# APPLICATIONS OF LAPLACE TRANSFORM FOR EVALUATING OCCUPATION TIME OPTIONS AND OTHER DERIVATIVES

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# Abstract


The present thesis provides an analysis of possible applications of the Laplace Transform (LT) technique to several pricing problems. In Finance this technique has received very little attention and for this reason, in the first chapter we illustrate with several examples why the use of the LT can considerably simplify the pricing problem. Observed that the analytical inversion is very often difficult or requires the computation of very complicated expressions, we illustrate also how the numerical inversion is remarkably easy to understand and perform and can be done with high accuracy and at very low computational cost.

In the second and third chapter we investigate the problem of pricing corridor derivatives, i.e. exotic contracts for which the payoff at maturity depends on the time of permanence of an index inside a band (corridor) or below a given level (hurdle). The index is usually an exchange or interest rate. This kind of bonds has evidenced a good popularity in recent years as alternative instruments to common bonds for short term investment and as opportunity for investors believing in stable markets (corridor bonds) or in non appreciating markets (hurdle bonds). In the second chapter, assuming a Geometric Brownian dynamics for the underlying asset and solving the relevant Feynman-Kac equation, we obtain an expression for the Laplace transform of the characteristic function of the occupation time. We then show how to use a multidimensional numerical inversion for obtaining the density function. In the third chapter, we investigate the effect of discrete monitoring on the price of corridor derivatives and, as already observed in the literature for barrier options and for lookback options, we observe substantial differences between discrete and continuous monitoring. The pricing problem with discrete monitoring is based on an appropriate numerical scheme of the system of PDE's.

In the fourth chapter we propose a new approximation for pricing Asian options based on the logarithmic moments of the price average. In particular, using a Maximum Entropy approach, we can recover with great accuracy the density of the average. For a given number of moments, the approximated distribution will be: a) always positive, b) uniquely determined by the infinite sequence of logarithmic moments. Moreover c) the sequence of approximants is converging to the true distribution in some norm, d) a bound to the error made in pricing Asian options is provided. We remark that the above properties a), c) and d) are not satisfied when we approximate the true density with an Edgeworth series using the algebraic moments.

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# Declaration of Inclusion of Published Work

The paper "Corridor Options and Arc-Sine Law", based on the material included in the second chapter has been published in *Annals of Applied Probability*, vol. 10, n.2, may 2000, 634-663. A preliminary version has been presented at the Quantitative Finance conference in Sydney, December 1998.

The paper "Pricing of occupation time derivatives: continuous and discrete monitoring", based on the third chapter, is joint work with Aldo Tagliani, U. of Trento, Italy. It has been presented at the conference on Numerical Methods in Finance, May 1999, Venice and it has been accepted for publication in *Journal of Computational Finance.*

The paper "Asian options: new insights", based on the fourth chapter of the thesis, is joint work with Aldo Tagliani, U. of Trento, Italy. It has been presented at the conference on Computational Finance, June 2000, Auronzo di Cadore and published in the proceedings. It has been accepted for publication in *International Journal of Theoretical and Applied Finance.*

# Introduction

The present thesis, "*Applications of the Laplace Transform for Evaluating Occupation Time Options and other Derivatives*", is composed of four chapters. The first illustrates general applications of the Laplace transform in finance. The second one applies the Laplace transform technique to the evaluation of corridor derivatives. In the third chapter we discuss the importance of discrete monitoring on the evaluation of corridor derivatives. In the fourth chapter we examine the pricing problem for Asian options giving new approximations.

The aim of this first chapter, *The use of Laplace Transform in Finance*, is to illustrate the importance of the Laplace transform for solving many pricing problems in finance. We do this applying the Laplace transform technique to several examples from option pricing theory: the classical Black-Scholes model and the CIR and Vasicek model (solving the pricing problem for plain vanilla options, double barriers options, Asian options, occupation time derivatives and other exotic options). Then, we discuss the problem of the numerical inversion, problem that is very rarely treated in the standard textbooks, in despite of the huge literature concerning this topic and of the fact that usually it is difficult to find an analytical expression for the Inverse Laplace Transform and if it is found it requires the computation of nonstandard functions. Very often the use of the numerical inversion is disbelieved generically referring to its "intrinsic instability" or for "its inefficiency from a computational point of view". So the aim of this chapter is also to illustrate with several examples that the numerical inversion is feasible, is accurate and is not computational intensive. For these reasons, we believe that this instrument will gain greater importance in the finance field, because it is well suited to solve many familiar problems.

In the second chapter[1], *Corridor options and Arc-Sine Law*, we obtain new results on a generalization of the Lévy arc-sine law. Lévy studied the density of the time spent by a Standard Brownian Motion (SBM) below a given level. We will provide results about the case of the Brownian Motion with drift below a given level and inside a given band. Akahori, Dassios, Takacs, Embrechts et al.. have solved the case of Brownian Motion with drift below a given level. So the main results are related to the case of the time spent by the Brownian motion (without drift and with drift) inside a band.

The results have also financial applications to the pricing of corridor and hurdle options, contracts whose payoff depends on the time spent by an index below a given level (hurdle or switch derivatives) or inside a band (corridor, Parisian and range options). In particular, we examine the case of the corridor option, where the option payoff depends on the time spent inside a given band up to maturity and the digital corridor option, where at the expiry the holder of the option receives a fixed amount if the occupation time of the interval has been greater than a prefixed level. The structure of these payoffs is common to FX range floaters boost and step structures as described in Hull, Linetsky, Pechtl, Tucker et al., Turnbull, Chacko and Das. Other applications are, as suggested in Taleb, to the management of a portfolio for the computation of the expected amount of time a trader is expected to spend in the red. A similar problem for the Standard Brownian Excursion can be found in chemistry as well and in particular in the theory of ring polymers as studied in Jansons.

Starting with the Feynman-Kac equation, we will obtain the expression for the Laplace

[1] On this chapter is based the paper "Corridor Options and Arc-Sine Law", published in *Annals of Applied Probability*, vol. 10, n.2, may 2000, 634-663.

transform of the characteristic function of the occupation time. Then we discuss the numerical inversion problem using different techniques: Fast Fourier transform and multivariate Laplace inversion. We provide also a numerical comparison with the Monte Carlo simulation method. We remark that this is the first work in Finance where the numerical inversion of a bidimensional Laplace transform is discussed. We show that this inversion, at least in the case studied in this paper, can be performed very quickly and with a great accuracy. Moreover the numerical methods adopted are remarkably easy to understand.

In the third chapter[2], *Pricing of occupation time derivatives: continuous and discrete monitoring*, the focus is on the differences between the price of the contract assuming continuous or, the more realistic, discrete time monitoring. Indeed, as observed in Broadie and Glasserman for barrier options, in Heynen and Kat for lookback options and as we confirm in the present case the consideration of discrete monitoring of the underlying can introduce significant price differences between contracts that are monitored discretely and those that are monitored continuously.

In the case of continuous time monitoring and describing the evolution of the index by a Geometric Brownian Motion, we use the results of the previous chapter. In the case of discrete time monitoring we follow a Partial Differential Equation approach à la Black and Scholes along the lines described in Wilmott et al.. In order to numerically solve the PDE, taking into account the actual performance of high-speed computers, we draw our attention toward a proper finite difference scheme that has to satisfy the following requirements: the numerical solution a) exists; b) is positive; c) converges to the exact one as the discretization steps tend to zero; d) is free of unwanted oscillations arising at the monitoring dates, due to the discontinuity introduced by the influx of new information.

[2]This chapter is joint work with Aldo Tagliani, U. of Trento, Italy. It has been presented at the conference on Numerical Methods in Finance, may 1999, Venice. It has been accepted for publication in Journal of Computational Finance.

We believe that these are minimal properties because they respect the physical nature (i.e. financial) of the problem. For this reason, in the present work, we resort to an upwind finite-difference scheme for the first spatial derivative and we show through a numerical analysis that it guarantees all of these requirements independently from the discretization step and the parameter values, i.e. all properties are satisfied unconditionally. In the final section we show the remarkable differences in the price and in the Greeks of the contracts assuming continuous or discrete monitoring.

In the fourth chapter[3], *Asian options: new insights*, we compare different and easy-to-implement methods for evaluating Asian options with fixed strike. For an Asian option the payoff depends on the arithmetic average value of the asset price over some time period. This kind of options has had a very large success because it reduces the possibility of market manipulation near the expiry date and offers a better hedge for firms having a stream of positions. Actually there is no closed form solution for the price of this option because in the usual Black-Scholes framework, the arithmetic average is a sum of correlated lognormal distributions for which there is no recognizable density function, although several tentatives have been done (Geman and Yor, Turnbull and Wakeman, Levy, Kemna and Vorst, Milevsky and Posner, Barraquand and Pudet, Zvan et al., Rogers and Shi, Clelow and Carverhill, Hull and White).

In this chapter, we show that the density law of the logarithm of the arithmetic average (henceforth logav) is uniquely determined by its moments. On the basis of this result and then using the logarithmic moments instead of the algebraic moments we try to recover completely the true density using the maximum entropy approach. The maximum entropy principle, Jaynes and Golan et al., is the most popular one in choosing the approximating

[3]This chapter is joint work with Aldo Tagliani, U. of Trento, Italy. It has been presented at the conference on Computational Finance, June 2000, Auronzo di Cadore and published in the proceedings. It has been accepted for publication in International Journal of Theoretical and Applied Finance.

distribution and is well known and widely diffused in statistics. In Finance has been already used for estimating the implied risk neutral distribution from the option prices, Buchen and Kelly, Avellaneda. Assuming the given moments as known information, the maximum entropy (ME) principle chooses, out of the distributions consistent with the given partial information, the one having maximum entropy, or equivalently the most uncertain, accomplishing a principle of scientific honesty.

For a given number of moments, the approximated distribution will be: a) always positive, b) uniquely determined by the infinite sequence of logarithmic moments. Moreover c) the sequence of approximants is converging to the true distribution in some norm, d) a bound to the error made in pricing Asian options is provided. We remark that the above properties a), b) and d) are not satisfied when we approximate the true density with an Edgeworth series using the algebraic moments.

We compare the proposed approximation with: a) the lognormal density and the Edgeworth series approximation proposed in Turnbull and Wakeman, in Levy and in Ritchken et al.; b) the reciprocal gamma density approximation as suggested in Milevsky and Posner; c) the approximation based on the Generalized Beta density of the second type with the same first four moments as the average; d) the lower and upper bounds given in Rogers and Shi and in Thompson, e) the numerical inversion of the Laplace transform given in Geman and Yor, using the inversion techniques discussed in the first chapter.

# Chapter 1. The Laplace transform and its use in finance

## 1. Introduction

Aim of this chapter is to illustrate the importance of the Laplace transform for solving many pricing problems in Finance. We do this with several examples from option pricing theory. We discuss also the problem of the numerical inversion, problem that is very rarely treated in the standard textbooks, despite the extensive literature concerning this topic and of the fact that usually is difficult to find analytical expression for the Inverse Laplace Transform and if they are found they are very difficult to implement. Very often the use of the numerical inversion is disbelieved generically referring to its "intrinsic instability" or for "its inefficiency from a computational point of view". So the aim of this chapter is also to illustrate that the numerical inversion is feasible, is accurate and is not computational intensive. For these reasons, we believe that this instrument will gain greater importance in the Finance field, because it is well suited to solve many familiar problems. Indeed in other fields such as engineering and physics the Laplace transform method is one of the most used techniques.

In the following section we give the main properties of the Laplace transform, in the third we illustrate why this technique can be useful for solving many problems in Finance. Then we examine the problem of the numerical inversion with several examples from

Finance. Finally, we apply the numerical inversion to these examples. We give also the Mathematica code for the inversion. Once the expression for the Laplace transform is available, the programming code consists of few lines (less than ten). In Appendix F it can be found a detailed description, for each section, of the main contributions of the Author.

## 2. The Laplace transform and its properties

In this section we give the basic definition and the properties of the Laplace transform. A good introduction to this topic can be found in Dyke [33][1], whilst a more advanced treatment in Doetsch [28].

**Definition 2.1.** *The Laplace transform $f(\gamma)$ of a function $F(\tau)$ is defined by the following integral:*

$$f(\gamma) = \mathcal{L}(F(\tau)) = \int_0^{+\infty} e^{-\gamma\tau} F(\tau)\, dt \tag{2.1}$$

*where $F(\tau)$ is any function which, for some value of $\gamma$, gives the integral meaning, that is $F(\tau)$ is of exponential order. The integral then exists for a whole interval of values of $\gamma$, so that the function $f(\gamma)$ is defined. The integral converges in a right-plane $\gamma > \gamma_0$ and diverges for $\gamma < \gamma_0$. The number $\gamma_0$, which may be $+\infty$ or $-\infty$, is called the abscissa of convergence.*

Not every function of $\tau$ has a Laplace transform, because the defining integral can fail to converge. For example, the functions $1/\tau$, $\exp(\tau^2)$, $\tan(\tau)$ do not possess Laplace transforms. A large class of functions that possess a Laplace transform are of exponential

[1]The bibliography is after the conclusions in the present chapter.

order, i.e. there exist two constants $M$ and $k$ for which:

$$|F(\tau)| \leq Me^{k\tau},\ \tau > 0$$

Then the Laplace transform of $F(\tau\ )$ surely exists if the real part of $\gamma$ is greater than $k$. In this case, $k$ is the abscissa of convergence that we have called $\gamma_0$. Also there are certain functions that cannot be Laplace transforms, because they do not satisfy the property $f(+\infty) = 0$, e.g. $f = \gamma$. An important fact is the uniqueness of the representation (2.1), i.e. a function $f(\gamma)$ cannot be the transform of more than one continuous function $F(\tau)$. We have indeed the following theorem:

**Theorem 2.2.** *Let $F(\tau)$ be a continuous function, $0<\tau < \infty$; and $f(\gamma) \equiv 0$, for $\gamma_0 < \gamma < \infty$. Then we have $F(\tau) \equiv 0$.*

The formula for the inverse Laplace transform is given by the following theorem that is based on complex integration together with residue calculus:

**Theorem 2.3.** *If the Laplace Transform of $F(\tau)$ exists, then for $\tau > 0$:*

$$F(\tau) = \mathcal{L}^{-1}(f(\gamma)) = \lim_{R\to\infty} \frac{1}{2\pi i} \int_{c-iR}^{c+iR} f(\gamma)\, e^{\tau\gamma} d\gamma$$

*where $|F(\tau)| \leq Me^{k\tau}$ for some positive real number $k$ and $c$ is another real number such that $c > k$.*

Sometimes, in order to be clear respect which variable we are considering the transform or the inverse, we will use the notation $\mathcal{L}(F(\tau), \tau- > \gamma)$ and $\mathcal{L}^{-1}(f(\gamma), \gamma- > \tau)$, or if no confusion is possible $\mathcal{L}(F(\tau))$ and $\mathcal{L}^{-1}(f(\gamma))$.

In Table 1a we have the most important properties of the Laplace transform. We note the importance of the convolution property for which the convolution of two functions has as Laplace transform the product of the respective Laplace transforms.

In Table 2b we have the analytical expression for the original function, given the Laplace transform.

Table 1a: Basic Properties of the Laplace transform

| | Function | Laplace transform |
|---|---|---|
| Definition | $F(\tau)$ | $f(\gamma)=\int_0^{+\infty} e^{-\gamma\tau} F(\tau)\, d\tau$ |
| Linearity | $aF_1(\tau)+bF_2(\tau)$ | $af_1(\gamma)+bf_2(\gamma)$ |
| Scale | $aF(a\tau)$ | $f(\gamma/a)$ |
| Shift | $e^{a\tau}F(\tau)$ | $f(\gamma-a)$ |
| Shift | $\begin{cases} F(\tau-a) & \tau>a \\ 0 & \tau<a \end{cases}$ | $e^{-a\gamma}f(\gamma)$ |
| Time Derivative | $\frac{\partial F(\tau)}{\partial\tau}$ | $\gamma f(\gamma)-F(\tau)\vert_{\tau=0}$ |
| Differentiation | $\frac{\partial^n F(\tau)}{\partial\tau^n}$ | $\gamma^n f(\gamma)-\gamma^{n-1}F(0)+$ <br> $-\gamma^{n-2}F'(0)-\ldots-F^{(n-1)}(0)$ |
| Integral | $\int_0^\tau F(s)\, ds$ | $\frac{f(\gamma)}{\gamma}$ |
| Multiplication by polynomials | $\tau^n F(\tau)$ | $(-1)^n f^{(n)}(\gamma)$ |
| Convolution | $\int_0^\tau F(s)\, G(\tau-s)\, ds$ | $f(\gamma)\, g(\gamma)$ |
| Mean Value | $\frac{F(\tau)}{\tau}$ | $\int_\gamma^\infty f(x)\, dx$ |
| Ratio of Polyn. | $\sum\limits_{k=1}^{n}\frac{P(\alpha_k)}{Q'(\alpha_k)}e^{\alpha_k\tau}$ | $\sum\limits_{k=1}^{n}\frac{P(\gamma)}{Q(\gamma)}$ |

P(x) polynomial of degree <n; Q(x)=(x-$a_1$)(x-$a_2$)...(x-$a_n$)

where $a_1 \neq a_1 \neq \ldots \neq a_n$

| | Function | Laplace transform |
|---|---|---|
| Final Value | $\lim\limits_{\tau->\infty} F(\tau)$ | $\lim\limits_{\gamma->0}\gamma f(\gamma)$ |
| Initial Value | $\lim\limits_{\tau->0} F(\tau)$ | $\lim\limits_{\gamma->\infty}\gamma f(\gamma)$ |
| Inversion | $\lim\limits_{k\to\infty}\frac{1}{2\pi i}\int_{c-ik}^{c+ik} f(\gamma)\, e^{\tau\gamma} d\gamma$ | $f(\gamma)$ |

where c is the real part of the rightmost singularity in the image function

| Table 1b: Some Laplace transforms and their inverses | | |
|---|---|---|
| | $f(\gamma) = \int_0^{+\infty} e^{-\gamma\tau} F(\tau)\, dt$ | $F(\tau)$ |
| 1 | $1$ | $\delta(\tau)$ |
| 2 | $e^{-a\gamma}$ | $\delta(\tau - a)$ |
| 3 | $\frac{1}{\gamma}$ | $1$ |
| 4 | $\frac{1}{\gamma^2}$ | $\tau$ |
| 5 | $\frac{1}{\gamma^n}, n > 0$ | $\frac{\tau^{n-1}}{\Gamma(n)}$ |
| 6 | $\frac{1}{(\gamma-a)^n}, n > 0$ | $\frac{\tau^{n-1}e^{a\tau}}{(n-1)!}$ |
| 7 | $\frac{1}{\sqrt{\gamma-a}+b}$ | $e^{a\tau}\left(\frac{1}{\sqrt{\pi\tau}} - be^{b^2\tau}\,\mathrm{Erfc}\,(b\sqrt{\tau})\right)$ |
| 8 | $\frac{e^{-\lvert a\rvert\sqrt{\gamma}}}{\sqrt{\gamma}}$ | $\frac{e^{-a^2/4\tau}}{\sqrt{\pi\tau}}$ |
| 9 | $e^{-\lvert a\rvert\sqrt{\gamma}}$ | $\frac{ae^{-a^2/4\tau}}{2\sqrt{\pi\tau^3}}$ |
| 10 | $\frac{e^{-a\sqrt{\gamma}}}{\sqrt{\gamma}(\sqrt{\gamma}+b)}$ | $e^{b(b\tau+a)}\,\mathrm{Erfc}\left(b\sqrt{\tau} + \frac{a}{2\sqrt{\tau}}\right)$ |
| 11 | $\frac{e^{-a/\gamma}}{\gamma^{n+1}}$ | $\left(\frac{\tau}{a}\right)^{n/2} J_n\,(2\sqrt{a\tau})$ |

The function $\delta(t)$ is the delta-Dirac function, whilst the function $J_n(x)$ is the Bessel function of the first kind and of order $n$.

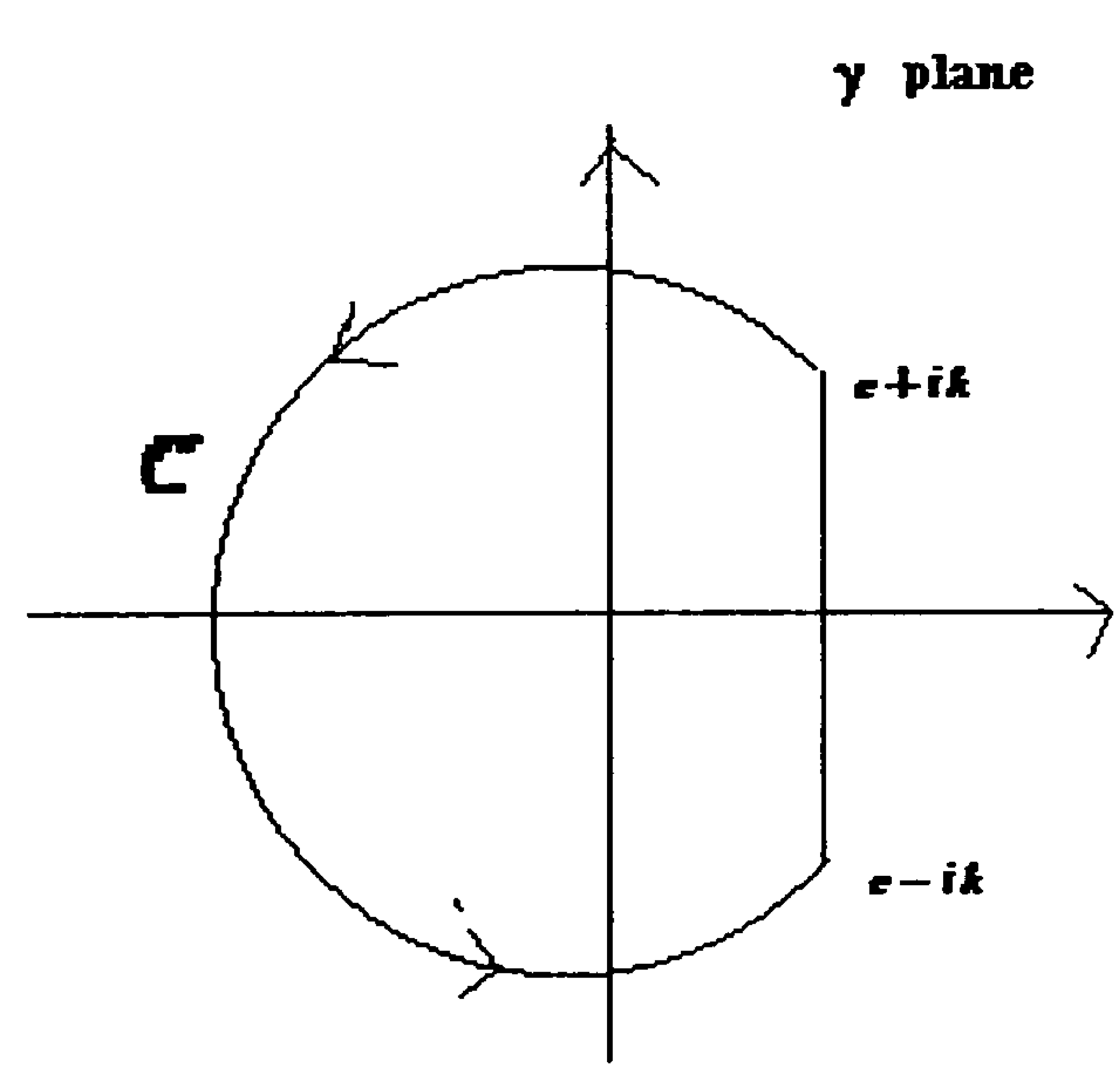


Figure 2.1: The Bromwich contour

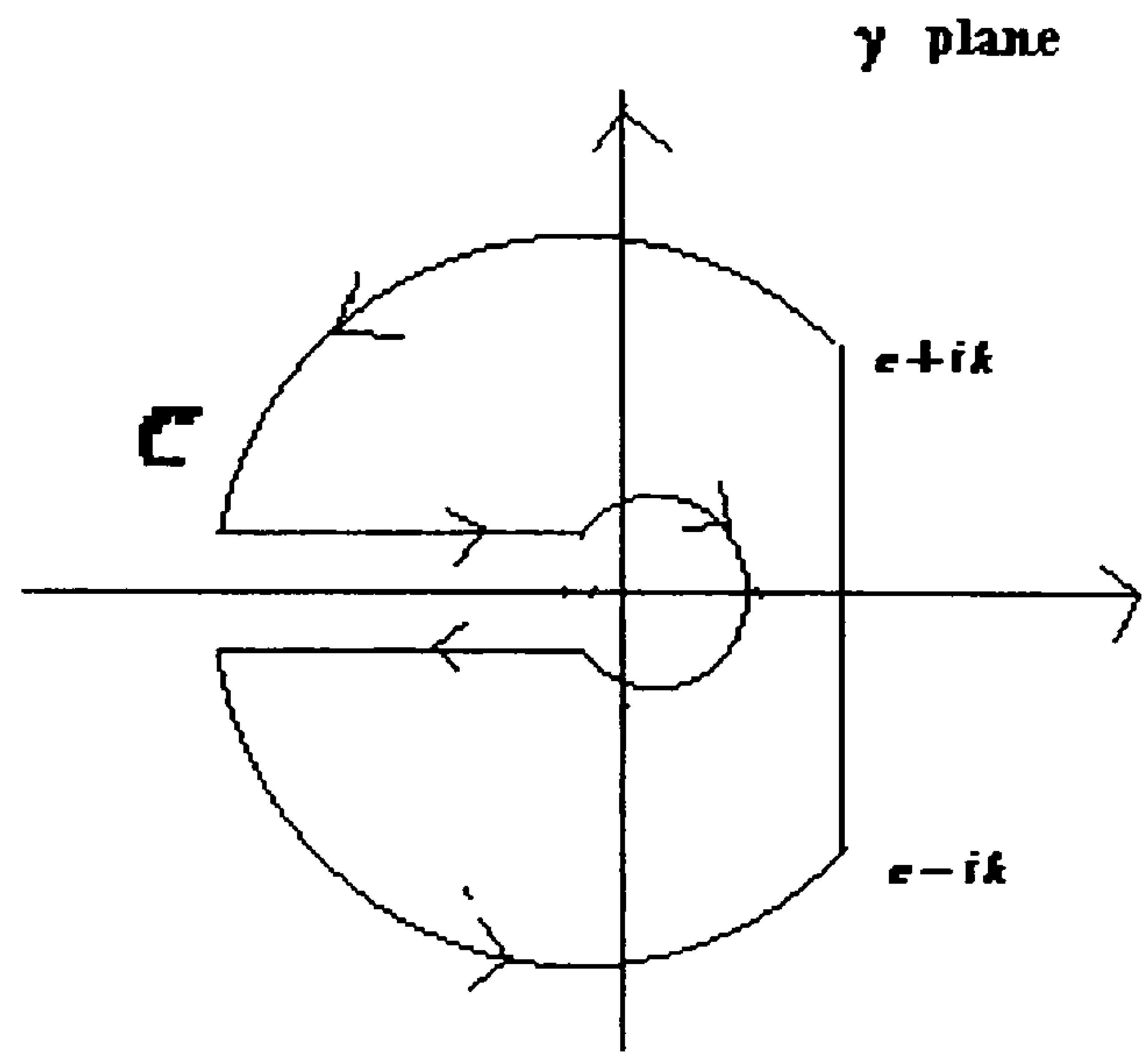


Figure 2.2: The cut Bromwich contour

The inversion integral gives us the possibility of writing, as a contour integral, an expression for the original function, once its Laplace transform is known.

The contour integral is usually transformed in a closed contour (Bromwich contour) as in Figure, consisting of a circle of radius $R$ together with a straight line segment connecting the two points $c-iR$ and $c+iR$. This transformation requires that the contribution around the infinite semicircle $C$ to be zero. The real number $c$ must be selected so that all the singularities of the image function $f(\gamma)$ are to the left of this lines. The integral

$$\int_C f(\gamma)\, e^{t\gamma} d\gamma$$

where $C$ is the Bromwich contour is evaluated using Cauchy's residue theorem. If the Laplace transform presents branch points the Bromwich contour has to be distorted with some cuts at the branch point, as shown in Figure 2.

**Theorem 2.4.** *(Residue Theorem) If $f(\gamma)$ is analytic inside $C$ except at points $\gamma_1$, $\gamma_2$,...,$\gamma_N$ where $f(\gamma)$ has poles then*

$$\int_C f(\gamma)\, e^{\tau\gamma} d\gamma = 2\pi i \sum_{k=1}^{N} (\text{sum of residues of } e^{\gamma\tau} f(\gamma) \text{ at } \gamma = \gamma_k)$$

For example, we will use this result in order to invert the Laplace transform for Barrier options.

For the fact that sometimes it is difficult to find an analytical inverse formula, it is of interest to examine methods that yield approximate expressions for inverse Laplace transforms. In this sense, it can be useful to resort to the *Abelian and Tauberian asymptotic theories* for Laplace transforms, Zauderer [108] pag. 293. These theorems relate the values of the transform $f(\gamma)$ as $\gamma \to 0$ or as $\gamma \to \infty$ to the values of the function $F(\tau)$ as $\tau \to \infty$ and $\tau \to 0$, respectively. If the original function is reasonably smooth and for the presence of the exponential in the definition of the transform, the main contributions to the integral for $f(\gamma)$ come from small values of $\gamma\tau$. Thus if $\gamma$ is large, $\tau$ must be small and viceversa.

An Abelian result states that if as $\tau \to 0$ we have:

$$F(\tau) \approx \sum_{n=0}^{\infty} \alpha_n^n \tau^n,\ \tau \to 0, \tag{2.2}$$

then for the transform $f(\gamma)$ we have, as $\gamma \to \infty$,

$$f(\gamma) \approx \sum_{n=0}^{\infty} \alpha_n \frac{n!}{\gamma^{n+1}},\ \gamma \to \infty, \tag{2.3}$$

and the converse is also true, so that (2.3) implies (2.2). The series need not converge, but may have only asymptotic validity, i.e. for small $\tau$ and large $\gamma$, $F(\tau)$ and $f(\gamma)$ are well approximated by the leading terms in the given series even if the series diverge.

For small values of $\gamma$, we have a Tauberian result. Let $F(\tau)$ be a real function. If $f(\gamma)$ is analytic for $\mathrm{Re}(\gamma) > c$ and it has isolated real and simple poles in the finite $\gamma$-plane located at the set of points $\gamma_n$, $\gamma_1 > \gamma_2 >$ ...and with corresponding residues $\beta_i$, then we have:

$$F(\tau) = \sum_{n=1}^{\infty} \beta_n e^{\gamma_n \tau}$$

which is valid as $\tau \rightarrow \infty$.

## 3. The Laplace transform in Finance

As we can see from the definition of the Laplace transform, the image function has also a financial interpretation as present value at rate $\gamma$, of a deterministic cash flow $F(t)$. In this sense the properties of the Laplace transform such as linearity (present value of different cash flows is equal to the sum of the present values), shifting (reduction in the discount rate) and scaling (a modification in the time schedule of the original cash flows implies a multiplicative change in the discount factor) can be understood also from a financial point of view. At this regard compare the discussion in Buser [11]. However, this interpretation is useful only as an introductory step, because the real usefulness of the Laplace transform goes well beyond this point of view that can be useful only in a deterministic world. Indeed, in this section our intention is to exploit not the financial interpretation of the Laplace transform, but its utility as instrument for solving many pricing problems that arise quite naturally in Finance.

In Finance usually we assume a (risk-neutral) dynamics for the price of an underlying asset $S_t$, (asset, interest rate, exchange rate) and then the problem is to price a derivative contract paying an amount $\phi(S_T)$ at the expiry $T$.

Suppose that the risk-neutral dynamics of $S_t$ is given by:

$$dS = \mu(S,t)\,dt + \sigma(S,t)\,dW_t,$$

If $r$ is the risk-free interest rate, following the standard Black-Scholes approach, the price $C(S,t)$ of the derivative contract has to satisfy the partial differential equation:

$$C_t + \mu(S,t)\,C_s + \frac{1}{2}\sigma^2(S,t)\,C_{ss} = rC \tag{3.1}$$

characterised by a terminal condition:

$$C(S,T) = \phi(S)$$

and by boundary conditions expressing the behavior of the solution at the extremes of the domain of $S$. For example, in the case of a call option $\phi(S) = (S-K)^+$ and assuming a Geometric Brownian motion for the underlying, $\mu(S,t) = rS$ and $\sigma(S,t) = \sigma S$, the asset price can assume values only in $(0,+\infty)$ so the boundary conditions, that arise from the financial aspects of the contract, are $lim_{S\to 0}C(S,t) = 0$ and the first derivative bounded as $S$ goes to infinity, $lim_{S\to\infty}|\partial C(S,t)/\partial S| < M$.

Also barrier options can enter in this framework, because if we consider a double knock-out barrier option paying a rebate $R$ if the asset hits before maturity a lower barrier $L$ or an upper barrier $U$, or the amount $(S-K)^+$, $L < K < U$, at maturity, the price of the barrier option has to satisfy the PDE (3.1), but with different boundary conditions, given now by:

$$C(L,t) \quad = \quad R, 0 < t < T$$

$$C(U,t) = R, 0 < t < T$$

The PDE (3.1) is termed (second-order) parabolic equation: in Finance usually this is the type of PDE that we can encounter. On a practical point of view, we recognize a parabolic PDE when the two-variable function $C(S,t)$ has to satisfy a PDE where there appears only the first partial derivative with respect to $t$, usually interpreted as time, and the first two partial derivatives with respect to $S$, the state variable. This fact is very important because Laplace transform is useful in solving parabolic and some hyperbolic PDE's, but not in general useful for solving elliptic PDE's.

Many often, when we have to treat path-dependent derivative contracts it is useful to introduce a new state variable $Y$, in such a way that the augmented system $(S,Y)$ is no more path-dependent. Two examples that fall in this case, and studied in this thesis, are the problem of pricing Asian options with arithmetic average, where $Y_t$ is given by:

$$Y_t = \int_0^t S_u du$$

and then it follows that:

$$dY_t = S_t dt$$

and corridor derivatives, where $Y_t$ is the time spent by the asset inside a given interval, and then:

$$\begin{aligned} Y_t &= \int_0^t 1_{(L<S_u<U)} du \\ dY &= 1_{(L<S_t<U)} dt \end{aligned}$$

Many other examples can be found in the book by Wilmott et al. [104], and in general we have that:

$$dY_t = g\left(S, t\right) dt$$

for some known function $g\left(S, t\right)$.

In this case the price of the derivative contract depends on three variables $C\left(S, Y, t\right)$ and has to satisfy the PDE:

$$C_t + \mu\left(S, t\right) C_s + g\left(S, t\right) C_y + \frac{1}{2}\sigma^2\left(S, t\right) C_{ss} = rC \tag{3.2}$$

with the relevant terminal and boundary conditions for $S$ and $Y$. This PDE is called a degenerate second-order parabolic equation, for the fact that the second derivative with respect to $Y$ does not appear.

Finally the third type of PDE's that usually we encounter in Finance is the multivariate second-order parabolic equation that arise when we have more than one state variable, such as when we assume stochastic volatility or when we have to price options on several assets, such as spread options. For example, in the case of stochastic volatility, we have:

$$\begin{aligned}
dS &= \mu\left(S, t\right) dt + \sigma\left(S, \xi, t\right) dW_1 \\
d\xi &= \alpha\left(S, \xi, t\right) dt + \beta\left(S, \xi, t\right) dW_2 \\
E\left(dW_1 dW_2\right) &= \rho dt
\end{aligned}$$

and the PDE for the price of the derivative contract becomes:

$$\begin{gathered}
C_t + \mu\left(S, t\right) C_s + \alpha\left(S, \xi, t\right) C_\xi + \tfrac{1}{2}\sigma^2\left(S, \xi, t\right) C_{ss} + \\
\rho\sigma\left(S, \xi, t\right) \beta\left(S, \xi, t\right) C_{s\xi} + \tfrac{1}{2}\beta^2\left(S, \xi, t\right) C_{ss} = rC
\end{gathered} \tag{3.3}$$

where we see the appearance of the derivative $C_{s\xi}$. Also in this case we need to specify the terminal condition and the boundary conditions that depend on the nature of the contract and on the assumed dynamics for the asset.

Through the use of the Laplace transform we can solve linear parabolic equations, i.e. PDE where the different functions $\mu(S,t)$, $\sigma^2(S,t)$, $\alpha(S,\xi,t)$ and $\beta^2(S,\xi,t)$ are only functions of the state variables $S$ and $\xi$ (or $Y$) and do not depend on the time. Only in very particular cases, we can consider time dependent coefficients.

Let us consider the PDE (3.1) and we assume that the drift and diffusion coefficients are functions of $S$ only. Instead of considering the time $t$, $t < T$, it is convenient to consider the time to maturity $\tau$ of the contract, $\tau = T - t$, so that $\tau > 0$. As consequence, in all above PDE's the term $C_t$ has to be changed with -$C_\tau$ and the terminal condition becomes now the initial condition:

$$-C_\tau + \mu(S)\, C_s + \frac{1}{2}\sigma^2(S)\, C_{ss} = rC \tag{3.4}$$

$$C(S,0) = \phi(x) \tag{3.5}$$

This change of perspective is very useful, because now the symbol $\tau$, for the fact that ranges from 0 to infinity, coincides with the range of the Laplace transform.

Denoting with $c(S,\gamma)$ the Laplace transform of $C(S,\tau)$ with respect to $\tau$ and remembering the properties:

$$\begin{aligned}
\mathcal{L}(C(S,\tau)) &= \int_0^\infty e^{-\gamma\tau} C(S,\tau)\, d\tau = c(S,\gamma) \\
\mathcal{L}\left(\frac{\partial C(S,\tau)}{\partial \tau}\right) &= \int_0^\infty e^{-\gamma\tau} \frac{\partial C(S,\tau)}{\partial \tau} d\tau = \gamma c(S,\gamma) - C(S,0) \\
\mathcal{L}\left(\frac{\partial C(S,\tau)}{\partial S}\right) &= \int_0^\infty e^{-\gamma\tau} \frac{\partial C(S,\tau)}{\partial S} d\tau = \frac{\partial c(S,\gamma)}{\partial S} \\
\mathcal{L}\left(\frac{\partial^2 C(S,\tau)}{\partial S^2}\right) &= \int_0^\infty e^{-\gamma\tau} \frac{\partial^2 C(S,\tau)}{\partial S^2} d\tau = \frac{\partial^2 c(S,\gamma)}{\partial S^2}
\end{aligned}$$

we have the means of turning the PDE (3.1), for the linearity of the Laplace transform, into the second order differential equation:

$$\frac{1}{2}\sigma^2(S)\,c_{ss}(S,\gamma)+\mu(S)\,c_s(S,\gamma)+(\gamma-r)\,c(S,\gamma)-C(S,0)=0 \tag{3.6}$$

with boundary conditions that are given by the Laplace transform of the boundary conditions of the original PDE, whilst the initial condition of the PDE has been included inside the ODE by the term $C(S,0)$. So the resolution of the PDE has been reduced to the solution of a second order differential equation that usually is simpler to solve[2]. Then the problem will be to recover the solution of the PDE from the solution of the ODE, i.e. to find the inverse Laplace transform, as illustrated in Table 2. In many cases, unfortunately, it is not easy to find an analytical inversion formula for the Laplace transform, and so we have to resort to numerical inversion methods. These methods will be described in section 4.

Table 2: Laplace Transform and PDE

| | | | | | |
|---|---|---|---|---|---|
| Original Space | $\left\{\begin{array}{c}\text{PDE}\\ +\text{ IC}\\ +\text{ BC's}\end{array}\right\}$ | | | Solution <br> $\nearrow\nwarrow$ | |
| | | | analytical | | numerical |
| | $\downarrow$ | | | $\nwarrow\nearrow$ | |
| | $\mathcal{L}$-transform | | | $\mathcal{L}^{-1}$-transform | |
| | $\downarrow$ | | | $\uparrow$ | |
| Image Space | $\left\{\begin{array}{c}\text{ODE}\\ +\text{BC's}\end{array}\right\}$ | $\rightarrow$ | | Solution | |

From the above procedure, we can also understand why it can be a problem to have time dependent coefficients. For the particular case of polynomial coefficients in $t$, from

[2] A very useful technique in this case is the method of Green's functions.

the properties of the Laplace transform:

$$\mathcal{L}\left(\left(-1\right)^{n}\tau^{n}C\left(S,\tau\right)\right)=\frac{\partial^{n}c\left(S,\gamma\right)}{\partial\gamma^{n}}.$$

So if we take the Laplace transform wrt $\tau$ when $\mu$ and $\sigma^2$ are polynomials in $\tau$, instead of getting an ODE, we get a new (non-linear) PDE, with the partial derivatives respect to $\gamma$ and $S$.

Instead of solving the PDE using the Laplace transform, we can always use the transform technique but coupled with the martingale approach. For example, in the case of the GBM we know that the transition density of $\ln S_{t+\tau}/S_t$ has a normal distribution with mean $m\tau$, $m=\left(r-\sigma^2/2\right)$, and variance $\sigma^2\tau$ and so the price of the derivative contract is obtained by integrating the payoff over the conditional distribution of $S_{t+\tau}$. However, it can happen that in much more complex cases, we do not know the transition density of the asset price, but we can obtain in some way its Laplace transform with respect to $\tau$, i.e. we know the function $g\left(\gamma,s,S\right)$:

$$g\left(\gamma,s,S\right)=\int_{0}^{+\infty}e^{-\gamma\tau}\Pr\left(S_{t+\tau}\in\left[S,S+dS\right]\left|S_t=s\right.\right)d\tau$$

From the expression of $g\left(\gamma,s,S\right)$ we can then obtain an expression for the Laplace transform wrt $\tau$ of the option price with residual life $\tau$ and payoff $\varphi\left(S_{t+\tau}\right)$:

$$\begin{aligned}
&\mathcal{L}\left[e^{-r\tau}C\left(s,\tau\right)\right]=\\
&=\textstyle\int_{0}^{+\infty}e^{-\gamma\tau}e^{-r\tau}\int_{-\infty}^{+\infty}\varphi\left(S\right)\Pr\left(S_{t+\tau}\in dS\left|S_t=s\right.\right)d\tau\\
&=\textstyle\int_{-\infty}^{+\infty}\varphi\left(S\right)\int_{0}^{+\infty}e^{-\left(\gamma+r\right)\tau}\Pr\left(S_{t+\tau}\in dS\left|S_t=s\right.\right)d\tau\\
&=\textstyle\int_{-\infty}^{+\infty}\varphi\left(S\right)g\left(\gamma+r,s,S\right)dS
\end{aligned}$$

The Laplace transform $g\left(\gamma,s,S\right)$ is also called *resolvent kernel* of the transition probability density and turns out to be useful for computing quantities like the moment generating functions of the hitting times. We will discuss this aspect in the example related to the problem of pricing barrier options. Particularly important is the expression for the resolvent kernel in the case of the Arithmetic Brownian motion with drift $m$ and diffusion coefficient $\sigma$ :

$$g(\gamma, s, S) = \frac{e^{\left(m \ln S/s - \sqrt{2\gamma\sigma^2 + m^2}|\ln S/s|\right)/\sigma^2}}{\sqrt{2\gamma\sigma^2 + m^2}} \tag{3.7}$$

The derivation can be found in Karlin and Taylor [59] pag. 288, for the case of standard Brownian motion and the expression above for the general Brownian motion follows from that. Compare also the derivation based on a probabilistic argument in Feller [36], pagg. 476-7.

Note that the Laplace transform wrt $\tau$ is equivalent, taken into account a division factor $\gamma$, to assume that the time $\tau$ is a random variable independent on the Brownian process and with exponential distribution. This fact explains the affirmation found in Geman and Yor [42], pag. 368, that it can be easier to handle a functional of a Brownian motion taken at an independent exponential time rather than at a fixed time.

A natural question at this point is the following. If we consider an asset price, this can assume only values in the range $(0, \infty)$ and this range coincides with the range of the Laplace transform as well, so why do not consider the Laplace transform wrt $S$? The reason is quite simple: for the fact that in the PDE there appear the terms $\mu(S,t)\, C_s$ and $\sigma^2(S,t)\, C_{ss}$, only for particular forms of the coefficients $\mu$ and $\sigma^2$ we are able to obtain the Laplace transforms of these terms wrt $S$. Indeed the Laplace transform respect to $S$ can be used to solve linear parabolic equations where the coefficients of the partial derivatives $C_s$ and $C_{ss}$ are polynomial in $S$. However, in general the application of the Laplace transform respect to $S$ is useful if the degree of the polynomial is less than the maximum order of the derivative appearing in the PDE. In the case that the degree of the polynomial is equal or greater of the order of the PDE the Laplace transform can be useful only in the case that the transformation leads to an equation with known solution. For parabolic equations the maximum order derivative is the second, so the Laplace transform respect to $S$ can be useful only if $\mu(S,t)$ and $\sigma^2(S,t)$ are linear in $S$. A discussion of this fact can be found in the classical book by Doetsch [28], pag. 89. An example can be found in Feller [35], that for obtaining the density law of the square-root process, later on used by Cox, Ingersoll and Ross [20] for modelling the dynamics of the instantaneous interest rate, considered the forward Kolmogorov equation and then Laplace transformed wrt to $S$ and not wrt time. In effect in this case:

$$\begin{aligned} \mu(S,t) &= a - bS \\ \sigma^2(S,t) &= \sigma^2 S \end{aligned}$$

i.e. the drift and the variance coefficients are linear in $S$. This is not the case of the BS model where $\sigma^2$ is quadratic in $S$ and it is not possible to give a simple expression to the Laplace transform of $SC_s$ and $S^2C_{ss}$ in terms of the Laplace transform of $C$ and so we cannot transform the PDE in a simpler equation.

The Laplace transform wrt to $S$ is related to the determination of the moment generating function of $S_\tau$ defined as:

$$\begin{aligned} m(s,\tau;\gamma) &= E_{t,s}\left(e^{-\gamma S_{t+\tau}}\right) \\ &= \int_0^{+\infty} e^{-\gamma S_{t+\tau}} \Pr\left(S_{t+\tau} \in [S, S+dS] \,|S_t = s\right) dS_{t+\tau} \end{aligned}$$

Respect to the definition of Laplace transform, the existence of $m(s,\tau;\gamma)$ is not always guaranteed because it is required that $m(s,\tau;\gamma)$ is defined in a complete neighborhood of the origin. For example, a lognormal random variable does not admits the mgf: its density function decays slower than the exponential function and then the integral defining the mgf does not exist finite for $\gamma \in (-\varepsilon, \varepsilon)$. From the definition, it results that the **mgf** can be thought of as the (undiscounted) price of a derivative asset paying $\exp(-\gamma S)$ at the expiry.

A famous property of the mgf is that the moments of the r.v. $S_{t+\tau}$, when they exist finite, can be determined differentiating the mgf:

$$E_{t,s}\left(S_{t+\tau}^n\right) = (-1)^n \left.\frac{\partial^n E_{t,s}\left(e^{-\gamma S_{t+\tau}}\right)}{\partial \gamma^n}\right|_{\gamma=0}$$

We will exploit this property in the computation of the price of exotic contracts on interest rates. Another important use of the mgf is the computation of the Laplace transform wrt to the strike of the price of a call or put option. The call price is given by:

$$C(s,\tau,K) = \int_K^{+\infty} (S_{t+\tau} - K) \Pr\left(S_{t+\tau} \in [S, S+dS] \,|S_t = s\right) dS_{t+\tau}$$

If we consider the Laplace transform wrt to $K$, we have:

$$\begin{aligned}
&\int_0^{+\infty} e^{-\gamma K} C(s,\tau,K)\, dK = \\
&= \int_0^{+\infty} e^{-\gamma K} \int_K^{+\infty} (S_{t+\tau} - K) \Pr\left(S_{t+\tau} \in [S, S+dS] \,|S_t = s\right) dS_{t+\tau} dK
\end{aligned}$$

and doing a change of integration, we have:

$$\begin{aligned}
&= \int_0^{+\infty} \left(\int_0^{S_{t+\tau}} e^{-\gamma K} (S_{t+\tau} - K)\, dK\right) \Pr\left(S_{t+\tau} \in [S, S+dS] \,|S_t = s\right) dS_{t+\tau} \\
&= \int_0^{+\infty} \left(S_{t+\tau} \tfrac{1-e^{-\gamma S_{t+\tau}}}{\gamma} + \tfrac{e^{-\gamma S_{t+\tau}} \gamma S_{t+\tau} + e^{-\gamma S_{t+\tau}} - 1}{\gamma^2}\right) \Pr\left(S_{t+\tau} \in [S, S+dS] \,|S_t = s\right) dS_{t+\tau} \\
&= \int_0^{+\infty} \left(S_{t+\tau} \tfrac{1}{\gamma} + \tfrac{e^{-\gamma S_{t+\tau}}}{\gamma^2} - \tfrac{1}{\gamma^2}\right) \Pr\left(S_{t+\tau} \in [S, S+dS] \,|S_t = s\right) dS_{t+\tau} \\
&= \tfrac{1}{\gamma} E_{t,s}\left(S_{t+\tau}\right) - \tfrac{1}{\gamma^2} + \tfrac{1}{\gamma^2} m(s,\tau;\gamma).
\end{aligned}$$

i.e.:

$$\int_0^{+\infty} e^{-\gamma K} C(s,\tau,K)\,dK = \frac{1}{\gamma} E_{t,s}(S_{t+\tau}) - \frac{1}{\gamma^2} + \frac{1}{\gamma^2} m(s,\tau;\gamma) \tag{3.8}$$

Note that if we need to invert numerically the above quantity it is convenient to invert only the term $m(s,\tau;\gamma)/\gamma^2$, because the inverse of the first two terms is $E_{t,s}(S_{t+\tau}) - K$.

The mgf can be determined solving the Feynman-Kac equation. Let us consider a stochastic differential equation for the asset price:

$$dS = \mu(S,t)\,dt + \sigma(S,t)\,dW_t$$

and consider the following expression:

$$v(\tau,s;\gamma,\mu) = E_{t,s}\left[\exp\left(-\gamma S_{t+\tau} - \lambda \int_t^{t+\tau} h(S(u))\,du\right)\right]$$

Then the Feynman-Kac equation, compare Karatzas and Shreve [58] pag. 267 and ff.., says that the unique solution $v(\tau,s;\gamma,\mu)$, under regularity conditions on the coefficients of the sde, is solution of the following PDE:

$$-v_\tau + \mu(s,t)\,v_s + \tfrac{1}{2}\sigma^2(s,t)\,v_{ss} = \lambda h(s)\,v$$
$$v(0,s;\gamma,\lambda) = \exp(-\gamma s)$$

In particular $v(\tau,s;\gamma,\lambda)$is the Laplace transform or moment generating function of the joint density $\left(S_\tau, \int_0^\tau h(S(u))\,du\right)$. When $\gamma = 0$, solving the above PDE we find the mgf of the r.v. $\int_0^\tau h(S(u))\,du$, whilst when $\lambda = 0$ we find the mgf of the r.v. $S_\tau$. Quantities like $\int_0^\tau h(S(u))\,ds$ are very important in Finance, because they are related to the payoff of exotic contracts such as interest rate derivatives, arithmetic averages and occupation time derivatives. For example, if $S$ represents the interest rate, and $h(S) = S$ then $v(\tau,S;0,1)$

represents the price of a zero-coupon bond with residual life $\tau$. Similarly, in pricing fixed strike Asian options, we need the density of the time average, so $h(S) = S$, and if we solve for $v(\tau, S; 0, \lambda)$ we find the Laplace transform (mgf) of the arithmetic average. In pricing floating strike Asian options we are always interested in $h(S) = S_t$, but now we need the joint density of the terminal asset price and of its time average, and then the problem is related to the knowledge of $v(\tau, S; \gamma, \lambda)$. $v(\tau, S; \gamma, \lambda)$ is also an example of a double Laplace transform.

Multidimensional Laplace transforms arise also when we consider the Laplace transform wrt to time in the PDE above and we obtain a second order differential equation. An example will be given for pricing hurdle options.

The Handbook by Borodin and Salminen[9] is a very useful source where to find expressions for the resolvent kernel and for the moment generating functions. Let us consider now several examples where the Laplace transform method can be successfully applied.

