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Web-based Supplementary Materials for "Estimation of Covariate-Specific Time-Dependent ROC Curves in the Presence of Missing Biomarkers" by Shanshan Li and Yang Ning

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Appendix A: Regularity Conditions and Preliminary Results

To prove the asymptotic results, we introduce more notations. Define

$s^{(k)}(\theta, t) = E\{Y_i(t) Q_i^{\otimes k} \exp(\theta^T Q_i)\}$, $k = 0, 1, 2$;

$e(\theta, t) = s^{(1)}(\theta, t)/s^{(0)}(\theta, t)$;

$v(\theta, t) = s^{(2)}(\theta, t)/s^{(0)}(\theta, t) - \{s^{(1)}(\theta, t)/s^{(0)}(\theta, t)\}^{\otimes 2}$.

Let α_0 and θ_0 denote the true parameter values for α, θ respectively. Let $M_i(t) = N_i(t) - \int_0^t Y_i(u) \exp(\theta_0^T Q_i) d\Lambda_0(u)$ be the martingale process associated with the counting process $N_i(t)$, and $M_{Q,i}^* = \int_0^\tau [Q_i - e(\theta_0, t)] dM_i(t)$ be the martingale transform with mean $E[M_{Q,i}^*] = 0$ and variance $\Sigma = E[{M_{Q,i}}^{*\otimes 2}]$. For any function $f(\cdot)$, we use $F'(\cdot)$ to denote the first derivative. Let $\mathcal{Z}$ and $\mathcal{X}$ be the supports of Z and X, respectively.

Suppose the following regularity conditions hold:

(A1) $\Lambda_0(\tau) < \infty$.

(A2) $P\{Y(\tau) = 1\} > 0$.

(A3) $Q = (Z, X^T)^T$ is time-independent and bounded.

(A4) $I_{\theta_0} = \int_0^\tau v(\theta_0, t) s^{(0)}(\theta_0, t) d\Lambda_0(t)$ is positive definitive.

(A5) There exists $\epsilon > 0$ such that, the selection probability satisfies $\pi \geqslant \epsilon > 0$.

(A6) For $(z, x) \in \mathcal{Z} \times \mathcal{X}$, the survival function $S(t \mid z, x)$ is absolutely continuous for $t \in [0, \tau]$.

(A7) $H(u)$ is bounded and has bounded first- and second-order derivatives for $u \in (-\infty, +\infty)$.

(A8) $\Gamma = E(XX^T)$ is positive definite.

(A9) The conditional densities

$$f_1(c; t, x) = -\frac{d\text{TPR}_{\mathbb{C}}(c; t, x)}{dc} = \frac{\{1 - S(t \mid c, x)\} H'(c - \alpha_0^T x)}{\int_{-\infty}^{\infty} \{1 - S(t \mid u, x)\} dP(Z \leqslant u \mid X = x)}$$

$$f_1^*(c; t, x) = -\frac{d\text{TPR}_{\mathbb{I}}(c; t, x)}{dc} = \frac{f(t \mid c, x) H'(c - \alpha_0^T x)}{\int_{-\infty}^{\infty} f(t \mid u, x) dP(Z \leqslant u \mid X = x)}$$

$$f_0(c; t, x) = -\frac{d\text{FPR}_{\mathbb{D}}(c; t, x)}{dc} = \frac{S(t \mid c, x) H'(c - \alpha_0^T x)}{\int_{-\infty}^{\infty} S(t \mid u, x) dP(Z \leqslant u \mid X = x)}$$

exist, and $f_0(c; t, x)$ is positive for $y \in [F^{(-1)}(p) - \epsilon, F^{(-1)}(q) + \epsilon]$ for some constants p and q, $0 < p < q < 1$, and $\epsilon > 0$.

(A10) $I_{\phi_0} = nE\{U_\phi(\phi_0)\}^{\otimes 2}$ is positive definitive.

(A11) $I_{\eta_0} = nE\{U_\eta(\alpha_0, \theta_0, \eta_0)\}^{\otimes 2}$ is positive definitive.

(A12) For $k = 1, 2$, $\sup_{t \in [0,\tau]} ||n^{-1/2} \sum_{i=1}^{n} \frac{V_i - \pi_i}{\pi_i} g_k(t, W_i)|| = O_p(1)$, where

$$\begin{aligned} g_1(t, W_i) &= \left[\int \exp\{\theta_0^T Q_i \delta_i - \Lambda_0(T_i)\exp(\theta_0^T Q_i)\} f(Z_i|X_i) dZ_i\right]^{-2} \\ &\times \int Z_i M_i(t) f(Z_i|X_i) \exp\{\theta_0^T Q_i \delta_i - \Lambda_0(T_i)\exp(\theta_0^T Q_i)\} \\ &\times \left[\int \exp\{\theta_0^T Q_i(\delta_i + 1) - \Lambda_0(T_i)\exp(\theta_0^T Q_i)\} f(Z_i|X_i) dZ_i \right. \\ &\left. - \exp(\theta_0^T Q_i) \times \int \exp\{\theta_0^T Q_i \delta_i - \Lambda_0(T_i)\exp(\theta_0^T Q_i)\} f(Z_i|X_i) dZ_i\right] dZ_i, \end{aligned}$$

and

$$g_2(t, W_i) = Y_i(t) dN_i(t) \int \exp(\theta_0^T Q_i) \frac{\partial f(Z_i \mid W_i)}{\partial \Lambda_0(R_i)} dZ_i.$$

(A13) Assume that $||n^{-1/2} \sum_{i=1}^{n} \frac{V_i - \pi_i}{\pi_i} g_3(W_i)|| = O_p(1)$, where

$$g_3(W_i) = \int (Z_i - \alpha_0^T X_i) X_i \frac{\partial f(Z_i \mid W_i)}{\partial \Lambda_0(R_i)} dZ_i.$$

(A14) Assume that $\sup_{z \in \mathcal{Z}} ||n^{-1/2} \sum_{i=1}^{n} \frac{V_i - \pi_i}{\pi_i} g_4(z, W_i)|| = O_p(1)$, where

$$g_4(z, W_i) = \int I(Z_i - \alpha_0^T X_i \leqslant z) \frac{\partial f(Z_i \mid W_i)}{\partial \Lambda_0(R_i)} dZ_i.$$

Conditions (A1)–(A5) are typical assumptions for handling missing variables in the Cox model using simple weighted estimators, see Wang and Chen (2001); Qi et al. (2005); Luo et al. (2009); Xu et al. (2009). Conditions (A6)-(A9) are assumptions for validity of the ROC estimators, see Song and Zhou (2008). Conditions (A10)-(A12) guarantee the theoretical properties of augmented weighted estimators for θ and $\Lambda_0(t)$ in the Cox model. Conditions (A13) and (A14) guarantee the theoretical properties of augmented weighted estimators for α and $H(\cdot)$ in the semiparametric location model.

Under the regularity conditions (A1)-(A5), Qi et al. (2005) showed that

$$\sqrt{n}(\hat{\theta} - \theta_0) = I_{\theta_0}{}^{-1} n^{-1/2} \sum_{i=1}^{n} (V_i/\pi_i) M^*_{Q,i} + o_p(1), \tag{1}$$

where I_{θ_0} is given in Condition (A4). Denote $\Sigma_{\text{SW}} = E[\pi_i^{-1} M_{Q,i}{}^{*\otimes 2}]$.

When the parametric model $\pi_i(\phi_0)$ is correctly specified, similar to Xu et al. (2009), under regularity conditions (A1)-(A5) and (A10), we obtain

$$\sqrt{n}(\hat{\theta}_{\text{P}} - \theta_0) = I_{\theta_0}{}^{-1} n^{-1/2} \sum_{i=1}^{n} \left\{ (V_i/\pi_i) M^*_{Q,i} - I_{\phi_0\theta_0} I_{\phi_0}^{-1} U_{\phi,i}(\phi_0) \right\} + o_p(1), \tag{2}$$

where $I_{\phi_0\theta_0} = E\{\pi^{-1}(W_i;\phi_0) M^*_{Q,i} \frac{\partial}{\partial \phi^T} \pi(W_i;\phi_0)\}$, and $I_{\phi_0} = E\{U_{\phi,i}(\phi_0)\}^{\otimes 2}$.

Under regularity conditions (A1)-(A5) and (A10)-(A12), similar to Xu et al. (2009), we obtain

$$\sqrt{n}(\hat{\theta}_{\text{A}} - \theta_0) = I_{\theta_0}{}^{-1} n^{-1/2} \sum_{i=1}^{n} \left\{ (V_i/\pi_i) M^*_{Q,i} + (1 - V_i/\pi_i) M^{*o}_{Q,i} \right\} + o_p(1), \tag{3}$$

where $M^{*o}_{Q,i} = E[M^*_{Q,i}|W_i]$. Furthermore, when both $\pi_i(\phi_0)$ and $f(Z_i \mid W_i; \chi_0)$ are correctly specified, we have $\sqrt{n}(\hat{\theta}_{\text{AP}} - \theta_0) = \sqrt{n}(\hat{\theta}_{\text{A}} - \theta_0) + o_p(1)$.

Appendix B: Proofs of Lemmas 1-9

We assume that the regularity conditions in Appendix A hold. Proof of Lemma 1 follows similar techniques to those in Qi et al. (2005). Proof of Lemma 2 follows similar techniques to those in Song and Zhou (2008).

Lemma 1: *Given* $t \in [0, \tau]$, $n^{1/2}\{\hat{\Lambda}_0(t, \hat{\theta}) - \Lambda_0(t)\}$ *converges weakly to a zero-mean Gaussian process and admits an asymptotic i.i.d. representation* $n^{-1/2} \sum_{i=1}^{n} \xi_{\Lambda,i}(t;\theta_0)$, *where* $\xi_{\Lambda,i}(t;\theta_0)$ *is given in* (B.1).

Proof. To establish the asymptotic normality of $\hat{\Lambda}_0(t, \hat{\theta})$, we write

$$\sqrt{n}\{\hat{\Lambda}_0(t, \hat{\theta}) - \Lambda_0(t)\} = \sqrt{n}\{\hat{\Lambda}_0(t, \hat{\theta}) - \hat{\Lambda}_0(t;\theta_0)\} + \sqrt{n}\{\hat{\Lambda}_0(t;\theta_0) - \Lambda_0(t)\}. \tag{4}$$

For the first term in the right hand side of (4), by the mean value theorem,

$$\sqrt{n}\{\hat{\Lambda}_0(t, \hat{\theta}) - \hat{\Lambda}_0(t;\theta_0)\} = \sqrt{n}\hat{G}(t;\theta^*)(\hat{\theta} - \theta_0),$$

where

$$\hat{G}(t;\theta) = -\frac{1}{n}\sum_{i=1}^{n}\frac{V_i}{\pi(W_i)}\int_0^t \left\{\frac{S^{(1)}(\theta,u)}{S^{(0)}(\theta,u)^{\otimes 2}}\right\} dN_i(u),$$

and θ^* is on the line segment between $\hat{\theta}$ and θ_0. By the proof of Theorem 1 in Qi et al. (2005), we can show that

$$\sup_{u\in[0,\tau],\theta\in\Theta} ||S^{(k)}(\theta,u) - s^{(k)}(\theta,u)|| \to 0 \ a.s.$$

where Θ is a compact neighborhood of θ. By the law of large numbers,

$$\frac{1}{n}\sum_{i=1}^{n}\frac{V_i}{\pi(W_i)} dN_i(u) \to E[dN_i(u)] \ a.s.$$

Because the functional defined by $\hat{G}$ is continuous with respect to the supreme norm topology, following the same techniques as in Huang and Wang (2000), we can show that almost surely $\hat{G}(t;\theta^*)$ converges to $G(t;\theta_0) = \int_0^t \{s^{(1)}(\theta_0,u)/s^{(0)}(\theta_0,u)\} d\Lambda_0(u)$, uniformly in $t \in [0,\tau]$ and $\theta^* \in \Theta$. By (1), one can write

$$\sqrt{n}\big\{\hat{\Lambda}_0(t,\hat{\theta}) - \hat{\Lambda}_0(t;\theta_0)\big\} = G(t;\theta_0) I_{\theta_0}^{-1} n^{-1/2}\sum_{i=1}^{n}\frac{V_i}{\pi(W_i)} M_{Q,i}^* + o_p(1).$$

For the second term in the right hand side of (4), we have

$$\sqrt{n}\big\{\hat{\Lambda}_0(t;\theta_0) - \Lambda_0(t)\big\} = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{V_i}{\pi(W_i)}\int_0^t \frac{1}{S^{(0)}(\theta_0,u)} dM_i(u) + o_p(1).$$

Let $\bar{M}_n(t) = n^{-1/2}\sum_{i=1}^{n} V_i \pi_i^{-1} M_i(t)$. By Example 2.11.16 of Van Der Vaart and Wellner (1996), $\bar{M}_n(u)$ converges weakly to a process $W_M(u)$ with continuous sample paths. By the strong embedding theorem (Shorack and Wellner, 1986), there exists a new probability space where $(\bar{M}_n^*(u), S^{*(0)}(\theta_0,u)) \to_{a.s.} (W_M^*(u), s^{*(0)}(\theta_0,u))$. We added $*$ to the original notation to denote the processes in the new space, which are equal in law to those in the old space. Therefore, it can be shown that

$$\int_0^t \frac{1}{S^{*(0)}(\theta_0,u)} d\bar{M}_i^*(u) \to_{a.s.} \int_0^t \frac{1}{s^{*(0)}(\theta_0,u)} dW_M^*(u),$$

$$\int_0^t \frac{1}{s^{*(0)}(\theta_0,u)} d\bar{M}_i^*(u) \to_{a.s.} \int_0^\tau \frac{1}{s^{*(0)}(\theta_0,u)} dW_M^*(u).$$

This implies that $|\int_0^t \frac{1}{S^{*(0)}(\theta_0,u)} d\bar{M}_i^*(u) - \int_0^t \frac{1}{s^{*(0)}(\theta_0,u)} d\bar{M}_i^*(u)| \to_{a.s.} 0$ in the new space, and thus the convergence is in probability back in the original space, i.e.,

$$\left| \int_0^t \frac{1}{S^{(0)}(\theta_0, u)} d\bar{M}_i(u) - \int_0^t \frac{1}{s^{(0)}(\theta_0, u)} d\bar{M}_i(u) \right| = o_p(1).$$

We have

$$\begin{aligned} \sqrt{n}\big\{\hat{\Lambda}_0(t;\theta_0) - \Lambda_0(t)\big\} &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{V_i}{\pi(W_i)} \int_0^t \frac{1}{S^{(0)}(\theta_0, u)} dM_i(u) + o_p(1) \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \frac{V_i}{\pi(W_i)} \int_0^t \frac{1}{s^{(0)}(\theta_0, u)} dM_i(u) + o_p(1). \end{aligned}$$

Thus $\sqrt{n}(\hat{\Lambda}_0(t, \hat{\theta}) - \Lambda_0(t)) = n^{-1/2} \sum_{i=1}^n \xi_{\Lambda,i}(t;\theta_0) + o_p(1)$, where

$$\xi_{\Lambda,i}(t;\theta_0) = \{V_i/\pi(W_i)\} G(t;\theta_0) I_{\theta_0}^{-1} M^*_{Q,i} + \{V_i/\pi(W_i)\} \int_0^t \{1/s^{(0)}(\theta_0, u)\} dM_i(u). \qquad \text{(B.1)}$$

LEMMA 2: *Given $x \in \mathcal{X}$, $n^{1/2}\{\hat{H}(z - \hat{\alpha}^T x) - H(z - \alpha_0^T x)\}$ converges weakly to a zero-mean Gaussian process and admits an asymptotic i.i.d representation $n^{-1/2}\sum_{i=1}^n h_i(z, x; \alpha_0)$, where $h_i(z, x; \alpha_0)$ is given in* (B.2).

