Connection problem for the sine-Gordon/Painlevé III tau function and irregular conformal blocks

A. Its[1], **O. Lisovyy**[2], **Yu. Tykhyy**[3]

[1] Department of Mathematical Sciences, Indiana University-Purdue University, 402 N. Blackford St., Indianapolis, IN 46202-3267, USA

[2] Laboratoire de Mathématiques et Physique Théorique CNRS/UMR 7350, Université de Tours, 37200 Tours, France

[3] Bogolyubov Institute for Theoretical Physics, 03680, Kyiv, Ukraine

E-mail: `itsa@math.iupui.edu`, `lisovyi@lmpt.univ-tours.fr`, `tykhyy@bitp.kiev.ua`

Abstract. The short-distance expansion of the tau function of the radial sine-Gordon/Painlevé III equation is given by a convergent series which involves irregular $c = 1$ conformal blocks and possesses certain periodicity properties with respect to monodromy data. The long-distance irregular expansion exhibits a similar periodicity with respect to a different pair of coordinates on the monodromy manifold. This observation is used to conjecture an exact expression for the connection constant providing relative normalization of the two series. Up to an elementary prefactor, it is given by the generating function of the canonical transformation between the two sets of coordinates.

1. Introduction

Since the late seventies, there has been an explosive growth in the application of Painlevé equations to an extraordinary variety of problems, both mathematical and physical. Of particular importance is the role that the Painlevé functions play in problems related to random matrices and random processes, orthogonal polynomials, string theory, and in exactly solvable statistical mechanics and quantum field models.

Most recently, Painlevé transcendents have been understood to play a role in the two-dimensional conformal field theory [17, 20, 28, 35]. This connection produces, in particular, a new type of representation of the Painlevé functions in the form of explicit series which have a meaning of the Fourier transforms of the $c = 1$ Liouville conformal blocks and their irregular counterparts [17, 18]. It also proved to be very useful in tackling one of the most difficult problems of the analytic theory of Painlevé functions, the problem of evaluation of the constant factors in the asymptotics of the Painlevé tau functions [21].

Usually, it is not the Painlevé functions *per se* but the related *tau functions* that are objects which actually appear in applications, notably in the description of the

This is the author's manuscript of the article published in final edited form as:
Its, A., Lisovyy, O., & Tykhyy, Y. (2015). Connection problem for the sine-Gordon/Painlevé III tau function and irregular conformal blocks. International Mathematics Research Notices, 2015(18), 8903-8924. http://dx.doi.org/10.1093/imrn/rnu209

correlation functions of integrable statistical mechanics and quantum field models. The main analytic issue in these applications is the large time and distance asymptotics with a particular focus on the relevant connection formulae between different critical expansions and the evaluation of the above mentioned constant factors. The latter, very often, contain the most important information of the models in question. Starting from the seminal works of Onsager and Kaufman on the Ising model whose mathematical needs led to the birth of the Strong Szegö Theorem in the theory of Toeplitz matrices (see e.g. [10] for more on the history of the matter), the evaluation of the constant terms in the asymptotics of different correlation and distribution functions of the random matrix theory and of the theory of solvable statistical mechanics models has always been a great challenge in the field. In addition to the Strong Szegö Theorem we mention the work of Tracy [36], where the "constant problem" related to the Ising model was solved. This was the first rigorous solution of a "constant problem" for Painlevé equations (a special Painlevé III transcendent). Further developments [26], [12], [11] were devoted to the rigorous derivation of Dyson's constant [13] in the asymptotics of the sine-kernel determinant describing the gap probability in large random matrices. This determinant represents the tau function of a particular solution of the fifth Painlevé equation. Other "constant" problems were also considered and solved in the works [2], [6], [9], [27] and [1].

A natural framework for the global asymptotic analysis of the solutions of Painlevé equations is provided by the Isomonodromy-Riemann-Hilbert method, see monograph [14] for more detail and for the history of the subject. However, the Riemann-Hilbert technique addresses directly the Painlevé functions and not the associated tau functions. The latter are logarithmic antiderivatives of certain rational functions of Painlevé transcendents — the Hamiltonians of Painlevé equations. Hence the problem: one should be able to evaluate integrals of certain combinations of Painlevé transcendents and their derivatives. So far, this problem has been successfully handled only for very special solutions of the Painlevé equations. The tau functions evaluated on these solutions admit additional representations in terms of certain Fredholm or Toeplitz determinants. It is this extra property that, in spite of the difference in the approaches, is a principal reason of the success in the solution of the "constant problem" in all of the works mentioned in the previous paragraph.

The situation with the "constant problem" has changed very recently. Conformal block representations of the generic Painlevé VI tau function [17] imply that the connection constants satisfy certain functional relations. They have been solved in [21] using the hamiltonian interpretation of the monodromy parametrization of Painlevé VI transcendents provided via the Isomonodromy-Riemann-Hilbert framework. Indeed, as every other Painlevé equation, the sixth Painlevé equation is a classical hamiltonian system, and the monodromy data form a canonically conjugated pair of coordinates on the space of its solutions. The parametrization of the asymptotics of the tau function in terms of monodromy data implies in turn that the correspondence between the asymptotic parameters at the different critical points can be interpreted as a canonical map between different sets of canonical coordinates. The asymptotic constants in question can be thought of as the *generating functions* of these maps. This observation has allowed to formulate detailed conjectures about the dependence of the Painlevé VI asymptotic constants on the monodromy data [21], i.e., practically, to evaluate them in explicit form.

In the present paper we extend the method of [21] to a special case of the third Painlevé equation, denoted as Painlevé III_3 or $\mathrm{P}_{\mathrm{III}}\left(D_8\right)$ [33],

$$u_{rr} + \frac{u_r}{r} + \sin u = 0, \tag{1.1}$$

which appears in physical applications as a reduction of the sine-Gordon equation.

Following the definitions of [23], the Painlevé III_3 tau function is given by the formula

$$\frac{d}{dr} \ln \tau \left(2^{-12} r^4\right) = \frac{r}{4} \left[\left(\frac{i u_r}{2} + \frac{1}{r} \right)^2 + \frac{\cos u}{2} \right]. \tag{1.2}$$

We are concerned with two representations of the tau function. The first one is its short-distance ($r \to 0$) expansion. This expansion has been known already [18, 19] and is given by the convergent series

$$\tau(t) = \sum_{n \in \mathbb{Z}} e^{4\pi i n \eta} \mathcal{F}\left(\sigma + n, t\right), \tag{1.3}$$

where the function $\mathcal{F}\left(\sigma, t\right)$ is the irregular $c = 1$ Virasoro conformal block normalized as indicated in the equations (3.4)–(3.5) below. The parameters σ and η are related to the monodromy of the associated linear system, see Section 2 for their exact definition. The AGT correspondence identifies $\mathcal{F}\left(\sigma, t\right)$ with an instanton partition function given by a sum over pairs of Young diagrams [29, 30], see formulae (3.6)–(3.7). One of our results (Proposition 1) is the proof of convergence of this sum and the series (1.3).

