NUMERICAL SOLUTION OF LAMINAR COMPRESSIBLE FLOW OVER A CIRCULAR CYLINDER

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Heng-wai Hsu

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NOMENCLATURE

English Notation

C	=	$[f''(0)]^{1/3}$, transformation parameter, function of α only
$C_\alpha(\alpha)$	=	curvature parameter
f	=	$\eta - 4\alpha$, function of α and χ
f'	=	$\partial f/\partial\chi = \Lambda$, function of α and χ
g	=	f/C, a transformed function, function of α and ζ
g'	=	f'/C^2, function of α and ζ
$G(\alpha)$	=	$\alpha^{\frac{1}{2}}$
h_α	=	metric measurement along α-line
$h_{\alpha o}$	=	value of h_α at the surface
h_β	=	metric measurement along β-line
K	=	$P_o - P_\infty + U^2/2$, Bernoulli constant
$N(\alpha)$	=	$\lambda(\partial \ln h_\alpha/\partial\beta^*)$, an integration constant of equation (III-3)
P	=	pressure
P_o	=	pressure at the boundary of the boundary-layer
P_s	=	pressure at the surface
P_∞	=	pressure at free stream
R_e	=	Reynolds number based on radius R
R_α, R_β	=	radii of curvature of α-lines and β-lines, respectively
S	=	distance from the forward stagnation point measured along the surface of the cylinder

English Notation

u, v	=	velocity components along α-lines and β-lines, respectively
U	=	potential flow velocity at the boundary of the boundary-layer = 2 sin ϕ
U_∞	=	free stream velocity
x, y	=	components of two-dimensional rectangular coordinates

Greek Notation

α	=	orthogonal coordinate in the direction of flow
β	=	orthogonal coordinate perpendicular to the direction of flow
β^*	=	$\beta(R_e)^{\frac{1}{2}}$
δ	=	boundary-layer thickness
ζ	=	$C\chi$, a transformed independent variable
η	=	$\beta^*/\alpha^{\frac{1}{2}}$
λ	=	h_α/h_β, a function of α and β^*
Λ	=	λ, but a function of α and η
μ	=	viscosity of the fluid
σ	=	distance normal to the surface
τ	=	skin friction
ϕ	=	angle formed with the line joining the forward stagnation point and the center of cylinder with the latter as the axis of rotation.
χ	=	$\int_0^\eta [1/\Lambda(\alpha,\eta)]\, d\eta$

Greek
Notation

ψ	=	stream function
ω	=	vorticity in the direction of the axis of the cylinder

SUMMARY

This thesis constitutes an investigation of a laminar boundary-layer flow of a compressible fluid around a circular cylinder with the application of numerical methods. Suitable equations of motion are developed and approximated for the boundary-layer in a curvilinear orthogonal coordinate system. Simplification has been achieved by the application of order of magnitude comparison. Difficulties encountered and their effects on the solution are discussed. A search has been made to obtain numerical solutions of velocity profiles and skin friction at several positions along the surface of the cylinder. Results are compared with other existing solutions.

CHAPTER I

INTRODUCTION

Laminar boundary-layer flow around a circular cylinder for a compressible fluid has been a topic of research for many years. The problem of placing a cylindrical body in a fluid stream moving perpendicular to its axis, with the consideration of boundary-layer, has been investigated by many researchers.

The method of solution was first given by H. Blasius [1]. It was developed further by H. Hiemanz [2] and L. Howarth [2]. The velocity of the potential flow is assumed to have the form of a power series in x, the distance from the stagnation point along surface. The velocity profile in the boundary-layer is also represented as a similar power series in x, where the coefficients are assumed to be functions of the coordinate y, measured at right angles to the wall (Blasius series). Although the solutions are exact and of high accuracy, the methods are quite tedious and time-consuming. It is, therefore, important to devise approximate methods which would in such cases quickly lead to an answer even if their accuracy were to be inferior to that of the numerical methods and exact solutions. Following Theodore von Kármán and K. Pohlhausen [2], it is possible to devise such simplified methods if it is agreed to satisfy the differential equations of boundary-layer flow only in the average and over the boundary-layer thickness rather than to try to satisfy the boundary conditions for every individual fluid particle. With the use of the digital computers for scientific research, numerical methods have been developed by W. Schoenauer, Hartree and Womersley [3], Smith and Clutter [4], etc. Numerical methods minimize the time of solution greatly while retaining its accuracy. However, sometimes due to the lack of appropriate methods, difficulties develop in the attempts to produce numerical results.

The purpose of this thesis is to investigate the flow field around a circular cylinder for a compressible laminar fluid by applying numerical methods to the solution. Theories in this direction have been developed by many researchers. However, very few have tried to obtain solutions for this particular case with the consideration of curvature effects. Although due to certain difficulites, accurate solutions were not obtained in this thesis, the development of theories and methods of solution can be used for further research.

The problem is formulated with the use of orthogonal coordinates (see Appendix A). Constant β-lines are taken to be along and parallel to the surface. Constant α-lines are orthogonal to the β-lines and the surface. Instead of solving for the pressure distribution, which is the usual approach, an experimental surface pressure distribution (see Appendix B) is used as a known quantity. Therefore, the Reynolds number of 53000 at which the pressure distribution was measured is used throughout this thesis for the numerical calculations. Velocity profiles and skin friction are obtained at certain positions and are compared with other existing results.

CHAPTER II

THEORY OF LAMINAR COMPRESSIBLE FLOW AROUND A CIRCULAR CYLINDER

The problem to be examined is the laminar flow of a compressible fluid around a circular cylinder. Its flow direction is perpendicular to the axis of the cylinder. A set of orthogonal coordinates is used which must satisfy the Gauss orthogonality equation. Radii of curvature of the α-lines and β-lines can be given from the above relation. The Navier-Stokes equations are applied to govern the flow field in the vicinity of the cylinder. The elimination of pressure terms from both equations result in the vorticity equation, where the vorticity in the direction of the axis of the cylinder is already defined. A new variable λ is introduced as the ratio of h_α to h_β. It turns out that the value for λ can be used as velocity profile. An equation of λ can be obtained and it leads to the solution with appropriate boundary conditions and transformations.

A. Gauss Orthogonality Equation [5]

Let $\alpha(x,y)$ and $\beta(x,y)$ be two functions with

$$\left(\frac{\partial\alpha}{\partial x}\right)\left(\frac{\partial\beta}{\partial x}\right) + \left(\frac{\partial\alpha}{\partial y}\right)\left(\frac{\partial\beta}{\partial y}\right) = 0 \qquad \text{(II-1)}$$

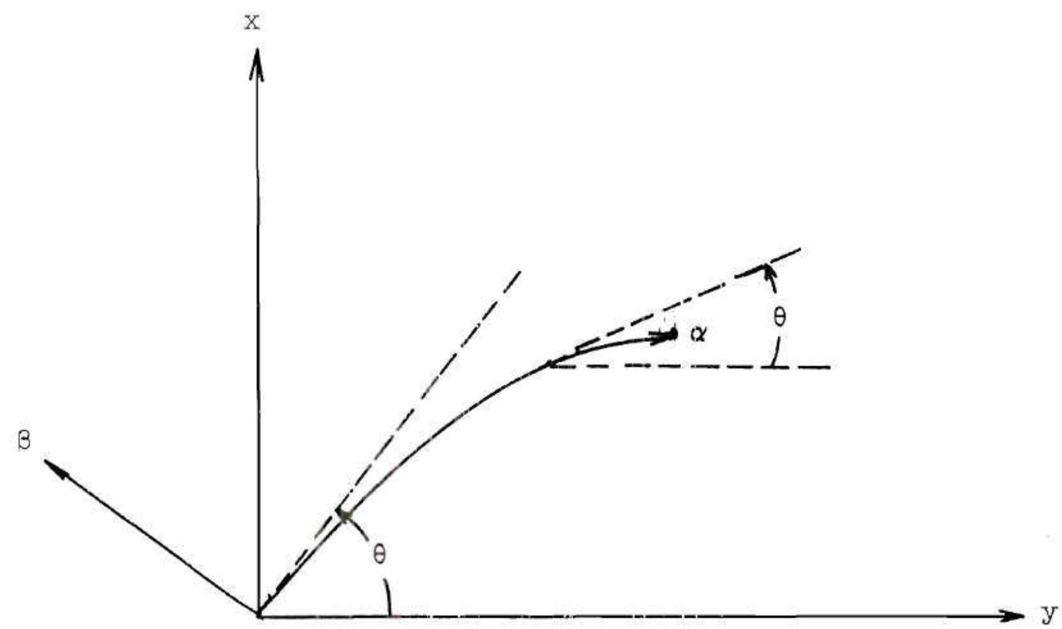

Fig. 1 Transformation of Coordinates

Let $h_\alpha d_\alpha$ and $h_\beta d_\beta$ be the elements of length along the α-lines and β-lines respectively. One can show that

$$h_\alpha = \left[\left(\frac{\partial x}{\partial \alpha}\right)^2 + \left(\frac{\partial y}{\partial \alpha}\right)^2\right]^{\frac{1}{2}} \tag{II-2}$$

$$h_\beta = \left[\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2\right]^{\frac{1}{2}} \tag{II-3}$$

$$\frac{1}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \beta} = R_\alpha^{-1} \tag{II-4}$$

$$\frac{1}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \alpha} = R_\beta^{-1} \tag{II-5}$$

where R_α and R_β are the radii of curvature of α-lines and the β-lines, respectively. The two functions h_α and h_β satisfy the Gauss equation

$$\frac{\partial}{\partial \beta}\left(\frac{1}{h_\beta} \frac{\partial h_\alpha}{\partial \beta} \right) + \frac{\partial}{\partial \alpha}\left(\frac{1}{h_\alpha} \frac{\partial h_\beta}{\partial \alpha} \right) = 0 \tag{II-6}$$

B. Navier-Stokes Equations [6]

If the α-lines are so chosen to be along the stream lines, β can be equalized to the stream function, $\beta = \psi$. The equation of continuity which is identically satisfied by the stream function can be substituted by the following expressions:

$$u \equiv \frac{1}{h_\beta} \frac{\partial \psi}{\partial \beta} = \frac{1}{h_\beta} \tag{II-7}$$

$$v = - \frac{1}{h_\alpha} \frac{\partial \psi}{\partial \alpha} = 0 \tag{II-8}$$

Then the Navier-Stokes equations of motion are given by

$$\frac{u}{h_\alpha}\frac{\partial u}{\partial \alpha} = -\frac{1}{h_\alpha}\frac{\partial p}{\partial \alpha} + \frac{1}{Re}\frac{1}{h_\beta}\left[\frac{1}{h_\alpha h_\beta}\frac{\partial (h_\alpha u)}{\partial \beta}\right] \qquad \text{(II-9)}$$

$$\frac{u}{h_\beta}\frac{\partial u}{\partial \beta} - \frac{1}{h_\alpha h_\beta}\frac{\partial (h_\alpha u)}{\partial \beta} = -\frac{1}{h_\beta}\frac{\partial p}{\partial \beta} - \frac{1}{Re}\frac{1}{h_\alpha}\frac{\partial}{\partial \alpha}\left[\frac{1}{h_\alpha h_\beta}\frac{\partial (h_\alpha u)}{\partial \beta}\right] \qquad \text{(II-10)}$$

C. The Vorticity Equation [5]

The vorticity in the direction of the cylinder axis is given by

$$\omega = -\frac{1}{h_\alpha h_\beta}\cdot\frac{\partial (h_\alpha u)}{\partial \beta} \qquad \text{(II-11)}$$

Eliminating p from equation (II-9) and (II-10) the vorticity equation is obtained

$$\frac{\partial}{\partial \alpha}\left[\frac{h_\beta}{h_\alpha}\frac{\partial \omega}{\partial \alpha}\right] + \frac{\partial}{\partial \beta}\left[\frac{h_\alpha}{h_\beta}\frac{\partial \omega}{\partial \beta}\right] = Re\frac{\partial \omega}{\partial \alpha} \qquad \text{(II-12)}$$

D. Introduction of λ [5]

For the convenience of operations a new variable λ is introduced.

It is defined as

$$\lambda \equiv \frac{h_\alpha}{h_\beta} = \left(\frac{u}{1/h_\alpha} \right) \tag{II-13}$$

Therefore, the vorticity equation becomes

$$\frac{\partial}{\partial\alpha}\left[\lambda^{-1} \frac{\partial\omega}{\partial\alpha} \right] + \frac{\partial}{\partial\beta}\left[\lambda \frac{\partial\omega}{\partial\beta} \right] = \mathrm{Re}\, \frac{\partial\omega}{\partial\alpha} \tag{II-14}$$

and the Gauss equation can be written as

$$\frac{\partial}{\partial\alpha}\left[\lambda^{-1} \frac{\partial \ln h_\alpha}{\partial\alpha} \right] + \frac{\partial}{\partial\beta}\left[\lambda \frac{\partial \ln h_\alpha}{\partial\beta} \right] = \frac{\partial}{\partial\alpha}\left[\lambda^{-2} \frac{\partial\lambda}{\partial\alpha} \right] \tag{II-15}$$

For potential flow, $\lambda = 1$, equations (II-14) and (II-15) are satisfied automatically.

Thus if a viscous flow over a rigid surface, on which ψ chosen to be zero, approaches a potential flow at free stream, the proper boundary conditions to be used for λ are

$$\lambda = 0 \text{ at } \psi = 0 \tag{II-16}$$

$$\lambda \rightarrow 1 \text{ as } \psi \rightarrow \infty \tag{II-17}$$

and in view of equation (II-11)

$$\omega = - \frac{\lambda}{h_\alpha^2} \frac{\partial \lambda}{\partial \beta} \tag{II-18}$$

$$\frac{\partial \lambda}{\partial \beta} \rightarrow 0 \text{ as } \psi \rightarrow \infty \tag{II-19}$$

The boundary conditions for h_α may be chosen to be

$$h_\alpha = \text{known value at } \psi = 0 \tag{II-20}$$

and from equation (II-2)

$$\frac{\lambda}{h_\alpha^2} \frac{\partial h_\alpha}{\partial \beta} = 1 \quad \text{at } \psi = 0 \text{ for a circular cylinder} \tag{II-21}$$

CHAPTER III

DEVELOPMENT OF EQUATIONS FOR TWO-DIMENSIONAL LAMINAR COMPRESSIBLE FLOW AROUND A CYLINDER

After the establishment of the theories the next necessary step is to develop a general equation which can be applied in the case of a two-dimensional, laminar, compressible flow around a cylinder. The order of magnitude of each term in the equations formulated in Chapter II are compared. The terms of order of $R_e^{\frac{1}{2}}$ are to be neglected while those of order of R_e or larger are retained. By doing so, the curvature effects are being considered. It is important to find an expression for h_α, which appears in the equations frequently, such that h_α can be expressed in terms of the independent variables. Therefore, the appearance of h_α in the equations will not create a problem during the process of solution. Moreover, a curvature parameter $C_\alpha(\alpha)$ must be defined in order to emphasize the curvature effects. The basic equations are obtained from the Navier-Stokes equations. Three steps of transformation are performed to simplify the basic equation to a neater form. Solutions then are obtained from this equation.

A. Boundary-Layer Approximations [5]

The boundary-layer approximation is carried out in the generalized coordinates by introducing the transformation

$$\beta^{*} = R_e^{\frac{1}{2}} \beta \qquad \text{(III-1)}$$

From equations (II-14) and (II-15), the first-order boundary-layer equations are obtained by neglecting small magnitude terms

$$\frac{\partial}{\partial\alpha}\left(\frac{\lambda}{h_\alpha^2}\frac{\partial\lambda}{\partial\beta^*}\right) = \frac{\partial}{\partial\beta^*}\left[\lambda\frac{\partial}{\partial\beta^*}\left(\frac{\lambda}{h_\alpha^2}\frac{\partial\lambda}{\partial\beta^*}\right)\right] \qquad \text{(III-2)}$$

$$\frac{\partial}{\partial\beta^*}\left(\lambda\frac{\partial \ln h_\alpha}{\partial\beta^*}\right) = 0 \qquad \text{(III-3)}$$

The boundary condition (II-21) becomes

$$\frac{\lambda}{h_\alpha^2}\frac{\partial h_\alpha}{\partial\beta^*} = R_e^{-\frac{1}{2}}, \quad \beta^* = 0 \qquad \text{(III-4)}$$

Thus, the surface curvature would appear in the first order boundary-layer approximations (the C-approximation).

