NUTTALL'S THEOREM WITH ANALYTIC WEIGHTS ON ALGEBRAIC S-CONTOURS

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Dedicated to the memories of Herbert Stahl, brilliant mathematician and a kind friend, and Andrei Alexandrovich Gonchar, great visionary and a wonderful teacher.

ABSTRACT. Given a function f holomorphic at infinity, the n-th diagonal Padé approximant to f, denoted by $[n/n]_f$, is a rational function of type (n, n) that has the highest order of contact with f at infinity. Nuttall's theorem provides an asymptotic formula for the error of approximation $f - [n/n]_f$ in the case where f is the Cauchy integral of a smooth density with respect to the arcsine distribution on $[-1, 1]$. In this note, Nuttall's theorem is extended to Cauchy integrals of analytic densities on the so-called algebraic S-contours (in the sense of Nuttall and Stahl).

1 INTRODUCTION

Let

(1) $$f(z) = \sum_{k \geqslant 0} f_k z^{-k}$$

be a convergent power series. A diagonal Padé approximant to f at infinity is a rational function that has the highest order of contact with f at infinity [18, 5]. More precisely, let (P_n, Q_n) be a pair of polynomials, each of degree at most n, satisfying

(2) $$R_n(z) := (Q_n f - P_n)(z) = O\left(1/z^{n+1}\right) \quad \text{as} \quad z \to \infty.$$

It is not hard to verify that the above relation can be equivalently written as a linear system in terms of the Laurent coefficients of f, P_n, and Q_n with one more unknown than equations. Therefore the system is always solvable and no solution of it can be such that $Q_n \equiv 0$ (we may thus assume that Q_n is monic). In general, a solution of (2) is not unique. However, if (P_n, Q_n) and $(\tilde{P}_n, \tilde{Q}_n)$ are two distinct solutions, then $P_n \tilde{Q}_n - \tilde{P}_n Q_n \equiv 0$ since this difference must behave like $O(1/z)$ near the point at infinity as easily follows from (2). Thus, each solution of (2) is of the form (LP_n, LQ_n), where (P_n, Q_n) is the unique solution of minimal degree. Hereafter, (P_n, Q_n) will always stand for this unique pair of polynomials. A *diagonal Padé approximant* to f of type (n, n), denoted by $[n/n]_f$, is defined as $[n/n]_f := P_n/Q_n$.

We say that a function f of the form (1) belongs to the class $\mathcal{S}$ if it has a meromorphic continuation along any arc originating at infinity that belongs to $\mathbb{C} \setminus E_f$, $\mathsf{cp}(E_f) = 0$, and there are points in $\mathbb{C} \setminus E_f$ to which f possesses distinct continuations.[1] Given $f \in \mathcal{S}$, a compact set K is called *admissible* if $\overline{\mathbb{C}} \setminus K$ is connected and f has a meromorphic and single-valued extension there. The following theorems summarize one of the fundamental contributions of Herbert Stahl to complex approximation theory [21, 22, 23, 24].

Theorem (Stahl). *Given $f \in \mathcal{S}$, there exists the unique admissible compact Δ_f such that $\mathsf{cp}(\Delta_f) \leqslant \mathsf{cp}(K)$ for any admissible compact K and $\Delta_f \subseteq K$ for any admissible K satisfying $\mathsf{cp}(\Delta_f) = \mathsf{cp}(K)$. Furthermore, Padé approximants $[n/n]_f$ converge to f in logarithmic capacity in $D_f := \overline{\mathbb{C}} \setminus \Delta_f$.*

[1] $\mathsf{cp}(\cdot)$ stands for logarithmic capacity [20].

The domain D_f is optimal in the sense that the convergence does not hold in any other domain D such that $D \setminus D_f \neq \varnothing$.

The minimal capacity set Δ_f, the boundary of the extremal domain D_f, has a rather special structure.

Theorem (Stahl). *It holds that*

$$\Delta_f = E_0 \cup E_1 \cup \bigcup \Delta_j,$$

where $E_0 \subseteq E_f$, E_1 consists of isolated points to which f has unrestricted continuations from the point at infinity leading to at least two distinct function elements, and Δ_j are open analytic arcs.

Moreover, the set Δ_f possesses Stahl's symmetry property.

Theorem (Stahl). *Let $\partial/\partial n^{\pm}$ be the one-sided normal derivatives on $\bigcup \Delta_j$. It holds that*

$$\frac{\partial g_{\Delta_f}}{\partial n^+} = \frac{\partial g_{\Delta_f}}{\partial n^-} \quad on \quad \bigcup \Delta_j,$$

where g_{Δ_f} is Green's function with pole at infinity for D_f.

Finally, the arcs Δ_j can be described as trajectories of a certain quadratic differential.

Theorem (Stahl). *Let $h_{\Delta_f}(z) = 2\partial_z g_{\Delta_f}(z)$, where $2\partial_z := \partial_x - i\partial_y$. The function $h_{\Delta_f}^2$ is holomorphic in D_f, has a zero of order 2 at infinity, and the arcs Δ_j are negative critical trajectories of the quadratic differential $h_{\Delta_f}^2(z)dz^2$. That is, for any smooth parametrization $z(t) : (0,1) \to \Delta_j$ it holds that $h_{\Delta_f}^2(z(t))\big(z'(t)\big)^2 < 0$ for all $t \in (0,1)$.*

If $f \in \mathcal{S}$ is an algebraic function, then the set E_f is finite and so is the collection $\bigcup \Delta_j$. This motivated the following definition.

Definition 1. *A compact set Δ is called an algebraic S-contour if the complement of Δ, say D, is connected,*

$$\Delta = E_0 \cup E_1 \cup \bigcup \Delta_j,$$

where $\bigcup \Delta_j$ is a finite union of open analytic arcs, $E_0 \cup E_1$ is a finite set of points such that each element of E_0 is an endpoint of exactly one arc Δ_j while each element of E_1 is an endpoint of at least three arcs, and

$$\frac{\partial g_\Delta}{\partial n^+} = \frac{\partial g_\Delta}{\partial n^-} \quad on \quad \bigcup \Delta_j,$$

where g_Δ is Green's function for D with pole at infinity.

Any algebraic S-contour is a minimal capacity contour for some algebraic function f. Given Δ, an eligible function $f_\Delta \in \mathcal{S}$ can be constructed in the following way. Denote by m the number of connected components of Δ, by E_{0j} the intersection of E_0 with the j-th connected component, and by m_j the cardinality of E_{0j}. Then one can take $f_\Delta(z) = \sum_{j=1}^m \big(\prod_{e \in E_{0j}} (z-e) \big)^{-1/m_j}$.

Algebraic S-contours admit a description via critical trajectories of *rational* quadratic differentials. For such a contour Δ, set

$$(3) \qquad\qquad h_\Delta(z) := 2\partial_z g_\Delta(z).$$

For each point $e \in E_0 \cup E_1$ denote by $i(e)$ the *bifurcation index* of e, that is, the number of different arcs Δ_j incident with e. It follows immediately from the definition of an algebraic S-contour that $i(e) = 1$ for $e \in E_0$ and $i(e) \geqslant 3$ for $e \in E_1$. Denote also by E_2 the set of *critical points* of g_Δ with $j(e)$ standing for the *order* of $e \in E_2$, i.e., $\partial_z^j g_\Delta(e) = 0$ for $j \in \{1, \ldots, j(e)\}$ and $\partial_z^{j(e)+1} g_\Delta(e) \neq 0$. The set E_2 is necessarily finite.

Theorem (Perevoznikova-Rakhmanov) [19]. *Let Δ be an algebraic S-contour. Then the arcs Δ_j are negative critical trajectories of the quadratic differential $h_\Delta^2(z)dz^2$. Moreover,*

$$h_\Delta^2(z) = \prod_{e \in E_0 \cup E_1} (z-e)^{i(e)-2} \prod_{e \in E_2} (z-e)^{2j(e)}$$

and $h_\Delta^2(z) = z^{-2} + O\big(z^{-3}\big)$ as $z \to \infty$.