### 3.1. Example 1: Black-Scholes model

We consider this problem for his instructive value. Note that the first reference where we have found the use of the Laplace transform for option pricing in the Black-Scholes world goes back to 1983 in an Appendix of the PH.D. thesis by M. Selby [86].

In this model we have $\mu(S) = rS$ and $\sigma(S) = \sigma S$ and payoff $\varphi(S)$ at the expiry. We show in two different ways, solution of the PDE and martingale approach, how to get the Laplace transform of the price. First of all, it is convenient to transform the Black-Scholes PDE into a standard diffusion equation, Wilmott et al. [104], with the substitutions:

$$\begin{aligned} C &= e^{\alpha x + \beta \tau} V(x, \tau) \\ x &= \ln S, \tau = \frac{\sigma^2}{2}(T - t) \end{aligned}$$

$$k_1 = \frac{2r}{\sigma^2}, \alpha = -\frac{1}{2}(k_1 - 1), \beta = -\frac{1}{4}(k_1 - 1)^2 - \frac{2r}{\sigma^2}$$

and so we obtain the standard diffusion equation:

$$-\frac{\partial V}{\partial \tau} + \frac{\partial^2 V}{\partial x^2} = 0 \tag{3.9}$$

with initial condition:

$$V(x, 0) = e^{-\alpha x} \varphi(e^x)$$

We remark that the domain $(0, \infty)$ for the asset price has been now transformed into a domain $(-\infty, +\infty)$ for $x$ and we have to solve the heat equation in an infinite region. The solution can be found in several ways and is given by, compare Strauss [92] pag. 47:

$$V(x, \tau) = \frac{1}{\sqrt{4\pi\tau}} \int_{-\infty}^{+\infty} e^{-(x-\xi)^2/4\tau} e^{-\alpha\xi} \varphi\left(e^{\xi}\right) d\xi \tag{3.10}$$

and then:

$$C(S, T-t) = e^{\alpha x + \beta\tau} V\left(\ln S, \frac{\sigma^2}{2}(T-t)\right)$$

In the case of the call option, $\varphi(S) = (S-K)^+$ and with some algebraic manipulation we obtain the standard BS formula:

$$\begin{aligned} C(S, T-t) &= SN(d_1) - e^{-r(T-t)} K N(d_2) \\ d_1 &= \frac{\ln(S/K) + \left(r + \sigma^2/2\right)(T-t)}{\sigma\sqrt{T-t}} \\ d_2 &= d_1 - \sigma\sqrt{T-t} \end{aligned} \tag{3.11}$$

The use of the Laplace transform is one method for solving the BS PDE. We follow two methods:

a) consider the LT wrt $\tau$ in (3.9) and solve the related 2nd order differential equation;

b) use the expression for the resolvent kernel.

a) If we take the Laplace transform of (3.9) wrt $\tau = \frac{\sigma^2}{2}(T-t)$ we obtain the ODE:

$$v_{xx} - \gamma v + e^{-\alpha x}\varphi(e^x) = 0 \tag{3.12}$$

in the region $(-\infty, +\infty)$ requiring the consistent boundary values $v(x,\gamma) \to x/\gamma$ as $x \to \infty$ and $v(x,\gamma) \to 0$ as $x \to -\infty$. The general solution of the equation is given by:

$$v(x,\gamma) = \int_{-\infty}^{+\infty} k(x,\xi;\gamma)\, e^{-\alpha\xi}\varphi\left(e^{\xi}\right) d\xi \tag{3.13}$$

where the function $k(x,y;\lambda)$ is a bounded function satisfying the ODE:

$$k_{xx} - \gamma k = \delta(x-y)$$

and in the theory of ODE's is called the Green function of (3.12) and its inverse Laplace transform gives the Green function associated to the PDE (3.9).

Solving separately the problem for $x > y$ and for $x < y$, and imposing the continuity condition in $x = y$, the function $k(x,y;\gamma)$ is given by:

$$k(x,y;\gamma) = \begin{cases} c_\gamma e^{-\sqrt{\gamma}x + 2\sqrt{\gamma}y} & x > y \\ \\ c_\gamma e^{\sqrt{\gamma}x} & x < y \end{cases}$$

and the constant $c_\gamma$ depends on $\gamma$ and can be found observing that if $e^{-\alpha x}\varphi(e^x)$ is equal to 1 a solution of the ODE (3.12) is given by $v(x,\gamma) = 1/\gamma$, so that:

$$\frac{1}{\gamma} = \int_{-\infty}^{+\infty} k(\xi, y; \gamma)\, 1 d\xi$$

$$\frac{1}{\gamma} = c_\gamma \left( \int_{-\infty}^{y} e^{\sqrt{\gamma}\xi} d\xi + \int_{y}^{+\infty} e^{-\sqrt{\gamma}\xi + 2\sqrt{\gamma}y} d\xi \right)$$

so that:

$$c_\gamma = \frac{e^{-\sqrt{\gamma}y}}{2\sqrt{\gamma}}$$

and we obtain:

$$k(x,y;\gamma) = \begin{cases} \frac{e^{-\sqrt{\gamma}(x-y)}}{2\sqrt{\gamma}} & x > y \\ \\ \frac{e^{-\sqrt{\gamma}(y-x)}}{2\sqrt{\gamma}} & x < y \end{cases} = \frac{e^{-\sqrt{\gamma}|x-y|}}{2\sqrt{\gamma}}$$

The Laplace transform of the option price is then given by:

$$v(x,\gamma) = \int_{-\infty}^{+\infty} \frac{e^{-\sqrt{\gamma}|x-\xi|}}{2\sqrt{\gamma}} e^{-\alpha\xi} \varphi\left(e^{\xi}\right) d\xi \tag{3.14}$$

b) A second way to obtain the above expression (3.14) is using the resolvent kernel for the ABM given in equation (3.7). Indeed, the Laplace transform of the option price wrt to time to maturity is exactly the integral of the payoff function wrt the resolvent kernel.

The Inverse Laplace Transform can be found once we look up in the tables the following inversion formula:

$$\mathcal{L}^{-1}\left(\frac{e^{-|x|\sqrt{\gamma}}}{\sqrt{\gamma}}\right) = \frac{e^{-x^2/4\tau}}{\sqrt{\pi\tau}}$$

so, if the integral in the solution of the ODE can be changed with the $\mathcal{L}^{-1}$ transformation, we obtain:

$$V(x,\tau) = \int_{-\infty}^{+\infty} \frac{e^{-(x-\xi)^2/4\tau}}{\sqrt{4\pi\tau}} e^{-\alpha\xi} \varphi\left(e^{\xi}\right) d\xi$$

i.e. expression (3.10).

In the case of a call option, where $\varphi(e^x) = (e^x - K)^+$, we can take a more direct approach solving directly equation (3.12), that becomes $v_{xx} - \gamma v + e^{-\alpha x}(e^x - K)^+ = 0$. We can solve separately for $x > \ln K$ and for $x < \ln K$ and then impose the smooth pasting conditions at $x = \ln K$. For $x > \ln K$, we obtain:

$$v(x,\gamma) = \frac{e^{-(\alpha-1)x}}{\gamma - (\alpha-1)^2} - K\frac{e^{-\alpha x}}{\gamma - \alpha^2} + Be^{-\sqrt{\gamma}x}$$

whilst for $x < \ln K$, we obtain $v(x,\gamma) = Ae^{\sqrt{\gamma}x}$. Imposing the continuity and differentiability condition at $x = \ln K$, we find a linear system for the constants $A$ and $B$. The expression for the option price is then given by:

$$C(S,T-t) = e^{\beta\frac{\sigma^2}{2}(T-t)}\mathcal{L}^{-1}\left(e^{\alpha\ln S}v(\ln S,\gamma);\gamma \to \frac{\sigma^2}{2}(T-t)\right)$$

where:

$$e^{\alpha x}v(x,\gamma) = \begin{cases} \frac{e^{-(\sqrt{\gamma}+\alpha-1)\ln K}}{2\sqrt{\gamma}(\sqrt{\gamma}+\alpha-1)(\sqrt{\gamma}+\alpha)}e^{(\sqrt{\gamma}+\alpha)x} & x < \ln K \\ \\ \frac{e^{(\sqrt{\gamma}-\alpha+1)\ln K}}{2\sqrt{\gamma}(\sqrt{\gamma}-\alpha+1)(\sqrt{\gamma}-\alpha)}e^{-(\sqrt{\gamma}-\alpha)x} + \frac{S}{\gamma-(\alpha-1)^2} - \frac{K}{\gamma-\alpha^2} & x > \ln K \end{cases} \tag{3.15}$$

with $\sqrt{\gamma} > \max[|1-\alpha|;|\alpha|]$.

It is easy to obtain the Greeks of the contract. For the delta, we have:

$$\Delta(S,T-t) = \frac{\partial C(S,T-t)}{\partial S} =$$

$$= \begin{cases} e^{\beta\frac{\sigma^2}{2}(T-t)}\mathcal{L}^{-1}\left(\frac{e^{-(\sqrt{\gamma}+\alpha-1)\ln K}}{2\sqrt{\gamma}(\sqrt{\gamma}+\alpha-1)}e^{(\sqrt{\gamma}+\alpha-1)\ln S}\right) & S < K \\ \\ e^{\beta\frac{\sigma^2}{2}(T-t)}\mathcal{L}^{-1}\left(-\frac{e^{(\sqrt{\gamma}-\alpha+1)\ln K}}{2\sqrt{\gamma}(\sqrt{\gamma}-\alpha+1)}e^{(-\sqrt{\gamma}+\alpha-1)\ln S} + \frac{1}{\gamma-(\alpha-1)^2}\right) & S > K \end{cases}$$

and for the gamma:

$$\Gamma(S,T-t)=\frac{\partial^2 C(S,T-t)}{\partial S^2}=$$
$$=\begin{cases} e^{\beta\frac{\sigma^2}{2}(T-t)}\mathcal{L}^{-1}\left(\frac{e^{-(\sqrt{\gamma}+\alpha-1)\ln K}}{2\sqrt{\gamma}}e^{(\sqrt{\gamma}+\alpha-2)\ln S}\right) & S<K \\ \\ e^{\beta\frac{\sigma^2}{2}(T-t)}\mathcal{L}^{-1}\left(\frac{e^{(\sqrt{\gamma}-\alpha+1)\ln K}}{2\sqrt{\gamma}}e^{-(\sqrt{\gamma}-\alpha+2)\ln S}\right) & S>K \end{cases}$$

### 3.2. Example 2: Barrier options

The Laplace transform method is well suited to price barrier options. Indeed this way has been followed by several authors. In general the different techniques used have been solution of the PDE, probabilistic method, Laplace transform and combination of them:

a) probabilistic approach Kunitomo and Ikeda [60], Geman and Yor [42], Pelsser [78], Sbuelz [84], Jamshidian [56], with Sbuelz giving very interesting insights on the hedging problem;

b) solution of the PDE using the method of separation of variables Hui [48] and Hui et al. [49];

c) solution of the PDE with analytical, Linetsky [66], and numerical inversion of the Laplace transform, Davydov and Linetsky [26] and Fusai in the present thesis;

d) binomial trees, compare the discussion in Heynen and Kat [45].

A double barrier option is characterized by a payoff $\varphi(S,T)$ at the expiry date $T$ if no barrier has been hit before, $\varphi_L(T-t)$ if the asset price hits the lower barrier $L$ at time $t$ and before hitting the upper barrier, payoff $\varphi_U(T-t)$ if the asset price hits the upper barrier $U$ before maturity and before hitting the lower barrier.

Before proceeding, let us discuss which kind of solution we should expect and why the application of the Laplace transform can be useful.

We will follow the following steps: 1) obtain an expression for the expected value of the different components of the option payoff, 2) compute their Laplace transform, 3) relate these LT's to the resolvent kernel of the process.

**Step 1**: The payoff of the double barrier option has three components:

a) the payoff $\varphi_L(T-t)$ if the asset hits the lower barrier before hitting the upper one and before maturity;

b) the payoff $\varphi_U(T-t)$ if the asset hits the upper barrier before hitting the lower one and before maturity;

c) the payoff $\varphi(S,T)$ if the asset does not hit either barriers before maturity.

The price of the option will be given by sum of the expected value of these three components.

Define $x = \ln(S/L)$ and corresponding the barriers become 0 and $\ln(U/L)$. We call $H = H_{L,U}$ the first exit time of the process $x_t$ out of the interval $(0, \ln U/L)$, i.e.:

$$H = H_{0,\ln(U/L)} = \min\left(\zeta : x_\zeta \notin (0, \ln U/L)\right)$$

If at time $\zeta$, $\zeta < T$, the asset touches the lower barrier without before having hit the upper barrier we will receive at time $\zeta$ the amount $\varphi_L(T-\zeta)$. The instantaneous payoff due to the lower barrier will be then this amount times the probability of hitting the lower barrier in the time interval $(\zeta, \zeta + d\zeta)$ before hitting the upper barrier:

$$\varphi_L(T-\zeta)\Pr_{t,x}(H \in d\zeta, x_H = 0)\, d\zeta$$

and the expected discounted payoff over the time window $(t,T)$ is given by:

$$\int_t^T e^{-r(\zeta-t)}\varphi_L(T-\zeta)\Pr_{t,x}(H \in d\zeta, x_H = 0)$$

Similarly for the upper barrier: if we hit in the time interval $(\zeta.\zeta+d\zeta)$ the upper barrier before hitting the lower one, we obtain the payoff

$$\varphi_U(T-\zeta).$$

with probability:

$$\Pr_{t,x}(H\in d\zeta, x_H=\ln(U/L))$$

and so the payoff in the time interval $(\zeta,\zeta+d\zeta)$ is:

$$\varphi_U(T-t)\Pr_{t,x}(H\in d\zeta, x_H=\ln(U/L))$$

and over the time interval $(t,T)$ the expected discounted payoff is given by:

$$\int_t^T e^{-r(\zeta-t)}\varphi_U(T-\zeta)\Pr_{t,x}(H\in d\zeta, x_H=0)$$

There is also the possibility of not hitting both barriers before maturity. In this case, at the expiry, we obtain the payoff $\varphi(Le^x)$. This happens with probability $\Pr(H>T, x_T\in dx)$ and so the discounted terminal payoff in the case of not having touched both barriers and $x_T$ being at the level $dx$ is given by:

$$e^{-r(T-t)}\varphi(Le^{x_T})\Pr_{t,x}(H>T, x_T\in dx)$$

and the expected value of this amount is obtained integrating with respect to $x$. Note that the term $\Pr_{t,x}(H>T, x_T\in dx)$ can be expressed as:

$$\Pr_{t,x}\left(\inf_{t\le s\le T} x_s>0,\ \sup_{t\le s\le T} x_s<b, x_T\in dx\right)$$

and this quantity has been used by He et al. [44] for pricing the so called Double Lookbacks.

In conclusion, the expected payoff will be obtained summing the three components:

$$\begin{aligned}
&\int_t^T e^{-r(\zeta-t)}\varphi_L(T-\zeta)\Pr_{t,x}(H\in d\zeta, x_H=0)\\
&+\int_t^T e^{-r(\zeta-t)}\varphi_U(T-\zeta)\Pr_{t,x}(H\in d\zeta, x_H=\ln(U/L))\\
&+e^{-r(T-t)}\int_0^{\ln(U/L)}\varphi(Le^w)\Pr_{t,x}(H>T, w_T\in dw)
\end{aligned}$$

and we can write it as $\exp(-r(T-t))\,V(x,T-t)$ where $V(x,T-t)$:

$$\begin{aligned}
&V(x,T-t)=\int_0^{T-t} e^{r(T-t-\zeta)}\varphi_L(T-t-\zeta)\Pr_{0,x}(H\in d\zeta, x_H=0)\\
&+\int_0^{T-t} e^{r(T-t-\zeta)}\varphi_U(T-t-\zeta)\Pr_{0,x}(H\in d\zeta, x_H=\ln(U/L))\\
&+\int_0^{\ln(U/L)}\varphi(Le^w)\Pr_{0,x}(H>T-t, w\in dw)
\end{aligned}$$

and we have assumed the strong Markov property of the process.

**Step 2**: Now if we take the Laplace transform wrt $\tau=T-t$ of these three components, exploiting the convolution property of the Laplace Transform so that the integrals are transformed in products, we obtain:

$$\begin{aligned}
&v(x,\gamma)=\mathcal{L}(V(x,\tau))=\textstyle\int_0^{+\infty}e^{-\gamma\tau}e^{-rs}V(x,s)\,ds\\
&=\mathcal{L}\left(e^{r(T-t)}\varphi_L(T-t)\right)\mathcal{L}\left(\Pr_{0,x}(H\in d(T-t), x_H=0)\right)\\
&+\mathcal{L}\left(e^{r(T-t)}\varphi_U(T-t)\right)\mathcal{L}\left(\Pr_{0,x}(H\in d(T-t), x_H=\ln(U/L))\right)\\
&+\textstyle\int_0^{\ln(U/L)}\varphi(Le^w)\,\mathcal{L}\left(\Pr_{0,x}(H>T-t, w\in dw)\right)
\end{aligned}$$

and if we define:

$$l(\gamma) = \mathcal{L}(\varphi_L(T-t))$$

$$u(\gamma) = \mathcal{L}(\varphi_U(T-t))$$

and

$$\begin{aligned} h^+(x,\gamma) &= \mathcal{L}(\mathrm{Pr}_{0,x}(H \in d(T-t), x_H = 0)) \\ h^-(x,\gamma) &= \mathcal{L}(\mathrm{Pr}_{0,x}(H \in d(T-t), x_H = \ln(U/L))) \\ h_d(X,x,\gamma) &= \mathcal{L}(\mathrm{Pr}_{0,x}(H > T-t, X_{T-t} \in dX)) \end{aligned}$$

then the Laplace transform of the price of the option with respect to $T-t$, is given by:

$$\begin{aligned} &\mathcal{L}\left(e^{-r(T-t)}V(T-t)\right) \\ &= l(\gamma)\,h^-(x,\gamma) + u(\gamma)\,h^+(x,\gamma) + \textstyle\int_0^{\ln(U/L)} \varphi(Le^w)\,h_d(X,x,\gamma)\,dX \end{aligned}$$

and in the case of time-independent payoffs at the barrier, if we think to the Laplace transform as a discounting factor, we have:

$$\text{Laplace transform} = \sum \text{payoff at barrier x probability of discounted stopping times}$$

In the case of zero rebates, i.e. the case of the double knock-out option, the above Laplace simplifies in:

$$\mathcal{L}\left(e^{-r(T-t)}V(T-t)\right) = \int_0^{\ln(U/L)} \varphi(Le^w)\,h_d(X,x,\gamma)\,dX$$

**Step 3**: The determination of these expressions requires the knowledge of the moment generating function of the hitting time. Specifying a particular model, i.e. Geometric Brownian motion or a mean-reverting process, where the drift and diffusion coefficient depend only on $x$, we can find an expression for the mgf solving a differential equation.

compare Karlin and Taylor [59], pag. 202-3 and Ito and McKean pag.130.

Indeed if we consider a process with barriers at $a$ (lower barrier) and $b$ (upper barrier) and define the mgf's of the hitting time densities:

$$\begin{aligned}
h^{+}(x,\gamma) &\equiv h^{+}(x,\gamma;a,b) = \int_{0}^{+\infty} e^{-\gamma s}\,\mathrm{Pr}_{0,x}(H \in ds, x_H = b) \\
h^{-}(x,\gamma) &\equiv h^{-}(x,\gamma;a,b) = \int_{0}^{+\infty} e^{-\gamma s}\,\mathrm{Pr}_{0,x}(H \in ds, x_H = a) \\
h(X,x,\gamma) &\equiv h(X,x,\gamma;a,b) = \int_{0}^{+\infty} e^{-\gamma s}\,\mathrm{Pr}_{0,x}(H > s, X_\tau \in dX)\,ds
\end{aligned}$$

then these mgf's are continuous solutions of the differential equations:

$$\begin{aligned}
&\tfrac{1}{2}\sigma^2(x)\,h^{+}_{xx}(x,\gamma) + \mu(x)\,h^{+}_{x}(x,\gamma) - \gamma h^{+}(x,\gamma) = 0 \\
&h^{+}(a,\gamma) = 0; h^{+}(b,\gamma) = 1
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
&\tfrac{1}{2}\sigma^2(x)\,h^{-}_{xx}(x,\gamma) + \mu(x)\,h^{-}_{x}(x,\gamma) - \gamma h^{-}(x,\gamma) = 0 \\
&h^{-}(a,\gamma) = 1; h^{-}(b,\gamma) = 0
\end{aligned} \tag{3.17}$$

and $h_d$ (we omit the index $d$) is the unique continuous function in $(a,b)$ with a jump in the first derivative at $x = X$:

$$\begin{aligned}
&\tfrac{1}{2}\sigma^2(x)\,h_{xx}(X,x,\gamma) + \mu(x)\,h_{x}(X,x,\gamma) - \gamma h(X,x,\gamma) = \delta(x - X) \\
&h(X,a,\gamma) = 0; h(X,b,\gamma) = 0 \\
&h(X,X^{-},\gamma) - h(X,X^{+},\gamma) = 0 \\
&h_x(X,X^{-},\gamma) - h_x(X,X^{+},\gamma) = -\tfrac{2}{\sigma^2(X)}
\end{aligned} \tag{3.18}$$

Instead of solving these differential equations, Jamshidian [56], assuming the strong Markov property of the process, has shown that $h^{+}$, $h^{-}$ and $h_d$ can be expressed in terms of the *resolvent kernel* of the process, i.e. in terms of the Laplace transform with respect to time of the transition density. In particular, if we define the resolvent kernel of an arbitrary time homogenous Markov process with the strong Markov property:

$$g(\gamma, x, X) = \int_0^{+\infty} e^{-\gamma\tau} f(X, \tau; x)\, d\tau$$

where:

$$f(X, \tau; x)\, dX = \Pr\left(X_{t+\tau} \in [X, X + dX] \,|X_t = x\right)$$

then Jamshidian has shown that in the case of a single barrier $b$:

$$\begin{aligned}
h^+(x, \gamma; -\infty, b) &= \frac{g(\gamma, x, b)}{g(\gamma, b, b)} 1_{(x<b)} &\quad (3.19)\\
h^-(x, \gamma; b, +\infty) &= \frac{g(\gamma, x, b)}{g(\gamma, b, b)} 1_{(b<x)} &\quad (3.20)\\
h_d(X, x, \gamma; -\infty, b) &= g(\gamma, b, X)\, h^+(x, \gamma; -\infty, b)\, 1_{(x<b)} &\quad (3.21)\\
h_d(X, x, \gamma; b, +\infty) &= g(\gamma, b, X)\, h^-(x, \gamma; -\infty, b)\, 1_{(b<x)} &\quad (3.22)
\end{aligned}$$

whilst in the case of two barriers $a$ and $b$, $a < x < b$:

$$\begin{aligned}
h^+(x, \gamma; a, b) &= \frac{g(\gamma, x, b)\, g(\gamma, a, a) - g(\gamma, x, a)\, g(\gamma, a, b)}{g(\gamma, b, b)\, g(\gamma, a, a) - g(\gamma, b, a)\, g(\gamma, a, b)}, &\quad (3.23)\\
h^-(x, \gamma; a, b) &= \frac{g(\gamma, x, a)\, g(\gamma, b, b) - g(\gamma, x, b)\, g(\gamma, b, a)}{g(\gamma, b, b)\, g(\gamma, a, a) - g(\gamma, b, a)\, g(\gamma, a, b)}, &\quad (3.24)
\end{aligned}$$

and if $a < X < x < b$, $a < x < X < b$:

$$h_d(X, x, \gamma; a, b) = g(\gamma, b, X)\, h^+(x, \gamma; a, b) + g(\gamma, a, X)\, h^-(x, \gamma; a, b) \quad (3.25)$$

The computation of the barrier option price is reduced to the determination of the resolvent kernel. We investigate this aspect in the following subsection for different processes: Geometric Brownian motion, Square Root process, Ornstein-Uhlenbeck process and CEV process. Table 3 gives the number of equation in the following sections relevant for the different processes:

| Table 3: The resolvent kernel for different processes | |
|---|---|
| GBM | equation (3.7) |
| Square Root | equation (3.28) |
| O.-U. | equations (3.34) and (3.29) |
| CEV | equations (3.28) and (3.39) |

### 3.2.1. Geometric Brownian Motion

If the asset price evolves according to a GBM, then the asset returns $x = \ln S$ are described by an ABM with drift $m = \left(r - \sigma^2/2\right)$ and diffusion coefficient $\sigma$ and $h^+$ and $h^-$ solve:

$$\frac{1}{2}\sigma^2 h_{xx} + m h_x - \gamma h = 0 \tag{3.26}$$

Substituting $\exp\left(\theta\left(\gamma\right)x\right)$ in the equation, we find that $\theta\left(\gamma\right)$ is solution of:

$$\frac{1}{2}\sigma^2\theta^2 + m\theta - \gamma = 0$$

i.e.

$$\theta^{\pm} = \frac{-m \pm \sqrt{m^2 + 2\sigma^2\gamma}}{\sigma^2}$$

and the general solution is given by:

$$h\left(x,\gamma\right) = Ae^{\theta^+(\gamma)x} + Be^{\theta^-(\gamma)x}$$

Using the appropriate boundary conditions, we find for the functions[3] $h^{\pm}$:

$$h^{+}(x,\gamma) = e^{\frac{m}{\sigma^2}(b-x)} \frac{\sinh\left(\frac{\sqrt{m^2+2\sigma^2\gamma}}{\sigma^2}(x-a)\right)}{\sinh\left(\frac{\sqrt{m^2+2\sigma^2\gamma}}{\sigma^2}(b-a)\right)}$$

$$h^{-}(x,\gamma) = e^{-\frac{m}{\sigma^2}(x-a)} \frac{\sinh\left(\left(\frac{\sqrt{m^2+2\sigma^2\gamma}}{\sigma^2}\right)(b-x)\right)}{\sinh\left(\frac{\sqrt{m^2+2\sigma^2\gamma}}{\sigma^2}(b-a)\right)}$$

i.e. the expressions used in Geman and Yor, Jamshidian and Pelsser.

The function $h_d(X,x,\gamma)$ can be found or using the procedure shown in Appendix or removing the drift term in (3.26) using the substitution $h_d(X,x,\gamma) = e^{\beta x} v(X,x,\gamma)$, $\beta = -m/\sigma^2$, and then using the results related to the case of BM with zero drift that can be found in Borodin and Salminen [9], pag. 147 eq. 1.15.6, where $\lambda$ has to be substituted by $(m^2+2\sigma^2\gamma)/(2\sigma^4)$ :

$$h_d(X,x,\gamma) = e^{-\frac{m}{\sigma^2}x} \frac{\cosh\left((b-a-|X-x|)\frac{\sqrt{m^2+2\sigma^2\gamma}}{\sigma^2}\right) - \cosh\left((b+a-X-x)\frac{\sqrt{m^2+2\sigma^2\gamma}}{\sigma^2}\right)}{\sqrt{2\mu}\sinh\left((b-a)\frac{\sqrt{m^2+2\sigma^2\gamma}}{\sigma^2}\right)}$$

that is valid for $a < x/\backslash X, x\backslash/X < b$.

We obtain the same results substituting in (3.23) and (3.25) the expression for the resolvent kernel for the ABM given in (3.7). Setting $b = +\infty$ or $a = -\infty$, depending on the initial level $x$, we obtain the solution for the single barrier case. Setting $m = 0$, we are considering the case of BM with zero drift.

In Appendix B.1 we discuss the analytical inversion of the above quantities and we show that we can obtain two equivalent representations, one as Fourier sine series and the other in terms of the fundamental solutions or Green's function. Both solution forms

[3] The expression for $h_d(X,x,\gamma)$ can be found removing the drift term using the substitution $h_d(X,x,\gamma) = e^{bx} v(X,x,\gamma)$, $b = -m/\sigma^2$, and then using the results related to the case of BM with zero drift, Borodin and Salminen [9], pag. 147 eq. 1.15.6. where $\lambda$ has to be substituted by $(m^2+2\sigma^2\gamma)/(2\sigma^4)$. Moreover in Borodin, we need to divide the expression by $\lambda$.

are useful in evaluating the solution for different ranges of values of $T - t$. Because of the rapid exponential decay of the terms in the Fourier series, the series converges very rapidly for large values of $T - t$. Indeed, if we truncate the Fourier series representation the boundary conditions are exactly satisfied for all times, but the initial condition will be described only approximately. Thus, the truncated Fourier series should provide a good approximation for $T - t$ large. Instead, the series of fundamental solutions converges most rapidly for small values of $T - t$. In particular, the boundary conditions on the barriers are only approximately satisfied with the truncated series, and this approximation deteriorates as $T - t$ gets large. Thus for the purposes of computation each of the series is useful.

### 3.2.2. Square Root Process

In this case, we can follow different ways:

a) find the mgf of the hitting time densities solving the ODE's (3.17) and (3.16);

b) find the resolvent kernel of the process and then use the results in Jamshidian.

On a practical side the two approaches are equivalent, because in order to find the resolvent kernel we have always to solve the same ODE, although with different boundary conditions.

If we assume that the asset price process evolves as a square-root process:

$$dS_t = (m - kS_t)\,dt + \sigma\sqrt{S}dW_t$$

the kernel resolvent is solution of:

$$\frac{1}{2}\sigma^2 s g_{ss} + (m - ks)\,g_s - \gamma g = \delta\,(s - S)$$

If we operate the change of variable $y{=}2ks/\sigma^2$, we obtain the following ODE for the function $w\left(\gamma, 2ks/\sigma^2, 2kS/\sigma^2\right) = g\left(\gamma, s, S\right)$:

$$y w_{yy} + \left(\frac{2m}{\sigma^2} - y\right) w_y - \frac{\gamma}{k} w = \delta\left(y - Y\right)$$

The solution of the homogeneous equation is given, compare Appendix A, by:

$$w\left(\gamma, y, Y\right) = A\Upsilon\left(\frac{\gamma}{k}, \frac{2m}{\sigma^2}; y\right) + B\,\bar{\Upsilon}\left(\frac{\gamma}{k}, \frac{2m}{\sigma^2}; y\right) \tag{3.27}$$

where $\Upsilon$ and $\bar{\Upsilon}$ are the confluent hypergeometric function and the confluent hypergeometric function of the second kind, respectively.

Then the Green's function is given by:

$$w\left(\gamma, y, Y\right) = \begin{cases} A\Upsilon\left(\frac{\gamma}{k}, \frac{2m}{\sigma^2}; y\right) & 0 < y \leq Y \\ & \\ B\,\bar{\Upsilon}\left(\frac{\gamma}{k}, \frac{2m}{\sigma^2}; y\right) & Y \leq y \end{cases}$$

where we have used the fact that for small values of the arguments, the function $\bar{\Upsilon}$ behaves like $y^{1-\frac{2m}{\sigma^2}}$. So if we assume that $\frac{2m}{\sigma^2} > 1$, the only bounded solution in $(0, Y)$ is given by $\Upsilon\left(\frac{\gamma}{k}, \frac{2m}{\sigma^2}; \frac{2ky}{\sigma^2}\right)$. For large arguments, only the function $\bar{\Upsilon}$ stays bounded.

In order to find the constants $A$ and $B$, the fundamental solution has to fulfill certain boundary conditions at $y = Y$:

| | |
|---|---|
| continuity | $w\left(\gamma, Y^-, Y\right) - w\left(\gamma, Y^+, Y\right) = 0$ |
| jump in the derivative | $\left.\frac{\partial w(\gamma,y,Y)}{\partial y}\right\rvert_{y=Y^-} - \left.\frac{\partial w(\gamma,y,Y)}{\partial y}\right\rvert_{y=Y^+} = -1$ |

These two conditions amount to solve the following linear system:

$$\begin{aligned}&A\Upsilon\left(\tfrac{\gamma}{k},\tfrac{2m}{\sigma^2};Y\right)-B\,\bar{\Upsilon}\left(\tfrac{\gamma}{k},\tfrac{2m}{\sigma^2};Y\right)=0\\ &A\left.\frac{\partial\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};y\right)}{\partial y}\right|_{y=Y^-}-B\left.\frac{\partial\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};y\right)}{\partial y}\right|_{y=Y^+}=-1\end{aligned}$$

and we obtain:

$$\begin{aligned}A&=-\frac{\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};Y\right)}{W(Y)}\\ B&=-\frac{\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};Y\right)}{W(Y)}\end{aligned}$$

where $W(Y)$, the Wronskian of the system, is, Lebedev pag. 265 eq. 9.10.10:

$$W(Y)=-\frac{\Gamma\left(\frac{2m}{\sigma^2}\right)}{\Gamma\left(\frac{\gamma}{k}\right)}e^{Y}Y^{-\frac{2m}{\sigma^2}}$$

Finally, we obtain the kernel resolvent of the square root process:

$$\begin{aligned}g(\gamma,s,S)&=w\left(\gamma,\frac{\sigma^2}{2k}s,\frac{\sigma^2}{2k}S\right)\\ &=\begin{cases}-\dfrac{\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2kS}{\sigma^2}\right)\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ks}{\sigma^2}\right)}{W\left(\frac{2kS}{\sigma^2}\right)} & 0<s\le S\\[2ex] -\dfrac{\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2kS}{\sigma^2}\right)\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ks}{\sigma^2}\right)}{W\left(\frac{2kS}{\sigma^2}\right)} & S\le s\end{cases}\end{aligned}\tag{3.28}$$

If we have a single barrier at the level $a$, we can use (3.19) and find the mgf of the hitting times and we obtain the same expression as in Leblanc and Scaillet [64], equation (9), although they consider only the case $0\le s\le a$:

$$h^{+}(s,\gamma;0,a)=\frac{\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ks}{\sigma^2}\right)}{\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ka}{\sigma^2}\right)}1_{(0<s\le a)}$$

and:

$$h^{-}(s,\gamma;a,+\infty)=\frac{\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ks}{\sigma^2}\right)}{\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ka}{\sigma^2}\right)}1_{(a\leq s)}$$

Moreover, we have:

$$\begin{aligned} h_d(S,s,\gamma;0,a) &= g(\gamma,a,S)\,h^{+}(s,\gamma;0,a)\,1_{(s<a)} \\ h_d(S,s,\gamma;a,+\infty) &= g(\gamma,a,S)\,h^{-}\left(s,\gamma;a,+1_{(s<a)}\right)1_{(a<s)} \end{aligned}$$

For the double barrier case, $a<s<b$, we can use equations (3.19) and (3.23) or solve the ODE:

$$\begin{aligned} &\tfrac{1}{2}\sigma^2 s h^{\pm}_{ss}+(m-ks)\,h^{\pm}_{s}-\gamma h^{\pm}=0 \\ &h^{+}(a,\gamma;a,b)=0;\,h^{+}(b,\gamma;a,b)=1 \\ &h^{-}(a,\gamma;a,b)=1;\,h^{-}(b,\gamma;a,b)=0 \end{aligned}$$

The solution of the homogeneous equation is:

$$h^{\pm}(s,\gamma;a,b)=A\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ks}{\sigma^2}\right)+B\,\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ks}{\sigma^2}\right)$$

and choosing $A$ and $B$ in order to satisfy the above boundary conditions, we find:

$$\begin{aligned} h^{+}(x,\gamma;a,b) &= \frac{\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ka}{\sigma^2}\right)\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ks}{\sigma^2}\right)-\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ka}{\sigma^2}\right)\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ks}{\sigma^2}\right)}{\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ka}{\sigma^2}\right)\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2kb}{\sigma^2}\right)-\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2kb}{\sigma^2}\right)\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ka}{\sigma^2}\right)} \\ h^{-}(x,\gamma;a,b) &= \frac{\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2kb}{\sigma^2}\right)\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ks}{\sigma^2}\right)-\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2kb}{\sigma^2}\right)\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ks}{\sigma^2}\right)}{\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ka}{\sigma^2}\right)\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2kb}{\sigma^2}\right)-\Upsilon\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2kb}{\sigma^2}\right)\bar{\Upsilon}\left(\frac{\gamma}{k},\frac{2m}{\sigma^2};\frac{2ka}{\sigma^2}\right)} \end{aligned}$$

With regard to the determination of the function $h_d(S,s,\gamma;a,b)$, it is convenient to use equation (3.25).

### 3.2.3. Ornstein-Uhlenbeck process

In this case the asset returns are described by an Ornstein-Uhlenbeck process:

$$d\ln S_t = (m - k\ln S)\,dt + \sigma dW_t$$

and we fix the barriers at the levels $a < b$. It is convenient to define a new process $X_t = \alpha(\ln S_t - m/k)$, $\alpha^2 = k/\sigma^2$, with dynamics:

$$dx = -kxdt + \sqrt{k}dW_t$$

The resolvent kernel $g(\gamma, s, S)$ of the process $\ln S_t$ is then related to the resolvent kernel $w(\gamma, \alpha(\ln x - m/k), \alpha(\ln x - m/k))$of the new process by:

$$g(\gamma, s, S) = w(\gamma, \alpha(\ln x - m/k), \alpha(\ln x - m/k)) \tag{3.29}$$

The resolvent kernel $w(\gamma, x, X)$is solution of:

$$\begin{aligned} w_{xx} - 2xw_x - 2\frac{\gamma}{k}w &= \delta(x - X) && (3.30)\\ \lim_{x\to-\infty} w(\gamma, x, X) &= 0 && (3.31)\\ \lim_{x\to+\infty} w(\gamma, x, X) &= 0 && (3.32)\end{aligned}$$

We solve the ODE imposing the continuity and the jump condition for the first derivative at $x = X$:

$$\begin{array}{lll} \text{continuity} & w(\gamma, x, X)|_{x=X^-} - w(\gamma, x, X)|_{x=X^+} = 0 & \\ & & (3.33) \\ \text{jump in the derivative} & \left.\frac{\partial w(\gamma,x,X)}{\partial x}\right|_{x=X^-} - \left.\frac{\partial w(\gamma,x,X)}{\partial x}\right|_{x=X^+} = -1 & \end{array}$$

We can recognize in (3.30) the differential equation for the Hermite polynomials, Lebedev pag. 284. The general solution is then given by, Lebedev pag. 286 eq. 10.2.17:

$$w(\gamma, x, X) = \begin{cases} AH_{-\gamma/k}(-x) + BH_{-\gamma/k}(x) & x \leq X \\ \\ CH_{-\gamma/k}(-x) + DH_{-\gamma/k}(x) & x > X \end{cases}$$

where $H_v(z)$ is the Hermite polynomial related to the confluent hypergeometric function by, Lebedev pag. 285 footnote 4:

$$H_v(x) = 2^v \,\bar{\Upsilon}\left(-\frac{v}{2}, \frac{1}{2}; x^2\right)$$

Moreover, $H_v(z)$ has the following integral representation, Lebedev pag. 290, eq. 10.5.2:

$$H_v(x) = \frac{1}{2\Gamma(-v)} \int_0^\infty e^{-s^2-2sx} s^{-v-1} ds$$

Using this integral representation, in order to obtain a bounded solution as $x \to \pm\infty$, we need to set $B = C = 0$. Then using the conditions (3.33), we find:

$$A = \frac{H_{-\gamma/k}(X)}{W(X)}; D = \frac{H_{-\gamma/k}(-X)}{W(X)}$$

where $W(X)$ is the Wronskian of the system and is given by. Lebedev pag. 286 eq. 10.2.11.:

$$W(z) = W(H_v(-z); H_v(z)) = \frac{2^{v+1}\sqrt{\pi}}{\Gamma(-v)} e^{z^2}$$

We find:

$$w(\gamma, x, X) = \begin{cases} \frac{H_{-\gamma/k}(X)H_{-\gamma/k}(-x)}{W(X)} & x \leq X \\ \\ \frac{H_{-\gamma/k}(-X)H_{-\gamma/k}(x)}{W(X)} & x > X \end{cases}$$

Exploiting the relationship between Hermite function and confluent function, we also obtain a different form of the solution:

$$w(\gamma, x, X) = \frac{1}{\sqrt{\pi}} \Gamma\left(\frac{\gamma}{k}\right) e^{-X^2} 2^{-\gamma/k-1} \bar{\Upsilon}\left(\frac{\gamma}{2k}, \frac{1}{2}; X^2\right) \bar{\Upsilon}\left(\frac{\gamma}{2k}, \frac{1}{2}; x^2\right) \tag{3.34}$$

In the case of only one barrier at the level $\hat{b} = \alpha(b - m/k)$, using (3.19) we have:

$$h^+\left(x, \gamma; -\infty, \hat{b}\right) = \frac{H_{-\gamma/k}(-x)}{H_{-\gamma/k}\left(-\hat{b}\right)} 1_{\left(x<\hat{b}\right)} \tag{3.35}$$

$$h^-\left(x, \gamma; \hat{b}, +\infty\right) = \frac{H_{-\gamma/k}(x)}{H_{-\gamma/k}\left(\hat{b}\right)} 1_{\left(x>\hat{b}\right)} \tag{3.36}$$

or, using the relationship with the confluent hypergeometric function:

$$h^+\left(x, \gamma; -\infty, \hat{b}\right) = h^-\left(x, \gamma; \hat{b}, +\infty\right) = \frac{\bar{\Upsilon}\left(\frac{\gamma}{2k}, \frac{1}{2}; \frac{kx^2}{\sigma^2}\right)}{\bar{\Upsilon}\left(\frac{\gamma}{2k}, \frac{1}{2}; \frac{k\hat{b}^2}{\sigma^2}\right)}$$

i.e. the expression in Leblanc and Scaillet [64].

In the case of two barriers at the levels $\hat{a}=\alpha\left(a-m/k\right)$, and $\hat{b}=\alpha\left(b-m/k\right)$, we find:

$$h^{+}\left(x,\gamma;\hat{a},\hat{b}\right)=\frac{H_{-\gamma/k}\left(\hat{b}\right)H_{-\gamma/k}(-x)-H_{-\gamma/k}\left(-\hat{b}\right)H_{-\gamma/k}(x)}{H_{-\gamma/k}\left(-\hat{a}\right)H_{-\gamma/k}\left(\hat{b}\right)-H_{-\gamma/k}(a)H_{-\gamma/k}\left(-\hat{b}\right)}$$
$$h^{+}\left(x,\gamma;\hat{a},\hat{b}\right)=\frac{H_{-\gamma/k}(x)H_{-\gamma/k}\left(-\hat{a}\right)-H_{-\gamma/k}(-x)H_{-\gamma/k}\left(\hat{a}\right)}{H_{-\gamma/k}\left(-\hat{a}\right)H_{-\gamma/k}\left(\hat{b}\right)-H_{-\gamma/k}\left(\hat{a}\right)H_{-\gamma/k}\left(-\hat{b}\right)} \tag{3.37}$$

Expression (3.35) is the same as in Borodin and Salminen [9], pag. 429, eq. 2.0.1. where the Laplace transform is expressed in terms of the Weber function $D_v\left(x\right)$, related to the Hermite function, Lebedev [63] pag. 284 footnote 3:

$$D_v\left(x\right)=2^{-v/2}e^{-z^2/4}H_v\left(\frac{z}{\sqrt{2}}\right)$$

Expression (3.37) can be found in Borodin and Salminen [9], pag. 434, eq. 3.0.1. where the Laplace transform is expressed in terms of a two parameter function $S\left(v,x,y\right)$ connected to the Weber function $D_v\left(x\right)$, $v>0$, by the relationship:

$$S\left(v,x,y\right)=\frac{\Gamma\left(v\right)}{\pi}e^{\left(x^2+y^2\right)/4}\left(D_{-v}\left(-x\right)D_{-v}\left(y\right)-D_{-v}\left(x\right)D_{-v}\left(-y\right)\right)$$

The function $h_d\left(X,x,\gamma;\hat{a},\hat{b}\right)$, with eventually $\hat{a}=-\infty$ or $\hat{b}=+\infty$, is obtained combining (3.35) and (3.37) with (3.34), using (3.19) and (3.25).

Expressions (3.35) and (3.37) can be also obtained solving the appropriate ODE's. In the case of one barrier:

$$\begin{aligned}&h_{xx}-2xh_x-2\tfrac{\gamma}{k}h=0\\&h\left(\hat{b},\gamma\right)=1\\&h\left(x,\gamma\right)=0\text{ as }x\to+\infty\text{ if }x>\hat{b}\\&h\left(x,\gamma\right)=0\text{ as }x\to-\infty\text{ if }x<\hat{b}\end{aligned} \tag{3.38}$$

and using as general solution $AH_{-\gamma/k}\left(-x\right)+BH_{-\gamma/k}\left(x\right)$ and then imposing the boundary conditions, we find (3.35). Similarly for the two barriers case.

### 3.2.4. Cev process

This model has been introduced by Cox and Ross [21] and analysed in more detail in Cox [19]. The idea is to construct a model where the variance of the rate of return varies inversely with the stock price, a feature that Black noted as characterising the stock prices (leverage effect). The dynamics for the asset price assuming a CEV process is given by:

$$dS = rSdt + \sigma S^{\lambda/2} dW_t$$

where $0 \leq \lambda < 2$. In the limiting case where $\lambda = 2$, we obtain the lognormal models. Moreover, if $\lambda \neq 0$, the process has an absorbing barrier at zero.

In order to obtain a solution to the option pricing problem, it is convenient to consider the following transformation:

$$Y = S^{2-\lambda} \tag{3.39}$$

and then using the Ito's lemma it is easy to show that $Y$ follows a process with instantaneous mean

$$r(2-\lambda)y + \frac{1}{2}\sigma^2\left(\lambda^2 - 3\lambda + 2\right)$$

and instantaneous variance

$$\sigma^2(2-\lambda)^2 y.$$

Note that the drift and the variance are linear in $y$, so that this process is given by the square root process seen in the previous section and then we can easily obtain the Laplace transform of the hitting time changing the barrier level from $a$ and $b$ to $a^{2-\lambda}$ and $b^{2-\lambda}$ respectively.

### 3.3. Example 3: Asian options

In this case the problem is the pricing of a payoff:

$$\left(\frac{1}{T}\int_0^T S(u)\,du - K\right)^+$$

for the fixed strike Asian option, and of the payoff:

$$\left(\frac{1}{T}\int_0^T S(u)\,du - S_T\right)^+$$

for the floating strike Asian option. Let us consider the fixed strike case. The other case can be treated in the same manner.

#### 3.3.1. The Geman-Yor approach

The first possibility is to follow the approach in Geman Yor [41]: with advanced probabilistic techniques and exploiting the representation of the Geometric Brownian motion in terms of a random-time changed Bessel process they are able to obtain the Laplace transform of the price of the Asian options. Indeed Geman and Yor show that the Asian option price can be expressed as:

$$c_{gy} = \frac{e^{-r(T-t)}}{T}\frac{4S}{\sigma^2}c(h,q) \tag{3.40}$$

and they are able to obtain the Laplace transform wrt $h = \sigma^2(T-t)/4$ of the function $c(h,q)$:

$$\begin{aligned} C(\gamma,q) &= \int_0^{+\infty} e^{-\gamma h} c(h,q)\,dh \\ &= \frac{\int_0^{1/(2q)} e^{-x} x^{\frac{1}{2}(\mu-v)-2}(1-2qx)^{\frac{1}{2}(\mu+v)+1}dx}{\gamma(\gamma-2-2v)\Gamma\left(\frac{1}{2}(\mu-v)-1\right)} \\ &= \frac{\int_0^1 e^{-\frac{w}{2q}} w^{\frac{1}{2}(\mu-v)-2}(1-w)^{\frac{1}{2}(\mu+v)+1}dw}{(2q)^{\frac{1}{2}(\mu-v)-1}\Gamma\left(\frac{1}{2}(\mu-v)-1\right)\gamma(\gamma-2-2v)} \end{aligned} \tag{3.41}$$

where:

$$\begin{aligned} v &= \frac{2r}{\sigma^2} - 1; q = \frac{\sigma^2}{4S}\left(K * T - t * a\right) \\ \mu &= \sqrt{2\gamma + v^2} \end{aligned} \tag{3.42}$$

Instead of discussing their original approach, we report here a simple derivation found in Yor [100], pag. 82. The derivation is based on a remarkable expression of the law of the arithmetic average taken at an independent exponential time.

Assuming for simplicity that $t = 0$ and $a = 0$ in (3.42), we define:

$$D_h = \int_0^h e^{2(W_s + vs)} ds$$

Yor shows that $D_h$ taken at an exponentially distributed random time $T_\gamma$, independent of $W$, with mean $1/\gamma$, is characterized by:

$$2D_{T_\gamma} \stackrel{\mathcal{L}}{=} \frac{B_{1,\alpha}}{G_\beta}$$

where $\alpha = (v + \mu)/2$, $\beta = \alpha - v$, and the random variable $B_{1,\alpha} \sim Beta\,(1, \alpha)$ and $G_\beta \sim Gamma\,(\beta, 1)$ are independent.

For pricing Asian options we are interested in the following quantity:

$$A_t = \int_0^t e^{\sigma W_s + \left(r - \frac{\sigma^2}{2}\right)s} ds$$

that is related to the expression of $D_h$ by[4]:

$$A_t = \frac{4}{\sigma^2} D_{\frac{\sigma^2 t}{4}}$$

[4]The relationship is due to the time-scaling property of the Brownian motion.

In particular we need to compute:

$$\begin{aligned} e^{-rT}E_0\left(\tfrac{S_0}{T}A_T-K\right)^+ &= e^{-rT}E_0\left(\tfrac{4}{\sigma^2}\tfrac{S_0}{T}D_{\frac{\sigma^2T}{4}}-K\right)^+ \\ &= e^{-rT}\tfrac{4}{\sigma^2}\tfrac{S_0}{T}E_0\left(D_{\frac{\sigma^2T}{4}}-\tfrac{\sigma^2T}{4S_0}K\right)^+ \\ &= e^{-rT}\tfrac{4}{\sigma^2}\tfrac{S_0}{T}E_0\left(D_h-q\right)^+ \end{aligned}$$

and then if we compute the Laplace transform with respect to $h$ of the above quantity, exploiting the relationship between Laplace transform and expectation wrt to an exponential distributed random variable, we obtain:

$$\int_0^{+\infty} e^{-\gamma h}E\left(D_h-q\right)^+dh = \frac{1}{2\gamma}E\left(2D_{T_\gamma}-2q\right)^+$$

Now using the above result due to Yor, we have:

$$\begin{aligned} E\left(2D_{T_\gamma}-2q\right)^+ &= E\left(\tfrac{B_{1,\alpha}}{G_\beta}-2q\right)^+ \\ &= \int_0^1\int_0^{+\infty}\left(\tfrac{u}{x}-2q\right)^+\tfrac{x^{\beta-1}}{\Gamma(\beta)}e^{-x}dx\alpha\left(1-u\right)^{\alpha-1}du \\ &= \int_0^1\int_0^{u/2q}\left(\tfrac{u}{x}-2q\right)\tfrac{x^{\beta-1}}{\Gamma(\beta)}e^{-x}dx\alpha\left(1-u\right)^{\alpha-1}du \\ &= \int_0^{1/2q}\tfrac{x^{\beta-1}}{\Gamma(\beta)}e^{-x}dx\int_{2qx}^1\left(\tfrac{u}{x}-2q\right)\alpha\left(1-u\right)^{\alpha-1}du \\ &= \tfrac{1}{1+\alpha}\int_0^{1/2q}\tfrac{x^{\beta-2}}{\Gamma(\beta)}e^{-x}\left(1-2qx\right)^{\alpha+1}dx \end{aligned}$$

where we have used:

$$\int_{2qx}^1\left(\frac{u}{x}-2q\right)\alpha\left(1-u\right)^{\alpha-1}du = \frac{\left(1-2qx\right)^{\alpha+1}}{\left(1+\alpha\right)x}$$

So using $\alpha = (\mu+v)/2$, $\beta = \alpha - v = (\mu-v)/2$, and $\Gamma(\beta) = (\beta-1)\Gamma(\beta-1)$ and

$(1+\alpha)(\beta-1) = (\gamma - 2 - 2v)/2$ we finally obtain

$$\begin{aligned}
&\int_0^{+\infty} e^{-\gamma h} E\left(D_h - q\right)^+ dh \\
&= \tfrac{1}{2\gamma} \tfrac{1}{(1+\alpha)(\beta-1)} \int_0^{1/2q} \tfrac{x^{\beta-2}}{\Gamma(\beta-1)} e^{-x} \left(1 - 2qx\right)^{\alpha+1} dx \\
&= \tfrac{1}{\gamma(\gamma-2-4v)i.} \int_0^{1/2q} \tfrac{x^{(\mu-v)/2-2}}{\Gamma((\mu-v)/2-1)} e^{-x} \left(1 - 2qx\right)^{(\mu+v)/2+1} dx
\end{aligned}$$

i.e. the expression (3.40). Obviously, this derivation is strongly based on the representation of the quantity $D_{T_\gamma}$ as ratio of two independent r.v.'s.