B. Curvature Parameter and Metric Measurement

Equation (III-3) is integrated and with $N(\alpha)$ as the integration constant, a function of α only, one obtains

$$\lambda\frac{\partial \ln h_\alpha}{\partial\beta^*} = N(\alpha) \qquad \text{(III-5)}$$

Further integration of equation (III-5), gives

$$\ln h_\alpha = \int_0^{\beta^*} \frac{N(\alpha)}{\lambda}\, d\beta^* + \ln h_{\alpha o} \qquad \text{(III-6)}$$

where $h_{\alpha o}$ is a function of α only and is the value of h_α at $\beta^* = 0$ for a given β^*-line. Therefore,

$$h_\alpha = h_{\alpha o}(\alpha) \exp\left[N(\alpha) \int_0^{\beta^*} \frac{d\beta^*}{\lambda} \right] \qquad \text{(III-7)}$$

From boundary condition (III-4), one has

$$\frac{\lambda}{h_{\alpha o}} \left(\frac{\partial \ln h_\alpha}{\partial \beta^*} \right)_{\beta^* = 0} = R_e^{-\frac{1}{2}}, \text{ at } \beta^* = 0 \qquad \text{(III-8)}$$

The above relation is substituted into equation (III-5), to obtain

$$N(\alpha) = \frac{h_{\alpha o}}{R_e^{\frac{1}{2}}} \qquad \text{(III-9)}$$

Define

$$C_\alpha(\alpha) \equiv \sqrt{\alpha}\, N(\alpha) \qquad \text{(III-10)}$$

As the curvature parameter and with equation (III-9) it becomes

$$C_\alpha(\alpha) = \frac{\sqrt{\alpha}\, h_{\alpha o}}{R_e^{\frac{1}{2}}} \qquad \text{(III-11)}$$

The effect of $C_\alpha(\alpha)$ on the solution will be discussed later.

C. Derivation of Momentum Equation [6]

With the use of boundary-layer approximations, equation (II-9) and (II-10) become

$$\frac{u}{h_\alpha}\frac{\partial u}{\partial \alpha} = -\frac{1}{h_\alpha}\frac{\partial p}{\partial \alpha} + \frac{1}{h_\beta}\frac{\partial}{\partial \beta^*}\left[\frac{1}{h_\alpha h_\beta}\frac{\partial(h_\alpha u)}{\partial \beta^*}\right] \qquad \text{(III-12)}$$

$$\frac{u}{h_\beta}\frac{\partial u}{\partial \beta^*} - \frac{u}{h_\alpha h_\beta}\frac{\partial(h_\alpha u)}{\partial \beta^*} = -\frac{1}{h_\beta}\frac{\partial p}{\partial \beta^*} \qquad \text{(III-13)}$$

and equation (III-13) can be further simplified to

$$\frac{u^2}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \beta^*} = \frac{1}{h_\beta} \frac{\partial p}{\partial \beta^*} \qquad \text{(III-14)}$$

by expanding the differentiation of the second term.

The effects of curvature become significant when the pressure gradient $\frac{\partial p}{\partial \beta^*}$ grows up to the order of one. By retaining terms of order one or larger, curvature effects are considered.

Equation (II-11), the definition of vorticity is expanded to give

$$\omega = - \frac{1}{h_\beta} \frac{\partial u}{\partial \beta} - \frac{u}{h_\alpha h_\beta} \frac{\partial h_\alpha}{\partial \beta} \qquad \text{(III-15)}$$

and equations (III-14) and (III-15) are combined to obtain

$$\omega = - \frac{1}{h_\beta} \frac{\partial u}{\partial \beta} - \frac{1}{u h_\beta} \frac{\partial p}{\partial \beta} \qquad \text{(III-16)}$$

From equation (II-7), it is given that

$$u = \frac{1}{h_\beta}$$

therefore,

$$\omega = - u \frac{\partial u}{\partial \beta} - \frac{\partial p}{\partial \beta} \qquad \text{(III-17)}$$

The expressions for ω from equations (III-17) and (II-18) are equated, and

$$- u \frac{\partial u}{\partial \beta} - \frac{\partial p}{\partial \beta} = - \frac{\lambda}{h_\alpha^2} \frac{\partial \lambda}{\partial \beta} \qquad \text{(III-18)}$$

or

$$- u \frac{\partial u}{\partial \beta^*} - \frac{\partial p}{\partial \beta^*} = - \frac{\lambda}{h_\alpha^2} \frac{\partial \lambda}{\partial \beta^*} \qquad \text{(III-19)}$$

The above equation is integrated in the region inside the boundary-layer, and one obtains

$$\int_0^\delta \frac{\lambda}{h_\alpha^2} \frac{\partial \lambda}{\partial \beta^*} d\beta^* = \left[\tfrac{1}{2} u^2 + p \right]_{\beta^*=0}^{\beta^*=\delta} \qquad \text{(III-20)}$$

where δ is the boundary-layer thickness.

The boundary conditions to the above equation are

$$u = U, \quad \text{at} \quad \beta^* = \delta \tag{III-21}$$

$$u = 0, \quad \text{at} \quad \beta^* = 0 \tag{III-22}$$

$$p = p_o, \quad \text{at} \quad \beta^* = \delta \tag{III-23}$$

$$p = p_s, \quad \text{at} \quad \beta^* = 0 \tag{III-24}$$

where U is the potential velocity on the boundary and p_o and p_s are pressures at the boundary and surface respectively. With these boundary conditions equation (III-20) is evaluated, thus

$$\int_o^\delta \frac{\lambda}{h_\alpha^2} \frac{\partial\lambda}{\partial\beta^*} \, d\beta^* = \tfrac{1}{2} U^2 + P_o - P_s \tag{III-25}$$

or

$$\int_o^\delta \frac{\lambda}{h_\alpha^2} \frac{\partial\lambda}{\partial\beta^*} \, d\beta^* = \tfrac{1}{2} U^2 + (P_o - P_\infty) - (P_s - P_\infty) \tag{III-26}$$

where P_∞ is the pressure at free stream. The boundary conditions for λ and h_α are the same as already specified by equations (II-16), (II-17) and (II-19).

The Bernoulli equation is

$$\frac{1}{2} U^2 + P_o = \text{constant} \qquad \text{(III-27)}$$

for a potential flow and, therefore a constant is defined as

$$K = \frac{1}{2} U^2 + (P_o - P_\infty) \qquad \text{(III-28)}$$

since P_∞ is also a constant.

Finally the momentum equation is obtained

$$\int_o^\delta \frac{\lambda}{h_\alpha^2} \frac{\partial \lambda}{\partial \beta^*} d\beta^* = K - (P_s - P_\infty) \qquad \text{(III-29)}$$

The paremeter λ goes to unity at the boundary and retains the same value outside the boundary layer. Consequently, the derivative $\frac{\partial \lambda}{\partial \beta^*}$ goes to zero on and outside the boundary. Therefore the upper limit of the integral can be extended to infinity and equation (III-29) becomes

$$\int_0^\infty \frac{\lambda}{h_\alpha^2} \frac{\partial\lambda}{\partial\beta^*} d\beta^* = K - (P_s - P_\infty) \tag{III-30}$$

At the stagnation point h_α approaches infinity and this point is considered as a singular point. Therefore, its neighborhood is not being considered in the solution.

With the condition that

$$h_\alpha \to \infty \quad \text{as} \quad \alpha \to 0 \tag{III-31}$$

the left-hand side of equation (III-30) becomes zero.

Therefore,

$$K = (P_s - P_\infty)_{\alpha=0} \tag{III-32}$$

Let

$$\Delta P_i = P_i - P_\infty \tag{III-33}$$

where P_i is any pressure inside and on the boundary-layer. Consequently equation (III-30) becomes

$$\int_{0}^{\infty} \frac{\lambda}{h_{\alpha}^{2}} \frac{\partial\lambda}{\partial\beta^{*}} \, d\beta^{*} = K - \Delta P_{s} \qquad \text{(III-34)}$$

The values for ΔP_s, the surface pressure distribution, are given in a tabulated form and they are to be used in the solution.

D. First Transformation

In order to reduce the boundary-layer equation, equation (III-2), to a neater form, the following transformation is carried out. Let

$$\eta = \frac{\beta^{*}}{G(\alpha)} \qquad \text{(III-35)}$$

where $G(\alpha)$ is a function of α. The function $\lambda(\alpha, \beta^{*})$ becomes a new function $\Lambda(\alpha, \eta)$. Let

$$G(\alpha) = \sqrt{\alpha} \qquad \text{(III-36)}$$

then

$$\eta = \frac{\beta^{*}}{\sqrt{\alpha}} \qquad \text{(III-37)}$$

The new variable η is substituted for β^*, and equation (III-2) becomes

$$[2\Lambda(\Lambda^2)'' + (\eta - 4C_\alpha)\,(\Lambda^2)']' - 2\,\frac{C_\alpha}{\Lambda}\,[2\Lambda(\Lambda^2)'' + (\eta - 4C_\alpha)\,(\Lambda^2)']$$

$$= 2\alpha\,[\,\frac{\partial}{\partial\alpha}\,(\Lambda^2)' - 2(\Lambda^2)'\,\frac{\partial}{\partial\alpha}\,(\ln h_\alpha)\,] \qquad \text{(III-38)}$$

with boundary conditions

$$\Lambda = 0 \quad \text{at} \quad \eta = 0$$

$$\Lambda = 1 \quad \text{at} \quad \eta = \infty \qquad \text{(III-39)}$$

$$\Lambda' = 0 \quad \text{at} \quad \eta = \infty$$

where the prime (') indicates differentiation with respect to η. Therefore

$$h_\alpha = h_{\alpha o}(\alpha)\,\exp\left\{\,C_\alpha(\alpha)\int_o^\eta \frac{d\eta}{\Lambda}\right\} \qquad \text{(III-40)}$$

E. Second Transformation

A further simplification in the equation can be achieved through another transformation. A new variable χ is introduced as follows:

$$\chi = \int_0^{\eta} \frac{d\eta}{\Lambda} \qquad \text{(III-41)}$$

as well as a new function $f(\alpha,\chi)$

$$f(\alpha,\chi) = \eta - 4\ C_{\alpha} \qquad \text{(III-42)}$$

Therefore,

$$f' = \Lambda \qquad \text{(III-43)}$$

$$f'' = \Lambda\Lambda' = \tfrac{1}{2}(\Lambda^2)' \qquad \text{(III-44)}$$

$$f''' = \tfrac{1}{2}\ \Lambda(\Lambda^2)'' \qquad \text{(III-45)}$$

where the prime (') indicates differentiation with respect to χ. Equation (III-40) can now be written as

$$h_\alpha = h_{\alpha o}(\alpha) \exp [C_\alpha(\alpha)\chi] \qquad \text{(III-46)}$$

Then equation (III-38) becomes

$$[2f''' + ff'']' - 2C_\alpha [2f''' + ff'']$$

$$= 2\alpha f' \left[\frac{\partial f''}{\partial \alpha} - 2f'' \left(\frac{d}{d\alpha} \ln h_{\alpha o} + \chi \frac{dC_\alpha}{d\alpha} \right) \right] \qquad \text{(III-47)}$$

and the boundary conditions become

$$\begin{aligned} f &= -4C_\alpha && \text{at } \chi = 0 \\ f' &= 0 && \text{at } \chi = 0 \\ f' &= 1 && \text{as } \chi \to \infty \\ f'' &= 0 && \text{as } \chi \to \infty \end{aligned} \qquad \text{(III-48)}$$

Finally, equation (III-34) transforms to

$$h_{\alpha o}^{-2} \int_{o}^{\infty} e^{-2C_{\alpha}(\alpha)\chi} f' f'' d\chi = K - \Delta P_s \qquad \text{(III-49)}$$

With equations (III-47) and (III-49) and the specified boundary conditions (III-48) the problem is ready to be solved.

CHAPTER IV

METHODS AND SOLUTIONS

Although the fundamental equations (III-47) and (III-49) have been derived, some further simplifications will be carried out to achieve a numerical solution. Such simplifications are necessary for two reasons. First of all, some terms are comparatively of small magnitude and their elimination will reduce the numerical effort. In the second instance, it has not been possible to find a proper method to numerically evaluate some of the terms in the differential equations. For this case, the solution obtained with these terms results in certain deviation and the effects will be discussed later.

Another problem appears when it is observed that a differential equation must be accompanied by all its initial conditions in order to be solved by numerical methods. Therefore, a transformation is needed to transfer some of the boundary conditions at infinity to the starting point. The remainder unknown initial conditions will be assumed and a trial and error solution will be applied. An inverse transformation is also necessary in order to represent the solution in a useful form.

A. First Simplification

As it was discussed above, equation (III-47) will be simplified by eliminating terms of comparatively small magnitude. The term $\chi(dC_\alpha/d_\alpha)$ in equation (III-47) is then compared with $\partial(\ln h_{\alpha o})/\partial\alpha$. The order of magnitude of $(dC_\alpha/d\alpha)$ is approximately .0005 for s greater than $\pi/180$, and χ ranges from 0 to approximately 5. Therefore,

$$\max\ [\chi(d(\alpha)d\alpha)] \cong .0025 \qquad \text{(IV-1)}$$

While for the other term,

$$O\ [d(\ln h_{\alpha_o}/d\alpha] = 5 \qquad \text{(IV-2)}$$

Therefore,

$$O\ [d(\ln h_{\alpha o})/d\alpha] \gg O\ [\chi(dC_\alpha/d_\alpha)] \qquad \text{(IV-3)}$$

Evidently, the term $\chi(dC_\alpha/d\alpha)$ can be neglected. However, the term was retained in the solution in some trials, but there was no noticeable change in the solution.

With the above simplification, the basic equation becomes

$$[2f''' + ff'']' - 2C_\alpha\ [2f''' + ff'']$$

$$= 2\alpha f' \left[\frac{\partial f''}{\partial\alpha} - 2f'' \frac{\partial}{\partial\alpha} \ln h_{\alpha o} \right] \qquad \text{(IV-4)}$$

B. Second Simplification

With the term $\frac{\partial f''}{\partial \alpha}$ included in equation (IV-4), it is a partial differential equation. Although many numerical approximations for the partial differentiation term have been developed, it has been recommended that such an approach be avoided [7]. Under certain situations, the approximations give large error and therefore the solution diverges. It is a great risk to leave the term in the equation while trying to solve it. The function to be differentiated is first approximated by a multi-degree polynomial. Even if the polynomial approximates the function closely, the slopes at the same coordinate can be substantially different.