Proof. Under conditions (A3) and (A8), the least square estimator $\hat{\alpha}$ satisfies

$$n^{1/2}(\hat{\alpha} - \alpha_0) = n^{-1/2} \sum_{i=1}^n \Gamma^{-1} \frac{V_i}{\pi(W_i)} (X_i Z_i - X_i X_i^T \alpha_0) + o_p(1),$$

where $\Gamma = E(XX^T)$. Let $\mathcal{N}(\alpha_0)$ be a compact neighborhood of α_0, and $B(\alpha) = EH\{z + \alpha^T(X_i - x) - \alpha_0^T X_i\}$. By the functional central limit theorem and Slutsky's theorem, $n^{1/2}[\hat{H}(z - \alpha^T x) - B(\alpha)]$ converges to a zero-mean Gaussian process on $(z, \alpha) \in \mathcal{Z} \times \mathcal{N}(\alpha_0)$. It follows from the equicontinuity of the foregoing process and the consistency of $\hat{\alpha}$, that

$$\sup_{z \in \mathcal{Z}} \left| n^{1/2}\big\{\hat{H}(z - \hat{\alpha}^T x) - B(\hat{\alpha})\big\} - n^{1/2}\big\{\hat{H}(z - \alpha_0^T x) - H(z - \alpha_0^T x)\big\} \right| = o_p(1),$$

which implies $n^{1/2}\big\{\hat{H}(z - \hat{\alpha}^T x) - \hat{H}(z - \alpha_0^T x)\big\} = n^{1/2}\big\{B(\hat{\alpha}) - H(z - \alpha_0^T x)\big\} + o_p(1)$. Under condition (A7), the Taylor expansion yields

$$n^{1/2}\big\{B(\hat{\alpha}) - H(z - \alpha_0^T x)\big\} = H'(z - \alpha_0^T x)\{E(X_i) - x\}^T n^{-1/2}(\hat{\alpha} - \alpha_0) + o_p(1),$$

where $H'(\cdot)$ denotes the derivative of $H(\cdot)$. After straightforward algebra, we have

$$\begin{aligned} & n^{1/2}\{\hat{H}(z-\hat{\alpha}^T x)-H(z-\alpha_0^T x)\} \\ = \;& n^{1/2}\{\hat{H}(z-\hat{\alpha}^T x)-\hat{H}(z-\alpha_0^T x)\}+n^{1/2}\{\hat{H}(z-\alpha^T x)-H(z-\alpha_0^T x)\} \\ = \;& n^{-1/2}\sum_{i=1}^n h_i(z,x;\alpha_0)+o_p(1), \end{aligned}$$

where

$$\begin{aligned} h_i(z,x;\alpha_0)= \;& H'(z-\alpha_0^T x)\{E(X_i)-x\}^T\Gamma^{-1}\{V_i/\pi(W_i)\}(X_iZ_i-X_iX_i^T\alpha_0) \\ & +\{V_i/\pi(W_i)\}\{I(Z_i-\alpha_0^T X_i\leqslant z-\alpha_0^T x)-H(z-\alpha_0^T x)\}. \end{aligned} \quad \text{(B.2)}$$

Lemma 3: *Given $t\in[0,\tau]$, $n^{1/2}\{\hat{\Lambda}_0^{\mathrm{P}}(t;\hat{\theta}_P,\hat{\phi})-\Lambda_0(t)\}$ converges weakly to a zero-mean Gaussian process and admits an asymptotic i.i.d. representation $n^{-1/2}\sum_{i=1}^n\xi_{\Lambda,i}^{\mathrm{P}}(t;\theta_0,\phi_0)$, where $\xi_{\Lambda,i}^{\mathrm{P}}(t;\theta_0,\phi_0)$ is given in* (B.3).

Proof. To establish the asymptotic normality of $\hat{\Lambda}_0^{\mathrm{P}}(t;\hat{\theta}_P,\hat{\phi})$, we consider the following decomposition

$$\sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{P}}(t;\hat{\theta}_P,\hat{\phi})-\Lambda_0(t)\big\} = \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{P}}(t;\hat{\theta}_P,\hat{\phi})-\hat{\Lambda}_0^{\mathrm{P}}(t;\theta_0,\phi_0)\big\}+\sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{P}}(t;\theta_0,\phi_0)-\Lambda_0(t)\big\}.$$

For the first term, the Taylor expansion yields

$$\sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{P}}(t;\hat{\theta}_P,\hat{\phi})-\hat{\Lambda}_0^{\mathrm{P}}(t;\theta_0,\phi_0)\big\}=\sqrt{n}\tilde{G}(t;\theta_t^*,\phi_t^*)(\hat{\theta}_P-\theta_0)+\sqrt{n}\tilde{L}(t;\theta_t^*,\phi_t^*)(\hat{\phi}-\phi_0)+o_p(1),$$

where θ_t^* is on the line segment between $\hat{\theta}_P$ and θ_0, ϕ_t^* is on the line segment between $\hat{\phi}$ and ϕ_0, and

$$\tilde{G}(t;\theta,\phi)=-\frac{1}{n}\sum_{i=1}^n\frac{V_i}{\pi(W_i;\phi)}\int_0^t\left\{\frac{S^{(1)}(\theta,u)}{S^{(0)}(\theta,u)^{\otimes 2}}\right\}dN_i(u),$$

$$\tilde{L}(t;\theta,\phi)=-\frac{1}{n}\sum_{i=1}^n\frac{V_i}{\pi(W_i;\phi)}\frac{\partial\pi(W_i;\phi)/\partial\phi}{\pi(W_i;\phi)}\int_0^t\left\{\frac{1}{S^{(0)}(\theta,u)}\right\}dN_i(u).$$

Similar to the proof of Lemma 1, we can show that $\tilde{L}(t;\theta_t^*,\phi_t^*)$ converges uniformly to $L(t;\theta_0,\phi_0)=E[\pi^{-1}(W_i;\phi_0)\frac{\partial}{\partial\phi^T}\pi(W_i;\phi_0)\int_0^t\big\{1/s^{(0)}(\theta_0,u)\big\}dN_i(u)]$, and $\tilde{G}(t;\theta_t^*,\phi_t^*)$ converges uniformly to $G(t;\theta_0)$. In addition, similar to the proof of Lemma 1, we have

$$\begin{aligned}\sqrt{n}\big\{\hat\Lambda_0^{\mathrm{P}}(t;\theta_0,\phi_0)-\Lambda_0(t)\big\} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{V_i}{\pi(W_i)}\int_0^t\frac{1}{S^{(0)}(\theta_0,u)}dM_i(u)+o_p(1)\\ &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\frac{V_i}{\pi(W_i)}\int_0^t\frac{1}{s^{(0)}(\theta_0,u)}dM_i(u)+o_p(1).\end{aligned}$$

The above results, together with (2), lead to the following expression

$$\sqrt{n}\big\{\hat\Lambda_0^{\mathrm{P}}(t;\hat\theta_P,\hat\phi)-\Lambda_0(t)\big\}=n^{-1/2}\sum_{i=1}^{n}\xi^{\mathrm{P}}_{\Lambda,i}(t;\theta_0,\phi_0)+o_p(1),\quad\text{where}$$

$$\begin{aligned}\xi^{\mathrm{P}}_{\Lambda,i}(t;\theta_0,\phi_0) &= G(t;\theta_0){I_{\theta_0}}^{-1}\big\{\frac{V_i}{\pi(W_i)}M^*_{Q,i}-I^T_{\phi_0\theta_0}I^{-1}_{\phi_0}U_{\phi,i}(\phi_0)\big\}+L(t;\theta_0,\phi_0)I^{-1}_{\phi_0}U_{\phi,i}(\phi_0)\\ &\quad+\frac{V_i}{\pi(W_i)}\int_0^t\frac{1}{s^{(0)}(\theta_0,u)}dM_i(u). \qquad \text{(B.3)}\end{aligned}$$

The asymptotic normality of $\sqrt{n}\big\{\hat\Lambda_0^{\mathrm{P}}(t;\hat\theta_P,\hat\phi)-\Lambda_0(t)\big\}$ can be established consequently.

LEMMA 4: *Given $x\in\mathcal{X}$, $n^{1/2}\{\hat H_P(z-\hat\alpha_P^T x;\hat\phi)-H(z-\alpha_0^T x)\}$ converges weakly to a zero-mean Gaussian process and admits an asymptotic i.i.d. representation $n^{-1/2}\sum_{i=1}^n h_i^{\mathrm{P}}(z,x;\alpha_0,\phi_0)$, where $h_i^{\mathrm{P}}(z,x;\alpha_0,\phi_0)$ is given in* (B.4).

Proof. The first derivative matrix of the estimating equations $\{U_\alpha(\alpha,\phi),U_\phi(\phi)\}$ is of the form

$$\begin{pmatrix} -\frac{\partial}{\partial\alpha^T}U_\alpha(\alpha,\phi) & -\frac{\partial}{\partial\phi^T}U_\alpha(\alpha,\phi)\\ -\frac{\partial}{\partial\alpha^T}U_\phi(\phi) & -\frac{\partial}{\partial\phi^T}U_\phi(\phi)\end{pmatrix}.$$

Clearly, $\frac{\partial}{\partial\alpha^T}U_\phi(\phi)=0$. Let $\mathcal{N}(\phi_0)$ be a compact neighborhood of ϕ_0. After straightforward calculation, we have

$$\sup_{(\alpha,\phi)\in(\mathcal{N}(\alpha_0),\mathcal{N}(\phi_0))}\left\|-\frac{\partial}{\partial\alpha^T}U_\alpha(\alpha,\phi)-\Gamma\right\|\xrightarrow{a.s.}0,$$

$$\sup_{\phi\in\mathcal{N}(\phi_0)}\left\|-\frac{\partial}{\partial\phi^T}U_\phi(\phi)-I_{\phi_0}\right\|\xrightarrow{a.s.}0,$$

$$\sup_{(\alpha,\phi)\in(\mathcal{N}(\alpha_0),\mathcal{N}(\phi_0))}\left\|-\frac{\partial}{\partial\phi^T}U_\alpha(\alpha,\phi)-I_{\phi_0\alpha_0}\right\|\xrightarrow{a.s.}0.$$

where $I_{\phi_0\alpha_0}=nE\{U_\alpha(\alpha_0,\phi_0)U_\phi^T(\phi_0)\}=E\{(X_iZ_i-X_iX_i^T\alpha_0)\pi^{-1}(W_i,\phi_0)\partial\pi(W_i;\phi_0)/\partial\phi^T\}$, and $I_{\phi_0}=nE\{U_\phi(\phi_0)U_\phi^T(\phi_0)\}$, By the Taylor expansion, it can be shown that

$$
\begin{aligned}
n^{1/2}(\hat\alpha_P - \alpha_0) &= n^{-1/2}\sum_{i=1}^{n}\Big\{\Gamma^{-1}\frac{V_i}{\pi(W_i;\phi_0)}(X_iZ_i - X_iX_i^T\alpha_0) \\
&\quad -\Gamma^{-1}I_{\alpha_0\phi_0}I_{\phi_0}^{-1}\frac{V_i-\pi(W_i;\phi_0)}{\pi(W_i;\phi_0)\{1-\pi(W_i;\phi_0)\}}\frac{\partial\pi(W_i;\phi_0)}{\partial\phi^T}\Big\} + o_p(1).
\end{aligned}
$$

To spell out the dependence of $\hat H_P(z)$ on $\hat\phi$, we denote $\hat H_P(z)$ by $\hat H_P(z;\hat\phi)$. In the following, we establish the large sample distribution of $\hat H_P(z-\hat\alpha_P^T X;\hat\phi)$. Let $B(\alpha,\phi) = E[\frac{V_i}{\pi_i(\phi)}H\{z+\alpha^T(X_i-x)-\alpha_0^TX_i\}]$. By the functional central limit theorem and Slutsky's theorem, $n^{1/2}[\hat H_P(z-\alpha^Tx;\phi) - B(\alpha,\phi)]$ converges to a zero-mean Gaussian process on $(z,\alpha,\phi)\in\mathcal{Z}\times\mathcal{N}(\alpha_0)\times\mathcal{N}(\phi_0)$. It follows from the equicontinuity of the foregoing process and the consistency of $\hat\alpha_P$, that

$$
\sup_{z\in\mathcal{Z}}\Big| n^{1/2}\big\{\hat H_P(z-\hat\alpha_P^Tx;\hat\phi) - B(\hat\alpha_P,\hat\phi)\big\} - n^{1/2}\big\{\hat H_P(z-\alpha_0^Tx;\hat\phi) - B(\alpha_0,\hat\phi)\big\}\Big| = o_p(1),
$$

which implies that

$$
n^{1/2}\big\{\hat H_P(z-\hat\alpha_P^Tx;\hat\phi) - \hat H_P(z-\alpha^Tx;\hat\phi)\big\} = n^{1/2}\big\{B(\hat\alpha_P,\hat\phi) - B(\alpha_0,\hat\phi)\big\} + o_p(1).
$$

The Taylor expansion yields

$$
\begin{aligned}
B(\hat\alpha_P,\hat\phi) - B(\alpha_0,\hat\phi) &= H'(z-\alpha_0^Tx)\big[E\{\frac{V_i}{\pi_i(\hat\phi)}(X_i-x)\}\big]^T(\hat\alpha_P-\alpha_0) + o_p(n^{-1/2}) \\
&= H'(z-\alpha_0^Tx)\big[E\{\frac{V_i}{\pi_i(\phi_0)}(X_i-x)\}\big]^T(\hat\alpha_P-\alpha_0) + o_p(n^{-1/2}) \\
&= H'(z-\alpha_0^Tx)\big[E(X_i-x)\big]^T(\hat\alpha_P-\alpha_0) + o_p(n^{-1/2}).
\end{aligned}
$$

