^{(1)}(u)$ (resp. $\mathcal{F}^{(3)}(u)$) for the $\mathcal{E}^{(1)}_1$ (resp. $\mathcal{E}^{(3)}_1$) Fock modules.

Theorem 4.4. *We have an isomorphism of $\mathcal{E}^{(1)}_1\otimes\mathcal{E}^{(3)}_1$ modules*

$$\mathcal{F}^{(2;0)}(u)=\bigoplus_{s\in\mathbb{Z}} x^{s(2s+1)}z^s\ \mathcal{F}^{(1)}(-qq_1^{2s}u)\boxtimes\mathcal{F}^{(3)}(-qq_3^{2s}u). \tag{4.1}$$

In particular, this isomorphism identifies the space $\mathcal{F}^{(1)}(-qq_1^{2s}u)\boxtimes\mathcal{F}^{(3)}(-qq_3^{2s}u)$ with the subspace of $\mathcal{F}^{(2;0)}(u)$ of cweight s.

Here the factor $x^{s(2s+1)}z^s$ signifies the cweight and degree of the top vector of the subspace $\mathcal{F}^{(1)}(-qq_1^{2s}u)\boxtimes\mathcal{F}^{(3)}(-qq_3^{2s}u)$ in $\mathcal{F}^{(2;0)}(u)$.

Proof. The algebras $\mathcal{E}^{(1)}_1$ and $\mathcal{E}^{(3)}_1$ are defined in terms of the perpendicular generators of $\mathcal{E}_2$ and the action of $\mathcal{E}_2$ in $\mathcal{F}^{(2;0)}(u)$ is given in terms of usual generators. Therefore, in general, it is not easy to compute the action of algebras $\mathcal{E}_1^{(1)}$ and $\mathcal{E}_1^{(3)}$ in $\mathcal{F}^{(2;0)}(u)$. However, at least we have the following formula, see Lemma 3.3, which turns out to be sufficient for our purposes.

$$H_{0,1}^{(1)} = (-q) \lim_{s\to\infty} q_1^{-2s} T^s (q_1^{-1} H_{1,1} + H_{0,1}), \tag{4.2}$$

$$H_{0,1}^{(3)} = (-q) \lim_{s\to\infty} q_3^{-2s} T^s (q_3^{-1} H_{1,1} + H_{0,1}). \tag{4.3}$$

It follows that we can compute

$$\langle v|H_{0,1}^{(1)}|v\rangle = -q \lim_{s\to\infty} \langle T^{-s} v|q_1^{-2s}(q_1^{-1} H_{1,1} + H_{0,1})|T^{-s} v\rangle, \tag{4.4}$$

$$\langle v|H_{0,1}^{(3)}|v\rangle = -q \lim_{s\to\infty} \langle T^{-s} v|q_3^{-2s}(q_3^{-1} H_{1,1} + H_{0,1})|T^{-s} v\rangle.$$

We use these formulas to establish

$$H^{(1)}_{0,1}|\Lambda^s\rangle = q^2uq_1^{2s}|\Lambda^s\rangle, \qquad H^{(3)}_{0,1}|\Lambda^s\rangle = q^2uq_3^{2s}|\Lambda^s\rangle. \tag{4.5}$$

Now note, that for all $s\in\mathbb{Z}$ the vectors $|\Lambda^s\rangle$ are lowest weight vectors with respect to $\mathcal{E}_1^{(1)}$ and $\mathcal{E}_1^{(3)}$ for the degree reasons. Since the levels of $\mathcal{E}_1^{(1)}$ and $\mathcal{E}_1^{(3)}$ coincide with the level of $\mathcal{E}_2$, the vector $|\Lambda^s\rangle$ generates a level q module for both of these algebras. Such a module necessarily contains a Fock module. The evaluation parameter of the Fock module is now obtained from (4.5).

It follows that $\mathcal{F}^{(2;0)}(u)$ contains the right hand side of (4.1). Then the equality follows from Lemma 4.3. □

Remark 4.5. Using (4.4), we can compute the action of operators $H^{(1)}_{0,1}$ and $H^{(3)}_{0,1}$ on basis $v^s_{\lambda,\mu}$. Moreover, in fact, the spectrum of the operator $H_{0,1}$ is simple in the $\mathcal{E}_1$ Fock module. It allows us to identify (up to a constant) the vector $v^s_{\lambda,\mu}$ with the vector $|\lambda\rangle\boxtimes|\mu\rangle$.

4.2. **Fock modules for $\mathcal{E}_n$.** In this section we generalize the results of Section 4.1 for all Fock modules.

Fix $p\in\{0,\dots,n-1\}$, and consider the $\mathcal{E}_n$ Fock module $\mathcal{F}^{(p)}(u)$. Then we have a picture, similar to Figure 1, where the lattice of roots is now $\mathbb{Z}_{n-1}$. The top vectors (or extremal vectors) are obtained by the action of the braid group on the $|\emptyset\rangle$.