The reason to restrict our attention from all possible S-contours to the algebraic ones is that one might hope for a stronger convergence than convergence in capacity. Indeed, it was suggested by Nuttall [15] that if

(4)
$$f_\rho(z) := \frac{1}{2\pi i} \int_\Delta \frac{(\rho/w_\Delta^+)(t)}{t-z} dt, \quad z \in \overline{\mathbb{C}} \setminus \Delta,$$

where ρ is a Hölder continuous and non-vanishing function on an algebraic S-contour Δ and

(5)
$$w_\Delta^2(z) := \prod_{e \in E_\Delta} (z-e)$$

with E_Δ being the subset of $E_0 \cup E_1$ consisting of points having odd bifurcation index, then the diagonal Padé approximants $[n/n]_{f_\rho}$ converge to f_ρ *"nearly"* uniformly in D (uniformly if Δ is an interval). The absence of the uniform convergence is due to the presence of a finite number of *"wandering"* or *"spurious"* poles [13, 25], see the discussion after Theorem 5 further below. The presence of these poles was already observed by Akhiezer [2, Section 53] and [1], who considered the case of Δ being a union of several real intervals and ρ being a positive polynomial on Δ (the so-called Bernstein-Szegő case). Nuttall himself, in the joint work with Singh [17], extended Akhiezer's method to an arbitrary algebraic S-contour and an arbitrary non-vanishing polynomial (getting rid of positivity). Later, Nuttall showed the validity of his claim on an interval [16] using the method of the singular integral equations. With the help of this method, Nuttall's claim has been verified by Suetin [26, 27] when Δ is a disjoint union of analytic arcs and by Baratchart and the author [7] when Δ is a union of three arcs meeting at one point. Martínez Finkelshtein, Rakhmanov, and Suetin also considered the case of connected algebraic S-contours and semi-classical weights using WKB analysis [9]. In this note we prove Nuttall's theorem on an arbitrary algebraic S-contour but only when ρ in (4) is holomorphic and non-vanishing in a neighborhood of Δ. The proof of the full Nuttall's theorem will appear elsewhere [29].

This note is complimentary to [4] by Aptekarev and the author, where the same problem is considered but it is only required that ρ is holomorphic across each Δ_j and can vanish or blow up at the points of $E_0 \cup E_1$. However, [4] places the restriction on the algebraic S-contours requiring the bifurcation index $i(e)$ to be either 1 or 3 (no such restriction is present here). This note as well as [4] uses the matrix Riemann-Hilbert approach that requires local analysis around the points in $E_0 \cup E_1$ unless the weight is precisely as in (4) with ρ non-vanishing and holomorphic (this was first observed by Aptekarev and Van Assche for the case of an interval [3]). This is the sole reason for the difference in assumptions between [4] and this note.

This paper is organized as follows. In the next section we construct the Riemann surface of h_Δ, which turns out to be the "correct" domain of definition for the functions describing the asymptotics of Padé approximants. The latter functions are then introduced as solutions to a certain family of boundary value problems on the constructed surface. With these preliminaries out of the way, we prove the main result in the last section.

2 BOUNDARY VALUE PROBLEM

Fix an algebraic S-contour Δ with complement D and let h_Δ be given by (3).

2.1 *Riemann Surface*

Denote by $\mathfrak{R}$ the Riemann surface defined by h_Δ or equivalently by w_Δ. We represent $\mathfrak{R}$ as a two-sheeted ramified cover of $\overline{\mathbb{C}}$ constructed in the following manner. Two copies of $\overline{\mathbb{C}}$ are cut along each arc Δ_j. These copies are clipped together at the elements of $E_\Delta \subseteq E_0 \cup E_1$ (branch points of h_Δ). They are further glued together along the cuts in

such a manner that the right (resp. left) side of the arc Δ_j belonging to the first copy, say $\mathfrak{R}^{(0)}$, is joined with the left (resp. right) side of the same arc Δ_j only belonging to the second copy, $\mathfrak{R}^{(1)}$. The genus of $\mathfrak{R}$, which we denote by g, satisfies the equality $2(g+1) = |E_\Delta|$.

According to the above construction, each arc Δ_j together with its endpoints corresponds to a cycle, say $\boldsymbol{\Delta}_j$, on $\mathfrak{R}$. We set $\boldsymbol{\Delta} := \bigcup_j \boldsymbol{\Delta}_j$, denote by π the canonical projection $\pi : \mathfrak{R} \to \overline{\mathbb{C}}$, and define

$$D^{(k)} := \mathfrak{R}^{(k)} \cap \pi^{-1}(D) \quad \text{and} \quad z^{(k)} := D^{(k)} \cap \pi^{-1}(z)$$

for $k \in \{0, 1\}$ and $z \in D$. We further set $\mathbf{E}_\Delta := \pi^{-1}(E_\Delta)$, which is comprised exactly of the ramification points of $\mathfrak{R}$. The cycles $\boldsymbol{\Delta}_j$ are oriented so that $D^{(0)}$ remains on the left when $\boldsymbol{\Delta}_j$ is traversed in the positive direction. We designate the symbol $\cdot^*$ to stand for the conformal involution acting on the points of $\mathfrak{R}$ that fixes the ramification points $\mathbf{E}_\Delta$ and sends $z^{(k)}$ into $z^{(1-k)}$, $k \in \{0, 1\}$. We use bold lower case letters such as $\mathbf{z}, \mathbf{t}$ to indicate points on $\mathfrak{R}$ with canonical projections z, t.

Since h_Δ has only square root branching, each connected component of Δ contains even number of branch points. This allows us to number these points, $E_\Delta = \{e_0, e_1, \ldots, e_{2g+1}\}$, in the following fashion. If we consider ∂D as a positively oriented Jordan curve (this way it contains two copies of each Δ_j) and traverse it in the positive direction starting at e_{2k}, the next encountered branch point should be e_{2k+1}, $k \in \{1, \ldots, g\}$.

Denote by $\boldsymbol{\alpha}_k$, $k \in \{1, \ldots, g\}$, a smooth involution-symmetric, i.e., $\boldsymbol{\alpha}_k = \{\mathbf{z}^* | \mathbf{z} \in \boldsymbol{\alpha}_k\}$, Jordan curve that passes through $\mathbf{e}_1$ and $\mathbf{e}_{2k}$, and no other point of $\boldsymbol{\Delta}$ (until the end of the subsection we assume that $g \geqslant 1$), which is oriented so that the positive direction in $D^{(0)}$ goes from $\mathbf{e}_1$ to $\mathbf{e}_{2k}$. We require that $\boldsymbol{\alpha}_k \cap \boldsymbol{\alpha}_j = \{\mathbf{e}_1\}$ for each pair $k \neq j$. We further denote by $\boldsymbol{\beta}_k$ a smooth involution-symmetric Jordan curve that passes through $\mathbf{e}_{2k}$ and $\mathbf{e}_{2k+1}$ and is oriented so that at the point of intersection the tangent vectors to $\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k$ form the right pair. Again, we suppose that $\boldsymbol{\Delta} \cap \boldsymbol{\beta}_k = \{\mathbf{e}_{2k}, \mathbf{e}_{2k+1}\}$ and also assume that $\boldsymbol{\beta}_j$ has empty intersection with any cycle $\gamma \in \{\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k\}_{k=1}^g$ except for itself and $\boldsymbol{\alpha}_j$ with which it has only one point in common, necessarily $\mathbf{e}_{2j}$. Set

$$\widetilde{\mathfrak{R}} := \mathfrak{R} \setminus \bigcup_{k=1}^g (\boldsymbol{\alpha}_k \cup \boldsymbol{\beta}_k) \quad \text{and} \quad \widehat{\mathfrak{R}} := \mathfrak{R} \setminus \bigcup_{k=1}^g \boldsymbol{\alpha}_k.$$

The constructed collection $\{\boldsymbol{\alpha}_k, \boldsymbol{\beta}_k\}_{k=1}^g$ forms a homology basis on $\mathfrak{R}$ and so defined $\widetilde{\mathfrak{R}}$ is simply connected. In the case $g = 0$ these definitions are void and the whole surface is conformally equivalent to the Riemann sphere $\overline{\mathbb{C}}$.

2.2 *Differentials on $\mathfrak{R}$*

Denote by $d\vec{\Omega} := (d\Omega_1, \ldots, d\Omega_g)^\mathsf{T}$ the column vector of g linearly independent holomorphic differentials normalized so that $\oint_{\boldsymbol{\alpha}_k} d\vec{\Omega} = \vec{e}_k$, $k \in \{1, \ldots, g\}$, where $\{\vec{e}_k\}_{k=1}^g$ is the standard basis for $\mathbb{R}^g$ and $\vec{e}^\mathsf{T}$ is the transpose of $\vec{e}$. Since the genus of $\mathfrak{R}$ is g, the differentials $d\Omega_k$ form a basis for the space of holomorphic differentials on $\mathfrak{R}$. Set

$$(6) \qquad \mathbf{B} := \left[\oint_{\boldsymbol{\beta}_j} d\Omega_k \right]_{j,k=1}^g .$$

It is known that the matrix $\mathbf{B}$ is symmetric and has a positive definite imaginary part. Put

$$(7) \qquad w(z^{(k)}) := (-1)^k w_\Delta(z), \quad z \in D,$$

which is continuous across Δ and therefore is rational on $\mathfrak{R}$. It can be argued that

$$(8) \qquad d\Omega_j(\mathbf{z}) = (L_j/w)(\mathbf{z})d\mathbf{z},$$

for some L_j, which is a polynomial in z lifted to $\mathfrak{R}$ of degree at most $g-1$.