The second representation corresponds to the long-distance ($r \to \infty$) expansion. We suggest that the *same* function $\tau(t)$ can be represented as

$$\tau\left(2^{-12} r^4\right) = \chi\left(\sigma, \nu\right) \sum_{n \in \mathbb{Z}} e^{4\pi i n \rho} \mathcal{G}\left(\nu + i n, r\right). \tag{1.4}$$

The structure and normalization of the function $\mathcal{G}\left(\nu, r\right)$ is described in the equations (3.14)–(3.15) of the main text. The parameters ν and ρ are explicit elementary functions of σ and η, see (3.10) and (3.16). The series (1.4) is our *first conjecture* regarding the structure of the long-distance asymptotics of the Painlevé III_3 tau function. It is based on the analysis of several first terms in the long-distance asymptotics of the function $u(r)$ which are available via the Riemann-Hilbert method. The irregular expansion (1.4) suggests the existence of novel distinguished bases in spaces of irregular conformal blocks [16]. An important open problem would be to understand the representation-theoretic origin of such bases.

The quantity $\chi(\sigma,\nu)$, independently of the conjecture (1.4), has a well-defined meaning of the constant prefactor in the long-distance asymptotics of the tau function. Our *second conjecture* concerns this prefactor. The periodicity of asymptotic expansions around both singularities enables one to develop a hamiltonian-based scheme similar to Painlevé VI arguments outlined above. This scheme produces an explicit conjectural expression for the constant $\chi(\sigma,\nu)$ (equation (4.12) in the end of the paper), which is further confirmed by numerics. In our future work we hope to be able to prove this conjecture using the theory of non-isomonodromy deformations of the tau functions of integrable models, recently developed in [5].

We want to conclude this introduction by another remark on the structure of the expansions (1.3) and (1.4). The periodicity with respect to monodromy data is completely lost in the leading terms of the asymptotics of the function $u(r)$ at $r=0$ and at $r=\infty$. The series (1.3) and (1.4) tell us that full asymptotic expressions can be obtained by summing up all the integer shifts of the leading terms. More precisely, in order for this procedure to work, one has to switch from the Painlevé function $u(r)$ to the tau function $\tau(t)$ and replace the leading term of the asymptotics by the appropriate conformal block. It is interesting to note that very similar effect of "loosing" the periodicity with respect to the relevant parameter and its "recovering" by summing up over all the integer shifts had already emerged in the 1991's Basor-Tracy conjecture [3] concerning the general Fisher-Hartwig asymptotics in the theory of Toeplitz determinants. For more details, we refer the reader to [8] where the Basor-Tracy conjecture was proven and [34] where it was used in a concrete application.

Acknowledgements. We are grateful to N. Iorgov, V. Novokshenov and J. Teschner for helpful discussions. The present work was supported by the NSF Grant DMS-1001777, Ukrainian SFFR project F54.1/019 and the IRSES project "Random and integrable models in mathematical physics".

2. Background

2.1. Auxiliary linear problem

Let us briefly outline the relation of the radial sine-Gordon equation to the theory of monodromy preserving deformations of linear ODEs with rational coefficients. The reader is referred to [14, 31] for more details.

Consider the following linear system:

$$\frac{\partial\Phi}{\partial z}=A(z)\,\Phi, \tag{2.1}$$

$$A(z)=-\frac{ir^2\sigma_3}{16}-\frac{iv\sigma_1}{4z}+\frac{ie^{-\frac{iu\sigma_1}{2}}\sigma_3 e^{\frac{iu\sigma_1}{2}}}{z^2}. \tag{2.2}$$

Here $\Phi(z)$ is a multivalued 2×2 matrix function on $\mathbb{P}^1\backslash\{0,\infty\}$, the scalars r,u,v are independent of z, and $\sigma_{1,2,3}$ denote the Pauli matrices,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

The system (2.1) has two irregular singular points $z = 0, \infty$ of Poincaré rank 1. The formal fundamental solutions around these points have the form

$$\Phi^{(0)}(z) \ = e^{-\frac{iu\sigma_1}{2}} \left[1 + \sum_{k=1}^{\infty} \phi_k^{(0)} z^k \right] e^{-\frac{i\sigma_3}{z}}, \tag{2.3}$$

$$\Phi^{(\infty)}(z) = \left[1 + \sum_{k=1}^{\infty} \phi_k^{(\infty)} z^{-k} \right] e^{-\frac{ir^2\sigma_3 z}{16}}. \tag{2.4}$$

The asymptotics (2.3)–(2.4) uniquely determines the canonical solutions $\Phi_{1,2,3}^{(0,\infty)}(z)$ in the Stokes sectors $\mathcal{S}_{1,2,3}^{(0,\infty)}$ defined by

$$\begin{aligned} \mathcal{S}_k^{(0)} = & \ \left\{ z : (k-2)\pi < \arg z < k\pi, \ |z| < R \right\}, \\ \mathcal{S}_k^{(\infty)} = & \ \left\{ z : \left(k - \frac{3}{2} \right) \pi - 2\epsilon < \arg z < \left(k - \frac{1}{2} \right) \pi + 2\epsilon, \ |z| > R \right\}, \end{aligned}$$

with $k = 1, 2, 3$ and small finite $\epsilon > 0$.

2.2. Monodromy data

The set of monodromy data consists of four Stokes matrices

$$S_{k \to k+1}^{(p)} = \Phi_k^{(p)^{-1}}(z) \, \Phi_{k+1}^{(p)}(z), \qquad p = 0, \infty, \quad k = 1, 2,$$

and one connection matrix

$$C = \Phi_1^{(0)^{-1}}(z) \, \Phi_1^{(\infty)}(z).$$

The familiar triangular structure of the Stokes matrices and the symmetry $\sigma_1 A(z) \sigma_1 = -A(-z)$ of the linear system (2.1) imply that

$$S_{1\to 2}^{(0)} = \sigma_1 S_{2\to 3}^{(0)} \sigma_1 = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}, \tag{2.5}$$

$$S_{1\to 2}^{(\infty)} = \sigma_1 S_{2\to 3}^{(\infty)} \sigma_1 = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}. \tag{2.6}$$

The same symmetry constraints the form of the connection matrix. Indeed, we have the following relations

$$\sigma_1 C \sigma_1 = \left(S_{1\to 2}^{(0)} \right)^{-1} C \, S_{1\to 2}^{(\infty)} = S_{2\to 3}^{(0)} \, C \left(S_{2\to 3}^{(\infty)} \right)^{-1}. \tag{2.7}$$

Also, since $\det \Phi(z)$ is independent of z, the normalization of the formal solutions (2.3)–(2.4) implies that $\det C = 1$. Now it follows from (2.7) that in the generic case the connection matrix can be parameterized as

$$C = \frac{1}{\sin 2\pi\sigma} \begin{pmatrix} \sin 2\pi\eta & -i \sin 2\pi(\eta+\sigma) \\ i \sin 2\pi(\eta-\sigma) & \sin 2\pi\eta \end{pmatrix}, \tag{2.8}$$

and, moreover, the Stokes factors a, b in (2.5)–(2.6) are given by

$$a = b = -2i \cos 2\pi\sigma. \tag{2.9}$$

Thus the pair $(\sigma, \eta) \in \mathbb{C}^2$ determines the monodromy of the linear system (2.1). Since the connection matrix and Stokes factors remain invariant under the transformation $(\sigma, \eta) \to (-\sigma, -\eta)$, it can be assumed without loss of generality that $0 \le \Re\sigma \le \frac{1}{2}$ and $-\frac{1}{2} < \Re\eta \le \frac{1}{2}$.

2.3. Monodromy preserving deformation

The above construction defines the monodromy map $\mathfrak{m}$ from the parameter set $\mathcal{P}$ of the linear system (2.1) to the moduli space $\mathcal{M}$ of monodromy data. The triple (r, u, v) and the pair (σ, η) can be seen as local coordinates on these two spaces.