Denote the simple roots of $\mathfrak{sl}_n$ by α_j, $j=1,\cdots,n-1$. Let η, $\eta^{(p)}$, η_i be the following $\mathfrak{sl}_n$ roots:

$$\eta=\sum_{j=1}^{n-1} j\alpha_j, \qquad \eta^{(p)}=\sum_{j=1}^{p-1} j\alpha_j + p\sum_{j=p}^{n-1}\alpha_j, \qquad \eta_i=\sum_{j=1}^{i}(i-j+1)\alpha_{n-j}.$$

Here $i=0,\dots,n-2$.

Given an $\mathfrak{sl}_n$ root γ, there is unique $s,a_1,\dots,a_{n-2}\in\mathbb{Z}$, $i\in\{0,1,\dots,n-2\}$, such that

$$\gamma=\eta^{(p)}+\eta_i+s\eta+\sum_{j=1}^{n-2}a_j\alpha_j. \tag{4.6}$$

For $s\equiv i+p \pmod{n-1}$ denote $v^{s,i}$ the extremal vector of cweight

$$w(s,i,p):=\eta^{(p)}+\eta_i+\frac{s-i-p}{n-1}\eta.$$

Lemma 4.6. *For $ns\geq i+p$, we have $v^{s,i}=|\Lambda^{s,i}\rangle$, where the partition $\Lambda^{s,i}$ has $ns-p$ non-trivial parts given by*

$$(\Lambda^{s,i})_j=\frac{ns-i-p}{n-1}-\left[\frac{j-i-1}{n-1}\right],$$

see Figure 3.

Proof. The principal degree (or total number of boxes) of $\Lambda^{s,i}$ is easily computed and is given by

$$\rho(s,i,p)=\frac{(ns+n-i-p-1)(ns+i-p)}{2(n-1)}.$$

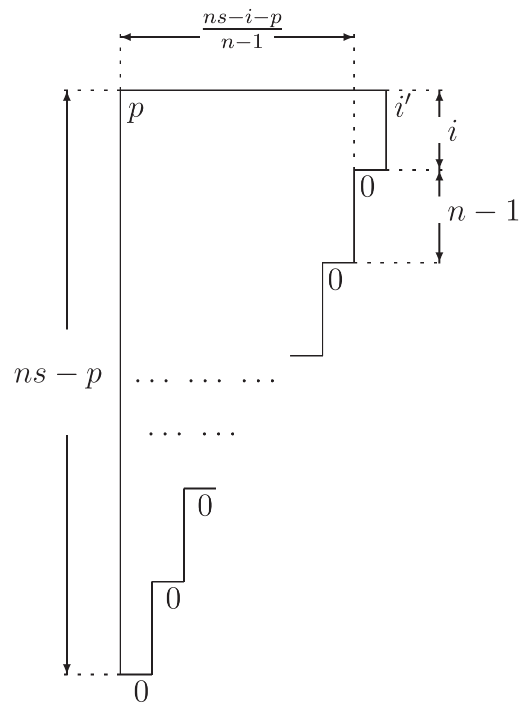

FIGURE 3. The staircase partition $\Lambda^{s,i}$. Here the color i' is $n-i-1$.

Then one checks by a straightforward computation that

$$\rho(s,i,p)=\frac{n}{2}\big((w(s,i,p)-\mathbf{\Lambda}_p,w(s,i,p)-\mathbf{\Lambda}_p)-(\mathbf{\Lambda}_p,\mathbf{\Lambda}_p)\big)+(w(s,i,p),\sum_{l=0}^{n-1}\mathbf{\Lambda}_l)$$

Here $\mathbf{\Lambda}_p$ denotes the p-th fundamental $\widehat{\mathfrak{sl}}_n$ weight. The lemma follows. □

Lemma 4.7. *The vector $|\Lambda^{s,i}\rangle$ has the smallest principal degree among all vectors of cweights* (4.6) *with the given s,i and various a_i.*

Proof. Note that the Fock module $\mathcal{F}^{(p)}(u)$ is irreducible, and $F_{i,k}|\Lambda^{s,i}\rangle=0$ for $i=0,\dots,n-2$. The statement of the lemma follows. □

Let T be the automorphism of $\mathcal{E}_n$ given by $T=T^n_{n-1|0}$, see (2.15). After restriction to the horizontal subalgebra $U_q^{hor}\big(\widehat{\mathfrak{sl}}_n\big)$, T becomes the translation operator in the braid group. in terms of the Lusztig simple reflections we have, see Lemma 2.5,

$$T^{-1}=(s_{n-1}\dots s_2s_1s_0)^{n-1}.$$

Note also that T act as identity on the Heisenberg algebra $\mathfrak{a}^{hor}$. Hence the operator T^{-1} acts on $\mathcal{F}^{(p)}(u)$ and changes cweight by η. In particular, we have

$$T^{-1}v^{s,i}=a_{n,s,i}v^{s+n-1,i},$$

for some non-zero constants $a_{n,s,i}$.

We recall that we have the subalgebra $\mathcal{E}^{n-1|0}_{n-1}\otimes\mathcal{E}^{n-1||0}_1\subset\mathcal{E}_n$.

