ANALYSIS OF CARTESIAN STIFFNESS AND COMPLIANCE WITH APPLICATIONS

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ANALYSIS OF CARTESIAN STIFFNESS AND COMPLIANCE WITH APPLICATIONS

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It has been a long journey and I have been lucky. Because, I always had the comfort of knowing that I had a life-long friend whom I could confide with and rely upon. Goodness flows from her to anybody, as if by nature; but, I know, I must have a special place in her heart. She relentlessly supported me, tried to make things easier for me. Most of all, she believed in me. She taught me those invaluable qualities of our lives such as love, respect and passion by demonstrating them. So, I feel privileged to dedicate this work to the person without whom it would not be what it is now; my beautiful wife, Meral.

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SUMMARY

Many elastic systems can be modelled by a 6×6 Cartesian stiffness or compliance matrix. Using spatial vector (screw) algebra, spatial stiffness and compliance are defined. Investigation of the linear elastic behavior is achieved by analyzing the geometric and constitutive properties of the stiffness and compliance matrices. The results are applicable in the analysis, design and control of elastic systems such as serial and parallel robotic manipulators, robotic grasp problems, assembly automation devices, spatial structures, and so on. The geometric and constitutive properties of an elastic system can be understood in terms of suitable eigenvalue problems. However, construction of physically and geometrically intuitive eigenvalue problems for stiffness and compliance in screw space is neither unique nor straightforward.

First, a set of singular eigenvalue problems from earlier studies is shown to be related to free-vectors. Closed form solutions for the location of centers of elasticity, stiffness and compliance are found in terms of quantities related to free-vector eigenvalue problems. Then, the constitutive nature and other properties of the centers of stiffness and compliance are presented, which were previously unknown. The centers of elasticity, stiffness and compliance are shown to be geometrically related. Considering line-vectors, instead of free-vectors, a new set of singular eigenvalue problems is proposed and solved. Every point in space generates a distinct set. Similar to the free-vector case, line-vector decompositions of stiffness and compliance are found and co-centers of elasticity are identified. The free-vector and line-vector results lead to generalized definitions of compliant axes and a refined compliance hierarchy.

The stiffness matrix of parallel spatial connections with line and torsional springs is found in closed form. The skew-symmetric part of stiffness for line springs is described completely, which explains previously observed asymmetry. The observation in earlier studies that the stiffness of line

springs is symmetric in a special reference frame is explained. There exist infinitely many such frames forming a 2-parameter family. In contrast, there is no such frame for torsional springs.

A theory is developed to determine orthogonal sets of isotropic vectors of a symmetric matrix, which, together with the stiffness equation for spring systems, leads to the synthesis of stiffness by springs. A general synthesis solution had not been found until now. The necessary and sufficient condition is that the off-diagonals of stiffness matrix have a zero trace . Algorithms and examples support the theory.

The free-vector and line-vector results are applied to rotational symmetry devices such as the remote center of compliance (RCC) device used in automated assembly operations. Previously unavailable and more accurate design equations are determined. Optimum device configurations are demonstrated. The conditions for the construction of RCC devices with beams and springs are found. Definitions of RCC-like devices are generalized.

The theory for the elastic systems is shown to be applicable in the dynamics of single rigid body. The mass matrix replaces the stiffness matrix. The free-vector and line-vector eigenvalue problems are explicitly solved for the mass matrix. Special axes resulting from the line-vector case explains the center of percussion phenomenon. A practical optimum design of sport equipment involving the center of percussion is presented. Combination of the elastic and kinetic cases leads to the determination of the necessary and sufficient conditions for the existence of special free vibration modes.

Numerical examples are provided for each topic to verify the theoretical results.

Part I

INTRODUCTION

CHAPTER I

INTRODUCTION

The purpose of this study is an analysis of linearly elastic systems using an elastically suspended rigid body as a model. The model is applicable in many areas of engineering such as structural mechanics, robot mechanics, mechanisms, etc. The main advantage of the model is the reduction of the problem to the analysis of 6×6 stiffness and compliance matrices. This enables one to extract beneficial information that can be used for simplified analytical equations, design rules, control strategies, etc.

The elastically suspended rigid body model naturally suggests the use of screw theory and related concepts as the mathematical description. Screw theory is a well developed and systematic geometric tool. Consequently, almost every result in this study has an intuitive geometric meaning. Screw theory brings a unification of many seemingly unrelated quantities and leads to simplified results. Applications of the theory in this study has solved problems that would have been much more difficult otherwise.

The following sections are presented by anticipating the questions that may first occur to the reader. The definitions of most terms and concepts are postponed until Chapter 2. Here, they are either assumed to be known or are briefly described.

1.1 Why Does One Need to Know About Elastic Systems?

Virtually all mechanical systems exhibit elastic behavior of some degree. In science and engineering, the elastic properties of a system may be beneficial, harmful, or unimportant.

For example, in mechanism analysis or design the elements are usually assumed rigid. Since this is never the case, one would like know the elastic response of the mechanism in order to estimate errors or to modify its design so that the elastic effects are minimized. In this case, the elastic effects are undesirable.

On the contrary, there are mechanisms that take advantage of elasticity. These are called compliant mechanisms which may not have distinct joints. In this case, the elastic effects are very beneficial and the knowledge of the elastic behavior is central to the design.

There are many other examples. A serial robotic arm may seriously suffer from elastic effects. A parallel robotic manipulator can be safely used in high precision positioning tasks simply due to the fact that its elasticity is very low. A space structure, such as a bridge, can be designed so that the deflections are predictable, thus safer, if the elastic behavior is known accurately. Passive devices that have very special elastic behavior can be designed to be used in assembly tasks as inexpensive control alternatives (Chapter 9). Combined with the dynamic effects, the elastic effects give rise to the vibration phenomenon which can controlled, predicted and used beneficially if the elastic properties are known.

However, there is a more general reason behind the study presented here. The model and methods proposed in this study can also be applied to the dynamics of rigid bodies (Chapter 10). This brings together the notions from elasticity and dynamics, providing a common methodology and therefore improving the current understanding of the notion of elastic behavior.

1.2 What Does Analysis of Elastic Systems Mean?

The phrase "analysis of elastic systems" refers to a broad range of subjects and methods. Elastic systems range from a simple spring to very complicated distributed systems with kinematical constraints. Methods of analysis include continuum mechanics, discrete theories, statics, kinematics, dynamics, linear and non-linear theories of elasticity, and so on. But, there are a few common aspects that are central. One of them is the quantification of the elastic behavior.

Once a quantification is achieved, one can extract important information that is beneficial in the prediction, design, control, etc. of the elastic system. However, for a systematic analysis, one has to agree on the mathematical tools to be used in the quantification. This is best explained with a simple example as follows.

Perhaps, the simplest elastic system is an ideal, linear line spring, Figure 1.1. In many simple applications a line spring is assumed to operate only along its axis. In other words, the axis remains unchanged. Then, a simple and relative quantification of the elastic behavior of a line spring is provided by terms such as strong, weak, stiff, compliant, very stiff, very compliant, etc. Each of these terms refers to a relative degree of quantification as experienced by humans. For example a stiff or strong spring means that the deflection is relatively small under nominal axial forces, or relatively large axial forces are needed for nominal deflections, etc. This hints that the ratio of applied force to the resulting deflection, or its inverse, may be important. This is the basis of a more precise quantification of the elastic response of linear line springs given by the well known **Hooke's law**

$$\frac{f}{y} = k \qquad \text{or} \qquad f = ky \tag{1.1}$$

$$\frac{y}{f} = c \qquad \text{or} \qquad y = cf \tag{1.2}$$

where f is the axial force, $k = \frac{1}{c}$ is a constant and y is the deflection, see Figure 1.1.

For a given deflection, the axial force is directly proportional to k. So, k is a direct measure of how *stiff* the spring is and is called the simple line spring stiffness or spring rate. On the other hand, for a given force, the deflection is directly proportional to c. So, c is a direct measure of how *compliant* the spring is and is called the simple line spring compliance.

The simple spring force-deflection relation is globally linear. That is, it linearly relates finite forces to finite deflections. Yet, such globally linear elastic systems are almost non-existent in practice. Even a helical coil spring exhibits a linear relation only in a limited range of forces and

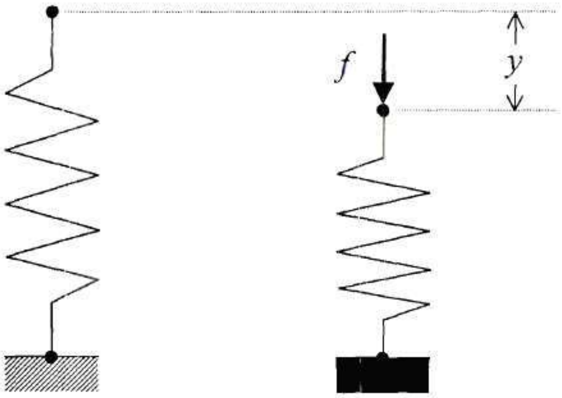


Figure 1.1: The elastic behavior of a simple line spring can be quantified by the ratio of the applied force to the resulting deflection, f/y.

deflections. To overcome this, one notes that the rate equation can actually be given as $k = \frac{\Delta f}{\Delta x}$, where x is the coordinate of the end point of the spring with respect to any point on the spring axis. Δf is the change in the applied force and Δx is the change in the location of the end point. This leads to the generalization of the simple stiffness as

$$k = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \frac{df}{dx} \tag{1.3}$$

which generates the Hooke's law by assuming linearity. However, an important difference is that (1.3) can also model a line spring with a non-linear force-deflection ratio. The simple stiffness becomes dependent on x (or f),

$$\delta f = k(x)\delta x \tag{1.4}$$

where $\delta()$ denotes a small change in its argument.

What happens if there are more than one line spring connected in parallel and oriented in different directions? Figure 1.2 illustrates such a connection. In general, this systems reacts to forces in any direction with differing degrees of stiffness. Moreover, it can also react to moments. In this case, a scalar number for the whole system cannot sufficiently describe the elastic behavior.

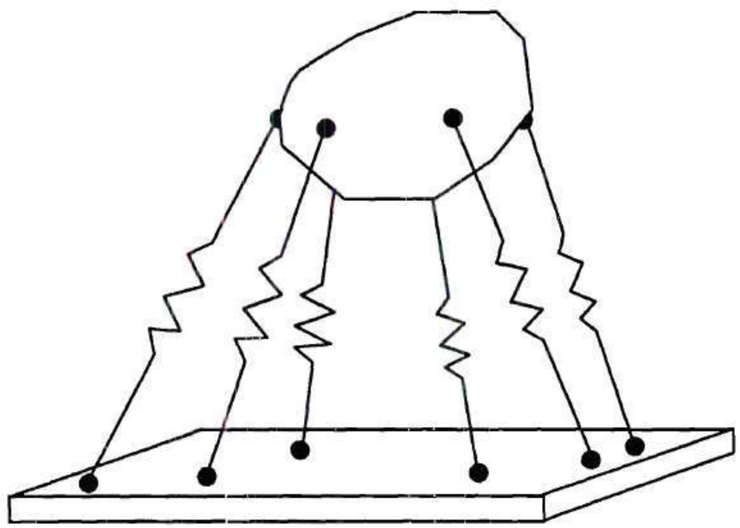


Figure 1.2: A parallel connection of many springs.

Nevertheless, the single spring case provides a model using the form of (1.4). In words, the stiffness is a linear relation between small changes in configuration and small changes in loads. This general definition of stiffness and compliance presented in Chapter 2. Also, small changes in the configuration of and the loads on an elastically suspended rigid body are shown to be described by 6×1 vectors. Then, the linear relation (1.4) and its inverse are generalized to a linear equation relating two such vectors.

The most general linear relations between two vectorial quantities of the same dimension are given by *second order tensors* and can be represented by square matrices. As a result, $k(x)$ in (1.4) is replaced by a 6×6 matrix. The central topic of this study is the analysis of linearly elastic systems via the analysis of stiffness and compliance matrices of elastically suspended rigid body systems.

1.3 An Overview

This section is aimed at providing the reader with a brief overview of the material contained in each main chapter of the thesis. For simplicity, each chapter is made as independent as possible from the others.

For the reader not familiar with screw theory, Chapter 2 presents basic notions. Infinitesimal displacement or velocity, and loads are shown to be screw quantities, which are used throughout this study. Terms and quantities such as spatial vectors, elastic system, elastically suspended rigid body, etc. are defined. These ultimately lead to the definition of stiffness and compliance. Finally, a survey of previous studies is presented.

Chapter 3 investigates a set of eigenvalue problems for symmetric stiffness and compliance proposed by Lipkin and Patterson [30]. First, these eigenvalue problems are shown to be related to special screw subspaces of free-vectors (Chapter 2). Closed form equations for the location of the center of elasticity (Lipkin and Patterson) are determined. Previously unknown physical properties the centers of stiffness and compliance (Loncaric [32]) are found. These lead to the following principal results: 1) *if a compliant axis exists, it must pass through all three centers*, and 2) *if two compliant axes exist, the centers of elasticity, stiffness and compliance coalesce.* Additional physical properties of the centers are presented. The theory is applied to an RCC (remote center of compliance, see Chapter 9) device and a dexterous robotic hand.

Chapter 4 presents a new set of eigenvalue problems which are complementary to the free-vector eigenvalue problems. The new eigenvalue problem is called the line-vector eigenvalue problem due to its relation to special screws called line-vectors. Unlike the free-vector eigenvalue problems, each point in space generates a pair of line-vector eigenvalue problems. The free-vector eigenvalue problem leads to unique decompositions of stiffness and compliance. The line-vector eigenvalue problems provide different decompositions of stiffness and compliance for every point. Both free-vector and line-vector decompositions are used in Chapter 8 to achieve the syntheses of stiffnesses by springs using a minimum number of springs.

Chapter 5 is based on a missing link between the free-vector and line-vector eigenvalue problems. The center of elasticity is identified based solely on the free-vector eigenvalue problems. Since the line-vector eigenvalue problems are complementary to the free-vector problems, the question is what the analogous center of elasticity for the line-vector case is. Chapter 5 establishes the correct analogy by redefining the center of elasticity. The result is the identification of points called the co-centers of elasticity which have properties analogous to the center of elasticity. Unlike the center of elasticity, the co-center may not be unique. Numerical examples are provided.

Chapter 6 brings together the results of the free-vector and line-vector eigenvalue problems in the investigation of compliant axes. Until now, the compliant axis definitions were based on the free-vector eigenvalue problems. The incorporation of the line-vector eigenvalue quantities completes the picture and leads to generalized special axes such as force-translation axis, force-rotation axis, etc. The hierarchy of these special axes, first introduced by Patterson and Lipkin ??, ??, is thus refined. Also, it is shown that a compliant axes pass through at least one and at most three co-centers, and if there exists two or more compliant axes all centers coalesce. Numerical examples are given.

Chapter 7 focuses on a different problem. First, the stiffness matrix of an arbitrary parallel connection of line springs is determined. This new result is later used in the synthesis problem (Chapter 8). Interestingly, the stiffness matrix of line springs is asymmetric when it is not in an unloaded equilibrium. The fundamental contribution is the reduction of the skew-symmetric part of the stiffness matrix to its simplest and most understandable form. This has practical significance since very compliant systems usually operate away from the unloaded equilibrium configuration resulting in asymmetric stiffness matrices. For example, asymmetry can be caused by the effect of gravity forces alone. Immediate results of the skew-symmetric reduction are: the stiffness matrix is symmetric if and only if it is at unloaded equilibrium, stiffness matrices composed of line springs in parallel are characterized by 26 independent parameters (not 36), and the stiffness matrices viewed in the fixed and moving frames are transposes. The analysis is then extended to torsional spring systems. Unlike line springs, torsional spring systems do not have a simple skew-symmetric part. Therefore, the stiffness can be asymmetric even in the unloaded state, or it can symmetric in a loaded state. Nevertheless, for both line and torsional springs, the stiffness is symmetric if all the springs are individually unloaded. For planar parallel connections, Pigoski et al. [43] showed the existence of a moving body with respect to which the stiffness is always symmetric. In Chapter 7, this result is extended to arbitrary spatial connections. For this, first the differential rule of spatial vectors with respect to different frames is developed and applied to the stiffness formula. The results show that there exists infinitely many moving bodies for which the line spring stiffness is symmetric. On the contrary, there is no such body for the torsional case in general.

Chapter 8 uses the results of Chapter 7 to solve the previously unsolved problem of stiffness synthesis by using springs. First, a seemingly totally unrelated and general eigenvector problem is proposed and solved. This is called the isotropic vector problem which is applicable to any square matrix. A classical eigenvector of a matrix is only scaled under the action of the matrix. The isotropic vector of a matrix, however, is rotated to an orthogonal direction and scaled. In Chapter 8, a simple formula is found for the determination of isotropic vectors. However, the most important result is that a matrix has an orthogonal basis of isotropic vectors if and only if its trace vanishes. Equally important is the development of a recursive and numerically stable algorithm for the determination of orthogonal isotropic vectors. This is what makes the synthesis problem solvable. The solution for the three dimensional case is explicitly determined. Then, examples from continuum mechanics and screw theory involving isotropic vectors are presented. For the synthesis, the line and torsional springs are used. From Chapter 7, the form of the stiffness is known in terms of spring axes and rates.

First, it is shown that the minimum number of springs is equal to the rank r of the stiffness. Then, assuming r springs in the synthesis and using theorems from linear algebra, the synthesis problem is reduced to finding orthogonal sets of isotropic vectors of a matrix related to the stiffness. The necessary and sufficient condition turns out to be the vanishing trace of the off-diagonals of stiffness. So, only and all such stiffnesses can be realized by springs. It is shown that there exists infinitely many syntheses in general. The theory is extended to synthesis by $n > r$ springs by presenting a method that uses r spring synthesis. Synthesis by $n > r$ springs requires $r \geq 3$. For $r = 1$ there exists a unique synthesis. For $r = 2$ there exist syntheses by $n > 2$ springs only for special stiffnesses. Finally, the results of Chapters 3 and 4, the free-vector and line-vector decompositions, are applied to the synthesis problem. This reduces the related isotropic vector problem to the three dimensional case. In general, the free-vector decomposition yields three line springs and three torsional springs. The line-vector decomposition yields six line springs, three of which intersect at a point. Algorithms and numerical examples are provided for all cases.

Chapter 9 applies the theoretical results of Chapters 3 and 4 to devices with rotational symmetry. An early example is the RCC device with three beams developed by Whitney and coworkers [34], [35], [51], [52], [53]. The rotational symmetry was exploited by Ciblak and Lipkin [10] for closed form design equations. This result is presented in Chapter 9. The equations allow for $n \geq 3$ beams to be symmetrically placed on a cone. The theory is generalized to allow any stiffness element to be used instead of beams. The result is the most general rotational symmetry device obtained by revolving a generator stiffness along an axis. General properties of such devices are investigated and necessary and sufficient conditions for classical RCCs are found. Devices having a symmetric Stewart platform construction are shown to belong to the general set. As a theoretical confirmation of the theory, an O-ring is modelled by infinitely many infinitesimal beams symmetrically placed about the normal. The theoretical stiffness of the O-ring is calculated. The stiffness in the normal direction is shown to be equal to the usual spring rate for small angle helical springs.

Chapter 10 demonstrates that the results and methods of this study can be applied to the dynamics of a single rigid body. Instead of infinitesimal spatial displacements and loads, one uses spatial velocity and spatial momentum to arrive at the definition of a spatial mass matrix. Then, the stiffness and mass matrices are shown to be analogous. Therefore, all the results of previous chapters apply in the mass case. The free-vector and line-vector eigenvector structure is simpler due to the special form of the mass matrix. This enables one to solve the geometric structure completely. The center of mass is analogous to the combined centers of elasticity, stiffness, compliance and the co-center. A very practical result is obtained by applying the newly found special axes theorems (Chapter 6) to the mass matrix. It is shown that the force-rotation and rotation-force axes defined in Chapter 6 generalize the concept of center of percussion. The axis and joint of percussion are defined with closed form equations. A practical optimum case that can be used in sport equipment design is identified. Finally, by considering both elastic and dynamic effects, the stiffness and mass matrices of an elastically suspended rigid body are used to describe the free vibration case. The necessary and sufficient conditions for the existence of special free vibration modes are investigated.

Chapter 11 presents a discussion of the results of the study. Each result is separately discussed and conclusions are derived. Suggestions for future research are offered.

A bibliography of references made in this study is provided at the end.

1.4 Principal Contributions

This study makes significant contributions to the theoretical and applied analysis of stiffness and compliance. Results lead to a better understanding of stiffness and compliance of elastic systems, practical design rules for a class of robotic devices, and solutions of previously unsolved problems. The following list briefly presents the main results.

- Closed form equations for the location of the center of elasticity proposed in earlier studies are determined. These equations are used in many other results of this study.
- Physical and geometrical meanings of the centers of stiffness and compliance proposed in earlier studies are established. Previously, these centers were looked upon as merely convenient points where stiffness and compliance have special forms.
- The geometric relations between the centers of elasticity, stiffness and compliance are found using compliant axes. This leads to a better classification of compliant systems and design of elastic devices.
- A set of eigenvalue problems proposed in earlier studies is interpreted as a property of mappings from the free-vector subspaces to the general screw space (free-vector eigenvalue problems).
- A new set of eigenvalue problems, complementary to the free-vector eigenvalue problems, is proposed and solved (line-vector eigenvalue problems). This lead to previously unknown decompositions of stiffness and compliance into their geometric and constitutive parts. Every point in space generates a distinct decomposition.
- A convenient redefinition of the center of elasticity leads to complementary centers based on the line-vector eigenvalue problems.
- The compliant axes and other special axes are explained in terms of free-vector and line-vector eigenvalue problems. Generalized definitions are presented. Previously unknown special axes are predicted. Results provide a better classification of complaint systems. A compliant axis is shown to pass through all centers.

- Stiffness matrices of systems of line springs and torsional springs in parallel are determined in closed form for general spatial connections. Previously, only limited solutions were available for planar cases. The stiffness matrix of line spring systems under loading is shown to be asymmetric, which explains the observations made in other studies.

- The skew-symmetric part of stiffness of line springs is shown to be equal to minus one half the applied wrench in spatial cross product form. This is in general not true for torsional springs. In other studies, researcher identified a third frame for which the stiffness is symmetric. Here, it is shown that for line spring stiffness there exists a 2-parameter family of bodies. Whereas for torsional springs, there is no such frame of reference in general.

- The closed form stiffness equations for line and torsional springs are used in solving the unsolved problem of stiffness synthesis by springs. For this, first a theory is developed to determine orthogonal sets of isotropic vectors of arbitrary square and symmetric matrices. Then, the synthesis problem is reduced to the determination of orthogonal isotropic vector sets. The necessary and sufficient condition for synthesis is the zero trace for off-diagonal matrices of stiffness. Any such stiffness is shown to be realizable by r springs, where r is the rank of the stiffness. The theory is extended to synthesis by more than r springs. Algorithms for synthesis are developed.

- The new results about the centers of elasticity, stiffness and compliance are applied to elastic devices with rotational symmetry such as a remote center of compliance device (RCC). The theory leads to new and more accurate closed form design equations. The results are very beneficial in the design and control of robotic manipulators.

- The developed theory is shown to be applicable to the dynamics of a single rigid body. The analog of the stiffness matrix is the mass matrix. The free-vector and line-vector eigenvalue problems are explicitly solved in the mass case. The special axes defined in the elastic case are re-interpreted. A special pair of axes are shown to generalize the center of percussion concept. The generalization leads to the definitions of axes and joints of percussion. Explicit equations for these axes are determined. Results are applied to optimize the design of sport equipment such a golf club and tennis racket.

- The necessary and sufficient conditions for the existence of special free vibration modes proposed in other studies are determined by using the results of the free-vector and line-vector eigenvalue problems as applied to stiffness and mass matrices.

CHAPTER II

BACKGROUND

This chapter is designed to give the reader a minimum familiarity with the concepts, definitions and tools that are used throughout this study. First, some essential mathematical objects, definitions and conventions are introduced in an increasing order of complexity. Then, a survey and discussion of previous studies relevant to this study is presented. They provide a connection to the research in this area and demonstrate the relevance to practical applications.

2.1 Preliminary Concepts

This section first introduces elementary concepts and definitions involving the screw theory. The development is based on an investigation of well known objects such as displacement, velocity, force and couple, etc. Then screws are generalized as elements of a six dimensional vector space. The important concepts of screw axis and pitch are developed both geometrically and algebraically. Special screws called free-vectors and line-vectors are investigated. All possible metrics for the general screw space are determined. Then, spatial vectors are defined as screws referred to a common frame of reference. An elastic system is defined along with underlying assumptions. The concepts of equilibrium and stability are introduced. Finally, the definitions for stiffness and compliance are presented based on spatial quantities.

The reader who is already familiar with these concepts may prefer skipping this section.

2.1.1 Screws

Screws are basically geometric objects that can be comprehended rather intuitively. Screws are best introduced by the well known examples of velocity and load on a rigid body. The curious thing about such objects is their double nature. Almost any engineering or science student may realize at once that the velocity of a rigid body has two seemingly distinct components, namely translational and rotational. Similarly, a load may consist of a force or a couple, or both. This composite feature of such quantities is what gives rise to the abstraction of them as screws.

Velocity Consider a body in motion. In order to fully describe its motion it is sufficient to give its rotational velocity and the translational velocity of a point in the body. Let the point be labeled as P. Then the velocity of the body is fully described by the pair $(\vec{\mathbf{v}}_P, \vec{\omega})$ where $\vec{\mathbf{v}}_P$ is the translational velocity of point P and $\vec{\omega}$ is the rotational velocity. One may consider the two vectors as attached to the point P, as markers designating the motion of the whole body. But, it is clear that the motion of the whole body cannot be dependent on a particular point P. So, another point Q should serve equally well by describing the motion with a pair $(\vec{\mathbf{v}}_Q, \vec{\omega})$. Since both representations mean the same thing, that is the motion of the body, they must be dependent on each other. The difference in the representations of the motion at distinct points is due to the difference in the translational velocities. This difference is induced by the rotation of the body. Given $(\vec{\mathbf{v}}_P, \vec{\omega})$, consider the point P fixed temporarily only as to prevent its translation. Then, another point Q will be rotating about an axis through P by $\vec{\omega}$. Its translational velocity due to this rotation can be given by $\vec{\omega} \times \overrightarrow{PQ}$. Hence the relation between representations of the motion at different points of the body can be given by

$$(\vec{\mathbf{v}}_Q, \vec{\omega}) = (\vec{\mathbf{v}}_P + \vec{\omega} \times \overrightarrow{PQ}, \vec{\omega}) = (\vec{\mathbf{v}}_P - \overrightarrow{PQ} \times \vec{\omega}, \vec{\omega}) \tag{2.1}$$

Figure 2.1 depicts this relation pictorially. This is a general instance of a well known theorem in kinematics called the **Chasles' theorem** which states that the instantaneous motion of a body can always be given as a rotation about a certain axis (instantaneous axis of rotation) plus a translation parallel to the axis. This is proven mathematically later in this section, leading to the definition of screw axis, pitch, etc.

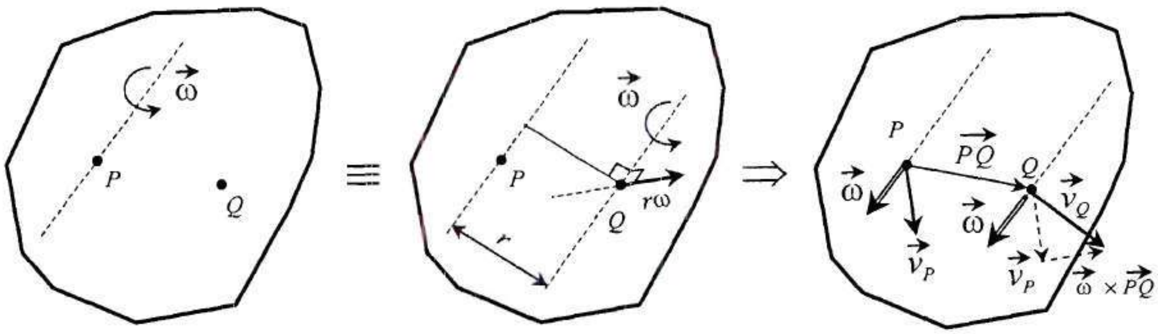


Figure 2.1: The difference in the translational velocities of two points is given by the cross product of rotation and the relative position vector.

<u>Load and Poinsot's Theorem</u> There exists a complete analogy in terms of loads on the rigid body, which is frequently encountered in statics. Figure 2.2 is a well known example of equivalent loads on a rigid body at distinct points. It is immediately seen that force is analogous to rotation. The force $\vec{\mathbf{f}}$ at P induces a couple at Q. This couple, combined with the force $\vec{\mathbf{f}}$ as applied at Q, is equivalent to applying $\vec{\mathbf{f}}$ at P alone. The action of $\vec{\mathbf{f}}$ at Q is given by $\vec{\mathbf{f}} \times \overrightarrow{PQ}$ which reveals the complete analogy. This is why the rotation in Figure 2.1 and the force in Figure 2.2 are both shown by double line arrows. In the most general case, a couple also exists at P which is just added to the couple generated at Q by the force to find the resultant couple at Q. Therefore, if $(\vec{\mathbf{f}}, \vec{\mathbf{m}}_P)$ represents the loads on the rigid body with respect to point P, then the equivalent representation at Q is simply given by

$$(\vec{\mathbf{f}}, \vec{\mathbf{m}}_Q) = (\vec{\mathbf{f}}, \vec{\mathbf{m}}_P + \vec{\mathbf{f}} \times \overrightarrow{PQ}) = (\vec{\mathbf{f}}, \vec{\mathbf{m}}_P - \overrightarrow{PQ} \times \vec{\mathbf{f}}) \tag{2.2}$$

Equation (2.2) is a general instance of a theorem called the **Poinsot's theorem** which states that any load (force plus moment) can be represented by an equivalent load system consisting of a force along a certain axis plus a moment about the axis.

One can easily see the analogy between the translations and couples. If a body is in pure

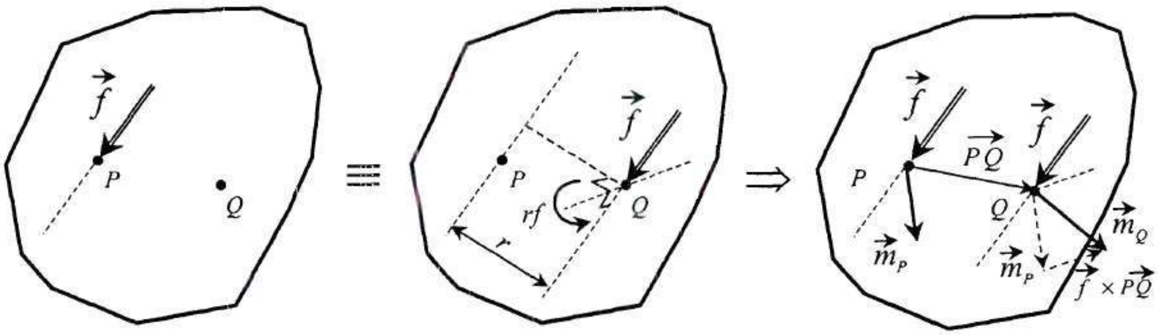


Figure 2.2: The difference between the moments of two equivalent loads at two points is given by the cross product of the force and the relative position vector.

translation then its velocity is given by $(\vec{\mathbf{v}}_0, \vec{\mathbf{0}})$ at every point, which simply means that all points move with the same translational velocity. Similarly, if there exists a pure moment on the body then the load is given by $(\vec{\mathbf{0}}, \vec{\mathbf{m}}_0)$ at every point. These are well known observations in kinematics and statics.

General Terminology The reader should note that the linear component of the velocity (translation) is analogous to the angular component of the load (couple) and the angular component of the velocity (rotation) is analogous to the linear component of the load (force). This indicates that the velocity and load are different objects although there is a complete analogy between them.

This has its roots in the manifold theory, see, for example, Bishop and Goldberg [2], Schutz [45]. In manifold theory, or tensor calculus, velocities correspond to objects called *vectors* and loads correspond to complementary objects called *1-forms*. This is an unfortunate and sometimes confusing terminology. Because both velocities and loads form *vector spaces*. A vector space is a very general, abstract and algebraic construct. Saying that *vectors* and *1-forms* each form vector spaces is confusing since the elements of vectors spaces are commonly called vectors.

Screw theory brings a clearer terminology in which velocities are instances of elements called *twists* and loads are instances of elements called *wrenches*. The definitions of these are presented later.

Ray- and Axis-Coordinates There is some degree of arbitrariness in the representation of velocities or loads as pairs at a given point of a body. Recall that for velocities the term $(\vec{\mathrm{v}}, \vec{\omega})$ has the point dependent part as the first entry, whereas for loads $(\vec{\mathbf{f}}, \vec{\mathrm{m}})$, it is the second entry. On the contrary, both terms have the linear components as the first entry and angular components as the second. These are not essential issues as long as a consistent usage is decided on beforehand. For example, it would be completely legitimate to use $(\vec{\omega}, \vec{\mathrm{v}})$ or $(\vec{\mathrm{m}}, \vec{\mathbf{f}})$.

The representations like $(\vec{\mathbf{f}}, \vec{\mathrm{m}})$ and $(\vec{\omega}, \vec{\mathrm{v}})$, where the first entry is point independent, are called as the **representations in ray-coordinates** and the representations like $(\vec{\mathrm{v}}, \vec{\omega})$ and $(\vec{\mathrm{m}}, \vec{\mathbf{f}})$, where the first entry is point dependent, are called as the **representations in axis-coordinates**.

In this study, the velocity and other similar quantities are represented in axis-coordinates, whereas the load and other similar quantities are represented in ray-coordinates. There are two main reasons for this. First, in both cases, the linear and angular parts will be in the same position leading to easier expressions such as work done. Second, which is more important, the higher order quantities such as stiffness, mass, etc. become symmetric in their matrix forms. Still, however, it should be stressed that the difference is superficial. One could very well use all ray- or all axis-coordinates and still obtain the same results. The difference is a matter of simplicity.

The matrix forms for the 3-vector pairs describing velocity and load are introduced and used throughout the study.

$$\hat{V}_P = \begin{bmatrix} \vec{\mathbf{v}}_P \\ \vec{\omega} \end{bmatrix} \qquad \hat{W}_P = \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\mathbf{m}}_P \end{bmatrix} \tag{2.3}$$

Here $\hat{V}_P$ is the velocity of the body represented at P in axis-coordinates and $\hat{W}_P$ is the load on the body represented at P in ray-coordinates. A 6×6 matrix defined by

$$\hat{\Delta} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{2.4}$$

provides a transformation between the ray- and axis-coordinates representations of a such an object in both directions. This is demonstrated below.

$$\begin{bmatrix} \vec{\omega} \\ \vec{\mathbf{v}}_P \end{bmatrix}_{\text{ray}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{v}}_P \\ \vec{\omega} \end{bmatrix}_{\text{axis}} \qquad \begin{bmatrix} \vec{\mathbf{m}}_P \\ \vec{\mathbf{f}} \end{bmatrix}_{\text{axis}} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\mathbf{m}}_P \end{bmatrix}_{\text{ray}} \tag{2.5}$$

Important properties of $\hat{\Delta}$ are

$$\hat{\Delta}^T = \hat{\Delta} \qquad \hat{\Delta}^{-1} = \hat{\Delta} \qquad \det(\hat{\Delta}) = -1 \tag{2.6}$$

$\hat{\Delta}$ has a much deeper importance and interpretation than given here. Later, it is shown that $\hat{\Delta}$ provides a *metric* for the space of screws.

<u>Screw Axis, Pitch, Twist and Wrench</u> It is clear now from (2.1) and (2.2) that, apart from superficial differences, both velocities and load exhibit an essential property related to their representations at different points in the body. This suggests the definition of a general object which encompasses all such instances.

The rule of representation at different points, as given by (2.1) and (2.2), can now be stated in matrix form as

$$\begin{bmatrix} \vec{\mathbf{v}}_Q \\ \vec{\omega} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & -\overrightarrow{PQ}\times \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{v}}_P \\ \vec{\omega} \end{bmatrix} \quad \rightarrow \quad \hat{V}_Q = \hat{X}_a \hat{V}_P \tag{2.7}$$

$$\begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\mathbf{m}}_Q \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ -\overrightarrow{PQ}\times & \mathbf{I} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\mathbf{m}}_P \end{bmatrix} \quad \rightarrow \quad \hat{W}_Q = \hat{X}_r \hat{W}_P \tag{2.8}$$

$\hat{X}_a$ and $\hat{X}_r$ are essentially the same thing only operating on different representations, namely ray- and axis-coordinates. Frequently, the subscripts $_a$ and $_r$ will be omitted whenever the context is clear or a general situation is concerned. The matrix $\hat{X}$ is a transformation matrix.

Definition 1 *Any quantity that transforms according to (2.7) or (2.8) is called a* ***screw quantity****.*

The above definition is somewhat implicit since it is based on a transformation rule. There can be more transparent definitions. The reason this is selected here is that it follows naturally from the Chasles' and Poinsot's theorems. It is possible to define screws as completely geometric objects in Euclidean space with some invariance requirements. These geometric objects are directed lines. The screw description of lines follows naturally from the requirements that the lengths and angles are preserved under rigid body transformations of Euclidean space.

The problem with the definition of a screw as given above is that, rather than the object, its transformation properties are indicated. At this point, a screw appears as a vague quantity which has representations at all points related in a certain manner. The antidote to this approach is to present the geometric meaning of screws and, possibly, fix them as objects comprehensible by common geometrical notions.

Consider a screw $\hat{S}_P = \begin{bmatrix} \vec{\mathbf{a}}^T & \vec{\mathbf{b}}_P^T \end{bmatrix}^T$ in ray-coordinates. In general, this means that $\hat{S}_P$ refers to a pair of 3-vectors, not necessarily parallel, emanating from point P. In search of a simpler form, one may ask whether there exists a point Q where the 3-vectors of $\hat{S}$ become parallel to each other. First, assume the existence of such a point Q. Then,

$$\vec{\mathbf{b}}_Q = h\vec{\mathbf{a}} \qquad h \in R \tag{2.9}$$

due to parallelism. Hence, the question becomes about the existence of $\vec{\mathbf{r}} = \overrightarrow{PQ}$ such that

$$\begin{bmatrix} \vec{\mathbf{a}} \\ h\vec{\mathbf{a}} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\vec{\mathbf{r}}\times & \mathbf{I} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{a}} \\ \vec{\mathbf{b}}_P \end{bmatrix} \tag{2.10}$$

the first row of which is satisfied automatically. Hence, the question reduces to finding solutions to

$$h\vec{\mathbf{a}} = \vec{\mathbf{b}}_P - \vec{\mathbf{r}} \times \vec{\mathbf{a}} \qquad \text{or} \qquad \vec{\mathbf{b}}_P = h\vec{\mathbf{a}} + \vec{\mathbf{r}} \times \vec{\mathbf{a}} \tag{2.11}$$

But, the latter is simply the decomposition of $\vec{\mathbf{b}}_P$ into components parallel and perpendicular to $\vec{\mathbf{a}}$, which is always possible. Since the component of $\vec{\mathbf{r}}$ parallel to $\vec{\mathbf{a}}$ does not matter due to the properties of cross product, one can write

$$\vec{\mathbf{r}} = \alpha\vec{\mathbf{a}} + \vec{\mathbf{r}}_\perp \qquad\qquad \vec{\mathbf{b}}_P = h\vec{\mathbf{a}} + \vec{\mathbf{r}}_\perp \times \vec{\mathbf{a}} \tag{2.12}$$

where $\vec{\mathbf{r}}_\perp$ is component of $\vec{\mathbf{r}}$ perpendicular to $\vec{\mathbf{a}}$. By forming the dot and cross product with $\vec{\mathbf{a}}$ of the second equation in (2.12) one finds

$$h = \frac{\vec{\mathbf{a}}^T\vec{\mathbf{b}}_P}{\vec{\mathbf{a}}^T\vec{\mathbf{a}}} \qquad\qquad \vec{\mathbf{r}}_\perp = \frac{\vec{\mathbf{a}} \times \vec{\mathbf{b}}_P}{\vec{\mathbf{a}}^T\vec{\mathbf{a}}} \tag{2.13}$$

where the vector identity $-\vec{\mathbf{a}} \times \vec{\mathbf{a}} \times \vec{\mathbf{r}}_\perp = \left(\vec{\mathbf{a}}^T\vec{\mathbf{a}}\right)\vec{\mathbf{r}}_\perp$ is used. Equations (2.13) give the complete description of $\vec{\mathbf{b}}_P$ provided that $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$. The complete solution for the locations of the point Q is

$$\overrightarrow{PQ} = \vec{\mathbf{r}} = \alpha\vec{\mathbf{a}} + \vec{\mathbf{r}}_\perp \tag{2.14}$$

where α is an arbitrary real number. Clearly Q is not unique. But, since $\vec{\mathbf{r}}_\perp$ is uniquely found, (2.14) now describes locus of all such points as a line parallel to $\vec{\mathbf{a}}$ whose perpendicular vector from P is given by $\vec{\mathbf{r}}_\perp$. Figure 2.3 summarizes these solutions.

Now, consider the case when $\vec{\mathbf{a}}$ is a rotation and $\vec{\mathbf{b}}_P$ is a translation. Then, the representation at any point Q satisfying (2.10) is a rotation $\vec{\mathbf{a}}$ and a parallel translation $h\vec{\mathbf{a}}$. This means that with respect to an observer at Q the body translates by an amount of h units in the direction of the

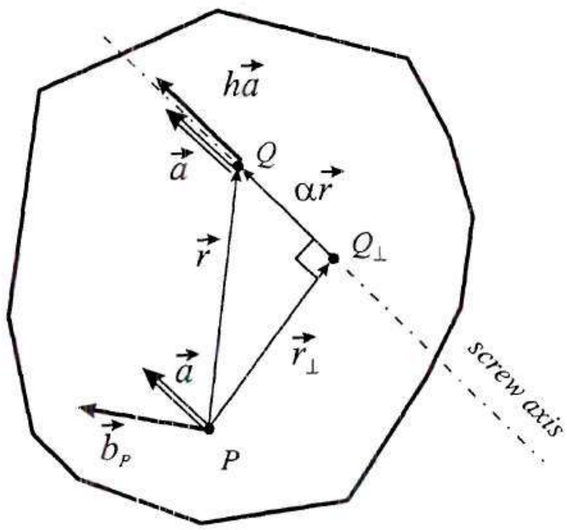


Figure 2.3: The geometric depiction of a screw.

rotation per unit rotation. This is exactly what happens in case of a mechanical screw with a pitch of h which involves the teeth geometry. For example, in a bolt and nut assembly, the bolt advances in the direction of the bolt axis by an amount equal to its pitch per unit rotation about the same axis.

Similarly, if $\vec{\mathbf{a}}$ is a force and $\vec{\mathbf{b}}_P$ is a couple, then the equivalent load system at Q is a force $\vec{\mathbf{a}}$ and a parallel couple $h\vec{\mathbf{a}}$. Again, this is exactly the same as the force and torque relation for tightening or loosening of a bolt and nut system (a screw) which involves the teeth geometry of the screw, etc.

Definition 2 *A screw representing velocity or displacement is called a* ***twist****, and a screw representing the load is called a* ***wrench****.*

These observations and analogy to screws explains the term *screw* used for such objects. The factor h and the locus of points where this behavior is observed are analogous to the pitch and axis of a bolt-and-nut system. Hence, the following definition is sensible.

Definition 3

1. *The coefficient h in equation (2.11) of a screw is called the* **pitch**.

2. *The locus of points where the point dependent part of a screw is parallel to its point independent part is called the* **screw axis** *(Figure 2.3).*

It seems that (2.13) involves the particular point P upon which the analysis is based. This is not desirable if the screw is required to be a well defined quantity independent of particular points of the body. This is indeed the case as shown in the following lemma.

Lemma 4 *The pitch and the screw axis of a screw are independent of the point of representation.*

Proof. Let P and Q be two arbitrary points at which the screw is given by $\begin{bmatrix} \vec{\mathbf{a}}^T & \vec{\mathbf{b}}_P^T \end{bmatrix}^T$ and $\begin{bmatrix} \vec{\mathbf{a}}^T & \vec{\mathbf{b}}_Q^T \end{bmatrix}^T$, respectively. From the transformation rule one can write $\vec{\mathbf{b}}_Q = \vec{\mathbf{b}}_P - \overrightarrow{PQ} \times \vec{\mathbf{a}}$. Then, the fact that $h = \frac{\vec{\mathbf{a}}^T \vec{\mathbf{b}}_Q}{\vec{\mathbf{a}}^T \vec{\mathbf{a}}} = \frac{\vec{\mathbf{a}}^T \vec{\mathbf{b}}_P}{\vec{\mathbf{a}}^T \vec{\mathbf{a}}}$ immediately follows since $\vec{\mathbf{a}}^T \left(\overrightarrow{PQ} \times \vec{\mathbf{a}} \right) = 0$. This shows that the pitch is independent of the point of representation.

Now, the screw axis as obtained from P is the locus of points given by $\vec{\mathbf{r}}_P = \alpha_P \vec{\mathbf{a}} + \frac{\vec{\mathbf{a}} \times \vec{\mathbf{b}}_P}{\vec{\mathbf{a}}^T \vec{\mathbf{a}}}$. The same by using Q is given by $\vec{\mathbf{r}}_Q = \alpha_Q \vec{\mathbf{a}} + \frac{\vec{\mathbf{a}} \times \vec{\mathbf{b}}_Q}{\vec{\mathbf{a}}^T \vec{\mathbf{a}}}$. Here, α_P and α_Q act as parameters for the description of points on the screw axis. If they describe the same line then for any point on the line there must exist values of α_P and α_Q such that both $\vec{\mathbf{r}}_P$ and $\vec{\mathbf{r}}_Q$ locate the same point. This means that there exist α_P and α_Q such that the vector triangle $\vec{\mathbf{r}}_P = \overrightarrow{PQ} + \vec{\mathbf{r}}_Q$ is satisfied for every point on the screw axis. That is,

$$\begin{aligned} \overrightarrow{PQ} &= \vec{\mathbf{r}}_P - \vec{\mathbf{r}}_Q = (\alpha_P - \alpha_Q)\,\vec{\mathbf{a}} + \frac{\vec{\mathbf{a}} \times \left(\vec{\mathbf{b}}_P - \vec{\mathbf{b}}_Q \right)}{\vec{\mathbf{a}}^T \vec{\mathbf{a}}} && (2.15) \\ &= (\alpha_P - \alpha_Q)\,\vec{\mathbf{a}} + \frac{\vec{\mathbf{a}} \times \overrightarrow{PQ} \times \vec{\mathbf{a}}}{\vec{\mathbf{a}}^T \vec{\mathbf{a}}} && (2.16) \end{aligned}$$

where $\vec{\mathbf{b}}_P - \vec{\mathbf{b}}_Q = \overrightarrow{PQ} \times \vec{\mathbf{a}}$ follows from screw properties. But, equation (2.16) is only a decomposition of $\overrightarrow{PQ}$ into components parallel and perpendicular to $\vec{\mathbf{a}}$. Since, this is always possible, given any α_P one can find a α_Q such that (2.16) is satisfied. Therefore, both representations, at P and Q, result in the same screw axis.

■

The meaning of Lemma 4 is that a screw as an object is sufficiently defined by an axis and a pitch (up to the magnitude of $\vec{\mathbf{a}}$), and is independent of any point of representation. This suggests that a screw is a geometric object in its own right which can be viewed as a directed line with an associated scalar. For example, the velocity of a rigid body as a screw is regarded as a rotation about a certain axis plus a translation parallel to the axis. Similarly, the load on a rigid body is a force along a certain axis plus a couple parallel to the axis. In general, a screw in ray-coordinates can be written without ambiguity at any point P in the following form.

$$\hat{S}_P = \begin{bmatrix} \vec{\mathbf{a}} \\ h\vec{\mathbf{a}} + \vec{\mathbf{r}}_\perp \times \vec{\mathbf{a}} \end{bmatrix} \tag{2.17}$$

where $\vec{\mathbf{r}}_\perp$ is the perpendicular vector to the screw axis from P and h is the pitch of the screw.

The matrix $\hat{\Delta}$ and the matrix representation of screws allows one to express the pitch in a simple manner. To do this the following matrix, in ray- and axis-coordinates, is introduced first.

$$\hat{\Gamma}_r = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \hat{\Gamma}_a = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{2.18}$$

Then, pitch equation in (2.13) can be compactly written as

$$h = \begin{cases} \frac{1}{2}\frac{\hat{S}^T\hat{\Delta}\hat{S}}{\hat{S}^T\hat{\Gamma}_r\hat{S}} & \text{for ray-coordinates} \\ \frac{1}{2}\frac{\hat{S}^T\hat{\Delta}\hat{S}}{\hat{S}^T\hat{\Gamma}_a\hat{S}} & \text{for axis-coordinates} \end{cases} \tag{2.19}$$

For unit screws, that is either $\hat{S}^T\hat{\Gamma}_r\hat{S} = 1$ or $\hat{S}^T\hat{\Gamma}_a\hat{S} = 1$, equation (2.19) reduces to

$$h = \frac{1}{2}\hat{S}^T\hat{\Delta}\hat{S} \tag{2.20}$$

One can easily confirm the properties

$$\hat{X}^T\hat{\Delta}\hat{X} = \hat{\Delta} \qquad \hat{X}_r^T\hat{\Gamma}_r\hat{X}_r = \hat{\Gamma}_r \qquad \hat{X}_a^T\hat{L}_a\hat{X}_a = \hat{L}_a \tag{2.21}$$

Hence, for any transformation of screws, the invariance of the pitch follows from (2.19) and (2.21). It is demonstrated later that, similar to $\hat{\Delta}$, $\hat{\Gamma}$ also provides a metric for the screw space.

Zero and Infinite Pitch Screws A pure force or rotation has the distinction that its representation is $\begin{bmatrix} \vec{\mathbf{a}}^T & \vec{\mathbf{0}}^T \end{bmatrix}^T$ at any point on the screw axis. For such screws, the pitch is clearly zero due to (2.13). The pure forces and rotations are not alone in being distinctive. At the other end of the spectrum are the pure couples and translations. In this case, the screw is simply $\begin{bmatrix} \vec{\mathbf{0}}^T & \vec{\mathbf{b}}^T \end{bmatrix}^T$ at every point. It is not too difficult to say that in the limit $\vec{\mathbf{a}} \to \vec{\mathbf{0}}$, the pitch becomes infinite, i.e. $h \to \infty$. All other screw types have finite non-zero pitches. Note that, these cases can be separated according to whether the scalar $\vec{\mathbf{a}}^T\vec{\mathbf{b}}$ is zero or not. For $\vec{\mathbf{a}}^T\vec{\mathbf{b}} = 0$, one has either a zero pitch or an infinite pitch screw, depending on whether $\vec{\mathbf{a}}$ is zero or not. These are summarized in the following definition.

Definition 5 *A non-zero screw* $\begin{bmatrix} \vec{\mathbf{a}}^T & \vec{\mathbf{b}}^T \end{bmatrix}^T$ *such that* $\vec{\mathbf{a}}^T\vec{\mathbf{b}} = 0$ *is called*

1. *a **zero pitch screw** if* $\vec{\mathbf{a}} \neq 0$,

2. *an **infinite pitch screw** if* $\vec{\mathbf{a}} = 0$.

This classification of screws is very important and central to many results of this study. In later sections, the zero and infinite pitch screws are redefined as free-vectors and line-vectors. These redefinitions provide a better geometric view of zero and infinite pitch screws. Both definitions are interchangeably used in this study.

Unit Screws Any screw $\begin{bmatrix} \vec{\mathbf{a}}^T & \vec{\mathbf{b}}^T \end{bmatrix}^T$ such that $\vec{\mathbf{a}}^T\vec{\mathbf{a}} \neq 0$ can be written as

$$\hat{S}_O = \beta \begin{bmatrix} \vec{\mathbf{u}} \\ h\vec{\mathbf{u}} + \vec{\mathbf{r}} \times \vec{\mathbf{u}} \end{bmatrix} \qquad \vec{\mathbf{u}}^T\vec{\mathbf{u}} = 1 \tag{2.22}$$

where $\vec{\mathbf{u}}$ is a unit vector parallel to $\vec{\mathbf{a}}$, $\vec{\mathbf{r}}$ is the vector from O to a point on the screw axis and h is the pitch. The term $\begin{bmatrix} \vec{\mathbf{u}}^T & h\vec{\mathbf{u}}^T + \vec{\mathbf{r}} \times \vec{\mathbf{u}} \end{bmatrix}_A^T$ is called a **unit screw**, where h has the units of length and $\vec{\mathbf{u}}$ is unitless. Hence the unit screw, which is the essential part of a general screw, is completely geometric and uniquely determines a screw axis. A finite pitch unit screw is defined by five parameters. This is sensible since a general screw is described by six parameters, and a unit screw is obtained from this by applying the constraint $\vec{\mathbf{a}}^T\vec{\mathbf{a}} = 1$.

For infinite pitch screws $\vec{\mathbf{a}} = \vec{\mathbf{0}}$ and the following factoring is possible.

$$\hat{S}_O = \beta \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{u}} \end{bmatrix} \qquad \vec{\mathbf{u}}^T\vec{\mathbf{u}} = 1 \tag{2.23}$$

In this case, the unit screw is $\begin{bmatrix} \vec{\mathbf{0}}^T & \vec{\mathbf{u}}^T \end{bmatrix}^T$ with respect to any point and there is no definitive screw axis. An infinite pitch unit screw is defined by two parameters.

In equations (2.22) and (2.23), the factor β can be any scalar with appropriate units. If the screw is of finite pitch, then β has the units of angular speed and force for velocities and loads, respectively. If the screw is of infinite pitch, then β has the units of translational speed and moment for velocities and loads, respectively. So, the physical (constitutive) nature is provided by β and the geometric content is provided by the appropriate unit screw.

<u>Screw Representations of Lines</u> Another geometric result concerns the association of lines in 3-dimensional space with screws. Since every finite pitch screw has a unique screw axis, one associates the line defined by the screw axis with the screw itself by taking the pitch as zero since it is irrelevant to the line. Therefore, there exists a 1-1 association of lines with zero pitch screws. Then, given a coordinate system at O, any line can be given as

$$\begin{bmatrix} \vec{\mathbf{u}} \\ \vec{\mathbf{r}} \times \vec{\mathbf{u}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{u}} \\ \vec{\mathbf{r}}_\perp \times \vec{\mathbf{u}} \end{bmatrix} \tag{2.24}$$

where $\vec{\mathbf{u}}$ is unit vector along the line, $\vec{\mathbf{r}}$ is the vector from O to any point on the line and $\vec{\mathbf{r}}_{\perp}$ is the perpendicular vector from O to the line. Any of the matrices in (2.24) sufficiently determines the line. The numbers in the column matrices of (2.24) are known in literature as the **Plücker coordinates of a line.**

2.1.2 Screw Systems

The well known fact that quantities like wrenches (loads) and twists (velocities) are additive can be generalized to the additivity of screws. The additivity of wrenches is meaningful when they are applied to the same body. The equations of statics, i.e. the sums of forces and couples, are united in terms of the sum of wrenches. If the pure forces on a rigid body are given by $\vec{\mathbf{f}}_i$ and the pure couples are given by $\vec{\mathbf{m}}_j$. Then, the net load on the body is given in statics by

$$\vec{\mathbf{f}} = \sum_i \vec{\mathbf{f}}_i \qquad \vec{\mathbf{m}}_O = \sum_j \vec{\mathbf{m}}_j + \sum_i \vec{\mathbf{r}}_i \times \vec{\mathbf{f}}_i \tag{2.25}$$

where $\vec{\mathbf{m}}_O$ is the net moment at O and $\vec{\mathbf{r}}_i$ are the position vectors from O of some points A_i on the line of action of the i^{th} force. By definition of the wrench, net load on the body with respect to point O can be written as

$$\hat{W}_O = \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\mathbf{m}}_O \end{bmatrix} = \begin{bmatrix} \sum_i \vec{\mathbf{f}}_i \\ \sum_j \vec{\mathbf{m}}_j + \sum_i \vec{\mathbf{r}}_i \times \vec{\mathbf{f}}_i \end{bmatrix} \tag{2.26}$$

A trivial way of representing the right side of (2.26) is

$$\hat{W}_O = \sum_i \begin{bmatrix} \vec{\mathbf{f}}_i \\ \vec{\mathbf{r}}_i \times \vec{\mathbf{f}}_i \end{bmatrix} + \sum_j \begin{bmatrix} \vec{0} \\ \vec{\mathbf{m}}_j \end{bmatrix} = \sum_i \hat{W}_O^0 + \sum_j \hat{W}^\infty \tag{2.27}$$

where the first sum in (2.27) is that of zero pitch wrenches and the second is that of infinite pitch wrenches. One may also combine one or more couples with one or more forces, in many different ways, so that the sum in (2.27) becomes that of wrenches with finite and infinite pitches. So, in general the net load on a rigid body can be given as

$$\hat{W}_O = \sum_i \hat{W}_{Oi} \tag{2.28}$$

where, now, $\hat{W}_{Oi}$ are individual wrenches with general pitches.

In case of twists, a different situation is considered since one can speak of only one velocity twist for a rigid body at a time. Consider a certain number of rigid bodies such that the twist of i^{th} body is specified relative to the $(i-1)^{\text{th}}$ body as $\hat{T}_{O(i/i-1)}$. At this point, disregard the question as to which body the point O belongs to. Then, the net twist of the i^{th} body with respect to the 0^{th} body (e.g., fixed frame) can be given as

$$\hat{T}_{O(i/0)} = \sum_{j=1}^{i} \hat{T}_{O(j/j-1)} \tag{2.29}$$

Such a situation arises in the kinematical analysis of serial manipulators. See, for example, Featherstone [20].

The screws were defined considering the points on a single rigid body only. Yet, as the twists case demonstrated, it is not meaningful to consider more than one twist at a time for a single body. On the other hand, for multi-body cases, summing twists belonging to distinct bodies is ambiguous. Because, until now it is assumed that the point of representation belongs to the body whose motion is described by the twist. But, then it is not clear as to what is meant by the summation of twists represented at points belonging to distinct bodies. Hence, the question about the point of representation is a legitimate and important one. This ambiguity will be resolved in the next section, the result of which will be the concept of spatial vectors. The point here is that additivity of screws is a sensible property.

Screw Space and n-Systems If n screws $\hat{S}_{Oi}$ are given, then one defines a linear combination of them as the sum

$$\hat{S}_O = \sum_{i=1}^{n} \alpha_i \hat{S}_{Oi} = \sum_{i=1}^{n} \alpha_i \begin{bmatrix} \vec{\mathbf{a}}_i \\ \vec{\mathbf{b}}_{Oi} \end{bmatrix} \tag{2.30}$$

where α_i are real scalars. That the resultant, $\hat{S}_O$, is a screw follows from the linearity of the transformation rule when applied to the sum in (2.30). If one considers a set of all possible screws, then (2.30) indicates that the linear combination of any number of screws belongs to the set. Such a set is called a **vector space**.

Definition 6 *The vector space of all screws is called the **screw space**.*

Since screws are represented by 6×1 matrices the screw space is a 6-dimensional vector space, a well known fact in linear algebra. This is restated as the following lemma.

Lemma 7 *The screw space is a 6-dimensional vector space.*

Given any number of elements of a vector space, the set of all linear combinations formed solely from these elements form a **vector subspace** and these elements are said to **span** this subspace. If the number of elements that span a vector space is a minimum then they are called a **basis** for the vector space and this minimum number is the **dimension** or *rank* of the vector space. The dimension of a vector space is unique, but there may be infinitely many bases that span the same vector space. Elements of a basis for the screw space are called the **basis screws**.

An arbitrary number of screws $\hat{S}_{Oi}$ are said to be **linearly independent** of each other if

$$\sum_i \alpha_i \hat{S}_{Oi} = \hat{0} \qquad \Rightarrow \qquad \alpha_i = 0 \qquad \text{for all } i \tag{2.31}$$

The elements in any basis form a linearly independent set. Linear independence simply means that *none* of the elements in such a set can be represented as a linear combination of others.

Given $n \leq 6$ linearly independent screws, the screw subspace spanned by these elements is n-dimensional. Historically, these are given a special name as in the following definition.

Definition 8 *A vector subspace of screws spanned by n linearly independent screws is called an **n-system**.*

An n-system simply contains all linear combinations of its n basis elements. For example, a **1-system** consists of only one screw and its multiples. Clearly, in a 1-system all screws have the same pitch and the same screw axis. The properties of n-systems has been investigated extensively in the literature. See, for example, Ball [1], Hunt [28]. Most important to this study are the **3-systems** of screws. However, in the next section, the 2-system of screws is presented since they are easier to comprehend and therefore help understand the structure of 3-systems.

The Geometry of 2-systems If all n-systems are ordered by their geometric complexity based on human visualization, from trivial to humanly impossible, then the 2-systems are found in the middle. They are neither trivial nor too complex to visualize. Therefore, the study of their geometric structure provides invaluable insight into the structure of higher order systems.

In general, the pitches of the elements of a 2-system are finite. In particular, a 2-system may contain one or two linearly independent infinite pitch screws. However, the finite pitch 2-systems are better suited for presentation in this section. Thus, in the rest of this section a 2-system is assumed to contain only finite pitch screws.

Consider a 2-system spanned by two linearly independent screws $\hat{S}_{O1}$ and $\hat{S}_{O2}$. Any screw in this 2-system is given by

$$\hat{V}_O = \alpha_1 \hat{S}_{O1} + \alpha_2 \hat{S}_{O2} = \alpha_1 \begin{bmatrix} \vec{\mathbf{a}}_1 \\ \vec{\mathbf{b}}_{O1} \end{bmatrix} + \alpha_2 \begin{bmatrix} \vec{\mathbf{a}}_2 \\ \vec{\mathbf{b}}_{O2} \end{bmatrix} \tag{2.32}$$

The direction of the resulting screw is given by the unit vector of $\alpha_1 \vec{\mathbf{a}}_1 + \alpha_2 \vec{\mathbf{a}}_2$. Therefore, all

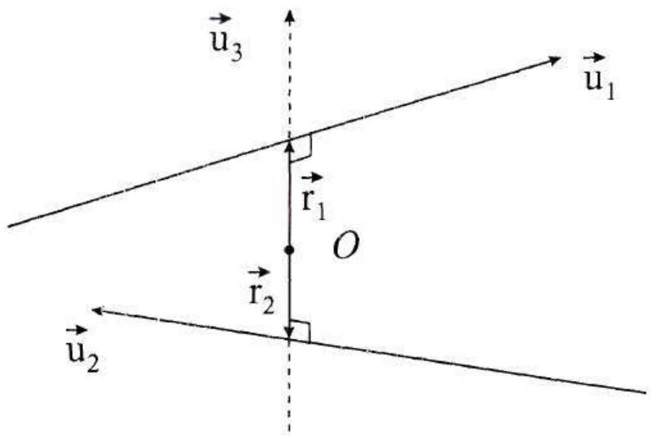


Figure 2.4: Basis for a 2-system of screws.

screws have directions parallel to a plane spanned by $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$, and there exists at least one screw in any given direction parallel to the plane. Therefore, there is no loss of generality in taking $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$ as perpendicular unit vectors, which simplifies the matter greatly. Furthermore, the point O can be taken at the mid-point of the common perpendicular of the basis screws. Figure 2.4 illustrates this arrangement. The unit vectors $\vec{\mathbf{u}}_i$ form a right-handed orthonormal system. Clearly, $\vec{\mathbf{r}}_1 = -\vec{\mathbf{r}}_2 = r\vec{\mathbf{u}}_3$ and the 2-system can be written as

$$\hat{V}_O = \alpha_1 \begin{bmatrix} \vec{\mathbf{u}}_1 \\ h_1\vec{\mathbf{u}}_1 + \vec{\mathbf{r}}_1 \times \vec{\mathbf{u}}_1 \end{bmatrix} + \alpha_2 \begin{bmatrix} \vec{\mathbf{u}}_2 \\ h_2\vec{\mathbf{u}}_2 + \vec{\mathbf{r}}_2 \times \vec{\mathbf{u}}_2 \end{bmatrix} \tag{2.33}$$

$$= \alpha_1 \begin{bmatrix} \vec{\mathbf{u}}_1 \\ h_1\vec{\mathbf{u}}_1 + r\vec{\mathbf{u}}_2 \end{bmatrix} + \alpha_2 \begin{bmatrix} \vec{\mathbf{u}}_2 \\ h_2\vec{\mathbf{u}}_2 + r\vec{\mathbf{u}}_1 \end{bmatrix} \tag{2.34}$$

Since only a geometric picture of a 2-system is essential one may require $\hat{V}_O$ to be a unit screw, i.e.

$$(\alpha_1\vec{\mathbf{u}}_1 + \alpha_2\vec{\mathbf{u}}_2)^T (\alpha_1\vec{\mathbf{u}}_1 + \alpha_2\vec{\mathbf{u}}_2) = \alpha_1^2 + \alpha_2^2 = 1 \tag{2.35}$$

so that a substitution such as $\alpha_1 = \cos\theta$ and $\alpha_2 = \sin\theta$ is sensible. Then from (2.13) the pitch and the perpendicular vector from O of $\hat{V}$ are found as

$$h_V = \frac{h_1 + h_2}{2} + \frac{h_1 - h_2}{2}\cos 2\theta + r\sin 2\theta \qquad \vec{\mathbf{r}}_V = \left[\frac{h_2 - h_1}{2}\sin 2\theta + r\cos 2\theta\right]\vec{\mathbf{u}}_3 \tag{2.36}$$

This well known and interesting result shows that all screws in a 2-system have a common perpendicular. In case of Figure 2.4, all screws intersect the shown common perpendicular of the generating screws and are in the $\vec{\mathbf{u}}_1\vec{\mathbf{u}}_2$-plane. Another observation is that both h_V and $\vec{\mathbf{r}}_V$ are bounded. The extrema are

$$\mathrm{ext}(\vec{\mathbf{r}}_V) = \pm\sqrt{r^2 + \left(\frac{h_2 - h_1}{2}\right)^2}\,\vec{\mathbf{u}}_3 \qquad \text{at} \qquad \tan 2\theta = \frac{h_2 - h_1}{2r} \tag{2.37}$$

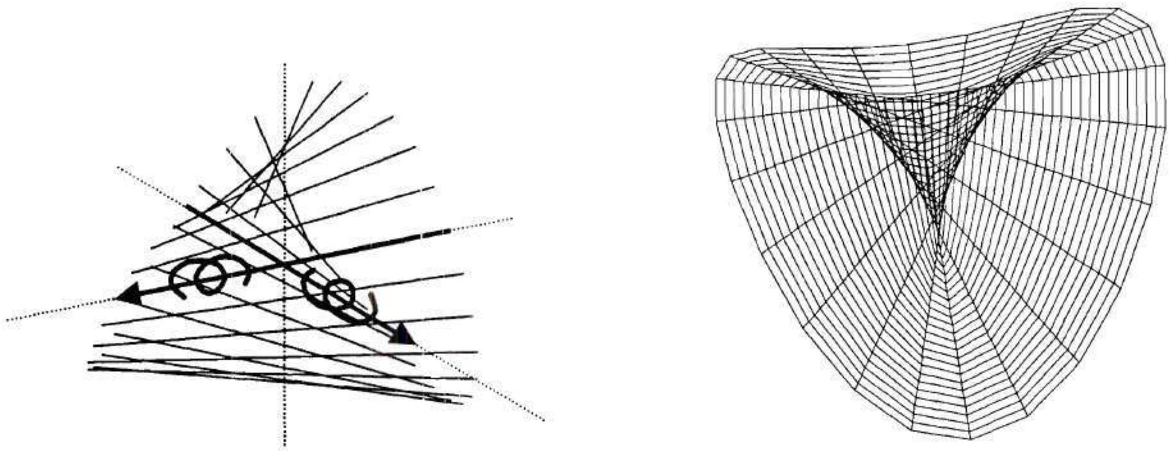
$$\mathrm{ext}(h_V) = \frac{h_1 + h_2}{2} \pm \sqrt{\left(\frac{h_1 - h_2}{2}\right)^2 + r^2} \qquad \text{at} \qquad \tan 2\theta = -\frac{2r}{h_2 - h_1} \tag{2.38}$$

where $(+)$ sign is for $\vec{\mathbf{u}}_3$ direction and $(-)$ sign is for $-\vec{\mathbf{u}}_3$ direction. One can easily confirm that the parameter θ is the angle between the generated screw and the basis screw $\hat{S}_1$. So, as θ takes on values from 0 to π, the generated screws starts from $\hat{S}_1$ and moves on the common perpendicular as it spirals. It reaches to a maximum distance indicated by (2.37) and traces back, passes through $\hat{S}_2$ at $\theta = \frac{\pi}{2}$, continuing the spiral until it reaches the other maximum distance and returns back to $\hat{S}_1$.This process reveals that there exist exactly two screws passing through any given point of the common perpendicular in the range of $\vec{\mathbf{r}}_V$, except at ends. The screws at ends have pitches equal to $\frac{h_1+h_2}{2}$. Each pair of screws through a point have distinct orientations and pitches, in general.

An interesting situation occurs when the screw pair passing through O is considered. In this case, $\vec{\mathbf{r}}_V = \vec{\mathbf{0}}$ which means

$$\frac{h_2 - h_1}{2}\sin 2\theta + r\cos 2\theta = 0 \qquad \Rightarrow \qquad \tan 2\theta = -\frac{2r}{h_2 - h_1} \tag{2.39}$$

There are two solutions in the range $\theta \in (0, \pi)$, corresponding to two screws passing through O. But, condition (2.39) is the same as (2.38), meaning that the screws with stationary pitches pass through O. Furthermore, these special screws have orthogonal directions. To see this consider the two



(a) (b)

Figure 2.5: A finite pitch two system. (a) The thick arrows are two perpendicular principal screws intersecting at the center. All screws are parallel to the plane spanned by the principal screws and intersect the normal through the center. (b) The screw axes of all screws of 2-system generate a surface called a **cylindroid**.

solutions to (2.39) which can be written as $\tan 2\theta = -\frac{2r}{h_2 - h_1} = t \Rightarrow \theta = \frac{1}{2}\arctan(t) + k\frac{\pi}{2} = \theta_p + k\frac{\pi}{2}$, $k = (0, 1)$, where θ_p is half the principal argument of $\tan(\cdot)$ in $(0, \pi)$. Then, the directions of the screws passing through O are given by

$$\vec{\mathbf{p}}_1 = \alpha_1 \vec{\mathbf{u}}_1 + \alpha_2 \vec{\mathbf{u}}_2 = \cos\theta_p \vec{\mathbf{u}}_1 + \sin\theta_p \vec{\mathbf{u}}_2 \tag{2.40}$$

$$\vec{\mathbf{p}}_2 = \cos(\theta_p + \frac{\pi}{2})\vec{\mathbf{u}}_1 + \sin(\theta_p + \frac{\pi}{2})\vec{\mathbf{u}}_2 = -\sin\theta_p \vec{\mathbf{u}}_1 + \cos\theta_p \vec{\mathbf{u}}_2 \tag{2.41}$$

from which it follows that $\vec{\mathbf{p}}_1^T \vec{\mathbf{p}}_2 = 0$, proving the orthogonality. These results are sufficient to render a picture of a generic 2-system as given in Figure 2.5.

So, in general, in a 2-system there exists two unique screws with maximum and minimum pitches, which also intersect perpendicularly at a point P on the common perpendicular of all screws of the 2-system. This point and its associated screws are important in this study. These special screws and their intersecting point are completely geometric properties. The special screws are called the *principal screws* and the intersection point is called the *center* of the 2-system. Formal definitions are presented in the next section considering the 3-systems.

Principal Screws and Center of 3-systems The special screws and points of 2-systems are also found in 3-systems, see Ball [1], Hunt [28] for details. Actually, it is possible to completely determine these special screws by just requiring that their pitches be stationary. The fact that they intersect each other at right angles at a point follows from that. Since these screws can also be used as a basis for an 3-system, it is legitimate to pose the problem as a search for a basis with stationary pitches. Since any element of this special basis must be linear combination of the current basis elements, one can write

$$\hat{P} = \begin{bmatrix} \hat{S}_1 & \hat{S}_2 & \hat{S}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_n \end{bmatrix} = \hat{S}_{6\times 3}\bar{\alpha}_{3\times 1} \tag{2.42}$$

where $\hat{P}$ is an element of the required basis. The pitch of $\hat{P}$ is

$$h_P = \frac{1}{2}\hat{P}^T\hat{\Delta}\hat{P} = \frac{1}{2}\bar{\alpha}^T\hat{S}^T\hat{\Delta}\hat{S}\bar{\alpha} \tag{2.43}$$

if $\hat{P}$ is a unit screw. One can apply the following constraint to make $\hat{P}$ a unit screw.

$$\hat{P}^T\hat{\Gamma}\hat{P} = \bar{\alpha}^T\hat{S}^T\hat{\Gamma}\hat{S}\bar{\alpha} = 1 \tag{2.44}$$

To find the stationary values of h_P in (2.43) as a function of $\bar{\alpha}$, one forms the Lagrangian form incorporating the constraint (2.44) and proceeds in the usual manner of finding extrema.

$$\mathrm{ext}(h_P): \quad \frac{\partial}{\partial\bar{\alpha}}\left[\frac{1}{2}\bar{\alpha}^T\hat{S}^T\hat{\Delta}\hat{S}\bar{\alpha} - \lambda\left(\bar{\alpha}^T\hat{S}^T\hat{\Gamma}\hat{S}\bar{\alpha} - 1\right)\right] = 0 \tag{2.45}$$

$$\frac{\partial}{\partial\lambda}\left[\frac{1}{2}\bar{\alpha}^T\hat{S}^T\hat{\Delta}\hat{S}\bar{\alpha} - \lambda\left(\bar{\alpha}^T\hat{S}^T\hat{\Gamma}\hat{S}\bar{\alpha} - 1\right)\right] = 0 \tag{2.46}$$

The result is

$$\left[\frac{1}{2}\hat{S}^T\hat{\Delta}\hat{S}\right]_{3\times 3}\bar{\alpha} = \lambda\left[\hat{S}^T\hat{\Gamma}\hat{S}\right]_{3\times 3}\bar{\alpha} \tag{2.47}$$

along with the constraint (2.44). Equation (2.47) is a generalized eigenvalue problem. In this case, it involves two symmetric matrices. It is important to note that, from (2.21), the matrices $\frac{1}{2}\hat{S}^T\hat{\Delta}\hat{S}$ and $\hat{S}^T\hat{\Gamma}\hat{S}$, hence (2.47), are invariant origin transformations. Therefore, these matrices and (2.47) are essential to 3-systems. Existence and nature of solutions depend on the particular properties of the matrices.

For a 3-system of finite pitch screws $\hat{S}^T\hat{\Gamma}\hat{S}$ is positive definite. Then, from linear algebra, the generalized eigenvalue equation (2.47) has three real and independent solutions. Let the eigenvalues and eigenvectors be h_i^* and $\bar{\alpha}_i$, respectively.

It is also well known in linear algebra that for any generalized eigenvalue problem with symmetric matrices such as (2.47) there exist orthogonality conditions which, in this case, are given by

$$\bar{\alpha}_j^T\hat{S}^T\hat{\Delta}\hat{S}\bar{\alpha}_i = h_i^*\delta_{ij} \qquad\qquad \bar{\alpha}_j^T\hat{S}^T\hat{\Gamma}\hat{S}\bar{\alpha}_i = \delta_{ij} \tag{2.48}$$

where

$$\delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \tag{2.49}$$

is the **Kronecker's delta**. Note that $\hat{P}_i = \hat{S}\bar{\alpha}_i$ is the i^{th} special screw. If the direction vector of $\hat{P}_i$ is denoted by $\vec{\mathbf{p}}_i$ then the second equation in (2.48) simply means

$$\vec{\mathbf{p}}_i^T\vec{\mathbf{p}}_j = \delta_{ij} \tag{2.50}$$

Thus, the three special screws form an *orthonormal system of screw axes*, as claimed earlier.

The following is needed to show that the $\vec{\mathbf{p}}_i$ also intersect at a common point.

Theorem 9 *Given two screws* $\hat{V}_1 = \left[\begin{array}{cc} \vec{\mathbf{a}}_1^T & \vec{\mathbf{b}}_1^T \end{array}\right]^T$ *and* $\hat{V}_2 = \left[\begin{array}{cc} \vec{\mathbf{a}}_2^T & \vec{\mathbf{b}}_2^T \end{array}\right]^T$ *such that* $\vec{\mathbf{a}}_1 \perp \vec{\mathbf{a}}_2$*, then* *the two screws intersect if and only if*

$$\hat{V}_1^T \hat{\Delta} \hat{V}_2 = 0 \tag{2.51}$$

Proof. First assume that they intersect at a point A which must be unique since the screw axes are perpendicular. Then, $\hat{V}_{1/A} = \begin{bmatrix} \vec{\mathbf{a}}_1^T & h_1\vec{\mathbf{a}}_1^T \end{bmatrix}^T$ and $\hat{V}_{2/A} = \begin{bmatrix} \vec{\mathbf{a}}_2^T & h_2\vec{\mathbf{a}}_2^T \end{bmatrix}^T$ since A is on the screw axes of both screws. Clearly, $\hat{V}_{1/A}^T \hat{\Delta} \hat{V}_{2/A} = 0$ holds simply due to $\vec{\mathbf{a}}_1^T \vec{\mathbf{a}}_2 = 0$. Then, by invariance of $\hat{\Delta}$ as given in (2.21), $\hat{V}_1^T \hat{\Delta} \hat{V}_2 = 0$ follows. Hence, it is a necessary condition.

Next, assume that (2.51) is true. Then, at a point A on the screw axis of $\hat{V}_1$ one must have

$$\vec{\mathbf{a}}_1^T \vec{\mathbf{b}}_2 + h_1 \vec{\mathbf{a}}_2^T \vec{\mathbf{a}}_1 = \vec{\mathbf{a}}_1^T \vec{\mathbf{b}}_2 = 0 \tag{2.52}$$

But this means $\vec{\mathbf{b}}_2 = \alpha_2 \vec{\mathbf{a}}_2 + \alpha_3 \vec{\mathbf{a}}_3$, where $\vec{\mathbf{a}}_3$ is a vector perpendicular to both $\vec{\mathbf{a}}_1$ and $\vec{\mathbf{a}}_2$, such that $\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2 = \vec{\mathbf{a}}_3$. Now, by shifting the representation point from A to another point on the axis of $\hat{V}_1$, B, say, given by $\overrightarrow{AB} = r\vec{\mathbf{a}}_1$, one obtains

$$\hat{V}_{2/B} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \overrightarrow{AP}\times & \mathbf{I} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{a}}_2 \\ \vec{\mathbf{b}}_2 \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{a}}_2 \\ r\vec{\mathbf{a}}_1 \times \vec{\mathbf{a}}_2 + \alpha_2 \vec{\mathbf{a}}_2 + \alpha_3 \vec{\mathbf{a}}_3 \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{a}}_2 \\ \alpha_2 \vec{\mathbf{a}}_2 + (r + \alpha_3)\vec{\mathbf{a}}_3 \end{bmatrix} \tag{2.53}$$

Clearly, there exists a point P on the screw axis of $\hat{V}_1$, corresponding to $r = -\alpha_3$, at which $\hat{V}_{2/B}$ has the form $\begin{bmatrix} \vec{\mathbf{a}}_2^T & h_2\vec{\mathbf{a}}_2^T \end{bmatrix}^T$. Thus, there exists a unique point B through which both screws pass as claimed.

■

Now, for $i \neq j$ (??) is $\bar{\alpha}_j^T \hat{S}^T \hat{\Delta} \hat{S} \bar{\alpha}_i = \hat{P}_j^T \hat{\Delta} \hat{P}_i = 0$. Therefore, $\hat{P}_i$ and $\hat{P}_j$ intersect each other at a unique point P since $\vec{\mathbf{p}}_i^T \vec{\mathbf{p}}_j = 0$ is known. This is true for all $\hat{P}_i$. But, three mutually orthogonal axes can intersect each other only at a unique point. This completes the proof of the following theorem.

Theorem 10 *In any 3-system of screws with finite pitches there exist three unique, mutually perpendicular and intersecting unit basis screws with stationary pitches.*

Definition 11

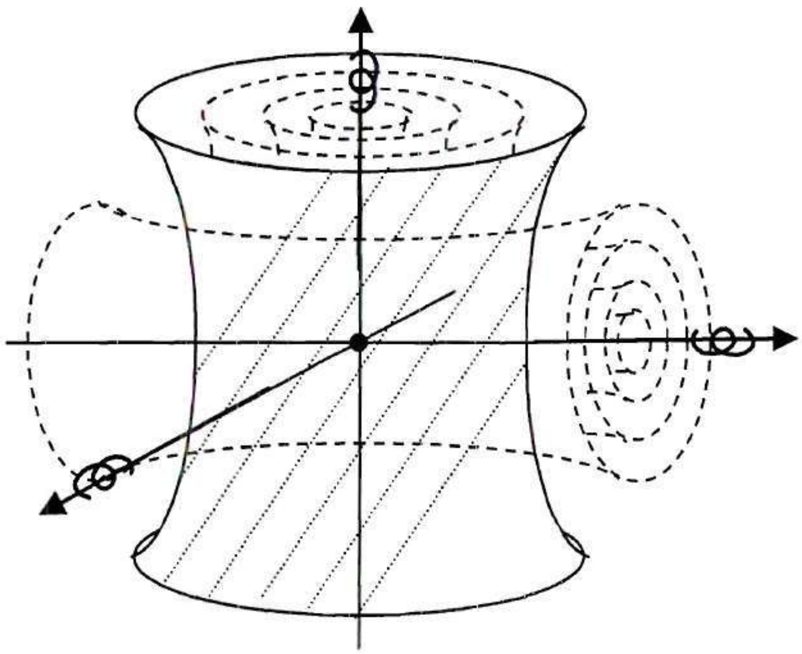


Figure 2.6: A picture of a 3-system of screws. The three mutually orthogonal principal screws intersect at the center of the 3-system. Axes of all screws with identical pitch form a surface called a **hyperboloid**. Thus a hyperboloid is a constant pitch surface. Each principal screw is the central axis of concentric hyperboloids representing smoothly changing pitch from the stationary value (principal pitch) on a principal screw axis (a degenerate hyperboloid) to infinity.

1. *The three unique basis screws of a 3-system with stationary pitches are called the* ***principal screws****.*

2. *The stationary pitches of the principal screws are called the* ***principal pitches****.*

3. *The point of intersection of the principal screws is called the* ***center of the 3-system****.*

Theorem 10 and Definition 11 can also be stated for 2-systems as shown before. 1-systems consist of only multiples of a unit screw which can be taken as the principal screw with the exception that every point on the screw axis can be taken as the center.

Metric Tensors and Scalar Products Given any vector space, one can define a scalar product operation between any two elements. In general, a scalar product is an association of a real number with any two given elements of a vector space provided that the associated number is invariant under a change of representation.

The notion of scalar product leads to the notion of length of a vector when the involved elements are the same. If the vector elements are represented by matrices, then a classical example of scalar product is given by $\bar{a}^T\bar{b}$ for elements $\bar{a}$ and $\bar{b}$. Consider a change of representation given by $\bar{a}' = A\bar{a}$, where A is an invertible square matrix so that $\bar{a} = A^{-1}\bar{a}'$ is also defined. If the scalar product is to be invariant then $\bar{a}^T\bar{b} = \bar{a}'^T\bar{b}' = \bar{a}^T\left(A^TA\right)\bar{b}$ for any two vectors. Therefore, $A^TA = I$ is required. Such matrices are called *orthogonal matrices* and play an important role in many familiar vector spaces. The point here is that the scalar product $\bar{a}^T\bar{b}$ is invariant only if the change of representation in the vector space is performed by orthogonal matrices.

Considering the scalar product of an element with itself, $\bar{a}^T\bar{a}$, one is lead to the classical definition of *length of a vector*, $\|\bar{a}\| = \sqrt{\bar{a}^T\bar{a}}$, due to its well known geometrical interpretation as the Euclidean distance between two points in a Euclidean space. The obvious advantage is that the length of an element can always be taken as a real positive number and is zero if and only if the element is the zero vector.

What happens if the change of representation is performed by a non-orthogonal matrix? This is clearly the case in screw space. A possible answer is the generalization of the scalar product to a bilinear form $\bar{a}^TG\bar{b}$, where G is a symmetric square matrix. The symmetry is needed for $\bar{a}^TG\bar{b} = \bar{b}^TG\bar{a}$. Then, if the change of representation is performed by a matrix A, as above, the requirement for invariant scalar product becomes $A^TGA = G$, so that it is no longer necessary for A to be orthogonal. The scalar product of a vector with itself becomes $\bar{a}^TG\bar{a}$. Such expressions are well known and called *quadratic forms.* The sign of the number $\bar{a}^T G\bar{a}$ depends on the eigenvalues of the matrix G which can be classified as follows.

1. $\bar{a}^T G\bar{a} > (<)0$ for all $\bar{a} \neq \bar{0}$ if G is positive definite (negative definite).

2. $\bar{a}^T G\bar{a} \geq (\leq)0$ for all $\bar{a} \neq \bar{0}$ if G is semi-positive definite (semi-negative definite).

3. $\bar{a}^T G\bar{a} < 0$ and $\bar{b}^T G\bar{b} > 0$ for some $\bar{a}, \bar{b}$ if G is indefinite.

The first case above is well known. Only definiteness of G is important because if G is negative-definite one easily introduces $-G$ as matrix of scalar product, which is positive-definite. Hence, negative- and positive-definite G are treated as equivalent. In such cases, the length of a vector can be defined as the square root of the positive number $\bar{a}^T G\bar{a}$ which can be zero only when the vector argument is the zero vector. The example in previous paragraphs is equivalent to taking $G = I$ which results in the *standard Euclidean length measure.*

On the contrary, in cases (2) and (3) there are non-zero elements whose scalar products with themselves vanish. In these cases, a geometric interpretation of the scalar product similar to the Euclidean length is not meaningful. There are two distinct subclasses. In one, G has zero eigenvalues, which may correspond to semi-definite or indefinite cases. Then, there exists a subspace of vectors whose scalar products with themselves vanish. Algebraically, these vectors belong to the null space of G. In the other, G is non-singular, but indefinite. Then, there exists a set of vectors, not necessarily a vector subspace, for which the self scalar product vanishes.

In the second case, where G is indefinite, the self scalar product takes on both negative and positive values. The notion of a length becomes much more unsuitable. Instead, the real number $\bar{a}^T G\bar{a}$ is considered as a measure of energy because of its relation to the energy integrals involving velocity vectors in an n-dimensional space, see Schutz [45], for example. A famous example is from the special theory of relativity which deals with 4-vectors, composed of spatial and temporal quantities. The matrix G of the special theory of relativity is taken as $\mathrm{diag}(1,1,1,-c^2)$, where c is the speed of light. A 4-vector can be thought as connecting two points in the 4-dimensional space-time. Therefore, $\bar{a}^T G \bar{a}$ is usually called an *interval.* Positive and negative intervals have special meanings. More importantly, there are zero intervals corresponding to non-zero vectors, or distinct points in space-time, which are ultimately related to the paths that light rays follow.

In general, the matrix G is known as a **metric tensor** for the vector space. A positive definite metric is usually called a **Riemannian metric**. A singular metric is called a **degenerate metric**. If a degenerate metric is semi-definite it is sometimes called a **semi-Riemannian** metric. Otherwise, a metric is called an **indefinite metric**. The metric tensor of the special theory of relativity is an indefinite metric known as *Lorentz* or *Minkowski metric.* The scalar product defined by Minkowski metric is invariant under *Lorentz transformations* which applies to the 4-dimensional space-time.

The following theorem applies these ideas to the vector space of screws.

Theorem 12 *There exist exactly four distinct families of metric tensors for the screw space:*

$$\hat{G} = \left\{ \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \alpha \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix}, \beta \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \begin{bmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \beta\mathbf{I} & \mathbf{0} \end{bmatrix} \right\} \tag{2.54}$$

where $\mathbf{0}, \mathbf{I}$ are the 3×3 zero and unit matrices, and α, β are arbitrary real numbers.

Proof. The most general form of a scalar product rule for the screw space is $\hat{S}_1^T \hat{G} \hat{S}_2$. It is required that the matrix $\hat{G}$ be such that $\hat{S}_1^T \hat{G} \hat{S}_2$ is left invariant under screw transformations. The screw transformation was given in (2.7) and (2.8), which is simply $\hat{S}' = \hat{X}\hat{S}$, assuming all quantities are in ray-coordinates. Hence, the invariance can be stated as

$$\hat{S}_1^T \hat{G} \hat{S}_2 = \hat{S}_1'^T \hat{G} \hat{S}_2' = \hat{S}_1^T \left(\hat{X}^T \hat{G} \hat{X} \right) \hat{S}_2 \qquad \text{for all } \hat{S}_1, \hat{S}_2 \tag{2.55}$$

This can be satisfied if only if

$$\hat{X}^T \hat{G} \hat{X} = \hat{G} \tag{2.56}$$

Clearly, $\hat{G} = \hat{0}$ is a solution which is the first matrix of (2.54). This metric assigns the number zero to all pairs of screws, which is invariant under screw transformations, and is called the *trivial metric tensor.*

To find all non-trivial solutions one has to use the form of $\hat{X}$ explicitly as follows.

$$\begin{bmatrix} \mathbf{R} & \mathbf{0} \\ -\vec{\mathbf{r}}\times\mathbf{R} & \mathbf{R} \end{bmatrix}^T \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ -\vec{\mathbf{r}}\times\mathbf{R} & \mathbf{R} \end{bmatrix} = \begin{bmatrix} \mathbf{G}_{11} & \mathbf{G}_{12} \\ \mathbf{G}_{21} & \mathbf{G}_{22} \end{bmatrix} \tag{2.57}$$

where $\mathbf{G}_{ij}$ are the 3×3 partitions of $\hat{G}$. The equation (2.57) is equivalent to

$$\begin{aligned} \mathbf{R}^T\mathbf{G}_{11}\mathbf{R} - \mathbf{R}^T\mathbf{G}_{12}\vec{\mathbf{r}}\times\mathbf{R} + \mathbf{R}^T\vec{\mathbf{r}}\times\mathbf{G}_{21}\mathbf{R} - \mathbf{R}^T\vec{\mathbf{r}}\times\mathbf{G}_{22}\vec{\mathbf{r}}\times\mathbf{R} &= \mathbf{G}_{11} && (2.58) \\ \mathbf{R}^T\mathbf{G}_{12}\mathbf{R} + \mathbf{R}^T\vec{\mathbf{r}}\times\mathbf{G}_{22}\mathbf{R} &= \mathbf{G}_{12} && (2.59) \\ \mathbf{R}^T\mathbf{G}_{21}\mathbf{R} - \mathbf{R}^T\mathbf{G}_{22}\vec{\mathbf{r}}\times\mathbf{R} &= \mathbf{G}_{21} && (2.60) \\ \mathbf{R}^T\mathbf{G}_{22}\mathbf{R} &= \mathbf{G}_{22} && (2.61) \end{aligned}$$

In what follows, α, β and γ are arbitrary non-zero real numbers. Equations (2.58) through (2.61) must hold for all possible pairs of $\vec{\mathbf{r}}$ and $\mathbf{R}$. Therefore, one immediately sees that $\mathbf{G}_{22} = \mathbf{0}, \gamma\mathbf{I}$. Letting $\mathbf{R} = \mathbf{I}$ in (2.60), it can be deduced that $\mathbf{G}_{22} = \mathbf{0}$ in order (2.60) to be valid for all $\vec{\mathbf{r}}$. But this, in turn, requires that $\mathbf{G}_{21} = \mathbf{0}, \beta_1\mathbf{I}$. Similar conclusions force the result that $\mathbf{G}_{12} = \mathbf{0}, \beta_2\mathbf{I}$. So, now, all the equations above, except (2.58) are satisfied simultaneously. Considering the case $\vec{\mathbf{r}} = \vec{\mathbf{0}}$, (2.58) forces the conclusion that $\mathbf{G}_{11} = 0, \alpha\mathbf{I}$. This reduces (2.58) to

$$\vec{\mathbf{r}}\times\mathbf{G}_{21} = \mathbf{G}_{12}\vec{\mathbf{r}}\times \qquad \mathbf{G}_{21} = \mathbf{0}, \beta_1\mathbf{I} \qquad \mathbf{G}_{12} = \mathbf{0}, \beta_2\mathbf{I} \tag{2.62}$$

which can be satisfied for all $\vec{\mathbf{r}}$ and $\mathbf{R}$ only if $\mathbf{G}_{21} = \mathbf{G}_{12}$, therefore $\beta_1 = \beta_2$. Hence, the solutions are

$$\mathbf{G}_{11} = \mathbf{0}, \alpha\mathbf{I} \qquad \mathbf{G}_{12} = \mathbf{G}_{21} = \mathbf{0}, \beta\mathbf{I} \qquad \mathbf{G}_{22} = \mathbf{0} \tag{2.63}$$

All possible combinations of (2.63) results in the distinct set of solutions as given in (2.54). This proves the theorem.

■

It is interesting to note that the second and the third metrics in (2.54) are none other than the matrices $\hat{\Gamma}$ and $\hat{\Delta}$, respectively, since the scalar product can be easily scaled by α or β. Hence, these families are represented by $\hat{\Gamma}$ and $\hat{\Delta}$. The forth metric, which is relatively new, is a linear combination of $\hat{\Gamma}$ and $\hat{\Delta}$, and inherits the invariance due to the linearity of the scalar product. It was also recently demonstrated by Kumar et al. [54] in a different way. These show the importance of the matrices $\hat{\Gamma}$ and $\hat{\Delta}$ since, now, any metric for the screw space can be given as $\alpha\hat{\Gamma} + \beta\hat{\Delta}$. Interestingly, this makes the set of metrics for the screw space a two dimensional vector space spanned by $\hat{\Gamma}$ and $\hat{\Delta}$.

Theorem 13 *All non-trivial metrics for the screw space are semi-definite or indefinite with two distinct eigenvalues, each with an algebraic multiplicity of three.*

Proof. Allowing the cases $\alpha, \beta = 0$, all metrics for the screw space can be given as

$$\hat{G} = \begin{bmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \beta\mathbf{I} & \mathbf{0} \end{bmatrix} \tag{2.64}$$

It is straightforward to show that the characteristic polynomial of $\hat{G}$ is

$$\left(\lambda^2 - \alpha\lambda - \beta^2\right)^3 \tag{2.65}$$

whose roots are

$$\mathrm{eig}(\hat{G}) = \frac{\alpha \pm \sqrt{\alpha^2 + 4\beta^2}}{2} \tag{2.66}$$

each repeated three times.

The case $\alpha = \beta = 0$ corresponds to the trivial metric with six repeated zero eigenvalues. For $\alpha = 0, \beta \neq 0$ one gets the metric $\hat{\Delta}$ after a scaling. The eigenvalues of $\hat{\Delta}$ are ± 1 repeated three times. Therefore, $\hat{\Delta}$ is an indefinite non-singular metric. For $\alpha \neq 0, \beta = 0$ one gets the metric

$\hat{\Gamma}$, again after a scaling. The eigenvalues of $\hat{\Gamma}$ are 0 and 1, each repeated three times. Therefore, $\hat{\Gamma}$ is a degenerate (semi-Riemannian) metric. Finally, if $\alpha, \beta \neq 0$ then, since $\sqrt{\alpha^2 + 4\beta^2} > \|\alpha\|$, one always have three repeated non-zero eigenvalues and three repeated non-zero eigenvalues with opposite signs, which corresponds to indefinite, non-singular metrics.

■

The above theorem have the following consequence.

Corollary 14 *There exists no positive definite (Riemannian) metric for the screw space.*

It is well known that any metric tensor can be reduced, by a suitable change of basis, to a diagonal form such that the diagonal elements are either one of $+1, 0, -1$, [45]. These correspond to the signs of the eigenvalues. The diagonal form is called the **canonical form** of the metric tensor. The sum of the diagonal elements is called the **signature** of the metric tensor. For example, any Euclidean metric for an n-dimensional vector space has a signature n, the Minkowski metric has a signature $3 - 1 = 2$, the signature of $\hat{\Delta}$ is 0 and that of $\hat{\Gamma}$ is 3. Any two metric tensors that can be reduced to the same diagonal form are considered equivalent. Therefore, there are only two non-trivial metric classes for the screw space: 1) that represented by $\hat{\Gamma}$, which is already in diagonal form and gives all possible semi-definite metrics, and, 2) that represented by $\hat{\Delta}$, whose diagonal form contains three $+1$'s and three -1's, and is equivalent to all possible indefinite metrics, since Theorem 13 showed that all indefinite metrics have three positive and three negative eigenvalues.

The scalar product defined by $\hat{\Delta}$ has a special importance since it also defines a transformation from ray- to axis-coordinates, and vice versa. For any two screws, the scalar product $\hat{S}_1^T \hat{\Delta} \hat{S}_2 = \hat{S}_{1r}^T \hat{S}_{2a}$, where the subscripts denote the ray- and axis-coordinates. So, for example, if $\hat{W}$ and $\hat{T}$ are a wrench in ray-coordinates and a twist in axis-coordinates, then their scalar product under the metric $\hat{\Delta}$ is simply $\hat{W}^T \hat{T}$ which is related to the work done. This shows why using ray- and axis-coordinates simultaneously leads to simplified and familiar expressions.

Reciprocal Systems The notion of scalar product for a vector space has a very well known geometric interpretation when the metric is Euclidean. Two elements of such a space whose scalar product vanishes are perpendicular or orthogonal to each other in the sense of Euclidean plane geometry. In such a case, the scalar product is related to the notion of angle between the lines coinciding with the two elements. However, in a general vector space with non-Euclidean metric tensor these notions are blurred. The concept of orthogonality in an arbitrary vector space is generalized to the concept of *reciprocity*. Two vectors, whose scalar product with respect to a given metric tensor vanishes, are said to be **reciprocal** to each other. In screw space, two screws are said to be reciprocal to each other if

$$\hat{S}_1^T \hat{\Delta} \hat{S}_2 = 0 \tag{2.67}$$

It is not difficult to show that there exist screws which are reciprocal to themselves, or self-reciprocal. For example, all zero pitch and infinite pitch screws are self-reciprocal. This is why the concept of orthogonality cannot be applied to the screw space as is.

Given an n-system of screws defined by the matrix of basis $\hat{S} = \left[\hat{S}_1, \hat{S}_2, ..., \hat{S}_n\right]$, consider a screw $\hat{V}$ that is reciprocal to all $\hat{S}_i$. Then, by linearity, $\hat{V}$ is reciprocal to every screw in the n-system. The set of all screws that are reciprocal to a given n-system is called the **reciprocal system** of the n-system. That the set is a screw system is evident from the linearity of the scalar product, since the linear combination of any number of reciprocal screws is still a reciprocal screw, etc. From a linear algebra point of view, the condition for a screw, $\hat{V}$, to be in the reciprocal system can be given explicitly as

$$\hat{S}^T \hat{\Delta} \hat{V} = \hat{0} \tag{2.68}$$

But, this means that the column matrix $\hat{\Delta}\hat{V}$ must be in the null space of $\hat{S}^T$, or left-null space of $\hat{S}$. Since $\hat{S}$ is a $6 \times n$ matrix with linearly independent columns, its rank is exactly n. From linear algebra, it is well known that the rank of the left-null space of $\hat{S}$ is given by $6 - n$. Therefore, the

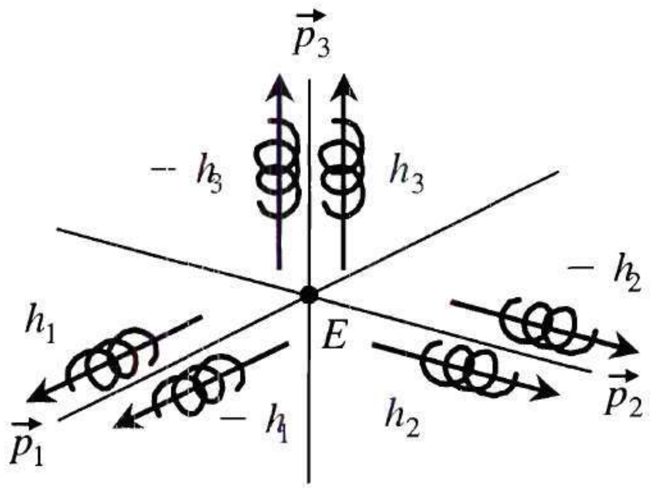


Figure 2.7: The principal screws of a 3-system and its reciprocal 3-system are coincident with pitches of equal magnitude and opposite signs. Both systems have a common center.

null space of $\hat{S}^T$ is $6-n$ dimensional. If the null space basis matrix is designated by $N(\hat{S}^T)$, then the basis matrix for the reciprocal system is given by $\hat{\Delta}^{-1}N(\hat{S}^T) = \hat{\Delta}N(\hat{S}^T)$, which is a $(6-n)$-system of screws. For example, the reciprocal system of a 1-system is a 5-system, that of a 2-system is a 4-system, etc. It is clear that the reciprocity relation is symmetric. That is, if a 1-system is reciprocal to a 5-system, then that 5-system is reciprocal to the 1-system. As mentioned before, since there may exist self-reciprocal screws in a screw system, the intersection of a screw system with its reciprocal system may contain non-zero elements. Therefore, combination of an n-system with its reciprocal system may not span the whole screw space, although their dimensions may falsely seem to suggest so.

The important screw systems in this study are 3-systems whose reciprocal systems are also 3-systems. Here, they are shown to have important roles in the analysis of stiffness-like quantities. It has been shown that the center of an 3-system coincides with the center of its reciprocal 3-system [1], [28], [30]. Moreover, the reciprocal 3-system's principal screws are parallel to those of the given

3-system's, with principal pitches of equal magnitude and opposite signs. Figure 2.7 summarizes these facts.

Free-Vectors and Line-Vectors The action of a pure force on a rigid body is completely determined by giving its magnitude and line of action. If the line of action is changed the resulting force has different effects, even if the direction is kept constant. This is in contrast to what happens in pure couple case. A couple is determined by its magnitude and direction. Yet, there is no preferred point of application. That is, one can shift the line of couple, without changing the direction, and still get the same effect. These can be deduced from the screw representations of these quantities.

Consider two points on the body, say A and B. Let two parallel lines l_A and l_B pass through A and B, respectively. Two pure forces along l_A and l_B have representations, say with respect to a point O, given by

$$\hat{F}_O(A) = \begin{bmatrix} \vec{\mathbf{f}} \\ \overrightarrow{OA} \times \vec{\mathbf{f}} \end{bmatrix} \qquad \hat{F}_O(B) = \begin{bmatrix} \vec{\mathbf{f}} \\ \overrightarrow{OB} \times \vec{\mathbf{f}} \end{bmatrix} \tag{2.69}$$

which are not equal unless $\left(\overrightarrow{OA} - \overrightarrow{OB}\right) \times \vec{\mathbf{f}} = \vec{0}$, or $\overrightarrow{AB} = \alpha\vec{\mathbf{f}}$, meaning that both points must be on the same line. So, unless $l_A \equiv l_B$, the forces, which differs from each other only by a parallel shift, are not equivalent.

Now, consider two pure couples instead of the two forces. The representations are

$$\hat{C}_O(A) = \begin{bmatrix} \vec{0} \\ \vec{\mathbf{m}} \end{bmatrix} \qquad \hat{C}_O(B) = \begin{bmatrix} \vec{0} \\ \vec{\mathbf{m}} \end{bmatrix} \tag{2.70}$$

which can be obtained by applying screw transformations. Hence, the couples are equivalent regardless of the point of application. Briefly stated, a pure force depends on the line that describes its screw axis, whereas a pure couple does not.

Similar situation is observed in the case of displacements (velocities). If a body is in pure translation then it does not matter which point is said to have it, as all points of the body have the exact same translation. On the contrary, a rotation through a point A along l_A is completely different from that through B along l_B, unless $l_A \equiv l_B$. One can visualize this by considering a rotating rectangle; first fixed at one corner, then fixed at another. The two describe completely different velocities. Therefore, a pure rotation depends on the line that describes its screw axis, whereas a pure translation does not.

The above observations suggest two essentially distinct classes of screw objects: 1) those that are *line independent*, and 2) those that are *line dependent*. One should also immediately notice that the line independent quantities are infinite pitch screws and the line dependent quantities are zero pitch screws. The following definition naturally follows.

Definition 15

1. *An infinite pitch screw is called a* ***free-vector****. Examples are translations and couples.*

2. *A zero pitch screw is called a* ***line-vector****. Examples are rotations and forces.*

Theorem 16 *A self-reciprocal screw is either a free-vector or a line-vector.*

Proof. Consider a screw $\hat{S} = \begin{bmatrix} \vec{\mathbf{a}}^T & \vec{\mathbf{b}}^T \end{bmatrix}^T$. From (2.67), $\hat{S}$ is self-reciprocal if

$$\hat{S}^T \hat{\Delta} \hat{S} = 2\vec{\mathbf{a}}^T \vec{\mathbf{b}} = 0 \tag{2.71}$$

which can hold only if either $\vec{\mathbf{a}} = \vec{\mathbf{0}}$, $\vec{\mathbf{b}} = \vec{\mathbf{0}}$ or $\vec{\mathbf{a}} \perp \vec{\mathbf{b}}$. In case $\vec{\mathbf{a}} = \vec{\mathbf{0}}$ the screw is a free-vector. In cases $\vec{\mathbf{b}} = \vec{\mathbf{0}}$ or $\vec{\mathbf{a}} \perp \vec{\mathbf{b}}$ it is line vector. These follow from the definition of pitch.

■

Later chapters demonstrate the importance of free- and line-vectors more explicitly. Since a twist is a combination of a translation and a rotation, and, a wrench is that of a force and couple, a natural thing to consider is the decomposition of an arbitrary screw into such constituents. An immediate way of doing this comes from equation (2.17) which is repeated here as follows.

$$\hat{S}_O = \begin{bmatrix} \vec{\mathbf{a}} \\ \vec{\mathbf{r}}_\perp \times \vec{\mathbf{a}} \end{bmatrix} + \begin{bmatrix} \vec{0} \\ h\vec{\mathbf{a}} \end{bmatrix} \tag{2.72}$$

where h is the pitch and $\vec{\mathbf{r}}_\perp$ is the perpendicular vector from O to the screw axis. Clearly, the first term on the right of (2.72) is a line-vector, whereas the second is a free-vector. However, this decomposition is by no means unique. For example, the following, though somewhat trivial, is also a legitimate decomposition

$$\hat{S}_O = \begin{bmatrix} \vec{\mathbf{a}} \\ \vec{0} \end{bmatrix} + \begin{bmatrix} \vec{0} \\ \vec{\mathbf{b}} \end{bmatrix} \tag{2.73}$$

The difference between the two can be traced to the line-vector part. In (2.72), the line-vector is coincident with the screw axis, whereas in (2.73) it passes through O, which is not necessarily the screw axis although it is parallel to it. The most general expression for the decomposition into free- and line-vector components can be given as

$$\hat{S}_O = \begin{bmatrix} \vec{\mathbf{a}} \\ \overrightarrow{OA} \times \vec{\mathbf{a}} \end{bmatrix} + \begin{bmatrix} \vec{0} \\ \vec{\mathbf{b}} - \overrightarrow{OA} \times \vec{\mathbf{a}} \end{bmatrix} \tag{2.74}$$

where A is an arbitrary point. The first term on the right of (2.74) is a line-vector parallel to $\vec{\mathbf{a}}$ and passes through the point A. The second term is a free-vector. The decomposition (2.72) is special in that the components become parallel. Since all possible decompositions involve line-vectors which are parallel to the screw axis, but pass through distinct points, there exist as many decompositions as there are distinct lines parallel to the screw axis.

The decomposition of screws into line- and free-vector components is shown to be important in later chapters. Also, the concept will be explained in the light of new definitions such as line- and free-vector subspaces of the general screw space.

Free-Vector and Line-Vector Subspaces A single non-zero free-vector defines a 1-dimensional screw subspace, namely all scalar multiples of it. A unit 3-vector is sufficient to describe such a subspace. So, there exists as many distinct 1-dimensional free-vector subspaces as there are distinct unit 3-vectors, namely ∞^2. Equivalently, all 1-dimensional free-vector subspaces have a 1-1 association with all lines through a point.

The sum of two free-vectors is a free-vector itself. If $\begin{bmatrix} \vec{0}^T & \vec{a}_1^T \end{bmatrix}^T$ and $\begin{bmatrix} \vec{0}^T & \vec{a}_2^T \end{bmatrix}^T$ are two linearly independent free-vectors, then all linear combinations of them give a set of free-vectors with all the directions in the plane spanned by $\vec{a}_1$ and $\vec{a}_2$. This is clearly a 2-dimensional screw subspace. To every distinct plane through a point, there corresponds a distinct 2-dimensional free-vector subspace. Since all planes through a point have a 1-1 association with all lines through the point (normals to the planes), there exist as many 2-dimensional free-vector subspaces as there are distinct lines through a point, again ∞^2.

In general, the linear combinations of n free-vectors is a free-vector subspace. However, since all are in the form $\begin{bmatrix} \vec{0}^T & \vec{a}^T \end{bmatrix}^T$, the maximum dimension of such spaces is three. Any three linearly independent free-vectors is a basis for this 3-dimensional free-vector subspace. Since any four 3-vectors are linearly dependent this subspace is the largest and unique free-vector subspace. These are summarized in the following theorem.

Theorem 17 *There exist only 1, 2 and 3-dimensional free-vector subspaces of the general screw space. The 3-dimensional free-vector subspace is unique.*

For line-vectors, a single line vector trivially forms a 1-dimensional line-vector subspace. To every distinct line in space there corresponds a distinct 1-dimensional line-vector subspace. Since a general line is described by four parameters, the multitude of distinct 1-dimensional line-vector subspaces is ∞^4.

Higher dimensions are not as straightforward. A linear combination of two line-vectors is not necessarily a line-vector. An example of this is two forces with equal magnitudes and in opposite directions, acting on two distinct parallel lines. The result is well known to be a pure couple, a free-vector. The following lemma shows this for $n > 3$ dimensions.

Lemma 18 *Every n-system of screws such that $n > 3$ contains a free-vector subspace.*

Proof. Let $\hat{S} = [\hat{S}_1, \hat{S}_2, ..., \hat{S}_n]$ be a set of basis screws for the given system. Any screw $\hat{Y}$ in the system is given by $\hat{Y} = \hat{S}\bar{\alpha}$, where $\bar{\alpha}$ is an $n \times 1$ matrix of real numbers. The condition for the existence of a free-vector can be written as $\hat{Y}^T\hat{\Gamma}\hat{Y} = \bar{\alpha}^T \left[\hat{S}^T\hat{\Gamma}\hat{S}\right] \bar{\alpha} = 0$. The rank of the matrix $\hat{\Gamma}$ is three. It is well known in linear algebra that the rank of a product of matrices is less than or equal to rank of the factors. So, $\text{rank}(\hat{S}^T\hat{\Gamma}\hat{S}) \leq 3$, where $\hat{S}^T\hat{\Gamma}\hat{S}$ is an $n \times n$ matrix. Therefore, if $n > 3$ there always exists an $\bar{\alpha}$ such that $\bar{\alpha}^T \left[\hat{S}^T\hat{\Gamma}\hat{S}\right] \bar{\alpha} = 0$ indicating the existence of a free-vector.

■

Lemma 18 shows that only $n \leq 3$-dimensional line-vector subspaces can exist. Consider a number of line-vectors through a common point O. All such line-vectors have a representation at O similar to $\begin{bmatrix} \vec{\mathbf{a}}^T & \vec{0}^T \end{bmatrix}^T$. So, as in the case of free-vectors, their linear combinations lead to a line-vector subspace. Together with Lemma 18, this gives the following.

Corollary 19 *There exist only one, two and three dimensional line-vector subspaces.*

Proof. Non-existence of line-vector subspaces with dimensions higher than three is shown by Lemma 18. The existence of lower dimensional line-vector subspaces comes by example. A single line-vector with all its multiples make a 1-system of line-vectors. Linear combinations of two orthogonally intersecting line-vectors make a 2-system of line-vectors. Finally, linear combinations of three orthogonally intersecting line-vectors make a 3-system of line-vectors.