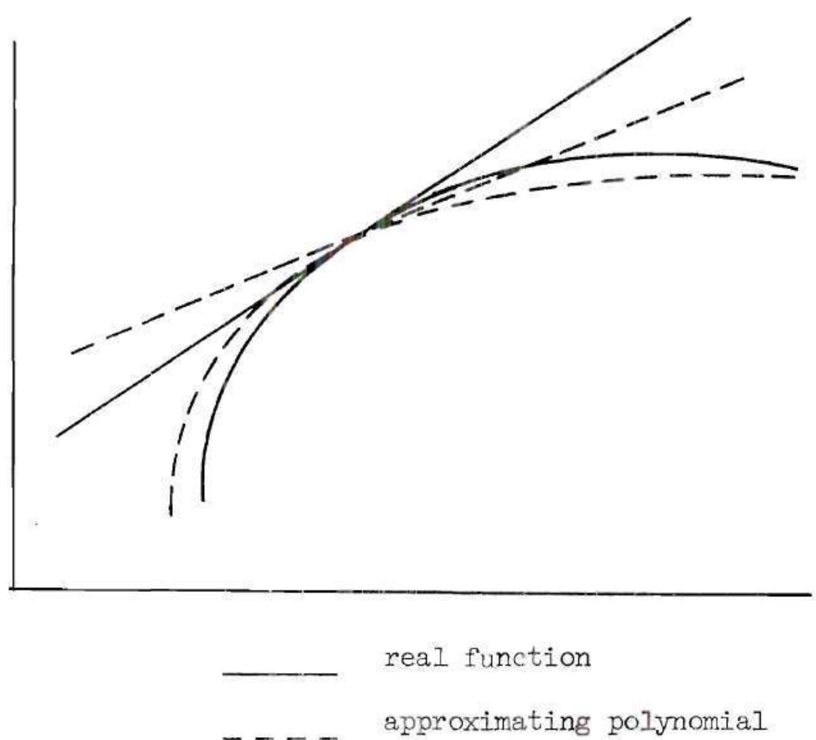

Fig. 2 Discrepancy of Numerical Differentiation

It can be seen from Figure 2 that the slopes of the real function and the approximating polynomical are quite different. Thus if the partial differentiation is of equal order of magnitude as the other terms, the approximated value will effect the solution considerably. Sometimes, it is possible that the solution diverges. With the above consideration, the term $\frac{\partial f''}{\partial \alpha}$ is dropped from equation (IV-4).

The differentiation with respect to α at a given point is approximated by the values at the previous two solutions. Therefore, the solution should be from the stagnation point. Since the stagnation point is a singularity, it is not apparent where the solution should start so that the term to be calculated is not effected. It will be seen later from the results that even at 1°, the profile diviation from the Blasius solution is considered to be very small. With the removal of term $\frac{\partial f''}{\partial \alpha}$, equation (IV-4) becomes

$$[2f''' + ff'']' - 2C_\alpha [2f''' + ff''] = - 4\alpha\, f'\, f'' \frac{\partial}{\partial \alpha} \ln h_{\alpha o} \quad \text{(IV-5)}$$

C. Final Transformation

In the use of numerical methods for solving differential equations, it is required that all the boundary conditions at the starting point are known. However, a numerical solution can be carried out even if some of the known boundary conditions are at infinity. In this instance, a trial and error solution is necessary. The rest of the unknown boundary conditions at zero are assumed values and trials are performed until the known boundary conditions at infinity satisfy the solution. For the present case, it is desired to transform the original equation to a state where all except one of the boundary conditions at zero are known. The remaining unknown boundary condition is then assumed a value and the trial and error solution can be initiated. Therefore, a transformation is applied to equation (IV-5). Let

$$\zeta = C\chi \qquad \text{(IV-6)}$$

and

$$g = f/C \qquad \text{(IV-7)}$$

then

$$g' = f'/C^2 \qquad \text{(IV-8)}$$

$$g'' = f''/C^3 \qquad \text{(IV-9)}$$

$$g''' = f'''/C^4 \qquad \text{(IV-10)}$$

where C is a function of α only and the prime (') for f denotes differentiation with respect to χ whereas the prime (') for g denotes differentiation with respect to ζ. The above relations are substituted into equation (III-47), and give

$$(2g''' + gg'')' - 2(C_\alpha/C)\,(2g''' + gg'')$$

$$= 2\alpha g' \left[\frac{\partial g''}{\partial \alpha} - 2g'' \frac{d}{d\alpha} \ln h_{\alpha o} \right] \qquad \text{(IV-11)}$$

The boundary conditions then become

$$\begin{aligned} g &= -4C_\alpha/C && \text{at} \quad \zeta = 0 \\ g' &= 0 && \text{at} \quad \zeta = 0 \\ g' &= 1/C^2 && \text{as} \quad \zeta \to \infty \\ g'' &= 0 && \text{as} \quad \zeta \to \infty \end{aligned} \qquad \text{(IV-12)}$$

Also if $$C = [f''(0)]^{1/3} \qquad \text{(IV-13)}$$

an additional boundary condition is obtained at the surface as follows:

$$g'' = 1 \quad \text{at} \quad \zeta = 0 \tag{IV-14}$$

Therefore, in summary, the boundary conditions have been rearranged as

$$\begin{aligned} g &= -4C_\alpha/C && \text{at} \quad \zeta = 0 \\ g' &= 0 && \text{at} \quad \zeta = 0 \\ g'' &= 1 && \text{at} \quad \zeta = 0 \\ g'' &= 0 && \text{as} \quad \zeta \to \infty \end{aligned} \tag{IV-15}$$

If $g'''(0)$ is known, then equation (IV-11) can be solved. Therefore, the value of $g'''(0)$ must be assumed. The solution is then checked by the condition

$$g' = [f''(0)]^{-2/3} = C^{-2} \quad \text{as} \quad y \to \infty \tag{IV-16}$$

for a particular C. The trial and error process is continued until the condition (IV-16) is satisfied.

D. Numerical Solution

1. Estimation of $h_{\alpha o}$ and α

The evaluation of $h_{\alpha o}$ and α is a trial and error process. For this it is necessary to use equations (III-49) and (IV-11). However, as a first step, it is necessary to transform equation (III-49) by the use of equations (IV-6) through (IV-10)

$$C^4 h_{\alpha o}^{-2} \int_0^\infty e^{-2\left(\frac{C_\alpha}{C}\right)\zeta} g'g'' \, d\zeta = K - \Delta Ps \qquad \text{(IV-17)}$$

To initiate the evaluating process for $h_{\alpha o}$ and α, initial values are assumed for both quantities. Equations (IV-11) is then solved and the values of g' and g" are substituted into equation (IV-17) to obtain a new value of $h_{\alpha o}$. The trial and error process is continued until the assumed $h_{\alpha o}$ and the calculated $h_{\alpha o}$ agree within 1 percent of error.

After the computation for the first few profiles, it is found that the integral at the left hand side of equation (IV-17) is approximately .53 in each case. Therefore, for subsequent profiles, it can initially be assumed that

$$h_{\alpha o}(s) = \left[\frac{.497049 - \Delta Ps(s)}{.53}\right]^{-\frac{1}{2}} \qquad \text{(IV-18)}$$

where s is the diatance along the surface of the cylinder from the forward stagnation point, and the corresponding $\alpha(s)$ value can be evaluated by

$$\alpha(s) = \int_0^s h_{\alpha o}^{-1}(s)\, ds \qquad \text{(IV-19)}$$

With this process, the values of $h_{\alpha o}$ and α that will satisfy both equations (IV-11) and (IV-17) can be obtained.

2. Calculation of g'''(0) by Iteration

Equation (IV-11) is solved directly by use of the Runge-Kutta Method, and an initially assumed value for $g'''(0)$. The Newton-Raphson iteration method is then used to evaluate the true $g'''(0)$ which will yield a solution that can satisfy the boundary condition

$$g'' = 0 \quad \text{as} \quad \zeta \to \infty \qquad \text{(IV-15)}$$

The method of iteration is discussed in detail in Appendix D. For each particular value of $g(0)$, a corresponding value for $g'''(0)$ should be obtained through iteration. At the same time, a corresponding Reynolds number can be computed.

The Reynolds number is calculated by

$$Re = \frac{g'(\infty)\ \alpha\ h_{\alpha o}^{\ 2}}{(C_\alpha/C)^2} \qquad \text{(IV-20)}$$

where $C_\alpha/C = -\ (1/4)\ g(0)$. Since the known surface pressure distribution selected for this problem was measured at a Reynolds number of 53,000 [8], it is necessary that the R_e calculated from (IV-20) agrees with 53,000 within 5. A set of values for $g(0)$ and $h_{\alpha o}$ which will give a R_e of 53,000 is searched. The process of finding a proper set of $g(0)$ and $h_{\alpha o}$ may be summarized as follows:

(1) A value of $g(0)$ is assumed.

(2) Values of $h_{\alpha o}$ and α are then assumed through the use of equations (IV-18) and (IV-19).

(3) Equation (IV-11) is solved.

(4) Values of g' and g'' obtained from the previous step are substituted into equation (IV-17), and a new value of $h_{\alpha o}$ is obtained.

(5) Going through the trial and error process mentioned in the previous section, a set of values of $h_{\alpha o}$ and α, which satisfy equations (IV-11) and (IV-17), is obtained.

(6) The Reynolds number is calculated through the use of equation (IV-20), and it is compared with 53,000.

(7) If they agree within 5, the process is accomplished. Otherwise, a new value of $g(0)$, which can be approximated by

$$g(0) = 4h_{\alpha_o} \; [g'(\infty) \; \alpha/53{,}000]^{\frac{1}{2}} \qquad \text{(IV-21)}$$

is used. The process is then repeated from the first step until the calculated R_e is within 5 of 53,000.

3. Inverse Transformation

After the solution for g, g', g", and g''' have been obtained, they are transformed back to f, f', f", and f'''. From equation (IV-16) C is given by the relation

$$C = [g'(\infty)]^{-\frac{1}{2}} \qquad \text{(IV-22)}$$

and consequently the values of f, f', f", and f''' can be calculated without difficulty. The inverse transformation is based on the transformations given by equations (IV-6) through (IV-10).

4. Change of the Independent Variable

In order to be able to compare the results calculated by the method presented here with existing solutions, the independent variable χ is changed to n where

$$n = R_e^{\frac{1}{2}} \int_o^{\beta} h_\beta d_\beta \qquad \text{(IV-23)}$$

and is proportional to $\sigma R_e^{\frac{1}{2}}$ where σ is the non-dimensionalized distance normal to the surface. With the use of equation (II-13) h_α and λ are substituted for h_β and

$$n = R_e^{\frac{1}{2}} \int_o^\beta \frac{h_\beta}{\lambda} \, d\beta \qquad \text{(IV-24)}$$

The parameter β is also eliminated with the help of equations (III-1) and (III-36) to obtain the following expression in terms of η

$$\eta = \alpha^{\frac{1}{2}} \int_o^\eta \frac{h_\alpha}{\Lambda(\alpha,\eta)} \, d\eta \qquad \text{(IV-25)}$$

Finally, through the use of equations (III-40) and (III-41) the expression in equation (IV-25) is transformed with respect to χ and λ

$$\eta = \alpha^{\frac{1}{2}} \, h_{\alpha_o} \int_o^\chi \frac{\text{Exp}\,[C_\alpha \chi]}{\Lambda(\alpha_n,\chi)} \, \Lambda d\chi \qquad \text{(IV-26)}$$

which is integrated to give

$$\eta = \frac{\alpha^{\frac{1}{2}} h_{\alpha o}}{C_\alpha(\alpha)} \left[\mathrm{Exp}\ (C_\alpha \chi) - 1 \right] \tag{IV-27}$$

The definition of $C_\alpha(\alpha)$ was given by equation (III-11) as

$$\frac{\alpha^{\frac{1}{2}}\ h_{\alpha o}}{R_e^{\frac{1}{2}}} = C_\alpha$$

and therefore

$$\eta = R_e^{\frac{1}{2}} \left[\mathrm{Exp}\ (C_\alpha \chi) - 1 \right] \tag{IV-28}$$

With these changes carried out, it is now possible to plot f' versus η profiles and compare these with existing results.

5. <u>Skin Friction</u>

Skin friction is defined as

$$\tau_o = R_e^{-1} \left(\frac{\partial u}{\partial \sigma} \right)_{\sigma=0} \tag{IV-29}$$

which can also be written as

$$\tau_o = \frac{U}{R_e}\left[\frac{\partial(u/U)}{\partial \chi}\right]_{\chi=0}\frac{\partial \chi}{\partial \sigma} \qquad \text{(IV-30)}$$

where U is the potential velocity.

In terms of f" and C_α from equations (IV-11), (IV-9) and (III-11)

$$\tau_o = \frac{U}{R_e}\ \frac{f''(0)}{C_\alpha(\alpha)} \qquad \text{(IV-31)}$$

Since

$$U = 2 \sin s \qquad \text{(IV-32)}$$

where s is the non-dimensionalized distance along the surface from the forward stagnation point, then

$$\tau_o = \frac{2\ f''(0)}{R_e C_\alpha} \qquad \text{(IV-33)}$$

and

$$\tau_o \sqrt{R_e} = \frac{2\ f''(0)}{R_e^{\frac{1}{2}}\ C_\alpha}\ \sin s \qquad \text{(IV-34)}$$

CHAPTER V

RESULTS AND DISCUSSION

The numerical results are calculated using the procedure outlined in Chapter IV and they are converted to a suitable form for comparison with some results published in the literature. The results used for comparison are the Blasius solution and the Pohlhausen solution. The former is a series solution with very high accuracy while the latter is an approximation method with some departure from the exact solution. The results are given in tabulated form in section A and they are compared in Figures 3 through 8 at 1°, 5°, 15°, 30°, 45° and 60° from the forward stagnation point respectively. Profiles exceeding 60° were not obtained due to the calculation. The skin friction is compared in both tabular form, Table 4, and praphical form, Figure 9.

A. Tabulated Results

Table 1 The Blasius Series Solution

Table 2 The Pohlhausen Approximate Solution

Table 3 Present Numerical Solution

Table 4 Skin Friction Comparison

Table 1 The Blasius Series Solution

n	u/U					
	1 Degree	5 Degree	15 Degree	30 Degree	45 Degree	60 Degree
0.1414	0.226586	0.226265	0.223464	0.213949	0.197699	0.173953
0.2828	0.414478	0.413945	0.409484	0.394158	0.367607	0.327983
0.4243	0.566274	0.565654	0.560448	0.542418	0.510663	0.462102
0.5657	0.685874	0.685248	0.679989	0.661604	0.628626	0.576816
0.7071	0.777876	0.777300	0.772445	0.755307	0.723940	0.673187
0.8485	0.846679	0.846183	0.841990	0.827029	0.799048	0.752310
0.9899	0.896783	0.896379	0.892951	0.880580	0.856895	0.815959
1.2728	0.959790	0.956556	0.954559	0.947162	0.932252	0.904469
1.4142	0.973193	0.973026	0.971594	0.966215	0.955055	0.933376
1.6971	0.990497	0.990421	0.989763	0.987212	0.981574	0.969585
2.1214	0.998399	0.998382	0.998227	0.997594	0.996030	0.992136
2.5456	0.999800	0.999797	0.999768	0.999646	0.999306	0.998321
2.8284	1.000000	0.999999	0.999995	0.999972	0.999891	0.999685

Table 2 The Pohlhausen Approximate Solution

	u/U				
	5 Degree	15 Degree	30 Degree	45 Degree	60 Degree
0.02	0.033264	0.032899	0.031270	0.028684	0.024955
0.05	0.081691	0.090820	0.076894	0.070664	0.061643
0.10	0.158574	0.156973	0.149592	0.137883	0.120825
0.13	0.202468	0.200490	0.191252	0.176597	0.155168
0.16	0.244730	0.242419	0.231480	0.214123	0.188647
0.20	0.298602	0.295911	0.282934	0.262341	0.231958
0.30	0.421426	0.418073	0.401076	0.374090	0.333719
0.50	0.620655	0.616940	0.595863	0.562291	0.510464
0.75	0.795411	0.792365	0.771748	0.738595	0.684843
1.00	0.904600	0.902695	0.886346	0.859505	0.813015
1.25	0.965024	0.964162	0.953685	0.935684	0.901344
1.50	0.992045	0.991827	0.986962	0.977625	0.956664
1.75	0.999582	0.999575	0.998552	0.995649	0.986282
2.00	1.000000	1.000000	1.000000	0.999908	0.997976