Analogously to (7), the function

(9)
$$h(z^{(k)}) := (-1)^k h_\Delta(z), \quad z \in D,$$

extends to Δ by continuity and is rational on $\mathfrak{R}$. By setting $dG(z) = h(z)dz$, we obtain the so-called *Green's differential* on $\mathfrak{R}$. That is, all the periods (integrals over cycles on $\mathfrak{R}$) of dG are purely imaginary and dG is meromorphic having two simple poles at $\infty^{(1)}$ and $\infty^{(0)}$ with respective residues 1 and -1 (it holds that $dG(z^{(k)}) = ((-1)^{k+1}/\zeta + \text{holomorphic})d\zeta$ in local coordinates $\zeta = 1/z^{(k)}$). Thus, we can define two vectors of real constants $\vec{\omega} = (\omega_1, \dots, \omega_g)^\mathsf{T}$ and $\vec{\tau} = (\tau_1, \dots, \tau_g)^\mathsf{T}$ by

(10)
$$\omega_k := -\frac{1}{2\pi i} \oint_{\beta_k} dG \quad \text{and} \quad \tau_k := \frac{1}{2\pi i} \oint_{\alpha_k} dG.$$

2.3 *Mapping Function*

Define

(11)
$$\Phi(z) := \exp\left\{ \int_{e_0}^z dG \right\}, \quad z \in \widetilde{\mathfrak{R}}.$$

The function Φ is holomorphic and non-vanishing on $\widetilde{\mathfrak{R}}$ except for a simple pole at $\infty^{(0)}$ and a simple zero at $\infty^{(1)}$. Furthermore, it possesses continuous traces on both sides of each cycle of the canonical basis that satisfy

(12)
$$\Phi^+ = \Phi^- \begin{cases} \exp\{2\pi i \omega_k\} & \text{on} \quad \alpha_k, \\ \exp\{2\pi i \tau_k\} & \text{on} \quad \beta_k. \end{cases}$$

In the case $g = 0$, Φ is a rational function well-defined on the whole Riemann surface.

Observe that the path of integration in (9) always can be chosen so that it completely belongs to either $\mathfrak{R}^{(0)}$ or $\mathfrak{R}^{(1)}$. Thus, it readily follows from (9) and (3) that

(13)
$$\Phi(z^{(k)}) = \exp\left\{ (-1)^k \int_{e_0}^z h_\Delta(t)dt \right\} \quad \text{and} \quad |\Phi(z^{(k)})| = \exp\left\{ (-1)^k g_D(z) \right\}$$

for $z \in D$. This computation has a trivial but remarkably important consequence, namely,

(14)
$$\Phi(z^{(0)})\Phi(z^{(1)}) \equiv 1 \quad \text{and} \quad |\Phi(z^{(0)})| > |\Phi(z^{(1)})|, \quad z \in D.$$

When $g = 0$, the pull back of Φ from $D^{(0)}$ to D is nothing else but the conformal map of D onto $\{|z| > 1\}$ fixing the point at infinity and sending e_0 to 1.

2.4 *Cauchy Kernel*

Let γ be an involution-symmetric, piecewise-smooth oriented chain on $\mathfrak{R}$ that has only finitely many points in common with the α-cycles. Further, let λ be a Hölder continuous function on γ. That is, for each $z \in \gamma$, $\lambda \circ \phi_z$ is Hölder continuous on $\phi_z^{-1}(\gamma)$ where ϕ_z is a holomorphic local parametrization around z.

Denote by $d\Omega_{z,z^*}$ the normalized abelian differential of the third kind (i.e., it is a meromorphic differential with two simple poles at z and z^* with respective residues 1 and -1 normalized to have zero periods on the α-cycles). Set

$$\Lambda(z) := \frac{1}{4\pi i} \oint_\gamma \lambda d\Omega_{z,z^*}, \quad z \notin \gamma.$$

It is known [30, Eq. (2.7)–(2.9)] that Λ is a holomorphic function in $\widehat{\mathfrak{R}} \setminus \gamma$, $\Lambda(z) + \Lambda(z^*) \equiv 0$ there, the traces $\Lambda^\pm$ are continuous and satisfy

$$\Lambda^+(z) - \Lambda^-(z) = \frac{1}{2} \begin{cases} \lambda(z) + \lambda(z^*), & z \in \gamma, \\ -2\oint_\gamma \lambda d\Omega_k, & z \in \alpha_k \setminus \gamma. \end{cases}$$

That is, the differential $d\Omega_{z,z^*}$ is a discontinuous Cauchy kernel on $\mathfrak{R}$ (it is discontinuous as Λ has additional jumps across the α-cycles).

2.5 *Auxiliary Functions, I*

To remove the jumps of Φ across the β-cycles, define $\lambda_{\vec{\tau}}$ to be the function on $\gamma = \cup\beta_k$ such that $\lambda_{\vec{\tau}} \equiv -2\pi i\tau_k$ on β_k and set

(15)
$$S_{\vec{\tau}}(z) := \exp\{\Lambda_{\vec{\tau}}(z)\}, \quad z \in \widetilde{\mathfrak{R}}.$$

Then $S_{\vec{\tau}}$ is a holomorphic function in $\widetilde{\mathfrak{R}}$ with continuous traces that satisfy

(16)
$$S_{\vec{\tau}}^+ = S_{\vec{\tau}}^- \begin{cases} \exp\{2\pi i(\mathbf{B}\vec{\tau})_k\} & \text{on} \quad \alpha_k, \\ \exp\{-2\pi i\tau_k\} & \text{on} \quad \beta_k, \end{cases}$$

where the upper equality follows straight from (6) and we adopt the convention $(\vec{c})_k = c_k$ for $\vec{c} = (c_1, \ldots, c_g)$.

Let now ρ be a non-vanishing holomorphic function on Δ. Select a smooth branch of $\log \rho$ and lift it to Δ, $\lambda_\rho := -\log \rho \circ \pi$. Define

(17)
$$S_\rho(z) := \exp\{\Lambda_\rho(z)\}, \qquad \vec{c}_\rho := -\frac{1}{2\pi i}\oint_\Delta \lambda_\rho d\vec{\Omega}.$$

Then S_ρ is a holomorphic and non-vanishing function in $\widehat{\mathfrak{R}} \setminus \Delta$ with continuous traces that satisfy

(18)
$$S_\rho^+ = S_\rho^- \begin{cases} \exp\{2\pi i(\vec{c}_\rho)_k\} & \text{on} \quad \alpha_k, \\ 1/\rho \circ \pi & \text{on} \quad \Delta. \end{cases}$$

By gathering together (12), (16), (18) and setting $S_{n\vec{\tau}} := S_{\vec{\tau}}^n$, we deduce that

(19)
$$(\Phi^n S_\rho S_{n\vec{\tau}})^+ = (\Phi^n S_\rho S_{n\vec{\tau}})^- \begin{cases} \exp\{2\pi i(\vec{c}_\rho + n(\vec{\omega} + \mathbf{B}\vec{\tau}))_k\} & \text{on} \quad \alpha_k, \\ 1/\rho \circ \pi & \text{on} \quad \Delta. \end{cases}$$

2.6 *Jacobi Inversion Problem*

To remove the jump of $\Phi^n S_\rho S_{n\vec{\tau}}$ from the α-cycles, let us digress into explaining what a Jacobi inversion problem is.