■

The special line-vector subspaces given as examples in the proof of Corollary 19 are well known

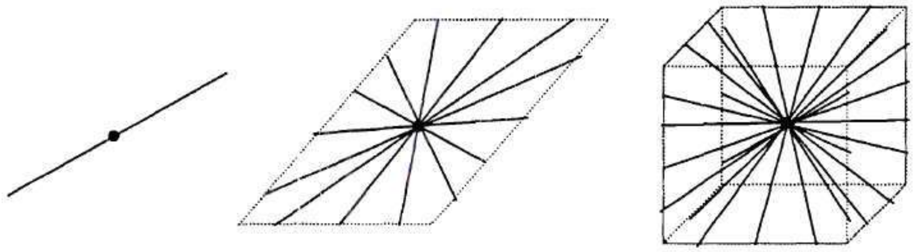


Figure 2.8: 1-, 2- and 3-dimensional line-vector spaces.

and given special names when only the lines through the common point are considered. Figure 2.8 illustrates these special subspaces.

The set of all lines intersecting at a common point and forming a plane is called a **pencil** of lines. The set of all lines through a point is called a **bundle** of lines. If the scalar multiples of lines are also included in the set then a pencil becomes a 2-system of line-vectors and a bundle becomes a 3-system of line-vectors. One dimensional line-vector subspaces are considered as degenerate cases, in which the screws intersect each other at all points of the single axis of the 1-system. Then, a common property of these examples is that all line-vectors in such systems intersect at a common point. It is interesting to inquire if there can be line-vector subspaces without such a property. The following theorem proves the contrary.

Theorem 20 *All screws of a line-vector subspace pass through a common point A.*

Proof. The proof is for 3-systems. The 2-system case is similar and the 1-system case is trivial. Let $\hat{S} = [\hat{S}_1, \hat{S}_2, \hat{S}_3]$ be a set of basis screws with perpendicular directions. Then, as in Lemma 18, any screw in the system is given as $\hat{Y} = \hat{S}\bar{\alpha}$, where $\bar{\alpha}$ is an 3×1 matrix of real numbers. By the zero pitch constraint, one writes

$$\hat{Y}^T \hat{\Delta} \hat{Y} = \bar{\alpha}^T \left[\hat{S}^T \hat{\Delta} \hat{S}\right] \bar{\alpha} = 0 \qquad \text{for all } \bar{\alpha} \tag{2.75}$$

This means $\hat{S}^T \hat{\Delta} \hat{S} = \mathbf{0}$ which can be written explicitly as

$$\begin{bmatrix} \hat{S}_1^T \\ \hat{S}_2 \\ \hat{S}_3^T \end{bmatrix} \hat{\Delta} \left[\begin{array}{ccc} \hat{S}_1 & \hat{S}_2 & \hat{S}_3 \end{array} \right] = \left[\hat{S}_i^T \hat{\Delta} \hat{S}_j \right] = \mathbf{0} \tag{2.76}$$

But, by Theorem 9, if two perpendicular screws $\hat{S}_i$ and $\hat{S}_j$ satisfy $\hat{S}_i^T \hat{\Delta} \hat{S}_j = 0$ then they intersect. Therefore, any two basis screws intersect each other. Then, all basis screws intersect at a single point because they are perpendicular to each other. The linear combinations of such screws yield screws which also pass through the same point. Hence, all screws in such systems intersect at a common point.

■

Theorem 20 shows that distinct 2-dimensional line-vector subspaces are described by distinct planes in space. In 3-dimensional space, planes are described by three parameters. Therefore, the multitude of distinct 2-dimensional line-vector subspaces is ∞^3.

The 3-dimensional line-vector subspaces have a 1-1 association with each and every point of the 3-dimensional space, Theorem 20. Then, the multitude of 3-dimensional line-vector subspace is ∞^3. Specifying a point is sufficient to identify the associated 3-dimensional line-vector subspace, the bundle of lines. Therefore, the following definition is proposed.

Definition 21 *Given a 3-dimensional line-vector subspace, the common intersection point is called the **generator** G.*

The three dimensional free-vector line-vector subspaces are of special importance in this study. So, from here on, the names *free-vector subspace* and *line-vector subspace* are assumed to mean the 3-dimensional cases unless otherwise is specified. The intersection property of line-vectors can be extended to free-vectors by using the concepts of projective geometry. Briefly stated, the intersection point of a free-vector subspace can taken as a point at infinity. See Chapter 5 for details on projective spaces.

Definition 22

1. *The unique free-vector subspace of twists is called the* ***translation bundle*** *and denoted by* V_f.

2. *The unique free-vector subspace of couples is called the* ***couple bundle*** *and denoted by* V_f^*.

3. *A line-vector subspace of twists generated by* G *is called a* ***rotation bundle*** *at* G *and denoted by* $V_{l/G}$.

4. *A line-vector subspace of wrenches generated by* G *is called a* ***force bundle*** *at* G *and denoted by* $V_{l/G}^*$.

The subscripts $_f$ and $_l$ dencte the free-vector and line-vector properties, respectively.

2.1.3 Spatial Vectors

Spatial vectors are essentially screws. However, they arise due to a distinction between the frames (bodies) of representations used in describing screws. In previous section, a screw is defined as a quantity associated with a given rigid body, that obeys a certain rule of transformation from a point of the body to another. The key point is that the screw is represented with respect to a coordinate system that is attached to a point of the body. It is also demonstrated in previous section that this results in an ambiguity if the sum of two twists belonging to distinct bodies are to be defined. The resolution of this ambiguity requires the definition of spatial vectors.

Material Representation Versus Spatial Representation Consider two bodies $\mathfrak{F}$ and $\mathfrak{B}$. The body $\mathfrak{F}$ is referred to as the **fixed body**, although it doesn't really have to be fixed in any sense. The body $\mathfrak{B}$ is the one for which the descriptions of displacement and load are sought. Any two instantaneously

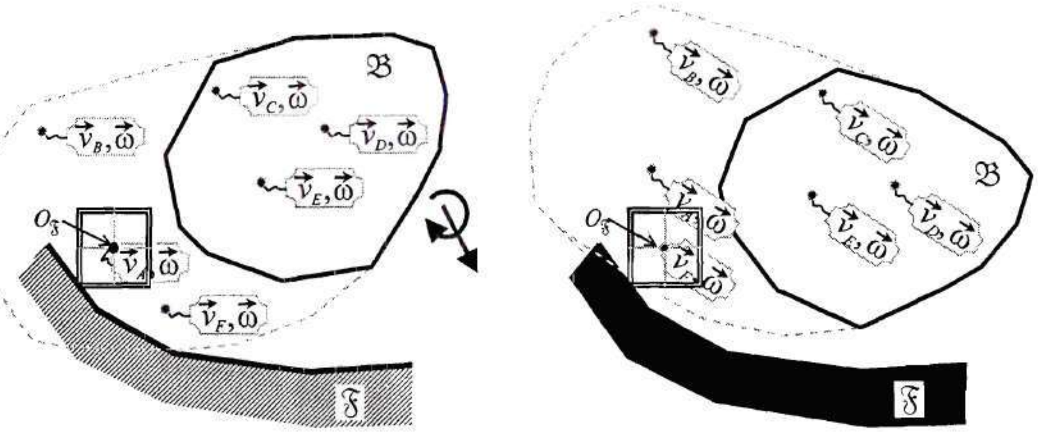


Figure 2.9: A visual interpretation of the spatial representation of a screw field on a body $\mathfrak{B}$ in motion with respect to $\mathfrak{F}$.

coincident points are marked by the same capital letter, but with different subscripts. Thus, if $O_{\mathfrak{B}}$ is a point of $\mathfrak{B}$, then $O_{\mathfrak{F}}$ is a point of $\mathfrak{F}$ coincident with the former.

If $\hat{S}_{O_{\mathfrak{B}}}$ is a screw defined on $\mathfrak{B}$, then one can consider a screw defined on $\mathfrak{F}$ given by $\hat{S}_{O_{\mathfrak{F}}} = \hat{S}_{O_{\mathfrak{B}}}$, which is with respect a coordinate frame attached to $\mathfrak{F}$ and parallel to that on $\mathfrak{B}$. The representation $\hat{S}_{O_{\mathfrak{B}}}$ is called the **material representation** and the representation $\hat{S}_{O_{\mathfrak{F}}}$ is called the **spatial representation**. If the body $\mathfrak{B}$ is at rest with respect to $\mathfrak{F}$ then both representations are equivalent at all times. However, if $\mathfrak{B}$ is in motion with respect to $\mathfrak{F}$, then the representations are equivalent only at the instant $O_{\mathfrak{B}} = O_{\mathfrak{F}}$. Since for any given motion of $\mathfrak{B}$ and a point $O_{\mathfrak{B}}$ one can always find a point $O_{\mathfrak{F}}$ that is instantaneously coincident with $O_{\mathfrak{B}}$, any screw field on $\mathfrak{B}$ have a spatial representation that is instantaneously equivalent. However, as $\mathfrak{B}$ moves the two representations become different. Nevertheless, if the motion of $\mathfrak{B}$ is known, an observer on $\mathfrak{F}$ can always deduce the material representation by only observing the spatial one. For this, the observer records $\hat{S}_{O_{\mathfrak{F}}}$ at a given time t. Then, since the motion is known, he finds $O_{\mathfrak{B}}(t)$ that is coincident with $O_{\mathfrak{F}}$ at time t, so that $\hat{S}_{O_{\mathfrak{B}}}(t) = \hat{S}_{O_{\mathfrak{F}}}$. This sufficient to know the screw field on $\mathfrak{B}$.

In Figure 2.9, the screw field on $\mathfrak{B}$ is pictured as markers attached to every point of $\mathfrak{B}$. An observer on $\mathfrak{F}$ has a coordinate frame attached at $O_{\mathfrak{F}}$ and is interested in the spatial representation. He sets up a measurement window at the origin and records the screw marker that is seen through the window. Even if the screw field is constant on $\mathfrak{B}$, the observer on $\mathfrak{F}$ records a change due to the motion of $\mathfrak{B}$. This is a well known phenomenon in continuum mechanics. In accordance with the terminology of fluid mechanics, for example, such changes are considered to be due to *convection.* The window is actually similar to the control volume concept of fluid mechanics.

The apparent change in the spatial representation is related to the motion of $\mathfrak{B}$ with respect to $\mathfrak{F}$. The connection is given by the relations between spatial differentiations with respect to different bodies, which is explored in more detail in Chapter 7.

Spatial representation resolves the ambiguity in the addition of twists of distinct bodies. Now, one body, say $\mathfrak{F}$ can be selected with respect to which all twists are represented. Then, (2.29) can be written as

$$\hat{T}_{O_{\mathfrak{F}}(i/0)} = \sum_{j=1}^{i} \hat{T}_{O_{\mathfrak{F}}(j/j-1)} \tag{2.77}$$

Similar conclusions apply to the wrenches case.

Spatial Vectors and Transformations The coordinate frame on $\mathfrak{F}$ does not have to be attached to the point $O_{\mathfrak{F}}$. One can choose any other point on $\mathfrak{F}$. Using the screw transformation rules on $\mathfrak{B}$, as given in (2.7) and (2.8), and the fact that $\hat{S}_{O_{\mathfrak{B}}} = \hat{S}_{O_{\mathfrak{F}}}$ one gets

$$\hat{S}_{P_{\mathfrak{F}}} = \begin{bmatrix} \mathbf{I} & 0 \\ -\overrightarrow{Q_{\mathfrak{F}}P_{\mathfrak{F}}}\times & \mathbf{I} \end{bmatrix} \hat{S}_{Q_{\mathfrak{F}}} \tag{2.78}$$

in ray-coordinates, for any two points Q and P on $\mathfrak{F}$. This shows that the spatial representation on $\mathfrak{F}$ is essentially a screw field. Therefore, the following definition is sensible.

Definition 23 *The 6-dimensional vector space of screws is also called the* ***spatial vector space****.*

The transformation (2.78) accounts for the shifting the origin of coordinates while maintaining the orientation. The most general screw transformation should also include a rotation of the coordinate axes. Let $\hat{S}_O$ and $\hat{S}'_O$ be the representations of the same screw in two coordinates frames at O, differing from each other by a rotation only. If the rotation is given by a 3×3 rotation matrix $\mathbf{R}$, such that $\mathbf{R}^T\mathbf{R} = \mathbf{I}$, then the two representation are related as

$$\hat{S}'_O = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \hat{S}_O \tag{2.79}$$

Each $\mathbf{R}$ on the diagonal acts on the corresponding 3-vector part to perform the rotation.

One can combine the origin and rotational transformations. First, let $\vec{\mathbf{r}} = \overrightarrow{Q_{\mathfrak{F}}P_{\mathfrak{F}}}$ the position vector from Q to P on $\mathfrak{F}$. Again, let $\mathbf{R}$ be the rotation matrix. Consider a change of origin and a change of orientation performed consecutively. Then,

$$\hat{S}'_P = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\vec{\mathbf{r}}\times & \mathbf{I} \end{bmatrix} \hat{S}_Q = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ -\mathbf{R}\vec{\mathbf{r}}\times & \mathbf{R} \end{bmatrix} \hat{S}_Q \tag{2.80}$$

If the rotation is applied before the origin change, the let $\vec{\mathbf{r}}'$ be the representation of $\vec{\mathbf{r}}$ in the new coordinate system. That is, $\vec{\mathbf{r}}' = \mathbf{R}\vec{\mathbf{r}}$. Then,

$$\hat{S}'_P = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\vec{\mathbf{r}}'\times & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \hat{S}_Q = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ -\vec{\mathbf{r}}' \times \mathbf{R} & \mathbf{R} \end{bmatrix} \hat{S}_Q \tag{2.81}$$

Both (2.80) and (2.81) describe the same transformation. They only differ in the order of origin and rotational transformations. In this study, usually the form (2.81) is used. The primes are dropped since one can immediately see the order from the matrix itself. The same transformations in axis-coordinates are

$$\hat{S}'_P = \begin{bmatrix} \mathbf{R} & -\mathbf{R}\vec{\mathbf{r}}\times \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \hat{S}_Q = \begin{bmatrix} \mathbf{R} & -\vec{\mathbf{r}}' \times \mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \hat{S}_Q \tag{2.82}$$

These result lead to the following definition.

Definition 24 *The most general form of the screw transformations is given by*

$$\hat{X}_r = \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ -\vec{\mathbf{r}} \times \mathbf{R} & \mathbf{R} \end{bmatrix} \qquad \hat{X}_a = \begin{bmatrix} \mathbf{R} & -\vec{\mathbf{r}} \times \mathbf{R} \\ \mathbf{0} & \mathbf{R} \end{bmatrix} \qquad \textit{such that } \mathbf{R}^T\mathbf{R} = \mathbf{I} \tag{2.83}$$

in ray- and axis-coordinates, respectively. The matrix $\hat{X}$ is also called the ***spatial transformation matrix****.*

■

The material and the spatial representations can be transformed to each other in a similar way. If $\hat{S}$ is a screw in ray-coordinates, then the relation between its representations at $P_{\mathfrak{F}}$ and $Q_{\mathfrak{B}}$ is

$$\hat{S}_{Q_{\mathfrak{F}}} = \begin{bmatrix} \mathbf{R}_{\mathfrak{B}/\mathfrak{F}} & \mathbf{0} \\ -\overrightarrow{Q_{\mathfrak{F}}P_{\mathfrak{B}}} \times \mathbf{R}_{\mathfrak{B}/\mathfrak{F}} & \mathbf{R}_{\mathfrak{B}/\mathfrak{F}} \end{bmatrix} \hat{S}_{P_{\mathfrak{B}}} \tag{2.84}$$

which is the same as a screw transformation on a body, except that this involves coordinate frames attached to two distinct bodies. However, the subscripts identify an important difference. The term $\mathbf{R}_{\mathfrak{B}/\mathfrak{F}}$ gives the rotation of the body $\mathfrak{B}$ with respect to $\mathfrak{F}$, and, the term $\overrightarrow{Q_{\mathfrak{F}}P_{\mathfrak{B}}}$ gives the displacement of the body $\mathfrak{B}$ with respect to $\mathfrak{F}$. So, the transformation matrix in (2.84) completely describes the configuration of $\mathfrak{B}$ with respect to $\mathfrak{F}$. For this reason, it is usually called the **rigid body transformation** matrix. The difference is important when a differentiation is involved. The differentials of $\mathbf{R}_{\mathfrak{B}/\mathfrak{F}}$ and $\overrightarrow{Q_{\mathfrak{F}}P_{\mathfrak{B}}}$ are related to the angular and translational velocities of $\mathfrak{B}$ with respect to $\mathfrak{F}$. See Chapter 7 for details.

For simplicity of notation, the body subscripts are seldom used in this study. They are beneficial when a precise geometric and physical description is needed, mostly in Chapter 7. Other than that, the plain capital letters are used to denoted the points on any given body. Sometimes, primed capital letters are preferred for points on moving bodies. In any case, the declaration of the points sufficiently clarifies which body is meant.

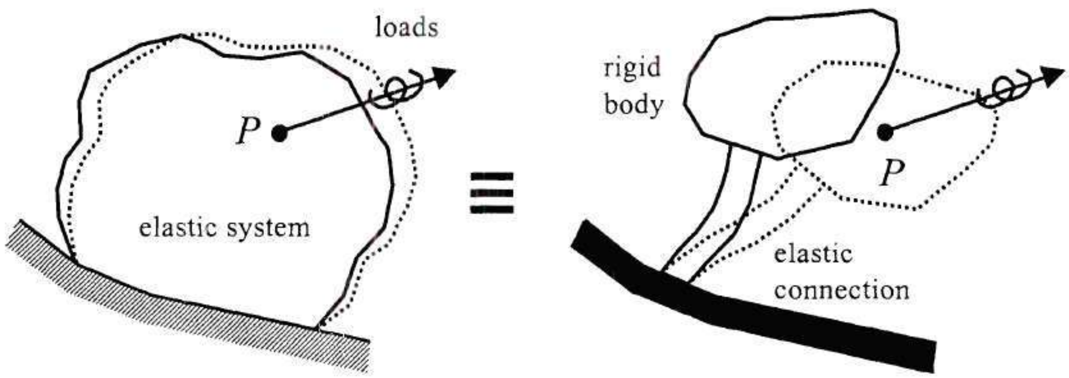


Figure 2.10: The displacement response of a general elastic system to loads applied at a point can be modelled by the elastically suspended rigid body model.

2.1.4 Elastic System Model

Figure 2.10 shows a general elastic system under the action of a load acting at a point P' of the system. A rigid body attached to an infinitesimally small region including the point P' can be used to measure the displacement caused by the load. Also, the load is considered as applied to the rigid body. This effectively provides a model for the analysis of general elastic systems as far as their reaction to loads at a single point is considered. This model, frequently called the **elastically suspended rigid body** system, is used throughout this study.

All inertial, damping, energy dissipation, plasticity, etc. effects are neglected. Therefore, the system is considered to be only and fully elastic. There are countless examples of elastic systems that can be modelled by an elastically suspended rigid body. In all such systems there exists a point where the displacements and loads are important. Some practical examples are shown in Figure 2.11.

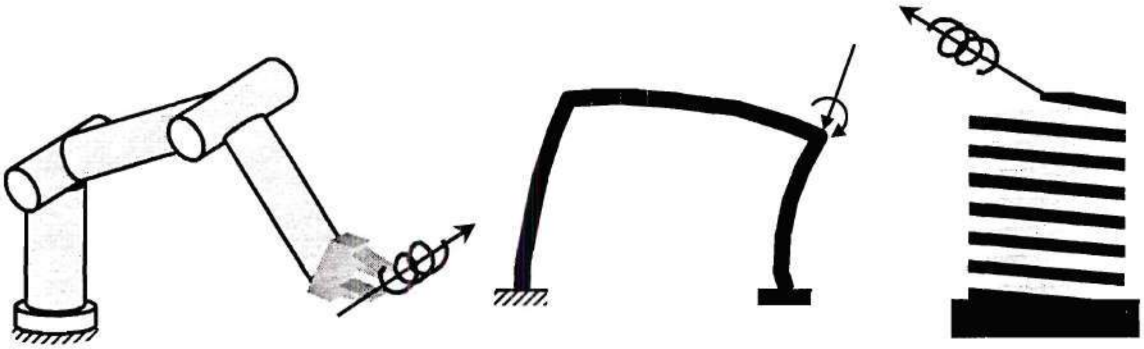


Figure 2.11: Generic examples of practical systems that can be modelled by elastically suspended rigid body model. (a) robotic systems, (b) spatial structures, (c) springs.

2.1.5 Configuration and Infinitesimal Spatial Displacement

Let an elastic system be modelled by an elastically suspended rigid body as in Figure 2.10. A **configuration** of the rigid body roughly means the location of its material points with respect to a coordinate system on the fixed body. Therefore, by rigid body constraints, if a coordinate frame $\{x_i\}_O$ on the fixed body and another coordinate frame $\{x'_i\}_{P'}$ on the rigid body are chosen, then all configurations of the rigid body with respect to the fixed body are fully described by the translation OP' and the rotation $\mathbf{R} : x_i \to x'_i$. Unlike translations, rotations cannot be represented as vectors since finite rotations do not commute in general. Thus, there exists no vector description for the configurations of a rigid body in general.

Given a configuration, an *infinitesimal change in the configuration* is described by an infinitesimal translation and an infinitesimal rotation of the rigid body. Contrary to the finite case, both infinitesimal translations and rotations can be represented by 3-vectors. So, let $d\overrightarrow{OP'}$ be the infinitesimal translation of the point P' and $\delta\vec{\theta}_{\mathfrak{B}/\mathfrak{F}}$ be the infinitesimal rotation in 3-vector form of the rigid body with respect to the fixed body. Components of $\delta\vec{\boldsymbol{\theta}}_{\mathfrak{B}/\mathfrak{F}}$ are the infinitesimal rotation amounts about the coordinate axes. The notation $\delta(\cdot)$ is the acknowledgment of the fact that this infinitesimal vector is not a differential of any finite rotation 3-vector. If the change in configuration is parametrized by time t, then $\frac{d\overrightarrow{OP'}}{dt}$ is the linear velocity of the point P' and $\frac{\delta\vec{\boldsymbol{\theta}}_{\mathfrak{B}/\mathfrak{F}}}{dt}$ is the angular velocity of the rigid body with respect to the fixed body. But, the linear and angular velocities of a rigid body are described by screws or spatial vectors. So,

$$\hat{V}_P = \begin{bmatrix} \frac{d\overrightarrow{OP'}}{dt} \\ \\ \frac{\delta\vec{\boldsymbol{\theta}}_{\mathfrak{B}/\mathfrak{F}}}{dt} \end{bmatrix} \tag{2.85}$$

is the spatial velocity with respect to the point P of the fixed body. By letting $P' \to O'$, one gets

$$\hat{V}_O = \begin{bmatrix} \frac{d\vec{\mathbf{r}}_O}{dt} \\ \\ \frac{\delta\vec{\boldsymbol{\theta}}}{dt} \end{bmatrix} \tag{2.86}$$

as the spatial velocity with respect to the coordinate frame at O, where $d\vec{\mathbf{r}}_O$ is the infinitesimal translation of a point O', instantaneously coincident with O, and the subscript on $\delta\vec{\boldsymbol{\theta}}$ is dropped for simplicity. Since dt does nothing more than a scaling, the vector defined by

$$\delta\hat{q}_O = \begin{bmatrix} d\vec{\mathbf{r}}_O \\ \\ \delta\vec{\boldsymbol{\theta}} \end{bmatrix} \tag{2.87}$$

is a spatial vector. This leads to the following definition.

Definition 25 *The spatial vector $\delta\hat{q}_O$, such that $\hat{V}_O = \frac{\delta\hat{q}_O}{dt}$, is called the* ***infinitesimal spatial displacement****.*

2.1.6 Equilibrium and Stability

It is assumed in this study that to every configuration of the rigid body there corresponds a unique wrench such that the rigid body is in *static equilibrium*, considering both the external wrench and elastic reactions. This is not the case, for example, for a body with a kinematic constraint. A body constrained to surface is in static equilibrium no matter what the magnitude of the normal force is. Such cases are excluded in this study.

It is also assumed that the rigid body is in static equilibrium at all times. This is sensible since there is no notion of inertial forces at this point. Even if the body is in motion, at any given time it is statically balanced. So, as the configuration changes gradually the applied wrench also changes as to keep the static equilibrium. Sometimes, these equilibria are called as the *quasi-static equilibria.*

These assumptions lead to the conclusion that the applied wrench can be given as a function of the configuration. The converse is not true. For any given wrench, there may be finitely or infinitely many configurations for which the body is in static equilibrium. For example, if a pure axial force is applied to a line spring, it may assume one of two configurations. One is the extended, and the other is the compressed state. Another well known example is the buckling of columns in which there exist infinitely many solutions for the deflection corresponding to a unique axial compressive force.

Definition 26 *Any static equilibrium configuration of an elastically rigid body system corresponding to a given wrench $\hat{W}_O$ is called an* ***equilibrium configuration*** *of $\hat{W}_O$. Any configuration corresponding to the zero wrench is called an* ***unloaded equilibrium****.*

Let $\hat{W}_O = [\ \vec{\mathbf{F}}^T \quad \vec{\mathbf{M}}_O^T\ ]^T$ be the spatial representation of the wrench with respect to a point O of the fixed body, where $\vec{\mathbf{F}}$ is the force and $\vec{\mathbf{M}}_O$ is the moment with respect to O. Using the definition of the infinitesimal spatial displacement, the **infinitesimal work done** is given by

$$\delta\omega = \hat{W}_O^T \delta\hat{q}_O = \vec{\mathbf{F}}^T \mathbf{d}\vec{\mathbf{r}}_O + \vec{\mathbf{M}}_O^T \delta\vec{\boldsymbol{\theta}} \tag{2.88}$$

For any given equilibrium configuration of $\hat{W}_O$, the set of all possible infinitesimal displacements, $\{\delta\hat{q}_O\}$, is a 6-dimensional vector space. Then, the infinitesimal work done can be given as a scalar-valued vector function over $\{\delta\hat{q}_O\}$, $\delta\omega(\delta\hat{q}_O) = \hat{W}_O^T\delta\hat{q}_O$, where $\delta\hat{q}_O$ is taken as an indeterminate.

If $\delta\omega(\delta\hat{q}_O) > 0$ for all $\delta\hat{q}_O$ then a positive work is needed to effect an infinitesimal change in the configuration. Such a configuration is called a **stable equilibrium**. If there exist some $\delta\hat{q}_O$ for which $\delta\omega(\delta\hat{q}_O) = 0$, the equilibrium is said to be **singular** in those directions. Finally, if $\delta\omega(\delta\hat{q}_O) < 0$ for some $\hat{q}_O$, the configuration is said to be an **unstable equilibrium**.

2.1.7 Definition of Stiffness and Compliance

Figure 2.12-(a) shows an elastically suspended rigid body in equilibrium under the action of the wrench $\hat{W}$. For stable equilibrium, a small change in the wrench results in a corresponding small change in the configuration. This is illustrated in Figure 2.12-(b) where the infinitesimal change in the wrench is $d\hat{W}$ and the corresponding change in the configuration is the infinitesimal spatial displacement $\delta\hat{q}$. The causal relationship can be reversed. That is, one can consider $\delta\hat{q}$ as an applied infinitesimal change in the configuration and $d\hat{W}$ as the resulting change in the wrench in order to maintain the static equilibrium.

In general, in any equilibrium configuration the set of all possible infinitesimal spatial displacements $\{\delta\hat{q}\}$ form a six dimensional vector space, namely the twist space. Similarly, the set of all possible infinitesimal variations in $\hat{W}$, $\{d\hat{W}\}$, form a six dimensional vector space, namely the wrench space. Then, for an elastic system in this configuration, the response to infinitesimal spatial displacements can be considered as a mapping from $\{\delta\hat{q}\}$ to $\{d\hat{W}\}$, or the response to infinitesimal change in the wrench can be considered as a mapping from $\{d\hat{W}\}$ to $\{\delta\hat{q}\}$. That is, for any elastic system in configuration x, there exist mappings

$$k(x) \quad : \quad \{\delta\hat{q}\} \longrightarrow \{d\hat{W}\} \tag{2.89}$$

$$c(x) \quad : \quad \{d\hat{W}\} \longrightarrow \{\delta\hat{q}\} \tag{2.90}$$

Given a particular $\delta\hat{q}$, the corresponding $d\hat{W}$ is the necessary change in the applied wrench to sustain the static equilibrium. This is similar to the Hooke's law for line springs, $\Delta f = k\ \Delta l$, where

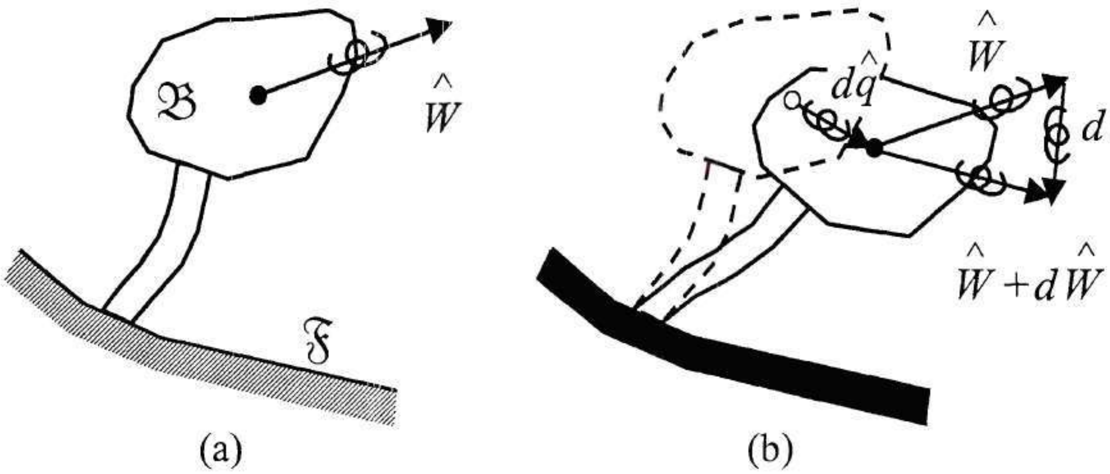


Figure 2.12: An elastically suspended rigid body system in static equilibrium: (a) equilibrium under the load $\hat{W}$, (b) new equilibrium under the load $\hat{W} + d\hat{W}$. The two equilibrium configurations are infinitesimally close to each other. They are separated by an infinitesimal spatial displacement $\delta\hat{q}$ which cause the change in the wrench $d\hat{W}$, and vice versa.

Δf is the change in the axial force (scalar), Δl change in the length (scalar) and k is the spring rate (scalar stiffness). So, the Hooke's law is actually a mapping between 1-dimensional quantities. The only difference is that in the spatial case $\delta\hat{q}$ and $d\hat{W}$ are 6-vectors. So, by analogy, the mapping $k(x)$ is a measure of how stiff the elastic system is. For example, if $\delta\hat{q}$ is a pure translation with magnitude dr, then the components of $d\hat{W}$ are the forces and couples that resist this translation. Also, the components of the rate vector $\frac{d\hat{W}}{dr}$ are similar to spring rates. However, there is no simple notion of magnitude for general $\delta\hat{q}$. Therefore, further analogy should not be proposed at this point. The theory developed in Chapters 3 and 4 presents correct generalizations.

A similar argument can be made for the mapping $c(x)$. Clearly, $c(x)$ is a reverse mapping similar to $\Delta l = \frac{1}{k}\,\Delta f$. So, by analogy, $c(x)$ is a measure of how compliant the elastic system is. These observations suggest the following definitions.

Definition 27

1. *The mapping $k(x) : \{\delta\hat{q}\} \longrightarrow \{d\hat{W}\}$ is called the* ***stiffness mapping.***

2. *The mapping $c(x) : \{d\hat{W}\} \longrightarrow \{\delta\hat{q}\}$ is called the* ***compliance mapping.***

Since both $\delta\hat{q}$ and $d\hat{W}$ are infinitesimal quantities, the relation between them can be taken to be linear. That is, the mapping $k(x)$ is such that $d\hat{W}$ is a linear function of $\delta\hat{q}$, and $c(x)$ is such that $\delta\hat{q}$ is a linear function of $d\hat{W}$. In other words, $k(x)$ and $c(x)$ can be taken as linear mappings. It is well known that linear mappings between two vector spaces can be represented by matrices of appropriate sizes. That is, in spatial case, there exist 6×6 matrices $\hat{K}(x)$ and $\hat{C}(x)$ such that

$$k(x) \; : \; \{\delta\hat{q}\} \longrightarrow \{d\hat{W}\} \quad \text{such that} \quad d\hat{W} = \hat{K}(x)\delta\hat{q} \tag{2.91}$$

$$c(x) \; : \; \{d\hat{W}\} \longrightarrow \{\delta\hat{q}\} \quad \text{such that} \quad \delta\hat{q} = \hat{C}(x)d\hat{W} \tag{2.92}$$

A linear map between two vector spaces of the same dimension is called a **tensor**. Therefore, $\hat{K}$ and $\hat{C}$ are matrix representation of tensors. These matrices are the main subject of this study. For any elastic system in a configuration x, $\hat{K}$ and $\hat{C}$ characterize the elastic behavior in terms of how stiff or compliant the system is. Therefore, the following definitions are proposed.

Definition 28 *For an elastic system in equilibrium under the action of a wrench $\hat{W}$,*

1. *the matrix $\hat{K}$ that linearly maps a given infinitesimal spatial displacement to a corresponding infinitesimal change in the wrench so that the static equilibrium is preserved is called the* ***spatial stiffness matrix.***

2. *the matrix $\hat{C}$ that linearly maps a given infinitesimal change in the wrench to a corresponding infinitesimal spatial displacement so that the static equilibrium is preserved is called the* ***spatial compliance matrix.***

For simplicity, throughout this study the spatial stiffness matrix and spatial compliance matrix are frequently referred to as stiffness and compliance matrices, or simply stiffness and compliance.

2.1.8 Stiffness and Compliance Transformations

Definition 24 presented the spatial vector transformations between different reference frames. If the wrenches are in ray-coordinates and the twists are in axis-coordinates, then

$$d\hat{W}_P = \hat{X}_r d\hat{W}_O \qquad \delta\hat{q}_P = \hat{X}_a \delta\hat{q}_O \tag{2.93}$$

are the transformation rules from a reference frame with origin at O to another with origin at P. Here, $\hat{X}_r$ and $\hat{X}_a$ are the ray- and axis- coordinate representations of the same spatial transformation matrix. From Definition 24, it is not difficult to show that

$$\hat{X}_a = \hat{X}_r^{-T} \tag{2.94}$$

So, if the stiffness and compliance relations are given at O as

$$d\hat{W}_O = \hat{K}_O \delta\hat{q}_O \qquad \delta\hat{q}_O = \hat{C}_O d\hat{W}_O \tag{2.95}$$

then, using (2.93) one gets

$$\hat{X}_r^{-1} d\hat{W}_P = \hat{K}_O \hat{X}_a^{-1} \delta\hat{q}_P \qquad \hat{X}_a^{-1} \delta\hat{q}_P = \hat{C}_O \hat{X}_r^{-1} d\hat{W}_P \tag{2.96}$$

or,

$$d\hat{W}_P = \left(\hat{X}_r \hat{K}_O \hat{X}_r^T\right) \delta\hat{q}_P \qquad \delta\hat{q}_P = \left(\hat{X}_a \hat{C}_O \hat{X}_a^T\right) d\hat{W}_P \tag{2.97}$$

Hence, by definition

$$\hat{K}_P = \hat{X}_r \hat{K}_O \hat{X}_r^T \qquad \hat{C}_P = \hat{X}_a \hat{C}_O \hat{X}_a^T \tag{2.98}$$

are the transformation rules for stiffness and compliance. The equations (2.98) are in agreement with the classical tensor transformations.

2.2 A Survey of Previous Studies

Ball [1] introduced the use of screw theory for rigid body dynamics. One of the ideas he presented was the concept of principal screws of inertia. In this, a wrench applied to a rigid body about a principal screw of inertia produces an instantaneous twist on the same screw. The screw axes are along the conventional principal axes and the pitches are equivalent to the radii of gyration about the axes. Ball also applied the same approach to a rigid body in a potential field. A rigid body in a potential field, subjected to a twist on a principal screw of the potential produces a wrench on the same screw. The results of this problem were much less elegant than in the inertial case. For example, in general none of the principal screws of the potential are orthogonal or intersect.

Dimentberg [15] employed screw theory in the analysis of the statics and small vibrations of a rigid body elastically suspended by line springs. He characterized the structure of an elastic suspension by a special set of six wrenches. These are the reaction wrenches obtained by applying to the system three special orthogonal rotations and three translations parallel to the rotations. The resultant wrenches are generally non-intersecting. However when they intersect, Dimentberg calls the point of intersection the *center of elasticity*. In this case, the linear and angular static equations of the system decouple. No other meaning is attached to the center. In the small vibrations case of an elastically suspended rigid body, he showed that the equation of motion can be given in terms of the small displacement screw of the body with suitable introduction of inertia. However, the equations of motion generally do not decouple unless the principal axes of the inertia coincide with the special directions of the orthogonal translations mentioned above, which he termed as the principal directions of the suspension rigidity, and the system has a 'center-of-elasticity'.

Hunt [28] applied the screw algebra to analyze the kinematics of mechanisms. Bokelberg and Hunt [4] examined the infinitesimal motion using screws. They showed that a differential screw can be defined to describe the difference between two successive screws

Loncaric [32] examined the stiffness and compliance matrices of robotic mechanisms using Lie Algebra. He introduced two generally distinct centers called the *centers of compliance* and *stiffness*. At these points the off-diagonal blocks of the respective 6×6 stiffness or compliance matrix become symmetric. Loncaric's approach is motivated by the desire to find the simplest possible forms of stiffness and compliance. A general symmetric matrix can be diagonalized by using orthogonal transformations. However, only rigid body transformations (spatial transformations) are allowable for stiffness and compliance. Loncaric showed that rigid body transformations are insufficient to diagonalize the stiffness or compliance matrices. First, he searched for a point where the stiffness matrix would have symmetric off-diagonals. He showed the existence of this point and proved that it was unique. Then, one could apply rotational transformations leading to the diagonalization of the now symmetric off-diagonals. In this way, the linear and angular parts of the stiffness map are maximally decoupled. A similar procedure yielded another point for compliance. Loncaric called the resulting forms of stiffness and compliance the *normal forms*. However, no other meaning is given to the centers. Their geometric or constitutive properties remained unknown. In a different direction, Loncaric also investigated the synthesis of stiffness by springs [33]. He showed that any stiffness matrix with a zero trace off-diagonal can be realized by springs. However, an explicit solution of the problem remained unsolved. Loncaric also discussed the construction of unstable springs using loaded stable springs. Recently, Huang and Schimmels [27] studied the synthesis problem and verified Loncaric's realizability theorem results by using screw algebra. They presented a synthesis algorithm based on the Cholesky decomposition that generally requires seven springs, with three through the origin, for nonsingular stiffnesses.

Lipkin and Patterson [30], [31] extended and generalized the center of elasticity concept of Dimentberg [15]. They formulated two singular eigenvalue problems to yield three *eigenwrenches* and three *eigentwists*. An eigenwrench produces a pure translation parallel to the force part of the eigenwrench and an eigentwist produces a pure couple parallel to the rotation part of the eigentwist, see Chapter 3. The three eigenwrenches are the reactions due to the special orthogonal translations introduced by Dimentberg. But, the eigentwists were new and are the reactions due to special orthogonal couples. The eigenwrenches and eigentwists define two 3-systems. Lipkin and Patterson showed that the centers of these 2-systems coincide and is a generalization of the center of elasticity proposed by Dimentberg. Lipkin and Patterson also called the center as the center of elasticity. Another important result was that the eigenwrenches and eigentwists lead to decompositions of stiffness and compliance matrices into geometric and constitutive parts. Later, Patterson and Lipkin [41], [42] used the results from the new eigenvalue problems to classify robot compliances. They defined special axes called force-compliant, rotation-compliant and compliant axes. A general stiffness may not have any of these special axes. However, when exist, they imply special stiffnesses. These lead to a compliance hierarchy. Patterson and Lipkin found the geometric relations between the compliant axes and the eigenwrenches, eigentwists, the center of elasticity, etc. Patterson and Lipkin [40] also applied the theory to constrained systems (singular stiffness or compliance).

Ciblak and Lipkin [9] further analyzed the eigenwrenches and eigentwists. They showed that there exists a simple formula for the location of the center of elasticity in terms of eigenwrench and eigentwist properties. The centers of stiffness and compliance are shown to have properties analogous to those of the center of elasticity, although the former are shown to be of constitutive nature, whereas the latter has a purely geometric character. Then, Ciblak and Lipkin showed that compliant axes pass through all centers, establishing a geometric significance for the centers of stiffness and compliance.

Fasse and Broenink [19] applied spatial vectors (screws) to the stiffness and compliance control of robotic manipulators. Although stiffness, compliance, impedance and admittance control are well known and can be achieved by other means, Fasse and Broenink show that the use of spatial quantities makes the parameter selection easier and intuitive. Other controllers use different mathematical descriptions that conceal the geometric information and the control law is difficult to choose for complex spatial motions.

Nguyen [36] investigated the properties of force-closure grasp of an object by using stiffness properties. He introduced virtual springs to model the desired stiffness. The main problem was to synthesize the springs at the contact points so that a stable grasp is achieved. While the object is manipulated the fingers are continuously controlled to maintain the stable grasp. This amounts to controlling the contact points (where the virtual springs are attached) and the virtual spring rates.

Cutkosky and Kao [14] expressed the compliance of a grasp of a robotic hand as a function of grasp geometry, contact conditions, and mechanical properties of the fingers. They showed that a force-closure grasp, which resists any kind of loading, is indicated by the rank of the stiffness matrix, and, the positive definiteness of the stiffness matrix is a measure of the grasp stability. They also presented a method to control the joint variables in order to achieve a desired stiffness of the hand. Their example of a robotic hand inserting a rivet is used in numerical examples of this study.

Whitney and coworkers [16], [17], [34], [35], [51], [52], [53] analyzed and experimented on the problem of successfully mating rigid parts which are elastically supported. The use of the *remote center of compliance* (RCC) device, developed at Draper Laboratories by Whitney and coworkers, attached at the hand of a robotic manipulator is explained. An RCC device responds to pure forces through a certain point E by parallel pure translations, and to pure couples by parallel pure rotations through the same point E, see Chapter 9 for details. The point E is historically known as the center of compliance. In an RCC device, E is outside the physical boundaries of the device, hence the name remote-center-of-compliance. These devices are well known and are commercially available. The advantages of the use of RCC device are exclusively due to the special form of the stiffness matrix. A special construction proposed and implemented by the researches uses three beams symmetrically placed on a cone. Whitney and coworkers attempted to calculate the location of E as well as device stiffness, since these are the essential factors in the performance of an RCC. However, their design equations were based on assumptions, such as parallel beams, which are proven in this study to be very coarse. Their experiments on a prototype actually indicated that their predictions contained considerable errors. Although they attributed these discrepancies to experimental and modeling errors, which may always exist, in Chapter 9 it is shown that most of the error can be traced to the assumptions they adopted.

Ciblak and Lipkin [10] analyzed the RCC devices made of beams. Using $n \geqslant 3$ beams symmetrically placed on a cone, they found a closed form equation for the location of the elastic center and the scalar stiffnesses (eigenvalues). The position of the elastic center was shown to be extremely sensitive to the cone angle. This partially explains the failure to accurately predict the center and stiffnesses. Also, the assumption of parallel beams is proven to be inadequate.

Kim and coworkers [29] studied the stiffness control of a three degree of freedom planar parallel mechanism. The mechanism used in the study was tried to be given the stiffness characteristics of an RCC device by actively controlling the configuration. The advantage of this kind of devices is the on-line adjustability of the RCC device center, which is a feature that the passive RCCs lack.

Featherstone [20] investigated the dynamics of serially connected rigid bodies by using spatial vector algebra. He used the concept of articulated inertia (the effective inertia of a serial chain) to construct a recursive algorithm that may be used in the control and dynamic simulation of serial manipulators. The Newton-Euler form of the equation of motion was determined in spatial form.

The studies presented above assume symmetric stiffness and compliance matrices. An elastic system always has symmetric stiffness at an unloaded equilibrium configuration [48]. When the system is away from the unloaded equilibrium the stiffness mapping may or may not be symmetric depending on the type of coordinates used. Simo ??, Nour-Omid and Rankin [38], and Bufler [7] investigated the structure of the stiffness matrix and showed that if the covariant derivative is used then the stiffness matrix is always symmetric for conservative systems. The covariant derivative is a tensor calculus concept and is beyond the scope of this study. Interested reader may refer to Schutz [45], Bishop and Goldberg [2], and Edwards [18].

Griffis and Duffy [22], [21] derived an non-symmetric stiffness matrix for a special Stewart platform device modelled by six springs connecting the upper platform directly to the ground. Pigoski, Griffis and Duffy [43] investigated the nonsymmetric stiffness properties by using a planar, three-spring, 3×3 stiffness matrix. These results are limited but constitute good examples for testing the results of the current research.

Ciblak and Lipkin [8] investigated the stiffness of parallel spatial connection of an arbitrary number of line springs. They showed that the skew-symmetric part of the spatial stiffness is minus one half of the spatial cross product operator form of the applied wrench. Thus, the stiffness of such systems is symmetric only in an unloaded equilibrium. They also showed that the stiffnesses referenced to the fixed and body frames are transposes of each other.

Zefran and Kumar [54] analyzed the Cartesian stiffness for general conservative elastic systems. Using the methods of manifold theory, tensor algebra and calculus, and Lie algebra they showed that the results of Ciblak and Lipkin [8] are valid for any conservative system.

This study is based on the theory and methods presented in the above mentioned studies. Therefore, the terminology and notation are made as close to those in the previous studies as possible in order to help the reader in comparing and interpreting the current results. Some of these studies provided the general theoretical setting of this study such as linear spaces, screws, kinematics, elasticity, etc. The rest is directly related to the current research through particular problems, theorems or experiments.

Part II

THEORY

CHAPTER III

FREE-VECTOR EIGENVALUE PROBLEMS FOR STIFFNESS AND COMPLIANCE

The stiffness and compliance matrices for conservative systems at unloaded equilibrium are symmetric [8], [25]. Therefore, symmetric stiffness and compliance are essential in the analysis of elastic systems. Since these mappings are represented by tensorial quantities, there exists no obvious, simple number, such as the stiffness of a linear line spring, that can give a quantification of the system behavior. In many similar problems, i.e. when a tensor is involved, a way of describing the system in terms of simpler objects is to try to separate the quantity into constitutive and geometric contents. This results in a family of problems collectively known as **eigenvalue problems**. The main topic of this chapter is the identification and solution of meaningful eigenvalue problems for spatial stiffness and compliance matrices. Many other important results follow. Applications are presented in Chapter 9.

Consider an elastically suspended rigid body. If a pure force is applied to the body, it will deflect. Clearly, the translational part of the deflection and its direction will be different for different forces. It may be of interest to know the direction of the unit force which will cause the largest translation, or that which will cause the smallest. Then, a designer would interpret this by saying that the largest deflection direction is the *most compliant direction* and the smallest deflection direction is the *most stiff direction*. These are sometimes called the *weak-* and *strong-axes* in structural analysis [39]. This is equivalent to extracting some geometric information along with quantifying numbers. In this example, the directions are the geometric information. A number can be associated by each direction as to quantify how stiff or compliant the system is in that direction. This can be formed, for example, by the ratio of the applied force to the resulting translation. Better known examples are from the theory of elasticity where one speaks of the principal stresses and strains, and their associated directions. Such problems are collectively known as **eigenvalue/eigenvector** problems. eigenvalue problems are about the maximum or minimum, or, in general, the stationary values of a quantity related to the system.

This chapter shows that, first of all, the spatial eigenvalue problems are not as straightforward as the usual eigenvalue problems one encounters in other fields. After briefly restating the set of eigenvalue problems proposed by Lipkin and Patterson [30], new properties of eigenscrews are presented which lead to previously unknown expressions for the location of the *center of elasticity*. In this context, the *centers of stiffness and compliance*, proposed by Loncaric [32], are shown to have previously unknown properties. Their relations to the center of elasticity are explained in terms of eigenscrews and compliant axes.

To show that the eigenvalue problems are not straightforward, let $\hat{T}$ be a twist, λ a scalar and $\hat{K}$ the spatial stiffness. Then, an eigenvalue equation such as $\hat{K}\hat{T} = \lambda\hat{T}$ cannot be satisfied physically. The easiest way to show this is to check the units of both sides. Let the units be taken as meters (m) and Newtons (N).

$$\begin{array}{ccccc} \hat{K} & \hat{T} & = & \lambda\hat{T} & \\ \begin{bmatrix} [\mathrm{N/m}] & [\mathrm{N}] \\ [\mathrm{N}] & [\mathrm{N\text{-}m}] \end{bmatrix} & \begin{bmatrix} [\mathrm{m}] \\ [1] \end{bmatrix} & = & \lambda \begin{bmatrix} [\mathrm{m}] \\ [1] \end{bmatrix} & \\ & \begin{bmatrix} [\mathrm{N}] \\ [\mathrm{N\text{-}m}] \end{bmatrix} & = & \lambda \begin{bmatrix} [\mathrm{m}] \\ [1] \end{bmatrix} & \begin{array}{l} \Rightarrow \quad \lambda \sim [\mathrm{N/m}] \\ \Rightarrow \quad \lambda \sim [\mathrm{N\text{-}m}] \end{array} \end{array} \tag{3.1}$$

The first row of (3.1) requires units of λ to be [N/m], whereas the second requires it to be [N-m], which indicates a contradiction. That is, there is no λ which can make (3.1) physically consistent and meaningful. In general terms, the difficulty originates from the fact that stiffness maps a twist to a wrench which cannot be equated to another twist. The problem can resolved by introducing another mapping on the right side of (3.1) which maps a twist to a wrench. For example, a meaningful eigenvalue equation is obtained by using the mapping

$$\hat{\Delta} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{3.2}$$

which maps a twist to a wrench that is about the same screw, and vice versa. The modified equation

$$\hat{K}\hat{T} = \lambda\hat{\Delta}\hat{T} \tag{3.3}$$

correctly associates a wrench with another wrench. $\hat{K}\hat{T}$ and $\hat{\Delta}\hat{T}$ are about the same screw. The scalar λ has the units of force, which leads to a consistent equation. The new mapping $\hat{\Delta}$ is actually an indefinite metric on the screw space, Chapter 2. This problem has been investigated by Ball [1]. The existence and uniqueness of these kind of eigenvalues and eigenscrews are easily shown. However, Ball's eigenscrews, while they are interesting, have not proven useful in applications. In this and the following chapter, the role of $\hat{\Delta}$ is played by some other mappings resulting in two distinct kinds of eigenvalue problems whose constitutive and geometric meanings are more profound and useful in applications.

3.1 Definition

An alternative and useful eigenvalue problem was proposed by Lipkin and Patterson [30]. They first considered the following two problems.

1. Determine a wrench that only causes a parallel translation (a free-vector).

2. Determine a twist that only causes a parallel couple (a free-vector).

In equation form, these are

$$\hat{C}\begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\tau} \end{bmatrix} = a_f \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\mathbf{0}} \end{bmatrix} \qquad \hat{K}\begin{bmatrix} \vec{\delta} \\ \vec{\gamma} \end{bmatrix} = k_\gamma \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\gamma} \end{bmatrix} \tag{3.4}$$

where $\hat{C} = \hat{K}^{-1}$ are the compliance and stiffness matrices with respect to some origin O, $\vec{\mathbf{f}}$ is a unit force and $\vec{\tau}$ is the accompanying couple of the wrench at O, $\vec{\gamma}$ is a unit rotation and $\vec{\boldsymbol{\delta}}$ is the accompanying translation of the twist at O, a_f is a **linear compliance**, and k_γ is an **angular stiffness**. See Figure 3.1. If [L] and [F] respectively denote the length and force units, the units of the quantities in (3.4) are

$$\begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\tau} \end{bmatrix} \sim \begin{bmatrix} [1] \\ [L] \end{bmatrix} \qquad a_f \sim [L/F] \qquad \begin{bmatrix} \vec{\delta} \\ \vec{\gamma} \end{bmatrix} \sim \begin{bmatrix} [L] \\ [1] \end{bmatrix} \qquad k_\gamma \sim [F^*L] \tag{3.5}$$

Now consider a mapping $\hat{\Gamma}$ whose representations in ray- and axis-coordinates are

$$\hat{\Gamma}_r = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \hat{\Gamma}_a = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{3.6}$$

Using $\hat{\Gamma}$, (3.4) can be expressed as a pair of generalized eigenvalue problems as

$$\hat{C}\hat{W}_f = a_f \hat{\Gamma}_r \hat{W}_f \qquad \hat{K}\hat{T}_\gamma = k_\gamma \hat{\Gamma}_a \hat{T}_\gamma \tag{3.7}$$

where $\hat{W}_f$ and $\hat{T}_\gamma$ are a unit wrench and a unit twist, respectively.

Definition 29 *The generalized eigenvalue problems (3.7), or equivalently (3.4), are called the **free-vector eigenvalue problems** for stiffness and compliance.*

1. *The wrench $\hat{W}_f$ in (3.7) that causes only a pure translation parallel to the force part is called an **eigenwrench** (Figure 3.1).*

2. *The twist $\hat{T}_\gamma$ in (3.7) that causes only a pure couple parallel to the rotation part is called an **eigentwist** (Figure 3.1).*

■

The free-vector eigenvalue problems are also obtainable by minimizing the potential energy of the system. To see this, let $\hat{W}$ and $\hat{T}$ denote a unit wrench and a unit twist, respectively, and Φ be

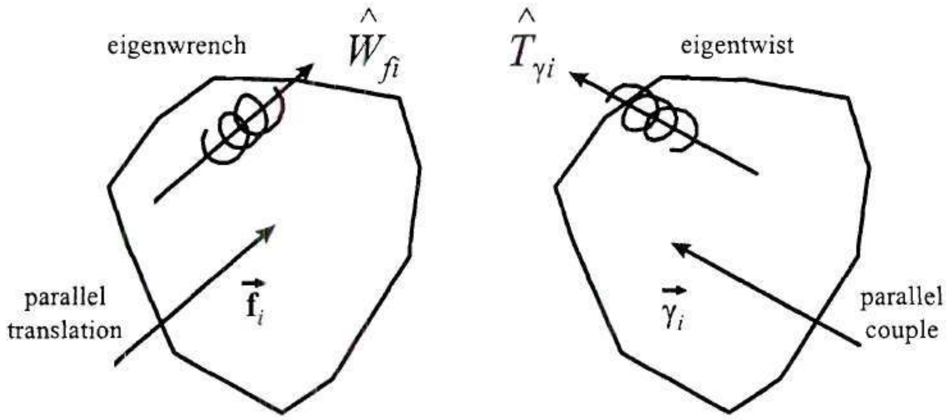


Figure 3.1: Eigenwrench and eigentwist systems.

the potential energy. Using (4.3), the minimization problems are given as

$$\begin{array}{cc} \underline{\text{minimize}} & \underline{\text{constraint}} \\ \Phi_W = \frac{1}{2}\hat{W}^T\hat{C}\hat{W} & \hat{W}^T\hat{\Gamma}_r\hat{W} = 1 \\ \Phi_T = \frac{1}{2}\hat{T}^T\hat{K}\hat{T} & \hat{T}^T\hat{\Gamma}_a\hat{T} = 1 \end{array} \tag{3.8}$$

For the wrenches and the twists the constraints can be written explicitly as

$$\vec{\mathbf{f}}^T\vec{\mathbf{f}} = 1 \qquad \vec{\gamma}^T\vec{\gamma} = 1 \tag{3.9}$$

Solving the minimization problems yields the eigenvalue problems (3.4). To do this, one first transforms the constrained minimization problem to an unconstrained one by introducing the Lagrangian form. For wrenches, this is $\Phi_W^* = \frac{1}{2}\hat{W}^T\hat{C}\hat{W} - a_f\left(\hat{W}^T\hat{\Gamma}_r\hat{W} - 1\right)$ where a_f is a scalar. Then, by taking the derivative with respect to $\hat{W}$ and equating to zero, one gets the first equation in (3.7).

An alternative development is presented here, which establishes a strong parallelism with the second eigenvalue problems presented in Chapter 4.

Let V and V^* be the twist and wrench spaces. Consider the free-vector subspaces $V_f \subset V$ and $V_f^* \subset V^*$, where V_f is the subspace of *translations* and V_f^* is the subspace of *couples*. The subscript $_f$ denotes the free-vector property. It is shown in Chapter 2 that V_f and V_f^* are unique 3-dimensional subspaces of V and V^*, respectively.

The stiffness $\hat{K}$ and the compliance $\hat{C}$ respectively act on V_f and V_f^*. The action of $\hat{K}$ on the translation subspace V_f results in a set of wrenches $V_W^* \subset V^*$. Similarly, the action of $\hat{C}$ on the couple subspace V_f^* results in a set twists $V_T \subset V$. In short, $\hat{K}$ and $\hat{C}$ induce the following mappings.

$$\hat{K}(V_f) : V_f \rightarrow V_W^* \subset V^* \qquad\qquad \hat{C}(V_f^*) : V_f^* \rightarrow V_T \subset V \tag{3.10}$$

If the mappings are non-singular then it follows, by linearity, that

Theorem 30 *V_W^* and V_T are 3-system of screws. That is, they are 3-dimensional subspaces.*

By inverting the equations in (3.4),

$$\hat{K}\begin{bmatrix} \vec{\mathbf{f}} \\ \vec{0} \end{bmatrix} = k_f \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\tau} \end{bmatrix} \qquad\qquad \hat{C}\begin{bmatrix} \vec{0} \\ \vec{\gamma} \end{bmatrix} = a_\gamma \begin{bmatrix} \vec{\delta} \\ \vec{\gamma} \end{bmatrix} \tag{3.11}$$

one shows that the eigenscrews are the images of free-vectors. Then, from (3.10), $\hat{W}_f \in V_W^*$ and $\hat{T}_\gamma \in V_T$. Lipkin and Patterson showed that there exist three independent eigenscrews for each equation of (3.11). Therefore, the eigenwrenches and eigentwists form bases for V_W^* and V_T, respectively. Together with Theorem 30, this leads to the following definition.

Definition 31 *V_W^* is called the* ***eigenwrench 3-system*** *and V_T is called the* ***eigentwist 3-system****.*

3.2 Uniqueness and Existence

It is assumed throughout that the elastic system is *nonsingular* and it is at a *stable, unloaded equilibrium.* The material in this section is required subsequently. Further details are in Lipkin and Patterson [30], [31].

$\hat{K}$ and $\hat{C}$ are symmetric 6×6 stiffness and the compliance matrices. In terms of 3×3 partitions

$$\hat{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \qquad \hat{C} = \begin{bmatrix} \mathbf{D} & \mathbf{E}^T \\ \mathbf{E} & \mathbf{F} \end{bmatrix} \tag{3.12}$$

Equations (3.11) can be expressed in terms of these 3×3 submatrices as

$$k_f \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\tau} \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{0} \end{bmatrix} \qquad a_\gamma \begin{bmatrix} \vec{\delta} \\ \vec{\gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{D} & \mathbf{E}^T \\ \mathbf{E} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \vec{0} \\ \vec{\gamma} \end{bmatrix} \tag{3.13}$$

which yield a pair of classical eigenvalue problems,

$$k_f \vec{\mathbf{f}} = \mathbf{A}\vec{\mathbf{f}} \qquad a_\gamma \vec{\gamma} = \mathbf{F}\vec{\gamma} \tag{3.14}$$

for the directions of the eigenwrenches and eigentwists. Since the system is assumed to be nonsingular and at a stable, unloaded equilibrium, $\mathbf{A}$ and $\mathbf{F}$ are symmetric and positive definite. Thus each relation yields three positive eigenvalues that are respectively the stationary (maximum, saddle, minimum) values of linear stiffness $k_{fi} > 0$ and rotational compliance $a_{\gamma i} > 0$. Each set of unit eigenvectors $\vec{\mathbf{f}}_i$ and $\vec{\gamma}_i$ form a unique orthogonal set when the eigenvalues are distinct. When there are repeated eigenvalues, an orthogonal set of three linearly independent eigenvectors can always be selected.

Back substitution of the eigenvalues and eigenvectors into (3.13) yield three unit eigenwrenches $\hat{W}_{fi} = \begin{bmatrix} \vec{\mathbf{f}}_i^T & \vec{\tau}_i^T \end{bmatrix}^T$ and three unit eigentwists $\hat{T}_{\gamma i} = \begin{bmatrix} \vec{\delta}_i^T & \vec{\gamma}_i^T \end{bmatrix}^T$. Since the $\vec{\mathbf{f}}_i$ are orthogonal, the three eigenwrenches $\hat{W}_{fi}$ are along three orthogonal lines which are generally skew. Similarly, since the $\vec{\gamma}_i$ are orthogonal, the three eigentwists $\hat{T}_{\gamma i}$ are along another three orthogonal lines which are generally skew.

Lipkin and Patterson then showed that the eigenwrenches and eigentwists form reciprocal spaces,

$$\hat{T}_{\gamma i}^T \hat{W}_{fj} = 0 \qquad \text{for all } i, j = 1, 2, 3 \tag{3.15}$$

3.3 Free-Vector Decomposition of Stiffness and Compliance

The eigenscrews and eigenvalues ultimately lead to unique decompositions of the stiffness and compliance matrices. First the following 3×3 matrices are defined.

$$\mathbf{f} = \begin{bmatrix} \vec{\mathbf{f}}_1 & \vec{\mathbf{f}}_2 & \vec{\mathbf{f}}_3 \end{bmatrix} \qquad \tau = \begin{bmatrix} \vec{\tau}_1 & \vec{\tau}_2 & \vec{\tau}_3 \end{bmatrix} \tag{3.16}$$

$$\gamma = \begin{bmatrix} \vec{\gamma}_1 & \vec{\gamma}_2 & \vec{\gamma}_3 \end{bmatrix} \qquad \delta = \begin{bmatrix} \vec{\delta}_1 & \vec{\delta}_2 & \vec{\delta}_3 \end{bmatrix} \tag{3.17}$$

$$\mathbf{k}_f = \mathbf{a}_f^{-1} = \text{diag}(k_{f1}, k_{f2}, k_{f3}) \tag{3.18}$$

$$\mathbf{k}_\gamma = \mathbf{a}_\gamma^{-1} = \text{diag}(k_{\gamma 1}, k_{\gamma 2}, k_{\gamma 3}) \tag{3.19}$$

Note that $\mathbf{f}$ and γ are orthogonal matrices

$$\mathbf{f}^T\mathbf{f} = \mathbf{I} \qquad \gamma^T\gamma = \mathbf{I} \tag{3.20}$$

Then, equations (3.13) and (3.14) lead to the 3×3 submatrices of the stiffness and compliance as follows.

$$\mathbf{A}\vec{\mathbf{f}}_i = k_{fi}\vec{\mathbf{f}}_i \qquad \text{for} \qquad i = 1, 2, 3 \tag{3.21}$$

$$\mathbf{A}\begin{bmatrix} \vec{\mathbf{f}}_1 & \vec{\mathbf{f}}_2 & \vec{\mathbf{f}}_3 \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{f}}_1 & \vec{\mathbf{f}}_2 & \vec{\mathbf{f}}_3 \end{bmatrix} \begin{bmatrix} k_{f1} & 0 & 0 \\ 0 & k_{f2} & 0 \\ 0 & 0 & k_{f3} \end{bmatrix} \tag{3.22}$$

$$\mathbf{A}\mathbf{f} = \mathbf{f}\mathbf{k}_f \tag{3.23}$$

$$\mathbf{A} = \mathbf{f}\mathbf{k}_f\mathbf{f}^T \tag{3.24}$$

Others are obtained similarly,

$$\mathbf{A} = \mathbf{f}\mathbf{k}_f\mathbf{f}^T \qquad \mathbf{B} = \mathbf{f}\mathbf{k}_f\boldsymbol{\tau}^T \tag{3.25}$$

$$\mathbf{F} = \boldsymbol{\gamma}\mathbf{a}_\gamma\boldsymbol{\gamma}^T \qquad \mathbf{E} = \boldsymbol{\gamma}\mathbf{a}_\gamma\boldsymbol{\delta}^T \tag{3.26}$$

$\mathbf{C}$ and $\mathbf{D}$ are obtained by using the inverse relation $\hat{K}^{-1}\hat{C} = \hat{I}$ as

$$\mathbf{C} = \boldsymbol{\tau}\mathbf{k}_f\boldsymbol{\tau}^T + \boldsymbol{\gamma}\mathbf{k}_\gamma\boldsymbol{\gamma}^T \tag{3.27}$$

$$\mathbf{D} = \boldsymbol{\delta}\mathbf{a}_\gamma\boldsymbol{\delta}^T + \mathbf{f}\mathbf{a}_f\mathbf{f}^T \tag{3.28}$$

By substituting (3.25) through (3.28) in (3.12), the following forms for stiffness and compliance are obtained.

$$\hat{K} = \begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \boldsymbol{\tau} & \boldsymbol{\gamma} \end{bmatrix} \begin{bmatrix} \mathbf{k}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_\gamma \end{bmatrix} \begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \boldsymbol{\tau} & \boldsymbol{\gamma} \end{bmatrix}^T \tag{3.29}$$

$$\hat{C} = \begin{bmatrix} \mathbf{f} & \boldsymbol{\delta} \\ \mathbf{0} & \boldsymbol{\gamma} \end{bmatrix} \begin{bmatrix} \mathbf{a}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_\gamma \end{bmatrix} \begin{bmatrix} \mathbf{f} & \boldsymbol{\delta} \\ \mathbf{0} & \boldsymbol{\gamma} \end{bmatrix}^T \tag{3.30}$$

In (3.29), the columns of $[\mathbf{f}^T\ \boldsymbol{\tau}^T]^T$ are the unit eigenwrenches, and in (3.30), the columns of $[\mathbf{f}^T\ \mathbf{0}^T]^T$ are the corresponding unit translations. In (3.30), the columns of $[\boldsymbol{\delta}^T\ \boldsymbol{\gamma}^T]^T$ are the unit eigentwists, and in (3.29), the columns of $[\mathbf{0}^T\ \boldsymbol{\gamma}^T]^T$ are the corresponding unit couples.

One may wonder whether the decomposition based on the solution of the free-vector eigenvalue problems at another point, say A, would be different. To see this one simply applies the transformation rules to the decompositions above. The stiffness case is considered. Let the position vector of A from O be given by $\overrightarrow{OA} = \vec{\mathbf{r}}$. Then,

$$\hat{K}_A = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\vec{\mathbf{r}}\times & \mathbf{I} \end{bmatrix} \hat{K}_O \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\vec{\mathbf{r}}\times & \mathbf{I} \end{bmatrix}^T \tag{3.31}$$

$$= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\vec{\mathbf{r}}\times & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \tau_O & \gamma \end{bmatrix} \begin{bmatrix} \mathbf{k}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_\gamma \end{bmatrix} \begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \tau_O & \gamma \end{bmatrix}^T \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\vec{\mathbf{r}}\times & \mathbf{I} \end{bmatrix}^T \tag{3.32}$$

$$= \begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \tau_O - \vec{\mathbf{r}}\times\mathbf{f} & \gamma \end{bmatrix} \begin{bmatrix} \mathbf{k}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_\gamma \end{bmatrix} \begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \tau_O - \vec{\mathbf{r}}\times\mathbf{f} & \gamma \end{bmatrix}^T \tag{3.33}$$

$$= \begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \tau_A & \gamma \end{bmatrix} \begin{bmatrix} \mathbf{k}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_\gamma \end{bmatrix} \begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \tau_A & \gamma \end{bmatrix}^T \tag{3.34}$$

which clearly shows that the $\hat{K}_A$ is decomposed by the same eigenscrews in the same manner, just represented at A. The same is true for the compliance. That is, no matter where the problem is posed the same eigenscrews form the basis of the decomposition.

3.4 Center of Elasticity

In a 3-system of screws, any three linearly independent screws form a basis. In general, such bases contain screws which are neither orthogonal nor intersecting. Eigenscrews are examples of mutually orthogonal basis elements. For any given 3-system, Ball [1] showed the existence of three special basis screws which are mutually orthogonal and intersect at a point. These screws are called the **principal screws** of the 3-system, Chapter 2. The pitches of the principal screws, principal pitches, are stationary values of all pitches in the 3-system. The axes of the principal screws of the reciprocal system are coincident with those of the given 3-system and intersect at exactly the same point in the space [1], [28]. The principal pitches of a 3-system and its reciprocal system have the equal magnitudes and opposite signs. Lipkin and Patterson showed that eigenscrews lead to principal screws for the stiffness and compliance systems and termed the unique intersection point the **center of elasticity**. In this study, the center of elasticity is denoted by E. Figure 3.2 illustrates the principal screws of the eigenscrew systems.

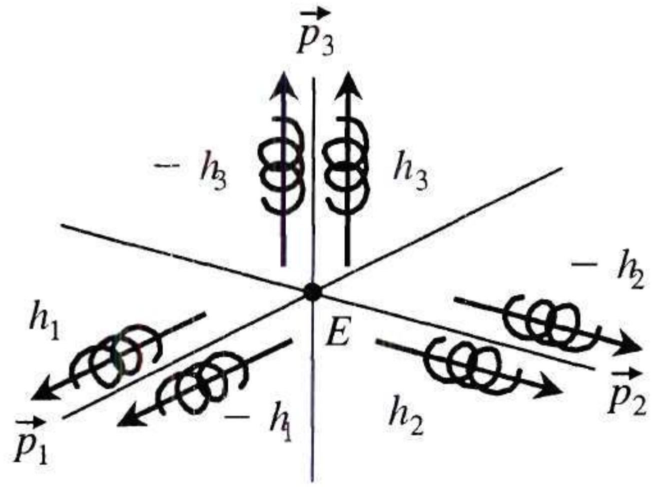


Figure 3.2: The principal screws of the eigenwrench and eigentwist systems have equal and opposite pitches, and orthogonally intersect at the center of elasticity.

Linear combinations of the eigenwrenches yield the principal screws of V_W^* and linear combinations of the eigentwists yield the principal screws of V_T, i.e.,

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{q}_\tau \end{bmatrix} = \begin{bmatrix} \mathbf{f} \\ \boldsymbol{\tau} \end{bmatrix} \boldsymbol{\mu}_f \qquad \begin{bmatrix} \mathbf{q}_\delta \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\delta} \\ \boldsymbol{\gamma} \end{bmatrix} \boldsymbol{\mu}_\gamma \tag{3.35}$$

where the orthogonal directions of the principal screws are

$$\mathbf{p} = [\ \vec{\mathbf{p}}_1 \quad \vec{\mathbf{p}}_2 \quad \vec{\mathbf{p}}_3\ ] \tag{3.36}$$

At E, the principal screws can be written as

$$\begin{bmatrix} \mathbf{p} \\ \mathbf{q}_\tau \end{bmatrix}_E = \begin{bmatrix} \mathbf{p} \\ \mathbf{ph} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{p}}_1 & \vec{\mathbf{p}}_2 & \vec{\mathbf{p}}_3 \\ h_1\vec{\mathbf{p}}_1 & h_2\vec{\mathbf{p}}_2 & h_3\vec{\mathbf{p}}_3 \end{bmatrix} \tag{3.37}$$

where $\mathbf{h} = \mathrm{diag}(h_1, h_2, h_3)$ and h_i are the principal pitches. In (3.35), $\mathbf{q}_\tau$, $\mathbf{q}_\delta$ are 3×3 matrices, and the coefficients form 3×3 orthogonal matrices

$$\boldsymbol{\mu}_f^T \boldsymbol{\mu}_f = \mathbf{I} \quad \boldsymbol{\mu}_\gamma^T \boldsymbol{\mu}_\gamma = \mathbf{I} \tag{3.38}$$

since $\mathbf{f}$, $\boldsymbol{\gamma}$ and $\mathbf{p}$ are orthogonal matrices. This leads to the following theorem that is used subsequently.

Theorem 32 *An eigenwrench (eigentwist) and a principal screw are equal if and only if their axes are parallel.*

Proof. It is sufficient to prove the eigenwrench case since the eigentwist case is similar. Assume that the first eigenwrench and first principal screw are parallel so $\vec{\mathbf{f}}_1 = \vec{\mathbf{p}}_1$. Since $\vec{\mathbf{f}}_1$ is orthogonal to $\vec{\mathbf{f}}_2$ and $\vec{\mathbf{f}}_3$ then from (3.35) the first column of $\boldsymbol{\mu}_f$ is $[\ 1 \quad 0 \quad 0\ ]^T$making the first eigenwrench and principal screw equal. The converse is trivial since if the eigenwrench and principal screw are equal then the directions are identical.

∎

The center of elasticity also has several important properties. The following is proved in [30], [31], and used in the sequel.

Theorem 33 (Lipkin and Patterson) *The perpendicular vectors from the center of elasticity E to the eigenwrenches (eigentwists) sum to zero,*

$$\sum_i \vec{\mathbf{r}}_{fi}^E = \vec{\mathbf{0}} \qquad \left(\sum_i \vec{\mathbf{r}}_{\gamma i}^E = \vec{\mathbf{0}}\right) \tag{3.39}$$

Theorem (33) also implies that each set of perpendicular vectors are coplanar (Figure 3.3).

Ciblak and Lipkin [9] generalized Theorem 33 in terms of an arbitrary point O as follows.

Theorem 34 *The position vector to E from any point O, $\vec{\mathbf{r}}_E$, is equal to one-half the sum of the perpendicular vectors from O to the eigenwrenches (eigentwists),*

$$\vec{\mathbf{r}}_E = \frac{1}{2}\sum_i \vec{\mathbf{r}}_{fi} \qquad \left(\vec{\mathbf{r}}_E = \frac{1}{2}\sum_i \vec{\mathbf{r}}_{\gamma i}\right) \tag{3.40}$$

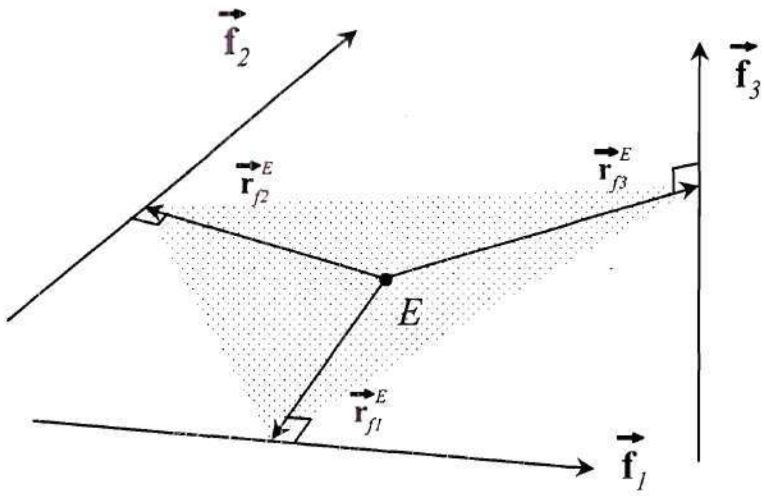


Figure 3.3: Perpendicular vectors from the center of elasticity to the eigenscrews are coplanar and sum to zero.

Proof. The proof is given only for the eigenwrenches since the proof for the eigentwists is similar. Referring to Figure 3.4, the.position vector $\vec{\mathbf{r}}_E = \overrightarrow{OE}$ from some point O to E is

$$\vec{\mathbf{r}}_E = \vec{\mathbf{r}}_{fi} + \alpha_{fi}\,\vec{\mathbf{f}}_i - \vec{\mathbf{r}}_{fi}^E \tag{3.41}$$

where $\vec{\mathbf{r}}_{fi}$ is the perpendicular distance from O and $\alpha_{fi}\vec{\mathbf{f}}_i$ is the projection of $\vec{\mathbf{r}}_E$ on $\vec{\mathbf{f}}_i$. Forming the product with $\vec{\mathbf{f}}_i^T$ gives

$$\alpha_{fi} = \vec{\mathbf{f}}_i^T \vec{\mathbf{r}}_E \tag{3.42}$$

Multiplying (3.42) by $\vec{\mathbf{f}}_i$ and summing over the index gives

$$\sum_i \alpha_{fi}\,\vec{\mathbf{f}}_i = \left[\sum_i \vec{\mathbf{f}}_i\vec{\mathbf{f}}_i^T\right]\vec{\mathbf{r}}_E = \vec{\mathbf{r}}_E \tag{3.43}$$

where $\sum_i \vec{\mathbf{f}}_i\vec{\mathbf{f}}_i^T = \mathbf{I}$ since $\vec{\mathbf{f}}_i$ are orthogonal. Summing (3.41) over the index i

$$3\vec{\mathbf{r}}_E = \sum_i \vec{\mathbf{r}}_{fi} + \sum_i \alpha_{fi}\,\vec{\mathbf{f}}_i - \sum_i \vec{\mathbf{r}}_{fi}^E \tag{3.44}$$

Using (3.39) and (3.43) yields the desired result (3.40).

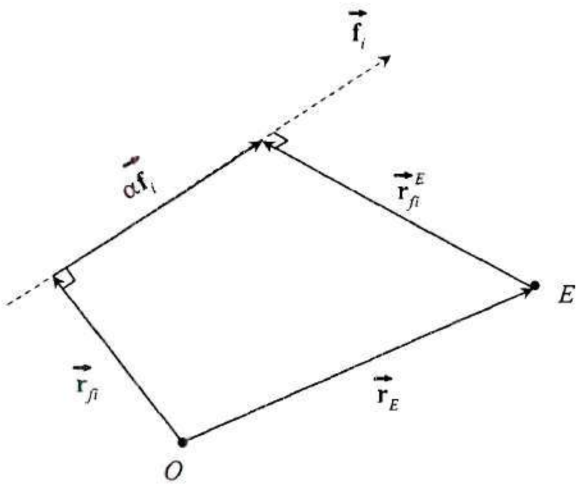


Figure 3.4: Vector loop from the center of elasticity to any arbitrary point O, along an eigenscrew and related perpendicular vectors.

■

3.5 Centers of Stiffness and Compliance

Loncaric's [32] identification of the centers of stiffness and compliance is basically motivated by the desire to obtain simplest possible forms of the matrices. A way to do this is to look for points where, for example, the stiffness matrix is maximally decoupled with respect to its translational and rotational parts. Loncaric first showed that a possible approach is to require the off-diagonal submatrices be symmetric, so that they can consequently be diagonalized by pure rotations of the coordinate frame. He showed the existence of such points, centers of stiffness and compliance, where the respective matrices have this property. He called these forms at the centers the **normal forms** of the stiffness and compliance matrices. In his work, however, the centers seem to have only a formal importance. Here, it is shown that they actually have unique, intrinsic constitutive and geometric properties which complement the center of elasticity. Also shown here are their relations to eigenscrews and compliant axes, which were unknown previously. In this study the center of stiffness is denoted by S and the center of compliance is denoted by C.

In the following it is assumed that all quantities that are origin dependent are with respect to origin O unless otherwise noted.

Let $\vec{\mathbf{r}}_S = \overrightarrow{OS}$ be the position vector to the center of stiffness S and $\vec{\mathbf{r}}_C = \overrightarrow{OC}$ be the position vector to the center of compliance C. Representations of the stiffness at S and the compliance at C are obtained by the transformations

$$\hat{K}_S = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\vec{\mathbf{r}}_S\times & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\vec{\mathbf{r}}_S\times & \mathbf{I} \end{bmatrix}^T \tag{3.45}$$

$$\hat{C}_C = \begin{bmatrix} \mathbf{I} & -\vec{\mathbf{r}}_C\times \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{D} & \mathbf{E}^T \\ \mathbf{E} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\vec{\mathbf{r}}_C\times \\ \mathbf{0} & \mathbf{I} \end{bmatrix}^T \tag{3.46}$$

For *normal* forms of the matrices, the off-diagonal blocks must be symmetric

$$\mathbf{B} + \mathbf{A}\vec{\mathbf{r}}_S\times = \mathbf{B}^T - \vec{\mathbf{r}}_S \times \mathbf{A} \qquad \mathbf{E}^T - \vec{\mathbf{r}}_C \times \mathbf{F} = \mathbf{E} + \mathbf{F}\vec{\mathbf{r}}_C\times \tag{3.47}$$

Letting

$$\vec{\mathbf{b}}\times = \frac{1}{2}(\mathbf{B} - \mathbf{B}^T) \qquad \vec{\mathbf{e}}\times = \frac{1}{2}(\mathbf{E} - \mathbf{E}^T) \tag{3.48}$$

gives

$$2\vec{\mathbf{b}}\times = -(\mathbf{A}\vec{\mathbf{r}}_S \times + \vec{\mathbf{r}}_S \times \mathbf{A}) \qquad 2\vec{\mathbf{e}}\times = -(\mathbf{F}\vec{\mathbf{r}}_C \times + \vec{\mathbf{r}}_C \times \mathbf{F}) \tag{3.49}$$

The expressions (3.49) are respectively equivalent to

$$[\mathbf{A} - \mathrm{tr}(\mathbf{A})\mathbf{I}]\,\vec{\mathbf{r}}_S = 2\vec{\mathbf{b}} \qquad [\mathbf{F} - \mathrm{tr}(\mathbf{F})\mathbf{I}]\,\vec{\mathbf{r}}_C = 2\vec{\mathbf{e}} \tag{3.50}$$

The forms of the solutions in (3.50) are given in [32], but not derived. The following is a derivation based on index notation.

Derivation of equations (3.50): In index notation define

$$\vec{\mathbf{b}} = b_i \qquad \mathbf{A} = a_{ij} \qquad \vec{\mathbf{r}} = r_i \qquad \vec{\mathbf{r}}\times = -\epsilon_{ijk} r_k \tag{3.51}$$

where ϵ_{ijk} is the permutation symbol. Then, the first equation in (3.49) can be written as

$$-2\epsilon_{ijk} b_k = \epsilon_{ipq} r_q a_{pj} + a_{ip} \epsilon_{pjq} r_q \tag{3.52}$$

where the Einstein summation convention is on repeated indices, i.e. $\epsilon_{ijk} b_k = \sum_k \epsilon_{ijk} b_k$, etc. Multiplying both sides by ϵ_{ijm} and using the identities

$$\epsilon_{ipq}\epsilon_{imn} = \delta_{pm}\delta_{qn} - \delta_{pn}\delta_{qm} \qquad \epsilon_{ijm} = \epsilon_{jmi} \qquad \epsilon_{pjq} = \epsilon_{jqp} \tag{3.53}$$

where δ_{ij} is the Kronecker delta, yields

$$\begin{aligned} -2\epsilon_{ijm}\epsilon_{ijk} b_k &= \left[\epsilon_{ijm}\epsilon_{ipq} a_{pj} + \epsilon_{ijm}\epsilon_{pjq} a_{ip}\right] r_q & (3.54) \\ -2(\delta_{jj}\delta_{km} - \delta_{jm}\delta_{jk}) b_k &= \left[(\delta_{jp}\delta_{mq} - \delta_{jq}\delta_{mp}) a_{pj} + (\delta_{mq}\delta_{ip} - \delta_{mp}\delta_{iq}) a_{ip}\right] r_q & (3.55) \end{aligned}$$

By definition of the Kronecker delta, for any quantity $g_{ij\ldots}$

$$\delta_{ij} g_{\ldots i \ldots} = g_{\ldots j \ldots} \qquad \delta_{ii} = n \tag{3.56}$$

where n is the number of all possible choices for i or the dimension of the space over which δ_{ij} is defined. In this case $n = 3$. So, (3.55) simplifies to

$$\begin{aligned} -2(3\delta_{km} - \delta_{km}) b_k &= \left[a_{pp}\delta_{mq} - \delta_{mp} a_{pq} + a_{pp}\delta_{mq} - \delta_{mp} a_{qp}\right] r_q & (3.57) \\ -4b_m &= \left[2a_{pp}\delta_{mq} - (a_{mq} + a_{qm})\right] r_q & (3.58) \\ 2b_m &= \left[\frac{1}{2}(a_{mq} + a_{qm}) - a_{pp}\delta_{mq}\right] r_q & (3.59) \end{aligned}$$

which corresponds to the matrix equation

$$2\vec{\mathbf{b}} = \left[\frac{1}{2}(\mathbf{A} + \mathbf{A}^T) - \mathrm{tr}(\mathbf{A})\mathbf{I}\right] \vec{\mathbf{r}} \tag{3.60}$$

Equation (3.60) is equivalent to the first equation in (3.50), since $\mathbf{A}$ is symmetric. The second equation in (3.50) follows similarly.

■

A physical interpretation of these equations is developed by expressing $\vec{\mathbf{b}}$ and $\vec{\mathbf{e}}$ in terms of eigenscrews by using the decompositions (3.25) and (3.26), and the partitions (3.12). First,

$$\mathbf{B} = \mathbf{f}\mathbf{k}_f\boldsymbol{\tau}^T = \sum_i k_{fi}\vec{\mathbf{f}}_i\vec{\boldsymbol{\tau}}_i^T \qquad \mathbf{E} = \boldsymbol{\gamma}\mathbf{a}_\gamma\boldsymbol{\delta}^T = \sum_i a_{\gamma i}\vec{\boldsymbol{\gamma}}_i\vec{\boldsymbol{\delta}}_i^T \tag{3.61}$$

so that

$$2\vec{\mathbf{b}}\times = \mathbf{B} - \mathbf{B}^T = \sum_i k_{fi}(\vec{\mathbf{f}}_i\vec{\boldsymbol{\tau}}_i^T - \vec{\boldsymbol{\tau}}_i\vec{\mathbf{f}}_i^T) \tag{3.62}$$

$$2\vec{\mathbf{e}}\times = \mathbf{E} - \mathbf{E}^T = \sum_i a_{\gamma i}(\vec{\boldsymbol{\gamma}}_i\vec{\boldsymbol{\delta}}_i^T - \vec{\boldsymbol{\delta}}_i\vec{\boldsymbol{\gamma}}_i^T) \tag{3.63}$$

Using the vector identity

$$\vec{\mathbf{u}}\vec{\mathbf{v}}^T - \vec{\mathbf{v}}\vec{\mathbf{u}}^T = (\vec{\mathbf{v}} \times \vec{\mathbf{u}})\times \tag{3.64}$$

yields

$$2\vec{\mathbf{b}}\times = \sum_i k_{fi}(\vec{\boldsymbol{\tau}}_i \times \vec{\mathbf{f}}_i)\times \qquad 2\vec{\mathbf{e}}\times = \sum_i a_{\gamma i}(\vec{\boldsymbol{\delta}}_i \times \vec{\boldsymbol{\gamma}}_i)\times \tag{3.65}$$

However the perpendicular vectors from O to the eigenwrenches and eigentwists are (for example, see Brand [6] or Chapter 2),

$$\vec{\mathbf{r}}_{fi} = \frac{\vec{\mathbf{f}}_i \times \vec{\boldsymbol{\tau}}_i}{\vec{\mathbf{f}}_i^T\vec{\mathbf{f}}_i} \qquad \vec{\mathbf{r}}_{\gamma i} = \frac{\vec{\boldsymbol{\gamma}}_i \times \vec{\boldsymbol{\delta}}_i}{\vec{\boldsymbol{\gamma}}_i^T\vec{\boldsymbol{\gamma}}_i} \tag{3.66}$$

Since $\vec{\mathbf{f}}_i$ and $\vec{\boldsymbol{\gamma}}_i$ are unit vectors (3.65) become

$$\vec{\mathbf{b}} = -\frac{1}{2}\sum_i k_{fi}\vec{\mathbf{r}}_{fi} \qquad \vec{\mathbf{e}} = -\frac{1}{2}\sum_i a_{\gamma i}\vec{\mathbf{r}}_{\gamma i} \tag{3.67}$$

These expressions are very similar to ones for the center of elasticity (3.40) of Theorem 34 except that they involve *weighted* sums of the perpendicular vectors. Eliminating $\vec{\mathbf{b}}$ and $\vec{\mathbf{e}}$ in (3.50) and using (3.67) gives

$$[\mathrm{tr}(\mathbf{A})\mathbf{I} - \mathbf{A}]\,\vec{\mathbf{r}}_S = \sum_i k_{fi}\vec{\mathbf{r}}_{fi} \qquad [\mathrm{tr}(\mathbf{F})\mathbf{I} - \mathbf{F}]\,\vec{\mathbf{r}}_C = \sum_i a_{\gamma i}\vec{\mathbf{r}}_{\gamma i} \tag{3.68}$$

By moving from O to S (C) so that $\vec{\mathbf{r}}_S = \vec{0}$ $\left(\vec{\mathbf{r}}_C = \vec{0}\right)$ the following theorem is proven.

Theorem 35 *At the center of stiffness S (center of compliance C) the perpendicular vectors to the eigenwrenches (eigentwists) weighted by the stationary values of linear stiffness (rotational compliance) sum to zero, viz.*

$$\sum_i k_{fi}\vec{\mathbf{r}}^S_{fi} = \vec{0} \qquad \left(\sum_i a_{\gamma i}\vec{\mathbf{r}}^C_{\gamma i} = \vec{0}\right) \tag{3.69}$$

Since the weighted perpendicular vectors sum to zero then the following results as an immediate corollary.

Corollary 36 *The perpendicular vectors from the center of stiffness S (center of compliance C) to the eigenwrenches (eigentwists) are coplanar.*

There is a strong similarity between the relations for the center of elasticity in (3.39) and those for the centers of stiffness and compliance in (3.69). The difference is that the relation for the E is *purely geometric* whereas the relations for S and C involve geometric quantities weighted by the stationary values of *constitutive* properties. With this view, the decompositions (3.29) and (3.30) decouple the stiffness and compliance matrices into purely geometric quantities (the eigenscrews) and purely constitutive quantities (the eigenvalues).

For stable systems, the matrices in the brackets in (3.68) are invertible. To see this consider (3.68) in which $\mathbf{A}$ is positive definite symmetric matrix. Then $\mathrm{tr}(\mathbf{A}) > k_{fi} > 0$ for all i. If $[\mathrm{tr}(\mathbf{A})\mathbf{I} - \mathbf{A}]$ is singular then there exists a vector $\vec{\mathbf{u}}$ such that $[\mathrm{tr}(\mathbf{A})\mathbf{I} - \mathbf{A}]\,\vec{\mathbf{u}} = \vec{0}$ or $\mathbf{A}\vec{\mathbf{u}} = \mathrm{tr}(\mathbf{A})\vec{\mathbf{u}}$. This means that $\mathrm{tr}(\mathbf{A})$ is an eigenvalue of $\mathbf{A}$, therefore $\mathrm{tr}(\mathbf{A}) = k_{fi}$ for some i which contradicts the requirement that $\mathrm{tr}(\mathbf{A}) > k_{fi}$. Then, for positive definite $\mathbf{A}$ the matrix $[\mathrm{tr}(\mathbf{A})\mathbf{I} - \mathbf{A}]$ is non-singular, hence invertible. This allows one to solve (3.68) for the locations of S and C,

$$\vec{\mathbf{r}}_S = [\mathrm{tr}(\mathbf{A})\mathbf{I} - \mathbf{A}]^{-1}\sum_i k_{fi}\vec{\mathbf{r}}_{fi} \qquad \vec{\mathbf{r}}_C = [\mathrm{tr}(\mathbf{F})\mathbf{I} - \mathbf{F}]^{-1}\sum_i a_{\gamma i}\vec{\mathbf{r}}_{\gamma i} \tag{3.70}$$

To demonstrate the results in eigenscrews and related terms, one uses (3.25) and (3.26), which yields

$$\vec{\mathbf{r}}_S = \mathbf{f}\left[\operatorname{tr}(\mathbf{k}_f)\mathbf{I} - \mathbf{k}_f\right]^{-1}\mathbf{f}^T \sum_i k_{fi}\vec{\mathbf{r}}_{fi} \qquad \vec{\mathbf{r}}_C = \boldsymbol{\gamma}\left[\operatorname{tr}(\mathbf{a}_\gamma)\mathbf{I} - \mathbf{a}_\gamma\right]^{-1}\boldsymbol{\gamma}^T \sum_i a_{\gamma i}\vec{\mathbf{r}}_{\gamma i} \tag{3.71}$$

where $\operatorname{tr}(\mathbf{k}_f) = \sum_i k_{fi}$ and $\operatorname{tr}(\mathbf{a}_\gamma) = \sum_i a_{\gamma i}$.

As an aside, (3.68) are useful for examining the singular cases. The terms in the parentheses in (3.68) represent the projections of the position vectors onto the directions of the respective eigenvectors. Since the matrices in the brackets are diagonal, if the rank is two then there exists a line of centers in the direction of an eigenvector, and if the rank is one then there exists a plane of centers normal to an eigenvector.

3.6 Compliant Axes

Some elastic systems exhibit characteristics that are desirable in practical applications. For example, line springs respond to a pure force by a pure translation in the same direction, torsional springs respond to a pure couple by a parallel pure rotation. A well known example of a more complicated elastic system with such characteristics is the *remote center of compliance* (RCC) device, see Chapter 9. The RCC device has the characteristics of line and torsional springs in all directions. For this reason, RCCs are successfully used in robotic applications [16]. By definition of eigenscrews, a force causing a parallel translation is an eigenwrench and a rotation causing a parallel couple is an eigentwist. In particular such eigenscrews are line-vectors (zero pitch screws).

Patterson and Lipkin [42] investigated such special cases in detail and presented a classification. In their study, a zero pitch eigenwrench direction is called a **force-compliant axis** and a zero pitch eigentwist is called a **rotation-compliant axis**, see Figure 3.5.

There also exist more specialized cases. For an RCC, not only do the eigenscrews have zero pitch, but they are also parallel to each other. Lipkin and Patterson described these special cases in the following definition.

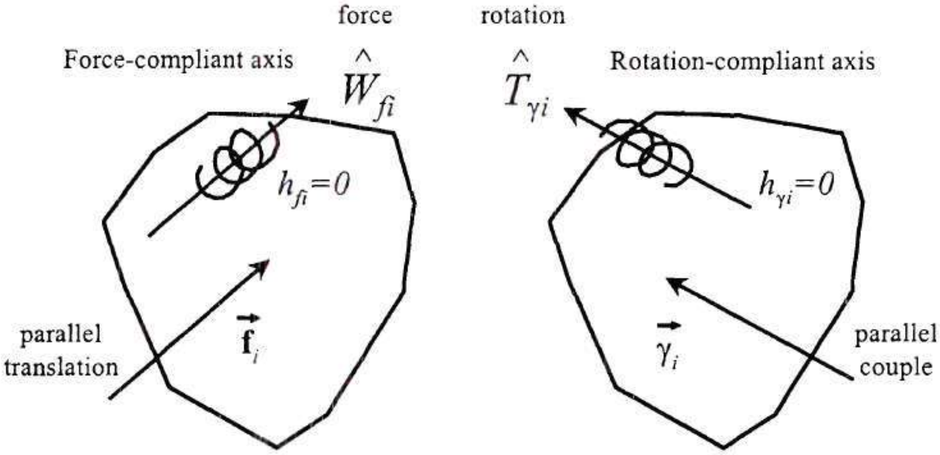


Figure 3.5: Zero pitch eigenscrews characterize well known behavior of simple elastic systems. A force yields a translation, a rotation yields a couple. The axes of such forces and rotations are called the force-compliant and rotation compliant axes, respectively.

Definition 37 (Lipkin and Patterson) *A pair of collinear force- and rotation-compliant axes is said to form a* ***compliant axis*** *(see Figure 3.6).*

An elastic system may or may not have a compliant axis. When it does, any pure force along the compliant axis produces a pure parallel translation and any pure rotation produces a pure parallel couple. For example, RCC device has compliant axes in all directions, all through a certain point. This is investigated later in Chapter 9 in more detail. A compliant axis has special properties. The following theorem is due to Lipkin and Patterson [31].

Theorem 38 (Lipkin and Patterson) *A compliant axis passes through the center of elasticity.*

It follows that for an RCC all compliant axes intersect at the center of elasticity. This is what gives the device its unique behavior with respect to the forces and rotations through the center. It is shown in this section that compliant axes have a far richer structure than previously thought,

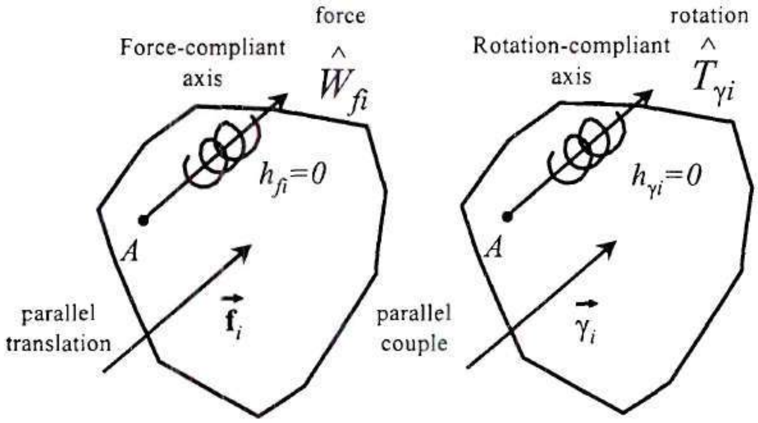


Figure 3.6: A pair of collinear force- and rotation-compliant axes form a compliant axis.

relating the center of elasticity to centers of stiffness and compliance, and eigenscrews. In Chapter 5, the definition of compliant axes is generalized in a more systematic way, leading to a better classification of compliant systems.

3.7 Geometrical Relations Between the Centers and Compliant Axes

This section establishes the fundamental relationships between the three centers, principal screws, the eigenwrenches and eigentwists, and the existence of compliant axes. The results are summarized in four theorems.

From (3.39) and (3.69) the following forms are deduced.

$$\begin{aligned}
\vec{\mathbf{r}}^E_{f1} &= \alpha_2\vec{\mathbf{f}}_2 - \alpha_3\vec{\mathbf{f}}_3 & \qquad k_{f1}\vec{\mathbf{r}}^S_{f1} &= \beta_2\vec{\mathbf{f}}_2 - \beta_3\vec{\mathbf{f}}_3 \\
\vec{\mathbf{r}}^E_{f2} &= \alpha_3\vec{\mathbf{f}}_3 - \alpha_1\vec{\mathbf{f}}_1 & \qquad k_{f2}\vec{\mathbf{r}}^S_{f2} &= \beta_3\vec{\mathbf{f}}_3 - \beta_1\vec{\mathbf{f}}_1 \\
\vec{\mathbf{r}}^E_{f3} &= \alpha_1\vec{\mathbf{f}}_1 - \alpha_2\vec{\mathbf{f}}_2 & \qquad k_{f3}\vec{\mathbf{r}}^S_{f3} &= \beta_1\vec{\mathbf{f}}_1 - \beta_2\vec{\mathbf{f}}_2
\end{aligned} \tag{3.72}$$

The coefficients α_i and β_i are respectively the projections of $\vec{\mathbf{r}}^E_{fi}$ and $k_{fi}\vec{\mathbf{r}}^S_{fi}$ onto $\vec{\mathbf{f}}_i$. Expressing

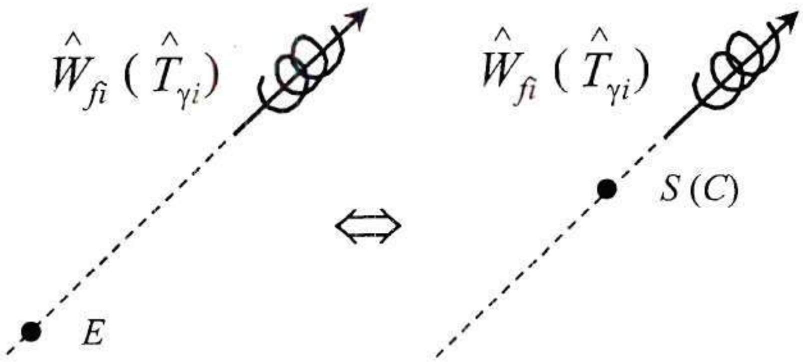


Figure 3.7: An eigenwrench (eigentwist) always passes through both E and S (C).

(3.40) at S and (3.71) at E, instead of O, yields two expressions for the vectors between E and S

$$\vec{\mathbf{r}}_E^S = \frac{1}{2}\sum_i \vec{\mathbf{r}}_{fi}^S \tag{3.73}$$

$$\vec{\mathbf{r}}_S^E = \mathbf{f}\left[tr(\mathbf{k}_f)\mathbf{I} - \mathbf{k}_f\right]^{-1}\mathbf{f}^T \sum_i k_{fi}\vec{\mathbf{r}}_{fi}^E \tag{3.74}$$

where $\vec{\mathbf{r}}_E^S = -\vec{\mathbf{r}}_S^E$. Substitution of (3.72) in (3.73) and (3.74) yields

$$\vec{\mathbf{r}}_S^E = \frac{k_{f3} - k_{f2}}{k_{f3} + k_{f2}}\alpha_1\vec{\mathbf{f}}_1 + \frac{k_{f1} - k_{f3}}{k_{f1} + k_{f3}}\alpha_2\vec{\mathbf{f}}_2 + \frac{k_{f2} - k_{f1}}{k_{f2} + k_{f1}}\alpha_3\vec{\mathbf{f}}_3 \tag{3.75}$$

$$\vec{\mathbf{r}}_E^S = \frac{1}{2}\left((a_{f3} - a_{f2})\beta_1\vec{\mathbf{f}}_1 + (a_{f1} - a_{f3})\beta_2\vec{\mathbf{f}}_2 + (a_{f2} - a_{f1})\beta_3\vec{\mathbf{f}}_3\right) \tag{3.76}$$

Similar procedure for the eigentwists gives

$$\vec{\mathbf{r}}_C^E = \frac{a_{\gamma 3} - a_{\gamma 2}}{a_{\gamma 3} + a_{\gamma 2}}\eta_1\vec{\gamma}_1 + \frac{a_{\gamma 1} - a_{\gamma 3}}{a_{\gamma 1} + a_{\gamma 3}}\eta_2\vec{\gamma}_2 + \frac{a_{\gamma 2} - a_{\gamma 1}}{a_{\gamma 2} + a_{\gamma 1}}\eta_3\vec{\gamma}_3 \tag{3.77}$$

$$\vec{\mathbf{r}}_E^C = \frac{1}{2}\left((k_{\gamma 3} - k_{\gamma 2})\xi_1\vec{\gamma}_1 + (k_{\gamma 1} - k_{\gamma 3})\xi_2\vec{\gamma}_2 + (k_{\gamma 2} - k_{\gamma 1})\xi_3\vec{\gamma}_3\right) \tag{3.78}$$

The preceding results lead to the following theorems,

Theorem 39 *An eigenwrench (eigentwist) passes through E if and only if it also passes through S (C). See Figure 3.7.*

Proof. The proof is for the eigenwrench case. Assume the 3rd eigenwrench passes through E so $\vec{\mathbf{r}}_{f3}^{E} = \vec{\mathbf{0}}$. From (3.72)

$$\alpha_1 = \alpha_2 = 0 \tag{3.79}$$

and substitution into (3.75) yields

$$\vec{\mathbf{r}}_{S}^{E} = \frac{k_{f2} - k_{f1}}{k_{f2} + k_{f1}} \alpha_3 \vec{\mathbf{f}}_3 \tag{3.80}$$

Thus the 3rd eigenwrench axis passes through S since it is in the direction from E to S. Note that the cases $k_{f2} = k_{f1}$ and $\alpha_3 = 0$ are trivial. For the converse, assume the 3rd eigenwrench axis passes through S so $\vec{\mathbf{r}}_{f3}^{S} = \vec{\mathbf{0}}$. Again from (3.72)

$$\beta_1 = \beta_2 = 0 \tag{3.81}$$

and substitution into (3.76) yields

$$\vec{\mathbf{r}}_{E}^{S} = \frac{1}{2}(a_{f2} - a_{f1}) \beta_3 \vec{\mathbf{f}}_3 \tag{3.82}$$

Thus the 3rd eigenwrench axis passes through E since it is in the direction from S to E. The cases $a_{f2} = a_{f1}$ and $\beta_3 = 0$ are trivial.

■

Theorem 40 *E and S (C) are coincident if and only if the eigenwrenches (eigentwists) are principal screws.*

Proof. The proof is for the eigenwrenches. First assume that the eigenwrenches are principal screws and thus intersect at E. From (3.72) $\alpha_i = 0$. So, from (3.75) $\vec{\mathbf{r}}_{S}^{E} = \vec{\mathbf{0}}$ making E and S coincident. The converse considers three distinct cases with $\vec{\mathbf{r}}_{S}^{E} = \vec{\mathbf{0}}$.

1. k_{fi} are distinct. From (3.75) $\alpha_i = 0$ so from (3.72) the eigenwrenches pass through E and are thus principal screws.

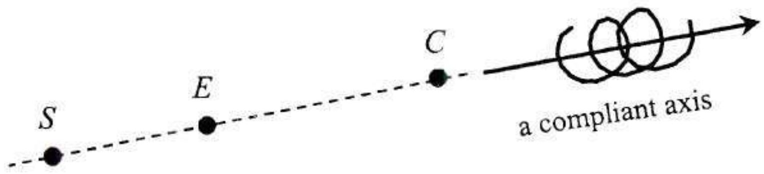


Figure 3.8: A compliant axis always passes through all three centers.

2. There exists a double eigenwrench stiffness. Let $k_{f1} = k_{f2} \neq k_{f3}$. From (3.75) $\alpha_1 = \alpha_2 = 0$ and from (3.72) $\vec{\mathbf{r}}^E_{f3} = \vec{\mathbf{0}}$ so that the third eigenwrench passes through E and is thus a principal screw. Since now $\vec{\mathbf{f}}_3 = \vec{\mathbf{p}}_3$ then from (3.35) the vector directions $\vec{\mathbf{f}}_1$, $\vec{\mathbf{f}}_2$ are linear combinations of $\vec{\mathbf{p}}_1$, $\vec{\mathbf{p}}_2$. However, since the eigenvalues k_{f1}, k_{f2} have a repeated value then from (3.14) $\vec{\mathbf{f}}_1$, $\vec{\mathbf{f}}_2$ are not unique and any pair of orthogonal directions normal to $\vec{\mathbf{f}}_3$ may be selected. In particular, select $\vec{\mathbf{f}}_1$, $\vec{\mathbf{f}}_2$ parallel to $\vec{\mathbf{p}}_1$, $\vec{\mathbf{p}}_2$ so that from Theorem 32 the eigenwrenches are principal screws.

3. All eigenwrench stiffnesses are equal: $k_{f1} = k_{f2} = k_{f3}$. Since there is only one distinct eigenvalue then from (3.14) the directions of the eigenvectors $\vec{\mathbf{f}}_i$ are arbitrary. Thus selecting them parallel to the $\vec{\mathbf{p}}_i$ makes the eigenwrenches principal screws.

■

Theorem 41 *A compliant axis passes through all three centers E, S and C. Thus if a compliant axis exists then E, S and C are collinear. See Figure 3.8.*

Proof. Let the pair $\hat{W}_{fi}$, $\hat{T}_{\gamma i}$ form a compliant axis. By Theorem 38 it must pass through E. By Theorem 39, $\hat{W}_{fi}$ must also pass through S and $\hat{T}_{\gamma i}$ must also pass through C. But, since their axes

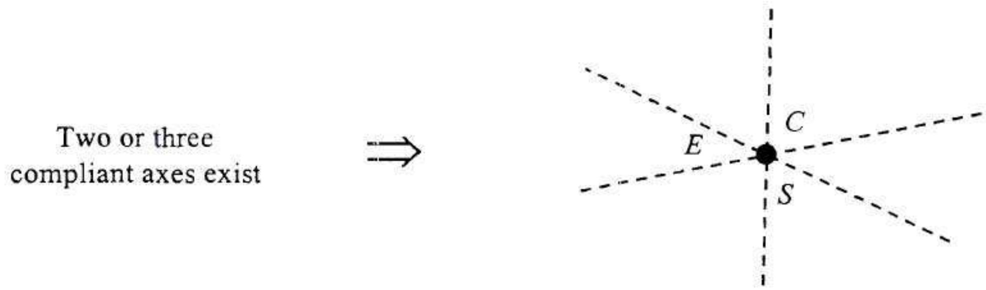


Figure 3.9: Existence of more than one compliant axes implies coalescing centers. Yet, the converse is not true.

are, by definition, collinear they pass through all centers E, S and C. Consequently, E, S and C must be collinear.

■

Since compliant axes are along orthogonal eigenwrenches (or eigentwists) then, as an immediate consequence of the above theorem,

Corollary 42 *If two compliant axes exist then E, S and C coalesce.*

Converse of Corollary 42 is not true. For example, the system

$$\hat{K} = \begin{bmatrix} 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 2 & 0 & 0 & 2 & 0 \\ 0 & 0 & 3 & 0 & 0 & -3 \\ 2 & 0 & 0 & 8 & 0 & 0 \\ 0 & 2 & 0 & 0 & 7 & 0 \\ 0 & 0 & -3 & 0 & 0 & 9 \end{bmatrix} \qquad \hat{C} = \begin{bmatrix} 2 & 0 & 0 & -\frac{1}{2} & 0 & 0 \\ 0 & \frac{7}{10} & 0 & 0 & -\frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 & \frac{1}{6} \\ -\frac{1}{2} & 0 & 0 & \frac{1}{4} & 0 & 0 \\ 0 & -\frac{1}{5} & 0 & 0 & \frac{1}{5} & 0 \\ 0 & 0 & \frac{1}{6} & 0 & 0 & \frac{1}{6} \end{bmatrix} \tag{3.83}$$

has the following structure

$$\hat{W}_f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{k}_f = \mathrm{diag}(1,2,3) \qquad \hat{T}_\gamma = \begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{a}_\gamma = \mathrm{diag}(\tfrac{1}{4}, \tfrac{1}{5}, \tfrac{1}{6}) \tag{3.84}$$

which shows that the eigenwrenches $\hat{W}_f$ and the eigentwists $\hat{T}_\gamma$ are also the principal screws with pitches $\mathbf{h}_f = -\mathbf{h}_\gamma = \mathrm{diag}(2, 1, -1)$. The given origin is the center of elasticity which coincides with the centers of stiffness and compliance since the submatrices are symmetric. So, E, S and C coalesce. However, there is no compliant axis since none of the principal screws are of zero pitch.

3.8 Examples

Results obtained in this chapter provide additional tools to understand and predict the properties of stiffness and compliance. Important practical applications are presented in Chapter 9. The following two application examples are presented as demonstrations. The first example uses the compliance matrix measured for a remote center of compliance (RCC) device. There is good agreement with the intended design and the existence of an RCC center is confirmed. It also illustrates Corollary 42 where the three centers coalesce. The second example concerns a dexterous hand grasping a rivet for insertion. The existence of a single compliant axis illustrates Theorem 41 where the centers become collinear.

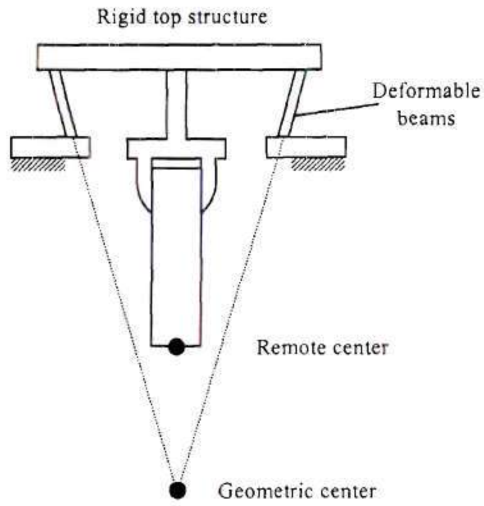


Figure 3.10: Simplified schematics of a practical RCC device. The remote center corresponds to the three combined centers.