Table 3 Present Numerical Solution

1 Degree		5 Degree		15 Degree	
n	u/U	n	u/U	n	u/U
0.052765	0.84247	0.054336	0.084114	0.054425	0.083181
0.105543	0.163582	0.108685	0.163334	0.108867	0.161597
0.211130	0.307730	0.217422	0.307308	0.217782	0.304331
0.316769	0.433153	0.326210	0.432620	0.326752	0.438852
0.422456	0.540919	0.435050	0.540329	0.435770	0.536154
0.528191	0.632354	0.543941	0.631752	0.544843	0.627477
0.633974	0.708955	0.652883	0.708373	0.653964	0.704233
0.739805	0.772308	0.761877	0.77169	0.763140	0.767928
0.845685	0.824024	0.870923	0.823542	0.872367	0.820100
0.951615	0.865680	0.980020	0.965262	0.981646	0.862270
1.057594	0.898780	1.089168	0.898427	1.090976	0.895895
1.163617	0.924718	1.198368	0.924428	1.200354	0.922338
1.269691	0.944757	1.307616	0.944523	1.309787	0.942838
1.375818	0.960016	1.416918	0.959832	1.419275	0.958502
1.481988	0.971464	1.526276	0.971322	1.528812	0.970294
1.588210	0.979927	1.635692	0.979817	1.638397	0.979039
1.694477	0.986081	1.745139	0.986002	1.748038	0.985424
1.800755	0.990493	1.854647	0.990435	1.857732	0.990015
1.907162	0.993604	1.964207	0.993563	1.967474	0.993262
2.013579	0.995762	2.073822	0.995734	2.077270	0.995523
2.120042	0.997236	2.183485	0.997217	2.187119	0.997072
2.333117	0.99879	2.457874	0.999108	2.451964	0.999055
2.599739	0.999669	2.732585	0.999742	2.737138	0.999725
3.026961	0.999963	3.062669	0.999950	3.012635	0.999928

Table 3 (Continued)

30 Degree		45 Degree		60 Degree	
n	u/U	n	u/U	n	u/U
0.055761	0.080207	0.058768	0.075794	0.064552	0.070632
0.111535	0.156048	0.117558	0.147786	0.129121	0.138080
0.223126	0.294781	0.235158	0.280457	0.258314	0.263466
0.334771	0.416712	0.352828	0.398345	0.387579	0.376322
0.446467	0.522634	0.470559	0.501991	0.581612	0.522792
0.558222	0.613565	0.588349	0.592112	0.711058	0.605896
0.670028	0.690686	0.706197	0.667579	0.840577	0.678056
0.781890	0.755284	0.824111	0.735377	0.970168	0.739985
0.893802	0.808706	0.942079	0.790571	1.099834	0.792483
1.005773	0.852307	1.060100	0.836276	1.229569	0.836415
1.117796	0.887417	1.178200	0.973621	1.359380	0.872689
1.229873	0.915298	1.296351	0.903716	1.489262	0.902223
1.342005	0.937126	1.414563	0.927626	1.619217	0.925924
1.454193	0.953969	1.532836	0.946346	1.749247	0.944662
1.566431	0.966772	1.651168	0.960783	1.879349	0.959251
1.678728	0.976357	1.769562	0.971748	2.009523	0.970433
1.791076	0.983423	1.888014	0.979946	2.139769	0.978866
1.903479	0.988550	2.006527	0.985979	2.270090	0.985125
2.015939	0.992211	2.125103	0.990346	2.400484	0.989692
2.128451	0.994783	2.303078	0.994644	2.530949	0.992970
2.241019	0.996560	2.540589	0.997691	2.661490	0.995282
2.353638	0.997767	2.956819	0.999548	2.922789	0.997977
2.748245	0.999567	3.194996	0.999838	3.184382	0.999189
3.087013	0.999910	3.552722	0.999969	3.774038	0.999919

Table 4 Skin Friction Comparison

	$\tau_o \sqrt{R_e}$		
Angle	Numerical Solution	Blasius Solution	Pohlhausen Solution
1 Degree	0.055879	0.060719	
5 Degree	0.278020	0.303347	0.293362
15 Degree	0.814715	0.889271	0.854228
30 Degree	1.479218	1.635111	1.580910
45 Degree	1.871789	2.118596	2.048763
60 Degree	1.938523	2.254226	2.178529

B. Graphical Results

Figure 3 Velocity Profiles at 1 Degree

Figure 4 Velocity Profiles at 5 Degree

Figure 5 Velocity Profiles at 15 Degree

Figure 6 Velocity Profiles at 30 Degree

Figure 7 Velocity Profiles at 45 Degree

Figure 8 Velocity Profiles at 60 Degree

Figure 9 Skin Friction Comparison

C. Discussion of Results

The results obtained in this thesis involve several approximations and simplifications. Some of them were done to transform the fundamental equation (IV-4) to a form such that it can be solved more easily. These did not cause significant discrepancy in the solution. However, the other did. They were done simply because no known method could be applied to retain the term $\frac{\partial f''}{\partial \alpha}$ in the equation when the solution was sought. Without any doubt, this omission caused a significant discrepancy. Nevertheless, $\frac{\partial f''}{\partial \alpha}$ was neglected and the discrepancy can be