An *integral divisor* is a formal symbol of the form $\mathcal{D} = \sum n_j z_j$, where $\{z_j\}$ is an arbitrary finite collection of distinct points on $\mathfrak{R}$ and $\{n_j\}$ is a collection of positive integers. The sum $\sum n_j$ is called the *degree* of the divisor $\mathcal{D}$. Let $\mathcal{D}_1 = \sum n_j z_j$ and $\mathcal{D}_2 = \sum m_j w_j$ be integral divisors. A divisor $\mathcal{D}_1 - \mathcal{D}_2$ is called *principal* if there exists a rational function on $\mathfrak{R}$ that has a zero at every z_j of multiplicity n_j, a pole at every w_j of order m_j, and otherwise is non-vanishing and finite. By Abel's theorem, $\mathcal{D}_1 - \mathcal{D}_2$ is principal if and only if the divisors $\mathcal{D}_1$ and $\mathcal{D}_2$ have the same degree and

$$\vec{\Omega}(\mathcal{D}_1) - \vec{\Omega}(\mathcal{D}_2) \equiv \vec{0} \pmod{\text{periods } d\vec{\Omega}},$$

where $\vec{\Omega}(\mathcal{D}_1) := \sum n_j \int_{e_0}^{z_j} d\vec{\Omega}$ and the equivalence of two vectors $\vec{c}, \vec{e} \in \mathbb{C}^g$ is defined by $\vec{c} \equiv \vec{e} \pmod{\text{periods } d\vec{\Omega}}$ if and only if $\vec{c} - \vec{e} = \vec{j} + \mathbf{B}\vec{m}$ for some $\vec{j}, \vec{m} \in \mathbb{Z}^g$.

Set $\mathcal{D}_* = g\infty^{(1)}$. We are seeking a solution of the following Jacobi inversion problem: find an integral divisor $\mathcal{D}$ of degree g such that

(20)
$$\vec{\Omega}(\mathcal{D}) - \vec{\Omega}(\mathcal{D}_*) \equiv \vec{c}_\rho + n(\vec{\omega} + \mathbf{B}\vec{\tau}) \pmod{\text{periods } d\vec{\Omega}},$$

where the vectors $\vec{\omega}$ and $\vec{\tau}$ were defined in (10). This problem is always solvable and the solution is unique up to a principal divisor. That is, if $\mathcal{D} - \{\text{principal divisor}\}$ is an integral divisor, then it also solves (20). Immediately one can see that the subtracted principal divisor should have an integral part of degree at most g. As $\mathfrak{R}$ is hyperelliptic, such divisors come solely from rational functions on $\overline{\mathbb{C}}$ lifted to $\mathfrak{R}$. In particular, such

principal divisors are involution-symmetric. Hence, if a solution of (20) contains at least one pair of involution-symmetric points, then replacing this pair by another such pair produces a different solution of (20). However, if a solution does not contain such a pair, then it solves (20) uniquely.

2.7   Solutions of the JIP

In what follows, we denote by $\mathcal{D}_n$ either the unique solution of (20) or the solution where each conjugate-symmetric pair is replaced by $\infty^{(0)} + \infty^{(1)}$. We further set $\mathbb{N}_*$ to be the subsequence of all indices for which (20) is uniquely solvable. Non-unique solutions are related to unique solutions in the following manner:

$$(21) \qquad \mathcal{D}_n = \sum_{i=1}^{g-l} t_i + k\infty^{(0)} + (l-k)\infty^{(1)} \quad \Leftrightarrow \quad \mathcal{D}_{n+j} = \mathcal{D}_n + j\big(\infty^{(0)} - \infty^{(1)}\big),$$

for $j \in \{-k, \ldots, l-k\}$, where $l > 0$, $k \in \{0, \ldots, l\}$, and $|t_i| < \infty$. Indeed, Riemann's relations state that

$$\oint_{\beta_k} d\Omega_{\infty^{(1)}, \infty^{(0)}} = 2\pi i \int_{\infty^{(0)}}^{\infty^{(1)}} d\Omega_k$$

for each $k \in \{1, \ldots, g\}$, where the path of integration lies entirely in $\widetilde{\mathfrak{R}}$. Since the differentials $d\Omega_{\infty^{(1)}, \infty^{(0)}}$ and dG have the same poles with the same residues, they differ by a holomorphic differential. Their normalizations imply that

$$dG = d\Omega_{\infty^{(1)}, \infty^{(0)}} + 2\pi i \sum_{k=1}^{g} \tau_k d\Omega_k.$$

Combining the last two equations with (6) and (10) we get that

$$(22) \qquad \qquad \vec{\Omega}\big(\infty^{(0)}\big) - \vec{\Omega}\big(\infty^{(1)}\big) = \vec{\omega} + \mathbf{B}\vec{\tau},$$

which immediately implies that

$$\vec{\Omega}(\mathcal{D}_n) - \vec{\Omega}(\mathcal{D}_*) + j\left(\vec{\Omega}\big(\infty^{(0)}\big) - \vec{\Omega}\big(\infty^{(1)}\big)\right) \equiv \vec{c}_\rho + (n+j)\big(\vec{\omega} + \mathbf{B}\vec{\tau}\big)$$

from which (21) easily follows. In particular, (21) implies the unique solvability of (20) for the indices $n - k$ and $n + l - k$.

In another connection, if $\mathcal{D}_n$ is a unique solution of (20) that does not contain $\infty^{(k)}$, $k \in \{0, 1\}$, then $\mathcal{D}_{n-(-1)^k}$ is also a unique solution of (20) as otherwise it would contain at least one pair $\infty^{(1)} + \infty^{(0)}$, which would imply that $\mathcal{D}_n$ contains $\infty^{(k)}$ by (21). Moreover, the divisors $\mathcal{D}_n$ and $\mathcal{D}_{n-(-1)^k}$ have no points in common. Indeed, denote by $\mathcal{D}$ the common part. Then

$$(23) \quad \vec{\Omega}(\mathcal{D}_n) - \vec{\Omega}\big(\mathcal{D}_{n-(-1)^k}\big) - (-1)^k\left(\vec{\Omega}\big(\infty^{(0)}\big) - \vec{\Omega}\big(\infty^{(1)}\big)\right) \equiv \vec{0} \quad (\text{mod periods } d\vec{\Omega})$$

and therefore the divisor $\mathcal{D}_n - \mathcal{D}_{n-(-1)^k} - (-1)^k\infty^{(0)} + (-1)^k\infty^{(1)}$ is principal. However, if the degree of $\mathcal{D}$ were strictly positive, the integral part of the constructed divisor would be at most g. Such divisors come solely from rational functions on $\overline{\mathbb{C}}$ lifted to $\mathfrak{R}$ and are involution-symmetric. Hence, the divisor $\mathcal{D}_n - \mathcal{D}$ would contain an involution-symmetric pair or $\infty^{(k)}$. As both conclusions are impossible, the claim indeed takes place.

2.8   Limit Points

One can consider integral divisors of degree g as elements of $\mathfrak{R}^g/\Sigma_g$, the quotient of $\mathfrak{R}^g$ by the symmetric group Σ_g, which is a compact topological space. Thus, it make sense to talk about the limit points of $\{\mathcal{D}_n\}$. The considerations of the previous section extend to them in the following manner.

Let $\mathbb{N}' \subseteq \mathbb{N}$ be such that $\mathcal{D}_n \to \mathcal{D}'$, $n \in \mathbb{N}'$, for some divisor $\mathcal{D}'$. In the most general form the divisor $\mathcal{D}'$ can be written as

$$\mathcal{D}' = \mathcal{D} + \sum_{i=1}^{k} \left(z_i^{(0)} + z_i^{(1)} \right) + l_0 \infty^{(0)} + l_1 \infty^{(1)},$$

where the integral divisor $\mathcal{D}$ has degree $g - 2k - l_0 - l_1$, is non-special, and contains neither $\infty^{(0)}$ nor $\infty^{(1)}$. Let further $\mathbb{N}'' \subseteq \mathbb{N}'$ be another subsequence such that the divisors $\mathcal{D}_{n+l_1+k}$, $n \in \mathbb{N}''$, converge to some divisor, say $\mathcal{D}''$. Then the continuity of $\vec{\Omega}$ implies that

$$\lim_{\mathbb{N}'' \ni n \to \infty} \vec{\Omega}(\mathcal{D}_n) = \vec{\Omega}(\mathcal{D}') \quad \text{and} \quad \lim_{\mathbb{N}'' \ni n \to \infty} \vec{\Omega}(\mathcal{D}_{n+l_1+k}) = \vec{\Omega}(\mathcal{D}'')$$

with all the paths of integration belonging to $\widetilde{\mathfrak{R}}$. That is,

$$\begin{cases} \lim\limits_{\mathbb{N}'' \ni n \to \infty} \left(\vec{c}_\rho + n(\vec{\omega} + \mathbf{B}\vec{\tau}) \right) & \equiv \quad \vec{\Omega}(\mathcal{D}') - \vec{\Omega}(\mathcal{D}_*), \\ \lim\limits_{\mathbb{N}'' \ni n \to \infty} \left(\vec{c}_\rho + (n + l_1 + k)(\vec{\omega} + \mathbf{B}\vec{\tau}) \right) & \equiv \quad \vec{\Omega}(\mathcal{D}'') - \vec{\Omega}(\mathcal{D}_*), \end{cases}$$

since $\mathcal{D}_n$ solves (20). Hence, it holds by (22) that

$$\vec{\Omega}(\mathcal{D}'') \equiv \vec{\Omega}(\mathcal{D}') + (l_1 + k) \left(\vec{\Omega}(\infty^{(0)}) - \vec{\Omega}(\infty^{(1)}) \right).$$