The subalgebra $\mathcal{E}_{n-1}^{n-1|0}$ is generated by currents $E^{\perp}_{n-1|0}(z)$, $F^{\perp}_{n-1|0}(z)$,$K^{\pm,\perp}_{n-1|0}(z)$ and $E^{\perp}_i\big(q_1^{\frac{i}{n-1}}z\big)$, $F^{\perp}_i\big(q_1^{\frac{i}{n-1}}z\big)$, $K^{\pm,\perp}_i\big(q_1^{\frac{i}{n-1}}z\big)$ with $i=1,\dots,n-2$.

The subalgebra $\mathcal{E}_1^{n-1||0}$ is generated by currents $E^{\perp}_{n-1||0}(z)$,$F^{\perp}_{n-1||0}(z)$,$K^{\pm,\perp}_{n-1||0}(z)$, see (3.13).

By Theorem 3.4 the subalgebras $\mathcal{E}_{n-1}^{n-1|0}$ and $\mathcal{E}_1^{n-1||0}$ commute inside $\mathcal{E}_n$.

The following lemma follows from the construction of the subalgebras and Lemma 2.4.

Lemma 4.8. *Let* $\deg^{(n-1)}$ *and* $\deg^{(n)}$ *denote the degree in* $\mathcal{E}_{n-1}$ *and* $\mathcal{E}_n$*, respectively. The embedding of* $\mathcal{E}_{n-1}$ *is graded. Namely, if* $x\in\mathcal{E}_{n-1}$ *is a graded element such that* $\deg^{(n-1)}x=(\ell,\ell+d_1,\dots,\ell+d_{n-2},k)$*, then the embedded element, which we denote also by* x*, is graded and* $\deg^{(n)}x=(\ell,\ell+d_1,\dots,\ell+d_{n-2},\ell,k)$*. Similarly, if* $x\in\mathcal{E}_1$ *is such that* $\deg^{(1)}=(\ell,k)$ *then* $\deg^{(n)}=(\ell,\ell,\dots,\ell,k)$*.*

Now we are in a position to describe the decomposition of the $\mathcal{E}_n$-module $\mathcal{F}^{(p)}(u)$.

We write $\mathcal{F}^{(n;p)}(u)$ for the $\mathcal{E}_n$ Fock module, similarly we write $\mathcal{F}^{(n-1;k)}(u)$ (resp. $\mathcal{F}^{(1)}(u)$) for the $\mathcal{E}_{n-1}^{n-1|0}$ (resp. $\mathcal{E}_1^{n-1||0}$) Fock modules.

Theorem 4.9. *We have an isomorphism of* $\mathcal{E}_{n-1}^{n-1|0}\otimes\mathcal{E}_1^{n-1||0}$ *modules*

$$\mathcal{F}^{(n;p)}(u)=\bigoplus_{i=0}^{n-2}\ \bigoplus_{\substack{s\in\mathbb{Z},\\ s\equiv p+i\,(\mathrm{mod}\,n-1)}} x^{\rho(s,i,p)}z^{w(s,i,p)}\ \mathcal{F}^{(n-1;n-i-1)}(-qq_1^{\frac{ns-p}{n-1}}u)\boxtimes\mathcal{F}^{(1)}(-qq_3^{ns-p}u).$$

Proof. The theorem is proved similarly to Theorem 4.4. The first step is to show that the vector $\Lambda^{s,i}$ is the lowest weight vectors for both of the actions of $\mathcal{E}_{n-1}$ and $\mathcal{E}_1$. In the proof we use Lemma 4.7 and Lemma 4.8.

Let us indicate the combinatorial picture. Similarly to $n=2$ case, we define the basis $|\Lambda^{s,i}_{\lambda,\mu}\rangle$ and the partition $\Lambda^{s,i}_{\lambda,\mu}$ is obtained from $\Lambda^{s,i}$ by adding legs in the shape of μ, and arms in the shape of λ. Moreover, each box of μ is replaced with vertical strip of n boxes colored $0,1,2,\dots,n-1$ (from top to bottom). The partition λ is colored by $n-1$ colors, with the top left box being $n-i-1$. Then 0 boxes of λ are replaced in $\Lambda^{s,i}_{\lambda,\mu}$ by horizontal dominoes colored $0,n-1$ from left to right. The other boxes of λ go to the boxes of the same color in $\Lambda^{s,i}_{\lambda,\mu}$.

Then we use Lemma 3.3 in place of (4.2), (4.3).

We leave the rest of the details to the reader. □

4.3. **Generic tensor products of Fock modules.** Our next goal is to establish the decomposition of the module $\mathcal{F}^{(n,p_1)}(u_1)\otimes\mathcal{F}^{(n,p_2)}(u_2)\otimes\cdots\otimes\mathcal{F}^{(n,p_k)}(u_k)$ with generic evaluation parameters $u_1,\dots,u_k$ as $\mathcal{E}_{n-1}^{n-1|0}\otimes\mathcal{E}_1^{n-1||0}$ module.

We prepare the following lemma.