3.8.1 Eigenscrew Structure of an RCC Device

Whitney [52] explains the development and properties of the RCC device. The calibration of the device is detailed in Drake [16] and a schematic is shown in Figure 3.10. Using a numerically controlled machine, three translations and three rotations about the x, y, and z axes were applied to the tip of the device at the expected compliance center. The resulting six wrenches were measured by a six degrees-of-freedom force/torque cell attached to the RCC base. The wrench and twist data are listed in [16] and used to form a 6×6 matrix equation to solve for the compliance as

$$\hat{C} = \left[\begin{array}{ccc|ccc} 0.0898 & 0.0058 & 0.0007 & -0.0292 & -0.1937 & -0.1955 \\ -0.0034 & 0.0924 & -0.0004 & 0.2255 & -0.0614 & 0.1955 \\ -0.0001 & 0.0004 & 0.0007 & 0.0047 & 0.0036 & 0.0916 \\ \hline -0.0374 & 0.1651 & 0.0023 & 2.3246 & -0.1127 & 0.7121 \\ -0.1655 & -0.0580 & 0.0007 & -0.0426 & 2.4148 & 2.4551 \\ -0.0122 & -0.2169 & -0.0260 & -1.6724 & 0.0816 & 130.9437 \end{array}\right] \tag{3.85}$$

where the units of force, length, and angle are newtons, millimeters, and milliradians. Note that the resulting matrix is asymmetric. The author ascribes this to measurement error. Using the symmetric part of $\hat{C}$ results in the following stationary values of linear stiffness k_{fi} and angular stiffness $k_{\gamma i}$,

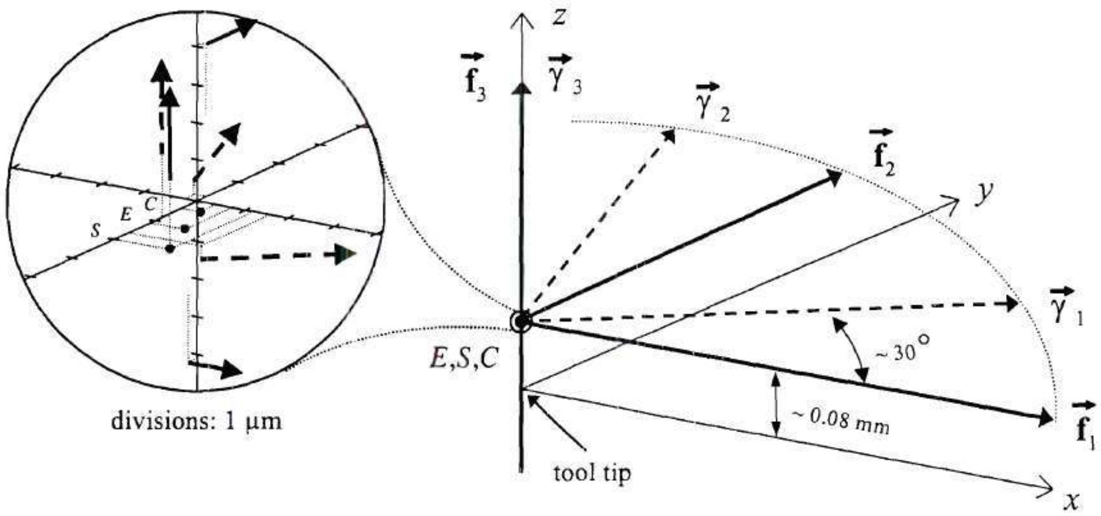


Figure 3.11: Eigenwrenches (solid) and eigentwists (dashed) of the RCC device approximately intersect at a point on the z-axis. A pair along z-axis forms a compliant axis and the three centers coalesce.

along with eigenwrench and eigentwist pitches,

$$\mathbf{k}_f = \text{diag}[\ 13.2 \quad 13.4 \quad 1498.1\ ] \qquad \mathbf{k}_\gamma = \text{diag}[\ 0.4387 \quad 0.4090 \quad 0.0076\ ] \tag{3.86}$$

$$\mathbf{h}_f = \text{diag}[\ 0.0167 \quad 0.0224 \quad -0.0002\ ] \qquad \mathbf{h}_\gamma = \text{diag}[\ 0.0025 \quad -0.0246 \quad 0.0003\ ] \tag{3.87}$$

The eigenscrews of the device come very close to having zero pitches (shown in millimeters), particularly, the third eigenwrench and eigentwist. In Figure 3.11, the eigenwrenches (solid lines) and eigentwists (dashed lines) are shown. They nearly intersect at a point that is *approximately* on the z axis, 0.0791 mm above the origin where the centers of elasticity, stiffness, and compliance virtually coalesce. There is a compliant axis in the z direction since a zero pitch eigenwrench and a zero pitch eigentwist are coincident along the z axis.

The remaining two eigenwrenches are nearly in the x and y directions with approximately zero pitches indicating that they are forces. The eigentwists are rotated about 30° and approximately have zero pitches indicating that they are pure rotations. Note that the two linear stiffnesses and two rotational stiffnesses in the xy-plane are nearly equal. If each pair had identical stiffnesses then there would be compliant axes in all xy directions and the centers would coalesce *exactly* on the z axis. While this is not the actual case, the RCC device comes very close to this goal, as the design intends. Axes through the centers but *not* in the xy-plane are *not* compliant axes since the stiffnesses in the z direction are different. For example a force in the xz-plane will cause a translation in a different direction of the xz-plane.

The insert in Figure 3.11 shows a blow-up view around the centers, at a scale of 1 μm. All centers are in xy-plane and collinear. The furthest eigenwrenches are approximately 0.0041 mm away from the centers plane. Considering the numerical and experimental inaccuracies, these suggest that the device practically behaves as an RCC for resolutions as small as 10 μm.

In this example only the symmetric part of the compliance matrix was used and the asymmetry was associated with measurement error. As a test, the asymmetric compliance matrix was inverted and then made symmetric yielding similar results. For example, the centers are located at 0.0795 mm (instead of 0.0791 mm) on the z axis and the maximum difference in the stiffness (z angular) is less than 4%. This would tend to support the claim that the asymmetry was primarily due to artifacts in the data.

3.8.2 Eigenscrew Analysis of a Robotic Hand

Stiffness properties are important for controlling dexterous robotic hands. Cutkosky and Kao [14] consider using two fingers from the Stanford/JPL hand to manipulate a 0.02 m (20 mm) rivet for insertion, see Figure 3.12. Each finger has a *soft* fingertip that transmits forces in three directions through the contact and a moment along the normal to the contact plane. This is sufficient to provide full force closure for the hand and rivet. Using representative compliance values measured for the cables, joints, links, and fingertips and including the effect of the servo system, the combined structural and servo stiffness matrix for the grasped rivet at the tip is given as

$$\hat{K} = \left[\begin{array}{ccc|ccc} 2490 & 0 & 0 & 0 & 258 & 0 \\ 0 & 28900 & 0 & 191 & 0 & 0 \\ 0 & 0 & 61610 & 0 & 0 & 0 \\ \hline 0 & 191 & 0 & 22 & 0 & 0 \\ 258 & 0 & 0 & 0 & 37 & 0 \\ 0 & 0 & 0 & 0 & 0 & 35 \end{array}\right] \tag{3.88}$$

where the units of force is newtons, length is meters, and angle is radians. The stationary values of stiffness are all distinct

$$\mathbf{k}_f = \mathrm{diag}(\ 2490 \quad 28900 \quad 61610\ ) \tag{3.89}$$

$$\mathbf{k}_\gamma = \mathrm{diag}(\ 10.268 \quad 20.738 \quad 35.000\ ) \tag{3.90}$$

Figure 3.13 shows the eigenwrenches (solid lines), eigentwists (dotted lines), and the three centers. All eigenwrenches and eigentwists are along the coordinate directions. They have zero pitches indicating pure forces and rotations respectively. The z axis contains a collinear eigenwrench-eigentwist pair that indicates a compliant axis. From the symmetry of Figure 3.12 this is reasonable. k_{f3} and $k_{\gamma 3}$ indicate that z-direction is the most stiff, both for translational and rotational cases. All three centers lie on the compliant axis as predicted in Theorem 41. Point E is 0.0485 m (48.5 mm) above the origin and S is at 0.0021 m (2.1 mm) and C is at 0.0671 m (67.1 mm). Since E is a geometric center it occupies a symmetrical position with respect to the eigenwrenches and eigentwists.

The eigenwrench $\vec{\mathbf{f}}_2$ (y-direction) and the eigentwist $\vec{\gamma}_1$ (x-direction) near the tip are useful in the insertion process:

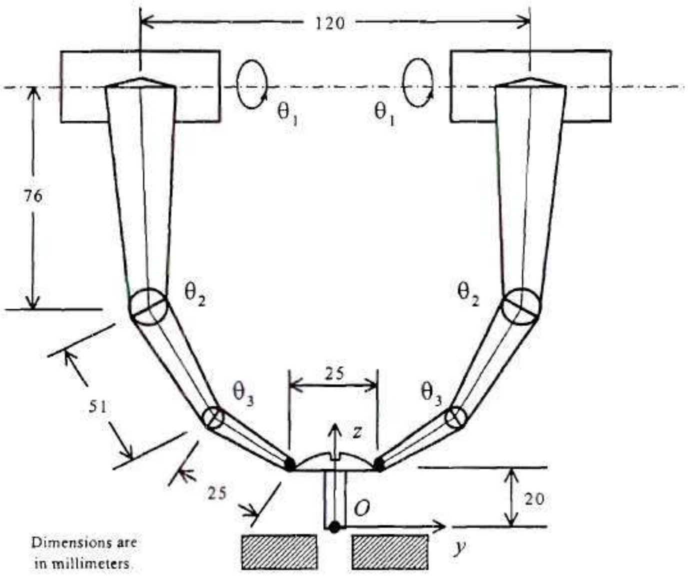


Figure 3.12: Simplified schematics of a dexterous robotic hand inserting a rivet, Cutkosky and Kao (1989).

- Both are zero pitch screws and intersect the z-axis at about 6.6 mm below the rivet's tip. An ideal RCC would have zero pitch eigenscrews through the tip.

- Due to the closeness to the tip of $\vec{f}_2$, an y-force at the tip creates an x translation with only a negligible rotation, which corrects any linear positional misalignment. For example, a 1 N y-force causes a 3.7×10^{-2} mm y-translation of the tip with less than $2° \times 10^{-2}$ of x-rotation. The translation amount, which may not be as large as desired in this case, can be adjusted by decreasing the y-direction linear stiffness.

- Similarly, due to the closeness to the tip of $\vec{\gamma}_1$, an x-moment creates a parallel rotation with negligible translation at the tip, which corrects any angular misalignment that would lead to jamming. For example, a 1 N-m x-moment causes about 2.8° x-rotation, accompanied by a

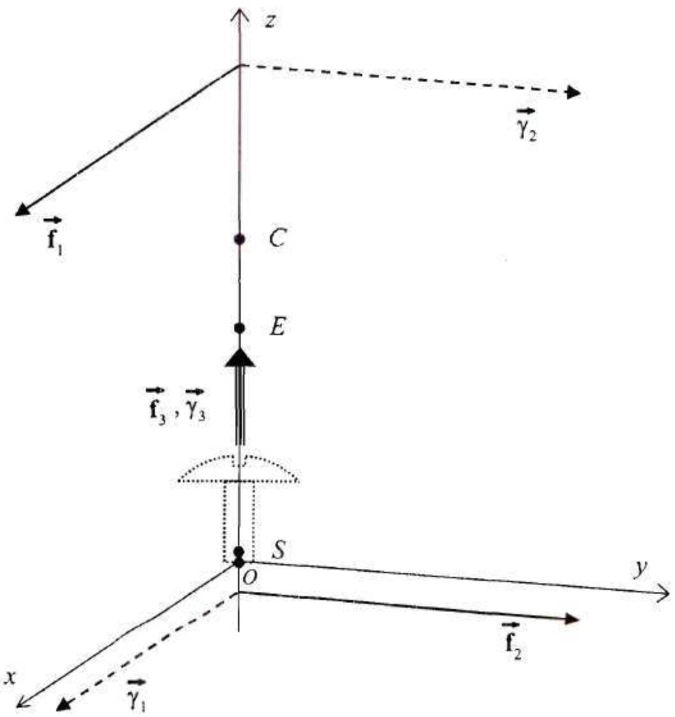


Figure 3.13: Eigenwrenches (solid) and eigentwists (dashed) for the dexterous hand. A pair along z form the only compliant axis. The three centers are collinear on z.

0.3 mm y-translation of the tip. The rotation is sufficiently large as a corrective action. The translation is small enough in order not to promote any jamming.

On the contrary, the eigenwrench $\vec{f}_1$ (x-direction) and the eigentwist $\vec{\gamma}_2$ (y-direction) are not as desirable:

- Although both are zero pitch screws which is necessary for an RCC behavior, they intersect the z-axis at about 0.1036 m (103.6 mm) above the tip, too far to be beneficial.

- An x-force at the tip generates a corrective parallel translation, but it also causes a significant rotation. For example, a 1 N x-force causes a 1.5 mm x-translation of the tip, but also causes a y-rotation of about $0.6°$, which may increase the possibility of jamming.

- Similarly, a y-moment generates a corrective rotation, but also causes a significant translation of the tip. For example, a 1 N-m y-moment causes a $5.6°$ of y-rotation, which is a good corrective action, but it also causes a 10.1 mm x-translation, which is very likely to lead to jamming or wedging.

The poor performance of the grasp due to $\vec{\mathbf{f}}_1$ and $\vec{\gamma}_2$ may partially be attributed to relatively low servo *stiffness* of the actuators at the revolute joints of θ_1-axis. To see this clearly, just assume that there are no actuators on θ_1-axis. This makes the fingers free to rotate about that axis. Consequently, even the smallest force at the tip will cause infinite rotation of the rivet about the θ_1-axis.

As a result, for better performance of the robotic hand, the actuators on θ_1-axis should be stiffened. This can be compensated by sufficiently loosening the soft contact, in order to avoid a too stiff grasp. For a better RCC behavior, the distances of the three centers to the tip of the rivet should be minimized. This can be achieved by adjusting the configuration variables θ_2 and θ_3, and the force applied to the contact. For off-line adjustment, different materials for the fingertips may be considered as to vary the friction properties.

CHAPTER IV

LINE-VECTOR EIGENVALUE PROBLEMS AND DECOMPOSITION

The free-vector eigenvalue problems introduced in Chapter 3 are related to the free-vector subspaces, namely translations and couples. Naturally, one asks whether a parallel development is possible using the line-vector subspaces, namely rotations and forces. In this chapter this question is answered completely. A new set of eigenvalue problems are obtained and shown to be complementary to the free-vector eigenvalue problems. Unlike the latter, the new eigenvalue problems are not unique since every point in space generates a distinct line-vector subspace. Similar to the free-vector eigenvalue problems, the new ones lead to decompositions of stiffness and compliance, distinct at every point.

To indicate the complementary nature, any quantity related to the new eigenvalue problem is named by adding the prefix **co-** to the corresponding name of the analogous quantity of the free-vector eigenvalue problem.

4.1 Definition

In parallel to the statements of the free-vector eigenvalue problems, consider the following. For any given point G:

1. Determine a wrench that causes a pure rotation (a line-vector) through G and parallel to the couple part at G.

2. Determine a twist that causes a pure force (a line-vector) through G and parallel to the translation part at G.

In equation form, these are

$$\hat{C}_G \begin{bmatrix} \vec{\mathbf{n}} \\ \vec{\mathbf{m}} \end{bmatrix}_G = a_m \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{m}} \end{bmatrix}_G \qquad \hat{K}_G \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{w}} \end{bmatrix}_G = k_t \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{0}} \end{bmatrix}_G \tag{4.1}$$

where $\hat{C} = \hat{K}^{-1}$ are the compliance and stiffness matrices with respect to G, $\vec{\mathbf{n}}$ is a unit force and $\vec{\mathbf{m}}$ is the accompanying couple of the wrench at G, $\vec{\mathbf{w}}$ is a unit rotation and $\vec{\mathbf{t}}$ is the accompanying translation of the twist at G, a_m is an **angular compliance**, and k_t is a **linear stiffness**. See Figure 4.1. If [L] and [F] respectively denote the length and force units, the units of the quantities in (4.1) are

$$\begin{bmatrix} \vec{\mathbf{n}} \\ \vec{\mathbf{m}} \end{bmatrix} \sim \begin{bmatrix} [1] \\ [\mathrm{L}] \end{bmatrix} \qquad a_m \sim [1/\mathrm{F}^*\mathrm{L}] \qquad \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{w}} \end{bmatrix} \sim \begin{bmatrix} [\mathrm{L}] \\ [1] \end{bmatrix} \qquad k_t \sim [\mathrm{F}/\mathrm{L}] \tag{4.2}$$

Now consider a mapping $\hat{\mathbf{L}}_G$ whose representations in ray- and axis-coordinates are

$$\hat{\mathbf{L}}_{rG} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \qquad \hat{\mathbf{L}}_{aG} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \tag{4.3}$$

Using $\hat{\mathbf{L}}_G$, (4.1) can be expressed as a pair of generalized eigenvalue problems as

$$\hat{C}\hat{W}_m = a_m \hat{\mathbf{L}}_r \hat{W}_m \qquad \hat{K}\hat{T}_t = k_t \hat{\mathbf{L}}_a \hat{T}_t \tag{4.4}$$

where all quantities are with respect to G. The eigenvalue a_m is called the **angular co-compliance**, and k_t is called the **linear co-stiffness**.

Definition 43 *The generalized eigenvalue problems (4.4) are called the* ***line-vector eigenvalue problems*** *at G for stiffness and compliance.*

1. *The wrench $\hat{W}_m$ in (4.4) that causes only a pure rotation through G parallel to the couple part at G is called a* ***co-eigenwrench*** *(Figure 4.1).*

2. *The twist $\hat{T}_t$ in (4.4) that causes only a pure force through G parallel to the translation part at G is called a* ***co-eigentwist*** *(Figure 4.1).*

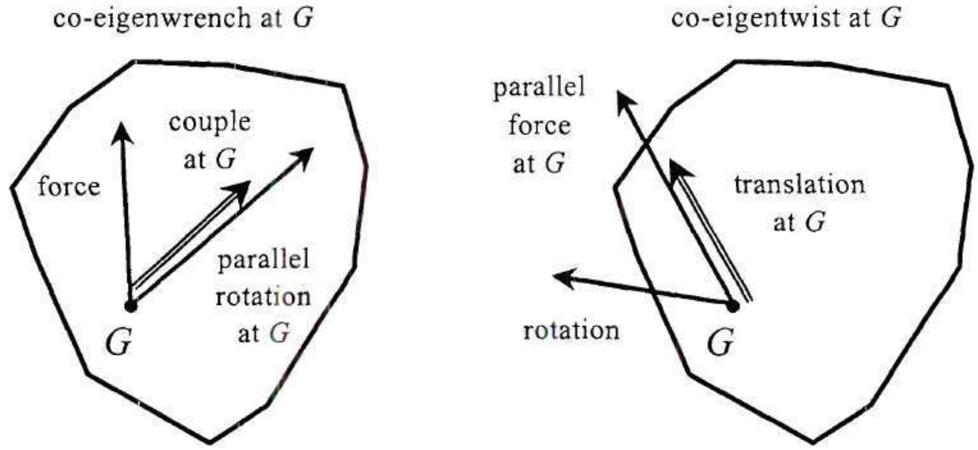


Figure 4.1: A co-eigenwrench produces a rotation through G, parallel to its couple part at G. A co-eigentwist produces a force through G, parallel to its translation part at G.

The line-vector eigenvalue problems are also obtainable by minimizing the potential energy of the system. To see this, let $\hat{W}$ and $\hat{T}$ denote a wrench and a twist, respectively, and Φ be the potential energy. The minimization problems are given at G as

minimize	constraint
$\Phi_W = \frac{1}{2}\hat{W}^T\hat{C}\hat{W}$	$\hat{W}^T\hat{\mathrm{L}}_r\hat{W} = 1$
$\Phi_T = \frac{1}{2}\hat{T}^T\hat{K}\hat{T}$	$\hat{T}^T\hat{\mathrm{L}}_a\hat{T} = 1$

(4.5)

Solving the minimization problems yields the line-vector eigenvalue problems (4.4). For the wrenches and the twists the constraints can be written explicitly as

$$\vec{\mathbf{m}}_G^T\vec{\mathbf{m}}_G = 1 \qquad \vec{\mathbf{t}}_G^T\vec{\mathbf{t}}_G = 1 \tag{4.6}$$

If the equations in (4.1) are inverted, one gets,

$$\hat{K}_G\begin{bmatrix}\vec{\mathbf{0}}\\ \vec{\mathbf{m}}\end{bmatrix}_G = k_m\begin{bmatrix}\vec{\mathbf{n}}\\ \vec{\mathbf{m}}\end{bmatrix}_G \qquad \hat{C}_G\begin{bmatrix}\vec{\mathbf{t}}\\ \vec{\mathbf{0}}\end{bmatrix}_G = a_t\begin{bmatrix}\vec{\mathbf{t}}\\ \vec{\mathbf{w}}\end{bmatrix}_G \tag{4.7}$$

which show that the co-eigenwrenches and co-eigentwists are the images of corresponding line-vectors through G. Recall from Chapter 2 that $V_{l/G}$ and $V^*_{l/G}$ are respectively the 3-dimensional rotation and force subspaces generated by G, namely the rotation and force bundles at G. The stiffness $\hat{K}$ and the compliance $\hat{C}$ respectively act on $V_{l/G}$ and $V^*_{l/G}$. The action of $\hat{K}$ on the rotation subspace $V_{l/G}$ results in a set of wrenches $V^*_{cW/G} \subset V^*$. Similarly, the action of $\hat{C}$ on the force subspace $V^*_{l/G}$ results in a set twists $V_{cT/G} \subset V$. In short, $\hat{K}$ and $\hat{C}$ induce the following mappings.

$$\hat{K}(V_{l/G}) : V_{l/G} \to V^*_{cW/G} \subset V^* \qquad \hat{C}(V^*_{l/G}) : V^*_{l/G} \to V_{cT/G} \subset V \tag{4.8}$$

If the mappings are non-singular then it follows, by linearity, that

Theorem 44 *The image of the rotation bundle at G, $V^*_{cW/G}$, and the image of the force bundle at G, $V_{cT/G}$ are 3-systems of screws.*

Definition 45 *$V^*_{cW/G}$ is called the* ***co-eigenwrench 3-system*** *generated by G, and $V_{cT/G}$ is called the* ***co-eigentwist 3-system*** *generated by G.*

By transforming the spatial quantities to an arbitrary origin O, the line-vector eigenvalue problems with generator G can be stated as

$$\hat{C}_O \begin{bmatrix} \vec{\mathbf{n}} \\ \overrightarrow{OG} \times \vec{\mathbf{n}} + \vec{\mathbf{m}} \end{bmatrix} = a_m \begin{bmatrix} \overrightarrow{OG} \times \vec{\mathbf{m}} \\ \vec{\mathbf{m}} \end{bmatrix} \tag{4.9}$$

$$\hat{K}_O \begin{bmatrix} \overrightarrow{OG} \times \vec{\mathbf{w}} + \vec{\mathbf{t}} \\ \vec{\mathbf{w}} \end{bmatrix} = k_t \begin{bmatrix} \vec{\mathbf{t}} \\ \overrightarrow{OG} \times \vec{\mathbf{t}} \end{bmatrix} \tag{4.10}$$

The above eigenvalue problems are equivalent to (4.1). This is easily demonstrated by taking $O = G$ in (4.10). For analytical simplicity, it is assumed throughout this study that the problems are specified as in (4.1) and the subscripts are dropped with the understanding that the *matrices are specified at the generator point.* There are as many different solutions for a_m and k_t as there are generators. So, these too depend on the generator. However, again it is preferable not to use any markers for them since the context makes it sufficiently clear as to which solution is meant.

4.2 Reciprocity

Similar to the free-vector eigenvalue problems, the co-eigenwrenches and co-eigentwists can be shown to be reciprocal to each other. Let any co-eigenwrench and co-eigentwist be given as

$$\hat{W}_{mi} = \begin{bmatrix} \vec{\mathbf{n}}_i \\ \vec{\mathbf{m}}_i \end{bmatrix} \qquad \hat{T}_{tj} = \begin{bmatrix} \vec{\mathbf{t}}_j \\ \vec{\mathbf{w}}_j \end{bmatrix} \tag{4.11}$$

Consider the scalar product

$$\hat{W}_{mi}^T \hat{T}_{tj} = \left[a_{mi}\hat{K} \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{m}}_i \end{bmatrix} \right]^T \left[k_{tj}\hat{C} \begin{bmatrix} \vec{\mathbf{t}}_j \\ \vec{\mathbf{0}} \end{bmatrix} \right] \tag{4.12}$$

$$= a_{mi}k_{tj} \begin{bmatrix} \vec{\mathbf{0}}^T & \vec{\mathbf{m}}_i^T \end{bmatrix} \hat{K}^T \hat{C} \begin{bmatrix} \vec{\mathbf{t}}_j \\ \vec{\mathbf{0}} \end{bmatrix} \tag{4.13}$$

$$= a_{mi}k_{tj} \begin{bmatrix} \vec{\mathbf{0}}^T & \vec{\mathbf{m}}_i^T \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}}_j \\ \vec{\mathbf{0}} \end{bmatrix} = 0 \tag{4.14}$$

for all i, j. This proves the following theorem.

Theorem 46 *For any given generator G, the co-eigenwrench 3-system and the co-eigentwist 3-system are the reciprocal spaces of each other.*

4.3 Uniqueness and Existence

Consider the 3×3 partitions of the stiffness and compliance matrices, respectively,

$$\hat{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \qquad \hat{C} = \begin{bmatrix} \mathbf{D} & \mathbf{E}^T \\ \mathbf{E} & \mathbf{F} \end{bmatrix} \tag{4.15}$$

The line-vector eigenvalue problems with generator as the origin are

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \vec{0} \\ \vec{\mathbf{m}} \end{bmatrix} = k_m \begin{bmatrix} \vec{\mathbf{n}} \\ \vec{\mathbf{m}} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{D} & \mathbf{E}^T \\ \mathbf{E} & \mathbf{F} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{0} \end{bmatrix} = a_t \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{w}} \end{bmatrix} \tag{4.16}$$

resulting in

$$\mathbf{B}\vec{\mathbf{m}} = k_m \vec{\mathbf{n}} \qquad \mathbf{C}\vec{\mathbf{m}} = k_m \vec{\mathbf{m}} \tag{4.17}$$

$$\mathbf{D}\vec{\mathbf{t}} = a_t \vec{\mathbf{t}} \qquad \mathbf{E}\vec{\mathbf{t}} = a_t \vec{\mathbf{w}} \tag{4.18}$$

It is seen that

i) k_m and $\vec{\mathbf{m}}$ are the classical eigenvalues and eigenvectors of $\mathbf{C}$,

ii) a_t and $\vec{\mathbf{t}}$ are the classical eigenvalues and eigenvectors of $\mathbf{D}$.

The symmetry of $\mathbf{C}$ and $\mathbf{D}$ ensures the existence of three real eigenvalues and eigenvectors, and the orthogonality of $\{\vec{\mathbf{m}}_i\}$, $\{\vec{\mathbf{t}}_i\}$. The orthogonality results in the linear independence of the corresponding set of co-eigenscrews. This proves the following theorem.

Theorem 47 *The line-vector eigenvalue problems have unique solutions for every generator in terms of three independent co-eigenwrenches and three independent co-eigentwists.*

Existence of three linearly independent co-eigenscrews means that each set forms a basis for the corresponding co-eigenscrew subspace, $V_{cT/G}$ or $V^*_{cW/G}$. Although, the set of vectors $\{\vec{\mathbf{m}}_i\}$ and $\{\vec{\mathbf{t}}_i\}$ are orthogonal, the sets obtained by $\{\vec{\mathbf{n}}_i\} = \{a_{mi}\mathbf{B}\vec{\mathbf{m}}_i\}$ and $\{\vec{\mathbf{w}}_i\} = \{k_{ti}\mathbf{E}\vec{\mathbf{t}}_i\}$ are not, in general. The vectors $\vec{\mathbf{n}}_i$ and $\vec{\mathbf{w}}_i$ give the directions of co-eigenwrenches and co-eigentwists, respectively. Thus, in contrast to the free-vector problems, in general the co-eigenscrews do not have orthogonal directions.

4.4 Line-Vector Decomposition of Stiffness and Compliance

Each set of eigenvectors of $\mathbf{C}$ and $\mathbf{D}$ forms an orthogonal set of 3-vectors. With this property (4.17) and (4.18) can be solved for the 3×3 matrices as follows.

$$\mathbf{B}\,[\vec{\mathbf{m}}_1, \vec{\mathbf{m}}_2, \vec{\mathbf{m}}_3] = [\vec{\mathbf{n}}_1, \vec{\mathbf{n}}_2, \vec{\mathbf{n}}_3] \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \quad (4.19)$$

$$\mathbf{Bm} = \mathbf{nk}_m \quad \text{, etc.} \quad (4.20)$$

so that

$$\mathbf{B} = \mathbf{nk}_m\mathbf{m}^{-1} \qquad \mathbf{C} = \mathbf{mk}_m\mathbf{m}^{-1} \quad (4.21)$$

$$\mathbf{D} = \mathbf{ta}_t\mathbf{t}^{-1} \qquad \mathbf{E} = \mathbf{wa}_t\mathbf{t}^{-1} \quad (4.22)$$

where $\mathbf{n}, \mathbf{m}, \mathbf{t}, \mathbf{w}$ are formed using the corresponding vectors as columns, and, $\mathbf{k}_m$ and $\mathbf{a}_t$ are diagonal matrices formed from the corresponding eigenvalues.

Let the co-eigenscrews be normalized with respect to the units of the line-vector components of the screws (force and rotation). Then, $\mathbf{n}$ and $\mathbf{w}$ have units of [1], and, the units of $\mathbf{m}$ and $\mathbf{t}$ have units of length [L]. However, unlike in the free-vector decomposition, the inverted matrices in (4.21) and (4.22) have units of inverse length, $[\mathrm{L}^{-1}]$. Therefore, $\mathbf{m}^{-1}$ and $\mathbf{m}^T$ have different physical units although they are numerically equal because of orthogonality. This leads to confusion if $\mathbf{m}^T$ is used in equations (4.21) and (4.22) in place of $\mathbf{m}^{-1}$. Such a problem is not encountered in the free-vector eigenvalue problem since only the normalized quantities are inverted there. The difficulty really originates from incomplete formulations of the eigenvalue problems. The next section demonstrates a geometrically sensible resolution of this problem.

4.4.1 Modified Forms the Line-Vector Eigenvalue Problems

The problem of units can be rectified if the eigenvalue problems are proposed as follows.

$$\hat{K}\begin{bmatrix}\vec{\mathbf{0}}\\ \\ \vec{\mathbf{m}}'\end{bmatrix} = k_m'\begin{bmatrix}\vec{\mathbf{n}}'\\ \\ l_m\vec{\mathbf{m}}'\end{bmatrix} \qquad \hat{C}\begin{bmatrix}\vec{\mathbf{t}}'\\ \\ \vec{\mathbf{0}}\end{bmatrix} = a_t'\begin{bmatrix}l_t\vec{\mathbf{t}}'\\ \\ \vec{\mathbf{w}}'\end{bmatrix} \tag{4.23}$$

where all vectors are unitless, and, l_m and l_t are non-zero scaling factors with length units. Now, all the screws on the right of equations in (4.23) have units $\begin{bmatrix}[1] & [L]\end{bmatrix}^T$ for wrenches and $\begin{bmatrix}[L] & [1]\end{bmatrix}^T$ for twists. The screws on the left of equations in (4.23) also have the same units. For example, on the left of equations in (4.23), $\vec{\mathbf{m}}'$ is a unit rotation and $\vec{\mathbf{t}}'$ is a unit force. If the scaling factors l_m and l_t are taken as unity then the resulting eigenvalues and vectors are numerically equal to what would be obtained from the unmodified forms. Using the submatrices of the stiffness and compliance in (4.23), the following equations result.

$$\mathbf{B}\vec{\mathbf{m}}' \;=\; k_m'\vec{\mathbf{n}}' \qquad \mathbf{C}\vec{\mathbf{m}}' = k_m' l_m \vec{\mathbf{m}}' \tag{4.24}$$

$$\mathbf{D}\vec{\mathbf{t}}' \;=\; a_t' l_t \vec{\mathbf{t}} \qquad \mathbf{E}\vec{\mathbf{t}}' = a_t'\vec{\mathbf{w}}' \tag{4.25}$$

Comparing (4.24) and (4.25) to (4.17) and (4.18) one shows that

$$k_m \;=\; k_m' l_m \qquad a_t = a_t' l_t \tag{4.26}$$

$$\vec{\mathbf{m}} \;\equiv\; \vec{\mathbf{m}}' \qquad \vec{\mathbf{t}} \equiv \vec{\mathbf{t}}' \qquad \text{(numerically)} \tag{4.27}$$

The units of both $k_m' = (a_m')^{-1}$ and $k_t' = (a_t')^{-1}$ become [F].

As was done for the unmodified eigenvalue equations, the 3×3 submatrices of the stiffness and compliance can be solved as

$$\mathbf{B} \;=\; \mathbf{n}'\mathbf{k}_m'(\mathbf{m}')^T \qquad \mathbf{C} = \mathbf{m}'\mathbf{k}_m'\mathbf{l}_m(\mathbf{m}')^T \tag{4.28}$$

$$\mathbf{D} \;=\; \mathbf{t}'\mathbf{a}_t'\mathbf{l}_t(\mathbf{t}')^T \qquad \mathbf{E} = \mathbf{w}'\mathbf{a}_t'(\mathbf{t}')^T \tag{4.29}$$

where, as before, $\mathbf{n}', \mathbf{m}', \mathbf{t}', \mathbf{w}'$ are 3×3 matrices with columns corresponding to respective 3×1 vectors obtained from the solutions to (4.24) and (4.25). $\mathbf{k}_m', \mathbf{l}_m, \mathbf{a}_t', \mathbf{l}_t$ are 3×3 diagonal matrices with respective entries. The difference is that, now, $\mathbf{m}'$ and $\mathbf{t}'$ are unitless, therefore $(\mathbf{m}')^{-1} = (\mathbf{m}')^T$ and $(\mathbf{t}')^{-1} = (\mathbf{t}')^T$ are consistent both numerically and physically.

To solve for other submatrices the reciprocity relation (4.14) is used. Let $\hat{W}_{mi}$ and $\hat{T}_{ti}$ be the co-eigenwrenches and the co-eigentwists, respectively. The following is a compact form of the reciprocity relation in terms of 3×3 matrices.

$$\left[\hat{W}_{m1}, \hat{W}_{m2}, \hat{W}_{m3}\right]^T \left[\hat{T}_{t1}, \hat{T}_{t2}, \hat{T}_{t3}\right] = \mathbf{0} \tag{4.30}$$

$$\begin{bmatrix} \mathbf{n}' \\ \mathbf{m}'\mathbf{l}_m \end{bmatrix}^T \begin{bmatrix} \mathbf{t}'\mathbf{l}_t \\ \mathbf{w}' \end{bmatrix} = \mathbf{0} \tag{4.31}$$

$$(\mathbf{n}')^T\mathbf{t}'\mathbf{l}_t + \mathbf{l}_m(\mathbf{m}')^T\mathbf{w}' = \mathbf{0} \tag{4.32}$$

Using the condition $\hat{K}\hat{C} = \hat{I}$ gives $\mathbf{AD} + \mathbf{BE} = \mathbf{I}$ and $\mathbf{B}^T\mathbf{E}^T + \mathbf{CF} = \mathbf{I}$, which can be solved for $\mathbf{A}$ and $\mathbf{F}$ in terms of others. The procedure for $\mathbf{A}$ is as follows. From $\mathbf{AD} + \mathbf{BE} = \mathbf{I}$, (4.28) and (4.29)

$$\mathbf{A} = \mathbf{D}^{-1} - \mathbf{BED}^{-1} \tag{4.33}$$

$$= \mathbf{t}'\mathbf{k}'_t\mathbf{l}_t^{-1}(\mathbf{t}')^T - \mathbf{n}'\mathbf{k}'_m(\mathbf{m}')^T\mathbf{w}'\mathbf{a}'_t(\mathbf{t}')^T\mathbf{t}'\mathbf{k}'_t\mathbf{l}_t^{-1}(\mathbf{t}')^T \tag{4.34}$$

$$= \mathbf{t}'\mathbf{k}'_t\mathbf{l}_t^{-1}(\mathbf{t}')^T - \mathbf{n}'\mathbf{k}'_m(\mathbf{m}')^T\mathbf{w}'\mathbf{l}_t^{-1}(\mathbf{t}')^T \tag{4.35}$$

But, from (4.32) $(\mathbf{m}')^T\mathbf{w}' = -\mathbf{l}_m^{-1}(\mathbf{n}')^T\mathbf{t}'\mathbf{l}_t$, so (4.35) becomes

$$\mathbf{A} = \mathbf{t}'\mathbf{k}'_t\mathbf{l}_t^{-1}(\mathbf{t}')^T + \mathbf{n}'\mathbf{k}'_m\mathbf{l}_m^{-1}(\mathbf{n}')^T\mathbf{t}'\mathbf{l}_t\mathbf{l}_t^{-1}(\mathbf{t}')^T \tag{4.36}$$

$$\mathbf{A} = \mathbf{t}'\mathbf{k}'_t\mathbf{l}_t^{-1}(\mathbf{t}')^T + \mathbf{n}'\mathbf{k}'_m\mathbf{l}_m^{-1}(\mathbf{n}')^T \tag{4.37}$$

Note that the units of $\mathbf{k}'_t\mathbf{l}_t^{-1}$ and $\mathbf{k}'_m\mathbf{l}_m^{-1}$ are [F/L] as expected. Similar procedure for $\mathbf{F}$ yields

$$\mathbf{F} = \mathbf{m}'\mathbf{a}'_m\mathbf{l}_m^{-1}(\mathbf{m}')^T + \mathbf{w}'\mathbf{a}'_t\mathbf{l}_t^{-1}(\mathbf{w}')^T \tag{4.38}$$

The units of $\mathbf{a}'_m\mathbf{l}_m^{-1}$ and $\mathbf{a}'_t\mathbf{l}_t^{-1}$ are $[\mathrm{F*L}]^{-1}$ as expected. Reconstruction of $\hat{K}$ and $\hat{C}$ results in the following decompositions.

$$\hat{K} = \begin{bmatrix} \mathbf{n}' & \mathbf{t}' \\ \mathbf{m}'\mathbf{l}_m & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{k}'_m\mathbf{l}_m^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{k}'_t\mathbf{l}_t^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{n}' & \mathbf{t}' \\ \mathbf{m}'\mathbf{l}_m & \mathbf{0} \end{bmatrix}^T \tag{4.39}$$

$$\hat{C} = \begin{bmatrix} \mathbf{0} & \mathbf{t}'\mathbf{l}_t \\ \mathbf{m}' & \mathbf{w}' \end{bmatrix} \begin{bmatrix} \mathbf{a}'_m\mathbf{l}_m^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{a}'_t\mathbf{l}_t^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{t}'\mathbf{l}_t \\ \mathbf{m}' & \mathbf{w}' \end{bmatrix}^T \tag{4.40}$$

It is interesting to note that the constitutive matrix in the middle of (4.39) has units of [F/L], where that of (4.40) has units of $[\text{F*L}]^{-1}$. The form (4.39) is shown to be essential in the synthesis of stiffnesses with all line springs which have stiffness coefficients with units [F/L], see Chapter ??.

If the scaling factors are taken as unity, the decompositions in (4.39) and (4.40) become numerically equivalent to what would be obtained from the unmodified forms. That is, for $\mathbf{l}_m = \mathbf{l}_t = \mathbf{I}$ [L],

$$\hat{K} = \begin{bmatrix} \mathbf{n} & \mathbf{t} \\ \mathbf{m} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{k}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_t \end{bmatrix} \begin{bmatrix} \mathbf{n} & \mathbf{t} \\ \mathbf{m} & \mathbf{0} \end{bmatrix}^T \tag{4.41}$$

$$\hat{C} = \begin{bmatrix} \mathbf{0} & \mathbf{t} \\ \mathbf{m} & \mathbf{w} \end{bmatrix} \begin{bmatrix} \mathbf{a}_m & \mathbf{0} \\ \mathbf{0} & \mathbf{a}_t \end{bmatrix} \begin{bmatrix} \mathbf{0} & \mathbf{t} \\ \mathbf{m} & \mathbf{w} \end{bmatrix}^T \tag{4.42}$$

But, what do the scaling factors do? Do they change the co-eigenscrews? To answer this it suffices to relate the vector $\vec{\mathbf{n}}'$ to $\vec{\mathbf{n}}$. It is already known that numerically $\vec{\mathbf{m}}' \equiv \vec{\mathbf{m}}$, since they are the unit eigenvectors of the same matrix. Let $\vec{\mathbf{m}} = \alpha\vec{\mathbf{m}}'$, where $\alpha = 1$ [L]. Recall that $k_m = k'_m l_m$. Then, $\mathbf{B}\vec{\mathbf{m}}' = k'_m\vec{\mathbf{n}}'$ becomes $\frac{1}{\alpha}\mathbf{B}\vec{\mathbf{m}} = \frac{k_m}{l_m}\vec{\mathbf{n}}'$. But, from the unmodified form, $\mathbf{B}\vec{\mathbf{m}} = k_m\vec{\mathbf{n}}$. So, $\vec{\mathbf{n}}' = \frac{l_m}{\alpha}\vec{\mathbf{n}}$. Therefore, the unmodified co-eigenwrenches are related to the modified forms as

$$\begin{bmatrix} \vec{\mathbf{n}}' \\ l_m\vec{\mathbf{m}}' \end{bmatrix} = \begin{bmatrix} \frac{l_m}{\alpha}\vec{\mathbf{n}} \\ \frac{l_m}{\alpha}\vec{\mathbf{m}} \end{bmatrix} = \frac{l_m}{\alpha}\begin{bmatrix} \vec{\mathbf{n}} \\ \vec{\mathbf{m}} \end{bmatrix} = \beta\begin{bmatrix} \vec{\mathbf{n}} \\ \vec{\mathbf{m}} \end{bmatrix} \tag{4.43}$$

where β is an arbitrary non-zero real number without units. This shows that co-eigenwrenches of the unmodified and modified forms are scalar multiples of each other. The same is true for the co-eigentwists. As a result, introduction of non-zero scaling factors does not essentially destroy the geometric content. This is analogous to what happens in classical eigenvalue problems where one can use any scalar multiple of a preferred eigenvector (usually the one with unit magnitude) as the eigenvector.

For simplicity, an explicit distinction between unmodified and modified quantities is not made in the rest of this treatment. Either it is implicitly assumed that $l_{mi} = l_{ti} = 1$ [L] or l_{mi}, l_{ti} are explicitly used indicating that the modified forms are being used.

A very similar modification can be made to the free-vector eigenvalue problems. However, the results become identical to the unmodified forms, as one can verify easily. Therefore, the free-vector eigenvalue problems are used in their unmodified forms.

4.5 Stiffness and Compliance Ellipsoids

The two three-system of screws corresponding to the co-eigenscrews are formed by all possible linear combinations of the basis elements. These can be given as

$$\hat{W}_m = \sum_i \mu_i \hat{W}_{mi} \qquad \hat{T}_t = \sum_j \nu_j \hat{T}_{tj} \tag{4.44}$$

for all $\mu_i, \nu_j \in R$. It is straightforward to show that any pure rotation of the body is due to an element of the co-eigenwrench three-system, $\hat{W}_m$, and similarly, any pure force reaction of the elastic connection is due to an element of the co-eigentwist three-system, $\hat{T}_t$. The corresponding pure rotation and pure force are given by

$$\hat{T}_m = \sum_i \mu_i \begin{bmatrix} \vec{0} \\ a_{mi}\vec{\mathbf{m}}_i \end{bmatrix} \qquad \hat{W}_t = \sum_j \nu_j \begin{bmatrix} k_{tj}\vec{\mathbf{t}}_j \\ \vec{0} \end{bmatrix} \tag{4.45}$$

As an example, the co-eigenwrench system is examined. The couple component in (4.44) can be given by

$$\vec{\mathbf{m}} = \sum_i \mu_i \vec{\mathbf{m}}_i = \mathbf{m}\bar{\mu} \tag{4.46}$$

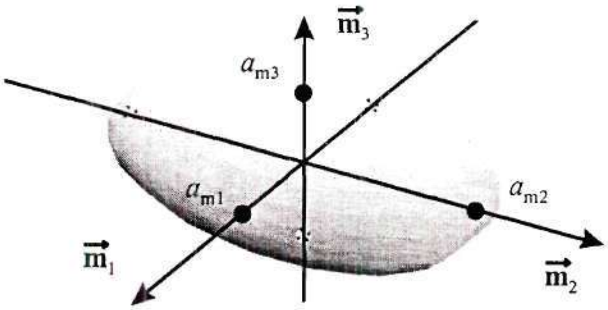


Figure 4.2: Angular co-compliance ellipsoid.

where $\bar{\mu}$ is the column matrix with entries μ_i. The resulting rotation in (4.45) becomes

$$\vec{\mathbf{w}} = \sum_i \mu_i a_{mi} \vec{\mathbf{m}}_i = \mathbf{m}\mathbf{a}_m \bar{\mu} \tag{4.47}$$

Eliminating $\bar{\mu}$ between (4.46) and (4.47) one gets

$$\vec{\mathbf{m}} = \mathbf{m}\mathbf{a}_m^{-1}\mathbf{m}^T \vec{\mathbf{w}} \tag{4.48}$$

which could also be obtained easily from (4.39). To find the reaction to a unit couple one applies the constraint $\vec{\mathbf{m}}^T\vec{\mathbf{m}} = 1$ which, after aligning the coordinates parallel to $\vec{\mathbf{m}}_i$, yields from (4.48) the following equation for the components of the rotation $\vec{\mathbf{w}}$ along $\vec{\mathbf{m}}_i$.

$$\left(\frac{w_1}{a_{m1}}\right)^2 + \left(\frac{w_2}{a_{m2}}\right)^2 + \left(\frac{w_3}{a_{m3}}\right)^2 = 1 \tag{4.49}$$

This is called the **angular co-compliance ellipsoid** and depicted in Figure 4.2.

A radial vector to the surface represents a pure rotation $\vec{\mathbf{w}}$ and is due to an element of the co-eigenwrench system. Only along the axes of the ellipsoid is the rotation parallel to the couple component. That is, the axes of the ellipsoid are parallel to the co-eigenwrenches. An interpretation is that the radial distance to the surface represents an equivalent angular compliance relating the projection of the resulting rotation in the direction of the couple to the couple, i.e.,

$$a_m = \frac{\vec{\mathbf{m}}^T\vec{\mathbf{w}}}{\vec{\mathbf{m}}^T\vec{\mathbf{m}}} \tag{4.50}$$

The analysis of the co-eigentwist system is similar. Any pure force reaction of the system is due to an element of the co-eigentwist system. So, if $\vec{\mathbf{t}}$ is the translational component of a twist in the co-eigentwist three-system and $\vec{\mathbf{n}}$ is the corresponding pure force reaction, the following is obtained.

$$\vec{\mathbf{t}} = \mathbf{t}\mathbf{k}_t^{-1}\mathbf{t}^T\vec{\mathbf{n}} \tag{4.51}$$

which could also be obtained from (4.40). To find the reaction to a unit translation one applies the constraint $\vec{\mathbf{t}}^T\vec{\mathbf{t}} = 1$ which, after aligning the coordinates parallel to $\vec{\mathbf{t}}_i$, yields from (4.51) the following equation for the components of the force $\vec{\mathbf{n}}$ along $\vec{\mathbf{t}}_i$.

$$\left(\frac{n_1}{k_{t1}}\right)^2 + \left(\frac{n_2}{k_{t2}}\right)^2 + \left(\frac{n_3}{k_{t3}}\right)^2 = 1 \tag{4.52}$$

This is called the **linear co-stiffness ellipsoid**.

As before, a radial vector to the surface represents a pure force reaction, and is due to an element of the co-eigentwist system. Only along the ellipsoid axes is the force parallel to the translation. The radial distance to the surface is an equivalent linear stiffness relating the projection of the force in the direction of the translation to the translation. i.e.

$$k_t = \frac{\vec{\mathbf{t}}^T\vec{\mathbf{n}}}{\vec{\mathbf{t}}^T\vec{\mathbf{t}}} \tag{4.53}$$

The free-vector eigenvalue problem identifies the linear compliance and angular stiffness ellipsoids, see Lipkin and Patterson [30], whereas the line-vector eigenvalue problem identifies the angular co-compliance and linear co-stiffness ellipsoids as complementary results.

CHAPTER V

CO-CENTER OF ELASTICITY

The free-vector eigenvalue problems lead to the definition of the center of elasticity, a unique geometric center. In Chapter 3, the relations between the centers of elasticity, stiffness and compliance are presented. The complaint axes relate these centers in a geometric manner. In Chapter 4, the line-vector eigenvalue problems are proposed as complementary problems. However, the analogy seems incomplete since no concept of a center complementary to the center of elasticity has been presented. This chapter focuses on that question.

First, the center of elasticity is redefined independently from the free-vector eigenvalue problems. This enables one to analogously define a new, complementary center. Then, the new center is shown to be non-unique in general. Both general and special cases are investigated. The results lead to a better classification of compliant systems.

5.1 Center of Elasticity Redefined

For eigenscrew 3-systems, there exists a unique point, the center of elasticity E, which is the center of both eigentwist and eigenwrench 3-systems. A distinguishing property of this center is that a set of three orthogonally intersecting screws with stationary pitches pass through the point and span the eigenscrew 3-systems, namely the principal screws. One set of principal screws spans the eigentwist system, the other set with opposite pitches spans the eigenwrench system. For an eigenscrew 3-system, there exists *no* other point through which another set of three mutually orthogonal and intersecting screws pass. This property is related to the submatrices of stiffness and compliance as in the following theorem.

Theorem 48 *For the eigentwist and eigenwrench systems, three mutually orthogonal screws pass through a point P if and only if $\mathbf{A}^{-1}\mathbf{B} = -\mathbf{E}^T\mathbf{F}^{-1}$ is symmetric at P.*

Proof. Consider the eigentwist case. Assume there exists an element of the eigentwist system passing through P. Then, at P

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} h\vec{\mathbf{a}} \\ \vec{\mathbf{a}} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{\kappa} \end{bmatrix} \tag{5.1}$$

where $[\ h\vec{\mathbf{a}}^T \quad \vec{\mathbf{a}}^T\ ]^T$ is an element of the eigentwist 3-system. The first row of (5.1) can be written as

$$h\mathbf{A}\vec{\mathbf{a}} = -\mathbf{B}\vec{\mathbf{a}} \tag{5.2}$$

This is a generalized eigenvalue problem and can also be given as

$$h\vec{\mathbf{a}} = -\mathbf{A}^{-1}\mathbf{B}\vec{\mathbf{a}} \tag{5.3}$$

If three mutually orthogonal screws pass through P, then there exist three mutually orthogonal $\vec{\mathbf{a}}_i$ satisfying (5.3), which implies that $\mathbf{A}^{-1}\mathbf{B}$ is symmetric at P. Conversely, if $\mathbf{A}^{-1}\mathbf{B}$ is symmetric at P, then there exist a set of three mutually orthogonal eigenvectors $\vec{\mathbf{a}}_i$, which indicates the existence of three mutually orthogonal and intersecting screws through P. From the identity $\hat{K}\hat{C} = \hat{I}$ one finds that $\mathbf{A}^{-1}\mathbf{B} = -\mathbf{E}^T\mathbf{F}^{-1}$, which proves the same result for the eigenwrench case at the same point P.

■

From the free-vector decomposition theorems, $\mathbf{A} = \mathbf{f}\mathbf{k}_f\mathbf{f}^T$ and $\mathbf{B} = \mathbf{f}\mathbf{k}_f\boldsymbol{\tau}^T$. So, $\mathbf{A}^{-1}\mathbf{B} = \mathbf{f}\boldsymbol{\tau}^T$. Similarly, $\mathbf{E}^T\mathbf{F}^{-1} = \boldsymbol{\delta}\boldsymbol{\gamma}^T$. Lipkin and Patterson [31] showed that $\mathbf{f}\boldsymbol{\tau}^T = -\boldsymbol{\delta}\boldsymbol{\gamma}^T$ is symmetric at E, although they did not show the uniqueness of this property.. So, the point P of Theorem 48 is expected to be E. The following theorem is a proof of the existence and uniqueness of this point, which is later used as a model for the co-eigenscrews case.

Theorem 49 *There exists a unique point where* $\mathbf{A}^{-1}\mathbf{B} = -\mathbf{E}^T\mathbf{F}^{-1}$ *is symmetric.*

Proof. Let the stiffness and compliance be given at an arbitrary point O. Assume that there exists a point P where the matrix $\mathbf{A}^{-1}\mathbf{B}_P$ is symmetric. Let $\vec{\mathbf{r}} = \overrightarrow{OP}$. The condition of symmetry at P is

$$\mathbf{A}^{-1}\mathbf{B}_P - (\mathbf{A}^{-1}\mathbf{B}_P)^T = \mathbf{0} \tag{5.4}$$

Using the spatial transformation of stiffness one gets $\mathbf{B}_P = \mathbf{B}_O + \mathbf{A}\vec{\mathbf{r}}\times$, which reduces (5.4) to

$$\mathbf{A}^{-1}\mathbf{B}_O - \mathbf{B}_O^T\mathbf{A}^{-1} + 2\vec{\mathbf{r}}\times = \mathbf{0} \tag{5.5}$$

which is the condition of symmetry in terms of quantities at O. Now, given O, the point P can be determined uniquely from (5.5) as

$$\vec{\mathbf{r}}\times = \overrightarrow{OP}\times = -\frac{1}{2}\left(\mathbf{A}^{-1}\mathbf{B}_O - \mathbf{B}_O^T\mathbf{A}^{-1}\right) \tag{5.6}$$

which always exists since both sides of (5.6) are skew-symmetric.

■

The following corollary follows from the above theorem and can be used as an alternative definition of the center of elasticity.

Corollary 50 $\mathbf{A}^{-1}\mathbf{B} = -\mathbf{E}^T\mathbf{F}^{-1}$ *is symmetric only at the center of elasticity, E.*

Proof. From the free-vector decomposition $\mathbf{A} = \mathbf{f}\mathbf{k}_f\mathbf{f}^T$ and $\mathbf{B} = \mathbf{f}\mathbf{k}_f\boldsymbol{\tau}^T$. So, (5.6) becomes

$$\begin{aligned} \overrightarrow{OP}\times &= -\frac{1}{2}\left(\mathbf{f}\boldsymbol{\tau}_O^T - \boldsymbol{\tau}_O\mathbf{f}^T\right) = -\frac{1}{2}\sum_i\left(\vec{\mathbf{f}}_i\vec{\boldsymbol{\tau}}_{Oi}^T - \vec{\boldsymbol{\tau}}_{Oi}\vec{\mathbf{f}}_i^T\right) && (5.7)\\ &= -\frac{1}{2}\sum\left(\vec{\boldsymbol{\tau}}_{Oi}\times\vec{\mathbf{f}}_i\times - \vec{\mathbf{f}}_i\times\vec{\boldsymbol{\tau}}_{Oi}\times\right) = \left(\frac{1}{2}\sum\vec{\mathbf{f}}_i\times\vec{\boldsymbol{\tau}}_{Oi}\right)\times && (5.8)\end{aligned}$$

where the identity $\vec{\mathbf{f}}_i\vec{\boldsymbol{\tau}}_{Oi}^T = \vec{\boldsymbol{\tau}}_{Oi}\times\vec{\mathbf{f}}_i\times + \left(\vec{\mathbf{f}}_i^T\vec{\boldsymbol{\tau}}_{Oi}\right)\mathbf{I}$ is used. But, by the fact that $\vec{\mathbf{f}}_i^T\vec{\mathbf{f}}_i = 1$ for all i, the expression under the summation sign in (5.8) is recognized as $\vec{\mathbf{f}}_i\times\vec{\boldsymbol{\tau}}_{Oi} = \vec{\mathbf{r}}_{f_i}$, the perpendicular vectors to the eigenwrenches from O. Therefore, the location of the point P from O is uniquely given as

$$\overrightarrow{OP} = \frac{1}{2}\sum \vec{\mathbf{r}}_{f_i} \tag{5.9}$$

Now, comparing this result to Theorem 34, Chapter 3, one sees that (5.9) is the equation for the location of the center of elasticity. Therefore, $P = E$ and the corollary is proven.

■

Theorem 49 and Corollary 50 yield, as an aside, an alternative equation for the location of the center of elasticity, as summarized in the following corollary.

Corollary 51 *The position vector $\vec{\mathbf{r}}_E$ from any point O to the center of elasticity is given by*

$$\vec{\mathbf{r}}_E = -\frac{1}{2}\text{vector}\left[\mathbf{A}^{-1}\mathbf{B}_O - \mathbf{B}_O^T\mathbf{A}^{-1}\right] = -\text{vector}\left[\mathbf{A}^{-1}\mathbf{B}_O\right]_{skew} \tag{5.10}$$

$$= -\frac{1}{2}\text{vector}\left[\mathbf{F}^{-1}\mathbf{E}_O - \mathbf{E}_O^T\mathbf{F}^{-1}\right] = -\text{vector}\left[\mathbf{F}^{-1}\mathbf{E}_O\right]_{skew} \tag{5.11}$$

where the operator vector$(\cdot)$ *is the inverse of the* 3×3 *cross product operator such that if* $(\cdot)\times : \vec{\mathbf{v}} \mapsto \vec{\mathbf{v}}\times$ *for any 3-vector then* vector$(\cdot) : \vec{\mathbf{v}}\times \mapsto \vec{\mathbf{v}}$.

These results prove that the following re-definition of the center of elasticity is equivalent to the original definition.

Definition 52 *The unique point where $\mathbf{A}^{-1}\mathbf{B}$ (or $\mathbf{E}^T\mathbf{F}^{-1}$) is symmetric is called the center of elasticity.*

5.2 Co-Center of Elasticity

A parallel development for the co-eigenscrew systems is not as straightforward. There are infinitely many co-eigentwist/co-eigenwrench system pairs, each corresponding to a distinct generator. Being reciprocal 3-systems, each pair in general have a point where there exists a set of mutually orthogonal and intersecting screws. By definition, these are the center and the principal screws of the particular co-eigenscrew system. However, there seems to be no general analogy between these infinitely many centers and the center of elasticity. This section shows that for some generators there is a close analogy.

5.2.1 Definition

A straightforward calculation shows that, in general, the generator and the center of the related 3-systems are distinct points. It is therefore natural to inquire if there is a generator that is also the center. The following theorem answers this question and establishes the analogy to the center of elasticity.

Theorem 53 *Let a generator G and its related co-eigentwist and co-eigenwrench systems be given. Then, three mutually orthogonal screws of the co-eigentwist and co-eigenwrench 3-systems pass through G if and only if $\mathbf{C}^{-1}\mathbf{B}^T = -\mathbf{E}\mathbf{D}^{-1}$ is symmetric at G.*

Proof. For the co-eigentwist case assume that there exists an element of the co-eigentwist system passing through the generator G. Then, at G,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} h\vec{\mathbf{a}} \\ \vec{\mathbf{a}} \end{bmatrix} = \begin{bmatrix} \vec{\eta} \\ \vec{\mathbf{0}} \end{bmatrix} \tag{5.12}$$

where $[\ h\vec{\mathbf{a}}^T \quad \vec{\mathbf{a}}^T\ ]^T$ is an element of the co-eigentwist 3-system. The second row of (5.12) is equivalent to

$$h\mathbf{B}^T\vec{\mathbf{a}} = -\mathbf{C}\vec{\mathbf{a}} \tag{5.13}$$

This is a generalized eigenvalue problem which can also be given as

$$h^{-1}\vec{\mathbf{a}} = -\mathbf{C}^{-1}\mathbf{B}^T\vec{\mathbf{a}} \tag{5.14}$$

The rest of the proof is the same as that for Theorem 48. The equality $\mathbf{C}^{-1}\mathbf{B}^T = -\mathbf{E}\mathbf{D}^{-1}$ is due to the identity $\hat{K}\hat{C} = \hat{I}$.

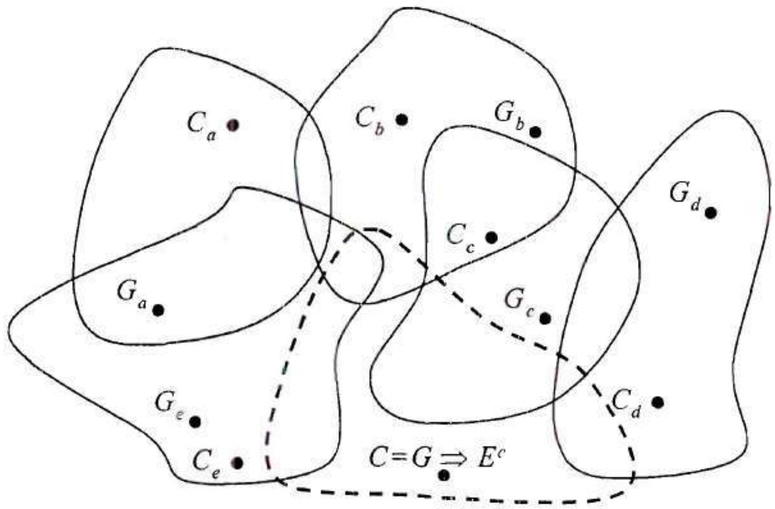


Figure 5.1: Eigensystems generated by distinct generators G_i. The center of each system is C_i. The co-center-of-elasticity is a generator which is also the center of its own eigensystem: $G = C = E^c$.

■

Note that, in Theorem 53 the screws can have infinite pitch. In such a case only the orthogonality constraint needs to be satisfied. Infinite screws exist if $\mathbf{B}$ is singular. This is best described by (5.14) which involves h^{-1} instead of h so that for infinite pitch screws one gets $h^{-1} = 0$.

The symmetry of $\mathbf{C}^{-1}\mathbf{B}^T$ and $\mathbf{E}\mathbf{D}^{-1}$ is analogous to what happens at the center of elasticity, i.e. the symmetry of $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{E}^T\mathbf{F}^{-1}$. Hence, the following definition is proposed.

Definition 54 *A point where $\mathbf{C}^{-1}\mathbf{B}^T$ (or $\mathbf{E}\mathbf{D}^{-1}$) is symmetric is called a* ***co-center of elasticity*** *and denoted by E^c.*

For brevity, in the rest of the treatment the co-center of elasticity is frequently called the **co-center**.

5.2.2 Co-center Equations

For the center of elasticity case, the symmetry condition is $\mathbf{A}^{-1}\mathbf{B} - \mathbf{B}^T\mathbf{A}^{-1} = \mathbf{0}$, where $\mathbf{B}$ is a linear function of $\vec{\mathbf{r}}\times$ and $\mathbf{A}$ is invariant under origin transformations. This yields a linear solution for $\vec{\mathbf{r}}$ which shows the existence and uniqueness of E.

For the co-center case, the symmetry condition is $\mathbf{C}^{-1}\mathbf{B}^T - \mathbf{B}\mathbf{C}^{-1} = \mathbf{0}$, where all matrices are origin dependent and $\mathbf{C}$ a quadratic function of $\vec{\mathbf{r}}\times$. This leads to a third order matrix polynomial in $\vec{\mathbf{r}}\times$. Therefore, an explicit solution for the co-centers is essentially difficult. However, the existence of the co-center can be proven by using the properties of polynomial equations. The following are some simplifications needed in subsequent theorems.

Lemma 55 $\mathbf{C}^{-1}\mathbf{B}^T$ *is symmetric if and only if* $\mathbf{CB}$ *is symmetric.*

Proof. Since $\mathbf{C}$ is symmetric, then if $\mathbf{C}^{-1}\mathbf{B}^T$ symmetric, it follows that $\mathbf{C}\left(\mathbf{C}^{-1}\mathbf{B}^T\right)\mathbf{C} = \mathbf{B}^T\mathbf{C}$ is symmetric and so its transpose, $\mathbf{CB}$. Converse is similar.

■

Lemma 55 leads to the following corollary, as a simplified condition for a co-center that does not involve inverses.

Corollary 56 *A point O is a co-center if and only if*

$$\mathbf{CB} - \mathbf{B}^T\mathbf{C} = \mathbf{0} \tag{5.15}$$

Let unsubscripted quantities belong to any given arbitrary point O. Then, the stiffness matrix at any other point A is

$$\hat{K}_A = \hat{X}\hat{K}_O\hat{X}^T = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\vec{\mathbf{r}}\times & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \vec{\mathbf{r}}\times \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{5.16}$$

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}_A \\ \mathbf{B}_A^T & \mathbf{C}_A \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} + \mathbf{A}\vec{\mathbf{r}}\times \\ \mathbf{B}^T - \vec{\mathbf{r}}\times\mathbf{A} & \mathbf{C} + \mathbf{B}^T\vec{\mathbf{r}}\times - \vec{\mathbf{r}}\times\mathbf{B} - \vec{\mathbf{r}}\times\mathbf{A}\vec{\mathbf{r}}\times \end{bmatrix} \tag{5.17}$$

where $\vec{\mathbf{r}} = \overrightarrow{OA}$. Now, one can express $\mathbf{C}_A\mathbf{B}_A$ in terms of the quantities at O by using (5.17),

$$\begin{aligned}
\mathbf{C}_A\mathbf{B}_A &= \left(\mathbf{C} + \mathbf{B}^T\vec{\mathbf{r}}\times - \vec{\mathbf{r}}\times\mathbf{B} - \vec{\mathbf{r}}\times\mathbf{A}\vec{\mathbf{r}}\times\right)(\mathbf{B} + \mathbf{A}\vec{\mathbf{r}}\times) && (5.18)\\
&= \mathbf{CB} + \mathbf{CA}\vec{\mathbf{r}}\times + \mathbf{B}^T\vec{\mathbf{r}}\times\mathbf{B} + \mathbf{B}^T\vec{\mathbf{r}}\times\mathbf{A}\vec{\mathbf{r}}\times + \\
&\quad -\vec{\mathbf{r}}\times\mathbf{B}^2 - \vec{\mathbf{r}}\times\mathbf{BA}\vec{\mathbf{r}}\times - \vec{\mathbf{r}}\times\mathbf{A}\vec{\mathbf{r}}\times\mathbf{B} - \vec{\mathbf{r}}\times\mathbf{A}\vec{\mathbf{r}}\times\mathbf{A}\vec{\mathbf{r}}\times && (5.19)
\end{aligned}$$

Since $\mathbf{A}$ is a positive definite symmetric matrix and independent of $\vec{\mathbf{r}}$, the following substitutions are possible.

$$\begin{aligned}
\vec{\mathbf{r}}\times &= \mathbf{A}^{-\frac{1}{2}}\vec{\rho}\times\mathbf{A}^{-\frac{1}{2}} & \mathbf{B} &= \mathbf{A}^{\frac{1}{2}}\mathbf{P}\mathbf{A}^{-\frac{1}{2}} & \mathbf{C} &= \mathbf{A}^{-\frac{1}{2}}\mathbf{Q}\mathbf{A}^{-\frac{1}{2}} && (5.20)\\
\vec{\rho}\times &= \mathbf{A}^{\frac{1}{2}}\vec{\mathbf{r}}\times\mathbf{A}^{\frac{1}{2}} & \mathbf{P} &= \mathbf{A}^{-\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}} & \mathbf{Q} &= \mathbf{A}^{\frac{1}{2}}\mathbf{C}\mathbf{A}^{\frac{1}{2}} && (5.21)
\end{aligned}$$

where $\mathbf{A}^{\frac{1}{2}}$ is the symmetric positive definite square root of $\mathbf{A}$ such that $\mathbf{A} = \mathbf{A}^{\frac{1}{2}}\mathbf{A}^{\frac{1}{2}}$. Using (5.20) in (5.19) yields

$$\begin{aligned}
\mathbf{Q}_A\mathbf{P}_A &= \mathbf{QP} + \mathbf{Q}\vec{\rho}\times + \mathbf{P}^T\vec{\rho}\times\mathbf{P} + \mathbf{P}^T\vec{\rho}\times\vec{\rho}\times + && (5.22)\\
&\quad -\vec{\rho}\times\mathbf{P}^2 - \vec{\rho}\times\mathbf{P}\vec{\rho}\times - \vec{\rho}\times\vec{\rho}\times\mathbf{P} - \vec{\rho}\times\vec{\rho}\times\vec{\rho}\times
\end{aligned}$$

Note that $\mathbf{Q}$ is always symmetric. Also, since $\mathbf{QP} = \mathbf{A}^{\frac{1}{2}}\mathbf{C}\mathbf{A}^{\frac{1}{2}}\mathbf{A}^{-\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}} = \mathbf{A}^{\frac{1}{2}}\mathbf{CB}\mathbf{A}^{\frac{1}{2}}$, for a point A, $\mathbf{CB}$ is symmetric if and only if $\mathbf{QP}$ is symmetric. Therefore, the symmetry condition in terms of the transformed quantities is

$$\begin{aligned}
\mathbf{Q}_A\mathbf{P}_A - \mathbf{P}_A^T\mathbf{Q}_A &= \mathbf{0} && (5.23)\\
[\mathbf{QP} + \mathbf{Q}\vec{\rho}\times + \mathbf{P}^T\vec{\rho}\times\mathbf{P} + \mathbf{P}^T\vec{\rho}\times\vec{\rho}\times - \vec{\rho}\times\mathbf{P}^2 + &\\
-\vec{\rho}\times\mathbf{P}\vec{\rho}\times - \vec{\rho}\times\vec{\rho}\times\mathbf{P} - \vec{\rho}\times\vec{\rho}\times\vec{\rho}\times] &\\
-[\mathbf{QP} + \mathbf{Q}\vec{\rho}\times + \mathbf{P}^T\vec{\rho}\times\mathbf{P} + \mathbf{P}^T\vec{\rho}\times\vec{\rho}\times - \vec{\rho}\times\mathbf{P}^2 + &\\
-\vec{\rho}\times\mathbf{P}\vec{\rho}\times - \vec{\rho}\times\vec{\rho}\times\mathbf{P} - \vec{\rho}\times\vec{\rho}\times\vec{\rho}\times]^T &= \mathbf{0} && (5.24)
\end{aligned}$$

which simplifies to

$$
\begin{aligned}
&2\vec{\rho}\times\vec{\rho}\times\vec{\rho}\times+\vec{\rho}\times\left(\mathbf{P}-\mathbf{P}^T\right)\vec{\rho}\times+2\vec{\rho}\times\vec{\rho}\times\mathbf{P}-2\mathbf{P}^T\vec{\rho}\times\vec{\rho}\times+ \\
&-2\mathbf{P}^T\vec{\rho}\times\mathbf{P}+\vec{\rho}\times\mathbf{P}^2+\left(\mathbf{P}^T\right)^2\vec{\rho}\times-\mathbf{Q}\vec{\rho}\times-\vec{\rho}\times\mathbf{Q}-\left(\mathbf{QP}-\mathbf{P}^T\mathbf{Q}\right) = \mathbf{0} \qquad (5.25)
\end{aligned}
$$

This is a third order matrix polynomial in $\vec{\rho}\times$.

The following theorem is also needed for further simplification.

Theorem 57 *The matrix $\mathbf{P}=\mathbf{A}^{-\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}$ is symmetric if and only if $O=E$, the center of elasticity.*

Proof. Assume $\mathbf{P}$ is symmetric at a point O. Then, $\mathbf{A}^{-\frac{1}{2}}\mathbf{P}\mathbf{A}^{-\frac{1}{2}}$ is symmetric due to the symmetry of $\mathbf{A}^{-\frac{1}{2}}$. But, $\mathbf{A}^{-\frac{1}{2}}\mathbf{P}\mathbf{A}^{-\frac{1}{2}}=\mathbf{A}^{-\frac{1}{2}}\left(\mathbf{A}^{-\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}\right)\mathbf{A}^{-\frac{1}{2}}=\mathbf{A}^{-1}\mathbf{B}$, which is symmetric only at the center of elasticity (Corollary 50). Therefore, O is the center of elasticity. For the converse assume $O=E$. Then $\mathbf{A}^{-1}\mathbf{B}$ is symmetric. But, then so is $\mathbf{A}^{\frac{1}{2}}\left(\mathbf{A}^{-1}\mathbf{B}\right)\mathbf{A}^{\frac{1}{2}}=\mathbf{A}^{-\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}=\mathbf{P}$.

■

The co-center condition can be expressed an any point. So if $O=E$ is chosen, then by Theorem 57 $\mathbf{P}^T=\mathbf{P}$. Furthermore, one can also align the coordinate system at E such that $\mathbf{P}$ is diagonal due to its symmetry. As a result, the following forms can be used without loss of generality.

$$
\mathbf{P}=\begin{bmatrix} p_1 & 0 & 0 \\ 0 & p_2 & 0 \\ 0 & 0 & p_3 \end{bmatrix} \qquad \mathbf{Q}=\begin{bmatrix} q_1 & q_4 & q_5 \\ q_4 & q_2 & q_6 \\ q_5 & q_6 & q_3 \end{bmatrix} \qquad (5.26)
$$

Using $\vec{\rho}=\begin{bmatrix} x & y & z \end{bmatrix}^T$ and (5.26) in (5.25) one obtains the following 3$^{\text{rd}}$ degree polynomial equations in terms of the components of $\vec{\rho}$.

$$
\begin{aligned}
2x^3+\left(2y^2+2z^2-p_{23}^2+q_2+q_3\right)x+p_{23}\left(q_6-2yz\right)-q_4y-q_5z &= 0 \\
2y^3+\left(2z^2+2x^2-p_{31}^2+q_3+q_1\right)y+p_{31}\left(q_5-2zx\right)-q_6z-q_4x &= 0 \qquad (5.27) \\
2z^3+\left(2x^2+2y^2-p_{12}^2+q_1+q_2\right)z+p_{12}\left(q_4-2xy\right)-q_5x-q_6y &= 0
\end{aligned}
$$

where $p_{ij} = p_i - p_j$, $(i \neq j)$.

Every real solution of (5.27) corresponds to a co-center.

Conditions for a Co-center at E Before going into more detail, one should make the following important observations.

Theorem 58 *There exists a co-center at E, that is, $x = y = z = 0$ is a solution of (5.27), if and only if*

$$(p_2 - p_3)\, q_6 = (p_3 - p_1)\, q_5 = (p_1 - p_2)\, q_4 = 0 \tag{5.28}$$

Proof. Assume that there exists a co-center at E. Then, $x = y = z = 0$ and (5.28) follows from (5.27). Conversely, if (5.28) is true then $x = y = z = 0$ is a solution to (5.27).

■

Theorem 58 is important in cases where it is desirable to have or to know that a co-center exists at the center of elasticity. However, it also enumerates the conditions on matrices **P** and **Q** to have a co-center at the center of elasticity. The following corollary describes the cases of (5.28).

Corollary 59 *There exists a co-center at E if and only if at least one of the following is true.*

1. $p_1 = p_2 = p_3$
2. $\{p_1 = p_2, q_5 = q_6 = 0\}$ or $\{p_2 = p_3, q_4 = q_5 = 0\}$ or $\{p_1 = p_3, q_4 = q_6 = 0\}$.
3. $q_4 = q_5 = q_6 = 0$.

All cases in Corollary 78 correspond to simultaneous diagonalizability of **P** and **Q**. In case (1), $\mathbf{P} = p\mathbf{I}$ and, therefore, one can rotate the coordinate axes such that $\mathbf{P} = p\mathbf{I}$ and **Q** is diagonal. In case (2), **P** has a double eigenvalue corresponding to one of the xy-, yz- and zx-planes. Consider the case when the double eigenvalue corresponds to xy-plane, $p_1 = p_2$, so that z-axis becomes the direction of the eigenvector of p_3. Then, since $q_5 = q_6 = 0$, $\mathbf{Q}$ has an eigenvector parallel to z-axis, too. So, there exists a rotation about the z-axis which makes both $\mathbf{P}$ and $\mathbf{Q}$ diagonal. In case (3), both $\mathbf{P}$ and $\mathbf{Q}$ are diagonal already. This proves the following corollary.

Corollary 60 *There exists a co-center at E if and only if $\mathbf{P}$ and $\mathbf{Q}$ have a common set of eigenvectors.*

Note that Corollary 60 can also be obtained in a coordinate free way by using the co-center equations in matrix form, (5.25), which directly indicates that there exists a co-center at E if and only if $\mathbf{QP}$ is symmetric there, because the constant terms vanish. But, since $\mathbf{P}$ is already symmetric at E, this means that $\mathbf{QP} = \mathbf{PQ}$, or, $\mathbf{P}$ and $\mathbf{Q}$ commute. One can show that this result is equivalent to the statement of Corollary 60. The importance of Corollary 60 is that it is a purely geometric condition. Corollary 78 is more detailed. It gives distinct cases which are separated by multiplicities in the eigenvalues of $\mathbf{P}$, which are of constitutive nature. These cases are treated later in this chapter after investigating the total solutions of the co-center equations.

It should be stressed that the coordinate rotations in the transformed space, $\vec{\rho}$-space, may or may not correspond to rotations in the original Cartesian space. At this point, the development is performed completely in the transformed space. Later, the meaning of the substitutions (5.20) is presented and results are imported back to the original space.

5.2.3 Existence of Co-centers

Among the topics of *algebraic geometry* is the totality of solutions to multivariate polynomial equations, such as those in (5.27). A solution of the three cubic equations in (5.27) is a point in $\vec{\rho}$-space. In general, there may be a finite number of isolated points, an infinite number of points or a combination of both. These points may be real or complex. Each co-center equation represents a surface in $\vec{\rho}$-space and the total solution is composed of points in the common intersection of these three surfaces.

Curves, surfaces, hypersurfaces, etc. in an n-dimensional space are usually expressed in a form such as $F(x_1, ..., x_n) = 0$ called a *nonhomogeneous* form. Equations (5.27) are examples in 3-dimensions. Considering that the variables can take on any value in the set of complex numbers, $\mathcal{C}$, the actual n-dimensional space is $\mathcal{C}^n$, Cartesian product of n copies of the set of complex numbers.

In 1-dimension, a polynomial equation represents a finite set of isolated points, rather than curves, surfaces, etc. The points are the roots of the polynomial and are completely contained in $\mathcal{C}$. This is a well known and important theorem in algebra. The number of points is equal to the degree of the polynomial equation. However, a polynomial equation of any degree in two dimensions is more complicated. Consider, for example, $3x_1^2 + 2x_2^2 - 1 = 0$. This defines an ellipse, in the real plane, which is an infinite set of points. If, in addition, the complex solutions are also considered, the picture becomes further complicated. The nature of the set of solutions, i.e. whether it is a set of isolated points or a set describing curves or surfaces, is important in the analysis of such systems.

For a systematic analysis of such systems, *homogeneous coordinates* are introduced. This is achieved by introducing an extra coordinate u_0 and the substitutions

$$x_1 = \frac{u_1}{u_0},\ x_2 = \frac{u_2}{u_0},\ \dots,\ x_n = \frac{u_n}{u_0} \tag{5.29}$$

which puts the non-homogeneous form $F(x_1, ..., x_n) = 0$ into the *homogeneous form*

$$f(u_1, ..., u_n, u_0) = 0 \tag{5.30}$$

Switching between the two forms is easy. For $u_0 = 1$, the homogeneous coordinates are equivalent to the non-homogeneous ones. The advantage of the homogeneous coordinates is in the representation of *points at infinity*. For $u_0 = 0$, each point $(u_1, ..., u_n, 0)$ is a point at infinity. For details, reader is referred to Hodge and Pedoe [24]. The space of homogeneous coordinates becomes what is known as the complex **projective space**, $\mathcal{P}_n(\mathcal{C})$. For simplicity $\mathcal{P}_n$ is used to denote the projective n-space over complex numbers. Because of (5.30), a point $(u_1, ..., u_n, u_0)$ is equivalent to

$\rho(u_1,...,u_n,u_0) = (\rho u_1,...,\rho u_n,\rho u_0)$ for any $\rho \neq 0$. For this reason, any point $(u_1,...,u_n,u_0)$ such that $u_0 \neq 0$ is equivalent to a unique point $(u_1/u_0,...,u_n/u_0,1)$, a finite point.

The original space from which the projective space is obtained is called the **affine space**, usually denoted by $\mathcal{A}^n$. Here, the original affine space is $\mathcal{C}^n$. The original affine space is composed of points with $u_0 = 1$. One can also consider the collection of points with $u_i = 1$ for any i. In this way, different affine spaces are obtained and denoted by $\mathcal{A}_i^n$. Clearly, $\mathcal{P}_n$ has $n+1$ associated affine spaces. It is required that every point of $\mathcal{P}_n$ belong to at least one affine space $\mathcal{A}_i^n$, including $i = 0$ for the original affine space. For this reason, the point $(0,0,...,0)_{n+1}$ cannot belong to the projective space since it does not have a corresponding point in any of the associated affine spaces.

Now, returning to the co-center equations, the following are the homogeneous forms obtained after substitutions (5.29) and modifications.

$$\begin{aligned}
F_1: &\quad \left(u_1^2+u_2^2+u_3^2\right)u_1 + \left(\alpha_1 u_1 - q_4 u_2 - q_5 u_3 + \beta_1 u_0\right)u_0^2 + \alpha_{23}u_2u_3u_0 = 0 \\
F_2: &\quad \left(u_1^2+u_2^2+u_3^2\right)u_2 + \left(\alpha_2 u_2 - q_6 u_3 - q_4 u_1 + \beta_2 u_0\right)u_0^2 + \alpha_{31}u_1u_3u_0 = 0 \\
F_3: &\quad \left(u_1^2+u_2^2+u_3^2\right)u_3 + \left(\alpha_3 u_3 - q_5 u_1 - q_6 u_2 + \beta_3 u_0\right)u_0^2 + \alpha_{12}u_1u_2u_0 = 0
\end{aligned} \tag{5.31}$$

where u_0 is the homogeneous variable, and α_i, β_j, etc. are redefined coefficients. The following theorem can be found in Shafarevich, [46].

Theorem 61 (Shafarevich) *A system of n homogeneous real equations in $n+1$ variables has a non-zero real solution if the degree of each equation is odd.*

The system of equations in (5.31) clearly satisfies the conditions of Theorem 61. Therefore, they have at least one real solution such that not all u_i are zero, possibly including points at infinity.

For the co-center equations (5.31), the points at infinity are found by setting $u_0 = 0$.

$$F_1^\infty: \quad \left(u_1^2+u_2^2+u_3^2\right)u_1 = 0$$

$$F_2^\infty: \qquad \left(u_1^2 + u_2^2 + u_3^2\right) u_2 = 0 \tag{5.32}$$

$$F_3^\infty: \qquad \left(u_1^2 + u_2^2 + u_3^2\right) u_3 = 0$$

This is satisfied if $u_1^2 + u_2^2 + u_3^2 = 0$ or $u_1 = u_2 = u_3 = 0$. The latter gives a point which is not in $\mathcal{P}_n$. Hence, all the points at infinity must satisfy $u_1^2 + u_2^2 + u_3^2 = 0$. In the literature, this is called as the **spherical circle at infinity** and at least one of u_1, u_2, u_3 must always be complex. Therefore, the solutions to the co-center equations at infinity are all complex. Together with Theorem 61, this proves the following.

Theorem 62 (Existence of co-centers) *For any given stiffness/compliance, there always exists at least one real and finite co-center of elasticity.*

5.2.4 Classification and Solutions of Co-center Equations

None of the previous results exclude the possibility of having more than one real and finite co-center, which may or may not be isolated points. To investigate this, it is necessary to deal with the total solutions of the co-center equations. In what follows are definitions and theorems that are essential to understanding the nature of solutions for general systems of multivariate equations. Then, the particular case of the system (5.27) is analyzed.

A system of equations is called an **algebraic variety**. Strictly speaking, an algebraic variety is actually a set of points that simultaneously satisfy a system of equations. An algebraic variety is called an **affine variety** if its points are considered to belong to an affine space and a **projective variety** if its points belong to a projective space. Since the algebraic varieties are point sets, they can be treated in a set-theoretic manner leading to concepts such as *intersection, union*, etc. Topologically, the set represented by an algebraic variety may consist of *smaller* sets which may or may not be disconnected. For example, a set of n isolated points is made up of n disconnected sets, each containing only one point. The set of points on two non-intersecting circles has two disconnected subsets, whereas that on two intersecting circles has two subsets that are not disconnected. Such subsets of an algebraic variety are called the **components**. So, an algebraic variety is equivalent to the union of its components.

Let F_1 and F_2 be two algebraic varieties. Then, the **intersection** of F_1 with F_2 is defined as the set of points on which belong to both F_1 and F_2. So, the intersection of F_1 and F_2 is denoted as $F_1 \cap F_2$. The points in $F_1 \cap F_2$ simultaneously satisfy equations of both F_1 and F_2, hence the intersection is an algebraic variety itself. It is clear that $F_1 \cap F_2 \subseteq F_1, F_2$.

The **sum** of two algebraic varieties F_1 and F_2 is defined as the union of the point sets of F_1 and F_2, and denoted by $F_1 \cup F_2$. In this case, $F_1 \cup F_2 \supseteq F_1, F_2$ is an algebraic variety whose equations are satisfied by the points of either F_1 or F_2. As an example, assume that each of F_1 and F_2 is represented by a single equation, say $f = 0$ and $g = 0$, respectively. Then, $F_1 \cup F_2$ is represented by $fg = 0$.

An algebraic variety F is called **reducible** if there exist F_1 and F_2, distinct from F, such that $F = F_1 \cup F_2$. If an algebraic variety is not reducible then it is called **irreducible**. For example, if $f = 0$ is a single polynomial equation representing F, then F is reducible if there exists two polynomials g, h such that $f = gh$ and $g, h \neq f$, excluding constants, since then F is the sum of two algebraic varieties represented by $g = 0$ and $h = 0$. Every algebraic variety can be given as a unique sum of finitely many irreducible algebraic varieties, called the **irreducible components**, [24].

The terms irreducibility and reducibility are dependent on the *algebraic field* used to define the algebraic varieties, that is, the field used to form the coefficients in the polynomials in the equations. For example, whereas it may not be possible to factor a univariate real polynomial into real factors, it can always be factored into linear factors defined over the field of complex numbers. The original field over which a given variety is defined is called the *ground field*. In the case of (5.27), the ground field is taken as the set of complex numbers and the reducibility is understood as having proper factor polynomials with complex coefficients.

For the co-center equations, all of the polynomials in (5.27) are of third degree. Therefore, if any of them is reducible then its proper factors can only be first and second degree polynomials. Thus, in the most general form, the factoring must be

$$\left(ax^2+by^2+cz^2+dxy+exz+fyz+gx+hy+kz+m\right)(nx+ry+sz+t) \tag{5.33}$$

where $a, b, ..., t$ are complex numbers in general. Note that this is the union of a quadric surface and a plane. One requires this form to be identically equal to any of the polynomials in (5.27). The result is a system of equations in $a, b, ..., t$, which has no solution *unless* the conditions in the following theorem are true.

Theorem 63 *All of the polynomials in (5.27) are in general irreducible (in $\mathcal{C}$). Each is reducible (in $\mathcal{R}$) if and only if*

$$\begin{aligned} p_2 - p_3 &= q_4 = q_5 = 0 \qquad \textit{for } F_1 \\ p_3 - p_1 &= q_6 = q_4 = 0 \qquad \textit{for } F_2 \\ p_1 - p_2 &= q_5 = q_6 = 0 \qquad \textit{for } F_3 \end{aligned} \tag{5.34}$$

This leads to the following essential result.

Theorem 64 *If the co-center equations are reducible then there exists a co-center at the center of elasticity.*

Proof. Theorems 58 and 63 can be used for the proof. If any of the co-center equations is reducible then one of (5.34) is true. No matter which one is true, $(p_2 - p_3)\, q_6 = (p_3 - p_1)\, q_5 = (p_1 - p_2)\, q_4 = 0$ is satisfied. Thus, by Theorem 58, there exists a co-center at E. Converse is not true. For example,

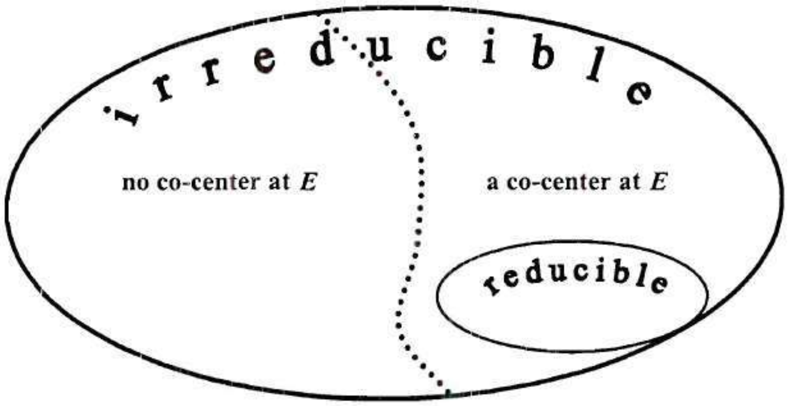


Figure 5.2: Classification of stiffnesses with respect to their co-centers.

for $q_6 = q_5 = q_4 = 0$, there exists a co-center at E (Theorem 58), but none of the co-center equations are reducible (Theorem 63), unless $p_i = p_j$ for some $i \neq j$.

∎

These results classify stiffnesses in two ways: 1) reducible versus irreducible, 2) co-center at E versus no co-center at E. Using Theorem 63 gives rise to three distinct classes of stiffness concerning the co-centers. In a decreasing order of generality, these classes are:

1. No co-center at E (the co-center equations are irreducible).

2. A co-center at E and the co-center equations are irreducible.

3. A co-center at E and the co-center equations are reducible.

Figure 5.2 represents these classes pictorially. Each case is separately analyzed in the subsequent sections.

No Co-center at E If a co-center does not exist at E then the co-center equations are irreducible and none of the conditions in (5.34) is true. This is the most general case since existence of a co-center at E is provided only by simultaneous vanishing of some of $(p_i - p_j)_{i \neq j}$, q_4, q_5 and q_6.

One of the well known theorems about projective varieties is the *Bezout's theorem*, see, for example, [24], [46]. Bezout's theorem gives the degree of the intersection of two varieties as the product of their degrees if some conditions are satisfied. The most important condition is that the dimension of the intersection must exactly be $n_1 + n_2 - N$, where n_i are the dimensions of the intersecting varieties and N is the dimension of the projective space. If this is satisfied then the degree of the intersection is $d_1 d_2$, where d_i are the degrees of the intersecting varieties. For example, in general, the intersection of any two ellipses in the projective plane is $1 + 1 - 2 = 0$ dimensional, meaning a finite number of isolated points. In this case, Bezout's theorem indicates that the degree of intersection is $2 * 2 = 4$, meaning that there are exactly four points in the projective plane. However, if the intersection is 1-dimensional, coinciding ellipses, then the Bezout's theorem cannot be applied.

For the co-center equations, consider first taking F_1 and F_2. One can show that $F_1 \cap F_2$ is exactly $2+2-3 = 1$ dimensional (a curve). Thus, by Bezout's theorem, its degree is $3*3 = 9$. Now, if $(F_1 \cap F_2) \cap F_3$ were exactly $1 + 2 - 3 = 0$ dimensional (isolated points) then Bezout's theorem would predict the degree to be $9*3 = 27$. This would then indicate that the solutions of the co-center equations contain 27 isolated points in the projective 3-space. However, if one puts $u_0 = 0$ in (5.31) then it is easily seen that all of the equations contain the points given by $\{u_1^2 + u_2^2 + u_3^3 = 0, u_0 = 0\}$, the spherical circle at infinity, which is 1-dimensional. Hence, Bezout's theorem cannot be applied. Another problem with Bezout's theorem is that it concerns all types of solutions: real, complex, finite and infinite, whereas here only the real and finite solutions are sought.

Any attempt at determining the real and finite solutions to the co-center equations reveals that the problem is inherently difficult. It is also beyond the scope of this study. Therefore, this case is left as an open problem.

It is interesting that the numerical construction of stiffnesses with more than one co-center is also difficult. However, there exist such stiffnesses. For example, the stiffness corresponding to

$$P = \begin{bmatrix} 0.4 & -3.2729 & -1.6235 \\ -2.3409 & 1.4 & -0.7807 \\ -0.9356 & -0.4946 & -1.6 \end{bmatrix} \qquad Q = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 20 & 0 \\ 0 & 0 & 30 \end{bmatrix} \tag{5.35}$$

has three real co-centers whose position vectors in $\vec{\rho}$ space are

$$\vec{\rho}_{1,2} = \pm \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \qquad \vec{\rho}_3 = \begin{bmatrix} 0.7759 \\ 0.5959 \\ -1.2692 \end{bmatrix} \tag{5.36}$$

none of which is the center of elasticity whose position vector is $\begin{bmatrix} 0.1431 & -0.3439 & 0.4660 \end{bmatrix}^T$.

Therefore, unlike the centers of stiffness, compliance and elasticity, the co-center of elasticity is not unique in general. However, from numerical studies it appears that if a stiffness is picked at random it is highly probable that it will have a unique co-center.

<u>A Co-center at E with Irreducible Equations</u> If a co-center exists at the center of elasticity E then, by Corollary 60, the matrices $\mathbf{P}$ and $\mathbf{Q}$ are simultaneously diagonalizable. So, assume that $\mathbf{P}$ and $\mathbf{Q}$ are both diagonal in a certain coordinate system. Then, the co-center equations written at E become

$$\begin{aligned} \left(2x^2 + 2y^2 + 2z^2 - p_{23}^2 + q_2 + q_3\right) x - 2p_{23}yz &= 0 \\ \left(2y^2 + 2z^2 + 2x^2 - p_{31}^2 + q_3 + q_1\right) y - 2p_{31}zx &= 0 \\ \left(2z^2 + 2x^2 + 2y^2 - p_{12}^2 + q_1 + q_2\right) z - 2p_{12}xy &= 0 \end{aligned} \tag{5.37}$$

where $p_{ij} = p_i - p_j$. Since $\mathbf{Q}$ is diagonal $q_4 = q_5 = q_6 = 0$. However, the co-center equations are irreducible by assumption, therefore $p_{ij} \neq 0$ for all $i \neq j$ follows from Theorem 63. The following general property is needed in the subsequent proofs.

Theorem 65 *Any two stiffnesses corresponding to* $(\mathbf{P}, \mathbf{Q})$ *and* $(\mathbf{P} + \alpha\mathbf{I}, \mathbf{Q})$, *where* α *is an arbitrary real number, have identical sets of co-centers.*

Proof. For a stiffness described by $(\mathbf{P}' = \mathbf{P} + \alpha\mathbf{I}, \mathbf{Q})$, $\mathbf{P}'_A = \mathbf{P} + \alpha\mathbf{I} + \vec{\mathbf{r}}_A\times = \mathbf{P}_A + \alpha\mathbf{I}$ for any point A such that $\overrightarrow{EA} = \vec{\mathbf{r}}_A$. Also, $\mathbf{Q}'_A = \mathbf{Q}_A$. Then, $\mathbf{Q}'_A\mathbf{P}'_A = \mathbf{Q}_A\mathbf{P}_A + \alpha\mathbf{Q}_A$. Since $\mathbf{Q}_A$ is always symmetric, then $\mathbf{Q}'_A\mathbf{P}'_A$ is symmetric if and only if $\mathbf{Q}_A\mathbf{P}_A$ is symmetric. This proves the theorem.

Note that $\mathbf{P} + \alpha\mathbf{I}$ amounts to a shifting of all eigenvalues of $\mathbf{P}$ by the same constant. Therefore, the differences $(p_i - p_j)$ remain unchanged. The general co-center equations involve only $(p_i - p_j)$. Hence, the solutions remain the same. This is clearly demonstrated in (5.37).

■

The following theorem indicates all allowable solutions to (5.37).

Theorem 66 *If there exists a co-center at* E, *then every co-center is on a line through* E, *which is parallel to an eigenvector of* $\mathbf{P}$.

Proof. If there exists a co-center at E, then $\mathbf{QP} = \mathbf{PQ}$, and the co-center equation in matrix form is

$$\begin{aligned} 2\vec{\rho}\times\vec{\rho}\times\vec{\rho}\times + 2\vec{\rho}\times\vec{\rho}\times\mathbf{P} - 2\mathbf{P}\vec{\rho}\times\vec{\rho}\times + & \\ -2\mathbf{P}\vec{\rho}\times\mathbf{P} + \vec{\rho}\times\mathbf{P}^2 + \mathbf{P}^2\vec{\rho}\times - \mathbf{Q}\vec{\rho}\times - \vec{\rho}\times\mathbf{Q} &= \mathbf{0} \end{aligned} \tag{5.38}$$

Note that, if the terms in the above equation are grouped with respect to the degree of $\vec{\rho}\times$ they contain, each group is skew-symmetric by itself. So, it is possible to define the following vectors.

$$\vec{\mathbf{a}} = 2\mathrm{vector}(\vec{\rho}\times\vec{\rho}\times\vec{\rho}\times) = -2\rho^2\mathrm{vector}(\vec{\rho}\times) = -2\rho^2\vec{\rho} \tag{5.39}$$

$$\vec{\mathbf{b}} = 2\text{vector}(\vec{\rho}\times\vec{\rho}\times\mathbf{P} - \mathbf{P}\vec{\rho}\times\vec{\rho}\times) \tag{5.40}$$

$$\vec{\mathbf{c}} = \text{vector}(-2\mathbf{P}\vec{\rho}\times\mathbf{P} + \vec{\rho}\times\mathbf{P}^2 + \mathbf{P}^2\vec{\rho}\times - \mathbf{Q}\vec{\rho}\times - \vec{\rho}\times\mathbf{Q}) \tag{5.41}$$

Clearly, (5.38) means $\vec{\mathbf{a}}+\vec{\mathbf{b}}+\vec{\mathbf{c}}=\vec{\mathbf{0}}$. Therefore, the vectors $\vec{\mathbf{a}},\vec{\mathbf{b}},\vec{\mathbf{c}}$ are either coplanar or collinear. Hence, a necessary condition for the existence of co-centers is $\vec{\mathbf{a}}^T\vec{\mathbf{b}}\times\vec{\mathbf{c}}=0$. This is trivially satisfied for $\vec{\rho}=\vec{\mathbf{0}}$. So, assume $\vec{\rho}\neq\vec{\mathbf{0}}$.

If the vectors are collinear then $\vec{\mathbf{a}}\times\vec{\mathbf{b}}=\vec{\mathbf{0}}$, etc. for all pairs. Using the identity $\vec{\mathbf{u}}\times\vec{\mathbf{v}}\times = \vec{\mathbf{v}}\vec{\mathbf{u}}^T - (\vec{\mathbf{u}}^T\vec{\mathbf{v}})\,\mathbf{I}$ for three-vectors in (5.40) twice, one shows that $\vec{\mathbf{b}} = -2\vec{\rho}\times\mathbf{P}\vec{\rho}$. So, for collinearity one must have $\vec{\mathbf{a}}\times\vec{\mathbf{b}} = -4\rho^2\vec{\rho}\times\vec{\rho}\times\mathbf{P}\vec{\rho}=\vec{\mathbf{0}}$. But, this can be true if and only if $\mathbf{P}\vec{\rho}=\lambda\vec{\rho}$. Hence, in this case, the line from the elastic center to any co-center must be parallel to an eigenvector of $\mathbf{P}$.

On the other hand, if the vectors are coplanar, then $\vec{\mathbf{a}}^T\vec{\mathbf{c}}\times\vec{\mathbf{b}}=0$ gives

$$\vec{\rho}^T(\mathbf{G}+\mathbf{H})\vec{\rho}=0 \tag{5.42}$$

where

$$\mathbf{G} = -2\mathbf{P}\vec{\rho}\times\mathbf{P}\vec{\rho}\times\mathbf{P} + \mathbf{P}^2\vec{\rho}\times\vec{\rho}\times\mathbf{P} \tag{5.43}$$

$$\mathbf{H} = -\mathbf{Q}\vec{\rho}\times\vec{\rho}\times\mathbf{P} \tag{5.44}$$

The equations (5.42) can be true if and only if $\mathbf{G}+\mathbf{H}$ is *not* definite.

By Theorem 65, any solution to (5.42) must also be a solution when $\mathbf{P}+\alpha\mathbf{I}$ is used instead of $\mathbf{P}$, where α is arbitrary. This can be used to show that there exists an α which makes both $\mathbf{G}$ and $\mathbf{H}$ semi-positive definite, thus leaving only the singular solutions. To do this each matrix is analyzed separately.

Matrix G: From (5.43) $\mathbf{G}=\mathbf{P}(-2\vec{\rho}\times\mathbf{P}\vec{\rho}\times+\mathbf{P}\vec{\rho}\times\vec{\rho}\times)\mathbf{P}$. Using $\mathbf{P}+\alpha\mathbf{I}$ one gets

$$\mathbf{G}(\alpha) = (\mathbf{P}+\alpha\mathbf{I})\left[-2\vec{\rho}\times\mathbf{P}\vec{\rho}\times+\mathbf{P}\vec{\rho}\times\vec{\rho}\times-\alpha\vec{\rho}\times\vec{\rho}\times\right](\mathbf{P}+\alpha\mathbf{I}) \tag{5.45}$$

Letting $\mathbf{P}(\alpha) = \mathbf{P} + \alpha\mathbf{I}$ and $\Psi(\alpha) = -2\vec{\rho}\times\mathbf{P}\vec{\rho}\times + \mathbf{P}\vec{\rho}\times\vec{\rho}\times - \alpha\vec{\rho}\times\vec{\rho}\times$, yields

$$\mathbf{G}(\alpha) = \mathbf{P}(\alpha)\Psi(\alpha)\mathbf{P}(\alpha) \tag{5.46}$$

The eigenvalues of $\mathbf{P}(\alpha)$ are linearly shifted by α. Therefore, there exists an α_1 such that for all $\alpha > \alpha_1$ the matrix $\mathbf{P}(\alpha)$ is positive definite. For all such α, $\mathbf{G}(\alpha)$ and $\Psi(\alpha)$ are similar matrices. That is their eigenvalues have identical signs. For the eigenvalues of $\Psi(\alpha)$, assume that the coordinate system is aligned so that $\vec{\rho} = [\ 1 \quad 0 \quad 0\ ]^T$. Then, the form of $\mathbf{P}$ is

$$\mathbf{P} = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \tag{5.47}$$

The eigenvalues of $\Psi(\alpha)$ can be verified to be

$$\lambda_1 = 0 \tag{5.48}$$

$$\lambda_{2,3} = \alpha + \frac{1}{2}(b+c) \pm \frac{3}{2}\sqrt{(b-c)^2 + 4f^2} \tag{5.49}$$

which are all real. Furthermore, since $\lambda_{2,3}$ change linearly with α, there exists an α_2 such that for all $\alpha > \alpha_2$ the eigenvalues $\lambda_{2,3}$ are positive. Thus, for all $\alpha > \alpha_2$, the matrix $\Psi(\alpha)$ is semi-positive definite. Now, for all $\alpha > \alpha_1, \alpha_2$, $\mathbf{G}(\alpha)$ and $\Psi(\alpha)$ are similar and $\Psi(\alpha)$ is semi-positive definite. Therefore, by similarity, for all $\alpha > \alpha_1, \alpha_2$, $\mathbf{G}(\alpha)$ is semi-positive definite. Let $\alpha_G = \max(\alpha_1, \alpha_2)$. Then, the condition for the semi-positive definiteness of $\mathbf{G}(\alpha)$ is $\alpha > \alpha_G$.

Matrix H: Again, assume that the coordinate system is aligned so that $\vec{\rho} = [\ 1 \quad 0 \quad 0\ ]^T$. Then, the form of $\mathbf{P}$ is as in (5.47) and the form of $\mathbf{Q}$ is

$$\mathbf{Q} = \begin{bmatrix} q_1 & q_4 & q_5 \\ q_4 & q_2 & q_6 \\ q_5 & q_6 & q_3 \end{bmatrix} \tag{5.50}$$

By introducing $\mathbf{P}(\alpha) = \mathbf{P} + \alpha\mathbf{I}$ in (5.44) one gets

$$\mathbf{H}(\alpha) = -\mathbf{Q}\vec{\rho} \times \vec{\rho} \times \mathbf{P}(\alpha) = -\mathbf{Q}\vec{\rho} \times \vec{\rho} \times \mathbf{P} - \alpha\mathbf{Q}\vec{\rho} \times \vec{\rho}\times \tag{5.51}$$

whose eigenvalues are of the form

$$\lambda_1 = 0 \tag{5.52}$$

$$\lambda_{2,3} = h_1(\alpha) \pm \sqrt{h_2(\alpha)} \tag{5.53}$$

where $h_1(\alpha)$ and $h_2(\alpha)$ are respectively linear and quadratic functions of α. The coefficient of α in $h_1(\alpha)$ is $\frac{1}{2}(q_2+q_3)$, which is always positive due to the positive definiteness of $\mathbf{Q}$. Thus, there exists an α_3 such that $h_1(\alpha) > 0$ for all $\alpha > \alpha_3$. The coefficient of α^2 in $h_2(\alpha)$ is $\frac{1}{4}(q_2+q_3)^2+q_6^2$, which is always positive, too. So, there exists an α_4 such that $h_2(\alpha) > 0$ for all $\alpha > \alpha_4$, which means that $\lambda_{2,3}$ are real. Therefore, for $\alpha > \alpha_3, \alpha_4$ the eigenvalue $h_1(\alpha) + \sqrt{h_2(\alpha)}$ is positive. For the eigenvalue $h_1(\alpha) - \sqrt{h_2(\alpha)}$, one investigates the sign of $h_2(\alpha) - h_1^2(\alpha)$, which is a quadratic function of α with leading coefficient $(q_2q_3 - q_6^2)$. But, $(q_2q_3 - q_6^2)$ is the determinant of the lower 2×2 principal minor of $\mathbf{Q}$, which is always positive due to the positive definiteness of $\mathbf{Q}$. Consequently, there exists an α_5 such that $h_1(\alpha) - \sqrt{h_2(\alpha)} > 0$ for all $\alpha > \alpha_3, \alpha_5$. Let $\alpha_H = \max(\alpha_3, \alpha_4, \alpha_5)$. Then, for all $\alpha > \alpha_H$, the matrix $\mathbf{H}(\alpha)$ is semi-positive definite.

The sum of two semi-positive definite matrices is at least semi-positive definite. Therefore, since $\mathbf{G}(\alpha)$ is semi-positive definite for all $\alpha > \alpha_G$, and, $\mathbf{H}(\alpha)$ is semi-positive definite for all $\alpha > \alpha_H$, one concludes that

$$\mathbf{G}(\alpha) + \mathbf{H}(\alpha) \text{ is at least semi-positive definite for all } \alpha > \max(\alpha_G, \alpha_H) \tag{5.54}$$

This means that only the singular solutions can exists for (5.42). In both $\mathbf{G}(\alpha)$ and $\mathbf{H}(\alpha)$ the zero eigenvalue corresponds to the same eigenvector since the coordinate directions were taken the same. Hence, $\mathbf{G}(\alpha) + \mathbf{H}(\alpha)$ is singular for all α, and $\mathbf{G}(\alpha) + \mathbf{H}(\alpha)$ is semi-positive definite for all $\alpha > \max(\alpha_G, \alpha_H)$.

From (5.46) and (5.51), one shows that for both $\mathbf{G}(\alpha)$ and $\mathbf{H}(\alpha)$, the zero eigenvalue of $\mathbf{G}(\alpha)+\mathbf{H}(\alpha)$ corresponds to the vectors $\mathbf{P}(\alpha)\vec{\rho} = \lambda\vec{\rho}$, which is equivalent to $\mathbf{P}\vec{\rho} = (\alpha+\lambda)\,\vec{\rho}$. In other words, $\vec{\rho}$ must be an eigenvector of $\mathbf{P}$. But, as was shown earlier, this corresponds to the collinearity of $\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}$. Thus, the collinear case is the only solution. This proves the theorem.

■

The above theorem basically shows that all solutions to (5.37) must be in one of the forms $(x,0,0),(0,y,0)$ and $(0,0,z)$, when $\mathbf{P}$ is diagonal. For each of these forms, the co-center equations are solved as

$$x = \pm\sqrt{\frac{1}{2}(p_{23}^2 - q_2 - q_3)} \qquad y,z=0 \tag{5.55}$$

$$y = \pm\sqrt{\frac{1}{2}(p_{31}^2 - q_3 - q_1)} \qquad z,x=0 \tag{5.56}$$

$$z = \pm\sqrt{\frac{1}{2}(p_{12}^2 - q_1 - q_2)} \qquad x,y=0 \tag{5.57}$$

which is a total of **six** points, real and complex. Together with the trivial solution $x=y=z=0$, the co-center equations have exactly **seven** finite solutions. The signs of the quantities $p_{ij}^2 - q_i - q_j, (i \neq j)$ determine whether the points are real or complex. Note that the points are found on three mutually perpendicular lines, each line containing two non-trivial co-centers that are equally separated from E in opposite directions.

For positive definite stiffnesses, there is an important inequality relation between the matrices $\mathbf{P}$ and $\mathbf{Q}$. The compliance matrix in $\vec{\rho}$-space is

$$\hat{C}_O = \begin{bmatrix} \mathbf{I}+\mathbf{P}_O\left(\mathbf{Q}_O-\mathbf{P}_O^T\mathbf{P}_O\right)^{-1}\mathbf{P}_O^T & -\mathbf{P}_O\left(\mathbf{Q}_O-\mathbf{P}_O^T\mathbf{P}_O\right)^{-1} \\ -\left(\mathbf{Q}_O-\mathbf{P}_O^T\mathbf{P}_O\right)^{-1}\mathbf{P}_O^T & \left(\mathbf{Q}_O-\mathbf{P}_O^T\mathbf{P}_O\right)^{-1} \end{bmatrix} \tag{5.58}$$

at any point O. The matrix $\left(\mathbf{Q}_O-\mathbf{P}_O^T\mathbf{P}_O\right)^{-1}$, whose eigenvalues are the eigentwist compliances, is *invariant* under origin transformations. Therefore, $\left(\mathbf{Q}_O-\mathbf{P}_O^T\mathbf{P}_O\right)^{-1}$ and $\mathbf{Q}_O-\mathbf{P}_O^T\mathbf{P}_O$ are positive definite matrices for positive definite stiffnesses. By the invariance, $\mathbf{Q}_O-\mathbf{P}_O^T\mathbf{P}_O = \mathbf{Q}-\mathbf{P}^2$. So, for any non-zero vector $\vec{u}$ one must have $\vec{u}^T\left(\mathbf{Q}-\mathbf{P}^2\right)\vec{u} > 0$. By using unit vectors along the eigendirections of $\mathbf{P}$, with $\mathbf{P}$ diagonal, one shows that

$$q_i > p_i^2 \qquad \text{for all } i \tag{5.59}$$

Let $q_i = p_i^2 + \varepsilon_i$ where $\varepsilon_i > 0$. For real roots of (5.55) through (5.57)

$$p_{ij}^2 - q_i - q_j = (p_i - p_j)^2 - p_i^2 - p_j^2 - \varepsilon_i - \varepsilon_j > 0 \tag{5.60}$$

$$p_i p_j < -\tfrac{1}{2}(\varepsilon_i + \varepsilon_j) < 0 \tag{5.61}$$

Therefore, for any solution corresponding to p_{ij} to be real it is necessary, but not sufficient, that p_i and p_j have opposite signs. The only essential sign combination is $(+,+,-)$. So, one pair must always have the same sign yielding a complex co-center. The remaining two pairs may yield from none to 4 real co-centers. The following theorem summarizes these results.

Theorem 67 *If a co-center exists at E and the co-center equations are irreducible, then there exist either none, two or four additional co-centers. For every additional co-center there exists another located at an equal but opposite distance from E. Any line containing three co-centers is parallel to an eigenvector of* $\mathbf{P}$.

Figure 5.3 illustrates the statement of Theorem 67.

Example 68 *Any stiffness corresponding to*

$$\mathbf{P} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 66 & 0 & 0 \\ 0 & 40 & 0 \\ 0 & 0 & 7 \end{bmatrix} \tag{5.62}$$

has a co-center at the center of elasticity and four more co-centers at

$$\{x = \pm 1, y = 0, z = 0\}, \{x = 0, y = \pm 2, z = 0\} \tag{5.63}$$

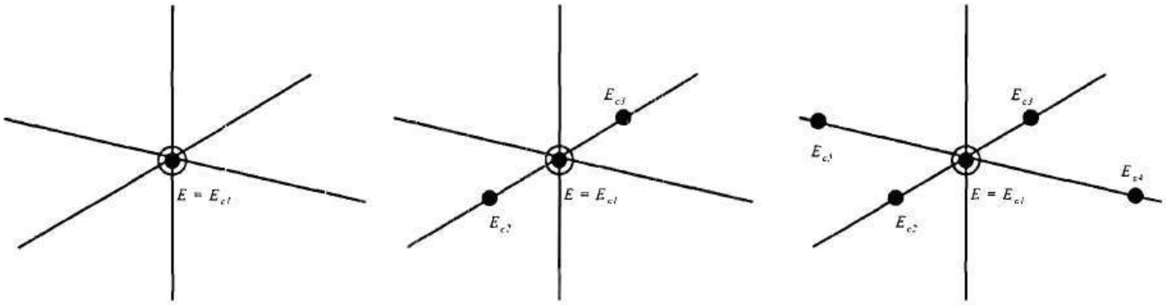


Figure 5.3: If there exists one co-center at the center of elasticity and the co-center equations are irreducible, then thre can be none, two or four non-trivial co-centers. Any line through E and a non-trivial co-center is parallel to a common eigenvector of $\mathbf{P}$ and $\mathbf{Q}$.

which can be verified numerically.