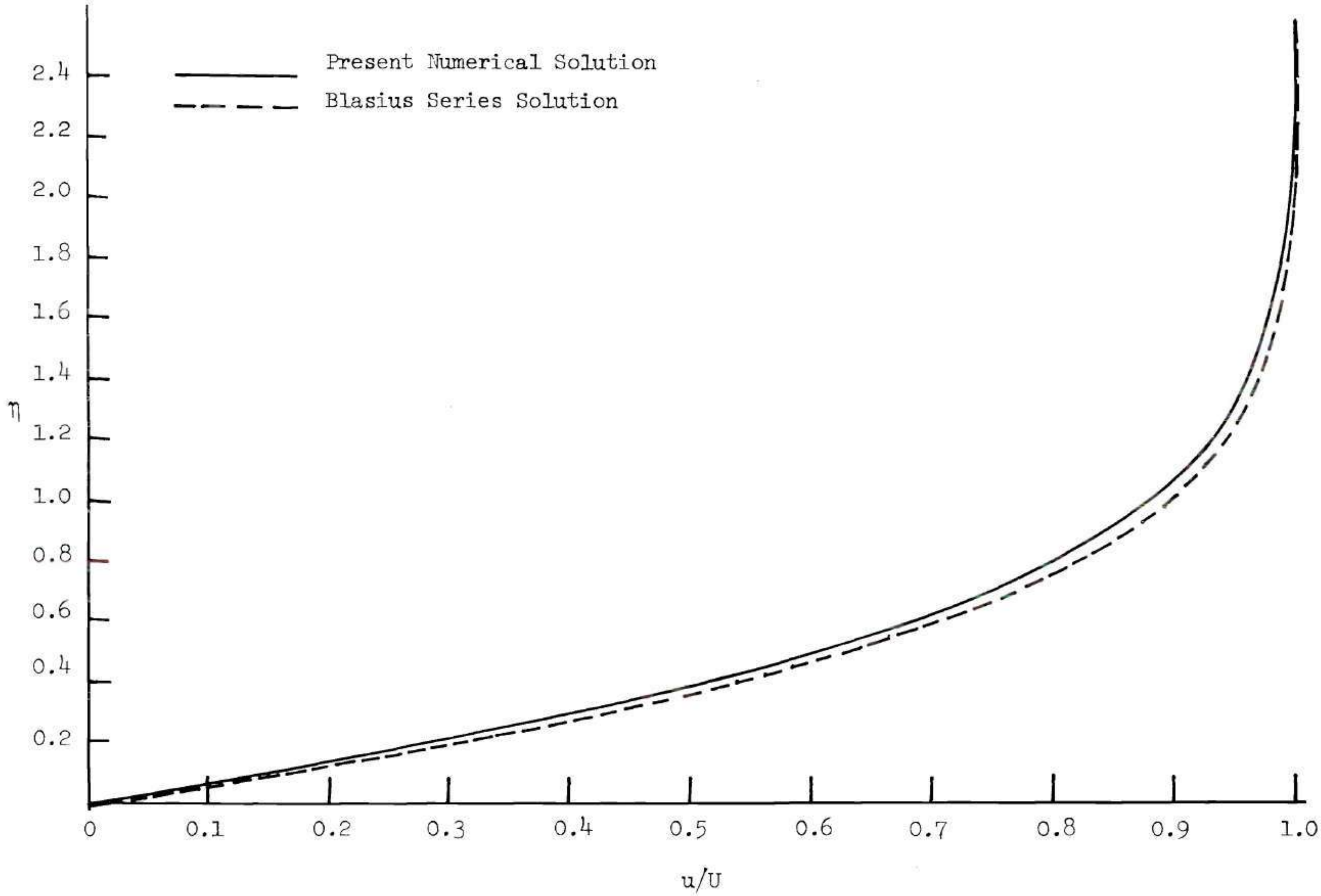

Fig. 3 Velocity Profiles at 1 Degree

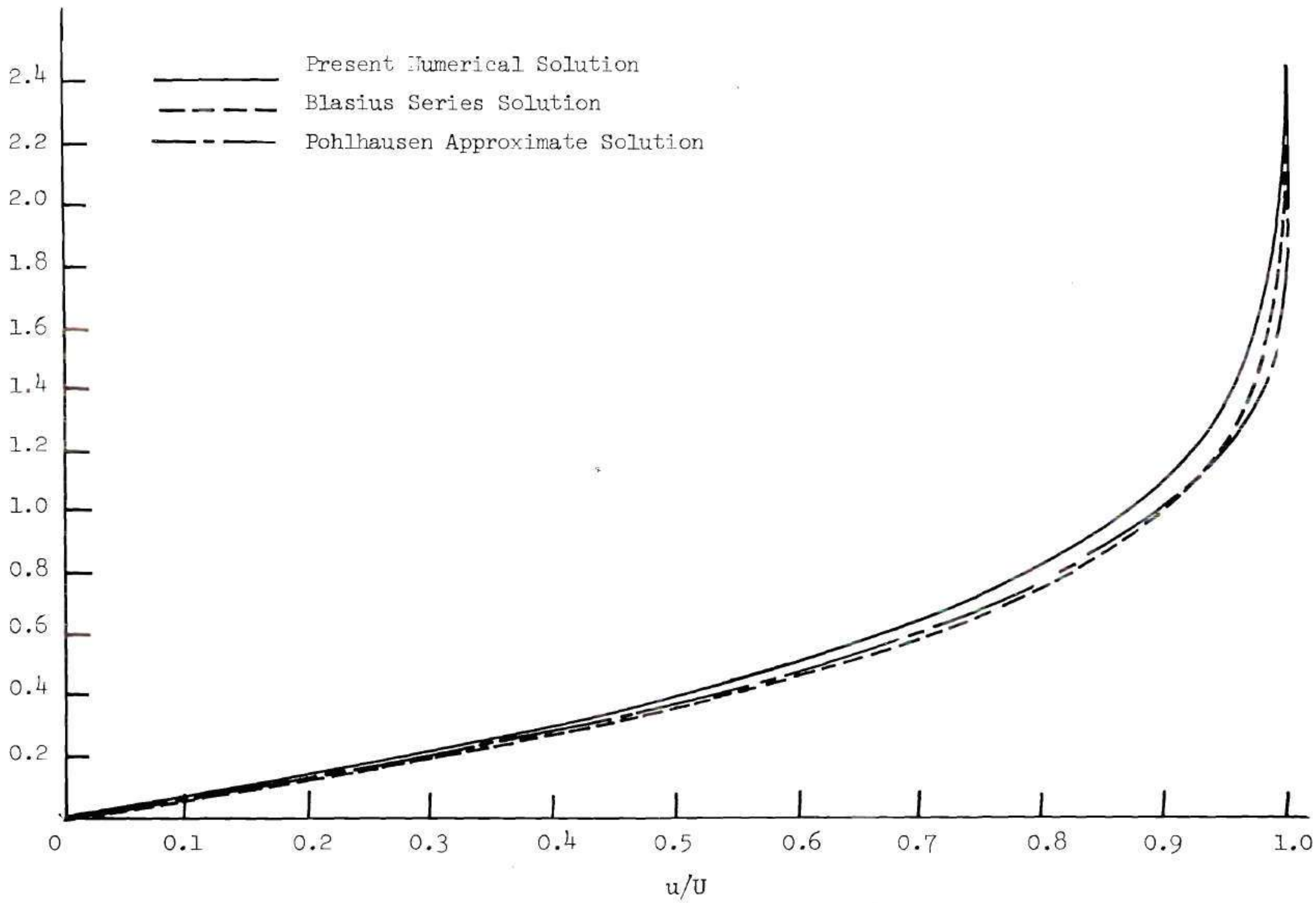

Fig. 4 Velocity Profiles at 5 Degree

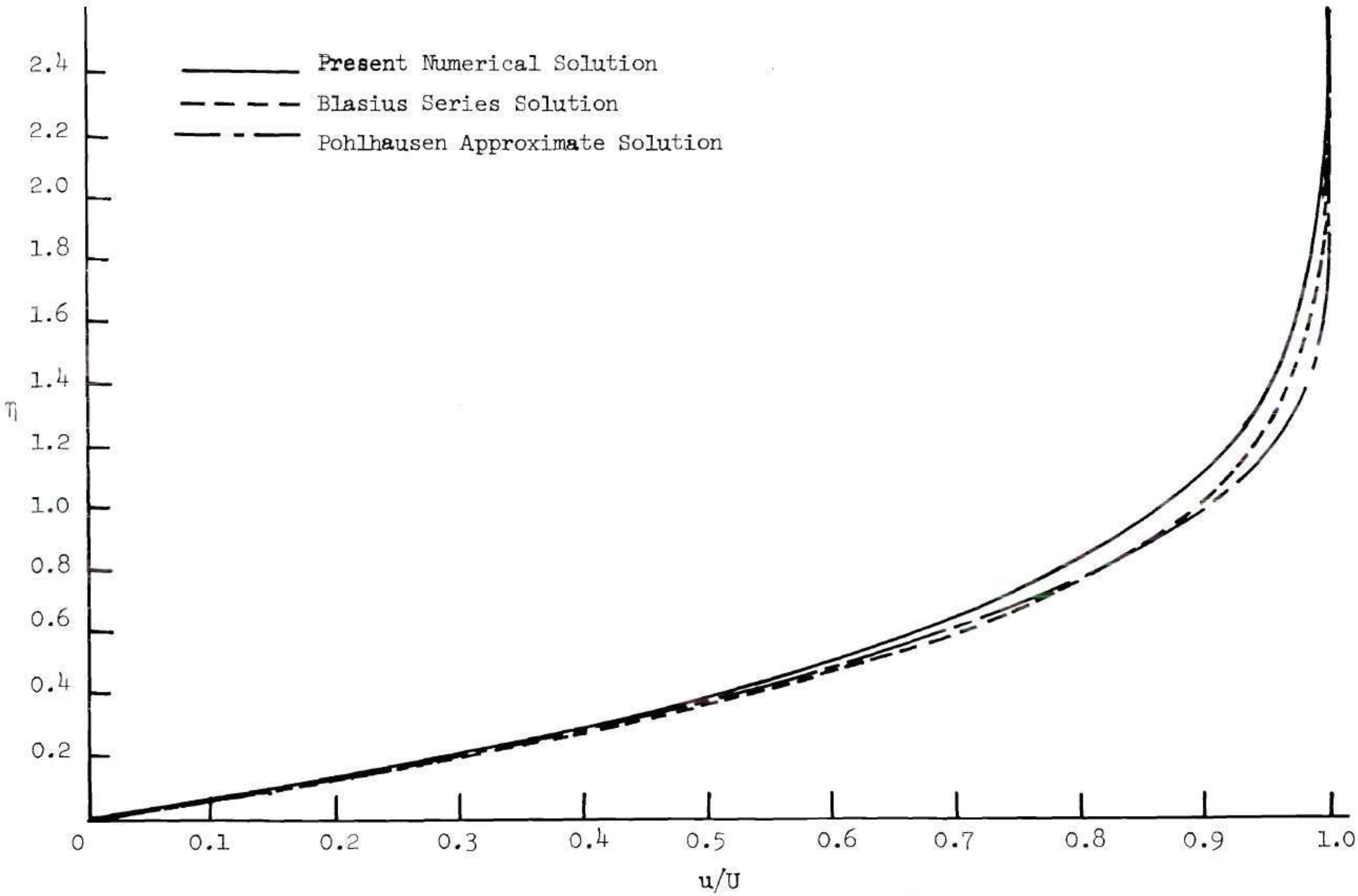

Fig. 5 Velocity Profiles at 15 Degree

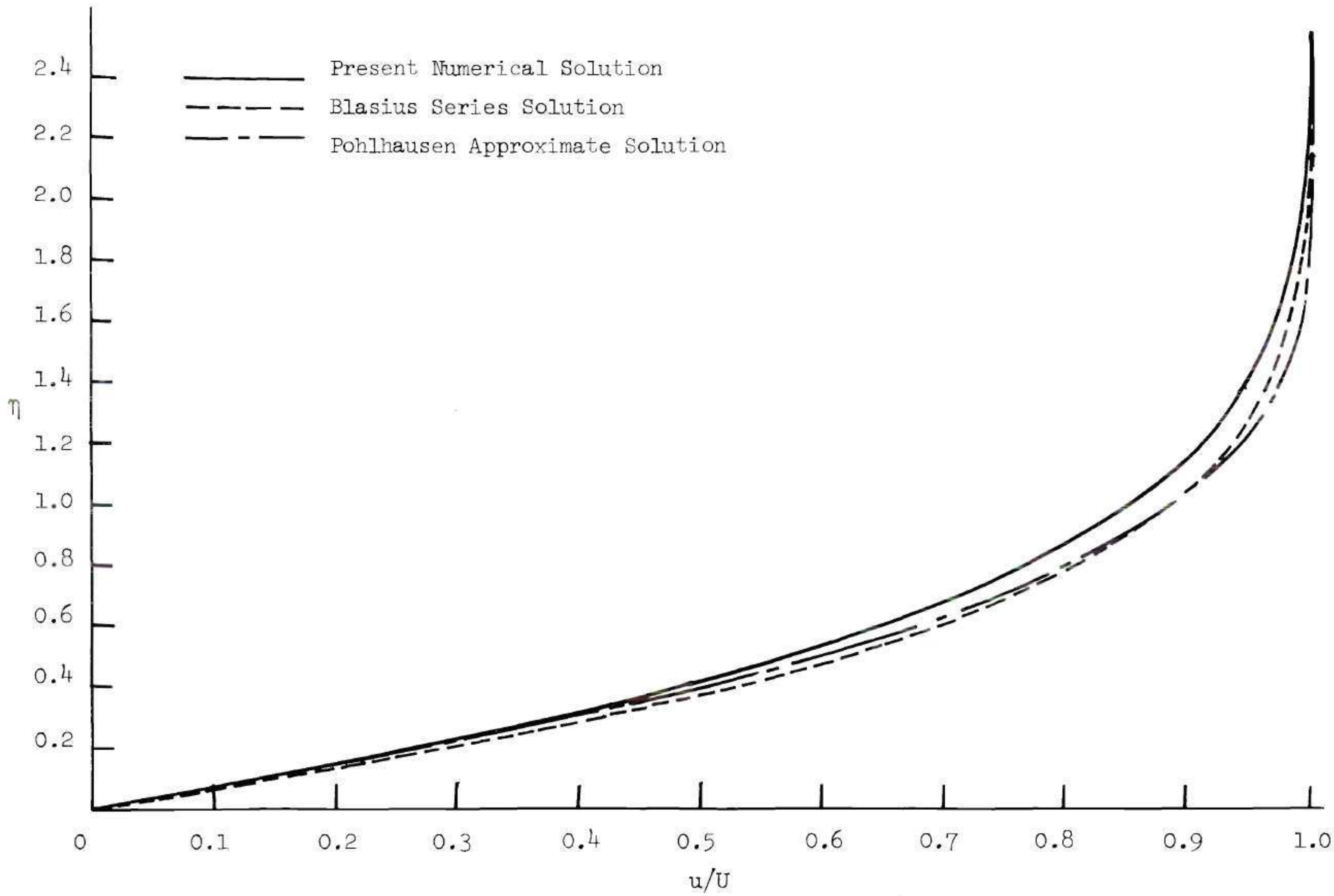

Fig. 6 Velocity Profiles at 30 Degree

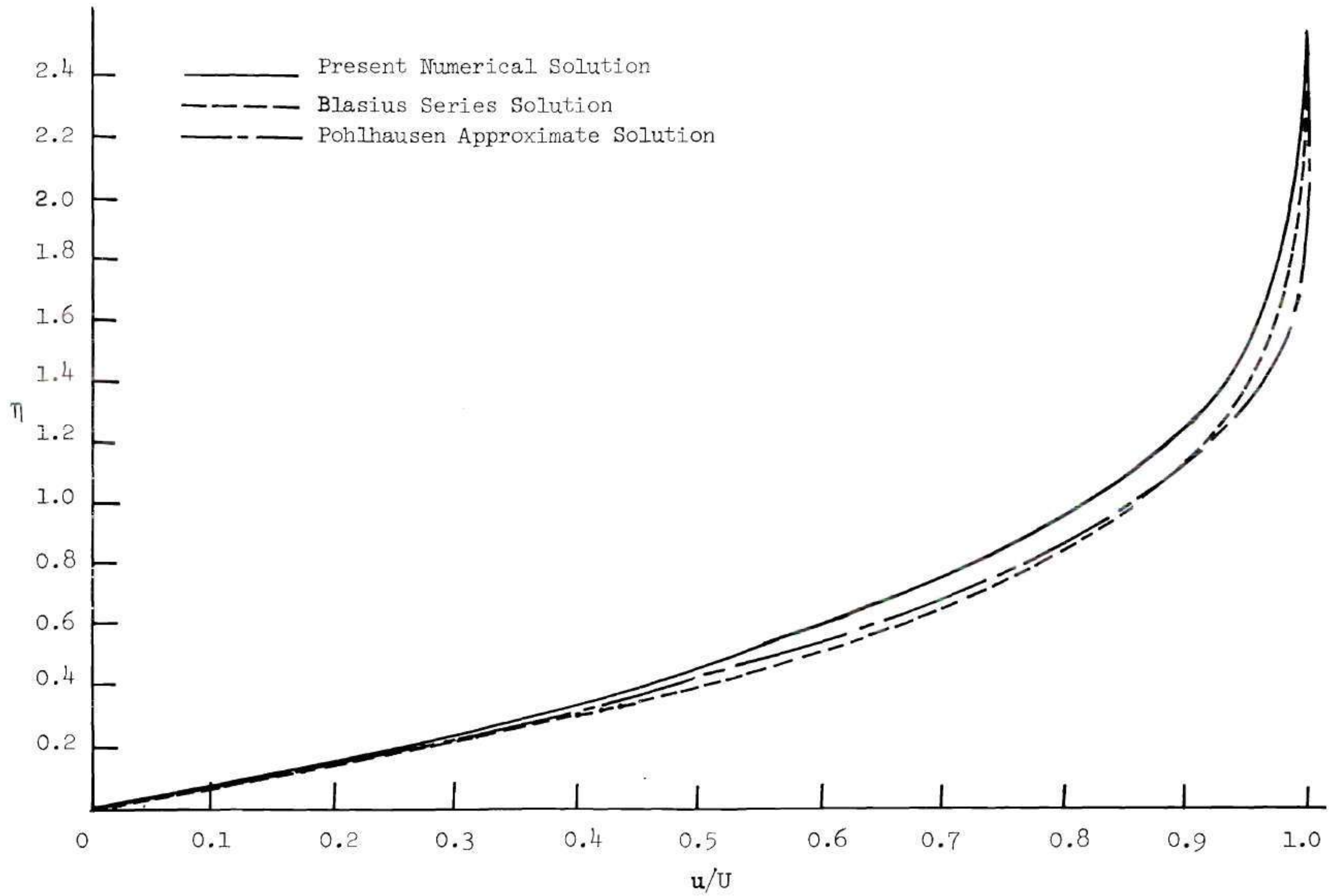

Fig. 7 Velocity Profiles at 45 Degree

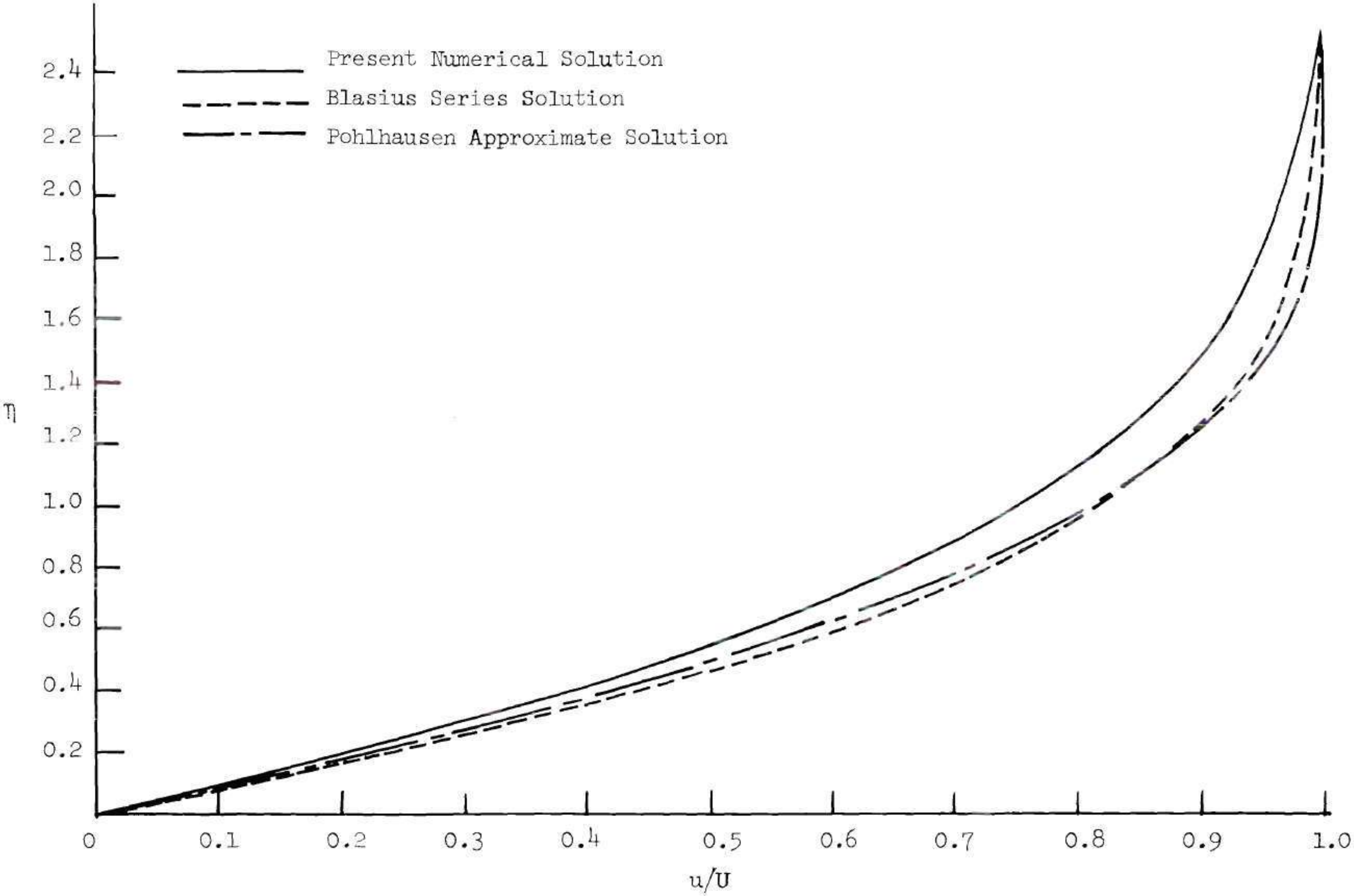

Fig. 8 Velocity Profiles at 60 Degree

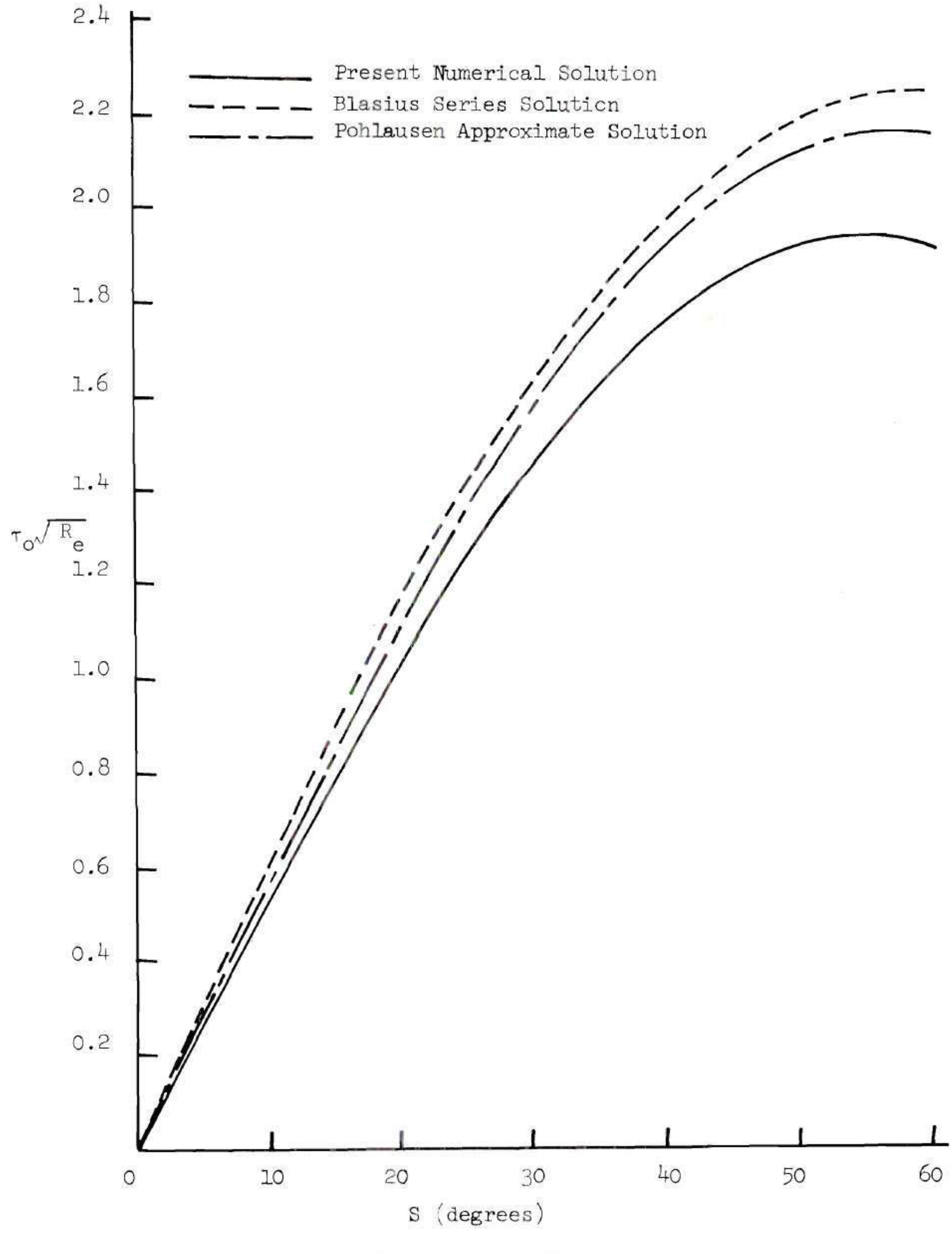

Fig. 9 Skin Friction Comparison

observed when it is compared with other existing results. Generally speaking, three major approximations were carried out in order to make the solution of equation (IV-4) possible. First of all, the partial differentiation term $\frac{\partial f''}{\partial \alpha}$ was dropped. Secondly, f' was used in place of u/U. Thirdly, U, the reference velocity, is the potential velocity at each individual location within the boundary layer instead of the potential velocity at the boundary of the boundary layer which is used in the other results. The effects of these approximations are discussed in more detail below.