Lemma 4.10. *Let* $W=\mathcal{F}^{(p_1)}(u_1)\otimes\mathcal{F}^{(p_2)}(u_2)\otimes\cdots\otimes\mathcal{F}^{(p_k)}(u_k)$ *be a tensor product of* $\mathcal{E}_n$*-Fock modules. Assume that* $u_1,\dots,u_k,q_1,q_2$ *are algebraically independent over* $\mathbb{Q}$*, in particular, that* W *is irreducible.*

Let V *be an* $\mathcal{E}_n$ *module such that*

- V *is an irreducible lowest weight* $\mathcal{E}_n$*-module* V *with lowest weight vector* v*.*

- *W and V have the same graded character in the principal gradation. Let $w_1, \dots, w_k$ be a basis of the subspace of V of vectors of the principal degree one. We choose a basis consisting of eigenvectors of operators K_i and $H_{i,1}$, $i = 0, \dots, n-1$.*
- *The eigenvalues of K_i, $H_{i,1}$, $i = 0, \dots, n-1$, on vectors $v, w_1, \dots, w_k$ coincide with the corresponding eigenvalues of vectors $|\emptyset\rangle^{\otimes k}$, $|\emptyset\rangle^{\otimes i-1} \otimes |\{1\}\rangle \otimes |\emptyset\rangle^{\otimes k-i}$, $i = 1, \dots, k$, in W.*

Then $\mathcal{E}_n$-modules V and W are isomorphic.

Proof. Let $K_i v = q^{-k_i} v$. Then k_i is the number of Fock representations $\mathcal{F}^{(p_j)}(u_j)$ in W such that $p_j = i$. Then the number of w_j such that $K_i w_j = q^{-k_i+2} w_j$ is k_i. It follows that the i-th component of the lowest weight of v has at most k_i zeros and at most k_i poles.

Therefore the lowest weight of V coincides with that of a product of vacuum Macmahon modules $\mathcal{M}^{(p_1)}(\tilde{u}_1, \kappa_1) \otimes \mathcal{M}^{(p_2)}(\tilde{u}_2, \kappa_2) \otimes \cdots \otimes \mathcal{M}^{(p_k)}(\tilde{u}_k, \kappa_k)$, see [FJMM2], for some levels κ_i and some evaluation points parameters $\tilde{u}_j$. On the other hand, for the Fock module $\mathcal{F}^{(i)}(u)$, the difference of eigenvalues of $H_{i,1}$ on $|\emptyset\rangle$ and on $|(1)\rangle$ is given by $-(q+q^{-1})u$. The same holds also for the Macmahon module $\mathcal{M}^{(i)}(u, \kappa)$. From the hypothesis of the lemma, we then conclude that $\{\tilde{u}_i\}_{i=1}^k = \{u_i\}_{i=1}^k$. In particular, the tensor product of Macmahon modules is well-defined.

Note that given lowest weight of V, the choice of the Macmahon modules (the choice of κ_j) is not unique, it is defined by the choice of pairing up the factors in the numerator with factors in the denominator. Since zeros in the denominator are algebraically independent, for each factor in the numerator, there is at most one zero in the denominator, such that the corresponding Macmahon module is not irreducible. Let us choose the pairing such that as many Macmahon modules as possible are not irreducible. We can also choose the order of factors in such a way that the tensor product is cyclic.

Then in the tensor product we need to have k linearly independent singular vectors of principal degree 2. One can see that it is possible only if $\kappa_i = q$ for all i. The lemma follows. □

Note that the subalgebras $\mathcal{E}_{n-1}^{n-1|0}$ and $\mathcal{E}_1^{n-1||0}$ are not Hopf subalgebras. However, the decomposition formula looks as a tensor product formula.

Theorem 4.11. *We have an isomorphism of $\mathcal{E}_{n-1}^{n-1|0} \otimes \mathcal{E}_1^{n-1||0}$ modules*

$$\bigotimes_{j=1}^{k} \mathcal{F}^{(n,p_j)}(u_j) = \bigoplus_{i_1,\dots,i_k=0}^{n-2} \bigoplus_{\substack{s_1,\dots,s_k \in \mathbb{Z}, \\ s_j \equiv p_j + i_j (\operatorname{mod} n-1)}} x^{\sum_{j=1}^k \rho(s_j,i_j,p_j)} z^{\sum_{j=1}^k w(s_j,i_j,p_j)}$$

$$\times \left(\bigotimes_{j=1}^{k} \mathcal{F}^{(n-1;n-i_j-1)}(-q q_1^{\frac{ns_j-p_j}{n-1}} u_j) \right) \boxtimes \left(\bigotimes_{j=1}^{k} \mathcal{F}^{(1)}(-q q_3^{ns_j-p_j} u_j) \right). \tag{4.7}$$

Proof. Clearly it is enough to show that the vectors $\otimes_{j=1}^k |\Lambda^{s_j,i_j}\rangle$ are lowest weight vectors ?with respect to $\mathcal{E}_{n-1}^{n-1|0}$ and $\mathcal{E}_1^{n-1||0}$ and to compute their lowest weights in accordance with (4.7).