■

<u>A Co-center at E with Reducible Equations</u> In this case, the equations (5.37) still apply. Further, if any of those equations are reducible then, by Theorem 63, $p_{ij} = 0$ for some $i \neq j$.

Assume first that only one equation is reducible. Consider the first equation, for example. Then, $p_{23} = 0$, that is $p_2 = p_3$, which indicates that $p_{12} = -p_{31}$ and

$$\begin{aligned}
\left(2x^2 + 2y^2 + 2z^2 + q_2 + q_3\right) x &= 0 \\
\left(2y^2 + 2z^2 + 2x^2 - p_{12}^2 + q_3 + q_1\right) y + 2p_{12}zx &= 0 \\
\left(2z^2 + 2x^2 + 2y^2 - p_{12}^2 + q_1 + q_2\right) z - 2p_{12}xy &= 0
\end{aligned} \tag{5.64}$$

Since $\mathbf{Q}$ is positive definite for positive definite stiffnesses, $q_1, q_2, q_3 > 0$. Thus, for real solutions, the first equation gives $x = 0$. The equations reduce to

$$x = 0 \tag{5.65}$$

$$\left(2y^2 + 2z^2 - p_{12}^2 + q_3 + q_1\right) y = 0 \tag{5.66}$$

$$\left(2z^2 + 2y^2 - p_{12}^2 + q_1 + q_2\right) z \;=\; 0 \tag{5.67}$$

There are four distinct cases:

1. $y = z = 0$

2. $z = 0$ and $2y^2 + 2z^2 - p_{12}^2 + q_3 + q_1 = 0$

3. $y = 0$ and $2z^2 + 2y^2 - p_{12}^2 + q_1 + q_2 = 0$

4. $2y^2 + 2z^2 - p_{12}^2 + q_3 + q_1 = 0$ and $2z^2 + 2y^2 - p_{12}^2 + q_1 + q_2 = 0$

Case (1), $x = y = z = 0$, corresponds to the co-center at E. Cases (2) and (3) lead to the following solutions.

$$y \;=\; \pm\sqrt{\frac{1}{2}(p_{12}^2 - q_3 - q_1)} \qquad z, x = 0 \tag{5.68}$$

$$z \;=\; \pm\sqrt{\frac{1}{2}(p_{12}^2 - q_1 - q_2)} \qquad x, y = 0 \tag{5.69}$$

These give sets of co-centers similar to those of irreducible case, i.e. none, two or four additional real co-centers, except that all co-centers are in the yz-plane. Note that the yz-plane corresponds to the multiple eigenvalues of $\mathbf{P}$, namely $p_2 = p_3$.

In case (4), the equations to be satisfied represent concentric circles centered at E. Two concentric circles do not have finite common points unless they coincide. Therefore, for an allowable solution their radii must be the same. This condition is equivalent to $q_2 = q_3$. The solution becomes

$$y^2 + z^2 = \frac{1}{2}(p_{12}^2 - q_1 - q_2) \tag{5.70}$$

which is a real or complex circle of radius $\sqrt{\frac{1}{2}(p_{12}^2 - q_1 - q_2)}$, centered at E in yz-plane. Note that this also indicates that $\mathbf{Q}$ too has a double eigenvalue corresponding to the yz-plane.

Finally, if two of the co-center equations are reducible then $p_{23} = p_{31} = 0$. But this means $p_{12} = 0$, so the third equation must also be reducible. Therefore, $p_1 = p_2 = p_3$ and the co-center equations become

$$
\begin{aligned}
\left(2x^2+2y^2+2z^2+q_2+q_3\right)x &= 0 \\
\left(2y^2+2z^2+2x^2+q_3+q_1\right)y &= 0 \\
\left(2z^2+2x^2+2y^2+q_1+q_2\right)z &= 0
\end{aligned}
\tag{5.71}
$$

for which the only real solution is $x = y = z = 0$. So, the co-center at E is unique. These results are summarized in the following theorem and illustrated in Figure 5.4.

Theorem 69 *If a co-center exists at E and the co-center equations are reducible, then either*

1. *there exist none, two or four additional co-centers in the plane corresponding to a double eigenvalue of $\mathbf{P}$, or*
2. *there exists a circle of co-centers centered at E, in the plane corresponding to double eigenvalues of $\mathbf{P}$ and $\mathbf{Q}$, or*
3. *the co-center at E is unique and $\mathbf{P} = p\mathbf{I}$.*

Example 70 *Any stiffness corresponding to*

$$
\mathbf{P} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \qquad \mathbf{Q} = \begin{bmatrix} 18 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} \tag{5.72}
$$

has a co-center at E. In addition, every point on the unit circle in yz-plane, centered at E, is also a co-center. To verify this, let $\vec{\mathbf{r}}_A = [0, \sin\theta, \cos\theta]^T$ be the position vector of a generic point A on the

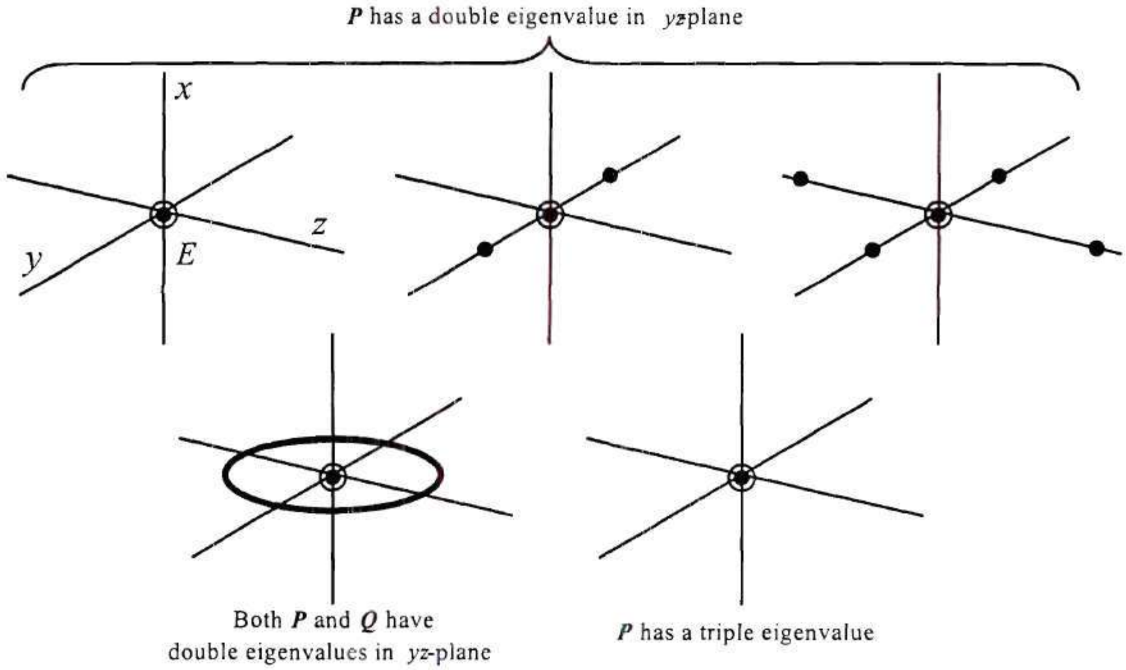


Figure 5.4: Arrangement of co-centers when there exists a co-center at E and the co-center equations are reducible.

unit circle. Then,

$$\mathbf{P}_A = \mathbf{P} + \vec{\mathbf{r}}_A \times = \begin{bmatrix} 4 & -\cos\theta & \sin\theta \\ \cos\theta & -1 & 0 \\ -\sin\theta & 0 & -1 \end{bmatrix} \tag{5.73}$$

$$\mathbf{Q}_A = \mathbf{Q} + \mathbf{P}\vec{\mathbf{r}}_A \times - \vec{\mathbf{r}}_A \times \mathbf{P} - \vec{\mathbf{r}}_A \times \vec{\mathbf{r}}_A \times \tag{5.74}$$

$$= \begin{bmatrix} 19 & -5\cos\theta & 5\sin\theta \\ -5\cos\theta & 5+\cos^2\theta & -\cos\theta\sin\theta \\ 5\sin\theta & -\cos\theta\sin\theta & 5+\sin^2\theta \end{bmatrix} \tag{5.75}$$

and calculate $\mathbf{Q}_A\mathbf{P}_A$. After simplifications, this is obtained as

$$\mathbf{Q}_A\mathbf{P}_A = \begin{bmatrix} 71 & -14\cos\theta & 14\sin\theta \\ -14\cos\theta & 2\cos 2\theta - 3 & -2\sin 2\theta \\ 14\sin\theta & -2\sin 2\theta & -2\cos 2\theta - 3 \end{bmatrix} \tag{5.76}$$

which is symmetric regardless of θ. Hence, every point on the unit circle is a co-center as claimed.

■

5.3 Interpretation of Results

The co-center equations have been analyzed in a space different from the original Cartesian space. The transformation from the original space to the $\vec{\rho}$-space was given by (5.21), namely $\vec{\rho}\times = \mathbf{A}^{\frac{1}{2}}\vec{\mathbf{r}} \times \mathbf{A}^{\frac{1}{2}}$. To understand what this transformation means the following lemma is first presented.

Lemma 71 *For any non-singular matrix* $\mathbf{G}$,

$$\mathbf{G}\vec{\mathbf{r}} \times \mathbf{G}^T = \det(\mathbf{G})[\mathbf{G}^{-T}\vec{\mathbf{r}}]\times \tag{5.77}$$

Proof. Any square matrix $\mathbf{G}$ can be written as $\mathbf{G} = \mathbf{F}\mathbf{R}$, where $\mathbf{F}$ is symmetric and $\mathbf{R}$ is an orthogonal matrix. The solutions are

$$\mathbf{F} = \left(\mathbf{G}\mathbf{G}^T\right)^{\frac{1}{2}} \qquad \mathbf{R} = \left(\mathbf{G}\mathbf{G}^T\right)^{-\frac{1}{2}}\mathbf{G} \tag{5.78}$$

Clearly, $\mathbf{F}$ is symmetric and one can verify $\mathbf{R}\mathbf{R}^T = \mathbf{I}$ so that $\mathbf{R}$ is orthogonal. Then,

$$\mathbf{G}\vec{\mathbf{r}} \times \mathbf{G}^T = \mathbf{F}\left(\mathbf{R}\vec{\mathbf{r}} \times \mathbf{R}^T\right)\mathbf{F} \tag{5.79}$$

It is well known that for orthogonal matrices $\mathbf{R}\vec{\mathbf{r}} \times \mathbf{R}^T = (\mathbf{R}\vec{\mathbf{r}})\times$. So, letting $\vec{\mathbf{r}}' = \mathbf{R}\vec{\mathbf{r}}$, one gets $\mathbf{G}\vec{\mathbf{r}} \times \mathbf{G}^T = \mathbf{F}\vec{\mathbf{r}}' \times \mathbf{F}$. Let the coordinate axes be oriented in such a way that $\mathbf{F}$ is diagonal. Let f_i

be the diagonal elements of $\mathbf{F}$ and r_i be the components of $\vec{\mathbf{r}}'$. Then,

$$\mathbf{F}\vec{\mathbf{r}}' \times \mathbf{F} = \begin{bmatrix} 0 & -f_1 f_2 r_3 & f_1 f_3 r_2 \\ f_1 f_2 r_3 & 0 & -f_2 f_3 r_1 \\ -f_1 f_3 r_2 & f_2 f_3 r_1 & 0 \end{bmatrix} = f_1 f_2 f_3 \begin{bmatrix} 0 & -\frac{r_3}{f_3} & \frac{r_2}{f_2} \\ \frac{r_3}{f_3} & 0 & -\frac{r_1}{f_1} \\ -\frac{r_2}{f_2} & \frac{r_1}{f_1} & 0 \end{bmatrix} \tag{5.80}$$

which is equivalent to $\det(\mathbf{F})\left[\mathbf{F}^{-1}\vec{\mathbf{r}}'\right]\times$. Thus

$$\mathbf{F}\vec{\mathbf{r}}' \times \mathbf{F} = \det(\mathbf{F})\left[\mathbf{F}^{-1}\vec{\mathbf{r}}'\right]\times = \det(\mathbf{F})\left[\mathbf{F}^{-1}\mathbf{R}\vec{\mathbf{r}}\right]\times \tag{5.81}$$

Using the facts that $\det(\mathbf{G}) = \det(\mathbf{FR}) = \det(\mathbf{F})$ and $\mathbf{G}^{-T} = \mathbf{F}^{-1}\mathbf{R}$, one gets the equation (5.77) proving the lemma.

■

Applying the above lemma to the stiffness space transformation one finds that

$$\vec{\rho}\times = \mathbf{A}^{\frac{1}{2}}\vec{\mathbf{r}} \times \mathbf{A}^{\frac{1}{2}} = \det(\mathbf{A}^{\frac{1}{2}})\left[\mathbf{A}^{-\frac{1}{2}}\vec{\mathbf{r}}\right]\times \tag{5.82}$$

$$\vec{\rho} \;=\; \det(\mathbf{A}^{\frac{1}{2}})\mathbf{A}^{-\frac{1}{2}}\vec{\mathbf{r}} \qquad\qquad \vec{\mathbf{r}} = \det(\mathbf{A}^{-\frac{1}{2}})\mathbf{A}^{\frac{1}{2}}\vec{\rho} \tag{5.83}$$

Therefore, the transformation from the original Cartesian space to the $\vec{\rho}$-space is a linear map performed by $\det(\mathbf{A}^{\frac{1}{2}})\mathbf{A}^{-\frac{1}{2}}$. It is well known that such transformations map lines to lines, planes to planes, etc. The eigenvectors of $\mathbf{A}^{\frac{1}{2}}$ are the directions of eigenwrenches, $\vec{\mathbf{f}}_i$. Therefore, any vector parallel to a $\vec{\mathbf{f}}_i$ is only scaled. Any other vector is both scaled and rotated under such transformations.

Let $\vec{\rho}$ be an eigenvector of $\mathbf{P}$ corresponding to an eigenvalue p. Then, $\mathbf{P}\vec{\rho} = p\vec{\rho}$. Using $\mathbf{P} = \mathbf{A}^{-\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}$ and premultiplying by $\mathbf{A}^{\frac{1}{2}}$, one gets $\mathbf{B}\mathbf{A}^{\frac{1}{2}}\vec{\rho} = p\mathbf{A}^{\frac{1}{2}}\vec{\rho}$, which shows that p and $\mathbf{A}^{\frac{1}{2}}\vec{\rho}$ are the eigenvalue and eigenvector of $\mathbf{B}$. So, $\mathbf{B}$ and $\mathbf{P}$ have identical eigenvalues. The eigenvectors of $\mathbf{B}$, say $\vec{\mathbf{u}}$, are related to those of $\mathbf{P}$ via the map $\vec{\mathbf{u}} = \mathbf{A}^{\frac{1}{2}}\vec{\rho}$ or $\vec{\mathbf{u}} = \det(\mathbf{A}^{-\frac{1}{2}})\mathbf{A}^{\frac{1}{2}}\vec{\rho}$, after a scaling. Thus, the mapping from $\vec{\rho}$-space to the original space maps the eigenvectors of $\mathbf{P}$ into those of $\mathbf{B}$. Note that, in general, the eigenvectors of $\mathbf{B}$ are oblique to each other.

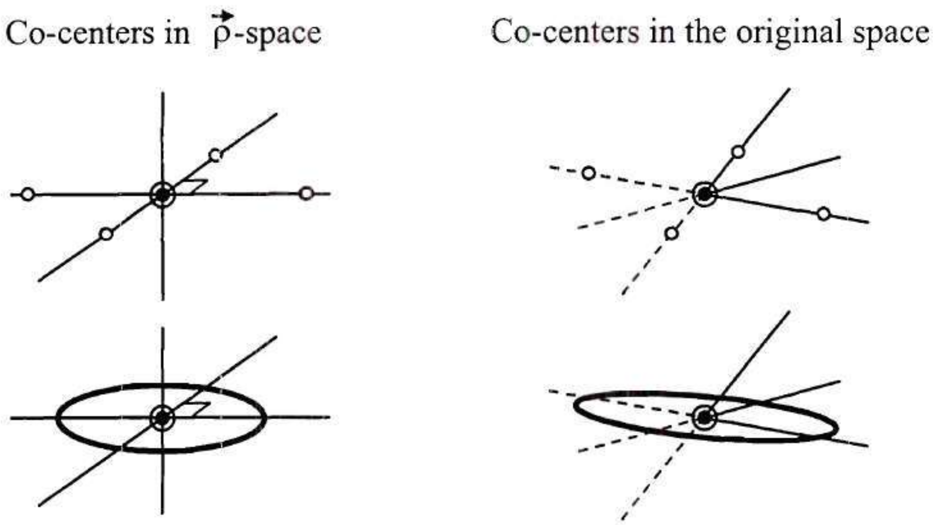


Figure 5.5: The positive definite linear map from the $\vec{\rho}$-space to the original space preserves the collinearity. However, initially perpendicular lines are mapped to oblique lines. A circle is mapped to an ellipse in general.

The above identifications enable one to characterize the distribution of the co-centers in the original Cartesian space. Consider the special cases corresponding to the existence of a co-center at E. Note that E is the fixed point of the mapping. If a co-center exists at E, then either it is unique or,

1. there exist two or four additional co-centers, each pair on a line through E and parallel to an eigenvector of $\mathbf{B}$ (Figure 5.5), and, at equal and opposite distances from E (four additional co-centers are in general on an oblique pair of lines), or

2. every point on an ellipse centered at E is a co-center (Figure 5.5). A necessary, but not sufficient, condition is that $\mathbf{B}$ has a double eigenvalue. The ellipse is in the plane spanned by the two eigenvectors of $\mathbf{B}$ corresponding to the double eigenvalue.

The eigenvalues and eigenvectors of $\mathbf{Q}$ and $\mathbf{C}$ are not as easily related to each other as those of $\mathbf{P}$ and $\mathbf{B}$. For example, a double eigenvalue of $\mathbf{Q}$ does not necessarily indicate a double eigenvalue of $\mathbf{C}$. So, results cannot be easily interpreted using the eigen properties of $\mathbf{C}$. This proves the advantage of performing the analyses in the $\vec{\rho}$-space.

In the next chapter, physical meanings of these special cases are explained using the concept of compliant axes.

CHAPTER VI

ISOTROPIC SCREW PAIRS

The free-vector and line-vector eigenvalue problems presented in the previous chapters lead to the eigenscrew and co-eigenscrew subspaces of stiffness and compliance. The eigentwist and eigenwrench subspaces are respectively the images of the couple and translation bundles and are unique. Couples and translations are free-vectors, i.e. infinite pitch screws. The unique common center of these two eigensystems is the center of elasticity. On the other hand, the co-eigentwist and co-eigenwrench subspaces are respectively the images of force and rotation bundles through a point called the generator. Therefore, for every generator in space there exists a pair of co-eigensystems. Forces and rotations are line-vectors, i.e. zero pitch screws. In Chapter 5, some of the centers of these pairs of co-eigensystems are shown to be analogous to the center of elasticity and therefore called the co-center of elasticity.

Patterson and Lipkin [42] introduced the force- and rotation-compliant axes definitions based on the eigenscrew systems. Then, a collinear pair of a force- and rotation-compliant axes gives a compliant axis. Note that the force- and rotation-compliant axes are zero pitch screws whose actions are infinite pitch screws. In Chapter 3, various relations between the centers of elasticity, stiffness and compliance are presented.

The purpose of this chapter stems from the following observations:

- The zero and infinite pitch screws play a central role in the definition of compliant axes, as well as the free- and line-vector eigenvalue problems.

- The current definitions of force-compliant, rotation-compliant and compliant axes do not include any reference to the co-eigenscrew systems.

- The relations between compliant axes and centers do not include the co-center of elasticity.

This chapter demonstrates that the existence of zero and infinite pitch screws in the eigen- and co-eigenscrew systems are related to each other. This leads to generalized and systematic re-definitions of compliant and other special axes, which naturally exhibit the complementary features of the eigen- and co-eigensystems. Furthermore, a new kind of special axis is found. Also, it is shown that the co-centers are equally involved in the existence of compliant axes.

6.1 Infinite and Zero Pitch Screws in Eigen- and Co-eigensystems

First, the existence of infinite pitch screws in the eigensystems is shown to be possible only for non-positive definite systems. Then, as a complementary result, the existence of zero pitch screws in the co-eigensystems intersecting their actions is shown to be possible only for non-positive definite systems. In particular, for positive definite systems, the co-eigensystems do *not* contain zero pitch screws through their generators. Finally, the existence of zero pitch screws in the eigensystems is shown to be equivalent to the existence of infinite pitch screws in the co-eigensystems.

Recall that if a screw $\hat{S}$ has zero or infinite pitch then $\frac{1}{2}\hat{S}^T\Delta\hat{S} = 0$. In Chapter 8, a vector whose scalar product with respect to a matrix vanishes is called an *isotropic vector* of that matrix. Therefore, the zero and infinite pitch screws are the isotropic vectors of the matrix $\hat{\Delta}$. Chapter 8 presents a detailed analysis of isotropic vectors.

Both the eigen- and co-eigenwrench systems are obtained as the images of special screw sub-spaces under the stiffness mapping. Consider the general mapping $\hat{W} = \hat{K}\hat{T}$. For the eigenwrench system, $\hat{T}$ is an infinite pitch twist (a translation) and, for the co-eigenwrench system, $\hat{T}$ is a zero pitch twist (a rotation). The condition for $\hat{W}$ to have zero or infinite pitch is

$$\frac{1}{2}\hat{W}^T\Delta\hat{W} = \frac{1}{2}\hat{T}^T\left(\hat{K}\Delta\hat{K}\right)\hat{T} = 0 \tag{6.1}$$

Thus, $\hat{T}$ is an isotropic vector of $\hat{K}\Delta\hat{K}$ which causes $\hat{W}$, an isotropic vector of Δ. Such pairs of twists and wrenches are the focus of this chapter. Accordingly, they are called the **isotropic screw pairs**.

Similarly, for the eigen- and co-eigentwist systems the general mapping $\hat{T} = \hat{C}\hat{W}$. For the eigentwist system, $\hat{W}$ is an infinite pitch wrench (a couple) and, for the co-eigentwist system, $\hat{W}$ is a zero pitch wrench (a force). The condition for $\hat{T}$ to have zero or infinite pitch is

$$\frac{1}{2}\hat{T}^T \Delta \hat{T} = \frac{1}{2}\hat{W}^T \left(\hat{C}\Delta\hat{C}\right) \hat{W} = 0 \tag{6.2}$$

Again, $\hat{W}$ is an isotropic vector of $\hat{C}\Delta\hat{C}$ which causes $\hat{T}$, an isotropic vector of Δ. So, $\hat{W}$ and $\hat{T}$ form an isotropic screw pair.

Equation (6.1) means that the eigenwrench (co-eigenwrench) system contains a zero or infinite pitch screw if it is due to a translation (rotation) which is also an isotropic vector of the matrix $\frac{1}{2}\hat{K}\Delta\hat{K}$. Similarly, (6.2) means that the eigentwist (co-eigentwist) system contains a zero or infinite pitch screw if it is due to a couple (force) which is also an isotropic vector of the matrix $\frac{1}{2}\hat{C}\Delta\hat{C}$. In the following sections, all cases resulting from (6.1) and (6.2) are presented separately. The following expressions are used subsequently.

$$\frac{1}{2}\hat{K}\Delta\hat{K} = \frac{1}{2}\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathbf{A}\mathbf{B}^T + \mathbf{B}\mathbf{A} & \mathbf{A}\mathbf{C} + \mathbf{B}\mathbf{B} \\ \mathbf{B}^T\mathbf{B}^T + \mathbf{C}\mathbf{A} & \mathbf{B}^T\mathbf{C} + \mathbf{C}\mathbf{B} \end{bmatrix} \tag{6.3}$$

$$\frac{1}{2}\hat{C}\Delta\hat{C} = \frac{1}{2}\begin{bmatrix} \mathbf{D} & \mathbf{E}^T \\ \mathbf{E} & \mathbf{F} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{D} & \mathbf{E}^T \\ \mathbf{E} & \mathbf{F} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \mathbf{D}\mathbf{E} + \mathbf{E}^T\mathbf{D} & \mathbf{D}\mathbf{F} + \mathbf{E}^T\mathbf{E}^T \\ \mathbf{E}\mathbf{E} + \mathbf{F}\mathbf{D} & \mathbf{E}\mathbf{F} + \mathbf{F}\mathbf{E}^T \end{bmatrix} \tag{6.4}$$

where $\mathbf{A}, \mathbf{B}, \ldots$,etc. are the submatrices of stiffness and compliance.

6.1.1 Zero and Infinite Pitch Screws in Eigensystems

The existence of infinite pitch screws in the eigenscrews systems is related to the singularity of diagonal submatrices as the following theorem shows.

Theorem 72 *E_T (E^*_W) contains infinite pitch screws if and only if the matrix* $\mathbf{A}$ *(*$\mathbf{F}$*) is singular.*

Proof. Assume that there exists an infinite pitch twist (a translation) $\begin{bmatrix} \vec{\delta}^T & \vec{0}^T \end{bmatrix}^T \in E_T$. Since every element of E_T is due to a couple, it follows that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \vec{\delta} \\ \vec{0} \end{bmatrix} = \begin{bmatrix} \mathbf{A}\vec{\delta} \\ \mathbf{B}^T\vec{\delta} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{\mathbf{v}} \end{bmatrix} \tag{6.5}$$

where $\vec{\mathbf{v}}$ is the couple. The first equation in (6.5) is $\mathbf{A}\vec{\delta} = \vec{0}$ from which the theorem follows. Proof for E_W^* is similar and is based on $\mathbf{F}\vec{\tau} = \vec{0}$, where $\vec{\tau}$ is a couple.

■

Corollary 73 *For a positive definite elastic system, the eigenscrew systems do not contain infinite pitch screws.*

Proof. For positive definite systems, both $\mathbf{A}$ and $\mathbf{F}$ are positive definite, hence non-singular. Then, the corollary follows due to Theorem 72.

■

Physically, Corollary 73 states that, for a positive definite elastic system, a pure couple *cannot* produce a pure translation, and vice versa. Note that the existence of pure translations in the eigentwist system requires a singular $\mathbf{A}$ so that at least one of the linear stiffnesses, k_{fi}, is zero. On the other hand, the existence of pure couples in the eigenwrench system requires a singular $\mathbf{F}$ so that at least one of angular compliances, $a_{\gamma i}$, is zero.

Since the eigensystems are the images of infinite pitch screws, if one takes $\hat{T} = [\ \vec{\mathbf{u}}^T \quad \vec{0}^T\ ]^T$ in (6.1) and $\hat{W} = [\ \vec{0}^T \quad \vec{\mathbf{v}}^T\ ]^T$ in (6.2) then

$$\frac{1}{2}\hat{T}^T\left(\hat{K}\Delta\hat{K}\right)\hat{T} = \frac{1}{2}\vec{\mathbf{u}}^T(\mathbf{A}\mathbf{B}^T + \mathbf{B}\mathbf{A})\vec{\mathbf{u}} = \vec{\mathbf{u}}^T(\mathbf{A}\mathbf{B}^T)_{\mathrm{sym}}\vec{\mathbf{u}} = 0 \tag{6.6}$$

$$\frac{1}{2}\hat{W}^T\left(\hat{C}\Delta\hat{C}\right)\hat{W} = \frac{1}{2}\vec{\mathbf{v}}^T(\mathbf{E}\mathbf{F} + \mathbf{F}\mathbf{E}^T)\vec{\mathbf{v}} = \vec{\mathbf{v}}^T(\mathbf{F}\mathbf{E}^T)_{\mathrm{sym}}\vec{\mathbf{v}} = 0 \tag{6.7}$$

where $\vec{\mathbf{u}}$ is a translation that yields either a pure force or a pure couple in the eigenwrench system, and $\vec{\mathbf{v}}$ is a couple that yields either a pure translation or a pure rotation in the eigentwist system. Using Corollary 73 and equations (6.6) and (6.7) one gets

Theorem 74 *For a positive definite elastic system,*

1. *the eigenwrench system, E_W^*, contains zero pitch screws (pure forces) if and only if $(\mathbf{AB}^T)_{sym}$ has isotropic vectors, i.e. $(\mathbf{AB}^T)_{sym}$ is indefinite or singular, and*

2. *the eigentwist system, E_T, contains zero pitch screws (pure rotations) if and only if $(\mathbf{FE}^T)_{sym}$ has isotropic vectors, i.e. $(\mathbf{FE}^T)_{sym}$ is indefinite or singular.*

■

Let A and B be two points, then $\mathbf{AB}_B^T = \mathbf{AB}_A^T - \mathbf{A}\overrightarrow{AB} \times \mathbf{A}$. But, since $\mathbf{A}\overrightarrow{AB} \times \mathbf{A}$ is always skew-symmetric, one gets $(\mathbf{AB}_A^T)_{\mathrm{sym}} = (\mathbf{AB}_B^T)_{\mathrm{sym}}$. Thus, the matrix $(\mathbf{AB}^T)_{\mathrm{sym}}$ is independent of origin. Also, note that the matrices $(\mathbf{AB}^T)_{\mathrm{sym}}$ and $\mathbf{A}^{-1}(\mathbf{AB}^T)_{\mathrm{sym}}\mathbf{A}^{-1}$ have identical signatures. Now, since

$$\mathbf{A}^{-1}(\mathbf{AB}^T)_{\mathrm{sym}}\mathbf{A}^{-1} = (\mathbf{B}^T\mathbf{A}^{-1})_{\mathrm{sym}} = (\mathbf{A}^{-1}\mathbf{B})_{\mathrm{sym}} \tag{6.8}$$

$(\mathbf{A}^{-1}\mathbf{B})_{\mathrm{sym}}$ is origin independent and has the same definiteness with $(\mathbf{AB}^T)_{\mathrm{sym}}$. But, if E is the center of elasticity, then $\mathbf{A}^{-1}\mathbf{B}_E$ is always symmetric, so $(\mathbf{A}^{-1}\mathbf{B})_{\mathrm{sym}} = \mathbf{A}^{-1}\mathbf{B}_E$, see Chapter 5.1. The eigenvalues of $\mathbf{A}^{-1}\mathbf{B}_E$, and therefore of $(\mathbf{A}^{-1}\mathbf{B})_{\mathrm{sym}}$, are simply the principal pitches of the principal screws. As a result, if $(\mathbf{A}^{-1}\mathbf{B})_{\mathrm{sym}}$ is indefinite then the minimum principal pitch is negative and the maximum is positive. But, from Ball's work [1], these are the necessary and sufficient conditions for the 3-system of eigenscrews to contain zero pitch screws. Conclusions for $(\mathbf{FE}^T)_{\mathrm{sym}}$ are similar. Hence, Theorem 74 is a restatement of Ball's result on 3-systems of screws. However, Theorem 74 is preferable since it provides an explicit and simple method to calculate the zero pitch screws in the eigensystems, namely by finding appropriate isotropic vectors.

The following example demonstrates the use of Theorem 74 in finding a zero pitch screw of the eigenwrench system.

Example 75 *Consider the following randomly selected, positive definite stiffness.*

$$\hat{K} = \begin{bmatrix} 21 & 13 & 12 & 13 & 24 & 20 \\ 13 & 20 & 7 & 16 & 20 & 12 \\ 12 & 7 & 13 & 13 & 15 & 15 \\ 13 & 16 & 13 & 20 & 20 & 16 \\ 24 & 20 & 15 & 20 & 30 & 23 \\ 20 & 12 & 15 & 16 & 23 & 23 \end{bmatrix} \tag{6.9}$$

For the eigenwrenches, one first calculates $(\mathbf{AB}^T)_{\text{sym}}$,

$$(\mathbf{AB}^T)_{\text{sym}} = \begin{bmatrix} 825 & 764.5 & 616 \\ 764.5 & 692 & 531 \\ 616 & 531 & 456 \end{bmatrix} \tag{6.10}$$

This is an indefinite matrix and therefore satisfies the requirements of Theorem 74. Using the methods explained in Chapter 8, an isotropic vector of $(\mathbf{AB}^T)_{\text{sym}}$ is found as

$$\vec{\mathbf{u}} = [\ -0.6817 \quad 0.4928 \quad 0.5409\ ]^T \tag{6.11}$$

This is a translation that maps to an element of the eigenwrench system as explained earlier,

$$\hat{W} = \hat{K} \begin{bmatrix} \vec{\mathbf{u}} \\ \vec{\mathbf{0}} \end{bmatrix} = \begin{bmatrix} -1.4188 & 4.7795 & 2.3005 & 6.0536 & 1.6080 & 0.3927 \end{bmatrix}^T \tag{6.12}$$

One easily verifies that $\hat{W}^T \hat{\Delta} W = 0$. Therefore, $\hat{W}$ is a zero pitch element of the eigenwrench system.

■

As it is shown in Chapter 5, the identity $\hat{K}\hat{C} = \hat{I}$ leads to $\mathbf{A}^{-1}\mathbf{B} = -\mathbf{E}^T\mathbf{F}^{-1}$. By also considering (6.8) one finds that, in Theorem 74, $\mathbf{A}\mathbf{B}^T$ is indefinite or singular if and only if so is $\mathbf{E}^T\mathbf{F}$. Therefore, it follows that a zero pitch screw exists in the eigenwrench system if and only if a zero pitch screw exists in the eigentwist system. This is sensible since the eigenvalues of $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{E}^T\mathbf{F}^{-1}$ are respectively the principal pitches in the eigenwrench and eigentwist systems, which are equal in magnitude and opposite in sign. However, there is more to this relation. To see this let there be a zero pitch screw in the eigenwrench system, i.e. a pure force, given as $[\ \vec{\mathbf{f}}^T \quad \vec{\mathbf{0}}^T\ ]^T$ at a point on the screw axis. Then, by definition, this force must be due to a pure translation, i.e.

$$\hat{K}\begin{bmatrix} \vec{\mathbf{u}} \\ \vec{\mathbf{0}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\mathbf{0}} \end{bmatrix} \tag{6.13}$$

The essential part of this equation is $\mathbf{B}^T\vec{\mathbf{u}} = \vec{\mathbf{0}}$. But, from linear algebra, if $\mathbf{B}^T$ is singular then so is $\mathbf{B}$. Therefore, there exists a vector $\vec{\gamma}$ such that $\mathbf{B}\vec{\gamma} = \vec{\mathbf{0}}$ or

$$\hat{K}\begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\gamma} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{v}} \end{bmatrix} \tag{6.14}$$

where $\vec{\mathbf{v}}$ is a couple. Then, by definition, $[\ \vec{\mathbf{0}}^T \quad \vec{\gamma}^T\ ]^T$ is a zero pitch element of the eigentwist system through the same point. This result is summarized in the following theorem.

Theorem 76 *The eigenwrench system, E_W^*, contains a zero pitch screw (pure force) through a point O if and only if the eigentwist system, E_T, contains a zero pitch screw (pure rotation) through O.*

Note that in the proof of Theorem 76, the discussion is focused on an arbitrary point on one of the zero pitch eigensystem element. Since one can take any other point on the screw axis, it follows that if a zero pitch screw $\hat{W}$ exists in, say, E_W^*, then E_T contains zero pitch screws through every point on the screw axis of $\hat{W}$. Then, reversing the argument, one concludes that the same is true for a zero pitch screw in E_T. In summary, every zero pitch screw of an eigenscrew system is

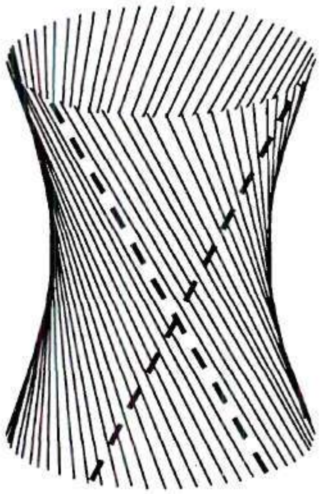


Figure 6.1: A hyperboloid of one sheet is a surface of revolution generated by a line. For 3-systems, the lines are the axes of equal pitch screws. Here, all zero pitch screws of an eigensystem are shown. The dashed lines are the generators, one for the eigensystem, the other for the reciprocal system.

intersected at every point by zero pitch screws of the reciprocal system. This is a well known result for n-system of screws. For a 3-system, the axes of all zero pitch screws form a surface called a **hyperboloid** of one sheet, which is a surface of revolution generated by any one of the axes, see Figure 6.1. Thus, the zero pitch screws of an eigenscrew system do not intersect each other. The zero pitch screws of the reciprocal system also forms the same surface. Note that since the isotropic vectors of a 3×3 matrix is in general a 1-parameter family, the zero pitch screws of eigensystems can also be generated as a 1-parameter family.

6.1.2 Zero and Infinite Pitch Screws in Co-eigensystems

In the previous section, it is shown that the existence of infinite pitch screws in the eigensystems requires the singularity of the origin independent diagonal submatrices. The analogous result for the co-eigensystems concerns zero pitch screws. However, it is essentially different since, instead of existence, it restricts the intersection of zero pitch screws with their actions.

Theorem 77 *$E_{cT/G}$ ($E^{\star}_{cW/G}$) contains zero pitch screws intersecting their actions (zero pitch screws through G) if and only if the matrix $\mathbf{C}$ ($\mathbf{D}$) is singular at the intersection.*

Proof. Assume that there exists a zero pitch twist (a pure rotation) in $E_{cT/G}$. Every element of $E_{cT/G}$ is due to a pure force (a zero pitch wrench). If the rotation and the force intersect each other at a point A, then they can be given at A as $\hat{T}_A = \begin{bmatrix} \vec{0}^T & \vec{\gamma}^T \end{bmatrix}_A^T$ and $\hat{W}_A = \begin{bmatrix} \vec{\mathbf{f}}^T & \vec{0}^T \end{bmatrix}_A^T$, respectively. Since the stiffness maps the rotation to the force, it follows that

$$\hat{K}_A \hat{T}_A = \begin{bmatrix} \mathbf{A} & \mathbf{B}_A \\ \mathbf{B}_A^T & \mathbf{C}_A \end{bmatrix} \begin{bmatrix} \vec{0} \\ \vec{\gamma} \end{bmatrix} = \begin{bmatrix} \mathbf{B}_A \vec{\gamma} \\ \mathbf{C}_A \vec{\gamma} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{0} \end{bmatrix} \tag{6.15}$$

The second equation in (6.15) means $\mathbf{C}_A \vec{\gamma} = \vec{0}$ from which the theorem follows. Proof for $E^{\star}_{cW/G}$ is similar and is based on $\mathbf{D}_A \vec{\mathbf{f}} = \vec{0}$, where $\vec{\mathbf{f}}$ is a force.

■

Since the actions of all co-eigenscrews are all zero pitch screws through the generator (force and rotation bundles), Theorem 77 leads to the following corollary in particular.

Corollary 78 *For a positive definite elastic system, $E_{cT/G}$ and $E^{\star}_{cW/G}$ do not contain zero pitch screws through the generator G.*

Proof. If a zero pitch element of $E_{cT/G}$ or $E^{\star}_{cW/G}$ pass through G, then it intersects its action since all co-eigenscrews are the images of zero pitch screws through G. Then, by Theorem 77 either $\mathbf{C}_G$ or $\mathbf{D}_G$ must be singular. However, for positive definite systems $\mathbf{C}$ and $\mathbf{D}$ are positive definite at all points. Thus, there cannot be such zero pitch screws.

■

Physically, Corollary 73 states that if a positive definite elastic system only rotates under the action of a pure force, then the axes of the rotation and the force cannot intersect. For positive definite systems, existence of infinite pitch screws in the eigensystems is not allowed, but the existence of zero pitch screws in the co-eigensystems is allowed as long as they do not intersect their actions. Note that the existence of pure rotations in the co-eigentwist system requires a singular $\mathbf{C}$ so that at least one of the angular stiffnesses, k_{mi}, is zero. On the other hand, the existence of pure forces in the co-eigenwrench system requires a singular $\mathbf{D}$ so that at least one of linear compliances, a_{ti}, is zero.

For the general existence of infinite and zero pitch screws, the equations (6.1) and (6.2) are used. Since the co-eigensystems are the images of zero pitch screws through G, if one takes $\hat{T} = [\ \vec{\mathbf{0}}^T \quad \vec{\mathbf{u}}^T\ ]^T$ in (6.1) and $\hat{W} = [\ \vec{\mathbf{v}}^T \quad \vec{\mathbf{0}}^T\ ]^T$ in (6.2) then

$$\frac{1}{2}\hat{T}^T\left(\hat{K}\Delta\hat{K}\right)\hat{T} = \frac{1}{2}\vec{\mathbf{u}}^T(\mathbf{B}^T\mathbf{C}+\mathbf{C}\mathbf{B})\vec{\mathbf{u}} = \vec{\mathbf{u}}^T(\mathbf{B}^T\mathbf{C})_{\mathrm{sym}}\vec{\mathbf{u}} = 0 \tag{6.16}$$

$$\frac{1}{2}\hat{W}^T\left(\hat{C}\Delta\hat{C}\right)\hat{W} = \frac{1}{2}\vec{\mathbf{v}}^T(\mathbf{D}\mathbf{E}+\mathbf{E}^T\mathbf{D})\vec{\mathbf{v}} = \vec{\mathbf{v}}^T(\mathbf{D}\mathbf{E})_{\mathrm{sym}}\vec{\mathbf{v}} = 0 \tag{6.17}$$

where $\vec{\mathbf{u}}$ is a rotation through G that yields either a pure force or a pure couple in the co-eigenwrench system, and $\vec{\mathbf{v}}$ is a force through G that yields either a pure translation or a pure rotation in the co-eigentwist system. Again, $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are identified as the isotropic vectors of $(\mathbf{B}^T\mathbf{C})_{\mathrm{sym}}$ and $(\mathbf{D}\mathbf{E})_{\mathrm{sym}}$, respectively. Unlike the eigenscrews case, these matrices are origin dependent. The following is a summary of these results.

Theorem 79 *For an elastic system,*

1. *the co-eigenwrench system, $E^*_{cW/G}$, contains zero or infinite pitch screws if and only if $(\mathbf{B}^T\mathbf{C})_{sym}$ has isotropic vectors, i.e. $(\mathbf{B}^T\mathbf{C})_{sym}$ is indefinite or singular, and*

2. *the co-eigentwist system, $E_{cT/G}$, contains zero or infinite pitch screws if and only if $(\mathbf{D}\mathbf{E})_{sym}$ has isotropic vectors, i.e. $(\mathbf{D}\mathbf{E})_{sym}$ is indefinite or singular.*

The infinite pitch screws in co-eigensystems can be found directly as the following theorem shows.

Theorem 80 *$E^*_{cW/G}$ ($E_{cT/G}$) contains infinite pitch screws if and only if* $\mathbf{B}$ *(*$\mathbf{E}$*) is singular. Every rotation (force) in the null space of* $\mathbf{B}$ *yields a pure couple (translation) in $E^*_{cW/G}$ ($E_{cT/G}$).*

Proof. Consider the co-eigenwrench system which is the image of rotation bundle at G. Then, for an infinite pitch element (a pure couple),

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \vec{0} \\ \vec{\gamma} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{\tau} \end{bmatrix} \tag{6.18}$$

which means that $\mathbf{B}\vec{\gamma} = \vec{0}$, so that $\mathbf{B}$ is singular. Converse is trivial. Clearly, $\vec{\gamma}$ is a rotation in the null space of $\mathbf{B}$. The co-eigentwist case is similar.

■

As a demonstration, a zero pitch element of the co-eigenwrench system in Example 75 is found below.

Example 81 *One first calculates* $(\mathbf{B}^T\mathbf{C})_{sym}$,

$$(\mathbf{B}^T\mathbf{C})_{sym} = \begin{bmatrix} 788 & 1079.5 & 877.5 \\ 1079.5 & 1425 & 1147 \\ 877.5 & 1147 & 941 \end{bmatrix} \tag{6.19}$$

This is an indefinite matrix and therefore satisfies the requirements of Theorem 79. Using the methods explained in Chapter 8, an isotropic vector of $(\mathbf{B}^T\mathbf{C})_{sym}$ *is found as*

$$\vec{u} = [\ -0.7999 \quad 0.3199 \quad 0.5078\ ]^T \tag{6.20}$$

This is a rotation through G which maps to an element of the co-eigenwrench system,

$$\hat{W} = \hat{K} \begin{bmatrix} \vec{0} \\ \vec{u} \end{bmatrix} = \begin{bmatrix} 7.4350 & -0.3067 & 2.0171 & -1.4749 & 5.2786 & 6.2390 \end{bmatrix}^T \tag{6.21}$$

One easily verifies that $\hat{W}^T\hat{\Delta}W = 0$. Therefore, $\hat{W}$ is a zero pitch element of the co-eigenwrench system.

■

Corollary 82 *For a positive definite system,*

1. $E^*_{cW/G}$ *contains an infinite pitch screw if and only if* $E_{cT/G}$ *contains an infinite pitch screw.*

2. $E^*_{cW/G}$ *contains a zero pitch screw if and only if* $E_{cT/G}$ *contains a zero pitch screw.*

Proof. (1) If $E^*_{cW/G}$ contains an infinite pitch screw then $\mathbf{B}$ is singular, Theorem 80. But, by $\hat{K}\hat{C} = \hat{I}$, $\mathbf{C}^{-1}\mathbf{B}^T = -\mathbf{E}\mathbf{D}^{-1}$, so $\mathbf{B}$ is singular if and only if $\mathbf{E}$ is singular. Thus, by Theorem 80, $E_{cT/G}$ contains an infinite pitch screw. (2) If $E^*_{cW/G}$ contains a zero pitch screw then $(\mathbf{B}^T\mathbf{C})_{\mathrm{sym}} = (\mathbf{CB})_{\mathrm{sym}}$ has isotropic vectors, Theorem 79. By the properties of isotropic vectors, $\mathbf{CB}$ also has isotropic vectors, Chapter 8. The null spaces of $\mathbf{B}$ and $\mathbf{CB}$ are identical. Since the null space of $\mathbf{B}$ gives the infinite pitch screws, any zero pitch screw must be due to an isotropic vector which is not in the null space of $\mathbf{B}$, and therefore $\mathbf{CB}$. In other words, if $E^*_{cW/G}$ contains zero pitch screws, they must be due to the isotropic vectors in the range space of $\mathbf{CB}$. So, $\mathbf{CB}$ must be indefinite. Since $\mathbf{C}$ is non-singular due to positive definiteness, $\mathbf{C}^{-1}\mathbf{CBC}^{-1} = \mathbf{BC}^{-1}$ is indefinite, too. Clearly, this indicates that $\mathbf{C}^{-1}\mathbf{B}^T$ is also indefinite. Again, by $\hat{K}\hat{C} = \hat{I}$, one concludes that $\mathbf{ED}^{-1}$ is indefinite. So, as in the case of $\mathbf{CB}$, $\mathbf{DE}$ too has isotropic vectors not in the null space of $\mathbf{E}$, which give rise to zero pitch screws in $E_{cT/G}$. Converses are similar.

■

These theorems demonstrate the interesting and distinct behavior of the co-eigenscrew subspaces. In other words, whereas Theorem 72 shows that a pure moment cannot cause a pure translation, Theorem 77 states that there may be infinitely many pure forces that can cause pure rotations, not both through the same point. The latter quantities belong to the co-eigenscrew subspaces.

6.1.3 Pitch Relations Between Eigen- and Co-eigensystems

Inclusion of zero pitch screws in eigensystems and infinite pitch screws in co-eigensystems are closely related. This is demonstrated by the following theorem.

Theorem 83 *The following are equivalent.*

a) E_T *contains zero pitch screws through* G.

b) E^*_W *contains zero pitch screws through* G.

c) $E_{cT/G}$ *contains infinite pitch screws.*

d) $E^*_{cW/G}$ *contains infinite pitch screws.*

Proof. (a)⇔(b) follows from Theorem 76, and (c)⇔(d) follows from Corollary 82. So, it suffices to show, say, (a)⇔(d). Consider the point G as the origin. Then,

$$\hat{C}\begin{bmatrix} \vec{0} \\ \vec{\mathbf{m}} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{\gamma} \end{bmatrix} \quad \Leftrightarrow \quad \hat{K}\begin{bmatrix} \vec{0} \\ \vec{\gamma} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{\mathbf{m}} \end{bmatrix} \tag{6.22}$$

But, by definition, $[\ \vec{0}^T \quad \vec{\gamma}^T\ ]^T$ is a zero pitch element of the eigentwist system since it is due to a pure couple $\vec{\mathbf{m}}$, and $[\ \vec{0}^T \quad \vec{\mathbf{m}}^T\ ]^T$ is an infinite pitch element of the co-eigenwrench system at G since it is due to a rotation $\vec{\gamma}$ through G. This proves the theorem.

■

Consider the zero pitch screw subspaces of E_T or E^*_W, whose elements must intersect at a point G. So, G is the generator of these subspaces. Let $E^0_{T/G}$ and $E^{*0}_{W/G}$ be the zero pitch screw subspaces of E_T and E^*_W, respectively, with generator G. Also, let $E^{\infty}_{cT/G}$ and $E^{*\infty}_{cW/G}$ be the infinite pitch screw subspaces of $E_{cT/G}$ and $E^*_{cW/G}$, respectively. Then, an immediate consequence of Theorem 83 is the following.

Corollary 84 $\dim(E^0_{T/G}) = \dim(E^{*0}_{W/G}) = \dim(E^{\infty}_{cT/G}) = \dim(E^{*\infty}_{cW/G})$. *In words, there are as many pure rotations in E_T through G as there are pure forces in E^*_W through G or pure translations in $E_{cT/G}$ or pure couples in $E^*_{cW/G}$, and vice versa.*

6.2 Compliant Axes: A New Perspective

The zero pitch elements of the eigenscrew spaces and the infinite pitch elements of the co-eigenscrew spaces are special and deserve more attention. They are also rarities in real elastic systems. The existence of a zero pitch element of the eigenwrench system means that there exists a pure force that causes only a pure translation. Similarly, a zero pitch element of the eigentwist system indicates the existence of a pure rotation that causes only a pure couple. An infinite pitch element of a co-eigenwrench system is a couple that causes a pure rotation, and that of a co-eigentwist system is a translation that causes a pure force. Moreover, existence of any of such special screws is related to the others, Theorem 83.

These isotropic screw pairs have close ties to compliant axes, Section 3.6. A compliant axis is a collinear pair of zero pitch eigenwrench and eigentwist. A zero pitch eigenwrench is called a force-compliant axis and a zero pitch eigentwist is called a rotation-compliant axis. The difference of these quantities from those implied in Theorem 83 is just that a zero pitch eigenscrew is parallel to its action. So, Patterson and Lipkin's definitions isolate particular cases of zero pitch screws in the eigenscrew systems.

There are two points to be improved in these classifications. Firstly, the definitions do not include references to co-eigenscrew spaces, concealing the symmetry introduced by Theorem 83 concerning eigen- and co-eigenscrew spaces. In other words, Theorem 83 shows that the infinite pitch screws of co-eigenscrew systems are as important as the zero pitch screws of eigenscrew systems, equivalent indeed. Secondly, the uses of the suffix *-compliant*, a historical remnant, and the *force-*

and *rotation-* adjectives do not explain the causal relationship and put an unnecessary emphasis on zero pitch screws.

6.2.1 Definition of Special Axes

The following generalizations of compliant axes terminology are proposed to clarify the causal relations between, and to preserve the symmetry within, the eigen- and co-eigenscrew systems.

Definition 85

- *A zero pitch element of the eigenwrench system is called the* ***generalized force-translation axis****.*
- *The action of an infinite pitch element of the co-eigentwist system is called the* ***generalized translation-force axis****.*

Equivalency of the above definitions is clearly seen from Theorem 83. So, the first usage is preferable since the pure force directly determines the axis. The translation is a free-vector, so only its action determines the axis. Definition 85 simply points to a case in which *a general pure force causes a general pure translation*, and vice versa. The adjective *generalized* only indicates that the force and translation parts need not be parallel, see Figure 6.2. When the force is parallel to the translation one gets the *force-compliant axis.*

Definition 86

- *A zero pitch element of the eigentwist system is called the* ***generalized rotation-couple axis****.*
- *The action of an infinite pitch element of the co-eigenwrench system is called the* ***generalized couple-rotation axis****.*

Again, the above definitions are equivalent and the first one is preferable. Definition 86 simply point to a case in which *a general pure rotation causes a general pure couple*, and vice versa. When

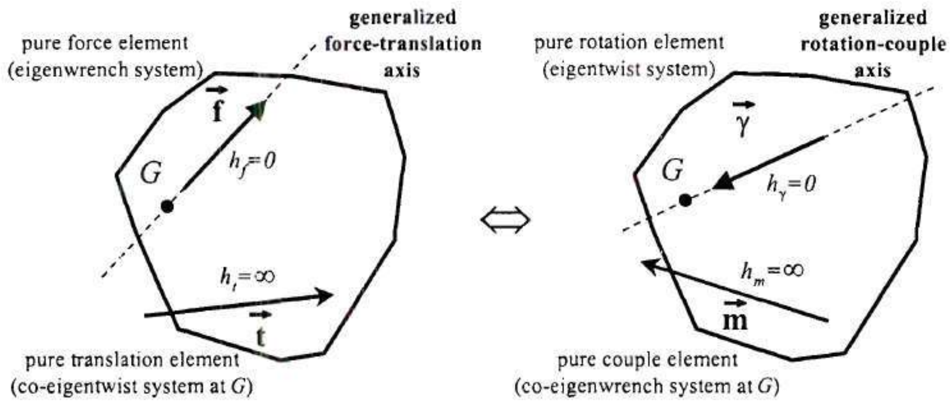


Figure 6.2: Generalized force-translation and rotation-couple axes.

the rotation is parallel to the couple one gets the *rotation-compliant axis.* Figure 6.2 demonstrates these definitions as well as Theorem 83. Note that, by Theorem 83, a generalized force-translation axis through G exists if and only if a generalized rotation-couple axis exists through G.

Force- and rotation-compliant axes appear as particular cases of Definitions 85 and 86. For a force-compliant axis, the force part becomes a zero pitch eigenwrench which necessarily makes its action, the translation, an infinite pitch co-eigentwist by definition. Similar things apply for the rotation-compliant axis. These suggest the following definitions that are intended to replace the force- and rotation-compliant axes definitions.

Definition 87

- *A zero pitch eigenwrench whose action is an infinite pitch co-eigentwist is called the* ***force-translation axis.***
- *A zero pitch eigentwist whose action is an infinite pitch co-eigenwrench is called the* ***rotation-couple axis.***

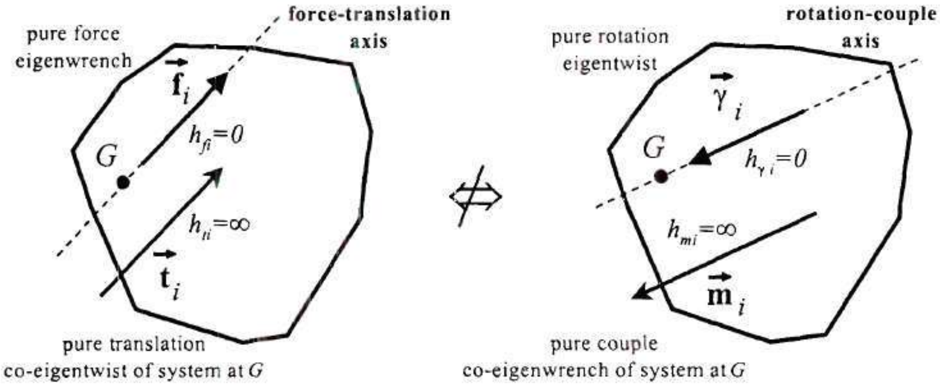


Figure 6.3: Force-translation and rotation-couple axes. Existence of one does not imply the existence of the other.

As can be concluded from the above definitions, a force-compliant axis is redefined as a force-translation axis which corresponds to a zero pitch eigenwrench through a point G, parallel to an infinite pitch co-eigentwist of $E_{cT/G}$, and, a rotation-compliant axis is redefined as a rotation-couple axis which corresponds to a zero pitch eigentwist through G, parallel to an infinite pitch co-eigenwrench of $E^*_{cW/G}$. These are special cases of the generalized definitions and occur whenever an action is parallel to its reaction. Figure 6.3 demonstrates the force-translation and rotation-couple axes. An important observation is that one does not imply the other. Rather, if a force-translation axis through G exists then there exists at least a generalized rotation-couple axis through G, and so on.

6.2.2 Pairs of Zero Pitch Screws

All of the definitions in the previous section concern pairs of screws one with zero pitch and one with infinite pitch. What about pairs with zero-zero or infinite-infinite pitches? For positive definite mappings, Theorem 72 shows that a pure couple cannot cause a pure translation, ruling out the possibility of infinite-infinite pitch screw pairs. Therefore, as an analogy to the above definitions, a *translation-couple axis* cannot exist. On the other hand, Theorem 79 shows the possibility of having a pure force causing a pure rotation, though not through the same point. It is interesting that these force/rotation pairs are solely due to co-eigenscrew systems. The eigenscrew systems have nothing to do with such pairs since, by definition, their actions are always infinite pitch screws. These force/rotation pairs have not been noticed in the literature, although there exists a well known topic in dynamics, namely the *center of percussion*, which involves a special case of these pairs. This is explained in more detail in Chapter 10.

By Theorem 77, the force/rotation pairs are always non-intersecting. The following corollary shows that they are never parallel, either.

Corollary 88 *For positive definite systems, the zero pitch screws, if any, of the co-eigenscrew subspaces are skew with respect to their actions.*

Proof. Contrary to the statement, assume that there exists a zero pitch screw in the co-eigenwrench system generated by G that is parallel to its action. Then, one can write, at G,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \vec{0} \\ \alpha\vec{\mathbf{a}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{a}} \\ \vec{\mathbf{r}} \times \vec{\mathbf{a}} \end{bmatrix} \quad \Rightarrow \quad \begin{array}{c} \alpha\mathbf{B}\vec{\mathbf{a}} = \vec{\mathbf{a}} \\ \alpha\mathbf{C}\vec{\mathbf{a}} = \vec{\mathbf{r}} \times \vec{\mathbf{a}} \end{array} \tag{6.23}$$

However, these equations imply that $\vec{\mathbf{a}}^T\mathbf{C}\vec{\mathbf{a}} = 0$ contradicting the assumption that the mappings are positive definite. Hence, a zero pitch eigenwrench cannot be parallel to its action as the corollary states. The proof for co-eigentwist case is similar. Since Theorem 77 proved that the zero pitch screw cannot intersect its action, this completes the proof.

■

Note that, for non-positive definite systems one could have the parallelism or even the intersection. These pairs complete the description of special screws in eigen- and co-eigenscrew systems, and their relationship with each other. So, the following definitions are proposed.

Definition 89

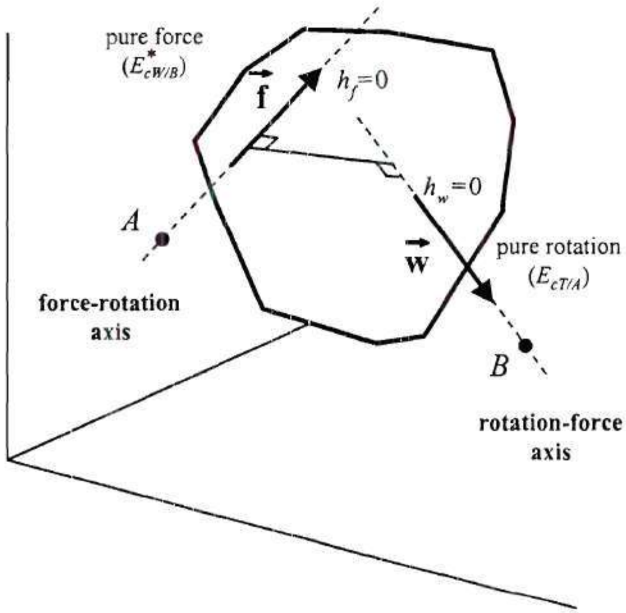


Figure 6.4: Force-rotation and rotation-force axes. These are distinct axes, but one implies the other.

- *A zero pitch wrench of the co-eigenwrench system is called a* ***force-rotation axis****.*
- *A zero pitch twist of the co-eigentwist system is called a* ***rotation-force axis****.*

Figure 6.4 shows a pair of such axes. In Figure 6.4, $\vec{\mathbf{f}}$ is a pure force through that generates a pure rotation $\vec{\mathbf{w}}$ through B. Thus, the rotation is a zero pitch screw of $E_{cT/A}$ by definition. Reversing the perspective, the pure rotation $\vec{\mathbf{w}}$ through B causes the pure force $\vec{\mathbf{f}}$ through A. So, by definition, the force is a zero pitch screw of $E^*_{cW/B}$. Unlike in Definitions 86 and 87, the above definitions are not equivalent. The existence of one guaranties the existence of the other. Nevertheless, they describe distinct axes which are always skew, Corollary 88.

6.2.3 Compliant Axis Redefined

Next and last in the hierarchy of special axes are the compliant axes. In the new terminology proposed here, a compliant axis is a pair of collinear force-translation and rotation-couple axes. Hence, the following redefinition of a compliant axis is proposed.

Definition 90 *The common axis of a collinear pair of force-translation and rotation-couple axes is called a* ***compliant axis****.*

Again, the term *compliant axis* is historic and does not tell much about the actual character of this special axis. However, in order to be in agreement with the current literature it is continued to be used in this study. The most important property of a compliant axis is that the stiffness and compliance matrices are decoupled with respect to linear and angular quantities along this axis.

6.2.4 Compliant Axes and Centers

In Section 3.6, it was shown that a compliant axis passes through the centers of stiffness, compliance and elasticity. Naturally, one would wonder why the co-centers of elasticity are missing in this picture. For an elastic system with three compliant axes, such as an RCC device, all three previous centers coalesce, and the off-diagonals of the stiffness and compliance matrices vanish at the common point. For this case, it is not difficult to see that all co-centers of elasticity also coalesce with the others since the matrix $\mathbf{CB} = \mathbf{0}$ and the co-center equations have only one solution, see Chapter 5 for details. So, looking for a relation between a single compliant axis and the co-centers is not unfounded. The following theorem establishes this relation and is analogous of Theorem 38.

Theorem 91 *If a compliant axis exists, it passes through at least one and at most three co-centers of elasticity.*

Proof. It is easier to prove this explicitly. Assume that there exists a compliant axis. Since, by Theorem 41, a compliant axis must pass through S, the center of stiffness, the following must hold at S,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{a}} \\ \vec{0} \end{bmatrix} = k_{\mathrm{ft}} \begin{bmatrix} \vec{\mathbf{a}} \\ \vec{0} \end{bmatrix} \qquad \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & \mathbf{C} \end{bmatrix} \begin{bmatrix} \vec{0} \\ \vec{\mathbf{a}} \end{bmatrix} = k_{\mathrm{rc}} \begin{bmatrix} \vec{0} \\ \vec{\mathbf{a}} \end{bmatrix} \tag{6.24}$$

where $\vec{\mathbf{a}}$ is a unit vector along the compliant axis, and, k_{ft} and k_{rc} are the appropriate eigenstiffnesses. The subscripts $_{\mathrm{ft}}$ and $_{\mathrm{rc}}$ indicate the force-translation and rotation-couple axes. Note that $\mathbf{B}$ is symmetric at S. It is clear that $\vec{\mathbf{a}}$ is an eigenvector of both $\mathbf{A}$ and $\mathbf{C}$, and is in the null space $\mathbf{B}$. Let the coordinate axes be aligned such that $\vec{\mathbf{a}} = \vec{\mathbf{e}}_1 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$, and $\vec{\mathbf{e}}_2$ and $\vec{\mathbf{e}}_3$ are along the remaining eigenvectors of $\mathbf{A}$. Then, the forms of the submatrices must be

$$\mathbf{A} = \begin{bmatrix} k_{f1} & 0 & 0 \\ 0 & k_{f2} & 0 \\ 0 & 0 & k_{f3} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & b_{22} & b_{23} \\ 0 & b_{23} & b_{33} \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} k_{\gamma 1} & 0 & 0 \\ 0 & c_{22} & c_{23} \\ 0 & c_{23} & c_{33} \end{bmatrix} \tag{6.25}$$

where $k_{f1} = k_{\mathrm{ft}}$ and $k_{\gamma 1} = k_{\mathrm{rc}}$. Now, with these definitions, assume that the origin is shifted to another point A on the compliant axis given by $\vec{\mathbf{r}} = r\vec{\mathbf{e}}_1$ from S. By using the stiffness transformations, one can determine the matrix $\mathbf{C}_A\mathbf{B}_A$ which must be symmetric for A to be a co-center. The form of $\mathbf{C}_A\mathbf{B}_A$ is

$$\mathbf{C}_A\mathbf{B}_A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \bullet & \bullet \\ 0 & \bullet & \bullet \end{bmatrix} \tag{6.26}$$

The skew-symmetric part of this matrix has only one non-trivial component, which is a function of r and stiffness matrix coefficients, of the form

$$\begin{aligned} 2\left[(\mathbf{C}_A\mathbf{B}_A)_{\mathrm{skew}}\right]_{32} &= 2k_{3f}k_{2f}r^3 + 3b_{23}\left(k_{2f} - k_{3f}\right)r^2 \\ &\quad - \left((b_{22} - b_{33})^2 + 4b_{23}^2 - c_{33}k_{3f} - c_{22}k_{2f}\right)r \\ &\quad + \left(c_{33} - c_{22}\right)b_{23} + \left(b_{22} - b_{33}\right)c_{23} \end{aligned} \tag{6.27}$$

Since this is a third order polynomial in r there is at least one and at most three real solutions. This gives at least one co-center on the compliant axis. For three real solutions, if only two are distinct then one has two co-centers, and if all three are distinct one has three co-centers on the compliant axis. This proves the theorem. Note that there can be other co-centers which are not on the compliant axis.

■

Theorem 92 *If two compliant axes exist then E, S, C coalesce and there exists a co-center at E. In general, on each compliant axis there can be either none or two more co-centers at equal and opposite distances from E. In particular, there can be an ellipse of co-centers centered at E.*

Proof. The fact that E, S, C coalesce when two or more compliant axes exist has been proven earlier, Chapter 3. As in the previous theorem, one first determines the forms of the submatrices. In this case,

$$\mathbf{A} = \begin{bmatrix} k_{f1} & 0 & 0 \\ 0 & k_{f2} & 0 \\ 0 & 0 & k_{f3} \end{bmatrix} \qquad \mathbf{B} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & b_{33} \end{bmatrix} \qquad \mathbf{C} = \begin{bmatrix} k_{\gamma 1} & 0 & 0 \\ 0 & k_{\gamma 2} & 0 \\ 0 & 0 & k_{\gamma 3} \end{bmatrix} \tag{6.28}$$

which is at E. Clearly, E is a co-center since $\mathbf{BC}$ is already symmetric. The compliant axis directions are the eigenvectors of $\mathbf{B}$ corresponding to zero eigenvalues. Also, due to the double eigenvalue of $\mathbf{B}$, the co-center equations are reducible. So, by Theorem 69 there can be none or two co-centers on each compliant axis, which are equally separated from E. In Chapter 5, the matrix $\mathbf{Q}$ is defined as $\mathbf{Q} = \mathbf{A}^{\frac{1}{2}}\mathbf{C}\mathbf{A}^{\frac{1}{2}}$. It is shown there that if the matrix $\mathbf{Q}$ has a double eigenvalue corresponding to the double eigenvalue of $\mathbf{B}$, then there can be an ellipse of co-centers. In terms of eigenstiffnesses this means $\frac{k_{f1}}{k_{\gamma 1}} = \frac{k_{f2}}{k_{\gamma 2}}$.

■

Finally, when there exist three compliant axes, the matrix $\mathbf{B}$ vanishes. Chapter 5 also defines a matrix $\mathbf{P}$ as $\mathbf{P} = \mathbf{A}^{-\frac{1}{2}}\mathbf{B}\mathbf{A}^{\frac{1}{2}}$. So, for three compliant axes, the matrix $\mathbf{P}$ vanishes, too. Then, by Theorem 69 there exists a unique co-center at E, which leads to the following.

Theorem 93 *If there exist three compliant axes all centers coalesce:* $S = C = E = E^c$.

Example 94 *Robotic hand stiffness co-centers. A unique co-center exists collinear with* E, S, C *on a compliant axis.*

The stiffness of the robotic hand example of the previous chapters is given at the tip of the tool as

$$\hat{K} = \begin{bmatrix} 2490 & 0 & 0 & 0 & 258 & 0 \\ 0 & 28900 & 0 & 191 & 0 & 0 \\ 0 & 0 & 61610 & 0 & 0 & 0 \\ 0 & 191 & 0 & 22 & 0 & 0 \\ 258 & 0 & 0 & 0 & 37 & 0 \\ 0 & 0 & 0 & 0 & 0 & 35 \end{bmatrix} \tag{6.29}$$

The center of elasticity is calculated to be $\vec{\mathbf{r}}_E = \begin{bmatrix} 0 & 0 & 0.0485 \end{bmatrix}^T$. First $\hat{K}$ is transformed into $\hat{K}_E$. Then, the matrices $\mathbf{P}$ and $\mathbf{Q}$ are calculated using (5.21), with $\mathbf{P}$ in its diagonal form after a suitable rotation. The results are

$$\mathbf{P} = \begin{bmatrix} 467.5 & 0 & 0 \\ 0 & -467.5 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \mathbf{Q} = \begin{bmatrix} 392750.7 & 122546.5 & 0 \\ 122546.5 & 392750.7 & 0 \\ 0 & 0 & 2156350.0 \end{bmatrix} \tag{6.30}$$

Referring to (5.26) one finds the co-center equations which are of the form

$$\left(2r^2 - p^2 + q_2 + q_3\right) x + 2pyz - q_4 y = 0$$

$$\begin{aligned}\left(2r^2 - p^2 + q_3 + q_1\right) y + 2pzx - q_4 x &= 0 \\ \left(2r^2 - 4p^2 + q_1 + q_2\right) z + 2p\left(q_4 - 2xy\right) &= 0\end{aligned} \tag{6.31}$$

where $r^2 = x^2 + y^2 + z^2$, $p = 467.5$, $q_1 = q_2 = 392750.7$, $q_3 = 2156350.0$, $q_4 = 122546.5$. An immediate solution is $x = y = 0$, which satisfies the first two of (6.31), and $z = -423.8$. Further analysis shows that the remaining solutions are complex. Therefore, the only real solution is $\vec{\rho} = \begin{bmatrix} 0 & 0 & -423.8 \end{bmatrix}^T$. In original coordinates this corresponds to $\vec{\mathbf{r}} = \begin{bmatrix} 0 & 0 & -0.04995 \end{bmatrix}^T$.

It is numerically verified that at the point indicated by this vector the matrix **CB** is symmetric as expected. So, in the case of the robotic hand, there exists a unique co-center located ~50.0 mm below the elastic center. With respect to the rivet tip, the original reference, the co-center is about 1.5 mm below the origin. This is very close to the center of stiffness which was found to be 2.1 mm above the rivet tip. It was demonstrated earlier that E, S, C are on the z-axis. This examples shows that a unique co-center exists which is on the compliant axis. So, E, S, C and the unique E^c are collinear on the compliant axis.

■

CHAPTER VII

ELASTIC CONNECTIONS WITH SPRINGS

Springs are common machine elements. However, their importance to this study stems from a different point of view. Springs can easily be said to be the simplest elastic elements. Using such simple elements to understand more complex elastic behavior is sensible. For example, in many cases, when one talks about some eigenstiffness of an elastic system in a certain direction, one usually tends to imagine a line spring modeling the elastic behavior in that direction with the given scalar stiffness. A generalization of this approach involves the synthesis of stiffnesses by springs (see Chapter 8). To do this one has to have a solid description of the stiffness of elastic connections with springs. Many researchers use springs to describe complicated stiffness situations, such as those occurring in complaint robots, grasp problems, etc. [22], [32], [43], [36], [29].

In this chapter, the stiffness matrix of elastic connections with springs is determined in a closed and compact form for all configurations of the body. It is very likely that such a closed form stiffness expression is possible only for systems involving springs. For example, it is not possible to find a closed form expression for systems containing beams because of their non-linear behavior for finite displacements. Therefore, elastic connections with springs constitutes a very valuable example demonstrating the structure of stiffness. Another important result is that the stiffness matrix of parallel spring connections is non-symmetric [8].

Some applications of the results obtained in this section are presented in Chapter 8. However, the potential for applications in many other areas of mechanisms, kinematics, robotics, etc. cannot be overstated if one only considers the sheer number of the uses of springs in these areas.

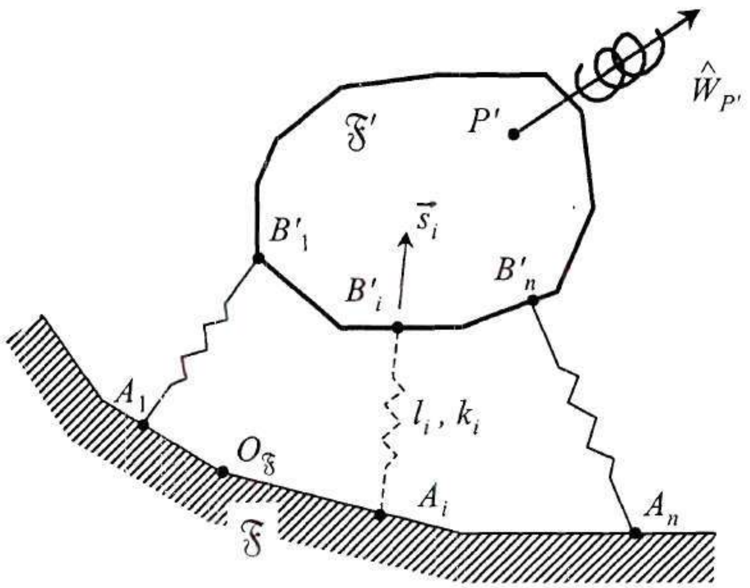


Figure 7.1: A rigid body connected to a fixed body via line springs. The springs are shown in equilibrium with the applied load.

Also presented in this chapter is the differentiation rules for screw quantities referred to different frames of reference. This is needed in subsequent sections. It also reaffirms the analogy between the spatial vector algebra and the classical 3-vector algebra.

7.1 Stiffness Matrix of Line Springs

Figure 7.1 shows a wrench $\hat{W}_{P'}$ applied at P' in to an elastically suspended rigid body. It is desired to express all quantities at O in $\mathfrak{F}$ and explicitly determine the differential expression

$$d\hat{W}_O = \hat{K}_O \delta\hat{q}_O \tag{7.1}$$

where $\hat{K}_O$ is the stiffness matrix, $\delta\hat{q}_O$ is an infinitesimal spatial displacement and $d\hat{W}_O$ is the corresponding change in the wrench. For simplicity, (7.1) is derived for a *single* line spring and then subsequently generalized for an arbitrary number of springs using the additivity of stiffnesses in parallel connections.

7.1.1 Stiffness of a Single Line Spring

First an expression for $\delta\hat{q}_O$ is derived. Let $\delta\hat{q}_{P'} = [\ d\vec{\mathbf{r}}_{P'}^T,\ \ \delta\vec{\boldsymbol{\theta}}^T\ ]^T$ be the spatial displacement of $\mathfrak{F}'$ with respect to $\mathfrak{F}$, where $d\vec{\mathbf{r}}_{P'}$ is the infinitesimal translation of P' and $\delta\vec{\boldsymbol{\theta}}$ is the infinitesimal rotation. Transforming $\delta\hat{q}_{P'}$ to O gives

$$\delta\hat{q}_O = \hat{X}_{OP'}\delta\hat{q}_{P'} = \begin{bmatrix} \mathbf{I} & \overrightarrow{OP'}\times \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} d\vec{\mathbf{r}}_{P'} \\ \delta\vec{\boldsymbol{\theta}} \end{bmatrix} \tag{7.2}$$

$$= \begin{bmatrix} \overrightarrow{OP'} \times \delta\vec{\boldsymbol{\theta}} + d\vec{\mathbf{r}}_{P'} \\ \delta\vec{\boldsymbol{\theta}} \end{bmatrix} \tag{7.3}$$

The position vector from O to P' is

$$\vec{\mathbf{r}}_{P'} = \overrightarrow{OP'} = \overrightarrow{OA} + \overrightarrow{AB'} + \overrightarrow{B'P'} \tag{7.4}$$

$$= \overrightarrow{OA} + l\vec{\mathbf{s}} + \overrightarrow{B'P'} \tag{7.5}$$

where A and B' are the spring connection points, l is the spring length, and $\vec{\mathbf{s}}$ is a unit vector in the direction of the spring axis.

An infinitesimal change in the configuration of the body results in corresponding changes in spring lengths, directions, etc. Two observers, one on the fixed body the other on the moving body, observe different changes. Let $d()$ denote the changes observed by the fixed observer and $d'()$ denote that by the moving observer. For example, $d(\overrightarrow{A_1A_i}) = \vec{\mathbf{0}}$ since both points belong to the rigid body $\mathfrak{F}$, hence the vector connecting them stays unchanged with respect to an observer on $\mathfrak{F}$. However, $d'(\overrightarrow{A_1A_i}) \neq \vec{\mathbf{0}}$, since the observer on $\mathfrak{F}'$ sees the vector rotating due to the relative motion. The following differentiation formula for 3-vectors is well known.

$$d\vec{\mathbf{u}} = d'\vec{\mathbf{u}} + \delta\vec{\boldsymbol{\theta}} \times \vec{\mathbf{u}} \tag{7.6}$$

The equation (7.6) relates the changes in $\vec{\mathbf{u}}$ observed in two reference frames with a relative motion $\delta\vec{\boldsymbol{\theta}}$.

The infinitesimal changes in any quantity can be related to the infinitesimal displacement of the moving body. Differentiating (7.5) in the fixed frame and using (7.6) to reduce $d\overrightarrow{B'P'}$ yields

$$\begin{aligned} d\vec{\mathbf{r}}_{P'} &= d\overrightarrow{OA} + dl\vec{\mathbf{s}} + ld\vec{\mathbf{s}} + d\overrightarrow{B'P'} \\ &= dl\vec{\mathbf{s}} + ld\vec{\mathbf{s}} + \delta\vec{\boldsymbol{\theta}} \times \overrightarrow{B'P'} \end{aligned} \tag{7.7}$$

where $d\overrightarrow{OA}, d'\overrightarrow{B'P'} = \vec{\mathbf{0}}$ are used. Substituting (7.7) into (7.3) gives the kinematic relation,

$$\delta\hat{q}_O = \begin{bmatrix} \overrightarrow{OB'} \times \delta\vec{\boldsymbol{\theta}} + dl\vec{\mathbf{s}} + ld\vec{\mathbf{s}} \\ \delta\vec{\boldsymbol{\theta}} \end{bmatrix} \tag{7.8}$$

Next $\hat{W}_O$ is determined by introducing the spring constant k and free length l_0,

$$\hat{W}_O = \begin{bmatrix} \vec{\mathbf{f}} \\ \overrightarrow{OA} \times \vec{\mathbf{f}} \end{bmatrix} = k(l - l_0)\hat{S}_O \tag{7.9}$$

where $\hat{S}_O$ is a unit spatial line vector along the spring,

$$\hat{S}_O = \begin{bmatrix} \vec{\mathbf{s}} \\ \overrightarrow{OA} \times \vec{\mathbf{s}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{s}} \\ \overrightarrow{OB'} \times \vec{\mathbf{s}} \end{bmatrix} \tag{7.10}$$

Using (7.10), (7.8), and $\vec{\mathbf{s}}^T d\vec{\mathbf{s}} = 0$ (since $\vec{\mathbf{s}}^T\vec{\mathbf{s}} = 1$) yields an important kinematic relation

$$\hat{S}_O^T \delta\hat{q}_O = dl \tag{7.11}$$

While this relation is expressed at point O it is also valid at any arbitrary point.

From (7.7) and (7.10) a second important kinematic relation is derived,

$$\begin{aligned} d(l\hat{S}_O) &= \begin{bmatrix} d(l\vec{\mathbf{s}}) \\ \overrightarrow{OA} \times d(l\vec{\mathbf{s}}) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & \overrightarrow{B'P'}\times \\ \overrightarrow{OA}\times & \overrightarrow{OA} \times \overrightarrow{B'P'}\times \end{bmatrix} \delta\hat{q}_{P'} \end{aligned} \tag{7.12}$$

Using the inverse relation of (7.2) gives the relation,

$$d(l\hat{S}_O) = \hat{M}_O \delta\hat{q}_O \tag{7.13}$$

where

$$\hat{M}_O \equiv \begin{bmatrix} \mathbf{I} & -\overrightarrow{OB'}\times \\ \overrightarrow{OA}\times & -\overrightarrow{OA}\times\overrightarrow{OB'}\times \end{bmatrix} \tag{7.14}$$

The stiffness matrix maps a differential displacement to the differential load, so from (7.9),

$$\begin{aligned} d\hat{W}_O &= d\left(k(l-l_0)\hat{S}_O\right) \\ &= k\hat{S}_O dl + k(1-\rho)l d\hat{S}_O \\ &= k\hat{S}_O dl + k(1-\rho)\left(d(l\hat{S}_O) - dl\hat{S}_O\right) \end{aligned} \tag{7.15}$$

where

$$\rho = \frac{l_0}{l} \tag{7.16}$$

Substituting kinematic relations (7.11) and (7.13) in (7.15) yields the desired constitutive relation

$$d\hat{W}_O = \hat{K}_O \delta\hat{q}_O \tag{7.17}$$

where $\hat{K}_O$ is the stiffness matrix represented at O and given by

$$\hat{K}_O = k\rho\hat{S}_O\hat{S}_O^T + k(1-\rho)\hat{M}_O \tag{7.18}$$

Note that the first term in (7.18) is symmetric while the latter is not. If the spring is unloaded then $\rho = 1$, the second term vanishes, and the stiffness matrix becomes symmetric.

7.1.2 Skew-Symmetric Properties

The symmetric and skew-symmetric parts of the stiffness matrix are

$$\hat{K}_{\mathrm{sym}O} = \frac{1}{2}\left(\hat{K}_O + \hat{K}_O^T\right) = \frac{1}{2}k\rho\hat{S}_O\hat{S}_O^T + \frac{1}{2}k(1-\rho)\left(\hat{M}_O + \hat{M}_O^T\right)$$

$$= \frac{1}{2}k\rho\hat{S}_O\hat{S}_O^T - \frac{1}{2}k(1-\rho)\begin{bmatrix} 2\mathbf{I} & (\overrightarrow{OB'}+\overrightarrow{OA})\times \\ -(\overrightarrow{OB'}+\overrightarrow{OA})\times & \overrightarrow{OA}\times\overrightarrow{OB'}\times+\overrightarrow{OB'}\times\overrightarrow{OA}\times \end{bmatrix} \tag{7.19}$$

$$\hat{K}_{\text{skew}O} = \frac{1}{2}\left(\hat{K}_O - \hat{K}_O^T\right) = \frac{1}{2}k(1-\rho)\left(\hat{M}_O - \hat{M}_O^T\right)$$

$$= -\frac{1}{2}k(1-\rho)\begin{bmatrix} \mathbf{0} & (\overrightarrow{OB'}-\overrightarrow{OA})\times \\ (\overrightarrow{OB'}-\overrightarrow{OA})\times & -(\overrightarrow{OB'}\times\overrightarrow{OA}\times-\overrightarrow{OA}\times\overrightarrow{OB'}\times) \end{bmatrix} \tag{7.20}$$

The skew-symmetric part is reducible to a simple form. Using $\overrightarrow{OB'} - \overrightarrow{OA} = \overrightarrow{AB'} = l\vec{s}$ and Jacobi's identity for the lower right hand block gives

$$\begin{aligned}
\hat{K}_{\text{skew}O} &= -\frac{1}{2}k(1-\rho)\begin{bmatrix} \mathbf{0} & \overrightarrow{AB'}\times \\ \overrightarrow{AB'}\times & (\overrightarrow{OA}\times\overrightarrow{OB'})\times \end{bmatrix} \\
&= -\frac{1}{2}k(1-\rho)\begin{bmatrix} 0 & l\vec{s}\times \\ l\vec{s}\times & (\overrightarrow{OA}\times l\vec{s})\times \end{bmatrix} \\
&= -\frac{1}{2}\begin{bmatrix} 0 & k(l-l_0)\vec{s}\times \\ k(l-l_0)\vec{s}\times & (\overrightarrow{OA}\times k(l-l_0)\vec{s})\times \end{bmatrix} \\
&= -\frac{1}{2}\begin{bmatrix} \mathbf{0} & \vec{\mathbf{f}}\times \\ \vec{\mathbf{f}}\times & (\overrightarrow{OA}\times\vec{\mathbf{f}})\times \end{bmatrix}
\end{aligned} \tag{7.21}$$

The off-diagonal blocks are skew-symmetric forms of the spring force and the lower right hand block is the skew-symmetric form of the moment of the force about point O. Thus, (7.21) is negative one-half of the applied load in spatial cross product operator form

$$\hat{K}_{\text{skew}O} = -\frac{1}{2}\hat{W}_O\times \tag{7.22}$$

Since there is only one spring, it is also equivalent to one-half of the spring force.

7.1.3 Stiffness of n Line Springs in Parallel

The generalization of the above formulations to n springs in parallel is simple. Spring related quantities are indicated by the subscript $_i$. The net load on the body is equivalent to the sum of the individual springs forces. The differential of the sum is

$$
\begin{aligned}
d\hat{W}_O &= d\left(\sum_i \hat{W}_{Oi}\right) = \sum_i \left(\hat{K}_{Oi}\delta\hat{q}_O\right) \\
&= \left(\sum_i \hat{K}_{Oi}\right)\delta\hat{q}_O \\
&= \hat{K}_O\delta\hat{q}_O
\end{aligned}
\tag{7.23}
$$

where

$$
\hat{K}_O = \left(\sum_i \hat{K}_{Oi}\right) \tag{7.24}
$$

$$
\hat{K}_{Oi} = k_i\rho_i\hat{S}_{Oi}\hat{S}_{Oi}^T + k_i(1-\rho_i)\hat{M}_i \tag{7.25}
$$

$$
\hat{M}_{Oi} = \begin{bmatrix} \mathbf{I} & -\overrightarrow{OB'_i}\times \\ \overrightarrow{OA_i}\times & -\overrightarrow{OA_i}\times\overrightarrow{OB'_i}\times \end{bmatrix} \tag{7.26}
$$

When every spring is unloaded, that is, $l = l_i$ for all i, the following theorem is obtained.

Theorem 95 *The spatial stiffness matrix of a group of parallel line springs is symmetric if every spring is in the unloaded configuration.*

Proof. If every spring in the group is unloaded then $\rho_i = 1$ for all i. So,

$$
\hat{K}_O(\rho_i = 1) = \sum_{i=1}^{n} k_i\hat{S}_{Oi}\hat{S}_{Oi}^T \tag{7.27}
$$

which is symmetric.

■

The form (7.28) is instrumental in solving the stiffness synthesis problem which is presented in Chapter 8.

The symmetric and skew-symmetric parts of the stiffness are

$$
\hat{K}_{\mathrm{sym}O} = \sum_i \hat{K}_{\mathrm{sym}Oi} = \sum_i \left(k_i\rho_i\hat{S}_{Oi}\hat{S}_{Oi}^T + k_i(1-\rho_i)\hat{M}_{\mathrm{sym}i}\right) \tag{7.28}
$$

$$
\hat{K}_{\mathrm{skew}O} = \sum_i \hat{K}_{\mathrm{skew}Oi} = -\frac{1}{2}\sum_i \hat{W}_{Oi}\times = -\frac{1}{2}\hat{W}_O\times \tag{7.29}
$$

so that

$$\hat{K}_O = \hat{K}_{\mathrm{sym}\,O} - \frac{1}{2}\hat{W}_O \times \tag{7.30}$$

Note that $\hat{W}_O$ is the external load applied to the body whereas $-\hat{W}_O$ is the net load applied to the body by the n springs. The above relation is one of the *central results* of this chapter and is summarized as the following theorem,

Theorem 96 *The skew-symmetric part of the spatial stiffness matrix is one-half the net spring load represented in spatial cross product operator form.*

Two corollaries follow as an immediate consequence. At an unloaded equilibrium $\hat{W}_O = \hat{0}$ and it follows from (7.30) that

Corollary 97 *The spatial stiffness matrix is symmetric if and only if the spring system is in an* unloaded *equilibrium.*

By comparing Corollary 97 to Theorem 95 one can see that the converse of Theorem 95 is not true. A line spring system can be in unloaded equilibrium and, therefore, can have a symmetric stiffness (Corollary 97), while some or all springs are loaded. As an example, consider two line springs, both extended and pulling the rigid body in the middle in opposite directions by equal forces. The net load is zero, but the springs are individually loaded. This shows that the symmetry of stiffness does not imply unloaded springs, but rather it implies an unloaded system.

For 6×6 matrices, a *general* one has 36 independent elements, a symmetric one has 21 independent elements, and a skew-symmetric one has 15 independent elements. However for the class of symmetric stiffness matrices realizable by springs, Loncaric [32] has determined an additional constraint. The symmetric stiffness matrix of spring systems has vanishing traces of the off-diagonal blocks, leaving 20 independent parameters. It is easy to see that this observation is confirmed by the form of the symmetric part of stiffness matrix derived here. The off-diagonal block of the symmetric part is obtained by using (7.10) in (7.28) whose trace is calculated as follows.

$$\begin{aligned} \mathrm{tr}(\hat{K}_{\text{off-diag}}) &= \mathrm{tr}\sum_i \left[k_i\rho_i\vec{s}_i(\overrightarrow{OA_i}\times\vec{s}_i)^T - \frac{1}{2}k_i(1-\rho_i)\left(\overrightarrow{OB_i'}+\overrightarrow{OA_i}\right)\times\right] \\ &= \sum_i k_i\rho_i\left[\mathrm{tr}\left(\vec{s}_i(\overrightarrow{OA_i}\times\vec{s}_i)^T\right)\right] \\ &= \sum_i k_i\rho_i\left(\vec{s}_i^{\,T}(\overrightarrow{OA_i}\times\vec{s}_i)\right) = 0 \end{aligned} \tag{7.31}$$

But, since the off-diagonals of the skew-symmetric part of the spatial stiffness already have vanishing traces, the above fact is generalized to the asymmetric case in the following corollary.

Corollary 98 *For an elastic system comprised of line springs, the traces of the off-diagonal matrices of the asymmetric spatial stiffness are always zero.*

From (7.29), the skew-symmetric part of a stiffness matrix only requires 6 independent elements. Therefore every stiffness matrix (formed from line springs in parallel) only requires a maximum of 26 independent elements. It is simple to show that they form a closed set under addition which combines the stiffnesses of springs in parallel.

Corollary 99 *Spatial stiffnesses of line springs form a 26-parameter set which is closed under addition.*

Proof. Let $\hat{K}_O^1$ and $\hat{K}_O^2$ be the spatial stiffnesses of two groups of springs. If the spring groups are connected to the same rigid body in parallel then the net stiffness of the connection is found by addition. So,

$$\hat{K}_O = \hat{K}_O^1 + \hat{K}_O^2 \tag{7.32}$$

$$= \left(\hat{K}_{\mathrm{sym}O}^1 + \hat{K}_{\mathrm{sym}O}^2\right) - \frac{1}{2}\left(\hat{W}_O^1\times + \hat{W}_O^2\times\right) \tag{7.33}$$

$$= \hat{K}_{\mathrm{sym}O} - \frac{1}{2}\hat{W}_O\times \tag{7.34}$$

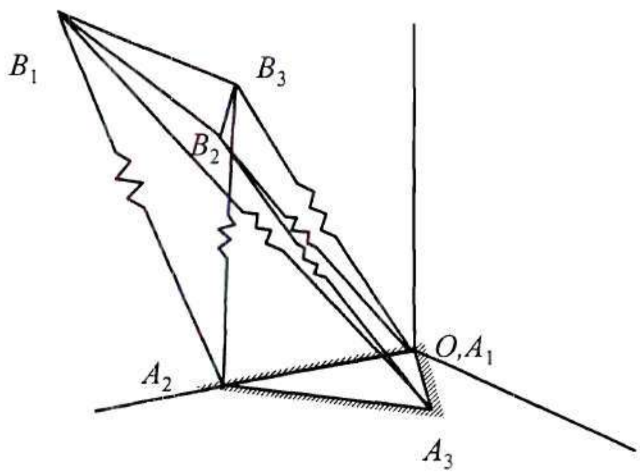


Figure 7.2: Stewart platform with line springs in loaded configuration.

where $\hat{W}_O^k$ are the spatial forces on each group of springs. The fact that the sum of two spatial cross product matrices is a spatial cross product matrix is obvious from the form (7.21). Also, since the trace of a sum of matrices is equal to the sum of traces, $\hat{K}_{\text{sym}O}$ has off-diagonals with zero trace.

■

7.1.4 Numerical Example

An example by Griffis and Duffy (1993) illustrates the relation of the external load to the skew-symmetric portion of the stiffness matrix.

A Stewart platform, Figure 7.2, is modelled as two bodies that are connected by six springs in parallel using the following spring stiffnesses k_i, free lengths l_{0i}, fixed connection points $\vec{\mathbf{a}}_i$, and moving connection points $\vec{\mathbf{b}}_i$.

$$[k_1 \ldots k_6] = \begin{bmatrix} 10 & 20 & 30 & 40 & 50 & 60 \end{bmatrix} \tag{7.35}$$

$$[l_{01} \ldots l_{06}] = \begin{bmatrix} 11 & 12 & 13 & 14 & 15 & 16 \end{bmatrix} \tag{7.36}$$

$$[\vec{\mathbf{a}}_1 \dots \vec{\mathbf{a}}_6] = \begin{bmatrix} 0 & 7 & 7 & 3.500 & 3.500 & 0 \\ 0 & 0 & 0 & 6.062 & 6.062 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{7.37}$$

$$[\vec{\mathbf{b}}_1 \dots \vec{\mathbf{b}}_6] = \begin{bmatrix} 14.041 & 14.041 & 14.496 & 14.496 & 10 & 10 \\ 8.041 & 8.041 & 1.071 & 1.071 & 4 & 4 \\ 16.041 & 16.041 & 16.496 & 16.496 & 12 & 12 \end{bmatrix} \tag{7.38}$$

Calculating the spring lengths l_i, lines of spring force action $\hat{\mathbf{s}}_i$ from $\vec{\mathbf{b}}_i - \vec{\mathbf{a}}_i = l_i \vec{\mathbf{s}}_i$, and the resulting stiffness matrix $\hat{K}_O$ from (7.24)-(7.26) gives

$$[l_1 \dots l_6] = [\ 22.784 \quad 19.276 \quad 18.151 \quad 20.444 \quad 13.802 \quad 16.125\ ] \tag{7.39}$$

$$[\hat{\mathbf{s}}_1 \dots \hat{\mathbf{s}}_6] = \left[\begin{array}{cccccc} 0.616 & 0.365 & 0.413 & 0.538 & 0.471 & 0.620 \\ 0.353 & 0.417 & 0.059 & -0.244 & -0.149 & 0.248 \\ 0.704 & 0.832 & 0.909 & 0.807 & 0.869 & 0.744 \\ \hline 0 & 0 & 0 & 4.891 & 5.270 & 0 \\ 0 & -5.825 & -6.362 & -2.824 & -3.043 & 0 \\ 0 & 2.920 & 0.413 & -4.115 & -3.378 & 0 \end{array}\right] \tag{7.40}$$

$$\hat{K}_O = \left[\begin{array}{ccc|ccc} 80.00 & 5.21 & 75.56 & 206.94 & 303.54 & -239.53 \\ 5.21 & 39.32 & 5.21 & -581.46 & 5.27 & 516.66 \\ 75.56 & 5.21 & 150.61 & 466.52 & -836.51 & -212.21 \\ \hline 206.94 & -75.50 & 407.24 & 2141.45 & -2094.74 & -1518.72 \\ -202.43 & 5.27 & -532.15 & -1725.14 & 4107.41 & 608.60 \\ -180.24 & 212.01 & -212.21 & -3895.11 & -336.51 & 3263.92 \end{array}\right] \tag{7.41}$$

The skew-symmetric part of $\hat{K}_O$ is

$$\hat{K}_{\text{skew}O} = \left[\begin{array}{ccc|ccc} 0 & 0 & 0 & 0 & 252.98 & -29.64 \\ 0 & 0 & 0 & -252.98 & 0 & 152.33 \\ 0 & 0 & 0 & 29.64 & -152.33 & 0 \\ \hline 0 & 252.98 & -29.64 & 0 & -184.80 & 1188.20 \\ -252.98 & 0 & 152.33 & 184.80 & 0 & 472.55 \\ 29.64 & -152.33 & 0 & -1188.20 & -472.55 & 0 \end{array}\right] \tag{7.42}$$

The net wrench on the moving body is given by summing the spring forces in (7.9)

$$\hat{W}_O = \sum_{i=1}^{6} k_i(l_i - l_{0i})\hat{S}_{Oi} = \begin{bmatrix} 304.653 \\ 59.284 \\ 505.963 \\ 945.102 \\ -2376.391 \\ -369.605 \end{bmatrix} \tag{7.43}$$

Negative one-half of the external force is

$$-\frac{1}{2}\hat{W}_O = \begin{bmatrix} -152.327 \\ -29.642 \\ -252.982 \\ -472.551 \\ 1188.196 \\ 184.803 \end{bmatrix} \tag{7.44}$$

These values are identical to the ones that form the elements of $\hat{K}_{\text{skew}O}$. Thus, it is shown that $\hat{K}_{\text{skew}O} = -\frac{1}{2}\hat{W}_O\times$.

7.2 Stiffness Matrix of Torsional Springs

The construction of the spatial stiffness of a torsional spring has intrinsic difficulties, mainly due to the fact that the rotations play an important role. Finite rotations of a rigid body cannot be represented as a vectorial quantity. Yet, it is the finite rotation of the rigid body which causes a torsional spring to deform, via a suitable kinematical connection.

Consider the rigid body connected to the ground via the kinematic joints shown in Figure 7.3. First, the case of one torsional spring is developed, then it is generalized to many springs case.

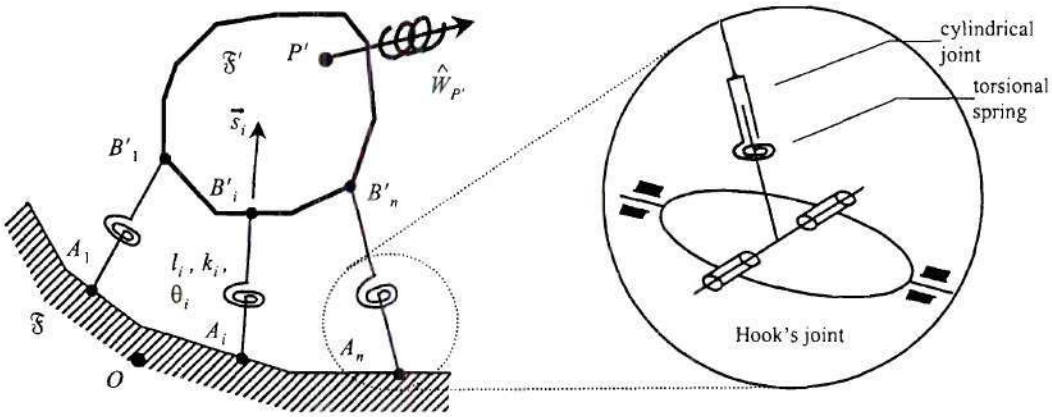


Figure 7.3: An elastically suspended system using torsional springs in parallel.

7.2.1 Stiffness of a Single Torsional Spring

The torsional spring attached to the cylindrical joint resists the rotations of one part of the joint relative to the other about the joint axis. The joints on the body and the ground only allow torques about the cylindrical joint axis to be transmitted. Consequently, only torques about the joint axis are resisted by the spring, so the wrench on the body in a loaded equilibrium is given by.

$$\hat{W}_O = k\,(\theta - \theta_0)\begin{bmatrix} \vec{0} \\ \vec{\mathrm{s}} \end{bmatrix} \tag{7.45}$$

Here, $\vec{\mathrm{s}}$ is the unit vector along the joint axis, k is the torsional spring constant. To understand what θ means consider a dial attached to the torsional spring that measures the number of revolutions about the joint axis. θ_0 is not essential and added to preserve the parallelism to the line spring case. θ_0 can be thought as the number of turns of physical spring in unloaded case. Then, θ is the number of turns in the torsional spring in loaded case. In this way, $(\theta - \theta_0)$ plays the same role that $(l - l_0)$ does in the case of line springs. If $\delta\vec{\theta}$ is the infinitesimal rotation vector of the rigid body, then the rotation about the cylindrical joint axis is

$$d\theta = \vec{\mathrm{s}}^T \delta\vec{\theta} \tag{7.46}$$

The relations (7.2) through (7.8) are still valid. Now, by differentiating (7.45) one gets

$$d\hat{W}_O = kd\theta \begin{bmatrix} \vec{0} \\ \vec{s} \end{bmatrix} + k(\theta - \theta_0) \begin{bmatrix} \vec{0} \\ d\vec{s} \end{bmatrix} \tag{7.47}$$

$$= kd\theta \begin{bmatrix} \vec{0} \\ \vec{s} \end{bmatrix} + k\frac{(\theta - \theta_0)}{l} \begin{bmatrix} \vec{0} \\ ld\vec{s} \end{bmatrix} \tag{7.48}$$

Using (7.46) and the first row of (7.8) in (7.48) yields

$$d\hat{W}_O = k \begin{bmatrix} \vec{0} \\ \vec{s}\vec{s}^T \delta\vec{\theta} \end{bmatrix} + k\frac{(\theta - \theta_0)}{l} \begin{bmatrix} \vec{0} \\ d\vec{r}_O - \overrightarrow{OB'} \times \delta\vec{\theta} - dl\vec{s} \end{bmatrix} \tag{7.49}$$

The identity (7.11) can be rewritten explicitly as

$$dl = \vec{s}^T d\vec{r}_O - \vec{s}^T \overrightarrow{OB'} \times \delta\vec{\theta} \tag{7.50}$$

Eliminating dl in (7.49) by using (7.50) and applying the identity $\vec{s} \times \vec{s}\times = \vec{s}\vec{s}^T - \mathbf{I}$, and after manipulations, one gets

$$d\hat{W}_O = k \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\frac{(\theta-\theta_0)}{l}\vec{s} \times \vec{s}\times & \vec{s}\vec{s}^T + \frac{(\theta-\theta_0)}{l}\vec{s} \times \vec{s} \times \overrightarrow{OB'}\times \end{bmatrix} \delta\hat{q}_O \tag{7.51}$$

As a result, the stiffness matrix of a single torsional spring connected as in Figure 7.3 is

$$\hat{K}_O = k \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\frac{(\theta-\theta_0)}{l}\vec{s} \times \vec{s}\times & \vec{s}\vec{s}^T + \frac{(\theta-\theta_0)}{l}\vec{s} \times \vec{s} \times \overrightarrow{OB'}\times \end{bmatrix} \tag{7.52}$$

This is a sensible result. From (7.51), a linear force cannot be generated by any displacement of the rigid body, that is the system is in equilibrium under a pure couple load. This explains the zero submatrices of $\hat{K}_O$. For the non-zero terms, note that the changes in the wrench can occur due to two reasons:

1. The infinitesimal rotation component about the joint axis $\vec{s}$. This is accounted for by the $\vec{s}\vec{s}^T$ term, which produces the projection of the rotation onto the joint axis.

2. The motion of the cylindrical joint axis. Any translation of the body perpendicular to the joint axis moves the axis into a different orientation which results in a different wrench. This manifests itself in the lower off-diagonal of the stiffness. The term $-\vec{s} \times \vec{s}\times$ projects any translation onto a direction perpendicular to $\vec{s}$. Similarly, any rotation of the body about an axis not through B' changes the joint axis causing a change in the wrench. This is because such rotations induce a non-zero translation of B'. This manifests itself in the stiffness through the term $\overrightarrow{OB'}\times$, since $-\overrightarrow{OB'} \times \delta\vec{\theta}$ is the translation at B' induced by the rotation. Again, only the perpendicular translation component is important, which is produced by the remaining $-\vec{s} \times \vec{s}\times$ term.

The symmetric and the skew-symmetric parts of the stiffness matrix are

$$\hat{K}_{\mathrm{sym}O} = \frac{k}{2} \begin{bmatrix} 0 & -\frac{(\theta-\theta_0)}{l}\vec{s} \times \vec{s}\times \\ -\frac{(\theta-\theta_0)}{l}\vec{s} \times \vec{s}\times & 2\vec{s}\vec{s}^T - \frac{(\theta-\theta_0)}{l}\left(\vec{b} \times \vec{s} \times \vec{s} \times -\vec{s} \times \vec{s} \times \vec{b}\times\right) \end{bmatrix} \tag{7.53}$$

$$\hat{K}_{\mathrm{skew}O} = k\frac{(\theta-\theta_0)}{2l} \begin{bmatrix} 0 & \vec{s} \times \vec{s}\times \\ -\vec{s} \times \vec{s}\times & \vec{s} \times \vec{s} \times \vec{b} \times +\vec{b} \times \vec{s} \times \vec{s}\times \end{bmatrix} \tag{7.54}$$

where $\vec{b} = \overrightarrow{OB'}$ is used for simplicity. It seems from (7.54) that, unlike in the case of line springs, $\hat{K}_{\mathrm{skew}O}$ is not a simple function of the wrench.

7.2.2 Stiffness of n Torsional Springs in Parallel

The stiffness matrix of n torsional springs connected to a rigid body in parallel is found straightforwardly, as in the case of line springs,

$$\hat{K}_O = \sum_{i=1}^{n} k_i \begin{bmatrix} 0 & 0 \\ -\frac{\theta_i-\theta_{0i}}{l_i}\vec{s}_i \times \vec{s}_i\times & \vec{s}_i\vec{s}_i^T + \frac{\theta_i-\theta_{0i}}{l_i}\vec{s}_i \times \vec{s}_i \times \overrightarrow{OB'_i}\times \end{bmatrix} \tag{7.55}$$

where the subscript i refers to the i^{th} spring. An immediate theorem is the following.

Theorem 100 *The spatial stiffness matrix of a group of parallel torsional springs is symmetric if every spring is in the unloaded configuration.*

Proof. If every spring in the group is unloaded then $\theta_i = \theta_{0i}$ for all i. So,

$$\hat{K}_O(\theta_i = \theta_{0i}) = \sum_{i=1}^{n} k_i \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \vec{\mathbf{s}}_i \vec{\mathbf{s}}_i^T \end{bmatrix} \tag{7.56}$$

which is symmetric.

■

Theorem 100 also gives a very useful form of the stiffness matrix for unloaded configurations. If $\hat{S}_i = \begin{bmatrix} \vec{\mathbf{0}}^T & \vec{\mathbf{s}}_i^T \end{bmatrix}^T$ is the screw along the wrench applied by a torsional spring, then (7.56) can be written as

$$\hat{K}_O(\theta_i = \theta_{0i}) = \sum_{i=1}^{n} k_i \hat{S}_i \hat{S}_i^T \tag{7.57}$$

This is exactly the same form as that in (7.27). Equation (7.57) proves to be essential in solving the stiffness synthesis problem.

As in the case of line springs, the converse of Theorem 100 is not true in general. That is, the symmetry of $\hat{K}_O$ does not imply unloaded torsional springs. To see this assume that $\hat{K}_O$ is symmetric. Then, the skew-symmetric part must vanish, i.e.

$$\hat{K}_{\text{skew}O} = \sum_{i=1}^{n} \alpha_i \begin{bmatrix} \mathbf{0} & \vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times \\ -\vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times & \vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times \overrightarrow{OB_i'} \times + \overrightarrow{OB_i'} \times \vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times \end{bmatrix} = \hat{0} \tag{7.58}$$

Thus, for symmetry,

$$\sum \alpha_i \vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times \; = \; \mathbf{0} \tag{7.59}$$

$$\sum \alpha_i \left(\vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times \overrightarrow{OB_i'} \times + \overrightarrow{OB_i'} \times \vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times \right) \; = \; \mathbf{0} \tag{7.60}$$

where $\alpha_i = k_i \frac{\theta_i - \theta_{0i}}{2 l_i}$. For unloaded springs $\alpha_i = 0$ $(\theta_i = \theta_{0i})$. Yet, (7.59) does not imply $\alpha_i = 0$ unless all of the matrices $\vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times$ are independent of each other, which cannot be assumed in general.

Similarly, in general, the matrices in the parenthesis in (7.60) cannot be assumed to be independent of each other. Thus, (7.59) does not imply $\alpha_i = 0$, either. Hence, as for the line springs, the symmetry of stiffness does not imply unloaded torsional springs in general.

Theorem 100 is the torsional spring analogous of Theorem 95. However, the analogy does not extend further. For line springs, the symmetry of stiffness implies unloaded system and vice versa. On the contrary, for torsional springs, these properties are not directly related. For an unloaded torsional spring system, the zero wrench means

$$\sum k_i \left(\theta_i - \theta_{0i}\right) \vec{\mathrm{s}}_i = \vec{0} \tag{7.61}$$

which does not imply any of (7.59) and (7.60). Further, (7.59) and (7.60) do not imply (7.61), either. A quick way of seeing this is to note that (7.59) and (7.60) are functions of l_i and $\overrightarrow{OB'_i}$ which are not found in (7.61). As a result, unlike line springs, for torsional springs one can have symmetric stiffness in a loaded equilibrium, or asymmetric stiffness in an unloaded configuration.

7.3 Stiffness with Respect to Different Frames

A tensorial quantity is a coordinate-free object. However, the representation of a tensorial quantity is dependent on the reference frame. For 3-vectors and related tensors, the representation depends on the orientation of the particular coordinate axes. It does not depend on the origin of the coordinate axes. For example, a force 3-vector has different representations in two distinct reference frames only if the frames have distinct orientations. For spatial vectors and tensors, however, the representations additionally depend on the origin of the coordinates. The representations of a screw in two distinct coordinate frames are in general distinct, even if the two coordinate frames have the same orientation.

Another issue is the effects of the motion of a coordinate system on the representations of tensorial quantities. A general motion of a reference frame induces a change in the representation of a tensorial quantity. This is a change in addition to the change that may be *intrinsic* to the tensorial quantity itself. For example, if there exists a time-varying force 3-vector, the intrinsic change is solely due to the time dependence. The motion of a reference frame adds to this intrinsic change an additional change which manifests itself in the representation of the force vector in this frame. So, one may call this additional change an *apparent change* since it is completely due to the motion of the observer. However, these concepts are at best vague since an observer cannot ultimately ascertain its motion in relation to other bodies. The reader is referred to well known discussions in theoretical physics concerning non-accelerating (inertial) frames of reference. What is more important is the total change observed in a given reference frame as compared to the total change observed in another given reference frame. This is the main topic of the rest of this chapter.

The spatial stiffness and compliance are spatial tensors. For elastic connections between two bodies which have distinct motions the representations of these tensors are in general different. Pigoski et al. [43] first considered the relations between representations of asymmetric spatial stiffness with respect to the ground and body frames in planar case. They showed that the asymmetric stiffnesses referred to the frames on both bodies of the elastic connection were transposes of each other. Then, they showed the existence of a third body with respect to which the stiffness representation is symmetric. Later, Ciblak and Lipkin [8] extended the transpose relation to general spatial connections.

The aim of the rest of this chapter is to present these relations in the most general manner. It must be noted that the results are not dependent on the use of line springs, or any springs for that matter, in the connection. The results generally apply to any tensorial quantity. However, the line spring stiffness is shown to be a very valuable example demonstrating important special cases such as the transpose relation and symmetric bodies. Unlike line springs, torsional spring connections are shown not to obey the transpose relation and in general do not admit a reference frame with respect to which the stiffness is symmetric.

7.3.1 Differentiation in Moving Frames

Consider two rigid bodies denoted by $\mathfrak{F}$ and $\mathfrak{F}'$. Points in $\mathfrak{F}'$ are labeled by primed letters, such as O'. Any two points denoted with the same letter are assumed to be instantaneously coincident. For example, O and O' are instantaneously coincident points, the former on $\mathfrak{F}$ and the latter on $\mathfrak{F}'$. When necessary, the body markers, $\mathfrak{F}$ and $\mathfrak{F}'$, are used as subscripts to indicate the body that a given quantity belongs or is referred to.