Suppose that (r, u, v) vary in such a way that the monodromy remains constant. It is convenient to consider r as a parameter and u, v as smooth functions of r. The isomonodromy condition implies that $\dfrac{\partial\Phi}{\partial r}\,\Phi^{-1}$ is a meromorphic function on $\mathbb{P}^1$. In fact, from (2.3)–(2.4) one obtains

$$\frac{\partial \Phi}{\partial r} = B\,(z)\,\Phi, \tag{2.10}$$

$$B\,(z) = -\frac{ir\sigma_3}{8}\,z - \frac{iu_r\sigma_1}{2}. \tag{2.11}$$

Matrix equations (2.1) and (2.10) provide a Lax pair for the radial sine-Gordon equation. Indeed, the zero-curvature condition $[\partial_z - A, \partial_r - B] = 0$ is equivalent to two ODEs:

$$v = r u_r, \tag{2.12}$$

$$u_{rr} + \frac{u_r}{r} + \sin u = 0. \tag{2.13}$$

These equations can be rewritten as a non-autonomous Hamiltonian system

$$u_r = \frac{\partial \mathcal{H}}{\partial v}, \qquad v_r = -\frac{\partial \mathcal{H}}{\partial u},$$

with the Hamiltonian given by

$$\mathcal{H} = \frac{v^2}{2r} - r \cos u. \tag{2.14}$$

Note that the monodromy parameters (σ, η) can be interpreted as two integrals of motion for the sine-Gordon equation (2.13). The formulae for the inverse monodromy map $\mathfrak{m}^{-1} : (\sigma, \eta) \mapsto u(r)$ in terms of explicitly defined series will be provided in Section 3.

2.4. Tau function and relation to Painlevé III

The tau function of the radial sine-Gordon equation (2.13) will be defined by

$$\frac{d}{dr}\ln\tau\left(2^{-12}r^4\right) = \frac{r}{4}\left[\left(\frac{iu_r}{2}+\frac{1}{r}\right)^2+\frac{\cos u}{2}\right] = \tag{2.15}$$
$$= -\frac{\mathcal{H}}{8}+\frac{1}{4}\frac{d}{dr}\ln re^{iu}.$$

Its logarithmic derivative $\zeta(t) = t\dfrac{d}{dt}\ln\tau(t)$ satisfies the σ-form of Painlevé III$_3$:

$$\left(t\zeta''\right)^2 = 4\left(\zeta'\right)^2\left(\zeta - t\zeta'\right) - 4\zeta'. \tag{2.16}$$

Also, setting $s = 2^{-6}r^2$, $q(s) = -e^{iu(r)}$, we get Painlevé III$_3$ in the standard form,

$$q'' = \frac{\left(q'\right)^2}{q} - \frac{q'}{s} + \frac{8\left(q^2-1\right)}{s}. \tag{2.17}$$

Converse relations between u, τ and ζ are given by

$$e^{-iu(r)} = 4r^{-1}\frac{d}{dr}r\frac{d}{dr}\ln\tau\left(2^{-12}r^4\right) = 2^{-6}r^2\,\zeta'\left(2^{-12}r^4\right). \tag{2.18}$$

It is important to emphasize that (2.15) determines the tau function up to a constant factor. In applications, however, there is often a distinguished normalization coming from the physical context which fixes this ambiguity.

3. Critical expansions

3.1. Short-distance expansion

Let us assume that $-\frac{\pi}{4}+\epsilon < \arg r < \frac{\pi}{4}-\epsilon$ and consider generic situation where the Stokes parameter σ belongs to the strip $0 < \Re\sigma < \frac{1}{2}$. The asymptotic behavior of $u(r)$ as $r \to 0$ is then given by [24, 32, 22]

$$e^{iu(r)} = -e^{4\pi i\eta}\frac{\Gamma^2\left(1-2\sigma\right)}{\Gamma^2\left(2\sigma\right)}\left(\frac{r}{8}\right)^{8\sigma-2}\left[1+o(1)\right], \tag{3.1}$$

see also [31] and [14, Chapter 14]. Subleading corrections may be calculated using the following ansatz:

$$u(r) = \alpha\ln r + \beta + \sum_{k=1}^{\infty}\sum_{l<k}\sum_{\epsilon=\pm} a_{kl}^{\epsilon}r^{2k+i\epsilon l\alpha}. \tag{3.2}$$

Here the first coefficients α, β are read off the asymptotics (3.1), whereas $a_{kl}^{\pm}$ can be determined recursively from the sine-Gordon equation.

It was discovered very recently [18, 19] that the whole short-distance expansion can be written down *explicitly* if one uses the tau function instead of $u(r)$. The result is given by a Fourier transform

$$\tau(t) = \sum_{n\in\mathbb{Z}} e^{4\pi i n\eta} \mathcal{F}\left(\sigma+n,t\right), \tag{3.3}$$

of the irregular $c=1$ Virasoro conformal block $\mathcal{F}\left(\sigma,t\right)$ normalized as

$$\mathcal{F}\left(\sigma,t\right) = \frac{t^{\sigma^2}\mathcal{B}\left(\sigma,t\right)}{G\left(1+2\sigma\right)G\left(1-2\sigma\right)}, \tag{3.4}$$

$$\mathcal{B}\left(\sigma,t\right) = 1 + \frac{t}{2\sigma^2} + \frac{\left(8\sigma^2+1\right)t^2}{4\sigma^2\left(4\sigma^2-1\right)^2} + \frac{\left(8\sigma^4-5\sigma^2+3\right)t^3}{24\sigma^2\left(\sigma^2-1\right)^2\left(4\sigma^2-1\right)^2} + \ldots \tag{3.5}$$

Here $G(z)$ denotes the Barnes G-function, which satisfies the functional equation $G(z+1) = \Gamma(z)G(z)$. The coefficients of the series (3.5) are rational functions of σ^2; their poles $\sigma\in\mathbb{Z}/2$ correspond to zeros of $c=1$ Kac determinant.

Irregular conformal blocks involve coherent states (Whittaker vectors) on which the annihilation part of the Virasoro algebra acts diagonally [15, 16]. We are dealing here with one of the simplest cases where conformal block coincides with the norm of such state, $\mathcal{F}\left(\sigma,t\right) = {}_W\langle\sigma|\sigma\rangle_W$, satisfying

$$\begin{aligned}
&L_0|\sigma\rangle_W = 2t\,\partial_t|\sigma\rangle_W,\\
&L_1|\sigma\rangle_W = \sqrt{t}\,|\sigma\rangle_W,\\
&L_{n\geq 2}|\sigma\rangle_W = 0.
\end{aligned}$$

Painlevé III_3 independent variable is thus related to the only nontrivial eigenvalue.