Recall that we defined a basis of the Fock space $|\Lambda^{s,i}_{\lambda,\mu}\rangle$, see proofs of Theorems 4.1 and 4.9. Consider the basis of $\bigotimes_{j=1}^{k} \mathcal{F}^{(n,p_j)}(u_j)$ given by $\bigotimes_{j=1}^{k} |\Lambda^{s_j,i_j}_{\lambda_j,\mu_j}\rangle$. Here $i_j = 0, \dots, n-2$, $s_j \in \mathbb{Z}$ and λ_j, μ_j are arbitrary partitions.

Consider $\mathcal{E}_{n-1}^{n-1|0}$. By definition, any given $\tilde{g} \in \mathcal{E}_{n-1}^{n-1|0}$ acts as a limit of operators $q_1^{rs} T^s g$ for some r and some $g \in \mathcal{E}_n$. We have $(T^s g)v = T^s \circ g \circ T^{-s} v$. Also, note that action of T in the tensor product of modules is given by the tensor product of action of T in factors.

Therefore to compute action of $\tilde{g}$ we have to apply g to an element shifted far to the stable zone, that is to $\otimes_{j=1}^k |\Lambda_{\lambda_j,\mu_j}^{s_j,i_j}\rangle$ with large s_j.

If all s_j are large the vector $\otimes_{j=1}^k |\Lambda_{\lambda_j,\mu_j}^{s_j,i_j}\rangle$ corresponds to a vector given by a product of partitions of the type shown on Figure 2. Then the operator g acting on this vector produces a linear combination of vectors corresponding to the product of partitions where a fixed total amount of boxes has been removed and added.

For any fixed r, there exist some $M > 0$ such that if all $s_j > M$, there is no vector of degree less then $\otimes_{j=1}^k |\Lambda^{s_j,i_j}\rangle$ which corresponds to a set of partitions that can be obtained from Λ^{s_j,i_j} by a change of r boxes. It follows that this vector is a lowest weight vector with respect to $\mathcal{E}_{n-1}^{n-1|0}$ and similarly with respect to $\mathcal{E}_1^{n-1||0}$.

Moreover, for the same reason, the span of all vectors $\otimes_{j=1}^k |\Lambda_{\lambda_j,\mu_j}^{s_j,i_j}\rangle$ with fixed s_j, i_j and all tuples of partitions λ_j, μ_j is stable under the action of the $\mathcal{E}_{n-1}^{n-1|0}$.

To determine the lowest weight of $\otimes_{j=1}^k |\Lambda^{s_j,i_j}\rangle$, we check the conditions of Lemma 4.10. By a similar argument as above, the vectors of principal degree 1 are obtained from $\otimes_{j=1}^k |\Lambda^{s_j,i_j}\rangle$ by adding one box or a domino. The action of $H_{i,1}$ is computed then as before. □

4.4. **The modules $\mathcal{N}_{\alpha,\beta}^{(p)}$.** In this section we deduce the decomposition of modules $\mathcal{N}_{\alpha,\beta}^{(p)}(u)$ from Theorem 4.11.

Recall that $\mathcal{N}_{\alpha,\beta}^{(p)}(u)$ is a submodule in the tensor product $\mathcal{F}^{(p_1)}(u_1) \otimes \cdots \otimes \mathcal{F}^{(p_k)}(u_k)$, see (2.18), where p_i, u_i are given by (2.17). Recall also, that a_i, b_i are given by (2.20).

Assume that $p_i \in \{0, \ldots, n-1\}$, and define the numbers m_i by

$$p_i = p_{i+1} + b_i - a_i - m_i n. \tag{4.8}$$

Theorem 4.12. *We have an isomorphism of $\mathcal{E}_{n-1}^{n-1|0} \otimes \mathcal{E}_1^{n-1||0}$ modules*

$$\begin{aligned}\mathcal{N}_{\alpha,\beta}^{(n;p)}(u) = \bigoplus_{i_1,\ldots,i_k=0}^{n-2} \bigoplus_{\substack{s_1,\ldots,s_k\in\mathbb{Z},\\ s_j\equiv p_j+i_j \,(\mathrm{mod}\, n-1)}} & x^{\sum_{j=1}^k \rho(s_j,i_j,p_j)} z^{\sum_{j=1}^k w(s_j,i_j,p_j)} \\ & \times \mathcal{N}_{\gamma(s),\beta}^{(n-1;n-1-i_k)}(-qq_1^{\frac{ns_1-p_1}{n-1}} u) \boxtimes \mathcal{N}_{\gamma(s),\alpha}^{(1)}(-qq_3^{ns_1-p_1} u),\end{aligned} \tag{4.9}$$

where

$$l_j(s) = s_j - s_{j+1} + m_j, \qquad \gamma_j(s) - \gamma_{j+1}(s) = l_j(s),$$

and the summation is over $s_1, \ldots, s_k$ such that $l_j(s) \geq 0$, $j = 1, \ldots, k-1$.

Proof. Theorem 4.12 is deduced from Theorem 4.11. The module $\mathcal{N}_{\alpha,\beta}^{(n;p)}(u)$ is the submodule of a tensor product of Fock modules, see (2.18), which is described by conditions (2.19).