**A long-term fixed-strike Asian option** We now investigate the behavior of the quantity

$$A_t = \int_0^t e^{\sigma W_s + \left(r - \frac{\sigma^2}{2}\right)s} ds$$

as $t$ tends to infinity. We use the expression in (3.41) and the Abelian result that relates the behavior of the function in the $h$-domain as $h \to \infty$ to the behavior of its Laplace transform as $\gamma \to 0$. We do not take into account the discounting factor, otherwise the price of the Asian option tends to 0. We have:

$$\begin{aligned}
\lim_{h\to\infty} c(h,q) &= \lim_{\gamma\to 0} \gamma \frac{\int_0^{1/(2q)} e^{-x} x^{\frac{1}{2}(\mu-v)-2} \left(1-2qx\right)^{\frac{1}{2}(\mu+v)+1} dx}{\gamma\left(\gamma - 2 - 2v\right)\Gamma\left(\frac{1}{2}(\mu - v) - 1\right)} \\
&= \lim_{\gamma\to 0} \frac{\int_0^{1/(2q)} e^{-x} x^{\frac{1}{2}(\mu-v)-2} \left(1-2qx\right)^{\frac{1}{2}(\mu+v)+1} dx}{-2\left(1+v\right)\Gamma\left(\frac{1}{2}(\mu - v) - 1\right)} \\
&= \lim_{\gamma\to 0} \frac{\int_0^{1/(2q)} e^{-x} x^{\frac{1}{2}(\mu-v)-2} \left(1-2qx\right)^{\frac{1}{2}(\mu+v)+1} dx}{-2\left(1+v\right)\Gamma\left(\frac{1}{2}(\mu - v)\right) / \left(\frac{1}{2}(\mu - v) - 1\right)}
\end{aligned}$$

where we have used the fact that $\Gamma(z-1) = \Gamma(z)/(z-1)$. Note that

$$\lim_{\gamma\to 0} \mu = |v|$$

and in order to chose the sign, we require the argument of the Gamma function to be positive. This implies that $|v| = -v$ and this can happen only when $r < \sigma^2/2$. If $r > \sigma^2/2$ the limit does not exist. Then we have:

$$\lim_{h\to\infty} c(h,q) = \frac{\int_0^{1/(2q)} e^{-x} x^{-v-2} (1-2qx)\,dx}{2\Gamma(-v)}$$

If we operate the change of variable $z = 2/(\sigma^2 x)$, $x$=$2/(\sigma^2 z)$, the price of the Asian option becomes:

$$\begin{aligned}
&\frac{4S}{\sigma^2 T}\frac{\int_{4q/\sigma^2}^{+\infty} e^{-2/(\sigma^2 z)} \left(\frac{2}{\sigma^2 z}\right)^{-v-2} \left(1-2q\frac{2}{\sigma^2 z}\right)\left(\frac{2}{\sigma^2 z^2}\right) dz}{2\Gamma(-v)} \\
=\ & \frac{4S}{\sigma^2 T}\left(\frac{2}{\sigma^2}\right)^{-v-1} \frac{\int_{4q/\sigma^2}^{+\infty} e^{-2/(\sigma^2 z)} (z)^{v-1} \left(z-\frac{4q}{\sigma^2}\right) dz}{2\Gamma(-v)} \\
=\ & \left(\frac{2}{\sigma^2}\right)^{-v} \frac{\int_{KT/S}^{+\infty} e^{-2/(\sigma^2 z)} z^{v-1} \left(\frac{S}{T}z - K\right) dz}{\Gamma(-v)} \\
=\ & \left(\frac{2}{\sigma^2}\right)^{-v} \frac{\int_0^{+\infty} e^{-2/(\sigma^2 z)} z^{v-1} \left(\frac{S}{T}z - K\right)^+ dz}{\Gamma(-v)}
\end{aligned}$$

So we have found that when $r < \sigma^2/2$, the density law of the quantity

$$A_\infty = \lim_{t\to\infty} A_t = \lim_{h\to\infty} \frac{4}{\sigma^2} D_h$$

is given by:

$$\Pr\left(A_\infty \in (a, a+da)\right) = \left(\frac{2}{\sigma^2}\right)^{-v} \frac{e^{-2/(\sigma^2 a)} a^{v-1}}{\Gamma(-v)}$$

Then the stationary density for $A_\infty$ is given by a reciprocal gamma density, i.e. $1/A_\infty$ has a gamma density, as it is easy to verify. The same result, using different techniques, have been also obtained by Dufresne [32], De Schepper et al. [27], Yor [99], Milevsky and Posner [73].

### 3.3.2. The PDE approach

The solution given in Geman and Yor can be obtained using a PDE approach and exploiting a self-similarity property of the solution, Lipton [67]. Indeed, for the two payoff structures, instead of considering directly the degenerate PDE with two state variables (asset price and average), Ingersoll [53] for the floating-strike case and Rogers and Shi ([82]) for the fixed strike case find a self-similar solution that allows to represent the price of the Asian call in terms of a new Markovian state variable.

**Fixed strike Asian option.** The PDE to be solved is given by:

$$C_t + \frac{1}{2}\sigma^2 S^2 C_{ss} + rSC_s + SC_A - rC = 0 \tag{3.43}$$

with terminal condition:

$$C(S, t = T) = \left(\frac{1}{T}\int_0^t S(u)\,du - K\right)^+$$

If $\int_0^t S(u)\,du > KT$, the option will be surely exercised at the maturity, so the current price will be given by:

$$\begin{aligned} C(S, T-t) &= e^{-r(T-t)} E_{t,S}\left[\frac{1}{T}\left(\int_t^T S(u)\,du + \int_0^t S(u)\,du\right) - K\right] \\ &= S\frac{1-e^{-r(T-t)}}{Tr} + \left(\frac{1}{T}\int_0^t S(u)\,du - K\right)e^{-r(T-t)} \\ &= S\left(\frac{1-e^{-r(T-t)}}{Tr} + \left(\frac{1}{T}\int_0^t S(u)\,du - K\right)\frac{e^{-r(T-t)}}{S}\right) \\ &= S\left(\frac{1-e^{-r(T-t)}}{Tr} + ye^{-r(T-t)}\right) \end{aligned} \tag{3.44}$$

It is useful to try the following representation:

$$C(S, T-t) = S\Phi(Y, \tau)$$

where $\tau = \sigma^2 (T-t)$ and where $Y$ is the new Markovian state variable and is given by:

$$Y(t) = \frac{\frac{1}{T}\int_0^t S(u)\,du - K}{S(t)}$$

In this way the two state variables, asset price and current average, are synthesized in a single state variable. This transformation is related to a change of numeraire: instead of considering the money account as numeraire, in this context it is convenient to use the asset price as numeraire. The prices of all contracts expressed in terms of the price of the underlying asset are martingales.

Substituting the expression for $C(S, T-t)$ in the Black-Scholes PDE, it can be shown that $\Phi(Y, \tau)$ satisfies the parabolic PDE:

$$\begin{aligned}
&\Phi_\tau + \left(\hat{r}\, y - 1/\hat{T}\right)\Phi_y - \tfrac{1}{2}y^2\Phi_{yy} = 0 \\
&-\infty < y < \infty \\
&\Phi(y, 0) = y^+ \\
&\Phi(y, \tau) \to 0 \text{ when } y \to -\infty \\
&\Phi(y, \tau) \to e^{-\hat{r}\tau} \text{ when } y \to +\infty
\end{aligned} \tag{3.45}$$

where $\hat{r} = r/\sigma^2$ and $\hat{T} = T * \sigma^2$. The PDE has to be solved on the entire real axis, but it is easy to realize that if $y \geq 0$, i.e. $\int_0^t S(u)\,du > KT$ the Asian option will be exercised and the price $C$ is given by (3.44) and then the function $\Phi(y,\tau)$ becomes:

$$\Phi(y,\tau) = \left( e^{-\hat{r}\tau} y + \frac{1 - e^{-\hat{r}\tau}}{\hat{r}\hat{T}} \right) 1_{y \geq 0}$$

In order to solve the PDE in the case $y < 0$, we can use the Laplace transform wrt $\tau = \sigma^2 (T - t)$:

$$\Psi(y,\gamma) = \int_0^{+\infty} e^{-\gamma\tau} \Phi(y,\tau)\, d\tau,$$

and we obtain the following second-order differential equation when $-\infty < y < 0$:

$$\begin{aligned}
&\tfrac{1}{2} y^2 \Psi_{yy} + \left( 1/\hat{T} - \hat{r}\, y \right) \Psi_y - \left( \gamma + \hat{r} \right) \Psi = 0 \\
&\Psi(y,\gamma) \to 0 \text{ as } y \to -\infty \qquad\qquad (3.46) \\
&\Psi(y,\gamma) \to \mathcal{L}(\Phi(0,\tau)) = \frac{1}{\left(\gamma + \hat{r}\right)\gamma\hat{T}} \text{ as } y \to 0^-
\end{aligned}$$

The solution of this ODE is found in Appendix A and C and is given by:

$$\Psi(y,\gamma) = \frac{\Gamma\left(2\left(\hat{r} - b + 1\right) + b\right)\left(\frac{-y}{2}\right)^b}{\Gamma\left(2\left(\hat{r} - b + 1\right)\right)\left(\gamma + \hat{r}\right)\gamma\left(\hat{T}\right)^{1-b}} \Upsilon\left(-b, 2\left(\hat{r} - b + 1\right); \frac{2}{\hat{T}\, y}\right) \qquad (3.47)$$

where $\Upsilon(\alpha,\beta;z)$ is the confluent hypergeometric function and $b = 1/2 + \hat{r} - \sqrt{2\gamma + \left(1/2 + \hat{r}\right)^2}$

In order to find the same expression as found in Geman and Yor, equation (3.41) above, we can exploit the integral representation for the function $\Upsilon(\alpha,\beta;x)$, Lebedev pag. 266 eq. 9.11.1:

$$\Upsilon(\alpha,\beta;x) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)\Gamma(\alpha)} \int_0^1 e^{xh} h^{\alpha-1} (1-h)^{\beta-\alpha-1}\, dh$$

and to take into account that in Geman-Yor the Laplace transform is wrt $\sigma^2 (T-t)/4$, whilst here is wrt $\sigma^2 (T-t)$. Then the two results can be related by the scaling property of the Laplace transform. Moreover, the quantity $q$ in GY is related to the quantity $y$ in Lipton through the relationship $y = -4q/\left(\sigma^2 T\right)$, and $\hat{r} = (v+1)/2$.

**Floating strike Asian options** In this case the payoff is given by:

$$\left(\int_t^T S(u)\,du - S_T\right)^+$$

and no result is currently available. We are able to find the Laplace transform of the price.

Following Ingersoll, pag. 377, we define the variable $Y(t)$:

$$Y(t) = \frac{S(t)}{\int_0^t S(u)\,du}$$

and because of the linear homogeneity, we can write the price of the floating strike Asian option as:

$$C\left(S, \int_0^t S(u)\,du, T-t\right) = \left(\int_0^t S(u)\,du\right)\Phi(y,\tau)$$

where $\tau = T - t$ and where $\Phi(y,\tau)$solves the following PDE for $y > 0$:

$$-\Phi_\tau + \tfrac{1}{2}\sigma^2 y^2 \Phi_{yy} + \left(ry - y^2\right)\Phi_y + (y - r)\Phi = 0; 0 < y$$

$$\Phi(0,\tau) = 0$$

$$\Phi_y(\infty,\tau) = 1$$

$$\Phi(y,0) = \left(y - \tfrac{1}{T}\right)^+$$

Ingersoll does not solve the PDE. Instead if we consider the Laplace transform wrt $\tau = T - t$:

$$\Psi(y,\gamma) = \int_0^{+\infty} e^{-\gamma\tau}\Phi(y,\tau)\,d\tau$$

we obtain the following second order differential equation for the function $\Psi(y,\gamma)$ :

$$
\begin{aligned}
&-\left(\gamma\Psi-\left(y-\tfrac{1}{T}\right)^{+}\right)+\tfrac{1}{2}\sigma^2y^2\Psi_{yy}+\left(ry-y^2\right)\Psi_y+\left(y-r\right)\Psi=0\\
&\Psi(y,\gamma)\to 0 \text{ when } y\to 0\\
&\Psi_y(\infty,\gamma)\to \tfrac{1}{\gamma}\text{when } y\to\infty
\end{aligned}
$$

when $y>0$. We define a new function $\Psi^*(x,\gamma)$:

$$\Psi(y,\gamma)=y^b\Psi^*(ay,\gamma)$$

and we have:

$$
\begin{aligned}
\Psi_y &= by^{b-1}\Psi^*+ay_x^b\Psi^*\\
\Psi_{yy} &= b(b-1)\,y^{b-2}\Psi^*+2aby^{b-1}\Psi_x^*+a^2y^b\Psi_{xx}^*
\end{aligned}
$$

and then:

$$
\begin{aligned}
&\tfrac{1}{2}\sigma^2a^2y^{b+2}\Psi_{xx}^*+\left(a\left(\sigma^2b+r\right)y^{b+1}-ay^{b+2}\right)\Psi_x^*\\
&+\left(\left(-\gamma+\tfrac{1}{2}\sigma^2b(b-1)+rb-r\right)y^b+(1-b)\,y^{b+1}\right)\Psi^*+\left(y-\tfrac{1}{T}\right)^{+}=0
\end{aligned}
$$

If we choose $b$ such that:

$$\frac{1}{2}\sigma^2b^2+\left(r-\frac{1}{2}\sigma^2\right)b-(r+\gamma)=0$$

we obtain the two roots:

$$
\begin{aligned}
b_1 &= \frac{-\left(r-\frac{1}{2}\sigma^2\right)+\sqrt{2\gamma+\left(r+\frac{1}{2}\sigma^2\right)^2}}{\sigma^2}\\
b_2 &= \frac{-\left(r-\frac{1}{2}\sigma^2\right)-\sqrt{2\gamma+\left(r+\frac{1}{2}\sigma^2\right)^2}}{\sigma^2}
\end{aligned}
\tag{3.48}
$$

We will see later on that we have to set $b = b_1$. After dividing by $\frac{1}{2}\sigma^2 a y^{b+1}$, and using the fact that $x = ay$, with $a = 2/\sigma^2$, we obtain:

$$x\Psi^*_{xx} + (\alpha - x)\,\Psi^*_x + \beta\Psi^* + \left(\frac{x}{a}\right)^{-(b+1)}\left(\frac{x}{a} - \frac{1}{T}\right)^+ = 0. \tag{3.49}$$

where we have set:

$$\alpha = a\left(\sigma^2 b + r\right); \beta = 1 - b$$

The solution of the homogenous equation is described in Appendix A and can be expressed in terms of hypergeometric functions:

$$\Psi^*(x, \gamma) = A\Upsilon(-\beta, \alpha; x) + B\,\bar{\Upsilon}(-\beta, \alpha; x)$$

where $\bar{\Upsilon}(,;x)$ is the confluent function of the second kind, Lebedev [63], pag. 263 eq.9.10.3. $\bar{\Upsilon}(,;x)$ admits the following integral representation, Lebedev [63], pag. 268 eq.9.11.6, when $\mathrm{Re}(-\beta) > 0, \mathrm{Re}(x) > 0$:

$$\bar{\Upsilon}(-\beta, \alpha; x) = \frac{1}{\Gamma(-\beta)}\int_0^\infty e^{-x\xi}\xi^{-\beta-1}(1+\xi)^{\alpha+\beta-1}\,d\xi$$

In order that $\mathrm{Re}(-\beta) > 0$, we need $b - 1 > 0$ and this implies that we have to choose the positive root in (3.48).

The general solution of equation can be found using the method of variation of parameters, and using the expression of the Wronskian of the functions $\Upsilon(-\beta, \alpha; x)$ and $\bar{\Upsilon}(-\beta, \alpha; x)$ given in Lebedev, pag. 265 eq. 9.10.10:

$$W(x) \equiv W\left(\Upsilon(-\beta, \alpha; x), \bar{\Upsilon}(-\beta, \alpha; x)\right) = -\frac{\Gamma(\alpha)}{\Gamma(-\beta)}x^{-\alpha}e^x$$

and we obtain:

$$\Psi^*(x,\gamma) = A\Upsilon(-\beta,\alpha;x) + B\,\bar{\Upsilon}(-\beta,\alpha;x) + \int_0^\infty G(x,s;\gamma)\left(\frac{s}{a}\right)^{-(b+1)}\left(\frac{s}{a}-\frac{1}{T}\right)^+ ds$$

where $G(x,s;\gamma)$(the Green's function associated to equation (3.49)) is given by:

$$\begin{aligned} G(x,s;\gamma) &= \begin{cases} g_1(x,s;\gamma) & 0 \le x \le s < +\infty \\ \\ g_2(x,s;\gamma) & 0 \le s \le x < +\infty \end{cases} \\ &= \begin{cases} -\frac{\Upsilon(-\beta,\alpha;x)\bar{\Upsilon}(-\beta,\alpha;s)}{sW(s)} & 0 \le x \le s < +\infty \\ \\ -\frac{\Upsilon(-\beta,\alpha;s)\bar{\Upsilon}(-\beta,\alpha;x)}{sW(s)} & 0 \le s \le x < +\infty \end{cases} \end{aligned}$$

and the constants $A$ and $B$ have to be chosen in order to satisfy the boundary conditions for the function $\Psi(y,\gamma)$. In particular we can use $\Psi(y,\gamma) = y^b\Psi^*(ay,\gamma)$ and the integral representation of the function $\Upsilon$ for large $|y|$, compare Appendix A:

$$\Upsilon(\beta,\alpha;x) = \frac{\Gamma(\alpha)}{\Gamma(-\beta)}e^x x^{-(\alpha+\beta)}\left[1 + O\left(|x|^{-n-1}\right)\right]$$

We have:

$$y^b\Upsilon(-\beta,\alpha;ay) \approx Cy^{b-(\alpha+\beta)}e^{ay}$$

and in order to have a bounded derivative we need to set $A = 0$. For large $|y|$, we have

$$y^b\,\bar{\Upsilon}(-\beta,\alpha;ay) \approx y^b y^{1-b} \approx y$$

and the first derivative is bounded as desired

For small values of $y$, $y^b\Upsilon(-\beta,\alpha;ay)$ tends to 0, whilst $y^b\,\bar{\Upsilon}(-\beta,\alpha;ay)$, $b = b_1$, behaves like:

$$y^b y^{1-\alpha} = y^{b+1-\alpha} = y^{1-b-\frac{2}{\sigma^2}r} = y^{\beta-\frac{2}{\sigma^2}r},$$

but we have required Re $(-\beta) > 0$ so that the exponent of $y$ is negative and $y^b\ \bar{\Upsilon}\ (-\beta, \alpha; ay)$ is unbounded as $y \longrightarrow 0$. So we require $B = 0$.

The expression for the Laplace transform simplifies in:

$$\begin{aligned}
\Psi(y,\gamma) &= \int_{a/T}^{\infty} G(ay,s;\gamma)\left(\left(\frac{s}{a}\right)^{-b} - \left(\frac{s}{a}\right)^{-(b+1)}\frac{1}{T}\right) ds \\
&= \begin{cases} \int_{a/T}^{\infty} g_1(ay,s;\gamma)\left(\left(\frac{s}{a}\right)^{-b} - \left(\frac{s}{a}\right)^{-(b+1)}\frac{1}{T}\right) ds & y < \frac{1}{T} \\ \\ \int_{a/T}^{ay} g_2(ay,s;\gamma)\left(\left(\frac{s}{a}\right)^{-b} - \left(\frac{s}{a}\right)^{-(b+1)}\frac{1}{T}\right) ds + & \\ \int_{ay}^{\infty} g_1(ay,s;\gamma)\left(\left(\frac{s}{a}\right)^{-b} - \left(\frac{s}{a}\right)^{-(b+1)}\frac{1}{T}\right) ds & \frac{1}{T} < y \end{cases}
\end{aligned}$$

Unfortunately, in this case the computation of the Laplace transform can be time consuming and so it is preferable to use different methods, such as numerical solution of the PDE or Monte Carlo simulation.

### 3.4. Example 4: Interest rate models with a square root process

Let us now consider the square root process for the instantaneous interest rate:

$$dr_t = (a - br_t)\, dt + \sigma\sqrt{r} dW_t \tag{3.50}$$

and where we assume that the parameters $a$, $b$ and $\sigma$ are non-negative. This process has been introduced by Feller [35] that studied the existence and uniqueness of the solution, compare also Lamberton and Lapeyre [62] pag. 128, and in Ikeda and Watanabe [51], pag. 221. Using the Fokker-Planck Kolmogorov equation for the transition density, Feller showed that if $2a/\sigma^2 \geq 1$, $r(t)$ always remains positive, and if $2a/\sigma^2 < 1$, it can reach zero but will never become negative. Shiga and Watanabe [87] showed that the law of the process when $4a/\sigma^2$ is a positive integer is that of the squared norm of the Ornstein-Uhlenbeck process. Maghsoodi [72] showed that the above process is a.s. equivalent to a time changed lognormal process. Cox, Ingersoll and Ross [20] used this process for the description of the dynamics of term structure of interest rates.

For the present model, it is more convenient to look for the moment generating function (Laplace transform) of $r$ and not the Laplace transform with respect to time. Indeed, this convenience is due to the linearity of the drift and variance coefficients that give rise to a particularly simple form of the mgf. Feller obtained an expression for the mgf applying the Laplace transform wrt $r$ to the forward Kolmogorov equation. But it is more convenient to follow a procedure based on an idea in Cox, Ingersoll and Ross, compare also Lamberton and Lapeyre [62].

Let us consider the following quantity:

$$v(\tau, r; \gamma, \mu) := v(\tau, r) = E_{0,r}\left[\exp\left(-\gamma r_\tau - \mu \int_0^\tau r(s)\, ds\right)\right] \tag{3.51}$$

and then by the Feynman-Kac equation the unique solution $v(\tau, r)$ satisfies the following PDE:

$$-v_\tau + (a - br)\, v_r + \tfrac{1}{2}\sigma^2 r v_{rr} = \mu r v$$

$$v(0, r) = \exp(-\gamma r)$$

In particular if we set $\mu = 0$ in the above PDE, we obtain the Laplace transform of $r_\tau$; if we set $\gamma = 0$ we obtain the Laplace transform of the quantity $\int_0^\tau r(s)\, ds$; if $\mu$ and $\gamma$ are both different from 0 we obtain the Laplace transform of the joint density of $\left(r_\tau, \int_0^\tau r(s)\, ds\right)$.

In solving the above PDE, we exploit the linearity of the drift and variance coefficients and try a solution of the type:

$$v(\tau, r) = \exp\left(-a\phi(\tau;\gamma,\mu) - r\psi(\tau;\gamma,\mu)\right) \tag{3.52}$$

and substituting this expression in the PDE we obtain, compare Appendix D:

$$\psi(\tau;\gamma,\mu) = \frac{\gamma\left(\lambda + b + (\lambda - b)e^{\tau\lambda}\right) + 2\mu\left(e^{\tau\lambda} - 1\right)}{\sigma^2\gamma\left(e^{\tau\lambda} - 1\right) + \lambda - b + (b + \lambda)e^{\tau\lambda}} \tag{3.53}$$

and:

$$\phi(\tau;\gamma,\mu) = -\frac{2}{\sigma^2}\ln\frac{2\lambda e^{(\lambda+b)\tau/2}}{\left(\gamma\sigma^2\left(e^{\tau\lambda} - 1\right) - b + \lambda + (b + \lambda)e^{\tau\lambda}\right)} \tag{3.54}$$

where:

$$\lambda = \sqrt{b^2 + 2\sigma^2\mu}$$

In the following we derive: a) the transition density law for the instantaneous rate $r_t$, b) the price of a zcb, c) the Laplace transform wrt to the strike of the price of an option on a zcb, d) the Laplace transform wrt to the strike of the price of an option on a spot rate, e) the Laplace transform wrt to the strike of the price of an option on the difference between spot rates (spread option), e) the Laplace transform wrt the strike of the price of an Asian option on the interest rate, f) Asian options on the spread of spot rates, g) we describe other possible applications. In deriving the expression for the Laplace transforms, it will be very useful to exploit the fact that:

$$\begin{aligned}
\frac{\partial^n v(\tau, r;\gamma,\mu)}{\partial\gamma^n} &= (-1)^n E_{0,r}\left(r_\tau^n e^{-\gamma r_\tau - \mu y_\tau}\right) \\
\frac{\partial^n v(\tau, r;\gamma,\mu)}{\partial\mu^n} &= (-1)^n E_{0,r}\left(y_\tau^n e^{-\gamma r_\tau - \mu y_\tau}\right)
\end{aligned}$$

We remark that all formulas we will obtain below are applicable also to the case of the

Ornstein-Uhlenbeck process and we have to change only the expression for the functions $\psi$ and $\phi$.

### 3.4.1. The transition density law of the square root process

If we set $\mu = 0$, we obtain the Laplace transform (moment generating function) of $r_\tau$ :

$$E_{0,r}e^{-\gamma r_\tau} = \exp\left(-a\phi\left(\tau;\gamma,0\right) - r\psi\left(\tau;\gamma,0\right)\right)$$

where, using the fact that $\lambda = b > 0$, for the hypothesis on $b$:

$$\begin{aligned}
\psi\left(\tau;\gamma,0\right) &= \frac{\gamma b\exp\left(-\tau b\right)}{\left(1-\exp\left(-\tau b\right)\right)\gamma\sigma^2/2+b} \\
\phi\left(\tau;\gamma,0\right) &= -\frac{2}{\sigma^2}\ln\frac{b}{\left(\left(1-e^{-b\tau}\right)\gamma\sigma^2/2+b\right)}
\end{aligned}$$

and then:

$$\begin{aligned}
E_{0,r}e^{-\gamma r_\tau} &= \int_0^{+\infty} e^{-\gamma\xi} f_{r_\tau}\left(\xi,\tau;r,0\right)d\xi \qquad (3.55)\\
&= \exp\left(\frac{2a}{\sigma^2}\ln\frac{b}{\left(\left(1-e^{-b\tau}\right)\gamma\sigma^2/2+b\right)} - r\frac{\gamma b\exp\left(-\tau b\right)}{\left(1-\exp\left(-\tau b\right)\right)\gamma\sigma^2/2+b}\right)\\
&= \left(\frac{b}{\left(\left(1-e^{-b\tau}\right)\gamma\sigma^2/2+b\right)}\right)^{\frac{2a}{\sigma^2}}\exp\left(-r\frac{\gamma b\exp\left(-\tau b\right)}{\left(1-\exp\left(-\tau b\right)\right)\gamma\sigma^2/2+b}\right)\\
&= \left(\frac{1}{\left(\left(1-e^{-b\tau}\right)\gamma\sigma^2/\left(2b\right)+1\right)}\right)^{\frac{2a}{\sigma^2}}\exp\left(-r\frac{\gamma\exp\left(-\tau b\right)}{\gamma\left(1-\exp\left(-\tau b\right)\right)\sigma^2/\left(2b\right)+1}\right)
\end{aligned}$$

Before finding the inverse, we can investigate the behavior of the density as $r_\tau \to 0$. For this we can exploit the Abelian result that relates the behavior of the original function as $r_\tau \to 0$ to the behavior of the Laplace transform as $\gamma \to \infty$, i.e.:

$$\lim_{r_\tau\to 0} f_{r_\tau}\left(r_\tau,\tau;r,0\right) = \lim_{\gamma\to\infty}\gamma\exp\left(-a\phi\left(\tau;\gamma,0\right) - r\psi\left(\tau;\gamma,0\right)\right).$$

In particular we have:

$$\lim_{\gamma\to\infty}\gamma\left(\frac{1}{\left(\left(1-e^{-b\tau}\right)\gamma\sigma^2/\left(2b\right)+1\right)}\right)^{\frac{2a}{\sigma^2}}\exp\left(-r\frac{\gamma\exp\left(-\tau b\right)}{\gamma\left(1-\exp\left(-\tau b\right)\right)\sigma^2/\left(2b\right)+1}\right)$$
$$=\left(\frac{1}{\left(1-e^{-b\tau}\right)\sigma^2/\left(2b\right)}\right)^{\frac{2a}{\sigma^2}}\exp\left(-r\frac{\exp\left(-\tau b\right)}{\left(1-\exp\left(-\tau b\right)\right)\sigma^2/\left(2b\right)}\right)\lim_{\gamma\to\infty}\gamma^{1-\frac{2a}{\sigma^2}}$$

and then we can conclude that:

$$\begin{aligned}
&\text{if } \tfrac{2a}{\sigma^2}<1 \quad \lim_{r_\mathcal{T}\to 0} f_{r_\mathcal{T}}\left(r_\mathcal{T},\mathcal{T};r,0\right)=\infty\\
&\text{if } \tfrac{2a}{\sigma^2}=1 \quad \lim_{r_\mathcal{T}\to 0} f_{r_\mathcal{T}}\left(r_\mathcal{T},\mathcal{T};r,0\right)=c\\
&\text{if } \tfrac{2a}{\sigma^2}>1 \quad \lim_{r_\mathcal{T}\to 0} f_{r_\mathcal{T}}\left(r_\mathcal{T},\mathcal{T};r,0\right)=0
\end{aligned} \tag{3.56}$$

where the finite value $c\neq 0$ is given by:

$$c=\left(\frac{1}{\left(1-e^{-b\tau}\right)\sigma^2/\left(2b\right)}\right)\exp\left(-r\frac{\exp\left(-\tau b\right)}{\left(1-\exp\left(-\tau b\right)\right)\sigma^2/\left(2b\right)}\right). \tag{3.57}$$

This result is related to the behavior of the diffusion process near the origin as found by Feller [35].

In order to find the Inverse Laplace transform of (3.55), it is convenient to define the following quantity that does not depend on the Laplace variable $\gamma$:

$$L=\left(1-\exp\left(-\tau b\right)\right)\sigma^2/4b$$

so that:

$$
\begin{aligned}
&E_{0,r}e^{-\gamma r_\tau}\\
&=\left(\tfrac{1}{2\gamma L+1}\right)^{\frac{2a}{\sigma^2}}\exp\left(-r\exp\left(-\tau b\right)\tfrac{\gamma}{2\gamma L+1}\right)\\
&=\left(\tfrac{1}{2\gamma L+1}\right)^{\frac{2a}{\sigma^2}}\exp\left(-r\tfrac{\exp(-\tau b)}{2L}\tfrac{2\gamma L+1-1}{2\gamma L+1}\right)\\
&=\left(\tfrac{1}{2\gamma L+1}\right)^{\frac{2a}{\sigma^2}}\exp\left(-r\tfrac{\exp(-\tau b)}{2L}\left(1-\tfrac{1}{2\gamma L+1}\right)\right)\\
&=e^{-r\frac{\exp(-\tau b)}{2L}}\left(\tfrac{1}{2\gamma L+1}\right)^{\frac{2a}{\sigma^2}}\exp\left(r\tfrac{\exp(-\tau b)}{2L}\tfrac{1}{2\gamma L+1}\right)\\
&=e^{-r\frac{\exp(-\tau b)}{2L}}\left(\tfrac{1}{(\gamma+1/(2L))2L}\right)^{\frac{2a}{\sigma^2}}\exp\left(r\tfrac{\exp(-\tau b)}{4L^2}\tfrac{1}{(\gamma+1/(2L))}\right)
\end{aligned}
$$

Using the following Laplace transform formula:

$$
\mathcal{L}\left(\left(\frac{x}{a}\right)^{n/2}J_n\left(2\sqrt{ax}\right),x\to\gamma\right)=\frac{e^{-a/\gamma}}{\gamma^{n+1}},n>-1
$$

where $J_n\left(x\right)$ is the Bessel function of the first kind and of order $n$, and exploiting the shifting property of the Laplace transform,

$$
\mathcal{L}\left(e^{-\beta x}G\left(x\right),x\to\gamma\right)=g\left(\gamma+\beta\right)
$$

then the inverse Laplace transform of $E_{0,r}e^{-\gamma r_\tau}$ is given by, setting $q=2a/\sigma^2-1$:

$$
\begin{aligned}
&f_{r_\tau}\left(r_\tau,\tau;r,0\right)\\
&=e^{-r_\tau/(2L)}e^{-r\frac{\exp(-\tau b)}{2L}}\frac{1}{(2L)^{\frac{2a}{\sigma^2}}}\left(\frac{r_\tau}{-r\frac{\exp(-\tau b)}{4L^2}}\right)^{q/2}J_q\left(2\sqrt{-r\frac{\exp(-\tau b)}{4L^2}r_\tau}\right)
\end{aligned}
$$

We note that the quantities $-\left(r_\tau 4L^2\right)/\left(r\exp\left(-\tau b\right)\right)$ and $-r\exp\left(-\tau b\right)r_\tau/4L^2$ are negative so that we can write the above expression using the imaginary number $i=\sqrt{-1}$ in the following way:

$$
=\frac{e^{-(r\exp(-\tau b)+r_\tau)/(2L)}}{(2L)^{1+q}}\left(-\frac{r_\tau 4L^2}{r\exp\left(-\tau b\right)}\right)^{q/2}i^{q/2}J_q\left(2i\sqrt{r\frac{\exp\left(-\tau b\right)}{4L^2}r_\tau}\right)
$$

and now we can exploit the relationship between the Bessel function of the first kind of order $n$, $J_n(x)$, with the modified Bessel function of the first kind of order $n$:

$$I_n(x) = (-i)^n J_n(ix)$$

where[5]:

$$I_n(x) = \left(\frac{x}{2}\right)^n \sum_{j=1}^{\infty} \frac{(x^2/4)^j}{j!\Gamma(q+j+1)}$$

Then the expression for the transition density is given by:

$$= \frac{e^{-(r\exp(-\tau b)+r_T)/(2L)}}{2L} \left(\frac{r_T 4L^2}{r\exp(-\tau b)(2L)^2}\right)^{q/2} I_q\left(2\sqrt{r\frac{\exp(-\tau b)}{4L^2}r_T}\right)$$

i.e.:

$$f(r_T, T; r, 0) = \frac{e^{-(r\exp(-\tau b)/L+r_T/L)/2}}{2L} \left(\frac{r_T}{r\exp(-\tau b)}\right)^{q/2} I_q\left(2\sqrt{r\frac{\exp(-\tau b)}{4L^2}r_T}\right)$$

and we can compare with the same expression as in CIR, eq. 18, pag. 391-2, where it has been used the following notation:

$$c = \frac{1}{2L}; u = cre^{-\lambda\tau}; v = \frac{r_T}{2L}; q = \frac{2a}{\sigma^2} - 1$$

Finally, we remark that the probability density function of a noncentral $\chi^2$ r.v. with $v$ degrees of freedom and noncentrality parameter $\delta$ is given by, compare Johnson and Kotz [57], pag. 133:

$$f_{\chi^2}(x; v, \delta) = e^{-(\delta+x)/2}\frac{1}{2}\left(\frac{x}{\delta}\right)^{(v-2)/4} I_{(v-2)/2}\left(\sqrt{\delta x}\right), x > 0$$

[5] For an integral representation of the Bessel function compare Hochstadt [46] pag. 210.

and then comparing with the transition density law of $r_\tau$ we can conclude that $r_\tau/L$ has a noncentral $\chi^2$ density law with $2q+2$ degrees of freedom and noncentrality parameter $r\exp(-\tau b)/L$.

### 3.4.2. Pricing of a zero-coupon bond

From the expression of the double Laplace transform, we can obtain at no cost the price of a zero-coupon bond with residual life $\tau$. Indeed this price is given by:

$$E_{0,r}\left(e^{-\int_0^\tau r(s)ds}\right)$$

where the expected value is calculated under the risk-neutral measure[6]. Comparing with the expression of the double Laplace transform, the price of the zcb is obtained setting $\gamma = 0$ and $\mu = 1$, so that it is given by:

$$v(\tau, r; 0, 1) = \exp(-a\phi(\tau; 0, 1) - r\psi(\tau; 0, 1))$$

where $\phi(\tau; 0, 1)$ and $\psi(\tau; 0, 1)$ are obtained from 3.54 and 3.53:

$$\psi(\tau; 0, 1) = \frac{2(\exp(\tau\lambda) - 1)}{\lambda - b + (b + \lambda)\exp(\tau\lambda)} \tag{3.58}$$

$$\phi(\tau; 0, 1) = -\frac{2}{\sigma^2}\ln\frac{2\lambda e^{(\lambda+b)\tau/2}}{(-b + \lambda + (b + \lambda)e^{\tau\lambda})} \tag{3.59}$$

$$\lambda = \sqrt{b^2 + 2\sigma^2} \tag{3.60}$$

[6] We have assumed that the process for the instantaneous interest rate in equation 3.50 is given under the risk-neutral measure.

### 3.4.3. Pricing of options on zcb

The payoff of the option with residual life $\tau_1$ on a zcb with residual life $\tau_2$, $\tau_2 > \tau_1$, is given by:

$$(v(\tau_2 - \tau_1, r_{\tau_1}; 0, 1) - k)^+$$

and the price is given by:

$$E_{0,r}\left(e^{-\int_0^{\tau_1} r(s)ds}\left(v(\tau_2 - \tau_1, r_{\tau_1}; 0, 1) - k\right)^+\right)$$

Unfortunately, within the stochastic interest framework, it is difficult to apply directly the risk-neutral argument, because the stochastic discount factor is correlated with the payoff function, and directly evaluating the expectation of the discount factor and the payoff function can be a difficult task on the analytical side. We can follow different ways:

a) To use the so-called change of measure where the numeraire is the zcb with same residual life as the option. In this case if we know the dynamics of the interest rate under the forward measure, i.e. the measure which make date $\tau_1$ forward prices martingales, then the option price can be expressed as:

$$v(\tau_1, r; 0, 1) = E_{0,r}^{\tau_1}\left[\frac{(v(\tau_2 - \tau_1, r_{\tau_1}; 0, 1) - k)^+}{v(\tau_1 - \tau_1, r_{\tau_1}; 0, 1)}\right] \tag{3.61}$$

and where $v(0, r) = 1$. In this way, instead of computing an expected value of a discounted payoff, we can compute the present value of an expected payoff. The computation of the expected value under the forward measure is solved once we have recovered the dynamics of the interest rate under the forward measure and this can be accomplished using the results in Jamshidian [55], pag. 115 or the "separation theorem" in Longstaff [69], pagg. 102-3. These results allow us to write the value of the contingent claim as above in equation

(3.61) and obtain the dynamics of $r$ under the $\tau_1$ forward measure:

$$dr = \left(a - \left(b + \sigma^2 \psi\left(\tau_1; 0, 1\right)\right) r\right) dt + \sigma\sqrt{r} dW$$

Using the result on the distribution of $r$ under the risk-neutral measure, we can remark that the change of measure has only changed the coefficient of $r$ in the drift, whilst the volatility term is not changed. If we repeat the same procedure as in the case of the risk-neutral measure, it can be proved, compare the appendix in Jamshidian [55] or Longstaff [69], that the distribution of $r_\tau$ under the $\tau_1$ measure is again related to the noncentral $\chi^2$ distribution with the same degrees of freedom, but a different non-centrality parameter. The price of the option on zcb can be expressed in terms of the noncentral $\chi^2$ distribution, and different approximations to the noncentral $\chi^2$ have been suggested. A review can be found in Johnson and Kotz [57], chapter 28, where can also be found a particularly good approximation derived by Sankaran [83]. Compare also Schroder [85] for applications to option pricing problems.

b) Once we have observed that $v\left(\tau_2 - \tau_1, r_{\tau_1}; 0, 1\right)$ can be expressed as deterministic function of $r_{\tau_1}$, we can obtain from the expression for the double Laplace transform of $\int_0^{\tau_1} r\left(s\right) ds$ and $r_{\tau_1}$, using the numerical inversion, their joint density and then integrate it over the payoff. This task can be time-consuming because we need to call the Laplace inversion routine as many times as the number of points at which we want the joint density. A possibility, that we do not explore, can be to exploit the efficiency of the Fast Fourier transform.

c) Instead of computing trough inversion the joint density and then integrate over the payoff, we can follow the following strategy. Let us write the option price as expected value of the discounted payoff:

$$
\begin{aligned}
& E_{0,r}\left(e^{-\int_0^{\tau_1} r(s)ds}\left(e^{-a\phi(\tau_2-\tau_1;0,1)-r_{\tau_1}\psi(\tau_2-\tau_1;0,1)}-k\right)^+\right) \\
= & \int_0^{+\infty} e^{-x}\int_0^{+\infty}\left(e^{-a\phi(\tau_2-\tau_1;0,1)-y\psi(\tau_2-\tau_1;0,1)}-k\right)^+ f(x,y)\,dydx \\
= & \int_0^{+\infty} e^{-x}\int_0^{y^*}\left(e^{-a\phi(\tau_2-\tau_1;0,1)-y\psi(\tau_2-\tau_1;0,1)}-k\right) f(x,y)\,dydx
\end{aligned}
$$

where $f(x,y)$ is the joint density when $\left(\int_0^{\tau_1} r(s)\,ds = x; r_{\tau_1} = y\right)$ and $y^*$ is given by:

$$
y^* = -\frac{a\phi(\tau_2-\tau_1;0,1)+\log k}{\psi(\tau_2-\tau_1;0,1)}
$$

Now let us consider the Laplace transform of the option price wrt $y^*$ :

$$
\begin{aligned}
& \textstyle\int_0^{\infty} e^{-\gamma y^*}\int_0^{+\infty} e^{-x}\int_0^{y^*}\left(e^{-a\phi(\tau_2-\tau_1;0,1)-y\psi(\tau_2-\tau_1;0,1)}-k\right) f(x,y)\,dydxdy^* \\
& = \textstyle\int_0^{+\infty} e^{-x}\int_0^{\infty}\left(\int_y^{+\infty} e^{-\gamma y^*}dy^*\right)\left(e^{-a\phi(\tau_2-\tau_1;0,1)-y\psi(\tau_2-\tau_1;0,1)}-k\right) f(x,y)\,dxdy \\
& = \textstyle\int_0^{+\infty} e^{-x}\int_0^{\infty}\frac{e^{-\gamma y}}{\gamma}\left(e^{-a\phi(\tau_2-\tau_1;0,1)-y\psi(\tau_2-\tau_1;0,1)}-k\right) f(x,y)\,dxdy \\
& = \frac{e^{-a\phi(\tau_2-\tau_1;0,1)}}{\gamma}E_{0,r}\left[e^{-\int_0^{\tau_1} r(s)ds-(\gamma+\psi(\tau_2-\tau_1;0,1))r_{\tau_1}}\right]-\frac{k}{\gamma}E_{0,r}\left[e^{-\int_0^{\tau_1} r(s)ds-\gamma r_{\tau_1}}\right]
\end{aligned}
$$

and we can substitute in the last row the expression of the double Laplace transform that we have defined in equation (3.52), so we obtain the expression for the Laplace transform wrt to $y^*$ of the price of the call option on a zcb:

$$
\frac{e^{-a\phi(\tau_2-\tau_1;0,1)}}{\gamma}v\left(\tau,r;\gamma+\psi(\tau_2-\tau_1;0,1),1\right)-\frac{k}{\gamma}v(\tau,r;\gamma,1) \tag{3.62}
$$

The analytical inversion can be done using the convolution property of the Laplace transform and the quantity

$$
\mathcal{L}^{-1}\left(v(\tau,r;\gamma,1);\gamma\to y\right)
$$

can be found analytically in a similar way to the previous section. If we do the inversion numerically, it is convenient to invert directly the expression (3.62) avoiding the computation of the integrals coming from the convolution.

### 3.4.4. Pricing of options on spot rates

Following the strategy described above, we can easily obtain the Laplace transform of the price of options on spot rates of maturity $w$. Indeed in the CIR model, the spot rate with residual life $w$, $R(\tau, \tau + w)$ is linear in $r_\tau$:

$$R(\tau, \tau + w) = a\frac{\phi(w; 0, 1)}{w} + r(\tau)\frac{\psi(w; 0, 1)}{w}$$

so that:

$$\begin{aligned} &(R(\tau, \tau + w) - K)^+ \\ &= \left(a\tfrac{\phi(w;0,1)}{w} + r(\tau)\tfrac{\psi(w;0,1)}{w} - K\right)^+ \\ &= \tfrac{\psi(w;0,1)}{w}\left(r(\tau) - K^*\right)^+ \end{aligned}$$

where $K^* = \frac{w}{\psi(w;0,1)}\left(K - a\frac{\phi(w;0,1)}{w}\right)$. The price of the option is obtained by:

$$\begin{aligned} &\int_0^\infty \int_0^\infty e^{-y}(x - K^*)^+ f(x, y)\,dxdy \\ = &\int_0^\infty \int_{K^*}^\infty e^{-y}(x - K^*)^+ f(x, y)\,dxdy \end{aligned}$$

and considering the Laplace transform wrt $K^*$, after some algebra, we obtain:

$$\begin{aligned} &\textstyle\int_0^\infty e^{-\gamma K^*} \int_0^\infty \int_{K^*}^\infty e^{-y}(x - K^*)^+ f(x, y)\,dxdydK^* \\ &= \textstyle\int_0^\infty \int_0^\infty \left(\frac{e^{-y-\gamma x} - e^{-y}}{-\gamma} x f(x, y) + \frac{e^{-y-\gamma x}\gamma x + e^{-y-\gamma x} - e^{-y}}{\gamma^2} f(x, y)\right) dxdy \\ &= \textstyle\int_0^\infty \int_0^\infty \left(\frac{e^{-y}}{\gamma} x f(x, y) + \frac{e^{-y-\gamma x} - e^{-y}}{\gamma^2} f(x, y)\right) dxdy \\ &= \frac{1}{\gamma}\left.\frac{\partial v(\tau, r; \zeta, 1)}{\partial \zeta}\right|_{\zeta=0} + \frac{1}{\gamma^2} v(\tau, r; \gamma, 1) - \frac{1}{\gamma^2} v(\tau, r; 0, 1) \end{aligned}$$

where we have exploited the fact that:

$$\frac{\partial v(\tau,r;\gamma,\mu)}{\partial\gamma}=-E_{0,r}\left(re^{-\gamma r_\tau-\mu y_\tau}\right)$$

The inversion of the above expression is given by:

$$\begin{aligned}&\left.\tfrac{\partial v(\tau,r;\zeta,1)}{\partial\zeta}\right|_{\zeta=0}-K^*v(\tau,r;0,1)+\mathcal{L}^{-1}\left(\tfrac{1}{\gamma^2}v(\tau,r;\gamma,1)\right)\\ =&\left.\tfrac{\partial v(\tau,r;\zeta,1)}{\partial\zeta}\right|_{\zeta=0}-K^*v(\tau,r;0,1)+\textstyle\int_0^{K^*}(K^*-\kappa)\,\mathcal{L}^{-1}\left(v(\tau,r;\gamma,1);\gamma\to\kappa\right)d\kappa\end{aligned}$$

On the numerical side it is convenient to invert the expression $v(\tau,r;\gamma,1)/\gamma^2$ directly and to avoid the computation of the integral.

### 3.4.5. Pricing of options on the spread of spot rates

Using the results of previous section, we can easily find an expression for the Laplace transform wrt the strike of a spread option, i.e. an option on the difference of spot rates. The payoff at time $\tau$, is given by:

$$\left(R(\tau,\tau+w_1)-R(\tau,\tau+w_2)\right)^+$$

and exploiting the linearity of the spot rates respect to the instantaneous interest rate, we have:

$$\begin{aligned}&\left(a\frac{\phi(w_1;0,1)}{w_1}+r(\tau)\frac{\psi(w_1;0,1)}{w_1}-a\frac{\phi(w_2;0,1)}{w_2}-r(\tau)\frac{\psi(w_2;0,1)}{w_2}\right)^+\\ =&\left(r(\tau)\left(\frac{\psi(w_1;0,1)}{w_1}-\frac{\psi(w_2;0,1)}{w_2}\right)+a\left(\frac{\phi(w_1;0,1)}{w_1}-\frac{\phi(w_2;0,1)}{w_2}\right)\right)^+\\ =&\;M\left(r(\tau)-K\right)^+\end{aligned}$$

i.e. a spread option is equivalent, in the CIR model, to $M$ options on the instantaneous interest rate $r(\tau)$ with strike $K$, where:

$$\begin{aligned} M &= \left(\frac{\psi(w_1;0,1)}{w_1} - \frac{\psi(w_2;0,1)}{w_2}\right) \\ K &= \frac{a}{M}\left(\frac{\phi(w_2;0,1)}{w_2} - \frac{\phi(w_1;0,1)}{w_1}\right) \end{aligned}$$

The Laplace transform wrt the strike can be easily obtained using the same procedure as in previous section.

### 3.4.6. Pricing of interest rate Asian options

This kind of contract has been studied by Longstaff [70] in the case of a Vasicek model for the interest rate and by Leblanc and Scaillet [64] and Chacko and Das [13] in the case of the CIR model. For an Asian option on the interest rate, the payoff is given by:

$$\left(\frac{1}{\tau}\int_0^\tau r(s)\,ds - k\right)^+$$

and then the price is (setting $y_\tau = \int_0^\tau r(s)\,ds$):

$$\frac{1}{\tau}E_{0,r}\left(e^{-y_\tau}(y_\tau - k\tau)^+\right)$$

Let us take the Laplace transform wrt the strike $K = k\tau$ :

$$\begin{aligned} &\int_0^\infty e^{-\mu K} E_{0,r}\left(e^{-y_\tau}(y_\tau - K)^+\right) dK \\ &= \int_0^\infty e^{-\mu K}\int_0^\infty e^{-y}(y-K)^+ f_{y_\tau}(y)\,dydK \\ &= \int_0^\infty e^{-\mu K}\int_K^\infty e^{-y}(y-K)\, f_{y_\tau}(y)\,dydK \end{aligned}$$

and changing the integration order, we obtain:

$$
\begin{aligned}
&= \int_0^\infty e^{-y}\left(\int_0^y e^{-\mu K} y dK\right) f_{y_\tau}(y)\,dy - \int_0^\infty e^{-y}\left(\int_0^y e^{-\mu K} K dK\right) f_{y_\tau}(y)\,dy \\
&= \int_0^\infty e^{-y}\frac{1-e^{-\mu y}}{\mu} y f_{y_\tau}(y)\,dy + \int_0^\infty e^{-y}\left(\frac{e^{-\mu y}\mu y + e^{-\mu y} - 1}{\mu^2}\right) f_{y_\tau}(y)\,dy \\
&= \frac{1}{\mu}\int_0^\infty e^{-y} y f_{y_\tau}(y)\,dy + \frac{1}{\mu^2}\int_0^\infty \left(e^{-(\mu+1)y} - e^{-y}\right) f_{y_\tau}(y)\,dy
\end{aligned}
$$

and then the Laplace transform wrt $K = k\tau$ of the price of the Asian option is given by:

$$
\begin{aligned}
&\tfrac{1}{\tau}\textstyle\int_0^\infty e^{-\mu K} E_{0,r}\left(e^{-y_\tau}\left(y_\tau - K\right)^+\right) dK \\
&= \tfrac{1}{\tau}\left(-\tfrac{1}{\mu}\left.\tfrac{\partial v(\tau,r;0,\zeta)}{\partial\zeta}\right|_{\zeta=1} + \tfrac{1}{\mu^2} v\left(\tau,r;0,\mu+1\right) - \tfrac{1}{\mu^2} v\left(\tau,r;0,1\right)\right)
\end{aligned}
\tag{3.63}
$$

and where we have used the expression for the Laplace transform (mgf) of $y_\tau$ :

$$
v\left(\tau,r;0,\mu\right) = E_{0,r} e^{-\mu\int_0^\tau r(s)ds}
$$

Note that in order to find the price of the Asian option we have to invert numerically only the term $v\left(\tau,r;0,\mu+1\right)/\mu^2$, for the fact that:

$$
L^{-1}\left(\frac{1}{\mu}\left.\frac{\partial v\left(\tau,r;0,\zeta\right)}{\partial\zeta}\right|_{\zeta=1} + \frac{1}{\mu^2} v\left(\tau,r;0,1\right)\right) = \left.\frac{\partial v\left(\tau,r;0,\zeta\right)}{\partial\zeta}\right|_{\zeta=1} + K\tau v\left(\tau,r;0,1\right)
$$

In the above expressions there appears the first derivative of the mgf of $y_\tau$. In this way, instead of having to invert the Laplace transform $v\left(r,\tau;0,\mu\right)$ for obtaining the density function and then integrating it over quantity $ye^{-y}$, we can calculate the first derivative of the mgf of $y$, analytically using packages like Mathematica or Maple or numerically through the use of the Cauchy integral formula. Instead, Leblanc and Scaillet [64] solve for the Laplace transform, invert numerically it and then integrate over the desired payoff. In this way they have also the problem of checking for the positivity of the density and for the fact that the area under the density is equal to 1. Chacko and Das [13] solve for the price of binary Asian options and then express the formula for the Asian call as sum of Asian binary calls at ascending strikes. Moreover they use Fourier inversion. Our expression is clearly more immediate and simpler to calculate.