Let $\hat{W}$ be a spatial vector. By the properties of spatial vectors, the relation between the representations in frames at O and P' is given by

$$\hat{W}_O = \hat{X}_{OP'}\hat{W}_{P'} \tag{7.62}$$

$$\hat{X}_{OP'} = \begin{bmatrix} \mathbf{R} & 0 \\ \mathbf{R}\left(\overrightarrow{OP'}\right)_{\mathfrak{F}'} \times & \mathbf{R} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & 0 \\ \mathbf{R}\vec{\mathbf{r}}' \times & \mathbf{R} \end{bmatrix} \tag{7.63}$$

$$\hat{X}_{OP'} = \begin{bmatrix} \mathbf{R} & 0 \\ \left(\overrightarrow{OP'}\right)_{\mathfrak{F}} \times \mathbf{R} & \mathbf{R} \end{bmatrix} = \begin{bmatrix} \mathbf{R} & 0 \\ \vec{\mathbf{r}} \times \mathbf{R} & \mathbf{R} \end{bmatrix} \tag{7.64}$$

where $\hat{X}_{OP'}$ performs the spatial transformation from a frame attached to $\mathfrak{F}'$ at P', to one attached to $\mathfrak{F}$ at O. $\left(\overrightarrow{OP'}\right)_{\mathfrak{F}} = \vec{\mathbf{r}}$ indicates that the vector $\overrightarrow{OP'}$ is represented in $\mathfrak{F}$ frame. $\left(\overrightarrow{OP'}\right)_{\mathfrak{F}'} = \vec{\mathbf{r}}'$ indicates that the vector $\overrightarrow{OP'}$ is represented in $\mathfrak{F}'$ frame. $\mathbf{R}$ is the rotation matrix relating the orientations of the reference frames at O and P'. In (7.63), the translation is performed before the rotation, so it is represented in $\mathfrak{F}'$ by $\vec{\mathbf{r}}'$. In (7.64), the translation is performed after the rotation, so it is represented in $\mathfrak{F}$ by $\vec{\mathbf{r}}$.

Note that, if the coordinate frames on the two bodies are instantaneously coincident then $\hat{X} = \hat{I}$. In such a case, the representations of a spatial vector are identical no matter what the relative motion is. The difference will be in the observed changes in the spatial vector. To analyze this phenomenon, first the following well known lemma is needed.

Lemma 101 *Let $\vec{\omega}_{\mathfrak{F}'/\mathfrak{F}}(t)$ be the angular velocity 3-vector of $\mathfrak{F}'$ relative to $\mathfrak{F}$ and $\mathbf{R}(t)$ be the rotation transformation matrix for 3-vectors from $\mathfrak{F}'$ to $\mathfrak{F}$, where t is the time as a parameter. Then,*

$$\frac{d\mathbf{R}}{dt} = \vec{\omega}_{\mathfrak{F}'/\mathfrak{F}} \times \mathbf{R} \tag{7.65}$$

Proof. The proof of this lemma can be found in many theoretical kinematics references. For example, see [5].

■

Let the spatial velocity of $\mathfrak{F}'$ relative to $\mathfrak{F}$ be $\hat{V}_{\mathfrak{F}'/\mathfrak{F}}$ whose representation with respect to the frame at O is denoted as $\hat{V}_O$. Similarly, let the spatial velocity of $\mathfrak{F}$ relative to $\mathfrak{F}'$ be $\hat{V}_{\mathfrak{F}/\mathfrak{F}'}$ whose representation with respect to the frame at P' is denoted as $\hat{V}_{P'}$.

$$\hat{V}_O = \begin{bmatrix} \vec{\mathbf{v}}_O \\ \vec{\omega}_{\mathfrak{F}'/\mathfrak{F}} \end{bmatrix} \qquad \hat{V}_{P'} = \begin{bmatrix} \vec{\mathbf{v}}_{P'} \\ \vec{\omega}_{\mathfrak{F}/\mathfrak{F}'} \end{bmatrix} \tag{7.66}$$

where $\vec{\mathbf{v}}_O$ is the linear velocity of a point O' of $\mathfrak{F}'$ instantaneously coincident with O, and $\vec{\mathbf{v}}_{P'}$ is the linear velocity of a point P of $\mathfrak{F}$ instantaneously coincident with P'. See Chapter 2 or [20] for details.

The following is analogous to Lemma 101 for spatial vectors.

Lemma 102

$$\frac{d}{dt}\left(\hat{X}_{OP'}\right) = \hat{V}_O \times \hat{X}_{OP'} \tag{7.67}$$

$$\frac{d}{dt}\left(\hat{X}_{P'O}\right) = \hat{V}_{P'} \times \hat{X}_{P'O} \tag{7.68}$$

where $\hat{V}_O\times$ and $\hat{V}_{P'}\times$ are the spatial cross product forms.

Proof. It is easier to use (7.63) than (7.64) to show this lemma, since in the former the position vector is not affected by the rotation. So,

$$\frac{d}{dt}\left(\hat{X}_{OP'}\right) = \begin{bmatrix} \frac{d\mathbf{R}}{dt} & \mathbf{0} \\ \frac{d\mathbf{R}}{dt}\vec{\mathbf{r}}' \times + \mathbf{R}\frac{d\vec{\mathbf{r}}' \times}{dt} & \frac{d\mathbf{R}}{dt} \end{bmatrix} \tag{7.69}$$

Next, by using the relation

$$\frac{d\vec{\mathbf{r}}'}{dt} = \frac{d\left(\overrightarrow{OP'}\right)_{\mathfrak{F}'}}{dt} = \frac{d\left(\overrightarrow{OO'} + \overrightarrow{O'P'}\right)_{\mathfrak{F}'}}{dt} = \frac{d\left(\overrightarrow{OO'}\right)_{\mathfrak{F}'}}{dt} = \vec{\mathbf{v}}'_O \tag{7.70}$$

where $\vec{\mathbf{v}}'_O$ is the representation of $\vec{\mathbf{v}}_O$ in $\mathfrak{F}'$, and Lemma 101 in (7.69) one gets

$$\frac{d}{dt}\left(\hat{X}_{OP'}\right) = \begin{bmatrix} \vec{\omega}_{\mathfrak{F}'/\mathfrak{F}} \times \mathbf{R} & \mathbf{0} \\ \vec{\omega}_{\mathfrak{F}'/\mathfrak{F}} \times \mathbf{R}\vec{\mathbf{r}}' \times + \mathbf{R}\vec{\mathbf{v}}'_O \times & \vec{\omega}_{\mathfrak{F}'/\mathfrak{F}} \times \mathbf{R} \end{bmatrix} \tag{7.71}$$

By the properties of the 3-vector cross-product operator one can write

$$\mathbf{R}\vec{\mathbf{v}}'_O \times = \mathbf{R}\vec{\mathbf{v}}'_O \times \mathbf{R}^T\mathbf{R} = \left(\mathbf{R}\vec{\mathbf{v}}'_O\right) \times \mathbf{R} = \vec{\mathbf{v}}_O \times \mathbf{R} \tag{7.72}$$

Therefore, (7.71) becomes

$$\begin{aligned} \frac{d}{dt}\left(\hat{X}_{OP'}\right) &= \begin{bmatrix} \vec{\omega}_{\mathfrak{F}'/\mathfrak{F}} \times \mathbf{R} & \mathbf{0} \\ \vec{\omega}_{\mathfrak{F}'/\mathfrak{F}} \times \mathbf{R}\vec{\mathbf{r}}' \times + \vec{\mathbf{v}}_O \times \mathbf{R} & \vec{\omega}_{\mathfrak{F}'/\mathfrak{F}} \times \mathbf{R} \end{bmatrix} && (7.73) \\ &= \begin{bmatrix} \vec{\omega}_{\mathfrak{F}'/\mathfrak{F}} \times & \mathbf{0} \\ \vec{\mathbf{v}}_O \times & \vec{\omega}_{\mathfrak{F}'/\mathfrak{F}} \times \end{bmatrix} \begin{bmatrix} \mathbf{R} & \mathbf{0} \\ \mathbf{R}\,\vec{\mathbf{r}}' \times & \mathbf{R} \end{bmatrix} && (7.74) \\ &= \hat{V}_O \times \hat{X}_{OP'} && (7.75) \end{aligned}$$

One also demonstrates the equation (7.68) in a similar manner.

■

Let $\hat{W}$ be any spatial vector whose representations in $\mathfrak{F}$ and $\mathfrak{F}'$ frames are $\hat{W}_O$ and $\hat{W}_{P'}$, respectively. Also let $\hat{W}(t)$ imply that $\hat{W}$ is a smooth function of time. Consider two snapshots in time separated by Δt, say t and $t+\Delta t$. Then, the spatial vectors and transformation matrices observed in $\mathfrak{F}$ and $\mathfrak{F}'$ frames at these instances are as follows.

$$\begin{array}{lcc} & \mathfrak{F} & \mathfrak{F}' \\ \hline t: & \hat{W}_O(t), \hat{X}_{OP'}(t) & \hat{W}_{P'}(t), \hat{X}_{P'O}(t) \\ t+\Delta t: & \hat{W}_O(t+\Delta t), \hat{X}_{OP'}(t+\Delta t) & \hat{W}_{P'}(t+\Delta t), \hat{X}_{P'O}(t+\Delta t) \end{array} \tag{7.76}$$

At the two instances the transformation rule gives

$$\hat{W}_O(t) = \hat{X}_{OP'}(t)\hat{W}_{P'}(t) \tag{7.77}$$

$$\hat{W}_O(t+\Delta t) = \hat{X}_{OP'}(t+\Delta t)\hat{W}_{P'}(t+\Delta t) \tag{7.78}$$

The changes in the representations of $\hat{W}$ in both frames are defined and denoted as

$$\left(\Delta\hat{W}\right)_O = \hat{W}_O(t+\Delta t) - \hat{W}_O(t) \tag{7.79}$$

$$\left(\Delta'\hat{W}\right)_{P'} = \hat{W}_{P'}(t+\Delta t) - \hat{W}_{P'}(t) \tag{7.80}$$

Note that, by definition of a vector space, the change in a spatial vector is a spatial vector itself. So, $\Delta\hat{W}$ and $\Delta'\hat{W}$ are spatial vectors. They are the changes in $\hat{W}$ as observed in $\mathfrak{F}$ and $\mathfrak{F}'$ frames, respectively. Then, $\left(\Delta\hat{W}\right)_O$ is the change in $\hat{W}$ as observed and represented in $\mathfrak{F}$. Similarly, $\left(\Delta'\hat{W}\right)_{P'}$ is the change in $\hat{W}$ as observed and represented in $\mathfrak{F}'$ (Conceptually, it is possible to have $\left(\Delta'\hat{W}\right)_O$ which is the change in $\hat{W}$ observed in $\mathfrak{F}'$ but represented in $\mathfrak{F}$). From (7.77) through (7.80) one gets

$$\hat{W}_O(t+\Delta t) - \hat{W}_O(t) = \hat{X}_{OP'}(t+\Delta t)\hat{W}_{P'}(t+\Delta t) - \hat{X}_{OP'}(t)\hat{W}_{P'}(t) \tag{7.81}$$

$$\begin{aligned} \left(\Delta\hat{W}\right)_O &= \hat{X}_{OP'}(t+\Delta t)\hat{W}_{P'}(t+\Delta t) - \hat{X}_{OP'}(t)\hat{W}_{P'}(t) + \\ &\quad +\hat{X}_{OP'}(t)\hat{W}_{P'}(t+\Delta t) - \hat{X}_{OP'}(t)\hat{W}_{P'}(t+\Delta t) \end{aligned} \tag{7.82}$$

$$= \left[\hat{X}_{OP'}(t+\Delta t) - \hat{X}_{OP'}(t)\right]\hat{W}_{P'}(t+\Delta t) + \hat{X}_{OP'}(t)\left(\Delta'\hat{W}\right)_{P'} \tag{7.83}$$

Now, letting Δt to approach zero in the limit yields

$$\lim_{\Delta t \to 0} \frac{\left(\Delta \hat{W}\right)_O}{\Delta t} = \left(\frac{d\hat{W}}{dt}\right)_O \qquad \lim_{\Delta t \to 0} \frac{\left(\Delta' \hat{W}\right)_{P'}}{\Delta t} = \left(\frac{d'\hat{W}}{dt}\right)_{P'} \tag{7.84}$$

$$\lim_{\Delta t \to 0} \frac{\hat{X}_{OP'}(t+\Delta t) - \hat{X}_{OP'}(t)}{\Delta t} = \frac{d\hat{X}_{OP'}}{dt} = \hat{V}_O \times \hat{X}_{OP'} \tag{7.85}$$

Dividing both sides of (7.83) Δt and applying the limit $\Delta t \to 0$ gives

$$\left(\frac{d\hat{W}}{dt}\right)_O = \hat{V}_O \times \hat{X}_{OP'} \hat{W}_{P'} + \hat{X}_{OP'} \left(\frac{d'\hat{W}}{dt}\right)_{P'} \tag{7.86}$$

$$\left(\frac{d\hat{W}}{dt}\right)_O = \hat{V}_O \times \hat{W}_O + \left(\frac{d'\hat{W}}{dt}\right)_O \tag{7.87}$$

The operator $\frac{d()}{dt}$ symbolizes the differentiation of a spatial vector in $\mathfrak{F}$ frame,and $\frac{d'()}{dt}$ symbolizes the differentiation in $\mathfrak{F}'$ frame. Now, since all quantities in (7.87) are represented in the same coordinate frame, the subscripts can be dropped to give the following theorem as the coordinate-free version of (7.87).

Theorem 103 *Let $\hat{W}$ be a spatial vector and $\mathfrak{F}$ and $\mathfrak{F}'$ be two frames of reference. If $\hat{V}_{\mathfrak{F}'/\mathfrak{F}}$ is the spatial velocity of $\mathfrak{F}'$ relative to $\mathfrak{F}$, then*

$$D\hat{W} = D'\hat{W} + \hat{V}_{\mathfrak{F}'/\mathfrak{F}} \times \hat{W} \tag{7.88}$$

where $D = \frac{d}{dt}$ and $D' = \frac{d'}{dt}$.

A different derivation of (7.88) can be found in Featherstone [20], although it is not as general as it is here. The equation (7.88) is analogous to the well known differentiation rule of 3-vectors with respect to different frames of reference. For a 3-vector $\vec{\mathbf{f}}$ this rule can be stated as

$$D\vec{\mathbf{f}} = D'\vec{\mathbf{f}} + \vec{\omega}_{\mathfrak{F}'/\mathfrak{F}} \times \vec{\mathbf{f}} \tag{7.89}$$

where $D\vec{\mathbf{f}}$ and $D'\vec{\mathbf{f}}$ are the time derivatives of $\vec{\mathbf{f}}$ in $\mathfrak{F}$ and $\mathfrak{F}'$, respectively.

7.3.2 Stiffness of General Systems with Respect to Different Frames of Reference

For the rest of the treatment, it is preferable to put (7.88) into a form more useful in dealing with stiffness relations. This is done by removing the explicit time dependency to get the differential relation

$$d\hat{W} = d'\hat{W} + \delta\hat{q} \times \hat{W} \tag{7.90}$$

The spatial velocities become infinitesimal spatial displacements. In the literature, the differentiation with respect to different frames usually involves frames moving relative to each other. For this reason, in this section the quantities such as $\delta\hat{q}$ are still referred to as spatial velocities.

In previous works [43], [8], the two connected bodies were termed the **fixed** and **moving** bodies. This terminology is followed here also. However, since the stiffness relations are considered for arbitrary moving bodies, an additional term is needed. Thus, any body in motion (other than the fixed and moving bodies) is called the **intermediate** body.

Consider any body $\mathfrak{F}^p$ with a velocity $\delta\hat{p}$ relative to the fixed body. Quantities related to $\mathfrak{F}^p$ are denoted by the superscript ^{p}. The primed quantities are reserved for the first moving body. Then, the changes observed in the fixed body and $\mathfrak{F}^p$ are related by

$$d\hat{W} = d^p\hat{W} + \delta\hat{p} \times \hat{W} \tag{7.91}$$

By "the stiffness observed in $\mathfrak{F}^p$" is meant the following linear map

$$d^p\hat{W}_{A^p} = \hat{K}^p_{A^p}\delta\hat{q}_{A^p} \tag{7.92}$$

Here, A^p indicates a reference frame at point A^p $\mathfrak{F}^p$. Thus, $\delta\hat{q}_{A^p}$ and $d^p\hat{W}_{A^p}$ are respectively the velocity of the moving body and the change in the wrench as seen in $\mathfrak{F}^p$. Consequently, $\hat{K}^p_{A^p}$ is the stiffness of the connection observed in $\mathfrak{F}^p$ and represented with respect to a coordinate system at A^p.

It is possible to transform (7.91) to its representation in $\mathfrak{F}$ coordinates situated at O. This allows a coordinate free treatment. As a result,

$$d^p\hat{W} = \hat{K}^p\delta\hat{q} \tag{7.93}$$

where K^p is the stiffness of the connection observed in $\mathfrak{F}^p$ but represented in $\mathfrak{F}$, etc. Now, using (7.91) in (7.93) one gets

$$\begin{aligned} d\hat{W} - \delta\hat{p} \times \hat{W} &= \hat{K}^p\delta\hat{q} && (7.94) \\ d\hat{W} &= \hat{K}^p\delta\hat{q} + \delta\hat{p} \times \hat{W} && (7.95) \\ d\hat{W} &= \hat{K}^p\delta\hat{q} - \hat{W} \times \delta\hat{p} && (7.96) \end{aligned}$$

An interesting observation is the following. If for some $\delta\hat{p}$, $\hat{W} \times \delta\hat{p} = \hat{0}$ then $d\hat{W} = \hat{K}^p\delta\hat{q}$, and the stiffness with respect to the fixed and intermediate bodies become the same. An example of this is when the twist $\delta\hat{p}$ is about the same screw as $\hat{W}$. But, there exists a more general family of motions leading to the same result. To show this, the following theorem is needed.

Theorem 104 *Given a spatial vector $\hat{W}$, the vectors $\hat{Y}$ such that $\hat{W} \times \hat{Y} = \hat{0}$ form*

1. *a two-parameter family given by* $\hat{Y} = \begin{bmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \beta\mathbf{I} & \mathbf{0} \end{bmatrix} \hat{W}$ *for finite pitch* $\hat{W}$*, or*

2. *a four-parameter family given by* $\hat{Y} = \begin{bmatrix} \vec{\delta} \\ \vec{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I} \end{bmatrix} \hat{W}$ *for infinite pitch* $\hat{W}$*,*

where α and β are arbitrary scalars and $\vec{\boldsymbol{\delta}}$ is an arbitrary translation. The two-parameter family, a 2-system, contains all screws with the same screw axis of $\hat{W}$ and all possible pitches. The four-parameter family, a 4-system, contains all screws through every point with screw axes parallel to the moment $\hat{W}$ and all possible pitches.

Proof. Let $\hat{W} = [\vec{\mathbf{f}}^T, \vec{\mathbf{m}}^T]^T$ and $\hat{Y} = [\vec{\boldsymbol{\delta}}^T, \vec{\gamma}^T]^T$. Then,

$$\hat{W} \times \hat{Y} = \begin{bmatrix} \mathbf{0} & \vec{\mathbf{f}}\times \\ \vec{\mathbf{f}}\times & \vec{\mathbf{m}}\times \end{bmatrix} \begin{bmatrix} \vec{\delta} \\ \vec{\gamma} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{0} \end{bmatrix} \tag{7.97}$$

1. If $\hat{W}$ has finite pitch then $\vec{\mathbf{f}} \neq \vec{0}$, and the first row of (7.97) yields $\vec{\gamma} = \beta\vec{\mathbf{f}}$, $\beta \in R$. Then, the second row becomes

$$\vec{\mathbf{f}} \times \vec{\delta} - \beta\vec{\mathbf{f}} \times \vec{\mathbf{m}} = \vec{\mathbf{f}} \times (\vec{\delta} - \beta\vec{\mathbf{m}}) = \vec{0} \tag{7.98}$$

which implies that $\vec{\boldsymbol{\delta}} = \alpha\vec{\mathbf{f}} + \beta\vec{\mathbf{m}}$, $\alpha \in R$. So,

$$\hat{Y} = \begin{bmatrix} \alpha\vec{\mathbf{f}} + \beta\vec{\mathbf{m}} \\ \beta\vec{\mathbf{f}} \end{bmatrix} = \begin{bmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \beta\mathbf{I} & \mathbf{0} \end{bmatrix} \hat{W} \tag{7.99}$$

Note that the two-parameter vector $\hat{Y}$ can be given as

$$\hat{Y} = \alpha\hat{\Gamma}\hat{W} + \beta\hat{\Delta}\hat{W} \tag{7.100}$$

where $\beta\hat{\Delta}\hat{W}$ is a twist about the same screw as $\hat{W}$. On the other hand, $\alpha\hat{\Gamma}\hat{W}$ is a translation parallel to the force part. For each pair of α, β an element of the two-parameter family results, which is actually a 2-system of screws generated by a twist and a translation parallel to the rotation of the twist. To render a picture of this 2-system assume that $\hat{W}$ is given with respect to a point on its axis. Then,

$$\hat{Y} = \begin{bmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \beta\mathbf{I} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{f}} \\ h\vec{\mathbf{f}} \end{bmatrix} = \begin{bmatrix} (\alpha + h\beta)\vec{\mathbf{f}} \\ \beta\vec{\mathbf{f}} \end{bmatrix} \tag{7.101}$$

where h is the pitch of $\hat{W}$. Clearly, $\hat{Y}$ and $\hat{W}$ have the same screw axis. The pitch of $\hat{Y}$ is given by $h_{\hat{Y}} = \frac{(\alpha+h\beta)\beta}{\beta^2} = \frac{\alpha}{\beta} + h$, with $\beta \neq 0$ since due to the finite pitch of $\hat{W}$. Therefore, the 2-system is made of screws with the screw axis of $\hat{W}$ and all possible pitches.

2. If $\hat{W}$ has infinite pitch then $\vec{\mathbf{f}} = \vec{0}$, and the first row of (7.97) is trivial. Then, the second row

gives that $\vec{\delta}$ is arbitrary and $\vec{\gamma} = \beta\vec{\mathbf{m}}$, $\beta \in R$, so that

$$\hat{Y} = \begin{bmatrix} \vec{\delta} \\ \beta\vec{\mathbf{m}} \end{bmatrix} = \begin{bmatrix} \vec{\delta} \\ \vec{0} \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & \beta\mathbf{I} \end{bmatrix} \hat{W} \tag{7.102}$$

which is a four-parameter family, three from $\vec{\delta}$ and one from β. This is a 4-system of screws. In this case, $\hat{W}$ is a pure couple. Therefore, the second term on the right of (7.102) is a rotation parallel to the couple, where as the first term is an arbitrary translation. All the screws in the 4-system have screw axes passing through every point parallel to the couple. For every screw axis there exist screws with all possible pitches. This is similar to the finite pitch case, except that a 2-system with single screw axis and all possible pitches is duplicated at every point, giving a 4-system.

■

Theorem 104 leads to the following theorems and corollaries.

Theorem 105 *Given an elastic connection between a fixed and a moving body with velocity $\delta\hat{q}$, there exist infinitely many intermediate bodies whose velocities form a 2-system given by*

$$\delta\hat{p} = \begin{bmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \beta\mathbf{I} & \mathbf{0} \end{bmatrix} \hat{W} \tag{7.103}$$

if the wrench on the system has finite pitch, or a 4-system given by

$$\delta\hat{p} = \begin{bmatrix} \vec{\delta} \\ \vec{0} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I} \end{bmatrix} \hat{W} \tag{7.104}$$

if the wrench on the system has infinite pitch, with respect to which the stiffness is the same as that observed in the fixed frame. Here, α and β are arbitrary infinitesimal scalars, and $\vec{\delta}$ is an arbitrary infinitesimal translation.

Proof. By Theorem 121 and equations (7.103) or (7.104), $\hat{W} \times \delta\hat{p} = \hat{0}$. Therefore, (7.96) becomes $d\hat{W} = \hat{K}^p \delta\hat{q}$ so that $\hat{K}^p = \hat{K}$ for all such bodies.

■

Theorem 105 can be generalized to the following corollary.

Corollary 106 *Given an elastic connection between a fixed and a moving body, the stiffnesses observed in any two frames with velocities $\delta\hat{p}_1$ and $\delta\hat{p}_2$ are equivalent if and only if*

$$\delta\hat{p}_1 - \delta\hat{p}_2 = \begin{cases} \begin{bmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \beta\mathbf{I} & \mathbf{0} \end{bmatrix} \hat{W} & \text{for finite pitch } \hat{W} \\ \begin{bmatrix} \vec{\boldsymbol{\delta}} \\ \vec{\mathbf{0}} \end{bmatrix} + \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \beta\mathbf{I} \end{bmatrix} \hat{W} & \text{for infinite pitch } \hat{W} \end{cases} \tag{7.105}$$

where α, β are arbitrary infinitesimal scalars, and $\vec{\boldsymbol{\delta}}$ is an arbitrary infinitesimal translation.

Proof. For the two bodies, the equation (7.96) gives

$$d\hat{W} = \hat{K}^{p1}\delta\hat{q} - \hat{W} \times \delta\hat{p}_1 \tag{7.106}$$

$$d\hat{W} = \hat{K}^{p2}\delta\hat{q} - \hat{W} \times \delta\hat{p}_2 \tag{7.107}$$

Subtracting (7.107) from (7.106) one gets

$$(\hat{K}^{p1} - \hat{K}^{p2})\delta\hat{q} - \hat{W} \times (\delta\hat{p}_1 - \delta\hat{p}_2) = \hat{0} \tag{7.108}$$

which must be true for all $\delta\hat{q}$. Now, if $\delta\hat{p}_1 - \delta\hat{p}_2$ is as in (7.105) then $\hat{W} \times (\delta\hat{p}_1 - \delta\hat{p}_2) = \hat{0}$ which gives $(\hat{K}^{p1} - \hat{K}^{p2})\delta\hat{q} = \hat{0}$ forcing the result that $\hat{K}^{p1} = \hat{K}^{p2}$. If, on the other hand, $\hat{K}^{p1} = \hat{K}^{p2}$ then $\hat{W} \times (\delta\hat{p}_1 - \delta\hat{p}_2) = \hat{0}$ must hold which, by Theorem 104, yields (7.105).

■

The motions differing from each other by the two-parameter term $\begin{bmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \beta\mathbf{I} & \mathbf{0} \end{bmatrix} \hat{W}$ are equivalent in the sense that they see the same stiffness matrix. Moreover, it can be seen from (7.91) that this is a general result. That is, it is true for the differentiation of any spatial vector. This general phenomenon actually stems from the properties of the cross-product operator in a given vector space. For non-zero 3-vectors, $\vec{\mathbf{a}} \times \vec{\mathbf{b}}$ means that the vectors $\vec{\mathbf{a}}$ and $\vec{\mathbf{b}}$ are parallel. Thus, Theorem 104 may be interpreted as defining a parallelism with respect to spatial cross product. Then, as in Corollary 106, the set of motions with respect to which the differentiation of a given vector is the same can be viewed as *parallel* to the given vector (with respect to the spatial cross product). So, accordingly, such motions can be called as **×-parallel motions**.

7.3.3 Stiffness of Line Spring Systems with Respect to Different Frames

The special form of the stiffness matrix of line spring systems has important consequences. The results of previous sections lead to a very simple relationship which connects stiffness matrices of line springs when expressed in the fixed and moving frames. This was first shown by Pigoski et al., [43], for planar case and later generalized by Ciblak and Lipkin, [8], to the spatial case.

Theorem 107 *For an elastic connection of parallel line springs, the stiffness matrices referenced to the fixed and the moving bodies at coincident points are transposes of each other.*

Proof. Given a wrench $\hat{W}$ on the moving body, the moving observer's measurement of $d'\hat{W}$ is related to that of the fixed observer by

$$d\hat{W} = d'\hat{W} - \hat{W} \times \delta\hat{q} \tag{7.109}$$

Using the form of the stiffness for line spring systems given by (7.30) in (7.109) one gets

$$\begin{aligned} d'\hat{W} - \hat{W} \times \delta\hat{q} &= \left(\hat{K}_{\text{sym}} - \frac{1}{2}\hat{W}\times\right)\delta\hat{q} \\ &= \hat{K}_{\text{sym}}\delta\hat{q} - \frac{1}{2}\hat{W} \times \delta\hat{q} \end{aligned} \tag{7.110}$$

So,

$$d'\hat{W} = \hat{K}_{\text{sym}}\delta\hat{q} + \frac{1}{2}\hat{W} \times \delta\hat{q} = \left(\hat{K}_{\text{sym}} + \frac{1}{2}\hat{W}\times\right)\delta\hat{q} \tag{7.111}$$

which indicates that the stiffness as seen in the moving frame is

$$\hat{K}' = \hat{K}_{\text{sym}} + \frac{1}{2}\hat{W}\times \tag{7.112}$$

But, clearly, $\hat{K}' = \left(\hat{K}_{\text{sym}} - \frac{1}{2}\hat{W}\times\right)^T = \hat{K}^T$. This proves the corollary.

■

There is another interpretation of Theorem 107. Consider two observers, one situated on the fixed frame the other on the moving frame, who employ two respective coordinate axes with co-inciding origins O and O'. The fixed observer calculates the stiffness matrix by interpreting the connection point on the fixed body as fixed points and those on the moving body as the moving points. On the other hand, given the rule of computing the stiffness matrix, the moving observer interprets everything conversely. That is, according to the moving observer, it is the fixed body which is moving. From his perspective, points A_i are the moving connection points, so he has to replace A_i with B_i' and vice versa. The effect of this is that the moving observer replaces $\vec{\mathbf{s}}_i$ by $-\vec{\mathbf{s}}_i$ in the spatial stiffness expression. By equation (7.10), it is seen that $\hat{S}_i$ is replaced by $-\hat{S}_i$. So, the term $\hat{S}_i\hat{S}_i^T$ is the same with respect to both observers. Also, it is easily seen that the symmetric part of the matrix $\hat{M}$ is symmetric with respect to A_i and B_i', and is therefore interpreted as the same. Hence, both observers calculate the same symmetric part. The only change comes from the skew-symmetric part due to the replacement of $\vec{\mathbf{s}}_i$ by $-\vec{\mathbf{s}}_i$. The effect of this is that $-\hat{W}\times$ becomes $\hat{W}\times$. This is sensible since $-\hat{W}$ is the load on the fixed body according to the first moving observer. So, the moving observer calculates the total spatial stiffness of the connection as

$$\hat{K}' = \hat{K}_{\text{sym}} + \frac{1}{2}\hat{W}\times \tag{7.113}$$

which proves the fact that

$$\hat{K}' = \hat{K}^T \tag{7.114}$$

Note that, according to the moving observer, the fixed body moves by $-\delta\hat{q}$ and the change in the force is $-d\hat{W}$, the negative signs cancel each other in the stiffness expression. This second interpretation is logical since it shows that the stiffness expression is applicable in both bodies of the connection and gives consistent results.

Combining Corollary 97 and Theorem 107 gives,

Corollary 108 *The fixed body and moving body stiffnesses of line springs are equivalent if and only if the system is in an unloaded equilibrium.*

Now consider an arbitrary intermediate body with velocity $\delta\hat{p}$ relative to the fixed body. Using the special form of the stiffness of line springs, (7.30), and the differentiation formula (7.96) yields

$$d^p\hat{W} = \hat{K}_{\text{sym}}\delta\hat{q} + \hat{W} \times (\delta\hat{p} - \frac{1}{2}\delta\hat{q}) \tag{7.115}$$

Consider the constraint $\delta\hat{p} = \hat{P}\delta\hat{q}$, which is equivalent a kinematical connection between the moving body and the intermediate body. Then (7.115) becomes

$$d^p\hat{W} = \left[\hat{K}_{\text{sym}} + \hat{W} \times (\hat{P} - \frac{1}{2}\hat{I})\right]\delta\hat{q} \tag{7.116}$$

from which one concludes that the stiffness matrix of the connection observed in the intermediate body is

$$\hat{K}^p = \hat{K}_{\text{sym}} + \hat{W} \times (\hat{P} - \frac{1}{2}\hat{I}) \tag{7.117}$$

It seems from (7.117) that, in general, the spatial stiffness matrix with respect to an arbitrary moving body no longer has the simple form obtained for the fixed and moving bodies.

A special case worth noting is obtained for $\delta\hat{p} = \nu\delta\hat{q}$. That is, when the velocity of the intermediate body is a multiple of the velocity of the moving body. Note that, ν is not necessarily constant. It can be a scalar function of some parameters such as time, wrench, etc. The stiffness matrix becomes

$$\hat{K}^p = \hat{K}_{\text{sym}} + (\nu - \tfrac{1}{2})\hat{W}\times \tag{7.118}$$

Note that the symmetric part of $\hat{K}^p$ is equivalent to that of $\hat{K}$, and, the skew-symmetric part of $\hat{K}^p$, $(\nu - \frac{1}{2})\hat{W}\times$, is just a multiple of the skew-symmetric part of $\hat{K}$. This result is summarized in the following theorem.

Theorem 109 *Let an elastic connection with line springs be given between a fixed body and a moving body with velocity $\delta\hat{q}$. Then, the spatial stiffness matrix with respect to any body with velocity $\nu\delta\hat{q}$ has the form (7.118). The symmetric parts of the spatial stiffnesses with respect to all such bodies are equivalent.*

Note that, in (7.118), $\nu = 0$ correctly yields the stiffness with respect to the fixed body and $\nu = 1$ gives the stiffness with respect to the moving body. As ν changes from 0 to 1 the skew-symmetric part smoothly changes from $-\frac{1}{2}\hat{W}\times$ to $\frac{1}{2}\hat{W}\times$, while the symmetric part is constant, demonstrating the previous result that $\hat{K}' = \hat{K}^T$.

One immediately observes another interesting case for $\nu = \frac{1}{2}$, namely

$$\hat{K}^p = \hat{K}_{\text{sym}} \tag{7.119}$$

That is, with respect to a body moving with a velocity of $\frac{1}{2}\delta\hat{q}$ the stiffness matrix is symmetric. A special case of this result for parallel planar connections was obtained by Pigoski et al. [43]. Later, the case was generalized by Kumar et al. [?] to spatial connections. This body has been called the **symmetric body**. However, a natural question is about the uniqueness of such a body. That is, is there only one such body with respect to which the stiffness is symmetric? As one may now guess from Corollary 106, the answer is no.

Theorem 110 *Given an elastic connection with line springs between a fixed body and a moving body with velocity $\delta\hat{q}$, there exist infinitely many intermediate bodies, whose velocities in general form a* *two-parameter velocity field*

$$\delta\hat{p} = \begin{bmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \beta\mathbf{I} & \mathbf{0} \end{bmatrix} \hat{W} + \frac{1}{2}\delta\hat{q}, \tag{7.120}$$

with respect to which the stiffness matrices are symmetric and equivalent. In particular, if $\hat{W}$ is a pure couple then there exists a 4-parameter family.

Proof. By (7.119) the stiffness with respect to the body with velocity $\frac{1}{2}\delta\hat{q}$ is symmetric. Then, by Corollary 106 all bodies with velocities $\delta\hat{p}$ such that

$$\delta\hat{p} - \frac{1}{2}\delta\hat{q} = \begin{bmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \beta\mathbf{I} & \mathbf{0} \end{bmatrix} \hat{W} \tag{7.121}$$

see the same symmetric stiffness. In particular, if $\hat{W}$ is a pure couple then there exists a 4-parameter family of motions, as indicated in Corollary 106.

■

Equation (7.120) identifies a family of intermediate body motions that are equivalent in the sense that the spatial stiffness matrix is the same symmetric matrix. All such motions have a common component given by $\frac{1}{2}\delta\hat{q}$. They differ from each other by the first term in (7.120) which is a twist with the screw axis of $\hat{W}$ and an arbitrary pitch.

Since the motion $\nu\delta\hat{q}$ only affects the skew-symmetric part it can be superposed onto the motion (7.120) yielding the following corollary.

Corollary 111 *Given an elastic connection between a fixed and a moving body with velocity $\delta\hat{q}$, there exists an intermediate body with velocity*

$$\delta\hat{p} = \begin{bmatrix} \alpha\mathbf{I} & \beta\mathbf{I} \\ \beta\mathbf{I} & \mathbf{0} \end{bmatrix} \hat{W} + (\nu + \frac{1}{2})\delta\hat{q} \tag{7.122}$$

with respect to which the stiffness is

$$\hat{K}^p = \hat{K}_{sym} + \nu\hat{W}\times \tag{7.123}$$

The velocities of all such bodies form a three-parameter family of motions with respect which the symmetric part of the stiffness is $\hat{K}_{sym}$.

Note that, in this case, $\nu = -\frac{1}{2}, 0, \frac{1}{2}$ respectively correspond to the fixed, symmetric and moving bodies.

7.3.4 Torsional Spring Stiffness With Respect to Moving Bodies

In parallel to the development for line springs, one may wonder about the stiffness of torsional springs when referred to moving bodies. For torsional springs, the skew-symmetric part does not have a simple form. Consider again an intermediate body with velocity $\delta\hat{p}$. If the constraint $\delta\hat{p} = \hat{P}\delta\hat{q}$ is introduced, the stiffness with respect to the intermediate body is

$$\hat{K}^p = \hat{K} + \hat{W} \times \hat{P} \tag{7.124}$$

First consider the velocity field given by $\delta\hat{p}_O = \nu\delta\hat{q}_O$. Then, by using $\hat{P} = \nu\hat{I}$, (7.45) and (7.52) in (7.124), $\hat{K}^p$ becomes

$$\hat{K}^p = \sum_{i=1}^{n} k_i \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\frac{(\theta_i-\theta_{0i})}{l_i}\vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i\times & \vec{\mathbf{s}}_i\vec{\mathbf{s}}_i^T + \frac{(\theta_i-\theta_{0i})}{l_i}\vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times \overrightarrow{OB_i'} \times + \nu\left(\theta_i - \theta_{0i}\right)\vec{\mathbf{s}}_i\times \end{bmatrix} \tag{7.125}$$

Unlike the line springs case, no value of ν makes $\hat{K}^p$ symmetric. Hence, for these motions there is no *symmetric body* for torsional springs. For $\nu = 1$ one obtains the stiffness matrix with respect to the moving body.

$$\hat{K}' = \sum_{i=1}^{n} k_i \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\frac{(\theta_i-\theta_{0i})}{l_i}\vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i\times & \vec{\mathbf{s}}_i\vec{\mathbf{s}}_i^T + \frac{(\theta_i-\theta_{0i})}{l_i}\vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times \overrightarrow{OB_i'} \times + \left(\theta_i - \theta_{0i}\right)\vec{\mathbf{s}}_i\times \end{bmatrix} \tag{7.126}$$

Now, using the identity $\vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i\times = -\vec{\mathbf{s}}_i\times$ and the geometrical relation $\overrightarrow{OB_i'} = \overrightarrow{OA_i} + l_i\vec{\mathbf{s}}_i$ in (7.126) one gets

$$\hat{K}' = \sum_{i=1}^{n} k_i \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ -\frac{(\theta_i-\theta_{0i})}{l_i}\vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i\times & \vec{\mathbf{s}}_i\vec{\mathbf{s}}_i^T + \frac{(\theta_i-\theta_{0i})}{l_i}\vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times \overrightarrow{OA_i}\times \end{bmatrix} \tag{7.127}$$

This is what an observer on the moving body would observe as the stiffness. The symmetry in the expressions of the stiffness with respect to the fixed and the moving bodies is preserved, yet the matrices are not transposes of each other. One can easily see that

$$\hat{K} - \hat{K}' = -\hat{W}\times \tag{7.128}$$

But this is an identity that can be deduced from (7.91). Only in the line spring case does this difference equal twice the skew-symmetric part.

If instead of $\hat{P} = \nu\hat{I}$ one considers a general one specified by $\hat{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix}$, the condition for the symmetric stiffness is

$$2\hat{K}_{\text{skew}} + \hat{W} \times \hat{P} + P^T\hat{W}\times = 0 \tag{7.129}$$

where $\hat{W}$ is a pure couple. The lower diagonal of the matrix equation (7.129) is

$$\left[\sum k_i\left(\theta_i - \theta_{0i}\right)\vec{\mathbf{s}}_i\right] \times \mathbf{P}_{21} = -\sum k_i \frac{(\theta_i - \theta_{0i})}{l_i}\vec{\mathbf{s}}_i \times \vec{\mathbf{s}}_i \times \tag{7.130}$$

The left hand side of (7.130) is always singular, the right hand side, on the other hand, is in general non-singular. So, in general, there can be no solution for $\mathbf{P}_{21}$. Hence the following theorem.

Theorem 112 *In general, there is no intermediate body with respect to which the stiffness of a torsional spring connection is symmetric.*

Curiously, however, for a connection with one torsional spring there exists a solution given by

$$\hat{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \frac{1}{l}\vec{\mathbf{s}}\times & -\frac{1}{l}\vec{\mathbf{s}} \times \overrightarrow{OB'}\times \end{bmatrix} \tag{7.131}$$

where $\mathbf{P}_{11}$ and $\mathbf{P}_{12}$ are arbitrary. In terms of the velocity,

$$\delta\hat{p} \;=\; \hat{P}\delta\hat{q} = \begin{bmatrix} \vec{\boldsymbol{\delta}} \\ \frac{1}{l}\vec{\mathbf{s}} \times d\vec{\mathbf{r}} - \frac{1}{l}\vec{\mathbf{s}} \times \overrightarrow{OB'} \times \delta\vec{\boldsymbol{\theta}} \end{bmatrix} \tag{7.132}$$

$$= \begin{bmatrix} \vec{\delta} \\ \frac{1}{l}\vec{\mathbf{s}} \times \left(d\vec{\mathbf{r}} - \overrightarrow{OB'} \times \delta\vec{\theta} \right) \end{bmatrix} = \begin{bmatrix} \vec{\delta} \\ \frac{1}{l}\vec{\mathbf{s}} \times d\vec{\mathbf{r}}_B \end{bmatrix} \tag{7.133}$$

This is a body with an arbitrary translation. Note that $d\vec{r}_B$ is the translation of the moving body at B. Then, $\frac{1}{l}\vec{\mathbf{s}} \times d\vec{\mathbf{r}}_B$ is the rotation of the moving body if it were hinged at A and had a translation at B equal to the component of $d\vec{\mathbf{r}}_B$ perpendicular to $\vec{\mathbf{s}}$. Further interpretation of the rotation is possible but not very useful. The stiffness matrix with respect to such an intermediate body is simply given by the symmetric matrix of (7.56) for one spring.

As a final remark it can be noted that Corollary 106 also applies to the torsional spring case proving the existence of two parameter velocity fields, $\times$-parallel motions, with respect to which the stiffnesses are equivalent.

Part III

APPLICATIONS

CHAPTER VIII

SYNTHESIS OF STIFFNESS BY SPRINGS

The primary goal of this chapter is to present a systematic theory and method to solve the synthesis of stiffness by springs problem. The synthesis problem has remained unsolved until now.

First, a new eigenvector problem for general matrices is presented and solved. This new eigenvector problem is related to the concept of *isotropic vectors*. Accordingly, it is called the *isotropic vector problem* in this study. The results are used in synthesis of stiffnesses and in proving an existence theorem for classical RCC devices, see Chapter 9. However, the isotropic vector problem has very general uses as explained in a related section. Because of this generality, presentation of the isotropic vector problem may seem, and indeed is, unrelated to the synthesis problem, screws, stiffnesses, etc. So, those who are interested only in the synthesis problem may skip the discussion of the isotropic vectors.

Then, the synthesis of stiffness by springs is fully solved using the results from the isotropic vector problem. The method is applicable to any semi-positive definite stiffness. It is shown that any stiffness can be synthesized by $n \geq r$ springs, where r is the rank. Algorithms suitable for computer applications are provided along with numerical examples.

Finally, two minimum synthesis cases $(n = r)$ are shown to be obtainable by applying the free-vector and line-vector decompositions presented in Chapters 3 and 4. Algorithms and numerical examples are also presented.

It should be kept in mind that, more than anything, it is the spatial vector algebra which provides the essential platform that ultimately enables one to solve the synthesis problem. Although the development may seem complicated at times, the results are simple and sensible.

8.1 Isotropic Vector Problem

Eigenvalue problems in mathematics, from linear algebra to differential equations, almost always have physical correspondences. They help explain complicated physical phenomena in terms that make sense to the human mind. In engineering, the concepts of principal stress, principal strain, principal axes and values of inertia, etc. are among countless quantities that are derived from eigenvector problems with physical meanings.

In the analysis of stiffness and compliance, the eigenvalue problems provide a deeper understanding of these phenomena, especially using screw (spatial vector) algebra. The first example came from the work of Ball [1], who applied the screw theory to study rigid body motion. He investigated a generalized eigenvalue problem for stiffness, involving an indefinite metric, which lead to screws called the principal screws of potential. Then, Dimentberg [15] proposed a different eigenvalue problem. In Lipkin and Patterson's work [30], Dimentberg's proposal turned out to be only one half of their problem. Lipkin and Patterson generalized and completed Dimentberg's eigenvalue problem which yielded screws called the eigentwists and eigenwrenches with accompanying eigenstiffnesses. Further, in this study, the existence of a complementary eigenvalue problem for stiffness and compliance is presented and solved in earlier chapters.

In this section, a different class of eigenvector problems is proposed and solved. It is called an *eigenvector* problem rather than an *eigenvalue* problem because it is seen that there is no immediate meaning of an eigenvalue. Briefly stated, a vector $\vec{v}$ is an isotropic vector of matrix A if and only if $\vec{v}^T A\vec{v} = 0$. An explicit method is presented to determine arbitrary isotropic vectors of a given matrix. However, more important to this study is the existence of an orthonormal basis made of isotropic vectors of the matrix. This is needed in the synthesis problem. The necessary and sufficient condition for this is shown to be vanishing trace of the matrix. A recursive algorithm is also presented, which determines these bases for a given zero trace matrix. Explicit solutions are given for three dimensional matrices. Finally, subjects from different areas of science are shown to be related to the isotropic vectors.

8.1.1 Preliminaries

A generalized eigenvalue problem on a vector space V is given as

$$A\vec{u} = \lambda B\vec{u} \tag{8.1}$$

where $\vec{u} \in V$ and, A and B are matrices, and λ is a scalar. A matrix is a representation of a linear map on a vector space. This study is concerned only with linear maps represented by square matrices. Therefore, the generalized eigenvalue problem (8.1) is simply about the existence of a vector $\vec{u}$ which is transformed into the same vector under the actions of two linear operators A and λB.

Let the standard Euclidean norm be defined on $\mathcal{R}^n$ such that for any $\vec{v} \in \mathcal{R}^n$, $\vec{v}^T\vec{v} \geq 0$ is the square length of the vector $\vec{v}$. Any non-zero $\vec{u}$ such that $\vec{u}^T\vec{u} = 1$ is called a unit vector. Any two unit vectors $\vec{u}_1$ and $\vec{u}_2$ are said to be parallel if $\vec{u}_1^T\vec{u}_2 = 1$ and orthogonal (or perpendicular, reciprocal, etc.) if $\vec{u}_1^T\vec{u}_2 = 0$.

The action of a square matrix on a vector is composed of two parts: 1) a rotation and 2) a scaling. Rotation is understood in the sense of length invariance. For example, orthogonal matrices defined by $UU^T = I$ only perform rotations.

The eigenvalue problem in linear algebra is about the existence of vectors which are scaled but not rotated under the action of a given square matrix A,

$$A\vec{u} = \lambda\vec{u} \tag{8.2}$$

Such vectors $\vec{u}$ and scalars λ are the classical eigenvectors and eigenvalues of A. Equation (8.2) is a specialization of (8.1) by taking $B = I$, where I is the identity map. It is well known that for symmetric A there exist n real mutually orthogonal eigenvectors with real eigenvalues. Also well

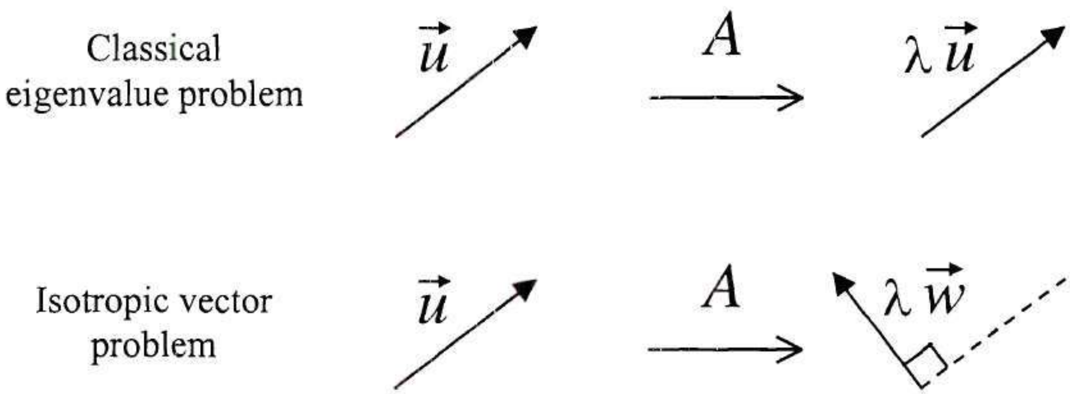


Figure 8.1: Comparison of the classical eigenvalue and isotropic vector problems.

known is the fact the classical eigenvectors, with the constraint $\vec{u}^T\vec{u} = 1$, make the quadratic form $\vec{u}^T A\vec{u}$ stationary, with eigenvalues being the stationary values.

In the proposed isotropic vector problem, a non-zero vector is scaled and rotated to an orthogonal direction. Figure 8.1 illustrates the classical eigenvector and isotropic vector problems geometrically.

8.1.2 Definition of Isotropic Vectors

Definition 113 *For a square matrix A, any vector $\vec{u}$ satisfying*

$$A\vec{u} = \vec{w} \qquad \vec{u}^T\vec{w} = 0 \tag{8.3}$$

*is called an **isotropic vector** of A.*

The following lemma follows directly from (8.3).

Lemma 114 *A vector $\vec{u}$ is an isotropic vector of A if and only if $\vec{u}^T A\vec{u} = 0$.*

Lemma 114 demonstrates the reason for the name *isotropic vector.* In tensor analysis, any vector $\vec{v}$ such that $\vec{v}^T G\vec{v} = 0$, where G is the metric tensor, is called an isotropic vector of the metric tensor.

Considering the quadratic form in Lemma 114 induced by A as a scalar valued vector function, *the classical eigenvectors are the stationary points and the isotropic vectors are the zero locus.*

Theorem 115 *The isotropic vectors of a matrix A are identical to those of its symmetric part.*

Proof. Let A be decomposed into its symmetric and skew-symmetric parts as $A = A_{\text{sym}} + A_{\text{skew}}$. Then (8.3) can be given as $A_{\text{sym}}\vec{u} = \vec{w} - A_{\text{skew}}\vec{u} = \vec{w}^*$. But,since $\vec{u}^T A_{\text{skew}}\vec{u} = 0$ for any $\vec{u}$, by Lemma 114 this is equivalent to $A_{\text{sym}}\vec{u} = \vec{w}^*$ such that $\vec{u}^T\vec{w}^* = 0$. So, by definition, $\vec{u}$ is an isotropic vector of A_{sym}.

■

This is an interesting distinction between the classical and isotropic vector problems, indicating that it is sufficient to restrict the analysis to symmetric matrices only. So, unless otherwise is noted, A is assumed to be symmetric throughout.

8.1.3 Existence of Isotropic Vectors

For any matrix A, if $\vec{v} \in \mathcal{N}(A)$, where $\mathcal{N}(A)$ is the null space of A, then $A\vec{v} = \vec{0}$ by definition, thus $\vec{v}^T A\vec{v} = \vec{0}$. So, any vector in $\mathcal{N}(A)$ is an isotropic vector. Such isotropic vectors are considered trivial. The following theorem gives the condition for the existence of non-trivial isotropic vectors.

Theorem 116 *A has non-trivial isotropic vectors if and only if it is indefinite.*

Proof. Let $\vec{v}$ be a non-trivial isotropic vector. Then, $\vec{v}^T A\vec{v} = \vec{0}$, $\vec{v} \notin \mathcal{N}(A)$ and, by definition, A is indefinite. Conversely, let A be an indefinite matrix with eigenvalues λ_i $(i = 1, ..., n)$ such that $\max(\lambda_i) = \lambda_1 > 0$ and $\min(\lambda_i) = \lambda_n < 0$. Consider a vector $\vec{v}$ as a point of $\mathcal{R}^n$. Then, the quadratic form $\vec{v}^T A\vec{v}$ is a continuous function from $\mathcal{R}^n$ into $\mathcal{R}$. It is well known that $\vec{v}_1^T A\vec{v}_1 = \lambda_1 > 0$ and $\vec{v}_n^T A\vec{v}_n = \lambda_n < 0$, where $\vec{v}_1$ and $\vec{v}_n$ are the unit eigenvectors corresponding to λ_1 and λ_n. Consider any smooth curve $\xi(t)$ from $\vec{v}_1$ to $\vec{v}_n$ parametrized by t. Then, $\vec{v}^T A\vec{v}(t)$ is a continuous function of t along the curve. So, by mean value theorem, $\vec{v}^T A\vec{v}(t)$ takes on every value in $[\vec{v}_n^T A\vec{v}_n, \vec{v}_1^T A\vec{v}_1]$.

Hence, there exists a t_0 such that $\vec{v}^T A\vec{v}(t_0) = 0 \in [\vec{v}_n^T A\vec{v}_n, \vec{v}_1^T A\vec{v}_1] = [\lambda_n, \lambda_1]$. In particular, for all smooth curves restricted to the plane formed by $\vec{v}_1$ and $\vec{v}_n$, the points on the curve represent vectors which are simply linear combinations of $\vec{v}_1$ and $\vec{v}_n$ and, therefore, not in the null space of A. Then, the points at which the quadratic form vanish give the non-trivial isotropic vectors.

■

A particular solution which is used the next section is the following.

Theorem 117 *If $\lambda_1 > 0$ and $\lambda_2 < 0$ are any two eigenvalues, and, $\vec{v}_1$ and $\vec{v}_2$ are any two corresponding eigenvectors of a symmetric A then two isotropic vectors are given by*

$$\vec{u}_{1,2} = \sqrt{-\lambda_2}\vec{v}_1 \pm \sqrt{\lambda_1}\vec{v}_2 \tag{8.4}$$

Proof. One only needs to use the well known properties of classical eigenvectors of symmetric matrices. Namely, $\vec{v}_i^T A\vec{v}_j = \delta_{ij}\lambda_i$, where δ_{ij} is the Kronecker's delta. Then

$$\begin{aligned}\vec{u}_i^T A\vec{u}_i &= \left(\sqrt{-\lambda_2}\vec{v}_1 \pm \sqrt{\lambda_1}\vec{v}_2\right)^T A\left(\sqrt{-\lambda_2}\vec{v}_1 \pm \sqrt{\lambda_1}\vec{v}_2\right)\\ &= -\lambda_2\vec{v}_1^T A\vec{v}_1 + \lambda_1\vec{v}_2^T A\vec{v}_2\\ &= -\lambda_2\lambda_1 + \lambda_1\lambda_2 = 0 \end{aligned}\tag{8.5}$$

■

Corollary 118 *The isotropic vectors of Theorem 117 are orthogonal if and only if $\lambda_1 = -\lambda_2$.*

Proof. By Theorem 117,

$$\begin{aligned}\vec{u}_1^T\vec{u}_2 &= \left(\sqrt{-\lambda_2}\vec{v}_1 + \sqrt{\lambda_1}\vec{v}_2\right)^T\left(\sqrt{-\lambda_2}\vec{v}_1 - \sqrt{\lambda_1}\vec{v}_2\right)\\ &= -(\lambda_2 + \lambda_1)\end{aligned}\tag{8.6}$$

from which the corollary follows.

■

Again for any $\vec{v}_1$ and $\vec{v}_2$ be such that $\vec{v}_1^T A\vec{v}_1 > 0$ and $\vec{v}_2^T A\vec{v}_2 < 0$, consider all vectors in the plane formed by $\vec{v}_1$ and $\vec{v}_2$. These can be given as $\vec{v} = a_1\vec{v}_1 + a_2\vec{v}_2$. This plane contains the origin and the quadratic form changes sign on it. Since $\vec{v}_1$ and $\vec{v}_2$ are not solutions to $\vec{v}^T A\vec{v} = 0$ one can take $a_i \neq 0$. Also, the magnitude of $\vec{v}$ is insignificant. Therefore, $\vec{v}$ is completely characterized by $\vec{v} = \vec{v}_1 + a\vec{v}_2$, up to the magnitude. Now, the quadratic equation becomes

$$\vec{v}^T A\vec{v} = \left(\vec{v}_2^T A\vec{v}_2\right) a^2 + 2\left(\vec{v}_1^T A\vec{v}_2\right) a + \left(\vec{v}_1^T A\vec{v}_1\right) = 0 \tag{8.7}$$

The discriminant of this equation is always positive since $\left(\vec{v}_2^T A\vec{v}_2\right)\left(\vec{v}_1^T A\vec{v}_1\right) < 0$. As a result there exist two distinct real solutions for a. Hence the following is proven.

Theorem 119 *If the quadratic form $\vec{v}^T A\vec{v}$ changes sign on a plane containing the origin then there exist two distinct lines on which it vanishes. The directions of these lines give two distinct isotropic vectors.*

Corollary 120 *For an indefinite matrix A, there exist infinitely many distinct isotropic vectors if $n > 2$, and there exist exactly two isotropic vectors if $n = 2$.*

Proof. For $n = 1$, A cannot be indefinite. Therefore, $n \geq 2$.

For $n = 2$, there exists a only one plane through the origin. The quadratic form changes sign on this plane since there are two lines on which it has opposite signs due to indefiniteness. Then, by Theorem 119, there exists two isotropic vectors.

For $n > 2$, there exist infinitely many planes through the origin. Since A is indefinite, the quadratic form changes sign on at least one plane, say π. Consider a plane π_1 which is obtained by an infinitesimal rotation of π about an axis through the origin. Then, by continuity, the quadratic form must also change sign π_1. In this way one obtains infinitely many planes on which the quadratic form changes sign. So, by Theorem 119, there exist infinitely many isotropic vectors for $n > 2$.

∎

From Corollary 120, every indefinite matrix of order n has at least n isotropic vectors. So, let $U = [\vec{u}_1, ..., \vec{u}_n]$ be a square matrix formed from any n isotropic vectors, and $W = [\vec{w}_1, ..., \vec{w}_n]$ be the corresponding vectors, $\vec{u}_i^T \vec{w}_i = 0$, where $\vec{w}_i = A\vec{u}_i$. Then, by definition, $AU = W$. Premultiplying both sides by U^T one gets

$$U^T A U = U^T W = \begin{bmatrix} \vec{u}_1^T \vec{w}_1 & \cdots & \vec{u}_1^T \vec{w}_n \\ \vdots & \ddots & \vdots \\ \vec{u}_n^T \vec{w}_1 & \cdots & \vec{u}_n^T \vec{w}_n \end{bmatrix} \tag{8.8}$$

But, the diagonal elements are $\vec{u}_i^T \vec{w}_i = 0$. Therefore,

Theorem 121 *Every indefinite matrix is congruent to a matrix whose diagonal elements are all zero.*

8.1.4 Orthonormal Sets of Isotropic Vectors

As an analogy with the classical eigenvalue problem, one may ask whether there can be n mutually orthogonal isotropic vectors of a matrix. Assume there exists such a set of isotropic vectors. Again, let $U = [\vec{u}_i]$ be an orthogonal matrix formed from these n elements. Then, by Theorem 121, $\Phi = U^T A U$ is a matrix with zero diagonal, and therefore zero trace. However, since U is orthogonal, $U^T U = I$. Therefore,

$$\begin{aligned} 0 &= \text{trace}(\Phi) = \text{trace}(U^T A U) & (8.9) \\ &= \text{trace}(U^T U A) = \text{trace}(A) & (8.10) \end{aligned}$$

Theorem 122 *If a matrix A of order n has an orthogonal set of n isotropic vectors then* $\text{trace}(A) = 0$.

Note that, if a matrix has n mutually orthogonal isotropic vectors then $U^T A U$ is an orthogonal transformation. In a basis formed by these vectors A has zero diagonals. This result is complementary to the classical eigenvalue problem, where a symmetric matrix has zero off-diagonal terms when expressed in an orthogonal basis comprised of eigenvectors. Theorem 122 is a necessary condition.

A zero trace matrix A is either zero or indefinite. If $n = 1$ then $A = 0$ and there exists a unique isotropic vector (trivial). If $n > 1$ and $A = 0$ then there exist infinitely many distinct isotropic vectors (all trivial). In other cases Corollary 120 applies. This proves the following.

Corollary 123 *A matrix A, such that* $\text{trace}(A) = 0$, *has infinitely many distinct isotropic vectors if $n > 2$, or, $n = 2$ and $A = 0$. It has exactly one and two isotropic vectors for $n = 1$ and $n = 2$ $(A \neq 0)$, respectively.*

These cases are separately analyzed below.

For $A = 0$ any orthonormal set of n vectors is an orthonormal set of isotropic vectors. So, the $A = 0$ cases are trivial for all n. For $n = 1$, $A = 0$. Therefore, the solution is trivial.

For $n = 2$ and $A \neq 0$, there exists only two distinct isotropic vectors, Corollary 123. However, $\text{trace}(A) = \lambda_1 + \lambda_2 = 0$ and the isotropic vectors are orthogonal by Corollary 118.

Only the cases with $n > 2$ and $A \neq 0$ are left to consider. By Corollary 123, there exist infinitely many isotropic vectors in this case. Given an indefinite matrix, one can always find an isotropic vector by using Theorem 117. Question is to find another that is orthogonal to the first when $n > 2$. If a suitable method is developed to do this, then it can be repeatedly applied to find a series of orthogonal isotropic vectors. The process should stop naturally when n such vectors are obtained, since there cannot be $n + 1$ orthogonal vectors in an n-dimensional vector space. The following discussion and theorems present a recursive algorithm that generates an orthonormal set of isotropic vectors.

For classical eigenvalue problems, there exists a method that sequentially and recursively generates the eigenvalues and corresponding eigenvectors. Let A be a symmetric matrix. A can be assumed to be positive definite. If it is not, one can always perform a shifting of the eigenvalues by $A' = A + kI$, where $k > |\min(\lambda_i)|$. The matrices A and A' have the same eigenvectors. Their eigenvalues are related by $\lambda_i' = \lambda_i + k$, so that A' is positive definite. Let the eigenvalues be ordered such that $\lambda_n \le \lambda_{n-1} \le \cdots \le \lambda_1$.

Many numerical procedures used to determine eigenvalues converge to the minimum. So, even if they are used with different initial guesses, chances of getting all eigenvalues are slim. The method used to overcome this difficulty is based on the fact that, for any symmetric positive definite matrix A, the matrix $A - \lambda_i \vec{v}_i \vec{v}_i^T$, where λ_i is any eigenvalue with the corresponding eigenvector $\vec{v}_i$, retains all the eigenvalues of A except λ_i which is replaced by zero. The eigenvectors are all identical. This is sometimes called *deflating* a matrix. So, if λ_n and $\vec{v}_n$ is found by any means, then one constructs $A_2 = A - \lambda_n \vec{v}_n \vec{v}_n^T$. Then, one performs an orthogonal transformation such that $\vec{v}_n$ is one of the standard basis vectors. In this system, $A_2 = \begin{bmatrix} 0 & \vec{0}^T \\ \vec{0} & A_2^* \end{bmatrix}$, where A_2^* is an $(n-1) \times (n-1)$ matrix whose eigenvalues are $\lambda_{n-1} \le \cdots \le \lambda_1$. So, by applying a numerical procedure, one finds λ_{n-1} and $\vec{v}_{n-1}^*$. Repeating the procedure in this way, one determines all eigenvalues and eigenvectors. It is this method that inspires the following theorem.

Theorem 124 *If $\vec{u}_1$ is a unit isotropic vector of A,* trace$(A) = 0$*, $A\vec{u}_1 = \vec{w}_1$, then there exists a symmetric matrix A^* given by*

$$A^* = A - \left(\vec{w}_1 \vec{u}_1^T + \vec{u}_1 \vec{w}_1^T\right) \tag{8.11}$$

such that

1. trace$(A^*) = 0$
2. *$\vec{u}_1$ is in the null space of A^*.*
3. *The isotropic vectors of A^* orthogonal to $\vec{u}_1$ are also the isotropic vectors of A, and vice versa.*

Proof.

(1) For any two vectors $\vec{a}$ and $\vec{b}$, trace$(\vec{a}\vec{b}^T) = \vec{a}^T\vec{b}$. Then, by using trace$(A) = 0$ and $\vec{w}_1^T\vec{u}_1 = 0$ in (8.11),

$$\begin{aligned} \text{trace}(A^*) &= \text{trace}(A) - \text{trace}\left(\vec{w}_1\vec{u}_1^T + \vec{u}_1\vec{w}_1^T\right) \\ &= -\vec{u}_1^T\vec{w}_1 - \vec{w}_1^T\vec{u}_1 = 0 \end{aligned} \tag{8.12}$$

(2) Next, by multiplying (8.11) by $\vec{u}_1$, and, using $\vec{u}_1^T\vec{u}_1 = 1$ and $\vec{w}_1^T\vec{u}_1 = 0$, one gets

$$A^*\vec{u}_1 = \left[A - \vec{w}_1\vec{u}_1^T - \vec{u}_1\vec{w}_1^T\right]\vec{u}_1 = A\vec{u}_1 - \vec{w}_1 \tag{8.13}$$

$$A^*\vec{u}_1 = \vec{0} \tag{8.14}$$

(3) Finally, if $\vec{u}_2$ is orthogonal to $\vec{u}_1$ then, from (8.11), $\vec{u}_2^T A^*\vec{u}_2 = \vec{u}_2^T A\vec{u}_2$. So, the quadratic forms of A and A^* have identical values in the subspace orthogonal to $\vec{u}_1$. Therefore, if $\vec{u}_2$ is an isotropic vector of A^* orthogonal to $\vec{u}_1$ then it is also an isotropic vector of A, and vice versa.

■

Corollary 125 *For any zero trace symmetric matrix A, there exists a finite sequence of zero trace symmetric matrices, $A = A_1, A_2, ..., A_n = 0$, recursively given by*

$$A_{i+1} = A_i - \left(\vec{u}_i\vec{w}_i^T + \vec{w}_i\vec{u}_i^T\right) \tag{8.15}$$

where $\vec{u}_i$ is an isotropic vector of A_i, $A_i\vec{u}_i = \vec{w}_i$, such that

1. *$A_i\vec{u}_j = \vec{0}$, for all $j < i$.*

2. *$\{\vec{u}_i\}$ is an orthonormal set.*

3. *$\vec{u}_i$ is an isotropic vector of all A_j, $j \leq i$.*

Proof. If $n = 1$ then $A_1 = 0$ and the sequence is determined. Statements of the theorem are trivially true. If $n = 2$, then the two orthogonal isotropic vectors of A_1 are given by Theorem 141, and, since A_2 has zero trace and a zero eigenvalue, $A_2 = 0$ which ends the sequence. Statements of the theorem are again verified.

So, assume $n > 2$. For the proof induction is used. Assume that (1), (2) and (3) are true for some i. Then,

1. for $j < i+1$

$$A_{i+1}\vec{u}_j = A_i\vec{u}_j - \vec{u}_i\vec{w}_i^T\vec{u}_j - \vec{w}_i\vec{u}_i^T\vec{u}_j \tag{8.16}$$
$$= \begin{cases} A_i\vec{u}_i - \vec{w}_i = \vec{0} & j = i \\ -\vec{u}_i\vec{w}_i^T\vec{u}_j = -\vec{u}_i\vec{u}_i^T\left(A_i\vec{u}_j\right) = \vec{0} & j < i \end{cases}$$

   which proves that $A_{i+1}\vec{u}_j = \vec{0}$ for all $j < i+1$. This means that all $\vec{u}_j$, $j < i+1$, are in the null space of A_{i+1}.

2. By Theorem 124, an isotropic vector of A_{i+1}, $\vec{u}_{i+1}$, exists which is not in the subspace spanned by $\{\vec{u}_i\}$. So, $\{\{\vec{u}_i\}, \vec{u}_{i+1}\}$ is an orthogonal set.

3. Finally, again by Theorem 124, one concludes that $\vec{u}_{i+1}$ is an isotropic vector of A_j for all $j \leq i+1$.

So, the theorem is true for $i+1$ if it is true for some i. But, it is true for $i = 2$ due to Theorem 124. Therefore, it is true for all i.

■

This corollary proves that the zero trace condition is sufficient for a symmetric matrix to have n orthonormal isotropic vectors. Combining with Theorem 122 yields the following main result of this section.

Theorem 126 *A matrix A of order n has an orthonormal set of n isotropic vectors if and only if* $\text{trace}(A) = 0$.

Corollary 127 Any matrix A *of order* n *is orthogonally congruent to a matrix with identical diagonal elements given by* $\frac{1}{n}\text{trace}(A)$.

Proof. For any A, $A' = A - \frac{1}{n}\text{trace}(A)I$ is a zero trace matrix. By Theorems 121 and 126, A' is orthogonally congruent to a matrix Φ with zero diagonals, i.e. $U^T A' U = \Phi$, $U^T U = I$. Therefore,

$$U^T A U = U^T A' U + \frac{1}{n}\text{trace}(A) U^T I U \tag{8.17}$$

$$= \Phi + \frac{1}{n}\text{trace}(A) I \tag{8.18}$$

which is a matrix with all diagonal entries equal to $\frac{1}{n}\text{trace}(A)$.

■

Corollary 125 also provides a recursive method for the construction of an orthonormal set of n isotropic vectors of arbitrary zero trace matrices. The following is a working algorithm.

1. Read $n \times n$ matrix M.
2. If $\text{trace}(M) \neq 0$ Stop. No solution.
3. Symmetric part: $A = \frac{1}{2}(M + M^T)$. (M and A have the same isotropic vectors, Theorem 115.)
4. Initialize: U =a basis for $\mathcal{N}(A)$. (Trivial isotropic vectors. Any orthogonal set must contain a basis for $\mathcal{N}(A)$.)
5. Loop: while $A \neq 0$,
6. $\quad$ λ_i =eigenvalues, $\vec{v}_i$ =eigenvectors of A.
7. $\quad$ Find the minimum eigenvalue and its corresponding eigenvector, λ_m and $\vec{v}_m$.
8. $\quad$ Find the maximum eigenvalue and its corresponding eigenvector, λ_p and $\vec{v}_p$.
9. $\quad$ Remark: $\lambda_m < 0$ and $\lambda_p > 0$ since $A \neq 0$ and $\text{trace}(A) = 0$.

10. $\vec{u} = \text{Normalize}\left[\sqrt{-\lambda_m}\vec{v}_p + \sqrt{\lambda_p}\vec{v}_m\right]$. (A unit isotropic vector, Theorem 117.)

11. Concatenate $U = [U : \vec{u}]$. ($\vec{u}$ is in the range space of A.Thus, it is orthonormal to vectors in $\mathcal{N}(A)$.)

12. $\vec{w} = A\vec{u}$

13. $A = A - \left(\vec{w}\vec{u}^T + \vec{u}\vec{w}^T\right)$. (Deflated matrix, see Theorem 124)

14. If $-\lambda_m = \lambda_p$ Then, (use Theorem 117 for the orthogonal companion of $\vec{u}$, Corollary 118.)

15. $\vec{u} = \text{Normalize}\left[\sqrt{-\lambda_m}\vec{v}_p + \sqrt{\lambda_p}\vec{v}_m\right]$

16. Concatenate $U = [U : \vec{u}]$

17. $\vec{w} = A\vec{u}$

18. $A = A - \left(\vec{w}\vec{u}^T + \vec{u}\vec{w}^T\right)$

19. End If.

20. $A = \frac{1}{2}(A + A^T)$. (Mask any numerical errors due to limited machine precision.)

21. End Loop.

The above algorithm has been tested numerically using MATLAB, a mathematics software package, with satisfactory results. The statement second from the bottom seems to be necessary for numerical stability. Note that whenever $\min(\lambda_i) = -\max(\lambda_i)$, both isotropic vectors predicted by Theorem 117 are used since they are already orthogonal in that case, Corollary 118. But, the actual reason is different. In general, the recursion in Corollary 125 gives $\text{rank}(A_{i+1}) = \text{rank}(A_i) - 1$. However, for any two eigenvalues with equal magnitudes and opposite signs, any of the isotropic vectors of Theorem 141 makes $\text{rank}(A - \vec{w}_1\vec{u}_1^T - \vec{u}_1\vec{w}_1^T) = \text{rank}(A) - 2$. Proof of this fact is not difficult, but omitted here. Then, one can also show that the other vector predicted by Theorem 117, $\vec{u}_2 \perp \vec{u}_1$, is also in the null space of the new matrix. Since the algorithm is basically a recursion based on the column spaces of current matrices, $\vec{u}_2$ must be added to the set or, otherwise, the algorithm returns a deficient set.

Multitude of Solutions It was shown earlier that a trace zero matrix has infinitely many isotropic vectors if $n > 2$, or, $n = 2$ and $A = 0$. So, a natural question is how many orthonormal set of isotropic vectors are there. Note that if U is an orthogonal matrix whose columns are isotropic vectors of A, then

$$U^T A U = \Phi = \begin{bmatrix} 0 & \bullet & \bullet \\ \bullet & \ddots & \bullet \\ \bullet & \bullet & 0 \end{bmatrix} \tag{8.19}$$

An $n \times n$ orthogonal matrix has $\frac{1}{2}n(n-1)$ independent parameters. Since only the diagonals are to be satisfied, the above matrix equation is equivalent to n scalar quadratic equations in terms of the parameters of U. However, due to the trace condition one of these equations is dependent on others, leaving $(n-1)$ independent equations. This gives a net total of $\frac{1}{2}n(n-1)-(n-1) = \frac{1}{2}(n-1)(n-2)$ free parameters. For $n = 1, 2$ this indicates 0 free parameters, meaning finitely many solutions as shown earlier ($n = 2$, $A = 0$ is a degenerate case). For $n = 3$ there is $\frac{1}{2}(3-1)(3-2) = 1$ free parameter. This is demonstrated in the next section.

That the number of free parameters is in general not less than $\frac{1}{2}(n-1)(n-2)$ can be shown by the intersection theory of algebraic varieties, see for example [24]. A sketch of the proof is as follows. Rather than the parameters of U, one uses the $\frac{1}{2}n(n-1)$ non-zero entries of Φ. This is beneficial since it eliminates the need to deal with any parametric form of U. From (8.19), A and Φ have the same characteristic polynomial. That is, the coefficients of their characteristic polynomials, known as **invariants**, must be the same. Since both are A and Φ zero trace matrices and the characteristic polynomials always have a unit leading coefficient, the condition reduces to the equivalency of $n-1$ coefficients. The result is $n-1$ equations in $\frac{1}{2}n(n-1)$ variables. When ordered with respect to the order of the invariants in the characteristic polynomial, these equations have distinct and increasing orders, from a 2^{nd} to an n^{th} order. Each equation represents a hypersurface in the space of $\frac{1}{2}n(n-1)$ variables. A theorem about the intersection of such surfaces, i.e. the common zero locus of such equations, is applicable, see [24]. The result is that there exist components of solutions whose dimension is not less than $\frac{1}{2}(n-1)(n-2)$. This is equivalent to saying that Φ (or U) is described by this many free parameters, though they may be bounded.

In degenerate cases the dimension of the solution space may be smaller. This is because the actual space of the algebraic varieties in question is the complex projective space. The set of solutions always has a component of dimension greater or equal to $\frac{1}{2}(n-1)(n-2)$ which include complex solutions. However, it is possible to have only finitely many real points. This case is illustrated in the next section for three dimensions.

The following theorem summarizes these results.

Theorem 128 *For any matrix A of order n, such that* $\operatorname{trace}(A) = 0$ *and $A \neq 0$, the space of orthogonal matrices $\{U|\ U^T AU = \Phi,\ U^T U = I\}$, whose columns are isotropic vectors of A, is $\frac{1}{2}(n-1)(n-2)$ dimensional in general.*

Closed Form Solutions for Three Dimensions In this section, it is shown that the orthogonal sets of isotropic vectors of 3×3 matrices can be found explicitly, i.e. without using the recursive algorithm presented earlier. To do this, one uses the well known fact that for orthogonal transformations such as $U^T AU = \Phi$, A and Φ must have the same characteristic equation. In three dimensions, the characteristic equation is

$$\det(A - \lambda I) = \det(\Phi - \lambda I) = \lambda^3 - I_1\lambda^2 + I_2\lambda - I_3 = 0 \tag{8.20}$$

whose solutions λ_i are the eigenvalues of A and Φ. The coefficients I_i are called the **invariants** of a matrix. Expressing A in diagonal form, which does not affect the invariants, one shows that

$$I_1 = \lambda_1 + \lambda_2 + \lambda_3 = \mathrm{trace}(A) = 0 \quad (8.21)$$

$$I_2 = \lambda_1\lambda_2 + \lambda_2\lambda_3 + \lambda_3\lambda_1 \quad (8.22)$$

$$I_3 = \lambda_1\lambda_2\lambda_3 = \det(A) \quad (8.23)$$

But, the form of Φ is already known to be

$$\Phi = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix} \quad (8.24)$$

which gives the invariants as

$$I_1 = 0 \quad (8.25)$$

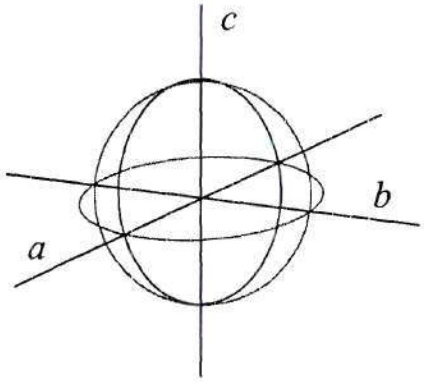
$$I_2 = -\left(a^2 + b^2 + c^2\right) \quad (8.26)$$

$$I_3 = 2abc \quad (8.27)$$

Given A, I_i can be calculated. Therefore, any solution Φ must satisfy (8.26) and (8.27) in terms of unknowns a, b, c.

In abc coordinates, (8.26) represents a *sphere*. Since $I_1 = 0$, one easily shows by $I_1^2 = 0$ that $I_2 \leq 0$ which is necessary and sufficient condition for the sphere to have real points. For $I_2 = 0$ and real a, b, c, the solution is trivial since $a = b = c = 0$, and both matrices must vanish. So, assume $I_2 < 0$. With this assumption, A can have at most one zero eigenvalue. Let λ_i be ordered so that $\lambda_1 \geq \lambda_2 \geq \lambda_3$.

Equation (8.27), on the other hand, is a *third order surface* that has four disconnected components if $I_3 \neq 0$. This is because if $\{a_0, b_0, c_0\}$ is a point on the third order surface then so is any



$$\Phi = \begin{bmatrix} 0 & a & b \\ a & 0 & c \\ b & c & 0 \end{bmatrix}$$

Figure 8.2: Solutions to the isotropic vector problem in 3-dimensions for the degenerate case: $\det(A) = 0$. The three circles are of the same radius and centered at the origin of abc-space. Each circle is in a distinct coordinate plane.

triplet obtained by reversing the signs of any two entries, such as $\{-a_0, -b_0, c_0\}$, giving a total of four points, each located in a distinct octant. These components are disconnected since it is not possible to trace any connected curve from $\{a_0, b_0, c_0\}$ to, say, $\{-a_0, -b_0, c_0\}$ without making $I_3 \neq 0$, which would be contrary to the assumption.

The intersection of the sphere and the third order surface is the solution set. Also, note that if $\{a_0, b_0, c_0\}$ is a solution, so is any permutation of it.

If there exists a zero eigenvalue it must be λ_2 due to ordering. In this case $I_3 = 0$. Then at least one of a, b, c is zero. Let $c = 0$. The others are obtained by permutations. Then, (8.27) is identically satisfied, and (8.26) reduces to $a^2 + b^2 = -I_2$. This is a circle. So, all solutions $\{a, b, c\}$are given by $\sqrt{-I_2}\{\cos\theta, \sin\theta, 0\}$, where θ is arbitrary. Sign reversals give the same circle and permutations give circles in different planes. A total of three circles exist. Figure 8.2 illustrates these circles.

Now, assume $I_3 \neq 0$. Treating $c \neq 0$ as a parameter one gets two equations in a and b as

$$a^2 + b^2 \quad = \quad -I_2 - c^2 \tag{8.28}$$

$$2ab = I_3/c \tag{8.29}$$

By adding and subtracting these two equations from each other one gets

$$(a+b)^2 = I_3/c - I_2 - c^2 = \alpha(c) \tag{8.30}$$

$$(a-b)^2 = -I_3/c - I_2 - c^2 = \beta(c) \tag{8.31}$$

For real solutions $\alpha, \beta \geq 0$. The solutions are

$$a = \frac{\pm\sqrt{\alpha} \pm \sqrt{\beta}}{2}, b = \frac{\pm\sqrt{\alpha} \mp \sqrt{\beta}}{2}, c = c \tag{8.32}$$

which are real if and only if $\alpha, \beta \geq 0$ (either upper or lower signs are to be used). For any given c these indicate two points in two distinct components. It is left to determine if there exist values of c for which $\alpha, \beta \geq 0$.

The conditions $\alpha, \beta \geq 0$ can be multiplied by c to give

$$c\alpha, c\beta \geq 0 \quad \text{for} \quad c > 0 \tag{8.33}$$

$$c\alpha, c\beta \leq 0 \quad \text{for} \quad c < 0 \tag{8.34}$$

Define functions $f^- = c^3 + I_2 c - I_3$ and $f^+ = c^3 + I_2 c + I_3$. Then, by using (9.89)and (8.31), and reversing signs, give

$$f^{\mp} = c^3 + I_2 c \mp I_3 \begin{cases} \leq 0 & \text{for } c > 0 \\ \geq 0 & \text{for } c < 0 \end{cases} \tag{8.35}$$

That is, the functions f^+ and f^- must be both negative for positive c and both positive for negative c. Let $f^0 = c^3 + I_2 c$. Then, f^+ and f^- are obtained by adding and subtracting the same constant, which amounts to vertical shifts of their graphs. Also, note that f^- is simply the characteristic polynomial. Therefore, it always has three real roots. It is also not difficult to show that if is λ_i a root of f^-, then $-\lambda_i$ is a root of f^+. Therefore, f^+ has three real roots, too. This is illustrated in Figure 8.3 for $I_3 > 0$. The solution set is denoted by thick line segments. $I_3 < 0$ case is similar.

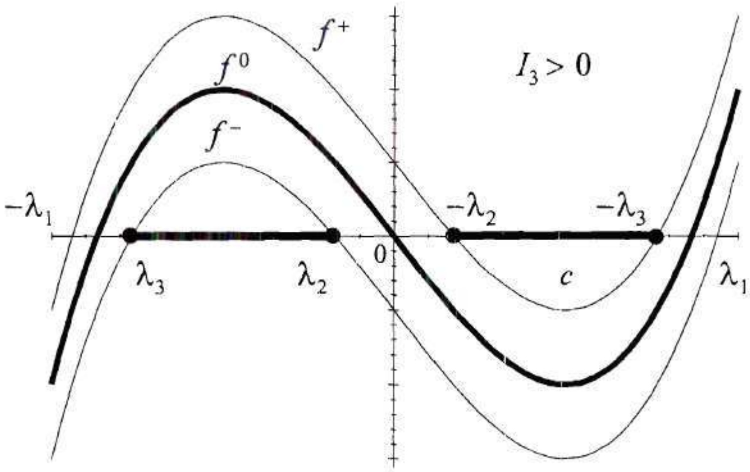


Figure 8.3: Solution regions for the parameter c. General case.

The solution regions for c are

$$c \in \left\{ \begin{array}{ll} [\lambda_3, \lambda_2] \cup [-\lambda_2, -\lambda_3] & \text{for } I_3 > 0 \\ [-\lambda_1, -\lambda_2] \cup [\lambda_2, \lambda_1] & \text{for } I_3 < 0 \end{array} \right\} \tag{8.36}$$

which has two disconnected regions in any case. Together with the solutions of a and b, the total real solution space is composed of four distinct components which are described by one free parameter c, as claimed in the previous section.

Any solution (a, b, c) means at least one 3×3 rotation matrix U such that $U^T AU = \Phi$. The columns of U give a set of three orthonormal isotropic vectors of A.

Degenerate cases exist for double eigenvalues which reduces the solutions to a finite number of isolated points, namely

$$\left[\begin{array}{l} \text{if } \lambda_3 = \lambda_2 \text{ and } I_3 > 0 \text{ or} \\ \text{if } \lambda_1 = \lambda_2 \text{ and } I_3 < 0 \end{array} \right] \quad \text{then} \qquad c \in \{\pm\lambda_2\} \tag{8.37}$$

In degenerate cases, a single point (a, b, c) corresponds infinitely many U. One can show that there exist four isolated solutions $(a, b, c)_{i=1,\ldots,4}$, such that $(a, b, c)_1 = -(a, b, c)_3$ and $(a, b, c)_2 = -(a, b, c)_4$.

In each solution, two of a, b, c are identical. This is the effect of the double eigenvalue in λ_i. Finally, if U is a solution corresponding to any of $(a, b, c)_{i=1,\dots,4}$, then all RU are solutions, where R is a rotation matrix about the axis corresponding to the single eigenvalue. This defines a 1-parameter family for U. As a result, although in the degenerate case the solutions (a, b, c) are isolated points, the space of corresponding U is still a 1-parameter family.

To see how U can be obtained from knowing A and Φ, let R_A be the rotation matrix formed by the classical eigenvectors of A and R_Φ be that for Φ. Since both A and Φ have the same eigenvalues then

$$R_A^T A R_A = R_\Phi^T \Phi R_\Phi = \mathrm{diag}[\ \lambda_1 \quad \lambda_2 \quad \lambda_3\ ] \tag{8.38}$$

Note that the eigenvalues must be ordered in the same way when determining R_A and R_Φ. Then,

$$\left(R_\Phi R_A^T\right) A \left(R_A R_\Phi^T\right) = \Phi \tag{8.39}$$

from which one concludes

$$U = R_A R_\Phi^T \tag{8.40}$$

Note again that if there is a double eigenvalue in λ_i, both R_A and R_Φ are non-unique and commonly defined by one parameter.

8.1.5 Examples

Continuum Mechanics The three dimensional problem has a physical explanation in the context of continuum mechanics. The stress tensor is taken as an example. However, the results trivially extend to strain tensor.

It is well known in continuum mechanics that the stress tensor at any point of a material is given by a 3×3 symmetric matrix. Any stress tensor σ can be decomposed into its hydrostatic, $\frac{1}{3}\mathrm{trace}(\sigma)\mathbf{I}$, and deviatoric, σ', components. This is identical to what is essentially done in Corollary 127. The hydrostatic component is a pure normal stress state of equal magnitude in every direction.

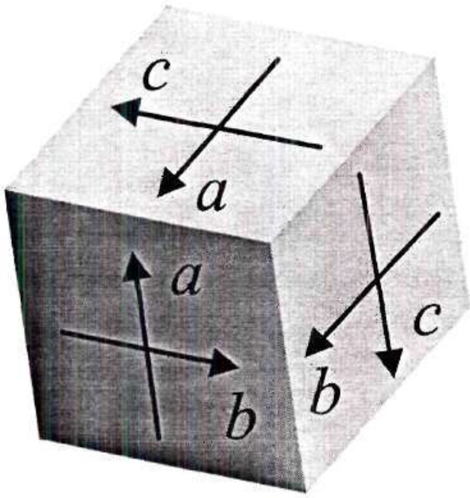


Figure 8.4: A stress element with pure shear stresses. The normals of the surfaces correspond to an orthonormal set of special eigenvectors. Shear values are given by a, b, c which make up the matrix Φ.

The deviatoric stress $\boldsymbol{\sigma}'$ is considered to correspond to pure shear loading. This is sensible since, by the fact that trace($\boldsymbol{\sigma}'$) $= 0$, there exists a coordinate system in which $\boldsymbol{\sigma}'$ has zero diagonals, Corollary 127. This means that in such coordinates a stress element has no normal stress component. Figure 8.4 illustrates such a state, which is a pure shear state. By the results obtained so far, these coordinates correspond to the isotropic vectors of $\boldsymbol{\sigma}'$ and, in general, there exists infinitely many of them described by one parameter. This is illustrated in Figure 8.5 which shows the Mohr's circle representation of three dimensional deviatoric stress. All possible stress states are in the shaded region. The thick lines correspond to distinct pure shear states which is in agreement with what is claimed here: a one parameter family.

Isotropic Vectors in Screw Systems There is no natural notion of length in the screw space that is geometrically or physically meaningful. Instead, the pitch is used as a scalar measure. Any scalar

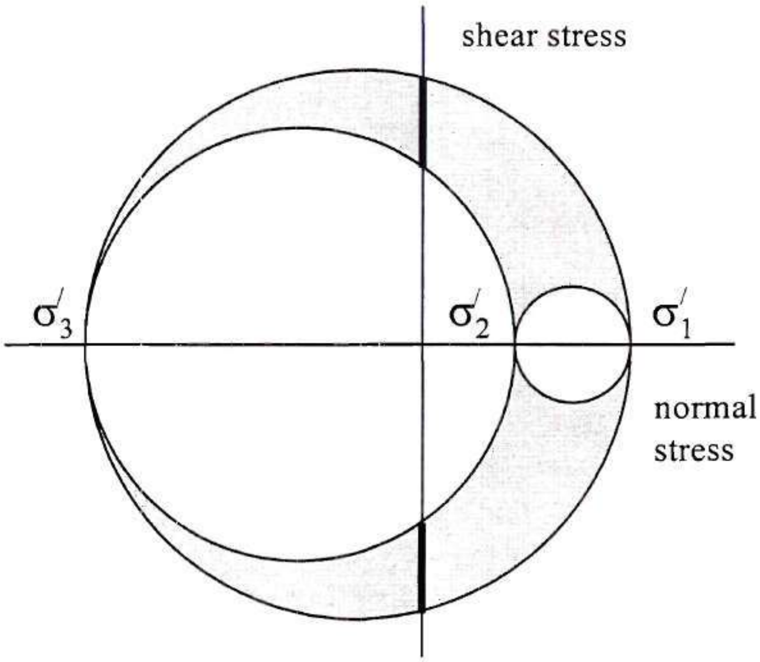


Figure 8.5: Mohr's circle for 3-dimensional deviatoric stress.

measure on a vector space can be defined by a *metric.* If $\vec{v}$ is any vector then a metric can be represented by a symmetric matrix G such that $\vec{v}^T G \vec{v}$ is a scalar measure. If G is definite then any non-zero vector has a non-zero scalar measure. If G is not definite, it is possible to have vectors whose scalar measure is zero. As mentioned earlier, such vectors are the *isotropic* vectors of the metric tensor. In screw space, the matrix

$$\hat{\Delta} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{8.41}$$

where all matrices are 3×3, defines an indefinite metric. Consider the scalar measure $\frac{1}{2}\hat{S}^T \hat{\Delta} \hat{S}$. If $\hat{S} = \begin{bmatrix} \vec{\mathbf{a}}^T & \vec{\mathbf{b}}^T \end{bmatrix}^T$ then $\frac{1}{2}\hat{S}^T \hat{\Delta} \hat{S} = \frac{1}{2}\vec{\mathbf{a}}^T \vec{\mathbf{b}}$, which is the pitch. Consequently, the zero and infinite pitch screws are the isotropic vectors of the screw space under the metric $\hat{\Delta}$.

A set of $n \leq 6$ independent screws spans an n-system of screws. An n-system may or may not contain isotropic screws (zero or infinite pitch screws). Now, if $\hat{V} = [\hat{S}_1, ..., \hat{S}_n]$ is a matrix of basis screws, then any other screw in the n-system can be given as the linear combination $\hat{S} = \hat{V}\bar{\alpha}$, where $\bar{\alpha}$ is an $n \times 1$ matrix of coefficients. Then, any isotropic screw $\hat{S}$ must satisfy

$$\hat{S}^T \hat{\Delta} \hat{S} = \bar{\alpha}^T (\hat{V}^T \hat{\Delta} \hat{V}) \bar{\alpha} = 0 \tag{8.42}$$

The matrix $A = \hat{V}^T \hat{\Delta} \hat{V}$ is a symmetric matrix of order n. Every solution $\bar{\alpha}$ corresponds to an isotropic screw in the n-system. But, from (8.42), $\bar{\alpha}$ must be the isotropic vectors of A by definition. So, by the existence theorem, such an n-system of screws contains zero or infinite pitch screws if and only if A is indefinite (non-trivial) or singular (trivial). If $A = 0$ then all the screws in the system are isotropic. For example, for a basis composed of only zero pitch screws through a common point, or a basis composed of only infinite pitch screws, the matrix A is identically zero indicating that any screw in the system is an isotropic vector.

8.1.6 Summary on Isotropic Vectors

The isotropic vector problem has complementary properties compared to the classical eigenvalue problems. Geometrically, the former is about orthogonality whereas the latter is about parallelism. From a functional point of view, the isotropic vectors are the zeros and the classical eigenvectors are the stationary points of the same quadratic form over a vector space.

Existence of isotropic vectors basically requires indefiniteness. On the other hand, for the existence of an orthonormal basis formed by isotropic vectors it is *necessary and sufficient* that the matrix has zero trace. For a matrix of order n, the space of all orthonormal bases formed by isotropic vectors is $\frac{1}{2}(n-1)(n-2)$ dimensional in general.

The recursive algorithm presented for the construction of particular orthonormal bases formed by isotropic vectors is well suited for computer applications.

8.2 Synthesis of Stiffnesses by Springs

Two types of stiffness problems exist; *analysis* and *synthesis*. In the analysis problem, the elements comprising the system are known and it is desired to determine the stiffness. Examples are spring systems, spatial structures, robotic manipulators or hands, Stewart platform-type parallel manipulators, remote center of compliance (RCC) devices, etc. In the synthesis problem, stiffness is known and it is desired to find suitable elements and their connections to yield the stiffness. A particular case is the synthesis of stiffnesses using springs only, which has previously remained unsolved.

The synthesis problem has applications in the design and control of parallel manipulators, robotic fingers grasping an object, RCC devices, etc. [14], [22], [27], [29], [32], [36]. Loncaric discussed the synthesis problem and showed that a necessary condition is that the stiffness matrix must have zero trace off-diagonals. Recently, Huang and Schimmels [27] studied the synthesis problem and verified Loncaric's results by using screw theory and by also allowing torsional springs.

This study provides a systematic approach to the synthesis problem using screw (spatial vectors) theory which results in complete solutions of the synthesis of semi-stable and stable stiffnesses using springs in parallel.

First, necessary conditions are presented and minimum configurations are identified. Then, the general equations for the synthesis problem are constructed and shown to be equivalent the existence of a special class of orthogonal matrices, namely orthogonal sets of isotropic vectors. The necessary condition for the existence of these orthogonal matrices is shown to be equivalent to the zero trace off-diagonals condition on stiffness matrix. Hence, the zero trace off-diagonals condition on stiffness is necessary and sufficient for the synthesis of stiffness by springs. In particular any such stiffness is shown to be realizable by exactly r springs, where r is the rank of the stiffness.

Then, the theory is extended to the synthesis by more than r springs. It is shown that any stiffness of rank three or higher can be synthesized by an arbitrarily large number of springs, provided that the zero trace condition is met. Interestingly, the rank one and two stiffnesses are in general synthesized by a unique combination of one and two springs, respectively. Only for a certain class of rank two stiffnesses do there exist syntheses by more than two springs.

Finally, two special minimum syntheses are identified and shown to follow naturally from the first and second decomposition theorems for stiffness and compliance. Then, corresponding algorithms are presented along with numerical and theoretical examples.

As before, the stiffness to be synthesized is assumed to be given with respect to a point O and have the following submatrices.

$$\hat{K}_O = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \tag{8.43}$$

8.2.1 Spring Systems

In Chapter 7, the stiffness matrices of line spring and torsional spring systems are determined. Here it is assumed that all springs in the parallel connection are unloaded.

For a system of unloaded line springs, the stiffness becomes,

$$\hat{K}_O = \sum_i k_i \hat{S}_{Oi}\hat{S}_{Oi}^T = \sum_i k_i \begin{bmatrix} \vec{\mathbf{s}}_i\vec{\mathbf{s}}_i^T & -\vec{\mathbf{s}}_i\vec{\mathbf{s}}_i^T\vec{\mathbf{a}}_i\times \\ \vec{\mathbf{a}}_i\times\vec{\mathbf{s}}_i\vec{\mathbf{s}}_i^T & -\vec{\mathbf{a}}_i\times\vec{\mathbf{s}}_i\vec{\mathbf{s}}_i^T\vec{\mathbf{a}}_i\times \end{bmatrix} \tag{8.44}$$

Here i refers to the i^{th} spring, k_i are the scalar spring stiffnesses and $\hat{S}_{Oi}$ are the screw representations of spring axes in ray-coordinates given by

$$\hat{S}_{Oi} = \begin{bmatrix} \vec{\mathbf{s}}_i \\ \overrightarrow{OA_i}\times\vec{\mathbf{s}}_i \end{bmatrix}, \quad \vec{\mathbf{s}}_i^T\vec{\mathbf{s}}_i = 1 \tag{8.45}$$

Similarly, for torsional spring systems,

$$\hat{K}_O = \sum_i k_i \hat{S}_{Oi}\hat{S}_{Oi}^T = \sum_i k_i \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \vec{\mathbf{s}}_i\vec{\mathbf{s}}_i^T \end{bmatrix} \tag{8.46}$$

where k_i is the torsional spring constant. The screw axes $\hat{S}_{Oi}$ are given in ray-coordinates as

$$\hat{S}_{Oi} = \begin{bmatrix} \vec{0} \\ \vec{s}_i \end{bmatrix}, \quad \vec{s}_i^T \vec{s}_i = 1 \tag{8.47}$$

Again, $\vec{s}_i$ are unit direction vectors along the spring axes.

Since (8.44) and (8.46) are of the same form and the stiffnesses in parallel are additive, one concludes that for a parallel connection composed of both line and torsional springs, all unloaded, the general formula for the total stiffness is

$$\hat{K}_O = \hat{S}_O \hat{k} \hat{S}_O^T \tag{8.48}$$

where the columns of $\hat{S}_O$ are $\hat{S}_{Oi}$ whose form is given by (8.45) and (8.47). The units of k_i, which are the diagonal entries of the diagonal matrix $\hat{k}$, are [F/L] for line springs and [F*L] for torsional springs, where [F] and [L] are some force and length units, respectively.

From here on the subscript O is omitted, since, unless otherwise is noted, the discussion is about a certain point O assumed to be specified.

8.2.2 Necessary Conditions

From (8.48), the spring axes in $\hat{S}$ form a basis for the range space of $\hat{K}$, $\mathcal{R}(\hat{K})$. This is the space of wrenches that do a finite non-zero work. The null space of $\hat{K}$, $\mathcal{N}(\hat{K})$, consists of twists that do zero work. $\mathcal{R}(\hat{K})$ and $\mathcal{N}(\hat{K})$ are reciprocal screw systems. For example, if $\hat{W} \in \mathcal{R}(\hat{K})$ and $\hat{T} \in \mathcal{N}(\hat{K})$, then $\hat{W}^T\hat{T} = 0$. Note that $\mathcal{R}(\hat{K}) = \mathcal{R}(\hat{K}^T)$ and $\mathcal{N}(\hat{K}) = \mathcal{N}(\hat{K}^T)$ due to symmetry.

Let $r = \text{rank}(\hat{K})$. Then, $\mathcal{R}(\hat{K})$ is r-dimensional and $\mathcal{N}(\hat{K})$ is $6 - r$ dimensional. Since $\{\hat{S}_i\}$ is a basis for $\mathcal{R}(\hat{K})$ then the minimum number of independent screws is equal to r. Thus, if n_l is the number of independent line springs and n_τ is the number of independent torsional springs, the following is a necessary condition

$$n_l + n_\tau \geq r \tag{8.49}$$

Theorem 129 *For any given stiffness $\hat{K}$ such that $r = \mathrm{rank}(\hat{K})$, any synthesis contains r or more springs.*

A general stiffness matrix of an elastic system in unloaded configuration is a symmetric 6×6 matrix and is therefore described by 21 independent parameters. Equations (8.44) and (8.46) show, however, that the trace of the off-diagonal 3×3 submatrix of the stiffness is always zero for parallel connections of springs. Therefore,

Theorem 130 (Loncaric) *The stiffness matrix of a connection comprised of springs in parallel has zero trace off-diagonal submatrices, $\mathrm{tr}(\mathbf{B}) = 0$. Accordingly, the zero trace off-diagonals is a necessary condition for a stiffness matrix to be realizable by springs.*

This leaves 20 parameters to be satisfied by synthesis using springs. A general screw $\hat{S} = \left[\vec{\mathbf{a}}^T, \vec{\mathbf{b}}^T\right]^T$ has 6 independent parameters. If $\hat{S}$ is in ray-coordinates then a unit screw is defined by the condition $\vec{\mathbf{a}}^T\vec{\mathbf{a}} = 1$ if $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$, and by $\vec{\mathbf{b}}^T\vec{\mathbf{b}} = 1$ if $\vec{\mathbf{a}} = \vec{\mathbf{0}}$. The pitch of the screw is $h = \frac{\vec{\mathbf{a}}^T\vec{\mathbf{b}}}{\vec{\mathbf{a}}^T\vec{\mathbf{a}}}$. If $\vec{\mathbf{a}} \neq \vec{\mathbf{0}}$ the screw has a **finite pitch**, if $\vec{\mathbf{a}} = \vec{\mathbf{0}}$ it has an **infinite pitch**. If $\vec{\mathbf{a}}^T\vec{\mathbf{b}} = 0$ for a finite pitch screw then it is a **zero pitch** screw.

It is seen from (8.45) that the screw axes for *line springs are zero pitch unit screws* and from (8.47) that for *torsional springs are infinite pitch unit screws.* A zero pitch unit screw requires 4 independent parameters and an infinite pitch unit screw requires 2 independent parameters.

A general line spring requires 5 independent parameters: 4 for $\hat{S}$, and 1 for k. A torsional spring requires 3 independent parameters: 2 for $\hat{S}$, and 1 for k. As a result, for a general stiffness matrix to be realizable

$$5n_l + 3n_\tau \geq 20 \tag{8.50}$$

Further, from (8.44) and (8.46), only the stiffnesses of line springs can contribute to the submatrices $\mathbf{A}$ and $\mathbf{B}$, which are described by $6 + (9-1) = 14$ parameters. Therefore, to synthesize a general stiffness one must have $5n_l \geq 14$, that is, there must be at least 3 line springs,

$$n_l \geq 3 \tag{8.51}$$

Equations (8.50) and (8.51) refer to general stiffnesses. If a given stiffness matrix has a special form it may be sufficient to use fewer springs. For example, consider the case in which the given stiffness is that of a single spring.

In general, the synthesis of arbitrary stiffnesses requires $r = 6$. So,

$$n_l + n_\tau \geq 6 \tag{8.52}$$

The general necessary conditions (8.50), (8.51), and (8.52) describe the allowable combinations of n_l and n_τ for the synthesis of arbitrary stiffnesses. Figure 8.6 illustrates the allowable region of n_l and n_τ as the shaded area. It also shows that (8.50) becomes redundant. The number of total springs is minimum on the line $n_l + n_\tau = 6$, shown in the figure. So, for minima $(n_l, n_\tau) \in \{(3,3); (4,2); (5,1); 6,0)\}$. The two border cases are shown in the figure by circles. They contain the minimum/maximum combinations of line/torsional springs. These two minima are especially important to this study. Later, they are shown to naturally follow from the free-vector and the line-vector decompositions for spatial stiffness. The following definition is proposed.

Definition 131 *For the synthesis of general stiffnesses, the two minima corresponding to*

$$\text{1}^{st} \textit{ minimum}: \qquad n_l = 3 \qquad n_\tau = 3 \tag{8.53}$$

$$\text{2}^{nd} \textit{ minimum}: \qquad n_l = 6 \qquad n_\tau = 0 \tag{8.54}$$

are respectively called the ***free-vector*** *and* ***line-vector syntheses*** *of general stiffnesses.*

Note that, among all minima, the free-vector synthesis contains the maximum number of torsional springs and the line-vector synthesis contains the maximum number of line springs. Whereas

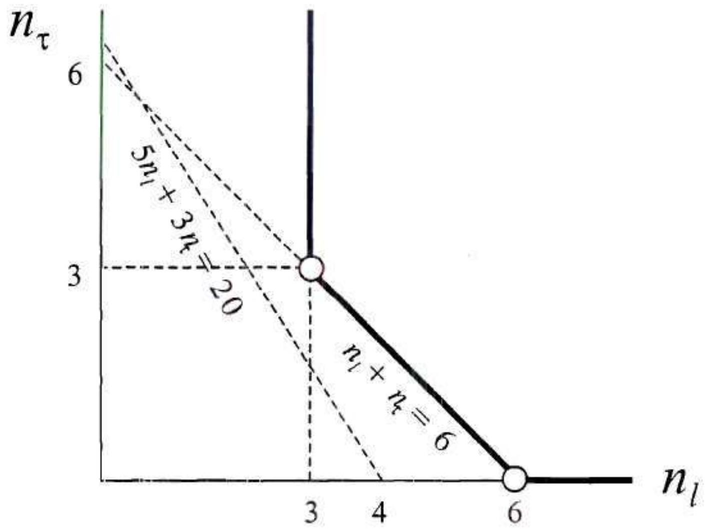


Figure 8.6: Allowable combinations of line (n_l) and torsional springs (n_τ) in the general case.

the free-vector synthesis provides a realization with $5 * 3 + 3 * 3 - 20 = 4$ excess parameters, the line-vector synthesis yields $5 * 6 - 20 = 10$ excess parameters. In later sections, these excess parameters are explained in both mathematical and physical terms.

The strict minima are really given by $n_l + n_\tau \geq r$, Theorem 129. This is important in using general algorithms for synthesis and is demonstrated later.

8.2.3 Solution of the Synthesis Problem

The following lemma develops some linear algebra, which is needed subsequently.

Lemma 132 *Let $\hat{K}$ be an $n \times n$ symmetric, positive or semi-positive definite, rank r matrix.*

1. *The space of $n \times n$ symmetric, positive or semi-positive definite, rank r matrices, such as $\hat{K}$, is described by $nr - r(r-1)/2$ parameters.*

2. There exists a real $n \times r$ matrix $\hat{G}$ such that

$$\hat{K} = \hat{G}\hat{G}^T \tag{8.55}$$

3. The space of all $\hat{G}$ is described by $r(r-1)/2$ parameters.

4. If $\hat{P}$ is any particular $n \times r$ matrix such that $\hat{K} = \hat{P}\hat{P}^T$, then any solution $\hat{G}$ is given by $\hat{G} = \hat{P}\hat{U}$, where $\hat{U}$ is an $r \times r$ real orthogonal matrix, i.e. $\hat{U}\hat{U}^T = \hat{I}_{r\times r}$.

Proof. A symmetric positive or semi-positive definite matrix of rank r is congruent to a diagonal matrix with positive or zero entries. That is, $\hat{K} = \hat{Q}\hat{d}\hat{Q}^T$, where $\hat{d}$ is diagonal and contains exactly $n-r$ zeros on the diagonal. Writing $\hat{K} = \sum_{i=1}^{n} d_i\hat{Q}_i\hat{Q}_i^T$, where $\hat{Q}_i$ are the columns of $\hat{Q}$, indicates that the terms $d_i = 0$ can be dropped, giving $\hat{K} = \sum_{i=1}^{r} d_i\hat{Q}_i\hat{Q}_i^T$, $(d_i > 0)$. This is equivalent to $\hat{K} = \hat{Q}^*\hat{d}^*\hat{Q}^{*T}$, where $\hat{Q}^*$ is a $n \times r$ matrix and $\hat{d}^*$ is an $r \times r$ diagonal matrix with positive entries.

1. In particular, $\hat{Q}_i^*$ can be taken as the eigenvectors of $\hat{K}$ corresponding to r positive eigenvalues d_i^*. This sufficiently characterizes the space of all rank r symmetric matrices. Since, for a symmetric matrix, $\{\hat{Q}_i^*\}$ is an orthonormal set of vectors, it is described by $nr - r(r-1)/2 - r$ parameters. Here, nr is introduced by taking r arbitrary $n \times 1$ vectors, $r(r-1)/2$ is the number of orthogonality equations, and r is the number of equations for normalizing the eigenvectors. However, since d_i^* introduces r extra parameters, the space of all rank r symmetric matrices is described by $nr - r(r-1)/2$ parameters.

2. By taking $\hat{G} = \hat{Q}^*\sqrt{\hat{d}^*}$ one finds a $n \times r$ matrix that satisfies (8.55).

3. The set of all $n \times r$ matrices is described by nr parameters. As a result, for a given $\hat{K}$, the number of free parameters in (8.55) is $r(r-1)/2$, which is the dimension of the space of all solutions $\hat{G}$ to (8.55).

4. If $\hat{U}$ is an $r \times r$ orthogonal matrix then, by definition, $\hat{U}\hat{U}^T = \hat{I}_{r\times r}$. So, if $\hat{K} = \hat{P}\hat{P}^T$ then $(\hat{P}\hat{U})(\hat{P}\hat{U})^T = \hat{P}\hat{U}\hat{U}^T\hat{P}^T = \hat{P}\hat{P}^T = \hat{K}$, hence $\hat{G} = \hat{P}\hat{U}$ is a solution. Since the set of all orthogonal matrices of order r is described by $r(r-1)/2$ parameters, the set $\{\hat{P}\hat{U},\ \hat{U}\hat{U} = \hat{I}\}$ sufficiently describes the set of solutions to (8.55). Therefore, every solution $\hat{G}$ can be given as $\hat{G} = \hat{P}\hat{U}$.

■

Consider the synthesis problem $\hat{K} = \hat{S}\hat{k}\hat{S}^T$, where $\hat{K}$ is a rank r, symmetric, positive or semi-positive definite stiffness. It is desired to determine the spring axes $\hat{S} = \left[\hat{S}_1, \hat{S}_2, ...\right]$ and corresponding spring rates $\hat{k} = \text{diag}([k_1, k_2, ...])$. In this section only the strict minimum syntheses are presented. That is, the number of springs is taken as r.

First, the synthesis problem is transformed to

$$\hat{K} = \hat{V}\hat{V}^T \tag{8.56}$$

where $\hat{V}$ is a $6 \times r$ matrix whose columns $\hat{V}_i$ are

$$\hat{V}_i = \sqrt{k_i}\hat{S}_i \qquad i = 1, ..., r \tag{8.57}$$

for a synthesis by r springs. Essentially, k_i are absorbed into $\hat{S}_i$ to give $\hat{V}_i$. But, (8.56) means that the matrix $\hat{V}$ must be solution to (8.55), Lemma 132. Then, by Lemma 132 again, there exists an orthogonal matrix $\hat{U}_V$ of order r such that

$$\hat{V} = \hat{P}\hat{U}_V \tag{8.58}$$

where $\hat{P}$ is any particular solution to (8.55).