Since the potential velocity U does not vary more than 1 percent within the boundary layer, there is not much sacrifice of accuracy if one treats U roughly constant within the boundary. Therefore, it does not matter whether the local potential velocity or the boundary potential velocity is used. Thus, the two sets of results can be compared without much discrepancy.

The velocity profile u/U is not exactly the same as f' or λ. It is found that the actual velocity profile is related as follows [9]

$$\frac{u}{U} \operatorname{Exp}\left[-C_\alpha(\alpha) \int_x^\infty \left(\frac{U}{u} - 1\right)\lambda dx\right] = f' = \lambda \qquad \text{(V-1)}$$

Thus, the ratio of u/U to f' is the exponential term $\operatorname{Exp}\left[C_\alpha(\alpha)\int_x^\infty\left(\frac{U}{u} - 1\right)\lambda dx\right]$. It is desirable that $C_\alpha(\alpha)\int_x^\infty\left(\frac{U}{u} - 1\right)\lambda dx$ be very small, and consequently for the exponential quantity to approach unity. Then f' can be used in place of $\frac{u}{U}$. It is estimated that C_α values range from 0 to .003, while the integral $\int_x^\infty \left(\frac{U}{u} - 1\right)\lambda dx$ is less than one. Therefore, the use of f' in place of u/U will produce at most a 0.3 percent error. If an error of less than 1 percent is considered to be small, then f' can be used in place of u/U as the velocity profile.

It can be seen from equation (III-49) that $h_{\alpha o}$ goes to infinity at the forward stagnation point. The stagnation point, then, becomes a singularity and solutions are not obtained at this point. It is needed, therefore, to consider how far from the stagnation point should one start to seek a solution. For the problem presented here solutions were sought starting from 1° from the stagnation point. The inability to obtain a solution starting from the stagnation point provides one with difficulty to solve equation (IV-4) with the partial differentiation term $\frac{\partial f''}{\partial \alpha}$ included. It can be explained more clearly in the following manner. The term $\frac{\partial f''}{\partial \alpha}$ can be approximated by

$$\left.\frac{\partial f''}{\partial \alpha}\right|_{\alpha_3 x_i} = \frac{3f''(\alpha_3,x_i) - 4f''(\alpha_2,x_i) + f''(\alpha_1,x_i)}{\alpha_3 - \alpha_1} \qquad \text{(V-2)}$$

where $f''(\alpha_3,x_i)$, $f''(\alpha_2,x_i)$ and $f''(\alpha_1,x_i)$ are values of f" along a constant x line at $\alpha = \alpha_1, \alpha_2, \alpha_3$, respectively. In order to evaluate $\left.\frac{\partial f''}{\partial \alpha}\right|_{\alpha_3,x_o}$, it is necessary to know the exact values of $f''(\alpha_3,x_i)$, $f(\alpha_2,x_i)$, and $f''(\alpha_1,x_i)$. Since the solution is not started from the beginning, one can only obtain solutions of the first two steps without the $\frac{\partial f''}{\partial \alpha}$ in there.

Therefore, the values of f'' obtained are not exact. There is no guarantee we can depend on those values to evaluate $\frac{\partial f''}{\partial \alpha}$. If the term $\frac{\partial f''}{\partial \alpha}$ is small compared with other terms for the region prior to 1°, neglecting of the term will not affect the solution significantly. However, from Figure 3, one can see that even at 1° the numerical and the Blasius solutions are well apart, especially in the middle range. In these calculations the term $\frac{\partial f''}{\partial \alpha}$ was neglected from equation (IV-4). One can see from Figures 3 through 8 that the discrepancy becomes larger at larger degrees. The neglect of $\frac{\partial f''}{\partial \alpha}$, indeed, cause some appreciable errors. Although the curves in each of the Figures 3 through 8 are very close together near the surface, there is significant difference in their slopes at $n = 0$. Looking at equation (IV-34), one can realize why the skin frictions differ as they are compared in Table 4 and Figure 9.

With $h_{\alpha o}^{-1}$ versus s, the distance along the surface from the forward stagnation point, a diagram is plotted as shown in Figure 10. It is observed that the curve is quite linear for $s < 26^\circ$. Therefore, for $s < 26^\circ$ we can let

$$h_{\alpha o}^{-1} = C_1 s \qquad \text{(V-3)}$$

where C_1 is a constant. Since

$$ds = h_{\alpha o}\, d\, \alpha \qquad \text{(V-4)}$$

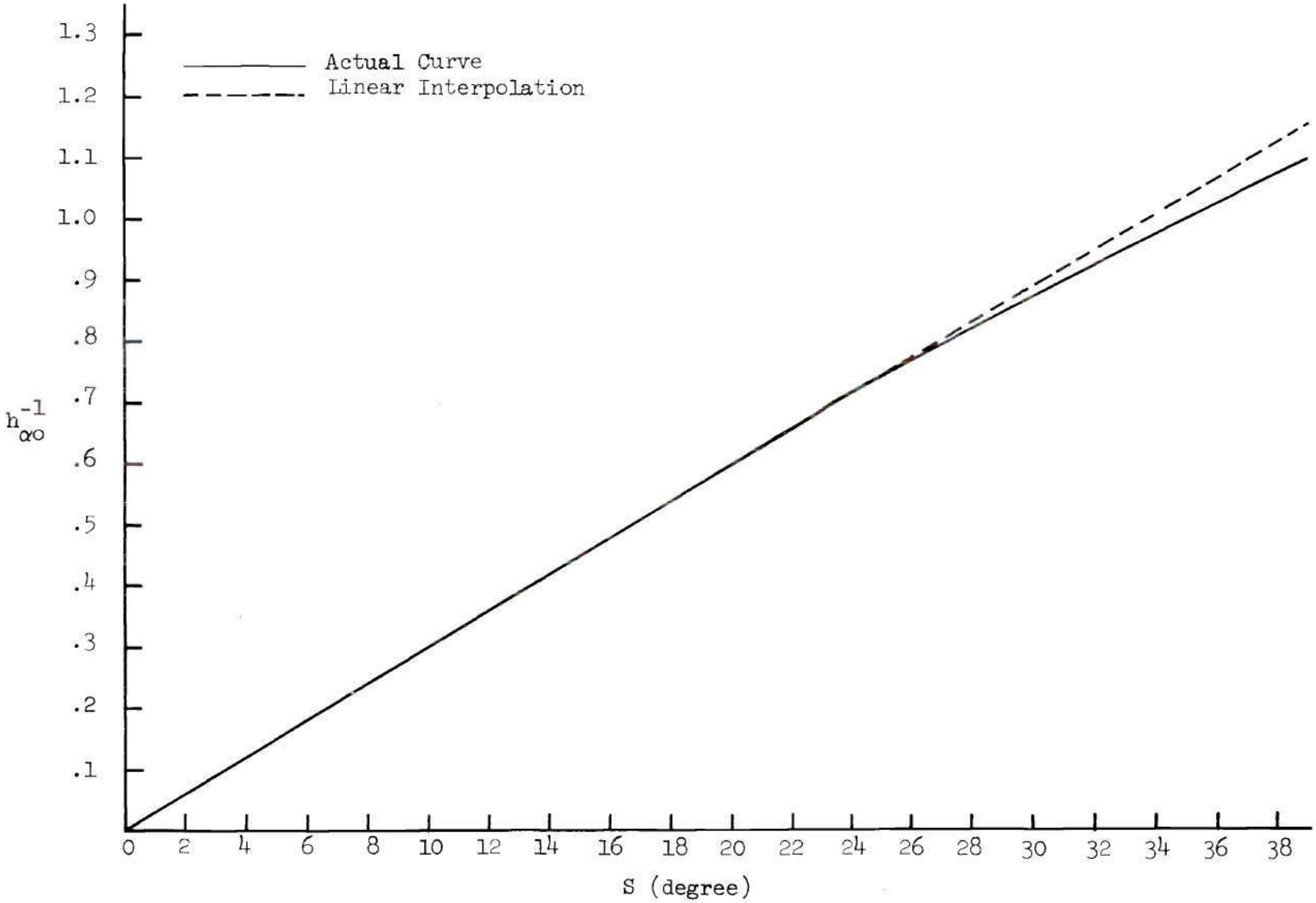

Fig. 10 Simpler Solution Investigation

it can also be written as

$$d\alpha = h_{\alpha_o}^{-1}\, ds \qquad \text{(V-5)}$$

Therefore,

$$\alpha = \int_o^s h_{\alpha o}\,(z)\; dz \qquad \text{(V-6)}$$

Where z is a dummy variable. Substituting relation (V-3) in equation (V-6), one finds

$$\alpha = \int_o^s C_1 z dz = \frac{C_1}{2}\, s^2 \qquad \text{(V-7)}$$

Reconsider equation (IV-4)

$$[2f''' + ff''] - 2C_\alpha\,[2f''' + ff''] = 2_\alpha f' \left[\frac{\partial f''}{\partial \alpha} - 2f'' \frac{d}{d\alpha} \ln h_{\alpha o} \right]$$

If one treats $\frac{\partial f''}{\partial \alpha}$ as small compared to $\partial f'' \frac{d}{d\alpha}$ ln $h_{\alpha o}$, then one can drop the term and the right hand side of the equation becomes

$$[-4\alpha \frac{d}{d\alpha} \ln h_{\alpha o}] \; f'f''$$

To evaluate $[-4\alpha \frac{d}{d\alpha} \ln h_{\alpha o}]$, the relation (V-3) is again used. One finds

$$h_{\alpha o} = \frac{1}{C_1} s^{-1} \qquad \text{(V-8)}$$

then

$$\ln h_{\alpha o} = \ln \frac{1}{C_1} - \ln s \qquad \text{(V-9)}$$

and

$$\frac{d}{d\alpha} \ln h_{\alpha o} = - \frac{1}{s} \frac{ds}{d\alpha} = - \frac{h_{\alpha o}}{s} \qquad \text{(V-10)}$$

Combining equations (V-7) and (V-10), one obtains

$$\alpha \frac{d}{d\alpha} \ln h_{\alpha o} = - \frac{C_1}{2} s \; h_{\alpha o} \qquad \text{(V-11)}$$

Since

$$h_{\alpha o}^{-1} = C_1 s$$

equation (V-11) becomes

$$\alpha \frac{d}{d\alpha} \ln h_{\alpha o} = - \frac{1}{2} \qquad (V-12)$$

Consequently,

$$-4\alpha \frac{d}{d\alpha} \ln h_{\alpha o} = 2 \qquad (V-13)$$

and equation (IV-4) becomes

$$[2f''' + ff''] - 2C_{\alpha} [2f''' + ff''] = 2f'f'' \qquad (V-14)$$

Equation (V-14) is an ordinary differential equation for a specified C_{α}. The boundary conditions in (III-48) are known for C_{α} being specified. Therefore, it can be solved directly by using the Runge-Kutta method.

The known pressure distribution used in this thesis fluctuates violently beyond 60°, and solutions beyond 60° were not sought. Therefore, flow separation was not reached.

Although neglecting the partial differentiation term $\frac{\partial f''}{\partial \alpha}$ gave comparatively less accurate results, there are certain other contributions from this thesis. First of all, a numerical method was generated as outlined in detail for the solution. It gives a good example of how to solve some flow problems by using numerical methods. As it was said in Chapter I, numerical methods are far more accurate and time-saving than the approximate solutions and series solution. Secondly, the equations were derived in generalized orthogonal coordinates corresponding to the flow direction. Therefore, there is only one velocity component. Also, the differential equation for this problem was derived and is available for solving. Further development may be based on this fundamental equation (IV-4). Thirdly, the curvature effect was considered in this thesis. Although, in some of the literature, people also discussed the curvature effect, they only discussed it on the surfaces where a similar solution can be obtained. However, along a cylinder a similar solution is not applicable.

Generally speaking, in this thesis the emphasis was placed on the methods of solution.

CHAPTER VI

CONCLUSIONS AND RECOMMENDATIONS

Due to the singularity property of the forward stagnation point a difficulty developed in obtaining exact solutions for the first two steps. It prevents the starting of the solutions at the stagnation point. Since exact solutions are not available during the first two steps, the numerical differentiation method does not generate a good approximation for the steps thereafter. The negligence of the term $\frac{\partial f''}{\partial \alpha}$ is done with relatively large sacrifice of accuracy. However, the approximation of f' for u/U and the treatment of U as local potential velocity is without noticeable effect on the solution.

A simpler process of soltuion can be applied to obtain profiles near the forward stagnation point. For each specified value of C_α, the fundamental equation (IV-4) can be solved directly using Runge-Kutta method. In other words, for a specified α, since C_α is a function of α, a profile can be obtained easily.

It is recommended that the solution be sought starting from a point very close to the forward stagnation point with an interval of about $.1^\circ$. For convenience, $\frac{\partial f''}{\partial \alpha}$ can be transformed to $\frac{\partial f''}{\partial s}$ as follows:

$$\frac{\partial f''}{\partial \alpha} = \frac{\partial f''}{\partial s} \frac{ds}{d\alpha} \qquad \text{(VI-1)}$$

Since

$$\frac{ds}{d\alpha} = h_{\alpha o} \qquad \text{(VI-2)}$$

then

$$\frac{\partial f''}{\partial \alpha} = h_{\alpha o} \frac{\partial f''}{\partial s} \qquad \text{(VI-3)}$$

Therefore, the numerical differentiation can be taken at an increment Δs instead of $\Delta\alpha$. The smaller the increment is taken, the more accurate the results will be. Because of the limitation of the computer time, it was not done for this thesis.

The pressure distribution used in this thesis was obtained in a situation where the cylinder is placed in a bounded stream. However, if the boundary is far enough from the cylinder surface, there will not be a noticeable difference in the pressure distribution. In this case, the boundary is about two feet away from the cylinder surface and it is sixteen times the cylinder radius. The curve fluctuates irregularly beyond 60°. Therefore, the results become doubtful beyond that point. These results were used because they are the best that could be found in the literature. It seems to the author that a search for a better pressure distribution is highly desirable.

Since the pressure distribution used in this thesis is experimental, readings available are only accurate to the third significant figures. Moreover, most of the pressure readings were obtained from interpolation.

In the region close to 0°, where the values $(K - \Delta P_s)$ in equation (III-49) are small, the accuracy of ΔP_s is deadly required. Therefore, the pressure distribution near the forward stagnation point has to be in a more precise form.

Although there still are things to be improved in order to achieve a more accurate solution, this thesis gives a general idea of how to solve this kind of flow problem. The required equations are derived. The consideration of the curvature effect of a circular cylinder contributes to the research in this field.

APPENDIX A

COORDINATE SYSTEM

The coordinate system used in this thesis is the generalized orthogonal coordinates [10]. Let the elements of length at (x_1, x_2, x_3) in the directions of increasing x_1, x_2, x_3, respectively, be h_1dx_1, h_2dx_2, and h_3dx_3. Let (a_1, a_2, a_3) denote the components of a vector $\underline{a}$ in the directions of increasing x_1, x_2, x_3, respectively. Then

$$\nabla \cdot \underline{a} = \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial x_1}(h_2h_3a_1) + \frac{\partial}{\partial x_2}(h_1h_3a_2) + \frac{\partial}{\partial x_3}(h_1h_2a_3)\right\} \qquad \text{(A-1)}$$

and the components of $\underline{b}$ = curl $\underline{a}$ are given by

$$b_1 = \frac{1}{h_1h_3}\left\{\frac{\partial}{\partial x_2}(h_3a_3) - \frac{\partial}{\partial x_3}(h_2a_2)\right\}, \text{ etc.} \qquad \text{(A-2)}$$

The components of the gradient of a scalar ϕ are

$$\frac{1}{h_1}\frac{\partial\phi}{\partial x_1}, \frac{1}{h_2}\frac{\partial\phi}{\partial x_2}, \frac{1}{h_3}\frac{\partial\phi}{\partial x_3} \qquad \text{(A-3)}$$

In this thesis only a two-dimensional problem is considered. Therefore, h_3 is treated as 1 while h_1 is h_α and h_2 is h_β. Then

$$\nabla \cdot \underline{a} = \frac{1}{h_\alpha h_\beta} \left\{ \frac{\partial}{\partial \alpha} (h_\beta a_1) + \frac{\partial}{\partial \beta} (h_\alpha a_2) \right\} \qquad (A\text{-}4)$$

while

$$b_3 = \frac{1}{h_\alpha h_\beta} \left\{ \frac{\partial}{\partial \alpha} (h_\beta a_2) - \frac{\partial}{\partial \beta} (h_\alpha a_1) \right\} \qquad (A\text{-}5)$$

APPENDIX B

DIMENSIONLESS GROUPS

Every variable in this thesis is non-dimensionalized. The radius of the cylinder, a, is taken to be the characteristic length. The free stream velcoity, U_∞, is taken to be the characteristic velocity. The pressures and stresses are non-dimensionalized by ℓU_∞^2. The Reynolds number is defined as $\frac{\ell U_\infty a}{\mu}$. The vorticity is non-dimensionalized by U_∞/a.