Similarly, the Taylor expansion yields

$$
\begin{aligned}
& n^{1/2}\big\{\hat H_P(z-\alpha_0^Tx;\hat\phi) - \hat H_P(z-\alpha_0^Tx;\phi_0)\big\} \\
=\ & -I_{H\phi_0}n^{1/2}(\hat\phi-\phi_0) + o_p(1) \\
=\ & -n^{-1/2}\sum_{i=1}^{n} I_{H\phi_0}I_{\phi_0}^{-1}\frac{V_i-\pi(W_i;\phi_0)}{\pi(W_i;\phi_0)(1-\pi(W_i;\phi_0))}\frac{\partial\pi(W_i;\phi_0)}{\partial\phi^T} + o_p(1),
\end{aligned}
$$

where $I_{H\phi_0} = E\{\pi(W_i;\phi_0)^{-1}\frac{\partial\pi(W_i;\phi_0)}{\partial\phi} I(Z_i - \alpha_0^T X_i \leqslant z - \alpha_0^T x)\}$. Therefore, we have

$$
\begin{aligned}
& n^{1/2}\big\{\hat{H}_P(z - \hat{\alpha}_P^T x; \hat{\phi}) - H(z - \alpha_0^T x)\big\} \\
= \;& n^{1/2}\big\{\hat{H}_P(z - \hat{\alpha}_P^T x; \hat{\phi}) - \hat{H}_P(z - \alpha_0^T x; \hat{\phi})\big\} \\
& + n^{1/2}\big\{\hat{H}_P(z - \alpha_0^T x; \hat{\phi}) - \hat{H}_P(z - \alpha_0^T X; \phi_0)\big\} + n^{1/2}\big\{\hat{H}_P(z - \alpha_0^T x; \phi_0) - H(z - \alpha_0^T x)\big\} \\
= \;& n^{-1/2}\sum_{i=1}^n h_i^{\mathrm{P}}(z, x; \alpha_0, \phi_0) + o_p(1),
\end{aligned}
$$

where

$$
\begin{aligned}
& h_i^{\mathrm{P}}(z, x; \alpha_0, \phi_0) = \\
& \left\{\frac{V_i}{\pi(W_i)} I(Z_i - \alpha_0^T X_i \leqslant z - \alpha_0^T x) - H(z - \alpha_0^T x)\right\} + H'(z - \alpha_0^T x)\big[E(X_i - x)\big]^T \times \\
& \left\{\Gamma^{-1}\frac{V_i}{\pi(W_i;\phi_0)}(X_i Z_i - X_i X_i^T \alpha_0) - \Gamma^{-1} I_{\alpha\phi_0} I_{\phi_0}^{-1} \frac{V_i - \pi(W_i;\phi_0)}{\pi(W_i;\phi_0)\{1 - \pi(W_i;\phi_0)\}} \frac{\partial\pi(W_i;\phi_0)}{\partial\phi^T}\right\} - \\
& I_{H\phi_0}\left\{I_{\phi_0}^{-1}\frac{V_i - \pi(W_i;\phi_0)}{\pi(W_i;\phi_0)\{1 - \pi(W_i;\phi_0)\}} \frac{\partial\pi(W_i;\phi_0)}{\partial\phi^T}\right\}. \qquad (B.4)
\end{aligned}
$$

LEMMA 5: *If both $\pi(W_i)$ and $f(Z_i \mid W_i)$ are correctly specified, then given $t \in [0, \tau]$, $n^{1/2}\{\hat{\Lambda}_0^{\mathrm{A}}(t; \hat{\theta}_{\mathrm{A}}) - \Lambda_0(t)\}$ converges weakly to a zero-mean Gaussian process and admits an asymptotic i.i.d. representation $n^{-1/2}\sum_{i=1}^n \xi_{\Lambda,i}^{\mathrm{A}}(t;\theta_0)$, where $\xi_{\Lambda,i}^{\mathrm{A}}(t;\theta_0)$ is given in* (B.5)*. Moreover, given t, $\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}})$ is consisent, provided either $\pi(W_i)$ or $f(Z_i \mid W_i)$ is correctly specified.*

Proof. To establish the asymptotic normality of $\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}})$, we write

$$\sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}}) - \Lambda_0(t)\big\} = \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}}) - \hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0)\big\} + \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0) - \Lambda_0(t)\big\}.$$

For the first term, by the Taylor expansion, we have

$$\sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}}) - \hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0)\big\} = \sqrt{n}\hat{G}^{\mathrm{A}}(t;\theta_t^*)(\hat{\theta}_{\mathrm{A}} - \theta_0) + o_p(1), \qquad (8)$$

where

$$\hat{G}^{\mathrm{A}}(t;\theta) = -\frac{1}{n}\sum_{i=1}^n \int_0^t \left\{\frac{S_{\mathrm{A}}^{(1)}(\theta, u)}{S_{\mathrm{A}}^{(0)}(\theta, u)^{\otimes 2}}\right\} dN_i(u),$$

and θ_t^* is on the line segment between $\hat{\theta}^{\mathrm{A}}$ and θ_0. As shown in Theorem 4 of Xu et al. (2009), we have

$$\sup_{u\in[0,\tau],\theta\in\Theta} ||S_{\mathrm{A}}^{(k)}(\theta,u)-s^{(k)}(\theta,u)|| \to 0 \ a.s.$$

Therefore, by (3)

$$\begin{aligned} & \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}})-\hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0)\big\} \\ = \ & n^{-1/2}\sum_{i=1}^{n} G(t;\theta_0){I_{\theta_0}}^{-1}\left[\{V_i/\pi(W_i)\}M_{Q,i}^*+\{1-V_i/\pi(W_i)\}E(M_{Q,i}^*|W_i)\right]+o_p(1). \end{aligned}$$

For the second term, similar to the proof of Lemma 1, we can show

$$\begin{aligned} & n^{1/2}\left\{\hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0)-\Lambda_0(t)\right\} \\ = \ & n^{1/2}\left\{\int_0^t \frac{n^{-1}\sum_{i=1}^n dN_i(u)}{S_{\mathrm{A}}^{(0)}(\theta_0,u)}-\Lambda_0(t;\theta_0)\right\} \\ = \ & \int_0^t \frac{n^{-1/2}\sum_{i=1}^n (V_i/\pi(W_i))dM_i(u)}{S_{\mathrm{A}}^{(0)}(\theta_0,u)}+\int_0^t \frac{n^{-1/2}\sum_{i=1}^n (1-V_i/\pi(W_i))E\{dM_i(u)\mid W_i\}}{S_{\mathrm{A}}^{(0)}(\theta_0,u)} \\ = \ & \int_0^t \frac{n^{-1/2}\sum_{i=1}^n (V_i/\pi(W_i))dM_i(u)}{s^{(0)}(\theta_0,u)}+\int_0^t \frac{n^{-1/2}\sum_{i=1}^n (1-V_i/\pi(W_i))E\{dM_i(u)\mid W_i\}}{s^{(0)}(\theta_0,u)}+o_p(1). \end{aligned}$$

Therefore, we have $n^{1/2}\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}})-\Lambda_0(t)\}=n^{-1/2}\sum_{i=1}^n \xi_{\Lambda,i}^{\mathrm{A}}(t;\theta_0)+o_p(1)$, where

$$\begin{aligned} \xi_{\Lambda,i}^{\mathrm{A}}(t;\theta_0) = \ & G(t;\theta_0){I_{\theta_0}}^{-1}\left[\{V_i/\pi(W_i)\}M_{Q,i}^*+\{1-V_i/\pi(W_i)\}E(M_{Q,i}^*|W_i)\right] \\ & +\int_0^t \{V_i/\pi(W_i)\}\{1/s^{(0)}(\theta_0,u)\}dM_i(u)+\int_0^t \{1-V_i/\pi(W_i)\}\{1/s^{(0)}(\theta_0,u)\}E\{dM_i(u)\mid W_i\}. \end{aligned} \tag{B.5}$$

The asymptotic normality of $\sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}})-\Lambda_0(t)\big\}$ can be established consequently.

Note that

$$\begin{aligned} \hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}})-\Lambda_0(t) &= \left\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}})-\hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0)\right\}+\left\{\hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0)-\Lambda_0(t)\right\} \\ &= I_1+I_2. \end{aligned}$$

By equation (8) and the double robustness of $\hat{\theta}_{\mathrm{A}}$, we have $I_1=o_p(1)$, provided either $\pi(W_i)$ or $f(Z_i\mid W_i)$ is correctly specified. By the double robustness of $S_{\mathrm{A}}^{(0)}(\theta_0,u)$, we have

$$\hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0)=\int_0^t n^{-1}\sum_{i=1}^n dN_i(u)/S_{\mathrm{A}}^{(0)}(\theta_0,u)=\int_0^t n^{-1}\sum_{i=1}^n dN_i(u)/s^{(0)}(\theta_0,u)+o_p(1).$$

That is $I_2 = o_p(1)$, provided either $\pi(W_i)$ or $f(Z_i \mid W_i)$ is correctly specified. Thus, $\hat{\Lambda}_0^{\mathrm{A}}(t; \hat{\theta}_{\mathrm{A}})$ is double robust.

LEMMA 6: *If both $\pi(W_i)$ and $f(Z_i \mid W_i)$ are correctly specified, then given $x \in \mathcal{X}$, $n^{1/2}\{\hat{H}_{\mathrm{A}}(z - \hat{\alpha}_{\mathrm{A}}^T x) - H(z - \alpha_0^T x)\}$ converges weakly to a zero-mean Gaussian process and admits an asymptotic i.i.d. representation $n^{-1/2}\sum_{i=1}^n h_i^{\mathrm{A}}(z, x; \alpha_0)$, where $h_i^{\mathrm{A}}(z, x; \alpha_0)$ is given in* (B.6)*. Moreover, given x and z, $\hat{H}_{\mathrm{A}}(z - \hat{\alpha}_{\mathrm{A}}^T x)$ is consistent, provided either $\pi(W_i)$ or $f(Z_i \mid W_i)$ is correctly specified.*

Proof. By the Taylor expansion, we have

$$\begin{aligned} n^{1/2}(\hat{\alpha}_{\mathrm{A}} - \alpha_0) &= \left\{ -\frac{\partial}{\partial \alpha} U_\alpha^{\mathrm{A}}(\alpha_0) \right\}^{-1} n^{1/2} U_\alpha^{\mathrm{A}}(\alpha_0) + o_p(1) \\ &= -n^{1/2}\Gamma^{-1} U_\alpha^{\mathrm{A}}(\alpha_0) + o_p(1). \end{aligned}$$

By the definition of $\hat{H}_{\mathrm{A}}(\cdot)$, we have

$$\begin{aligned} & n^{1/2}\{\hat{H}_{\mathrm{A}}(z - \hat{\alpha}_{\mathrm{A}}^T x) - H(z - \alpha_0^T x)\} \\ = \; & n^{-1/2} \sum_{i=1}^n \left\{ F(z + \hat{\alpha}_{\mathrm{A}}^T (X_i - x) \mid W_i) - H(z - \alpha_0^T x) \right\} \\ & + n^{-1/2} \sum_{i=1}^n \frac{V_i}{\pi(W_i)} \left\{ I(Z_i - \hat{\alpha}_{\mathrm{A}}^T X_i \leqslant z - \hat{\alpha}_{\mathrm{A}}^T x) - F(z + \hat{\alpha}_{\mathrm{A}}^T (X_i - x) \mid W_i) \right\} + o_p(1). \end{aligned}$$

Following similar arguments as in Lemma 2, we have

$$n^{-1/2} \sum_{i=1}^n \frac{V_i}{\pi(W_i)} \left\{ I(Z_i - \alpha^T X_i \leqslant z - \alpha^T x) - F(z + \alpha^T (X_i - x) \mid W_i) \right\}$$

converges weakly to a zero mean Gaussian process. By the equicontinuity property,

$$\begin{aligned} & n^{-1/2} \sum_{i=1}^n \frac{V_i}{\pi(W_i)} \left\{ I(Z_i - \hat{\alpha}_{\mathrm{A}}^T X_i \leqslant z - \hat{\alpha}_{\mathrm{A}}^T x) - F(z + \hat{\alpha}_{\mathrm{A}}^T (X_i - x) \mid W_i) \right\} \\ = \; & n^{-1/2} \sum_{i=1}^n \frac{V_i}{\pi(W_i)} \left\{ I(Z_i - \alpha_0 X_i \leqslant z - \alpha_0 x) - F(z + \alpha_0 (X_i - x) \mid W_i) \right\} + o_p(1). \end{aligned}$$

Applying the Taylor expansion, we can show that

$$\begin{aligned}
& n^{-1/2}\sum_{i=1}^{n}\Big\{F(z+\hat{\alpha}_{\mathrm{A}}^{T}(X_i-x)\mid W_i)-H(z-\alpha_0^{T}x)\Big\}\\
= & \; n^{-1/2}\sum_{i=1}^{n}\Big\{F(z+\alpha_0^{T}(X_i-x)\mid W_i)-H(z-\alpha_0^{T}x)\Big\}\\
& +n^{-1/2}\sum_{i=1}^{n}F'(z+\alpha_0^{T}(X_i-x)\mid W_i)(\hat{\alpha}_{\mathrm{A}}-\alpha_0)^{T}(X_i-x)+o_p(1)\\
= & \; n^{-1/2}\sum_{i=1}^{n}\Bigg(\Big\{F(z+\alpha_0^{T}(X_i-x)\mid W_i)-H(z-\alpha_0^{T}x)\Big\}\\
& -\Big(\Gamma^{-1}\frac{V_i}{\pi(W_i)}(X_iZ_i-X_iX_i^{T}\alpha_0)-\Gamma^{-1}\Big(1-\frac{V_i}{\pi(W_i)}\Big)E\{(X_iZ_i-X_iX_i^{T}\alpha)\mid W_i\}\Big)^{T}\\
& E\big\{F'(z+\alpha_0^{T}(X_i-x)\mid W_i)(X_i-x)\big\}\Bigg)+o_p(1).
\end{aligned}$$

Therefore, we have $n^{1/2}\{\hat{H}_{\mathrm{A}}(z-\hat{\alpha}_{\mathrm{A}}^{T}x)-H(z-\alpha_0^{T}x)\}=n^{-1/2}\sum_{i=1}^{n}h_i^{\mathrm{A}}(z,x;\alpha_0)$, where

$$\begin{aligned}
h_i^{\mathrm{A}}(z,x;\alpha_0) = & \frac{V_i}{\pi(W_i)}\Big\{I(Z_i-{\alpha_0}^{T}X_i\leqslant z-\alpha_0^{T}x)-F(z+\alpha_0^{T}(X_i-x)\mid W_i)\Big\}\\
& -(\Gamma^{-1}\frac{V_i}{\pi(W_i)}(X_iZ_i-X_iX_i^{T}\alpha_0)-\Gamma^{-1}\Big(1-\frac{V_i}{\pi(W_i)}\Big)\\
& E\{(X_iZ_i-X_iX_i^{T}\alpha)\mid W_i\})^{T}E\big\{F'(z+\alpha_0^{T}(X_i-x)\mid W_i)(X_i-x)\big\}\\
& +F(z+\alpha_0^{T}(X_i-x)\mid W_i)-H(z-\alpha_0^{T}x) \qquad \text{(B.6)}
\end{aligned}$$

The double robustness of $\hat{\alpha}_{\mathrm{A}}$ follows from Robins et al. (1994). Note that

$$\begin{aligned}
\hat{H}_{\mathrm{A}}(z-\hat{\alpha}_{\mathrm{A}}^{T}x) = & \; n^{-1}\sum_{i=1}^{n}\frac{V_i}{\pi(W_i)}I(Z_i-\hat{\alpha}_{\mathrm{A}}^{T}X_i\leqslant z-\hat{\alpha}_{\mathrm{A}}^{T}x)\\
& +n^{-1}\sum_{i=1}^{n}F(z+\hat{\alpha}_{\mathrm{A}}^{T}(X_i-x)\mid W_i)-n^{-1}\sum_{i=1}^{n}\frac{V_i}{\pi(W_i)}F(z+\hat{\alpha}_{\mathrm{A}}^{T}(X_i-x)\mid W_i)\\
= & \; I_1+I_2-I_3
\end{aligned}$$

When $\pi(W_i)$ is correctly specified, we have $I_1=H(z-\alpha_0^{T}x)+o_p(1)$ and $I_2-I_3=o_p(1)$, following from the fact that $E(V_i\mid W_i)=\pi(W_i)$, and the consistency of $\hat{\alpha}_{\mathrm{A}}$. On the other hand, if $f(Z_i\mid W_i)$ is correctly specified, we have $I_1-I_3=o_p(1)$ and $I_2=H(z-\alpha_0^{T}x)+o_p(1)$, following from the fact that $E\{F(z+\hat{\alpha}_0^{T}(X_i-x)\mid W_i)\}=H(z-\alpha_0^{T}x)$ and the consistency of $\hat{\alpha}_{\mathrm{A}}$. This completes the proof.