AGT duality relates conformal block $\mathcal{F}\left(\sigma,t\right)$ to the partition function of the $\mathcal{N}=2$ supersymmetric pure $SU(2)$ gauge theory. This allows to write the series in (3.5) as a Nekrasov instanton sum [29, 30] over pairs of Young diagrams:

$$\mathcal{B}\left(\sigma,t\right) = \sum_{\lambda,\mu\in\mathbb{Y}} \left(\frac{\dim\lambda\dim\mu}{|\lambda|!\,|\mu|!}\right)^2 \frac{t^{|\lambda|+|\mu|}}{\left[b_{\lambda,\mu}\left(\sigma\right)\right]^2}, \tag{3.6}$$

$$b_{\lambda,\mu}\left(\sigma\right) = \prod_{(k,l)\in\lambda}\left(\lambda'_l - k + \mu_k - l + 1 + 2\sigma\right)\prod_{(k,l)\in\mu}\left(\mu'_l - k + \lambda_k - l + 1 - 2\sigma\right). \tag{3.7}$$

Here the diagrams λ,μ are identified with partitions by $\lambda = \{\lambda_1\geq\lambda_2\geq\ldots\geq\lambda_{\ell(\lambda)}>0\}$, the size of $\lambda\in\mathbb{Y}$ is denoted by $|\lambda| = \sum_{k=1}^{\ell(\lambda)}\lambda_k$, and λ' corresponds to the transposed diagram. $\dim\lambda$ denotes the dimension of the irreducible representation of the symmetric group $S_{|\lambda|}$ associated to λ. It can be calculated using the hook-length formula

$$\frac{\dim\lambda}{|\lambda|!} = \frac{1}{\sqrt{b_{\lambda,\lambda}\left(0\right)}}.$$

The formulae (3.3)–(3.7) provide a series solution to the inverse monodromy problem for the radial sine-Gordon/Painlevé III_3 equation. They were obtained in [18, 19] as a limiting case of a similar statement for Painlevé VI equation [17]. Its CFT derivation and further generalizations were suggested in [20]. The approach of [20] is based on the braiding/fusion transformations of the Virasoro conformal blocks with degenerate fields. An alternative representation-theoretic proof of (3.3) was recently found by M. Bershtein and A. Shchechkin [4].

Conformal blocks are usually considered as formal multivariate power series. However, in the case we are interested in their meaning can be made more precise.

Proposition 1. *Let $2\sigma \notin \mathbb{Z}$. Then:*

(i) conformal block series (3.6)–(3.7) converges uniformly and absolutely on every bounded subset of $\mathbb{C}$,

(ii) tau function series (3.3) converges uniformly and absolutely on every bounded subset of the universal cover of $\mathbb{C}\backslash\{0\}$.

Proof. Introduce the notation $L = \min_{n\in\mathbb{Z}} |2\sigma - n|^2 > 0$. Then already the roughest estimate $|b_{\lambda,\mu}(\sigma)|^2 \geq L^{|\lambda|+|\mu|}$ implies that

$$|\mathcal{B}(\sigma,t)| \leq \left[\sum_{\lambda\in\mathbb{Y}} \left(\frac{\dim\lambda}{|\lambda|!}\right)^2 \left|\frac{t}{L}\right|^{|\lambda|}\right]^2.$$

Now according to the Burnside's formula $\sum_{\lambda\in\mathbb{Y},|\lambda|=n} (\dim\lambda)^2 = n!$, the previous inequality can be rewritten as $|\mathcal{B}(\sigma,t)| \leq \exp\frac{2|t|}{L}$.

The second assertion follows from the asymptotic behavior of the Barnes function coefficients in (3.4). For example, one has

$$G(1+2(\sigma+n))\,G(1-2(\sigma+n)) = \frac{G(1-2\sigma)}{G(1+2\sigma)}\left(\frac{i\sin 2\pi\sigma}{\pi}\right)^{2n} G^2(1+2(\sigma+n))\,. \tag{3.8}$$

The $n\to\infty$ asymptotics

$$\ln G(1+2(\sigma+n)) = 2n^2\ln n + O\left(n^2\right)$$

clearly dominates the factors $t^{(\sigma+n)^2}$ and $\dfrac{\sin^{2n} 2\pi\sigma}{\pi^{2n}}$ in (3.4) and (3.8). Together with the previous bound on $|\mathcal{B}(\sigma,t)|$, this suffices to complete the proof. □

In the Painlevé VI case, a slight modification of the above argument shows that the corresponding series have non-zero radius of convergence. The related issues will be discussed elsewhere.

3.2. Long-distance expansion

The formal long-distance ($r\to\infty$) expansion of $u(r)$ has the following form:

$$u(r) = \sum_{k,l=0}^{\infty}\sum_{\epsilon=\pm} b_{kl}^{\epsilon} e^{i\epsilon(2k+1)r} r^{(2k+1)\left(i\epsilon\nu-\frac{1}{2}\right)-l} + 2\pi n, \qquad n\in\mathbb{Z}, \tag{3.9}$$

where $\Im\nu \in (-1,1)$ and $r \in \mathbb{R}_{>0}$. The expansion coefficients and the parameter ν can be recursively determined from the sine-Gordon equation in terms of the first two coefficients $b_{00}^{\pm}$ which can be arbitrary. In particular, one finds that

$$\begin{aligned}
&\nu = -\frac{b_{00}^{+} b_{00}^{-}}{4}, \\
&b_{10}^{\pm} = -\frac{{b_{00}^{\pm}}^3}{2^4 \cdot 3}, \qquad b_{20}^{\pm} = \frac{{b_{00}^{\pm}}^5}{2^8 \cdot 5}, \qquad b_{30}^{\pm} = -\frac{{b_{00}^{\pm}}^7}{2^{12} \cdot 7}, \\
&b_{01}^{\pm} = \pm\frac{i b_{00}^{\pm}}{8}\left(6\nu^2 \pm 4i\nu - 1\right), \\
&b_{11}^{\pm} = \pm\frac{9 i b_{10}^{\pm}}{8}\left(2\nu^2 \pm 2i\nu - 1\right), \\
&b_{21}^{\pm} = \pm\frac{15 i b_{20}^{\pm}}{8}\left(2\nu^2 \pm 2i\nu - 1\right), \\
&b_{02}^{\pm} = -\frac{b_{00}^{\pm}}{128}\left(36\nu^4 \pm 128 i\nu^3 - 104\nu^2 \mp 56 i\nu + 9\right), \\
&b_{12}^{\pm} = -\frac{3 b_{10}^{\pm}}{128}\left(108\nu^4 \pm 296 i\nu^3 - 336\nu^2 \mp 236 i\nu + 71\right), \\
&b_{03}^{\pm} = \mp\frac{i b_{00}^{\pm}}{1024}\left(72\nu^6 \pm 624 i\nu^5 - 1788\nu^4 \mp 1824 i\nu^3 + 1522\nu^2 \pm 532 i\nu - 75\right).
\end{aligned}$$

The expression of the leading coefficients $b_{00}^{\pm}$ in terms of monodromy data (σ,η) was found in [22, 25, 32], see also [14, Chapter 14]. It reads

$$e^{\pi\nu} = \frac{\sin 2\pi\eta}{\sin 2\pi\sigma}, \tag{3.10}$$

$$b_{00}^{\pm} = -e^{\frac{\pi\nu}{2} \mp \frac{i\pi}{4}} 2^{1\pm 2i\nu} \frac{\Gamma\left(1 \mp i\nu\right)}{\sqrt{2\pi}} \frac{\sin 2\pi\left(\sigma \mp \eta\right)}{\sin 2\pi\eta}. \tag{3.11}$$