The expression for the Laplace transform can be easily generalised to consider Asian options on spot rates with maturity $w$. Indeed:

$$\begin{aligned}
&\left(\tfrac{1}{\tau}\int_0^\tau R(s,s+w)\,ds-K\right)^+ \\
&=\left(\tfrac{1}{\tau}\int_0^\tau\left(a\tfrac{\phi(w;0,1)}{w}+r(s)\tfrac{\psi(w;0,1)}{w}\right)ds-K\right)^+ \\
&=\left(a\tfrac{\phi(w;0,1)}{w}+\tfrac{\psi(w;0,1)}{w\tau}\int_0^\tau r(s)\,ds-K\right)^+ \\
&=\tfrac{\psi(w;0,1)}{w\tau}\left(\int_0^\tau r(s)\,ds-\bar{K}\right)^+
\end{aligned}\tag{3.64}$$

where

$$\bar{K}=\frac{w\tau}{\psi(w;0,1)}\left(K-a\frac{\phi(w;0,1)}{w}\right)$$

So that for obtaining the price of an Asian option on the spot rate with maturity $w$, we can multiply by $\psi(w;0,1)/w\tau$ the price of an Asian option on the instantaneous interest rate with strike $\bar{K}$.

### 3.4.7. Pricing of Asian options on the spread

An immediate and very simple generalization is to the case of the spread of time averages of spot rates. The payoff is given by:

$$\frac{1}{\tau}\left(\int_0^\tau R(s,s+w_1)\,ds-\int_0^\tau R(s,s+w_2)\,ds\right)^+$$

and by the linearity of the spot rates in the instantaneous interest rate, we can express it as:

$$
\begin{aligned}
&= \tfrac{1}{\tau}\left(\int_0^\tau \left(a\tfrac{\phi(w_1;0,1)}{w_1} + r(s)\tfrac{\psi(w_1;0,1)}{w_1}\right) ds - \int_0^\tau \left(a\tfrac{\phi(w_2;0,1)}{w_2} + r(s)\tfrac{\psi(w_2;0,1)}{w_2}\right) ds\right)^+ \\
&= M\left(\int_0^\tau r(s)\, ds - \bar{K}\right)^+
\end{aligned}
$$

where

$$
\begin{aligned}
M &= \frac{1}{\tau}\left(\frac{\psi(w_1;0,1)}{w_1} - \frac{\psi(w_2;0,1)}{w_2}\right) \\
\bar{K} &= \frac{\left(\frac{\phi(w_2;0,1)}{w_2} - \frac{\phi(w_1;0,1)}{w_1}\right)}{\left(\frac{\psi(w_1;0,1)}{w_1} - \frac{\psi(w_2;0,1)}{w_2}\right)} a\tau
\end{aligned}
$$

The expression of the Laplace transform wrt $\bar{K}$ can be easily obtained from (3.63).

### 3.4.8. Other applications and extensions

Other applications of the Laplace transform to interest rate models can be found in Cathcart [12], where it is discussed the pricing of floating rate instruments with various contractual features. The numerical routine for the inversion used by the author is based on the Gaussian quadrature formula as discussed in Piessens [79]. Respect to the cases discussed in the previous sections, Cathcart uses the Laplace transform wrt time and not the moment generating function. This way of proceeding is related to the determination of the resolvent kernel as discussed for barrier options. Duffie et al. [30] provides an analytical treatment of option valuation in the affine jump-diffusion model through analysis of various Fourier-like transforms of functionals of the state process, extending results in Chen and Scott [14].

### 3.5. Example 5: Occupation Time

Hurdle or range derivatives are contracts whose payoff depends on the time spent by some index (stock index, interest rate or exchange rate) below a predetermined level. The structure of the payoff is common to FX range floaters, step and boost options as described in Hull [50], Pechtl [77], Hugonnier [47], Linetsky [66], Turnbull[97] and Bregagnolio[10]. Chesney et al. [16] and [17] have used the LT technique to solve the pricing problem for the Parisian option. In this case the owner of an up-and-out Parisian option loses if the stock price reaches a level $L$ and remains constantly above this level for a time interval longer than $D$.

In order to calculate the density of the occupation time, we can observe that the time spent below the level $u$ by the GBM with instantaneous drift $r$ and volatility $\sigma$ is the same as the time spent below the level $U = \ln(u)/\sigma$ by an Arithmetic Brownian Motion with drift $\delta \equiv \left(r - \frac{\sigma^2}{2}\right)/\sigma$, diffusion coefficient unitary and starting at the level $z = \ln(x)/\sigma$. The distribution of the occupation time when $\delta = 0$ has been found by Lévy [65] and is known as arc-sine law. In the case $\delta \neq 0$, using the Feynman-Kac formula Akahori [5] provided for the first time the distribution of the occupation time but in the form of a double integral. Takacs [93], using a random walk approach, has given a simpler expression. We provide now the same expression as in [93], but solving again the Feynman-Kac formula.

Other interesting results can be found in Dassios [24] and in Embrechts et al. [34]. Hugonnier [47] and Linetsky [66] have found, using again the Laplace transform technique, the joint density of the Arithmetic Brownian Motion and of the Occupation time. They simplify the expressions that can be found in Borodin and Salminen, pag. 204-5, equations 1.5.8, 1.5.8.(1) and 1.5.8.(2).

We define the occupation time of the ABM in the time interval $[0,\tau]$ as:

$$Y_{0,x}(U,\tau) = \int_0^\tau \mathbf{1}_{(\delta s + W_s < U)} ds; W_0 = x \tag{3.65}$$

where $\mathbf{1}_{(A)}$ is the indicator function of the set $A$. We have the following result:

**Theorem 3.1.** : *The density function for the time spent below the level $U$=$(\ln u)/\sigma$ by a Arithmetic Brownian Motion starting at the level $z = (\ln x)/\sigma$, with drift $\delta \neq 0$ and diffusion coefficient unitary, is, for $0 < y < \tau$, given by:*

$$\Pr_{0,x}\left(Y_{0,x}(u,\tau) \in dy\right)$$
$$= \begin{cases} 1_{(x>u)} \left( \frac{e^{-\frac{\delta^2}{2}y}}{\sqrt{\pi y}} - \frac{\delta}{\sqrt{2}} \operatorname{Erfc}\left(\frac{\delta\sqrt{y}}{\sqrt{2}}\right)\right) \\ \times \left( \frac{e^{-\frac{1}{2}\frac{\left(\left(\frac{1}{\sigma}\ln\frac{x}{u}+\delta(\tau-y)\right)\right)^2}{\tau-y}}}{\sqrt{\pi(\tau-y)}} + \frac{\delta}{\sqrt{2}} e^{-2\delta \ln\frac{x}{u}} \operatorname{Erfc}\left(\frac{\frac{1}{\sigma}\ln\frac{x}{u}-\delta(\tau-y)}{\sqrt{2(\tau-y)}}\right)\right) dy \\ \\ 1_{(x<u)} \left( \frac{e^{-\frac{\delta^2}{2}(\tau-y)}}{\sqrt{\pi(\tau-y)}} + \frac{\delta}{\sqrt{2}} \operatorname{Erfc}\left(-\frac{\delta\sqrt{\tau-y}}{\sqrt{2}}\right)\right) \\ \times \left( \frac{e^{-\frac{1}{2}\frac{\left(\left(\frac{1}{\sigma}\ln\frac{u}{x}-\delta y\right)\right)^2}{y}}}{\sqrt{\pi y}} - \frac{\delta}{\sqrt{2}} e^{+2\delta\ln\frac{u}{x}} \operatorname{Erfc}\left(\frac{\frac{1}{\sigma}\ln\frac{u}{x}+\delta y}{\sqrt{2y}}\right)\right) dy \end{cases} \tag{3.66}$$

*Moreover, we have:*

$$\Pr_{0,x}\left(Y_{0,x}(u,\tau) = 0\right) = 1_{(x>u)} \left[1 - \frac{1}{2}\left(\operatorname{Erfc}\left(\frac{\frac{1}{\sigma}\ln\frac{x}{u}+\delta\tau}{\sqrt{2\tau}}\right) + e^{-2\delta\ln\frac{x}{u}} \operatorname{Erfc}\left(\frac{\frac{1}{\sigma}\ln\frac{x}{u}-\delta\tau}{\sqrt{2\tau}}\right)\right)\right]$$

$$\Pr_{0,x}\left(Y_{0,x}(u,\tau) = \tau\right) = 1_{(x<u)} \left[1 - \frac{1}{2}\left(\operatorname{Erfc}\left(\frac{\frac{1}{\sigma}\ln\frac{u}{x}-\delta\tau}{\sqrt{2\tau}}\right) + e^{2\delta\ln\frac{u}{x}} \operatorname{Erfc}\left(\frac{\frac{1}{\sigma}\ln\frac{u}{x}+\delta\tau}{\sqrt{2\tau}}\right)\right)\right]$$

*where* $\operatorname{Erfc}(x) = 2\Phi\left(-\sqrt{2}x\right) = \left(2\int_x^{+\infty} e^{-w^2}dw\right)/\sqrt{\pi}$♦.

We can easily obtain the density of the occupation time in the case of zero drift setting $\delta = 0$ in the above theorem. In the case $x = u$, we obtain the famous arc-sine law

**Corollary 3.2.** *The density function for the time spent below the level $U$=$(\ln u)/\sigma$ by a*

*Standard Brownian Motion starting at the level $z = (\ln x)/\sigma$ is, for $0 < y < \tau$, given by:*

$$\mathrm{Pr}_{0,x}\left(Y_{0,x}\left(u,\tau\right) \in dy\right) = \begin{cases} 1_{(x>u)} \dfrac{e^{-\frac{1}{2}\frac{\left(\left(\frac{1}{\sigma}\ln\frac{x}{u}\right)\right)^2}{\tau-y}}}{\pi\sqrt{y(\tau-y)}} dy \\ 1_{(x=u)} \dfrac{1}{\pi\sqrt{y(\tau-y)}} dy \\ 1_{(x<u)} \dfrac{e^{-\frac{1}{2}\frac{\left(\left(\frac{1}{\sigma}\ln\frac{u}{x}-\delta y\right)\right)^2}{y}}}{\pi\sqrt{y(\tau-y)}} dy \end{cases} \tag{3.67}$$

*Moreover, we have:*

$$\mathrm{Pr}_{0,x}\left(Y_{0,x}(u,\tau) = 0\right) = 1_{(x>u)}\left[1 - \mathrm{Erfc}\left(\frac{\frac{1}{\sigma}\ln\frac{x}{u}}{\sqrt{2\tau}}\right)\right]$$

$$\mathrm{Pr}_{0,x}\left(Y_{0,x}(u,\tau) = \tau\right) = 1_{(x<u)}\left[1 - \mathrm{Erfc}\left(\frac{\frac{1}{\sigma}\ln\frac{u}{x}}{\sqrt{2\tau}}\right)\right]$$

♦

The result in the corollary can be checked with Borodin and Salminen [9], pag. 133, equation 1.5.4.

If we consider an ABM starting at $x$, with drift $\delta$ and diffusion coefficient unitary, and the barrier is set at the level $u$, then the mgf of the r.v. $Y_{0,x}(u,\tau)$:

$$\begin{aligned} &v(\tau,x) \equiv v(\tau,x;u) = \mathbf{E}_{0,x}\left[e^{-\mu Y(u,\tau)}\right] \\ &= \int_0^\tau e^{-\mu y}\,\mathrm{Pr}_{0,x}\left(Y(u,\tau) \in dy\right) + 1 \times \mathrm{Pr}_{0,x}\left[Y(u,\tau) = 0\right] + e^{-\mu\tau} \times \mathrm{Pr}_{0,x}\left[Y(u,\tau) = \tau\right], \end{aligned} \tag{3.68}$$

satisfies, by the Feynman-Kac formula, the equation:

$$-\frac{\partial v(\tau,x)}{\partial\tau} + \frac{1}{2}\frac{\partial^2 v(\tau,x)}{\partial^2 x} + \delta\frac{\partial v(\tau,x)}{\partial x} - \mu\mathbf{1}_{(x<u)}v(\tau,x) = 0; \tau > 0 \tag{3.69}$$

with initial condition:

$$v(0,x) = 1 \tag{3.70}$$

and boundary conditions:

$$v(\tau, +\infty) = 1 \tag{3.71}$$

$$v(\tau, -\infty) = e^{-\tau} \tag{3.72}$$

The first condition is due to the fact that if $x \to +\infty$, then the time spent below the level $u$ will be zero. Viceversa if $x \to -\infty$, then the time spent below the level $u$ will be equal to $\tau$ with probability 1. We observe that if $x > u$:

$$\begin{aligned}
&\Pr_{0,x}\left(Y_{0,x}(u,\tau) = 0\right) \\
&= \Pr_{0,x}\left(\inf_{0\le s\le\tau} \delta s + W(s) > u\right) \\
&= 1 - \left[\tfrac{1}{2}\operatorname{Erfc}\left(\tfrac{x-u+\delta\tau}{\sqrt{2\tau}}\right) + \tfrac{e^{-2m(x-u)}}{2}\operatorname{Erfc}\left(\tfrac{x-u-\delta\tau}{\sqrt{2\tau}}\right)\right]
\end{aligned}$$

and:

$$\begin{aligned}
&\Pr_{0,x}\left(Y_{0,x} = \tau\right) \\
&= \Pr_{0,x}\left(\sup_{0\le s\le\tau} \delta s + W(s) < u\right) \\
&= 1 - \left[\tfrac{1}{2}\operatorname{Erfc}\left(\tfrac{u-x-\delta\tau}{\sqrt{2\tau}}\right) + \tfrac{e^{2\delta(u-x)}}{2}\operatorname{Erfc}\left(\tfrac{u-x+m\tau}{\sqrt{2\tau}}\right)\right]
\end{aligned}$$

where we have used [9], formula 1.2.4 page 198 and formula 1.1.4 page 197.

In appendix E, equation (E.10), we obtain the expression for the Laplace transform wrt $\tau$ of the characteristic function:

$$\mathcal{L}\left(v(\tau,x);\tau\to\gamma\right) = \begin{cases} 1_{(x\ge u)}\left(\mathcal{L}\left(\Pr_{0,x\in(u;+\infty)}\left(Y(u,\tau)=0\right)\right) + \dfrac{e^{-\sqrt{\gamma}\sqrt{2}(x-u)-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)}\right) \\ \\ 1_{(x<u)}\left(\mathcal{L}\left(e^{-\mu\tau}\times\Pr_{0,x\in(-\infty;u)}\left(Y(u,\tau)=\tau\right)\right) + \dfrac{e^{-\sqrt{\gamma+\mu}\sqrt{2}(u-x)-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)}\right) \end{cases} \tag{3.73}$$

From the above expression, we obtain the double Laplace transform of the distribution function $\Pr_{0,x}(Y_{0,x} \in dy)$ when $0 < y < \tau$. Its expression can be then obtained through analytical inversion. Indeed we have:

$$\int_0^{+\infty} e^{-\gamma\tau} \int_0^{\tau} e^{-\mu y} \Pr_{0,x}\left(Y\left(u,\tau\right) \in dy\right) d\tau$$
$$= \begin{cases} 1_{(x\geq u)} \dfrac{e^{-\sqrt{\gamma}\sqrt{2}(x-u)-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)} \\ \\ 1_{(x<u)} \dfrac{e^{-\sqrt{\gamma+\mu}\sqrt{2}(u-x)-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)} \end{cases}$$

We observe that:

$$\frac{e^{-\sqrt{\gamma}\sqrt{2}(x-u)-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)}$$
$$= \frac{e^{-\sqrt{\gamma}\sqrt{2}(x-u)-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)} \frac{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}-\frac{\alpha}{\sqrt{2}}\right)}{\sqrt{\gamma}}$$
$$= \frac{e^{-\alpha u}}{\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)} \left(\frac{e^{-\sqrt{2\gamma}(x-u)}}{\sqrt{\gamma}} - \frac{\alpha}{\sqrt{2}} \frac{e^{-\sqrt{2\gamma}(x-u)}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}\right)$$

and similarly:

$$\frac{e^{-\sqrt{\gamma+\mu}\sqrt{2}(u-x)-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)}$$
$$= \frac{e^{-\sqrt{\gamma+\mu}\sqrt{2}(u-x)-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)} \frac{\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}+\frac{\alpha}{\sqrt{2}}\right)}{\sqrt{\gamma+\mu}}$$
$$= \frac{e^{-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} \left(\frac{e^{-\sqrt{2(\gamma+\mu)}(u-x)}}{\sqrt{\gamma+\mu}} + \frac{\alpha}{\sqrt{2}} \frac{e^{-\sqrt{2(\gamma+\mu)}(u-x)}}{\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)\sqrt{\gamma+\mu}}\right)$$

Using the Laplace transform in lines 7, 9 and 10 in the table of Laplace transform in section 2, we obtain:

$$
\begin{aligned}
&\int_0^\tau e^{-\mu y}\,\mathrm{Pr}_{0,x\in(u;+\infty)}\left(Y\left(u,\tau\right)\in dy\right)\\
&= e^{\alpha(x-u)-\frac{\alpha^2}{2}\tau}\int_0^\tau e^{-\mu\theta}\left(\frac{1}{\sqrt{\pi\theta}}+\frac{\alpha}{\sqrt{2}}e^{\frac{\alpha^2}{2}\theta}\,\mathrm{Erfc}\left(-\alpha\sqrt{\frac{\theta}{2}}\right)\right)\\
&\times\left(\frac{e^{-\frac{(x-u)^2}{2(\tau-\theta)}}}{\sqrt{\pi(\tau-\theta)}}-\frac{\alpha}{\sqrt{2}}e^{\frac{\alpha^2}{2}(\tau-\theta)+\alpha(x-u)}\,\mathrm{Erfc}\left(\alpha\sqrt{\frac{\tau-\theta}{2}}+\frac{x-u}{\sqrt{2(\tau-\theta)}}\right)\right)d\theta\\
&= \int_0^\tau e^{-\mu\theta}\left(\frac{e^{-\frac{\alpha^2}{2}\theta}}{\sqrt{\pi\theta}}+\frac{\alpha}{\sqrt{2}}\,\mathrm{Erfc}\left(-\alpha\sqrt{\frac{\theta}{2}}\right)\right)\\
&\times\left(\frac{e^{-\frac{(x-u)^2}{2(\tau-\theta)}-\frac{\alpha^2}{2}(\tau-\theta)+\alpha(x-u)}}{\sqrt{\pi(\tau-\theta)}}-\frac{\alpha}{\sqrt{2}}e^{+2\alpha(x-u)}\,\mathrm{Erfc}\left(\alpha\sqrt{\frac{\tau-\theta}{2}}+\frac{x-u}{\sqrt{2(\tau-\theta)}}\right)\right)d\theta
\end{aligned}
$$

and setting $\alpha=-\delta$, we obtain expression (3.66) for $x>u$ in theorem above, where $x$ and $u$ have been substituted by $\ln x$ and $\ln u$. When $x<u$ we get the same expression as for $x>u$ where $x-u$ has been substituted by $u-x$ and $\alpha$ by -$\alpha$.

The price of derivative contracts depending on $Y\left(u,\tau\right)$ is obtained calculating the relevant expected value.

## 4. The numerical inversion

"The inversion of the Laplace transform is well known to be an ill-conditioned problem. Numerical inversion is an unstable process and the difficulties often show up as being highly sensitive to round-off errors", Kwok and Barthez [61];

"The standard inversion formula is a contour integral, not a calculable expression.... These methods provide convergent sequences rather than formal algorithms; they are difficult to implement (many involve solving large, ill-conditioned systems of linear equations or analytically obtaining high-order derivatives of the transform) and none includes explicit, numerically computable bounds on error and computational effort", Platzman et al. [80].

Is it really worth to use the Laplace transform method for solving all the previous described problems, when the numerical inversion seems to be considered difficult[7]?

Although we have found the same criticism in many other papers, the aim of this section is to illustrate how we can use very simple numerical procedures for inverting the Laplace transform. The general opinion that the inversion of the Laplace transform is an ill-conditioned problem[8] is due to one of the first tentatives of inversion that were based on quadrature techniques to approximate the integral defining the (forward) Laplace transform $f(\gamma)$ of a function $F(\tau)$ :

$$f(\gamma) = \int_0^{+\infty} e^{-\gamma\tau} F(\tau)\, d\tau \tag{4.1}$$

Among these methods one the most well-known was proposed by Bellman et al. [7] that, using the quadrature approach, reduced the solution of the integral equation above to that of the solution of a system of linear algebraic equations, obtaining an approximate expression of the type:

$$\int_0^{+\infty} e^{-\gamma\tau} F(\tau)\, d\tau = \int_0^1 t^{\gamma-1} F(-\log t)\, dt = \sum_{i=1}^{N} w_i x_i^{\gamma-1} F(-\log x_i)$$

where $w_i$ and $x_i$ are the weights attached to the points $x_i$ at which $xF(x)$ is to be evaluated[9]. Letting $\gamma$ assume $N$ different values, the (4.1) yields a linear system of $N$ equations in the $N$ unknowns $g(x_i)$, $g(x_i) = F(-\log x_i)$:

$$\sum_{i=1}^{N} w_i x_i^k g(x_i) = f(k)\,;\ k = 0, 1, ..., N-1$$

[7] We have found the above quotations in Abate and Whitt [3].

[8] The concept of well-posedness was introduced by Hadamard and, simply stated, it means that a well-posed problem should have a solution, that this solution should be unique and that it should depend continuously on the problem's data. The first two requirements are minimal requirements for a reasonable problem, and the last ensures that perturbations, such errors in measurement, should not unduly affect the solution. For greater details compare Gustafsson and al. [43].

[9] In particular the points are determined by the zeros of Legendre polynomials.

The inversion is then very simple and fast, but Bellman et al. give several examples of unstable oscillatory behavior due to the fact that the matrix $\{w_i x_i^k\}$ is ill-conditioned and this problem rapidly worsens as $N$ increases. For the common situation of discretely sampled data over the real line, this approach requires a balance between the resolution of the recovered solutions, as determined by $N$, and the worsening numerical stability of the system. Therefore, if a high degree of accuracy is desired, the calculation must be carried out in multiple precision.

As explained clearly in Craig and Thompson [22], the ill-conditioning of the above inversion is common to all numerical routines that discretize the forward Laplace transform on the real line. The exponential kernel smooths out functions, and even takes noncontinuous functions into smooth transforms, so that the linear system cannot be solved with accuracy for arbitrary original functions. The "exponential filtering" property of the Laplace kernel allows only logarithmic improvements in recovery with increases in data accuracy. The underlying instability is then reflected in the poor conditioning of the resulting linear system and reliable inversions cannot be guaranteed even for machine-accurate data. These problems have given the idea that the Laplace inversion is not stable under reasonable perturbations.

Something similar occurs when we have the problem of numerically differentiating data. In this case the forward operation is the smoothing operation of integration, whilst the inverse problem is the differentiation one. If we do not have a closed analytic form for the function to be differentiated, the operation can be very inaccurate. If the function has a closed analytic form, the numerical differentiation can be improved greatly if done numerically on the complex plane using the Cauchy-Goursat theorem instead that only on the real line.

Similarly, the instability in the Laplace inversion can be in large part avoided when the image function is known as a closed analytic function in the complex plane. In this case instead of discretizing the forward Laplace transform we can compute the original function from values of the transform in the complex plane, i.e. operate the inversion using the Bromwich contour integral. In this way the inversion problem can be stabilized, although we can always encounter some difficulty in treating functions with spikes and poles. The different approach of inverting the Laplace transform on the real axis or on the complex plane is illustrated in Figure 3.

Letting the contour by any vertical line $\gamma = a$ such that $f(\gamma)$ has no singularities on or to the right of it, the original function $F(\tau)$ is given by the inversion formula:

$$F(\tau) = \frac{1}{2\pi i} \int_{a-i\infty}^{a+i\infty} e^{\gamma\tau} f(\gamma)\, d\gamma \tag{4.2}$$

where $i = \sqrt{-1}$. Alternatively, setting $a+iu = \gamma$ and exploiting the fact that $\operatorname{Im}((f(a+iu))) = -\operatorname{Im}((f(a-iu)))$:

$$F(\tau) = \frac{e^{a\tau}}{\pi} \int_0^{\infty} \left[\operatorname{Re}(f(a+iu)) \cos uh - \operatorname{Im}((f(a+iu))) \sin uh\right] du \tag{4.3}$$

Equations (4.2) and (4.3) can be replaced by the cosine and sine transform pairs:

$$\begin{aligned} \operatorname{Re}(f(a+iu)) &= e^{a\tau} \int_0^{+\infty} F(\tau) \cos(uh)\, dh \\ F(\tau) &= \frac{2e^{a\tau}}{\pi} \int_0^{+\infty} \operatorname{Re}(f(a+iu)) \cos(u\tau)\, du \end{aligned} \tag{4.4}$$

and:

$$\begin{aligned} \operatorname{Im}(f(a+iu)) &= -e^{a\tau} \int_0^{+\infty} c(\tau, q) \sin(u\tau)\, d\tau \\ F(\tau) &= -\frac{2e^{a\tau}}{\pi} \int_0^{+\infty} \operatorname{Re}(f(a+iu)) \sin(u\tau)\, du \end{aligned}$$

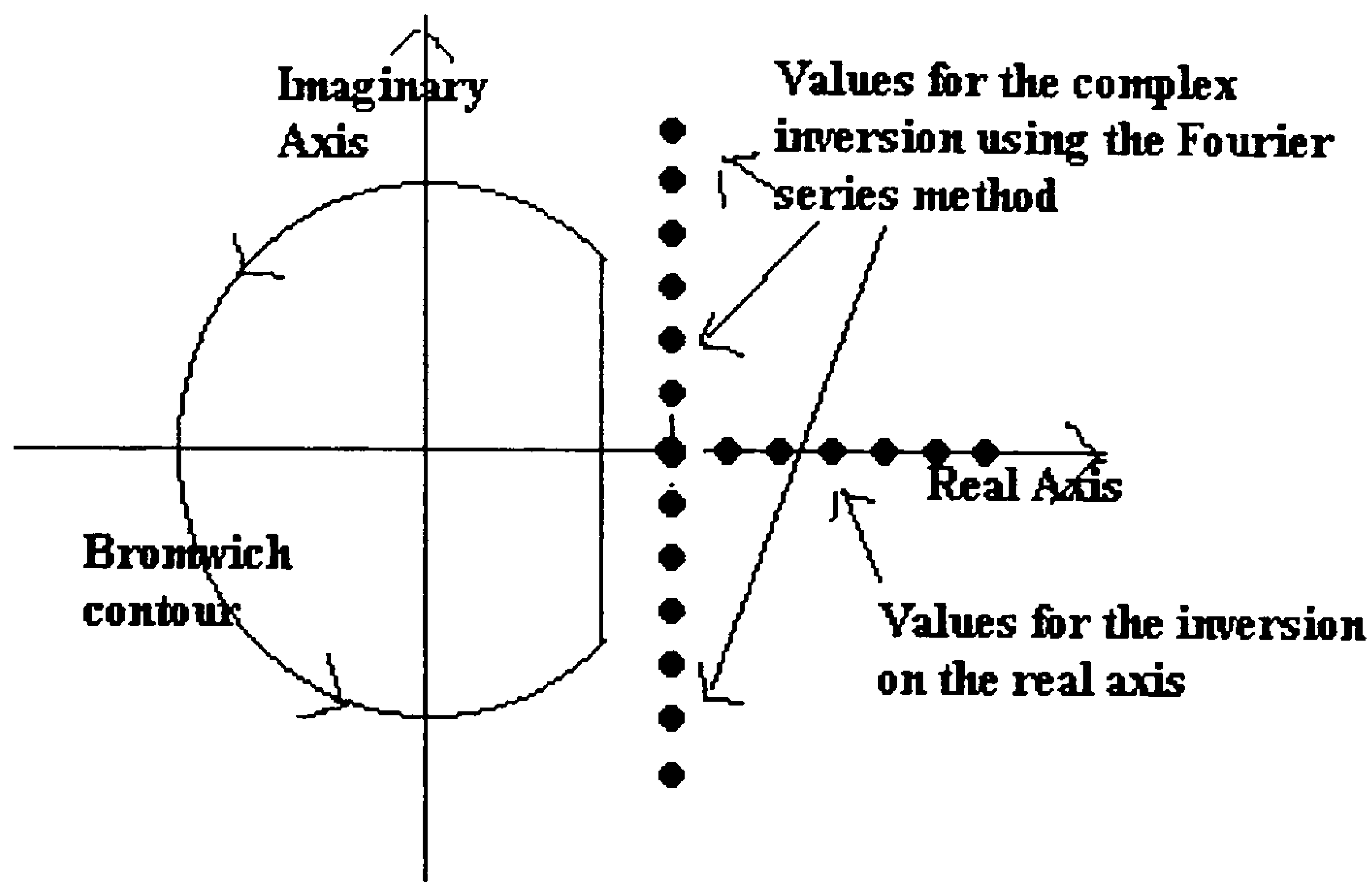


Figure 4.1: Sample points for the inversion with the Fourier series method using the Bromwich contour and sampla point for the inversion on the real axis using the definition of Laplace transfom.

### 4.1. Unidimensional inversion

In Davies and Martin [25] it can be found a review and a comparison of many numerical techniques available through 1979. More recently Duffy [31] compares three popular methods to numerically invert the Laplace transform[10].

The choice that we have done regarding which methods to use were due to the multidimensional inversion problem related to the pricing of corridor options studied in another chapter of the present thesis. In the literature the multidimensional inversion has been very rarely studied and indeed we have found very few references. The only unidimensional methods with a straightforward extension to the multidimensional case were the Fourier series and the Padé approximant methods. Moreover, both methods required very little computational effort in their implementation and, of greater importance, they gave very

[10]The methods examined in Duffy are a) the Crump inversion method based on a straightforward application of the trapezoidal rule to Bromwich's integral; b) the Weeks method that integrates the Bromwich's integral by using Laguerre polynomials; c) the Talbot method that deforms the Bromwich's contour so that it begins and ends in the third and second quadrant of the $\gamma$-plane. If the locations of the singularities are known, this scheme may provide accurate results at minimal computational expense.

successfully results. Then we have tried their application to the simple unidimensional inversion case, a very strange path indeed. It is however our intention to complete a more extensive study including other inversion methods, such as the Talbot [94] and the Weeks' method [102], that in the Duffy review appear very promising[11]. However, note that their implementation is more involved, because they require the user to provide a numerical value for some parameters and there is no automatic procedure or theoretical estimates for their computation. In particular, we have tried some preliminary tests with the Week-s's methods and its accuracy resulted to be very high when applied to the inversion of the Black-Scholes formula. But when we have tried other problems, such as Asian options and barrier options we encountered great difficulties in finding a good setting for the free parameters. So it is our intention to do much more research on this in future. At this regard, a recent paper by Weideman [103] seems to give more insights about the choice of the free parameters. Interesting extensions can be found in several other works, compare Abate et al. [1], and Garbow et al. [38], [39].

The Fourier series method is essentially a trapezoidal rule approximation to (4.3) or to (4.4). An essential feature of this method is that an expression for the error in the computed inverse transform is available which allows one to control the maximum error in the inversion technique. Since the trapezoidal rule is a quite simple integration procedure, its use can appear surprising. It turns out to be surprisingly effective in this context with periodic and oscillating integrands, because the errors tend to cancel. In particular, it turns out to be better than familiar alternatives such as Simpson's rule for inversion integrals. The third proposed method is based on the Padé approximant of the exponential function appearing in the Bromwich integral. The algorithm is very simple to implement,

[11] A real advantage of the Week's method respect to other methods is that it return an explicit form of the approximate solution. Consequently, if the inverse is desired at many values of $t$, these extra values can be obtained at little additional cost because most of the computational cost is spent in calculating the coefficients of the Laguerre expansions. In the other methods, we need to restart for each value of $\tau$.

although the control of the error is not so clear as for the other two methods.

### 4.1.1. The Fourier series method

Two of the most well-known Fourier series methods are due to Dubner and Abate, successively improved by Abate and Whitt, and to Crump. The underlying idea of the method is to discretize the Bromwich integral using the trapezoidal rule. Then the inversion is given as a sum of infinite terms. The two methods then differ in the use of an accelerating sequence (Euler and epsilon algorithm) for improving the convergence of the series that can be computed with great accuracy with a limited number of terms (in many of the examples studied in the present work no more than 30. This allows to perform the inversion at a reduced cost and with a very low computational time (in the examples usually less than 1").

**The Abate-Whitt inversion formula** This algorithm has been proposed by Dubner and Abate [29] and discussed in greater detail in Abate and Whitt [3]. It consists in applying the trapezoidal rule with interval size equal to $\Delta$ to the expression in (4.4), so considering only $\mathrm{Re}[F(\gamma)]$. We obtain:

$$F(\tau) \simeq F_{\Delta}^{DA}(\tau) = \frac{\Delta e^{a\tau}}{\pi} \mathrm{Re}(f(a)) + \frac{2\Delta e^{a\tau}}{\pi} \sum_{k=1}^{\infty} \mathrm{Re}(f(a+ik\Delta)) \cos(k\Delta\tau)$$

We can eliminate the cosine terms and obtain an alternating series by letting $\Delta = \pi/2\tau$ and setting at the same time $a = A/2\tau$, we obtain:

$$F_{\Delta}^{DA}(\tau) = \frac{e^{A/2}}{2\tau} \mathrm{Re}\left(f\left(\frac{A}{2\tau}\right)\right) + \frac{e^{A/2}}{\tau} \sum_{k=1}^{\infty} (-1)^k \mathrm{Re}\left(f\left(\frac{A+2k\pi i}{2\tau}\right)\right) \tag{4.5}$$

which is (21) of Dubner and Abate [29].

The choice of $A$ has to be made in such a way that $a$ falls at the right of the real part of all the singularities of the function $f(\gamma)$. In Abate and Whitt [3], pag. 30, is discussed how the value of A is chosen in order to control the discretization error[12]

$$E_d = \left| F(\tau) - F_{\Delta}^{DA}(\tau) \right|$$

that, assuming that $|F(\tau)| < M$, is given by

$$E_d < M \frac{e^{-A}}{1 - e^{-A}} \simeq M e^{-A}$$

so that we should set $A$ large in order to make $E_d$ small: in order to obtain a discretization error less than $10^{-\delta}$, we can set $A = \delta \ln 10$. However, increasing $A$ can make the inversion (4.5) harder to calculate requiring higher precision to avoid roundoff errors. Thus $A$ should not be chosen too large. In the numerical examples we have seen that a very good choice is to set $A = 18.4$ as suggested in Abate and Whitt. Figure 4.2 illustrates the behavior of the difference between the analytical Black-Scholes price and the inverse of the option as function of $A$. If we increase $A$ too much we can incur in round-off errors and the inversion can give results completely inaccurate. For the example in Figure, the optimal value of $A$ results to be near 25, although also with A=18.4 we obtain a five-digits accuracy (BS price= 5.28326898765, AW(A=25)=5.28326898815, AW(A=18.4)=5.28326913124, AW(A=55)=5.28270626068).

The remaining problem consists in computing the infinite sum in (4.5). If the term $\mathrm{Re}\left(f\left(\frac{A+2k\pi i}{2\tau}\right)\right)$ has a constant sign for all $k$, it can be convenient to consider an accelerating algorithm for alternating series and Euler summation is one of the more elementary

[12]The discretization error is obtained approximating the given function by a periodic function that can be represented by its Fourier series (aliasing). This explains the name Fourier-series method. Then it can be shown that this representation is just the trapezoidal rule approximation applied to the inversion integral, and comparing the two expressions we obtain an explicit expression for the discretization error associated with the trapezoidal rule approximation. Compare Abate and Whitt [3], pagg. 22 and following. The same idea is used for obtaininig a bound for the discretization error in the Crump inversion method.

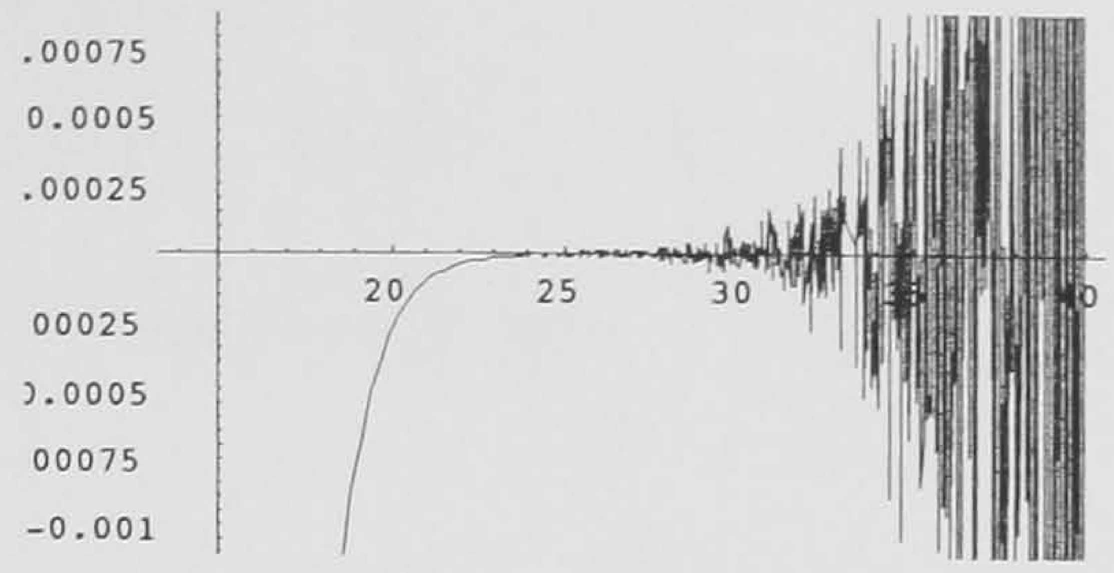


Figure 4.2: In figure we have represented 10000*( difference between the BS price and the numerical inversion of the Laplace transform), compare example 2, as function of A. Increasing A the difference can be made very small, but then we start to incurr in numerical problems. The inversion has been done using the Dubner Abate numerical inversion with the Euler algorithm, setting n=35, n+m+1=47. The parameters are spot price=100, strike=100, time to maturity=1 yr, volatility=0.05, interest rate=0.05.

acceleration techniques. Abate and Whitt suggests its use primarily for its simplicity and because, for practical purposes, it seems to provide adequate computational efficiency. We describe it in next section.

*The Euler algorithm.* For the series in (4.5) it is natural to apply an alternative method of summation well suited for alternating series. This approach was originally proposed by Simon, Stroot and Weiss [89] and consists in applying Euler summation. Practically the final result is a weighted average of the last $m$ partial sums by a binomial probability distribution with parameters $m$ and $p = 1/2$. In particular if we apply the Euler sum to $m$ terms after an initial $n$ terms, then the Euler sum[13] $E(\tau, m, n)$, as a function of $n$ and $m$, gives:

$$F_{\Delta}^{DA}(\tau) \approx E(\tau, m, n) = \sum_{k=0}^{m} \binom{m}{k} 2^{-m} s_{n+k}(\tau) \tag{4.6}$$

where $s_n(\tau)$ is the nth partial sum:

$$s_n(\tau) = \frac{e^{A/2}}{2\tau} \operatorname{Re}\left(f\left(\frac{A}{2\tau}\right)\right) + \frac{e^{A/2}}{\tau} \sum_{k=1}^{n} (-1)^k \operatorname{Re}\left(f\left(\frac{A + 2k\pi i}{2\tau}\right)\right) \tag{4.7}$$

An alternative way of expressing the Euler sum is through the difference operator $\Delta$.

[13]For details on the Euler sum compare Press and al. [81] pag. 167, and Abate and Whitt [3], pag. 46.

With $\{a_n : n \geq 1\}$ being a sequence, let $\Delta a_n = a_{n+1} - a_n$ and let $\Delta^k$ be the k-fold iterate. If we have an alternating series for which the $kth$ terms is $(-1)^k a_k$, $a_k \geq 0$, then (4.6) is equivalent to the more conventional definition:

$$E(\tau, m, n) = s_n(\tau) + (-1)^{n+1} \sum_{k=0}^{m-1} (-1)^k 2^{-(k+1)} \Delta^k a_{n+1}(\tau)$$

As pointed out in Abate and Whitt, pag. 46, in order for Euler summation to be effective, we need to have $a_k = \operatorname{Re}\left(f\left(\frac{A+2k\pi i}{2\tau}\right)\right)$ to have three properties for sufficiently large $k$: a) to be of constant sign, b) to be monotone, c) to have the higher-order differences $(-1)^m \Delta^m a_{n+k}$ be monotone. On a practical side, these properties are not checked and it will be hard to know, in the case they hold, when $k$ is large enough, so that the algorithm is used in a heuristic way. However, if the underlying function has appropriate smoothness the desired properties hold. Abate and Whitt suggests to use the absolute difference:

$$|E(\tau, m, n+1) - E(\tau, m, n)| = \left| \sum_{k=0}^{m} \binom{m}{k} 2^{-m} a_{n+k+1}(\tau) \right|$$

as estimate of the truncation error associated with Euler summation, although it does not always work. Usually, $E(\tau, m, n)$ computes the true sum with an error of the order of $10^{-13}$ or less with the choice $n = 38$ and $m = 11$, i.e. using a total of 50 terms against the direct computation of the infinite series that requires more than 10000 terms.

**The Crump inversion** The inversion formula due to Crump [23] is based on applying the trapezoidal rule approximation to (4.3) and setting $\Delta = \pi/H$, $H > \tau$:

$$\begin{aligned} F(\tau) \simeq F_{\Delta}^{CR}(\tau) &= \tfrac{e^{a\tau}}{2H} \operatorname{Re}(f(a)) \\ &+ \tfrac{e^{a\tau}}{2H} \sum_{k=1}^{\infty} \left[\operatorname{Re}(f(a + ik\pi/H)) \cos k\pi\tau/H - \operatorname{Im}((f(a + ik\Delta/H))) \sin k\pi\tau/H\right] \end{aligned} \tag{4.8}$$

We observe that this approximation contains all of the terms in (4.5) plus additional terms involving Im(C). Indeed it can be shown that $F_{\Delta}^{CR}(\tau)$ is simply the average of the approximate inverses using only cosine and sine terms, respectively. As consequence also the error is an average, so that in general it will significantly smaller because the two errors have different sign. Moreover, the parameter $a$ can be chosen so that $F_{\Delta}^{CR}(\tau)$ approximates $F(\tau)$ in the interval (0, 2H), whereas the error when using $F_{\Delta}^{DA}(\tau)$ can be made small only for $\tau$ in the interval (0, H).

Also in this case it is possible to exploit an accelerating algorithm for computing the sum in (4.8). The algorithm suggested is the epsilon-algorithm, MacDonald [71], given by:

$$\epsilon_{i+1}^{(m)} = \epsilon_{i-1}^{(m)} + \frac{1}{\epsilon_i^{(m+1)} - \epsilon_i^{(m)}}$$

where $\epsilon_{-1}^{(m)} = 0$ and $\epsilon_0^{(m)}$ is the m-th partial sum of 4.8. The quantity $\epsilon_n^{(0)}$, with $n$ even, represents the extrapolated sum. Under the assumption that $|F(\tau)| < Me^{\alpha\tau}$, the discretization error $E_d = |F(\tau) - F_{\Delta}^{CR}(\tau)|$ can be bounded by:

$$E_d \leq M\frac{e^{\alpha\tau}}{e^{2H(a-\alpha)} - 1}, 0 < \tau < 2\tau$$

and choosing $\alpha$ large we can make the error as small as desired. This error bound provides a simple algorithm for computing $F(\tau)$ to a predetermined accuracy. If we want to compute $F(\tau)$ over a range of values of which the largest is $\tau_{\max}$ and the relative error $E/Me^{\alpha\tau}$ is to be no greater than $E'$, then $H$ is chosen so that $2H > \tau_{\max}$ and

$$\alpha = a - \frac{\ln E}{2H}$$

where $a$ is a number slightly larger that the real part of the greatest pole of the Laplace transform.

*The epsilon algorithm.* The epsilon algorithm used in the Crump's inversion method originates with Wynn [106] and involves a two-dimensional table called the $\varepsilon - table$, given below. The subscript $k$ of $\varepsilon_k^{(j)}$ denotes the column and the superscript $j$ measures the progression down the column.

Table 4: The $\epsilon$−table

| | | | |
|---|---|---|---|
| $\varepsilon_{-1}^{(0)}$ | | | |
| | $\varepsilon_0^{(0)}$ | | |
| $\varepsilon_{-1}^{(1)}$ | | $\varepsilon_1^{(0)}$ | |
| | $\varepsilon_0^{(1)}$ | | $\varepsilon_2^{(0)}$ |
| $\varepsilon_{-1}^{(2)}$ | | $\varepsilon_1^{(1)}$ | |
| | $\varepsilon_0^{(2)}$ | | $\vdots$ |
| $\varepsilon_{-1}^{(3)}$ | | $\vdots$ | |
| $\vdots$ | $\vdots$ | | |

Table 4 is constructed iteratively from its first two columns. Define $\varepsilon_{-1}^{(j)}$ to be zero and $\varepsilon_0^{(j)}$ the given sequence for $j = 0, 1, 2...$ Then all the other elements may be calculated from the $\varepsilon$-algorithm, which is:

$$\varepsilon_{k+1}^{(j)} = \varepsilon_{k-1}^{(j+1)} + \left[\varepsilon_k^{(j+1)} - \varepsilon_k^{(j)}\right]^{-1}$$

In order to understand how this rule has to be applied, the following rombus scheme shows how to connect the elements in the table above.

Table 5: The $\epsilon$–algorithm

$$
\begin{array}{ccc}
 & \varepsilon_k^{(j)} & \\
\varepsilon_{k-1}^{(j+1)} & & \varepsilon_{k+1}^{(j)} \\
 & \varepsilon_k^{(j+1)} & \\
 & & \vdots
\end{array}
$$

It is remarkable that the sequence of the fourth column is the same as that obtained from Aitken's $\delta^2$ rule. The $\epsilon$–algorithm may be then regarded as a sequence-to-sequence transformation: it is an algorithm for transforming the elements of the given series in the second column to the elements on the principal diagonal. In other words, the sequence

$$\left\{\varepsilon_0^{(j)}, j = 0, 1, 2..\right\}$$

is transformed in a new sequence

$$\left\{\varepsilon_k^{(0)}, k = 0, 1, 2..\right\}$$

#### 4.1.2. The Padé approximant method

This inversion routine has been proposed by Singhal and Vlach [90]. Greater details can be found in Vlach and Singhal, chpt. 10 [98].

If we apply the transformation $z = \lambda\tau$ to (4.2) and we approximate the function $e^z$ by a rational Padé approximation:

$$R_{N,M}(z) = \frac{P_N(z)}{Q_M(z)} = \frac{\sum\limits_{k=0}^{N} (N+M-k)!\binom{N}{k} z^k}{\sum\limits_{k=0}^{M} (-1)^k (N+M-k)!\binom{M}{k} z^k}$$

the approximation $F^{PADE}(\tau)$ becomes:

$$F^{PADE}(\tau) = \frac{1}{2\pi i\tau}\int_{a'-i\infty}^{a'+i\infty} f\left(\frac{z}{\tau}\right) R_{N,M}(z)\, dz$$

Now the integral can be evaluated by residue calculus by closing the path of integration along an infinite arc either to the right or to the left. In order that the path along the infinite arc does not contribute to the integral, choose $M$ and $N$ such that the integrand has at least two more finite poles than zeros. Then we have the inversion formula:

$$F^{PADE}(\tau) = -\frac{1}{\tau}\sum_{k=1}^{M} f\left(\frac{z_k}{\tau}\right) K_k \tag{4.9}$$

where $z_k$ are the poles of $R_{N,M}(z)$ and $K_k$ are the corresponding residues. The poles and the residues can be previously calculated with high precision and the inversion can be done instantaneously. Unfortunately, in this case there is not a clear bound on the error so a heuristic way for estimating the size of the error is solving the same problem twice: first with a given $M$ and $N$ and the second time with the same $M$ but $N$ smaller by one. As long as the responses coincide the results are correct. Taking higher $M$ and $N$ means that a greater number of terms in the expansion are matched exactly. However increasing $M$ and $N$ too much we incur in numerical problems in the computation of the poles and residues. $N = M = 24$ is about the highest approximation which can be used with double precision without running into roundoff problems.

We have the following properties for the method:

a) it inverts exactly the function $\gamma^{-m}$, $N+2-M \leq m \leq M+N+1$ and the result is $F^{PADE}(\tau) = F(\tau) = \tau^{m-1}/(m-1)!$;

b) it inverts exactly the first $M+N+1$ terms of the Taylor series of any time response. Consequently, as long as the original function is approximated well by the initial portion of its Taylor series, the numerical inversion gives excellent results;

c) the inversion formula cannot be used to obtain $F(\tau)$ when $\tau$=0, because of the division by $\tau$. It is then convenient to use the initial value theorem

$$\lim_{\tau \to 0} F(\tau) = \lim_{\gamma \to \infty} \gamma f(\gamma)$$

and it can be shown that continuous functions $F(\tau)$ bounded at $\tau = 0$ are inverted exactly at $\tau = 0$ by using the above limiting procedure.

d) derivatives respect to $\tau$ can be obtained using the related and very simple inversion formula:

$$\frac{\partial^n F^{PADE}(\tau)}{\partial \tau^n} = -\frac{1}{\tau} \sum_{k=1}^{M} f\left(\frac{z_k}{\tau}\right) \left(\frac{z_k}{\tau}\right)^n K_k$$

We can also remark as the inversion formula recalls a Gaussian quadrature, a well-known method for the approximation of integrals. Indeed Wellekens [105] has shown the equivalence of the Padé inversion method with the Gaussian quadrature for the Bromwich integral. In these methods the original function can be approximated by the quadrature formula:

$$F(\tau) \approx \frac{1}{\tau} \sum_{k=1}^{M} f(p_k) A_k \tag{4.10}$$

where the weights $A_k$ and the tabular points $p_k$ are such as to ensure that the above inversion formula is an exact one for all polynomials of degree $2M - 1$.

As already said for the Padé inversion method, the basic shortcomings are that: a) there is no convenient rule for determining at the outset the order $N$ such that the desired accuracy is obtained; b) when $M$ is large, the weights are also large and this leads to considerable cancellation errors. Piessens [79] has shown how to avoid these two difficulties extending the $M$-point Gaussian rule of precision $2M - 1$ to a $(2M + 1)$-point quadrature formula of precision degree $3M + 1$, which makes use of the original $M$ Gauss abscissas.

These new quadrature formulas have weights which are much smaller than the weights of the Gaussian formulas with the same degree of precision. In this sense we can understand related inversion methods due to Piessens. Zakian [107] has studied the optimal way for choosing the weights and the points.

Finally, in the literature we have found also a considerable number of papers, Longman [68], where the Laplace transform, and not the exponential function, is replaced by Padé approximants in the inversion integral. But the application of this method requires the knowledge of the Taylor expansion of the Laplace transform about the origin and it is not possible to implement a procedure which uses only values of the Laplace transform.