LEMMA 7: *Assume that $g(W_i;\Lambda_0(R_i))$ is differentiable with respect to $\Lambda_0(R_i)$ and*

$$\sup_{t\in[0,\tau]} ||n^{-1/2}\sum_{i=1}^{n}\left(1-\frac{V_i}{\pi_i}\right)\frac{\partial g(W_i,t;\Lambda_0(R_i))}{\partial\Lambda_0(R_i)}|| = O_p(1).$$

Then we have

$$n^{-1/2}\sum_{i=1}^{n}\left(1-\frac{V_i}{\pi_i}\right)g(W_i,t;\tilde{\Lambda}_0(R_i,\phi_0)) = n^{-1/2}\sum_{i=1}^{n}\left(1-\frac{V_i}{\pi_i}\right)g(W_i,t;\Lambda_0(R_i)) + o_p(1).$$

Proof. Let $\tilde{\Lambda}_0(R_i)=\tilde{\Lambda}_0(R_i,\phi_0)$. By the Taylor expansion

$$\begin{aligned} & n^{-1/2}\sum_{i=1}^{n}\left(1-\frac{V_i}{\pi_i}\right)\left\{g(W_i,t;\tilde{\Lambda}_0(R_i)) - g(W_i,t;\Lambda_0(R_i))\right\} \\ = \ & n^{-1/2}\sum_{i=1}^{n}\left(1-\frac{V_i}{\pi_i}\right)\frac{\partial g(W_i,t;\Lambda_0(R_i))}{\partial\Lambda_0(R_i)}\{\tilde{\Lambda}_0(R_i)-\Lambda_0(R_i)\} + o_p(1). \end{aligned}$$

It follows from Lemma 1 that $\sup_{t\in[0,\tau]}||\tilde{\Lambda}_0(t)-\Lambda_0(t)|| = o_p(1)$. Similar to the arguments in Appendix A4 of Xu et al. (2009), we have

$$n^{-1/2}\sum_{i=1}^{n}\left(1-\frac{V_i}{\pi_i}\right)\left\{g(W_i,t;\tilde{\Lambda}_0(R_i)) - g(W_i,t;\Lambda_0(R_i))\right\} = o_p(1),$$

which completes the proof.

LEMMA 8: *Given $x\in\mathcal{X}$, the augmented estimator $\hat{H}_{\mathrm{AP}}(z-\hat{\alpha}_{\mathrm{AP}}^T x;\hat{\phi},\hat{\eta})$ has the same asymptotic distribution as $\hat{H}_{\mathrm{A}}(z-\hat{\alpha}_{\mathrm{A}}^T x)$ as long as both $\pi(W_i;\phi)$ and $f(Z_i\mid W_i;\alpha,\eta)$ are correctly specified.*

Proof. To emphasize the dependence of $f(Z_i\mid W_i)$ on $\Lambda_0(R_i)$, we use $\tilde{E}$ to indicate the expectations that are evaluated at the estimator $\tilde{\Lambda}(R_i,\phi)$ and E to indicate the expectations that are evaluated at the true value $\Lambda_0(R_i)$. For notational simplicity, let $\psi=(\phi,\eta)$. It is easily seen that $E\{\frac{\partial}{\partial\psi}U_\alpha^{\mathrm{A}}(\alpha_0,\phi_0,\eta_0)\}=0$ and $E\{\frac{\partial}{\partial\phi}U_\eta^{\mathrm{A}}(\phi_0,\eta_0)\}=0$. Thus, $\frac{\partial}{\partial\chi}U^{\mathrm{AP}}(\chi_0)$ converges in probability to a block diagonal matrix. By the Taylor expansion and block matrix inversion formula, we have

$$\begin{aligned} n^{1/2}(\hat{\alpha}_{\mathrm{AP}}-\alpha_0) &= \left\{-\frac{\partial}{\partial\alpha}U_\alpha^{\mathrm{AP}}(\alpha_0,\phi_0,\eta_0,\tilde{E})\right\}^{-1} n^{1/2}U_\alpha^{\mathrm{AP}}(\alpha_0,\phi_0,\eta_0,\tilde{E}) + o_p(1) \\ &= \left\{-\frac{\partial}{\partial\alpha}U_\alpha^{\mathrm{AP}}(\alpha_0,\phi_0,\eta_0,E)\right\}^{-1} n^{1/2}U_\alpha^{\mathrm{AP}}(\alpha_0,\phi_0,\eta_0,E) + o_p(1), \end{aligned}$$

where the second equation follows from the consistency of $\tilde{\Lambda}_0(t, \phi_0)$ and Lemma 7. Therefore, we have

$$\begin{aligned} n^{1/2}(\hat{\alpha}_{\mathrm{AP}} - \alpha_0) &= -n^{1/2}\Gamma^{-1}U_\alpha^{\mathrm{AP}}(\alpha_0, \phi_0, \eta_0, E) + o_p(1) \\ &= n^{1/2}(\hat{\alpha}_{\mathrm{A}} - \alpha_0) + o_p(1). \end{aligned}$$

This implies that, $\hat{\alpha}_{\mathrm{AP}}$ is asymptotically equivalent to $\hat{\alpha}_{\mathrm{A}}$. Note that

$$\begin{aligned} & n^{1/2}\Big\{\hat{H}_{\mathrm{AP}}(z - \hat{\alpha}_{\mathrm{AP}}^T x; \hat{\phi}, \hat{\eta}, \tilde{E}) - H(z - \alpha_0^T x)\Big\} \\ = \; & n^{1/2}\Big\{\hat{H}_{\mathrm{AP}}(z - \hat{\alpha}_{\mathrm{AP}}^T x; \hat{\phi}, \hat{\eta}, \tilde{E}) - \hat{H}_{\mathrm{AP}}(z - \hat{\alpha}_{\mathrm{AP}}^T x, \tilde{E})\Big\} \\ & + n^{1/2}\Big\{\hat{H}_{\mathrm{AP}}(z - \hat{\alpha}_{\mathrm{AP}}^T x, \tilde{E}) - \hat{H}_{\mathrm{AP}}(z - \hat{\alpha}_{\mathrm{AP}}^T x, E)\Big\} + n^{1/2}\Big\{\hat{H}_{\mathrm{AP}}(z - \hat{\alpha}_{\mathrm{AP}}^T x, E) - H(z - \alpha_0^T x)\Big\}. \end{aligned}$$

For the first term, by the Taylor expansion, we have

$$\begin{aligned} & n^{1/2}\Big\{\hat{H}_{\mathrm{AP}}(z - \hat{\alpha}_{\mathrm{AP}}^T x; \hat{\phi}, \hat{\eta}, \tilde{E}) - \hat{H}_{\mathrm{AP}}(z - \hat{\alpha}_{\mathrm{AP}}^T x, \tilde{E})\Big\} \\ = \; & \Big\{\frac{\partial}{\partial \psi}\hat{H}_{\mathrm{AP}}(z - \hat{\alpha}_{\mathrm{AP}}^T x, \tilde{E})\Big\} n^{1/2}(\hat{\psi} - \psi_0) + o_p(1) = o_p(1), \end{aligned}$$

where the last step follows from the consistency of $\tilde{\Lambda}_0(t, \phi_0)$ and $E\{\frac{\partial}{\partial \psi}\hat{H}_{\mathrm{AP}}(z - \alpha_0^T x, E)\} = 0$. By Lemma 7, the second term satisfies

$$n^{1/2}\Big\{\hat{H}_{\mathrm{AP}}(z - \hat{\alpha}_{\mathrm{AP}}^T x, \tilde{E}) - \hat{H}_{\mathrm{AP}}(z - \hat{\alpha}_{\mathrm{AP}}^T x, E)\Big\} = o_p(1).$$

Following the similar arguments to the proof of Lemma 6, we can show that the third term $n^{1/2}\{\hat{H}_{\mathrm{AP}}(z - \hat{\alpha}_{\mathrm{AP}}^T x) - H(z - \alpha_0^T x)\}$ is asymptotically equivalent to $n^{1/2}\{\hat{H}_{\mathrm{A}}(z - \hat{\alpha}_{\mathrm{A}}^T x) - H(z - \alpha_0^T x)\}$. This completes the proof.

LEMMA 9: *Given $t \in [0, \tau]$, $\hat{\Lambda}_0^{\mathrm{AP}}(t; \hat{\theta}_{\mathrm{AP}}, \hat{\phi}, \hat{\eta})$ has the same asymptotic distribution as $\hat{\Lambda}_0^{\mathrm{A}}(t; \hat{\theta}_{\mathrm{A}})$ as long as both $\pi(W; \phi)$ and $f(Z \mid W; \alpha, \eta)$ are correctly specified.*

Proof. The proof is similar to that of Lemma 5. Here, we only sketch the key steps. Note that

$$
\begin{aligned}
& \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{AP}}(t;\hat{\theta}_{\mathrm{AP}},\hat{\phi},\hat{\eta},\tilde{E}) - \Lambda_0(t)\big\} \\
= \; & \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{AP}}(t;\hat{\theta}_{\mathrm{AP}},\hat{\phi},\hat{\eta},\tilde{E}) - \hat{\Lambda}_0^{\mathrm{AP}}(t;\theta_0,\tilde{E})\big\} + \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{AP}}(t;\theta_0,\tilde{E}) - \hat{\Lambda}_0^{\mathrm{AP}}(t;\theta_0,E)\big\} \\
& \quad + \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{AP}}(t;\theta_0,E) - \Lambda_0(t)\big\}.
\end{aligned}
$$

Similar to Lemmas 5 and 8, we can show that the first term satisfies

$$
\begin{aligned}
\sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{AP}}(t;\hat{\theta}_{\mathrm{AP}},\hat{\phi},\hat{\eta},\tilde{E}) - \hat{\Lambda}_0^{\mathrm{AP}}(t;\theta_0,\tilde{E})\big\} &= \sqrt{n}\hat{G}^{\mathrm{A}}(t;\theta_0)(\hat{\theta}_{\mathrm{AP}} - \theta_0) + o_p(1) \\
&= \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}}) - \hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0)\big\} + o_p(1),
\end{aligned}
$$

where the second equation follows from the property that $n^{1/2}(\hat{\theta}_{\mathrm{AP}} - \theta_0) = n^{1/2}(\hat{\theta}_{\mathrm{A}} - \theta_0) + o_p(1)$, as established in Theorem 4 of Xu et al. (2009).

Following from the consistency of $\tilde{\Lambda}_0(t,\phi_0)$ and Lemma 7, the second term satisfies,

$$
\begin{aligned}
& \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{AP}}(t;\theta_0,\tilde{E}) - \hat{\Lambda}_0^{\mathrm{AP}}(t;\theta_0,E)\big\} \\
= \; & n^{-1/2}\sum_{i=1}^{n} dN_i(t)\frac{S_{\mathrm{A}}^{(0)}(\theta_0,\phi_0,\eta_0,t) - \tilde{S}_{\mathrm{A}}^{(0)}(\theta_0,\phi_0,\eta_0,t)}{S_{\mathrm{A}}^{(0)}(\theta_0,\phi_0,\eta_0,t)\tilde{S}_{\mathrm{A}}^{(0)}(\theta_0,\phi_0,\eta_0,t)} \\
= \; & n^{-1/2}\sum_{i=1}^{n} dN_i(t)\frac{S_{\mathrm{A}}^{(0)}(\theta_0,\phi_0,\eta_0,t) - \tilde{S}_{\mathrm{A}}^{(0)}(\theta_0,\phi_0,\eta_0,t)}{s^{(0)2}(\theta_0,\phi_0,\eta_0,t)} + o_p(1) \\
= \; & o_p(1).
\end{aligned}
$$

The arguments to the third term are identical to those of Lemma 5. The above results imply that $\hat{\Lambda}_0^{\mathrm{AP}}(t;\hat{\theta}_{\mathrm{AP}},\hat{\phi},\hat{\eta},\tilde{E})$ is asymptotically equivalent to $\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}})$, which completes the proof.

Appendix C: Proofs of Theorem 1-4

The proofs follow similar techniques to those in Song and Zhou (2008).

Proof of Theorem 1

By (1) and Lemma 1, under proportional hazards models, we have

$$\begin{aligned}
&\sqrt{n}\{\hat{S}(t \mid q) - S(t \mid q)\} \\
&\qquad = -\sqrt{n} S(t \mid q)\exp({\theta_0}^T q)\big[\{\hat{\Lambda}_0(t;\hat{\theta}_0) - \Lambda_0(t)\} + \Lambda_0(t) q^T(\hat{\theta} - \theta_0)\big] + o_p(1) \\
&\qquad = -n^{-\frac{1}{2}} S(t \mid q)\exp({\theta_0}^T q)\sum_{i=1}^{n}\{\xi_{\Lambda,i}(t;\theta_0) + \Lambda_0(t) q^T \xi_{\theta,i}\} + o_p(1),
\end{aligned}$$

where $q = (z, x^T)^T$, $\xi_{\theta,i} = \{V_i/\pi(W_i)\} I_{\theta_0}^{-1} M_{Q,i}^*$ denotes the asymptotic representation in (1), and $\xi_{\Lambda,i}(t;\theta_0)$ is given in Lemma 1.

Denote $\xi_{S,i}(t;\theta_0) = S(t \mid q)\exp({\theta_0}^T q)\{\xi_{\Lambda,i}(t;\theta_0) + \Lambda_0(t) q^T \xi_{\theta,i}\}$. Using the functional Taylor expansion, we have

$$\begin{aligned}
& n^{\frac{1}{2}}\{\widehat{\mathrm{FPR}}(c;t,x) - \mathrm{FPR}(c;t,x)\} \\
= \;& n^{\frac{1}{2}}\Big[\int_{-\infty}^{\infty} S(t \mid z,x) dH(z - \alpha_0^T x)\Big]^{-1} \times \\
& \Big[\int_{c}^{\infty}\{\hat{S}(t \mid z,x) - S(t \mid z,x)\} dH(z - \alpha_0^T x) + \int_{c}^{\infty} S(t \mid z,x) d\{\hat{H}(z - \hat{\alpha}^T x) - H(z - \alpha_0^T x)\}\Big] \\
& - n^{\frac{1}{2}}\Big[\int_{-\infty}^{\infty} S(t \mid z,x) dH(z - \alpha_0^T x)\Big]^{-2}\int_{c}^{\infty} S(t \mid z,x) dH(z - \alpha_0^T x) \times \\
& \Big[\int_{-\infty}^{\infty}\{\hat{S}(t \mid z,x) - S(t \mid z,x)\} dH(z - \alpha_0^T x) \\
& + \int_{-\infty}^{\infty} S(t \mid z,x) d\{\hat{H}(z - \hat{\alpha}^T x) - H(z - \alpha_0^T x)\}\Big] + o_p(1) \\
= \;& -n^{\frac{1}{2}}\sum_{i=1}^{n}\omega_i(c;t,x) + o_p(1),
\end{aligned}$$

where

$$\begin{aligned}
\omega_i(c;t,x) \;=\; & \Big[\int_{-\infty}^{\infty} S(t \mid z,x) dH(z - \alpha_0^T x)\Big]^{-1} \\
& \times \Big[\int_{c}^{\infty}\xi_{S,i}(t;\theta_0) dH(z - \alpha_0^T x) + \int_{c}^{\infty} S(t \mid z,x) dh_i(z,x;\alpha_0)\Big] \\
& - \Big[\int_{-\infty}^{\infty} S(t \mid z,x) dH(z - \alpha_0^T x)\Big]^{-2}\int_{c}^{\infty} S(t \mid z,x) dH(z - \alpha_0^T x) \\
& \times \Big[\int_{-\infty}^{\infty}\xi_{S,i}(t;\theta_0) dH(z - \alpha_0^T x) + \int_{-\infty}^{\infty} S(t \mid z,x) dh_i(z,x;\alpha_0)\Big]. \qquad \text{(C.1)}
\end{aligned}$$