APPENDIX C

SURFACE PRESSURE DISTRIBUTION ALONG A CYLINDER [8]

Degree	P_s	Degree	P_s	Degree	P_s
0	0.49700	21	0.28945	41	-0.18111
1	0.49624	22	0.27028	42	-0.20726
2	0.49456	23	0.25016	43	-0.23334
3	0.49196	24	0.23000	44	-0.25929
4	0.48844	25	0.20900	45	-0.28500
5	0.48400	26	0.18764	46	-0.31036
6	0.47864	27	0.16576	47	-0.33532
7	0.47236	28	0.14336	48	-0.36036
8	0.46516	29	0.12044	49	-0.38404
9	0.45704	30	0.09700	50	-0.40700
10	0.44800	31	0.07268	51	-0.42924
11	0.43800	32	0.04794	52	-0.45076
12	0.42716	33	0.02286	53	-0.47156
13	0.41536	34	-0.00248	54	-0.49164
14	0.40264	35	-0.02800	55	-0.51100
15	0.38900	36	-0.05320	56	-0.52964
16	0.37447	37	-0.07850	57	-0.54756
17	0.35901	38	-0.10390	58	-0.56476
18	0.34255	39	-0.12939	59	-0.58124
19	0.32564	40	-0.15500	60	-0.59700
20	0.30800				

APPENDIX D

NUMERICAL METHODS

1. Newton-Raphson Iteration Method [11]

This method is used to find the proper $g'''(0)$ for a specified $g(0)$ in equation (IV-11) with boundary conditions (IV-12).

If

$$g = \frac{dg}{dz} \qquad \text{(D-1)}$$

where $z = g'''(0)$, and define

$$g' = \frac{dq}{d\zeta} \qquad \text{(D-2)}$$

$$g' = \frac{\partial g}{\partial \zeta} \qquad \text{(D-3)}$$

then

$$\frac{dg'}{dz} = \frac{d}{d\zeta}\frac{dg}{dz} = g' \qquad \text{(D-4)}$$

$$\frac{dg''}{dz} = g'' \tag{D-5}$$

and so forth.

Differentiating equation (IV-11) with respect to z, gives

$$2q'^{V} + g'''q + gq''' - g'q'' - g''q' - 4(C_\alpha/C)\ q''' - 2\frac{C_\alpha}{C}gq''$$

$$- 2(C_\alpha/C)\ g''q = 0 \tag{D-6}$$

with additional boundary conditions

$$\begin{aligned} q(0) &= 0 \\ q'(0) &= 0 \\ q''(0) &= 0 \\ q'''(0) &= 1 \end{aligned} \tag{D-7}$$

First a value for $g'''(0)$ is obtained which may be called z_1. After computation, $g''(0)$ is obtained as ζ goes to infinity and it is compared with 1. If it is not satisfied, another value z_2 for $g'''(0)$ is assumed with the following relationship

$$z_2 = z_1 - \frac{g''(\infty)_1}{q''(\infty)_1} \qquad \text{(D-8)}$$

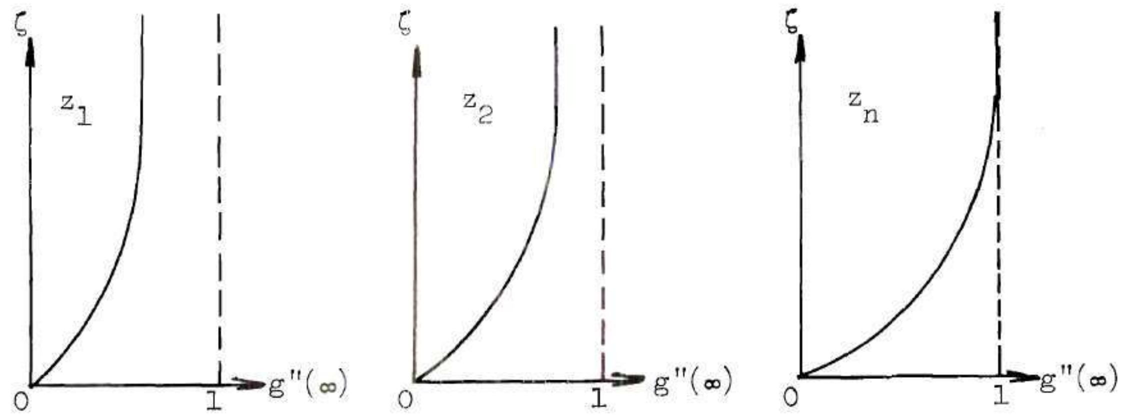

Fig. 11 The Determining of $g'''(0)$

It can be seen from Fig. 11 that for each new assumed value of $g'''(0)$, z_1, z_2, . . . , z_n, $g''(\infty)$ will approach a certain value. Only one value of $g'''(0)$, z_n, which makes $g''(\infty)$ approach 1 will be accepted. In order to btain z_n in a more rapid way, the Newton-Raphson Iteration Method is applied. Figure 12 represents a plot of $g''(\infty)$ versus z. It can be seen that if z_{m-1} is selected for $g'''(0)$, $[g''(\infty)]_{m-1}$ is obtained, which is different from 1. By drawing a tangent to the curve at the point where $z = z_{m-1}$, it intersects with the line $g''(\infty) = 1$ at the point $(z_m, 1)$. A line drawn parallel to the $g''(\infty)$-axis through the point $(z_m, 1)$ will intersect the curve at the point $(z_m, [g''(\infty)]_m)$.

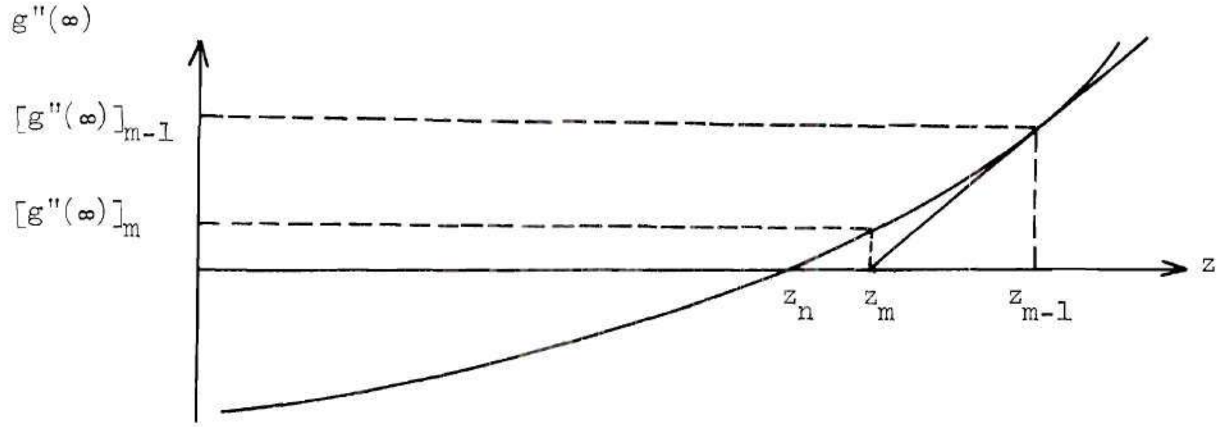

Fig. 12 Newton-Raphson Iteration Method

The value of z_m can always be calculated by

$$z_m = z_{m-1} - [g''(\infty)]_{m-1} / [q''(\infty)]_{m-1} \qquad \text{(D-9)}$$

By following the same procedure at this point, another value of $g'''(0)$ will be found closer to z_n. The iteration converges to z_n very radidly. In the computer program used in this thesis, at most six trials were needed to get z_n where $g''(\infty)$ would be within .0000005 from 1.

2. Numerical Integration [11]

In this thesis, quadratic numerical integration was used instead of linear numerical integration. It is considered that the quadratic form yields enough accuracy. In figure 13, $f(x)$, a function of x

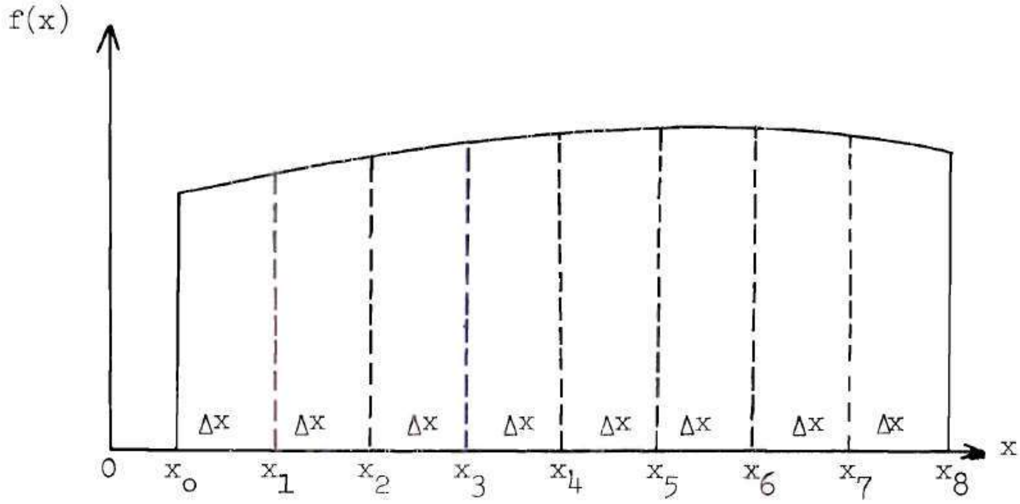

Fig. 13 Numerical Integration

is given. It is desired to integrate f(x) between the limits x_o and x_8

$$\int_{x_o}^{x_8} f(x)\ dx \qquad \text{(D-10)}$$

The interval is divided into eight subintervals of equal size each with a width of Δx. For a quadratic form the area under f(x) between $x = x_n$ and $x_n + 2$ are given by the relation

$$\int_{x_n}^{x_{n+2}} f(x)\ dx \cong \frac{1}{3}\,\Delta x\ [f(x_{n+2}) + 4f(x_{n+1}) + f(x_n)] \qquad \text{(D-11)}$$

Therefore, the integral

$$\int_{x_o}^{x_8} f(x)\ dx = \frac{1}{3}\ \Delta x \sum_{n=0,2,4,6,8}$$

$$[f(x_{n+2}) + 4f(x_{n+1}) + f(x_n)] \qquad \text{(D-12)}$$

The accuracy of the numerical integration depends largely on the inerement size Δx. In the present work $\Delta x = 0.1$ was used for each step. The same step size was lso used for the integration. The method was used to integrate a polynomial and the numerical result came out pretty much the same as the result obtained by applying the integration formula.

3. Runge-Kutta Method [12]

This is a subroutine stored in the Univac 1108 Math-pack. It computes the numerical solutions for a system of n first order differential equations using a modified Runge-Kutta method. The entry is

```
CALL RKDE (DY,Y,Z,H,W,Q,N)
```

where DY is the name of a function subprogram used to evaluate

$$\frac{dy_i}{dx} = f_i\Big(x,\ y_1(x),\ y_2(x),\ .\ .\ .\ ,\ y_n(x)\Big) \qquad \text{(D-13)}$$

Y is a one-dimensional array containing the initial conditions $y_i(x_o) = y_i$.

Y(1) is the initial value of abscissa (x_o). Therefore, for n equations there are n + 1 elements in the Y array. The dimension of array Y is N.

Z is the value of the abscissa at which the solutions are sought.

H is the step size used to increment x_o.

W is a one-dimensional array of N elements used as temporary storage for the K parameters.

Q is a one-dimensional array of N elements used as temporary storage for the q parameters.

N is one plus the number of equations to be solved (i.e., if there are n equations to be solved, N = n + 1). N is the dimension of arrays Y, W and Q.

For a nth order ordinary differential equation it can be divided into n simultaneous first order differential equations with n variables. Take the differential equation to be the form of

$$y^{(n)}(x) = a_{n-1}y^{(n-1)}(x) + a_{n-2}y^{(n-2)}(x) + \ .\ .\ .\ + a_o y(x) \qquad \text{(D-14)}$$

If we let

$$y(x) = y_1(x)$$

$$y'(x) = y_2(x)$$

$$\cdot$$

$$\cdot$$

$$y^{(n)}(x) = y_{n+1}(x) \qquad \text{(D-15)}$$

then we have n simultaneous first order differential equations with n variables,

$$y_1'(x) = y_2(x)$$

$$y_2'(x) = y_3(x)$$

$$\cdot$$

$$\cdot$$

$$y_{n-1}'(x) = y_n(x)$$

$$y_n'(x) = a_{n-1}y_n(x) + a_{n-2}y_{n-1}(x) + \ . \ . \ . + a_o y_1(x) \qquad \text{(D-16)}$$

Therefore, it is ready to be solved by using this method directly if all the initial conditions $y_i(x_o)$ are known.