Observe also that $\vec{\Omega}(z^{(0)}) = -\vec{\Omega}(z^{(1)})$ as follows from (7) and (8). Thus, the above congruence can be rewritten as

$$\vec{\Omega}(\mathcal{D}'') \equiv \vec{\Omega}(\mathcal{D}) + (l_0 + l_1 + 2k)\vec{\Omega}\left(\infty^{(0)} \right).$$

Therefore, it follows from Abel's theorem that the divisor $\mathcal{D} + (l_0 + l_1 + 2k)\infty^{(0)} - \mathcal{D}''$ is principal. However, it is also special and does not contain any involution-symmetric pair, which is possible only if it is identically zero. That is,

$$\mathcal{D}'' = \mathcal{D} + (l_0 + l_1 + 2k)\infty^{(0)}.$$

In fact, exactly as in the preceding subsection, we could take the second sequence to be $\mathcal{D}_{n+j}$ for any $j \in \{-l_0 - k, \ldots, l_1 + k\}$ and arrive at similar conclusions, see [4, Proposition 2].

Moreover, let now $\mathbb{N}''' \subseteq \mathbb{N}''$ be such that $\mathcal{D}_{n+l_1+k+1} \to \mathcal{D}'''$ for some divisor $\mathcal{D}'''$. It follows from the considerations as above and the argument used in (23) applied to $\mathcal{D}'''$ and $\mathcal{D}''$ that $\mathcal{D}'''$ is non-special and disjoined from $\mathcal{D}''$.

2.9 *Riemann's Theta Function*

The solution of the Jacobi inversion problem (20) helps us to remove the jump from the α-cycles in (19) via *Riemann's theta function*. The theta function associated with $\mathbf{B}$ is an entire transcendental function of g complex variables defined by

$$\theta(\vec{u}) := \sum_{\vec{n} \in \mathbb{Z}^g} \exp\left\{ \pi i \vec{n}^\top \mathbf{B} \vec{n} + 2\pi i \vec{n}^\top \vec{u} \right\}, \quad \vec{u} \in \mathbb{C}^g.$$

As shown by Riemann, the symmetry of $\mathbf{B}$ and positive definiteness of its imaginary part ensure the convergence of the series for any $\vec{u}$. It can be directly checked that θ enjoys the following periodicity properties:

$$(24) \qquad \theta\left(\vec{u} + \vec{j} + \mathbf{B}\vec{m} \right) = \exp\left\{ -\pi i \vec{m}^\top \mathbf{B}\vec{m} - 2\pi i \vec{m}^\top \vec{u} \right\} \theta(\vec{u}), \quad \vec{j}, \vec{m} \in \mathbb{Z}^g.$$

Specializing integral divisors to one point z, we reduce $\vec{\Omega}(z)$ to a vector of holomorphic functions in $\widetilde{\mathfrak{R}}$ with continuous traces on the cycles of the homology basis that satisfy

$$(25) \qquad \vec{\Omega}^+ - \vec{\Omega}^- = \begin{cases} -\mathbf{B}\vec{e}_k & \text{on} \quad \alpha_k, \\ \vec{e}_k & \text{on} \quad \beta_k, \end{cases}$$

$k \in \{1, \dots, g\}$. It readily follows from the relations above that each Ω_k is, in fact, holomorphic in $\widehat{\mathfrak{R}} \setminus \beta_k$. It is known that

$$\theta(\vec{u}) = 0 \quad \Leftrightarrow \quad \vec{u} \equiv \vec{\Omega}(\mathcal{D}_{\vec{u}}) + \vec{K} \quad \left(\text{mod periods } d\vec{\Omega}\right)$$

for some integral divisor $\mathcal{D}_{\vec{u}}$ of degree $g-1$, where $\vec{K}$ is the vector of Riemann constants defined by $(\vec{K})_j := ([\mathbf{B}]_{jj} - 1)/2 - \sum_{k \neq j} \oint_{\alpha_k} \Omega_j^- d\Omega_k$, $j \in \{1, \dots, g\}$.

For $n \in \mathbb{N}_*$ ($\mathcal{D}_n$ is unique, and hence does not contain involution-symmetric pairs), set

(26)
$$\Theta_n(z) := \frac{\theta\left(\vec{\Omega}(z) - \vec{\Omega}(\mathcal{D}_n) - \vec{K}\right)}{\theta\left(\vec{\Omega}(z) - \vec{\Omega}(\mathcal{D}_*) - \vec{K}\right)}.$$

Since the divisors $\mathcal{D}_n$ and $\mathcal{D}_*$ do not contain involution-symmetric pairs, $\vec{\Omega}(z) + \vec{\Omega}(z^*) \equiv 0$, and $\theta(-\vec{u}) = \theta(\vec{u})$, Θ_n is a multiplicatively multi-valued meromorphic function on $\mathfrak{R}$ with zeros at the points of the divisor $\mathcal{D}_n$ of respective multiplicities, a pole of order g at $\infty^{(1)}$, and otherwise non-vanishing and finite (there will be a reduction of the order of the pole at $\infty^{(1)}$ when the divisor $\mathcal{D}_n$ contains this point). In fact, it is meromorphic and single-valued in $\widehat{\mathfrak{R}}$ and

(27)
$$\begin{aligned} \Theta_n^+ &= \Theta_n^- \exp\left\{2\pi i\left(\Omega_k(\mathcal{D}_*) - \Omega_k(\mathcal{D}_n)\right)\right\} \\ &= \Theta_n^- \exp\left\{-2\pi i\left(\vec{c}_\rho + n(\vec{\omega} + \mathbf{B}\vec{\tau}) + \mathbf{B}\vec{m}_n\right)_k\right\} \end{aligned}$$

on α_k by (24) and (25), where $\vec{m}_n, \vec{j}_n \in \mathbb{Z}^g$ are such that

(28)
$$\vec{\Omega}(\mathcal{D}_n) - \vec{\Omega}(\mathcal{D}_*) = \vec{c}_\rho + n(\vec{\omega} + \mathbf{B}\vec{\tau}) + \vec{j}_n + \mathbf{B}\vec{m}_n.$$

2.10   *Auxiliary Functions, II*

Let $\lambda_{\vec{m}_n}$ be the function on $\gamma = \cup\beta_k$ such that $\lambda_{\vec{m}_n} \equiv -2\pi i(\vec{m}_n)_k$ on β_k. Set

(29)
$$S_{\vec{m}_n}(z) := \exp\left\{\Lambda_{\vec{m}_n}(z)\right\}, \quad z \in \widetilde{\mathfrak{R}}.$$

Since $\vec{m}_n \in \mathbb{Z}$, $S_{\vec{m}_n}$ is holomorphic across the β-cycles by the analytic continuation principle and therefore is holomorphic in $\widehat{\mathfrak{R}}$. It has continuous traces on the α-cycles that satisfy

(30)
$$S_{\vec{m}_n}^+ = S_{\vec{m}_n}^- \exp\left\{2\pi i\left(\mathbf{B}\vec{m}_n\right)_k\right\} \quad \text{on} \quad \alpha_k.$$

As $\mathbf{B}$ has positive definite imaginary part, any vector in $\vec{u} \in \mathbb{C}^g$ can be uniquely written as $\vec{x} + \mathbf{B}\vec{y}$ for some $\vec{x}, \vec{y} \in \mathbb{R}^g$. Write

$$\vec{c}_\rho =: \vec{x}_\rho + \mathbf{B}\vec{y}_\rho \quad \text{and} \quad \vec{\Omega}(\mathcal{D}_n) - \vec{\Omega}(\mathcal{D}_*) =: \vec{x}_n + \mathbf{B}\vec{y}_n, \quad n \in \mathbb{N},$$

Then, of course,

(31)
$$\vec{x}_n = \vec{x}_\rho + n\vec{\omega} + \vec{j}_n \quad \text{and} \quad \vec{y}_n = \vec{y}_\rho + n\vec{\tau} + \vec{m}_n$$

by (28). Since the image of the closure of $\widetilde{\mathfrak{R}}$ under $\vec{\Omega}$ is bounded in $\mathbb{C}^g$, so are the vectors $\vec{x}_n, \vec{y}_n$. Clearly, in this case (31) implies that the vectors $n\vec{\omega} + \vec{j}_n$ and $n\vec{\tau} + \vec{m}_n$ are bounded with n. Therefore,

(32)
$$C^{-1} \leqslant \left|S_\rho S_{n\vec{\tau} + \vec{m}_n}\right| \leqslant C$$

uniformly with n in $\widetilde{\mathfrak{R}}$ for some absolute constant $C > 1$.