Similarly, we can show that $\sqrt{n}\{\widehat{\mathrm{TPR}}_{\mathbb{C}}(\cdot;t,x) - \mathrm{TPR}_{\mathbb{C}}(\cdot;t,x)\}$ and $\sqrt{n}\{\widehat{\mathrm{TPR}}_{\mathbb{I}}(\cdot;t,x) -$

$\text{TPR}_{\mathbb{I}}(\cdot; t, x)\}$ converge to Gaussian processes $\mathcal{T}_C(\cdot; t, x)$ and $\mathcal{T}_I(\cdot; t, x)$ respectively, with zero-mean, and

$$\text{cov}\{\mathcal{T}_C(c_1; t, x), \mathcal{T}_C(c_2; t, x)\} = \text{cov}\{\eta_i(c_1; t, x), \eta_i(c_2; t, x)\},$$

$$\text{cov}\{\mathcal{T}_I(c_1; t, x), \mathcal{T}_I(c_2; t, x)\} = \text{cov}\{\eta_i^*(c_1; t, x), \eta_i^*(c_2; t, x)\}, \text{where}$$

$$\begin{aligned}
\eta_i(c; t, x) &= \Big[\int_{-\infty}^{\infty}\{1 - S(t \mid z, x)\}dH(z - \alpha_0^T x)\Big]^{-1} \\
&\quad \times\Big[-\int_c^{\infty}\xi_{S,i}(t; \theta_0)dH(z - \alpha_0^T x) + \int_c^{\infty}\{1 - S(t \mid z, x)\}dh_i(z, x; \alpha_0)\Big] \\
&\quad -\Big[\int_{-\infty}^{\infty}\{1 - S(t \mid z, x)\}dH(z - \alpha_0^T x)\Big]^{-2}\int_c^{\infty}\{1 - S(t \mid z, x)\}dH(z - \alpha_0^T x) \\
&\quad \times\Big[-\int_{-\infty}^{\infty}\xi_{S,i}(t; \theta_0)dH(z - \alpha_0^T x) + \int_{-\infty}^{\infty}\{1 - S(t \mid z, x)\}dh_i(z, x; \alpha_0)\Big] \qquad \text{(C.2)}
\end{aligned}$$

$$\begin{aligned}
\eta_i^*(c; t, x) &= \Big[\int_{-\infty}^{\infty}\exp(\beta_0 z)S(t \mid z, x)dH(z - \alpha_0^T x)\Big]^{-1} \times \Big[\int_c^{\infty} z\exp(\beta_0 z)\xi_{\beta,i}S(t \mid z, x)dH(z - \alpha_0^T x) \\
&\quad +\int_c^{\infty}\exp(\beta_0 z)\xi_{S,i}(t; \theta_0)dH(z - \alpha_0^T x) + \int_c^{\infty}\exp(\beta_0 z)S(t \mid z, x)dh_i(z, x; \alpha_0)\Big] \\
&\quad -\Big[\int_{-\infty}^{\infty}\exp(\beta_0 z)S(t \mid z, x)dH(z - \alpha_0^T x)\Big]^{-2}\int_c^{\infty}\exp(\beta_0 z)S(t \mid z, x)dH(z - \alpha_0^T x) \\
&\quad \times\Big[\int_{-\infty}^{\infty} z\exp(\beta_0 z)\xi_{\beta,i}S(t \mid z, x)dH(z - \alpha_0^T x) \\
&\quad +\int_{-\infty}^{\infty}\exp(\beta_0 z)\xi_{S,i}(t; \theta_0)dH(z - \alpha_0^T x) + \int_{-\infty}^{\infty}\exp(\beta_0 z)S(t \mid z, x)dh_i(z, x; \alpha_0)\Big],
\end{aligned} \qquad \text{(C.3)}$$

where $\xi_{\beta,i}$ is the first element of $\xi_{\theta,i}$.

Since $\text{ROC}_{\mathbb{C}/\mathbb{D}}(\cdot; t, x)$ is a composite functional of $S(t \mid z, x)$ and $H(z - \alpha^T x)$, under Assumption (A9), by the functional delta method, $\sqrt{n}\{\widehat{\text{ROC}}_{\mathbb{C}/\mathbb{D}}(\cdot; t, x) - \text{ROC}_{\mathbb{C}/\mathbb{D}}(\cdot; t, x)\}$ converges to a Gaussian process $\mathcal{G}_C(\cdot; t, x)$ on $[p, q]$. Specifically, using the functional Taylor expansion, we have $\sqrt{n}\{\widehat{\text{ROC}}_{\mathbb{C}/\mathbb{D}}(c; t, x) - \text{ROC}_{\mathbb{C}/\mathbb{D}}(c; t, x)\} = n^{\frac{1}{2}}\sum_{i=1}^n \phi_i(c; t, x) + o_p(1)$, where

$$\phi_i(c; t, x) = \eta_i\{\text{FPR}^{-1}(c; t, x); t, x\} - f_1\{\text{FPR}^{-1}(c; t, x); t, x\}\frac{\omega_i\{\text{FPR}^{-1}(c; t, x); t, x\}}{f_0\{\text{FPR}^{-1}(c; t, x); t, x\}},$$

and $f_1(c; t, x)$ and $f_0(c; t, x)$ are given in Condition (A9). Thus $\text{cov}\{\mathcal{G}_C(c_1; t, x), \mathcal{G}_C(c_2; t, x)\} = \text{cov}\{\phi_i(c_1; t, x), \phi_i(c_2; t, x)\}$.

Similarly, we can show that $\sqrt{n}\{\widehat{\mathrm{ROC}}_{\mathbb{I}/\mathbb{D}}(\cdot;t,x)-\mathrm{ROC}_{\mathbb{I}/\mathbb{D}}(\cdot;t,x)\}$ converges to a Gaussian process $\mathcal{G}_I(\cdot;t,x)$ on $[p,q]$ with zero-mean and covariance $\mathrm{cov}\{\mathcal{G}_I(c_1;t,x),\mathcal{G}_I(c_2;t,x)\} = \mathrm{cov}\{\phi_i^*(c_1;t,x),\phi_i^*(c_2;t,x)\}$, where

$$\phi_i^*(c;t,x)=\eta_i^*\{\mathrm{FPR}^{-1}(c;t,x);t,x\}-f_1^*\{\mathrm{FPR}^{-1}(c;t,x);t,x\}\frac{\omega_i\{\mathrm{FPR}^{-1}(c;t,x);t,x\}}{f_0\{\mathrm{FPR}^{-1}(c;t,x);t,x\}},$$

and $f_1^*(c;t,x)$ and $f_0(c;t,x)$ are given in Condition (A9).

Proof of Theorem 2

By Lemmas 3 and 4, the proof of Theorem 2 is similar to that of Theorem 1.

Let $\xi_{\theta,i}^{\mathrm{P}}={I_{\theta_0}}^{-1}\big\{(V_i/\pi_i)M_{Q,i}^*-I_{\phi_0\theta_0}I_{\phi_0}^{-1}U_{\phi,i}(\phi_0)\big\}$ denote the asymptotic representation in (2). Then we have $\sqrt{n}\{\hat{S}^{\mathrm{P}}(t\mid q)-S(t\mid q)\}=-n^{-\frac{1}{2}}\sum_{i=1}^n\xi_{S,i}^{\mathrm{P}}(t;\theta_0,\phi_0)+o_p(1)$, where

$$\xi_{S,i}^{\mathrm{P}}(t;\theta_0,\phi_0)=S(t\mid q)\exp({\theta_0}^Tq)\{\xi_{\Lambda,i}^{\mathrm{P}}(t;\theta_0,\phi_0)+\Lambda_0(t)q^T\xi_{\theta,i}^{\mathrm{P}}\}.$$

Similar to the proof of Theorem 1, we can show that $\sqrt{n}\{\widehat{\mathrm{TPR}_{\mathbb{C}}}^{\mathrm{P}}(\cdot;t,x)-\mathrm{TPR}_{\mathbb{C}}(\cdot;t,x)\}$, $\sqrt{n}\{\widehat{\mathrm{TPR}_{\mathbb{I}}}^{\mathrm{P}}(c;t,x)-\mathrm{TPR}_{\mathbb{I}}(c;t,x)\}$ and $\sqrt{n}\{\widehat{\mathrm{FPR}_{\mathbb{D}}}^{\mathrm{P}}(c;t,x)-\mathrm{FPR}_{\mathbb{D}}(c;t,x)\}$ have asymptotic representation $n^{-\frac{1}{2}}\sum_{i=1}^n\eta_i^{\mathrm{P}}(c;t,x)$, $n^{-\frac{1}{2}}\sum_{i=1}^n\eta_{i,*}^{\mathrm{P}}(c;t,x)$ and $n^{-\frac{1}{2}}\sum_{i=1}^n\omega_i^{\mathrm{P}}(c;t,x)$, respectively, where $\eta_i^{\mathrm{P}}(c;t,x)$ is obtained by replacing $\xi_{S,i}$, h_i with $\xi_{S,i}^{\mathrm{P}}$, h_i^{P} in (C.2), $\eta_{i,*}^{\mathrm{P}}(c;t,x)$ is obtained by replacing $\xi_{S,i}$, h_i with $\xi_{S,i}^{\mathrm{P}}$, h_i^{P} in (C.3), and $\omega_i^{\mathrm{P}}(c;t,x)$ is obtained by replacing $\xi_{S,i}$, h_i with $\xi_{S,i}^{\mathrm{P}}$, h_i^{P} in (C.1).

Using the functional Taylor expansion, we have $\sqrt{n}\{\widehat{\mathrm{ROC}}_{\mathbb{C}/\mathbb{D}}^{\mathrm{P}}(c;t,x)-\mathrm{ROC}_{\mathbb{C}/\mathbb{D}}(c;t,x)\}=n^{\frac{1}{2}}\sum_{i=1}^n\phi_i^{\mathrm{P}}(c;t,x)+o_p(1)$, where

$$\phi_i^{\mathrm{P}}(c;t,x)=\eta_i^{\mathrm{P}}\{\mathrm{FPR}^{-1}(c;t,x);t,x\}-f_1\{\mathrm{FPR}^{-1}(c;t,x);t,x\}\frac{\omega_i^{\mathrm{P}}\{\mathrm{FPR}^{-1}(c;t,x);t,x\}}{f_0\{\mathrm{FPR}^{-1}(c;t,x);t,x\}},$$

Similarly, $\sqrt{n}\{\widehat{\mathrm{ROC}}_{\mathbb{I}/\mathbb{D}}^{\mathrm{P}}(\cdot;t,x)-\mathrm{ROC}_{\mathbb{I}/\mathbb{D}}(\cdot;t,x)\}=n^{\frac{1}{2}}\sum_{i=1}^n\phi_{i,*}^{\mathrm{P}}(c;t,x)+o_p(1)$, where

$$\phi_{i,*}^{\mathrm{P}}(c;t,x)=\eta_{i,*}^{\mathrm{P}}\{\mathrm{FPR}^{-1}(c;t,x);t,x\}-f_1^*\{\mathrm{FPR}^{-1}(c;t,x);t,x\}\frac{\omega_i^{\mathrm{P}}\{\mathrm{FPR}^{-1}(c;t,x);t,x\}}{f_0\{\mathrm{FPR}^{-1}(c;t,x);t,x\}}.$$

Proof of Theorem 3

By Lemmas 5 and 6, the proof of Theorem 3 is similar to that of Theorem 1.

Let $\xi^{\mathrm{A}}_{\theta,i} = {I_{\theta_0}}^{-1}\big\{(V_i/\pi_i)M^{*}_{Q,i} + (1 - V_i/\pi_i)M^{*o}_{Q,i}\big\}$ denote the asymptotic representation in (3). Then we have $\sqrt{n}\{\hat{S}^{\mathrm{A}}(t \mid q) - S(t \mid q)\} = -n^{-\frac{1}{2}} \sum_{i=1}^{n} \xi^{\mathrm{A}}_{S,i}(t; \theta_0, \phi_0) + o_p(1)$, where

$$\xi^{\mathrm{A}}_{S,i}(t; \theta_0, \phi_0) = S(t \mid q)\exp({\theta_0}^T q)\{\xi^{\mathrm{A}}_{\Lambda,i}(t; \theta_0, \phi_0) + \Lambda_0(t) q^T \xi^{\mathrm{A}}_{\theta,i}\}.$$

Similar to the proof of Theorem 1, we can show that $\sqrt{n}\{\widetilde{\mathrm{TPR}_{\mathbb{C}}}(c; t, x) - \mathrm{TPR}_{\mathbb{C}}(c; t, x)\}$, $\sqrt{n}\{\widetilde{\mathrm{TPR}_{\mathbb{I}}}(c; t, x) - \mathrm{TPR}_{\mathbb{I}}(c; t, x)\}$ and $\sqrt{n}\{\widetilde{\mathrm{FPR}_{\mathbb{D}}}(c; t, x) - \mathrm{FPR}_{\mathbb{D}}(c; t, x)\}$ have asymptotic representation $n^{-\frac{1}{2}} \sum_{i=1}^{n} \eta^{\mathrm{A}}_i(c; t, x)$, $n^{-\frac{1}{2}} \sum_{i=1}^{n} \eta^{\mathrm{A}}_{i,*}(c; t, x)$ and $n^{-\frac{1}{2}} \sum_{i=1}^{n} \omega^{\mathrm{A}}_i(c; t, x)$, respectively, where $\eta^{\mathrm{A}}_i(c; t, x)$ is obtained by replacing $\xi_{S,i}$, h_i with $\xi^{\mathrm{A}}_{S,i}$, h^{A}_i in (C.2), $\eta^{\mathrm{A}}_{i,*}(c; t, x)$ is obtained by replacing $\xi_{S,i}$, h_i with $\xi^{\mathrm{A}}_{S,i}$, h^{A}_i in (C.3), and $\omega^{\mathrm{A}}_i(c; t, x)$ is obtained by replacing $\xi_{S,i}$, h_i with $\xi^{\mathrm{A}}_{S,i}$, h^{A}_i in (C.1).

Using the functional Taylor expansion, we have $\sqrt{n}\{\widetilde{\mathrm{ROC}}_{\mathbb{C}/\mathbb{D}}(c; t, x) - \mathrm{ROC}_{\mathbb{C}/\mathbb{D}}(c; t, x)\} = n^{\frac{1}{2}} \sum_{i=1}^{n} \phi^{\mathrm{A}}_i(c; t, x) + o_p(1)$, where

$$\phi^{\mathrm{A}}_i(c; t, x) = \eta^{\mathrm{A}}_i\{\mathrm{FPR}^{-1}(c; t, x); t, x\} - f_1\{\mathrm{FPR}^{-1}(c; t, x); t, x\}\frac{\omega^{\mathrm{A}}_i\{\mathrm{FPR}^{-1}(c; t, x); t, x\}}{f_0\{\mathrm{FPR}^{-1}(c; t, x); t, x\}},$$

Similarly, $\sqrt{n}\{\widetilde{\mathrm{ROC}}_{\mathbb{I}/\mathbb{D}}(\cdot; t, x) - \mathrm{ROC}_{\mathbb{I}/\mathbb{D}}(\cdot; t, x)\} = n^{\frac{1}{2}} \sum_{i=1}^{n} \phi^{\mathrm{A}}_{i,*}(c; t, x) + o_p(1)$, where

$$\phi^{\mathrm{A}}_{i,*}(c; t, x) = \eta^{\mathrm{A}}_{i,*}\{\mathrm{FPR}^{-1}(c; t, x); t, x\} - f^{*}_1\{\mathrm{FPR}^{-1}(c; t, x); t, x\}\frac{\omega^{\mathrm{A}}_i\{\mathrm{FPR}^{-1}(c; t, x); t, x\}}{f_0\{\mathrm{FPR}^{-1}(c; t, x); t, x\}}.$$

The double robustness of the augmented ROC estimators follows from the double robustness of $\hat{\theta}_{\mathrm{A}}$, $\hat{\Lambda}^{\mathrm{A}}_0(t; \hat{\theta}_{\mathrm{A}})$, and $\hat{H}_{\mathrm{A}}(z - \hat{\alpha}^T_{\mathrm{A}} x)$, which are established in Theorem 3 of Xu et al. (2009) and Lemma 5, Lemma 6 in Appendix B respectively.