As illustrated in Figure 8.7, the distinction of $\hat{V}$ from a general $\hat{G}$ that satisfies (8.55) is that its columns must be zero or infinite pitch screws. That is

$$\hat{V}_i^T \hat{\Delta}\hat{V}_i = 0 \qquad \text{(0 or } \infty \text{ pitch constraint)} \tag{8.59}$$

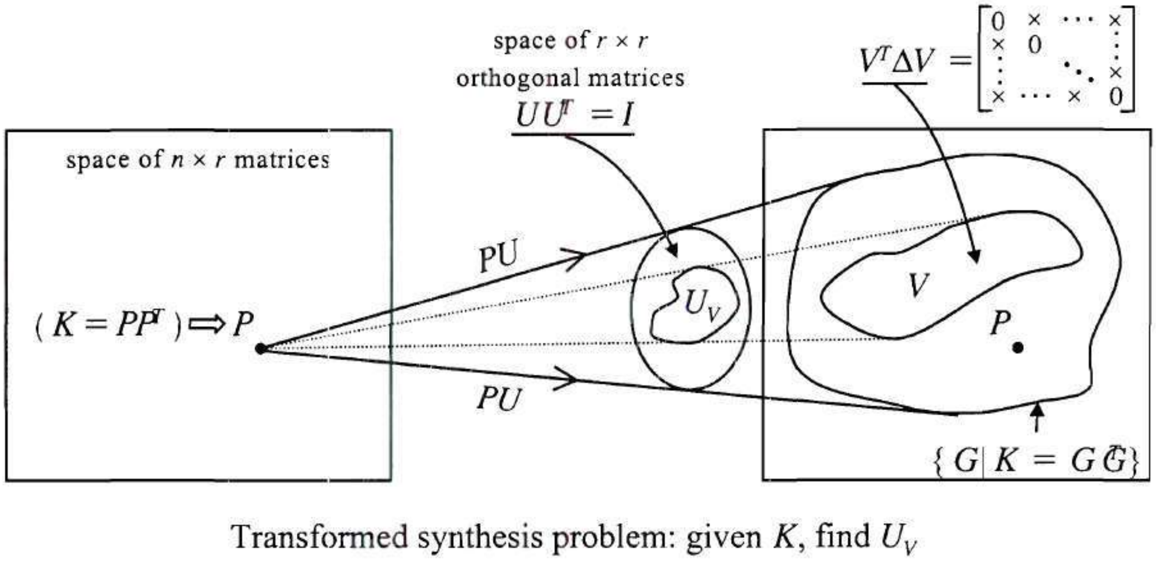


Figure 8.7: Graphical illustration of the synthesis by springs problem.

where $\hat{V}_i$ is a column of $\hat{V}$. The equation (8.59) can be compactly given for all springs as

$$\hat{V}^T \hat{\Delta} \hat{V} = \hat{\Phi} = \begin{bmatrix} 0 & \bullet & \bullet \\ \bullet & \ddots & \bullet \\ \bullet & \bullet & 0 \end{bmatrix}_{r \times r} \tag{8.60}$$

Now, using (8.58) in (8.60) yields

$$\hat{U}_V^T \left(\hat{P}^T \hat{\Delta} \hat{P} \right) \hat{U}_V = \hat{\Phi}_{r \times r} \tag{8.61}$$

where $\hat{P}$ is any particular $6 \times r$ matrix satisfying (8.55).

Thus, the synthesis problem is reduced to finding an orthogonal matrix $\hat{U}_V$ of order r that satisfies (8.61), Figure 8.7. These are found by solving the isotropic vector problem for $\hat{P}^T \hat{\Delta} \hat{P}$.

The equation (8.61) yields 6 scalar equations in terms of the components of $\hat{U}_V$, since only the diagonals of the matrix equation are required to be satisfied, which are the pitches of line and torsional springs. However, the trace is invariant under unitary transformations,

$$\text{trace}\left(\hat{P}^T \hat{\Delta} \hat{P}\right) = \text{trace}\left(\hat{\Phi}\right) = 0 \tag{8.62}$$

So, there cannot be a solution unless $\text{trace}\left(\hat{P}^T \hat{\Delta} \hat{P}\right) = 0$, as the following theorem guarantees.

Theorem 133 $\hat{K} = \hat{P}\hat{P}^T$ *has zero trace off-diagonals if and only if* $\text{trace}\left(\hat{P}^T \hat{\Delta} \hat{P}\right) = 0$.

Proof. From linear algebra, for any two matrices A and B, $\text{trace}(AB) = \text{trace}(BA)$. So,

$$\text{trace}\left(\hat{P}^T \hat{\Delta} \hat{P}\right) = \text{trace}\left(\hat{\Delta} \hat{P} \hat{P}^T\right) = \text{trace}\left(\hat{\Delta} \hat{K}\right) \tag{8.63}$$

Using (9.49) and (??) this gives

$$\text{trace}\left(\hat{P}^T \hat{\Delta} \hat{P}\right) = \text{trace}\left(\mathbf{B} + \mathbf{B}^T\right) = 2\ \text{trace}\left(\mathbf{B}\right) \tag{8.64}$$

which proves the theorem.

■

If the zero trace condition is satisfied then only $r-1$ of the r scalar equations, the diagonals of (8.60), are independent. Hence, one expects $r(r-1)/2 - (r-1) = (r-1)(r-2)/2$ free parameters after (8.61) is satisfied. The following theorem shows that the condition (8.62) is satisfied by the zero-trace off-diagonal stiffnesses.

As shown in the previous section, (8.61) means that the columns of $\hat{U}_V$ are the isotropic vectors of $\hat{P}^T \hat{\Delta} \hat{P}$ which form an orthonormal set. Any solution $\hat{U}_V$ yields a synthesis by r springs. Due to Theorem 126 of the previous section, there exist solutions $\hat{U}_V$ if and only if $\text{trace}\left(\hat{P}^T \hat{\Delta} \hat{P}\right) = 0$. Consequently, the following main result follows.

Theorem 134 *Any positive or semi-positive definite stiffness* $\hat{K}$ *of rank* r *can be synthesized by exactly* r *springs if and only if the off-diagonal matrices of* $\hat{K}$ *have zero trace. The solution space is in general* $(r-1)(r-2)/2$ *dimensional.*

For $r = 1, 2$ the dimension of the solution space is zero, indicating finitely many isolated solutions. The solution space is 1-dimensional for $r = 3$, 3-dimensional for $r = 4$, 6-dimensional for $r = 5$ and 10-dimensional for $r = 6$. These results are summarized in the following table in which the springs are assumed to be line springs. The torsional springs are considered as a subspace of line springs obtained by the vanishing of some parameters.

r	Parameters in $\hat{K}$ $[nr - r(r-1)/2]$	Spring parameters $[5r]$	Solution space $[(r-1)(r-2)/2)]$
1	$6 - 1 = 5$	5	$5 - 5 = 0$
2	$11 - 1 = 10$	10	$10 - 10 = 0$
3	$15 - 1 = 14$	15	$15 - 14 = 1$
4	$18 - 1 = 17$	20	$20 - 17 = 3$
5	$20 - 1 = 19$	25	$25 - 19 = 6$
6	$21 - 1 = 20$	30	$30 - 20 = 10$

This is in agreement with previous determinations made by counting the spring parameters, and therefore, explains the mathematical reasons behind these free parameters. If the inclusion of torsional springs is enforced, as in the free-vector synthesis, then $\hat{U}$ has to satisfy extra conditions which reduces the number of free parameters. So, for example, the 10 free parameters case is the most general for full rank stiffnesses.

Once a solution $\hat{V}$ is obtained then the spring axes and rates are determined from

$$k_i = \begin{cases} \hat{V}_i^T \hat{\Gamma} \hat{V}_i & \text{for line springs} \\ \hat{V}_i^T \hat{\mathrm{L}} \hat{V}_i & \text{for torsional springs} \end{cases} \tag{8.65}$$

$$\hat{S}_i = \frac{1}{\sqrt{k_i}} \hat{V}_i \tag{8.66}$$

or, after setting $\hat{k} = \operatorname{diag}(k_i)$

$$\hat{S} = \hat{V} \hat{k}^{-1} \tag{8.67}$$

The matrices $\hat{\Gamma}$ and $\hat{\mathrm{L}}$ are

$$\hat{\Gamma} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \qquad \hat{\mathrm{L}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \tag{8.68}$$

as presented in Chapters 3 and 4.

8.2.4 General Algorithm for Synthesis

The matrix $\hat{P}^T\hat{\Delta}\hat{P}$ is an $r\times r$ symmetric indefinite, zero trace, rank r matrix. The columns of the orthogonal matrix $\hat{U}_V$ are the isotropic vectors of $\hat{P}^T\hat{\Delta}\hat{P}$. In Section 8.1.4, a recursive method was presented for the construction of such matrices U_V. The following algorithm achieves the synthesis of a positive or semi-positive definite stiffness $\hat{K}$ using $r = \text{rank}(\hat{K})$ springs which is a minimum.

1. Read $\hat{K}$.
2. If $\hat{K}$ is not symmetric. No solution. $\hat{K}$ must be symmetric.
3. If trace($\hat{\Delta}\hat{K}$) = 0. No solution. $\hat{K}$ must have zero trace off-diagonals.
4. $\hat{R}$ =eigenvectors($\hat{K}$), $\hat{\lambda}$ =eigenvalues($\hat{K}$).
5. If $\min(\hat{\lambda}) < 0$. No solution. $\hat{K}$ must be positive or semi-positive definite.
6. $r = \text{rank}(\hat{K})$
7. Sort($\hat{\lambda}, \hat{R}$) such that $\max = \lambda_1, \min = \lambda_6$
8. Form $\hat{Q}^* = [\hat{R}_i], \hat{d}^* = \text{diag}(\lambda_i)$, $i = 1, ..., r$. ($\hat{Q}^*$ is a $6 \times r$ matrix.)
9. Let $\hat{P} = \hat{Q}^*\sqrt{\hat{d}^*}$
10. Find a $\hat{U}_V$ such that $\hat{U}_V^T\hat{A}\hat{U}_V = \hat{\Phi}$, $\hat{U}_V\hat{U}_V^T = \hat{I}$. (e.g., use the general algorithm presented on page 225.)
11. Let $\hat{V} = \hat{P}\hat{U}$

12. Use (8.65) and (8.66) to determine r springs $\hat{S}_i$ and corresponding rates k_i.

This algorithm has been implemented on MATLAB, a computer package for matrix manipulation and linear algebra, and it worked quite satisfactorily. Test runs were performed on randomly generated positive and semi-positive definite stiffnesses with zero trace off-diagonals. In all cases the algorithm successfully synthesized the given stiffnesses by r springs, where r is the rank. The errors in the syntheses were measured by $\max\left|\left[\hat{K} - \hat{S}\hat{k}\hat{S}^T\right]_{ij}\right|$, and were observed to be well within the machine precision. Examples are provided in later sections.

8.2.5 Excess Springs

The procedure outlined in the previous section for explicit synthesis is suitable for positive or semi-positive definite stiffnesses using r springs, where r is the rank of the stiffness. However, this section shows that if a stiffness is realizable by r springs than it is realizable by $n \geq r \geq 3$ springs.

If a stiffness matrix is synthesized using any number of springs, one can trivially increase the number of springs. For example, one may add an arbitrary number of zero stiffness springs with arbitrary axes. Another way is to consider a single spring with stiffness k_i as composed of m springs, on the same axis, with stiffnesses summing up to k_i. These are not considered as essential changes in the number of springs.

Here, it is shown that it is possible to increase the number of springs of the synthesis in a non-trivial way for $r \geq 3$ stiffnesses, and a certain class of $r = 2$ stiffnesses. Before this, the following theorem is needed.

Theorem 135 *Let K be any $n \times n$, positive or semi-positive definite, symmetric matrix. Let $\mathcal{R}(K)$ and $\mathcal{N}(K)$ be the column and null spaces. Let, for any vector $S \in \mathcal{R}(K)$, $K^* = K - kSS^T$, where k is a scalar. Then, there exists a sufficiently small positive value of k such that*

1. *K^* is positive or semi-positive definite,*

2. $\mathrm{rank}(K^*) = \mathrm{rank}(K)$.

Proof. Consider the quadratic forms $T^T K T$ and $T^T K^* T$, where T is any vector. A matrix is semi-positive definite if its quadratic form is non-negative for all T. Let $r = \mathrm{rank}(K)$.

1. The matrix K^* is investigated for $T \notin \mathcal{N}(K)$ and $T \in \mathcal{N}(K)$ separately.

   (a) Let $T \notin \mathcal{N}(K)$. If λ is a minimum eigenvalue of K, then $T^T K T \geq \lambda$ for all $T \notin \mathcal{N}(K)$ such that $T^T T = 1$. Let $S = \sigma Z$, where $Z^T Z = 1$. Then,

   $$T^T K^* T = T^T (K - kSS^T) T \geq \lambda - k\sigma^2 \left(T^T Z\right)^2 \geq \lambda - k\sigma^2 \tag{8.69}$$

   since $\left(T^T Z\right)^2 \leq 1$ for any two unit vectors. For any given positive numbers λ and σ, one can always find another positive number k such that $\lambda - ka^2 > 0$, namely all numbers $0 < k < \frac{\lambda}{\sigma^2}$. Therefore, there exists a $k > 0$ such that $T^T K^* T > 0$ for all $T \notin \mathcal{N}(K)$. This means that $\mathcal{R}(K^*) \supseteq \mathcal{R}(K)$, hence, $\mathrm{rank}(K^*) \geq r$.

   (b) For $T \in \mathcal{N}(K)$, $KT = 0$. However, since $S \in \mathcal{R}(K)$, and, for symmetric matrices $\mathcal{R}(K)$ and $\mathcal{N}(K)$ are orthogonal complements of each other, $S^T T = 0$. Therefore, $K^* T = (K - kSS^T)T = 0$. So, $\mathcal{N}(K^*) \supseteq \mathcal{N}(K)$. Thus, $\mathrm{rank}(K^*) \leq r$. For all such T, $T^T K^* T = 0$.

   As a result, $T^T K^* T \geq 0$ for all T. Hence, there exists a sufficiently small $k > 0$ such that K^* is positive or semi-positive definite.

2. In (1a) one gets $\mathrm{rank}(K^*) \geq r$, whereas in (1b) one gets $\mathrm{rank}(K^*) \leq r$. Therefore, $\mathrm{rank}(K^*) = \mathrm{rank}(K) = r$.

■

Figure 8.8 illustrates how Theorem 135 can used to achieve a synthesis by $n + 1$ springs if any stiffness can be synthesized by n springs. Let $\hat{K}$ be a given positive or semi-positive definite stiffness

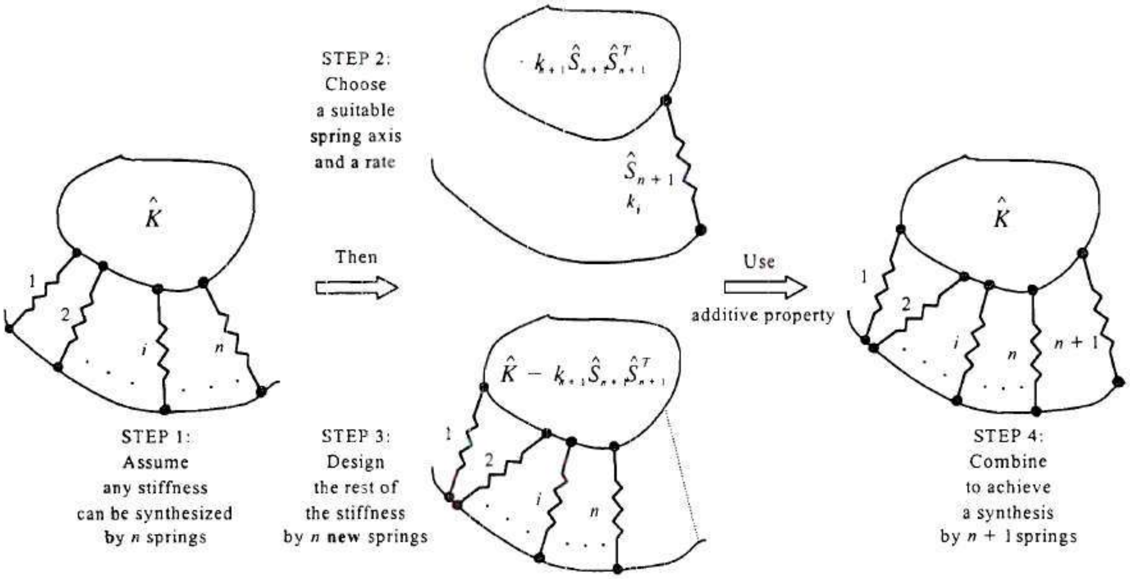


Figure 8.8: Graphical illustration of the algorithm that generates a synthesis by $n+1$ springs if any stiffness can be synthesized by n springs.

of rank r. Choose any zero or infinite pitch screw $\hat{S}_{n+1} \in \mathcal{R}(\hat{K})$ and consider the identity

$$\hat{K} = \left(\hat{K} - k_{n+1}\hat{S}_{n+1}\hat{S}_{n+1}^T\right) + k_{n+1}\hat{S}_{n+1}\hat{S}_{n+1}^T \tag{8.70}$$

For $\hat{S}_{n+1} \in \mathcal{R}(\hat{K})$, Theorem 135 proves that there exists a $k_{n+1} > 0$ such that $\hat{K}^*$ is at least semi-positive definite of rank r, where

$$\hat{K}^* = \hat{K} - k_{n+1}\hat{S}_{n+1}\hat{S}_{n+1}^T \tag{8.71}$$

Therefore,

$$\hat{K} = \hat{K}^* + k_{n+1}\hat{S}_{n+1}\hat{S}_{n+1}^T \tag{8.72}$$

simply represents an elastic connection composed of two stiffnesses in parallel; one whose stiffness is $\hat{K}^*$, the other is a stable spring $\left(k_{n+1}, \hat{S}_{n+1}\right)$. Since $\hat{K}^*$ is realizable by r springs then $\hat{K}$ is realizable by $r+1$ springs. One can continue this process to add an arbitrary number of springs. In general, if any stiffness can be realized by n springs then the above procedure can be used to achieve a synthesis with $n+1$ springs.

Note that, as also indicated in Figure 8.8, "increasing the number of springs in the synthesis" does not mean keeping n springs from a previous synthesis and adding more springs. The springs in a synthesis by $n+1$ springs have no direct relationship to any particular n-spring synthesis. What is needed is the hypothesis that any semi-positive definite stiffness can be realized by n springs. This is necessary to ensure an n-spring synthesis for $\hat{K}^*$ after choosing $\hat{S}_{n+1}$ and k_{n+1}.

The existence of zero or infinite pitch screws $\hat{S}_{n+1} \in \mathcal{R}(\hat{K})$ is not trivial. Since, by the synthesis theorem, Theorem 134, r spring axes always span $\mathcal{R}(\hat{K})$ if the stiffness is realizable by springs, $\mathcal{R}(\hat{K})$ contains at least r zero or infinite pitch screws. However, to increase the number of springs in the synthesis as described it is necessary for $\mathcal{R}(\hat{K})$ to contain more than r zero or infinite pitch screws.

A 1-system can contain only one zero or infinite pitch screw, so non-trivial addition of springs is not possible. In general, a 2-system can contain only two screws that may have zero or infinite pitches. So, in general, it is not possible to non-trivially increase the number of springs in the synthesis of rank 2 stiffnesses, unless there are special conditions as follows:

A 2-system contains infinitely many zero or infinite pitch screws only if it is spanned by,

1. two intersecting zero pitch screws (a pencil of lines), Figure 8.9-(1), or
2. two infinite pitch screws, Figure 8.9-(2), or
3. a zero pitch screw perpendicular to an infinite pitch screw, Figure 8.9-(3a,b).

In (1), all screws are of zero pitch and intersect at a point. In (2), all screws are of infinite pitch. In (3), screws are zero pitch parallel screws in addition to one independent infinite pitch screws perpendicular to the zero pitch screws. Figure 8.9 illustrates these cases with all possible spanning spring combinations.

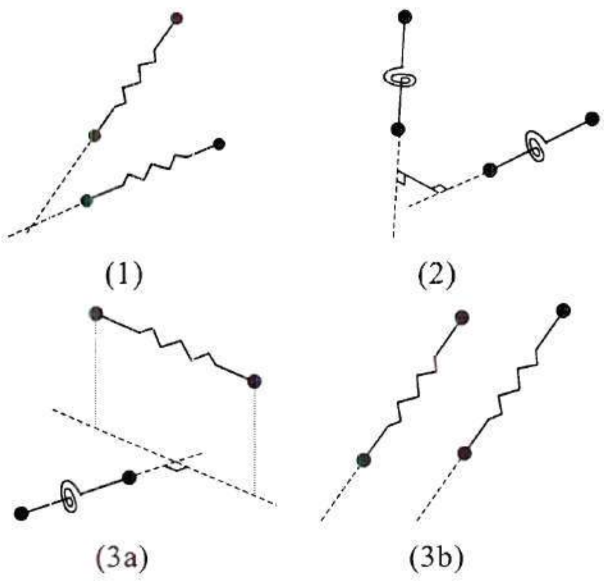


Figure 8.9: In special cases, a 2-system of screws contains infinitely many zero and infinite pitch screws. (1)-(3b) All possible bases: (1) All screws are zero pitch, through a common point and coplanar. (2) All screws are coplanar infinite pitch screws. (3a), (3b) Infinitely many zero pitch screws through every point that are all parallel, plus one infinite pitch screw perpendicular to the zero pitch screws.

Theorem 136 *Any positive or semi-positive definite stiffness can be synthesized by $n > r$ springs, where r is the rank if and only if the trace of the off-diagonals is zero, and*

1. $r \geq 3$, *or*

2. $r = 2$, *and the minimum synthesis exactly contains*

   - *two intersecting line springs, Figure 8.9-(1), or*
   - *two torsional springs, Figure 8.9-(2), or*
   - *a line spring perpendicular to a torsional spring, Figure 8.9-(3a), or*

- *two parallel line springs, Figure 8.9-(3b).*

For the special cases of $r = 2$ of the above theorem, by referring to Figure 8.9, all additional springs must respectively be either

1. line springs through the intersection point in the plane spanned by the original springs, or
2. torsional springs with directions in the plane spanned by the original springs, or
3. line springs parallel to the original line spring, or
4. line springs parallel or a torsional spring perpendicular to the original parallel pair of line springs.

Finding zero or infinite pitch screws in a given vector space is an isotropic vector problem itself, see Section 8.1.5. Let $\mathcal{R}(\hat{K})$ and $\mathcal{N}(\hat{K})$ respectively be the column and null spaces of a given stiffness. For an excess spring $\hat{S}_{n+1}$, it is required that $\hat{S}_{n+1} \in \mathcal{R}(\hat{K})$ and $\hat{S}_{n+1}^T \hat{\Delta} \hat{S}_{n+1} = 0$ (isotropic screw). Any vector in the column space of $\hat{K}$ is a linear combination of the columns of $\hat{K}$. Hence, $\hat{S}_{n+1} = \hat{K}\bar{\alpha}$, where $\bar{\alpha}$ is a 6×1 matrix of coefficients. Applying the spring condition yields

$$\bar{\alpha}^T \left(\hat{K}^T \hat{\Delta} \hat{K} \right) \bar{\alpha} = \hat{0} \tag{8.73}$$

Then, by definition, $\bar{\alpha}$ is an isotropic vector of $\hat{K}^T \hat{\Delta} \hat{K}$. For non-zero $\hat{S}_{n+1}$, $\bar{\alpha} \notin \mathcal{N}(\hat{K})$. Therefore, $\bar{\alpha}$ is a non-trivial isotropic vector of $\hat{K}^T \hat{\Delta} \hat{K}$. Section 8.1.3 presents a method to determine the isotropic vectors of indefinite matrices. If $\hat{K}$ is of rank r and realizable by springs (zero trace off-diagonals condition) then $\hat{K}^T \hat{\Delta} \hat{K}$ is necessarily indefinite and has at least r isotropic vectors (leading to r springs), see Theorem 134. In general, there exist finitely many solutions if $r = 1, 2$ and infinitely many solutions if $r > 2$. For $r = 2$, there exist infinitely many solutions only for the conditions outlined in Theorem 136. In this case, $\mathcal{R}(\hat{K})$ is completely made of zero or infinite pitch screws, which correspond to the case $\hat{K}^T \hat{\Delta} \hat{K} = \hat{0}$.

The above discussions leads to the following algorithm which synthesizes a given stiffness by $n \geq r$ springs. For simplicity, the algorithm selects all the springs automatically, i.e. without any user intervention.

1. Read $\hat{K}$.
2. If $\hat{K}$ is not symmetric. No solution. $\hat{K}$ must be symmetric.
3. If trace$(\hat{\Delta}\hat{K}) = 0$. No solution. $\hat{K}$ must have zero trace off-diagonals.
4. Read n. Desired number of springs in the synthesis.
5. $r = \text{rank}(\hat{K})$. If $n < r$. No solution. There must be at least r springs.
6. Determine if $n > r$ is possible:
7. $\hat{A} = \hat{K}^T \hat{\Delta} \hat{K}$
8. If $(n > r)$ AND $\left[(r = 1)\text{OR } \left[(r = 2)\text{AND } (\hat{A} \neq 0)\right]\right]$ Then,
9. Issue warning: "$\hat{K}$ can only be synthesized by r springs! Continuing with $n = r$"
10. $n = r$
11. End If
12. First determine suitable excess springs and their rates. There are $(n - r)$ of them.
13. $e = n - r$. Number of excess springs. A counter.
14. Initialize $\hat{E} = []_{6 \times e}$. Empty matrix. This is a holder for excess spring axes.
15. Initialize $\bar{c} = []_{e \times 1}$.Empty matrix. This is a holder for excess spring rates.
16. $\hat{K}^* = \hat{K}$. Initial value of $\hat{K}^*$ for the loop below.

17. Loop: While $e \neq 0$

18. Find a non-trivial isotropic vector of A, $\bar{\alpha}$. Use methods in Section 8.1.3.

19. $\hat{V}_e = \hat{K}^* \bar{\alpha}$. This is a zero or infinite pitch vector in $\mathcal{R}(\hat{K}^*)$.

20. $v = \sqrt{\hat{V}_e^T \hat{\Gamma} \hat{V}_e}$. Magnitude of the line-vector part.

21. If $v = 0$ Then $v = \sqrt{\hat{V}_e^T \hat{\mathrm{L}} \hat{V}_e}$. Infinite pitch. So, use the free-vector part for magnitude.

22. $\hat{S}_e = \frac{1}{v} \hat{V}_e$. This is an excess spring axis in screw form.

23. $\lambda =$ minimum eigenvalue of $\hat{K}$. Let $\sigma^2 = \hat{S}^T \hat{S}$.

24. $k_e =$ any number in $(0, \frac{\lambda}{\sigma^2})$.

25. $\hat{K}^* = \hat{K}^* - k_e \hat{S}_e \hat{S}_e^T$. $\hat{K}^*$ is still rank r, positive or semi-positive definite. See Theorem 135.

26. $\hat{E} =$ Concatenate$([\hat{E}, \hat{S}_e])$ and $\bar{c} =$ Concatenate$([\bar{c}, k_e])$. An excess spring and its rate is stored.

27. Remark: if $\hat{S}_e$ is identical to any column of $\hat{E}$, discard and repeat loop.

28. $e = e - 1$.

29. $\hat{A} = \hat{K}^{*T} \hat{\Delta} \hat{K}^*$.

30. End Loop.

31. At this point, one has $n - r$ springs with corresponding rates, and a remainder stiffness $\hat{K}^*$.

32. Use the algorithm on page 225, Section 8.1.4, to synthesize $\hat{K}^*$ with r springs: $\hat{S}_i, k_i$.

33. Put all springs and rates in proper matrices:

34. $\hat{S} = \text{Concatenate}([\hat{E}, \hat{S}_1, ..., \hat{S}_r])$ and $\hat{k} = \text{makediag}(\text{Concatenate}([\bar{c}, k_1, ..., k_r]))$.

The above algorithm has also been tested on MATLAB with satisfactory results.

8.2.6 Numerical Examples

Two numerical examples are presented. In the first, a rank six stiffness is synthesized by using six and seven springs. In the second, the same stiffness is made rank five and synthesis is demonstrated by using five and six springs.

General Synthesis Example 1: Rank Six Stiffness Consider the following randomly generated, integer, and positive definite stiffness matrix.

$$\hat{K} = \begin{bmatrix} 69 & -22 & 15 & 0 & 12 & 11 \\ -22 & 77 & 7 & 29 & 14 & 25 \\ 15 & 7 & 44 & 28 & 1 & -14 \\ 0 & 29 & 28 & 103 & 17 & -13 \\ 12 & 14 & 1 & 17 & 73 & 17 \\ 11 & 25 & -14 & -13 & 17 & 105 \end{bmatrix} \tag{8.74}$$

The off-diagonal traces are made zero without changing the definiteness. Applying the general algorithm for the synthesis by a minimum number of springs, six in this case, $\hat{K}$ is synthesized by the following six springs.

$$\hat{S} = \begin{bmatrix} 0.996 & 0.583 & 0.198 & 0.275 & -0.404 & 0.272 \\ 0.085 & -0.683 & 0.472 & 0.907 & 0.505 & -0.754 \\ 0.021 & 0.440 & 0.859 & 0.322 & 0.763 & -0.598 \\ -0.098 & 0.235 & -1.209 & 0.769 & 0.614 & -2.920 \\ 0.885 & -0.175 & -1.211 & -0.064 & 0.946 & 0.422 \\ 1.034 & -0.584 & 0.945 & -0.476 & -0.301 & -1.860 \end{bmatrix} \tag{8.75}$$

$$\hat{k} = \text{diag}\begin{bmatrix} 36.761 & 71.830 & 11.167 & 37.047 & 27.313 & 5.883 \end{bmatrix} \tag{8.76}$$

These results are rounded for presentation. For verification, the error term $\hat{K} - \hat{S}\hat{k}\hat{S}^T$ is less than 10^{-12} for all components. It is also verified that each column of $\hat{S}$ is a zero pitch screw since $\hat{S}_i^T \Delta \hat{S}_i = 0$ for all i. All the springs in this particular synthesis are line springs.

The same stiffness is synthesized by using seven springs, after a particular choice of an excess spring and applying the procedure for synthesis with excess springs. The results are,

$$\hat{S} = \begin{bmatrix} -0.11 & 0.59 & 0.75 & 0.75 & -0.81 & 0 & 1 \\ 0.81 & -0.35 & 0.59 & -0.28 & 0.33 & 0.96 & 0 \\ 0.58 & -0.73 & -0.31 & -0.60 & -0.49 & -0.28 & 0 \\ 1.12 & 1.53 & -0.10 & 1.14 & 0.07 & -0.59 & 0 \\ 0.21 & -2.24 & -0.45 & 2.06 & -0.10 & 0.44 & 0 \\ -0.07 & 2.33 & -1.09 & 0.46 & -0.19 & 1.49 & 0 \end{bmatrix} \tag{8.77}$$

$$\hat{k} = \text{diag}\begin{bmatrix} 55.46 & 3.57 & 10.66 & 10.28 & 69.56 & 30.47 & 10 \end{bmatrix} \tag{8.78}$$

The last column of $\hat{S}$ is the excess spring with a spring rate of 10. Note that the first six springs are different from those in the first synthesis, which should be expected. The maximum error is less than 10^{-12} and all springs are verified to be line springs.

<u>General Synthesis Example 2: Rank Five Stiffness</u> The previous matrix is made rank 5 by zeroing the fourth row and column.

$$\hat{K} = \begin{bmatrix} 69 & -22 & 15 & 0 & 12 & 11 \\ -22 & 77 & 7 & 0 & 14 & 25 \\ 15 & 7 & 44 & 0 & 1 & -14 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 12 & 14 & 1 & 0 & 73 & 17 \\ 11 & 25 & -14 & 0 & 17 & 105 \end{bmatrix} \tag{8.79}$$

Applying the general algorithm for the synthesis by minimum number of five springs gives,

$$\hat{S} = \begin{bmatrix} 0.147 & 0.998 & -0.117 & -0.354 & -0.587 \\ -0.875 & 0.053 & -0.618 & 0.898 & 0.081 \\ 0.461 & -0.028 & -0.778 & 0.260 & -0.806 \\ 0 & 0 & 0 & 0 & 0 \\ -0.871 & 0.053 & 1.470 & 0.124 & -0.762 \\ -1.655 & 0.100 & -1.168 & -0.428 & -0.076 \end{bmatrix} \tag{8.80}$$

$$\hat{k} = \mathrm{diag}\begin{bmatrix} 27.532 & 46.682 & 12.787 & 62.721 & 40.278 \end{bmatrix} \tag{8.81}$$

The same stiffness is synthesized by using six springs, after a particular choice of an excess spring. The results are,

$$\hat{S} = \begin{bmatrix} -0.950 & -0.439 & 0.637 & 0.026 & -0.312 & 1 \\ -0.277 & 0.795 & -0.682 & 0.721 & -0.479 & 0 \\ 0.146 & -0.418 & 0.359 & 0.692 & -0.820 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -0.276 & 0.791 & 0.247 & -0.142 & -2.970 & 0 \\ -0.524 & 1.500 & 0.470 & 0.149 & 1.730 & 0 \end{bmatrix} \tag{8.82}$$

$$\hat{k} = \operatorname{diag}\left[\begin{array}{cccccc} 34.862 & 29.352 & 52.502 & 57.878 & 5.406 & 10 \end{array}\right] \tag{8.83}$$

8.2.7 Synthesis by Free-Vector Decomposition: Free-Vector Synthesis

The free-vector decomposition for the stiffness matrix, see Lipkin and Patterson [30], presented in Chapter 3 is

$$\hat{K} = \begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \boldsymbol{\tau} & \boldsymbol{\gamma} \end{bmatrix} \begin{bmatrix} \mathbf{k}_f & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_\gamma \end{bmatrix} \begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \boldsymbol{\tau} & \boldsymbol{\gamma} \end{bmatrix}^T \tag{8.84}$$

The resemblance of (8.84) to (8.48) suggests the possibility of a solution. The columns of $[\ \mathbf{0}^T \quad \boldsymbol{\gamma}^T\ ]^T$ are already in the form of three torsional spring axes. The only difference is that the eigenwrenches, $[\ \mathbf{f}^T \quad \boldsymbol{\tau}^T\ ]^T$, in general do not have zero or infinite pitches as required by line springs. The rest of the analysis shows that there exists linear combinations of the eigenwrenches which result in three independent screws with zero pitches if the trace condition is satisfied by the stiffness matrix.

Choose the matrix $\hat{P}$ in (8.55) as

$$\hat{P} = \begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \boldsymbol{\tau} & \boldsymbol{\gamma} \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{k}_f} & \mathbf{0} \\ \mathbf{0} & \sqrt{\mathbf{k}_\gamma} \end{bmatrix} \tag{8.85}$$

and consider a special choice for $\hat{U}_V$

$$\hat{U}_V = \begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_2 \end{bmatrix} \tag{8.86}$$

where $\mathbf{R}_i$ are 3×3 orthogonal matrices, i.e. $\mathbf{R}_i\mathbf{R}_i^T = \mathbf{I}$. Using (8.85) and (8.86) in (8.61) one gets

$$\hat{U}_V^T \left(\hat{P}^T \hat{\Delta} \hat{P}\right) \hat{U}_V = \begin{bmatrix} \mathbf{R}_1^T \mathbf{H} \mathbf{R}_1 & \cdots \\ \cdots & \mathbf{0} \end{bmatrix} = \hat{\Phi} \tag{8.87}$$

where $\mathbf{H} = \sqrt{\mathbf{k}_f}(\boldsymbol{\tau}^T\mathbf{f} + \mathbf{f}^T\boldsymbol{\tau})\sqrt{\mathbf{k}_f}$. As can be seen the lower diagonal is automatically satisfied.

Then the problem is reduced to finding an $\mathbf{R}_1$ for a given $\mathbf{H}$ such that

$$\mathbf{R}_1^T\mathbf{H}\mathbf{R}_1 = \begin{bmatrix} 0 & \bullet & \bullet \\ \bullet & 0 & \bullet \\ \bullet & \bullet & 0 \end{bmatrix} \tag{8.88}$$

Since $\hat{K}$ has zero trace off-diagonals, trace$(\hat{P}^T\hat{\Delta}\hat{P}) = 0$, Theorem 133. So, trace$(\mathbf{H}) = 0$ follows necessarily from (8.87). Therefore, an $\mathbf{R}_1$ can always be found. In general, there exists a 1-parameter family of solutions for $\mathbf{R}_1$, as shown earlier. One may either use the general recursive construction algorithm or the explicit solutions given for the 3-dimensional case.

Once a particular solution $\mathbf{R}_1$ is selected, $\hat{V}$ is found from (8.58) as

$$\begin{aligned} \hat{V} &= \begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \boldsymbol{\tau} & \boldsymbol{\gamma} \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{k}_f} & \mathbf{0} \\ \mathbf{0} & \sqrt{\mathbf{k}_\gamma} \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_2 \end{bmatrix} && (8.89) \\ &= \begin{bmatrix} \mathbf{f}\sqrt{\mathbf{k}_f}\mathbf{R}_1 & \mathbf{0} \\ \boldsymbol{\tau}\sqrt{\mathbf{k}_f}\mathbf{R}_1 & \boldsymbol{\gamma}\sqrt{\mathbf{k}_\gamma}\mathbf{R}_2 \end{bmatrix} && (8.90) \end{aligned}$$

where now the first three columns are screws with zero pitches, and the last three are screws with infinite pitches. Clearly, the first three screws represent line springs, and the last three represent torsional springs.

The matrix $\mathbf{H}$, and therefore the solution space of $\mathbf{R}_1$, is invariant with respect to rigid body transformations. Solving the synthesis problem at a different origin does not yield distinct solutions. To see this assume that the coordinate system is rotated by $\mathbf{R}$. If the transformed quantities are given by primes, e.g. $\mathbf{f}' = \mathbf{R}\mathbf{f}$, then

$$\begin{aligned} \mathbf{H}' &= \sqrt{\mathbf{k}_f}(\boldsymbol{\tau}'^T\mathbf{f}' + \mathbf{f}'^T\boldsymbol{\tau}')\sqrt{\mathbf{k}_f} && (8.91) \\ &= \sqrt{\mathbf{k}_f}(\boldsymbol{\tau}^T\mathbf{R}^T\mathbf{R}\mathbf{f} + \mathbf{f}^T\mathbf{R}^T\mathbf{R}\boldsymbol{\tau})\sqrt{\mathbf{k}_f} && (8.92) \\ &= \sqrt{\mathbf{k}_f}(\boldsymbol{\tau}^T\mathbf{f} + \mathbf{f}^T\boldsymbol{\tau})\sqrt{\mathbf{k}_f} = \mathbf{H} && (8.93) \end{aligned}$$

For origin transformations, consider the shifting of the origin, from O to A by $\vec{\mathbf{r}}$. Then, by spatial transformations rules, only the torque part changes, $\vec{\tau}_{iA} = \vec{\tau}_{iO} + \vec{\mathbf{r}} \times \vec{\mathbf{f}}_i$, or $\boldsymbol{\tau}_A = \boldsymbol{\tau}_O + \vec{\mathbf{r}} \times \mathbf{f}$, so

$$\begin{aligned}
\mathbf{H}_A &= \sqrt{\mathbf{k}_f}(\boldsymbol{\tau}_A^T \mathbf{f} + \mathbf{f}^T \boldsymbol{\tau}_A)\sqrt{\mathbf{k}_f} && (8.94) \\
&= \sqrt{\mathbf{k}_f}\left[(\boldsymbol{\tau}_O^T - \mathbf{f}^T \vec{\mathbf{r}}\times)\mathbf{f} + \mathbf{f}^T(\boldsymbol{\tau}_O + \vec{\mathbf{r}} \times \mathbf{f})\right]\sqrt{\mathbf{k}_f} && (8.95) \\
&= \sqrt{\mathbf{k}_f}(\boldsymbol{\tau}^T \mathbf{f} + \mathbf{f}^T \boldsymbol{\tau})\sqrt{\mathbf{k}_f} = \mathbf{H} && (8.96)
\end{aligned}$$

This property is due to fact that the free-vector decomposition is origin independent.

The free-vector synthesis contains 4 free parameters . By counting the parameters resulting from the special choice for $\hat{U}_V$: one free parameter describes the one-parameter family of solutions for $\mathbf{R}_1$, the other three is due to $\mathbf{R}_2$ since it is an arbitrary 3×3 orthogonal matrix. This shows that the particular method used here represents all possible solutions of the problem of synthesis by three line and three torsional springs, since it includes all parameters. These results are summarized by the following theorem.

Theorem 137 *The free-vector decomposition of the stiffness completely characterizes all solutions to the free-vector synthesis problem of stiffness. In general, three line and three torsional springs are obtained.*

8.2.8 Synthesis by Line-Vector Decomposition: Line-Vector Synthesis

The line-vector decomposition for stiffness, as presented in Chapter 3, is

$$\hat{K} = \begin{bmatrix} \mathbf{n} & \mathbf{t} \\ \mathbf{m}\mathbf{h}_m & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{k}_m \mathbf{h}_m^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{k}_t \mathbf{h}_t^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{n} & \mathbf{t} \\ \mathbf{m}\mathbf{h}_m & \mathbf{0} \end{bmatrix}^T \quad (8.97)$$

where all submatrices are 3×3. This decomposition is obtained from a slightly modified form of the line-vector eigenvalue problem necessitated for unit consistency. The modification is the introduction of arbitrary positive scalars h_{ti} and h_{mi} with units of [length] which make up the diagonal matrices

$\mathbf{h}_t$ and $\mathbf{h}_m$. In this way, the quantities $\mathbf{n}, \mathbf{m}$ and $\mathbf{t}$ are rendered unitless. It was shown earlier that the modification does not result in any essential change in the co-eigenscrew structure. For all purposes, one can take $\mathbf{h}_t = \mathbf{h}_m = \mathbf{I}$ (length) so that the results of the unmodified and the modified eigenvalue problems are numerically the same. $\mathbf{k}_m = \mathbf{a}_m^{-1}$ and $\mathbf{k}_t = \mathbf{a}_t^{-1}$ are diagonal matrices with units of [force].

Unlike the free-vector decomposition, the line-vector decomposition is generator dependent. Generator is usually taken at the origin. The dependency has a useful physical meaning since it guarantees three line springs intersecting at the generator.

The procedure of synthesis by line-vector decomposition theorem follows a similar method to the one employed by synthesis by free-vector decomposition. Again, the resemblance of (8.97) to (8.48) suggests the possibility of a solution. But, unlike the free-vector decomposition, the diagonal matrix in the middle of (8.97) is composed of entries with units of [force/length], suggesting an all-line-springs solution. The columns of $[\ \mathbf{t}^T \quad \mathbf{0}^T\ ]^T$ are in the form of three mutually intersecting and perpendicular line spring axes. However, the co-eigenwrenches, $[\ \mathbf{n}^T \quad (\mathbf{m}\mathbf{h}_m)^T\ ]^T$, in general do not have zero pitches as required by line springs. The rest of the analysis shows that there exists linear combinations of the co-eigenwrenches which result in six independent screws with zero pitches if the trace condition is satisfied by the stiffness matrix.

Choose the matrix $\hat{P}$ as

$$\hat{P} = \begin{bmatrix} \mathbf{n} & \mathbf{t} \\ \mathbf{m}\mathbf{h}_m & \mathbf{0} \end{bmatrix} \begin{bmatrix} \sqrt{\mathbf{k}_m \mathbf{h}_m^{-1}} & \mathbf{0} \\ \mathbf{0} & \sqrt{\mathbf{k}_t \mathbf{h}_t^{-1}} \end{bmatrix} \tag{8.98}$$

Using the same form of $\hat{U}_V$ as in (8.86) gives exactly the same equation as (8.87) except that

$$\mathbf{H} = \sqrt{\mathbf{k}_m \mathbf{h}_m^{-1}}(\mathbf{n}^T \mathbf{m}\mathbf{h}_m + \mathbf{h}_m \mathbf{m}^T \mathbf{n})\sqrt{\mathbf{k}_m \mathbf{h}_m^{-1}} \tag{8.99}$$

Lower diagonal of $\hat{\Phi}$ is satisfied automatically. The problem reduces to finding solutions $\mathbf{R}_1$ to

(8.88). As before, the requirement that trace($\mathbf{H}$) = 0 is satisfied automatically. Therefore, there exists of a 1-parameter family of solutions for $\mathbf{R}_1$.

The springs are given by

$$\hat{V} = \begin{bmatrix} \mathbf{n}\sqrt{\mathbf{k}_m\mathbf{h}_m^{-1}}\mathbf{R}_1 & \mathbf{t}\sqrt{\mathbf{k}_t\mathbf{h}_t^{-1}}\mathbf{R}_2 \\ \mathbf{m}\sqrt{\mathbf{k}_m\mathbf{h}_m}\mathbf{R}_1 & \mathbf{0} \end{bmatrix} \tag{8.100}$$

In general, these represent six line springs. Last three springs pass through the origin. The units of the spring stiffnesses are equivalent to that of $\sqrt{\mathbf{kh}^{-1}}$ which is [force/length] as desired. If any of the first three springs is a torsional spring then the spring stiffnesses has the units of $\sqrt{\mathbf{k}_m\mathbf{h}_m}$, i.e. [force*length] as desired. The first three springs are linear combinations of the co-eigenwrenches and the last three are linear combinations of the reaction forces to the co-eigentwists.

In general, the line-vector synthesis, which uses 6 line springs, has 10 free parameters. The special choice for $\hat{U}_V$ together with the line-vector decomposition requires the three springs to intersect at a point, regardless of anything else. This introduces three constraints. Therefore, the line-vector decomposition theorem provides only seven free parameters. One free parameter comes from the solution of $\mathbf{R}_1$, as mentioned before. Three of the free parameters are due to the arbitrary orthogonal matrix $\mathbf{R}_2$. Finally, three more free parameters are provided by arbitrary generators (which has three parameters), since distinct generators result in distinct line-vector decompositions.

Since the line-vector decomposition method uses only 7 of 10 free parameters, it cannot encompass all solutions with 6 line springs. In other words, the line-vector decomposition theorem yields a subset of the whole solutions to the line-vector synthesis problem. For the most general solutions involving 6 line springs one must use the general algorithm for the construction of $\hat{U}_V$, instead of the special form allowed here. The following theorem summarizes the results of this section.

Theorem 138 *The line-vector decomposition of stiffness gives a subset of all solutions to the line-*

vector synthesis problem of stiffness described by 7 parameters. In general, all springs in the synthesis are line springs.

8.2.9 Examples and Algorithms

In this section numerical and theoretical examples are presented. For numerical examples, the stiffness matrix of the two fingered Stanford/JPL hand is taken, see Chapter 3, that was used by Cutkosky and Kao [14] in a study of robot grasp problem involving rivet insertion. The stiffness of this robotic hand, which contains the combined elastic effects of cables, joints, links, soft fingertips and the servo system, with respect to the tip of the grasped rivet is

$$\hat{K} = \left[\begin{array}{ccc|ccc} 2490 & 0 & 0 & 0 & 258 & 0 \\ 0 & 28900 & 0 & 191 & 0 & 0 \\ 0 & 0 & 61610 & 0 & 0 & 0 \\ \hline 0 & 191 & 0 & 22 & 0 & 0 \\ 258 & 0 & 0 & 0 & 37 & 0 \\ 0 & 0 & 0 & 0 & 0 & 35 \end{array}\right] \tag{8.101}$$

Free-vector Synthesis: Algorithm and Example The following is an algorithm describing a sequential method to achieve a synthesis by first decomposition.

1. Read $\hat{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix}$
2. If $\mathrm{tr}(\mathbf{B}) \neq 0$. Stop. No solution. Stiffness is not realizable.
3. Compute $\mathbf{f}, \boldsymbol{\tau}, \boldsymbol{\gamma}, \mathbf{k}_f, \mathbf{k}_\gamma$ (Free-vector eigenvalue problem).
4. If k_{fi} or $k_{\gamma i} < 0$ for some i. Stop. Unstable stiffness.
5. Construct $\mathbf{H} = \sqrt{\mathbf{k}_f}(\boldsymbol{\tau}^T\mathbf{f} + \mathbf{f}^T\boldsymbol{\tau})\sqrt{\mathbf{k}_f}$.

6. Find a solution $\mathbf{R}_1$ to $\mathbf{R}_1^T \mathbf{H} \mathbf{R}_1 = \begin{bmatrix} 0 & \bullet & \bullet \\ \bullet & 0 & \bullet \\ \bullet & \bullet & 0 \end{bmatrix}$.

7. Select and arbitrary 3×3 matrix $\mathbf{R}_2$ such that $\mathbf{R}_2 \mathbf{R}_2^T = \mathbf{I}$.

8. $\hat{V} = [\hat{V}_1, \hat{V}_2, ..., \hat{V}_6] = \begin{bmatrix} \mathbf{f}\sqrt{\mathbf{k}_f}\mathbf{R}_1 & \mathbf{0} \\ \tau\sqrt{\mathbf{k}_f}\mathbf{R}_1 & \gamma\sqrt{\mathbf{k}_\gamma}\mathbf{R}_2 \end{bmatrix}$

9. Loop $i = 1, ..., 6$

10. If $\hat{V}_i = \hat{0}$, Singular stiffness. Discard $\hat{V}_i$. Go to End Loop.

11. Compute $k_i = \hat{V}_i^T \hat{\Gamma} \hat{V}_i$. Assumes line spring.

12. If $k_i = 0$, Then $k_i = \hat{V}_i^T \hat{\mathrm{L}} \hat{V}_i$. Spring is torsional.

13. $\hat{S}_i = \frac{1}{\sqrt{k_i}} \hat{V}_i$.

14. End Loop.

Using the stiffness of (8.101), the eigenwrenches, reaction couples (in the direction of eigentwists), and related eigenstiffnesses are found as

$$\begin{bmatrix} \mathbf{f} & \mathbf{0} \\ \tau & \gamma \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0.0066 & 0 & 1 & 0 & 0 \\ 0.1036 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \quad (8.102)$$

$$\mathbf{k}_f = \mathrm{diag}\begin{bmatrix} 2490 & 28900 & 16610 \end{bmatrix} \quad (8.103)$$

$$\mathbf{k}_\gamma = \mathrm{diag}\begin{bmatrix} 20.74 & 10.27 & 35.00 \end{bmatrix} \quad (8.104)$$

Note that the eigenwrenches have zero pitches. So, a solution is already given by taking the columns in (8.102) as the spring axes and the corresponding k_{fi} and $k_{\gamma i}$ as the stiffnesses. This gives three mutually perpendicular line springs and three mutually perpendicular torsional springs. The first line spring is parallel to the x-axis and intersects the z-axis at 0.1036, the second line is parallel to the y-axis and intersects the z-axis at -0.0066, and, the third is coincident with the z-axis.

To demonstrate the method another synthesis is determined using the algorithm presented in this section. The $\mathbf{H}$ matrix is found as

$$\mathbf{H} = \begin{bmatrix} 0 & 935.02 & 0 \\ 935.02 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.105}$$

Using the explicit solution method for the isotropic vector problem in 3-dimensional case, a solution to (8.88) is found as

$$\mathbf{R}_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.5403 & 0.8415 \\ 0 & -0.8415 & 0.5403 \end{bmatrix} \tag{8.106}$$

Next, the following orthogonal matrix is chosen at random,

$$\mathbf{R}_2 = \begin{bmatrix} 0.9585 & 0.1755 & -0.2246 \\ -0.2815 & 0.7072 & -0.6486 \\ 0.0450 & 0.6849 & 0.7273 \end{bmatrix} \tag{8.107}$$

Then, the springs and their stiffnesses are determined using (8.65), (8.66) and (8.90). The results are

$$\hat{S} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0.403 & 0.730 & 0 & 0 & 0 \\ 0 & -0.915 & 0.684 & 0 & 0 & 0 \\ 0 & 0.003 & 0.005 & 0.978 & 0.170 & -0209 \\ 0.104 & 0 & 0 & -0.202 & 0.481 & -0.425 \\ 0 & 0 & 0 & 0.060 & 0.860 & 0.881 \end{bmatrix} \tag{8.108}$$

$$\hat{k} = \text{diag} \begin{bmatrix} 2490 & 52061 & 38449 & 19.937 & 22.191 & 23.877 \end{bmatrix} \tag{8.109}$$

Although the first spring is unchanged, all others are different and the perpendicular structure is lost. Results are rounded for presentation.

Line-Vector Synthesis: Algorithm and Example The following is an algorithm describing a sequential method to achieve a synthesis by second decomposition. Scaling factors are taken as unity for simplicity.

1. Read $\hat{K} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix}$
2. If $\text{tr}(\mathbf{B}) \neq 0$. Stop. No solution. Stiffness is not realizable.
3. Compute $\mathbf{n}, \mathbf{m}, \mathbf{t}, \mathbf{k}_m, \mathbf{k}_t$ (Line-vector eigenvalue problem).
4. If k_{mi} or $k_{ti} < 0$ for some i. Stop. Unstable stiffness.
5. Construct $\mathbf{H} = \sqrt{\mathbf{k}_m}(\mathbf{n}^T\mathbf{m} + \mathbf{m}^T\mathbf{n})\sqrt{\mathbf{k}_m}$.
6. Find a solution $\mathbf{R}_1$ to $\mathbf{R}_1^T\mathbf{H}\mathbf{R}_1 = \begin{bmatrix} 0 & \bullet & \bullet \\ \bullet & 0 & \bullet \\ \bullet & \bullet & 0 \end{bmatrix}$.

7. Select and arbitrary 3×3 matrix $\mathbf{R}_2$ such that $\mathbf{R}_2\mathbf{R}_2^T = \mathbf{I}$.

8. $\hat{V} = [\hat{V}_1, \hat{V}_2, ..., \hat{V}_6] = \begin{bmatrix} \mathbf{n}\sqrt{\mathbf{k}_m}\mathbf{R}_1 & \mathbf{t}\sqrt{\mathbf{k}_t}\mathbf{R}_2 \\ \mathbf{m}\sqrt{\mathbf{k}_m}\mathbf{R}_1 & \mathbf{0} \end{bmatrix}$

9. Loop $i = 1, ..., 6$

10. If $\hat{V}_i = \hat{0}$, Singular stiffness. Discard $\hat{V}_i$. Go to End Loop.

11. Compute $k_i = \hat{V}_i^T \hat{\Gamma} \hat{V}_i$

12. If $k_i = 0$, Then $k_i = \hat{V}_i^T \hat{\mathbf{L}} \hat{V}_i$

13. $\hat{S}_i = \frac{1}{\sqrt{k_i}} \hat{V}_i$.

14. End Loop.

Again, using the stiffness of (8.101), the co-eigenwrenches, reaction forces (in the direction of co-eigentwists), and related eigenstiffnesses are found as

$$\begin{bmatrix} \mathbf{n} & \mathbf{t} \\ \mathbf{m} & \mathbf{0} \end{bmatrix} = \begin{bmatrix} 0 & 6.9730 & 0 & 1 & 0 & 0 \\ 8.6818 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix} \tag{8.110}$$

$$\mathbf{k}_m = \text{diag}\begin{bmatrix} 22 & 37 & 35 \end{bmatrix} \tag{8.111}$$

$$\mathbf{k}_\gamma = \text{diag}\begin{bmatrix} 691 & 27242 & 61610 \end{bmatrix} \tag{8.112}$$

Note that the co-eigenwrenches have zero and infinite pitches. So, a solution is already given by taking the columns in (8.110) as the spring axes and the corresponding k_{mi} and k_{ti} as the stiffnesses (the first three have to be scaled). This gives three mutually perpendicular line springs intersecting at the origin. Interestingly, the other springs (the first three) have mutually perpendicular directions, but two are line springs and one is a torsional spring. Of these, the first line spring is parallel to the y-axis and intersects the z-axis at -0.1152, the second line spring is parallel to the x-axis and intersects the z-axis at 0.1434. The torsional spring is parallel to the z-axis. This is an example of 5 line springs plus 1 torsional spring synthesis.

To demonstrate the method another synthesis is determined using the algorithm presented in this section. The $\mathbf{H}$ matrix is found as

$$\mathbf{H} = \begin{bmatrix} 0 & 446.64 & 0 \\ 446.64 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.113}$$

As before, a solution $\mathbf{R}_1$ to (8.88) is found using the explicit method as

$$\mathbf{R}_1 = \begin{bmatrix} 0.5403 & 0.8415 & 0 \\ 0 & 0 & 1 \\ 0.8415 & -0.5403 & 0 \end{bmatrix} \tag{8.114}$$

Next, the $\mathbf{R}_2$ matrix of the first decomposition example is used, and, the springs and their stiffnesses are determined using (8.65), (8.66) and (8.90). The results are

$$\hat{S} = \begin{bmatrix} 0 & 0 & 1 & 0.446 & 0.022 & -0.028 \\ 1 & 1 & 0 & -0.860 & 0.566 & -0.510 \\ 0 & 0 & 0 & 0.207 & 0.824 & 0.860 \\ 0.115 & 0.115 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.143 & 0 & 0 & 0 \\ 0.226 & -0.093 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{8.115}$$

$$\hat{k} = \text{diag} \begin{bmatrix} 484.1 & 1174.1 & 1799.0 & 2918.2 & 42545 & 44080 \end{bmatrix} \tag{8.116}$$

These are all line springs. The first two are parallel to the y-axis, the third is parallel to the x-axis and intersects the z-axis at 0.1434. The remaining springs are line springs through the origin. Results are rounded for presentation.

CHAPTER IX

ROTATIONAL SYMMETRY DEVICES

First, a classical *remote-center-of-compliance* (RCC) device composed of beams is analyzed and closed form equations for the RCC center and stiffnesses are found. Generalization of the results leads to the concept of general symmetric devices that include classical RCCs, Stewart platforms, etc. Using infinitely many infinitesimally small elements, the stiffness matrices of continuous systems, such as an O-ring or a small pitch helical spring, are determined as confirmations of the theory.

The theory developed in this chapter is expected to be beneficial in both theoretical and practical applications since almost all results are new. Application areas are diverse and are explained in each major section.

9.1 Remote Center of Compliance (RCC) Devices

A *remote-center-of-compliance*, RCC, device is an elastic structure which responds by pure parallel translations and couples, respectively, to pure forces and rotations applied at a unique point. The point is called the **RCC center**.

RCC devices have found applications in automated assembly tasks involving mating of parts, especially when close tolerances are desired. For example, a common assembly task is the mating of parts by insertion, such as placing a motor shaft into a bearing. For close tolerances, jamming can occur when using automated equipment such as robots. Possibility of jamming is greatly reduced when the part to be inserted is held by the RCC device and the elastic center is around the region of contact. Figure 9.1 shows how the jamming is avoided during insertion.

Usually, the RCC device is placed at the wrist of a robot. If the compliance of the robot is negligible in comparison to that of the RCC device then the elastic behavior of the latter dominates

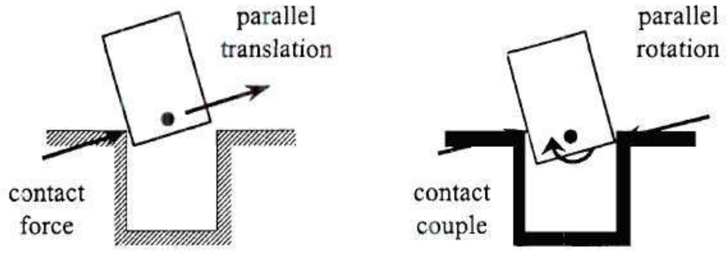


Figure 9.1: Contact forces/couples in insertion and the response of RCC device.

for loads applied to the workpiece. Because of this error correcting behavior, the device is regarded as a means of passive control and a cost effective alternative to active control strategies which usually involve complicated components due to sensing, feedback, etc. Moreover, the design, operation and maintenance of the RCC devices are relatively simpler.

9.1.1 Design Equations for a Classical RCC Device

A successful RCC device was developed at Draper Laboratories by Whitney and coworkers [16], [17], [34], [35], [51], [52], [53]. These devices are seemingly very well known and are described in most robotics textbooks. They have been sold commercially by several vendors.

Figure 9.1 shows an insertion strategy. When a linear misalignment occurs, a contact force translates the shaft away. When an angular misalignment occurs, a contact couple turns the shaft about the tip to straighten it. This type of action can be achieved by simulating linear and angular springs acting at the tip point. The point has been termed the *center-of-compliance* and is equivalent to what is called an *elastic center* in beam theory [39].

The importance of this device in this study comes from its suitability in the stiffness/compliance analyses developed here. There are many other aspects that have to be considered in the design of such devices for practical applications such as dynamic behavior, analysis of jamming modes, criteria for jamming avoidance. These require detailed analyses of particular geometries. Nevertheless, the special stiffness of the device, which makes it behave as self-correcting unit, is central to everything else. Therefore, a reliable analysis and prediction of the stiffness of the device is essential. This is the main topic of the section.

The primary feature of the device is that it responds to a pure force by a parallel pure translation and to a pure couple by a parallel pure rotation, all through a certain point. Clearly, this shows that the device has three compliant axes, and therefore, the point is where the centers of elasticity, stiffness, compliance, and the co-center of elasticity coincide. This justifies the historical identification of the point as a *center-of-compliance.* From the results in this study it is apparent that *the elastic center* is a better name. In a coordinate system aligned parallel to the principal screws, the stiffness/compliance matrices of such behavior are simply diagonal. So, the construction of RCC devices may be understood as construction of stiffnesses that are diagonal. This idea is generalized in later sections. The main point of this section is to show why RCC devices behave the way they do and how more accurate design equations can be obtained by using the results of this study.

Figure 9.2 is a planar representation of the basic device. The shaft is held by a gripper mounted on deformable rods at the top. The rods themselves are held by a sturdy ring that rests upon deformable wire legs fixed to a rigid annular mounting plate. Considering rods alone, their center point is known to be an elastic center. Considering the beam side structure alone, an elastic center was approximated by Whitney and coworkers to be at the intersection point [35], [51], [52], [53]. The net effect of the combination of the plates and the side structure is the shifting of the elastic center along the central symmetry axis. By adjusting the location of the intersection point and the relative stiffnesses, it is possible to place the resultant elastic center at the tip of the shaft. This is the premise of a practical RCC device.

Satisfactory operation of the RCC requires locating the elastic center such that the forces

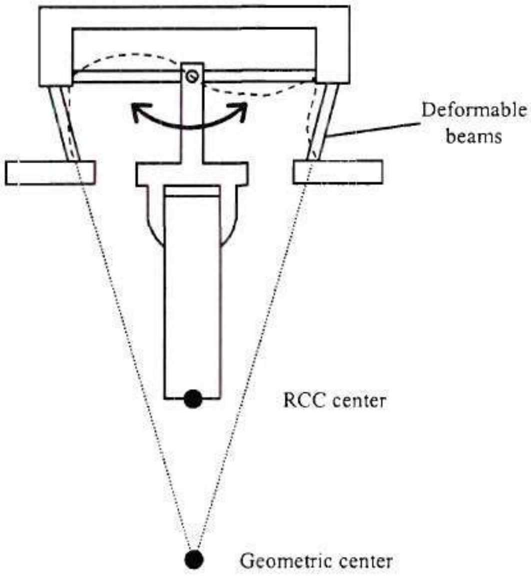


Figure 9.2: Simplifies schematics of the RCC device investigated by Drake et al.

resulting from excessive contact of the part due to positioning errors are acting through or about the elastic center. This is where the remoteness of the center is desired. Since the workpiece occupies the space between the device and the contact region, and the elastic center is desired to be in the contact region, the center should be designed to be outside of the physical body of the device. See Figure 9.3.

Whitney and coworkers, [35], [51], [52], [53], developed a successful construction method for RCC devices using three identical slender beams, called legs, connecting two relatively rigid plates in parallel, Figure 9.3. The beams are symmetrically located on a cone, and their longitudinal axes converges to the vertex of the cone. The elastic center is found on the cone axis, usually between the vertex and the bottom plate. In their analyses Whitney's team tried to account for the elastic effects of all substructures. Drake [16] modeled the beams and rods as simple beams and noted that the experimental properties of RCC devices did not match the theoretically predicted ones, especially

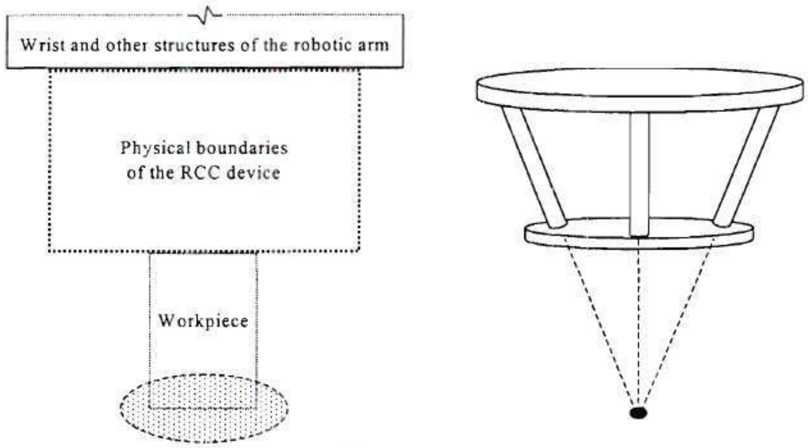


Figure 9.3: For satisfactory operation, the elastic center of an RCC must be outside the pyhsical boundaries and close to the region of expected contact. On the left is an idealized model of RCC with elastic beams only.

the elastic center location. In Nevins [34] this phenomena was attributed to previously unaccounted for compliances in the ring and the gripper or shaft. Further analysis of RCC properties is not evident in the literature. There are a few assumptions used in earlier analyses which must have contributed to this discrepancy. One of these, that is important in the present context, is that the geometric intersection point of the side beam axes is taken as the elastic center for the side structure. The analyses in this section prove this assumption to be quite coarse. Another assumption is that the side structure stiffness is taken as equivalent to that of a parallel arrangement. This is also shown to be very coarse.

The development here does not make these assumptions and concludes that the center location does not lie at the intersection point. In fact, small changes in the angle of the side structure have profound effects on various properties, especially center location. Since the interesting and new properties concern the side structure geometry of the RCC device, the compliance of the rods are

Figure 9.4: A generic beam element. Stiffness is defined considering the displacements and loads at the tip, O.

neglected, i.e. everything other than the side structure is considered rigid, Figure 9.3. If needed, the upper compliance of the rods can be easily accounted for as done in [16].

The analysis does not include commercial RCC devices that use shear pads in their design since they cannot be modelled as simple beams. It is believed that the results and analyses presented here are new. The fundamental focus is beyond a particular device and is concerned with the basic phenomena of remote elastic centers which may have application in areas other than assembly.

Beam Model Beam elements are represented by a general engineering model (see [44] for example) that discretizes the stiffness to relate loads and absolute displacements at the ends. The usual 12×12 structural element stiffness can be reduced to a 6×6 matrix by introducing zero displacement boundary conditions at one end so the displacements on the other end are relative deflections.

Figure 9.4 shows a coordinate frame referenced to an undeformed straight beam with a constant cross section. The origin O is at the free end of the beam, the z-axis is along the centroidal axis, and the x- and y-axes are along the principal axes of the cross-section. At O, the stiffness matrix has the form

$$\hat{K}_O = \begin{bmatrix} a & 0 & 0 & 0 & -h & 0 \\ 0 & b & 0 & g & 0 & 0 \\ 0 & 0 & c & 0 & 0 & 0 \\ 0 & g & 0 & d & 0 & 0 \\ -h & 0 & 0 & 0 & e & 0 \\ 0 & 0 & 0 & 0 & 0 & f \end{bmatrix} \tag{9.1}$$

where the terms $a, b, ..., h$ are defined as

$$\begin{array}{lll} a = \frac{12EI_x}{L^3(1+\Phi_y)} & b = \frac{12EI_y}{L^3(1+\Phi_x)} & c = \frac{AE}{L} \\ d = \frac{EI_y}{L}\frac{(4+\Phi_x)}{(1+\Phi_x)} & e = \frac{EI_x}{L}\frac{(4+\Phi_y)}{(1+\Phi_y)} & f = \frac{GJ}{L} \\ g = \frac{6EI_y}{L^2(1+\Phi_x)} & h = \frac{6EI_x}{L^2(1+\Phi_y)} & \end{array} \tag{9.2}$$

L is the longitudinal length; A is the cross–section area; I_x, I_y, J are the area and polar moments of inertia; E, G are the moduli of elasticity and shear with $G = \frac{E}{2(1+\nu)}$ and ν Poisson's ratio. Φ_x and Φ_y are shear correction factors defined as

$$\Phi_x = \frac{12EI_y}{GA_{sx}L^2} \qquad \Phi_y = \frac{12EI_x}{GA_{sy}L^2} \tag{9.3}$$

where A_{sx}, A_{sy} are the effective shear areas for the cross-section and define the shear form factors $f_{sx} = A_{sx}/A$, $f_{sy} = A_{sy}/A$ so

$$\Phi_x = 24(1+\nu)f_{sx}\left(\frac{\rho_y}{L}\right)^2 \qquad \Phi_y = 24(1+\nu)f_{sy}\left(\frac{\rho_x}{L}\right)^2 \tag{9.4}$$

The ρ_i are the radii of gyration defined by $\rho_i^2 = I_i\ /A$. The minimum of $\frac{L}{\rho_x}$, $\frac{L}{\rho_y}$ is called the **slenderness ratio.**

Using a rigid body translation, it is possible to represent the *same* load and displacement relationship for the end of the beam at a different origin location E,

$$d\hat{W}_E = \hat{K}_E \delta \hat{q}_E \qquad \hat{K}_E = \hat{X}_{OE} \hat{K}_O \hat{X}_{OE}^T \qquad \hat{X}_{OE} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\overrightarrow{OE}\times & \mathbf{I} \end{bmatrix} \tag{9.5}$$

To visualize the transformed stiffness consider a hypothetical rigid body attached to the end of the beam that contains the point E. The loads and displacements are measured at this point on the hypothetical rigid body. See Figure 9.5.

Beam Elastic Center Consider the stiffness matrix represented at a point E coincident with the mid-span of the beam

$$\overrightarrow{OE} = \begin{bmatrix} 0 & 0 & -\frac{L}{2} \end{bmatrix}^T \tag{9.6}$$

Using (7.21) in (9.5) yields a diagonal form in terms of beam parameters

$$\hat{K}_E = diag\left\{ \frac{12EI_x}{L^3(1+\Phi_y)}, \frac{12EI_y}{L^3(1+\Phi_x)}, \frac{AE}{L}, \frac{EI_y}{L}, \frac{EI_x}{L}, \frac{GJ}{L} \right\} \tag{9.7}$$

$$= diag\left\{ \lambda_x \quad \lambda_y \quad \lambda_z \quad \mu_x \quad \mu_y \quad \mu_z \right\} \tag{9.8}$$

Therefore with the origin at mid-span, the stiffness matrix is diagonal when the coordinate axes are aligned with the principal directions of the beam. The point E is known as the *elastic center* [39]. Clearly, E is the center of elasticity, stiffness, compliance and the co-center of elasticity. Referring to Figure 9.5, a force along a principal axis results in a parallel translation of the hypothetical rigid body (and hence of the free end). Similarly, a rotation about a principal axis of the hypothetical rigid body (and hence the free end) requires a parallel couple. These are the types of actions necessary to create an RCC device. The obvious problem with a single straight beam is that the elastic center E is coincident with beam material, i.e. it is not remote.

The following theorem summarizes the required results of this section,

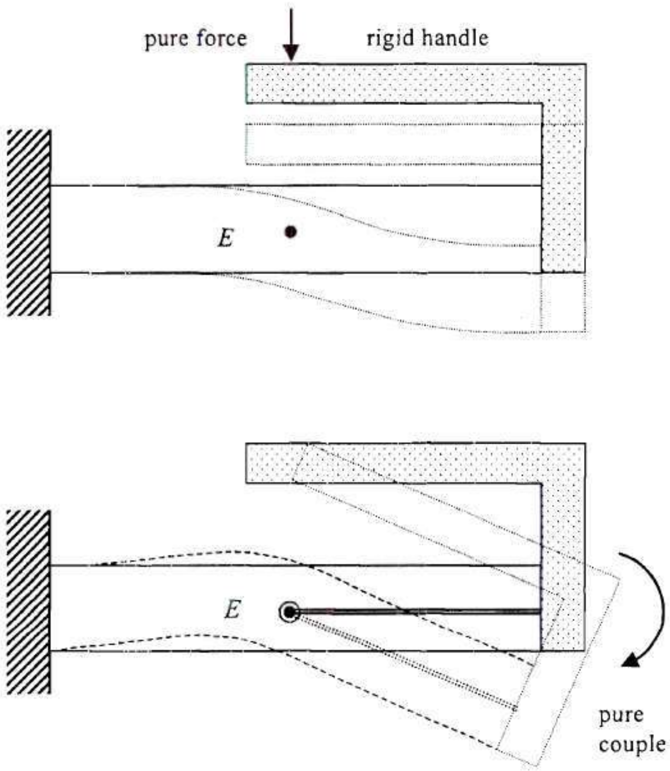


Figure 9.5: The mid point on the central axis of a beam is where all the centers coalesce. Therefore, a pure force through this point causes only a parallel translation, and, a pure couple causes only a parallel rotation.

Theorem 139 *For a beam with the stiffness matrix as given in (9.1), the centers of elasticity, stiffness and compliance, and the co-center coincide with the mid-point of the neutral axis in the undeformed state.*

Multiple Beams Consider n beams with elastic centers E_i. When the local coordinate axes through E_i are along the principal directions, the stiffness matrices are diagonal. Instead, let the coordinate axes be in an arbitrary global frame. The principal axis directions for the i^{th} beam are given by the orthogonal set $\vec{\mathbf{e}}_{xi}, \vec{\mathbf{e}}_{yi}, \vec{\mathbf{e}}_{zi}$ where $\vec{\mathbf{e}}_{zi}$ is along the centroidal axis and $\vec{\mathbf{e}}_{xi}, \vec{\mathbf{e}}_{yi}$ are along the

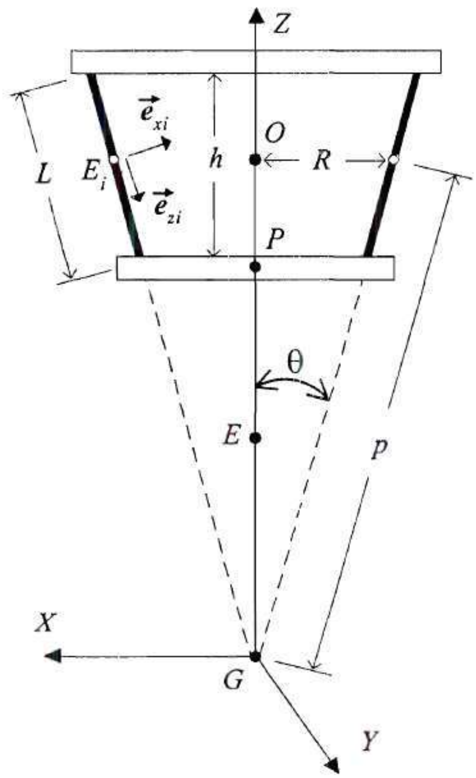


Figure 9.6: Geometry of RCC with beams symmetrically placed in a conical arrangement.

cross-section, see Figure 9.6. In these coordinates, each stiffness matrix has the form

$$\hat{K}_{E_i} = \begin{bmatrix} \mathbf{A}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_i \end{bmatrix} \tag{9.9}$$

where

$$\mathbf{A}_i = \lambda_{xi}\vec{\mathbf{e}}_{xi}\vec{\mathbf{e}}_{xi}^T + \lambda_{yi}\vec{\mathbf{e}}_{yi}\vec{\mathbf{e}}_{yi}^T + \lambda_{zi}\vec{\mathbf{e}}_{zi}\vec{\mathbf{e}}_{zi}^T \tag{9.10}$$

$$\mathbf{C}_i = \mu_{xi}\vec{\mathbf{e}}_{xi}\vec{\mathbf{e}}_{xi}^T + \mu_{yi}\vec{\mathbf{e}}_{yi}\vec{\mathbf{e}}_{yi}^T + \mu_{zi}\vec{\mathbf{e}}_{zi}\vec{\mathbf{e}}_{zi}^T \tag{9.11}$$

Note that one end of each beam be connected to a single rigid body and the other end of each beam be connected to a second rigid body. In this arrangement the stiffnesses are additive. Transforming the individual stiffnesses from E_i to an arbitrary point E and summing gives,

$$
\begin{aligned}
\hat{K}_E &= \sum_{i=1}^{n} \hat{X}_{E_i E}^T \hat{K}_{E_i} \hat{X}_{E_i E} \\
&= \sum_{i=1}^{n} \begin{bmatrix} \mathbf{A}_i & \mathbf{A}_i \overrightarrow{E_i E}\times \\ -\overrightarrow{E_i E}\times \mathbf{A}_i & \mathbf{C}_{E_i} - \overrightarrow{E_i E}\times \mathbf{A}_i \overrightarrow{E_i E}\times \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{A} & \mathbf{B}_E \\ \mathbf{B}_E^T & \mathbf{C}_E \end{bmatrix}
\end{aligned} \tag{9.12}
$$

where 3×3 partitions are introduced. $\hat{K}_E$ is the relative stiffness between the two rigid bodies.

Conical Symmetry It is desired to select a point E and an orientation of the global frame so that $\hat{K}_E$ becomes diagonal and E has the properties of an elastic center. The n beams could then potentially form a RCC device. However, since a general stiffness matrix cannot be diagonalized by rigid body transformations, it is necessary to impose additional constraints on the beams. This is achieved using symmetry.

Figure 9.6 suggests the symmetric arrangement of n identical beams on a surface of a cone. Let the global coordinates be XYZ, where Z is along the axis of the cone. The centroidal principal axes in the $\vec{\mathbf{e}}_{zi}$ directions all intersect at G. One of the other principal axes, say in the $\vec{\mathbf{e}}_{xi}$ directions, all intersect the Z-axis in another point. This is the generalized n-beam case of the symmetric RCC device investigated by Drake [16]. The half angle of the cone is θ and the angle between adjacent beams is $\frac{2\pi}{n}$, as measured on a circle parallel to the XY-plane. The X-axis is selected coplanar with the first beam and the Z-axis. The coordinate expressions for $\vec{\mathbf{e}}_{xi}, \vec{\mathbf{e}}_{yi}, \vec{\mathbf{e}}_{zi}$ are

$$\begin{aligned}
\vec{\mathbf{e}}_{xi} &= \begin{bmatrix} -\cos\theta\,\cos(\frac{2\pi}{n}(i-1)) \\ -\cos\theta\,\sin(\frac{2\pi}{n}(i-1)) \\ \sin\theta \end{bmatrix} \\
\vec{\mathbf{e}}_{yi} &= \begin{bmatrix} -\sin(\frac{2\pi}{n}(i-1)) \\ \cos(\frac{2\pi}{n}(i-1)) \\ 0 \end{bmatrix} \\
\vec{\mathbf{e}}_{zi} &= \begin{bmatrix} -\sin\theta\,\cos(\frac{2\pi}{n}(i-1)) \\ -\sin\theta\,\sin(\frac{2\pi}{n}(i-1)) \\ -\cos\theta \end{bmatrix}
\end{aligned} \tag{9.13}$$

and the following trigonometric identities are used in the subsequent analysis,

$$\begin{aligned}
&\sum_{i=1}^{n}\cos(\frac{2\pi}{n}(i-1)) = \sum_{i=1}^{n}\sin(\frac{2\pi}{n}(i-1)) = 0, \quad n>1 \\
&\sum_{i=1}^{n}\cos(\frac{2\pi}{n}(i-1))\sin(\frac{2\pi}{n}(i-1)) = 0, \quad n>1 \\
&\sum_{i=1}^{n}\cos^2(\frac{2\pi}{n}(i-1)) = \sum_{i=1}^{n}\sin^2(\frac{2\pi}{n}(i-1)) = \tfrac{1}{2}n, \ n>2
\end{aligned} \tag{9.14}$$

<u>Principal Linear Stiffnesses</u> The total stiffness matrix partitions $\mathbf{A}$, $\mathbf{B}_E$, and $\mathbf{C}_E$ are individually examined to determine the conditions for $\hat{K}_E$ to be diagonal. This yields the principal stiffnesses, directions, and elastic center for the symmetric, conical arrangement of n identical beams.

$\mathbf{A}$ is calculated from (9.10), (9.12) and (9.14). Due to symmetry, the summations $\sum \vec{\mathbf{e}}_{xi}\vec{\mathbf{e}}_{xi}^T$, $\sum \vec{\mathbf{e}}_{yi}\vec{\mathbf{e}}_{yi}^T$, and $\sum \vec{\mathbf{e}}_{zi}\vec{\mathbf{e}}_{zi}^T$ each reduce to diagonal matrices in the selected coordinate system,

$$\begin{aligned}
\mathbf{A} &= \sum \mathbf{A}_i \\
&= \lambda_x \sum \vec{\mathbf{e}}_{xi}\vec{\mathbf{e}}_{xi}^T + \lambda_y \sum \vec{\mathbf{e}}_{yi}\vec{\mathbf{e}}_{yi}^T + \lambda_z \sum \vec{\mathbf{e}}_{zi}\vec{\mathbf{e}}_{zi}^T \\
&= diag\{\ k_{XY} \quad k_{XY} \quad k_Z\ \}
\end{aligned} \tag{9.15}$$

where the linear stiffnesses are

$$k_{XY} = \tfrac{1}{2}n[\lambda_x \cos^2\theta + \lambda_y + \lambda_z \sin^2\theta] \tag{9.16}$$

$$k_Z = n[\lambda_x \sin^2\theta + \lambda_z \cos^2\theta] \tag{9.17}$$

The beam arrangement has a principal direction parallel to the Z-axis with linear stiffness k_Z. The linear stiffnesses in the X and Y directions are both k_{XY}. Therefore every direction parallel to the XY-plane is a linear principal direction with stiffness k_{XY} due to the symmetry of the beams. This fact agrees with experimental results from RCC devices.

The number of beams n only appears as a multiplier for the linear stiffnesses. Also, it is noted that the minimum number of beams required is three, due to the nature of the trigonometric identities (9.14).

Elastic Center At the elastic center, the off diagonal block of $\hat{K}_E$ must vanish, which from (9.12) is

$$\mathbf{B}_E = \sum_{i=1}^{n} \mathbf{A}_i \, \overrightarrow{E_iE}\times = \mathbf{0} \tag{9.18}$$

Introducing the vector relation (see Figure 9.6),

$$\overrightarrow{E_iE} = \overrightarrow{E_iG} + \overrightarrow{GE} = q\vec{\mathbf{e}}_{zi} + \overrightarrow{GE} \tag{9.19}$$

and (9.15) gives

$$\begin{aligned} 0 &= q\sum(\lambda_x \vec{\mathbf{e}}_{xi}\vec{\mathbf{e}}_{xi}^T \vec{\mathbf{e}}_{zi} \times + \lambda_y \vec{\mathbf{e}}_{yi}\vec{\mathbf{e}}_{yi}^T \vec{\mathbf{e}}_{zi}\times) + \sum \mathbf{A}_i \, \overrightarrow{GE} \times \\ &= q\sum(-\lambda_x \vec{\mathbf{e}}_{xi}\vec{\mathbf{e}}_{yi}^T + \lambda_y \vec{\mathbf{e}}_{yi}\vec{\mathbf{e}}_{xi}^T) + \mathbf{A}\overrightarrow{GE}\times \end{aligned} \tag{9.20}$$

which is solved as

$$\overrightarrow{GE}\times = -q\mathbf{A}^{-1}\sum(-\lambda_x \vec{\mathbf{e}}_{xi}\vec{\mathbf{e}}_{yi}^T + \lambda_y \vec{\mathbf{e}}_{yi}\vec{\mathbf{e}}_{xi}^T) \tag{9.21}$$

Using (9.14) and (9.15) yields

$$\overrightarrow{GE}\times = \tfrac{1}{2}nq(\lambda_x + \lambda_y)\cos\theta \begin{bmatrix} 0 & -\frac{1}{k_{XY}} & 0 \\ \frac{1}{k_{XY}} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{9.22}$$

Thus a point E exists that makes $\mathbf{B}_E = \mathbf{0}$ since both sides of (9.22) are skew-symmetric. Hence, the solution is

$$\overrightarrow{GE} = \frac{q(\lambda_x + \lambda_y)\cos\theta}{\lambda_x \cos^2\theta + \lambda_y + \lambda_z \sin^2\theta} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \tag{9.23}$$

so that E is on the Z-axis. This is a first in the literature of RCC devices; a closed form equation giving the location of the elastic center for a generic RCC with n beams.

Rather than using the variable intersection point G in locating E, point O, the fixed point at the physical center of the device is preferable. Using the distances in Figure 9.6 gives

$$GO = GE + EO \tag{9.24}$$

$$R = q\sin\theta \tag{9.25}$$

$$p = \frac{EO}{R} = \frac{(\lambda_z - \lambda_x)\cos\theta\sin\theta}{\lambda_x\cos^2\theta + \lambda_y + \lambda_z\sin^2\theta} \tag{9.26}$$

where the dimensionless ratio $p = \frac{EO}{R}$ is called the **projection ratio** and is the distance normalized by the central radius R which serves as a characteristic dimension of the beam arrangement. The projection ratio p is itself a function of beam structural property ratios. It indicates the amount of the projection of the elastic center from the center of the beams. Note that the location of E is independent of the number of beams n.

<u>Principal Angular Stiffness</u> For E to be an elastic center, $\mathbf{C}_E$ must also become diagonal in the global coordinate directions. Using (9.19) in (9.12) the expression for $\mathbf{C}_E$ becomes

$$\begin{aligned} \mathbf{C}_E &= \sum \mathbf{C}_i - q^2 \sum \vec{\mathbf{e}}_{zi} \times \mathbf{A}_i \vec{\mathbf{e}}_{zi} \times -\overrightarrow{GE} \times \mathbf{A}\overrightarrow{GE} \times \\ & \quad -q\overrightarrow{GE} \times (\sum \mathbf{A}_i \vec{\mathbf{e}}_{zi} \times) - q(\sum \vec{\mathbf{e}}_{zi} \times \mathbf{A}_i)\overrightarrow{GE} \times \end{aligned} \tag{9.27}$$

Next, using (9.13), (9.14), the condition for E in (9.22), and (9.25) reduces $\mathbf{C}_E$ to a diagonal matrix

$$\mathbf{C}_E = diag\{\ \kappa_{XY} \quad \kappa_{XY} \quad \kappa_Z\ \} \tag{9.28}$$

where the principal angular stiffnesses are

$$\begin{aligned} \kappa_{XY} &= \tfrac{1}{2} n \left[\mu_x \cos^2\theta + \mu_y + \mu_z \sin^2\theta + R^2 \tfrac{\lambda_x \lambda_y \sin^2\theta + \lambda_x \lambda_z + \lambda_y \lambda_z \cos^2\theta}{\lambda_x \cos^2\theta + \lambda_y + \lambda_z \sin^2\theta} \right] \\ \kappa_Z &= n \left[\mu_x \sin^2\theta + \mu_z \cos^2\theta + \lambda_y R^2 \right] \end{aligned} \tag{9.29}$$

This shows that the identified point E is indeed an elastic center since $\hat{K}_E$ is now diagonal.

The beam arrangement has a principal axis along the Z-axis with angular stiffness κ_Z. The angular stiffnesses in the X and Y directions are both κ_{XY}. Therefore every axis through E parallel to the XY-plane is an angular principal axis with stiffness κ_{XY} due to the symmetry of the beams. This fact also agrees with experimental results from RCC devices. The number of beams n only appears as a multiplier for the angular stiffnesses.

The main results of this section are summarized by,

Theorem 140 *A symmetric, conical arrangement of* three or more *identical beams has an elastic center on the cone axis. The location of the elastic center is independent of the number of beams. There is a compliant axis along the cone axis, and all lines through the elastic center perpendicular to the cone axis are compliant axes.*

9.1.2 Numerical Considerations

In this section the results of the theoretical model will be investigated in more detail. A few practically sensible approximations will prove helpful in understanding the numerical behavior of important parameters. To present the results in a compact form and minimize the number of variables, dimensionless ratios are introduced. This is facilitated by the following simplifying assumptions:

1. The beams have *symmetric cross-sections.*

2. The beams are *slender.*

These are consistent with most previous designs where the beams are quite thin. The exception being when shear pads are used instead of beams which requires a different analysis.

Symmetric Cross-sections For symmetric cross-sections several relations, regarding shear factors and stiffness entries, simplify to

$$f_s = f_{sx} = f_{sy}, \quad I = I_x = I_y, \quad \Phi = \Phi_x = \Phi_y \tag{9.30}$$

$$\lambda_{xy} = \lambda_x = \lambda_y = \frac{12EI}{L^3(1+\Phi)}, \quad \lambda_z = \frac{AE}{L} \tag{9.31}$$

$$\mu_{xy} = \mu_x = \mu_y = \frac{EI}{L}, \quad \mu_z = \frac{GJ}{L} \tag{9.32}$$

Slender Beams and Slenderness Ratio The **slenderness ratio** σ is defined as

$$\sigma = \frac{L}{\rho} \tag{9.33}$$

where $\rho = \sqrt{\frac{I}{A}}$ is the radius of gyration of the cross-section. σ is a measure of the maximum characteristic dimension to the minimum characteristic dimension. For example, a circular cross-section of diameter d_o has $\sigma = 4\frac{L}{d_o}$ and is usually considered slender when $\frac{L}{d_o} > 10$. The slender beam constraint used here is taken as

$$\sigma \geq 50 \tag{9.34}$$

The shear form factor f_s is $\frac{10}{9}$ for circular cross-sections and $\frac{3}{2}$ for square cross-sections. From (9.30) and (9.4) with Poisson's ratio $\nu \cong \frac{1}{3}$, which is typical for steels,

$$\Phi = \frac{32 f_s}{\sigma^2} \leq \begin{cases} 0.014 \text{ (circular)} \\ 0.019 \text{ (square)} \end{cases} \tag{9.35}$$

Considering (9.31), $\Phi \ll 1$ so it is reasonably assumed that $\Phi \cong 0$ for slender beams with practical cross-sections.

Projection Ratio Using the symmetrical cross-section and slenderness assumptions, the projection ratio (9.26) is expressed as a function of only two *geometrical* quantities, σ and θ,

$$p = \frac{(\sigma^2 - 12)\sin\theta\cos\theta}{12 + 12\cos^2\theta + \sigma^2\sin^2\theta} \tag{9.36}$$

Figure 9.7 shows p for $50 \leq \sigma \leq 250$. This range includes two representative experimental units for which $\sigma \cong 120$ in Drake [16] and $\sigma \cong 235$ in Nevins et al. [34].

The most striking features are the sharp peaks for relatively small beam angles. Here the maximum projection is attained at the cost of high sensitivity. This can actually be an advantage since a small adjustment of the angle can have a very large effect on the center location. After $\theta = 15°$ the projections are insensitive to the slenderness ratio σ. For $\sigma \geq 100$ this the range starts at 10°. Setting the derivative of (9.36) to zero leads to an interesting approximation for the peak value $p_{\max}$,

$$p_{\max} \cong \frac{\sigma}{10} \quad \text{at} \quad \theta \cong \frac{281°}{\sigma} \tag{9.37}$$

It is useful to introduce another dimensionless quantity

$$\alpha = \frac{L}{D} = \frac{L}{2R} \tag{9.38}$$

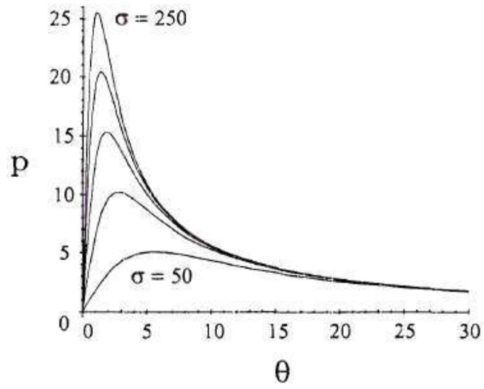


Figure 9.7: Projection ratio versus cone angle. For high slenderness ratios, the projection ratio is extremely sensitive around the maxium projection region.

which is defined as the **aspect ratio**. It is a measure of the device shape analogous to slenderness ratio. Roughly, large α indicates a slender structure and small α a compact one. Typical values in the literature for RCC devices fall within the range from 0.5 to 2.

The projection ratio p gives a good indication of the elastic center remoteness. However, since it is measured from the center of the structure it does not directly indicate whether the elastic center is actually remote. A more indicative measure is the based on the distance of the elastic center measured from the bottom of the beams at point P (refer to Figure 9.6),

$$EP = EO - \frac{L\cos\theta}{2} \tag{9.39}$$

To be a remote center EP must be positive. Normalizing by R gives

$$\frac{EP}{R} = p - \alpha\cos\theta \geq p - \alpha \tag{9.40}$$

Therefore $(p - \alpha)$ is a lower bound for the distance of the elastic center from the structure. This is also a very useful approximation for small θ since it includes the range in which the maximally remote distances occur.

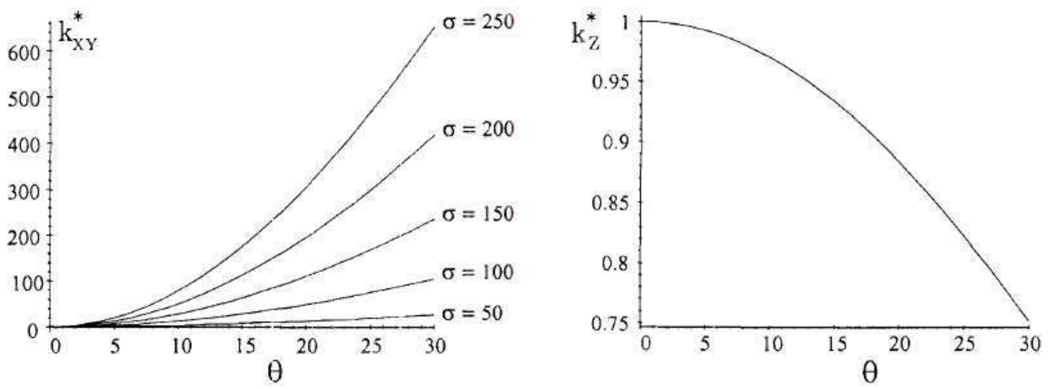


Figure 9.8: Nondimensionalized linear stiffnesses.

Nondimensional Stiffnesses The principal stiffnesses are made nondimensional by normalizing with respect to values at $\theta = 0$. These are also useful indicators of errors introduced by assuming that the conical structure stiffness is close to that of parallel beams. This compares the stiffnesses to a parallel and symmetric arrangement of beams on a cylinder with radius R. This is a limiting shape of the cone for which the elastic center occurs at the central point O. The principal stiffnesses at $\theta = 0$ are

$$k^0_{XY} = n\lambda_{xy} \qquad k^0_Z = n\lambda_z \tag{9.41}$$

$$\kappa^0_{XY} = n(\mu_{xy} + \tfrac{1}{2}\lambda_z R^2) \qquad \kappa^0_Z = n(\mu_z + \lambda_{xy} R^2) \tag{9.42}$$

The normalized principal stiffnesses are specified as $k^*_i = \frac{k_i}{k^0_i}$, etc. and for the linear stiffnesses they are

$$k^*_{XY} = 1 + \frac{\sigma^2 - 12}{24}\sin^2\theta \qquad k^*_Z = \frac{12}{\sigma^2}\sin^2\theta + \cos^2\theta \simeq \cos^2\theta \tag{9.43}$$

where the maximum error in the approximation of k^*_Z is less than 0.005 for $\sigma \geq 50$. The graphs of linear stiffnesses are shown in Figure 9.8.

The lateral linear stiffness increases very rapidly with increasing θ. This is not desirable since a low lateral stiffness allows an RCC device to more readily compensate for misalignment in assembly

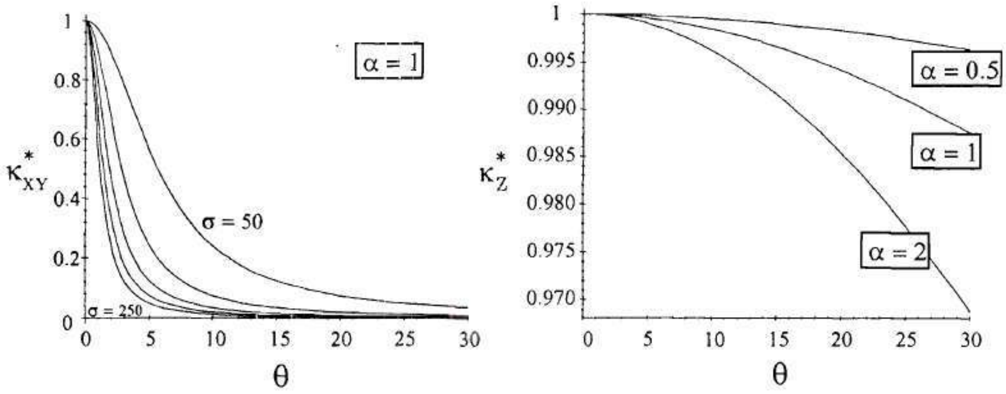


Figure 9.9: Nondimensional angular stiffnesses versus cone angle and aspect ratio. Note that κ_z^* is parctically insensitive.

tasks. However, for very small angles the increase can be manageable. For example, using the approximations (9.37) at the maximum projection $p_{\max}$ gives $k^*_{XY} \cong 2$ independently of σ.

In contrast, the longitudinal linear stiffness is relatively insensitive and decreases by less than 10% for $\theta \leq 15°$. For $\sigma \geq 50$, the longitudinal stiffness is essentially independent of σ.

The normalized lateral and longitudinal angular principal stiffnesses are

$$\begin{aligned} \kappa^*_{XY} &= \frac{\alpha^2(8-\sin^2\theta) + 12\frac{12\sin^2\theta+\sigma^2(1+\cos^2\theta)}{12(1+\cos^2\theta)+\sigma^2\sin^2\theta}}{\sigma^2+8\alpha^2} \\ &\cong \frac{24}{24+\sigma^2\sin^2\theta} \quad (\alpha \leq 4 \text{ and } \sigma \geq 50) \cong \frac{1}{k^*_{XY}+\frac{1}{2}\sin^2\theta} \cong \frac{1}{k^*_{XY}} \qquad (9.44) \\ \kappa^*_Z &= 1 + \frac{\alpha^2\sin^2\theta}{3\alpha^2+12} \qquad (9.45) \end{aligned}$$

and are plotted in Figure 9.9.

The approximation for κ^*_{XY} is inversely proportional to the lateral linear stiffness k^*_{XY}. This has significant design implications since low angular and linear lateral stiffnesses are desirable for RCC devices. One can only be achieved at the cost of the other. (The design by Drake, [16], compensates for this by introducing a compliant top structure in series with the side structure, see Figure 9.2. This decreases the total lateral angular stiffness with little effect on the linear stiffnesses.)

The approximation for lateral angular stiffness κ^*_{XY} is within 0.05 and is independent of the aspect ratio α. The stiffness drops very rapidly over the first few degrees. For example, using the approximations (9.37) at the maximum projection $p_{\max}$ gives $\kappa^*_{XY} \cong 0.5$ independently of σ.

The normalized longitudinal angular stiffness κ^*_Z is independent of σ and is essentially constant over the range for the curves $\alpha = 0.5, 1, 2, 4$.

In summary, for small values θ that occur near the maximum projection ratios $p_{\max}$, the lateral stiffnesses exhibit significant variations and the longitudinal stiffnesses are relatively constant. For increasing θ, the linear lateral stiffness increases which is undesirable for an RCC device. In contrast, the angular lateral stiffness decreases which is beneficial for the device.

Comparison to FEM Solutions A simple finite element package, called CADRE, has been used to predict the projection ratios for a sample RCC device. The package is a structural analysis program using beams modelled by linear elastic models. Non-linearity is not needed since only small deformations of the elements are considered.

The characteristics of the sample RCC are:

$$\begin{aligned} L &= 1000 \qquad R = 500 \qquad I = 3217 \qquad A = 201.0625 \\ E &= 1 \qquad \nu = \frac{1}{3} \qquad \sigma = 250 \qquad n = 3 \end{aligned}$$

The units are insignificant since this is not intended to be a practical example. Top ends of the beams with respect to the cone were fixed. The bottom ends are connected to a middle node via very stiff beams simulating the rigid bottom plate.

To determine the location of the elastic center, a small couple about a lateral axis is applied at the middle node. The corresponding displacement is solved by the FEM package. Resulting rotation

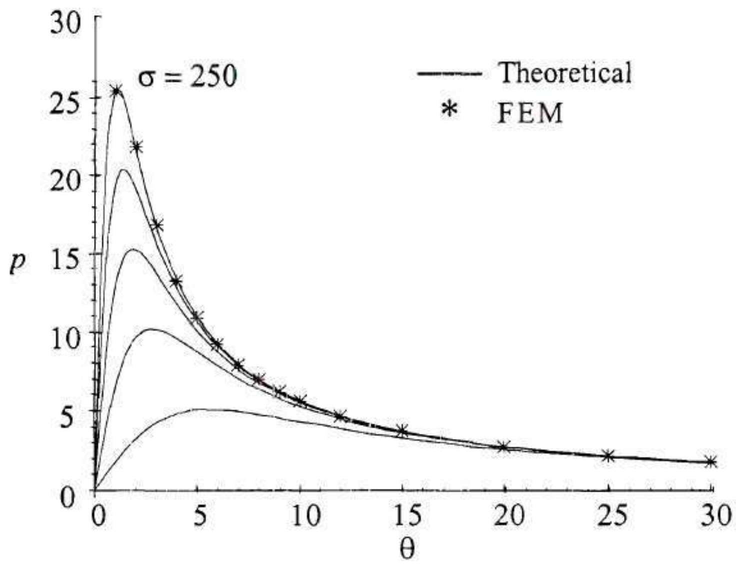


Figure 9.10: A graphical comparison of the theoretical results to FEM results for $\sigma = 250$ shows that they are virtually identical.

was shown to be a pure rotation parallel to the couple (maximum pitch was less than $2 * 10^{-4}$, occurring in the most sensitive region where the maximum projection ratios are found). The point where the axis of this rotation intersected the cone axis is the elastic center. Normalizing the result with R yielded the projection ratio. The procedure was repeated for the following θ values;

$$\theta = \left\{ 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 6, \quad 7, \quad 8, \quad 9, \quad 10, \quad 12, \quad 15, \quad 20, \quad 25, \quad 30, \right\}$$

Figure 9.10 is a graph of the projection ratio versus θ for various σ values. The percent error of theoretical projection ratios with respect to the numerical predictions is given in Figure 9.11. The maximum relative error occurs at $\theta = 1°$ and is less than 0.05%.

It is clear that the deviation between the FEM and theoretical solutions is vanishingly small and attributable wholly to numerical round-off errors. The reason for such a high degree of agreement is not surprising. Both the FEM package and the theoretical model use similar beam models and

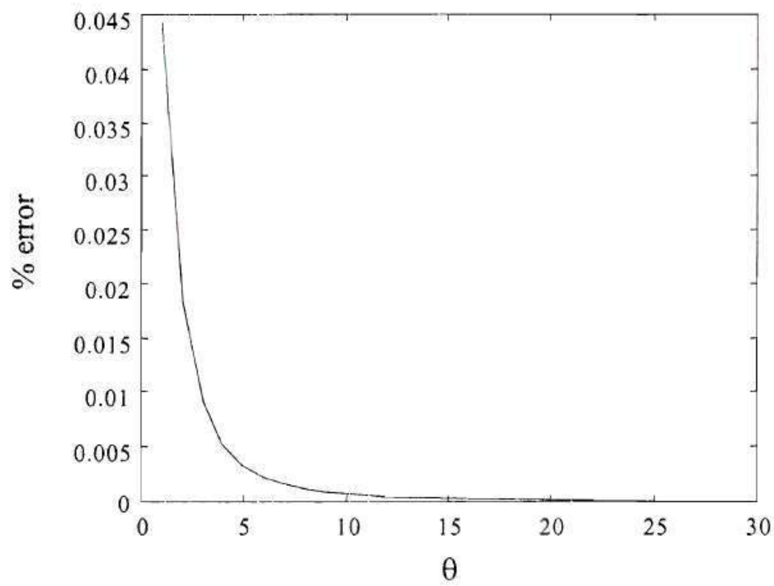


Figure 9.11: Percent error of the theoretical projection ratios with respect to the FEM predictions.

what the theoretical model achieves is simply the closed form expression of the result of assembling element stiffnesses that the FEM model does numerically. These results verify the closed form equations for the stiffnesses and the elastic center found in this study. Therefore, they can be used with confidence in the design of practical RCCs eliminating the need for sophisticated numerical packages or approximations which are now proven to be unacceptably coarse.

A detailed theoretical confirmation is presented in Section 9.2.4.

9.2 General Symmetric Constructions

In this section, the stiffness and compliance properties of general symmetric arrangements of arbitrary number of elements for parallel and serial connections are analyzed. This may be viewed as a rational generalization of the classical RCC device analyzed in the previous section. In the general case, the elements comprising the device are very general, in contrast to the use of beams in the construction of the classical RCC. In parallel connections the stiffnesses are additive, whereas in serial connections the compliances are additive. Other than this, the development and the analyses follow the same course. Since arbitrary element matrices will be used there is no immediate meaning of the term *conical.* Rather, one speaks of a symmetry axis which is analogous to the axis of revolution for generating surfaces of revolution, except the fact that generation process is discrete.

The elements are assumed to have identical stiffnesses in their local coordinates. Later, the condition on element stiffnesses are described in order to have an RCC device. Applications of the results include controlling the stiffnesses/compliances of robotic arms and/or fingers to achieve overall RCC characteristics, which can be used to aid in assembly operations.

9.2.1 Analysis of Symmetric Constructions

Construction of RCC devices with beams symmetrically placed on a cone naturally brings up the question of whether such arrangements with arbitrary elements have comparable properties. The symmetry of RCC device with beams is achieved by merely rotating an element by a constant amount about the symmetry axis, the cone axis, to obtain the next element. Figure 9.12 shows the case for three arbitrary but identical elements. The process is continued until the original element is reached. Therefore the constant angle of rotation equally divides a circle in the plane of symmetry into n equal slices (the number of elements). The original element is called the **generator**, although any of the elements can be used as a generator. The elastic centers of the elements, E_i, are equidistantly positioned on a circle in the symmetry plane.

The general process of generation is described as follows. Given an element as the generator, stiffness or compliance is selected as the element matrix depending on whether the connection is in parallel or serial. An axis of symmetry is also prescribed. Then, for a construction with n elements, the remaining element matrices are generated by rotations of the generator matrix about the symmetry axis. If the generator element is numbered as 1, then the i^{th} element is obtained by

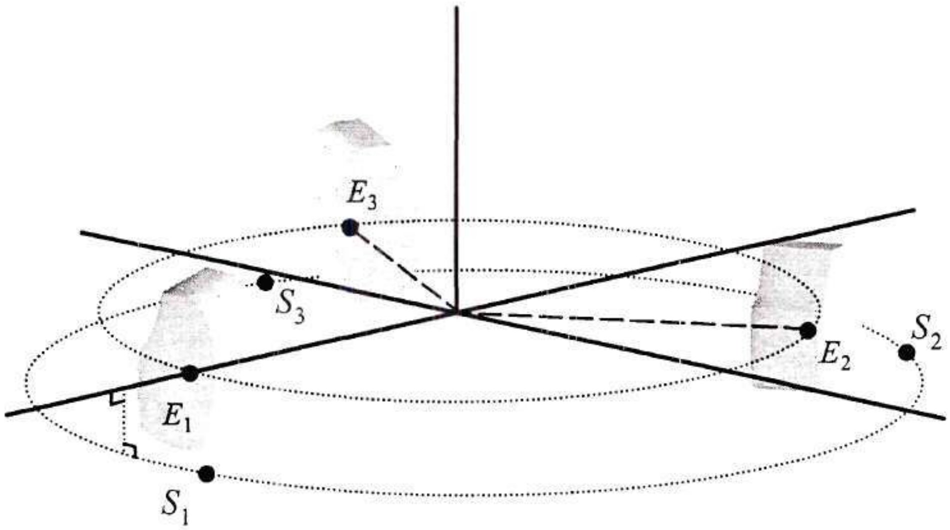


Figure 9.12: Rotational symmetry generation by using an arbitrary elastic element. Vertical axis is the symmetry axis. All centers are located on respective circles centered at a point on the symmetry axis.

a rotation through an angle of

$$\beta_i = \frac{2\pi}{n}(i-1) = \beta(i-1) \qquad\qquad i = 1, ..., n \tag{9.46}$$

where $\beta = \frac{2\pi}{n}$ is called the **angle of generation**. The centers of stiffness, compliance, etc. for all elements are found on circles on symmetry planes which are in general not coincident, Figure 9.12.

Let $\hat{G}_1$ be a generic element matrix for the generator, *which can either represent the stiffness or compliance.* Let O be a point on the symmetry axis which is assumed to be given. Let G_1 be given in terms of its 3×3 submatrices as

$$\hat{G}_{1/O} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \tag{9.47}$$

with respect to the point O.

Let $\vec{\mathbf{u}}_1$ be a unit vector along the symmetry axis. Then, the rotation matrix to generate the i^{th}

element, $\mathbf{R}_i$, is given by

$$\mathbf{R}_i = \mathbf{R}\left[\beta(i-1)\vec{\mathbf{u}}_1\right] = \mathbf{I} + \sin[\beta(i-1)]\vec{\mathbf{u}}_1 \times + (1-\cos[\beta(i-1)])\, \vec{\mathbf{u}}_1 \times \vec{\mathbf{u}}_1 \times \tag{9.48}$$

The rigid body rotations for spatial quantities is performed by a spatial transformation matrix, $\hat{X}_i$,

$$\hat{X}_i = \begin{bmatrix} \mathbf{R}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_i \end{bmatrix} \tag{9.49}$$

which, in this case, is the same for both stiffness and compliance. The constitutive matrix of the i^{th} element is

$$\hat{G}_i = \hat{X}_i \hat{G}_1 \hat{X}_i^T \tag{9.50}$$

The total constitutive matrix of the connection, whether parallel or serial, is found by summing all the element matrices.

$$\hat{G} = \sum_{i=1}^{n} \hat{X}_i \hat{G}_1 \hat{X}_i^T \tag{9.51}$$

$$= \sum_{i=1}^{n} \begin{bmatrix} \mathbf{R}_i\mathbf{A}\mathbf{R}_i^T & \mathbf{R}_i\mathbf{B}\mathbf{R}_i^T \\ \mathbf{R}_i\mathbf{B}^T\mathbf{R}_i^T & \mathbf{R}_i\mathbf{C}\mathbf{R}_i^T \end{bmatrix} = \begin{bmatrix} \mathbf{A}^\circ & \mathbf{B}^\circ \\ \mathbf{B}^{\circ T} & \mathbf{C}^\circ \end{bmatrix} \tag{9.52}$$

where $\hat{G}$ is the net constitutive matrix of the connection with respect to O. For parallel connections $\hat{G}$ is a stiffness matrix and for serial connections $\hat{G}$ is a compliance matrix.

Consider an arbitrary 3×3 matrix $\mathbf{Q}$. Then, the summation $\sum_{i=1}^{n} \mathbf{R}_i\mathbf{Q}\mathbf{R}_i^T$ with $\mathbf{R}_i$ as defined in (9.48) is performed only over scalar trigonometric functions, and the identities (9.14) simplify the summation to

$$\begin{aligned} \sum_{i=1}^{n} \mathbf{R}_i\mathbf{Q}\mathbf{R}_i^T &= n\,[\mathbf{Q} + \mathbf{Q}\vec{\mathbf{u}}_1 \times \vec{\mathbf{u}}_1 \times + \vec{\mathbf{u}}_1 \times \vec{\mathbf{u}}_1 \times \mathbf{Q} + \\ &\quad + \tfrac{3}{2}\vec{\mathbf{u}}_1 \times \vec{\mathbf{u}}_1 \times \mathbf{Q}\vec{\mathbf{u}}_1 \times \vec{\mathbf{u}}_1 \times - \tfrac{1}{2}\vec{\mathbf{u}}_1 \times \mathbf{Q}\vec{\mathbf{u}}_1 \times] \end{aligned} \tag{9.53}$$

Theorem 141 *For any arbitrary symmetric matrix* $\mathbf{Q}$ *let* $\mathbf{Q}^\circ = \sum_{i=1}^{n} \mathbf{R}_i\mathbf{Q}\mathbf{R}_i^T$ *as in (9.53). Then,*

1. $\vec{\mathbf{u}}_1$ *is an eigenvector of* $\mathbf{Q}^\circ$ *with a corresponding eigenvalue of* $q_1^\circ = n\left(\vec{\mathbf{u}}_1^T \mathbf{Q} \vec{\mathbf{u}}_1\right)$.

2. *The remaining two eigenvalues of* $\mathbf{Q}^\circ$ *are identical. Due to this double eigenvalue, any vector in the plane perpendicular to* $\vec{\mathbf{u}}_1$ *is also an eigenvector. If* $\vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3$ *is an orthonormal set of eigenvectors perpendicular to* $\vec{\mathbf{u}}_1$*, then* $q_2^\circ = q_3^\circ = \frac{1}{2}n(\vec{\mathbf{u}}_2^T \mathbf{Q}\vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{Q}\vec{\mathbf{u}}_3)$.