2.11   *A Family of BVPs*

By combining all the materials above, we obtain the following theorem.

Theorem 2. *For $n \in \mathbb{N}$, let $\tilde{n}$ be the greatest integer in $\mathbb{N}_*$ smaller or equal to n ($\tilde{n} = n - k$ using the notation from (21)). Then the function*

$$\Psi_n := \Phi^{\tilde{n}} S_\rho S_{\tilde{n}\vec{\tau}+\vec{m}_{\tilde{n}}} \Theta_{\tilde{n}} \tag{33}$$

is sectionally meromorphic in $\mathfrak{R} \setminus \Delta$ and its zeros and poles there[2] are described by the divisor

$$(n - g)\infty^{(1)} + \mathcal{D}_n - n\infty^{(0)}.$$

Moreover, it has continuous traces on $\Delta \setminus \mathbf{E}_\Delta$ that are bounded near the points in $\mathbf{E}_\Delta$ and satisfy

$$\Psi_n^+ = (1/\rho \circ \pi)\Psi_n^-.$$

Indeed, meromorphy in $\mathfrak{R} \setminus \Delta$ follows from (11), (15), (17), (26), and (29); the form of the divisor from the fact $\mathcal{D}_n = \mathcal{D}_{\tilde{n}} + k(\infty^{(0)} - \infty^{(1)})$; and the jump relations from (19), (27), and (30).

To describe the asymptotic properties of Ψ_n we need to further restrict $\mathbb{N}_*$.

Definition 3. *Given $\varepsilon > 0$, we say that an index n belongs to $\mathbb{N}_\varepsilon$ if and only if*

$$\pi\left(\mathfrak{R}^{(1)} \cap \mathcal{D}_{n-1}\right), \pi\left(\mathfrak{R}^{(0)} \cap \mathcal{D}_n\right) \subset \left\{z : |z| \leqslant \varepsilon^{-1}\right\},$$

where we consider a divisor as a subset of points on $\mathfrak{R}$.

The indices excluded from $\mathbb{N}_*$ are exactly the ones corresponding to the non-unique solutions of (20), that is, the solutions for which $\mathcal{D}_n$ contains at least one pair of $\infty^{(0)} + \infty^{(1)}$. Hence, if $n \in \mathbb{N}_\varepsilon$, then $n, n - 1 \in \mathbb{N}_*$. Furthermore, the subsequences $\mathbb{N}_\varepsilon$ are infinite for all ε small enough as follows from the considerations in Section 2.8 (this is the precise reason why this subsection is included).

It follows immediately from the definition of $\mathbb{N}_\varepsilon$ that the following constants are well defined:

$$1/\gamma_n := \lim_{z \to \infty^{(0)}} \Psi_n(z)z^{-n} \quad \text{and} \quad 1/\gamma_n^* := \lim_{z \to \infty^{(1)}} \Psi_{n-1}(z)z^{n-1-g}, \quad n \in \mathbb{N}_\varepsilon. \tag{34}$$

Lemma 4. *For each bounded $K \subset D^{(1)}$, there exists constant $C(K) > 1$ such that*

$$\max_K |\Psi_n| \leqslant C(K)^{-n}. \tag{35}$$

Moreover, for a given $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 1$ such that

$$C(\varepsilon)^{-1} \leqslant |\gamma_n \gamma_n^*| \leqslant C(\varepsilon), \quad n \in \mathbb{N}_\varepsilon. \tag{36}$$

Proof. To show (35), write

$$|\Psi_n| = \left|\Phi^{\tilde{n}-g}\right| \cdot \left|S_\rho S_{n\vec{\tau}+\vec{m}_n}\right| \cdot \left|\Phi^g \Theta_n\right|.$$

The first multiple in the decomposition above is locally uniformly geometrically small in $D^{(1)}$ by (14) and the second one is uniformly bounded by (32). Thus, it is enough to show that the functions $\left|\Phi^g \Theta_n\right|$ are uniformly bounded in $D^{(1)}$. It is, in fact, a family of continuous functions in $\overline{D^{(1)}} \setminus \bigcup \alpha_k$ with uniformly bounded jumps on the α-cycles (boundedness of the jumps follows from (27) and the uniform boundedness of the vectors $n\vec{\tau} + \vec{m}_n$ concluded after (31)). Hence, each function is bounded in $\overline{D^{(1)}}$. As the family is indexed by the divisors $\mathcal{D}_n$ that belong to $\mathfrak{R}^g/\Sigma_g$ and the latter space is compact, the uniform boundedness follows.

It follows again from (27) and (32) that to show (36) it is sufficient to establish the uniform boundedness with $n \in \mathbb{N}_\varepsilon$ of the absolute values of

$$\Theta_n(\infty^{(0)}) \lim_{z \to \infty^{(1)}} \Theta_{n-1}(z)z^{-g}. \tag{37}$$

To this end, denote by $\mathfrak{C}_\varepsilon^0$ and $\mathfrak{C}_\varepsilon^1$ the closures of $\{\mathcal{D}_n\}_{n \in \mathbb{N}_\varepsilon}$ and $\{\mathcal{D}_{n-1}\}_{n \in \mathbb{N}_\varepsilon}$ in the $\mathfrak{R}^g/\Sigma_g$-topology. Neither of these sets contains special divisors. Indeed, both sequences

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[2] Ψ_n is non-vanishing and finite in $D^{(0)} \cup D^{(1)}$ except at the elements of its divisor that stand for zeros (resp. poles) if preceded by the plus (resp. minus) sign and the integer coefficients in front of them indicate multiplicity.

consist of non-special divisors and therefore we need to consider only the limiting ones. The limit points belonging to $\mathfrak{C}_\varepsilon^0$ are necessarily of the form

$$\mathcal{D} + \sum_{i=1}^k \left(z_i^{(0)} + z_i^{(1)} \right) + l\infty^{(1)},$$

where the integral divisor $\mathcal{D}$ has degree $g - 2k - l$, is non-special, and does contain $\infty^{(0)}$, $\infty^{(1)}$. If k were strictly positive, the considerations of Section 2.8 would imply that $\mathfrak{C}_\varepsilon^1$ should contain divisors of the form

$$\mathcal{D} + \sum_{i=1}^{k'} \left(w_i^{(0)} + w_i^{(1)} \right) + (k - k' - 1)\infty^{(0)} + (l + 1 + k - k')\infty^{(1)}$$

$0 \leqslant k' \leqslant k - 1$. In particular, it would be true that $l + 1 + k - k' \geqslant 2$, which is impossible by the very definition of $\mathbb{N}_\varepsilon$. Since the set $\mathfrak{C}_\varepsilon^1$ can be examined similarly, the claim follows. Thus, using (26), we can establish a quantity similar to (37), for the pairs of limit points in $\mathfrak{C}_\varepsilon^0 \times \mathfrak{C}_\varepsilon^1$. Moreover, all these quantities are finite and non-zero as all the divisors are non-special. The claim now follows from the compactness argument. $\qquad\square$

3 MAIN RESULTS

Fix an algebraic S-contour $\Delta = E_0 \cup E_1 \cup \bigcup \Delta_j$, see Definition 1, and let w_Δ be defined by (5), $z^{-g-1}w_\Delta(z) \to 1$ as $z \to \infty$. Let ρ be a function holomorphic and non-vanishing in a neighborhood of each connected component of Δ (in general, ρ is piecewise holomorphic). Recall (4) that we set

$$f_\rho(z) = \frac{1}{2\pi i} \int_\Delta \frac{(\rho/w_\Delta^+)(t)}{t - z} dt, \quad z \in \overline{\mathbb{C}} \setminus \Delta.$$

Further, let Ψ_n, which depends on ρ, be defined by (33). With a slight abuse of notation, put

(38) $$\Psi_n(z) := \Psi_n\left(z^{(0)}\right) \quad \text{and} \quad \Psi_n^*(z) := \Psi_n\left(z^{(1)}\right), \quad z \in D.$$

Then it follows from Theorem 2 that these functions are holomorphic in $\mathbb{C} \setminus \Delta$. Moreover, when $n \in \mathbb{N}_\varepsilon$, see Definition 3, it holds that Ψ_n has a pole of exact order n at infinity, Ψ_{n-1} has a pole of order at most $n-1$ there, Ψ_n^* vanishes at infinity, and Ψ_{n-1}^* has a zero of exact order $n - 1 - g$ there. Furthermore, it holds that

(39) $$\left(\Psi_n^*\right)^\pm = \rho\Psi_n^\mp \quad \text{on} \quad \bigcup \Delta_j,$$

where all the traces are continuous on $\bigcup \Delta_j$ and are bounded near $e \in E_0 \cup E_1$. Finally, let γ_n be defined by (34). Then the following theorem holds.