Proof of Theorem 4

Note that by Lemma 7, we find that, if $\tilde{\Lambda}$ in $\tilde{E}$ is replaced by any other uniformly consistent estimator, Lemmas 8 and 9 remains valid. This suggests that at any iteration, $\hat{\Lambda}^{\mathrm{AP}}_0(t; \hat{\theta}_{\mathrm{AP}}, \hat{\phi}, \hat{\eta})$ has the same asymptotic distribution as $\hat{\Lambda}^{\mathrm{A}}_0(t; \hat{\theta}_{\mathrm{A}})$ and $\hat{H}_{\mathrm{AP}}(z - \hat{\alpha}^T_{\mathrm{AP}} x; \hat{\phi}, \hat{\eta})$ has the same asymptotic distribution as $\hat{H}_{\mathrm{A}}(z - \hat{\alpha}^T_{\mathrm{A}} x)$. This completes the proof of Theorem 4.

Appendix D: Asymptotic Properties of the Fully Augmented Weighted Estimator when $\pi(W_i)$ is Misspecified

The asymptotic distribution of the fully augmented weighted estimator for ROC is derived in Theorem 4, if both $\pi(W_i)$ and $f(Z_i \mid W_i)$ are correctly specified. As pointed out by one referee, the limiting distribution of the fully augmented weighted estimator for ROC with misspecified $\pi(W_i)$ or $f(Z_i \mid W_i)$ is more useful for analyzing the real data. Although this question is important, there are limited results in this direction. To the best of our knowledge, the limiting distributions of the fully augmented estimators for θ and $\Lambda(t)$ in the proportional hazards model with misspecified $\pi(W_i)$ or $f(Z_i \mid W_i)$ have not been derived in the literature. In this Appendix, we focus on the situation that $\pi(W_i)$ is misspecified and $f(Z_i \mid W_i)$ is correctly specified. Under this situation, we establish the preliminary asymptotic results for the TPR, FPR and ROC estimators.

To emphasize the misspecification of $\pi(W_i)$, we use $\pi(W_i)$ to denote the misspecified missingness probability and $\pi^*(W_i)$ to denote the true missingness probability.

LEMMA 10: *Under the regularity conditions in Appendix A, we have*

$$\sqrt{n}(\hat{\theta}_{\mathrm{A}} - \theta_0) = -{I_{\theta_0}}^{-1} n^{-1/2} \sum_{i=1}^{n} \left\{ (V_i/\pi_i) M^*_{Q,i} + (1 - V_i/\pi_i) M^{*o}_{Q,i} \right\} + o_p(1),$$

and $\sqrt{n}(\hat{\theta}_{\mathrm{A}} - \theta_0) \to_d N(0, \Sigma_{m\pi})$, *where*

$$\Sigma_{m\pi} = {I_{\theta_0}}^{-1} E\left\{ \frac{\pi^*(W_i)}{\pi^2(W_i)} (M^*_{Q,i})^{\otimes 2} + \frac{\pi^2(W_i) - \pi^*(W_i)}{\pi^2(W_i)} (M^{*o}_{Q,i})^{\otimes 2} \right\} {I_{\theta_0}}^{-1}.$$

Proof. By the proof of Theorem 4 in Xu et al. (2009), even if $\pi(W_i)$ is misspecified, we still have

$$\sup_{u \in [0,\tau], \theta \in \Theta} ||S^{(k)}_A(\theta, u) - s^{(k)}(\theta, u)|| \to 0 \ a.s. \qquad (11)$$

In addition, we have

$$\begin{aligned} n^{1/2}U_A(\theta_0) &= \frac{1}{n^{1/2}}\sum_{i=1}^{n}\int_0^\tau \Bigg[\left(\frac{V_i}{\pi(W_i)}\right)\Big\{Q_i - e(\theta_0,t)\Big\}dN_i(t) \\ &\quad +\left(1-\frac{V_i}{\pi(W_i)}\right)\Big\{E(Q_i dN_i(t)\mid W_i) - e(\theta_0,t)E(dN_i(t)\mid W_i)\Big\}\Bigg] + o_p(1) \\ &= \frac{1}{n^{1/2}}\sum_{i=1}^{n}\int_0^\tau \Bigg[\left(\frac{V_i}{\pi(W_i)}\right)\Big\{Q_i - e(\theta_0,t)\Big\}dM_i(t) \\ &\quad +\left(1-\frac{V_i}{\pi(W_i)}\right)\Big\{E\Big((Q_i - e(\theta_0,t))dM_i(t)\mid W_i\Big)\Big\}\Bigg] + o_p(1) \\ &= \frac{1}{n^{1/2}}\sum_{i=1}^{n}\Big\{(V_i/\pi(W_i))M_{Q,i}^{*} + (1-V_i/\pi(W_i))M_{Q,i}^{*o}\Big\} + o_p(1). \end{aligned}$$

This implies that $n^{1/2}U_A(\theta_0)$ can be approximated by a sum of iid random variables. By the central limit theorem and the fact that $P(V_i = 1 \mid W_i) = \pi^*(W_i)$, we have

$$n^{1/2}U_A(\theta_0) \to_d N(0,\Sigma),$$

where

$$\Sigma = E\Big\{\frac{\pi^*(W_i)}{\pi(W_i)^2}(M_{Q,i}^{*})^{\otimes 2} + \frac{\pi^2(W_i)-\pi^*(W_i)}{\pi^2(W_i)}(M_{Q,i}^{*o})^{\otimes 2}\Big\}.$$

It is easily seen that

$$\begin{aligned} \frac{\partial U_A(\theta_0)}{\partial\theta} &= -\frac{1}{n}\sum_{i=1}^{n}\int_0^\tau \Bigg\{\frac{V_i}{\pi(W_i)}\frac{S_A^{(2)}(\theta_0,u)S_A^{(0)}(\theta_0,u) - S_A^{(1)\otimes 2}(\theta_0,u)}{(S_A^{(0)}(\theta_0,u))^2}dN_i(u) \\ &\quad +\left(1-\frac{V_i}{\pi(W_i)}\right)E\left(\frac{S_A^{(2)}(\theta_0,u)S_A^{(0)}(\theta_0,u) - S_A^{(1)\otimes 2}(\theta_0,u)}{(S_A^{(0)}(\theta_0,u))^2}dN_i(u)\mid W_i\right)\Bigg\} \\ &= -\frac{1}{n}\sum_{i=1}^{n}\int_0^\tau \Bigg\{\frac{V_i}{\pi(W_i)}v(\theta_0,u)dN_i(u) + \left(1-\frac{V_i}{\pi(W_i)}\right)v(\theta_0,u)E(dN_i(u)\mid W_i)\Bigg\} + o_p(1). \end{aligned}$$

We find that $\frac{\partial U_A(\theta_0)}{\partial\theta} = -I_{\theta_0} + o_p(1)$. Note that the weak convergence of $n^{1/2}U_A(\theta_0)$ implies that $U_A(\theta_0) = o_p(1)$. By the proof of theorem 2 of Foutz (1977), $\hat\theta_A$ is consistent for θ_0. Finally, by the Taylor expansion,

$$\begin{aligned} \sqrt{n}(\hat\theta_{\mathrm{A}} - \theta_0) &= -{I_{\theta_0}}^{-1}n^{1/2}U_A(\theta_0) + o_p(1) \\ &= -{I_{\theta_0}}^{-1}n^{-1/2}\sum_{i=1}^{n}\Big\{(V_i/\pi_i)M_{Q,i}^{*} + (1-V_i/\pi_i)M_{Q,i}^{*o}\Big\} + o_p(1). \end{aligned}$$

This completes the proof.

Lemma 11: *Under the regularity conditions in Appendix A, given* $t \in [0, \tau]$, $n^{1/2}\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}}) - \Lambda_0(t)\}$ *converges weakly to a zero-mean Gaussian process and admits an asymptotic i.i.d. representation* $n^{-1/2}\sum_{i=1}^n \xi_{\Lambda,mi}^{\mathrm{A}}(t;\theta_0)$*, where* $\xi_{\Lambda,mi}^{\mathrm{A}}(t;\theta_0)$ *is given in* (D.1).

Proof. Note that

$$\sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}}) - \Lambda_0(t)\big\} = \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}}) - \hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0)\big\} + \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0) - \Lambda_0(t)\big\}.$$

The second terms satisfies

$$\begin{aligned} & n^{1/2}\Big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0) - \Lambda_0(t)\Big\} \\ = \;& n^{-1/2}\sum_{i=1}^n \int_0^t \frac{(V_i/\pi(W_i))dM_i(u)}{S_{\mathrm{A}}^{(0)}(\theta_0,u)} + \frac{(1-V_i/\pi(W_i))E\{dM_i(u) \mid W_i\}}{S_{\mathrm{A}}^{(0)}(\theta_0,u)} \\ = \;& n^{-1/2}\sum_{i=1}^n \int_0^t \frac{(V_i/\pi(W_i))dM_i(u)}{s^{(0)}(\theta_0,u)} + \frac{(1-V_i/\pi(W_i))E\{dM_i(u) \mid W_i\}}{s^{(0)}(\theta_0,u)} + o_p(1). \end{aligned}$$

By the Taylor expansion, we have

$$\sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}}) - \hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0)\big\} = \sqrt{n}\hat{G}^{\mathrm{A}}(t;\theta_t^*)(\hat{\theta}_{\mathrm{A}} - \theta_0) + o_p(1),$$

where

$$\hat{G}^{\mathrm{A}}(t;\theta) = -\frac{1}{n}\sum_{i=1}^n \int_0^t \left\{\frac{S_{\mathrm{A}}^{(1)}(\theta,u)}{S_{\mathrm{A}}^{(0)}(\theta,u)^{\otimes 2}}\right\} dN_i(u),$$

and θ_t^* is on the line segment between $\hat{\theta}^{\mathrm{A}}$ and θ_0. By (11), $\hat{G}^{\mathrm{A}}(t;\theta) = G(t;\theta) + o_p(1)$ uniformly over t and θ. Therefore,

$$\begin{aligned} & \sqrt{n}\big\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}}) - \hat{\Lambda}_0^{\mathrm{A}}(t;\theta_0)\big\} \\ = \;& n^{-1/2}G(t;\theta_0){I_{\theta_0}}^{-1}\sum_{i=1}^n \Big[\{V_i/\pi(W_i)\}M_{Q,i}^* + \{1 - V_i/\pi(W_i)\}E(M_{Q,i}^*|W_i)\Big] + o_p(1). \end{aligned}$$

This implies that

$$n^{1/2}\{\hat{\Lambda}_0^{\mathrm{A}}(t;\hat{\theta}_{\mathrm{A}}) - \Lambda_0(t)\} = n^{-1/2}\sum_{i=1}^n \xi_{\Lambda,mi}^{\mathrm{A}}(t;\theta_0) + o_p(1),$$

where

$$\xi^{\mathrm{A}}_{\Lambda,mi}(t;\theta_0) = \int_0^t \{V_i/\pi(W_i)\}\{1/s^{(0)}(\theta_0,u)\}dM_i(u)$$
$$+ \int_0^t \{1 - V_i/\pi(W_i)\}\{1/s^{(0)}(\theta_0,u)\}E\{dM_i(u) \mid W_i\}$$
$$+ G(t;\theta_0){I_{\theta_0}}^{-1}\left[\{V_i/\pi(W_i)\}M^*_{Q,i} + \{1 - V_i/\pi(W_i)\}E(M^*_{Q,i}|W_i)\right]. \tag{D.1}$$

Lemma 12: *Under the regularity conditions in Appendix A,*

$$n^{1/2}(\hat\alpha_{\mathrm{A}} - \alpha_0) = -n^{1/2}\Gamma^{-1}U^{\mathrm{A}}_{\alpha}(\alpha_0) + o_p(1),$$

and $n^{1/2}(\hat\alpha_{\mathrm{A}} - \alpha_0) \to_d N(0,\Psi)$*, where*

$$\Psi = \Gamma^{-1}E\left[\frac{\pi^*(W_i)}{\pi(W_i)^2}\big\{(Z_i - \alpha_0^T X_i)X_i\big\}^{\otimes 2} + \frac{\pi^2(W_i) - \pi^*(W_i)}{\pi^2(W_i)}\big\{E((Z_i - \alpha_0^T X_i)X_i \mid W_i)\big\}^{\otimes 2}\right]\Gamma^{-1}.$$

Moreover, given $x \in \mathcal{X}$*,* $n^{1/2}\{\hat H_{\mathrm{A}}(z - \hat\alpha_{\mathrm{A}}^T x) - H(z - \alpha_0^T x)\}$ *converges weakly to a zero-mean Gaussian process and admits an asymptotic i.i.d. representation* $n^{-1/2}\sum_{i=1}^n h^{\mathrm{A}}_{mi}(z,x;\alpha_0)$*, where* $h^{\mathrm{A}}_{mi}(z,x;\alpha_0)$ *is given in* (D.2)*.*

Proof. Note that

$$\frac{\partial}{\partial\alpha}U^{\mathrm{A}}_{\alpha}(\alpha_0) = \frac{1}{n}\sum_{i=1}^n X_i X_i^T = \Gamma + o_p(1).$$

By the Taylor expansion, we have

$$\begin{aligned} n^{1/2}(\hat\alpha_{\mathrm{A}} - \alpha_0) &= \left\{-\frac{\partial}{\partial\alpha}U^{\mathrm{A}}_{\alpha}(\alpha_0)\right\}^{-1} n^{1/2}U^{\mathrm{A}}_{\alpha}(\alpha_0) + o_p(1) \\ &= -n^{1/2}\Gamma^{-1}U^{\mathrm{A}}_{\alpha}(\alpha_0) + o_p(1). \end{aligned}$$

It is easily seen that, when $\pi(W_i)$ is misspecified,

$$n^{1/2}U^{\mathrm{A}}_{\alpha}(\alpha_0) \to_d N(0,\Psi'),$$

where

$$\Psi' = E\left[\frac{\pi^*(W_i)}{\pi(W_i)^2}\big\{(Z_i - \alpha_0^T X_i)X_i\big\}^{\otimes 2} + \frac{\pi^2(W_i) - \pi^*(W_i)}{\pi^2(W_i)}\big\{E((Z_i - \alpha_0^T X_i)X_i \mid W_i)\big\}^{\otimes 2}\right].$$

Therefore, we have $n^{1/2}(\hat\alpha_{\mathrm{A}} - \alpha_0) \to_d N(0,\Psi)$, where $\Psi = \Gamma^{-1}\Psi'\Gamma^{-1}$.