## 4.2. The Mathematica Code

In this section we give the Mathematica code of the previous algorithms. In a paper by Cheng et al. [15] we have found the code for the Crump's algorithm, whilst the code for the other two methods has been written by us. We can observe that the code requires very few lines, because it is possible to exploit built-in functions, as the Euler summation for the Abate-Whitt code and the calculations of the residues and poles for the Padé approximants. However, we have written all the code also in Microsoft Visual C++ 5.0 using the C code and we have used the code in Mathematica mainly for testing purposes. For difficult inversions as for the Asian option case the inversion in Mathematica was very slow. Moreover, sometimes Mathematica incurred in not very clear numerical errors that we did not see in C. So in general it is preferable to use a compiled code. The Crump algorithm is available also as IMSL routine in Fortran.

The Crump algorithm requires some preliminary code (Durbin method). Once we have the expression for the terms in the inverse transform, we apply the accelerating sequence given by the epsilon algorithm. The computation of the inverse requires to the user the specification of a functional form of the function to be inverted, the Laplace transform variable and then the parameters to be used in the numerical routine: the even number of terms in the series for performing the epsilon algorithm, tmax the maximum value at which to compute the original function, alpha the leading pole of the image function (default value 0), and the tolerance level (default value 10^-8).

Table 6: **Crump algorithm (written by Cheng et al.)**

```
(* Durbin method *)

adurb[tmax_, alpha_, tol_] := adurb[tmax, alpha, tol] =alpha - Log[tol] / (8 tmax) //N


cdurb[F_, s_, tmax_, alpha_, tol_, 0] := F/2 /. s -> adurb[tmax, alpha, tol] //N


(* The inverse function *)

NLInvDurb[F_, s_, t_, j_, tmax_, alpha_:0, tol_:0.00000001] :=


NLInvDurb[F, s, t, j, tmax, alpha, tol] =( Exp[ adurb[tmax, alpha, tol] t ] / (4 tmax) )

 ( Sum[ Re[ cdurb[F, s, tmax, alpha, tol, k] ] Cos[ k Pi t / (4 tmax) ] -

 Im[ cdurb[F, s, tmax, alpha, tol, k] ] Sin[ k Pi t / (4 tmax) ], {k, 0, j} ] ) //N


(* Crump method *)

 (* Define the "epsilon algorithm " *)

eps[ F_, s_, t_, tmax_, alpha_, tol_, i_, m_] :=

eps[ F, s, t, tmax, alpha, tol, i, m] =  eps[ F, s, t, tmax, alpha, tol, i - 2, m + 1 ] +

1 / ( eps[ F, s, t, tmax, alpha, tol, i - 1, m + 1 ] - eps[ F, s, t, tmax, alpha, tol, i - 1, m ] )


eps[ F_, s_, t_, tmax_, alpha_, tol_, -1, m_] := 0


eps[ F_, s_, t_, tmax_, alpha_, tol_, 0, m_] := NLInvDurb[F, s, t, m, tmax, alpha, tol]


(* The inverse function *)

NLInvCrmp[F_, s_, t_, n_, tmax_, alpha_:0, tol_:0.00000001] :=

epsilon[F, s, t, 0.2 tmax, alpha, tol, n, 0]
```

Example: if we have to invert the function $1/(\gamma - b)$ then we have:

```
F[s_,b_]:=1/(s-b)

InverseCrmp[t_,b_,n_,tmax_,alpha_,tol_]:=

NLInvCrmp[F[s,b], s, t, n, tmax, alpha, tol]
```

and computing InverseCrmp[2, 3, 28, 4,3, 10^-12] we obtain 403.428793494 whilst computing the inversion with only 12 terms, InverseCrmp[2, 3, 12, 4,3, 10^-12], we obtain 402.381886739. The correct answer is $\exp(b * t) = \exp(3 * 2) = 403.428793493$.

In the Abate-Whitt algorithm we need the EulerSum algorithm that can be called from the library NLimit.

Table 7: **AW algorithm (written by G. Fusai)**

```
(*call the package NLimit*)

<<NumericalMath`NLimit`;


(*define the terms in the sum*)

akAW[F_, s_, t_, k_, A_]:= Re[F] /. s-> (A+ 2*k*Pi*I)/(2*t) //N


termAW[F_, s_, t_, k_, A_]:= Re[F] /. s-> A/(2*t) //N


(*do the inversion*)

NLInvAbate[F_, s_, t_, m_:15, n_:11, A_:18.4]:=

Exp[A/2]*termAW[F, s, t,k, A]/(2*t) +(Exp[A/2]/t)*

EulerSum[((-1)^k)*akAW[F, s, t,k, A], {k, 1, Infinity}, Terms->m, ExtraTerms->n] //N
```

Example: if we have to invert the function $1/(\gamma - b)$ then we have:

```
F[s_,b_]:=1/(s-b)

InverseAW[t_,b_,m_,n_,A_]:=

NLInvAbate[F[s,b], s, t, m, n, A]
```

and computing InverseAW[2, 3, 15, 11,32.4] we obtain 403.428793318 whilst computing the inversion with A=18.4, InverseAW[2, 3, 15, 11,18.4], we obtain 404.100229167, a result not very satisfactory.

In the Padé algorithm we need to call the libraries Calculus and NResidue for the computation of the Padé approximant to the exponential function and then for the determination of poles and residues. We remark that it is convenient to calculate once for all the poles and the residues of the Padé approximant and then store them. In this way when the computation of the Laplace transform is not difficult, the inversion is very fast and requires less than fractions of one second.

Example: with the same function to be inverted as in the other examples we have:

```
F[s_,b_]:=1/(s-b)

InversePade[t_,b_,num_,den_]:=

NLInvPade[F[s,b], s, t, num,den]
```

and computing InversePade[2, 3, 3, 18] we obtain 403.426190270194062 + I*2.273'*$10^{-13}$, the imaginary part has indeed to be equal to 0. Computing InversePade[2, 3, 14, 18] we obtain 405.622934503 + I*4.14446199313. The fact that the imaginary part is signficantly different from zero is signal of numerical problems in the computation of the poles and residues. This problem occurs increasing the degree of the numerator or of the denominator.

Table 8: **Padé algorithm (written by G. Fusai)**

```
(*call the package Nresidue*)

<<NumericalMath`NResidue`;

(*call the package Pade)

<< Calculus`Pade`;


(*define the routine for computing poles and residues*)

NLInvPade[F_, s_, t_, numk_,denk_]:=

 Block[{poli,residui,pade, sum,i},

 pade= Pade[Exp[x], {x, 0, numk, denk}]; (*define the Pade approximant*)

polipade= x /. NSolve[Denominator[pade]==0,x]; (*find the poles*)

i=1;

 sum=0;


(*compute the residues*)

 While[i<=denk,

 {respade= NResidue[pade, {x,polipade[[i]]}], (*find the residues*)

 term = F /. s-> polipade[[i]]/t, (*start the inversion*)

 sum = sum+respade*term,

 i=i+1}];

 Return[-sum/t] ] (**this is the inverse**)
```

## 4.3. Multidimensional inversion

In this section we illustrate how to adapt the previous numerical schemes to the problem of multidimensional inversion. This kind of problem arises when we solve the pricing PDE using a double Laplace transform. For example in the case of interest rate derivatives there was the problem of evaluating an Asian option with floating strike and we obtained the mgf of the joint density of the interest rate and of the time average, i.e. we had basically a double Laplace transform. In general, when we treat path dependent variables of the form:

$$\int_0^t h(W_s)\,ds$$

we obtain an expression for the mgf of this quantity solving the Feynman-Kac PDE. In this case, for example for occupation time derivatives or even for Asian options, it is convenient to take the Laplace transform with respect to time and so we obtain the Laplace transform of the mgf, i.e. again we have a double Laplace transform. For this reason, it is convenient to examine how the above numerical schemes can be adapted to cope with the multidimensional inversion problem. Unfortunately, in the numerical literature relatively little attention has been given to this topic. Except for the extension of the Padé inversion, other algorithms have begun to appear in the literature only in 1994.

Basically, we illustrate the extension of the Fourier inversion method due to Choudhury, Lucantoni and Whitt (CLW) [18] and the multidimensional Padé inversion due to Singhal, Vlach and Vlach [91]. We have found also an extension of the Crump inversion method due to Moorthy [74] but we had difficulties in his application to the case of occupation derivatives, for the presence of free parameters that cannot be fixed in a simple way, so we do not describe his work. Also multidimensional version of the Weeks method due to Abate et al. [2] and Moorthy [75] have appeared, but we do not have tested them.

### 4.3.1. The multidimensional Fourier inversion method

The algorithm due to CLW exploits the Fourier series method and consists in a multivariate extension of the Euler algorithm in Abate and Whitt, with an enhancement in order to control simultaneously the aliasing and roundoff errors. Their algorithm can be adapted to consider three types of two-dimensional transforms: i) continuous-continuous, ii) continuous-discrete and iii) discrete-discrete.

Let $f(\gamma_1, \gamma_2)$ be the two-dimensional Laplace transform of the function $F(\tau_1, \tau_2)$ :

$$f(\gamma_1, \gamma_2) = \int_0^{\infty} \int_0^{\infty} e^{-\gamma_1 \tau_1 - \gamma_2 \tau_2} F(\tau_1, \tau_2)\, d\tau_1 d\tau_2$$

and we assume that $f(\gamma_1, \gamma_2)$ is well defined, and $\mathrm{Re}(\gamma_1) > 0$ and $\mathrm{Re}(\gamma_2) > 0$.

The idea of the algorithm is to damp the given function by multiplying by a two dimensional decaying exponential function and approximate the damped function by a periodic function constructed by aliasing, i.e. the new function is constructed by adding translated versions of the original function. The two parameters in the exponential function control the aliasing error in the approximation using the periodic function. The inversion formula results to be the two dimensional Fourier series of the periodic function. Moreover, similarly to the one dimensional case, the periods of the periodic function can be chosen so that the two-dimensional Fourier series is a series nested within a second series, each of them being nearly an alternating series. This allows to calculate in an efficient way the infinite series by exploiting the Euler algorithm.

As in our cases, if $F(\tau_1, \tau_2)$ is a real function the inversion algorithm can be expressed as:

$$
\begin{aligned}
&F(\tau_1,\tau_2) \simeq F^{CLW}(\tau_1,\tau_2) = \frac{\exp\left(\frac{A_1}{2l_1}+\frac{A_2}{2l_2}\right)}{4\tau_1\tau_2 l_1 l_2} \\
&\times \Bigg\{ f\left(\tfrac{A_1}{2l_1\tau_1},\tfrac{A_2}{2l_2\tau_2}\right) + 2\sum_{k_1=1}^{l_2}\sum_{k=0}^{+\infty}(-1)^k \operatorname{Re}\left[e^{-\frac{ik_1\pi}{l_2}} f\left(\tfrac{A_1}{2l_1\tau_1},\tfrac{A_2}{2l_2\tau_2}-\tfrac{ik_1\pi}{\tau_2 l_2}-\tfrac{ik\pi}{\tau_2}\right)\right] \\
&+2\sum_{j_1=1}^{l_1}\sum_{j=0}^{+\infty}(-1)^j \operatorname{Re}\left[\sum_{k_1=1}^{l_2}\sum_{k=0}^{+\infty}(-1)^k e^{-\left(\frac{ij_1\pi}{l_1}+\frac{ik_1\pi}{l_2}\right)} f\left(\tfrac{A_1}{2l_1\tau_1}-\tfrac{ij_1\pi}{l_1\tau_1}-\tfrac{ij\pi}{\tau_1},\tfrac{A_2}{2l_2\tau_2}-\tfrac{ik_1\pi}{\tau_2 l_2}-\tfrac{ik\pi}{\tau_2}\right)\right] \\
&+2\sum_{j_1=1}^{l_1}\sum_{j=0}^{+\infty}(-1)^j \operatorname{Re}\Bigg[e^{-\left(\frac{ij_1\pi}{l_1}\right)} f\left(\tfrac{A_1}{2l_1\tau_1}-\tfrac{ij_1\pi}{l_1\tau_1}-\tfrac{ij\pi}{\tau_1},\tfrac{A_2}{2l_2\tau_2}\right) \\
&+\sum_{k_1=1}^{l_2}\sum_{k=0}^{+\infty}(-1)^k e^{-\left(\frac{ij_1\pi}{l_1}-\frac{ik_1\pi}{l_2}\right)} f\left(\tfrac{A_1}{2l_1\tau_1}-\tfrac{ij_1\pi}{l_1\tau_1}-\tfrac{ij\pi}{\tau_1},\tfrac{A_2}{2l_2\tau_2}+\tfrac{ik_1\pi}{\tau_2 l_2}+\tfrac{ik\pi}{\tau_2}\right)\Bigg]\Bigg\}
\end{aligned}
\tag{4.11}
$$

that basically consists in a trapezoidal approximation to the inversion formula.

In the inversion formula we have alternating sums, i.e. sums of the form $\sum_{k=0}^{+\infty}(-1)^k a_k$, where $a_k$ is real or complex. As described in the one-dimensional inversion formula, we can exploit the Euler transformation for computing infinite sums of alternating type and when $a_k$ is real.

The sources of error in the inversion formula are: a) discretization error in using the trapezoidal rule for the inversion, b) truncation error due to using a finite number of terms in the infinite series, c) roundoff error, which is due to multiplying large number by small ones. The first two errors are discussed also in the unidimensional case, whilst the third one is treated only in this case.

The discretization error can be reduced increasing the parameters $A_1$ and $A_2$, whilst the truncation error can be reduced to the order of $10^{-13}$ or less using the Euler algorithm with the choice $n = 38$ and $m = 11$. The roundoff error depends on the term $\exp\left(\frac{A_1}{2l_1}+\frac{A_2}{2l_2}\right)/(4\tau_1\tau_2 l_1 l_2)$ that can be large, so that there are four parameters to control it: $A_1$, $A_2$, $l_1$ and $l_2$. For the fact that $A_1$ and $A_2$ control the discretization error, $l_1$ and $l_2$ can be used for the roundoff error. As it appears clear from the above formula, we can make small the term $\exp\left(\frac{A_1}{2l_1}+\frac{A_2}{2l_2}\right)/(4\tau_1\tau_2 l_1 l_2)$ increasing $l_1$ and $l_2$, but this increases the computational time which is proportional to the product of $l_1$ and $l_2$. CLW say that with $l_1 = l_2 = 1$ we can usually achieve an overall accuracy of 5 or 6 digits and with $l_1 = l_2 = 2$ an overall accuracy of 10 or more digits. In the two-dimensional inversion usually this is an adequate choice.

### 4.3.2. The multidimensional Padé inversion

The extension of the unidimensional method is very straightforward and has been proposed by Singhal et al. [91]. This algorithm seems to have been for some time the only known inversion technique for the numerical inversion of multidimensional Laplace transforms. Following the same approach as for the single case inversion, i.e. approximating the exponential functions in the inversion integral by Padé rational functions and using the residue calculus and closing the path in the right half plane we get the final formula that is simply a generalization of (4.9):

$$F^{PADE}(\tau_1,\tau_2)=\frac{1}{\tau_1\tau_2}\sum_{i=1}^{m_1}\sum_{i=1}^{m_2}K_{1i}K_{2i}f\left(\frac{z_{1i}}{\tau_1},\frac{z_{2i}}{\tau_2}\right) \tag{4.12}$$

where $z_{ji}$, $j=1,2$, are the poles of the two Padé approximants of the exponential functions in the inversion contour and $K_{ji}$ are the corresponding residues.

The inversion reduces to a double summation and requires just $m_1 \times m_2$ function evaluations and the evaluation of the poles and the residues of the Padé approximant that have been already computed for the unidimensional case. So, on a programming point of view, the algorithm can be implemented very easily.

## 5. Numerical examples

In this section we illustrate the application of the numerical inversion of the Laplace transform to several problems that we have illustrated in the first part of this chapter.

### 5.1. Example: Forward contracts

Let us consider the case of a forward contract with forward price $K$, so that $\varphi_1(S) = S - K$. The actual value of the forward contract is given by $C_1(S,\tau) = S - Ke^{-r(T-t)}$, independently by any assumptions regarding the dynamics of $S$. Taking the Laplace transform wrt $T-t$ we obtain:

$$\mathcal{L}(C_1(S,\tau)) = \int_0^\infty e^{-\gamma\tau}\left(S - Ke^{-r\tau}\right)d\tau = \frac{S}{\gamma} - \frac{K}{\gamma + r}$$

We observe that the poles are located at 0 and $-r$ and so in the inversion the largest pole is given by max[0, $-r$]. In Table 9, for different values of time to maturity, we give the analytical value of a Forward contract and the percentage differences respect to the true value of the three inversion methods previously described.

Table 9: Value of a Forward Contract and % errors

Parameters setting: S=100, K=100, r=0.05.

| T-t | Analytical | %AW*$10^6$ | %Cr*$10^6$ | %Pade*$10^6$ |
|---|---|---|---|---|
| 0.02 | 0.0 9995 | -0.37% | 0.72% | -0.24% |
| 0.5 | 2. 4690 | -0.28% | 0.67% | -0.24% |
| 1 | 4. 8771 | -0.29% | 0.62% | -0.25% |
| 3 | 13. 9292 | -0.34% | 0.48% | -0.26% |
| 5 | 22. 1199 | -0.51% | 0.23% | -0.33% |

In Table 9 the settings for the different numerical inversion methods are: AW: A=18.4, m=15, n=11; Cr: a=3+maxpole+20/(2*H), H=5(T-t)/4, n=23; Padé: M=4 and N=18.

### 5.2. Example: Black-Scholes formula

The second example is given by the option contract. We substitute $\varphi_2(e^x) = (e^x - K)^+$ in (3.13) and the price of the call option is given by:

$$C(S, T-t) = e^{\alpha \ln S + \beta \frac{\sigma^2}{2}(T-t)} \mathcal{L}^{-1}\left(v(\ln S, \gamma), \gamma \to \frac{\sigma^2}{2}(T-t)\right)$$

where the expression for $v(x, \gamma)$ is given by 3.15:

Table 10 gives the maximum error for the different inversion methods.

Table 10: Maximum Error

| | Price*$10^6$ | Delta*$10^6$ | Gamma*$10^9$ |
|---|---|---|---|
| AW | -0.796149997 | -0.028435329 | -2.150113892 |
| Crump | -3.458221499 | -1.48082111 | 76.79226451 |
| Padé | 0.773100496 | 0.024909719 | 7.566674502 |

In the following three tables we compare for different values of the volatility the analytical expression for the option premium, the delta and the gamma as given by the BS formula in (3.11) and the corresponding values obtained by inversion of the Laplace transform. The parameter settings in AW are A=18.4, m=15, m+n+1=26, in Padé M=1 and N=20, in Crump we have set a=3+maxpole + 20/(2*H), H=5$\tau$/4 and n=29.

| Table 11: Black-ScholesOption Price | | | | | |
|---|---|---|---|---|---|
| $S = 100$ | $r = 0.05$ | $T - t = 1$ | difference*$10^6$ | | |
| $\sigma$ | $K$ | Analytical | AW | Crump | Padé |
| 0.05 | 70 | 33.4139 | -0.8 | -2.3310 | 0.5135 |
| 0.1 | 70 | 33.4140 | -0.0225 | -3.4582 | 0.7023 |
| 0.2 | 70 | 33.5401 | 0.1067 | -1.5419 | 0.7584 |
| 0.3 | 70 | 34.3953 | 0.0953 | -1.5620 | 0.7632 |
| 0.5 | 70 | 38.1162 | -0.0302 | -1.5818 | 0.7731 |
| 0.05 | 80 | 23.9016 | -0.6661 | -1.6224 | 0.3470 |
| 0.1 | 80 | 23.9100 | -0.0795 | -1.2149 | 0.4741 |
| 0.2 | 80 | 24.5888 | 0.0017 | -1.0790 | 0.5108 |
| 0.3 | 80 | 26.4621 | -0.0399 | -1.0958 | 0.5246 |
| 0.5 | 80 | 31.7925 | -0.1632 | -1.2068 | 0.5456 |
| 0.05 | 90 | 14.3906 | -0.5361 | -0.8410 | 0.1803 |
| 0.1 | 90 | 14.6288 | -0.1411 | -0.6359 | 0.2460 |
| 0.2 | 90 | 16.6994 | -0.1206 | -0.6276 | 0.2756 |
| 0.3 | 90 | 19.6974 | -0.1598 | -0.6969 | 0.2932 |
| 0.5 | 90 | 26.3727 | -0.2593 | -0.8924 | 0.3308 |
| 0.05 | 95 | 9.6708 | -0.4718 | -0.0849 | 0.0971 |
| 0.1 | 95 | 10.4053 | -0.1778 | -0.3980 | 0.1358 |
| 0.2 | 95 | 13.3465 | -0.1653 | -0.4295 | 0.1627 |
| 0.3 | 95 | 16.8012 | -0.2001 | -0.3089 | 0.1871 |
| 0.5 | 95 | 23.9829 | -0.2952 | -0.7581 | 0.2385 |
| 0.05 | 100 | 5.2833 | -0.4038 | -0.2352 | 0.0194 |
| 0.1 | 100 | 6.8050 | -0.1939 | -0.1436 | 0.0392 |
| 0.2 | 100 | 10.4506 | -0.1926 | -0.2313 | 0.0727 |
| 0.3 | 100 | 14.2313 | -0.2289 | -0.3224 | 0.1047 |
| 0.5 | 100 | 21.7926 | -0.3245 | -0.4863 | 0.1661 |
| 0.05 | 105 | 2.0543 | -0.2959 | -0.2017 | 0.0007 |
| 0.1 | 105 | 4.0461 | -0.1802 | 0.0351 | 0.0117 |
| 0.2 | 105 | 8.0214 | -0.2046 | -0.0437 | 0.0349 |
| 0.3 | 105 | 11.9769 | -0.2474 | -0.1434 | 0.0619 |
| 0.5 | 105 | 19.7898 | -0.3478 | -0.4308 | 0.1195 |
| 0.05 | 110 | 0.5078 | -0.1949 | -0.0237 | 0.0017 |
| 0.1 | 110 | 2.1739 | -0.1360 | -0.1907 | 0.0014 |
| 0.2 | 110 | 6.0401 | -0.1997 | 0.0266 | 0.0210 |
| 0.3 | 110 | 10.0201 | -0.2556 | -0.0226 | 0.0420 |
| 0.5 | 110 | 17.9623 | -0.3652 | -0.1911 | 0.0911 |
| 0.05 | 120 | 0.0067 | -0.0616 | -0.0003 | -0.0019 |
| 0.1 | 120 | 0.4625 | -0.0761 | -0.0078 | 0.0023 |
| 0.2 | 120 | 3.2475 | -0.1486 | -0.0036 | 0.0038 |
| 0.3 | 120 | 6.9040 | -0.2394 | 0.0522 | 0.0222 |
| 0.5 | 120 | 14.7833 | -0.3825 | 0.0149 | 0.0610 |
| 0.05 | 130 | 0.0000 | -0.0123 | -0.0001 | -0.0013 |
| 0.1 | 130 | 0.0673 | -0.0394 | -0.0055 | -0.0016 |
| 0.2 | 130 | 1.6396 | -0.1059 | -0.0354 | -0.0025 |
| 0.3 | 130 | 4.6734 | -0.1943 | 0.0099 | 0.0073 |
| 0.5 | 130 | 12.1591 | -0.3770 | 0.0860 | 0.0442 |

| Table 12: Black-Scholes Option Delta | | | | | |
|---|---|---|---|---|---|
| $S$=100 | $r = 0.05$ | $T - t = 1$ | difference*$10^6$ | | |
| Vol | Strike | Analytical | AW | Crump | Padé |
| 0.05 | 70 | 1 | -0.0171 | -0.0643 | 0.0168 |
| 0.1 | 70 | 1.0000 | 0.00 | -0.0530 | 0.0230 |
| 0.2 | 70 | 0.9836 | 0.0082 | -0.0526 | 0.0249 |
| 0.3 | 70 | 0.9339 | 0.0095 | -0.0492 | 0.0248 |
| 0.5 | 70 | 0.8562 | 0.0101 | -0.0426 | 0.0238 |
| 0.05 | 80 | 1.0000 | -0.0171 | -0.0631 | 0.0168 |
| 0.1 | 80 | 0.9973 | 0.0039 | -0.0567 | 0.0229 |
| 0.2 | 80 | 0.9286 | 0.0092 | -0.0496 | 0.0244 |
| 0.3 | 80 | 0.8555 | 0.0108 | -0.0433 | 0.0240 |
| 0.5 | 80 | 0.7871 | 0.0075 | -0.0385 | 0.0234 |
| 0.05 | 90 | 0.9991 | -0.0170 | -0.0616 | 0.0168 |
| 0.1 | 90 | 0.9456 | 0.0050 | -0.0495 | 0.0226 |
| 0.2 | 90 | 0.8097 | 0.0085 | -0.0780 | 0.0236 |
| 0.3 | 90 | 0.7479 | 0.0068 | -0.0385 | 0.0231 |
| 0.5 | 90 | 0.7125 | 0.0045 | -0.0385 | 0.0212 |
| 0.05 | 95 | 0.9799 | -0.0166 | -0.0621 | 0.0166 |
| 0.1 | 95 | 0.8561 | 0.0040 | -0.0506 | 0.0221 |
| 0.2 | 95 | 0.7279 | 0.0051 | 0.0396 | 0.0220 |
| 0.3 | 95 | 0.6871 | 0.0045 | -0.0328 | 0.0205 |
| 0.5 | 95 | 0.6746 | 0.0032 | -0.0678 | 0.0183 |
| 0.05 | 100 | 0.8473 | -0.0206 | -1.4808 | 0.0106 |
| 0.1 | 100 | 0.7088 | -0.0015 | -0.0366 | 0.0131 |
| 0.2 | 100 | 0.6368 | 0.0020 | -0.0041 | 0.0135 |
| 0.3 | 100 | 0.6243 | 0.0024 | -0.0299 | 0.0135 |
| 0.5 | 100 | 0.6368 | 0.0020 | -0.0282 | 0.0135 |
| 0.05 | 105 | 0.5196 | -0.0284 | -0.4627 | 0.0019 |
| 0.1 | 105 | 0.5248 | -0.0085 | -0.0083 | 0.0027 |
| 0.2 | 105 | 0.5422 | -0.0012 | -0.0182 | 0.0047 |
| 0.3 | 105 | 0.5612 | 0.0004 | -0.0212 | 0.0065 |
| 0.5 | 105 | 0.5996 | 0.0008 | -0.0254 | 0.0087 |
| 0.05 | 110 | 0.1891 | -0.0221 | -0.0044 | 0.0007 |
| 0.1 | 110 | 0.3434 | -0.0115 | 0.0005 | 0.0018 |
| 0.2 | 110 | 0.4496 | -0.0051 | -0.0043 | 0.0024 |
| 0.3 | 110 | 0.4996 | -0.0020 | -0.0112 | 0.0035 |
| 0.5 | 110 | 0.5633 | -0.0005 | 0.0447 | 0.0058 |
| 0.05 | 120 | 0.0044 | -0.0108 | 0.0000 | 0.0012 |
| 0.1 | 120 | 0.1015 | -0.0064 | 0.0010 | -0.0001 |
| 0.2 | 120 | 0.2872 | -0.0085 | 0.0043 | 0.0017 |
| 0.3 | 120 | 0.3855 | -0.0067 | 0.0011 | 0.0021 |
| 0.5 | 120 | 0.4942 | -0.0031 | -0.0103 | 0.0031 |
| 0.05 | 130 | 0.0000 | -0.0031 | 0.0000 | -0.0007 |
| 0.1 | 130 | 0.0191 | -0.0040 | -0.0003 | -0.0005 |
| 0.2 | 130 | 0.1681 | -0.0048 | 0.0009 | -0.0003 |
| 0.3 | 130 | 0.2885 | -0.0082 | 0.0054 | 0.0018 |
| 0.5 | 130 | 0.4306 | -0.0059 | -0.0016 | 0.0023 |

| Table 13: Black-Scholes Option Gamma | | | | | |
|---|---|---|---|---|---|
| $S = 100$ | $r = 0.05$ | $T - t = 1$ | difference*$10^9$ | | |
| Vol | Strike | Analytical | AW | Crump | Padé |
| 0.05 | 70 | 2.808E-16 | 0.0000 | 0.0000 | 0.0000 |
| 0.1 | 70 | 8.333E-06 | -0.0034 | -0.0001 | 0.0032 |
| 0.2 | 70 | 2.049E-03 | -0.0230 | 0.0061 | -0.0528 |
| 0.3 | 70 | 4.281E-03 | -0.1491 | 0.1356 | 0.0842 |
| 0.5 | 70 | 4.533E-03 | 0.0952 | -0.1307 | 0.0113 |
| 0.05 | 80 | 2.302E-08 | -0.0001 | 0.0000 | -0.0002 |
| 0.1 | 80 | 8.336E-04 | -0.0276 | -0.0118 | 0.0150 |
| 0.2 | 80 | 6.814E-03 | -0.2518 | -0.0945 | 0.1257 |
| 0.3 | 80 | 7.578E-03 | 0.1184 | -0.4592 | 0.0236 |
| 0.5 | 80 | 5.811E-03 | 0.2509 | 2.5231 | 0.0659 |
| 0.05 | 90 | 5.910E-04 | -0.0172 | -0.0468 | -0.0058 |
| 0.1 | 90 | 1.103E-02 | -0.3843 | -0.1532 | 0.1803 |
| 0.2 | 90 | 1.358E-02 | 0.5619 | -2.9657 | 0.0728 |
| 0.3 | 90 | 1.064E-02 | 0.4427 | -0.3346 | 0.2508 |
| 0.5 | 90 | 6.818E-03 | 0.2482 | -0.4638 | 0.3818 |
| 0.05 | 95 | 9.741E-03 | -0.3184 | -7.8147 | 0.1480 |
| 0.1 | 95 | 2.268E-02 | 0.9513 | -2.1308 | 0.1394 |
| 0.2 | 95 | 1.660E-02 | 0.6334 | -1.2026 | 0.7157 |
| 0.3 | 95 | 1.181E-02 | 0.4133 | 4.1068 | 0.8197 |
| 0.5 | 95 | 7.202E-03 | 0.2384 | -0.4750 | 0.7090 |
| 0.05 | 100 | 4.718E-02 | 1.5084 | 76.7923 | 7.5667 |
| 0.1 | 100 | 3.429E-02 | 1.0963 | -1.6232 | 5.4996 |
| 0.2 | 100 | 1.876E-02 | 0.5998 | -0.6682 | 3.0087 |
| 0.3 | 100 | 1.265E-02 | 0.4043 | -0.8975 | 2.0282 |
| 0.5 | 100 | 7.505E-03 | 0.2399 | -0.2046 | 1.2035 |
| 0.05 | 105 | 7.969E-02 | -2.1501 | -44.0335 | 1.0262 |
| 0.1 | 105 | 3.982E-02 | 1.7001 | -3.7753 | 0.3112 |
| 0.2 | 105 | 1.984E-02 | 0.7473 | -1.4540 | 0.9202 |
| 0.3 | 105 | 1.314E-02 | 0.4564 | -1.6618 | 0.9533 |
| 0.5 | 105 | 7.729E-03 | 0.26 | -0.51 | 0.78 |
| 0.05 | 110 | 5.412E-02 | -1.0532 | -10.4777 | -1.6064 |
| 0.1 | 110 | 3.678E-02 | -0.8720 | -1.4254 | 0.4330 |
| 0.2 | 110 | 1.979E-02 | 0.8492 | -1.4308 | 0.1718 |
| 0.3 | 110 | 1.330E-02 | 0.5385 | -0.9440 | 0.3925 |
| 0.5 | 110 | 7.878E-03 | 0.2811 | -1.9467 | 0.4924 |
| 0.05 | 120 | 2.569E-03 | -1.2734 | 0.0126 | 1.7376 |
| 0.1 | 120 | 1.774E-02 | -0.2468 | -0.5027 | -0.4991 |
| 0.2 | 120 | 1.704E-02 | -0.2867 | -0.1209 | 0.1659 |
| 0.3 | 120 | 1.275E-02 | 0.4520 | -0.6790 | 0.0318 |
| 0.5 | 120 | 7.978E-03 | 0.3350 | -0.5823 | 0.1711 |
| 0.05 | 130 | 1.073E-05 | -0.6125 | -0.0002 | 0.9300 |
| 0.1 | 130 | 4.647E-03 | -0.2677 | -0.7274 | 0.3224 |
| 0.2 | 130 | 1.256E-02 | -0.2957 | -0.1846 | 0.2219 |
| 0.3 | 130 | 1.138E-02 | -0.1132 | -0.0785 | 0.0907 |
| 0.5 | 130 | 7.858E-03 | 0.3259 | -1.3495 | 0.0430 |

### 5.3. Example: Double knock-out barrier options

The price of a double knock out option is given by the inverse Laplace transform of (B.7) or in an alternative way by the price of a standard call option minus the price of two knock-in options that enter in live when the asset price touches one of the two barriers.

Using the notation of Appendix B, we have that the price of a double knock out option can be expressed as:

$$\begin{aligned}
C(S,K,U,L) &= e^{\alpha \ln(S/L)+\beta\tau} \mathcal{L}^{-1}\left(v(\ln(S/L),\gamma)\,;\gamma \to \frac{\sigma^2}{2}(T-t)\right) \\
x &= \ln(S/L)\,, \tau = \frac{\sigma^2}{2}(T-t) \\
l &= \ln(U/L)\,, k = \ln(K/L) \\
k_1 &= \frac{2r}{\sigma^2}, \alpha = -\frac{1}{2}(k_1-1)\,, \beta = -\frac{1}{4}(k_1-1)^2 - \frac{2r}{\sigma^2}
\end{aligned}$$

where, using some algebra and equation (B.8) in Appendix B, it is possible to show that the function $v(x,\gamma)$ admits the following representation:

$$v(x,\gamma) = \begin{cases} \left(Lg_1(k,l,\gamma,\alpha-1) - Kg_1(k,l,\gamma,\alpha)\right) 1_{(S\leq K)} \\ \begin{pmatrix} Lg_1(x,l,\gamma,\alpha-1) - Kg_1(x,l,\gamma,\alpha) \\ +Lg_2(k,x,\gamma,\alpha-1) - Kg_2(k,x,\gamma,\alpha) \end{pmatrix} 1_{(S>K)} \end{cases}$$

where:

$$\begin{aligned}
g_1(m,M,\gamma,\alpha) &= \frac{\sinh\left(x\sqrt{\gamma}\right)}{\sqrt{\gamma}\sinh\left(l\sqrt{\gamma}\right)} \int_m^M e^{-\alpha\xi} \sinh\left((M-\xi)\sqrt{\gamma}\right) d\xi \\
g_2(m,M,\gamma,\alpha) &= \frac{\sinh\left((l-x)\sqrt{\gamma}\right)}{\sqrt{\gamma}\sinh\left(l\sqrt{\gamma}\right)} \int_m^M e^{-\alpha\xi} \sinh\left(\xi\sqrt{\gamma}\right) d\xi
\end{aligned}$$

If in the above formula, we let $U \to +\infty$ (or $L \to 0$), we obtain an expression for down and out (up and out) barrier option.

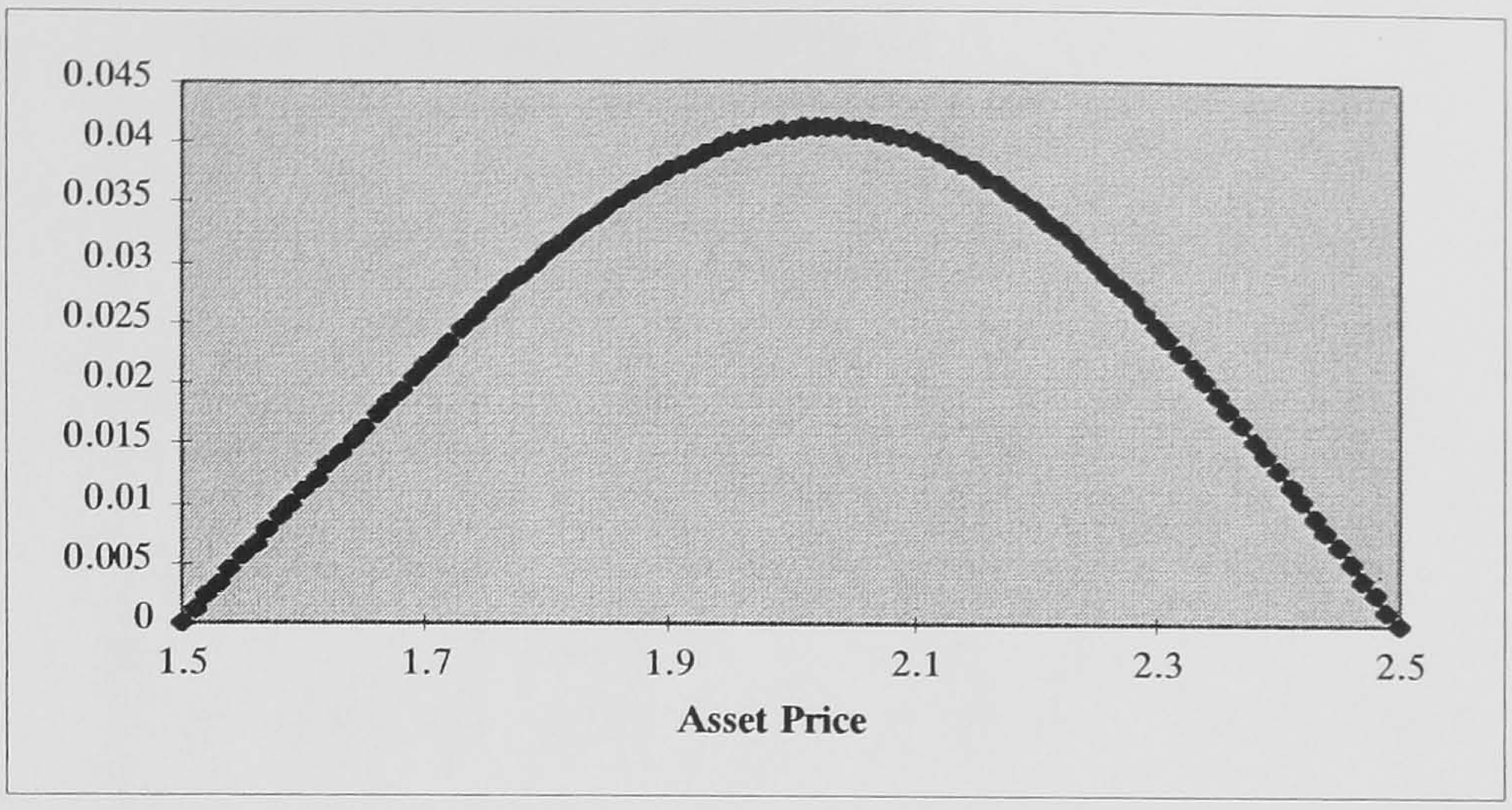


Figure 5.1: Price of the double knock-out barrier option as function of the price of the underlyng asset. The inversion has been made with the AW algorithm. Parameters: U=2.5, L=1.5, K=2, T-t=1, vol=0.2, r=0.05.

In Tables 14, 15 and 16 we report the prices of a double knock-out option as from Table 3.1. in Kunitomo and Ikeda. Practically, all different inversion methods agree up to the fourth digit.

Table 14: Double barrier knock-out option prices

Parameters setting: T-t=1/12, S=1000, K=1000, r=0.05

| $\sigma$ | U | L | KI | AW | Crump | Padé |
|---|---|---|---|---|---|---|
| 0.2 | 1500 | 500 | 25.12 | 25.1207 | 25.1207 | 25.1207 |
| 0.2 | 1200 | 800 | 24.76 | 24.7568 | 24.7568 | 24.7568 |
| 0.2 | 1050 | 950 | 2.15 | 2.1462 | 2.1462 | 2.1462 |
| 0.3 | 1500 | 500 | 36.58 | 36.5842 | 36.5842 | 36.5842 |
| 0.3 | 1200 | 800 | 29.45 | 29.4473 | 29.4473 | 29.4473 |
| 0.3 | 1050 | 950 | 0.27 | 0.2707 | 0.2707 | 0.2707 |
| 0.4 | 1500 | 500 | 47.85 | 47.8475 | 47.8475 | 47.8475 |
| 0.4 | 1200 | 800 | 25.84 | 25.8428 | 25.8428 | 25.8427 |
| 0.4 | 1050 | 950 | 0.02 | 0.0152 | 0.0152 | 0.0152 |

Table 15: Double barrier knock-out option prices

Parameters setting: T-t=1/4, S=1000, K=1000, r=0.05

| $\sigma$ | U | L | KI | AW | Crump | Padé |
|---|---|---|---|---|---|---|
| 0.2 | 1500 | 500 | 46.12 | 46.1163 | 46.1163 | 46.1163 |
| 0.2 | 1200 | 800 | 30.39 | 30.3916 | 30.3916 | 30.3916 |
| 0.2 | 1050 | 950 | 0.08 | 0.0795 | 0.0795 | 0.0795 |
| 0.3 | 1500 | 500 | 63.49 | 62.3388 | 62.3388 | 62.3388 |
| 0.3 | 1200 | 800 | 19.31 | 19.3105 | 19.3105 | 19.3104 |
| 0.3 | 1050 | 950 | 0.00 | 0.0002 | 0.0002 | 0.0002 |
| 0.4 | 1500 | 500 | 65.84 | 65.8435 | 65.8435 | 65.8434 |
| 0.4 | 1200 | 800 | 10.69 | 10.6920 | 10.6920 | 10.6919 |
| 0.4 | 1050 | 950 | 0.00 | 0.0000 | 0.0000 | -0.0000 |

Table 16: Double barrier knock-out option prices

Parameters setting: T-t=1/2, S=1000, K=1000, r=0.05

| $\sigma$ | U | L | KI | AW | Crump | Padé |
|---|---|---|---|---|---|---|
| 0.2 | 1500 | 500 | 66.13 | 66.1289 | 66.1289 | 66.1289 |
| 0.2 | 1200 | 800 | 22.08 | 22.0820 | 22.0820 | 22.0819 |
| 0.2 | 1050 | 950 | 0.00 | 0.0006 | 0.0006 | 0.0006 |
| 0.3 | 1500 | 500 | 67.88 | 67.8773 | 67.8773 | 67.8772 |
| 0.3 | 1200 | 800 | 9.26 | 9.2640 | 9.2640 | 9.2640 |
| 0.3 | 1050 | 950 | 0.00 | -0.0000 | -0.0000 | -0.0000 |
| 0.4 | 1500 | 500 | 53.35 | 53.3454 | 53.3454 | 53.3453 |
| 0.4 | 1200 | 800 | 3.14 | 3.1374 | 3.1374 | 3.1373 |
| 0.4 | 1050 | 950 | 0.00 | -0.0000 | 0.0000 | -0.0000 |

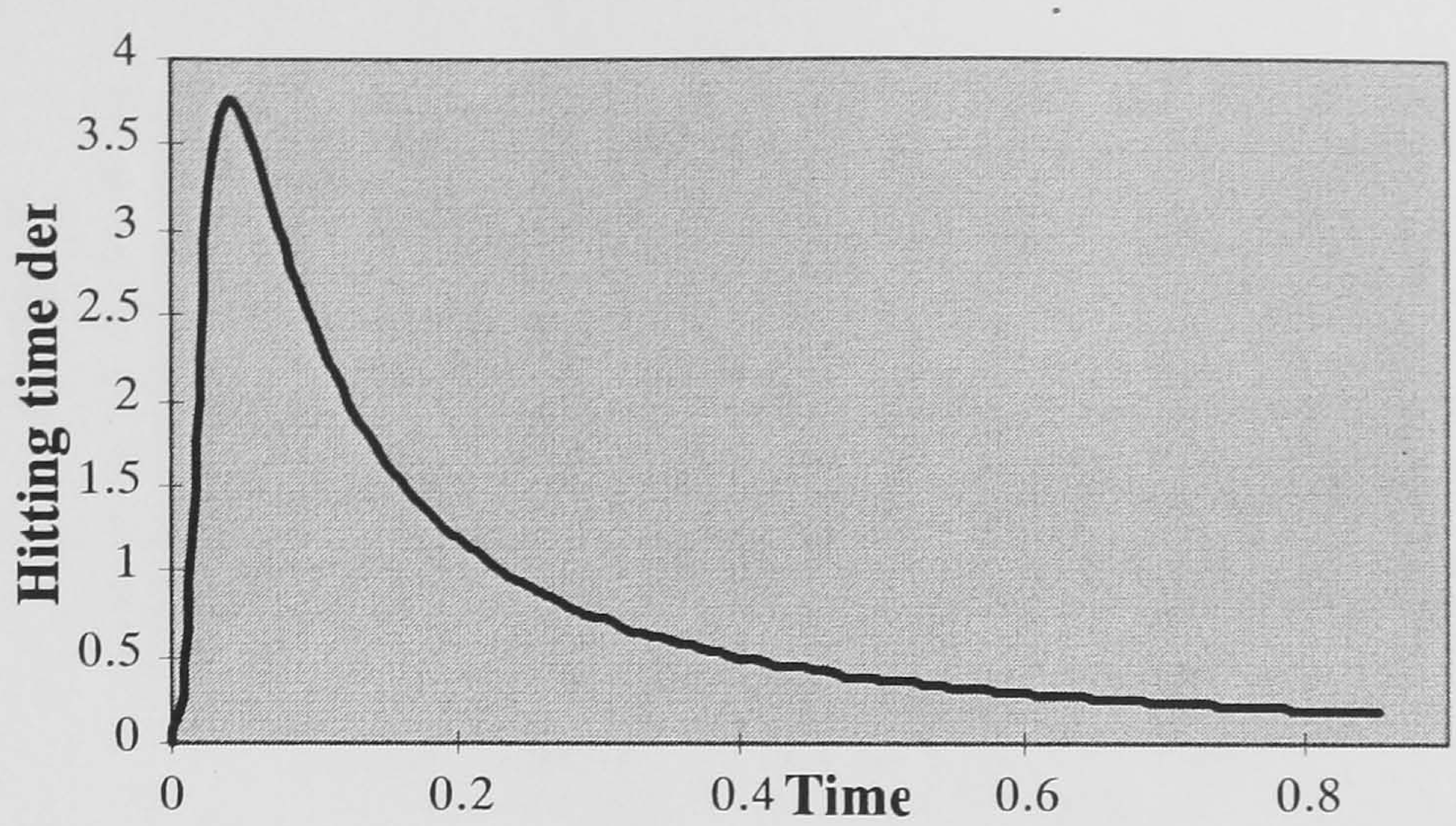


Figure 5.2: Density of the hitting time in the Vasicek model, where dr=(0.02-0.2 r) dt + 0.1414dW, $r_0$=0.15. The lower barrier is equal to 0.1. The inversion has been done using the Padé inversion method with num=3 and den=18. The grap h as been truncated at Time=0.9.

### 5.4. Example: Hitting Time density in the Vasicek model

Solving the pricing problem for barrier options, we have found that the distribution of the first hitting at the lower barrier $b$, given that the asset returns follow an O-U process:

$$d\ln S = (m - k\ln S)\,dt + \sigma dW$$

has not touched the lower barrier is obtained by the following Laplace inverse wrt $\tau = \sigma^2 (T-t)/2$:

$$\mathcal{L}^{-1}\left(\frac{H_{-\gamma/k}(x)}{H_{-\gamma/k}\left(\hat{b}\right)}1_{\left(x>\hat{b}\right)}; \gamma- > \tau\right) \tag{5.1}$$

$$x = \alpha(\ln S - m/k); \alpha = \sqrt{k}/\sigma; \hat{b} = \alpha(b - m/k) \tag{5.2}$$

In Figure (5.2) we represent the density of the first hitting time obtained numerically inverting the expression above. The shape is very similar to the case of the GBM.

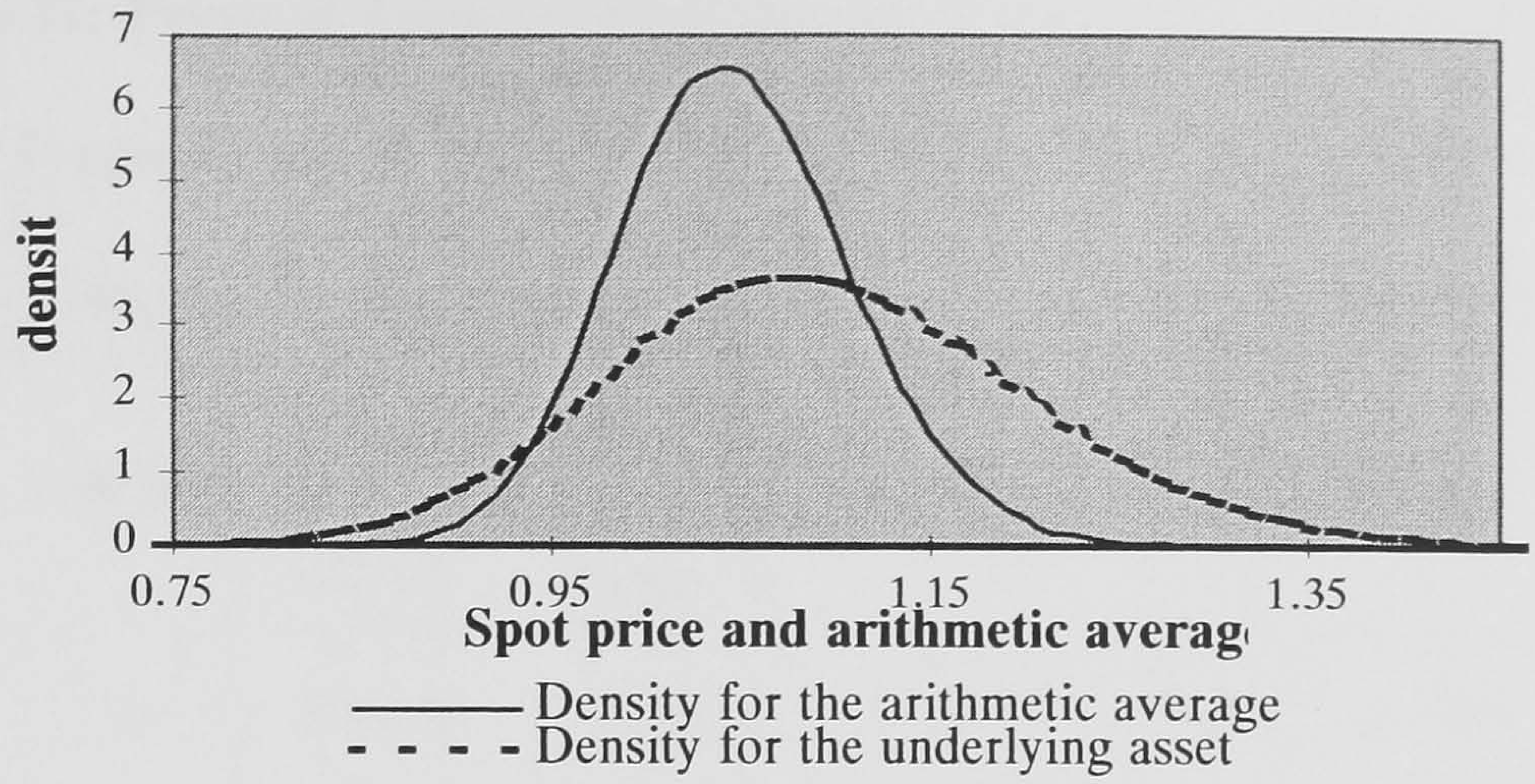


Figure 5.3: Density law of the average and of the underliyng asset. T-t=1, volatility =0.1, $S_0$=1, r=0.09. The process for the asset is GBM. The inversion has been done using the AW algorithm

### 5.5. Example: Asian Options

More details about this problem can be found in the chapter on the Asian option valuation. In Figure (5.3) we compare the densities of the average and of the underlying asset (i.e. a lognormal density). From Table 17, we note that the Fourier series inversion methods appear very reliable. Also Fu et al. [37] have tested the AW algorithm with satisfactory results and have shown the inefficiency of MonteCarlo simulation. Geman and Eydeland [40] have performed the inversion using the FFT, but their results do no seem to be very accurate. Respect to the lognormal approximation, the Laplace inversion gives prices inside the lower and upper bound found in Thompson [95], even when they are very tight. Moreover, the inversion is made with very few terms: just 26 (=15+11) in the A-W inversion and 29 in Crump, so that the computation was very fast usually less than 1". Also the computation of the Greeks does not represent a problem. The Padé method is not very satisfactory, probably because the Laplace transform is not well approximated by a rational series expansion. In the numerical inversion there were numerical problems due to truncation errors in the computer when $\sigma\sqrt{T-t} < 0.07$.