APPENDIX E

COMPUTER PROGRAM

```
@ RUN PRØGRM, 53C12015, HSU-H-W, 2, 100/500
@ FØR IS PRØGRM
      DIMENSIØN Y(9), W(9), Q(9), FØ(9), F(5), FF(5)
      CØMMØN S, B
      REAL INTGR
      EXTERNAL FN, RKDE
      M = 100
      N = 1
   38 H = .10
      MG = 2
      READ (5,50) S
   31 READ (5,601) S
   68 Y(2) = - 4.0 * S
      Y(1) = 0.0
      Y(3) = 0.0
      Y(4) = 1.0
      READ (5,601) Y(5)
      Y(6) = 0.0
      Y(7) = 0.0
      Y(8) = 0.0
      Y(9) = 1.0
```

```
      DØ 1000 I = 1,9
 1000 FØ(I) = Y(I)
      READ (5,601) ALPHA
      READ (5,601) HALØ
      B = 2. * EXP(-.50 * ALPHA)
   78 K = 1
   15 Z = 0
      Z = Z + H
      GØ TØ (701,702), MG
  701 WRITE (6,200) FØ(2), FØ(5), S, HALØ, ALPHA
      WRITE (6,251)
  702 J = 1
   20 CALL RKDE (FN, Y, Z, H, W, Q, 9)
      IF (ABS(Y(3) .GT. 100000.) GØ TØ 8
      GØ TØ (703,704), MG
  703 WRITE (6,300) (Y(I), I = 1,5)
  704 Z = Z + H
      J = J + 1
   30 IF (J-M) 20, 53, 53
   53 A = Y(4)
      IF (ABS(A) .LT. 0.0000005) GØ TØ 5
      FØ(5) = FØ(5) - A/Y(8)
      IF (K .GT. 6) GØ TØ 6
      DØ 35 I = 1,9
   35 Y(I) = FØ(I)
      K = K + 1
```

```
      GØ TØ 15
    5 RE = (Y(3) * ALPHA * HALØ ** 2)/S ** 2
      GØ TØ (705,706), MG
  705 WRITE (6,141) RE
  706 DRE = RE - 53000.
      IF (ABS(DRE) .LT. 5) GØ TØ 7
      S = HALØ * SQRT (ALPHA * Y(3) (53000.0)
      FØ(2) = - 4.0 * S
      DØ 5000 I = 1,9
 5000 Y(I) = FØ(I)
      GØ TØ 78
    6 WRITE (6,500)
    7 EN = HALØ/SQRT (RE)
      DELTAP = .497049 - PS
      C = 1.0/SQRT (Y(3))
      T = 2.0 * SQRT (ALPHA) * EN/C
      DØ 2000 I = 1,9
 2000 Y(I) = FØ(I)
  707 WRITE (6,200) FØ(2), FØ(5), S, HALØ, ALPHA
      WRITE (6,251)
  708 J = 1
      Z = H
      L = 2
      SUM = .03333333
   25 CALL RKDE (FN, Y, Z, H, W, Q, 9)
      INTGR = EXP(-T * Y(1)) * C ** 4 * Y(3) * Y(4)
```

```
      GØ TØ (55,60), L
   55 SUM = SUM + INTGR * H/3.0
  709 WRITE (6,350) (Y(I), I = 1,5), SUM
  710 SUM = SUM + INTGR * H/3.0
      L = 2
      GØ TØ 70
   60 SUM = SUM + 4.0 * INTGR * H/3.0
  711 WRITE (6,300) (Y(I), I = 1,5)
  712 L = 1
   70 Z = Z + H
      J = J + 1
      IF (J-M) 25,25,51
   51 HALØ 2 = SQRT (SUM/DELTAP)
      DHA = HALØ - HALØ 2
  713 WRITE (6,260) HALØ, HALØ 2, DHA
  715 DØ 3000 I = 1,9
 3000 Y(I) = FØ(I)
      H = .10 * C
      FF(2) = C * FØ(2)
      FF(4) = C ** 3
      FF(5) = C ** 4 * FØ(5)
      WRITE (6,130) FF(2), FF(4), FF(5), RE, ALPHA
   16 Z = 0
      Z = Z + H
      WRITE (6,250)
      J = 1
```

```
 17 CALL RKDE (FN, Y, Z, H, W, Q, 9)
    F(1) = Y(1)/C
    F(3) = Y(3) * C ** 2
    F(4) = Y(4) * C ** 3
    F(5) = Y(5) * C ** 4
    DELTA = (EXP (SQRT (ALPHA) * EN * F(1)) - 1.0) * SQRT (RE)
    WRITE (6,150) F(1), F(3), F(4), DELTA, F(5)
    Z = .10 * (J+2) * C
    J = J + 1
    IF (J-M) 14, 4, 4
  4 IF (N-70) 8, 18, 18
  8 N = N + 1
    GØ TØ 38
 18 STØP

141 FØRMAT (//, 10X, 16H REYNØLDS NUMBER = , F13.4)
130 FØRMAT (1H1, 40X, 58H SØLUTIØN ØF D (2*D3F + F*D2F) -2* CALPHA *
     (2*D3F + F*D2
  1 F) = B*DF*D2F,/,41X,58(1H-),//,9X,5HF(0)=,F12.8,3X,7HDF(0)=0,3X,
     7HD2
  2 F(0)=,F12.8,3X,7HD3F(0)=,F12.8,3X,3HRE=,F11.3,3X,6HALPHA=,F12.8)
250 FØRMAT (25X,1HX,18X,2HDF,18X,3HD2F,15X,5HDELTA,16X,3HD3F)
150 FØRMAT (10X,5F20.8)
 50 FØRMAT (F12.5)
601 FØRMAT (F13.8)
200 FØRMAT (1H1,40X,53HSØLUTIØN ØF D(2*D3Y+Y*D2Y)-2*S*(2*D3Y+Y*D2Y)-B*
```

```
      1 DY*D2Y,/,41X,53(1H-),//,9X,5HY(0)=,F13.8,3X,7HDY(0)=0,3X,8HD2Y(0)=
      2 1,3X,7HD3Y(0)=,F13.8,3X,2HS=,F13.8,2X,5HHALØ=,F13.8,2X,6HALPHA=,F1
      3 3.8)
  251 FØRMAT (21X,1HX,17X,1HY,16X,2HDY,16X,3HD2Y,15X,3HD3Y,15X,3HSUM)
  260 FØRMAT (//,10X,4HHALØ,11X,5HHALØ2,10X,3HDGA,/,3X,3F15.8)
  300 FØRMAT (10X,5F18.8)
  350 FØRMAT (10X,6F18.8)
  500 FØRMAT (10X,24X,24HNØ CØNVERGENCE UNTIL K = 6)
      END

@ FØR, IS SUBI
      SUBRØUTINE RKDE (DY, Y, Z, H, W, Q, N)
      DIMENSIØN Y(N), W(N), Q(N), A(4), C(4), B(4)
      DATA (A(I), C(I), B(I), I = 1,4) /2*.5,2.,2*.292893283,1., 2*
        1.70710671,
     1 1.,.166666666,.5,2./
C           ----------------------
C           DX- IS THE INTERVAL SIZE.
C           W- IS THE ARRAY USED TØ STØRE THE
C           VALUE ØF YPRIME (X). W(1) = FØ(X) = 1
C           ----------------------
      DX = H
      W(1) = 1.
C           ----------------------
C           FØR THE FIRST INTERVAL THE Q'S ARE SET TØ ZERØ
C           FØR SUBSEQUENT INTERVALS THE PREVIØISLY CØMPUTED
```

```
C          Q'S ARE USED
C          ----------------------
      DØ 5 J = 1,N
    5 Q(J) = 0.
   10 DØ 20 J = 1,4
      DØ 15 K = 2,N
   15 W(K) = DY(Y,K-1)
      DØ 20 K = 1,N
      Y(K) = Y(K) + DX*A(J)*(W(K)-B(J)*Q(K))
   20 Q(K) = Q(K)+3.*A(J)*(W(K)-B(J)*Q(K))-C(J)*W(J)
C          ----------------------
C          TEST IF VALUE ØF INEDPENDENT VARIABLE
C          HAS BEEN REACHED.
C          ----------------------
   25 RETURN
      END

@ FØR, IS FN
      REAL FUNCTIØN Y(1)
      CØMMØN S,B
      GØ TØ (35,40,45,50,55,60,65,70), I
   35 FN = Y(3)
      GØ TØ 100
   40 FN = Y(4)
      GØ TØ 100
   45 FN = Y(5)
```

```
      GØ TØ 100
   50 FN = (4.0*S*Y(5)+2.0*S*Y(2)*Y(4)+(B-1.0)*Y(3)*Y(4)-Y(2)*Y(5))/2.0
      GØ TØ 100
   55 FN = Y(7)
      GØ TØ 100
   60 FN = Y(8)
      GØ TØ 100
   65 FN= Y(9)
      GØ TØ 100
   70 FN = ((B-1.0)*Y(4)*Y(7)+(B-1.0)*Y(3)*Y(8)+4.0*S*Y(9)+2.0*S*Y(4)*Y(6)
    1 +2.*S*Y(2)*Y(8)-Y(5)*Y(6)-Y(2)*Y(9))/2.0
  100 RETURN
      END
```

APPENDIX F

GENERATION OF OTHER RESULTS

1. Blasius Series Solution [1]

The ideal velocity distribution in a potential irrotational flow past a circular cylinder of radius R and free-stream velcoity U_∞ parallel to the x-axis is given by

$$U(x) = 2U_\infty \sin\left(\frac{x}{R}\right) = 2U_\infty \sin\phi \qquad \text{(F-1)}$$

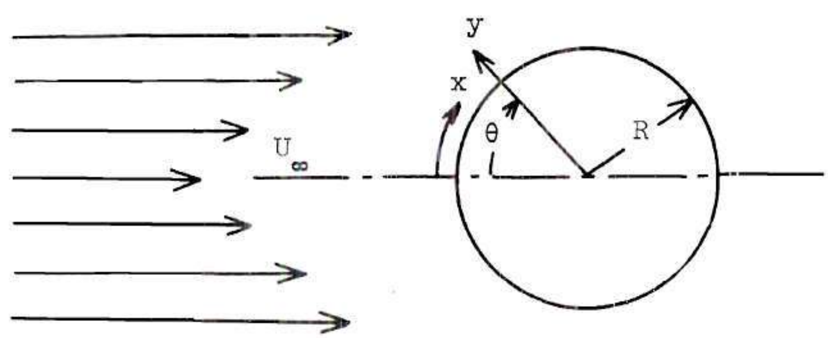

Fig. 14 Potential Flow Over a Circular Cylinder

The power expansion is obtained by expanding $\sin\left(\frac{x}{R}\right)$, so that

$$U(x) = 2U_\infty \left[\frac{x}{R} - \frac{1}{3!}\left(\frac{x}{R}\right)^3 + \frac{1}{5!}\left(\frac{x}{R}\right)^5 - \frac{1}{7!}\left(\frac{x}{R}\right)^7 + \ldots\right] \qquad \text{(F-2)}$$

The true velocity distribution is given by

$$\frac{u}{U_\infty} = 2\left[\left(\frac{x}{R}\right)f_1' - \frac{4}{3!}\,f_3'\left(\frac{x}{R}\right)^3 + \frac{6}{5!}\left(\frac{5}{R}\right)^5 f_5' - \frac{8}{7!}\left(\frac{x}{R}\right)^7 f_7' + \ldots\right] \quad \text{(F-3)}$$

where the values for f_1', f_3', g_5', h_5', etc. are tabulated with respect to η where η are given by

$$\eta = \sqrt{2}\left(\frac{y}{R}\right)\sqrt{R_e} \quad \text{(F-4)}$$

The values for f_5', f_7', etc. are to be evaluated by using the tabulated values.

The rest of f_i's are given by

$$f_5' = g_5' + \frac{10}{3}\,h_5'$$

$$f_7' = g_7' + 7\,h_7' + \frac{70}{3}\,k_7'$$

$$f_g' = g_q' + 12\,h_q' + \frac{126}{5}\,k_q' + 84\,j_q' + 280\,q_q' \quad \text{(F-5)}$$

$$f_{11}' = g_{11}' + \frac{55}{3}\,h_{11}' + 66\,k_{11}' + 220\,j_{11}' + 462\,q_{11}' \quad \text{(continued)}$$

$$+ 1540\, m'_{11} + \frac{15400}{3}\, \eta'_{11} \tag{F-6}$$

Therefore, the potential flow velocity is a function of x alone, while the true velocity is a function of both x and y. The velocity profile for a particular value of x can be determined.

Since we are going to plot u/U versus $\frac{y}{R}\sqrt{R_e}$, the values of $\eta/\sqrt{2}$ instead of η are used. This must be done in order that the results can be compared.

The shear stress, according to its definition

$$\tau_o = - \mu \frac{\partial u}{\partial y} \tag{F-7}$$

can be found to be

$$\frac{\tau_o}{\frac{1}{2}\rho U_\infty^2}\sqrt{\frac{U_\infty R}{2}} = 6.973\left(\frac{x}{R}\right) - 2.732\left(\frac{x}{R}\right)^3 + 0.292\left(\frac{x}{R}\right)^5 \tag{F-8}$$

$$- 0.0183\left(\frac{x}{R}\right)^7 + 0.000043\left(\frac{x}{R}\right)^9 - 0.000115\left(\frac{x}{R}\right)^{11} + \ . \ . \ . \tag{F-9}$$

where every variable is in dimensional form

It is required to plot $\frac{\tau_o}{\rho U_\infty^2}\sqrt{\frac{U_\infty R}{2}}$ versus $\left(\frac{x}{R}\right)$.

2. Karman, Pohlhausen Approximation Method [1]

This method is applied to the general problem of a two-dimensional boundary-layer with pressure gradient. The method in its original form was first indicated by K. Pohlhausen. The succeeding description of the method is based on its more modern form as developed by H. Holstein and T. Bohlen.

It is given that

$$U\,\delta_2^2\,R_e = \frac{0.470}{U^{5.1}}\int_o^x U^5 dx \qquad (F\text{-}10)$$

In the case of a flow over a cylinder, the potential flow velocity is

$$U = 2\sin x \qquad (F\text{-}11)$$

We obtain

$$\delta_2^2\,R_e = -\frac{0.047\cos x}{\sin^2 x} + \frac{.376}{3}\sin^{-6} x - \frac{.188\cos x}{\sin^6 x} \quad \text{(continued)}$$

$$+ \frac{.188 \cos^3 x}{\sin^6 x} \tag{F-12}$$

Since the second shape factor

$$K = R_e \, \delta_2^2 \frac{dU}{dx} \tag{F-13}$$

we can evaluate K for a given value of x.

From the following equation

$$K = \left(\frac{37}{315} - \frac{1}{945} \Lambda - \frac{1}{9072} \Lambda^2\right)^2 \Lambda \tag{F-14}$$

the corresponding value for Λ, the shape factor, can be found.

It is given that the velocity profile

$$\frac{u}{U} = (2\eta - 2\eta^3 + \eta^4) + \frac{\Lambda}{6} (\eta = 3\eta^2 + 3\eta^3 - \eta^4) \tag{F-15}$$

Table 5 Functional Coefficients for the Blasius Series

n	f_1'	f_3'	g_5'	h_5'	g_1'	h_1'	k_1'	g_g'	h_g'	k_g'
0	0	0	0	0	0	0	0	0	0	0
0.4	.4145	.2129	.1778	.0117	.1563	.0030	.0044	.1413	-.0079	-.0112
0.8	.0659	.2997	.2366	-.0177	.1994	-.0637	.0174	.0740	-.0760	-.0501
1.2	.8467	.3133	.2341	-.0442	.1896	-.1102	.0369	.1604	-.1157	-.0704
1.6	.9223	.2975	.2123	-.0504	.1665	-.1114	.0506	.1375	-.1101	-.0649
2.0	.9732	.2775	.1916	-.0406	.1469	-.0839	.0510	.1195	-.0798	-.0460
2.4	.9905	.2632	.1781	-.0257	.1349	-.0507	.0402	.1087	-.0470	-.0267
2.8	.9970	.2554	.1712	-.0133	.1288	-.0254	.0257	.1033	-.0231	-.0129
3.2	.9992	.2519	.1682	-.0057	.1263	-.0107	.0135	.1011	-.0096	-.0053
3.6	.9998	.2506	.1671	-.0021	.1254	-.0038	.0059	.1003	-.0034	-.0019
4.0	1.0000	.2501	.1668	-.0006	.1251	-.0011	.0021	.1001	-.0010	-.0006

Table 5 (Continued)

n	jq	q'_q	g'_{11}	h'_{11}	k'_{11}	j'_{11}	q'_{11}	m'_{11}	n'_{11}
0	0		0	0	0	0	0	0	0
0.4	.0288	-.0124	.1299	-.0145	-.0368	.0371	.0514	-.0721	.0206
0.8	.0833	-.0262	.1553	-.0816	-.1181	.0992	.1267	-.1489	.0406
1.2	.1480	-.0423	.1397	-.1152	-.1520	.1641	.1995	-.2288	.0595
1.6	.1829	-.0567	.1175	-.1055	-.1330	.1919	.2266	-.2874	.0747
2.0	.1718	-.0625	.1008	-.0746	-.0912	.1731	.2001	-.2982	.0822
2.4	.1290	-.0568	.0910	-.0432	-.0516	.1263	.1436	-.2575	.0790
2.8	.0795	-.0426	.0863	-.0210	-.0246	.0761	.0854	-.1858	.0655
3.2	.0406	-.0265	.0843	-.0087	-.0100	.0383	.0425	-.1121	.0463
3.6	.0173	-.0137	.0836	-.0030	-.0035	.0162	.0178	-.0565	.0275
4.0	.0062	-.0059	.0834	-.0009	-.0010	.0057	.0063	-.0238	.0137

where $\eta = y/\delta$.

It is found that the boundary layer thickness $\delta = \Lambda/2 \sqrt{R_e} \cos x$ and it can be calculated directly for a given x.

In order for the results to be able to be compared, it is necessary to use another variable

$$\eta^* = \frac{\eta \sqrt{R_e}}{\delta} \tag{F-16}$$

The shear stress at wall is given

$$\frac{\tau_o \, \delta \sqrt{R_e}}{U} = 2 + \frac{\Lambda}{6} \tag{F-17}$$

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