Therefore, we start from the generic tensor product of Fock modules as in Theorem 4.11. Note that the action of all operators depends on evaluation parameters algebraically. Therefore we can specialize the evaluation parameters to any values where the tensor product is well-defined. Let us specialize the evaluation parameters as in (2.17). Then we discard the representations of $\mathcal{E}_{n-1}^{n-1|0}\otimes\mathcal{E}_1^{n-1||0}$ whose lowest weight vectors do not satisfy (2.19). Next, we check that the surviving lowest weight vectors have exactly lowest weights of $\mathcal{N}^{(n-1;n-1-i_k)}_{\gamma(s),\beta}(-qq_1^{\frac{ns_1-p_1}{n-1}}u)\boxtimes\mathcal{N}^{(1)}_{\gamma(s),\alpha}(-qq_3^{ns_1-p_1}u)$. It shows that the left hand side of (4.9) contains the right hand side. It remains to see that both sides coincide which is readily done in the stable limit of large enough s_j. ☐

We do the following change of summation variables in formula (4.9). Let $y=(y_1,\dots,y_{k-1})$ be a vector with coordinates:

$$y_r=(ns_r-p_r)-(ns_{r+1}-p_{r+1})=nl_r+a_r-b_r,\qquad r=1,\dots,k-1. \tag{4.10}$$

Let also

$$\bar y=\sum_{r=1}^k(ns_r-p_r)=n(n-1)j+(n-1)\sum_{r=1}^k p_r+n\sum_{r=1}^k i_r,$$

where

$$j=\frac{1}{n-1}\sum_{r=1}^k(s_r-i_r-p_r)\in\mathbb{Z}.$$

Define

$$w(j,i,p,\alpha,\beta)=\sum_{r=1}^k\left(\eta^{(p_r)}+\eta_{i_r}\right)-j\eta,$$

where we used the notation $i=(i_1,\dots,i_k)$.

Let C_k be the Cartan matrix of $\mathfrak{sl}_k$. We have $(C_k^{-1})_{ir}=(C_k^{-1})_{ri}=i(k-r)/k$, where $1\le i\le r\le k-1$.

Corollary 4.13. *We have an isomorphism of $\mathcal{E}_{n-1}^{n-1|0}\otimes\mathcal{E}_1^{n-1||0}$ modules*

$$\mathcal{N}^{(n;p)}_{\alpha,\beta}(u)=\bigoplus_{i_1,\dots,i_k=0}^{n-2}\bigoplus_{j\in\mathbb{Z}}\left(x^{\frac{\bar y^2}{2(n-1)k}+\frac{\bar y}{2}+\frac{1}{2(n-1)}\sum_{r=1}^k(n-i_r-1)i_r}z^{w(j,i,p,\alpha,\beta)}\right)$$

$$\times\left(\bigoplus_{l_1,\dots,l_{k-1}=0}^{\infty}x^{\frac{y^tC_k^{-1}y}{2(n-1)}}\mathcal{N}^{(n-1;n-1-i_k)}_{\gamma(l),\beta}(-qq_1^{\frac{s}{n-1}}u)\boxtimes\mathcal{N}^{(1)}_{\gamma(l),\alpha}(-qq_3^{s}u)\right), \tag{4.11}$$

where

$$s=\frac{1}{k}(\bar y-\sum_{r=1}^{k-1}(r-k)y_r),$$

while the summation is over $l_1,\dots,l_{k-1}$ such that

$$l_r+a_r\equiv i_r-i_{r+1}+b_r\pmod{n-1}$$

and

$$\sum_{r=1}^{k-1}rl_r\equiv(n-1)j+\sum_{r=1}^k(i_r+p_r)+\sum_{r=1}^{k-1}rm_r\pmod{k}.$$

Proof. Formula (4.11) is obtained (4.9) by the straightforward change of variables. ☐

4.5. **Macmahon modules.** In this section we discuss the $k\to\infty$ limit of formula (4.11).

Fix partitions α,β. Adding zero parts we can think that α,β have k parts if k is sufficiently large. Then, one can define the analytic continuation of the module $\mathcal{N}^{(n;p)}_{\alpha,\beta}(u)$ with respect to parameter k, the result is the so called Macmahon module $\mathcal{M}^{(n;p)}_{\alpha,\beta,\emptyset}(u,K)$. The Macmahon module $\mathcal{M}^{(n;p)}_{\alpha,\beta,\emptyset}(u,K)$ is an admissible tame lowest weight $\mathcal{E}_n$-module of level K which is irreducible for generic values of K, and whose basis is labeled by plane partitions with boundary conditions $\alpha,\beta,\emptyset$, see [FJMM2].

We conjecture the decomposition of $\mathcal{M}^{(n;0)}_{\alpha,\beta,\emptyset}(u,K)$ as $\mathcal{E}^{n-1|0}_{n-1}\otimes\mathcal{E}^{n-1||0}_{1}$ module based on (4.11) as follows.

Fix non-negative integers $l_1,\dots,l_t,l'_1,\dots,l'_t$. Let L_k be the vector with k components of the form $L=(l_1,\dots,l_t,0,\dots,0,l'_t,l'_{t-1},\dots,l'_1)$.