Table 17: Prices of Asian Options (Approximations and Numerical Inversion)

Parameters settings: S=100, T-t=1, r=0.09; AW A=18.4, m=15, m+n+1=26;

Crump: n=29, tmax=vol*vol*0.8/4, $\alpha$=2*(r/sg^2-1)+4+20/(2*tmax); Pade num=3, den=18.

| Volat. | Strike | Lower Bound | Logn. Approx. | Padé | AW | Crump | Upper Bound |
|---|---|---|---|---|---|---|---|
| 0.1 | 90 | 13.38519 | 13.38629 | 12.74305 | 13.38537 | 13.38599 | 13.38603 |
| 0.1 | 95 | 8.911834 | 8.917207 | 8.631854 | 8.911854 | 8.911814 | 8.912955 |
| 0.1 | 100 | 4.915075 | 4.9231 | 5.265086 | 4.915119 | 4.915113 | 4.915409 |
| 0.1 | 105 | 2.069929 | 2.070451 | 2.746642 | 2.070066 | 2.070062 | 2.070383 |
| 0.1 | 110 | 0.630065 | 0.62338 | 1.074435 | 0.630273 | 0.630272 | 0.631017 |
| 0.2 | 90 | 13.83122 | 13.86167 | 13.57645 | 13.83150 | 13.83144 | 13.83721 |
| 0.2 | 95 | 9.99536 | 10.03043 | 9.85872 | 9.99565 | 9.99569 | 9.99807 |
| 0.2 | 100 | 6.77700 | 6.80355 | 6.79681 | 6.77735 | 6.77736 | 6.77866 |
| 0.2 | 105 | 4.29594 | 4.30408 | 4.42273 | 4.29647 | 4.29646 | 4.29798 |
| 0.2 | 110 | 2.54546 | 2.53453 | 2.69438 | 2.54622 | 2.54622 | 2.54854 |
| 0.3 | 90 | 14.98279 | 15.06704 | 14.88814 | 14.98396 | 14.98397 | 14.99285 |
| 0.3 | 95 | 11.65475 | 11.73287 | 11.57120 | 11.65589 | 11.65588 | 11.66128 |
| 0.3 | 100 | 8.82755 | 8.88576 | 8.77743 | 8.82876 | 8.82876 | 8.83329 |
| 0.3 | 105 | 6.51635 | 6.54628 | 6.50346 | 6.51779 | 6.51780 | 6.52257 |
| 0.3 | 110 | 4.69491 | 4.69511 | 4.71063 | 4.69671 | 4.69673 | 4.70265 |

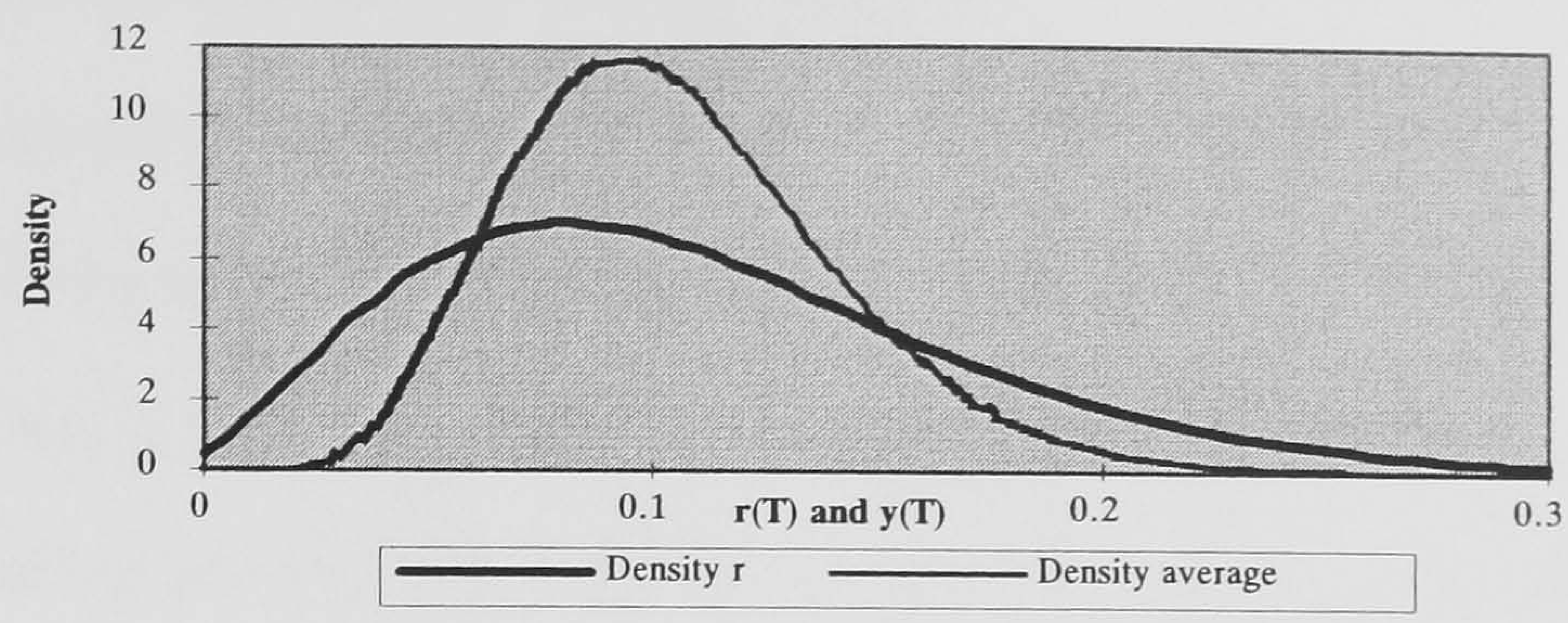


Figure 5.4: Density law for the interest rate and for its time average in the CIR model dr = (0.02-0.1*r) dt + 0.2 $r^{1/2}$ dW, $r_o$ =0.1, T-t=1. Both densities have been obtained inverting the mgf with the Padé method (num=3, den=20). Note that $2a/\sigma^2=1$ and then from (3.56) and (3.57) we have that the value of the density at $r_T = 0$ is equal to 0.45268. The numerical value obtained by inversion is 0.45269.

### 5.6. Example: Interest Rate Derivatives

In Table 18 we compare the prices of an option on a zcb given by the analytical formula in CIR, with those obtained inverting the Laplace transform as described in the section on options on zcb, equation (3.62). Note that for computing the formula in CIR we have numerically integrated the expression of the density of the non-central chi-squared density given in Lamberton and Lapeyre [62] pages 232-3. We compare this "analytical result" with the approximation to the distribution of a non-central chi-squared density due to Sankaran [83] and the numerical inversion of the Laplace transform. Particularly striking the accuracy of the Abate-Whitt inversion that gives a maximum percentage error of $1.8*10^{-9}$, providing an accuracy up to the tenth digit! The Sankaran approximation performs worst respect to the Fourier methods for numerical inversions, mainly when we increase the option maturity. Usually the different methods agree at least up to the fourth digit. In some cases (low option maturity and high interest rate) the Padé inversion gives a maximum percentage error of 3.5%: in all other cases it gives very reliable results.

In Figure (5.4) we represent the density of the instantaneous interest rate in the CIR model and the density of its time average.

In Table 19 we report the prices of Asian options on the spot rate and compare our results with those in Leblanc and Scaillet that find the density of the average and then integrate the payoff[14]. For the inversion they use the Abate and Whitt numerical procedure but without the accelerating Euler algorithm. They truncate the inversion integral using $n$ =1000 and set $A = 40$. In Table 19, we have just used 46 terms in the summation for the AW algorithm and only 26 for the Crump one. We do not report the results for the Padé inversion that do not seem to be very accurate for some parameter setting. The problem with the procedure followed in Leblanc and Scaillet is that they need to cope with two problems: area under the density not exactly equal to 1 (they report also values of 0.9819) and negative values for the density mainly in the tails. Instead we can compute the price of the Asian options directly, using the expression of the mgf of the average and differentiating it as described in equation (3.63) and (3.64), (the differentiation has been done analytically in Mathematica). Note that in Table 19, the last value in the column LS seems to be wrong, likely for a typo error in table 5 in Leblanc and Scaillet. Indeed they report a figure of 0.008131 that appears twice. Moreover the prices of the Asian option in the Vasicek and Cir model are rather close and in their table 4 they give a figure of 0.0047825 for the Vasicek model.

[14]The last value in the column LS seems to be wrong, likely for a typo error in table 5 in Leblanc and Scaillet. Indeed their value 0.008131 appears twice. Moreover the prices of the Asian option in the Vasicek and Cir model are rather close and in their table 4 there is the figure 0.0047825 for the Vasicek model.

Table 18: Prices of options on zcb in the CIR model

Parameters settings: dr=(0.01-0.1*r)dt+0.1$\sqrt{r}$dt; Strike has been set equal to the fwd price
AW: A=22.4, m=25, n=11; Padé: num=3, den=21; Crump: n=24, rmax=0.2, maxpole=0.3 (arbitrary).
Numerical Integration gives the price computed according to pages 132-3 in [62 ]
numerically integrating the non-central chi-squared law. Sankaran is the approximation to the non-central chi-squared law.

| | | | | | Percentage Difference: (Approximation/Num.Int.-1)*100 | | | |
|---|---|---|---|---|---|---|---|---|
| Option Maturity | Zcb Maturity | Strike | r | Numerical Integration | Sankaran | AW | Crump | Padé |
| 0.25 | 0.5 | 0.987133 | 0.05 | 0.0010645 | 0.0210 | 0.0000 | -0.0500 | 0.0161 |
| 0.5 | 1 | 0.973619 | 0.05 | 0.0028686 | 0.0717 | 0.0000 | 0.0057 | 0.0000 |
| 1 | 2 | 0.945136 | 0.05 | 0.0073248 | 0.2139 | 0.0000 | -0.0006 | 0.0000 |
| 2 | 4 | 0.885043 | 0.05 | 0.0165998 | 0.5086 | 0.0000 | -0.0027 | 0.0001 |
| 3 | 6 | 0.824389 | 0.05 | 0.0240526 | 0.7331 | 0.0000 | 0.0011 | 0.0002 |
| 4 | 8 | 0.765725 | 0.05 | 0.0289351 | 0.8888 | 0.0000 | -0.0010 | 0.0003 |
| 5 | 10 | 0.710266 | 0.05 | 0.0314441 | 0.9984 | 0.0000 | -0.0017 | 0.0003 |
| 0.25 | 0.5 | 0.975327 | 0.1 | 0.0014612 | 0.0059 | 0.0000 | 0.0001 | -1.2917 |
| 0.5 | 1 | 0.951357 | 0.1 | 0.0038267 | 0.0216 | 0.0000 | 0.0001 | 0.0153 |
| 1 | 2 | 0.905732 | 0.1 | 0.0092638 | 0.0753 | 0.0000 | 0.0007 | 0.0000 |
| 2 | 4 | 0.824186 | 0.1 | 0.0191333 | 0.2336 | 0.0000 | -0.0001 | 0.0000 |
| 3 | 6 | 0.754845 | 0.1 | 0.0256807 | 0.4126 | 0.0000 | -0.0002 | 0.0001 |
| 4 | 8 | 0.695713 | 0.1 | 0.0290005 | 0.5808 | 0.0000 | -0.0014 | 0.0002 |
| 5 | 10 | 0.644578 | 0.1 | 0.0298984 | 0.7246 | 0.0000 | 0.0006 | 0.0002 |
| 0.25 | 0.5 | 0.963663 | 0.15 | 0.0017433 | 0.0029 | 0.0000 | 0.0001 | 3.5068 |
| 0.5 | 1 | 0.929604 | 0.15 | 0.0044529 | 0.0103 | 0.0000 | 0.0000 | -0.3337 |
| 1 | 2 | 0.86797 | 0.15 | 0.0102914 | 0.0379 | 0.0000 | 0.0000 | 0.0015 |
| 2 | 4 | 0.767514 | 0.15 | 0.0196389 | 0.1314 | 0.0000 | 0.0000 | 0.0000 |
| 3 | 6 | 0.691167 | 0.15 | 0.0247452 | 0.2572 | 0.0000 | 0.0000 | 0.0001 |
| 4 | 8 | 0.632102 | 0.15 | 0.0265761 | 0.3964 | 0.0000 | 0.0000 | 0.0001 |
| 5 | 10 | 0.584964 | 0.15 | 0.0263234 | 0.5347 | 0.0000 | 0.0000 | 0.0002 |

Table 19: Prices of Asian options in the CIR model

Numerical Inversion and comparison with the results in LS [64], Table 5
Parameters settings: dr=(0.02-0.2*r)dt+$\sigma\sqrt{r}$dW, $r_0$=0.1;
AW: n+m+1=35+11+1, A=18.4; Crump: maxpole=1 (arbitrary), tmax=0.15*(T-t), n=26

| $\sigma^2$ | $T-t$ | Spot Rate Maturity | LS | AW | Crump |
|---|---|---|---|---|---|
| 0.0002 | 1 | 10 | 0.000332 | 0.000331639 | 0.000330629 |
| 0.02 | 1 | 10 | 0.000949 | 0.000949273 | 0.000949272 |
| 0.0002 | 0.25 | 10 | 0.000170 | 0.000169413 | 0.000169636 |
| 0.02 | 0.25 | 10 | 0.000120 | 0.000120193 | 0.000120192 |
| 0.0002 | 1 | 0.25 | 0.000842 | 0.000841902 | 0.000839777 |
| 0.02 | 1 | 0.25 | 0.008131 | 0.00813132 | 0.00813132 |
| 0.0002 | 0.25 | 0.25 | 0.00049 | 0.00047973 | 0.000475538 |
| 0.02 | 0.25 | 0.25 | 0.008131 | 0.00477464 | 0.00477464 |

## 5.7. Example: Hurdle Options

Using the results in section 3.5, we can compute the expected value of the payoff and we obtain[15]:

$$\mathbf{E}_{0,x}\left(\max\left[Y_{0,x}\left(u,t\right)-K;0\right]\right)$$
$$=\int_K^t\left(y-K\right)\mathrm{Pr}_{0,x}\left(Y\left(u,t\right)\in dy\right)+\left(t-K\right)\mathrm{Pr}_{0,x}\left(Y\left(u,t\right)=t\right)$$
$$=\begin{cases}1_{(x>u)}\left(\int_K^t\left(y-K\right)\mathrm{Pr}_{0,x\in(u;+\infty)}\left(Y\left(u,t\right)\in dy\right)\right)\\ 1_{(x<u)}\left(\begin{array}{l}\int_K^t\left(y-K\right)\mathrm{Pr}_{0,x\in(0,u)}\left(Y\left(u,t\right)\in dy\right)\\+\left(t-K\right)\left[1-\frac{1}{2}\left(\mathrm{Erfc}\left(\frac{\ln\frac{1}{\sigma}\frac{u}{x}-\delta t}{\sqrt{2t}}\right)+e^{2\delta\ln\frac{u}{x}}\mathrm{Erfc}\left(\frac{\ln\frac{1}{\sigma}\frac{u}{x}+\delta t}{\sqrt{2t}}\right)\right)\right]\end{array}\right)\end{cases} \tag{5.3}$$

In order to compute the prices trough the Laplace inversion, we can exploit the Laplace transform wrt to strike as in equation (3.8). Indeed we can write:

$$\mathbf{E}_{0,x}\left(\max\left[Y_{0,x}\left(u,t\right)-K;0\right]\right)$$
$$=E_{0,x}\left(Y\left(u,t\right)\right)-K\left(1-\mathrm{Pr}_{0,x}\left(Y\left(u,t\right)=0\right)\right)+\int_0^K\left(K-y\right)\mathrm{Pr}_{0,x}\left(Y\left(u,t\right)\in dy\right)$$

and if we consider the Laplace transform with respect to $K$ of the third term we obtain, using the convolution property of the Laplace transform:

$$\mathcal{L}\left(\int_0^K\left(K-y\right)\mathrm{Pr}_{0,x}\left(Y\left(u,\tau\right)\in dy\right);K\to\mu\right)$$
$$=\int_0^{+\infty}e^{-\mu K}\int_0^K\left(K-y\right)\mathrm{Pr}_{0,x}\left(Y\left(u,\tau\right)\in dy\right)dK$$
$$=\left(\int_0^{+\infty}e^{-\mu K}KdK\right)\left(\int_0^K e^{-\mu K}\mathrm{Pr}_{0,x}\left(Y\left(u,\tau\right)\in dK\right)\right)$$
$$=\frac{\int_0^K e^{-\mu K}\mathrm{Pr}_{0,x}(Y(u,\tau)\in dK)}{\mu^2}$$

where the expression for the Laplace transform wrt to $\tau$ of the numerator is obtained in

[15]The computation of the price and of its derivatives takes a fraction of second in Mathematica 3.0. However, we suggest a change of variable, $y->w=\sqrt{T-y}$,in the integral for eliminating the discontinuity at $y=T$.

(3.73) in Appendix E and is given by:

$$\begin{aligned}
&\mathcal{L}\left(\int_0^{\tau} e^{-\mu K}\operatorname{Pr}_{0,x}\left(Y\left(u,\tau\right)\in dK\right);\tau\to\gamma\right)\\
&=\mathcal{L}\left(\operatorname{Pr}_{0,x}\left(Y\left(u,\tau\right)\in dK\right);\tau\to\gamma,K\to\mu\right)\\
&=\begin{cases}
1_{(x>u)}\left(e^{-\sqrt{2\gamma}(x-u)}\dfrac{e^{-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)}\right)\\
\\
1_{(x<u)}\left(e^{-\sqrt{2(\gamma+\mu)}\sqrt{2}(u-x)}\dfrac{e^{-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)}\right)
\end{cases}
\end{aligned}\tag{5.4}$$

The expression above is then the double Laplace transform wrt to $\tau$ and $K$ of a component of the price of the hurdle option.

In Table 20, we compare the analytical value of the hurdle option price obtained discounting equation (5.3) with the numerical value obtained trough inversion of the double Laplace transform in (5.4). The maximum absolute difference is given by $0.5308^{*}10^{-5}$ for the Padé inversion and by $0.4104^{*}10^{-5}$ for the CLW algorithm.

Table 20: Analytical and Numerical Inversion for pricing Hurdle Options

Parameters setting: r=0.05, $\sigma$=0.2, K=0.2, L=100.

| Index Level | Analytical | (Analytical-LapInv)*$10^5$ | | | |
|---|---|---|---|---|---|
| | | Padé: | | Abate-Whitt: | |
| | (t=0.25) | num=3 den=18 | num=1 den=18 | n=20 n+m=30 | n=50 n+m=70 |
| 80 | 0.04872 | 0.0115 | -0.1396 | 0.0008 | 0.0000 |
| 90 | 0.03941 | 0.2682 | 0.1977 | 0.0056 | -0.0003 |
| 95 | 0.02646 | 0.4708 | 0.3903 | -0.0118 | -0.0005 |
| 100 | 0.00886 | 0.5308 | 0.4405 | 0.4104 | -0.0003 |
| 105 | 0.00080 | -0.1558 | -0.3432 | -0.0276 | -0.0001 |
| 110 | 0.00004 | 0.0231 | 0.0639 | 0.0007 | 0.0000 |
| 120 | 0.00000 | -0.0015 | -0.0049 | 0.0000 | 0.0000 |
| | (t=0.5) | | | | |
| 80 | 0.28157 | 0.0000 | 0.0000 | -0.0001 | 0.0000 |
| 90 | 0.22632 | 0.0002 | 0.0001 | -0.0027 | 0.0000 |
| 95 | 0.16909 | 0.0007 | 0.0001 | 0.0118 | 0.0000 |
| 100 | 0.09507 | 0.0044 | 0.0004 | 0.3436 | 0.0005 |
| 105 | 0.03977 | 0.0009 | 0.0001 | 0.0088 | 0.0000 |
| 110 | 0.01542 | 0.0002 | -0.0001 | -0.0034 | -0.0001 |
| 120 | 0.00188 | 0.0000 | 0.0000 | 0.0000 | 0.0000 |
| | (t=1) | | | | |
| 80 | 0.69227 | -0.0001 | -0.0001 | -0.0001 | -0.0001 |
| 90 | 0.54585 | 0.0003 | 0.0001 | -0.0049 | 0.0001 |
| 95 | 0.42937 | 0.0008 | 0.0001 | 0.0995 | 0.0000 |
| 100 | 0.29019 | 0.0041 | 0.0007 | 0.2418 | 0.0008 |
| 105 | 0.17332 | 0.0015 | 0.0003 | 0.1369 | 0.0002 |
| 110 | 0.10161 | 0.0005 | 0.0000 | -0.0225 | 0.0000 |
| 120 | 0.03322 | 0.0001 | 0.0000 | 0.0008 | 0.0000 |
| | (t=2) | | | | |
| 80 | 1.35469 | 0.0001 | 0.0001 | -0.0001 | 0.0001 |
| 90 | 1.06117 | 0.0007 | 0.0005 | 0.0165 | 0.0005 |
| 95 | 0.86813 | 0.0023 | 0.0016 | 0.0933 | 0.0016 |
| 100 | 0.65278 | 0.0039 | 0.0014 | 0.0912 | 0.0014 |
| 105 | 0.46342 | 0.0005 | -0.0008 | 0.2134 | -0.0007 |
| 110 | 0.32816 | 0.0008 | 0.0001 | 0.0289 | 0.0001 |
| 120 | 0.16357 | 0.0001 | -0.0001 | -0.0055 | -0.0001 |

## 6. Conclusions

In this chapter we have illustrated the application of the Laplace transform in Finance showing how to solve many different problems that can be encountered in pricing derivative assets. Then we have examined different methods for the numerical inversion and we have shown how this technique can be very reliable. In particular the Abate-Whitt inversion method appears the most preferable, because, respect to the other methods, the decision about the value of the parameters to be used in the inversion does not represent a problem. The Crump inversion method gives practically the same values as the Abate-Whitt algorithm being based on the same idea, but it requires the determination of free parameters and sometimes this is not easy. The Padé inversion method, particularly when we are dealing with smooth functions, gives good results with few exceptions. For all three different methods the computational time is always very low, because the operations required in the inversion are just sums. Some problems can occur when there are difficulties in computing the Laplace transform, such as for floating strike Asian options. More research has to be done regarding other inversion methods that in the literature have been well judged, e.g. the Talbot and Weeks methods. The difficulty in their implementation is that they require a not easy determination of free parameters. Another interesting research is the application of the Fast Fourier Transform either to solving pricing problems on the real axis either for the numerical inversion of the Laplace transform.

## References

[1] Abate, J. G. L. Choudhury and W. Whitt (1996), On the Laguerre method for numerically inverting Laplace transforms, *Informs . comput.*, vol. 8, 413-27.

[2] Abate, J. G. L. Choudhury and W. Whitt (1998), Numerical Inversion of multidimensional Laplace transforms by the Laguerre method, *Performance Evaluation*,

vol. 31, 229-43.

[3] Abate, J. and Whitt, W. (1992), The Fourier-series method for inverting transforms of probability distributions, *Queueing Systems Theory Appl.* 10 5-88.

[4] Abramowitz, M., and Stegun, I.A.(1972). *Mathematical Functions*, Dover Publications, Inc., New York.

[5] Akahori, J. (1995). Some formulae for a new type of path-dependent options, *Ann. Appl. Probab. 5*, 383-388.

[6] Bakshi G. and D. Madan (1999), Spanning and Derivative-Security Valuation, forthcoming in *Journal of Financial Economics.*

[7] Bellman R., R. E. Kalaba and J. Lockett (1966), Numerical Inversion of the Laplace Transform. Application to Biology, Economics, Engineering and Physics, American Elsevier, New York.

[8] Black F. and M. Scholes (1973), The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, vol. 81, pagg. 637-654.

[9] Borodin A. N. and P. Salminen (1996), Handbook of Brownian Motion - Facts and Formulae, Birkhauser.

[10] Bregagnolio T., Iori L. (1995), Il loro nome è bond, *Investire*, pp. 88-92, January.

[11] Buser S. A. (1986), Laplace Transforms as Present Value Rules: A Note, *Journal of Finance*, Vol. XLI, no. 1, march, pagg 243-7.

[12] Cathcart L. (1998), The Pricing of Floating Rate Instruments, *Journal of Computational Finance*, Vol. 1, no. 4, summer, pagg. 31-51.

[13] Chacko G. and S. R. Das (1997), Average Interest, w.p. Harvard Business School, may.

[14] Chen R. and L. Scott (1995), Interest Rate Options in Multifactor Cox-Ingersoll-Ross Models of the Term Structure, *Journal of Derivatives*, vol. 3, pagg. 53-72.

[15] Cheng A. H.-D, Sidauruk P. and Abousleiman Y. (1994), Approximate Inversion of the Laplace Transform, *Mathematica Journal*, Vol.4, No. 2.

[16] Chesney M., M. Jeanblanc-Picqué and M. Yor (1995), Brownian excursion and Parisian barrier options, Advances in Applied Probability,

[17] Chesney M., J. Cornwall, M. Jeanblanc-Picqué, G. Kentwell and M. Yor (1997), Parisian Pricing, Risk, Vol. 10, no. 1, january, pagg. 77-80.

[18] Choudhury G. L., Lucantoni D. M, and Whitt W. (1994), Multidimensional Transform Inversion with applications to the transient M/G/1 Queue, Ann. Appl. Probab. 5, 389-398.

[19] Cox J. C. (1996), The Constant Elasticity of Variance Option Pricing Model, *The Journal of Portfolio Management*, Special Issue, pagg.15-17.

[20] Cox J. C., J. E. Ingersoll and S. A. Ross (1985), A Theory of the Term Structure of Interest Rates, *Econometrica*, vol. 53, pagg. 363-84.

[21] Cox J. C. and S. Ross (1976), The Valuation of Options for Alternative Stochastic Processes, *Journal of Financial Economics*, pagg. 145-66.

[22] Craig I. and A. M. Thompson (1994), Why Laplace Transforms are Difficult to Invert Numerically, *Computers in Physics*, vol. 8, no. 6 nov-december.

[23] Crump, K. (1976), Numerical Inversion of Laplace Transform using Fourier Series Approximation, *J. Assoc. Comp. Mach.* 23, 89-96.

[24] Dassios, A. (1995). The Distribution of the Quantile of a Brownian Motion with Drift and the Pricing of Related Path-Dependent Options, ***Ann. Appl. Probab.*** **4**, 719-740.

[25] Davies B. and B. L. Martin (1970), Numerical Inversion of Laplace Transforms: a Critical Evaluation and Review of Methods, *J. Comp. Physics*, vol. 33, pagg. 1-32.

[26] Davydov D. and V. Linetsky, (1999), Structuring, Pricing and Hedging Double Barrier Step Options, w.p. University of Michigan, november 1999.

[27] De Schepper A., M. Teunen and M. Goovaerts (1994), An Analytical Inversion of a Laplace transform related to annuities certain, *Insurance: Math. Econonom.*, 14, 33-37.

[28] Doetsch G (1970), Introduction to the Theory and Application of the Laplace Transformation, Springer-Verlag.

[29] Dubner H. and Abate J. (1968), Numerical Inversion of Laplace Transforms by Relating them to the Finite Fourier Cosine Transform, *J. ACM* 15, 1 January, 115-123.

[30] Duffie D., J. Pan and K. Singleton (1999), Transform Analysis and Option Pricing for Affine Jump-Diffusions, w.p. Stanford University, February.

[31] Duffy D. A. (1993), On the Numerical Inversion of Laplace Transforms: Comparison of three new Methods on Characteristic Problems from Applications, *ACM Trans. on Math. Soft.*, vol. 19, no. 3, Sept., pagg. 333-59.

[32] Dufresne D. (1989), Weak Convergence of Random Growth Processes with Applications to Insurance, *Insurance: Math. Econonom.*, 8, 187-201.

[33] Dyke P. P. G. (1999), An Introduction to Laplace Transforms and Fourier Series, Springer.

[34] Embrechts, P., Rogers, L. C. G. and Yor, M. (1995). A proof of Dassios' representation of the $\alpha$-quantile of Brownian motion with drift, Ann. Appl. Probab. 5, 757-767.

[35] Feller W. (1951), Two Singular Diffusion Problems, *Annals of Mathematics*, vol. 54, No.1, pagg. 173-82.

[36] Feller W. (1971), An Introduction to Probability Theory and its Applications, vol. II, 2nd ed., Wiley, New York.

[37] Fu M. C., D. Madan and T. Wang (1998), Pricing Continuous Asian Options: a comparison of MonteCarlo and Laplace Transform Inversion Methods, *The Journal of Computational Finance,* Vol. 2, No. 1, 49-74.

[38] Garbow B. S., G. Giunta, J. N. Lyness and A. Murli (1988), Software for an implementation of Weeks' method for the inverse Laplace transform problem, *ACM Trans. Math. Software*, Vol. 14, 163-70.

[39] Garbow B. S., G. Giunta, J. N. Lyness and A. Murli (1988), Algorithm 662: A FORTRAN software package for numerical inversion of the Laplace transform based on Weeks' method, ACM *Trans. Math. Software,* Vol. 14, 171-6.

[40] Geman H. and A. Eydeland (1995), Domino Effect, *Risk*, Vol. 8, n. 4, april, 65-67.

[41] Geman H. and M. Yor (1993), Bessel Processes, Asian Options and Perpetuities, *Mathematical Finance*, vol. 3, no. 4, pagg. 349-375.

[42] Geman H. and M. Yor (1996), Pricing and hedging double barrier options: a probabilistic approach, *Mathematical Finance,* 6, n. 4, pagg. 365-78.

[43] Gustafsson B., H-O. Kreiss and J. Oliger (1996) Time Dependent Problems and Difference Methods, Wiley and Sons Inc.

[44] He H., Keirstead W. P. and Rebholz J. (1998), Double Lookbacks, *Mathematical Finance*, vol. 8, no. 3, July.

[45] Heynen R. C. and H. M. Kat (1997), Barrier options, in Exotic Options: the State of the Art, ed. Clelow and Strickland, International Thomson Business Press.

[46] Hochstadt H. (1986), The Functions of Mathematical Physics, Dover Publications, New York.

[47] Hugonnier J. N. (1999), The Feynman-Kac Formula and Pricing of Occupation Time Derivatives, *International Journal of Theoretical and Applied Finance*, vol. 2, no. 2, pagg. 153-78.

[48] Hui C. H. (1996), One-Touch Double Barrier Binary Option Values, *Appl. Financial Econ.* vol. 6, 343-6.

[49] Hui C. H., C. F. Lo and P.H. Yuen (2000), Comment on 'Pricing Double Barrier Options using Laplace Transforms' by Antoon Pelsser, *Finance and Stochastic* vol. 4, pagg. 105-7.

[50] Hull, J. (1997), Options, Futures, and Other Derivatives. Englewood Cliffs, NJ: Prenctice Hall.

[51] Ikeda N. and S. Watanabe (1981), Stochastic Differential Equations and Diffusion Processes, North-Holland.

[52] Ince (1964), Ordinary Differential Equations, Dover Publications, Inc., New York.

[53] Ingersoll J. E. (1986), Theory of Financial Decisions Making,

[54] Ito K. and McKean Jr. H. P. (1974), Diffusion Processes and Their Sample Paths, Springer Verlag.

[55] Jamshidian F. (1996), Pricing of Contingent Claims in the One-Factor Term Structure Model, republished in Vasicek and Beyond, edited by Lane Hughston, Risk Publications, London. Original w.p. dated 1987.

[56] Jamshidian F. (1997), A Note on Analytical Valuation of Double Barrier Options, w.p. Sakura Global Capital, February.

[57] Johnson N. L. and S. Kotz (1970), Distributions in Statistics: Continuous Univariate Distributions, vol. 2, Houghton Miffin Company, Boston.

[58] Karatzas I. and S. E. Shreve (1991), Brownian Motion and Stochastic Calculus, 2nd ed., Springer.

[59] Karlin S. and H. M. Taylor (1981), A Second Course in Stochastic Processes, Academic Press.

[60] Kunitomo N. and M. Ikeda (1992), Pricing options with curved boundaries, *Mathematical Finance* 2, pagg. 275-98.

[61] Kwok Y. K. and D. Barthez (1989), An Algorithm for the Numerical Inversion of Laplace Transforms, *Inverse Problems* vol. 5, pagg. 1089-95.

[62] Lamberton D. and B. Lapeyre (1996), Introduction to Stochastic Calculus Applied to Finance, Chapman and Hall.

[63] Lebedev N. N. (1972), Special functions and their applications, Dover Publications, New York.

[64] Leblanc B. and O. Scaillet (1998), Path dependent options on yields in the affine term structure model, *Finance and Stochastics*, vol. 2, pagg. 349-67.

[65] Lévy, P. (1939), Sur certain processus stochastiques homogènes, *Compositio Math.*, Vol. 7, pages 283-339.

[66] Linetski, V. (1999), Step options, *Mathematical Finance*, vol. 9, no. 1, January, pagg. 55-96.

[67] Lipton A. (1999), Similarities via self-similarities, *Risk*, september, pagg. 101-5.

[68] Longman I. M. (1975), Application of best Rational Function Approximation for Laplace Transform Inversion, *J. Comp. Appl. Math.*, vol. 1, pagg. 17-23.

[69] Longstaff F. A. (1990), The valuation of options on yields, *Journal of Financial Economics*, vol. 26, pagg. 97-121.

[70] Longstaff F. A. (1995), Hedging interest rate risk with options on average interest rates, *The Journal of Fixed Income*, vol. 4, no. 4, march, pagg. 37-45.

[71] MacDonald J. R. (1964), Accelerated convergence, divergence, iteration, extrapolation and curve fitting, *J. Appl. Phys.* 10, 3034-3041.

[72] Maghsoodi Y. (1996), Solution of the Extended CIR Term Structure and Bond Option Valuation, *Mathematical Finance*, vol. 6, no. 1, jan., pagg. 89-109.

[73] Milevsky M. A. and S. E. Posner (1998), Asian Options, the Sum of Lognormals and the Reciprocal Gamma Distribution, *Journal of Financial and Quantitative Analysis,* vol. 33, no. 3, pagg. 409-22, September.

[74] Moorthy, M. V. (1995), Numerical Inversion of two-dimensional Laplace transforms - Fourier series representation, *Applied Numerical Mathematics*, Vol. 17, pp. 119-127.

[75] Moorthy, M. V. (1995), Inversion of the multi-dimensional Laplace transform-expansion by Laguerre series, *Z angew Math Phys*, 46, 793-806.

[76] Necati Özişik M. (1989), Boundary Value Problems of Heat Conduction, Dover.

[77] Pechtl, A. (1995), Classified Information, in Over the Rainbow, ed by Robert Jarrow, Risk Publications, pag. 71-74

[78] Pelsser A. (2000), Pricing double barrier options using Laplace transforms, *Finance Stochastics,* vol. 4, pagg. 95-104.

[79] Piessens R. (1969), New quadrature formulas for the numerical inversion of the Laplace transform, *BIT*, vol. 9, pagg. 351-61.

[80] Platzman L. K., J. C. Ammons and J. J. Bartholdi (1988), A Simple and Efficient Algorithm to Compute Tail Probabilities from Transforms, *Oper. Res.,* vol. 26, pagg. 137-44.

[81] Press, W. H., Teukolsky S. A., Vetterling W. T. and Flannery B. P. (1997). *Numerical Recipes in C*, Cambridge University Press, version 2.08

[82] Rogers L. C. G. and Shi Z. (1992), The Value of an Asian option, *J. of Applied Probability*, Vol. 32, pagg. 1077-88.

[83] Sankaran M., 1963, Approximations to the Non Central Chi-Square Distribution, *Biometrika* 50, 199-204.

[84] Sbuelz A. (1999), A general treatment of barrier options and semi-static hedges of double barrier options, w.p. London Business School.

[85] Schroder, M. (1989), Computing the CEV option pricing formula, *Journal of Finance* vol. 44, pagg. 211-219.

[86] Selby, M. J. P. (1983), The Application of Option Theory to the Evaluation of Risky Debt, PH.D. Thesis, London Business School, May.

[87] Shiga, T. and S. Watanabe (1973), Bessel diffusions as a One-Parameter Family of Diffusion Processes, *Z. Wahrsch.*, 27, pagg. 37-46.

[88] Shimko, D. (199), Finance in continuous time: A Primer, Kolb Publishing Company.

[89] Simon R. M., Stroot M. T. and G. H. Weiss (1972), Numerical Inversion of Laplace Transforms with Applications to the Percentage Labeled Experiments, *Comput. Biomed. Res.* 6, pagg. 596-607.

[90] Singhal, K., Vlach, J. (1975), Computation of Time Domain Response by Numerical Inversion of the Laplace transform, Journal of the Franklin Institute, no. 2, pagg.110-27.

[91] Singhal, K., Vlach, J. and M. Vlach (1975), Numerical inversion of multidimensional Laplace transforms, *Proc. IEEE* 63, 1627-1628.

[92] Strauss, W.A. (1992), Partial Differential Equations: An Introduction, Wiley and Sons,

[93] Takàcs, L. (1996), On a generalization of the Arc-Sine Law, *The Annals of Applied Probability*, Vol. 6, No. 3, pagg. 1035-1040.

[94] Talbot, A. (1979), The Accurate Numerical Inversion of Laplace Transforms, *J. Inst. Math. Appl.* vol. 23, n. 1, pagg. 97-120.

[95] Thompson G. W. P. (1999), Topics in Mathematical Finance, PH.D. Thesis, Queens' College, January.

[96] Tucker, A. L. and Wei J. Z. (1997), The Latest Range, Advances in Futures and Options Research, Vol. 9, pages 287-296.

[97] Turnbull, S. M. (1995), Interest Rate Digital Options and Range Notes, *Journal of Derivatives* 3, 92-101.

[98] Vlach J. and K. Singhal (1993), Computer Methods for Circuit Analysis and Design, Van Nostrand Reinhold Company, New York, 2nd edition.

[99] Yor M. (1992), On some exponential functionals of Brownian Motion, *Adv. Appl. Prob.* 24, pagg. 509-531.

[100] Yor M. (1992), Some Aspects of Brownian Motion. Part I: Some Special Functionals. Birkhauser Verlag. Basel - Boston - Berlin.

[101] Yor M. (1992), Sur les lois des fonctionnelles exponentielles du mouvement brownien, considérées en certains instants aléatoires, C. R. Acad. Sci Paris Série I, 314, 951-956.

[102] Weeks W. T. (1966), Numerical inversion of Laplace transforms using Laguerre functions, *ACM* 13, 3, july, pagg. 419-29.

[103] Weideman J. A. C. (1999), Algorithms for Parameter Selection in the Weeks Method for Inverting the Laplace Transform, *SIAM J. Sci. Comput.*, vol. 21, no. 1, pagg. 111-28.

[104] Wilmott, P. Dewynne, J.N. and Howison, S. (1993), *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press

[105] Wellekens C. J. (1970), Generalization of Vlach's Method for the Numerical Inversion of the Laplace Transform, *Electronics Letter*, vol. 6, n. 23, November, pagg. 742–4.

[106] Wynn P. (1961), The $\epsilon$-algorithm and operational formulas of numerical analysis, *Math. Comp.* vol. 15, pagg. 151-8.

[107] Zakian V. (1970), Optimisation of Numerical Inversion of Laplace Transforms, *Electronics Letter*, vol. 6, n. 21, October, pagg. 677-9.

[108] Zauderer, E. (1989), *Partial Differential Equations of Applied Mathematics*, John Wiley & Sons, 2$^{\text{nd}}$ ed..

## A. The hypergeometric and the confluent differential equations

In this section we describe the solution of two differential equations we have to solve when we consider the Laplace transform with respect to time. As reference text we have used Lebedev [63].

The first differential equation is the so-called hypergeometric differential equation, where the function $f(x)$, eventually dependent on a Laplace parameter, satisfies the following second order differential equation:

$$x f_{xx} + (\beta - x) f_x - \alpha f = 0$$

where $\beta \neq 0, -1, -2, ....$ This ODE arises when we consider the Laplace transform wrt time of the PDE related to the CIR model. It is also useful for treating the pricing problem for Asian options.

The ODE can be recognized as a differential equation satisfied by the confluent hypergeometric function, Lebedev [63], pag. 262-3. Indeed the general solution can be written in the form:

$$f(x) = A\Upsilon(\alpha, \beta; x) + Bx^{1-\beta}\Upsilon(\alpha, \beta; x) \tag{A.1}$$

where $\Upsilon(\alpha, \beta; z)$ is the confluent hypergeometric function. It has the following series representation:

$$\Upsilon(\alpha, \beta; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k} \frac{x^k}{k!}, |x| < \infty, \beta \neq 0, -1, -2, ...$$

and $(\alpha)_k$ is the Pochammer symbol:

$$(\alpha)_0 = 1; (\alpha)_k = \alpha(\alpha+1)...(\alpha+k-1); k = 2.3...$$

and where the constants $A$ and $B$ have to be chosen in order to satisfy the boundary conditions.

It admits also an integral representation valid for an arbitrary $\alpha$ and for $\gamma \neq 0, -1, -2...$, given by:

$$\Upsilon(\alpha, \beta; x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)\Gamma(\beta-\alpha)} e^{x} \int_0^1 e^{-xs} s^{\beta-\alpha-1} (1-s)^{\alpha-1} ds,$$

Lebedev pag. 267.

Useful properties are the following, Lebedev pag. 261 eq. 9.9.4 and pag. 267 eq. 9.11.2:

$$\begin{aligned} \frac{d^m \Upsilon(\alpha, \beta; x)}{dx^m} &= \frac{(\alpha)_m}{(\beta)_m} \Upsilon(\alpha+m, \beta+m; x) \\ \Upsilon(\alpha, \beta; x) &= e^{z} \Upsilon(\beta-\alpha, \beta; x) \end{aligned}$$

and the following asymptotic representation for large $|x|$ and if $\arg x \leq \pi/2 - \epsilon$, where $\epsilon > 0$ and arbitrarily small:

$$\Upsilon(\alpha, \beta; x) = \frac{\Gamma(\beta)}{\Gamma(\alpha)} e^{x} x^{-(\beta-\alpha)} \left[ \sum_{k=0}^{n} \frac{(\beta-\alpha)_k (1-\alpha)_k}{k!} x^{-k} + O\left(|x|^{-n-1}\right) \right]$$

A different representation of the solution of equation (A.1) can be found again in Lebedev and is given by:

$$f(x) = A\Upsilon(\alpha, \beta; x) + B\,\bar{\Upsilon}(\alpha, \beta; x) \tag{A.2}$$

where $\bar{\Upsilon}(\alpha, \beta; x)$ is the confluent hypergeometric function of the second kind and is related to $\Upsilon(\alpha, \beta; x)$ by:

$$\bar{\Upsilon}(\alpha, \beta; x) = \frac{\Gamma(1-\beta)}{\Gamma(1+\alpha-\beta)} \Upsilon(\alpha, \beta; x) + \frac{\Gamma(\beta-1)}{\Gamma(\alpha)} x^{1-\beta} \Upsilon(1+\alpha-\beta, 2-\beta; x)$$

and then for small values of $x$, $\bar{\Upsilon}$ behaves as $x^{1-\beta}$.

The function $\bar{\Upsilon}\ (\alpha, \beta; x)$ satisfies:

$$\bar{\Upsilon}\ (\alpha, \beta; x) = x^{1-\beta}\ \bar{\Upsilon}\ (1 + \alpha - \beta, 2 - \beta; x)$$

and :

$$\frac{d^m\ \bar{\Upsilon}\ (\alpha, \beta; x)}{dx^m} = (-1)^m\ (\alpha)_m\ \bar{\Upsilon}\ (\alpha + m, \beta + m; x)$$

$\bar{\Upsilon}\ (\alpha, \beta; x)$ admits the following integral representation, Lebedev, pag. 268 eq. 9.11.6:

$$\bar{\Upsilon}\ (\alpha, \beta; x) = \frac{1}{\Gamma(\alpha)} \int_0^\infty e^{-x\xi} \xi^{\alpha-1} (1 + \xi)^{\beta-\alpha-1}\, d\xi$$

with $\mathrm{Re}\,(\alpha) > 0, \mathrm{Re}\,(x) > 0$.

In Lebedev, pag. 269 eq. 9.12.1, it is given an asymptotic representation for large $|x|$:

$$\bar{\Upsilon}\ (\alpha, \beta; x) = x^{-\alpha} \left[ \sum_{k=0}^{n} \frac{(-1)^k (\alpha)_k (1 + \alpha - \beta)_k}{k!} x^{-k} + O\left( |x|^{-n-1} \right) \right]$$

that is valid when $\mathrm{Re}\,(\alpha) > 0$, $|\arg x| \leq \pi/2 - \epsilon$, $n \geq \mathrm{Re}\,(\beta - \alpha) - 2$.

In the solution of the ODE with inhomogeneous boundary conditions, it is required the determination of the Wronskian of the homogeneous equation. It is given by:

$$W\left( \Upsilon\,(\alpha, \beta; x), \bar{\Upsilon}\ (\alpha, \beta; x) \right) = -\frac{\Gamma(\beta)}{\Gamma(\alpha)} x^{-\beta} e^x$$

for $|\arg x| \leq \pi$, $\alpha, \beta \neq 0, -1, -2, \ldots$.

The second important differential equation that turns out to be useful in treating the

Vasicek model, is the so-called confluent hypergeometric differential equation, given by:

$$f_{xx} + (\beta - x) f_x - \alpha f = 0$$

If we operate the change of variable $y = (\beta - x)^2 / 2$ , and we set $f(x) = v\left((\beta - x)^2 / 2\right)$, then $v(y)$ satisfies the following ODE:

$$y v_{yy} + \left(\frac{1}{2} - y\right) v_y - \frac{\alpha}{2} v = 0$$

and we recognize it as the hypergeometric differential equation. This result is due to the fact that the square of an Ornstein-Uhlenbeck process has the same law as a Bessel process of integer order, compare Maghsoodi [72]. The general solution of the above ODE is given by:

$$v = A\Upsilon\left(\frac{\alpha}{2}, \frac{1}{2}; y\right) + B\sqrt{y}\Upsilon\left(\frac{1+\alpha}{2}, \frac{3}{2}; y\right)$$

or:

$$v = A\Upsilon\left(\frac{\alpha}{2}, \frac{1}{2}; y\right) + B\,\bar{\Upsilon}\left(\frac{\alpha}{2}, \frac{1}{2}; y\right).$$

Another important equation that appears again with reference to the Vasicek model is the Hermite differential equation:

$$f_{xx} - 2x f_x + 2\alpha f = 0$$

whose solution is given by the Hermite functions that can be expressed in terms of the confluent hypergeometric function using $t = x^2$ as a new independent variable. For details compare Lebedev [63] pag. 284 and the section on the barrier options.

## B. The Laplace transform for barrier options

In this Appendix we price barrier options solving the Black-Scholes equation:

$$C_t + rSC_s + \frac{1}{2}\sigma^2 S^2 C_{ss} = rC \tag{B.1}$$

$$C(S,0) = \phi(S) \tag{B.2}$$

with boundary conditions:

$$\begin{aligned} C(U,t) &= \varphi_U(T-t) \\ C(L,t) &= \varphi_L(T-t) \end{aligned}$$

where the rebates can depend on time. This is the case of a double knock-in option, where $\varphi_L(T-t)$ and $\varphi_U(T-t)$ represent the time $t$ price of a standard call option with residual life $T-t$ and $\phi(S) = 0$. For a knock-out option, we have $\varphi_U(T-t) = \varphi_L(T-t) = 0$ and $\phi(S) = (S-K)^+$. It is convenient to make equation (B.1) dimensionless with the substitutions:

$$\begin{aligned} C &= e^{\alpha x + \beta\tau} V(x,\tau) \\ x &= \ln(S/L), \tau = \frac{\sigma^2}{2}(T-t) \\ k_1 &= \frac{2r}{\sigma^2}, \alpha = -\frac{1}{2}(k_1-1), \beta = -\frac{1}{4}(k_1-1)^2 - \frac{2r}{\sigma^2} \end{aligned}$$

and we obtain the standard diffusion equation:

$$-\frac{\partial V}{\partial \tau} + \frac{\partial^2 V}{\partial x^2} = 0 \tag{B.3}$$

but now with boundary conditions at $x = 0$ and $x = \ln(u/L)$:

$$\begin{aligned} V(0,\tau) &= e^{-\beta\tau}\varphi_L\left(\frac{2}{\sigma^2}\tau\right), \tau > 0 \\ V\left(\ln\frac{U}{L},\tau\right) &= e^{-\alpha\ln(U/L)-\beta\tau}\varphi_U\left(\frac{2}{\sigma^2}\tau\right), \tau > 0 \end{aligned} \tag{B.4}$$

and initial condition:

$$V(x,0) = e^{-\alpha x}\varphi(Le^x)$$

This kind of equation is known as one-dimensional nonhomogeneous heat equation in a finite region with Dirichlet boundary conditions. The solution is well known and can be found in several textbooks.

Necati Ozisik [76], pagg. 60-1, gives a Fourier series expansion of the solution:

$$\begin{aligned} &V(x,\tau) = \sum_{n=1}^{\infty} e^{-\lambda_n^2\tau}K(\lambda_n,x) \\ &\times\left[\int_0^{\ln U/L} K(\lambda_n,\xi)\,e^{-\alpha\xi}\varphi\left(Le^{\xi}\right)d\xi + \int_0^{\tau} e^{\lambda_n s}A(\lambda_n s)\,ds\right] \end{aligned} \tag{B.5}$$

where:

$$\begin{aligned} \lambda_n &= n\pi/\ln(U/L) \\ K(\lambda_n,x) &= \sqrt{\frac{2}{\ln(U/L)}}\sin\lambda_n x \\ A(\lambda_n,\tau) &= \left.\frac{\partial K(\lambda_n,x)}{\partial x}\right|_{x=0} e^{-\beta\tau}\varphi_L\left(\frac{2}{\sigma^2}\tau\right) \\ &\quad - \left.\frac{\partial K(\lambda_n,x)}{\partial x}\right|_{x=\ln(U/L)} e^{-\alpha\ln(U/L)-\beta\tau}\varphi_U\left(\frac{2}{\sigma^2}\tau\right). \end{aligned}$$

Zauderer [108] gives a solution using Green's functions.