The long-distance expansion of the tau function (somewhat surprisingly) exhibits a lot more structure than the series for $u(r)$, and looks similar to the short-distance series (3.3). Using the relation (2.15) and the above first coefficients of the expansion (3.9), one obtains

$$\begin{aligned}
\tau\left(2^{-12} r^4\right) = \mathrm{const}\cdot r^{\frac14} e^{\frac{r^2}{16}} \Bigg\{ & r^{\frac{\nu^2}{2}} e^{\nu r}\left[1 + \frac{\nu\left(2\nu^2+1\right)}{8r} + \frac{\nu^2\left(4\nu^4 - 16\nu^2 - 11\right)}{128 r^2}\right] + \\
&+ \frac{i b_{00}^{+}}{4} r^{\frac{(\nu+i)^2}{2}} e^{(\nu+i)r}\left[1 + \frac{(\nu+i)\left(2(\nu+i)^2+1\right)}{8r} + \frac{(\nu+i)^2\left(4(\nu+i)^4 - 16(\nu+i)^2 - 11\right)}{128 r^2}\right] \\
&+ \frac{i b_{00}^{-}}{4} r^{\frac{(\nu-i)^2}{2}} e^{(\nu-i)r}\left[1 + \frac{(\nu-i)\left(2(\nu-i)^2+1\right)}{8r} + \frac{(\nu-i)^2\left(4(\nu-i)^4 - 16(\nu-i)^2 - 11\right)}{128 r^2}\right] \\
&- \frac{{b_{00}^{+}}^2 (\nu+i)}{64} r^{\frac{(\nu+2i)^2}{2}} e^{(\nu+2i)r}\left[1 + \frac{(\nu+2i)\left(2(\nu+2i)^2+1\right)}{8r}\right] \\
&- \frac{{b_{00}^{-}}^2 (\nu-i)}{64} r^{\frac{(\nu-2i)^2}{2}} e^{(\nu-2i)r}\left[1 + \frac{(\nu-2i)\left(2(\nu-2i)^2+1\right)}{8r}\right] + O\left(r^{\frac{\nu^2}{2}-3} e^{\nu r}\right)\Bigg\}.
\end{aligned} \tag{3.12}$$

The manifest periodic pattern leads us to the following conjecture, cf (3.3):

Conjecture 2. *Long-distance expansion of the sine-Gordon/Painlevé III$_3$ tau function is given by a convergent series*

$$\tau\left(2^{-12}r^4\right)=\chi\left(\sigma,\nu\right)\sum_{n\in\mathbb{Z}}e^{4\pi in\rho}\mathcal{G}\left(\nu+in,r\right), \tag{3.13}$$

$$\mathcal{G}\left(\nu,r\right)=e^{\frac{i\pi\nu^2}{4}}2^{\nu^2}\left(2\pi\right)^{-\frac{i\nu}{2}}G\left(1+i\nu\right)r^{\frac{\nu^2}{2}+\frac14}e^{\frac{r^2}{16}+\nu r}\mathcal{D}\left(\nu,r\right), \tag{3.14}$$

where $\mathcal{D}\left(\nu,r\right)$ admits the asymptotic expansion

$$\mathcal{D}\left(\nu,r\right)\sim 1+\sum_{k=1}^{\infty}D_k\left(\nu\right)r^{-k},\qquad r\to\infty. \tag{3.15}$$

In these formulae, $G(z)$ again stands for the Barnes G-function. The parameters (ν,ρ) are related to monodromy data by (3.10) and

$$e^{4\pi i\rho}=\frac{\sin 2\pi\eta}{\sin 2\pi\left(\sigma+\eta\right)}. \tag{3.16}$$

Conjecture 2 is expected to hold for any $\sigma,\eta\notin\mathbb{Z}/2$. The most straightforward way to test it is to recursively compute next terms in the asymptotic expansion of $u(r)$ and $\tau\left(2^{-12}r^4\right)$ using the sine-Gordon equation. On one hand, this determines the coefficients $D_k\left(\nu\right)$:

$$\begin{aligned}
D_1\left(\nu\right)&=\frac{\nu\left(2\nu^2+1\right)}{8},\\
D_2\left(\nu\right)&=\frac{\nu^2\left(4\nu^4-16\nu^2-11\right)}{128},\\
D_3\left(\nu\right)&=\frac{\nu\left(8\nu^8-108\nu^6+402\nu^4+269\nu^2-24\right)}{3\cdot 2^{10}},\\
D_4\left(\nu\right)&=\frac{\nu^2}{3\cdot 2^{12}}\left(2\nu^{10}-56\nu^8+585\nu^6-2326\nu^4-\frac{7831}{8}\nu^2+612\right),\\
&\ldots\ldots\ldots
\end{aligned}$$

On the other hand, such a procedure should reproduce (and it does indeed!) the intriguing periodicity structure of (3.13).

We have implicitly set up the tau function normalization by (3.3)–(3.5). Therefore, the constant prefactor $\chi\left(\sigma,\nu\right)$ in (3.13) can no longer be chosen arbitrarily — in fact, it is completely determined by monodromy data. The crucial point is that the explicit form of $\chi\left(\sigma,\nu\right)$ cannot be derived from the asymptotics (3.9) and connection formulae (3.10)–(3.11) alone. We address this problem in the following section.

4. Connection coefficient

The pairs (σ,η) and (ν,ρ) were used to characterize the tau function behavior as $r\to 0$ and $r\to\infty$, respectively. Now it will become convenient to pick one parameter from each pair and use (σ,ν) as local coordinates on the space $\mathcal{M}$ of monodromy data. An important drawback of such labeling is that (σ,ν) does not fix η (and hence monodromy) uniquely (mod $\mathbb{Z}$): another solution of (3.10) would be given by $\frac{1}{2}-\eta$. We will specify which of the two solutions is chosen whenever it may lead to confusion.

Let us consider analytic continuation of the connection coefficient $\chi(\sigma,\nu)$ along a path in the space of monodromy data which joins the point $(\sigma,\nu;\eta)$ and $(\sigma+1,\nu;\eta)$ or $\left(\sigma,\nu+i;\eta\pm\frac{1}{2}\right)$. The periodicity of the expansions (3.3) and (3.13) implies that $\chi(\sigma,\nu)$ should satisfy two recurrence relations:

$$\frac{\chi(\sigma+1,\nu)}{\chi(\sigma,\nu)}=e^{-4\pi i\eta}, \tag{4.1}$$

$$\frac{\chi(\sigma,\nu+i)}{\chi(\sigma,\nu)}=e^{4\pi i\rho}. \tag{4.2}$$

Before we proceed with the construction of the general solution of (4.1)–(4.2), it is useful to make the following remark.

Proposition 3. *The pairs (σ,η) and (ν,ρ), connected by (3.10) and (3.16), provide two sets of canonically conjugate local coordinates on the monodromy manifold $\mathcal{M}$. More precisely, the pullback of the symplectic form $\Omega=dv\wedge du$ under the inverse monodromy map $\mathfrak{m}^{-1}$ is given by*

$$\Omega^*=32\pi i\,d\eta\wedge d\sigma=32\pi\,d\rho\wedge d\nu. \tag{4.3}$$

Proof. In the proof we shall follow the method which was used in [7] for the evaluation of the KdV symplectic form in terms of the relevant scattering data — the PDE analog of the monodromy data.