Similarly to the inductive construction of the Macmahon module, we expect the following.

Conjecture 4.14. *There exists an $\mathcal{E}_n$ lowest weight admissible tame module $\mathcal{M}^{(n;p),\gamma(l')}_{\gamma(l),\beta}(u,K)$ of level K which is the analytic continuation of $\mathcal{N}^{(p)}_{\gamma(L_k),\beta}(u)$ with respect to k.*

Note that the module $\mathcal{M}^{(n;p),\gamma(l')}_{\gamma(l),\beta}(u,K)$ does not change if sequences l and l' are extended by finitely many zeros. Namely, if $\tilde l=(l_1,\dots,l_t,0)$ and $\tilde l'=(l'_1,\dots,l'_t,0)$ then $\mathcal{M}^{(n;p),\gamma(l')}_{\gamma(l),\beta}(u,K)=\mathcal{M}^{(n;p),\gamma(l')}_{\gamma(\tilde l),\beta}(u,K)=\mathcal{M}^{(n;p),\gamma(\tilde l')}_{\gamma(l),\beta}(u,K)$.

If $l'=\emptyset$, this module is the Macmahon module: $\mathcal{M}^{(n;p),\gamma(\emptyset)}_{\gamma(l),\beta}(u,K)=\mathcal{M}^{(n;p)}_{\gamma(l),\beta}(u,K)$.

Recall that the parameters a_i,b_i are given by (2.20), p_i by (2.17), m by (4.8), and y by (4.10). If the partitions α,β have t non-zero parts, we set $a_r=b_r=m_r=p_r=0$ for $r>t$.

Let G be the Gordon matrix given by $G_{i,j}=\min\{i,j\}$.

Then we have the following decomposition formula.

Conjecture 4.15. *We have an isomorphism of $\mathcal{E}^{n-1|0}_{n-1}\otimes\mathcal{E}^{n-1||0}_{1}$ modules*

$$\mathcal{M}^{(n;0)}_{\alpha,\beta,\emptyset}(u,K)=\bigoplus_{\substack{0\le i_r,i'_r\le n-2\\ r=1,2,\cdots}}\ \bigoplus_{j=-\infty}^{+\infty} x^{\rho(i,i';j)}z^{w(i,i';j)}$$
$$\times\left(\bigoplus_{\substack{l_r,l'_r,\cdots\ge0\\ r=1,2,\cdots}} x^{\frac{y^tGy+(y')^tGy'}{2(n-1)}}\mathcal{M}^{(n-1;n-1-i'_1),\gamma(l')}_{\gamma(l),\beta}(-qq_1^{\frac{s}{n-1}}u,K)\boxtimes\mathcal{M}^{(1),\gamma(l')}_{\gamma(l),\alpha}(-qq_3^{s}u,K)\right),$$

where $y'_r=nl'_r$, $s=\sum_{r\ge1}y_r$,

$$\rho(i,i';j)=\tfrac{n(n-1)}{2}j+\tfrac{(n-1)}{2}\sum_{r\ge1}p_r+\tfrac{n+1}{2}\sum_{r\ge1}(i_r+i'_r)-\tfrac{1}{2(n-1)}\sum_{r\ge1}(i_r^2+(i'_r)^2),$$
$$w(i,i';j)=\textstyle\sum_{r\ge1}\left(\eta^{(p_r)}+\eta_{i_r}+\eta_{i'_r}\right)-j\eta,$$

and the summation is over $j\in\mathbb{Z}$, $i_r,i'_r\in\{0,\cdots,n-2\}$ and non-negative integers l_r,l'_r, $r\ge1$, such that only finitely many i_r,i'_r,l_r,l'_r are non-zero and

$$l_r+a_r\equiv i_r-i_{r+1}+b_r\ (\mathrm{mod}\,n-1),\qquad l'_r=i'_{r+1}-i'_r\ (\mathrm{mod}\,n-1),$$

and

$$\sum_{r\geq 1} rl_r - \sum_{r\geq 1} rl'_r = (n-1)j + \sum_{r\geq 1}(i_r + p_r + i'_r) + \sum_{r\geq 1} rm_r.$$

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References

[AGT] L. F. Alday, D. Gaiotto, and Y. Tachikawa, *Liouville correlation functions from four-dimensional gauge theories*, Lett. Math. Phys. **91** (2010) 167?-197

[BS] I. Burban and O. Schiffmann, *On the Hall algebra of an elliptic curve I*, Duke Math. J. **161** (2012) no.7, 1171–1231

[D] J. Dixmier, Enveloping Algebras, AMS, 1977

[FFJMM1] B. Feigin, E. Feigin, M. Jimbo, T. Miwa and E. Mukhin, *Quantum continuous $\mathfrak{gl}_\infty$: Semi-infinite construction of representations*, Kyoto J. Math. **51** (2011), no. 2, 337–364