In order to obtain the above solution we can consider the Laplace transform with respect to $\tau$ of the PDE (B.3) and we obtain the ODE:

$$v_{xx} - \gamma v + e^{-\alpha x} \varphi \left( L e^{x} \right) = 0 \tag{B.6}$$

with boundary conditions given by the Laplace transform of (B.4):

$$\begin{aligned} v\left(0,\gamma\right) &= \mathcal{L}\left(e^{-\beta\tau}\varphi_L\left(\frac{2}{\sigma^2}\tau\right)\right) =: a_0\left(\gamma\right) \\ v\left(\ln\frac{U}{L},\gamma\right) &= \mathcal{L}\left(e^{-\alpha\ln(U/L)-\beta\tau}\varphi_U\left(\frac{2}{\sigma^2}\tau\right)\right) =: a_1\left(\gamma\right) \end{aligned}$$

The solutions of the homogenous equations are $e^{\pm\sqrt{\gamma}x}$ and the solution of the ODE in a fixed strip with the above boundary conditions is given by, compare the elegant derivation in Doetsch [28] pagg. 87-9 and pagg. 288-9:

$$\begin{aligned} v\left(x,\gamma\right) &= a_0\left(\gamma\right)\frac{\sinh\left(\left(\ln\left(U/L\right)-x\right)\sqrt{\gamma}\right)}{\sinh\left(\ln\left(U/L\right)\sqrt{\gamma}\right)} + a_1\left(\gamma\right)\frac{\sinh\left(x\sqrt{\gamma}\right)}{\sinh\left(\ln\left(U/L\right)\sqrt{\gamma}\right)} \\ &\quad + \int_0^{\ln(U/L)} e^{-\alpha\xi}\varphi\left(Le^{\xi}\right)\Gamma\left(x,\xi;\gamma\right)d\xi \end{aligned} \tag{B.7}$$

where:

$$\Gamma\left(x,\xi;\gamma\right) = \begin{cases} \frac{\sinh\xi\sqrt{\gamma}\sinh\left(\left(\ln(U/L)-x\right)\sqrt{\gamma}\right)}{\sqrt{\gamma}\sinh\left(\ln(U/L)\sqrt{\gamma}\right)} & 0 \le \xi \le x \\ \\ \frac{\sinh x\sqrt{\gamma}\sinh\left(\left(\ln(U/L)-\xi\right)\sqrt{\gamma}\right)}{\sqrt{\gamma}\sinh\left(\ln(U/L)\sqrt{\gamma}\right)} & x \le \xi \le \ln\left(U/L\right) \end{cases} \tag{B.8}$$

If we replace the products in the numerators by differences thus:

$$\Gamma\left(x,\xi;\gamma\right) = \begin{cases} \frac{\cosh(x-\xi-\ln(U/L))\sqrt{\gamma}-\cosh(x+\xi-\ln(U/L))\sqrt{\gamma}}{2\sqrt{\gamma}\sinh\left(\ln(U/L)\sqrt{\gamma}\right)} & 0 \le \xi \le x \\ \\ \frac{\cosh(\xi-x-\ln(U/L))\sqrt{\gamma}-\cosh(\xi+x-\ln(U/L))\sqrt{\gamma}}{2\sqrt{\gamma}\sinh\left(\ln(U/L)\sqrt{\gamma}\right)} & x \le \xi \le \ln\left(L/U\right) \end{cases}$$

we obtain the same structure of the solution as in the main text, once we have taken into account that in the present case we have done a transformation that has removed the drift term and we have considered the Laplace transform wrt $\sigma^2 (T-t)/2$ and not simply wrt time to maturity.

$$\int_0^{\ln(U/L)} e^{-\alpha\xi} \varphi\left(Le^{\xi}\right) \Gamma(x, \xi; \gamma)\, d\xi$$

In the case of a double knock-out option, we have:

$$\begin{aligned}
&\frac{\sinh\left((b-x)\sqrt{\gamma}\right)}{\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)} \left(\int_0^x e^{-\alpha\xi}\left(Le^{\xi}-K\right)^+ \sinh\left(\xi\sqrt{\gamma}\right) d\xi\right) \\
&+\frac{\sinh x\sqrt{\gamma}}{\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)} \left(\int_x^b e^{-\alpha\xi}\left(Le^{\xi}-K\right)^+ \sinh\left((b-\xi)\sqrt{\gamma}\right) d\xi\right)
\end{aligned}$$

Let us define:

$$\begin{aligned}
p(\xi, \rho, \gamma) &= \textstyle\int e^{\rho\xi} \sinh\left(\xi\sqrt{\gamma}\right) d\xi \\
&= \tfrac{1}{2}\tfrac{\exp\left(\left(\sqrt{\gamma}+\rho\right)\xi\right)}{\sqrt{\gamma}+\rho} + \tfrac{1}{2}\tfrac{\exp\left(-\left(\sqrt{\gamma}-\rho\right)\xi\right)}{\sqrt{\gamma}-\rho}
\end{aligned}$$

$$\begin{aligned}
q(\xi, \rho, \gamma) &= \textstyle\int e^{\rho\xi} \sinh\left((b-\xi)\sqrt{\gamma}\right) d\xi \\
&= \tfrac{1}{2}\tfrac{\exp\left(\left(-\sqrt{\gamma}+\rho\right)\xi+b\sqrt{\gamma}\right)}{-\sqrt{\gamma}+\rho} + \tfrac{1}{2}\tfrac{\exp\left(\left(\sqrt{\gamma}+\rho\right)\xi-b\sqrt{\gamma}\right)}{-\sqrt{\gamma}-\rho}
\end{aligned}$$

and:

a) if $x < \ln(K/L) = c$, the Laplace transform of the double knock-out option is given by:

$$\begin{aligned}
&\frac{\sinh x\sqrt{\gamma}}{\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)} \left(\int_c^b e^{-\alpha\xi}\left(Le^{\xi}-K\right) \sinh\left((b-\xi)\sqrt{\gamma}\right) d\xi\right) \\
= &\frac{\sinh x\sqrt{\gamma}}{\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)} \left(\begin{array}{c} L\left(q(b, 1-\alpha, \gamma) - q(c, 1-\alpha, \gamma)\right) - \\ K\left(q(b, -\alpha, \gamma) - q(c, -\alpha, \gamma)\right) \end{array}\right)
\end{aligned}$$

b) if $x > \ln(K/L) = c$

$$\frac{\sinh\left((b-x)\sqrt{\gamma}\right)}{\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)}\left(\int_c^x\left(Le^{(1-\alpha)\xi}-Ke^{-\alpha\xi}\right)\sinh\left(\xi\sqrt{\gamma}\right)\right)d\xi$$
$$+\frac{\sinh x\sqrt{\gamma}}{\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)}\left(\int_x^b\left(Le^{(1-\alpha)\xi}-Ke^{-\alpha\xi}\right)\sinh\left((b-\xi)\sqrt{\gamma}\right)\right)d\xi$$
$$=\frac{\sinh\left((b-x)\sqrt{\gamma}\right)}{\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)}\left(\begin{array}{c}L\left(p\left(x,1-\alpha,\gamma\right)-p\left(c,1-\alpha,\gamma\right)\right)\\ -K\left(p\left(x,-\alpha,\gamma\right)-p\left(c,-\alpha,\gamma\right)\right)\end{array}\right)$$
$$+\frac{\sinh x\sqrt{\gamma}}{\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)}\left(\begin{array}{c}L\left(q\left(b,1-\alpha,\gamma\right)-q\left(x,1-\alpha,\gamma\right)\right)\\ -K\left(q\left(b,-\alpha,\gamma\right)-q\left(x,-\alpha,\gamma\right)\right)\end{array}\right)$$

### B.1. The analytical inversion

In this section we consider the inversion for the functions

$$h^{+}\left(\gamma,x\right)=\frac{\sinh\left(x\sqrt{\gamma}\right)}{\sinh\left(b\sqrt{\gamma}\right)} \tag{B.9}$$

and

$$h_d\left(X,x,\gamma\right)=\frac{\cosh\left((x-\xi-b)\sqrt{\gamma}\right)-\cosh\left((x+\xi-b)\sqrt{\gamma}\right)}{2\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)} \tag{B.10}$$

where $b=\ln U/L$. The other expressions can be inverted similarly and then the solution can be obtained exploiting the convolution property of the Laplace trasnform. The inversion can be done in two different ways conducting to a different representation of the same solution:

a) expand in series the expressions in (B.9) and in (B.10) and then invert term by term using known inversion formula, Doetsch[28] pagg. 199-200 and pag. 289;

b) if $f\left(\gamma\right)$ is the Laplace transform and $F\left(\tau\right)$ the original function, we can exploit the representation of the original functions in terms of the Bromwich's integral:

$$\mathcal{L}(\tau)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}f(\gamma)\,e^{\tau\gamma}d\gamma$$

where $c$ lie to the right of any of the singularities of the functions $f(\gamma)$. We can transform the line integral in a contour integral choosing an appropriate path and the value of the integral is then obtained applying the Cauchy 's Residue Theorem

$$2\pi i\sum_{k=1}^{N}\left(\text{sum of residues of } e^{\gamma_k\tau}f(\gamma_k)\right)$$

where $\gamma_k$ are the singularities of the Laplace transform. This method has been utilised by Pelsser [78].

The first method gives a solution in terms of the normal density and its first derivative, whilst the second give a solution expressed in terms of the sine function. The two final expressions are equivalent, the second being the Fourier series expansion of the first.

### B.1.1. Inversion in terms of normal densities

Let us write:

$$\begin{aligned}\frac{\sinh\left(x\sqrt{\gamma}\right)}{\sinh\left(b\sqrt{\gamma}\right)} &= \frac{e^{x\sqrt{\gamma}}-e^{-x\sqrt{\gamma}}}{e^{b\sqrt{\gamma}}-e^{-b\sqrt{\gamma}}}=\frac{e^{(x-b)\sqrt{\gamma}}-e^{-(x+b)\sqrt{\gamma}}}{1-e^{-2b\sqrt{\gamma}}}\\ &= \left(e^{(x-b)\sqrt{\gamma}}-e^{-(x+b)\sqrt{\gamma}}\right)\sum_{n=0}^{\infty}e^{-2bn\sqrt{\gamma}}\\ &= \sum_{n=0}^{\infty}\left(e^{-(2bn-x+b)\sqrt{\gamma}}-e^{-(2bn+x+b)\sqrt{\gamma}}\right)\end{aligned}$$

where we have used the fact that $1/(1-x)=\sum_{n=0}^{\infty}x^k$ which is valid for $|x|<1$, since it may be assumed that $\gamma$ is large enough so that $\exp\left(-2b\sqrt{\gamma}\right)\ll 1$. From a Table of Laplace transforms we can find that:

$$\mathcal{L}^{-1}\left(e^{-b\sqrt{\gamma}}\right) = \frac{b}{2\sqrt{\pi\tau^3}} e^{-b^2/4\tau},$$

we can invert term by terms so that:

$$\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{\sinh\left(x\sqrt{\gamma}\right)}{\sinh\left(b\sqrt{\gamma}\right)}\right) = \sum_{n=0}^{\infty} \mathcal{L}^{-1}\left(e^{-(2bn-x+b)\sqrt{\gamma}} - e^{-(2bn+x+b)\sqrt{\gamma}}\right) \\
&= \frac{1}{2\sqrt{\pi\tau^3}}\left(\sum_{n=0}^{\infty}(2bn-x+b)\, e^{-(2bn-x+b)^2/4\tau} - \sum_{n=0}^{\infty}(2bn+x+b)\, e^{-(2bn+x+b)^2/4\tau}\right)
\end{aligned}$$

We can modify the second sum in brackets, using $n = -v - 1$:

$$-\sum_{n=0}^{\infty}(2bn+x+b)\, e^{-(2bn+x+b)^2/4\tau} = \sum_{v=-1}^{-\infty}(2bv-x+b)\, e^{-(2bn+x+b)^2/4\tau}$$

and then we find:

$$\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{\sinh\left(x\sqrt{\gamma}\right)}{\sinh\left(b\sqrt{\gamma}\right)}\right) \\
&= \frac{1}{2\sqrt{\pi\tau^3}}\left(\sum_{n=0}^{\infty}(2bn-x+b)\, e^{-(2bn-x+b)^2/4\tau} + \sum_{v=-1}^{-\infty}(2bv-x+b)\, e^{-(2bn+x+b)^2/4\tau}\right) \\
&= \frac{1}{2\sqrt{\pi\tau^3}} \sum_{n=-\infty}^{\infty}(2bn-x+b)\, e^{-(2bn-x+b)^2/4\tau}
\end{aligned}$$

We can follow the same procedure for inverting the expressions involving the cosh functions:

$$\begin{aligned}
&\mathcal{L}^{-1}\left(\frac{\cosh\left(y\sqrt{\gamma}\right)}{2\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)}\right) = \mathcal{L}^{-1}\left(\frac{e^{y\sqrt{\gamma}}+e^{-y\sqrt{\gamma}}}{2\sqrt{\gamma}\left(e^{y\sqrt{\gamma}}-e^{-y\sqrt{\gamma}}\right)}\right) \\
&= \mathcal{L}^{-1}\left(\frac{e^{(y-b)\sqrt{\gamma}}+e^{-(y+b)\sqrt{\gamma}}}{2\sqrt{\gamma}\left(1-e^{-2b\sqrt{\gamma}}\right)}\right) \\
&= \mathcal{L}^{-1}\left(\frac{e^{(y-b)\sqrt{\gamma}}+e^{-(y+b)\sqrt{\gamma}}}{2\sqrt{\gamma}} \sum_{n=0}^{\infty} e^{-2bn\sqrt{\gamma}}\right) \\
&= \frac{1}{2}\mathcal{L}^{-1}\left(\sum_{n=0}^{\infty}\frac{e^{-(b-y+2bn)\sqrt{\gamma}}}{\sqrt{\gamma}} + \sum_{n=0}^{\infty}\frac{e^{-(y+b+2bn)\sqrt{\gamma}}}{\sqrt{\gamma}}\right)
\end{aligned}$$

and using the inversion formula:

$$\mathcal{L}^{-1}\left(\frac{e^{-b\sqrt{\gamma}}}{\sqrt{\gamma}}\right)=\frac{e^{-b^2/4\tau}}{\sqrt{\pi\tau}}$$

we obtain:

$$\mathcal{L}^{-1}\left(\frac{\cosh\left(y\sqrt{\gamma}\right)}{2\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)}\right)=\frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{e^{-(b-y+2bn)^2/4\tau}}{\sqrt{\pi\tau}}+\sum_{n=0}^{\infty}\frac{e^{-(y+b+2bn)^2/4\tau}}{\sqrt{\pi\tau}}\right)$$

and in the second sum using $n=-v-1$, we have:

$$\begin{aligned}
&= \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{e^{-(b-y+2bn)^2/4\tau}}{\sqrt{\pi\tau}}+\sum_{n=-1}^{-\infty}\frac{e^{-(y-b-2bv)^2/4\tau}}{\sqrt{\pi\tau}}\right)\\
&= \sum_{n=-\infty}^{\infty}\frac{e^{-(b-y+2bn)^2/4\tau}}{\sqrt{4\pi\tau}}
\end{aligned}$$

i.e. a series expansion in terms of the normal density. We can then obtain the inverse of the function $h_d\left(X,x,\gamma\right)$:

$$\begin{aligned}
&\mathcal{L}^{-1}\left(h_d\left(X,x,\gamma\right)\right)\\
&=\mathcal{L}^{-1}\left(\frac{\cosh\left((x-\xi-b)\sqrt{\gamma}\right)-\cosh\left((x+\xi-b)\sqrt{\gamma}\right)}{2\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)}\right)\\
&=\sum_{n=-\infty}^{\infty}\frac{e^{-(b-y+2bn)^2/4\tau}}{\sqrt{4\pi\tau}}-\sum_{n=-\infty}^{\infty}\frac{e^{-(b-y+2bn)^2/4\tau}}{\sqrt{4\pi\tau}}
\end{aligned}$$

The inversion formula above is then the sum of an infinite number of normal densities and we obtain the same result as found in Kunitomo and Ikeda [60] and He et al. [44].

### B.1.2. Inversion in terms of sine functions

This second method has been used by Pelsser [78]. The expression of the inverse of the Laplace transforms appearing in equation (B.9) and (B.10) can be found in many textbook concerning the Laplace transform, e.g. in the classical Abramowitz and Stegun [4].

We can use the identity $\sinh(z) = -i\sin(iz)$, and substituting we can see that the denominator of (B.9) is equal to zero when $\sinh\left(b\sqrt{\gamma}\right) = 0$, i.e. when $i\sin ib\sqrt{\gamma} = 0$ and this happens if

$$ib\sqrt{\gamma} = k\pi; k = 0, 1, ...$$

i.e. when:

$$\gamma_k = -\left(\frac{k\pi}{b}\right)^2$$

In this way we have identified all the singularities of the function $e^{\gamma\tau}h^+(\gamma, x)$. The residue for each singularity $\gamma_k$ is given by:

$$\begin{aligned}
\mathrm{Res}(\gamma_k) &= \lim_{\gamma\to\gamma_k} e^{\gamma\tau}(\gamma-\gamma_k)\frac{i\sin\left(ix\sqrt{\gamma}\right)}{i\sin\left(ib\sqrt{\gamma}\right)} \\
&= e^{\gamma_k\tau}\sin\left(ix\sqrt{\gamma_k}\right)\lim_{\gamma\to\gamma_k}\frac{(\gamma-\gamma_k)}{\sin\left(ib\sqrt{\gamma}\right)} \\
&= e^{\gamma_k\tau}\sin\left(x\frac{k\pi}{b}\right)\lim_{\gamma\to\gamma_k}\frac{1}{\cos\left(ib\sqrt{\gamma}\right)ib\gamma^{-1/2}/2} \\
&= e^{\gamma_k\tau}\sin\left(x\frac{k\pi}{b}\right)\frac{2\sqrt{\gamma_k}}{\cos(k\pi)\,ib} \\
&= -2(-1)^k e^{-(k\pi/b)^2\tau}\sin\left(\frac{xk\pi}{b}\right)\frac{k\pi}{b^2}
\end{aligned}$$

and summing up all of the residues, we find:

$$\mathcal{L}^{-1}\left(h^+(\tau, x)\right) = \frac{2\pi}{b^2}\sum_{k=1}^{\infty}(-1)^{k+1}ke^{-(k\pi/b)^2\tau}\sin\left(\frac{k\pi x}{b}\right)$$

We can proceed similarly, for the second function $\cosh\left(y\sqrt{\gamma}\right)/\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)$. Before calculating the residue, we verify that $\gamma = 0$ is a simple pole and not a branch point.

Indeed near the origin, we have[16]:

$$\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right) \approx \sqrt{\gamma}\sqrt{\gamma}b\sum_{n=0}^{\infty}\frac{b^{2n}\sqrt{\gamma}^{2n}}{(2n+1)!} = \gamma b\left(1+\sum_{n=1}^{\infty}\frac{b^{2n}\gamma^{n}}{(2n+1)!}\right)$$

and then $\gamma = 0$ is a simple pole. The residue is:

$$\operatorname{Res}(\gamma=0) = \lim_{\gamma\to 0} e^{\gamma\tau}\gamma\frac{\cosh\left(x\sqrt{\gamma}\right)}{\gamma b\left(1+\sum_{n=1}^{\infty}\frac{b^{2n}\gamma^{n}}{(2n+1)!}\right)} = \frac{1}{b}$$

Using the fact that $\sqrt{\gamma_k} = ik\pi/b$ and that $\cosh(z) = \cos(iz)$, the residues at the others poles are:

$$\begin{aligned}
\operatorname{Res}(\gamma_k) &= \lim_{\gamma\to\gamma_k} e^{\gamma\tau}(\gamma-\gamma_k)\frac{\cosh\left(x\sqrt{\gamma}\right)}{\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)} \\
&= \lim_{\gamma\to\gamma_k} e^{\gamma\tau}\cosh\left(x\sqrt{\gamma}\right)\frac{(\gamma-\gamma_k)}{\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)} \\
&= e^{\gamma_k\tau}\frac{\cosh\left(x\sqrt{\gamma_k}\right)}{\sqrt{\gamma_k}}\lim_{\gamma\to\gamma_k}\frac{(\gamma-\gamma_k)}{i\sin\left(ib\sqrt{\gamma}\right)} \\
&= e^{\gamma_k\tau}\frac{\cosh\left(x\sqrt{\gamma_k}\right)}{\sqrt{\gamma_k}}\lim_{\gamma\to\gamma_k}\frac{1}{i\cos\left(ib\sqrt{\gamma}\right)ib\gamma^{-1/2}/2} \\
&= 2e^{\gamma_k\tau}\frac{\cosh\left(x\sqrt{\gamma_k}\right)}{\sqrt{\gamma_k}}\frac{\sqrt{\gamma_k}}{i\cos\left(ib\sqrt{\gamma_k}\right)ib} \\
&= -2e^{\gamma_k\tau}\frac{\cos\left(ix\sqrt{\gamma_k}\right)}{\cos\left(ib\sqrt{\gamma_k}\right)b} = -2e^{\gamma_k\tau}\frac{\cos\left(x\frac{k\pi}{b}\right)}{\cos(k\pi)b} \\
&= \frac{2}{b}(-1)^{k+1}e^{\gamma_k\tau}\cos\left(x\frac{k\pi}{b}\right)
\end{aligned}$$

$$\operatorname{Res}(\gamma_k) = \frac{2}{b}(-1)^{k+1}e^{\gamma_k\tau}\cos\left(x\frac{k\pi}{b}\right)$$

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[16] We are using the series expansion:

$$\sinh x = \sum_{n=0}^{\infty}\frac{x^{2n+1}}{(2n+1)!}$$

and then the original function is given by:

$$\mathcal{L}^{-1}\left(\frac{\cosh\left(x\sqrt{\gamma}\right)}{\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)}\right)=\frac{1}{b}+\frac{2}{b}\sum_{k=1}^{\infty}(-1)^{k+1}e^{-(k\pi/b)^2\tau}\cos\left(xk\pi/b\right)$$

and then:

$$\begin{aligned}
&\mathcal{L}^{-1}\left(h_d\left(X,x,\gamma\right)\right)=\mathcal{L}^{-1}\left(\tfrac{\cosh\left(\left(x-\xi-b\right)\sqrt{\gamma}\right)-\cosh\left(\left(x+\xi-b\right)\sqrt{\gamma}\right)}{2\sqrt{\gamma}\sinh\left(b\sqrt{\gamma}\right)}\right)\\
&=\tfrac{1}{b}\left(\sum_{k=1}^{\infty}(-1)^{k+1}e^{-(k\pi/b)^2\tau}\cos\left(\left(x-\xi-b\right)k\pi/b\right)-\sum_{k=1}^{\infty}(-1)^{k+1}e^{-(k\pi/b)^2\tau}\cos\left(\left(x+\xi-b\right)k\pi/b\right)\right)\\
&=\tfrac{1}{b}\left(\sum_{k=1}^{\infty}(-1)^{k+1}e^{-(k\pi/b)^2\tau}\left(\cos\left(\left(x-\xi-b\right)k\pi/b\right)-\cos\left(\left(x+\xi-b\right)k\pi/b\right)\right)\right)\\
&=\tfrac{1}{b}\sum_{k=1}^{\infty}(-1)^{k+1}e^{-(k\pi/b)^2\tau}\left(2\sin\left(\tfrac{(x-\xi-b+(x+\xi-b))k\pi/b}{2}\right)\sin\left(\tfrac{(x+\xi-b-(x-\xi-b))k\pi/b}{2}\right)\right)\\
&=\tfrac{2}{b}\sum_{k=1}^{\infty}(-1)^{k+1}e^{-(k\pi/b)^2\tau}\sin\left(\left(x-b\right)k\pi/b\right)\sin\left(\xi k\pi/b\right)
\end{aligned}$$

The different terms we have found make up the solution given in terms of Fourier series in equation (B.5). This solution has a form quite different from that given in previous section, but by the uniqueness theorem for the initial and boundary value problem for the heat equation, both solutions must be identical. The equivalence can be proved using the Poisson summation formula, Zauderer [108] pag. 291.

## C. The Laplace transform for Asian options

In this appendix we solve the second-order differential equation (3.46) when $-\infty < y < 0$:

$$\begin{aligned}
&\tfrac{1}{2}y^2\Psi_{yy}+\left(1/\hat{T}-\hat{r}\,y\right)\Psi_y-\left(\gamma+\hat{r}\right)\Psi=0\\
&\Psi\left(y,\gamma\right)\to 0 \text{ as } y\to-\infty \qquad\qquad (\mathrm{C}.1)\\
&\Psi\left(y,\gamma\right)\to\mathcal{L}\left(\Phi\left(0,\tau\right)\right)=\frac{1}{\left(\gamma+\hat{r}\right)\gamma\hat{T}} \text{ as } y\to 0^-
\end{aligned}$$

In order to have an ODE on the positive axis, we make the change of variable $z=-y$, defining $\Psi^*(z,\gamma) = \Psi(-z,\gamma)$. This function solves the ODE for $0 < z < \infty$:

$$\frac{1}{2}z^2\Psi^*_{zz} - \left(1/\hat{T} + \hat{r}\,z\right)\Psi^*_z - \gamma\Psi^* = 0$$

$$\Psi^*(z,\gamma) \to 0 \text{ as } z \to \infty$$

$$\Psi^*(0,\gamma) = \frac{1}{\left(\gamma+\hat{r}\right)\gamma\hat{T}} \text{ as } z \to 0^+$$

Finally, we define the new function $\Lambda(x;\gamma)$, setting $x = a/z$:

$$\Psi^*(z,\gamma) = z^b\Lambda\left(\frac{a}{z};\gamma\right)$$

where $a$ and $b$ have to be determined. We obtain:

$$\begin{aligned}
\Psi^*_z &= bz^{b-1}\Lambda - az^{b-2}\Lambda_x \\
\Psi^*_{zz} &= b(b-1)z^{b-2}\Lambda - 2a(b-1)z^{b-3}\Lambda_x + a^2z^{b-4}\Lambda_{xx}
\end{aligned}$$

and substituting, we obtain:

$$\begin{aligned}
&\tfrac{1}{2}a^2z^{b-2}\Lambda_{xx} + \left(-a(b-1) + \tfrac{1}{\hat{T}}a\tfrac{z^{b-2}}{z^{b-1}} + \hat{r}\,a\right)z^{b-1}\Lambda_x \\
&+ \left(\left(\tfrac{1}{2}b(b-1) - \hat{r}\,b - \gamma\right)\tfrac{z^b}{z^{b-1}} - \tfrac{1}{\hat{T}}b\right)z^{b-1}\Lambda = 0
\end{aligned} \tag{C.2}$$

Now let us set $b$ st:

$$\frac{1}{2}b(b-1) - \hat{r}\,b - \gamma = 0$$

i.e.:

$$b = \left(\frac{1}{2} + \hat{r}\right) \pm \sqrt{2\gamma + \left(\frac{1}{2} + \hat{r}\right)^2} \tag{C.3}$$

and the choice between the two will be made later on. Then dividing in (C.2) by $z^{b-1}$, using the fact that $z = a/x$ and multiplying by $2/a$, we obtain:

$$x\Lambda_{xx} + \left(2\left(\hat{r} - b + 1\right) + \frac{2}{a\hat{T}}x\right)\Lambda_x - \frac{2b}{\hat{T}a}\Lambda = 0$$

Finally setting $a = -2/\hat{T} < 0$, we arrive at the following differential equation for $x$:

$$x\Lambda_{xx} + \left(2\left(\hat{r} - b + 1\right) - x\right)\Lambda_x + b\Lambda = 0.$$

As described in the Appendix A, the ODE can be recognized as a confluent hypergeometric differential equation, Lebedev [63], pag. 262-3, whose general solution can be written in the form:

$$\begin{aligned}&\Lambda(x,\gamma) = A\Upsilon\left(-b, 2\left(\hat{r} - b + 1\right); x\right)\\ &+Bx^{1-2\left(\hat{r}-b+1\right)}\Upsilon\left(1 - b - 2\left(\hat{r} - b + 1\right), 2 - 2\left(\hat{r} - b + 1\right); x\right)\end{aligned}$$

where $\Upsilon(\alpha, \beta; z)$ is the confluent hypergeometric function with the following series representation:

$$\Upsilon(\alpha, \beta; x) = \sum_{k=0}^{\infty} \frac{(\alpha)_k}{(\beta)_k}\frac{x^k}{k!}, |x| < \infty, \beta \neq 0, -1, -2, ...$$

and $(\alpha)_k$ is the Pochammer symbol:

$$\begin{aligned}(\alpha)_0 &= 1\\ (\alpha)_k &= \alpha(\alpha+1)...(\alpha+k-1); k = 2.3...\end{aligned}$$

and where the constants $b$, $A$ and $B$ have to be chosen in order to satisfy the boundary conditions. We have:

$$\Psi^*(z,\gamma) = z^b A \Upsilon\left(-b, 2\left(\hat{r} - b + 1\right); \tfrac{a}{z}\right)$$
$$+B^* z^{b-1+2\left(\hat{r}-b+1\right)} \Upsilon\left(1 - b - 2\left(\hat{r} - b + 1\right), 2 - 2\left(\hat{r} - b + 1\right); \tfrac{a}{z}\right)$$

with $B^* = Ba^{1-2\left(\hat{r}-b+1\right)}$.

In order to find the constants $A$, $B$, and $b$ we use the fact that:

$$\Upsilon(\alpha, \beta; 0) = 1$$

and the asymptotic representation for the function $\Upsilon(\alpha, \beta; z)$ for large $|x|$, (Lebedev pag. 271 eq. 9.12.8 and pag. 267 eq. 9.11.2):

$$\Upsilon(\alpha, \beta; x) = \frac{\Gamma(\beta)}{\Gamma(\beta - \alpha)} x^{-\alpha} \sum_{k=0}^{n} \frac{(-1)^k (\alpha)_k (1 + \alpha - \beta)_k}{k!} x^{-k} + O\left(|x|^{-n-1}\right)$$

Then:

$$\lim_{z \to +\infty} \Psi^*(z,\gamma) = \lim_{z \to +\infty} \left(z^b A + B^* z^{1+2\hat{r}-b}\right)$$

and:

$$
\begin{aligned}
&\lim_{z\to 0^+}\Psi^*(z,\gamma)\\
&=\lim_{z\to 0}\left(\begin{array}{l} Az^b\Upsilon\left(-b,2\left(\hat{r}-b+1\right);\frac{a}{z}\right)\\ +B^*z^{b-1+2\left(\hat{r}-b+1\right)}\Upsilon\left(1-b-2\left(\hat{r}-b+1\right),2-2\left(\hat{r}-b+1\right);\frac{a}{z}\right)\end{array}\right)\\
&=\lim_{z\to 0}\left(\begin{array}{l} Az^b\frac{\Gamma\left(2\left(\hat{r}-b+1\right)\right)}{\Gamma\left(2\left(\hat{r}-b+1\right)+b\right)}\frac{a^b}{z^b}\\ +B^*z^{b-1+2\left(\hat{r}-b+1\right)}\frac{\Gamma\left(2\left(\hat{r}-b+1\right)\right)}{\Gamma\left(2\left(\hat{r}-b+1\right)+b\right)}\left(\frac{a}{z}\right)^{-1+b+2\left(\hat{r}-b+1\right)}\end{array}\right)\\
&=A\frac{\Gamma\left(2\left(\hat{r}-b+1\right)\right)}{\Gamma\left(2\left(\hat{r}-b+1\right)+b\right)}a^b+B^*\frac{\Gamma\left(2\left(\hat{r}-b+1\right)\right)}{\Gamma\left(2\left(\hat{r}-b+1\right)+b\right)}a^{-1+b+2\left(\hat{r}-b+1\right)}
\end{aligned}
\tag{C.4}
$$

If we choose $b$ such that $\mathrm{Re}(b)<0$, i.e. in C.3 we choose the root with negative real part:

$$
b=\left(\frac{1}{2}+\hat{r}\right)-\sqrt{2\gamma+\left(\frac{1}{2}+\hat{r}\right)^2}
$$

then we observe that $\mathrm{Re}\left(1+2\,\hat{r}-b\right)>0$ and in order to have a bounded solution for $z\to+\infty$, we need to require $B^*=0$ and then the solution is given by[17]:

$$
Az^b\Upsilon\left(-b,2\left(\hat{r}-b+1\right);\frac{a}{z}\right)
$$

where $A$ is found (C.4) and using the condition for $z\to+0$:

$$
A=\frac{\Gamma\left(2\left(\hat{r}-b+1\right)+b\right)}{\Gamma\left(2\left(\hat{r}-b+1\right)\right)}\frac{a^{-b}}{\left(\gamma+\hat{r}\right)\gamma\,\hat{T}}
$$

[17] It is easy to verify that if choose the root with positive real part, then $\mathrm{Re}(1+2\hat{r}\text{-b})<0$ and we have to set $A=0$. However, we obtain the same solution.

and finally we obtain the Laplace transform wrt to $\sigma^2(T-t)$ of the price:

$$\Psi(y,\gamma)=\frac{\Gamma\left(2\left(\hat{r}-b+1\right)+b\right)\left(\frac{-y\hat{T}}{2}\right)^{b}}{\Gamma\left(2\left(\hat{r}-b+1\right)\right)\left(\gamma+\hat{r}\right)\gamma\hat{T}}\Upsilon\left(-b,2\left(\hat{r}-b+1\right);\frac{2}{\hat{T}y}\right).$$

i.e. equation (3.47) in the main text.

## D. The Laplace transform in the CIR model

In order to find an expression for:

$$v(\tau,r;\gamma,\mu):=v(\tau,r)=E_{0,r}\left[\exp\left(-\gamma r_\tau-\mu\int_0^\tau r(s)\,ds\right)\right] \tag{D.1}$$

we need to solve the Feynman-Kac equation:

$$-v_\tau+(a-br)v_r+\tfrac{1}{2}\sigma^2 r v_{rr}=\mu r v$$
$$v(0,r)=\exp(-\gamma r)$$

We exploit the linearity of the drift and variance coefficients and we try a solution of the type:

$$v(\tau,r)=\exp\left(-a\phi(\tau;\lambda,\mu)-r\psi(\tau;\lambda,\mu)\right)$$

Setting $\phi(\tau):=\phi(\tau;\lambda,\mu)$ and $\psi(\tau):=\psi(\tau;\lambda,\mu)$ and using the fact that:

$$\begin{aligned}\frac{\partial v(\tau,r)}{\partial\tau}&=\left(-a\phi_\tau-r\psi_\tau\right)v\\ \frac{\partial v(\tau,r)}{\partial r}&=-\psi v\\ \frac{\partial^2 v(\tau,r)}{\partial r^2}&=\psi^2 v\end{aligned}$$

and substituting these expressions in the above PDE, we obtain:

$$-\left(-a\phi_{\tau}-r\psi_{\tau}\right)v-\psi\left(a-br\right)v+\frac{1}{2}\sigma^{2}r\psi^{2}v=\mu rv$$

If we divide by $v$ and we collect the terms multiplying $r$, we obtain:

$$a\phi_{\tau}-a\psi+r\left(\psi_{\tau}+b\psi+\frac{1}{2}\sigma^{2}\psi^{2}-\mu\right)=0 \tag{D.2}$$

that has to be solved with respect to the functions $\phi(\tau)$ and $\psi(\tau)$. For (D.2) to be satisfied for all values of $r$, the terms multiplying $r$ and those independent of $r$ must each be identically zero.. Thus, we can split the above PDE in two separate ODE's:

$$-\psi_{\tau}=b\psi+\frac{1}{2}\sigma^{2}\psi^{2}-\mu \tag{D.3}$$

and:

$$\phi_{\tau}=\psi \tag{D.4}$$

In order to obtain the initial conditions, we observe that:

$$v\left(0,r\right)=\exp\left(-\gamma r\right)=\exp\left(-a\phi\left(0\right)-r\psi\left(0\right)\right)$$

and then the initial conditions are given by:

$$\begin{aligned}\phi\left(0\right) &= 0\\ \psi\left(0\right) &= \gamma\end{aligned}$$

The first ODE is known as Riccati differential equation, because it involves a non-linear term $\psi^2$. Fortunately, it is possible to give a closed form expression to the solution of the two ODE's. In deriving the solution we follow Ingersoll [53], pagg. 397-8. We rewrite (D.3) as:

$$\frac{d\psi}{\frac{1}{2}\sigma^2\psi^2 + b\psi - \mu} = -d\tau \tag{D.5}$$

and integrating and applying the initial condition, we obtain:

$$\begin{aligned}
\int \frac{d\psi}{\frac{1}{2}\sigma^2\psi^2+b\psi-\mu} &= \frac{2}{\sigma^2}\int \frac{d\psi}{(\psi-c_1)(\psi-c_2)} \\
&= \frac{2}{\sigma^2}\int \left(\frac{A}{(\psi-c_1)} + \frac{B}{(\psi-c_2)}\right) d\psi \\
&= \frac{\ln(\psi(\tau)-c_1)/(\psi(\tau)-c_2)}{\lambda}
\end{aligned}$$

where:

$$\begin{aligned}
c_1 &= \frac{-b+\lambda}{\sigma^2}; c_2 = \frac{-b-\lambda}{\sigma^2} \\
B &= 1/(c_2-c_1); A = 1/(c_1-c_2) \\
c_1+c_2 &= -2b/\sigma^2; c_1-c_2 = 2\lambda/\sigma^2; c_1c_2 = -\frac{2\mu}{\sigma^2}
\end{aligned}$$

and integrating both sides in (D.5), we obtain:

$$\frac{\ln(\psi(\tau)-c_1)/(\psi(\tau)-c_2)}{\lambda} = -\tau + K$$

where $K$ is chosen for satisfying the initial condition:

$$\frac{\ln(\gamma-c_1)/(\gamma-c_2)}{\lambda} = K$$

and then:

$$\ln(\psi(\tau)-c_1)/(\psi(\tau)-c_2) = -\lambda\tau + \ln(\gamma-c_1)/(\gamma-c_2)$$

i.e. solving wrt $\psi(\tau)$:

$$\begin{aligned}\psi(\tau) &= \frac{c_1 - c_2\frac{(\gamma-c_1)}{(\gamma-c_2)}\exp(-\tau\lambda)}{1-\frac{(\gamma-c_1)}{(\gamma-c_2)}\exp(-\tau\lambda)} \\ &= \frac{\gamma(c_1\exp(\tau\lambda)-c_2)+c_2c_1(1-\exp(\tau\lambda))}{\gamma(\exp(\tau\lambda)-1)-c_2\exp(\tau\lambda)+c_1}\frac{\exp(-\tau\lambda)}{\exp(-\tau\lambda)}\end{aligned}$$

Substituting the expressions for $c_1$ and $c_2$, finally we have:

$$\psi(\tau) = \frac{\gamma(\lambda + b + (\lambda - b)\exp(\tau\lambda)) + 2\mu(\exp(\tau\lambda) - 1)}{\sigma^2\gamma(\exp(\tau\lambda) - 1) + \lambda - b + (b + \lambda)\exp(\tau\lambda)}$$

where

$$\lambda = \sqrt{b^2 + 2\sigma^2\mu}$$

In order to find the function $\phi(\tau)$ let us integrate the expression of $\psi(\tau)$ given by:

$$\psi(\tau) = \frac{c_1 - c_2 a\exp(-\tau\lambda)}{1 - a\exp(-\tau\lambda)}; a = \frac{(\gamma - c_1)}{(\gamma - c_2)}$$

and we obtain:

$$\begin{aligned}\phi(\tau) &= \int_0^\tau \psi(s)\,ds = \int_0^\tau \frac{c_1 - c_2 a\exp(-s\lambda)}{1 - a\exp(-s\lambda)}ds \\ &= \frac{(c_1 - c_2)}{\lambda}\ln\frac{\left(-1 + ae^{-\tau\lambda}\right)}{(-1+a)} + c_1\tau\lambda\end{aligned}$$

and then using $c_1 = (-b + \lambda)/\sigma^2$ and $c_1 - c_2 = 2\lambda/\sigma^2$

$$\begin{aligned}\phi(\tau) &= -\frac{2}{\sigma^2}\ln\frac{(-1+a)}{\left(-1+ae^{-\tau\lambda}\right)} - \frac{2}{\sigma^2}\ln\left(e^{-\frac{\sigma^2}{2}c_1\tau}\right) \\ &= -\frac{2}{\sigma^2}\left(\ln\frac{(-1+a)}{\left(-1+ae^{-\tau\lambda}\right)} + \ln\left(e^{-\frac{\sigma^2}{2}c_1\tau}\right)\right) \\ &= -\frac{2}{\sigma^2}\left(\ln\frac{(-1+a)\,e^{-\frac{\sigma^2}{2}(c_1)\tau}}{\left(-1+ae^{-\tau\lambda}\right)}\right) = -\frac{2}{\sigma^2}\left(\ln\frac{(-1+a)\,e^{-\frac{\sigma^2}{2}(c_1)\tau}}{\left(-e^{\tau\lambda}+a\right)e^{-\tau\lambda}}\right)\end{aligned}$$

$$
\begin{aligned}
&= -\frac{2}{\sigma^2}\left(\ln\frac{(-1+a)\,e^{\left(\lambda-\frac{\sigma^2}{2}c_1\right)\tau}}{(-e^{\tau\lambda}+a)}\right) = -\frac{2}{\sigma^2}\left(\ln\frac{(-1+a)\,e^{\left(\lambda-\frac{\sigma^2}{2}\left(\frac{-b+\lambda}{\sigma^2}\right)\right)\tau}}{(-e^{\tau\lambda}+a)}\right) \\
&= -\frac{2}{\sigma^2}\left(\ln\frac{(-1+a)\,e^{(\lambda+b)\tau/2}}{(a-e^{\tau\lambda})}\right)
\end{aligned}
$$

and using the expression for $a$, and some algebra, we finally obtain:

$$
\phi(\tau) = -\frac{2}{\sigma^2}\ln\frac{2\lambda e^{(\lambda+b)\tau/2}/\sigma^2}{\left(\gamma\sigma^2\left(e^{\tau\lambda}-1\right)-b+\lambda+(b-\lambda)\,e^{\tau\lambda}\right)}
$$

## E. The Laplace transform for an occupation time

The mgf of the r.v. $Y_{0,x}(u,\tau)$, defined as:

$$
\begin{aligned}
&v(\tau,x) \equiv v(\tau,x;u) = \mathbf{E}_{0,x}\left[e^{-\mu Y(u,\tau)}\right] \\
&= \int_0^\tau e^{-\mu y}\,\mathrm{Pr}_{0,x}\left(Y(u,\tau)\in dy\right) + 1\times \mathrm{Pr}_{0,x}\left[Y(u,\tau)=0\right] + e^{-\mu\tau}\times \mathrm{Pr}_{0,x}\left[Y(u,\tau)=\tau\right] \\
&= \mathcal{L}\left[\mathrm{Pr}_{0,x}\left(Y(u,\tau)\in dy\right); y\to\mu\right] + 1\times \mathrm{Pr}_{0,x}\left[Y(u,\tau)=0\right] + e^{-\mu\tau}\times \mathrm{Pr}_{0,x}\left[Y(u,\tau)=\tau\right]
\end{aligned}
\tag{E.1}
$$

satisfies the equation:

$$
-\frac{\partial v(\tau,x)}{\partial\tau} + \frac{1}{2}\frac{\partial^2 v(\tau,x)}{\partial x^2} + \delta\frac{\partial v(\tau,x)}{\partial x} - \mu\mathbf{1}_{(x<u)}v(\tau,x) = 0 \tag{E.2}
$$

with initial condition:

$$
v(0,x) = 1 \tag{E.3}
$$

and boundary conditions:

$$
v(\tau,+\infty) = 1 \tag{E.4}
$$

$$
v(\tau,-\infty) = e^{-\tau} \tag{E.5}
$$

Considering the following transformation:

$$h(\tau, z) = e^{-\alpha(z+u)-\beta\tau} v(\tau, z+u) \tag{E.6}$$

$$z = x - u, \alpha = -\delta < 0, \beta = -\frac{\delta^2}{2} \tag{E.7}$$

we get:

$$-\frac{\partial h(\tau, z)}{\partial \tau} + \frac{1}{2}\frac{\partial^2 h(\tau, z)}{\partial z^2} - \mu \mathbf{1}_{(z<0)} h(\tau, z) = 0 \tag{E.8}$$

with $h(0, z) = e^{-\alpha(z+u)}$. We can consider now the Laplace transform respect to time of the function $h(\tau, z)$ defining:

$$H(\gamma, z) = \mathcal{L}[h(\tau, z); \tau \to \gamma] = \int_0^{+\infty} e^{-\gamma\tau} h(\tau, z)\, d\tau$$

and the above equation becomes:

$$-(\gamma H(\gamma, z) - h(0, z)) + \frac{1}{2} H_{zz}(\gamma, z) - \mu \mathbf{1}_{(z<0)} H(\gamma, z) = 0 \tag{E.9}$$

The general solution is:

$$H(\gamma, z) = \begin{cases} 1_{(z\geq 0)} \left( \frac{e^{-\alpha(z+u)}}{(\gamma - \alpha^2/2)} + Ae^{+\sqrt{2\gamma}z} + Be^{-\sqrt{2\gamma}z} \right) \\ \\ 1_{(z<0)} \left( \frac{e^{-\alpha(z+u)}}{(\gamma+\mu-\alpha^2/2)} + Ce^{+\sqrt{2(\gamma+\mu)}z} + De^{-\sqrt{2(\gamma+\mu)}z} \right) \end{cases}$$

The constants $A$, $B$, $C$ and $D$ have to be chosen in order to have continuity and differentiability of the solution at $z = 0$ and the boundary conditions as $x \to \pm\infty$ (i.e.

$z \to \pm\infty$). From (E.4), we have that:

$$\mathcal{L}\left[v\left(\tau,x\right);\gamma\right] = \begin{cases} \frac{1}{\gamma} & x \to +\infty \\ \\ \frac{1}{\gamma+\mu} & x \to -\infty \end{cases}$$

i.e. we should have:

$$e^{\alpha(z+u)}H\left(\gamma,z\right) = \begin{cases} \frac{1}{\gamma+\beta} & z \to +\infty \\ \\ \frac{1}{\gamma+\mu+\beta} & z \to -\infty \end{cases}$$

where $\beta = -\alpha^2/2$. This condition implies that for $z > 0$ we should have[18] $A = 0$ and when $z < 0$ we should set $D = 0$. The solution is then:

$$H\left(\gamma,z\right) = \begin{cases} 1_{(z\geq 0)}\left(\frac{e^{-\alpha(z+u)}}{(\gamma-\alpha^2/2)} + Be^{-\sqrt{2\gamma}z}\right) \\ \\ 1_{(z<0)}\left(\frac{e^{-\alpha(z+u)}}{(\gamma+\mu-\alpha^2/2)} + Ce^{+\sqrt{2(\gamma+\mu)}z}\right) \end{cases}$$

The constants $B$ and $C$ have to guarantee continuity and differentiability of the solution at $z = 0$, i.e.:

$$\begin{cases} \frac{e^{-\alpha u}}{(\gamma-\alpha^2/2)} + B = \frac{e^{-\alpha u}}{(\gamma+\mu-\alpha^2/2)} + C \\ \frac{-\alpha e^{-\alpha u}}{(\gamma-\alpha^2/2)} - \sqrt{2\gamma}B = \frac{-\alpha e^{-\alpha u}}{(\gamma+\mu-\alpha^2/2)} + \sqrt{2\left(\gamma+\mu\right)}C \end{cases}$$

Solving this linear system we obtain:

$$C = e^{-\alpha u}\left(\frac{1}{(\gamma+\mu-\alpha^2/2)} - \frac{1}{(\gamma-\alpha^2/2)}\right)\frac{\left(\alpha-\sqrt{2\gamma}\right)}{\left(\sqrt{2\left(\gamma+\mu\right)}+\sqrt{2\gamma}\right)}$$

[18] For the assumption that $\alpha < 0$. If $\alpha > 0$, we should set $B = 0$.

$$B = e^{-\alpha u}\left(\left(\frac{1}{(\gamma+\mu-\alpha^2/2)}-\frac{1}{(\gamma-\alpha^2/2)}\right)\left(\frac{(\alpha-\sqrt{2\gamma})}{\left(\sqrt{2\,(\gamma+\mu)}+\sqrt{2\gamma}\right)}+1\right)\right)$$

and then:

$$\begin{aligned}H(\gamma,0) &= \frac{e^{-\alpha u}}{(\gamma-\alpha^2/2)}+B\\ &= e^{-\alpha u}\left(\frac{1}{(\gamma-\alpha^2/2)}+\left(\frac{1}{(\gamma+\mu-\alpha^2/2)}-\frac{1}{(\gamma-\alpha^2/2)}\right)\left(\frac{(\alpha-\sqrt{2\gamma})}{\left(\sqrt{2(\gamma+\mu)}+\sqrt{2\gamma}\right)}+1\right)\right)\\ &= \frac{e^{-\alpha u}}{\left(\sqrt{\gamma+\mu}-\alpha/\sqrt{2}\right)\left(\sqrt{\gamma}+\alpha/\sqrt{2}\right)}\end{aligned}$$

The expression above looks simpler than the corresponding expression (1.4) in Akahori. This fact is due to the preliminary transformation that has allowed us to eliminate the drift of the process. Let us call $\zeta(\tau)$ the inverse Laplace of the quantity above, i.e.

$$\zeta(\tau) := h(\tau,0) = \mathcal{L}^{-1}\left(H(\gamma,0);\gamma\to\tau\right)$$

Using the transformation (E.6) and the density of the first hitting time of the BM with zero drift[19] at the level $z=0$, we can write:

$$h(\tau,z)=\begin{cases} 1_{(z>0)}\begin{pmatrix} e^{-\alpha(z+u)-\beta\tau}\times \Pr_{0,z\in(0;+\infty)}\left(Y(u,\tau)=0\right)\\ +e^{-\alpha(z+u)-\beta\tau}\int_0^\tau \frac{|z|}{\sqrt{2\pi(\tau-\theta)^3}}e^{-\frac{z^2}{2(\tau-\theta)}}\zeta(\theta)\,d\theta\end{pmatrix}\\ \\ 1_{(z<0)}\begin{pmatrix} e^{-\alpha(z+u)-\beta\tau}\times e^{-\mu\tau}\times \Pr_{0,z\in(-\infty;0)}\left(Y(u,\tau)=\tau\right)\\ +e^{-\alpha(z+u)-\beta\tau}\int_0^\tau \frac{|z|e^{-\frac{z^2}{2(\tau-\theta)}}}{\sqrt{2\pi(\tau-\theta)^3}}e^{-\mu(\tau-\theta)}\zeta(\theta)\,d\theta\end{pmatrix}\end{cases}$$

[19]We consider the BM with zero drift because we are working with the function $h(t,z)$ that depends on the BM with zero drift.

i.e.

$$v(\tau,x)=\begin{cases}1_{(x>u)}\left(\begin{array}{c}\Pr_{0,x\in(u;+\infty)}\left(Y(u,\tau)=0\right)\\+\int_0^\tau\frac{|x-u|}{\sqrt{2\pi(\tau-\theta)^3}}e^{-\frac{(x-u)^2}{2(\tau-\theta)}}\zeta(\theta)\,d\theta\end{array}\right)\\ \\ 1_{(x<u)}\left(\begin{array}{c}e^{-\mu\tau}\times\Pr_{0,x\in(-\infty;u)}\left(Y(u,\tau)=\tau\right)\\+\int_0^\tau\frac{|x-u|e^{-\frac{(x-u)^2}{2(\tau-\theta)}}}{\sqrt{2\pi(\tau-\theta)^3}}e^{-\mu(\tau-\theta)}\zeta(\theta)\,d\theta\end{array}\right)\end{cases}$$

Comparing the above expressions with (E.1), we have to find the quantity $\Pr_{0,z}(Y(u,\tau)\in dy)$. The Laplace transform wrt $\tau$ of $v(\tau,x)$ is then:

$$\mathcal{L}(v(\tau,x);\tau\to\gamma)=\begin{cases}1_{(x>u)}\left(\mathcal{L}\left(\Pr_{0,x\in(u;+\infty)}\left(Y(u,\tau)=0\right)\right)+\frac{e^{-\sqrt{\gamma}\sqrt{2}(x-u)-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)}\right)\\ \\ 1_{(x<u)}\left(\mathcal{L}\left(e^{-\mu\tau}\times\Pr_{0,x\in(-\infty;u)}\left(Y(u,\tau)=\tau\right)\right)+\frac{e^{-\sqrt{\gamma+\mu}\sqrt{2}(u-x)-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma+\mu}-\frac{\alpha}{\sqrt{2}}\right)}\right)\end{cases}\tag{E.10}$$

where we have used lines 7, 9 and 10 in Table 2 and the convolution property.

## F. Notes to the chapter

The results of the present chapter are in some part original and in other parts already known in the financial literature. In particular, Selby [86] is the first reference for the use of the Laplace transform for solving the Black Scholes model (Example 1), although the use of the LT for solving the heat equation is a very old technique.

In example 2 we have treated barrier options. The discussion of the utility of the LT for barrier options is new, although the results about the LT of the barrier option price can be found: a) for the lognormal case in Geman and Yor [42], Pelsser [78] and Sbuelz [84]; b) for the OU and the square root process some results about the single barrier case can be found in Leblanc and Scaillet [64]; c) the most general results (time homogeneous

Markov process) are however due to Jamshidian [56]. In every case the expressions for the LT of the hitting time distribution can be found in Borodin and Salminen [9].

Regard to the Asian option with fixed strike (example 3) in the lognormal case the first result is due to Geman and Yor [41]. More recently, Lipton [67] has obtained the same result using a PDE approach. The results for the floating strike case appear to be completely new, although we do not have discussed the numerical inversion.