Now, we consider the asymptotic expansion for $\hat{\alpha}_{\mathrm{A}}$. By the definition of $\hat{H}_{\mathrm{A}}(\cdot)$, we have

$$\begin{aligned} & n^{1/2}\{\hat{H}_{\mathrm{A}}(z-\hat{\alpha}_{\mathrm{A}}^T x)-H(z-\alpha_0^T x)\} \\ = \; & n^{-1/2}\sum_{i=1}^{n}\Big\{F(z+\hat{\alpha}_{\mathrm{A}}^T(X_i-x)\mid W_i)-H(z-\alpha_0^T x)\Big\} \\ & +n^{-1/2}\sum_{i=1}^{n}\frac{V_i}{\pi(W_i)}\Big\{I(Z_i-\hat{\alpha}_{\mathrm{A}}^T X_i\leqslant z-\hat{\alpha}_{\mathrm{A}}^T x)-F(z+\hat{\alpha}_{\mathrm{A}}^T(X_i-x)\mid W_i)\Big\}. \end{aligned}$$

We now consider the first term. The Taylor expansion yields

$$\begin{aligned} & n^{-1/2}\sum_{i=1}^{n}\Big\{F(z+\hat{\alpha}_{\mathrm{A}}^T(X_i-x)\mid W_i)-H(z-\alpha_0^T x)\Big\} \\ = \; & n^{-1/2}\sum_{i=1}^{n}\Big\{F(z+\alpha_0^T(X_i-x)\mid W_i)-H(z-\alpha_0^T x)\Big\} \\ & +n^{-1/2}\sum_{i=1}^{n}F'(z+\alpha_0^T(X_i-x)\mid W_i)(\hat{\alpha}_{\mathrm{A}}-\alpha_0)^T(X_i-x)+o_p(1) \\ = \; & n^{-1/2}\sum_{i=1}^{n}\Big\{F(z+\alpha_0^T(X_i-x)\mid W_i)-H(z-\alpha_0^T x)\Big\} \\ & +n^{1/2}(\hat{\alpha}_{\mathrm{A}}-\alpha_0)^T E\Big\{F'(z+\alpha_0^T(X_i-x)\mid W_i)(X_i-x)\Big\}+o_p(1). \end{aligned}$$

Note that

$$E\Big[\Big\{I(Z_i-\alpha_0^T X_i\leqslant z-\alpha_0^T x)-F(z+\alpha_0^T(X_i-x)\mid W_i)\Big\}\mid W_i\Big]=0.$$

Denote

$$G(\alpha)=n^{-1/2}\sum_{i=1}^{n}\frac{V_i}{\pi(W_i)}\Big\{I(Z_i-\alpha^T X_i\leqslant z-\alpha^T x)-F(z+\alpha^T(X_i-x)\mid W_i)\Big\}.$$

Following similar arguments as in Lemma 2, $G(\alpha)$ converges weakly to a zero mean Gaussian process indexed by α. By the equicontinuity property, we have $G(\hat{\alpha}_{\mathrm{A}})=G(\alpha_0)+o_p(1)$. This implies that

$$\begin{aligned} & n^{-1/2}\sum_{i=1}^{n}\frac{V_i}{\pi(W_i)}\Big\{I(Z_i-\hat{\alpha}_{\mathrm{A}}^T X_i\leqslant z-\hat{\alpha}_{\mathrm{A}}^T x)-F(z+\hat{\alpha}_{\mathrm{A}}^T(X_i-x)\mid W_i)\Big\} \\ = \; & n^{-1/2}\sum_{i=1}^{n}\frac{V_i}{\pi(W_i)}\Big\{I(Z_i-\alpha_0 X_i\leqslant z-\alpha_0 x)-F(z+\alpha_0(X_i-x)\mid W_i)\Big\}+o_p(1). \end{aligned}$$

Therefore, we have $n^{1/2}\{\hat{H}_{\mathrm{A}}(z-\hat{\alpha}_{\mathrm{A}}^T x)-H(z-\alpha_0^T x)\}=n^{-1/2}\sum_{i=1}^n h_{mi}^{\mathrm{A}}(z,x;\alpha_0)$, where

$$\begin{aligned} h_{mi}^{\mathrm{A}}(z,x;\alpha_0) = &\Big\{F(z+\alpha_0^T(X_i-x)\mid W_i)-H(z-\alpha_0^T x)\Big\} \\ &-\Gamma^{-1}(U_{i,\alpha}^A(\alpha_0))^T E\Big\{F'(z+\alpha_0^T(X_i-x)\mid W_i)(X_i-x)\Big\} \\ &+\frac{V_i}{\pi(W_i)}\Big\{I(Z_i-\alpha_0 X_i\leqslant z-\alpha_0 x)-F(z+\alpha_0(X_i-x)\mid W_i)\Big\}, \end{aligned} \tag{D.2}$$

where $U_{i,\alpha}^A(\alpha_0)=\frac{V_i}{\pi(W_i)}(X_iZ_i-X_iX_i^T\alpha_0)-(1-\frac{V_i}{\pi(W_i)})E\{(X_iZ_i-X_iX_i^T\alpha)\mid W_i\}$. This completes the proof.

THEOREM 1: *Under the regularity conditions in Appendix A, if $f(Z_i\mid W_i)$ is correctly specified, then given $(x,t)\in\mathcal{X}\times[0,\tau]$, $n^{1/2}\{\widetilde{\mathrm{TPR}_{\mathbb{C}}}^{\mathrm{P}}(\cdot;t,x)-\mathrm{TPR}_{\mathbb{C}}(\cdot;t,x)\}$, $n^{1/2}\{\widetilde{\mathrm{TPR}_{\mathbb{I}}}^{\mathrm{P}}(\cdot;t,x)-\mathrm{TPR}_{\mathbb{I}}(\cdot;t,x)\}$ and $n^{1/2}\{\widetilde{\mathrm{FPR}_{\mathbb{D}}}^{\mathrm{P}}(\cdot;t,x)-\mathrm{FPR}_{\mathbb{D}}(\cdot;t,x)\}$ converge weakly to zero-mean Gaussian processes on $\mathcal{Z}$. As a result, given $(x,t)\in\mathcal{X}\times[0,\tau]$, $n^{1/2}\{\widetilde{\mathrm{ROC}}^{\mathrm{P}}_{\mathbb{I}/\mathbb{D}}(\cdot;t,x)-\mathrm{ROC}_{\mathbb{I}/\mathbb{D}}(\cdot;t,x)\}$ and $n^{1/2}\{\widetilde{\mathrm{ROC}}^{\mathrm{P}}_{\mathbb{C}/\mathbb{D}}(\cdot;t,x)-\mathrm{ROC}_{\mathbb{C}/\mathbb{D}}(\cdot;t,x)\}$ converge weakly to zero-mean Gaussian processes on $[p,q]$.*

Proof. Let $\xi_{\theta,mi}^{\mathrm{A}}=I_{\theta_0}{}^{-1}\big\{(V_i/\pi_i)M_{Q,i}^{*}+(1-V_i/\pi_i)M_{Q,i}^{*o}\big\}$ denote the asymptotic representation for $\hat{\theta}_A$. Then we have

$$\sqrt{n}\{\hat{S}^{\mathrm{A}}(t\mid q)-S(t\mid q)\}=-n^{-\frac{1}{2}}\sum_{i=1}^n \xi_{S,mi}^{\mathrm{A}}(t;\theta_0,\phi_0)+o_p(1),$$

where

$$\xi_{S,mi}^{\mathrm{A}}(t;\theta_0,\phi_0)=S(t\mid q)\exp({\theta_0}^T q)\{\xi_{\Lambda,mi}^{\mathrm{A}}(t;\theta_0,\phi_0)+\Lambda_0(t)q^T\xi_{\theta,mi}^{\mathrm{A}}\}.$$

Similar to the proof of Theorems 1 and 3, we can show that $\sqrt{n}\{\widetilde{\mathrm{TPR}_{\mathbb{C}}}^{\mathrm{P}}(\cdot;t,x)-\mathrm{TPR}_{\mathbb{C}}(\cdot;t,x)\}$, $\sqrt{n}\{\widetilde{\mathrm{TPR}_{\mathbb{I}}}^{\mathrm{P}}(c;t,x)-\mathrm{TPR}_{\mathbb{I}}(c;t,x)\}$ and $\sqrt{n}\{\widetilde{\mathrm{FPR}_{\mathbb{D}}}^{\mathrm{P}}(c;t,x)-\mathrm{FPR}_{\mathbb{D}}(c;t,x)\}$ have asymptotic representation $n^{-\frac{1}{2}}\sum_{i=1}^n\eta_{mi}^{\mathrm{A}}(c;t,x)$, $n^{-\frac{1}{2}}\sum_{i=1}^n\eta_{mi,*}^{\mathrm{A}}(c;t,x)$ and $n^{-\frac{1}{2}}\sum_{i=1}^n\omega_{mi}^{\mathrm{A}}(c;t,x)$, respectively, where $\eta_{mi}^{\mathrm{A}}(c;t,x)$ is obtained by replacing $\xi_{S,i}$, h_i with $\xi_{S,mi}^{\mathrm{A}}$, h_{mi}^{A} in (C.2), $\eta_{mi,*}^{\mathrm{A}}(c;t,x)$ is obtained by replacing $\xi_{S,i}$, h_i with $\xi_{S,mi}^{\mathrm{A}}$, h_{mi}^{A} in (C.3), and $\omega_{mi}^{\mathrm{A}}(c;t,x)$ is obtained by replacing $\xi_{S,i}$, h_i with $\xi_{S,mi}^{\mathrm{A}}$, h_{mi}^{A} in (C.1).

Using the functional Taylor expansion, we have $\sqrt{n}\{\widetilde{\mathrm{ROC}}^{\mathrm{P}}_{\mathbb{C}/\mathbb{D}}(c;t,x) - \mathrm{ROC}_{\mathbb{C}/\mathbb{D}}(c;t,x)\} = n^{\frac{1}{2}} \sum_{i=1}^{n} \phi^{\mathrm{A}}_{mi}(c;t,x) + o_p(1)$, where

$$\phi^{\mathrm{A}}_{mi}(c;t,x) = \eta^{\mathrm{A}}_{mi}\{\mathrm{FPR}^{-1}(c;t,x);t,x\} - f_1\{\mathrm{FPR}^{-1}(c;t,x);t,x\}\frac{\omega^{\mathrm{A}}_{mi}\{\mathrm{FPR}^{-1}(c;t,x);t,x\}}{f_0\{\mathrm{FPR}^{-1}(c;t,x);t,x\}},$$

Similarly, $\sqrt{n}\{\widetilde{\mathrm{ROC}}^{\mathrm{P}}_{\mathbb{I}/\mathbb{D}}(\cdot;t,x) - \mathrm{ROC}_{\mathbb{I}/\mathbb{D}}(\cdot;t,x)\} = n^{\frac{1}{2}} \sum_{i=1}^{n} \phi^{\mathrm{A}}_{mi,*}(c;t,x) + o_p(1)$, where

$$\phi^{\mathrm{A}}_{mi,*}(c;t,x) = \eta^{\mathrm{A}}_{mi,*}\{\mathrm{FPR}^{-1}(c;t,x);t,x\} - f_1^*\{\mathrm{FPR}^{-1}(c;t,x);t,x\}\frac{\omega^{\mathrm{A}}_{mi}\{\mathrm{FPR}^{-1}(c;t,x);t,x\}}{f_0\{\mathrm{FPR}^{-1}(c;t,x);t,x\}}.$$

Appendix E: Additional Technical Details

Calculation of $\tilde{E}(\cdot \mid W_i;\chi)$

In Section 3 of the main manuscript, the estimating equations $U^{\mathrm{AP}}(\chi)$ involve $\tilde{E}(\cdot \mid W_i;\chi)$, where the expectation is taken with respect to the functions of X_i, $N_i(\cdot)$, and Z_i. Note that, X_i and $N_i(\cdot)$ are known given W_i. Therefore, the problem reduces to estimation of $\tilde{E}(g(Z_i) \mid W_i;\chi)$, where $g(Z_i)$ is an arbitrary function of Z_i. To calculate $\tilde{E}(g(Z_i) \mid W_i;\chi)$, we draw multiple samples $\{z_1, z_2, ...\}$ from the distribution $\hat{f}(Z_i \mid W_i;\chi)$ using the rejection sampling approach (Gilks et al., 1996), where the unknown parameters χ are replaced with the corresponding estimators in the previous iteration, and then we take the sample average of $\{g(z_1), g(z_2), ...\}$. For the first step in the iteration algorithm, the initial values for parameters in $f(Z_i \mid W_i;\chi)$ can be replaced with the simple weighted estimators. The advantage of the above procedure based on the rejection sampling is that it avoids the complexity of numerical integration. For instance, the calculation of $\tilde{E}(I(Z_i \leqslant z) \mid W_i;\chi)$ for a variety of different values of z using our method is computationally more efficient than the numerical integration.

Therectical SE Estimation

Although we have derived the asymptotic distributions of the proposed TPR, FPR and ROC estimators in Section 4 of the main manuscript, it is intractable to obtain the explicit analytic expressions for the variance-covariance processes. To alleviate this difficulty, we approximate the limiting distributions using resampling techniques, as proposed and used by Parzen et al. (1994); Cai and Pepe (2002) and others. We use $\text{TPR}_{\mathbb{C}}$ as an example for illustration. In Web Appendix C, we show that $n^{1/2}\{\widehat{\text{TPR}_{\mathbb{C}}}(c;t,x) - \text{TPR}_{\mathbb{C}}(c;t,x)\}$ admits an asymptotic i.i.d. representation $n^{-1/2}\sum_{i=1}^{n}\eta_i(c;t,x)$. The following are the main steps to generate stochastic processes that have the same asymptotic distributions as $n^{-1/2}\sum_{i=1}^{n}\eta_i(c;t,x)$ and to construct confidence intervals for $\text{TPR}_{\mathbb{C}}(c;t,x)$. First, we generate samples $\{\epsilon_{i,m}; i = 1,...,n, m = 1,...,M\}$ independently from the standard normal distribution N(0,1). Second, we calculate $\hat{q}_m(c;t,x) = n^{-1/2}\sum_{i=1}^{n}\hat{\eta}_i(c;t,x)\epsilon_{i,m}$, where $\hat{\eta}_i$ is obtained by replacing all the theoretical quantities by their empirical counterparts. Third, for any given c, we can use $V(c;t,x) = M^{-1}\sum_{m=1}^{M}\hat{q}_m^2(c;t,x)$ to approximate the asymptotic variance of $\eta_i(c;t,x)$, and the $(1-\alpha)\times 100\%$ confidence interval is given by $\widehat{\text{TPR}_{\mathbb{C}}}(c;t,x) \pm z_{\alpha/2}V^{1/2}(c;t,x)$, where $z_{\alpha/2}$ is the $(1-\alpha/2)\times 100$th quantile of N(0,1). The above procedure provides a practical way for variance estimation and confidence interval construction.

Appendix F: Additional Simulation Study

In Section 5 of the main manuscript, we conduct simulation studies under two settings. In the first simulation setting, the selection probability is $\pi_i = 0.7\delta_i + 0.3(1-\delta_i)$. In the second simulation setting, the selection probability is given by $\pi_i = 1/\{1+\exp(3-0.8R_i - 0.5X_{1i}^2 - X_{1i} - 0.5\delta_i)\}$. In this Web Appendix, we conduct more extensive simulations with varying sample sizes, censoring rates, missing proportions, and a nonconstant baseline hazard rate.

Tables 1 and 2 present the simulation results under the third simulation setting, where the data generation mechanism is the same as in the second setting but the sample size is 300. The empirical biases of the various estimators are similar to those under the second simulation setting in the main manuscript. In addition, due to reduced sample size, the

SEs of the various estimators are increased compared to those under the second simulation setting.