The form Ω is preserved by the Hamiltonian flow. Hence we can calculate the expression for Ω^* in terms of (σ,η) and (ν,ρ) by considering the limits

$$\Omega^*_0=\lim_{r\to 0}\Omega^*,\qquad \Omega^*_\infty=\lim_{r\to\infty}\Omega^*,$$

respectively. For instance, it follows from the asymptotic expansion (3.2) that

$$u(r\to 0)=\alpha\ln r+\beta+o(1),\qquad v(r\to 0)=\alpha+o(1),$$

and therefore $\Omega^*=\Omega^*_0=d\alpha\wedge d\beta$. The first equality in (4.3) can now be obtained by identifying $\alpha=2i(1-4\sigma)$ and $\frac{\partial\beta}{\partial\eta}\Big|_\sigma=4\pi$ with the help of (3.1). Similarly using the long-distance expansion (3.9), one obtains

$$\Omega^*=\Omega^*_\infty=2i\,db^+_{00}\wedge db^-_{00}.$$

The second equality in (4.3) then follows from the relation $b^+_{00}b^-_{00}=-4\nu$ and the differentiation formula $\frac{\partial}{\partial\rho}\ln b^-_{00}\Big|_\nu=-4\pi i$. □

It can be inferred from (4.3), or verified by straightforward differentiation of (3.10) and (3.16), that

$$\frac{\partial\eta}{\partial\nu}=i\frac{\partial\rho}{\partial\sigma}\,. \tag{4.4}$$

Furthermore, there exists a generating function $\mathcal{W}(\sigma,\nu)$ of the canonical transformation between the two pairs such that

$$\eta = \frac{\partial\mathcal{W}}{\partial\sigma}, \qquad \rho = -i\frac{\partial\mathcal{W}}{\partial\nu}. \tag{4.5}$$

Proposition 4. *The generating function $\mathcal{W}(\sigma,\nu)$ can be chosen as*

$$8\pi^2\mathcal{W}(\sigma,\nu) = \mathrm{Li}_2\left(-e^{2\pi i\left(\sigma+\eta-\frac{i\nu}{2}\right)}\right) + \mathrm{Li}_2\left(-e^{-2\pi i\left(\sigma+\eta+\frac{i\nu}{2}\right)}\right) - (2\pi\eta)^2 + (\pi\nu)^2, \tag{4.6}$$

where $\mathrm{Li}_2(z)$ denotes the classical dilogarithm and $\eta(\sigma,\nu) = \dfrac{\arcsin\left(e^{\pi\nu}\sin 2\pi\sigma\right)}{2\pi}$. The right side of (4.6) is understood as analytic continuation from an open neighborhood of the domain $0<\eta<\sigma<\frac{1}{4}$, $\nu\in\mathbb{R}_{<0}$, where the dilogarithms are defined by their principal branches.

Proof. The differentiation formula $\mathrm{Li}_2'(z) = -z^{-1}\ln(1-z)$ implies that

$$\frac{\partial}{\partial\sigma}\left[\mathrm{Li}_2\left(-e^{2\pi i\left(\sigma+\eta-\frac{i\nu}{2}\right)}\right) + \mathrm{Li}_2\left(-e^{-2\pi i\left(\sigma+\eta+\frac{i\nu}{2}\right)}\right)\right] =$$
$$= -2\pi i\left(1+\frac{\partial\eta}{\partial\sigma}\right)\left[\ln\left(1+e^{2\pi i\left(\sigma+\eta-\frac{i\nu}{2}\right)}\right) - \ln\left(1+e^{-2\pi i\left(\sigma+\eta+\frac{i\nu}{2}\right)}\right)\right].$$

On the other hand, from the easily verified identities

$$2\cos\pi\left(\sigma+\eta\pm\frac{i\nu}{2}\right) = e^{i\pi\left(\pm\sigma\mp\eta-\frac{i\nu}{2}-4\rho\right)} \tag{4.7}$$

it follows that

$$\ln\left(1+e^{2\pi i\left(\sigma+\eta-\frac{i\nu}{2}\right)}\right) - \ln\left(1+e^{-2\pi i\left(\sigma+\eta+\frac{i\nu}{2}\right)}\right) = 4\pi i\eta.$$

Combining this with the previous relation, we immediately deduce that $\eta = \frac{\partial\mathcal{W}}{\partial\sigma}$. The identity $\rho = -i\frac{\partial\mathcal{W}}{\partial\nu}$ is proven analogously. □

Now consider the following 1-form on $\mathcal{M}$ (more precisely, on its ramified cover whose different sheets are associated with the pairs (σ,ν) projecting down to the same monodromy):

$$\omega = \sigma d\eta + i\nu d\rho.$$

The relation (4.4) implies that this 1-form is closed. It will be integrated along the paths of two types:

(i) contours γ_σ going from the point $(\sigma,\nu;\eta)$ to $(\sigma+1,\nu;\eta \bmod \mathbb{Z})$,

(ii) contours γ_ν going from the point $(\sigma,\nu;\eta)$ to $\left(\sigma,\nu+i;\eta+\frac{1}{2} \bmod \mathbb{Z}\right)$

The derivatives

$$\frac{\partial\eta}{\partial\sigma} = \cot 2\pi\sigma\tan 2\pi\eta,$$
$$\frac{\partial\eta}{\partial\nu} = i\frac{\partial\rho}{\partial\sigma} = \frac{\tan 2\pi\eta}{2},$$
$$\frac{\partial\rho}{\partial\nu} = \frac{\sin 2\pi\sigma}{4i\cos 2\pi\eta\sin 2\pi(\sigma+\eta)},$$

are periodic under analytic continuation along the contours $\gamma_{\sigma,\nu}$. As a consequence, the function $\mathcal{A}(\sigma,\nu)$ defined by

$$\mathcal{A}(\sigma,\nu)=-\int^{(\sigma,\nu)}\omega=\mathcal{W}(\sigma,\nu)-\sigma\eta-i\nu\rho \tag{4.8}$$

under appropriate prescription of the integration/analytic continuation contours (in the first and second expression, respectively) satisfies

$$\begin{aligned}
\frac{\partial}{\partial\sigma}\Big[\mathcal{A}(\sigma+1,\nu)-\mathcal{A}(\sigma,\nu)\Big] &= -\frac{\partial\eta}{\partial\sigma},\\
\frac{\partial}{\partial\nu}\Big[\mathcal{A}(\sigma+1,\nu)-\mathcal{A}(\sigma,\nu)\Big] &= -\frac{\partial\eta}{\partial\nu},\\
\frac{\partial}{\partial\sigma}\Big[\mathcal{A}(\sigma,\nu+i)-\mathcal{A}(\sigma,\nu)\Big] &= \frac{\partial\rho}{\partial\sigma},\\
\frac{\partial}{\partial\nu}\Big[\mathcal{A}(\sigma,\nu+i)-\mathcal{A}(\sigma,\nu)\Big] &= \frac{\partial\rho}{\partial\nu}.
\end{aligned}$$

In other words, partial derivatives of $4\pi i\mathcal{A}(\sigma,\nu)$ satisfy the very same recurrence relations as the partial derivatives of $\ln\chi(\sigma,\nu)$, and hence the difference of the two quantities can only be a function with periodic derivatives. The latter does not need to be periodic itself and can in principle be very complicated. However, already this first approximation provides an insight that suffices to solve (4.1)–(4.2).

Proposition 4 and the identity

$$\mathrm{Li}_2\left(e^{2\pi iz}\right)=-2\pi i\ln\hat{G}(z)-2\pi iz\ln\frac{\sin\pi z}{\pi}-\pi^2z(1-z)+\frac{\pi^2}{6},$$

where $\hat{G}(z)=\dfrac{G(1+z)}{G(1-z)}$ and $z\in(0,1)$, imply that

$$4\pi i\mathcal{A}(\sigma,\nu)=\ln\frac{\hat{G}\left(\sigma+\eta+\frac{1-i\nu}{2}\right)}{\hat{G}\left(\sigma+\eta+\frac{1+i\nu}{2}\right)}+\begin{array}{c}\text{elementary}\\ \text{functions}\end{array}.$$

Now we have the following result:

Proposition 5. *The general solution of (4.1)–(4.2) is given by*

$$\begin{aligned}\chi(\sigma,\nu;\eta)=(2\pi)^{i\nu}\exp\left\{i\pi\left(\eta^2-2\sigma\eta-\sigma^2+\eta-\sigma-\frac{\nu^2}{4}\right)\right\}\times\\
\times\frac{\hat{G}\left(\sigma+\eta+\frac{1-i\nu}{2}\right)}{\hat{G}\left(\sigma+\eta+\frac{1+i\nu}{2}\right)}\chi_{\mathrm{per}}(\sigma,\nu;\eta),\end{aligned} \tag{4.9}$$

where $\chi_{\mathrm{per}}(\sigma,\nu;\eta)$ is an arbitrary periodic function of σ and ν.