Proof. To see that $\vec{\mathbf{u}}_1$ is an eigenvector, just postmultiply (9.53) by $\vec{\mathbf{u}}_1$ which results in

$$\left[\sum_{i=1}^{n} \mathbf{R}_i \mathbf{Q} \mathbf{R}_i^T\right] \vec{\mathbf{u}}_1 = n\left[\mathbf{Q} + \vec{\mathbf{u}}_1 \times \vec{\mathbf{u}}_1 \times \mathbf{Q}\right] \vec{\mathbf{u}}_1 \tag{9.54}$$

$$\mathbf{Q}^\circ \vec{\mathbf{u}}_1 = n\left[\mathbf{I} + \vec{\mathbf{u}}_1 \times \vec{\mathbf{u}}_1 \times\right] \mathbf{Q} \vec{\mathbf{u}}_1 \tag{9.55}$$

Considering the vector identity

$$\mathbf{I} + \vec{\mathbf{u}}_1 \times \vec{\mathbf{u}}_1 \times = \vec{\mathbf{u}}_1 \vec{\mathbf{u}}_1^T \tag{9.56}$$

reduces (9.55) to

$$\mathbf{Q}^\circ \vec{\mathbf{u}}_1 = n \vec{\mathbf{u}}_1 \vec{\mathbf{u}}_1^T \mathbf{Q} \vec{\mathbf{u}}_1 = (n \vec{\mathbf{u}}_1^T \mathbf{Q} \vec{\mathbf{u}}_1) \vec{\mathbf{u}}_1 \tag{9.57}$$

Therefore,

$$q_1^\circ = n\left(\vec{\mathbf{u}}_1^T \mathbf{Q} \vec{\mathbf{u}}_1\right) \tag{9.58}$$

is the eigenvalue corresponding to $\vec{\mathbf{u}}_1$ which proves the first part of the theorem. Now, since $\sum_{i=1}^{n} \mathbf{R}_i \mathbf{Q} \mathbf{R}_i^T$ is symmetric, by theorems of linear algebra, there exist two more eigenvectors orthogonal to each other and in the plane perpendicular to $\vec{\mathbf{u}}_1$. Let these be denoted by $\vec{\mathbf{u}}_2$ and $\vec{\mathbf{u}}_3$ and the corresponding eigenvalues be q_2° and q_3°. For the second part of the problem, it is sufficient to show the multiplicity of the remaining eigenvalues. To see this, first reduce (9.53) using the identity (9.56) to the following.

$$\mathbf{Q}^\circ = \tfrac{1}{2}n\left[\mathbf{Q} - \mathbf{Q}\vec{\mathbf{u}}_1\vec{\mathbf{u}}_1^T - \vec{\mathbf{u}}_1\vec{\mathbf{u}}_1^T\mathbf{Q} + 3\vec{\mathbf{u}}_1\vec{\mathbf{u}}_1^T\mathbf{Q}\vec{\mathbf{u}}_1\vec{\mathbf{u}}_1^T - \vec{\mathbf{u}}_1 \times \mathbf{Q}\vec{\mathbf{u}}_1 \times\right] \tag{9.59}$$

$$= \tfrac{1}{2}n\left[\mathbf{Q} + q_1^\circ\vec{\mathbf{u}}_1\vec{\mathbf{u}}_1^T - \vec{\mathbf{u}}_1 \times \mathbf{Q}\vec{\mathbf{u}}_1 \times\right] \tag{9.60}$$

The eigenvalues of $\mathbf{Q}^\circ$ can be given as $\vec{\mathbf{u}}_i^T\mathbf{Q}^\circ\vec{\mathbf{u}}_i$ since $\vec{\mathbf{u}}_i$ is an eigenvector. Applying $\vec{\mathbf{u}}_2, \vec{\mathbf{u}}_3$ results in the corresponding eigenvalues as

$$q_2^\circ = q_3^\circ = \tfrac{1}{2}n(\vec{\mathbf{u}}_2^T\mathbf{Q}\vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T\mathbf{Q}\vec{\mathbf{u}}_3) \tag{9.61}$$

which completes the proof.

■

The theorem allows one to express $\sum_{i=1}^n \mathbf{R}_i\mathbf{Q}\mathbf{R}_i^T$ in terms of $\vec{\mathbf{u}}_1$ only. Using the orthogonal decomposition and the identity $\vec{\mathbf{u}}_1\vec{\mathbf{u}}_1^T + \vec{\mathbf{u}}_2\vec{\mathbf{u}}_2^T + \vec{\mathbf{u}}_3\vec{\mathbf{u}}_3^T = \mathbf{I}$,

$$\sum_{i=1}^{n}\mathbf{R}_i\mathbf{Q}\mathbf{R}_i^T = \sum_j q_j^\circ\vec{\mathbf{u}}_j\vec{\mathbf{u}}_j^T = (q_1^\circ - q_2^\circ)\vec{\mathbf{u}}_1\vec{\mathbf{u}}_1^T + q_2^\circ\mathbf{I} \tag{9.62}$$

Note that this is exactly the same form encountered in the classical RCC analysis. However, unlike classical RCCs, (9.62) concerns more general structures obtained by arbitrary elements rather than beams.

Now the forms of $\mathbf{A}^\circ$ and $\mathbf{C}^\circ$ in equation (9.52) are determined. Let the eigenvalues of $\mathbf{A}^\circ$ and $\mathbf{C}^\circ$ be k_j and κ_j, respectively. Then,

$$k_1 = n\vec{\mathbf{u}}_1^T\mathbf{A}\vec{\mathbf{u}}_1 \qquad k_2 = k_3 = \tfrac{1}{2}n(\vec{\mathbf{u}}_2^T\mathbf{A}\vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T\mathbf{A}\vec{\mathbf{u}}_3) \tag{9.63}$$

$$\kappa_1 = n\vec{\mathbf{u}}_1^T\mathbf{C}\vec{\mathbf{u}}_1 \qquad \kappa_2 = \kappa_3 = \tfrac{1}{2}n(\vec{\mathbf{u}}_2^T\mathbf{C}\vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T\mathbf{C}\vec{\mathbf{u}}_3) \tag{9.64}$$

and

$$\mathbf{A}^\circ = (k_1 - k_2)\vec{\mathbf{u}}_1\vec{\mathbf{u}}_1^T + k_2\mathbf{I} \qquad \mathbf{C}^\circ = (\kappa_1 - \kappa_2)\vec{\mathbf{u}}_1\vec{\mathbf{u}}_1^T + \kappa_2\mathbf{I} \tag{9.65}$$

The off-diagonal block of (9.52), $\mathbf{B}^\circ$, is in general non-symmetric. But, if one considers the symmetric and skew-symmetric parts of the matrix $\mathbf{B}$, then the following can be written.

$$\mathbf{B}^\circ = \sum_i \mathbf{R}_i\mathbf{B}\mathbf{R}_i^T = \sum_i \mathbf{R}_i\mathbf{B}_{\text{sym}}\mathbf{R}_i^T + \sum_i \mathbf{R}_i\mathbf{B}_{\text{skew}}\mathbf{R}_i^T \tag{9.66}$$

$$= \sum_i \mathbf{R}_i\mathbf{B}_{\text{sym}}\mathbf{R}_i^T + \sum_i \mathbf{R}_i\vec{\mathbf{b}}\times\mathbf{R}_i^T \tag{9.67}$$

where $\mathbf{B}_{\text{skew}} = \vec{\mathbf{b}}\times$ can be written since the matrix is 3×3. The first summation in the last equation has the same form as $\mathbf{A}^\circ$ or $\mathbf{D}^\circ$. For the second, $\mathbf{R}_i\vec{\mathbf{b}}\times\mathbf{R}_i^T = \left[\mathbf{R}_i\vec{\mathbf{b}}\right]\times$. Using the expression for $\mathbf{R}_i$, (9.48), and the trigonometric identities (9.14) yields

$$\sum_i \mathbf{R}_i\vec{\mathbf{b}}\times\mathbf{R}_i^T = \left[\sum_i \mathbf{R}_i\vec{\mathbf{b}}\right]\times = n(\vec{\mathbf{u}}_1^T\vec{\mathbf{b}})\vec{\mathbf{u}}_1\times \tag{9.68}$$

Hence, the form of $\mathbf{B}^\circ$ is found to be

$$\mathbf{B}^\circ = (\mu_1-\mu_2)\vec{\mathbf{u}}_1\vec{\mathbf{u}}_1^T + \mu_2\mathbf{I} + n(\vec{\mathbf{u}}_1^T\vec{\mathbf{b}})\vec{\mathbf{u}}_1\times \tag{9.69}$$

where μ_j are the eigenvalues of $\mathbf{B}^\circ_{\text{sym}}$ and given by

$$\mu_1 = n\vec{\mathbf{u}}_1^T\mathbf{B}\vec{\mathbf{u}}_1 \qquad \mu_2=\mu_3=\tfrac{1}{2}n(\vec{\mathbf{u}}_2^T\mathbf{B}\vec{\mathbf{u}}_2+\vec{\mathbf{u}}_3^T\mathbf{B}\vec{\mathbf{u}}_3) \tag{9.70}$$

Here, $\mathbf{B}$ is used instead of $\mathbf{B}_{\text{sym}}$ since $\vec{\mathbf{u}}^T\mathbf{B}_{\text{sym}}\vec{\mathbf{u}} = \vec{\mathbf{u}}^T\mathbf{B}\vec{\mathbf{u}}$ for any $\vec{\mathbf{u}}$.

The location of the centers of stiffness, S, and compliance, C, are respectively given by

$$\vec{\mathbf{r}}_S^\circ = [\mathbf{A}^\circ - \text{tr}(\mathbf{A}^\circ)\mathbf{I}]^{-1}2\vec{\mathbf{b}}^\circ \tag{9.71}$$

$$\vec{\mathbf{r}}_C^\circ = -[\mathbf{C}^\circ - \text{tr}(\mathbf{C}^\circ)\mathbf{I}]^{-1}2\vec{\mathbf{b}}^\circ \tag{9.72}$$

depending on whether $\hat{G}$ represents the stiffness or the compliance, see Chapter 3 or Loncaric [32], Ciblak and Lipkin [?]. Here,

$$\vec{\mathbf{b}}^\circ\times = \tfrac{1}{2}(\mathbf{B}^\circ - \mathbf{B}^{\circ T}) = n(\vec{\mathbf{u}}_1^T\vec{\mathbf{b}})\vec{\mathbf{u}}_1\times \qquad \text{so} \qquad \vec{\mathbf{b}}^\circ = n(\vec{\mathbf{u}}_1^T\vec{\mathbf{b}})\vec{\mathbf{u}}_1 \tag{9.73}$$

Inserting (9.65) and (9.73) in (9.71) and (9.72), and performing the operations returns

$$\vec{\mathbf{r}}_S^\circ = -\frac{n\vec{\mathbf{u}}_1^T\vec{\mathbf{b}}}{k_2}\vec{\mathbf{u}}_1 = -\frac{\vec{\mathbf{u}}_1^T\vec{\mathbf{b}}}{k_2^*}\vec{\mathbf{u}}_1 \tag{9.74}$$

$$\vec{\mathbf{r}}_C^\circ = \frac{n\vec{\mathbf{u}}_1^T\vec{\mathbf{b}}}{\kappa_2}\vec{\mathbf{u}}_1 = \frac{\vec{\mathbf{u}}_1^T\vec{\mathbf{b}}}{\kappa_2^*}\vec{\mathbf{u}}_1 \tag{9.75}$$

$\vec{\mathbf{r}}_S^\circ$ is for parallel connections and $\vec{\mathbf{r}}_C^\circ$ is for serial connections. In either case the center is on the symmetry axis. The starred quantities are

$$k_2^* = \tfrac{k_2}{n} = \tfrac{1}{2}(\vec{\mathbf{u}}_2^T\mathbf{A}\vec{\mathbf{u}}_2+\vec{\mathbf{u}}_3^T\mathbf{A}\vec{\mathbf{u}}_3) = \tfrac{1}{2}(\text{tr}(\mathbf{A})-\vec{\mathbf{u}}_1^T\mathbf{A}\vec{\mathbf{u}}_1) \tag{9.76}$$

$$\kappa_2^* \;=\; \tfrac{\kappa_2}{n} = \tfrac{1}{2}(\vec{\mathbf{u}}_2^T \mathbf{C} \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{C} \vec{\mathbf{u}}_3) = \tfrac{1}{2}(\mathrm{tr}(\mathbf{C}) - \vec{\mathbf{u}}_1^T \mathbf{C} \vec{\mathbf{u}}_1) \tag{9.77}$$

showing that the center is independent of the number of elements used for construction. Also, the center is independent of $\mathbf{C}$ ($\mathbf{A}$) in case of stiffness (compliance). The starred quantities are the constitutive properties in the direction of the symmetry axis.

The equations for parallel and serial cases may now be given separately for simplicity. So, let the quantities for a parallel connection be denoted by the superscript p and those for a serial connection by the superscript s. Also, let the partitions for the generator matrices be given as

$$\hat{K}^p = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \qquad \hat{C}^s = \begin{bmatrix} \mathbf{D} & \mathbf{E}^T \\ \mathbf{E} & \mathbf{F} \end{bmatrix} \tag{9.78}$$

Note that $\hat{K}^p$ and $\hat{C}^s$ are not related quantities. The net stiffness of parallel connections is still described by (9.65) and 9.69. The net compliance for serial connections is described by

$$\mathbf{D}^\circ \;=\; (a_1 - a_2)\vec{\mathbf{u}}_1 \vec{\mathbf{u}}_1^T + a_2 \mathbf{I} \tag{9.79}$$

$$\mathbf{E}^\circ \;=\; (\eta_1 - \eta_2)\vec{\mathbf{u}}_1 \vec{\mathbf{u}}_1^T + \eta_2 \mathbf{I} + n(\vec{\mathbf{u}}_1^T \vec{\mathbf{e}})\vec{\mathbf{u}}_1 \times \tag{9.80}$$

$$\mathbf{F}^\circ \;=\; (\alpha_1 - \alpha_2)\vec{\mathbf{u}}_1 \vec{\mathbf{u}}_1^T + \alpha_2 \mathbf{I} \tag{9.81}$$

where

$$a_1 \;=\; n\vec{\mathbf{u}}_1^T \mathbf{D} \vec{\mathbf{u}}_1 \qquad a_2 = \tfrac{1}{2}n(\vec{\mathbf{u}}_2^T \mathbf{D} \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{D} \vec{\mathbf{u}}_3) \tag{9.82}$$

$$\eta_1 \;=\; n\vec{\mathbf{u}}_1^T \mathbf{E} \vec{\mathbf{u}}_1 \qquad \eta_2 = \tfrac{1}{2}n(\vec{\mathbf{u}}_2^T \mathbf{E} \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{E} \vec{\mathbf{u}}_3) \tag{9.83}$$

$$\alpha_1 \;=\; n\vec{\mathbf{u}}_1^T \mathbf{F} \vec{\mathbf{u}}_1 \qquad \alpha_2 = \tfrac{1}{2}n(\vec{\mathbf{u}}_2^T \mathbf{F} \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{F} \vec{\mathbf{u}}_3) \tag{9.84}$$

and

$$\vec{\mathbf{r}}_C^\circ = -\frac{n\vec{\mathbf{u}}_1^T \vec{\mathbf{e}}}{\alpha_2}\vec{\mathbf{u}}_1 = -\frac{\vec{\mathbf{u}}_1^T \vec{\mathbf{e}}}{\alpha_2^*}\vec{\mathbf{u}}_1 \tag{9.85}$$

The net connection matrices can now be transformed to their respective centers. The results are as follows.

$$\hat{K}_S^p = \begin{bmatrix} (k_1-k_2)\vec{\mathbf{u}}\vec{\mathbf{u}}^T + k_2\mathbf{I} & (\mu_1-\mu_2)\vec{\mathbf{u}}\vec{\mathbf{u}}^T + \mu_2\mathbf{I} \\ (\mu_1-\mu_2)\vec{\mathbf{u}}\vec{\mathbf{u}}^T + \mu_2\mathbf{I} & [\kappa_1 - (\kappa_2 - n\frac{(\vec{\mathbf{u}}^T\vec{\mathbf{b}})^2}{k_2^*})]\vec{\mathbf{u}}\vec{\mathbf{u}}^T + (\kappa_2 - n\frac{(\vec{\mathbf{u}}^T\vec{\mathbf{b}})^2}{k_2^*})\mathbf{I} \end{bmatrix} \tag{9.86}$$

$$\hat{C}_C^s = \begin{bmatrix} [a_1 - (a_2 - n\frac{(\vec{\mathbf{u}}^T\vec{\mathbf{e}})^2}{\alpha_2^*})]\vec{\mathbf{u}}\vec{\mathbf{u}}^T + (a_2 - n\frac{(\vec{\mathbf{u}}^T\vec{\mathbf{e}})^2}{\alpha_2^*})\mathbf{I} & (\eta_1-\eta_2)\vec{\mathbf{u}}\vec{\mathbf{u}}^T + \eta_2\mathbf{I} \\ (\eta_1-\eta_2)\vec{\mathbf{u}}\vec{\mathbf{u}}^T + \eta_2\mathbf{I} & [\alpha_1-\alpha_2]\,\vec{\mathbf{u}}\vec{\mathbf{u}}^T + \alpha_2\mathbf{I} \end{bmatrix} \tag{9.87}$$

where now the symmetry axis is $\vec{\mathbf{u}}$. All submatrices are symmetric as expected. But, what is more, all submatrices have the same eigenvectors. For both connections at their respective centers, when one coordinate axis is aligned parallel to $\vec{\mathbf{u}}$ the net constitutive matrices have the form

$$\hat{K}_S^p = \begin{bmatrix} k_1 & 0 & 0 & \mu_1 & 0 & 0 \\ 0 & k_2 & 0 & 0 & \mu_2 & 0 \\ 0 & 0 & k_2 & 0 & 0 & \mu_2 \\ \mu_1 & 0 & 0 & \kappa_1 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 & \kappa_2 - n\frac{(\vec{\mathbf{u}}^T\vec{\mathbf{b}})^2}{k_2^*} & 0 \\ 0 & 0 & \mu_2 & 0 & 0 & \kappa_2 - n\frac{(\vec{\mathbf{u}}^T\vec{\mathbf{b}})^2}{k_2^*} \end{bmatrix} \tag{9.88}$$

$$\hat{C}_C^s = \begin{bmatrix} a_1 & 0 & 0 & \eta_1 & 0 & 0 \\ 0 & a_2 - n\frac{(\vec{\mathbf{u}}^T\vec{\mathbf{e}})^2}{\alpha_2^*} & 0 & 0 & \eta_2 & 0 \\ 0 & 0 & a_2 - n\frac{(\vec{\mathbf{u}}^T\vec{\mathbf{e}})^2}{\alpha_2^*} & 0 & 0 & \eta_2 \\ \eta_1 & 0 & 0 & \alpha_1 & 0 & 0 \\ 0 & \eta_2 & 0 & 0 & \alpha_2 & 0 \\ 0 & 0 & \eta_2 & 0 & 0 & \alpha_2 \end{bmatrix} \tag{9.89}$$

From linear algebra, it can be shown that the inverse matrices, $\hat{C}_S^p$ and $\hat{K}_C^s$, also have the same form. Therefore, for both parallel and serial connections, the rotational symmetry devices have coinciding centers of stiffness and compliance. Since, in either case both $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{B}^T\mathbf{C}$ are symmetric, by definition, the center of elasticity and a co-center of elasticity coincide with the

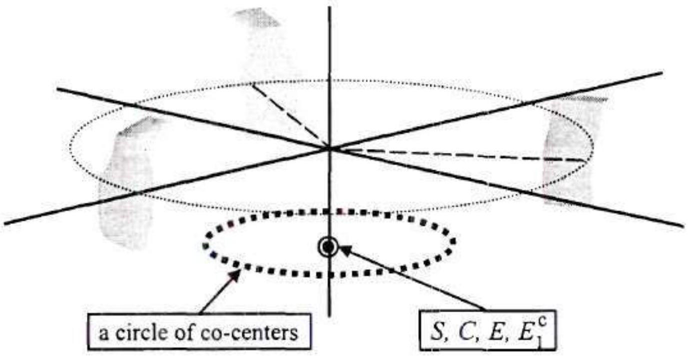


Figure 9.13: Distribution of centers for a general symmetric construction.

centers of stiffness/compliance. Also, due to the double eigenvalue of all submatrices there exists a circle of co-centers in $\vec{\mathbf{u}}_2\vec{\mathbf{u}}_3$-plane centered at the elastic center. The radius of the co-center circle is given by

$$r = \sqrt{\frac{1}{2k_1k_2}\left[(\mu_1-\mu_2)^2 - k_1\kappa_1 - k_2\kappa_2 + n^2(\vec{\mathbf{u}}^T\vec{\mathbf{b}})^2\right]} \qquad \text{(parallel)} \qquad (9.90)$$

$$r = \sqrt{\frac{1}{2\alpha_1\alpha_2}\left[(\eta_1-\eta_2)^2 - \alpha_1 a_1 - \alpha_2 a_2 + n^2(\vec{\mathbf{u}}^T\vec{\mathbf{e}})^2\right]} \qquad \text{(serial)} \qquad (9.91)$$

which, depending on particular values of the parameters, may be real or complex. The limit case of real co-center circles is $r = 0$, when all co-centers collapses to one point, E. Hence, the following theorem, illustrated in Figure 9.13, is proven.

Theorem 142 *For a rotational symmetry device with n arbitrary elements connected in series or parallel, all the centers of stiffness, compliance and elasticity coincide at a point E on the symmetry axis. There exists a co-center at E, and a circle of co-centers in the plane of symmetry centered at E, whose radius may range from zero to arbitrarily large numbers depending on the generator constitutive properties.*

9.2.2 Generalized RCC Definitions

A pure translation parallel to the symmetry axis (vertical axis in Figure 9.13) creates a parallel force through E and a parallel couple, see (9.88), which is an eigenwrench by definition. Similarly, any translation perpendicular to the symmetry axis creates a parallel force through E and a parallel couple. Therefore, there exist eigenwrenches, all with the same pitch (compare (9.93) and (9.94 below), through E in every direction perpendicular to the symmetry axis.

A pure couple parallel to the symmetry axis produces a parallel translation and a parallel rotation through E. Again, by definition, this is an eigentwist. Similarly, any couple perpendicular to the symmetry axis creates a parallel translation and a parallel rotation through E. Therefore, there exist eigentwists, all with the same pitch, through E in every direction perpendicular to the symmetry axis.

Unlike the classical RCCs, these eigenscrews of such devices do not necessarily have zero pitches. However, everything else is the same. Both the classical RCCs and general symmetric devices have eigenscrews through E. All centers coalesce at E. There exists a plane of constant pitch eigenscrews (zero pitch for RCCs). The eigenscrews are also the principal screws. The principal pitches for each connection are as follows.

$$h^p_{f1} = -h^p_{\gamma 1} = \frac{\mu_1}{k_1} = \frac{\vec{\mathbf{u}}_1^T \mathbf{B} \vec{\mathbf{u}}_1}{\vec{\mathbf{u}}_1^T \mathbf{A} \vec{\mathbf{u}}_1} \tag{9.92}$$

$$h^p_{f2} = -h^p_{\gamma 2} = \frac{\mu_2}{k_2} = \frac{\vec{\mathbf{u}}_2^T \mathbf{B} \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{B} \vec{\mathbf{u}}_3}{\vec{\mathbf{u}}_2^T \mathbf{A} \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{A} \vec{\mathbf{u}}_3} \tag{9.93}$$

$$h^p_{f3} = -h^p_{\gamma 3} = \frac{\mu_3}{k_3} = \frac{\vec{\mathbf{u}}_2^T \mathbf{B} \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{B} \vec{\mathbf{u}}_3}{\vec{\mathbf{u}}_2^T \mathbf{A} \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{A} \vec{\mathbf{u}}_3} \tag{9.94}$$

$$h^s_{f1} = -h^s_{\gamma 1} = \frac{\eta_1}{\alpha_1} = \frac{\vec{\mathbf{u}}_1^T \mathbf{E} \vec{\mathbf{u}}_1}{\vec{\mathbf{u}}_1^T \mathbf{F} \vec{\mathbf{u}}_1} \tag{9.95}$$

$$h^s_{f2} = -h^s_{\gamma 2} = \frac{\eta_2}{\alpha_2} = \frac{\vec{\mathbf{u}}_2^T \mathbf{E} \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{E} \vec{\mathbf{u}}_3}{\vec{\mathbf{u}}_2^T \mathbf{F} \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{F} \vec{\mathbf{u}}_3} \tag{9.96}$$

$$h^s_{f3} = -h^s_{\gamma 3} = \frac{\eta_3}{\alpha_3} = \frac{\vec{\mathbf{u}}_2^T \mathbf{E} \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{E} \vec{\mathbf{u}}_3}{\vec{\mathbf{u}}_2^T \mathbf{F} \vec{\mathbf{u}}_2 + \vec{\mathbf{u}}_3^T \mathbf{F} \vec{\mathbf{u}}_3} \tag{9.97}$$

The principal pitches are also independent of the number of elements used in the connections.

The co-eigenscrew structure at E is similar. A pure force through E parallel to the symmetry axis yields a parallel translation and a parallel rotation through E, which make up a co-eigentwist at E. Any force through E and perpendicular to the symmetry axis results in a parallel translation and a parallel rotation through E. So, for the co-eigentwist system generated by E, there exist co-eigentwists, all with the same pitch, through E and perpendicular to the symmetry axis.

Similarly, a pure rotation through E parallel to the symmetry axis yields a parallel force through E and a parallel couple, which make up a co-eigenwrench at E. Any rotation through E and perpendicular to the symmetry axis results in a parallel force through E and a parallel couple. So, for the co-eigenwrench system generated by E, there exist co-eigenwrenches, all with the same pitch, through E and perpendicular to the symmetry axis.

Again, in contrast to the classical RCCs, the co-eigenscrews at E of general rotational symmetry devices are not infinite pitch screws. The co-eigenscrew pitches can be easily determined from (9.88) and (9.89). Their forms are similar to those given in (9.92) through (9.97), except that one replaces $\mathbf{A}$ with $\mathbf{C}$, $\mathbf{F}$ with $\mathbf{D}$, and inverts each equation, so that all μ_i and η_i appear in the denominators. As a result, for classical RCCs, $\mu_i = \eta_i = 0$ leading to zero pitch eigenscrews and infinite pitch co-eigenscrews at E.

These observations hint that the classical RCC devices can be regarded as a special case of general symmetric devices, obtained by simply requiring the principal pitches of the eigenscrews to be zero.

One may go in the other direction, too, i.e. generalize it instead of specialize. For this, the double eigenvalue condition on (9.88) or (9.89) can be removed which gives a more general device with a constitutive matrix at E of the form

$$\hat{G}_E = \begin{bmatrix} \times & 0 & 0 & \times & 0 & 0 \\ 0 & \times & 0 & 0 & \times & 0 \\ 0 & 0 & \times & 0 & 0 & \times \\ \times & 0 & 0 & \times & 0 & 0 \\ 0 & \times & 0 & 0 & \times & 0 \\ 0 & 0 & \times & 0 & 0 & \times \end{bmatrix} \tag{9.98}$$

This device only lacks a symmetry. However, its eigenscrews, which are also the principal screws, are through E, and it responds to pure translations and couples in a manner similar to general symmetric devices.

From these results, the following definition is proposed.

Definition 143 *Consider an elastic connection with a constitutive matrix at E of the form (9.98).*

- *If the block matrices of (9.98) has general eigenvalues, then the device is called a* ***general RCC device.***
- *If the block matrices all have a double eigenvalue corresponding to the same plane, then it is called a* ***general symmetric RCC device.***
- *If all the block matrices have triple eigenvalues then the device is called* ***general spherical RCC device.***

Note that although the most general RCC is defined in Definition 143, the construction method outlined in this study can generate only general symmetric ($\lambda_2 = \lambda_3$) and spherical RCCs ($\lambda_1 = \lambda_2 = \lambda_3$), due to the symmetry inherent in the construction. The devices defined above can be specialized by requiring the eigenscrew pitches to be zero, or co-eigenscrew pitches at E to be infinite. This is equivalent to requiring the off-diagonals of the constitutive matrices to be zero. Hence, one gets the following.

Definition 144 *If an elastic connection defined in Definition (143) has zero off-diagonal blocks in (9.98), then the term* general *is dropped. Thus, one gets* ***RCC****,* ***symmetric RCC*** *and* ***spherical RCC devices****.*

Note that **symmetric RCC device** is synonymous with **classical RCC device**.

- A general RCC is described by **15** independent parameters: $3+3+3=9$ for the eigenvalues of the block matrices in (9.88), 3 for the orientation of principal axes, 3 for the location of the elastic center.

- A general symmetric RCC is described by **11** parameters: $2+2+2=6$ for the eigenvalues of the block matrices in (9.88), 2 for the direction of the symmetry axis, 3 for the location of the elastic center.

- A general spherical RCC is described by **6** parameters: $1+1+1=3$ for the eigenvalues of the block matrices in (9.88), 3 for the location of the elastic center. Every direction is a symmetry axis.

- An RCC is described by **12** parameters: $3+3=6$ for the eigenvalues of the diagonal block matrices in (9.88), 3 for the orientation of principal axes, 3 for the location of the elastic center.

- A symmetric (classical) RCC is described by **9** parameters: $2+2=4$ for the eigenvalues of the diagonal block matrices in (9.88), 2 for the direction of the symmetry axis, 3 for the location of the elastic center.

- A spherical RCC is described by only **5** parameters: $1+1=2$ for the eigenvalues of the diagonal block matrices in (9.88), 3 for the location of the elastic center.

9.2.3 Classical RCCs as Examples of Symmetric Constructions

For classical RCC devices the pitches of the eigenscrews must vanish, or equivalently the off-diagonal blocks of the net connection matrix must be zero. The zero pitch condition can be directly applied using equations (9.92) through (9.96), which results in the following.

$$\text{parallel} \quad : \qquad \vec{\mathbf{u}}^T\mathbf{B}\vec{\mathbf{u}} = 0 \qquad \mathrm{tr}(\mathbf{B}) = 0 \tag{9.99}$$

$$\text{serial} \quad : \qquad \vec{\mathbf{u}}^T\mathbf{E}\vec{\mathbf{u}} = 0 \qquad \mathrm{tr}(\mathbf{E}) = 0 \tag{9.100}$$

where $\vec{\mathbf{u}}$ is the symmetry axis. Note that, (9.99) is for parallel and (9.100) is for serial connections. The first equations in each of these, involving $\vec{\mathbf{u}}$, is the defining equation for the symmetry axis for constructing classical RCCs. The off-diagonal of a classical RCC always has a zero trace, and is a sum of the generator off-diagonal rotated n times. The trace is an invariant under rotations. So, $\mathrm{tr}(\mathbf{B}^\circ) = n\mathrm{tr}(\mathbf{B})$. So, it is sensible that all the element matrices and the generator must also have an off-diagonal block with a zero trace. Also, the point where the generator is represented is immaterial since the trace of the off-diagonal is invariant under origin transformations.

As presented earlier, the equations (9.99) and (9.100) are simply the definition of isotropic vectors of $\mathbf{B}$ and $\mathbf{G}$. The zero trace condition is sufficient for the existence of isotropic vectors. Therefore, the following theorem is proven.

Theorem 145 *A rotational symmetry device is a classical RCC if and only if the direction of the symmetry axis is an isotropic vector of the off-diagonal matrix that has a zero trace.*

Corollary 146 *Any stiffness with zero trace off-diagonals can generate infinitely many classical RCC devices.*

Proof. The off-diagonals of a stiffness matrix are 3×3. A zero trace 3×3 matrix is either indefinite or zero. In any case, there exist infinitely many isotropic vectors, see Chapter 8. Then, the corollary follows from Theorem 145.

■

EXAMPLE: Consider the following randomly selected matrix as that of a generator stiffness. The trace of the off-diagonal is made zero.

$$\hat{K}_O = \begin{bmatrix} 3.6808 & 1.2616 & 1.4952 & 0.9032 & 0.6154 & 0.7620 \\ 1.2616 & 4.5090 & 2.0173 & 1.4615 & -0.7921 & 1.2924 \\ 1.4952 & 2.0173 & 5.5979 & 1.6228 & 1.6225 & -0.1111 \\ 0.9032 & 1.4615 & 1.6228 & 5.3990 & 1.3046 & 0.2502 \\ 0.6154 & -0.7921 & 1.6225 & 1.3046 & 3.4652 & 0.6194 \\ 0.7620 & 1.2924 & -0.1111 & 0.2502 & 0.6194 & 4.9821 \end{bmatrix} \tag{9.101}$$

An isotropic vector of the upper off-diagonal is found by using the procedure outlined in Chapter 8, Section 8.1.4,

$$\vec{\mathbf{u}} = \begin{bmatrix} 0.7778 & 0.5263 & -0.3435 \end{bmatrix}^T \tag{9.102}$$

such that $\vec{\mathbf{u}}^T \mathbf{B} \vec{\mathbf{u}} = 0$. If this $\vec{\mathbf{u}}$ is used as the direction of a symmetry axis, the resulting construction should be a classical RCC no matter how many elements are used. For simplicity, the symmetry axis is taken to pass through O. Then, using *five* elements in the construction results in the following total stiffness at the elastic center.

$$\hat{K}_E^\circ = \text{diag}\begin{bmatrix} 18.205 & 25.367 & 25.367 & 27.623 & 20.746 & 20.746 \end{bmatrix} \tag{9.103}$$

which is clearly a classical RCC device. The location of the center with respect to O is

$$\vec{\mathbf{r}}_E = \begin{bmatrix} 0.0373 \\ 0.0252 \\ -0.0165 \end{bmatrix} = 0.047942\,45 \begin{bmatrix} 0.7778 \\ 0.5263 \\ -0.3435 \end{bmatrix} \simeq (0.048)\,\vec{\mathbf{u}} \tag{9.104}$$

which is along the symmetry axis.

9.2.4 Practical Examples

In this section, classical RCC devices made of beams or springs are investigated using the results of the general symmetric constructions presented in the previous section. The necessary and sufficient conditions for constructing classical RCCs by beams or springs are presented. Then, the theory is extended to show that some symmetric constructions beams or springs are special cases of what is known as Stewart platforms.

Classical RCCs with Beam Generators Analysis of classical RCCs using n number of beams has been presented in previous sections. But, it was assumed there that the beam axes intersect the symmetry axis and the construction was a parallel one. A question naturally comes to mind is whether these are the only cases. Previous sections showed that both parallel and serial connections would yield RCCs. So, it is yet to be shown whether the intersection constraint or conical symmetry is necessary or not. A beam is a classical RCC by itself, though the center is not so remote. The constitutive matrices for beams are presented in previous sections. The stiffness has the form

$$\hat{K}_{\text{beam/center}} = \begin{bmatrix} (\lambda_1 - \lambda_2)\vec{s}\vec{s}^T + \lambda_2 \mathbf{I} & \mathbf{0} \\ \mathbf{0} & (\lambda_3 - \lambda_4)\vec{s}\vec{s}^T + \lambda_4 \mathbf{I} \end{bmatrix} \tag{9.105}$$

The center is located at the mid-span of the beam. Consider the parallel connections for brevity, the serial connection case is similar. The stiffness of the beam at any other point, O, in space is

$$\hat{K}^p_{\text{beam}} = \begin{bmatrix} \mathbf{A} & -\mathbf{A}\vec{\mathbf{r}}\times \\ \vec{\mathbf{r}}\times\mathbf{A} & \mathbf{D}_O \end{bmatrix} \tag{9.106}$$

where $\vec{\mathbf{r}}$ is the position vector of the center with respect to O. The trace of the off-diagonal block of (9.106) is zero, already satisfying one condition for classical RCCs. The other condition is given in (9.99). Applying that to (9.106) one gets

$$\vec{\mathbf{u}}^T \left[(\lambda_1 - \lambda_2)\vec{s}\vec{s}^T + \lambda_2 \mathbf{I}\right] \vec{\mathbf{r}} \times \vec{\mathbf{u}} = (\vec{\mathbf{u}}^T\vec{s})\vec{s}^T\vec{\mathbf{r}} \times \vec{\mathbf{u}} = 0 \tag{9.107}$$

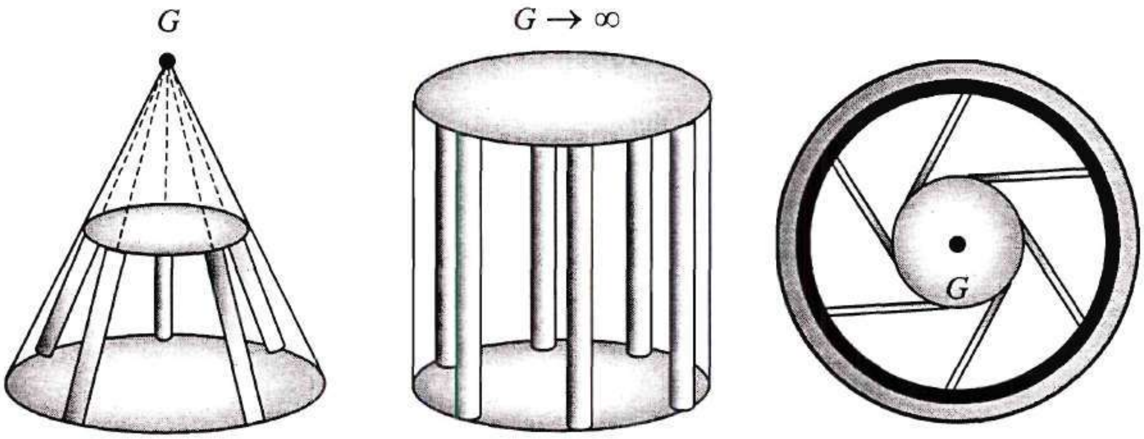
where $\vec{\mathbf{s}}$ is the longitudinal axis of the beam and $\vec{\mathbf{u}}$ is a symmetry axis needed for having a classical RCC. Note that, as expected, $\vec{\mathbf{u}}$ is an isotropic vector of $\vec{\mathbf{s}}\vec{\mathbf{s}}^T\vec{\mathbf{r}}\times$, which has a zero trace. In what follows, $\lambda_1 \neq \lambda_2$ is assumed as is the case for most practical situations. The condition (9.107) can be satisfied in two ways:

1) $\vec{\mathbf{s}}^T\vec{\mathbf{r}} \times \vec{\mathbf{u}} = 0$: Any axis parallel to the plane containing the generator beam and the generation point is a symmetry axis. All possible symmetry axes intersect the beam at a point G, Figure 9.14. In the case when the symmetry axis is parallel to the beam axis which generates elements on a cylinder, G is considered to be at infinity, Figure 9.14. This gives two types of classical RCCs: 1) conical RCC, 2) cylindrical RCC. The cylindrical type is a limiting case of the conical type. Therefore, they can commonly be called as conical RCCs. Note that conical RCCs also cover the possibility $\vec{\mathbf{r}} = \vec{\mathbf{0}}$.

2) $\vec{\mathbf{u}}^T\vec{\mathbf{s}} = 0$: All axes perpendicular to the beam are symmetry axes. The effect of this is that the longitudinal axes of all generated beams lie in one plane. The intersection of this plane of beams and the symmetry axis is the center of the resulting device. This can be regarded as a **planar RCC** device, Figure 9.14.

As a result, the planar and conical arrangements are the only two distinct classes of RCCs made of beams. The *geometric center* is the point of intersection of the symmetry axis with the plane of elements in the planar RCC case, and the vertex of the cone in the conical RCC case. It is seen that case (2) is the only way that the symmetry axis may not intersect the beam axes.

The above results also apply to serial connections. In this case, the classical RCC condition (9.107) would be applied to $-\mathbf{F}\vec{\mathbf{r}}\times$, instead of $-\mathbf{A}\vec{\mathbf{r}}\times$, which has the same form and leads to the same conditions on $\vec{\mathbf{u}}$. The conical, cylindrical and planar cases for serial connections are illustrated in Figure 9.15. The dark elements in the figure are rigid and used to connect the flexible beams (light tones) in series.



Conical RCC (parallel) Cylindrical RCC (parallel) Planar RCC (parallel)

Figure 9.14: All possible configurations of classical RCCs made of beams in parallel. G is the geometric center.

Theorem 147 *A symmetric construction with beam elements is a classical RCC if and only if the beam axes are on a cone or are coplanar.*

Remoteness of Elastic Centers in Parallel and Serial Cases For conical parallel arrangements, the off-diagonal block of the constitutive matrix has the form

$$\mathbf{B}_G = q\left[(\lambda_1 - \lambda_2)\vec{\mathbf{s}}\vec{\mathbf{s}}^T + \lambda_2\mathbf{I}\right]\vec{\mathbf{s}}\times = q\lambda_2\vec{\mathbf{s}}\times \qquad \vec{\mathbf{b}} = q\lambda_2\vec{\mathbf{s}} \tag{9.108}$$

where q is the distance between the geometric center, G, and the center of the beam. Therefore, using equations (??) through (9.77), the location of the elastic center from G is found as

$$\vec{\mathbf{r}}_S^{\circ} = -\frac{\vec{\mathbf{u}}^T\vec{\mathbf{b}}}{k_2^*}\vec{\mathbf{u}} = -\frac{q\lambda_2(\vec{\mathbf{u}}^T\vec{\mathbf{s}})}{\frac{1}{2}(\mathrm{tr}(\mathbf{A}) - \vec{\mathbf{u}}^T\mathbf{A}\vec{\mathbf{u}})}\vec{\mathbf{u}} \tag{9.109}$$

$$= -\frac{2q\cos\theta}{(1+\cos^2\theta) + \sin^2\theta\frac{\lambda_1}{\lambda_2}}\vec{\mathbf{u}} \tag{9.110}$$

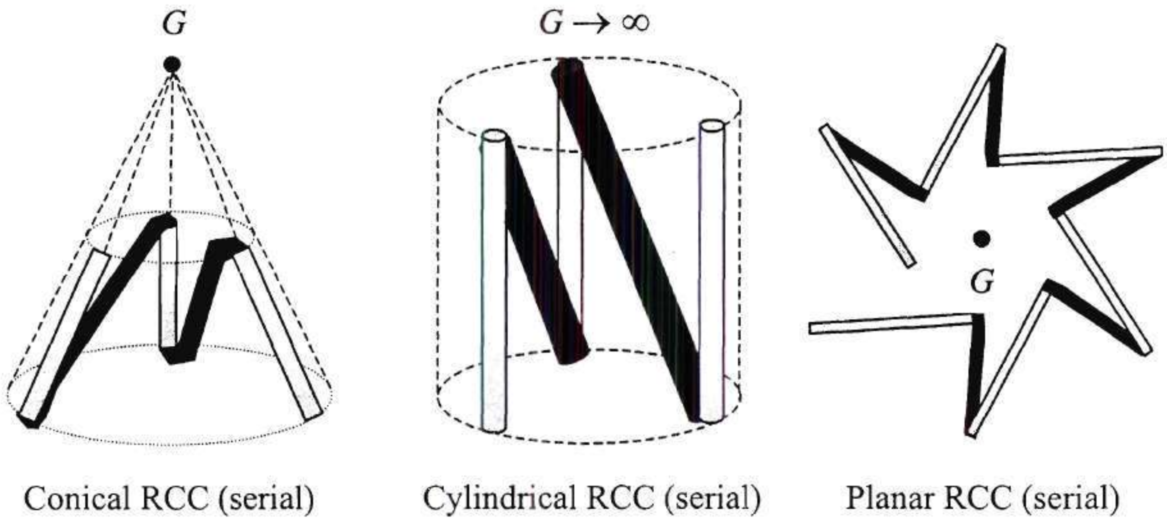


Figure 9.15: All possible serial connections of beams resulting in classical RCC devices.

where $\theta = \vec{\mathbf{u}}^T\vec{\mathbf{s}}$ is the half cone angle. This is a point inside the cone on the cone axis. This equation has been demonstrated previously. The procedure for serial connections is similar. The result, which involves the rotational compliances, is

$$\vec{\mathbf{r}}_C^{\circ} = -\frac{2q\cos\theta}{(1+\cos^2\theta)+\sin^2\theta\frac{\lambda_4}{\lambda_3}}\vec{\mathbf{u}} \tag{9.111}$$

A performance criterion for the classical RCCs may be taken as how far the center is from the physical body of the device. Since the geometric center is the point most distant from the beams among all other points on the axis inside the cone, closeness of the elastic center to the geometric center is a positive indication of the remoteness of the device. Ciblak and Lipkin [10] have investigated this measure extensively for parallel connections. So, how does the serial connection fare? Assuming slender beams with symmetrical cross-sections, the ratios of the stiffnesses and compliances in the equations above are given as

$$\lambda_1 = \tfrac{AE}{L} \qquad \lambda_2 = \tfrac{12EI}{L^3} \qquad \lambda_3 = \tfrac{GJ}{L} \qquad \lambda_4 = \tfrac{EI}{L} \tag{9.112}$$

$$\frac{\lambda_1}{\lambda_2} = \frac{AE/L}{12EI/L^3} = \frac{1}{12}\sigma^2 \qquad \frac{\lambda_4}{\lambda_3} = \frac{EI/L}{GJ/L} = 1 + \nu \simeq \frac{4}{3} \tag{9.113}$$

where A, L, E, I, G, ν are respectively the cross-sectional area, the length, the modulus of elasticity, the shear modulus and the Poisson's ratio of the beam geometry and material. $\sigma = \frac{L}{\sqrt{I/A}}$ is the slenderness ratio. The serial connections can be compared to the parallel ones by forming the ratio,

$$\frac{\|\vec{r}_S^{\circ}\|}{\|\vec{r}_C^{\circ}\|} = \frac{(1+\cos^2\theta) + \frac{4}{3}\sin^2\theta}{(1+\cos^2\theta) + \frac{1}{12}\sigma^2\sin^2\theta} = \begin{cases} < 1 & \text{for } \sigma > 4 \\ = 1 & \text{for } \sigma = 4 \text{ or } \theta = 0 \\ > 1 & \text{for } \sigma < 4 \end{cases} \tag{9.114}$$

So, unless the generator beam is very thick or short ($\sigma < 4$) or the connection is cylindrical ($\theta = 0$), the parallel connections have better projection ratios than the serial ones.

This result is explained by comparing (9.110) to (9.111). Both equations have the same form and terms, except that (9.110) has $\frac{\lambda_1}{\lambda_2}$ in the denominator, which is replaced by $\frac{\lambda_4}{\lambda_3}$ in (9.111). The numerators of the equations are identical. The greater the denominator the smaller the corresponding vector, or the closer the elastic center to the geometric center, or the better the projection ratio. But, unless $\theta = 0$, the greater $\frac{\lambda_1}{\lambda_2}$ or $\frac{\lambda_4}{\lambda_3}$ the greater the corresponding denominator. For slender beams, $\frac{\lambda_1}{\lambda_2} = \frac{1}{12}\sigma^2 >> 1$, but $\frac{\lambda_4}{\lambda_3} = \frac{4}{3}$. Therefore, $\frac{\lambda_1}{\lambda_2} >> \frac{\lambda_4}{\lambda_3}$, which results in much greater denominators for the parallel configuration, and therefore much better projection ratios, since usually slender beams are used in practise.

The border case of $\sigma = 4$, in which both serial and parallel connections have the same center location, may help one visualize how thick or short the beams must be for both arrangements to have equivalent projections. For a circular beam, $\sigma = 4$ means $L = d$, where d is the diameter. That is, the length and diameter of the beam are equal.

It is seen that for serial connections the location of the elastic center is dependent only on the geometry of the cone. For parallel connections, the center location also depends on the beam geometry via σ. Finally, the serial connection is much softer than the parallel one.

Classical RCCs with Spring Generators For a single unloaded line spring, the stiffness with respect to a point O has the form

$$\hat{K}_{\text{line-spring}} = k \begin{bmatrix} \vec{\mathbf{s}}\vec{\mathbf{s}}^T & -\vec{\mathbf{s}}\vec{\mathbf{s}}^T\vec{\mathbf{r}}\times \\ \vec{\mathbf{r}}\times\vec{\mathbf{s}}\vec{\mathbf{s}}^T & -\vec{\mathbf{r}}\times\vec{\mathbf{s}}\vec{\mathbf{s}}^T\vec{\mathbf{r}}\times \end{bmatrix} \tag{9.115}$$

where $\vec{\mathbf{s}}$ is the unit vector along the spring axis and $\vec{\mathbf{r}}$ is any vector from O to a point on the spring axis. The condition (9.99) results in an equation identical to (9.107). Therefore, the construction for having a classical RCC with line springs is geometrically identical to the beam case. If one transforms the stiffness to G, using $\vec{\mathbf{r}} = q\vec{\mathbf{s}}$ then the off-diagonal becomes

$$\mathbf{B}_G = qk\vec{\mathbf{s}}\vec{\mathbf{s}}^T\vec{\mathbf{s}}\times = \mathbf{0} \qquad \vec{\mathbf{b}} = \vec{0} \tag{9.116}$$

showing the *geometric center is the elastic center of such connections.* A three spring case was presented by Loncaric. This result generalizes that to n springs. Figure 9.14 still applies, except that the beams are replaced by line springs. Note that the actual locations of the springs, given by q, with respect to the geometric center do not matter. For parallel connections, the net stiffness is rank 3, having all zero rotational stiffnesses. For serial connections, compliances cannot be added simply since the element stiffnesses are singular. It is easy to see that a serial connection of line springs connected via ball joints will have more than six degrees of freedom (except for a finite number of singular configurations and regions in which it behaves like a single line spring either stable or non-stable). Therefore, serial connections of line springs are not considered here.

Theorem 148 *A symmetric construction with line spring elements is a classical RCC if and only if the arrangement is conical or coplanar, and the connection is parallel. The geometric center is the RCC center.*

It is worth noting that line springs actually have much less restrictive stiffnesses than beams in the sense that the symmetry condition is not needed to have an RCC constructed by using springs.

For example, any number of springs with arbitrary stiffnesses, intersecting at a point G without any symmetry comprise a rank 3 RCC device. To see this, it is sufficient to note that at G all springs have stiffnesses with vanishing off-diagonals and lower diagonals. Therefore, their sum too has vanishing off-diagonals and lower diagonal. Only the upper diagonal is non-zero. Since the upper diagonal is symmetric there exists a coordinate system in which it is diagonal, showing that the construction is an RCC (not classical) with three distinct linear stiffnesses in general and three zero angular stiffnesses. G is the elastic center.

It is not even necessary to have all intersecting line springs in order to have a classical or non-classical RCC. From the solution of the synthesis by springs problem, it can be shown that a classical RCC can be synthesized by non-intersecting line springs. In this case, however, the RCC center does not have an easy relation to the geometry of the connection. This shows that the advantage of intersecting line spring connections is the fact that the RCC center is known beforehand. The advantage of symmetric connections of line springs is the added benefit of the double eigenstiffness in the plane perpendicular to a known axis, the symmetry axis.

A Non-singular RCC with Springs It was shown earlier that the stiffness of m unloaded torsional springs in parallel is

$$\hat{K}_{\text{torsional-spring}} = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \sum_{i=1}^{n} \kappa_i \vec{\mathbf{s}}_i \vec{\mathbf{s}}_i^T \end{bmatrix} \tag{9.117}$$

with respect to any point O. So, any arbitrary parallel connection of torsional springs is already an RCC such that every point is an RCC center. Only the angular stiffnesses are non-zero.

Now, consider a classical RCC made of a symmetrical, parallel connection of n line springs. Let the RCC center be G and the symmetry axis be $\vec{\mathbf{u}}$. Next, consider a classical RCC made of a symmetrical, parallel connection of m torsional springs with the same symmetry axis $\vec{\mathbf{u}}$. If these two

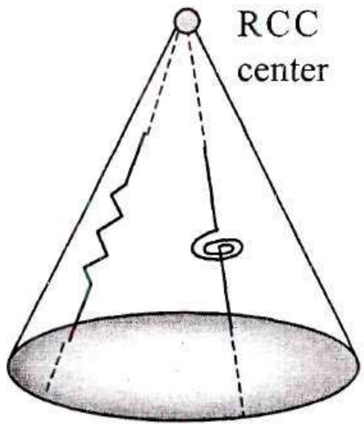


Figure 9.16: A model proposal for the construction of classical RCCs using springs. Only generators are shown. In general, the number of line and torsional springs can be different. Minimum three of each are needed for a non-singular stiffness.

devices are connected in parallel the resulting stiffness has the form

$$\hat{K}_{\text{line+torsional}/G} = \begin{bmatrix} \sum_{i=1}^{n} k_i \vec{\mathbf{s}}_{li}\vec{\mathbf{s}}_{li}^T & \mathbf{0} \\ \mathbf{0} & \sum_{i=1}^{m} \kappa_i \vec{\mathbf{s}}_{ti}\vec{\mathbf{s}}_{ti}^T \end{bmatrix} \tag{9.118}$$

where $\vec{\mathbf{s}}_{li}$ and $\vec{\mathbf{s}}_{ti}$ are the direction vectors for line and torsional springs, respectively. Clearly, the total connection is a classical RCC with symmetry axis of $\vec{\mathbf{u}}$ and double eigenstiffnesses in the plane perpendicular to $\vec{\mathbf{u}}$. The advantage of this construction is that the resulting stiffness is in general non-singular. This is proposed here as a theoretical or practical model for constructing general classical RCCs using springs. Figure 9.16 illustrates the arrangement.

Theorem 149 *A symmetrical parallel connection of line springs in parallel with a symmetrical connection of torsional springs is a classical RCC device if both connections have parallel symmetry axes. The device is non-singular if there exist at least three springs of each type. The common intersection of the line spring axes is the RCC center. See Figure 9.16.*

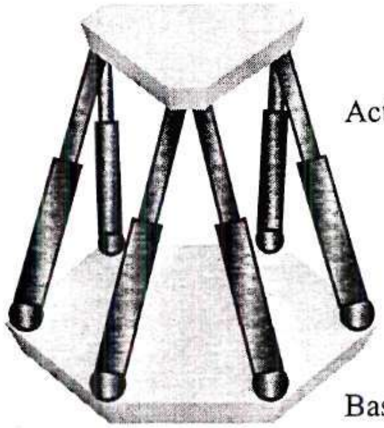


Figure 9.17: A Stewart platform is a special kind of 6-dof parallel manipulator. The actuators are incorporated into prismatic or cylindrical joints. Connections to the base and top plates are made via ball joints. Stewart platform can achieve very high degrees of accuracy and resolution in positioning tasks, usually in the order of microns.

Symmetric Stewart Platforms as RCCs A Stewart platform is a parallel manipulator, usually comprised of prismatic joints. A generic Stewart platform is shown in Figure 9.17. The ends of the actuators are incorporated into ball joints. Stewart platforms provide very high stiffness which is desired in high accuracy positioning tasks. Also, the positioning resolution can be very small compared to serial manipulators. Figure 9.18 shows simplified compliant models involving beam and line springs only. In reality, the symmetric configuration is rarely achieved. However, the model is important here for its stiffness characteristics rather than the kinematical. The two adjacent elements connecting one node of the base plate to the top can be considered as a single elastic element. Then, for a symmetric Stewart platform the other elements are obtained by consecutive constant rotations about the axis passing through the centers of the triangles, satisfying a circular symmetry, as shown in the figure. Therefore, by the results of previous section, it can be stated that such a

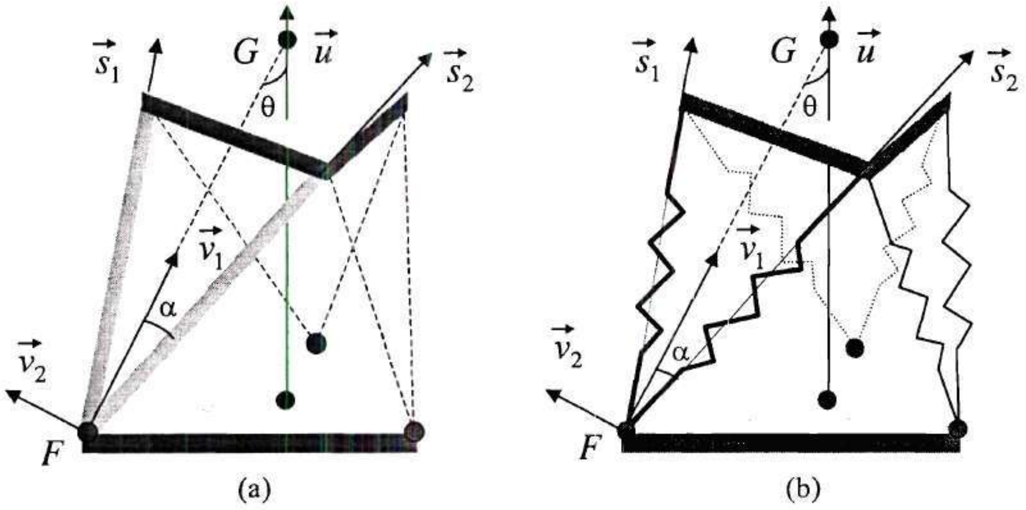


Figure 9.18: Simplified schematics of a Stewart platform in symmetric configuration: (a) parallel connection model with beams, (b) parallel connection model with line springs.

Stewart platform is at least a general RCC device. The following is an analysis of the beam model of the Stewart platform. The springs case is similar.

Symmetric Stewart Platform RCCs with Beams: The net constitutive matrix of the two elements, shown shaded in Figure 9.18, has two eigendirections in the plane containing the axes of the elements. If the elements are identical, one eigendirection bisects the angle between the axes. For simplicity, identical elements are assumed. The axis vectors of the elements can be represented in the $\vec{\mathbf{v}}_i$-system as follows

$$\vec{\mathbf{s}}_1 = \cos\alpha\ \vec{\mathbf{v}}_1 + \sin\alpha\ \vec{\mathbf{v}}_2 \tag{9.119}$$

$$\vec{\mathbf{s}}_2 = \cos\alpha\ \vec{\mathbf{v}}_1 - \sin\alpha\ \vec{\mathbf{v}}_2 \tag{9.120}$$

where α is half the angle between the axes as shown. The stiffnesses are most easily summed at the point of intersection of the two beams, F. The upper diagonal block of the stiffness of the two elements is

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 = (\lambda_1 - \lambda_2)(\vec{\mathbf{s}}_1\vec{\mathbf{s}}_1^T + \vec{\mathbf{s}}_2\vec{\mathbf{s}}_2^T) + 2\lambda_2\mathbf{I} \tag{9.121}$$

Using (9.119) and (9.120) yields

$$\mathbf{A} = 2(\lambda_1 - \lambda_2)(\cos^2\alpha\ \vec{\mathbf{v}}_1\vec{\mathbf{v}}_1^T + \sin^2\alpha\ \vec{\mathbf{v}}_2\vec{\mathbf{v}}_2^T) + 2\lambda_2\mathbf{I} \tag{9.122}$$

Hence, the $\vec{\mathbf{v}}_i$-system comprises the eigenvectors of $\mathbf{A}$, as expected due to the symmetry. The off-diagonals at F are simply

$$\mathbf{B}_{Fi} = \mathbf{A}_i(-\tfrac{L}{2}\vec{\mathbf{s}}_i)\times = -\tfrac{L}{2}\lambda_2\vec{\mathbf{s}}_i\ \times \tag{9.123}$$

$$\mathbf{B}_F = \mathbf{B}_{F1} + \mathbf{B}_{F2} = -\tfrac{L}{2}\lambda_2(\vec{\mathbf{s}}_1 + \vec{\mathbf{s}}_2)\ \times \tag{9.124}$$

$$= -L\lambda_2\cos\alpha\ \vec{\mathbf{v}}_1\times \tag{9.125}$$

The off-diagonal at any other point O is

$$\mathbf{B} = \mathbf{B}_F - \mathbf{A}\vec{\mathbf{r}}\times \tag{9.126}$$

where $\vec{\mathbf{r}}$ is the position vector of F with respect to O. For the symmetrical arrangement shown in Figure 9.18, the $\vec{\mathbf{v}}_{i1}$ axes intersect each other at a common point on the symmetry axis, G. This is similar to the conical symmetry in the classical RCCs with single beams as elements, except two beams here replace the one beam generator. Let O be this geometrical center, G. As in the single beam case, let $\vec{\mathbf{r}} = \overrightarrow{GF} = -q\vec{\mathbf{v}}_1$. Then,

$$\mathbf{B} = -L\lambda_2\cos\alpha\ \vec{\mathbf{v}}_1 \times +2q(\lambda_1 - \lambda_2)\sin^2\alpha\ \vec{\mathbf{v}}_2\vec{\mathbf{v}}_2^T\vec{\mathbf{v}}_1 \times +2\lambda_2 q\vec{\mathbf{v}}_1\ \times \tag{9.127}$$

$$= \lambda_2 2(q - \tfrac{L}{2}\cos\alpha)\vec{\mathbf{v}}_1 \times -2q(\lambda_1 - \lambda_2)\sin^2\alpha\ \vec{\mathbf{v}}_2\vec{\mathbf{v}}_3^T \tag{9.128}$$

$$\vec{\mathbf{b}}\times = \tfrac{1}{2}(\mathbf{B} - \mathbf{B}^T) = \left[2\lambda_2(q - \tfrac{L}{2}\cos\alpha) + q(\lambda_1 - \lambda_2)\sin^2\alpha\right]\vec{\mathbf{v}}_1\ \times \tag{9.129}$$

$$\vec{\mathbf{b}} = \left[2\lambda_2(q - \tfrac{L}{2}\cos\alpha) + q(\lambda_1 - \lambda_2)\sin^2\alpha\right]\vec{\mathbf{v}}_1 \tag{9.130}$$

The center of stiffness is found by using (9.74) as follows.

$$\vec{\mathbf{r}}_S^\circ = -\frac{\vec{\mathbf{u}}^T\vec{\mathbf{b}}}{k_2^*}\vec{\mathbf{u}} = -\frac{\left[2\lambda_2(q-\frac{L}{2}\cos\alpha)+q(\lambda_1-\lambda_2)\sin^2\alpha\right]\cos\theta}{\frac{1}{2}(\mathrm{tr}(\mathbf{A})-\vec{\mathbf{u}}^T\mathbf{A}\vec{\mathbf{u}})}\vec{\mathbf{u}} \tag{9.131}$$

where $\cos\theta = \vec{\mathbf{u}}^T\vec{\mathbf{v}}_1$ (Figure 9.18). For the denominator of the equation, first note that the symmetry axis is perpendicular to $\vec{\mathbf{v}}_2$. Then,

$$\vec{\mathbf{u}}^T\mathbf{A}\vec{\mathbf{u}} = 2(\lambda_1-\lambda_2)\cos^2\alpha\cos^2\theta + 2\lambda_2 \tag{9.132}$$

and with $\mathrm{tr}(\mathbf{A}) = 2\lambda_1 + 4\lambda_2$, (9.131) becomes

$$\vec{\mathbf{r}}_S^\circ = -\frac{\left[2\lambda_2(q-\frac{L}{2}\cos\alpha)+q(\lambda_1-\lambda_2)\sin^2\alpha\right]\cos\theta}{\lambda_1+2\lambda_2-\left[(\lambda_1-\lambda_2)\cos^2\alpha\cos^2\theta+\lambda_2\right]}\vec{\mathbf{u}} \tag{9.133}$$

$$= -\frac{\left[q(\lambda_1\sin^2\alpha+\lambda_2(1+\cos^2\alpha))-L\lambda_2\cos\alpha\right]\cos\theta}{\lambda_1+\lambda_2-(\lambda_1-\lambda_2)\cos^2\alpha\cos^2\theta}\vec{\mathbf{u}} \tag{9.134}$$

A way of testing this equation is to assign $\alpha = 0$ and check if the resulting equation corresponds to that of the classical RCC construction with single beam generators. Doing so yields

$$\vec{\mathbf{r}}_S^\circ(\alpha=0) = -\frac{2\lambda_2(q-\frac{L}{2})\cos\theta}{\lambda_1\sin^2\theta+\lambda_2(1+\cos^2\theta)}\vec{\mathbf{u}} \tag{9.135}$$

which is equivalent to (9.110) for symmetric cross-section beams, since $(q-\frac{L}{2})$ becomes the distance of the geometric center from the centers of the beams. The denominator of the equation (9.134) is always positive. When the numerator is negative the center is located in the opposite direction of $\vec{\mathbf{u}}$, toward the body of the device. When the numerator is positive, the elastic center is located in the direction of $\vec{\mathbf{u}}$, i.e. toward outside of the body of the device.

To check whether the device is a classical RCC or not, the condition (9.99) is checked for the particular symmetry axis in this problem. Using the fact that $\vec{\mathbf{u}} \perp \vec{\mathbf{v}}_2$ and equation (9.128), it is easily shown that $\vec{\mathbf{u}}^T\mathbf{B}\vec{\mathbf{u}} = 0$ is satisfied. Further, again (9.128) $\mathrm{tr}(\mathbf{B}) = 0$. Hence, the symmetric Stewart platform is a classical RCC device.

Figure 9.19 shows a geometrical model of the symmetric Stewart platform. To measure the performance of the device, the distance of the elastic center from the top plate can be used. The

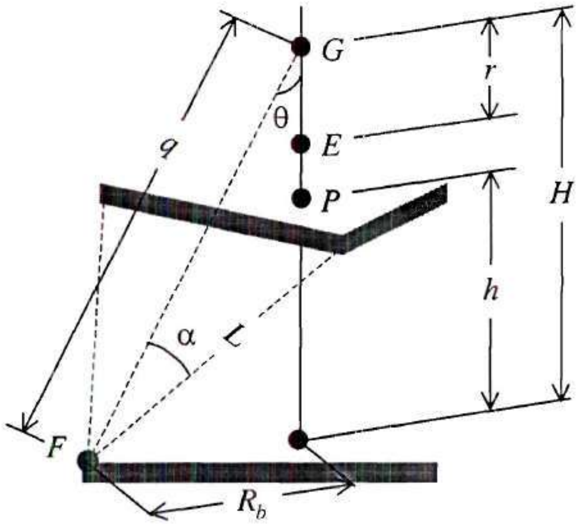


Figure 9.19: Geometrical definitions for the symmetric Stewart platform.

distance of the elastic center to the top plate, EP, can be given as

$$EP = H - h - r \tag{9.136}$$

$$= H - h - \frac{\left[q(\lambda_1 \sin^2\alpha + \lambda_2(1+\cos^2\alpha)) - L\lambda_2\cos\alpha\right]\cos\theta}{\lambda_1+\lambda_2-(\lambda_1-\lambda_2)\cos^2\alpha\cos^2\theta} \tag{9.137}$$

where H is the height of the geometric center from the bottom plate and h is the height of the device. From Figure 9.19

$$H = q\cos\theta \qquad h = L\cos\alpha\cos\theta \qquad q = \frac{R_b}{\sin\theta} \tag{9.138}$$

where R_b is the device base radius. Using (9.138) in (9.137) gives

$$EP = \frac{\cos^2\alpha\sin\theta\cos\theta\ \left(\lambda_1/\lambda_2 - 1\right)R_b + h}{\left(\lambda_1/\lambda_2+1\right) - \left(\lambda_1/\lambda_2-1\right)\cos^2\alpha\cos^2\theta} - h \tag{9.139}$$

Assuming slender beams as in the case of classical RCC with single beam generator gets rid of the constitutive properties by forming the ratio $\frac{\lambda_1}{\lambda_2} = \frac{1}{12}\sigma^2$, which, when used in (9.139) gives the result

$$p =: \frac{EP}{h} = \frac{\cos^2\alpha\sin\theta\cos\theta\ \left(\sigma^2-12\right)\chi + 12}{\left(\sigma^2+12\right) - \left(\sigma^2-12\right)\cos^2\alpha\cos^2\theta} - 1 \tag{9.140}$$

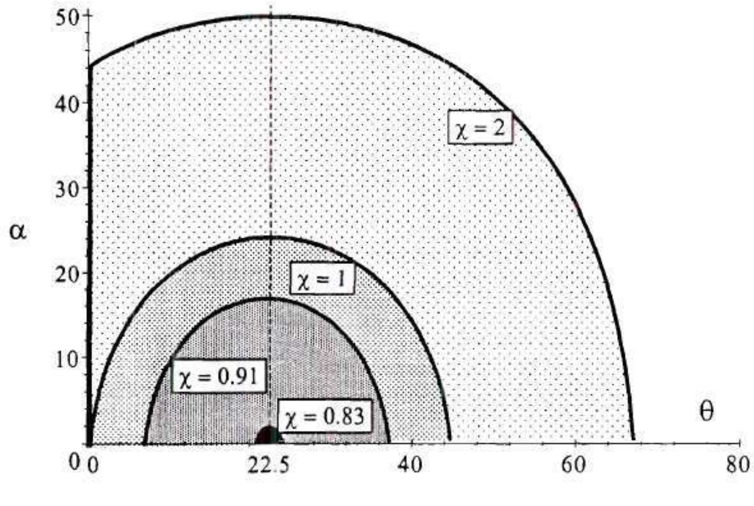


Figure 9.20: Regions of α and θ yielding non-negative projection ratios p for a symmetric Stewart platform type device. Each curve corresponds to a distinct aspect ratio, and zero projection ratio.

where all quantities are in dimensionless groups and p is a projection ratio. As in the analysis of the classical RCCs, the **aspect ratio** is defined as $\chi = \frac{R_b}{h}$, which is a measure of the overall device geometry. R_b is the radius of the base circle containing the connection points on the lower plate, Figure 9.19.

It is seen that the dimensionless projection ratio, p, is a function of four geometrical quantities, namely; θ, α, σ and χ. The definitions of dimensionless quantities here are taken differently from those for classical RCCs with beams to avoid complicated expressions.

A performance criterion for the symmetric Stewart platform compliant device is the value of the projection ratio. The higher p the more remote the elastic center from the top plate of the device. Equation (9.140) immediately hints the possibility of negative p, which not desirable. Using $p = 0$ in (9.140) one finds the critical combinations of α, θ and χ corresponding to $p = 0$. The result is not sensitive to σ for slender beams ($\sigma \geq 50$). Figure 9.20 shows the critical α, θ curves for a few values of the aspect ratio χ. The region under each curve contains the allowable combinations of α and θ

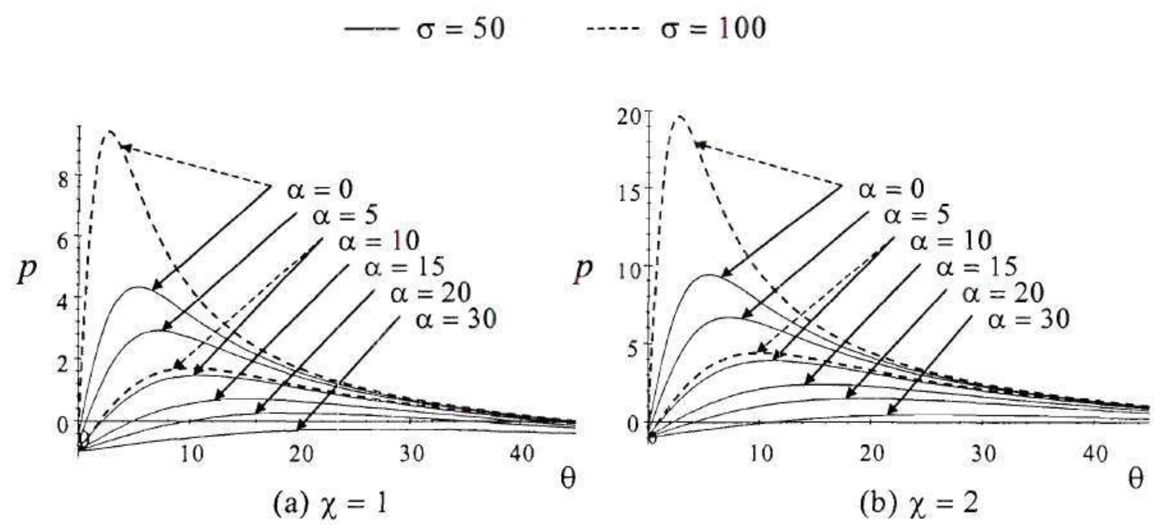


Figure 9.21: Projection ratio versus θ, α, χ and σ. For $\alpha > 10$, p is practically insensitive to σ. However, for $\alpha = 0$ (single beam RCC), p becomes exteremely sensitive to σ. High aspect ratios, χ, have very beneficial effect.

for positive projection ratios. It is clearly seen that as χ gets smaller than 1 the allowable region shrinks rapidly. One can solve (9.140) for this condition to show that a critical aspect ratio exists below which all combinations of α and θ give a negative projection. The approximate value (due to neglecting the effects of σ) of this critical aspect ratio is

$$\chi_{\text{critical}} = \frac{2}{1+\sqrt{2}} \cong 0.8284 \tag{9.141}$$

which is closely demonstrated in Figure 9.20. This value corresponds to $\alpha = 0, \theta = 22.5°$ and $p = 0$. Note that $\theta = 22.5°$ also corresponds to the maximum of each curve, which gives the upper limit of α for non-negative p. The positive effect of higher aspect ratios is clearly demonstrated in the figure, since the area of $p > 0$ increases. So, the higher χ, i.e. the shorter the device, the more choices of α and θ for positive projections.

Figure 9.21 presents the graphs of the projection ratio versus θ for selected values of α, χ and

σ. The beneficial effects of higher χ values are clearly seen when Figure 9.21-(a) is compared to Figure 9.21-(b) (The figures have different scales). In the first, $\chi = 1$, and in the second, $\chi = 2$. Higher values of α reduce the projection drastically. This shows that the single beam generator RCC is superior to the Stewart platform type. Also, for $\alpha \geq 10°$, the projection ratio is essentially independent of the slenderness ratio.

Symmetric Stewart Platform RCCs with Line Springs: The analysis of Stewart platform with springs as generators is similar to the beam case, but simpler. Upper diagonal block of the stiffness of the two spring elements is

$$\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2 = k(\vec{\mathbf{s}}_1\vec{\mathbf{s}}_1^T + \vec{\mathbf{s}}_2\vec{\mathbf{s}}_2^T) \tag{9.142}$$

where k is the spring constant. Using (9.119) and (9.120) yields

$$\mathbf{A} = 2k(\cos^2\alpha\ \vec{\mathbf{v}}_1\vec{\mathbf{v}}_1^T + \sin^2\alpha\ \vec{\mathbf{v}}_2\vec{\mathbf{v}}_2^T) \tag{9.143}$$

Hence, the $\vec{\mathbf{v}}_i$-system again comprises the eigenvectors of $\mathbf{A}$, and the eigenstiffnesses are $2k\cos^2\alpha$, $2k\sin^2\alpha$ and 0. The off-diagonals at F vanish for individual stiffnesses since this point is on the axes of both springs. So, net off-diagonal matrix of the two at any other point O is

$$\mathbf{B} = -\mathbf{A}\vec{\mathbf{r}}\times \tag{9.144}$$

where $\vec{\mathbf{r}}$ is the position vector of F with respect to O. Again, let O be the geometrical center, G. i.e., as in the beam case, $\vec{\mathbf{r}} = \overrightarrow{GF} = -q\vec{\mathbf{v}}_1$. Then,

$$\mathbf{B} = 2kq\sin^2\alpha\ \vec{\mathbf{v}}_2\vec{\mathbf{v}}_2^T\vec{\mathbf{v}}_1\ \times \tag{9.145}$$

$$= -2kq\sin^2\alpha\ \vec{\mathbf{v}}_2\vec{\mathbf{v}}_3^T \tag{9.146}$$

$$\vec{\mathbf{b}}\times = \tfrac{1}{2}(\mathbf{B} - \mathbf{B}^T) = kq\sin^2\alpha\ \vec{\mathbf{v}}_1\ \times \tag{9.147}$$

$$\vec{\mathbf{b}} = kq\sin^2\alpha\ \vec{\mathbf{v}}_1 \tag{9.148}$$

The center of stiffness is found as

$$\vec{\mathbf{r}}_S^{\circ} = -\frac{\vec{\mathbf{u}}^T\vec{\mathbf{b}}}{k_2^*}\vec{\mathbf{u}} = -\frac{kq\sin^2\alpha\cos\theta}{\frac{1}{2}(\mathrm{tr}(\mathbf{A}) - \vec{\mathbf{u}}^T\mathbf{A}\vec{\mathbf{u}})}\vec{\mathbf{u}} \tag{9.149}$$

where $\cos\theta = \vec{\mathbf{u}}^T\vec{\mathbf{v}}_1$ (Figure 9.18). Similar to the beams case, one gets

$$\vec{\mathbf{u}}^T\mathbf{A}\vec{\mathbf{u}} = 2k\cos^2\alpha\cos^2\theta \tag{9.150}$$

and with $\mathrm{tr}(\mathbf{A}) = 2k$, (9.149) becomes

$$\vec{\mathbf{r}}_S^{\circ} = -\frac{q\sin^2\alpha\cos\theta}{1-\cos^2\alpha\cos^2\theta}\vec{\mathbf{u}} \tag{9.151}$$

A way of testing this equation is to assign $\alpha = 0$ and check if the resulting equation corresponds to that of the classical RCC construction with single spring generators. In this case $\vec{\mathbf{r}}_S^{\circ} = \vec{\mathbf{0}}$ is obtained showing that the geometric center becomes the elastic center, confirming the previous results. Since there is no obvious concept of a device height in this case, q can be used as an indication of the device shape. An inspection of the equation above reveals that extrema occur only when $\alpha = \{0, \frac{\pi}{2}\}$ and $\theta = \{0, \frac{\pi}{2}\}$. Note that $\alpha = 0$ represents a classical RCC with single spring generator having the elastic center coincident with G, which is the maximum projection. However, the total stiffness is singular as shown before. When $\theta = 0$ the elastic center and the geometric center coincide at F, giving a similar result. Finally, $\alpha = \frac{\pi}{2}$ (generator springs are collinear) and $\theta = \frac{\pi}{2}$ (cone degenerates to a plane) cases result in planar RCCs. Apart from these cases, the distance of the elastic center from the geometric center varies as shown in Figure 9.22. Note that, the maximum and minimum projections correspond to $\frac{r}{q} = 0$ and $\frac{r}{q} = 1$, respectively. Again, better projections are obtained for smaller α.

As a final note on Stewart platforms, it is possible to construct a Stewart platform using any polygon as the base, instead of triangles. Also, for example, more than two elements can be used to connect a node on one plate to equally spaced points on the corresponding edge of

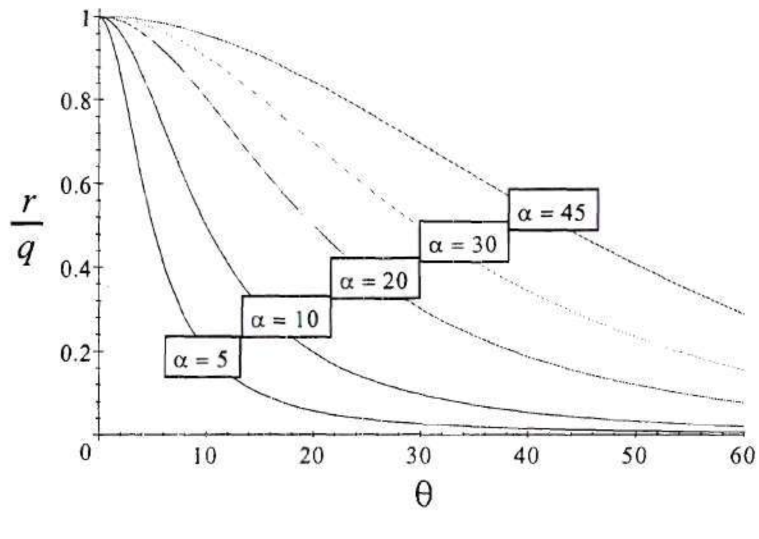


Figure 9.22: Location of the elastic center from the geometric center, r, for symmetric Stewart platform with line springs. The graphs are presented for the non-dimensional distance $\frac{r}{q}$.

the other. All such constructions are actually equivalent to superposition of an even number of general RCCs in symmetric groups such that the net effect is a classical RCC. For example, a classical Stewart platform has a two member generator. If one of these discarded, the resulting construction would yield a general RCC. The other member is obtained by a reflection with respect to a plane containing the symmetry axis. Hence, Stewart platforms or similar constructions incorporate reflective symmetry to the connection in addition to the existing rotational symmetry.

As Things Go To Infinity: Open End O-Ring A circular ring with open ends can be modelled by infinitesimally short beam elements in series, Figure 9.23. Due to the circular symmetry, such a ring would behave like a general RCC with an elastic center coinciding with the center of the ring. If, however, the general engineering model for the beams is accepted as sufficiently satisfactory, then the ring would become a classical planar RCC. In many finite element applications, curved beams, such as the O-ring, are approximated by straight beams with linear elasticity assumptions, which in

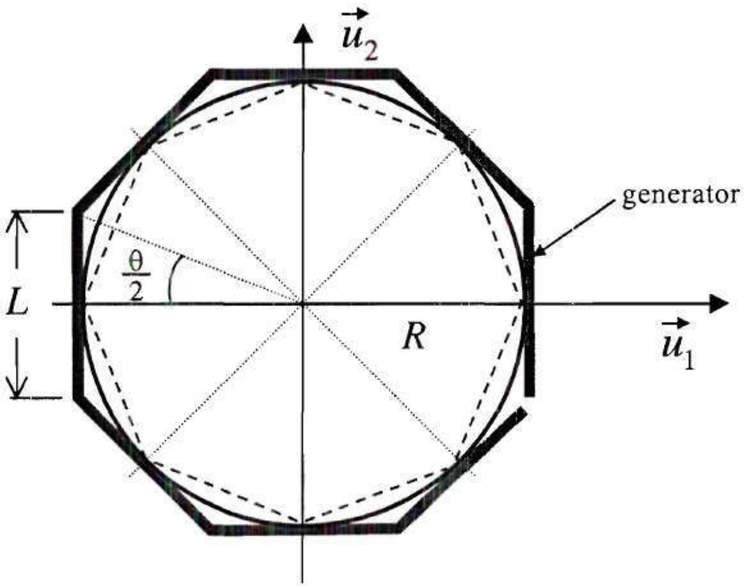


Figure 9.23: An open-end O-ring can be modelled by straight beams using the inner or outer polygons of the circle of the ring.

most cases are not applicable to short beams. Nevertheless, the results of such approximations have proven to be acceptable.

In Figure 9.23, the ring circle is approximated by its inner polygon (the outer polygon can also be used). Since the connection is serial, the compliances of the beams are additive. For simplicity, the generator element is aligned parallel to $\vec{\mathbf{u}}_2$. The symmetry axis is parallel to $\vec{\mathbf{u}}_3$ and the generation center is located at $-R\vec{\mathbf{u}}_1$, where R is the radius of the circle. The compliance of the generator at the center of the circle is found by first inverting the equation (9.8) and then transforming it to the center of the circle. The result is

$$\hat{C} = \begin{bmatrix} \mathbf{D} & \mathbf{E}^T \\ \mathbf{E} & \mathbf{F} \end{bmatrix} \tag{9.152}$$

where

$$\mathbf{D} \;=\; \lambda_1^{-1}\vec{\mathbf{u}}_1\vec{\mathbf{u}}_1^T + (\lambda_2^{-1} + R^2\lambda_4^{-1})\vec{\mathbf{u}}_2\vec{\mathbf{u}}_2^T + (\lambda_2^{-1} + R^2\lambda_3^{-1})\vec{\mathbf{u}}_3\vec{\mathbf{u}}_3^T \tag{9.153}$$

$$\mathbf{E} = R\left[(\lambda_3^{-1} - \lambda_4^{-1})\vec{\mathbf{u}}_2\vec{\mathbf{u}}_3^T - \lambda_4^{-1}\vec{\mathbf{u}}_1\times\right] \tag{9.154}$$

$$\mathbf{F} = (\lambda_3^{-1} - \lambda_4^{-1})\vec{\mathbf{u}}_2\vec{\mathbf{u}}_2^T + \lambda_4^{-1}\mathbf{I} \tag{9.155}$$

where λ_i^{-1} are the beam compliances. Although proven earlier, it may be confirmed here that $\mathrm{tr}(\mathbf{E}) = 0$ and $\vec{\mathbf{u}}_3^T\mathbf{E}\vec{\mathbf{u}}_3 = 0$, hence the resulting device is a classical RCC with $\vec{\mathbf{u}}_3$ as the symmetry axis. Only determining the eigencompliances for a construction with n beams is sufficient. For this, equations (9.82) and (9.84) are used. The results are as follows.

$$a_1 = n\vec{\mathbf{u}}_3^T\mathbf{E}\vec{\mathbf{u}}_3 = n(\lambda_2^{-1} + R^2\lambda_3^{-1}) \tag{9.156}$$

$$a_2 = \tfrac{1}{2}n(\vec{\mathbf{u}}_1^T\mathbf{E}\vec{\mathbf{u}}_1 + \vec{\mathbf{u}}_2^T\mathbf{E}\vec{\mathbf{u}}_2) = \tfrac{1}{2}n(\lambda_1^{-1} + \lambda_2^{-1} + R^2\lambda_4^{-1}) \tag{9.157}$$

$$\alpha_1 = n\vec{\mathbf{u}}_3^T\mathbf{H}\vec{\mathbf{u}}_3 = n\lambda_4^{-1} \tag{9.158}$$

$$\alpha_2 = \tfrac{1}{2}n(\vec{\mathbf{u}}_1^T\mathbf{H}\vec{\mathbf{u}}_1 + \vec{\mathbf{u}}_2^T\mathbf{H}\vec{\mathbf{u}}_2) = \tfrac{1}{2}n(\lambda_3^{-1} + \lambda_4^{-1}) \tag{9.159}$$

such that the total compliance of the ring with respect to $\vec{\mathbf{u}}_i$ is

$$\hat{C}_{\text{ring}} = \begin{bmatrix} (a_1 - a_2)\vec{\mathbf{u}}_3\vec{\mathbf{u}}_3^T + a_2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & (\alpha_1 - \alpha_2)\vec{\mathbf{u}}_3\vec{\mathbf{u}}_3^T + \alpha_2\mathbf{I} \end{bmatrix} \tag{9.160}$$

From geometry,

$$L = 2R\tan(\tfrac{\theta}{2}) \tag{9.161}$$

where $\theta = \frac{2\pi}{n}$ and n is the number of beams. Then, for sufficiently large n, the following can be written.

$$\lim_{n\to\infty} nL = 2R\lim_{n\to\infty} n\tan(\frac{\pi}{n}) = 2\pi R \tag{9.162}$$

as expected since nL approximates the circumference of the circle. The expressions for λ_i in terms of the generator geometry and constitutive properties are given by (9.7). Then, the following are obtained for the terms $n\lambda_i^{-1}$ by using (9.162);

$$\lim_{n\to\infty} n\lambda_1^{-1} = \tfrac{2\pi R}{AE} \qquad \lim_{n\to\infty} n\lambda_2^{-1} = \tfrac{1}{12EI}\lim_{n\to\infty} n\tan^3(\frac{\pi}{n}) = 0 \tag{9.163}$$

$$\lim_{n\to\infty} n\lambda_3^{-1} = \tfrac{2\pi R}{GJ} \qquad \lim_{n\to\infty} n\lambda_4^{-1} = \tfrac{2\pi R}{EI} \tag{9.164}$$

Using these in (9.156) through 9.159 yields

$$a_1 = \tfrac{4R^3}{Gr^4} \qquad a_2 = \tfrac{R}{Er^2}[1+4(\tfrac{R}{r})^2] \tag{9.165}$$

$$\alpha_1 = \tfrac{8R}{Er^4} \qquad \alpha_2 = \tfrac{4(2+\nu)R}{Er^4} \tag{9.166}$$

where r is the radius of the beam, $J = 2I = \frac{\pi r^4}{2}$ are the area moment of inertias, ν is the Poisson's ratio such that $E = 2(1+\nu)G$. The stiffness of the ring in the $\vec{\mathbf{u}}_i$-system of coordinates is determined by inverting the resulting compliance. The result is

$$\hat{K}_{\text{ring}} = \text{diag}\left\{\begin{array}{c} \frac{Er^2}{R}[1+4(\frac{R}{r})^2]^{-1} \\ \frac{Er^2}{R}[1+4(\frac{R}{r})^2]^{-1} \\ \frac{Gr^4}{4R^3} \\ \frac{Er^4}{4(2+\nu)R} \\ \frac{Er^4}{4(2+\nu)R} \\ \frac{Er^4}{8R} \end{array}\right\} \tag{9.167}$$

The scalar stiffness which relates the unit translational displacement of one end of the ring, with respect to the other, to a force applied along the symmetry axis is known as the axial stiffness, which is

$$k_{\text{axial}} = \frac{Gr^4}{4R^3} \tag{9.168}$$

This is the famous helical spring stiffness formula for a one-coil spring (see, for example, Shigley and Mischke [47]). One reason they match is that the analysis for the helical spring was approximated by assuming small helix angles, and the open ended O-ring is just such a spring. To see this, just assume that there are N coils. Since each coil would have the same compliance above, they can be added very simply with a net effect of multiplying the single coil compliances by N. So, for a N-coil spring with a very small helix angle, the approximate axial stiffness is given by

$$k_{\text{helix}} = \frac{Gr^4}{4R^3N} \tag{9.169}$$

This result confirms the performance of the stiffness model for beam elements used here. The ring can also approximate practical torsional springs. In this case, the angular stiffness corresponding to rotations/couples about the symmetry axis yields the well known torsional stiffness formula for springs with N coils,

$$k_{\text{torsion}} = \frac{Er^4}{8RN} \tag{9.170}$$

CHAPTER X

ANALYSIS OF SPATIAL MASS

In this chapter, the free-vector and line-vector eigenvalue problems are applied to the mass matrix of a single body. It is shown that the eigen and co-eigensystems have simplified structures due to the special form of the single rigid body mass matrix. This enables one to characterize the whole eigensystem structure, which is unique, and each co-eigensystem structure related to arbitrary points.

As a particular result, the concept of center-of-percussion is shown to be related to the co-eigenscrews of the mass matrix. This also leads to a generalization of the center of percussion in terms of *percussion axes.*

General results are applied to the free vibration problem of an elastically suspended rigid body. This leads to a set of necessary conditions for the existence of special free vibration modes identified in other studies. The free vibration modes of an RCC-like device are presented as examples.

10.1 Spatial Quantities of Dynamics

Consider a rigid body $\mathfrak{B}$ in motion with respect to a fixed frame $\mathfrak{F}$. Let the velocity twist of the body with respect to the fixed frame be

$$\hat{V}_O = \begin{bmatrix} \vec{\mathbf{v}}_O \\ \vec{\omega} \end{bmatrix} \tag{10.1}$$

represented in a coordinate system at an arbitrary point O of $\mathfrak{F}$.

Let M be the **center of mass** of $\mathfrak{B}$. Let m be the **total mass** and $\mathbf{J}$ be the **mass moment of inertia** with respect to M. Note that no distinction is made between the center of mass as a point of $\mathfrak{B}$, $M_{\mathfrak{B}}$, and that as a moving point of $\mathfrak{F}$, $M_{\mathfrak{F}}$. For simplicity, $M_{\mathfrak{B}}$ and $M_{\mathfrak{F}}$ are both labeled as M, since they are instantaneously coincident.

10.1.1 Spatial Momenta

Let $\vec{\mathbf{r}} = \overrightarrow{MO}$ be the position vector of O with respect to M. Then, the following definitions directly follow from elementary dynamics.

Definition 150 *The 3-vector*

$$\vec{\mathbf{p}} = m\vec{\mathbf{v}}_O - m\vec{\omega} \times \vec{\mathbf{r}} = m\vec{\mathbf{v}}_O + m\vec{\mathbf{r}} \times \vec{\omega} \tag{10.2}$$

is called the ***linear momentum****, and the 3-vector*

$$\vec{\mathbf{h}}_O = \mathbf{J}\vec{\omega} + \vec{\mathbf{p}} \times \vec{\mathbf{r}} = \mathbf{J}\vec{\omega} - m\vec{\mathbf{r}} \times \vec{\mathbf{v}}_O - m\vec{\mathbf{r}} \times \vec{\mathbf{r}} \times \vec{\omega} \tag{10.3}$$

is called the ***angular momentum*** *with respect to O.*

Note that, at the center of mass M, $\vec{\mathbf{p}} = m\vec{\mathbf{v}}_M$ and $\vec{\mathbf{h}}_M = \mathbf{J}\vec{\omega}$.

Lemma 151 *The pair $\left(\vec{\mathbf{p}}, \vec{\mathbf{h}}_O\right)$ transforms as a spatial vector (screw).*

Proof. Consider an arbitrary point A such that $\vec{\mathbf{r}}_{A/O} = \overrightarrow{OA}$ and $\vec{\mathbf{r}}_A = \vec{\mathbf{r}}_{A/M} = \overrightarrow{MA}$. Then, by Definition 150,

$$\vec{\mathbf{p}}_A = m\vec{\mathbf{v}}_A - m\vec{\omega} \times \vec{\mathbf{r}}_A \tag{10.4}$$

Since $\hat{V}$ is a spatial vector, one has the relation $\vec{\mathbf{v}}_A = \vec{\mathbf{v}}_O + \vec{\omega} \times \vec{\mathbf{r}}_{A/O}$. Also, from geometry, $\vec{\mathbf{r}}_A = \vec{\mathbf{r}} + \vec{\mathbf{r}}_{A/O}$. Using these in (10.4) one gets

$$\vec{\mathbf{p}}_A = m\vec{\mathbf{v}}_A - m\vec{\omega} \times \vec{\mathbf{r}}_A \tag{10.5}$$

$$= m\left(\vec{\mathbf{v}}_O + \vec{\omega} \times \vec{\mathbf{r}}_{A/O}\right) - m\vec{\omega} \times \left(\vec{\mathbf{r}} + \vec{\mathbf{r}}_{A/O}\right) \tag{10.6}$$

$$= m\vec{\mathbf{v}}_O - m\vec{\omega} \times \vec{\mathbf{r}} = \vec{\mathbf{p}} \tag{10.7}$$

which shows that $\vec{\mathbf{p}}$ is the same at all points. Then, again by Definition 150,

$$\vec{\mathbf{h}}_A = \mathbf{J}\vec{\omega} + \vec{\mathbf{p}} \times \vec{\mathbf{r}}_A = \mathbf{J}\vec{\omega} + \vec{\mathbf{p}} \times \left(\vec{\mathbf{r}} + \vec{\mathbf{r}}_{A/O}\right) = \vec{\mathbf{h}}_O + \vec{\mathbf{p}} \times \vec{\mathbf{r}}_{A/O} \tag{10.8}$$

Therefore, the transformation of the momentum pair from a point O to A is

$$\left(\vec{\mathbf{p}}, \vec{\mathbf{h}}_A\right) = \left(\vec{\mathbf{p}}, \vec{\mathbf{h}}_O + \vec{\mathbf{p}} \times \vec{\mathbf{r}}_{A/O}\right) \tag{10.9}$$

which is clearly a spatial vector transformation. ■

Definition 152 *The 6-vector*

$$\hat{P}_O = \begin{bmatrix} \vec{\mathbf{p}} \\ \vec{\mathbf{h}}_O \end{bmatrix} \tag{10.10}$$

*is called the **spatial momentum** of $\mathfrak{B}$ with respect to $\mathfrak{F}$ and represented at O.*

Note that the spatial momentum is defined in ray-coordinates, that is as a *wrench-like* vector. This is sensible since it is well known in dynamics that the time derivative of the momenta is equal to the force and moments. For a detailed discussion of the spatial case see Featherstone [20], who shows that $\frac{d\hat{P}}{dt} = \hat{W}$, where $\hat{W}$ is the wrench on the body. As expected, the spatial momentum transformation is exactly the same as that for wrenches,

$$\begin{bmatrix} \vec{\mathbf{p}} \\ \vec{\mathbf{h}}_A \end{bmatrix} = \begin{bmatrix} \mathbf{I} & 0 \\ -\vec{\mathbf{r}}_{A/O}\times & \mathbf{I} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{p}} \\ \vec{\mathbf{h}}_O \end{bmatrix} \tag{10.11}$$

Hence, the linear momentum is a line-vector, whereas the angular momentum is a free-vector.

This sets up a complete analogy between the elastic and kinetic cases. In the elastic case, one has the spatial vector spaces of $\delta\hat{q}$, infinitesimal spatial displacement, and $\hat{W}$, wrench (spatial force). In the kinetic case one has $\hat{V}$, spatial velocity, and $\hat{P}$, spatial momentum. Further, the relations $\hat{V} = \frac{\delta\hat{q}}{dt}$ and $\frac{d\hat{P}}{dt} = \hat{W}$ complement each other.

10.1.2 Spatial Mass

The fact that classical momenta are defined as vector functions of the velocities and mass properties indicates that it is sensible to introduce the following definition.

Definition 153 *The linear map from the spatial velocity space to the spatial momentum space is called the* **spatial mass** $\hat{M}_O$ *of a rigid body,*

$$\hat{P}_O = \hat{M}_O \hat{V}_O \tag{10.12}$$

In the rest of this study, the spatial mass is simply called the **mass matrix**.

Theorem 154 *The mass matrix represented at any point O, such that $\vec{\mathbf{r}} = \overrightarrow{MO}$, is a symmetric, positive definite matrix given by*

$$\hat{M}_O = \begin{bmatrix} m\mathbf{I} & m\vec{\mathbf{r}}\times \\ -m\vec{\mathbf{r}}\times & \mathbf{J} - m\vec{\mathbf{r}}\times\vec{\mathbf{r}}\times \end{bmatrix} \tag{10.13}$$

and transforms according to

$$\hat{M}_A = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\vec{\mathbf{r}}_{A/O}\times & \mathbf{I} \end{bmatrix} \hat{M}_O \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\vec{\mathbf{r}}_{A/O}\times & \mathbf{I} \end{bmatrix}^T = \hat{X}_{A/O}\hat{M}_O\hat{X}_{A/O}^T \tag{10.14}$$

Proof. From the equations of Definition 150,

$$\hat{P}_O = \begin{bmatrix} m\vec{\mathbf{v}}_O - m\vec{\omega}\times\vec{\mathbf{r}} \\ \mathbf{J}\vec{\omega} - m\vec{\mathbf{r}}\times\vec{\mathbf{v}}_O - m\vec{\mathbf{r}}\times\vec{\mathbf{r}}\times\vec{\omega} \end{bmatrix} = \begin{bmatrix} m\mathbf{I} & m\vec{\mathbf{r}}\times \\ -m\vec{\mathbf{r}}\times & \mathbf{J} - m\vec{\mathbf{r}}\times\vec{\mathbf{r}}\times \end{bmatrix} \hat{V}_O \tag{10.15}$$

and (10.13) follows by Definition 153. The spatial transformations for $\hat{P}$ and $\hat{V}$ are given by $\hat{P}_A = \hat{X}_{A/O}\hat{P}_O$ and $\hat{V}_A = \hat{X}_{A/O}^{-T}\hat{V}_O$, where $\hat{X}_{A/O}$ is as in (10.14). Then,

$$\hat{P}_A = \hat{M}_A\hat{V}_A \tag{10.16}$$

$$\hat{X}_{A/O}\hat{P}_O = \hat{M}_A\hat{X}_{A/O}^{-T}\hat{V}_O \tag{10.17}$$

$$\hat{P}_O = \left(\hat{X}_{A/O}^{-1}\hat{M}_A\hat{X}_{A/O}^{-T}\right)\hat{V}_O \rightarrow \hat{M}_O = \hat{X}_{A/O}^{-1}\hat{M}_A\hat{X}_{A/O}^{-T} \tag{10.18}$$

from which (10.14) follows.

∎

The above theorem clearly shows that the mass matrix is to the kinetic case what stiffness matrix is to the elastic case. The only distinction is that the mass matrix has a special form, rather than being any 6×6 symmetric matrix. This is more clearly indicated if one considers the case $O = M$, which leads to the following corollary.

Corollary 155 *The mass matrix at the center of mass is given by*

$$\hat{M}_M = \begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \tag{10.19}$$

10.1.3 Kinetic Energy

Corollary 156 *The kinetic energy of a rigid body moving with a velocity $\hat{V}$ is*

$$KE = \frac{1}{2}\hat{P}^T\hat{V} = \frac{1}{2}\hat{V}^T\hat{M}\hat{V} \tag{10.20}$$

Proof. The scalar product $\hat{P}_O^T\hat{V}_O$ is invariant under spatial transformations. Although this is a general fact, one may want to show this using $\hat{P}_A = \hat{X}_{A/O}\hat{P}_O$ and $\hat{V}_A = \hat{X}_{A/O}^{-T}\hat{V}_O$, which leads to $\hat{P}_A^T\hat{V}_A = \hat{P}_O^T\hat{V}_O$. This is sensible since the kinetic energy is a scalar which should be invariant under tensor transformations. Consequently, using Corollary 155 to express (10.20) at M gives

$$\frac{1}{2}\hat{P}^T\hat{V} = \frac{1}{2}\hat{P}_M^T\hat{V}_M = \frac{1}{2}\hat{V}_M^T\hat{M}_M\hat{V}_M = \frac{1}{2}m\vec{\mathbf{v}}_M^T\vec{\mathbf{v}}_M + \frac{1}{2}\vec{\omega}^T\mathbf{J}\vec{\omega} \tag{10.21}$$

which is recognized as the kinetic energy.

∎

Corollary 157 *The kinetic energy of a rigid body with a spatial momentum $\hat{P}$ is*

$$KE = \frac{1}{2}\hat{P}^T\hat{V} = \frac{1}{2}P^T\hat{M}^{-1}P \tag{10.22}$$

Proof. From Corollary 156 and $\hat{V} = \hat{M}^{-1}\hat{P}$

$$KE = \frac{1}{2}\hat{V}^T\hat{M}\hat{V} = \frac{1}{2}\hat{P}^T\hat{M}^{-1}\hat{M}\hat{M}^{-1}\hat{P} = \frac{1}{2}P^T\hat{M}^{-1}P \tag{10.23}$$

■

10.1.4 Spatial Equation of Motion

Featherstone [20] shows that, for a rigid body, the time derivative of the spatial momentum is equal to the wrench. In the context of this study, the time derivative can be performed with respect to either the fixed frame $\mathfrak{F}$ or the body frame $\mathfrak{B}$. As detailed in Chapter 7, let $D()$ and $D'()$ be the time derivatives with respect to $\mathfrak{F}$ and $\mathfrak{B}$, respectively. Then, using the differentiation formula presented in Chapter 7,

$$\hat{W} = D\hat{P} = D'\hat{P} + \hat{V} \times \hat{P} \tag{10.24}$$

where $\hat{W}$ is the wrench on a single rigid body. Using $\hat{P} = \hat{M}\hat{V}$ in the above equation one gets

$$\hat{W} = D'(\hat{M}\hat{V}) + \hat{V} \times \hat{M}\hat{V} \tag{10.25}$$

Since, the mass matrix of a rigid body is a constant with respect to an observer on the rigid body, $D'(\hat{M}\hat{V}) = \hat{M}(D'\hat{V})$. Also, again by differentiation formula, $D\hat{V} = D'\hat{V} + \hat{V} \times \hat{V} = D'\hat{V}$, since $\hat{V} \times \hat{V} = \hat{0}$ is a property of spatial cross product. Hence, (10.25) becomes

$$\hat{W} = \hat{M}D\hat{V} + \hat{V} \times \hat{M}\hat{V} \tag{10.26}$$

By linearity of the derivation, one can show that $D\hat{V}$ is a spatial vector. Featherstone [20] studies this vector in detail and shows that it is related to the classical acceleration vectors of the rigid body. The spatial vector $D\hat{V}$ is called the **spatial acceleration**. The spatial acceleration is denoted by $\hat{A}$ in this study. As a result, the following is obtained.

$$\hat{W} = \hat{M}\hat{A} + \hat{V} \times \hat{M}\hat{V} \tag{10.27}$$

Equation (10.27) is the spatial equivalent of the Newton-Euler equations of motion for a single rigid body. It compactly expresses both linear and angular equations. This equation was previously obtained by Featherstone [20] in a slightly different manner. Featherstone calls the term $\hat{V} \times \hat{M}\hat{V}$ as the *bias force*. He successfully uses (10.27) to investigate the dynamics of serial manipulators such as robotic arms.

10.1.5 Dynamics of a Rigid Body at *Rest*

The sentence "dynamics of a rigid body at rest" sounds like an oxymoron, because one normally tends to think that anything at rest is not *dynamic*. However, what is sometimes meant by "at rest" in the literature is a system with a zero velocity, but not necessarily a zero acceleration or jerk, etc.

The most well known practical examples come from impact or collision problems in which one of the bodies of collision is really-at-rest, meaning the velocity and all of its derivatives are zero. Right at the start of impact, the really-at-rest body is acted upon by impact loads. Theoretically, these impact loads can have a non-zero initial magnitudes. Such forces are usually called *impulsive loads*. Impulsive loads do not necessarily happen only in collision problems. A sudden creation of electrical, magnetic, gravitational, etc. fields has the same effect on an appropriate system (systems with charge, mass, etc.). Thus, it is conceivable to speak of the dynamics of a rigid body with $\hat{V} = \hat{0}$, but $\hat{A} \neq \hat{0}$. For such cases, the equation of motion becomes

$$\hat{W} = \hat{M}\hat{A} \tag{10.28}$$

In the rest of this study, the spatial mass is considered as a map from the spatial accelerations to wrenches. The impetus behind (10.28) is three-fold:

1. First, it shows that the mass matrix can be considered as a mapping from a twist-like space (spatial acceleration) to the wrench space. Therefore, (10.28) establishes a close analogy to the stiffness mapping. However, one should be strongly reminded that the analogy is already given

by $\hat{P} = \hat{M}\hat{V}$ in a natural way. Here, $\hat{V}$ is the twist-like object and $\hat{P}$ is the wrench-like object. The problem originates from the subjective reason that the momenta is much less intuitive than loads. For anyone who feels comfortable with the concept of momenta, many occurrences of "wrench" in the rest of this study can be replaced by "spatial momentum" without affecting the essential content.

2. Second, the impulsive force problems constitute good examples for the application of the mass matrix analysis. This is demonstrated in the analysis of percussion axes.

3. Finally, the introduction of wrenches becomes necessary in the analysis of small vibrations of elastically suspended bodies, presented as the last section of this chapter. Blanchet and Lipkin [3] show that the assumption of small amplitude leads to the fact that the term $\hat{V} \times \hat{M}\hat{V}$ becomes negligibly small compared to $\hat{M}\hat{A}$.

A disadvantage of (10.46) is that, instead of displacements or velocities, one has to deal with spatial accelerations which do not easily relate to the classical acceleration vectors, see Featherstone [20]. Nevertheless, for simplicity, the terms "translation" and "rotation" are still used in the results where they would actually mean "translational acceleration" and "rotational acceleration", respectively. The term "eigentwist" or "twist" should be understood as the screw about which the acceleration occurs.

10.2 Eigen- and Co-eigensystems of Mass Matrix

Due to its special form, the mass matrix is expected to have special eigen- and co-eigensystem structures. This is demonstrated in the following sections, where the mass matrix is assumed to be always positive definite and finite. That is, there are no kinematical joints or constraint surfaces.

10.2.1 Eigensystems at Center of Mass

Consider the free-vector eigenvalue problems for the mass matrix at M.

$$\begin{bmatrix} m^{-1}\mathbf{I} & 0 \\ 0 & \mathbf{J}^{-1} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{f}}_m \\ \vec{\tau}_m \end{bmatrix} = m_f^{-1} \begin{bmatrix} \vec{\mathbf{f}}_m \\ \vec{\mathbf{0}} \end{bmatrix} \tag{10.29}$$

$$\begin{bmatrix} m\mathbf{I} & 0 \\ 0 & \mathbf{J} \end{bmatrix} \begin{bmatrix} \vec{\delta}_m \\ \vec{\gamma}_m \end{bmatrix} = m_\gamma \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\gamma}_m \end{bmatrix} \tag{10.30}$$

where the subscript $_m$ is used to distinguish the eigenvectors from those of the stiffness matrix. It is seen from these equations that

$$m_{fi} = m, \qquad \vec{\mathbf{f}}_{mi} \text{ is arbitrary, and } \quad \vec{\tau}_{mi} = \vec{\mathbf{0}} \tag{10.31}$$

$$m_{\gamma_i}, \vec{\gamma}_{m_i} \text{ are the eigenvalues and vectors of } \mathbf{J}, \quad \text{and} \quad \vec{\delta}_{mi} = \vec{\mathbf{0}} \tag{10.32}$$

Hence,

Theorem 158 *For the mass matrix of a single rigid body,*

1. *The eigentwists and eigenwrenches are* zero pitch screws *(pure rotations and forces) through M. See Figure 10.1.*

2. *Every pure force through M is an eigenwrench with an eigenvalue m^{-1}. The eigenwrench 3-system is a force bundle generated by M.*

3. *Every pure rotation through M and parallel to any of the* principal inertia axes *is an eigentwist, corresponding to the principal inertia values. The eigentwist 3-system is a rotation bundle generated by M.*

Figure 10.1 illustrates the eigenscrew structure of the mass matrix.

Since the mass matrix has zero off-diagonals at M, the centers of the eigentwist and eigenwrench subspaces coincide with M. Clearly, M is analogous to the centers of elasticity, stiffness

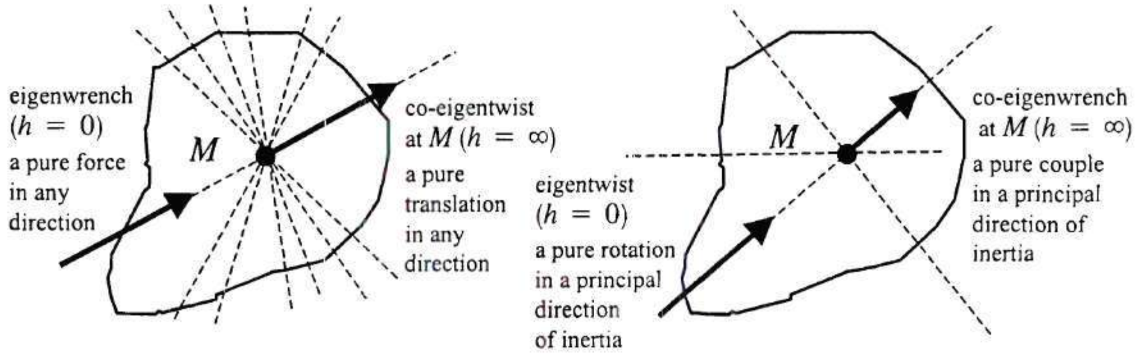


Figure 10.1: Eigen- and co-eigenscrews of the mass matrix at center of mass.

and compliance combined. In other words, M is to the mass matrix what E, S and C are to the stiffness matrix. Since the eigenscrews pass through M, they are also the principal screws of the mass matrix.

As in the case of stiffness, it is possible to show that the free-vector eigenvalue problems for the mass matrix also result from stationary values of the kinetic energy under certain constraints. The eigentwist problem is obtained by finding the stationary values of $KE = \frac{1}{2}\hat{V}^T\hat{M}\hat{V}$ subject to the constraint $\vec{\gamma}^T\vec{\gamma} = 1$. The eigenwrench problem is obtained by finding the stationary values of $KE = \frac{1}{2}\hat{P}^T\hat{M}^{-1}\hat{P}$ subject to the constraint $\vec{\mathbf{f}}^T\vec{\mathbf{f}} = 1$. For details of this method see Chapter 3.

10.2.2 Co-eigensystems at Center of Mass

Now consider the co-eigenscrew problems at M.

$$\begin{bmatrix} m\mathbf{I} & 0 \\ 0 & \mathbf{J} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}}_m \\ \vec{\mathbf{w}}_m \end{bmatrix} = m_t \begin{bmatrix} \vec{\mathbf{t}}_m \\ \vec{\mathbf{0}} \end{bmatrix} \tag{10.33}$$

$$\begin{bmatrix} m^{-1}\mathbf{I} & 0 \\ 0 & \mathbf{J}^{-1} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{n}}_m \\ \vec{\mathbf{m}}_m \end{bmatrix} = m_m^{-1} \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{m}}_m \end{bmatrix} \tag{10.34}$$

Again, it is clear that

$$m_{ti} = m, \qquad \vec{\mathbf{t}}_{mi} \text{ is arbitrary, and } \quad \vec{\mathbf{w}}_{mi} = \vec{\mathbf{0}} \tag{10.35}$$

$$m_{m_i}, \vec{\mathbf{m}}_{m_i} \text{ are the eigen-values and vectors of } \mathbf{J}, \quad \text{and} \quad \vec{\mathbf{n}}_{mi} = \vec{\mathbf{0}} \tag{10.36}$$

Hence,

Theorem 159 *For the mass matrix of a single rigid body,*

1. *The co-eigentwists and co-eigenwrenches at M are* infinite pitch screws *(pure translations and couples). See Figure 10.1.*

2. *Every pure translation is a co-eigentwist at M corresponding to an eigenvalue m.*

3. *Every pure couple parallel to any of the* principal inertia axes *is a co-eigenwrench at M, corresponding to the inverse of the principal inertia values.*

The co-eigenwrenches (co-eigentwists) at M are parallel to the eigentwists (eigenwrenches). Figure 10.1 illustrates the co-eigenscrews of the mass matrix at M.

Since the mass matrix has zero off-diagonals at M, the center of mass is also analogous to the co-center of elasticity and could be called as the **co-center of mass**, which is unique due to the zero off-diagonals, see Chapter 5.1.