Theorem 5. *Let $[n/n]_{f_\rho} = P_n/Q_n$ be the n-th diagonal Padé approximant to f_ρ defined by (4) with ρ holomorphic and non-vanishing on Δ and R_n be the linearized error of approximation given by (2). Then for all $n \in \mathbb{N}_\varepsilon$ large enough it holds that*

(40) $$\begin{cases} Q_n &= (1 + \upsilon_{n1})\gamma_n\Psi_n + \upsilon_{n2}\gamma_n\Psi_{n-1}, \\ w_\Delta R_n &= (1 + \upsilon_{n1})\gamma_n\Psi_n^* + \upsilon_{n2}\gamma_n\Psi_{n-1}^*, \end{cases}$$

locally uniformly in $\overline{\mathbb{C}} \setminus \Delta$, where $\upsilon_{nj}(\infty) = 0$ and $|\upsilon_{nj}| \leqslant C_\varepsilon^{-n}$ in $\overline{\mathbb{C}}$ for some constant $C_\varepsilon > 1$.

In the case where $g > 0$, formulae (40) clearly indicate the absence of uniform convergence of Padé approximants. Indeed, the error of approximation is equal to

$$f_\rho - [n/n]_{f_\rho} = \frac{R_n}{Q_n} = \frac{1}{w_\Delta} \frac{\Psi_n^*}{\Psi_n} \frac{1 + \upsilon_{n1} + \upsilon_{n2}\left(\Psi_{n-1}^*/\Psi_n^*\right)}{1 + \upsilon_{n1} + \upsilon_{n2}\left(\Psi_{n-1}/\Psi_n\right)}.$$

We do know from Lemma 4 that the functions Ψ_n^* are geometrically small on closed subsets of D. Similar argument can be used to show that the functions Ψ_n are geometrically

large in D except for possible zeros described by those elements of the divisor $\mathcal{D}_n$ that belong to $D^{(0)}$ (Rouché's theorem clearly implies that Q_n has a zero close to the canonical projection of each such element) and those zeros are the sole reason why the uniform convergence does not hold. In the "generic case", i.e., when 1 and the periods (10) of the Green differential dG are rationally independent, it is known [28, pages 190–191] that the divisors $\mathcal{D}_n$ are dense in $\mathfrak{R}^g/\Sigma_g$ and hence one will definitely observe the presence of wandering poles. However, in this generic case, there always exists a subsequence of indices such that the elements of the divisors $\mathcal{D}_n$ belong only to $D^{(1)}$ [26, Sec. 4.1] and therefore there always exists a subsequence along which Padé approximants $[n/n]_{f_\rho}$ converge to f_ρ locally uniformly in D.

The remaining part of this section is devoted to the proof of Theorem 5.

3.1 Initial R-H Problem

Below, we follow by now classical approach of Fokas, Its, and Kitaev [10, 11] connecting orthogonal polynomials to matrix Riemann-Hilbert problems. To this end, assume that the index n is such that

$$(41) \qquad \deg(Q_n) = n \quad \text{and} \quad R_{n-1}(z) \sim z^{-n} \quad \text{as} \quad z \to \infty.$$

Define

$$(42) \qquad Y = \begin{pmatrix} Q_n & R_n \\ m_{n-1}Q_{n-1} & m_{n-1}R_{n-1} \end{pmatrix},$$

where m_{n-1} is a constant such that $m_{n-1}R_{n-1}(z) = z^{-n}[1 + o(1)]$ near infinity. Then Y solves the following matrix Riemann-Hilbert problem (RHP-Y) :

(a) Y is analytic in $\overline{\mathbb{C}} \setminus \Delta$ and $\displaystyle\lim_{z \to \infty} Y(z)z^{-n\sigma_3} = I$, where $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$;

(b) Y has continuous traces on $\bigcup \Delta_j$ that satisfy $Y_+ = Y_- \begin{pmatrix} 1 & \rho/w_\Delta^+ \\ 0 & 1 \end{pmatrix}$;

(c) Y is bounded near each $e \in (E_0 \cup E_1) \setminus E_\Delta$ and the behavior of Y near each $e \in E_\Delta$ is described by $O\begin{pmatrix} 1 & |z-e|^{-1/2} \\ 1 & |z-e|^{-1/2} \end{pmatrix}$ as $D \ni z \to e$.

The property RHP-Y(a) follows immediately from (2) and (41). The property RHP-Y(b) is due to the equality

$$R_n^+ - R_n^- = Q_n\left(f_\rho^+ - f_\rho^-\right) = Q_n\rho/w_\Delta^+ \quad \text{on} \quad \bigcup \Delta_j,$$

which in itself is a consequence of (2), (4), and the Sokhotski-Plemelj formulae [12, Section 4.2]. Finally, to show RHP-Y(c), write $R_n = \sum_k R_{nk}$, where

$$R_{nk}(z) := \frac{1}{2\pi i} \int_{\Delta_k} \frac{(Q_n\rho/w_\Delta^+)(t)}{t-z} dt$$

and therefore the behavior of R_n near $e \in E_0 \cup E_1$ is deduced from the behavior R_{nk} there. If the endpoint e of Δ_k has an odd bifurcation index ($e \in E_\Delta$), then w_Δ^2 has a simple zero there and therefore $|R_{nk}(z)| \sim |z-e|^{-1/2}$ as $z \to e$, see [7, Section 3]. On the other hand, if e has an even bifurcation index ($e \in (E_0 \cup E_1) \setminus E_\Delta$), the respective function R_{nk} behaves as

$$\frac{\rho(e)}{2\pi i} \frac{w_{\Delta|\Delta_k}^+(e)}{w_\Delta^2(e)} \log(z-e) + R_{e,k}^*(z)$$

according to [12, Section 8.1], where the function $R_{e,k}^*$ has a definite limit at e and the logarithm is holomorphic outside of Δ_k. Since w_Δ does not have a branch point at e, it

holds that $\sum_k w^+_{\Delta|\Delta_k}(e) = 0$, where the sum is taken over all arcs Δ_k incident with e. Thus, we get that

$$R(z) = \frac{1}{2\pi}\frac{\rho(e)}{w^2_\Delta(e)}\sum_k \arg_{e,k}(z-e) + R^*_e(z),$$

where R^*_e has a definite limit at e, $\arg_{e,k}(z-e)$ has the branch cut along Δ_k, and the sum is again taken over all arcs incident with e. Thus, $\mathbf{Y}$ is bounded in the vicinity of each e with even bifurcation index.

To show that a solution of RHP-$\mathbf{Y}$, if exists, must be of the form (42) is by now a standard exercise, see for instance, [14, Lemma 2.3], [6], [4, Lemma 1]. Thus, we proved the following lemma.

Lemma 6. *If a solution of* RHP-$\mathbf{Y}$ *exists then it is unique. Moreover, in this case it is given by* (42) *where* Q_n *and* R_{n-1} *satisfy* (41). *Conversely, if* (41) *is fulfilled, then* (42) *solves* RHP-$\mathbf{Y}$.