Proof. Let us denote $\mu=\eta+\sigma-\frac{i\nu}{2}$. Different possible choices of parameter values, as well as analytic continuation along $\gamma_{\sigma,\nu}$, can only shift μ by integers. Rewrite the prefactor in the right side of (4.9) as

$$(2\pi)^{i\nu}\exp\left\{i\pi\left(\mu(\mu+i\nu+1-4\sigma)+2\sigma^2-\frac{(\nu-i)(\nu+4i\sigma)}{2}\right)\right\}\frac{\hat{G}\left(\mu+\frac12\right)}{\hat{G}\left(\mu+i\nu+\frac12\right)}. \tag{4.10}$$

We claim that this expression is invariant under integer shifts of μ. Indeed, it follows from the recurrence relation $\hat{G}(z+1) = -\pi(\sin\pi z)^{-1}\hat{G}(z)$ that the shift $\mu \mapsto \mu+1$ multiplies (4.10) by $e^{i\pi(2\mu+i\nu-4\sigma)}\dfrac{\cos\pi(\mu+i\nu)}{\cos\pi\mu}$. The latter quantity is equal to 1 thanks to the identities (4.7) used in the proof of Proposition 4.

Because of the last result the shift $\sigma \mapsto \sigma+1$ amounts to multiplication of (4.10) by $e^{i\pi(-4\mu+4\sigma-2i\nu)} = e^{-4\pi i\eta}$, which proves the functional relation (4.1). To demonstrate the remaining relation (4.2), it suffices to note that the shift $\nu \mapsto \nu+i$ produces an additional factor $\dfrac{e^{i\pi(-\mu+2\sigma-i\nu)}}{2\cos\pi(\mu+i\nu)}$, which is equal to $e^{4\pi i\rho}$ by (4.7). □

Numerical experiments show that the unknown periodic function $\chi_{\text{per}}(\sigma,\nu;\eta)$ is in fact a constant. This constant can be determined with the help of an elementary solution of Painlevé III$_3$ given by

$$u(r) = 0 \bmod 2\pi \qquad \Longleftrightarrow \qquad \tau\left(2^{-12}r^4\right) = \text{const}\cdot r^{\frac{1}{4}}\, e^{r^2/16}. \tag{4.11}$$

The relevant monodromy parameters can be chosen as

$$\sigma = \eta = \frac{1}{4}, \qquad \nu = 0, \qquad \rho \to -i\infty.$$

The connection coefficient computed directly from (4.11) is equal to

$$\chi\left(\frac{1}{4},0\right) = e^{-\frac{i\pi}{8}}\chi_{\text{per}} = \frac{2^{-\frac{3}{4}}}{\sqrt{\pi}\, G^2\left(\frac{1}{2}\right)}.$$

Hence we finally arrive at

Conjecture 6. *Connection coefficient $\chi(\sigma,\nu;\eta)$ for the Painlevé III$_3$ tau function has the following expression in terms of monodromy data:*

$$\chi(\sigma,\nu;\eta) = (2\pi)^{i\nu-\frac{1}{2}}\exp\left\{i\pi\left(\eta^2-2\sigma\eta-\sigma^2+\eta-\sigma-\frac{\nu^2}{4}+\frac{1}{8}\right)\right\}\times$$
$$\times\frac{2^{-\frac{1}{4}}}{G^2\left(\frac{1}{2}\right)}\frac{\hat{G}\left(\sigma+\eta+\frac{1-i\nu}{2}\right)}{\hat{G}\left(\sigma+\eta+\frac{1+i\nu}{2}\right)}, \tag{4.12}$$

where σ, η and ν are related by (3.10).

In addition to the functional relations (4.1)–(4.2), our answer (4.12) satisfies

- periodicity property $\chi(\sigma,\nu;\eta+1) = \chi(\sigma,\nu;\eta)$,
- reflection symmetry $\chi(-\sigma,\nu;-\eta) = \chi(\sigma,\nu;\eta)$.

This reflects the corresponding symmetries of monodromy parameterization. One also has an interesting symmetry which relates connection coefficients associated to two different solutions of (3.10) with fixed σ,ν:

$$\chi(\sigma,\nu;\eta)\,\chi(\sigma,\nu;1/2-\eta) = \frac{(2\pi)^{i\nu-1}\,e^{-\frac{i\pi\nu^2}{2}}}{\sqrt{2}\,G^4\left(\frac{1}{2}\right)\hat{G}(i\nu)}. \tag{4.13}$$

This is a Painlevé III_3 counterpart of an analogous result for Painlevé VI connection coefficients, see [20, formula (4.9)].

We performed numerical tests of Conjecture 6 for random values of monodromy parameters. Fig. 1 provides an illustration of such checks. There we plot the real and imaginary part of the short-distance (blue curve) and long-distance (red curve) expansions of $e^{-\frac{r^2}{16}-\nu r}\tau\left(2^{-12}r^4\right)$. The monodromy parameters are chosen as

$$(\sigma,\nu;\eta)=(0.12-0.25i, 0.34+0.29i; 0.23+0.42i)\,.$$

We keep the terms up to $O\left(t^{\sigma^2+15}\right)$ in the short-distance expansion (3.3) of $\tau(t)$ and the terms up to $O\left(r^{\frac{\nu^2}{2}-\frac{15}{4}}e^{\frac{r^2}{16}+\nu r}\right)$ in the long-distance asymptotic expansion (3.13)–(3.15) of $\tau\left(2^{-12}r^4\right)$.

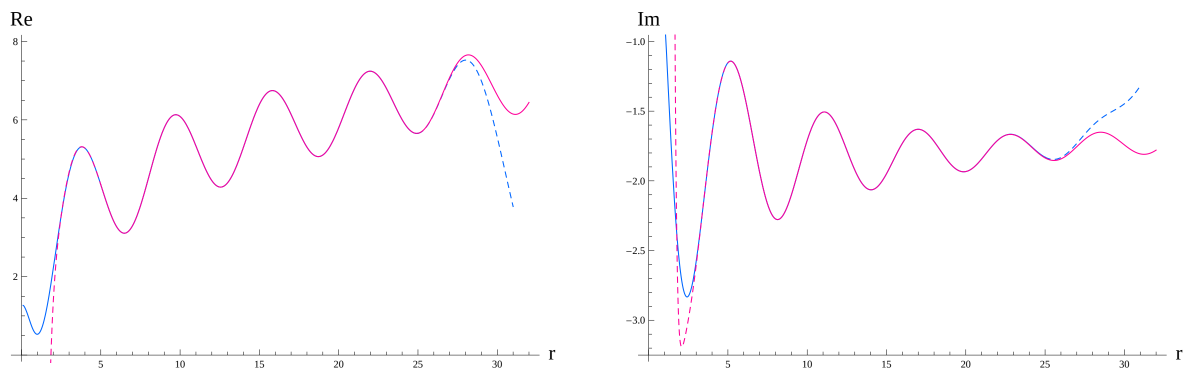

Fig. 1. Short-distance (blue curve) vs long-distance (red curve) expansion.