The line-vector eigenvalue problems for the mass matrix also result from stationary values of the kinetic energy under certain constraints. The co-eigentwist at M problem is obtained by finding the stationary values of $KE = \frac{1}{2}\hat{V}^T\hat{M}\hat{V}$ subject to the constraint $\vec{\mathbf{t}}_M^T\vec{\mathbf{t}}_M = 1$. The co-eigenwrench problem at M is obtained by finding the stationary values of $KE = \frac{1}{2}\hat{P}^T\hat{M}^{-1}\hat{P}$ subject to the constraint $\vec{\mathbf{m}}_M^T\vec{\mathbf{m}}_M = 1$. For details of this method see Chapter 3.

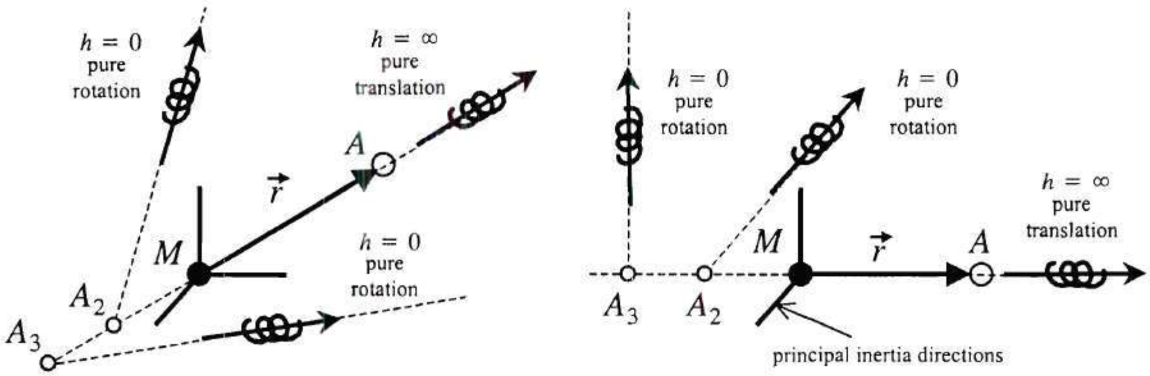


Figure 10.2: Co-eigentwists of mass at a point A away from the center of mass. The 3-system formed by these screws is a space of twists that only cause pure forces through A.

10.2.3 Co-eigensystems Away from Center of Mass

The properties of the co-eigenscrew subspaces at points other than M have curiously distinct properties. The following theorems demonstrate these.

Theorem 160 *Consider any point $A \neq M$ whose position from M is $\vec{\mathbf{r}}$. The co-eigentwist subspace of the mass matrix at A is formed by one infinite pitch co-eigenscrew parallel to $\vec{\mathbf{r}}$ and two zero pitch co-eigenscrews which intersect the line through A and M, Figure 10.2. If A is on a principal inertia axis then the co-eigentwists are parallel to the principal inertia directions.*

Proof. Using the transformation rules for the mass matrix, the co-eigentwist problem at A is given by

$$\begin{bmatrix} m\mathbf{I} & m\vec{\mathbf{r}}\times \\ -m\vec{\mathbf{r}}\times & \mathbf{J} - m\vec{\mathbf{r}}\times\vec{\mathbf{r}}\times \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}}_{m/A} \\ \vec{\mathbf{w}}_{m/A} \end{bmatrix} = m_{t/A} \begin{bmatrix} \vec{\mathbf{t}}_{m/A} \\ \vec{\mathbf{0}} \end{bmatrix} \quad \text{or} \tag{10.37}$$

$$\begin{bmatrix} m^{-1}\mathbf{I} - \vec{\mathbf{r}}\times\mathbf{J}^{-1}\vec{\mathbf{r}}\times & -\vec{\mathbf{r}}\times\mathbf{J}^{-1} \\ \mathbf{J}^{-1}\vec{\mathbf{r}}\times & \mathbf{J}^{-1} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}}_{m/A} \\ \vec{\mathbf{0}} \end{bmatrix} = m_{t/A}^{-1} \begin{bmatrix} \vec{\mathbf{t}}_{m/A} \\ \vec{\mathbf{w}}_{m/A} \end{bmatrix} \tag{10.38}$$

From the first equation in (10.37), it is seen that

$$\vec{\mathbf{r}} \times \vec{\mathbf{w}}_{m/A} = \left(\frac{m_{t/A}}{m} - 1\right) \vec{\mathbf{t}}_{m/A} \tag{10.39}$$

Note that $\vec{\mathbf{r}} \neq \vec{\mathbf{0}}$ and $\vec{\mathbf{t}}_{m/A} \neq \vec{\mathbf{0}}$. The left and right sides of (10.39) are either zero or non-zero, and correspond to the following solutions,

1. $m_{t/A} = m$ and $\vec{\mathbf{r}} \times \vec{\mathbf{w}}_{m/A} = \vec{\mathbf{0}}$. Thus, either $\vec{\mathbf{w}}_{m/A} = \vec{\mathbf{0}}$ or $\vec{\mathbf{w}}_{m/A} \parallel \vec{\mathbf{r}}$. In compact form, $\vec{\mathbf{w}}_{m/A} = \beta\vec{\mathbf{r}}$.

2. $\vec{\mathbf{w}}_{m/A}^T \vec{\mathbf{t}}_{m/A} = 0$.

If (1) is used in the second equation of (10.37), one gets

$$\beta \mathbf{J} \vec{\mathbf{r}} = m \vec{\mathbf{r}} \times \vec{\mathbf{t}}_{m/A} \tag{10.40}$$

which implies that

$$\vec{\mathbf{r}}^T \mathbf{J} \vec{\mathbf{r}} = 0 \qquad \text{or} \tag{10.41}$$

$$\beta = 0 \qquad \text{and} \qquad \vec{\mathbf{r}} \parallel \vec{\mathbf{t}}_{m/A} \tag{10.42}$$

The first condition is contrary to the assumption that $\mathbf{J}$ is positive definite. Therefore, (1) and (2) becomes

1. $m_{t/A} = m$, $\vec{\mathbf{w}}_{m/A} = \vec{\mathbf{0}}$ and $\vec{\mathbf{r}} \parallel \vec{\mathbf{t}}_{m/A}$. This gives an infinite pitch co-eigentwist parallel to $\vec{\mathbf{r}}$.

2. $\vec{\mathbf{w}}_{m/A}^T \vec{\mathbf{t}}_{m/A} = 0$. This gives a zero pitch co-eigentwist.

Hence, a co-eigentwist is found in (1) as

$$\hat{T}_{(c/A)_1} = \begin{bmatrix} \alpha\vec{\mathbf{r}} \\ \vec{\mathbf{0}} \end{bmatrix} \tag{10.43}$$

This is an infinite pitch co-eigenscrew parallel to $\vec{\mathbf{r}}$ as the theorem stated.

Now, the co-eigentwists have three perpendicular translations . Since, one is already found as parallel to $\vec{\mathbf{r}}$, the other two are perpendicular to the same. Then, by the second of equations (10.38), one concludes that $\vec{\mathbf{w}}_{(m/A)_2}, \vec{\mathbf{w}}_{(m/A)_3} \neq \vec{\mathbf{0}}$. Therefore, the other two eigentwists are zero pitch screws, as given in (2). Using (10.39) these zero pitch co-eigentwists can be written as

$$\hat{T}_{(c/A)_{2,3}} = \begin{bmatrix} \left(\frac{m_{t/A}}{m} - 1\right)^{-1} \vec{\mathbf{r}} \times \vec{\mathbf{w}}_{m/A} \\ \\ \vec{\mathbf{w}}_{m/A} \end{bmatrix}_{2,3} \tag{10.44}$$

At points A_2 and A_3, whose positions with respect to A are $\left(\frac{m_{(t/A)_2}}{m} - 1\right)^{-1} \vec{\mathbf{r}}$ and $\left(\frac{m_{(t/A)_3}}{m} - 1\right)^{-1} \vec{\mathbf{r}}$, respectively, the transformation rules gives the expressions for the screws as

$$\hat{T}_{(c/A_2)_2} = \begin{bmatrix} \vec{\mathbf{0}} \\ \\ \vec{\mathbf{w}}_{(m/A)_2} \end{bmatrix} \quad \text{and} \quad \hat{T}_{(c/A_3)_3} = \begin{bmatrix} \vec{\mathbf{0}} \\ \\ \vec{\mathbf{w}}_{(m/A)_3} \end{bmatrix} \tag{10.45}$$

Since A_2 and A_3 are on the line joining A and M, (10.45) proves that the zero pitch co-eigentwists intersect the line.

If A is on a principal inertia axis then $\vec{\mathbf{r}}$ is an eigenvector of $\mathbf{J}$. Therefore, infinite pitch co-eigentwist $\hat{T}_{(c/A)_1}$ is parallel to a principal inertia axis. The translational parts of the other co-eigentwists at A are perpendicular to $\vec{\mathbf{r}}$. From the first equation in (10.38), one concludes that these translations must be the eigenvectors of the matrix $\vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}}\times$. But, one can verify easily that the matrices $\vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}}\times$ and $\mathbf{J}$ have a common set of eigenvectors. Therefore, the translations of the other co-eigentwists are in the principal inertia directions. Consequently, from the second equation in (10.38), the rotations $\vec{\mathbf{w}}_{(m/A)_{2,3}}$ are also in the principal inertia directions. This concludes the proof of the theorem.

■

For any $A \neq M$, Theorem 160 identifies a pure translation parallel to the line passing through A and M which results in a parallel pure force through A, Figure 10.2. This is a well known result in the dynamics of a rigid body. Additionally, the theorem identifies, for any $A \neq M$, two pure rotations which result in pure forces through A that are perpendicular to each other and to the line joining A and M. Another interpretation is as follows. For any given $A \neq M$, there exist three mutually orthogonal pure forces through A, one also passing through M. The one that passes through M causes a pure parallel translation. The others cause two pure rotations whose axes intersect the line through A and M, and whose translations at A are parallel to the forces. Figure 10.2 illustrates the co-eigentwists at $A \neq M$.

Theorem 161 *Consider any point $A \neq M$, whose position from M is $\vec{\mathbf{r}}$. The co-eigenwrench subspace of the mass matrix at A is formed by*

1. *three zero pitch co-eigenwrenches perpendicular to $\vec{\mathbf{r}}$, if $\vec{\mathbf{r}}$ is not along a principal inertia axis.*

2. *one infinite pitch screw parallel to $\vec{\mathbf{r}}$ and two zero pitch screws perpendicularly intersecting $\vec{\mathbf{r}}$ in principal inertia directions, if $\vec{\mathbf{r}}$ is along a principal inertia axis.*

Proof. The co-eigenwrench problem at A is given by

$$\begin{bmatrix} m^{-1}\mathbf{I} - \vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}}\times & -\vec{\mathbf{r}} \times \mathbf{J}^{-1} \\ \mathbf{J}^{-1}\vec{\mathbf{r}}\times & \mathbf{J}^{-1} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{n}}_{m/A} \\ \vec{\mathbf{m}}_{m/A} \end{bmatrix} = m_{m/A}^{-1} \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{m}}_{m/A} \end{bmatrix} \quad \text{or} \tag{10.46}$$

$$\begin{bmatrix} m\mathbf{I} & m\vec{\mathbf{r}}\times \\ -m\vec{\mathbf{r}}\times & \mathbf{J} - m\vec{\mathbf{r}} \times \vec{\mathbf{r}}\times \end{bmatrix} \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{m}}_{m/A} \end{bmatrix} = m_{m/A} \begin{bmatrix} \vec{\mathbf{n}}_{m/A} \\ \vec{\mathbf{m}}_{m/A} \end{bmatrix} \tag{10.47}$$

From the first of equations (10.47) one gets

$$m\vec{\mathbf{r}} \times \vec{\mathbf{m}}_{m/A} = m_{m/A}\vec{\mathbf{n}}_{m/A} \tag{10.48}$$

$$\vec{\mathbf{m}}_{m/A}^{T}\vec{\mathbf{n}}_{m/A} = 0 \tag{10.49}$$

So, as before, the co-eigenwrenches are either zero or infinite pitch screws. From (10.48), the force parts are either zero or perpendicular to $\vec{\mathbf{r}}$.

If a co-eigenwrench is an infinite pitch screw then $\vec{\mathbf{n}}_{m/A} = \vec{\mathbf{0}}$. From (10.48), this implies that $\vec{\mathbf{m}}_{m/A} = \alpha\vec{\mathbf{r}}$. Then, by the second equation in (10.46) or (10.47)

$$\mathbf{J}\vec{\mathbf{m}}_{m/A} = m_{m/A}\vec{\mathbf{m}}_{m/A} \tag{10.50}$$

$$\mathbf{J}\vec{\mathbf{r}} = m_{m/A}\vec{\mathbf{r}} \tag{10.51}$$

That is, $\vec{\mathbf{r}}$ is an eigenvector of $\mathbf{J}$ and therefore it is parallel to a principal inertia axis. Then, $m_{m/A}$ is equal to the corresponding principal inertia. Converse is also true. That is, if $\vec{\mathbf{r}}$ is parallel to a principal inertia axis then there exists an infinite pitch co-eigenscrew parallel to the $\vec{\mathbf{r}}$. For the remaining co-eigenwrenches, $\vec{\mathbf{m}}_{m/A}$ is perpendicular to $\vec{\mathbf{r}}$. Then, by (10.48) and the identity $\vec{\mathbf{r}} \times \vec{\mathbf{r}} \times \vec{\mathbf{m}}_{m/A} = -r^2\vec{\mathbf{m}}_{m/A}$ (for $\vec{\mathbf{r}} \perp \vec{\mathbf{m}}_{m/A}$), one gets $\vec{\mathbf{m}}_{m/A} = \beta\vec{\mathbf{r}} \times \vec{\mathbf{n}}_{m/A}$. So, as before, there exists a point on the line through A and M at which $\vec{\mathbf{m}}_{m/A} = \vec{\mathbf{0}}$. So, the remaining co-eigenwrenches are pure forces intersecting $\vec{\mathbf{r}}$. Furthermore, from (10.48), these pure forces are parallel to the principal inertia directions.

At all other points $\vec{\mathbf{n}}_{m/A} \neq \vec{\mathbf{0}}$, $\vec{\mathbf{m}}_{m/A} \neq \alpha\vec{\mathbf{r}}$ and all co-eigenwrenches are zero pitch screws with directions perpendicular to $\vec{\mathbf{r}}$.

■

Theorem 161 identifies three mutually perpendicular pure rotations through a given point A, not on a principal inertia axis, which result in pure forces. If point A is on a principal inertia axis, then one rotation is parallel to the inertia axis and through M, and, causes a pure parallel moment. The other rotations cause pure forces that perpendicularly intersect the line through A and M.

An observation is that although the co-eigenwrenches in the above theorem are zero pitch screws when $A \neq M$, there exists an infinite pitch screw element in the co-eigenwrench subspace for any such A. This is easily seen from the theorem since it predicts that, for $A \neq M$, the co-eigenwrench directions are perpendicular to $\vec{\mathbf{r}}$. Therefore, their force vectors form a dependent set so that there exists a linear combination of the co-eigenwrenches which gives a zero force part, leading to a pure

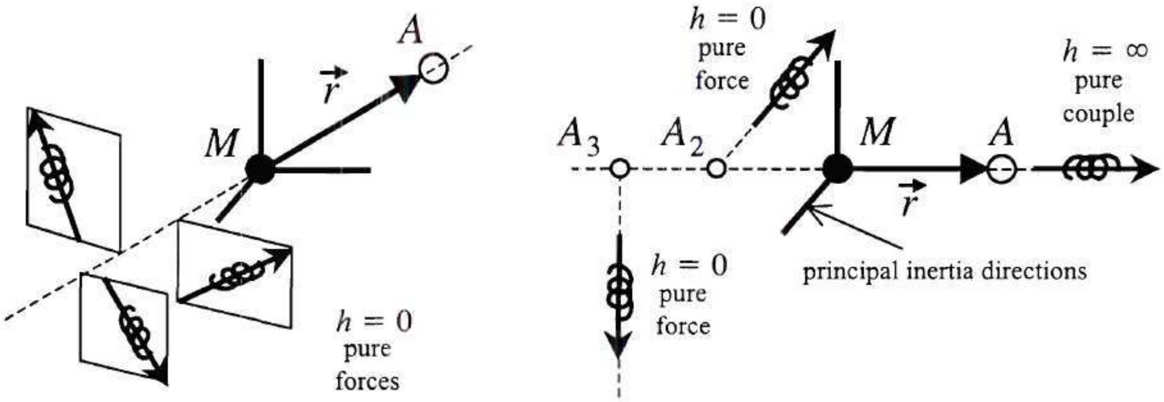


Figure 10.3: Co-eigenwrenches of mass for a point A away from the center of mass. The 3-system formed by these screws is a space of wrenches that only cause pure rotations through A.

couple. This can also be observed from equations (10.46) and (10.47) which shows that the off-diagonal submatrix $m\vec{\mathbf{r}}\times$ has a one-dimensional null space spanned by $\vec{\mathbf{r}}$. Therefore, there exists only one independent infinite screw. Additionally, the infinite pitch screw element is due to a rotation parallel to $\vec{\mathbf{r}}$. When $\vec{\mathbf{r}}$ coincides with one principal inertia axis, the infinite pitch screw becomes a co-eigenwrench. This allows the unification of Theorems 160 and 161 as,

Corollary 162 *At any point $A \neq M$, the co-eigentwist (co-eigenwrench) subspace of the mass matrix of single body is spanned by one infinite and two zero pitch screws. For co-eigentwists, the infinite pitch screw is a translation parallel to the line through A and M. For co-eigenwrenches, the infinite pitch screw is a pure couple whose action is a rotation parallel to the line through A and M. At M, the subspaces are spanned by three infinite pitch screws.*

10.3 Axes of Percussion

The zero pitch screws of Corollary 162 which cause zero pitch screws through some other points, have a practical explanation in the dynamics of rigid bodies. Figure 10.4 illustrates a well known

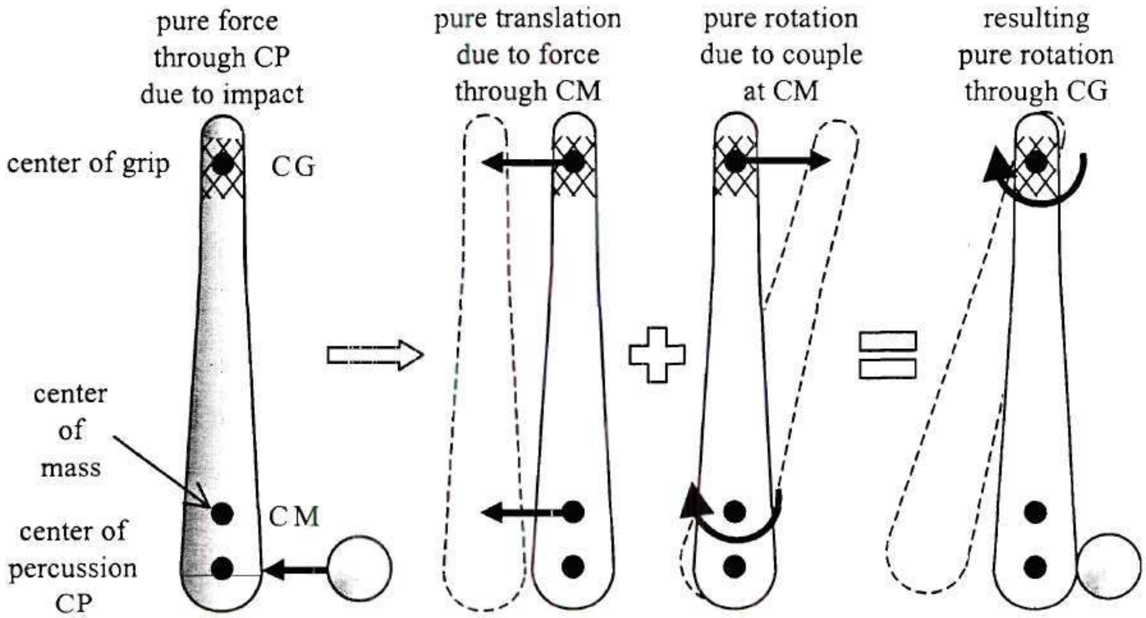


Figure 10.4: A ball hitting a baseball bat at the center of percussion creates negligible translation at another point inside the region where the bat is gripped. The linear impulsive force imparted to the grip is minimized so that the sensation of "sting" is reduced.

example. The figure shows a baseball bat about to hit a ball. The collision creates an impulsive pure force acting through the point of collision. The player holds the bat at other end. The region of grip is shown by crossed lines. The equivalent load system at the center of mass, indicated by CM in the figure, is a force parallel and equal to the force at the point of collision, and a couple perpendicular to the plane of figure. The force at CM tends to create a parallel translation, which is to the left in the figure and is the same for all points. The couple, on the other hand, tends to create a rotation about CM, which is clockwise in the figure. Consider the points on the line through the point of collision and CM. For all points above CM, the translation induced by the rotation cancels a portion of the translation induced by the linear force. Since the translation induced by the rotation increases linearly by the distance from CM, there exists a point on the line where it exactly cancels the effect of the rotation. That is, there exists a point which does not translate. In the literature, this point is sometimes called the instantaneous center of rotation. Right after the impact, the bat tends to rotate about an axis through this point without any translation.

Given the center of mass and an expected center of impact, a baseball bat (or a tennis racket, golf club, sledge hammer, etc.) is designed so that the area of the grip is centered around the instantaneous center of rotation. This reduces the sensation of sting or shock experienced by players in their hands. This sensation is actually due to the sudden change in the linear acceleration and jerk during and after the impact, which manifests itself as an impulsive force at the grip.

Historically, the point of impact is called the **center of percussion**, shown as CP in Figure 10.4. The center of percussion is one of the points known as the *sweet spots* in the terminology of sports instruments design. The other sweet spots have to do with the vibrations induced by the impact.

10.3.1 Center of Percussion Redefined and Generalized

To understand the physical correspondence, consider the co-eigentwist case for a general rigid body, as shown in Figure 10.5, where M is the center of mass. The pure forces through point $A \neq M$ generate the co-eigentwist system. From the results of previous sections, there exists two zero pitch co-eigentwists (pure rotations), which are shown as dashed lines perpendicular to $\vec{\mathbf{r}} = \overrightarrow{MO}$. Either of these two co-eigentwists is due to a pure force through A parallel to the translation at A induced by the co-eigentwist. Thus, a co-eigentwist and its corresponding pure force are perpendicular since the translation induced by a zero pitch twist is always perpendicular to the twist axis.

Figure 10.5 shows that the rigid body is attached to the ground at a point on the axis of a zero pitch co-eigentwist via a revolute joint co-axial with the axis. Therefore, if a pure force applied at A parallel to the translation at A induced by the co-eigentwist, then the body tends to only

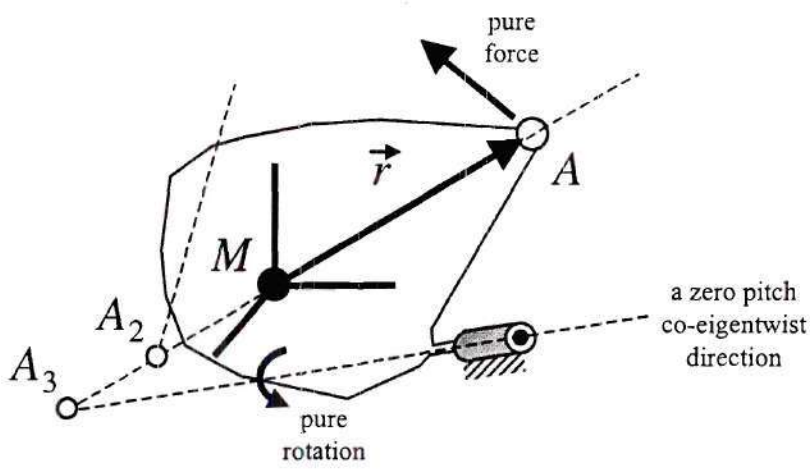


Figure 10.5: A zero pitch co-eigentwist with respect to point A is a pure rotation caused by a pure force through A. A revolute joint incorporated on the zero pitch co-eigentwist axis does not react to the instantaneous motion induced by the force through A.

rotate about the co-eigentwist axis, i.e. about the joint axis. Thus, the joint shows no reaction to the motion. As a result, the zero pitch co-eigentwists of a rigid body and their pure forces clearly correspond to the classical center of percussion phenomena.

The following analysis and theorem are needed in order to present a systematic definition of the center of percussion concept.

First consider a zero pitch co-eigenscrew due to a pure force through A. Clearly, only the line of force is important not the point A, Figure 10.6. This leads to the following general theorem, which applies to both stiffness and mass mappings.

Theorem 163 *The intersection of two co-eigenscrew systems generated by A and B, $A \neq B$, is a 1-system of screws due to a line vector through both A and B.*

Proof. Assuming non-singular mappings, every screw is the image of a unique screw. The co-

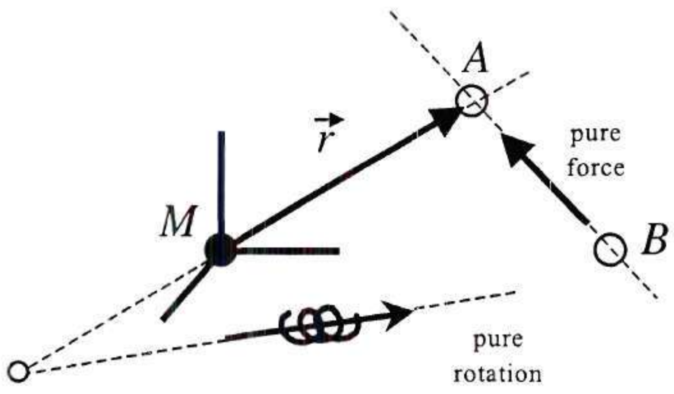


Figure 10.6: A pure force resulting in a pure rotation can be considered as acting at any point on the line of its action.

eigenscrew system generated by A is the image of the line-vector bundle at A. Also, the co-eigenscrew system generated by B is the image of the line-vector bundle at B. By the non-singularity, the intersection of the two co-eigensystems is the image of the intersection of the line-vector bundles at A and B. For, $A \neq B$, the two line-vector bundles have only one independent common element which pass through both A and B. This proves the theorem.

■

In the literature, the center of percussion is usually presented as a planar problem leading to identification of a *center* rather than a line of points. Hence, the co-eigentwist case presents the phenomenon as a spatial problem. Figure 10.5 indicates that there is no preferred point on a zero pitch co-eigenscrew axis for the joint location. Therefore, the center of percussion phenomenon is characterized by a pair of axes; one is the axis of a zero pitch co-eigentwists (pure rotation), the other is the axis of the zero pitch wrench (pure force) perpendicular to the co-eigentwist. The points on the axis of the pure force correspond to the centers of percussion, the points on the co-eigentwist correspond to the loci of joints.

A second observation is that in the definition of center of percussion, the axis of pure force and the axis of joint are equally important. In the literature, it seems as if the center of percussion is perceived as an independent property of a rigid body, similar to the center of mass. In actuality, the center of percussion is meaningless unless a joint location is specified. Therefore, the following definitions are proposed.

Definition 164 *Given the center of mass M and a point $A \neq M$ of a rigid body,*

1. *the axes of the pure forces through A, corresponding to the zero pitch co-eigentwists, are called the* ***axes of percussion*** *through A, and*

2. *the axes of the zero pitch co-eigentwists of $E_{cT/A}$ are called the* ***joint axes of percussion*** *generated by A.*

The duality is introduced by considering the co-eigenwrenches instead of co-eigentwists. From Theorem 161, for a point $A \neq M$, a rigid body generally has three zero pitch co-eigenwrenches. Since the co-eigenwrenches are the images of zero pitch twists, for a general point $A \neq M$ one has three pure rotations resulting in three pure forces. Therefore, the co-eigenwrenches of a rigid body also explain to the center of percussion phenomenon. The three zero pitch co-eigenwrenches correspond to axes of percussion and the resulting three rotations correspond to joint axes of percussion. Thus, Definition 164 can also be presented as,

Definition 165 *Given the center of mass M and a point $A \neq M$ of a rigid body,*

1. *the axes of the pure rotations through A, corresponding to the zero pitch co-eigenwrenches, are called the* ***joint axes of percussion*** *through A, and*

2. *the axes of the zero pitch co-eigenwrenches of $E^{*}_{cW/A}$ are called the* ***axes of percussion*** *generated by A.*

The following theorem shows that Definitions 164 and 165 are equivalent.

Theorem 166 *The co-eigentwist system generated by A has a zero pitch screw (a pure rotation) through another point B if and only if the co-eigenwrench system generated by B has a zero pitch screw through A (a pure force), which are the actions of each other.*

Proof. Consider a zero pitch screw of the co-eigentwist system at A, which passes through B. Then,

$$\hat{M}_A \begin{bmatrix} \overrightarrow{AB} \times \vec{\mathbf{w}} \\ \vec{\mathbf{w}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{n}} \\ \vec{0} \end{bmatrix} \tag{10.52}$$

By transforming all quantities to B one obtains

$$\hat{M}_B^{-1} \begin{bmatrix} \vec{\mathbf{n}} \\ \overrightarrow{BA} \times \vec{\mathbf{n}} \end{bmatrix} = \begin{bmatrix} \vec{0} \\ \vec{\mathbf{w}} \end{bmatrix} \tag{10.53}$$

which shows the existence of a zero pitch screw of the co-eigenwrench system at B that passes through A. Converse is similar. Since both screw are mapped to each other via the mass matrix and its inverse, they are the actions of each other.

■

This theorem is a general one and therefore applies to the stiffness case also. Theorem 166 means that choosing a center of percussion (point A) and finding the corresponding joint of percussion (point B via co-eigentwists) is equivalent to choosing a joint of percussion (point B) and finding a corresponding center of percussion (point A via co-eigenwrenches).

Definitions and theorems presented in this section lead to the following corollary as a summary.

Corollary 167 *For every point $A \neq M$ of a rigid body, in general*

1. *there exist two axes of percussion through A and two corresponding joint axes which intersect the line through A and M. Each joint axis is perpendicular to its axis of percussion.*

2. *there exist three joint axes of percussion through A and three corresponding axes of percussion. Each axis of percussion is perpendicular to both its joint axis and the line through A and M.*

10.3.2 An Improved Golf Club or Tennis Racket Design?

From the co-eigentwists problem, for any point $A \neq M$ generally there exist two axes of percussion corresponding to two joint axes of percussion (zero pitch co-eigentwists). Note that both axes of percussions pass through A. Then, a degenerate case occurs if the two joint axes are intersecting. In such a situation, any linear combination of the joint axis would be a zero pitch element of the co-eigentwist space, which is also the image of a pure force through A. Therefore, there would be a plane of axes of percussion and a corresponding plane of joint axes. So, a particular question is whether this situation can happen for a mass matrix, and if so, what are the necessary and sufficient conditions.

First, recall that the two zero pitch co-eigentwists intersect the line through M and A, Theorem 160. The intersection points are labeled as A_2 and A_3 in Figure 10.2. So, if the joint axes intersect each other then $A_2 = A_3$. From (10.44) the locations of these points are given as

$$\overrightarrow{AA_i} = \left(\frac{m_{(t/A)_i}}{m} - 1\right)^{-1} \vec{\mathbf{r}} \tag{10.54}$$

where $m_{(t/A)_i}$ are the linear co-eigenmasses corresponding to the zero pitch co-eigentwists generated by A, and $\vec{\mathbf{r}} = \overrightarrow{MA}$. From (10.54), $A_2 = A_3$ can happen if and only if $m_{(t/A)_2} = m_{(t/A)_3}$. From (10.38), $m^{-1}_{(t/A)_i}$ are the eigenvalues of the matrix $m^{-1}\mathbf{I} - \vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}}\times$. It is already known that an eigenvalue and its corresponding eigenvector of this matrix are m^{-1} and $\vec{\mathbf{r}}$, which corresponds to the infinite pitch co-eigentwist, Theorem 160. If the double eigenvalue is to correspond to the zero pitch co-eigentwists, they should also correspond to the eigenvectors of $m^{-1}\mathbf{I} - \vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}}\times$ that are perpendicular to $\vec{\mathbf{r}}$. Let $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ be these eigenvectors, and $\vec{\mathbf{r}} = r\vec{\rho}$, where $\vec{\rho}$ is a unit vector, such that $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\boldsymbol{\rho}}$ form a right handed orthonormal set. Then, letting $m^* = m_{(t/A)_2} = m_{(t/A)_3}$,

$$\left(m^{-1}\mathbf{I} - \vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}}\times\right)\vec{\mathbf{u}} = m^*\vec{\mathbf{u}} \tag{10.55}$$

$$\left(m^{-1}\mathbf{I} - \vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}}\times\right)\vec{\mathbf{v}} = m^*\vec{\mathbf{v}} \tag{10.56}$$

By using $\vec{\mathbf{r}} \times \vec{\mathbf{u}} = r\vec{\mathbf{v}}$ and $\vec{\mathbf{r}} \times \vec{\mathbf{v}} = -r\vec{\mathbf{u}}$, and, premultiplying (10.55) by $\vec{\mathbf{u}}^T$ and (10.56) by $\vec{\mathbf{v}}^T$, one gets

$$\vec{\mathbf{v}}^T\mathbf{J}^{-1}\vec{\mathbf{v}} = \vec{\mathbf{u}}^T\mathbf{J}^{-1}\vec{\mathbf{u}} = \frac{m^*}{r^2} \tag{10.57}$$

which shows that $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are the eigenvectors of $\mathbf{J}^{-1}$, and therefore of $\mathbf{J}$, corresponding to a double eigenvalue. Clearly, every linear combination of $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ is also an eigenvector. This necessarily makes $\vec{\mathbf{r}}$ an eigenvector of $\mathbf{J}$, too. The converse is trivial. Hence, the following theorem is proven.

Theorem 168 *For a point $A \neq M$, the zero pitch co-eigentwists of mass matrix intersect each other at a point on the line through A and M if and only if A is on a principal axis of inertia and there exists a double principal inertia corresponding to the plane perpendicular to the line through A and M. Then, every zero pitch twist through the intersection is a co-eigentwist giving a pencil of zero pitch screws.*

Note that in this case, the mass matrix becomes similar to the classical RCC devices presented in Chapter 9.

The importance of having a pencil of zero co-eigentwists and the corresponding pencil of pure forces is that now the rigid body responds to pure forces through A by pure rotations through $J_A = A_2 = A_3$. So, if the body is fixed to the ground at J_A by a joint that allows rotations in at least two perpendicular directions, such as a Hook's joint or a spherical joint, then it does not matter in what direction the force through A is applied as long as it is in the plane corresponding to the double principal inertia. One still gets a pure rotation response through J_A, hence no reaction. Such a device has superior performance in reducing the reactions at the joint. For example, in the

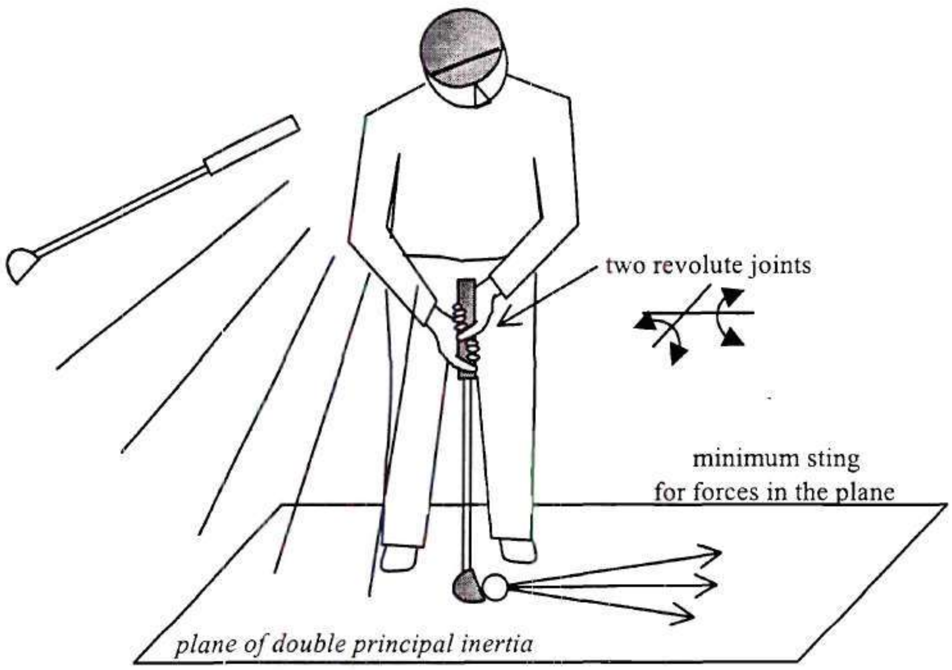


Figure 10.7: A golf club with a double principal inertia induces minimum sting for all directions of hit through the center of percussion and in the plane of the double principal inertia. This design can accommodate misalignments in the desired hit direction.

case of baseball bat there exists a double principal inertia due to the symmetry about the bat axis and homogenous mass distribution. Therefore, any ball hitting the bat perpendicular to the bat longitudinal axis and at the center of percussion will cause a minimum sting to the players hand. However, for a golf club or tennis racket this would not be the case unless the designer insures that the instrument has a double principal inertia in the plane normal to the axis through the hand and the collision point, see Figure 10.7.

10.3.3 Locations of Joints and Axes of Percussion

For unconstrained rigid bodies the mass matrix is positive definite. Therefore, the matrix in (10.55) is also a positive definite matrix, from which the following relation is concluded.

$$m_{t/A}^{-1} \geq m^{-1} \quad \Rightarrow \quad m \geq m_{t/A} \tag{10.58}$$

The intersection points for the zero pitch co-eigentwists, A_2 and A_3, are located from A by (10.54)

$$\overrightarrow{AA_i} = -\left(1 - \frac{m_{(t/A)_i}}{m}\right)^{-1} \vec{\mathbf{r}} = -\left(\frac{m}{m - m_{(t/A)_i}}\right) \vec{\mathbf{r}} \qquad i = 2, 3 \tag{10.59}$$

which are both on the far side, with respect to A, of the line joining M and A, since $\frac{m}{m - m_{(t/A)_i}} > 0$. This is also illustrated in Figure 10.2. Therefore, the center of mass is always found in the middle of the line segment from the center of percussion to the intersection of the joint axis with the line through A and M.

Sometimes in practice, the location of the joint with respect to the center of mass is known and the location of the center of percussion is desired. Consider that the centers of mass and percussion and the joint of percussion are collinear. Let $\vec{\mathbf{r}}_J$ be the position vector of a joint of percussion with respect to M. Then, if $\vec{\mathbf{r}}_P$ is the location of the center of percussion with respect to M, by using the equation (10.59) one gets

$$\vec{\mathbf{r}}_J - \vec{\mathbf{r}}_P = -\frac{m}{m - m_{(t/A)_i}} \vec{\mathbf{r}}_P \tag{10.60}$$

$$\vec{\mathbf{r}}_J = \left(1 - \frac{m}{m - m_{(t/A)_i}}\right) \vec{\mathbf{r}}_P = -\frac{m_{(t/A)_i}}{m - m_{(t/A)_i}} \vec{\mathbf{r}}_P \tag{10.61}$$

$$\vec{\mathbf{r}}_P = -\left(\frac{m}{m_{(t/A)_i}} - 1\right) \vec{\mathbf{r}}_J \qquad m_{(t/A)_i} \neq m \tag{10.62}$$

10.3.4 Classical Centers of Percussion

In classical center of percussion problems, the axis of rotation (joint axis) and the applied force (axis of percussion) are parallel to principal axes of inertia. In the generalized case presented in this study, this not necessarily true, although the same behavior is achieved. That is, the joint experiences no linear force. To find the conditions for a classical center of percussion one may employ (10.38), with the constraint that the rotation $\vec{\mathbf{w}}_{m/A}$ is an eigenvector of $\mathbf{J}$. So,

$$\begin{aligned}
\begin{bmatrix} m^{-1}\mathbf{I} - \vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}}\times & -\vec{\mathbf{r}} \times \mathbf{J}^{-1} \\ \mathbf{J}^{-1}\vec{\mathbf{r}}\times & \mathbf{J}^{-1} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}}_{m/A} \\ \vec{0} \end{bmatrix} &= m_{t/A}^{-1} \begin{bmatrix} \vec{\mathbf{t}}_{m/A} \\ \vec{\mathbf{w}}_{m/A} \end{bmatrix} && (10.63) \\
\mathbf{J}\vec{\mathbf{w}}_{m/A} &= m_{t/A}\vec{\mathbf{r}} \times \vec{\mathbf{t}}_{m/A} && (10.64) \\
\alpha\vec{\mathbf{w}}_{m/A} &= m_{t/A}\vec{\mathbf{r}} \times \vec{\mathbf{t}}_{m/A} && (10.65)
\end{aligned}$$

Now, using equation (10.39) in (10.65) one gets

$$\alpha\vec{\mathbf{w}}_{m/A} = m_{t/A}\left(\frac{m_{t/A}}{m} - 1\right)^{-1} \vec{\mathbf{r}} \times \vec{\mathbf{r}} \times \vec{\mathbf{w}}_{m/A} \qquad (10.66)$$

which simply states that $\vec{\mathbf{w}}_{m/A}$ is perpendicular to $\vec{\mathbf{r}}$. Since, $\vec{\mathbf{t}}_{m/A}$ is also perpendicular to $\vec{\mathbf{w}}_{m/A}$, the case that the rotation is along a principal inertia axis happens when both $\vec{\mathbf{r}}$ and $\vec{\mathbf{t}}_{m/A}$ (force direction) lie in the plane formed by two principal inertia axes. If, furthermore, $\vec{\mathbf{t}}_{m/A}$ (the direction of axes of percussion) is also parallel to a principal inertia axis, then $\vec{\mathbf{r}}$ must be parallel to the remaining principal inertia axis. Therefore, a complete classical percussion center is obtained.

Converse is also true and follows from Theorem 160. Same conclusions apply to the co-eigenwrench case. This proves the following theorem.

Theorem 169 *The co-eigenscrews and their actions with respect to a point $A \neq CM$ are parallel to the principal inertia axes if and only if the point A lies on a principal axis. The zero pitch co-eigentwists define classical joint axes of percussion on principal axes of inertia, and the zero pitch co-eigenwrenches define classical axes of percussion. All axes are parallel to principal inertia directions.*

Explicit solutions in the case of classical axes of percussion are possible. If $\vec{\mathbf{r}}$ is parallel to a principal inertia axis, then $\mathbf{J}^{-1}$ can be written as

$$\mathbf{J}^{-1} = \alpha \vec{\mathbf{r}}\vec{\mathbf{r}}^T + \beta \vec{\mathbf{u}}\vec{\mathbf{u}}^T + \gamma \vec{\mathbf{v}}\vec{\mathbf{v}}^T \tag{10.67}$$

where $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are unit vectors along the other two principal inertia axes. Then, using (10.67) in (10.63)

$$\begin{aligned}
\left(m^{-1}\mathbf{I} - \vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}}\times\right) \vec{\mathbf{t}}_{m/A} &= m_{t/A}\vec{\mathbf{t}}_{m/A} &\quad (10.68)\\
-\vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}} \times \vec{\mathbf{t}}_{m/A} &= \left(m_{t/A}^{-1} - m^{-1}\right) \vec{\mathbf{t}}_{m/A} &\quad (10.69)\\
r^2 \left(\beta \vec{\mathbf{v}}\vec{\mathbf{v}}^T + \gamma \vec{\mathbf{u}}\vec{\mathbf{u}}^T\right) \vec{\mathbf{t}}_{m/A} &= \left(m_{t/A}^{-1} - m^{-1}\right) \vec{\mathbf{t}}_{m/A} &\quad (10.70)
\end{aligned}$$

for which solutions are

$$\begin{aligned}
\vec{\mathbf{t}}_{(m/A)_2} &= \vec{\mathbf{u}} \qquad m_{(t/A)_2} = \frac{mI_2}{mr^2 + I_2} &\quad (10.71)\\
\vec{\mathbf{t}}_{(m/A)_3} &= \vec{\mathbf{v}} \qquad m_{(t/A)_3} = \frac{mI_3}{mr^2 + I_3} &\quad (10.72)
\end{aligned}$$

I_2 and I_3 are the principal inertias in directions perpendicular to $\vec{\mathbf{r}}$. These result can be used in the equations for the locations of joints and axes of percussion, (10.59) and (10.62) for design purposes.

10.3.5 Numerical Examples

Let the following mass matrix of a rigid body be given with respect to M.

$$\hat{M}_M = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 & 0 & 20 \end{bmatrix} \tag{10.73}$$

For simplicity, points on principal inertia axes are considered. So, let a point A, whose position with respect to M is $\vec{\mathbf{r}}_A = \begin{bmatrix} 0 & -1 & 0 \end{bmatrix}^T$, be considered. The mass matrix at A is

$$\hat{M}_A = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & -10 \\ 0 & 10 & 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 10 & 0 & 0 \\ 0 & 0 & 10 & 15 & 0 & 0 \\ 0 & 0 & 0 & 0 & 15 & 0 \\ -10 & 0 & 0 & 0 & 0 & 30 \end{bmatrix} \tag{10.74}$$

<u>Design by Co-eigentwists</u> Design by co-eigentwists amounts to solving the co-eigentwist problem at A. This means that the forces of percussion are constrained to be through point A. This is useful in problems where the point of impact is known, but the directions of the forces and joints are not.

The co-eigentwists and corresponding translational co-eigenmasses are

$$\hat{T}_{c/A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{2}{3} \\ 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 \end{bmatrix} \qquad \mathbf{m}_{t/A} = \begin{bmatrix} \frac{20}{3} & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & \frac{10}{3} \end{bmatrix} \tag{10.75}$$

The infinite pitch co-eigentwist is along y-axis as $\vec{\mathbf{r}}$ is and corresponding translational co-eigen mass is equal to the classical linear mass, as expected from Theorem 160. Furthermore, first co-eigentwist is a rotation about z-axis caused by a pure force through A parallel to x-axis, see Figure 10.8-(a). So, as expected, A corresponds to a classical center of percussion. The location of the joint corresponding to the first co-eigentwist is calculated with respect to M as

$$\vec{\mathbf{r}}_{J_{A1}/M} = -\frac{m_{(t/A)_1}}{m - m_{(t/A)_1}}\vec{\mathbf{r}}_A = -\frac{\frac{20}{3}}{10-\frac{20}{3}}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \tag{10.76}$$

The third co-eigentwist is also a pure rotation, but about z-axis and caused by a pure force through A parallel to z-axis. The location of its intersection of y-axis is

$$\vec{\mathbf{r}}_{J_{A2}/M} = -\frac{m_{(t/A)_3}}{m - m_{(t/A)_3}}\vec{\mathbf{r}}_A = -\frac{\frac{10}{3}}{10-\frac{10}{3}}\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 0 \end{bmatrix} \tag{10.77}$$

The design by co-eigentwists yields force directions and related joint axes as shown in Figure 10.8-(a).

Design by Co-eigenwrenches Design by co-eigenwrenches amounts to solving the co-eigenwrench problem at A. This means that the rotations, i.e. joint axes, of percussion are constrained to be through point A. This is useful in problems where the joint location is known, but the directions of the joint axes and the axes of percussions are not.

The co-eigenwrench and rotational co-eigenmasses are

$$\hat{W}_{c/A} = \begin{bmatrix} 0 & 0 & -\frac{1}{3} \\ 0 & 0 & 0 \\ \frac{2}{3} & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{m}_{m/A} = \begin{bmatrix} 15 & 0 & 0 \\ 0 & 15 & 0 \\ 0 & 0 & 30 \end{bmatrix} \tag{10.78}$$

The infinite pitch co-eigenwrench is along y-axis as $\vec{\mathbf{r}}$ is. The remaining co-eigenwrenches are pure forces in the principal inertia directions, as expected from Theorem 161. The first co-eigenwrench

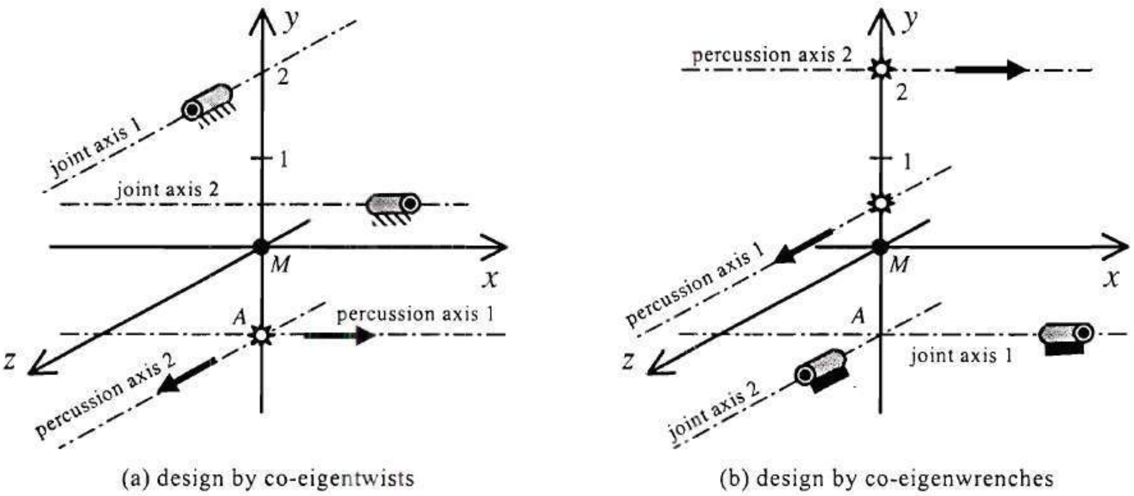


Figure 10.8: The axes and joint axes of percussion generated by a point A on a principal inertia direction (x-, y-, z-axes are the principal inertia directions). (a) Given A through which the force is known to pass, the co-eigentwists gives the permissible directions of the force and related joint axes. (b) Given A through which the joint axis is known to pass, the co-eigenwrenches gives the permissible joint directions and related axes of percussion.

is along z-axis caused by a pure rotation through A parallel to x-axis, Figure 10.8-(b). The third co-eigenwrench is along x-axis and caused by a pure rotation through A parallel to z-axis. The first co-eigenwrench intersects y-axis at 0.5 units above M and the third co-eigenwrench intersects y-axis at 2 units above M.

The design by co-eigenwrenches yields joint directions and related axes of percussions as shown in Figure 10.8-(b).

<u>General Axes of Percussion</u> Consider another point B not on a principal axis of inertia, say, $\vec{\mathbf{r}}_B = \begin{bmatrix} -2 & -1 & 0 \end{bmatrix}^T$. The mass matrix at B is

$$\hat{M}_B = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & -10 \\ 0 & 10 & 0 & 0 & 0 & 20 \\ 0 & 0 & 10 & 10 & -20 & 0 \\ 0 & 0 & 10 & 15 & -20 & 0 \\ 0 & 0 & -20 & -20 & 55 & 0 \\ -10 & 20 & 0 & 0 & 0 & 70 \end{bmatrix} \qquad (10.79)$$

The co-eigentwists and corresponding masses are

$$\hat{T}_{c/B} = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\frac{6}{17} \\ 0 & 0 & \frac{4}{17} \\ \frac{5}{7} & 0 & 0 \end{bmatrix} \qquad \mathbf{m}_{t/B} = \begin{bmatrix} \frac{20}{7} & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & \frac{30}{17} \end{bmatrix} \qquad (10.80)$$

The second co-eigentwist is the infinite pitch screw along $\vec{\mathbf{r}}$, whose co-eigenmass is equal to the classical mass, as Theorem 160 states. The first and the third co-eigentwists are a pure rotations. So, they define two joint axes of percussions. However, for the first co-eigentwist, the rotation is parallel to a principal direction, but the force is not. Similarly, for the third, the force is parallel to a principal direction, but the rotation is not. Thus, for both pairs of axis and joint axis of percussion, one axis is not parallel to a principal direction of inertia. Hence, a classical center of percussion is *not* generated by a point such as B. This verifies the statement of Theorem 169.

<u>Pencils of Axes and Joint Axes of Percussion</u> The mass matrix of previous examples is slightly modified so that the inertia matrix has a double eigenvalue. This is a necessary condition to have

a pencil of axes of percussion and corresponding joint axes, as proven in Theorem 168. So, at the center of mass

$$\hat{M}_M = \text{diag}\begin{pmatrix} 10 & 10 & 10 & 5 & 20 & 20 \end{pmatrix} \tag{10.81}$$

Theorem 168 also requires the generator point, say G, to be on the principal inertia axis perpendicular to the plane of double eigenvalue. So, in this case, $\vec{\mathbf{r}} = \overrightarrow{MG} = r[\ 1 \quad 0 \quad 0\ ]^T$. Let $r = 1$ for simplicity. Then, at G

$$\hat{M}_G = \begin{bmatrix} 10 & 0 & 0 & 0 & 0 & 0 \\ 0 & 10 & 0 & 0 & 0 & 10 \\ 0 & 0 & 10 & 0 & -10 & 0 \\ 0 & 0 & 0 & 5 & 0 & 0 \\ 0 & 0 & -10 & 0 & 30 & 0 \\ 0 & 10 & 0 & 0 & 0 & 30 \end{bmatrix} \tag{10.82}$$

The co-eigentwist problem yields

$$\hat{T}_{c/G} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{1}{3} \\ 0 & -\frac{1}{3} & 0 \end{bmatrix} \qquad \mathbf{m}_{t/G} = \begin{bmatrix} 10 & 0 & 0 \\ 0 & \frac{20}{3} & 0 \\ 0 & 0 & \frac{20}{3} \end{bmatrix} \tag{10.83}$$

As expected, there exists a double linear co-eigenmass. Also, the first co-eigentwist is the infinite pitch twist parallel to $\vec{\mathbf{r}}$, as predicted by Theorem 160. However, both the second and the third co-eigentwists are zero pitch screws through the same point on x-axis, namely $(2, 0, 0)$ with respect to M. Then, every linear combination of these is also a zero pitch twist. Therefore, these zero pitch screws form a pencil of zero pitch twists in yz-plane. It is not difficult to show that every member

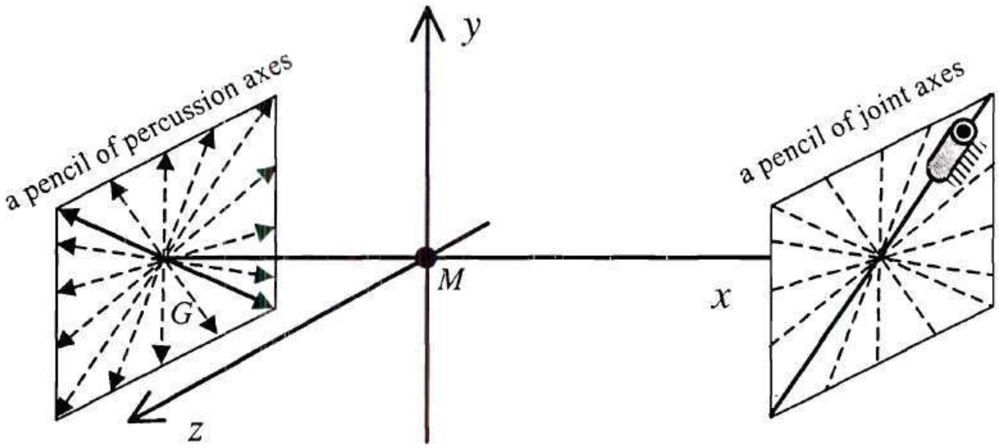


Figure 10.9: A double principal inertia and a point G on the principal inertia axis normal to the plane of double principal inertia lead to a pencil of percussion axes. The corresponding joint axes also form a pencil. To every axis of percussion a perpendicular joint axis corresponds. An example pair is shown by solid lines.

of the pencil is a co-eigentwist. This pencil is shown in Figure 10.9. Since these are zero pitch co-eigentwists, each correspond to a joint axis of percussion. Also shown in the figure is the pencil of pure forces through G that induces the pencil of zero pitch co-eigentwist subspace. Each element of the force pencil through G is an axis of percussion.

10.4 Free Vibration Modes

This section is concerned with the special free vibration modes about the equilibrium of an elastically suspended rigid body. Since both elastic and kinetic properties are responsible for vibrational behavior, stiffness and mass properties appear simultaneously. The special vibration modes are based on isotropic screws, namely pure translations, rotations, forces and couples. If a rigid body is in a harmonic motion that can be described by a screw, then the special vibrational modes are defined as

- **Pure Translation Mode**: The body translates only (∞ pitch).
- **Pure Rotation Mode**: The body rotates only (0 pitch).
- **Pure Force Mode**: The elastic force on the body is a pure force (0 pitch).
- **Pure Couple Mode**: The elastic force on the body is a pure couple (∞ pitch).

The analysis of stiffness and mass presented in this study enables one to investigate these vibration modes systematically. Blanchet and Lipkin [3] investigate the geometric properties of these modes for planar problems in detail.

10.4.1 Necessary Conditions

Blanchet and Lipkin [3] show that the equation of motion can be approximated by

$$\left(\hat{K} - \omega^2 \hat{M}\right) \hat{T} = \hat{0} \tag{10.84}$$

where, $\hat{K}$ is the stiffness matrix, $\hat{M}$ is the mass matrix, $\hat{T}$ is the twist and ω is the eigenfrequency. From linear algebra, (10.84) always has real solutions due to the symmetry and positive definiteness of the matrices. In general, there exists exactly six solutions. If there exists an algebraic multiplicity (in ω^2) of n, then corresponding eigenvectors $\hat{T}$ define an n-system whose every element is an eigenvector.

Let

E_{T_K}, E_{T_M} be the eigentwist subspaces of the stiffness and mass matrices.

E_{cT_K}, E_{cT_M} be the co-eigentwist subspaces of the stiffness and mass matrices.

$E^*_{W_K}$, $E^*_{W_M}$ be the eigenwrench subspaces of the stiffness and mass matrices.

$E^*_{cW_K}$, $E^*_{cW_M}$ be the co-eigenwrench subspaces of the stiffness and mass matrices.

Theorem 170

a) *Any pure translation mode is due to an element of* $E^*_{W_K}$ *and* $E^*_{W_M}$.

b) *Any pure couple mode is due to an element of* E_{T_K} *and* E_{T_M}.

c) *Any pure rotation mode is due to an element of* $E^*_{cW_K}$ *and* $E^*_{cW_M}$.

d) *Any pure force mode is due to an element of* E_{cT_K} *and* E_{cT_M}.

Proof. The proof is easily obtained from the definitions of the eigen- and co-eigenscrew subspaces.

a) The twist $\hat{T}$ is a pure translation. Then, $\hat{K}\hat{T} \in E^*_{W_K}$ and $\omega^2 \hat{M}\hat{T} \in E^*_{W_M}$.

b) The wrench $\hat{W}$ is a pure couple. Then, $\hat{K}^{-1}\hat{W} \in E_{T_K}$ and $\hat{M}^{-1}\hat{W} \in E_{T_M}$.

c) The twist is a pure rotation. Then, $\hat{K}\hat{T} \in E^*_{cW_K}$ and $\omega^2 \hat{M}\hat{T} \in E^*_{cW_M}$.

d) The wrench $\hat{W}$ is a pure force. Then, $\hat{K}^{-1}\hat{W} \in E_{cT_K}$ and $\hat{M}^{-1}\hat{W} \in E_{cT_M}$.

■

Note that, in (a), there exists a wrench which belongs to both $E^*_{W_K}$ and $E^*_{W_M}$, since $\hat{K}\hat{T} = \omega^2 \hat{M}\hat{T}$. Similar observations for (b), (c) and (d) lead to the following.

Corollary 171 *The necessary conditions for the existence of special vibrational modes are:*

a) *Pure Translation Mode* $\Rightarrow$ $E^*_{W_K} \cap E^*_{W_M} \neq \{\hat{0}\}$

b) *Pure Couple Mode* $\Rightarrow$ $E_{T_K} \cap E_{T_M} \neq \{\hat{0}\}$

c) *Pure Rotation Mode* $\Rightarrow$ $E^*_{cW_K} \cap E^*_{cW_M} \neq \{\hat{0}\}$

d) *Pure Force Mode* $\Rightarrow$ $E_{cT_K} \cap E_{cT_M} \neq \{\hat{0}\}$

Note that in the above Corollary, the converses are not true. This is due to the fact that, in general, the equation of motion cannot be satisfied by the intersection of the subspaces. That is, given two matrices $\hat{K}$ and $\hat{M}$, a non-empty intersection of eigenscrew (co-eigenscrew) subspaces of $\hat{K}$ and $\hat{M}$ cannot alone guarantee that the equation of motion (10.84) is satisfied for some elements in the intersection. For example, $E^*_{W_K} \cap E^*_{W_M}$ may contain non-zero elements, but not necessarily any pure force or couple.

In previous sections, it is shown that $E^*_{W_M}$ and E_{T_M} contain only zero pitch screws through M, and $E^*_{cW_M/M}$ and $E_{cT_M/M}$ are infinite pitch screw subspaces. Together with Corollary 171, these lead to

Corollary 172

a) *Pure Translation Mode* $\Rightarrow$ $E^*_{W_K}$ *contains pure forces through* M.

b) *Pure Couple Mode* $\Rightarrow$ E_{T_K} *contains pure rotations through* M.

c) *Pure Rotation Mode through* M $\Rightarrow$ $E^*_{cW_K/M}$ *contains pure couples.*

d) *Pure Force Mode through* M $\Rightarrow$ $E_{cT_K/M}$ *contains pure translations.*

In Corollary 172, (a) means a pure translation mode requires the wrench to be a pure force through M. But, by definition, this is a pure force mode through M, i.e. (d). Similarly, (b) implies (c). Converses are also true: (d) implies (a) and (c) implies (b). Hence,

Corollary 173

1. *There exists a pure translation mode if and only if there exists a pure force mode through* M.

2. *There exists a pure couple mode if and only if there exists a pure rotation mode through* M.

Corollary 173 indicates that pure translation and couple modes are equivalent to pure force and rotation modes through M. Therefore, in the rest of this chapter, only pure force and rotation modes not through M are considered.

10.4.2 Pure Translation or Force-through-M Modes

Assume that there exists a pure translation (force through M) mode given by $\hat{T}_M = [\ \vec{\mathbf{t}}^T \quad \vec{\mathbf{0}}^T\ ]^T$. Then, using the stiffness and mass submatrices, and (10.84),

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{0}} \end{bmatrix} = \omega^2 \begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{J} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{0}} \end{bmatrix} = m\omega^2 \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{0}} \end{bmatrix} \tag{10.85}$$

But, from Chapter 3, this means that $[\ \vec{\mathbf{t}}^T \quad \vec{\mathbf{0}}^T\ ]^T$ is an eigenwrench of the stiffness with a linear eigenstiffness of $k_f = m\omega^2$. So,

Theorem 174 *A pure translation and force-through-M mode exists if and only if the stiffness has a zero pitch eigenwrench through M. The axis and the direction of the zero pitch eigenwrench of the stiffness through M respectively give the pure force and pure translation modes. The corresponding eigenfrequency is* $\omega = \sqrt{\frac{k_f}{m}}$.

10.4.3 Pure Couple or Rotation-through-M Modes

Assume that there exists a pure couple (rotation through M) mode given by $\hat{W}_M = [\ \vec{\mathbf{0}}^T \quad \vec{\tau}^T\ ]^T$. Then, using the stiffness and mass submatrices, and (10.84),

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{w}} \end{bmatrix} = \omega^2 \begin{bmatrix} m\mathbf{I} & 0 \\ 0 & \mathbf{J} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{w}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\tau} \end{bmatrix} \tag{10.86}$$

where $[\ \vec{\mathbf{t}}^T \quad \vec{\mathbf{w}}^T\ ]^T$ is the corresponding twist at M. From the right side of (10.86), one gets $\vec{\mathbf{t}} = \vec{\mathbf{0}}$. That is, the twist must be a rotation through M, which is expected from previous theorems. By definitions, $[\ \vec{\mathbf{0}}^T \quad \vec{\mathbf{w}}^T\ ]^T$ is a zero pitch screw through M of the eigentwist system, but it is not necessarily an eigentwist. Similarly, $[\ \vec{0}^T \quad \vec{\tau}^T\ ]^T$ is an infinite pitch screw of the co-eigenwrench system at M, but it is not necessarily a co-eigenwrench. From (10.86) again,

$$\mathbf{C}\vec{\mathbf{w}} = \omega^2 \mathbf{J}\vec{\mathbf{w}} \tag{10.87}$$

which leads to

Theorem 175 *A pure couple and rotation-through-M mode exists if and only if the stiffness has a zero pitch screw through M in the eigentwist system, whose direction $\vec{\mathbf{w}}$ satisfies the generalized eigenvalue problem (10.87). The axis of the zero pitch element gives the pure rotation axis. The direction of $\mathbf{J}\vec{\mathbf{w}}$ gives the pure couple mode.*

10.4.4 Pure Rotation or Force Not-through-M Modes

In previous sections, the pure translation and couple modes, which correspond to pure force and rotation modes through M, are shown to be originating from the eigenscrew subspaces of stiffness and mass. Since the eigenscrew subspaces are origin independent and are well known for the mass matrix, the conditions for the existence of these modes are obtained without much difficulty. However, what about a pure force or rotation mode through a point $A \neq M$? Since these concern the co-eigenscrew subspaces which are distinct at every point, the answer is not straightforward.

Pure Rotation Not-through-M Modes Consider the equation of motion (10.84) at a point $A \neq M$. For a pure rotational mode through A, it is

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}_A \\ \mathbf{B}_A^T & \mathbf{C}_A \end{bmatrix} \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{w}} \end{bmatrix} - \omega^2 \begin{bmatrix} m\mathbf{I} & m\vec{\mathbf{r}}\times \\ -m\vec{\mathbf{r}}\times & \mathbf{J} - m\vec{\mathbf{r}}\times\vec{\mathbf{r}}\times \end{bmatrix} \begin{bmatrix} \vec{\mathbf{0}} \\ \vec{\mathbf{w}} \end{bmatrix} = \hat{0} \tag{10.88}$$

Then, by definition, the resulting wrench is in co-eigenwrench subspaces of both $\hat{K}_A$ and $\hat{M}_A$. That is, the intersection $E^*_{cW_K/A} \cap E^*_{cW_M/A}$ is not empty, as before. However, unlike that at M, $E^*_{cW_M/A}$

is no longer a space of infinite pitch screws only. By Corollary 162, it has a one dimensional subspace of infinite pitch screws and a two dimensional subspace of finite pitch screws.

If the wrench is in the infinite pitch subspace of $E^*_{cW_M/A}$ then, by definition, the mode also corresponds to a pure couple mode. But, by Theorem 175, this means that the pure rotation is through M. Therefore, for pure rotation modes not through M, the wrench must be a finite pitch screw. In other words,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}_A \\ \mathbf{B}_A^T & \mathbf{C}_A \end{bmatrix} \begin{bmatrix} \vec{0} \\ \vec{\mathbf{w}} \end{bmatrix} = \omega^2 \begin{bmatrix} m\mathbf{I} & m\vec{\mathbf{r}}\times \\ -m\vec{\mathbf{r}}\times & \mathbf{J} - m\vec{\mathbf{r}}\times\vec{\mathbf{r}}\times \end{bmatrix} \begin{bmatrix} \vec{0} \\ \vec{\mathbf{w}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\tau} \end{bmatrix} \tag{10.89}$$

such that $\vec{\mathbf{f}} \neq \vec{0}$, meaning $\mathbf{B}_A\vec{\mathbf{w}} \neq \vec{0}$ and $\vec{\mathbf{w}} \nparallel \vec{\mathbf{r}}$. Equations (10.89) can be separately given as

$$\mathbf{B}_A\vec{\mathbf{w}} = m\omega^2\vec{\mathbf{r}} \times \vec{\mathbf{w}} \tag{10.90}$$

$$\mathbf{C}_A\vec{\mathbf{w}} = \omega^2\left(\mathbf{J} - m\vec{\mathbf{r}} \times \vec{\mathbf{r}}\times\right)\vec{\mathbf{w}} = \omega^2\mathbf{J}_A\vec{\mathbf{w}} \tag{10.91}$$

Note that $\vec{\mathbf{w}}$ is a non-trivial isotropic vector of $\mathbf{B}_A$. The following theorem summarizes this result.

Theorem 176 *Let $A \neq M$ and $\vec{\mathbf{r}} = \overrightarrow{MA}$. A pure rotation $\vec{\mathbf{w}}$ not through M mode exists if and only if $\vec{\mathbf{w}}$ is a non-trivial isotropic vector of $\mathbf{B}_A$, such that $\mathbf{B}_A\vec{\mathbf{w}} = m\omega^2\vec{\mathbf{r}} \times \vec{\mathbf{w}}$, and satisfies the generalized eigenvalue equation $\mathbf{C}_A\vec{\mathbf{w}} = \omega^2\mathbf{J}_A\vec{\mathbf{w}}$.*

Pure Force Not-through-M Modes Consider the equation of motion (10.84) at a point $A \neq M$. For a pure force mode through A, it is

$$\begin{bmatrix} \mathbf{A} & \mathbf{B}_A \\ \mathbf{B}_A^T & \mathbf{C}_A \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{w}} \end{bmatrix} = \omega^2 \begin{bmatrix} m\mathbf{I} & m\vec{\mathbf{r}}\times \\ -m\vec{\mathbf{r}}\times & \mathbf{J} - m\vec{\mathbf{r}}\times\vec{\mathbf{r}}\times \end{bmatrix} \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{w}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{0} \end{bmatrix} \tag{10.92}$$

Then, by definition, the twist $[\ \vec{\mathbf{t}}^T \quad \vec{\mathbf{w}}^T\ ]^T$ is in co-eigentwist subspaces of both $\hat{K}_A$ and $\hat{M}_A$. That is, the intersection $E_{cT_K/A} \cap E_{cT_M/A}$ is not empty, as before. However, unlike that at M, $E_{cT_M/A}$ is no longer a space of infinite pitch screws only. By Corollary 162, it has a one dimensional subspace of infinite pitch screws and a two dimensional subspace of finite pitch screws.

If the twist is in the infinite pitch subspace of $E_{cT_M/A}$ then, by definition, the mode also corresponds to a pure translation mode. But, by Theorem 174, this means that the pure force is through M. Therefore, for pure force modes not through M, the twist must be a finite pitch screw. In other words, after inverting (10.92),

$$\begin{bmatrix} \mathbf{D}_A & \mathbf{E}_A^T \\ \mathbf{E}_A & \mathbf{F} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\mathbf{0}} \end{bmatrix} = \omega^{-2} \begin{bmatrix} m^{-1}\mathbf{I} - \vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}}\times & -\vec{\mathbf{r}} \times \mathbf{J}^{-1} \\ \mathbf{J}^{-1}\vec{\mathbf{r}}\times & \mathbf{J}^{-1} \end{bmatrix} \begin{bmatrix} \vec{\mathbf{f}} \\ \vec{\mathbf{0}} \end{bmatrix} = \begin{bmatrix} \vec{\mathbf{t}} \\ \vec{\mathbf{w}} \end{bmatrix} \tag{10.93}$$

such that $\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$, meaning $\mathbf{E}_A\vec{\mathbf{f}} \neq \vec{\mathbf{0}}$ and $\vec{\mathbf{f}} \nparallel \vec{\mathbf{r}}$. Equations (10.93) can be separately given as

$$\mathbf{D}_A\vec{\mathbf{f}} = \omega^{-2}\left(m^{-1}\mathbf{I} - \vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}}\times\right)\vec{\mathbf{f}} \tag{10.94}$$

$$\mathbf{E}_A\vec{\mathbf{f}} = \omega^{-2}\mathbf{J}^{-1}\vec{\mathbf{r}} \times \vec{\mathbf{f}} \tag{10.95}$$

Note that $\vec{\mathbf{f}}$ is a non-trivial isotropic vector of $\mathbf{E}_A$. The following theorem summarizes this result.

Theorem 177 *Let $A \neq M$ and $\vec{\mathbf{r}} = \overrightarrow{MA}$. A pure force $\vec{\mathbf{f}}$ not through M mode exists if and only if $\vec{\mathbf{f}}$ is a non-trivial isotropic vector of $\mathbf{E}_A$, such that $\mathbf{E}_A\vec{\mathbf{f}} = \omega^{-2}\mathbf{J}^{-1}\vec{\mathbf{r}} \times \vec{\mathbf{f}}$, and satisfies the generalized eigenvalue equation $\mathbf{D}_A\vec{\mathbf{f}} = \omega^{-2}\left(m^{-1}\mathbf{I} - \vec{\mathbf{r}} \times \mathbf{J}^{-1}\vec{\mathbf{r}}\times\right)\vec{\mathbf{f}}$.*

10.4.5 Examples: RCC-like Devices

The analysis of special free vibration modes is applied to systems with simple stiffnesses. This enables one to predict the existence and nature of special modes explicitly. First, a relatively general device is presented. Then, the case of robotic fingers in rivet insertion example is analyzed as a particular example.

Special Free Vibration Modes for an RCC-like Device Consider an elastically suspended rigid body system whose stiffness at the center of elasticity is

$$\hat{K}_E = \begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{C}_E \end{bmatrix} \tag{10.96}$$

Clearly, this stiffness very closely resembles that of a general RCC device, see Chapter 9. The only difference is that, unlike for general RCCs, $\mathbf{A}$ and $\mathbf{C}_E$ do not have common eigenvectors in general. However, such devices are the closest to general RCCs among all stiffnesses. Hence, they constitute a precursor to RCC behavior. The defining property is that the linear and the angular responses of the elastic system are decoupled.

From the results of Chapters 3 and 4, one determines that, for RCC-like devices in (10.96), the eigenscrew subspaces contain only zero pitch screws through E, and the co-eigenscrew subspaces at E are infinite pitch screw spaces.

There are two cases to consider: (1) $E = M$, (2) $E \neq M$.

(1) $E = M$: In this case, all stiffness eigenwrenches (co-eigenwrenches at M) are pure force through M (pure translation) modes, Theorem 174. In general, this gives three perpendicular pure force through M (pure translation) modes. Also, since the eigentwist system is composed of zero pitch screws through M in all directions, there exists three that satisfy the generalized eigenvalue problem (10.87), in general. Therefore, in general, there exists three pure rotation through M (pure couple) modes.

As a result, in general there exist six distinct free vibration modes, all special. Thus, there cannot be other modes. If there are any algebraic multiplicities, then any linear combination of the corresponding special modes gives a vibration mode, which are actually superpositions. These combined modes can be pure rotation or forces through points other than M, or other finite pitch vibration modes. See next section for a demonstration of these cases.

(2) $E \neq M$: In this case, eigensystems contain zero pitch screws through E in every direction, i.e. they are rotation and force bundles generated by E. However, Theorems 174 and 175 require any possible solution to pass through M. Since E and M are distinct, only one element from each eigenscrew system can pass through M. Therefore, there can be at most one pure force through M (pure translation) mode, and at most one pure rotation through M (pure couple) mode. Other vibration modes can be either pure rotation and force through points other than M, or general finite pitch modes.

Robotic Fingers in Rivet Insertion Figure 10.10 illustrates robotic fingers in a rivet insertion task. In previous chapters, it is discussed that the fingers are controlled on-line such that the net stiffness of the rivet system is that of a classical RCC. The elastic centers are located in a region close to the tip so that any contact force or moment results in a corrective action. The stiffness matrix at the center of elasticity E is

$$\hat{K}_E = \text{diag}\begin{bmatrix} k_1 & k_2 & k_2 & \kappa_1 & \kappa_2 & \kappa_2 \end{bmatrix} \tag{10.97}$$

where k_i and κ_i are the linear and angular eigenstiffnesses, respectively.

If the components of the robot other than the finger structure are satisfactorily rigid, then the mass of the system is contributed by those of fingers and the rivet itself. In the configuration shown, both finger and the rivet material is symmetrically distributed with respect to the vertical axis, $\vec{\mathbf{u}}$. So, it is conceivable that the inertia of the system have a principal direction parallel to $\vec{\mathbf{u}}$. Then, due to symmetry, all axes perpendicular to $\vec{\mathbf{u}}$ are also principal inertia directions, corresponding to a double principal inertia. Therefore, at the center of mass M, the net mass matrix is

$$\hat{M}_M = \text{diag}\begin{bmatrix} m & m & m & I_1 & I_2 & I_2 \end{bmatrix} \tag{10.98}$$

where m is the linear mass, I_1 is the principal inertia in the direction of $\vec{\mathbf{u}}$, and I_2 is the principal inertia in directions perpendicular to $\vec{\mathbf{u}}$.

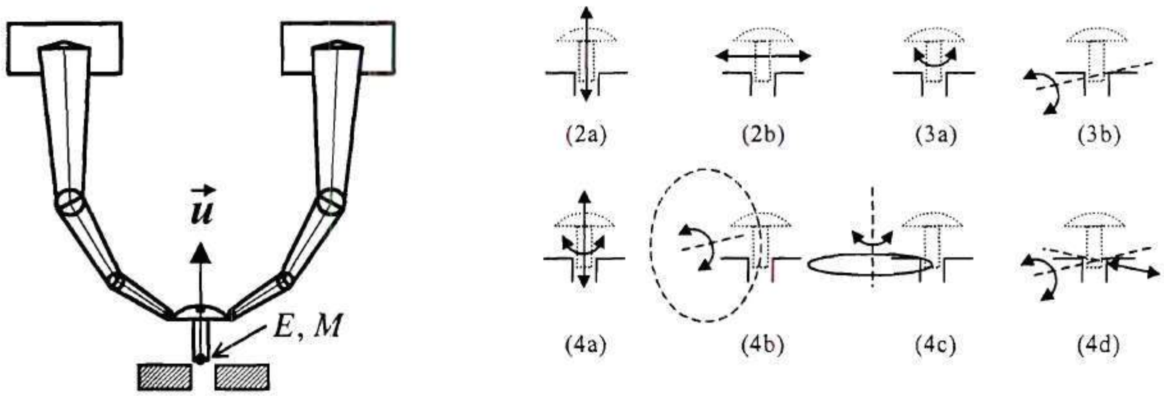


Figure 10.10: Free vibration modes for a rivet held by robotic fingers. (2a) and (2b) are pure translations (pure forces through M). (3a) and (3b) are pure rotations through M (pure couples). Others are superpositions of these simple modes. Except for (2a), (3a) and (4a), all others may interfere with satisfactory operation.

Figure 10.10 also shows some possible vibrational modes. It is seen that vibrations of types (2a), (3a) and (4a) do not seriously jeopardize the safety of the operation. These vibrations are pure translations or rotations along $\vec{u}$, or their combinations. Other modes interfere with the normal operation of the device and may increase the likelihood of jamming. In summary, any mode that contains a rotation through points other than E is not beneficial. In the previous section, it is shown that the existence of such undesirable modes is more likely if $E \neq M$. Therefore, for improved performance a designer should try to satisfy the constraint $E = M$.

The special free vibrations of RCC-like devices for $E = M$ is also explained in the previous section. However, due to further simplifications (classical RCC, double principal inertia) it is preferable to try to characterize all possible vibrational modes of the rivet system. It is instructive to do this explicitly.

The stiffness and mass matrices at $E = M$ can be given as

$$\hat{K} = \begin{bmatrix} (k_1 - k_2)\vec{\mathbf{u}}\vec{\mathbf{u}}^T + k_2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & (\kappa_1 - \kappa_2)\vec{\mathbf{u}}\vec{\mathbf{u}}^T + \kappa_2\mathbf{I} \end{bmatrix} \quad (10.99)$$

$$\hat{M} = \begin{bmatrix} m\mathbf{I} & \mathbf{0} \\ \mathbf{0} & (I_1 - I_2)\vec{\mathbf{u}}\vec{\mathbf{u}}^T + I_2\mathbf{I} \end{bmatrix} \quad (10.100)$$

Then, if $\hat{T} = [\ \vec{\mathbf{t}}^T \quad \vec{\mathbf{w}}^T\ ]^T$ is a twist, the equation of motion (10.84) yields two decoupled equations,

$$\left[(k_1 - k_2)\vec{\mathbf{u}}\vec{\mathbf{u}}^T + k_2\mathbf{I}\right]\vec{\mathbf{t}} - \omega^2 m\vec{\mathbf{t}} = \vec{\mathbf{0}} \quad (10.101)$$

$$\left[(\kappa_1 - \kappa_2)\vec{\mathbf{u}}\vec{\mathbf{u}}^T + \kappa_2\mathbf{I}\right]\vec{\mathbf{w}} - \omega^2\left[(I_1 - I_2)\vec{\mathbf{u}}\vec{\mathbf{u}}^T + I_2\mathbf{I}\right]\vec{\mathbf{w}} = \vec{\mathbf{0}} \quad (10.102)$$

which, after manipulations, yield

$$(k_1 - k_2)\left(\vec{\mathbf{u}}^T\vec{\mathbf{t}}\right)\vec{\mathbf{u}} = \left(m\omega^2 - k_2\right)\vec{\mathbf{t}} \quad (10.103)$$

$$\left[\kappa_1 - \kappa_2 - \omega^2(I_1 - I_2)\right]\left(\vec{\mathbf{u}}^T\vec{\mathbf{w}}\right)\vec{\mathbf{u}} = \left(I_2\omega^2 - \kappa_2\right)\vec{\mathbf{w}} \quad (10.104)$$

Any twist that satisfies both of these equations is a general free vibration mode. Also note that, if $\vec{\mathbf{t}}$ is any translation that satisfies (10.103) and $\vec{\mathbf{w}}$ is any rotation that satisfies (10.104), then all linear combinations of them are solutions to the vibration problem, provided that their corresponding ω are identical. Both $\vec{\mathbf{t}}$ and $\vec{\mathbf{w}}$ can be taken as unit vectors.

Equations (10.103) and (10.104) can be satisfied in two ways. Both sides of the equations are either zero or non-zero. For (10.103), one gets the following solutions.

1. $\vec{\mathbf{t}} = \vec{\mathbf{0}}$. Twist is a pure rotation through M.
2. $\vec{\mathbf{t}} = \vec{\mathbf{u}}$ and $(k_1 - k_2) = \left(m\omega^2 - k_2\right)$. Translation of twist at M is in $\vec{\mathbf{u}}$ direction.
3. $(k_1 - k_2)\left(\vec{\mathbf{u}}^T\vec{\mathbf{t}}\right) = 0$ and $\left(m\omega^2 - k_2\right) = 0$. Translation of twist at M is perpendicular to $\vec{\mathbf{u}}$ or $k_1 = k_2$.

For (10.104), one gets

a) $\vec{\mathbf{w}} = \vec{\mathbf{0}}$. Twist is a pure translation.

b) $\vec{\mathbf{w}} = \vec{\mathbf{u}}$ and $\left[\kappa_1 - \kappa_2 - \omega^2(I_1 - I_2)\right] = \left(I_2\omega^2 - \kappa_2\right)$. Rotation of twist is parallel to $\vec{\mathbf{u}}$.

c) $\left[\kappa_1 - \kappa_2 - \omega^2(I_1 - I_2)\right]\left(\vec{\mathbf{u}}^T\vec{\mathbf{w}}\right) = 0$ and $\left(I_2\omega^2 - \kappa_2\right) = 0$. Rotation is perpendicular to $\vec{\mathbf{u}}$ or $\kappa_1 - \kappa_2 - \omega^2(I_1 - I_2) = 0$.

Any combination of the above solutions for $\vec{\mathbf{t}}$ and $\vec{\mathbf{w}}$ is a solution. The following is a list of these combinations for which the item labels match those in Figure 10.10.

1a $\vec{\mathbf{t}} = \vec{\mathbf{w}} = \vec{\mathbf{0}}$. Trivial solution. No vibration.

2 $\vec{\mathbf{t}} \neq \vec{\mathbf{0}}$, $\vec{\mathbf{w}} = \vec{\mathbf{0}}$. Pure translation modes.

2a $\vec{\mathbf{t}} = \vec{\mathbf{u}}$ with $\omega_1 = \sqrt{k_1/m}$. See Figure 10.10-(2a). The wrench is $[\ k_1\vec{\mathbf{u}}^T \quad \vec{\mathbf{0}}^T\ ]^T$, a pure force through M as expected.

2b $\vec{\mathbf{t}} \perp \vec{\mathbf{u}}$ with $\omega_2 = \sqrt{k_2/m}$. See Figure 10.10-(2b). The wrench is $[\ k_2\vec{\mathbf{t}}^T \quad \vec{\mathbf{0}}^T\ ]^T$, a pure force through M. The other solution is $k_1 = k_2$. All axes through M are zero pitch eigenwrenches. So, any direction is a pure translation direction, and any axis through M is a pure force mode through M. This is due to the fact that in this case $\omega_1 = \omega_2$.

3 $\vec{\mathbf{t}} = \vec{\mathbf{0}}$, $\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$. Pure rotation modes through M.

3a $\vec{\mathbf{w}} = \vec{\mathbf{u}}$ with $\omega_3 = \sqrt{\kappa_1/I_1}$. See Figure 10.10-(3a). The wrench is $[\ \vec{\mathbf{0}}^T \quad \kappa_1\vec{\mathbf{u}}\ ]^T$, a pure couple as expected.

3b $\vec{\mathbf{w}} \perp \vec{\mathbf{u}}$ with $\omega_4 = \sqrt{\kappa_2/I_2}$. See Figure 10.10-(3b). The wrench is $[\ \vec{\mathbf{0}}^T \quad \kappa_2\vec{\mathbf{u}}\ ]^T$, a pure couple. The other solution is $\omega_4 = \sqrt{\kappa_2/I_2} = \sqrt{(\kappa_1 - \kappa_2)/(I_1 - I_2)}$, which gives $\kappa_1/I_1 = \kappa_2/I_2$. That is, $\omega_3 = \omega_4$, and every axes through M is a pure rotation mode.

The above solutions complete the expected vibrational modes since there can be only six such solutions, counting the multiplicities. However, the following is a list of other combinations leading to *other* solutions which are shown to be superpositions of the above simple modes.

4 $\vec{\mathbf{t}} \neq \vec{\mathbf{0}}$, $\vec{\mathbf{w}} \neq \vec{\mathbf{0}}$. Other possible modes.

4a $\vec{\mathbf{t}} = \vec{\mathbf{w}} = \vec{\mathbf{u}}$. See Figure 10.10-(4a). All finite pitch twists with screw axis through M and parallel to $\vec{\mathbf{u}}$. This is a combination of (2a) and (3a). The necessary and sufficient condition is $\omega_1 = \omega_3$, i.e. $k_1/m = \kappa_1/I_1$.

4b $\vec{\mathbf{t}} = \vec{\mathbf{u}}$, $\vec{\mathbf{w}} \perp \vec{\mathbf{u}}$. See Figure 10.10-(4b). Pure rotation mode not through M. The rotation is perpendicular to $\vec{\mathbf{u}}$. The necessary and sufficient condition is $\omega_1 = \omega_4$, i.e. $k_1/m = \kappa_2/I_2$.

4c $\vec{\mathbf{t}} \perp \vec{\mathbf{u}}$, $\vec{\mathbf{w}} = \vec{\mathbf{u}}$. See Figure 10.10-(4c). Pure rotation not through M. The rotation is parallel to $\vec{\mathbf{u}}$. The necessary and sufficient condition is $\omega_2 = \omega_3$, i.e. $k_2/m = \kappa_1/I_1$.

4d $\vec{\mathbf{t}} \perp \vec{\mathbf{u}}$, $\vec{\mathbf{w}} \perp \vec{\mathbf{u}}$. See Figure 10.10-(4d). All finite pitch screws perpendicularly intersecting the $\vec{\mathbf{u}}$-axis. This is called a **brush** in screw theory. The necessary and sufficient condition is $\omega_2 = \omega_4$, i.e. $k_2/m = \kappa_2/I_2$.

It is well known that the vibrational mode corresponding to the lowest natural frequency of a system is most likely to be excited in free vibrations case. Also, unwanted modes can be made hard to excite by making their natural frequencies larger. In the simple vibration modes (2a) through (3b), the undesirable ones are (2b) and (3b) respectively corresponding to ω_2 and ω_4. So, a better finger configuration would be that for which

$$\omega_1, \omega_3 << \omega_2, \omega_4 \qquad (10.105)$$

and

$$|\omega_1 - \omega_3| >> 0 \qquad \text{and} \qquad |\omega_2 - \omega_4| >> 0 \qquad (10.106)$$

These practically eliminate all combined modes (4a)-(4d), as well as making (2b) and (3b) much less likely.

Part IV

CONCLUSIONS

CHAPTER XI

CONCLUSIONS

This study presented a comprehensive and systematic analysis of the stiffness and compliance properties of elastic connections. The basic framework of spatial vector (screw) algebra is utilized throughout. It has been shown that the screw algebra is the essential and natural means of analysis of systems that has a connection to the rigid body mechanics. The fundamental benefit of the screw theory is to provide clear geometric meanings to the results, which, otherwise, would look like a complicated collection of equations and properties.

All principal results of this study are new or had been unsolved until now. In Chapter 3, the free-vector eigenvalue problems for stiffness and compliance were investigated in detail. The results included new relations between previously proposed centers, closed form equations for the locations of these centers, and, their relations to previously defined concept of compliant axes. The uses of these results were demonstrated in Chapter 9. In Chapter 4, the line-vector eigenvalue problems were proposed and solved. The results were shown to be complementary to those of the free-vector eigenvalue problems, which also led to a generalization of the compliant axes concept. In Chapter 5, a new center, the co-center of elasticity, was identified. The solutions were fully characterized and shown to lead to a geometric classification of the stiffness and compliance. Chapter 7 dealt with stiffness matrices of elastic connections made of springs. The results were the closed and compact forms of the stiffness matrices, explanation of the non-symmetry of the spatial stiffnesses and other properties. Applications of these results were presented Chapter 8.

Throughout this study, the emphasis has been on understanding the geometrical and constitutive ingredients of general elastic systems. More than anything, the theoretical implications and the systematic approaches were stressed. Nevertheless, the practical applications were shown to follow naturally, all of which were unsolved problems and expected to have very positive effects in robotics applications, ranging from kinematical analysis, design to control.

In what follows, the results of each chapter are summarized and discussed. Finally, a partial outline of related future work is attempted in hope of providing the future researchers a starting point.

11.1 Geometric and Constitutive Content of Elastic Systems

In this section, the results of Chapters 3, 4 and 5 are summarized. All of these chapters dealt with constructing a conceptual and theoretical framework for the analysis of stiffness and compliance using the free-vector and line-vector eigenvalue problems.

11.1.1 Centers of Elasticity, Stiffness and Compliance

The free-vector eigenvalue problems proposed by Lipkin and Patterson [30] led to the free-vector decomposition of stiffness and compliance matrices, and identification of the center of elasticity, a unique point with special properties. They also proposed a systematic definition of compliant axes which led to a geometric classification of the stiffness and compliance. On the other hand, there are the centers of stiffness and compliance identified by Loncaric [32], at which the respective matrices have simplest forms or maximum decoupling of linear and angular behavior.

In Chapter 3, closed form equations for the centers of elasticity, stiffness and compliance were found. It was shown that the centers of stiffness and compliance have properties that are analogous to those of the center of elasticity. Basically, the center of elasticity has a geometric nature, whereas the centers of stiffness and compliance have constitutive characters. These results led to previously unknown relations between the centers and compliant axes. That a compliant axis passes through all of these centers is a powerful geometric result. For more than one compliant axes, the centers coalesce. These results are expected to have applications in the design of elastic systems. An example was presented in Chapter 9 that led to closed form and compact equations for the design of RCC devices. This ultimately enabled an optimization of the performance of RCC devices.

The unified view of the centers, their relations with the compliant axes, etc. indicate that a classification of stiffness and compliance must involve all these centers.

Another useful result in Chapter 3 was to show that the free-vector eigenvalue problems are essentially obtained by restricting the stiffness and compliance mappings to the free-vector subspaces. Although, this seems to be only a change of perspective, the theoretical implications are quite important. Chapter 4 is a good example, in which the complementary eigenvalue problems were identified by simply considering the restriction of the stiffness and compliance mappings to the line-vector subspaces. Only after such a change of perspective does this discovery come naturally. For example, in a robotic task involving a rotational manipulation the loads are predominantly moments. This can happen in cases of robots with screwdrivers as end-effectors, or robots assembling circular pieces with tight tolerances, etc. If the loads are approximated as pure couples, then the overall displacement error due to the compliance of the manipulator is completely in the eigentwist subspace. Therefore, an active stiffness control algorithm may take the advantage of this knowledge by adjusting the stiffness so that the eigentwists are in or close to some preferred directions, and corresponding stiffnesses are maximized. This greatly reduces the required control action and constraints since the eigenwrenches and their stiffnesses need not be adjusted or monitored.

11.1.2 Line-Vector Eigenvalue Problems and Decomposition

In Chapter 4, a set of eigenvalue problems for stiffness and compliance were identified considering the restrictions of the stiffness and compliance mappings to the line-vector subspaces. This came as a natural extension of the theory of free-vector eigenvalue problems which concerned the free-vector subspaces. These new eigenvalue problems, the line-vector eigenvalue problems, are completely complementary to the free-vector eigenvalue problems. There are interesting differences, however.

First, they are not unique. Every point in space generates a related set of line-vector eigenvalue problems. Briefly, for every set of line-vector eigenvalue problems, there exist forces and rotations through the related point, called the generator, whose actions are screws (co-eigenscrews) with parallel translations and couples at the generator. Unlike the free-vector eigenvalue problems, the co-eigenscrews in general do not form orthogonal sets.

Similar to the free-vector eigenvalue problems, the line-vector eigenvalue problems led to decompositions of the stiffness and compliance matrices in terms of geometric and constitutive contents. However, for every generator one gets a distinct decomposition. Besides the theoretical importance, the practical uses of these decompositions, along with the free-vector decomposition, were presented in the synthesis of stiffnesses by springs problem, Chapter 8.

As an example of practical applications of the line-vector eigenvalue problems, one may consider a robotic hand interacting with the environment such that the induced loads or displacements are zero pitch screws (line-vectors). This happens, for example, in a task of curve tracing on a hard surface (plotting, measuring, etc.) where all the reaction loads are pure forces through the contact point. Then, any displacement error due to the total compliance of the manipulator and the hand is completely in the co-eigentwist subspace. Using this knowledge, a control algorithm may be targeted to adjust the stiffness so that the co-eigentwists and their stiffnesses are as desired.

11.1.3 Co-centers of Elasticity and Generalized Compliant Axes

In earlier studies, the center of elasticity was defined as the center of the 3-systems spanned by the eigenscrews. Then, if $\mathbf{A}$ and $\mathbf{B}$ are the upper diagonal and off-diagonals of the stiffness matrix, respectively, one deduces the property that $\mathbf{A}^{-1}\mathbf{B}$ is symmetric at the elastic center. In Chapter 5, this relation was reversed leading to an important re-definition of the center of elasticity. Thus, a point where $\mathbf{A}^{-1}\mathbf{B}$ is symmetric was defined as the elastic center. The uniqueness followed easily. Then, other known properties of the elastic centers are obtained as consequences. As side result, a very simple and compact equation for the location of the elastic center was obtained, which did not involve solving any eigenvalue problems.

A more profound effect of the re-definition of the elastic center was that it naturally suggested the complementary definition for the co-eigenscrews case. The result was the identification of the co-centers of elasticity. The defining equation for the co-centers turned out to be a very difficult one to solve. The existence was shown using the concepts from algebraic varieties. Non-uniqueness was demonstrated for general cases. Special cases were identified and fully solved, which have practical importance. These special cases were shown to be related to the existence of compliant axes. More importantly, a compliant axis passes through at least one co-center and the centers of elasticity, stiffness and compliance. These results lead to a more complete classification of stiffnesses and compliances in terms of the four centers.

In Chapter 6, close relations between the eigen- and co-eigenscrew subspaces were demonstrated in terms of the nature of screws they contain. This also led to a generalization of the concept of compliant axes. The new definitions take into account the co-eigenscrew systems. They are also more systematic and meaningful since they exhibit the complementary properties fully.

11.2 Stiffness of Parallel Connections

A parallel connection made up of springs can model a wide variety of elastic systems, from very general, to serial or parallel manipulators. The theory developed in Chapter 7 concentrated on two aspects. One was the determination of the stiffness matrix of spring systems in closed and compact forms. The other was the investigation of the resulting non-symmetric stiffness.

11.2.1 Elastic Systems Modeled by Springs

The closed form and compact equations for the stiffness of line and torsional spring systems were found in Chapter 7. This is a new result, which had been solved previously only for simple cases such as planar connections. The solutions were made possible by applying the screw theory and the geometry behind it. Applications are expected to be quite diverse, ranging from control theory to design. In Chapter 8, the synthesis problem was solved using these results.

11.2.2 The meaning of Non-symmetry

Another interesting result was the observation of the fact that, in general, the spatial stiffness is non-symmetric. This was a known phenomenon. However, this study first showed that the skew-symmetric part has a clear meaning in case of line springs, namely, it is equal to the one half of the total reaction wrench in spatial cross product form. This immediately led to the result that only unloaded line spring systems have symmetric stiffnesses. On the contrary, the skew-symmetric part of the stiffness of torsional springs is not related to the applied wrench in a simple way. It was shown that a torsional spring system may have asymmetric stiffness in unloaded state, or symmetric stiffness in a loaded state. For both line and torsional springs, the stiffness is symmetric if all the springs are individually unloaded.

For planar cases, previous researchers identified a body, the symmetric body, with respect to which the stiffness is symmetric. In this study, the result was generalized to spatial connections and shown to be caused by the properties of the spatial cross product and the special form of the stiffness matrix. Further, it was shown that there exist infinitely many bodies with respect to which the stiffness is symmetric. However, the results were shown not to extend to the torsional spring cases.

11.3 Solving the Synthesis of Stiffnesses by Springs

The problem of synthesizing a given stiffness by springs has long been known. However, except for a few partial results, the general problem had been unsolved until now. The solution of this problem required many of the theoretical results of this study as well as the solutions to a new general eigenvector problem.

11.3.1 A New Eigenvector Problem for General Matrices

Chapter 8 first presented a completely new eigenvector problem called the isotropic vector problem. In classical eigenvalue problems for general matrices, the matrix maps an eigenvector to a vector parallel to the eigenvector with respect to the standard Euclidean metric. As a complementary result, an isotropic vector of a general matrix is mapped by the matrix to a vector orthogonal to the isotropic vector under the Euclidean metric. As a result, the scalar product, with respect to the given matrix, of an isotropic vector by itself vanishes. This means that, when the matrix is taken as the metric on the vector space, an isotropic vector has a *zero length.* This is a well known fact in tensor analysis which is also the reason of the term *isotropic vector* used here. What is new in this study is a method to determine the isotropic vectors of a matrix. The necessary condition is that the matrix has to be indefinite or singular.

A case of special importance to this study is the existence of orthonormal bases made up of isotropic vectors. The necessary and sufficient condition for this was shown to be the vanishing trace of the given matrix. More importantly, a stable recursive algorithm to determine such bases were found. Also, the dimension of the space of solutions is determined, which, unlike the classical eigenvector bases, contains infinitely many bases in general.

Applications and examples of these result ranges from finding the zero or infinite pitch screws of arbitrary n-systems, the analysis of the deviatoric stress and strains, to the synthesis problem which was the main impetus behind this problem. Another well known example comes from the special theory of relativity in which the isotropic vectors in the 4-dimensional space-time vector space are due to the indefinite metric used by the theory, namely the Minkowski or Lorentz metric.

The author believes that the solution of the isotropic vector bases problem will have applications in a wide variety of subjects beside the aforementioned ones.

11.3.2 Systematic Approach to the Synthesis by Springs

A systematic formulation of the synthesis problem was presented in Chapter 8 using matrix and screw algebra. General necessary conditions were determined along with two special minimum syntheses. The results of Chapter 7 and the solution of the isotropic vector problem were applied yielding complete and general solutions to the synthesis problem. It was shown that in order to synthesize a rank r stiffness it is necessary and sufficient for the off-diagonal matrices of the stiffness to have a zero trace. In general, there exists infinitely many distinct syntheses for a given stiffness. Also, a method was presented to achieve a synthesis by $n > r$ springs, in general. A general algorithm and random examples were presented which very clearly verified the theory.

The free-vector and the line-vector decomposition theorems for stiffnesses were applied to the synthesis problem leading to the minimum syntheses. Further, they required the solution of the isotropic vector bases problem only for three dimensions. They also helped explain the physical reasons behind having infinitely many syntheses in general.

Applications of the results include the robotic grasp problems, active control of stiffness, smart design of elastic structures, etc.

11.4 Rotational Symmetry Devices

The results of the analysis of symmetric stiffness and compliance in terms of eigen- and co-eigenscrew structures, and centers, were applied and demonstrated in Chapter 9. The construction of rotational symmetry devices, such as the classical RCC devices, is detailed.

11.4.1 Solving the Equations of Classical RCCs with Beams

The classical RCC devices with beams, proposed and constructed by earlier researchers, lacked accurate design equations. This was evident both from the coarse approximations used to solve the device characteristics and non-matching experimental data.

In Chapter 9, the location of the center of elasticity, and others, for such classical RCCs was determined as a function of known device geometry and beam properties. The net stiffnesses of the device were also determined as functions of the same. Further, the theory was developed for $n \geq 3$ beams, instead of usual three beam case found in the literature. An important observation was the sensitivity of the location of the RCC center to the slenderness ratio of the beams and the cone angle of the device. This was in agreement with experimental observations. More importantly, the location of the RCC center, which does not depend on the number of beams, can be optimized by adjusting the cone angle. This gives a device with maximally remote RCC center. Also, it was shown that the device stiffnesses, which are now known in their functional forms, can be adjusted such that any possibility of jamming during insertion tasks is reduced. Examples, taken from previous experimental works as case studies, very clearly verified and demonstrated the theory.

These results are expected to greatly improve the existence design procedures for classical RCCs, as well as serve to achieve optimum configurations. Further applications include better control strategies for robotic hands, better designs of structures similar to RCC geometry, etc.

11.4.2 Generalized RCC Devices

The conical symmetry of the classical RCC devices is basically a form of rotational symmetry. Taking this view, in Chapter 9, a new construction technique was proposed which always yields a generalized RCC type stiffness. The new method still uses the rotational symmetry, but with arbitrary elements as generators instead of beams. The results led to a new classification of these rotational symmetry devices which contained the classical RCCs as a special case. Interestingly, it turned out that a general rotational symmetry device is a classical RCC if and only if the symmetry axis is parallel to an isotropic vector of the off-diagonal of the generator stiffness.

Applying the results of the general analysis back to the classical RCC case, it was shown that the conical symmetry of beams is in general the only way to construct a classical RCC. For classical RCCs with springs, if the construction has a conical symmetry (or is planar) then the device is a classical RCC. The converse is not true. From the synthesis problem, one shows by example that it is possible to make a classical RCCs using springs without any conical symmetry. However, if the construction is symmetric and has classical RCC characteristics then the springs must have a conical symmetry or be coplanar. This was presented as the converse.

Another important result was that symmetric Stewart platforms as compliant systems are actually classical RCCs. Both beam and spring cases were presented and analyzed. However, as classical RCCs their performance was shown to be less than the classical RCCs with single beam or spring generators. The advantage of a symmetric Stewart platform type device with springs is that, unlike the case of single spring generators, they provide a non-singular stiffness. This is a very promising result since earlier researchers, for example Loncaric [32], were able to synthesize classical RCCs with only specially arranged line springs which always had a singular stiffness. Further using both line and torsional springs a new configuration for classical RCCs with general springs were proposed.

11.4.3 Theoretical Confirmation

As a theoretical confirmation of the results, the theory of the rotational symmetry devices were applied to an open-end O-ring, which was considered as a continuous system. The goal was to determine the stiffness matrix using an infinitesimally small generator beam and infinitely many elements obtained by successive rotations. The process naturally necessitated the taking the limit of the involved functions as the beam length approaches to zero and the number of elements approaches to infinity. The resulting stiffness matrix was shown to be finite and that of a classical RCC, as expected. However, an open-end O-ring is actually used in literature as an approximation to helical springs with small helix angles. So, if any, the confirmation would be to show that the axial stiffness is equal or close to that of a helical spring with small helix angle. It turned out that the result was exact. Both axial and torsional stiffness formulae for helical springs, frequently given in machine element design textbooks, were recovered exactly. The author believes that this constitutes a very strong evidence for the validity of the theory and methods developed in this study for the analysis of rotational symmetry devices.

One should compare this procedure to the finite element methods used in 1-dimensional problems. Curved beams such as a helical spring are frequently approximated by taking sufficiently small straight beams. After finding a suitable stiffness model for the straight beams, the FE method only assembles the element matrices into a larger one representing every node on the curved beam. Then, the larger stiffness matrix is used, after applying appropriate boundary conditions, to determine the displacements corresponding to given loads. Frequently, one is concerned about the displacements of the end point versus loads applied at the same point. This is exactly the stiffness found in this study. Therefore, the method explained here is equivalent to assembling the stiffness matrices of infinitely many straight beams and reducing the result to a stiffness matrix related to the end displacements and loads.

11.5 Analysis of Cartesian Mass Matrix

Previous studies showed that the velocity and momenta of a single rigid body are spatial vectors [20]. By defining the spatial mass matrix as a mapping from the spatial velocity to the spatial momentum, one constructs a complete analogy between the stiffness and mass matrices. Mathematically, they are indistinguishable. Based on this, the current study showed that the theory developed for the analysis of stiffness is also applicable to the mass matrix, Chapter 10. For impulse and a certain class of problems, the mass matrix was shown to be a linear homogeneous map from the spatial acceleration to the wrench. For simplicity, the components of the acceleration are called translations and rotations. In this way, the theory for the stiffness analysis becomes directly applicable.

11.5.1 Eigen- and Co-eigenscrew Structure of Mass Matrix

Owing to the simple form of the mass matrix of a single body, the eigen- and co-eigenscrews were determined explicitly. The mass eigenwrenches are pure forces through the center of mass M. This is sensible since it is well known that only such forces can cause pure translations which is also required by the definition of eigenwrenches. Eigentwists, which cause pure couples by definition, are pure rotations through M in principal inertia directions. This is also a well known fact. Then, from the analysis presented in Chapter 3, M is analogous to the combined centers of elasticity, stiffness and compliance.

The co-eigenscrew structure is not that straightforward. The line-vector eigenvalue problems are different at different points. The co-eigentwists at M are pure translations in arbitrary directions. Note that the actions of the co-eigentwists, which are pure forces through M by definition, are eigenwrenches. The co-eigenwrenches at M are pure couples in principal inertia directions. The actions of the co-eigenwrenches are eigentwists. From the results of Chapter 5, M is also analogous to a unique co-center of elasticity.

The co-eigenscrews at points away from M have distinct properties. For a point $A \neq M$, a co-eigentwist is a pure translation parallel to $\overrightarrow{MA}$, which is also a co-eigentwist at M. The remaining two co-eigentwists are zero pitch screws, i.e. pure rotations. On the other hand, if $\overrightarrow{MA}$ is parallel to a principal inertia direction, then co-eigenwrenches are a pure couple parallel to $\overrightarrow{MA}$ and two pure forces, otherwise they are three pure forces which are coplanar and perpendicular to $\overrightarrow{MA}$. The fact that co-eigenscrews, which are due to zero pitch screws through a point, are themselves zero pitch screws is the most interesting property of the mass matrix leading to the generalization of the concept of percussion axes.

11.5.2 Axes and Joints of Percussion

The result that the co-eigenscrews away from M are zero pitch screws means that there exists pure forces causing pure rotations, and vice versa. These are the force-rotation and rotation-force axes defined in Chapter 6. In Chapter 10, these axes were identified as generalized forms of the well known center of percussion phenomenon. The duality is preserved by introducing the definitions of axes and joints of percussion.

Given a point $A \neq M$, the two zero pitch eigentwists are the joint axes of percussion about which the body would only rotate if the corresponding pure forces act through A. The axes of the forces become the axes of percussion and any point on them is a center of percussion. Similarly, given a point $A \neq M$, the three (or two) zero pitch eigenwrenches are the axes of percussion which cause only rotations about corresponding axes through A. These three (or two) rotations through A are the joint axes of percussion and any point on them can be used as a joint of percussion.

Chapter 10 also demonstrated methods to design center of percussion behavior. Then, conditions for having classical center of percussion are investigated. For a classical center of percussion the axes and joint of percussion are parallel to some principal inertia directions. The necessary and sufficient condition for a classical center of percussion is that A is on a principal inertia axis.

A previously overlooked case is the possibility of having a pencil of axes and joints of percussion. This is important in the sports equipment design, for example golf club design, where the center of percussion property is used to reduce the sting sensation. If the device has finite number of axes, then the ball must hit the club in a direction close to the axis of percussion. Any deviation increases the sting. However, if there exists a plane of axes of percussion, then the design accommodates any deviation in the direction of hit in the plane of the axes.

Finally, the results of the analysis were applied to the free vibrations of elastically suspended rigid bodies. In other studies, researchers identified special modes of free vibrations, namely pure translation, pure rotation, pure force and pure couple. Applying the results yielded the necessary and sufficient conditions for the existence of these modes. Their relations were determined. For example, a pure translation (couple) mode exists if and only if a pure force (rotation) through M mode exists. Pure force and rotation not-through-M were identified and analyzed separately. Results were demonstrated using the robotic hand in rivet insertion example in which some modes were identified as undesirable for the successful performance of the device.

11.6 Comments on Future Work

- The free-vector and line-vector eigenvalue problems completely characterize the principal subspaces of stiffness and compliance. Applications of these in robot control, structural design, mechanism design, automated assembly, etc. areas were hinted in this study. A more detailed investigation of these topics is suggested. Possible goals include efficient active stiffness control algorithms; design algorithms for spatial structures and mechanisms using minimum material; superior assembly techniques and devices; and computer applications. Consider the case of a robot interacting with the environment such that the external loads are predominantly forces rather than moments. Examples are welding robots, drawing robots, etc. If the loads can be approximated as pure forces then any deflection of the tool due to these forces completely belongs to the co-eigentwist system of the net stiffness.

- The co-centers problem remains unsolved for general cases as far as the number of co-centers are concerned. Future studies may focus on this problem, although it seems to be a purely theoretical problem. The goal can be a better classification of stiffnesses and compliances using the centers of elasticity, co-elasticity, stiffness and compliance. For example, given the centers of elasticity, stiffness and compliance, and, a configuration of co-centers, how many stiffnesses are there which have these centers in common? For rotational symmetry devices, it is possible to make the device have a circle of co-centers, along with the usual combined centers at the center of the circle. What happens if the device is used in insertion tasks such that the circle of co-centers coincides with the circumference of the peg? In this case, any initial contact with the hole will cause a pure force through a co-center.

- Another way to classify the stiffnesses and compliance is to use the generalized concept of compliant axes. For this, a better understanding of these entities and their properties is needed. Result can be used to construct structures or to control manipulators that have these axes which have predictable and meaningful stiffness properties.

- The spatial stiffness away from an unloaded equilibrium is non-symmetric. For line-springs, the skew-symmetric part is equal to the one half of the reaction wrench in spatial cross product form. Kumar and coworkers [25], [54] assumed conservative systems and proved the same result. The questions is whether the converse is true. That is, if the skew-symmetric part of the stiffness has that special form, is the system conservative? This is indicated by the results for line and torsional spring systems. The former has the special skew-symmetric relation and is conservative, whereas the latter does not have the special skew-symmetric relation and is not conservative.

- As for the applications presented in this study, the synthesis of stiffnesses by springs is fully solved, but their full grown applications in robot control problems (e.g., grasping tasks with shifting contacts), design of special spring systems such as the so-called perfect equilibrators, modeling of general structures, etc. are yet to come.

- Another application is the use of the analysis of rotational symmetry devices to construct compliant mechanisms that have general RCC characteristics. Design of structures or devices with such behavior may be combined with the mass matrix analysis to better predict and control the vibrational characteristics. A general RCC stiffness has a simple structure not unlike the mass matrix. Since a general RCC have combined centers, they can be designed to coincide with the center of mass. Then, by also making the elastic and mass principal screws coincide, one obtains a very simplified equation of motion that may lead to a simplified and relatively decoupled vibrational characteristics.

- The generalized concept of the axes of percussion is promising in achieving better design in sport equipment design. Although a simple analysis was presented in this study, a detailed analysis is needed for realistic designs. Applications can be extended to more complicated mechanisms. A relatively simple example is the door hinge mechanisms. Using the axes of percussion analysis one may be able to reduce the impact on joints during the collision of the door with the frame. Although in many cases dampers are used to reduce the impact, they sometimes malfunction or degrade in time.

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