3.2 Transformed R-H Problem

It can be directly verified that

$$\begin{pmatrix} 1 & 0 \\ -w^-_\Delta/\rho & 1 \end{pmatrix}\begin{pmatrix} 0 & \rho/w^+_\Delta \\ -w^+_\Delta/\rho & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ w^+_\Delta/\rho & 1 \end{pmatrix} = \begin{pmatrix} 1 & \rho/w^+_\Delta \\ 0 & 1 \end{pmatrix}.$$

This factorization of the jump matrix in RHP-$\mathbf{Y}$(b) suggests the following transformation of $\mathbf{Y}$:

$$(43) \qquad \mathbf{X} := \begin{cases} \mathbf{Y}\begin{pmatrix} 1 & 0 \\ -w_\Delta/\rho & 1 \end{pmatrix}, & \text{in } \Omega, \\ \mathbf{Y}, & \text{in } \mathbb{C}\setminus\overline{\Omega}, \end{cases}$$

where Ω is an open set bounded by Δ and Γ and Γ is a union of simple Jordan curves each encompassing one connected component of Δ and chosen so that ρ is holomorphic across Γ. It is trivial to verify that $\mathbf{X}$ solves the following Riemann-Hilbert problem (RHP-$\mathbf{X}$):

(a) $\mathbf{X}$ is analytic in $\mathbb{C}\setminus(\Delta\cup\Gamma)$ and $\lim_{z\to\infty}\mathbf{X}(z)z^{-n\sigma_3}=\mathbf{I}$;

(b) $\mathbf{X}$ has continuous traces on $\bigcup\Delta_j\cup\Gamma$ that satisfy

$$\mathbf{X}_+ = \mathbf{X}_-\begin{cases} \begin{pmatrix} 0 & \rho/w^+_\Delta \\ -w^+_\Delta/\rho & 0 \end{pmatrix} & \text{on } \bigcup\Delta_j \\ \begin{pmatrix} 1 & 0 \\ w_\Delta/\rho & 1 \end{pmatrix} & \text{on } \Gamma; \end{cases}$$

(c) $\mathbf{X}$ has the behavior near $e\in E_0\cup E_1$ described by RHP-$\mathbf{Y}$(c).

Then the following lemma can be easily checked.

Lemma 7. RHP-$\mathbf{X}$ *is solvable if and only if* RHP-$\mathbf{Y}$ *is solvable. When solutions of* RHP-$\mathbf{X}$ *and* RHP-$\mathbf{Y}$ *exist, they are unique and connected by* (43).

3.3 Asymptotics in the Bulk

Let Ψ_n, Ψ^*_n be defined by (38) and γ_n, γ^*_n be as in (34). Set

$$(44) \qquad \mathbf{N} := \begin{pmatrix} \gamma_n & 0 \\ 0 & \gamma^*_n \end{pmatrix}\widetilde{\mathbf{N}}, \quad \widetilde{\mathbf{N}} := \begin{pmatrix} \Psi_n & \Psi^*_n/w_\Delta \\ \Psi_{n-1} & \Psi^*_{n-1}/w_\Delta \end{pmatrix}.$$

Then $\mathbf{N}$ solves the following Riemann-Hilbert problem (RHP-$\mathbf{N}$):

(a) $\mathbf{N}$ is analytic in $\mathbb{C}\setminus\Delta$ and $\lim_{z\to\infty}\mathbf{N}(z)z^{-n\sigma_3}=\mathbf{I}$;

(b) $\mathbf{N}$ has continuous traces on $\bigcup\Delta_j$ that satisfy $\mathbf{N}_+ = \mathbf{N}_-\begin{pmatrix} 0 & \rho/w^+_\Delta \\ -w^+_\Delta/\rho & 0 \end{pmatrix}$;

(c) $\mathbf{N}$ has the behavior near $e\in E_0\cup E_1$ described by RHP-$\mathbf{Y}$(c).

Indeed, RHP-$\mathbf{N}$(a) follows immediately from the analyticity properties of Ψ_n, Ψ_n^* and (34). RHP-$\mathbf{N}$(b) can be easily checked by using (39). Finally, RHP-$\mathbf{N}$(c) is the consequence of the boundedness of $\Psi_n^\pm$ and $(\Psi_n^*)^\pm$ on $\bigcup \Delta_j$ and the definition of w_Δ.

Moreover, it can be readily checked that $\det(\mathbf{N})$ is a holomorphic function in $\overline{\mathbb{C}} \setminus (E_0 \cup E_1)$ and $\det(\mathbf{N})(\infty) = 1$. Since it is either bounded or behaves like $O\left(|z - e|^{-1/2}\right)$ near $e \in E_0 \cup E_1$, those points are in fact removable singularities and therefore $\det(\mathbf{N})$ is a bounded entire function. That is, $\det(\mathbf{N}) \equiv 1$ as follows from the normalization at infinity.

3.4 *Final R-H Problem*

Consider the following Riemann-Hilbert Problem (RHP-$\mathbf{Z}$):

(a) $\mathbf{Z}$ is a holomorphic matrix function in $\overline{\mathbb{C}} \setminus \Gamma$ and $\mathbf{Z}(\infty) = \mathbf{I}$;

(b) $\mathbf{Z}$ has continuous traces on Γ that satisfy $\mathbf{Z}_+ = \mathbf{Z}_- \widetilde{\mathbf{N}} \begin{pmatrix} 1 & 0 \\ w_\Delta/\rho & 1 \end{pmatrix} \widetilde{\mathbf{N}}^{-1}$.

Then the following lemma takes place.

Lemma 8. *The solution of* RHP-$\mathbf{Z}$ *exists for all* $n \in \mathbb{N}_\varepsilon$ *large enough and satisfies*

$$(45) \qquad \mathbf{Z} = \mathbf{I} + O\left(C_\varepsilon^{-n}\right)$$

for some constant $C_\varepsilon > 1$ *independent of* Γ, *where* $O(\cdot)$ *holds uniformly in* $\overline{\mathbb{C}}$.

Proof. Since $\det(\mathbf{N}) \equiv 1$ and therefore $\det(\widetilde{\mathbf{N}}) \equiv 1/(\gamma_n \gamma_n^*)$, the jump matrix for $\mathbf{Z}$ is equal to

$$\mathbf{I} + \frac{\gamma_n \gamma_n^*}{\rho w_\Delta} \begin{pmatrix} \Psi_n^* \Psi_{n-1}^* & -\left(\Psi_n^*\right)^2 \\ \left(\Psi_{n-1}^*\right)^2 & -\Psi_n^* \Psi_{n-1}^* \end{pmatrix} = \mathbf{I} + O\left(C_{\varepsilon,\Gamma}^{-2n}\right),$$

where the last equality follows from Lemma 4. The conclusion of the lemma follows now from the same argument as in [8, Corollary 7.108]. $\square$

3.5 *Asymptotics*

Let $\mathbf{Z}$ be a solution of RHP-$\mathbf{Z}$ granted by Lemma 8 and $\widetilde{\mathbf{N}}$ be the matrix function constructed in (44). Then it can be easily checked that

$$\mathbf{X} = \begin{pmatrix} \gamma_n & 0 \\ 0 & \gamma_n^* \end{pmatrix} \mathbf{Z} \widetilde{\mathbf{N}}$$

solves RHP-$\mathbf{X}$ and therefore

$$\mathbf{Y} := \begin{pmatrix} \gamma_n & 0 \\ 0 & \gamma_n^* \end{pmatrix} \mathbf{Z} \widetilde{\mathbf{N}} \begin{cases} \begin{pmatrix} 1 & 0 \\ w_\Delta/\rho & 1 \end{pmatrix}, & \text{in} \quad \Omega, \\ \mathbf{I}, & \text{in} \; \mathbb{C} \setminus \overline{\Omega}, \end{cases}$$

solves RHP-$\mathbf{Y}$ by Lemma 7. Given any closed set $K \subset \overline{\mathbb{C}} \setminus \Delta$, choose Ω so that $K \subset \overline{\mathbb{C}} \setminus \overline{\Omega}$. Write

$$\mathbf{Z} = \begin{pmatrix} 1 + \upsilon_{n1} & \upsilon_{n2} \\ \upsilon_{n3} & 1 + \upsilon_{n4} \end{pmatrix},$$

where we know from Lemma 8 that $|\upsilon_{nk}| \leqslant C_\varepsilon^{-n}$ uniformly in $\overline{\mathbb{C}}$ ($\upsilon_{nk}(\infty) = 0$ as $\mathbf{Z}(\infty) = \mathbf{I}$). Then

$$[\mathbf{Y}]_{1i} = (1 + \upsilon_{n1}) \gamma_n \left[\widetilde{\mathbf{N}}\right]_{1i} + \upsilon_{n2} \gamma_n \left[\widetilde{\mathbf{N}}\right]_{2i}, \quad i \in \{1, 2\}.$$

The claim of Theorem 5 now follows from (42) and (44).

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