References

[1] J. Baik, R. Buckingham, J. DiFranco, *Asymptotics of Tracy-Widom distributions and the total integral of a Painlevé II function*, Comm. Math. Phys. **280**, (2008), 463–497.

[2] E. L. Basor, C. A. Tracy, *Some problems associated with the asymptotics of τ-functions*, Surikagaku (Mathematical Sciences) **30**, no. 3, (1992), 71–76.

[3] E. L. Basor, C. A. Tracy. *The Fisher-Hartwig conjecture and generalizations*, Phys. A **177**, (1991), 167–173.

[4] M. Bershtein, A. Shchechkin, *Bilinear equations on Painlevé tau functions from CFT*, to appear.

[5] M. Bertola, *The dependence on the monodromy data of the isomonodromic tau function*, arXiv: 0902.4716 [nlin.SI].

[6] A. M. Budylin, V. S. Buslaev, *Quasiclassical asymptotics of the resolvent of an integral convolution operator with a sine kernel on a finite interval*, (Russian) Algebra i Analiz **7**, no. 6 (1995) 79–103; translation in St. Petersburg Math. J. **7**, no. 6 (1996), 925–942.

[7] V. S. Buslaev, L. D. Faddeev, L. A. Takhtajan, *Scattering theory for Korteweg-de Vries (KdV) equation and its Hamiltonian interpretation*, Physica **18D**, (1986), 255–266.

[8] P. Deift, A. Its, I. Krasovsky, *Asymptotics of Toeplitz, Hankel and Toeplitz+Hankel Determinants with Fisher-Hartwig Singularities*, Ann. Math. **174**, (2011), 1243–1299.

[9] P. Deift, A. Its, I. Krasovsky, *Asymptotics of the Airy-kernel determinant*, Comm. Math. Phys. **278**, (2008), no. 3, 643-678.

[10] P. Deift, A. Its, I. Krasovsky, *Toeplitz matrices and Toeplitz determinants under the impetus of the Ising model. Some history and some recent results*, arXiv:1207.4990 (2012).

[11] P. Deift, A. Its, I. Krasovsky, X. Zhou, *The Widom-Dyson constant for the gap probability in random matrix theory*, J. Comput. Appl. Math. **202**, (2007), no. 1, 26–47.

[12] T. Ehrhardt, *Dyson's constant in the asymptotics of the Fredholm determinant of the sine kernel*, Comm. Math. Phys. **262**, (2006), 317–341.

[13] F. Dyson, *Fredholm determinants and inverse scattering problems*, Commun. Math. Phys. **47**, (1976), 171–183.

[14] A. S. Fokas, A. R. Its, A. A. Kapaev, V. Yu. Novokshenov, *Painlevé transcendents: the Riemann-Hilbert approach*, Mathematical Surveys and Monographs **128**, AMS, Providence, RI, (2006).

[15] D. Gaiotto, *Asymptotically free $\mathcal{N} = 2$ theories and irregular conformal blocks*, arXiv:0908.0307 [hep-th].

[16] D. Gaiotto, J. Teschner, *Irregular singularities in Liouville theory and Argyres-Douglas type gauge theories, I*, arXiv:1203.1052 [hep-th].

[17] O. Gamayun, N. Iorgov, O. Lisovyy, *Conformal field theory of Painlevé VI*, JHEP **10**, (2012), 038; arXiv:1207.0787 [hep-th].

[18] O. Gamayun, N. Iorgov, O. Lisovyy, *How instanton combinatorics solves Painlevé VI, V and III's*, J. Phys. **A46**, (2013), 335203; arXiv:1302.1832 [hep-th].

[19] N. Iorgov, O. Lisovyy, A. Shchechkin, Yu. Tykhyy, *Painlevé functions and conformal blocks*, Constr. Approx. **39**, (2014), 255–272.

[20] N. Iorgov, O. Lisovyy, J. Teschner, *Isomonodromic tau-functions from Liouville conformal blocks*, arXiv:1401.6104 [hep-th].

[21] N. Iorgov, O. Lisovyy, Yu. Tykhyy, *Painlevé VI connection problem and monodromy of $c = 1$ conformal blocks*, J. High Energy Phys. **12**, (2013), 029; arXiv:1308.4092 [hep-th].

[22] A.R. Its, V.Yu. Novokshenov, *The Isomonodromy Deformation Method in the Theory of Painlevé Equations*, Lect. Notes in Math. **1191**, Springer-Verlag, (1986).

[23] M. Jimbo, T. Miwa, K. Ueno, *Monodromy preserving deformation of linear ordinary differential equations with rational coefficients*, Physica **D2**, (1981), 306–352.

[24] M. Jimbo, *Monodromy problem and the boundary condition for some Painlevé equations*, Publ. Res. Inst. Math. Sci. **18** (1982), no. 3, 1137–1161.

[25] A.V. Kitaev, *The method of isomonodromic deformations and the asymptotics of the solutions of the "complete" third Painlevé equation*, Mat. Sbornik **134** (176), (1987), no. 3, 421–444 (English Transl.: Math. USSR-Sb. **62**, (1989), no. 2, 421–444).

[26] I. Krasovsky, *Gap probability in the spectrum of random matrices and asymptotics of polynomials orthogonal on an arc of the unit circle*, Int. Math. Res. Not. **2004**, (2004), 1249–1272.

[27] O. Lisovyy, *Dyson's constant for the hypergeometric kernel*, in "New trends in quantum integrable systems" (eds. B. Feigin, M. Jimbo, M. Okado), World Scientific, (2011), 243–267; arXiv:0910.1914 [math-ph].

[28] A. Litvinov, S. Lukyanov, N. Nekrasov, A. Zamolodchikov, *Classical conformal blocks and Painlevé VI*, arXiv:1309.4700 [hep-th].

[29] N. A. Nekrasov, *Seiberg-Witten prepotential from instanton counting*, Adv. Theor. Math. Phys. **7**, (2004), 831–864; arXiv:hep-th/0206161.

[30] N. Nekrasov, A. Okounkov, *Seiberg-Witten theory and random partitions*, arXiv:hep-th/0306238.

[31] D. G. Niles, *The Riemann-Hilbert-Birkhoff inverse monodromy problem and connection formulae for the third Painlevé transcendents*, PhD Thesis, Purdue University, (2009).

[32] V.Yu. Novokshenov, *On the asymptotics of the general real solution of the Painlevé equation of the*

third kind, Sov. Phys. Dokl. **30**, (1985), 666–668.

[33] Y. Ohyama, H. Kawamuko, H. Sakai, K. Okamoto, *Studies on the Painlevé equations. V. Third Painlevé equations of special type* $P_{\mathrm{III}}(D_7)$ *and* $P_{\mathrm{III}}(D_8)$, J. Math. Sci. Univ. Tokyo **13**, (2006), 145–204.

[34] I. V. Protopopov, D. B. Gutman, A. D. Mirlin, *Luttinger liquids with multiple Fermi edges: Generalized Fisher-Hartwig conjecture and numerical analysis of Toeplitz determinants*, Lith. J. Phys. **52**, No. 2, (2012), 165–179; arXiv: 1203.6418 [cond-mat, str-el].

[35] J. Teschner, *Quantization of the Hitchin moduli spaces, Liouville theory and the geometric Langlands correspondence I.* Adv. Theor. Math. Phys. **15**, (2011), 471–564; arXiv:1005.2846 [hep-th].

[36] C. A. Tracy, *Asymptotics of the* τ*-function arising in the two-dimensional Ising model*, Comm. Math. Phys. **142**, (1991), 297–311.