[FFJMM2] B. Feigin, E. Feigin, M. Jimbo, T. Miwa and E. Mukhin, *Quantum continuous $\mathfrak{gl}_\infty$: Tensor product of Fock modules and $\mathcal{W}_n$ characters*, Kyoto J. Math. **51** (2011), no. 2, 365–392

[FFNR] B. Feigin, M. Finkelberg, A. Negut, and L. Rybnikov, *Yangians and cohomology rings of Laumon spaces*, Selecta Math. (N.S.) **17** (2011), no. 3, 573–607

[FF] B. Feigin, E. Frenkel, Quantization of the Drinfel'd-Sokolov reduction. Phys. Lett. B **246** (1990), no. 1-2, 75?-81

[FJMM1] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, *Quantum toroidal $\mathfrak{gl}_1$ algebra : plane partitions*, *Kyoto J. Math.* **52** (2012) 621–659

[FJMM2] B. Feigin, M. Jimbo, T. Miwa, and E. Mukhin, *Representations of quantum toroidal $\mathfrak{gl}_n$.*, J. Algebra **380** (2013), 78?-108

[FHHSY] B. Feigin, K. Hashizume, A. Hoshino, J. Shiraishi and S. Yanagida, *A commutative algebra on degenerate $\mathbb{C}P^1$ and Macdonald polynomials*, J. Math. Phys. **50** (2009), no. 9, 095215, 1–42

[FO] B. Feigin and A. Odesskii, *Vector bundles on an elliptic curve and Sklyanin algebras*, Topics in quantum groups and finite-type invariants, 6584, Amer. Math. Soc. Transl. Ser. 2, **185** (1998), 65?-84

[FT] B. Feigin and A. Tsymbaliuk, *Heisenberg action in the equivariant K-theory of Hilbert schemes via Shuffle Algebra*, Kyoto J. Math. **51** (2011), no. 4, 831?-854

[GKV] V. Ginzburg, M. Kapranov, and E. Vasserot *Langlands reciprocity for algebraic surfaces*, Math. Res. Lett. **2** (1995), no. 2, 147–160

[M99] K. Miki, *Toroidal braid group action and an automorphism of toroidal algebra $U_q(\mathfrak{sl}_{n+1,tor})$*, Lett. Math. Phys. **47** (1999), 365–378

[M00] K. Miki, *Representations of quantum toroidal algebra $U_q(\mathfrak{sl}_{n+1,tor})$ $(n \geq 2)$*, Journal of Math. Phys. **41** (2000), no. 10, 7079–7098.

[M01] K. Miki, *Quantum toroidal algebra $U_q(\mathfrak{sl}_{2,tor})$ and R matrices*, Journal of Math. Phys. **42** (2001), no. 5, 2293–2308

[M07] K. Miki, *A (q,γ) analog of the $W_{1+\infty}$ algebra*, Journal of Math. Phys. **48** (2007), no. 12, 1–35

[N] H. Nakajima, *Quiver varieties and quantum affine algebras*, translation of Sugaku **52** (2000), no. 4, 337–359; Sugaku Expositions **19** (2006), no. 1, 53–78

[O1] G. I. Olshanski, *Extension of the algebra U(g) for infinite-dimensional classical Lie algebras g, and the Yangians* $Y(gl(m))$. Soviet Math. Dokl. **36**, no. 3 (1988), 569–573

[O2] G. I. Olshanski, *Representations of infinite-dimensional classical groups, limits of enveloping algebras, and Yangians*, in "Topics in Representation Theory" (A. A. Kirillov, Ed.), Advances in Soviet Math. 2, AMS, Providence RI, 1991, 1?-66

[S] O. Schiffmann, *Drinfeld realization of the elliptic Hall algebra*, J. Algebraic Combin. **35** (2012), no. 2, 237?-262

[Sa] Y. Saito, *Quantum toroidal algebras and their vertex representations*, Publ. RIMS, Kyoto Univ. **34** (1998) no.2, 155–177

[STU] Y. Saito, K. Takemura and D. Uglov, *Toroidal actions on level* 1 *modules of* $U_q(\widehat{\mathfrak{sl}}_n)$, Transform. Groups **3** (1998) no.1, 75–102

[SV1] O. Schiffmann and E. Vasserot, *The elliptic Hall algebra, Cherednik Hecke algebras and Macdonald polynomials*, Compos. Math. **147** (2011), no. 1, 188–234

[SV2] O. Schiffmann and E. Vasserot, *The elliptic Hall algebra and the equivariant K-theory of the Hilbert scheme of* $\mathbb{A}^2$, Compos. Math. **147** (2011), no. 1, 188?-234

[TU] K. Takemura and D. Uglov, *Representations of the quantum toroidal algebra on highest weight modules of the quantum affine algebra of type* $\mathfrak{gl}_N$, Publ. Res. Inst. Math. Sci. **35** (1999), no. 3, 407–450

[VV1] M. Varagnolo and E. Vasserot, *Schur duality in the toroidal setting*, Comm. Math. Phys. **182** (1996), no. 2, 469–483

[VV2] M. Varagnolo and E. Vasserot, *Double-loop algebras and the Fock space*, Invent. Math. **133** (1998), no. 1, 133?159

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