Tables 3 and 4 present the simulation results under the fourth simulation setting. The sample size is 500. The censoring time follows an exponential distribution with mean $0.5X_2+5$, truncated at 10. The missing proportion is $\pi_i = 1/(1+\exp(2-R_i-0.5\delta_i))$. The baseline hazard $\lambda_0(t)$ follows a Weibull distribution $0.05t^{0.5}$. We compare estimators with misspecified $\hat{\pi}$, where π is estimated logistic regression with δ_i only, and also estimators with correctly specified $\hat{\pi}_i$, where π_i is estimated using logistic regression with covariates $G_i = (R_i, \delta_i)$. The finds are similar to those in the main manuscript.

[Table 1 about here.]

[Table 2 about here.]

[Table 3 about here.]

[Table 4 about here.]

References

Cai, T. and Pepe, M. S. (2002). Semiparametric receiver operating characteristic analysis to evaluate biomarkers for disease. *Journal of the American statistical Association* **97,** 1099–1107.

Foutz, R. V. (1977). On the unique consistent solution to the likelihood equations. *Journal of the American Statistical Association* **72,** 147–148.

Gilks, W. R., Richardson, S., and Spiegelhalter, D. J. (1996). Markov chain monte carlo in practice. *New York: Chapman Hall/CRC* .

Huang, Y. and Wang, C. Y. (2000). Cox regression with accurate covariates unascertainable: A nonparametric-correction approach. *Journal of the American Statistical Association* **95,** 1209–1219.

Luo, X., Tsai, W. Y., and Xu, Q. (2009). Pseudo-partial likelihood estimators for the Cox regression model with missing covariates. *Biometrika* **96,** 617–633.

Parzen, M. I., Wei, L. J., and Ying, Z. (1994). A resampling method based on pivotal estimating functions. *Biometrika* **81,** 341–350.

Qi, L., Wang, C. Y., and Prentice, R. L. (2005). Weighted estimators for proportional hazards regression with missing covariates. *Journal of the American Statistical Association* **100,** 1250–1263.

Robins, J. M., Rotnitzky, A., and Zhao, L. P. (1994). Estimation of regression coefficients when some regressors are not always observed. *Journal of the American Statistical Association* **89,** 846–866.

Shorack, G. R. and Wellner, J. A. (1986). *Empirical processes with applications to statistics.* New York: Wiley.

Song, X. and Zhou, X.-H. (2008). A semiparametric approach for the covariate specific ROC curve with survival outcome. *Statistica Sinica* **18,** 947–965.

Van Der Vaart, A. W. and Wellner, J. A. (1996). *Weak Convergence and empirical processes.* New York: Springer.

Wang, C. Y. and Chen, H. Y. (2001). Augmented inverse probability weighted estimator for Cox missing covariate regression. *Biometrics* **57,** 414–419.

Xu, Q., Paik, M. C., Luo, X., and Tsai, W. Y. (2009). Reweighting estimators for Cox regression with missing covariates. *Journal of the American Statistical Association* **104,** 1155–1167.

Table 1

Simulation results for ROC estimators evaluated at $t = 5$, $X_1 = 0$, and $X_2 = 1$ under the third scenario. G denotes (R, X_1, X_1^2, δ). B is the empirical bias (× 1000); SE is the sample standard error (× 1000); ASE is the average theoretical standard errors (× 1000); CP is the coverage probability of the 95% confidence interval (× 100).

	Incident ROC				Cumulative ROC			
Approach	B	SE	ASE	CP	B	SE	ASE	CP
$v = 0.1$								
Full cohort	0	22	21	93.8	2	31	31	96.1
CC	59	35	37	69.6	-119	53	55	42.6
SWE-π	-4	39	38	95.3	-7	82	67	89.3
SWE-$\hat{\pi}(\delta)$	-16	28	31	94.1	-173	59	59	13.2
SWE-$\hat{\pi}(G)$	-3	38	37	93.1	-6	81	65	88.9
AWE-π, $f_{Z\|X}$	0	45	42	92.2	2	80	70	94.6
AWE-$\hat{\pi}(\delta)$, $\hat{f}_{Z\|X}$	4	33	28	88.9	6	43	47	94.6
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X}$	-4	37	42	94.6	-6	73	74	97.2
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X_1}$	2	42	45	90.3	5	69	73	93.7
$v = 0.3$								
Full cohort	0	26	25	93.8	1	19	18	93.8
CC	57	35	36	65.1	-51	34	35	76.7
SWE-π	-5	42	43	96.1	-3	45	42	93.8
SWE-$\hat{\pi}(\delta)$	-15	31	36	93.8	-93	39	40	36.3
SWE-$\hat{\pi}(G)$	-5	43	44	96.9	-2	46	37	93.0
AWE-π, $f_{Z\|X}$	1	42	44	97.6	1	42	43	95.3
AWE-$\hat{\pi}(\delta)$, $\hat{f}_{Z\|X}$	4	36	32	90.7	4	24	29	96.9
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X}$	-1	41	45	96.1	-3	41	42	94.0
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X_1}$	3	41	48	93.9	2	38	47	96.3
$v = 0.5$								
Full cohort	-1	19	19	92.2	0	10	10	96.1
CC	39	24	23	58.9	-20	20	19	88.4
SWE-π	-2	31	31	94.6	-1	24	31	96.9
SWE-$\hat{\pi}(\delta)$	-10	25	27	95.3	-45	23	24	62.0
SWE-$\hat{\pi}(G)$	-4	30	31	96.8	-1	24	22	96.2
AWE-π, $f_{Z\|X}$	1	30	32	96.8	0	22	27	96.8
AWE-$\hat{\pi}(\delta)$, $\hat{f}_{Z\|X}$	3	26	24	91.4	2	14	16	96.1
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X}$	-1	30	32	95.3	-1	21	23	94.6
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X_1}$	2	30	33	93.3	1	20	27	97.4

Table 2

Simulation results for TPR and FPR estimators evaluated at $t = 5$, $(X_1, X_2) = (0, 1)$ under the third scenario. G denotes (R, X_1, X_1^2, δ). B is the empirical bias (× 1000); SE is the sample standard error (× 1000); ASE is the average theoretical standard errors (× 1000); CP is the coverage probability of the 95% confidence interval (× 100).

	Incident TPR				Cumulative TPR				FPR			
Approach	B	SE	ASE	CP	B	SE	ASE	CP	B	SE	ASE	CP
z = -1.4												
Full cohort	2	24	24	93.0	0	10	10	92.1	3	53	52	94.5
CC	-17	36	30	90.9	-41	28	21	61.2	-92	59	48	48.8
SWE-π	2	39	36	94.6	0	25	27	97.6	-5	63	65	95.3
SWE-$\hat{\pi}(\delta)$	5	34	27	90.5	-23	23	17	82.9	27	51	52	93.0
SWE-$\hat{\pi}(G)$	-5	39	34	93.7	-5	24	20	93.7	-1	61	64	97.7
AWE-π, $f_{Z\|X}$	-2	38	37	97.7	-1	19	26	98.9	-3	63	64	96.1
AWE-$\hat{\pi}(\delta)$, $\hat{f}_{Z\|X}$	1	32	37	97.6	0	14	18	96.8	-6	62	59	93.8
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X}$	-4	38	38	96.1	-2	22	23	96.9	-4	64	64	94.6
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X_1}$	4	37	43	95.6	1	19	28	97.5	5	58	69	95.7
z = -1.1												
Full cohort	0	35	37	95.3	-1	16	15	91.6	-1	45	49	97.6
CC	-22	48	45	94.6	-74	40	33	41.1	-92	47	43	51.7
SWE-π	4	51	52	93.0	-4	37	35	95.3	-3	57	63	97.6
SWE-$\hat{\pi}(\delta)$	13	47	43	90.7	-45	35	28	70.6	33	51	53	93.0
SWE-$\hat{\pi}(G)$	-5	51	52	93.7	-8	35	29	91.6	0	55	63	97.3
AWE-π, $f_{Z\|X}$	-5	50	55	96.1	-2	31	35	97.8	-7	58	60	93.5
AWE-$\hat{\pi}(\delta)$, $\hat{f}_{Z\|X}$	0	45	54	99.2	1	22	31	98.2	-7	54	58	97.5
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X}$	-7	53	55	93.0	-4	32	35	97.5	-6	58	61	93.8
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X_1}$	2	54	62	95.7	2	31	39	96.2	1	55	62	96.3
z = -0.8												
Full cohort	0	47	48	96.1	1	25	22	93.0	1	37	39	94.5
CC	-24	61	57	92.2	-122	55	46	25.6	-64	36	33	57.3
SWE-π	8	74	67	90.7	-2	62	53	89.4	1	55	53	90.8
SWE-$\hat{\pi}(\delta)$	22	61	58	93.0	-79	53	42	54.3	29	44	47	96.1
SWE-$\hat{\pi}(G)$	0	73	66	91.4	-6	59	48	89.6	3	53	53	94.6
AWE-π, $f_{Z\|X}$	-3	73	74	93.8	1	50	52	96.9	-3	55	49	89.4
AWE-$\hat{\pi}(\delta)$, $\hat{f}_{Z\|X}$	-5	59	69	98.4	2	34	47	99.2	-7	44	47	94.6
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X}$	-6	75	73	92.9	-3	55	51	92.2	-3	56	49	88.9
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X_1}$	5	74	80	96.4	1	52	57	94.5	3	47	55	96.1

Table 3

Simulation results for ROC estimators evaluated at $t = 5$, $X_1 = 0$, and $X_2 = 1$ under the fourth scenario. G denotes (R, δ). B is the empirical bias (× 1000); SE is the sample standard error (× 1000); ASE is the average theoretical standard errors (× 1000); CP is the coverage probability of the 95% confidence interval (× 100).

	Incident ROC				Cumulative ROC			
Approach	B	SE	ASE	CP	B	SE	ASE	CP
$v = 0.1$								
Full cohort	1	17	18	95.7	1	29	25	91.2
CC	55	29	32	66.3	-61	41	45	71.5
SWE-π	-4	29	31	95.8	2	52	49	94.7
SWE-$\hat{\pi}(\delta)$	5	25	32	98.9	-59	48	49	77.9
SWE-$\hat{\pi}(G)$	-3	30	31	96.8	-4	53	48	94.7
AWE-π, $f_{Z\|X}$	0	33	33	93.4	3	52	53	94.6
AWE-$\hat{\pi}(\delta)$, $\hat{f}_{Z\|X}$	-1	28	29	92.8	2	34	36	95.7
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X}$	4	33	32	94.7	-3	44	48	92.6
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X_1}$	4	36	36	91.4	4	53	53	94.3
$v = 0.3$								
Full cohort	-1	19	20	95.7	0	16	15	92.6
CC	53	29	32	64.2	-24	26	28	89.4
SWE-π	-3	34	36	97.7	0	30	30	95.7
SWE-$\hat{\pi}(\delta)$	6	27	36	97.8	-25	29	30	91.6
SWE-$\hat{\pi}(G)$	-2	32	35	95.7	1	30	28	92.6
AWE-π, $f_{Z\|X}$	-1	35	36	94.9	-1	29	30	95.4
AWE-$\hat{\pi}(\delta)$, $\hat{f}_{Z\|X}$	4	28	35	97.8	4	24	28	96.8
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X}$	5	34	35	96.6	3	32	30	91.6
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X_1}$	2	40	42	90.3	2	32	32	92.0
$v = 0.5$								
Full cohort	-1	14	15	95.7	0	9	8	94.7
CC	35	19	21	60.1	-8	14	15	97.9
SWE-π	-3	24	27	96.8	0	15	19	98.9
SWE-$\hat{\pi}(\delta)$	2	21	26	96.8	-14	16	17	95.7
SWE-$\hat{\pi}(G)$	-3	23	25	96.8	-1	16	16	95.7
AWE-π, $f_{Z\|X}$	0	26	27	93.9	0	16	18	96.2
AWE-$\hat{\pi}(\delta)$, $\hat{f}_{Z\|X}$	4	22	26	97.3	3	13	16	97.3
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X}$	2	25	27	95.7	0	17	17	96.7
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X_1}$	4	28	30	93.2	1	18	20	95.5

Table 4

Simulation results for TPR and FPR estimators evaluated at $t = 5$, $(X_1, X_2) = (0, 1)$ under the second scenario. G denotes (R, δ). B is the empirical bias (× 1000); SE is the sample standard error (× 1000); ASE is the average theoretical standard errors (× 1000); CP is the coverage probability of the 95% confidence interval (× 100).

	Incident TPR				Cumulative TPR				FPR			
Approach	B	SE	ASE	CP	B	SE	ASE	CP	B	SE	ASE	CP
z = -1.4												
Full cohort	0	22	20	92.6	0	9	8	92.6	-1	42	42	92.7
CC	1	26	26	94.7	-20	17	15	89.4	-55	51	47	81.2
SWE-π	0	32	32	91.6	-1	16	18	98.6	1	66	62	92.7
SWE-$\hat{\pi}(\delta)$	-7	28	32	96.5	-14	15	15	93.1	-15	51	53	97.9
SWE-$\hat{\pi}(G)$	-2	32	32	92.6	-2	16	14	92.4	1	65	62	93.8
AWE-π, $f_{Z\|X}$	0	35	32	92.1	0	16	19	96.2	3	65	61	91.7
AWE-$\hat{\pi}(\delta)$, $\hat{f}_{Z\|X}$	5	29	33	98.6	3	12	15	98.4	6	54	61	97.8
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X}$	1	33	34	95.6	0	15	16	95.7	-1	67	60	91.1
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X_1}$	0	40	46	98.2	0	19	22	97.3	1	74	72	95.1
z = -1.1												
Full cohort	0	32	31	93.6	0	13	12	92.1	1	41	39	93.6
CC	0	39	41	96.8	-70	25	25	75.7	-53	46	43	73.6
SWE-π	0	49	48	93.8	-3	26	26	95.7	-2	62	59	91.6
SWE-$\hat{\pi}(\delta)$	-13	41	50	96.5	-42	24	24	91.6	-20	50	51	94.7
SWE-$\hat{\pi}(G)$	-3	48	47	97.7	-2	25	22	92.6	-1	61	59	91.6
AWE-π, $f_{Z\|X}$	2	50	49	93.8	1	26	27	97.1	4	61	58	92.2
AWE-$\hat{\pi}(\delta)$, $\hat{f}_{Z\|X}$	4	48	52	98.7	2	21	26	97.3	3	56	62	98.5
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X}$	0	49	51	94.6	2	25	27	96.5	-2	63	56	91.8
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X_1}$	-1	59	62	97.7	1	29	33	98.1	0	70	69	94.4
z = -0.8												
Full cohort	1	40	41	97.8	1	21	19	90.8	-1	32	32	94.7
CC	-1	50	55	97.5	-115	38	36	61.2	-43	36	34	73.7
SWE-π	2	57	62	95.7	-4	35	34	94.7	2	49	48	94.7
SWE-$\hat{\pi}(\delta)$	-17	51	61	99.3	-75	34	36	81.2	-16	40	45	93.8
SWE-$\hat{\pi}(G)$	1	56	60	96.4	-4	34	32	91.6	3	48	49	94.7
AWE-π, $f_{Z\|X}$	4	59	62	95.6	2	36	36	94.7	4	50	48	93.2
AWE-$\hat{\pi}(\delta)$, $\hat{f}_{Z\|X}$	-3	57	59	97.2	0	30	36	98.3	2	45	51	96.8
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X}$	5	59	61	94.7	3	36	39	95.7	-3	51	44	93.7
AWE-$\hat{\pi}(G)$, $\hat{f}_{Z\|X_1}$	3	72	80	98.3	3	42	45	96.8	1	55	56	95.7