In example 3.4.1 we have presented the derivation of the density law for the square root process. The original derivation is due to Feller [35], although we have followed Lamberton and Lapeyre [62], mainly in the steps for the analytical inversion. The LT results in section 3.4.2 are new, but the expression of the option price in terms of a noncentral $\chi^2$ distribution is well known, also if its computation requires a numerical approximation. The results in examples 3.4.4 and 3.4.5 are a straightforward extension of example 3.4.2. The case of Asian options on interest rate has been discussed in Leblanc and Scaillet [64] and Chacko and Das [13], but they do not realise that the computation of the price can be simplified, avoiding the computation of the density function or the determination of the price of the Asian binary call options. The results in section 3.4.8 are obtained using section 3.4.7.

The results in the section on Occupation time derivative (example 3.5) simplify the results in Akahori [5] and Dassios [24] and agree with Takacs [93], although he does not use the LT technique. Hugonnier [47] and Linetsky [66] generalize the results to the joint density of the BM and occupation time. The application to hurdle options is original.

Finally, the algorithms for the numerical inversion are well known in engineering and physics, but have received little attention in Finance, except Fu et al. [37] and Davydov and Linetsky [26] that apply the AW algorithm and its multidimensional version. Numerical results can be found also in Leblanc and Scaillet [64] and in Cathcart [12].

# Chapter 2. The density law for an occupation time

## 1. Introduction

In this chapter we obtain new results on a generalization of the Lévy arc-sine law. Lévy studied the density of the time spent by a Standard Brownian Motion (SBM) below a given level. We will provide results about the case of the Brownian Motion with drift below a given level and inside a given band. This problem has been solved for the case of Brownian Motion with drift below a given level by Akahori [3][1], Dassios [8], Takacs [25], Embrechts et al.[10]. In this paper we derive the same expression as in [25] and simpler than that given in [3] and [8]. So the main results are related to the case of the time spent by the Brownian motion (without drift and with drift) inside a band.

The results have also financial applications to the pricing of corridor and hurdle options as we illustrate in section 2. Other applications are, as suggested in Taleb [27] (page. 66-67), to the management of a portfolio for the computation of the expected amount of time a trader is expected to spend in the red. A similar problem for the Standard Brownian Excursion can be found in chemistry as well and in particular in the theory of ring polymers as explained in Jansons [15].

We are interested in deriving the density law and the characteristic function of the

[1] The bibliography is after the conclusions in the present chapter.

time spent by a BM with drift inside two barriers. Special cases are obtained when the drift is zero or when the lower barrier goes to -$\infty$.

In next section we will illustrate the problem of the corridor derivative pricing. In order to price such a contract we need the knowledge of the distribution function of the occupation time. In section 3 we will give the expression for the Laplace transform of the characteristic function of the occupation time of the Brownian Motion of the interval $(-\infty, l]$ and $[l, u]$. In section 4, we discuss different inversion techniques (univariate and multidimensional) and we provide numerical examples with a comparison with the MonteCarlo simulation method.

## 2. The Corridor Derivative

Corridor derivatives are exotic options paying at expiry an amount that depends on the time spent by a reference index, usually an exchange rate or an interest rate, below a given level or inside a band. The structure of the payoff is similar to FX range floaters and boost structures as described in Hull [14], Linetski [19], Pechtl [22], Tucker et al.[28] and Turnbull[29]. This kind of product is suitable for investors believing in stable markets and then provide a way to sell volatility.

The popularity of this kind of contracts has increased over last years as reported in Navatte and Quittard-Pinon [20] where there is the description of new issues by the kingdom of Sweden and by Sociètè Gènèrale Acceptance.

In order to price the contract, we apply the well known result that in a arbitrage-free market, according to the well known Harrison-Kreps theorem of asset pricing [13], the price of any contract is just the expected value, under the risk-neutral measure, of the discounted payoff of the contract. So our aim will be to find the distribution function of the occupation time.

Let us suppose that the price of the underlying asset is described by a stochastic differential equation:

$$\begin{aligned} dP_t &= rP_t dt + \sigma P_t dW_t, \\ P_0 &= p \end{aligned}$$

where $r$ is the instantaneous risk free interest rate, so the dynamics of the asset price under the martingale measure is described by a Geometric Brownian Process and $\ln P(t)$ has normal distribution with mean $\ln p + \left(r - \sigma^2/2\right) t$ and variance $\sigma^2 t$.

If we define the random variable:

$$\tau(t, p; L, U) = \int_0^t \mathbf{1}_{(L<P(s)<U)} ds$$

then, a corridor option (hurdle option if $L = 0$) at maturity has payoff given by $\max[\tau - K; 0] := (\tau - K)^+$ and the price at time 0 of the contract having a residual life equal to $t$ and a strike $K < t$, is given by:

$$\begin{aligned} & e^{-rt}\mathbf{E}_{0,p}\left(\max\left[\tau(t, p; L, U) - K; 0\right]\right) \\ & = e^{-rt}\left(\int_0^t (s-K)^+ f_\tau(s, t, p; L, U)\, ds + (t-K)^+ \Pr_{0,p}\left[\tau(t, p; L, U) = t\right]\right) \end{aligned} \tag{2.1}$$

where $f_\tau(s, t, p; L, U)$ is the density function of the r.v. $\tau(t, p; L, U)$ calculated at $s$, when $0 < s < t$. Note that in order to calculate the price we need to take into account the fact that the index can always stay inside the band or below the level $U$ (when $L = 0$). This explains the presence of the term $(t - K)^+ \Pr\left[\tau(t, p; L, U) = t\right]$. In the following we write

$f_\tau(s,t,p)$ to mean $f_\tau(s,t,p;L,U)$. We observe that:

$$\begin{aligned}\mathbf{1}_{(L<P(s)<U)} &= \mathbf{1}_{\left(L<p\exp\left(\left(r-\frac{\sigma^2}{2}\right)s+\sigma W(s)\right)<U\right)} \\ &= \mathbf{1}_{\left(\frac{1}{\sigma}\ln\left(\frac{L}{p}\right)<\frac{1}{\sigma}\left(r-\frac{\sigma^2}{2}\right)s+W(s)<\frac{1}{\sigma}\ln\left(\frac{U}{p}\right)\right)}\end{aligned}$$

So we can calculate the density function of the r.v. $\tau(t,p;L,U)$ using the density of the occupation time of the SBM $W(t)$, if $\left(r-\frac{\sigma^2}{2}\right)/\sigma=0$, and of the BM with drift $W^{(m)}(t) := mt+W(t)$, where $m=\left(r-\sigma^2/2\right)/\sigma\neq 0$. In both cases the barriers are fixed at the levels $l=(\ln L)/\sigma$ and $u=(\ln U)/\sigma$ and with starting value $x=(\ln p)/\sigma$. We study this problem in the following sections.

It is natural to see that we have the same problem to solve if the barriers increase exponentially at rate $\delta: a(t)=Le^{\delta t}$ and $b(t)=Ue^{\delta t}$. Indeed:

$$\mathbf{1}_{\left(Le^{\delta t}<P(t)<Ue^{\delta t}\right)} = \mathbf{1}_{\left(l<\left(r-\delta-\frac{\sigma^2}{2}\right)\frac{1}{\sigma}t+W(t)<u\right)}$$

so the problem is equivalent to consider the occupation time for a Brownian Motion with an adjusted drift equal to $\left(r-\delta-\frac{\sigma^2}{2}\right)/\sigma$ and fixed barriers as above.

We observe that if the strike price is set to zero, we have a corridor bond (hurdle bond if $L=0$) and the price can be obtained discounting the following expression:

$$\mathbf{E}_{0,x}\left(\tau(t,p;L,U)\right) = \mathbf{E}_{0,x}\left[\int_0^t \mathbf{1}_{(l<ms+W(s)<u)}ds\right] \tag{2.2}$$

and applying the Fubini's theorem, we obtain:

$$\begin{aligned} &= \textstyle\int_0^t \mathbf{E}_{0,x}\left(\mathbf{1}_{(l<ms+W(s)<u)}\right)ds \\ &= \textstyle\int_0^t \mathrm{Pr}_{0,x}\left(l<ms+W(s)<u\right)ds \\ &= \textstyle\int_0^t \left(\Phi\left(h(x,u,s)\right)-\Phi\left(h(x,l,s)\right)\right)ds\end{aligned}$$

where $\Phi(x) = \int_{-\infty}^{x}\left(\exp\left(-\frac{w^2}{2}\right)/\sqrt{2\pi}\right)dw$ is the cumulative normal distribution and $h(x,l,t) = \frac{1}{\sqrt{t}}(l - x - mt)$.

## 3. The Characteristic Function of the Occupation Time of the interval $[l,u]$

In order to price corridor derivatives, we are interested in the evaluation of the distribution of the r.v.:

$$\begin{aligned}
\tau(t,x) \quad &: = \tau(t,x;u,l,m) = \int_0^t \mathbf{1}_{(l<W(s)+ms<u)}ds \\
W(0) \quad &= \quad x
\end{aligned}$$

representing the amount of time spent inside the interval $[l,u]$ up to time $t$ by a Brownian Motion with drift $m$ and starting at $x$.

If we define the characteristic function of the r.v. $\tau(t,x)$:

$$\begin{aligned}
&v(t,x,\mu) := v(t,x;u,l,m) = \mathbf{E}_{0,x}\left[e^{i\mu\tau(t,x)}\right] \\
&= \int_0^t e^{i\mu\tau} f_\tau(s,t,x;u,l,m)\,ds + 1 \times \mathrm{Pr}_{0,x}[\tau(t,x;u,l,m) = 0] + e^{i\mu t} \times \mathrm{Pr}_{0,x}[\tau(t,x;u,l,m) = t]
\end{aligned} \tag{3.1}$$

using the Feynman-Kac formula, [16] page. 366, it can be shown that $v(t,x,\mu)$ satisfies the following partial differential equation (PDE):

$$-\frac{\partial v(t,x,\mu)}{\partial t} + \frac{1}{2}\frac{\partial^2 v(t,x,\mu)}{\partial^2 x} + m\frac{\partial v(t,x,\mu)}{\partial x} + i\mu\mathbf{1}_{(l<x<u)}v(t,x,\mu) = 0 \tag{3.2}$$

with initial condition:

$$v(0,x) = 1,\ \forall x \in (-\infty;+\infty) \tag{3.3}$$

and boundary conditions:

$$v(t, \pm\infty) = 1, \, \forall t > 0 \tag{3.4}$$

### 3.1. The main result

Given a function of time $t$, we denote with $\mathcal{L}[\cdot; t \to \gamma]$ its Laplace transform with respect to the variable time $t$, and with $\mathcal{L}^{-1}[\cdot; \gamma \to t]$ the inverse Laplace transform. We have the following result:

**Theorem 3.1.** : *The characteristic function of the r.v. $\tau(t, x; u, l, m)$ admits the following representation:*

$$v(t, x, \mu; l, u, m) = \Omega(t, \mu, x; l, u, m) + \begin{cases} 1 \times \Pr_{0, x \in [u, +\infty)} \left( \inf_{0 \le s \le t} ms + W(s) > u \right) \\ \\ e^{i\mu t} \times \Pr_{0, x \in (u, l)} \left( \sup_{0 \le s \le t} ms + W(s) < u; \inf_{0 \le s \le t} ms + W(s) > l \right) \\ \\ 1 \times \Pr_{0, x \in (-\infty, l]} \left( \sup_{0 \le s \le t} ms + W(s) < l \right) \end{cases} \tag{3.5}$$

*where:*

$$\Omega(t, \mu, x; l, u, m) := \int_0^t e^{i\mu s} f_\tau(s, t, x; u, l, m) \, ds = e^{-mx - \frac{m^2}{2} t} \mathcal{L}^{-1}[\omega(\gamma, \mu, x; l, u, m); \gamma \to t]$$

*and:*

$$\omega(\gamma,\mu,x;l,u,m) = \begin{cases} \mathbf{1}_{(x\geq u)} e^{-\sqrt{2}(x-u)\sqrt{\gamma}} \mathcal{L}\left[y(t,1);t\to\gamma\right] \\ \\ \mathbf{1}_{(l<x<u)} \frac{\mathcal{L}[y(t,0);t\to\gamma]\sinh\left(a\pi\left(\frac{u-x}{u-l}\right)\right)+\mathcal{L}[y(t,1);t\to\gamma]\sinh\left(a\pi\left(\frac{x-l}{u-l}\right)\right)}{\sinh(a\pi)} \\ \\ \mathbf{1}_{(x\leq l)} e^{-\sqrt{2}(l-x)\sqrt{\gamma}} \mathcal{L}\left[y(t,0);t\to\gamma\right] \end{cases}$$

*and:*

$$\begin{aligned} \mathcal{L}\left[y(t,1);t\to\gamma\right] &= \frac{e^{mu}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{m}{\sqrt{2}}\right)} - \frac{c}{2\sqrt{\gamma}}\left(s(\gamma)+d(\gamma)\right) \\ \mathcal{L}\left[y(t,0);t\to\gamma\right] &= \frac{e^{ml}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{m}{\sqrt{2}}\right)} + \frac{c}{2\sqrt{\gamma}}\left(s(\gamma)-d(\gamma)\right) \end{aligned}$$

*with*[2]*:*

$$\begin{aligned} \frac{c}{\sqrt{\gamma}}d(\gamma) = & \frac{\sqrt{\gamma-i\mu}\sinh(a\pi)}{\left(\sqrt{\gamma-i\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1)\right)}\left(\frac{e^{mu}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{m}{\sqrt{2}}\right)}+\frac{e^{ml}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{m}{\sqrt{2}}\right)}\right. \\ & \left. +\frac{\left(-\frac{m}{\sqrt{2}}\left(e^{ml}-e^{mu}\right)(\cosh(a\pi)+1)-\sqrt{\gamma-i\mu}\left(e^{mu}+e^{ml}\right)\sinh(a\pi)\right)}{\sqrt{\gamma-i\mu}\sinh(a\pi)\left(\gamma-i\mu-\frac{m^2}{2}\right)}\right) \\ \frac{c}{\sqrt{\gamma}}s(\gamma) = & \frac{\sqrt{\gamma-i\mu}\sinh(a\pi)}{\sqrt{\gamma-i\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)-1)}\left(\frac{e^{mu}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{m}{\sqrt{2}}\right)}-\frac{e^{ml}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{m}{\sqrt{2}}\right)}\right. \\ & \left. -\frac{\left(-\frac{m}{\sqrt{2}}\left(e^{mu}+e^{ml}\right)(\cosh(a\pi)-1)+\sqrt{\gamma-i\mu}\left(e^{mu}-e^{ml}\right)\sinh(a\pi)\right)}{\sqrt{\gamma-i\mu}\sinh(a\pi)\left(\gamma-i\mu-\frac{m^2}{2}\right)}\right) \end{aligned} \tag{3.6}$$

$$a\pi = \sqrt{\frac{\gamma-i\mu}{c^2}};\quad \alpha=-m;\quad \beta=-\frac{m^2}{2}\quad c^2=\frac{1}{2(u-l)^2} \tag{3.7}$$

[2] *The function $\omega(\gamma,\mu,x;l,u,m)$ is analytic for values of $\gamma$ and $\mu$ such that Re[$\gamma$]> $m^2/2$ and Re[$\gamma-i\mu$]> $m^2/2$. If we replace the characteristic function with the Laplace transform we have to substitute the quantity $\gamma-i\mu$ with the quantity $\gamma+\mu$ and then we should require that Re[$\gamma$]> $m^2/2$ and Re[$\gamma+\mu$]> $m^2/2$.*

*Moreover we can express the density function $f(\tau, t, x; u, l, m)$ of the occupation time for a generic starting point $x$ and for $0<\tau<t$ in terms of the density function of the occupation time when $x = u$ and $x = l$ in the following way:*

$$f_\tau(\tau, t, x; u, l, m) = \begin{cases} 1_{(x\geq u)} e^{-m(x-u)} \int_\tau^t \frac{x-u}{\sqrt{2\pi(t-\eta)^3}} e^{-\frac{(x-u)^2}{2(t-\eta)} - \frac{m^2}{2}(t-\eta)} f_\tau(\tau, \eta, u)\, d\eta \\ \\ 1_{(l<x<u)} 2\pi c^2 \sum_{n=1}^{\infty} n \sin\left(n\pi\left(\frac{x-l}{u-l}\right)\right) \int_0^\tau e^{-\left(\frac{m^2}{2}+\lambda_n\right)\xi} \\ \times \left(e^{-m(x-l)} f_\tau(\tau-\xi, t-\xi, l) - (-1)^n e^{m(u-x)} f_\tau(\tau-\xi, t-\xi, u)\right) d\xi \\ \\ 1_{(x\leq l)} e^{m(l-x)} \int_\tau^t \frac{(l-x)}{\sqrt{2\pi(t-\eta)^3}} e^{-\frac{(l-x)^2}{2(t-\eta)} - \frac{m^2}{2}(t-\eta)} f_\tau(\tau, \eta, l)\, d\eta \end{cases} \tag{3.8}$$

### 3.2. Remarks

In the appendix, we solve the PDE and prove the theorem. We can make now some remarks.

1) A natural way of solving the PDE (3.2) could be to take the Laplace transform respect to $t$ and then obtain three second order differential equations. The continuity and differentiability of the solution at the barriers and its boundedness at $\pm\infty$, require then the determination of four constants, generalizing the example in Karatzas and Shreve [16] page. 273. However, using this approach we can incur in two problems.

a) The final expression of the solution will be the Laplace transform of the characteristic function $v(t, x, \mu)$ and then it will include the Laplace transform of the mass of probabilities concentrated at $\tau = 0$ and $\tau = t$. This fact as explained in Abate and Whitt [1] can create problems in the numerical inversion and it is advisable to remove the atoms of probability before the inversion. Attacking directly the PDE and using the Laplace transform only in a successive step, we avoid this problem. Indeed we are able to identify in the expression of the c.f. these probabilities and so we can give the Laplace transform of the function $\Omega(t, \mu, x; l, u, m)$ and not directly of the c.f..

b) If we want to use two univariate numerical inversions for limiting the programming effort and to use well-tested numerical inversion routines, as discussed in the next section, we should provide the Laplace transform of the real part and of the complex part of the function $\Omega(t, \mu, x; l, u, m)$. Using our approach, this consists in solving (A.11) separating the real and the complex part of the functions $D(t)$ and $S(t)$ and then, taking the Laplace transform, we obtain two linear systems of just two equations[3]. Instead if we take directly the Laplace transform of (3.2), we should solve three systems of two differential equations each and then the continuity and differentiability and boundedness of the solution will require the determination of eight constants in a linear system with eight equation.

2) The characteristic function is continuous and differentiable at $x = l$ and $x = u$, because as we will show later the PDE (3.2) has been solved requiring continuity and differentiability of the solution at these points. This property will be transmitted to the price of the corridor option.

3) Comparing expression (3.1) and (3.5), we obtain the natural results:

$$\begin{aligned}
&\mathrm{Pr}_{0,x}\left[\tau(t, x; u, l, m) = 0\right] \\
&= \begin{cases}
\mathbf{1}_{(x>u)}\,\mathrm{Pr}_{0,x}\left(\inf\limits_{0\le s\le t} ms + W(s) > u\right) \\
0;\ l \le x \le u \\
\mathbf{1}_{(x<l)}\,\mathrm{Pr}_{0,x}\left(\sup\limits_{0\le s\le t} ms + W(s) < l\right)
\end{cases}
\end{aligned}$$

[3]The relevant expressions are given in Appendix B.

and:

$$\Pr_{0,x}\left[\tau\left(t,x;u,l,m\right)=t\right]$$
$$=\begin{cases}\mathbf{1}_{(l<x<u)}\Pr_{0,x}\left(\sup\limits_{0\le s\le t} ms+W\left(s\right)<u;\inf\limits_{0\le s\le t} ms+W\left(s\right)>l\right)\\ 0;\ x\le l \text{ or } u\le x\end{cases}$$

The expressions for these quantities can be found in equations (A.20), (A.21) and (A.26) in the Appendix and can be compared with the same expressions in Borodin and Salminen (BS henceforth) [4].

4) We can show that from Theorem 1 we can recover known results. In particular, now we discuss the following cases: a) $m=0$, i.e. the case of the occupation time of the SBM of the interval $[l,u]$, b) $m=0$ and $l=-\infty$, i.e. the time spent below the level $u$ by the SBM and we obtain the Lévy arc-sine law, c) $l=-\infty$, i.e. the time spent by a BM with drift below the upper barrier $u$, case studied by Akahori [3], Dassios [8] and Takàcs[25].

a) If $m=0$, we are considering the occupation time of the SBM of the interval $[l,u]$. The expression for the functions $d\left(\gamma\right)$ and $s\left(\gamma\right)$, $\mathcal{L}\left[y\left(t,0\right);t\rightarrow\gamma\right]$ and $\mathcal{L}\left[y\left(t,1\right);t\rightarrow\gamma\right]$ simplify to:

$$d\left(\gamma\right)=\frac{1}{c}\frac{2\sqrt{\gamma}\sqrt{\gamma-i\mu}\sinh(a\pi)}{\left(\sqrt{\gamma-i\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1)\right)}\left(\frac{1}{\gamma}-\frac{1}{\gamma-i\mu}\right)$$

$$s\left(\gamma\right)=0$$

$$\mathcal{L}\left[y\left(t,0\right);t\rightarrow\gamma\right]=\mathcal{L}\left[y\left(t,1\right);t\rightarrow\gamma\right]=\frac{\sqrt{\gamma}\sinh(a\pi)+\sqrt{\gamma-i\mu}(\cosh(a\pi)+1)}{\left(\sqrt{\gamma-i\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1)\right)}\frac{1}{\sqrt{\gamma}\sqrt{\gamma-i\mu}}$$

and:

$$\omega\left(\gamma,\mu,x;l,u,m\right) = \begin{cases} \mathbf{1}_{(x\geq u)}\frac{e^{-\sqrt{2}(x-u)\sqrt{\gamma}}}{\sqrt{\gamma}}\frac{\sqrt{\gamma}\sinh(a\pi)+\sqrt{\gamma-i\mu}(\cosh(a\pi)+1)}{\sqrt{\gamma-i\mu}\left(\sqrt{\gamma-i\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1)\right)} \\ \\ \mathbf{1}_{(l<x<u)}\frac{\sinh\left(a\pi\left(\frac{u-x}{u-l}\right)\right)+\sinh\left(a\pi\left(\frac{x-l}{u-l}\right)\right)}{\sinh(a\pi)}\mathcal{L}\left[y\left(t,1\right);t\rightarrow\gamma\right] \\ \\ \mathbf{1}_{(x\leq l)}\frac{e^{-\sqrt{2}(l-x)\sqrt{\gamma}}}{\sqrt{\gamma}}\frac{\sqrt{\gamma}\sinh(a\pi)+\sqrt{\gamma-i\mu}(\cosh(a\pi)+1)}{\sqrt{\gamma-i\mu}\left(\sqrt{\gamma-i\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1)\right)} \end{cases} \tag{3.9}$$

This expression does not seem to admit a simple analytical inverse, although in BS [4], formula 1.7.4 pages 140-141, is given a very complicated expression. We remark that $\mathcal{L}\left[v\left(t,0\right);t\rightarrow\gamma\right]=\mathcal{L}\left[v\left(t,1\right);t\rightarrow\gamma\right]$, a consequence of the reflection principle.

b) If we let $l\rightarrow-\infty$ in (3.9), we are considering the time spent below the level $u$ by the SBM, so we should recover the Lévy arc-sine law. Indeed, we have:

$$\lim_{l\rightarrow-\infty}\mathcal{L}\left[y\left(t,1\right);t\rightarrow\gamma\right]=\frac{1}{\sqrt{\gamma}\sqrt{\gamma-i\mu}}$$

and then for $x>u$:

$$\omega\left(\gamma,\mu,x;l,u,m\right)=\mathbf{1}_{(x\geq u)}\frac{e^{-\sqrt{2}(x-u)\sqrt{\gamma}}}{\sqrt{\gamma}\sqrt{\gamma-i\mu}}$$

and inverting we obtain:

$$\lim_{l\rightarrow-\infty}f_{T}\left(\tau,t,x;u,l,0\right)=\begin{cases}\mathbf{1}_{(x>u)}\frac{1}{\pi}\frac{e^{-\frac{1}{2}\frac{(x-u)^2}{t-\tau}}}{\sqrt{\tau(t-\tau)}} \\ \mathbf{1}_{(x=u)}\frac{1}{\pi}\frac{1}{\sqrt{\tau(t-\tau)}} \\ \mathbf{1}_{(x<u)}\frac{e^{-\frac{1}{2}\frac{(u-x)^2}{\tau}}}{\pi\sqrt{\tau(t-\tau)}}\end{cases} \tag{3.10}$$

where the expression for the case $x<u$ has been found exploiting the symmetry property

$\tau(t,x;u,-\infty,m) = t - \tau(t,-x;-u,-\infty,-m)$, compare Takacs [25]. Expression (3.10) is the well known arc-sine law, Lévy [18] and BS [4], formula 1.4.4 page 129 where is given the time spent above the level $u$.

c) If we let $l \to -\infty$ in Theorem 1, we are considering the time spent by BM with positive drift below the level $u$. This case has been studied by Akahori [3], Dassios [8], although Takacs [25] provides a simpler expression.

From equation (3.6), we obtain[4]:

$$\begin{array}{l}\lim\limits_{l\to-\infty} \frac{c}{\sqrt{\gamma}} d(\gamma) = e^{-\alpha u} \frac{\sqrt{\gamma - i\mu}}{\sqrt{\gamma-i\mu}+\sqrt{\gamma}} \left( \frac{1}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} - \frac{\sqrt{\gamma-i\mu}+\frac{\alpha}{\sqrt{2}}}{\sqrt{\gamma-i\mu}\left(\gamma-i\mu-\frac{\alpha^2}{2}\right)} \right) \\ \lim\limits_{l\to-\infty} \frac{c}{\sqrt{\gamma}} s(\gamma) = e^{-\alpha u} \frac{\sqrt{\gamma - i\mu}}{\sqrt{\gamma-i\mu}+\sqrt{\gamma}} \left( \frac{1}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} - \frac{\sqrt{\gamma-i\mu}+\frac{\alpha}{\sqrt{2}}}{\sqrt{\gamma-i\mu}\left(\gamma-i\mu-\frac{\alpha^2}{2}\right)} \right)\end{array} \tag{3.11}$$

and then:

$$\begin{aligned}\lim_{l\to-\infty} \mathcal{L}\left[y(t,1);t\to\gamma\right] &= \frac{e^{-\alpha u}}{\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma-i\mu}-\frac{\alpha}{\sqrt{2}}\right)} \\ &= \frac{e^{-\alpha u}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}-\frac{\alpha}{\sqrt{2}}\right)}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)\left(\sqrt{\gamma-i\mu}-\frac{\alpha}{\sqrt{2}}\right)} \\ &= \frac{e^{-\alpha u}}{\left(\sqrt{\gamma-i\mu}-\frac{\alpha}{\sqrt{2}}\right)} \left( \frac{1}{\sqrt{\gamma}} - \frac{\alpha/\sqrt{2}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} \right)\end{aligned}$$

[4]The limits can appear to depend on the value of $m$, but we can suppose to have $m > 0$ without loss of generality. Indeed the key in the determination of the density is the symmetry property in [25] that allows us to find an expression for the density when $x < u$ in terms of the density when $x > u$. So if the drift is negative, we can suppose that $x < l$ and we let $u \to \infty$ so we consider the time spent above the level $l$. Then using the symmetry property, we find the expression for $x > l$ as well. So the result does not depend on the sign of $m$.

so when $x > u$:

$$
\begin{aligned}
&\Omega\left(t,\mu,x;l,u,m\right)\\
&= e^{-m(x-u)-\frac{m^2}{2}t}\mathcal{L}^{-1}\left[\frac{e^{mu}}{\left(\sqrt{\gamma-i\mu}+\frac{m}{\sqrt{2}}\right)}\left(\frac{e^{-\sqrt{2}(x-u)\sqrt{\gamma}}}{\sqrt{\gamma}}+\frac{m}{\sqrt{2}}\frac{e^{-\sqrt{2}(x-u)\sqrt{\gamma}}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{m}{\sqrt{2}}\right)}\right);\gamma\to t\right]\\
&= 1_{(x>u)}e^{-m(x-u)-\frac{m^2}{2}t}\left(\int_0^t e^{i\mu\theta}\left(\frac{1}{\sqrt{\pi\theta}}-\frac{m}{\sqrt{2}}e^{\frac{m^2}{2}\theta}\,\mathrm{Erfc}\left(\frac{m\sqrt{\theta}}{\sqrt{2}}\right)\right)\right)\\
&\times\left(\frac{e^{-\frac{(x-u)^2}{2(t-\theta)}}}{\sqrt{\pi(t-\theta)}}+\frac{m}{\sqrt{2}}e^{\frac{m^2}{2}(t-\theta)-m(x-u)}\,\mathrm{Erfc}\left(\frac{-m\sqrt{t-\theta}}{\sqrt{2}}+\frac{(x-u)}{\sqrt{2(t-\theta)}}\right)\right)d\theta
\end{aligned}
$$

where we have used the inversion formulas in Abramowitz and Stegun (AS henceforth) [2] and the convolution property of the Laplace transform.

For $x < u$ and $l \to -\infty$, we can again use the symmetry argument in Takacs [25], so we then obtain that the density function of the occupation time with only one barrier is given by:

$$
\begin{aligned}
&\lim_{l\to-\infty} f_\tau\left(\tau,t,x;u,l,m\right)\\
&=\begin{cases}
1_{(x\geq u)}\left(\frac{e^{-\frac{m^2}{2}\tau}}{\sqrt{\pi\tau}}-\frac{m}{\sqrt{2}}\,\mathrm{Erfc}\left(\frac{m\sqrt{\tau}}{\sqrt{2}}\right)\right)\\
\times\left(\frac{e^{-\frac{1}{2}\frac{((x-u+m(t-\tau)))^2}{t-\tau}}}{\sqrt{\pi(t-\tau)}}+\frac{m}{\sqrt{2}}e^{-2m(x-u)}\,\mathrm{Erfc}\left(\frac{(x-u)-m(t-\tau)}{\sqrt{2(t-\tau)}}\right)\right)\\
\\
1_{(x<u)}\left(\frac{e^{-\frac{m^2}{2}(t-\tau)}}{\sqrt{\pi(t-\tau)}}+\frac{m}{\sqrt{2}}\,\mathrm{Erfc}\left(-\frac{m\sqrt{t-\tau}}{\sqrt{2}}\right)\right)\\
\times\left(\frac{e^{-\frac{1}{2}\frac{((u-x-m\tau))^2}{\tau}}}{\sqrt{\pi\tau}}-\frac{m}{\sqrt{2}}e^{+2m(u-x)}\,\mathrm{Erfc}\left(\frac{(u-x)+m\tau}{\sqrt{2\tau}}\right)\right)
\end{cases}
\end{aligned}
\tag{3.12}
$$

In order to make comparable the expression above with equation (12) in Takacs [25], where is considered the average time spent below the level $x < u$, it is necessary to set in Takacs [25] $\alpha = (u-x)/\sqrt{t}$ and substitute $m$ with -$m\sqrt{t}$. We need to use as well the relationship $\mathrm{Erfc}(x) = 2\Phi\left(-\sqrt{2}x\right)$.

Moreover, if we have $m = 0$, the above density function reduces again to (3.10).

## 4. The numerical inversion

In this section we discuss the problem of the numerical inversion for obtaining the density function. Then we discuss how to obtain the moments of the occupation time.

### 4.1. The density law

From a computational point of view it is convenient to distinguish the problem of finding the density function from the problem of pricing the corridor option. Indeed, in the first case we like to use two single univariate inversions, whilst in the second it is better to use a multivariate inversion. The problem of the multivariate inversion and the pricing of the corridor option will be discussed in next chapter.

This choice is also due to the fact that up to now, relatively little attention has been given to inversion of multidimensional transforms, so in order to limit the programming effort a possibility is to obtain the density function using well tested univariate inversion routines as described in the following two steps:

1) Numerically find the inverse Laplace transform of the function $\Omega\left(t, \mu, x; l, u, m\right)$. We have solved this problem using the Crump's [7] method implemented in the IMSL-library subroutine FLINV[5].

2) Using the numerically computed $\Omega\left(t, \mu, x; l, u, m\right)$, find the function $f_{\tau}\left(\tau, t, x; U, L, m\right)$ through a further numerical Fourier transform inversion. We have computed this inverse using the Fast Fourier Transform and this allows us to reduce greatly the computational time.

The advantage of using the FFT routine is that we obtain simultaneously the entire density function, whilst if we use a different procedure (e.g. a Laplace inversion) we need to repeat as many times as the number of points at which we desire the density function.

[5] This method is ranked among the most accurate available numerical inversion techniques in the Davies and Martin [9] comparison.

Moreover, having the entire density function allows us to price more complex derivative products which depend on the occupation time.

The problem with using the two steps described above is that we cannot use directly the expressions given in Theorem 1, but we need separate expressions for the real and imaginary part of the function $\Omega(t, \mu, x; l, u, m)$. In this way using the first Laplace inversion, we obtain the real and imaginary part of the function $\varpi(z, \mu, \gamma)$ at which we can finally apply the FFT. In the Appendix B we provide the expressions for the real and imaginary part of the function $\Omega(t, \mu, x; l, u, m)$.

In Figure 1, we represent the density function of the occupation time obtained using this single univariate procedure. In Table 1 we compare the price of the corridor bond obtained using the density coming from the double inversion described above and the analytical formula for the corridor bond as from section 2.

The expected value in Table 4.1 has been calculated through a simple trapezoidal rule[6]. In the numerical inversion, we have to choose the Fourier transform maximum frequency, so from the table we can see that increasing it we obtain a great accuracy, but with the cost of increasing the computational time[7].

[6]The price of the corridor bond can be computed also calculating the first moment of the r.v. occupation time as discussed later on.

[7]All the calculations have been performed on a Pentium 133 machine.

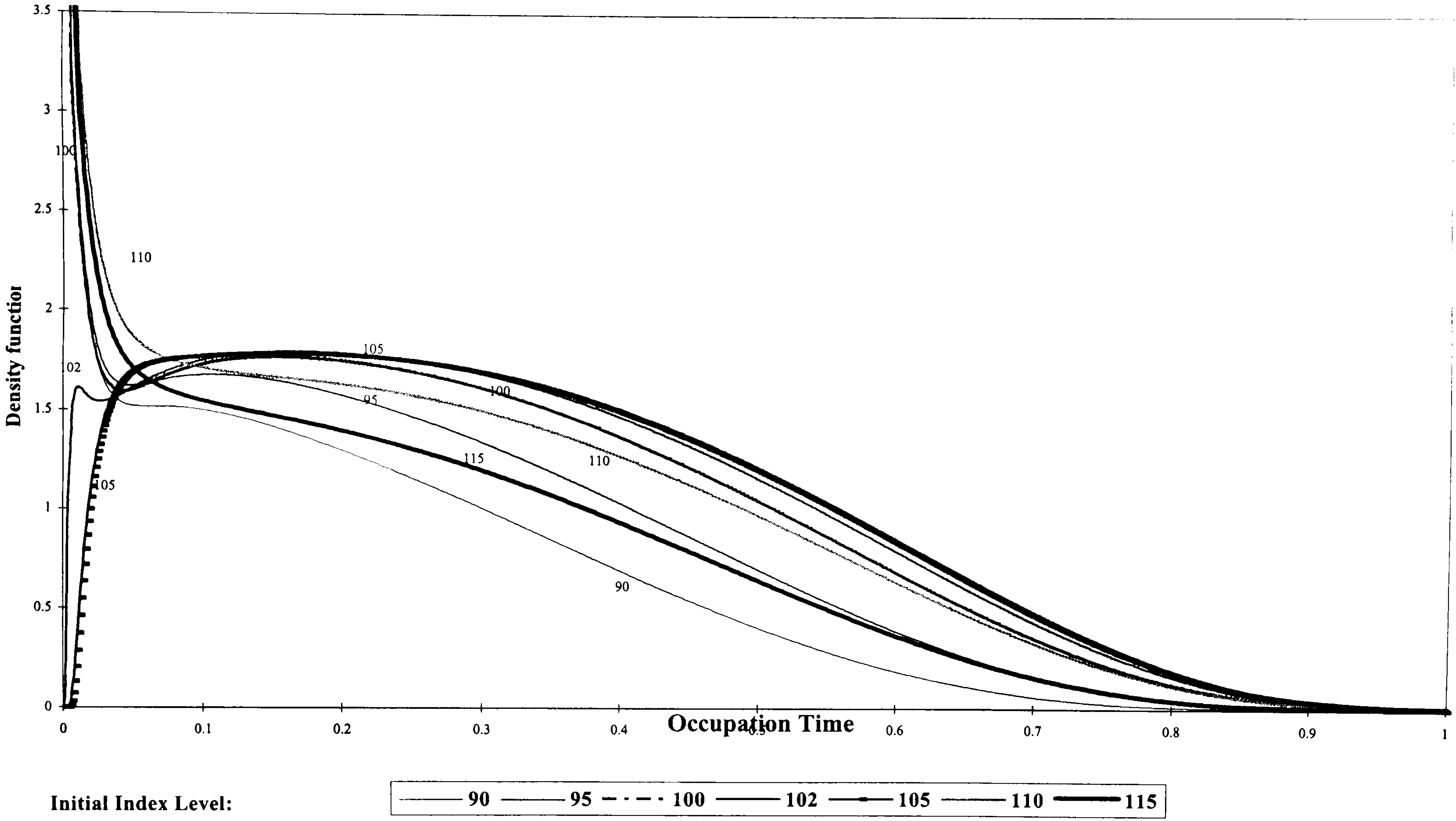
**Table 1. Price of the corridor bond**

Parameters setting: $r$=0.05,$\sigma = 0.2$, $L = 100$, $U = 110$, $t = 1yr$.

$n$ is the number of chosen points (a power of 2) in the FFT.

| Initial Index Level | Analytical Solution | Numerical integration of the density and two-step numerical inversion | | | |
|---|---|---|---|---|---|
| | | $n$=2048 | $n$=1024 | $n$=512 | $n$=256 |
| 80 | 0.04609 | 0.04607 | 0.04611 | 0.04620 | 0.04642 |
| 85 | 0.08149 | 0.08149 | 0.08155 | 0.08170 | 0.08204 |
| 90 | 0.13134 | 0.13137 | 0.13146 | 0.13169 | 0.13218 |
| 95 | 0.19606 | 0.19613 | 0.19627 | 0.19657 | 0.19723 |
| 100 | 0.27463 | 0.27471 | 0.27489 | 0.27528 | 0.27610 |
| 105 | 0.30959 | 0.30983 | 0.31006 | 0.31052 | 0.31145 |
| 110 | 0.25770 | 0.25772 | 0.25788 | 0.25824 | 0.25901 |
| 115 | 0.18058 | 0.18061 | 0.18073 | 0.18100 | 0.18159 |
| 120 | 0.12478 | 0.12478 | 0.12486 | 0.12506 | 0.12551 |
| 125 | 0.08509 | 0.08508 | 0.08514 | 0.08529 | 0.08562 |
| average time | 1" | 170" | 104" | 56" | 32" |

**Figure 1: Density of the occupation time for different index levels (L=100, U=110, r=0.05, σ=0.2, t=1)**



### 4.2. The corridor derivative

In Table 2 we present the prices of the corridor option comparing with the results of multidimensional Laplace inversion discussed in more detail in next chapter, i.e. the Fourier-series method firstly introduced for multidimensional transform inversion by Choudhury et al. [6], and the Padé approximation as suggested in Singhal et al. [24]. We compare the results with MonteCarlo simulation, that has been performed using 50000 simulation and using the antithetic variate technique, Boyle [5]. We remark that a limit of the MonteCarlo is the intrinsic discreteness of the simulation so we do not know if the process has crossed or not the barriers and we cannot compute exactly the time spent inside the barriers during each step of the simulation. However, the following table seems testify that this problem does not appear very important in the present context: indeed the different methods agree up to the third digit. In any case the MonteCarlo simulation requires a greater computational time and does not allow to compute in a precise way the Greeks of the contract.

In Figures 2a-b we represent the price and the delta of the corridor option for different strikes and index levels.

•

Table 2: Pricing of Corridor Options

Parameters setting: r=0.05, $\sigma$ =0.2, L=100, U=110, t=1; In the MC+AV column we report the Monte Carlo estimate, obtained using the Antithetic Variate method with 50000 simulations with 1200 steps each, and 1000*standard error. The poles and the residues in the Padé inversion have been preliminarily calculated in Mathematica 3.0 with the functions Pade[], NResidue[] and NSolve[].

| Index;Strike | Crump+FFT n=2048 | Fourier $l_1=l_2=2$ $A_1=A_2=20$ n=20, m=20 | Fourier $l_1=l_2=2$ $A_1=A_2=20$ n=100, m=20 | Padé num=3, den=18 | MC+AV |
|---|---|---|---|---|---|
| 90; 0.2 | 0.0462793 | 0.0463038 | 0.0463038 | 0.0463039 | 0.046606; 0.265 |
| 95; 0.2 | 0.0792503 | 0.0792444 | 0.0792444 | 0.0792445 | 0.078995; 0.308 |
| 100; 0.2 | 0.1247528 | 0.1247227 | 0.1247228 | 0.1247226 | 0.124951; 0.518 |
| 105; 0.2 | 0.1470881 | 0.1469239 | 0.1469239 | 0.1469239 | 0.147282; 0.665 |
| 110; 0.2 | 0.1161007 | 0.1161262 | 0.1161262 | 0.1161261 | 0.115834; 0.358 |
| 115; 0.2 | 0.0735259 | 0.0735554 | 0.0735554 | 0.0735555 | 0.073820; 0.311 |
| 120; 0.2 | 0.0456788 | 0.0457253 | 0.0457253 | 0.0457254 | 0.046075; 0.271 |
| average CPU | 170" | 8" | 39" | <1" | 337" |
| 90; 0.4 | 0.0100981 | 0.0101457 | 0.0101457 | 0.0101459 | 0.010328; 0.123 |
| 95; 0.4 | 0.0213042 | 0.0213357 | 0.0213358 | 0.0213361 | 0.021146; 0.175 |
| 100; 0.4 | 0.0400273 | 0.0400375 | r0.0400376 | 0.0400375 | 0.040250; 0.27 |
| 105; 0.4 | 0.0504331 | 0.0503482 | 0.0503483 | 0.0503485 | 0.050574; 0.376 |
| 110; 0.4 | 0.0372272 | 0.0372753 | 0.0372754 | 0.0372753 | 0.037048; 0.227 |
| 115; 0.4 | 0.0202404 | 0.0202948 | 0.0202948 | 0.0202951 | 0.020376; 0.176 |
| 120; 0.4 | 0.0107088 | 0.0107697 | 0.0107697 | 0.0107699 | 0.010868; 0.285 |
| average CPU | 170" | 8" | 38" | <1" | 337" |
| 90; 0.6 | 0.0008576 | 0.0009014 | 0.0009014 | 0.0009019 | 0.000927; 0.030 |
| 95; 0.6 | 0.0026490 | 0.0026893 | 0.0026893 | 0.0026899 | 0.002592; 0.053 |
| 100; 0.6 | 0.0067578 | 0.0067873 | 0.0067874 | 0.0067831 | 0.006865; 0.092 |
| 105; 0.6 | 0.0094893 | 0.0094617 | 0.0094618 | 0.0094619 | 0.009545; 0.131 |
| 110; 0.6 | 0.0062664 | 0.0063189 | 0.0063191 | 0.0063188 | 0.006258; 0.087 |
| 115; 0.6 | 0.0026123 | 0.0026664 | 0.0026664 | 0.0026670 | 0.002687; 0.056 |
| 120; 0.6 | 0.0010299 | 0.0010822 | 0.0010822 | 0.0010826 | 0.001093; 0.034 |
| average CPU | 170" | 8" | 37" | <1" | 337" |

**Figure 2a: Price of the corridor option for different strike prices and index levels:**

**L=100, U=110, σ=0.2, r=0.05, t=1**

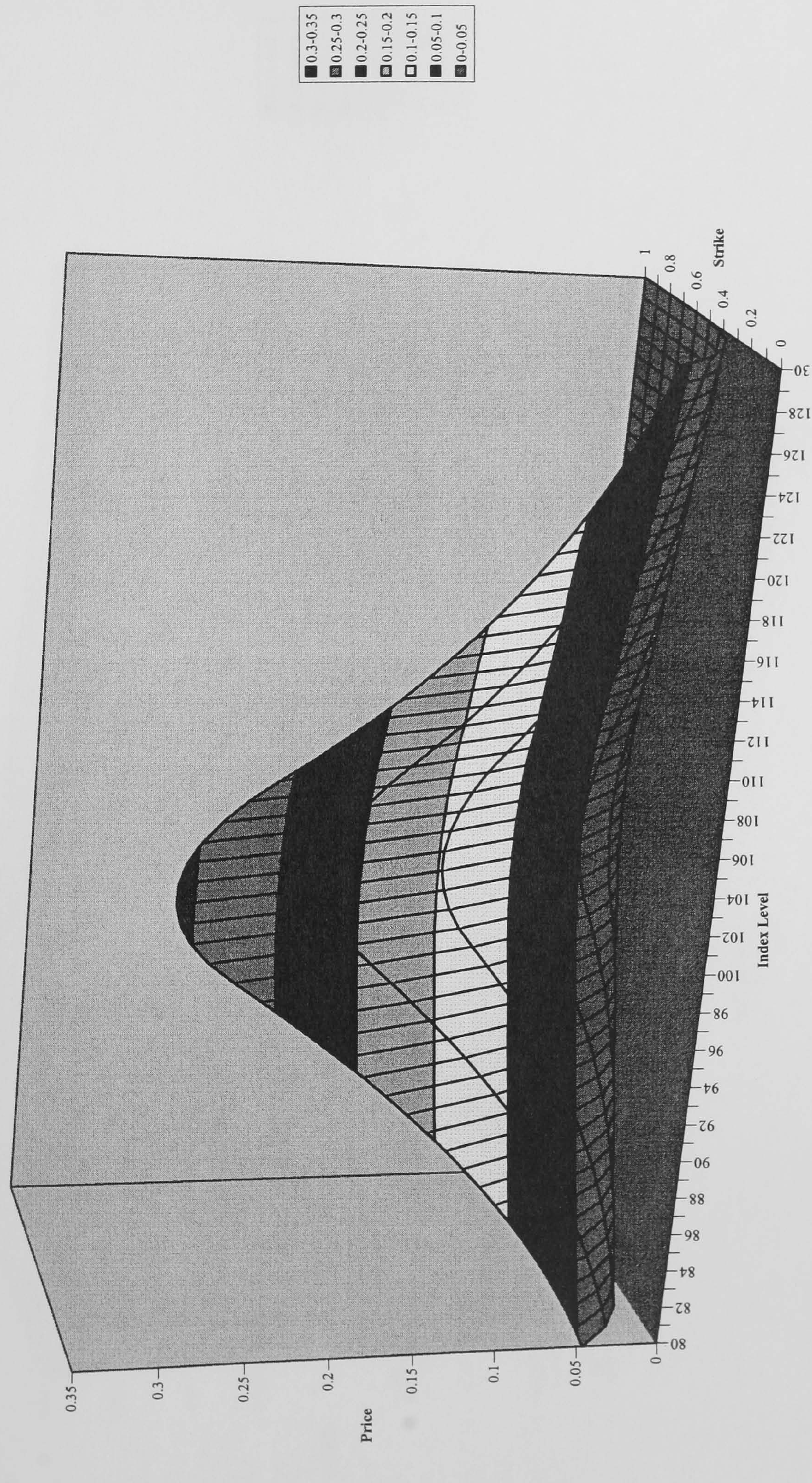
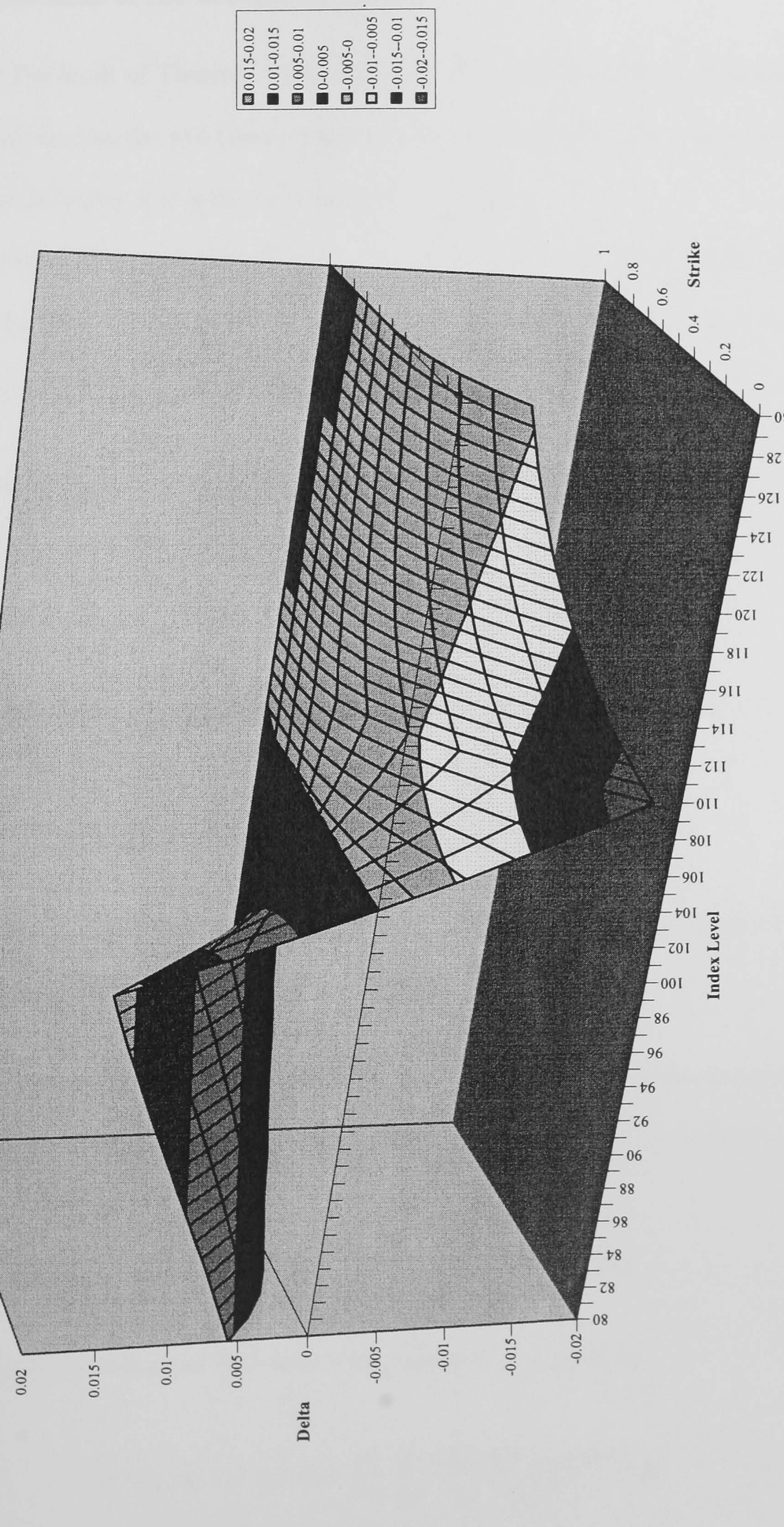


**Figure 2b: Delta of the corridor option for different strike prices and index levels (L=100, U=110, σ=0.2, r=0.05, t=1)**



### 4.3. The moments of the occupation time

We can use the result of Theorem 1 for computing the moments of the occupation time. The result is based on the well known Cauchy integral formula that allows to compute the derivative of all orders at a point for an analytic function.

The moments of a random variable can be obtained from the characteristic function deriving with respect to $\mu$ and then setting $\mu = 0$, through the following well known relationship:

$$
\begin{aligned}
&\mathbf{E}_{0,x}\left[\tau^n\left(t,x;l,u,m\right)\right] \equiv M_n\left(x,t\right) \\
&= (-1)^n \left.\tfrac{\partial\Omega^n(t,\mu,x;l,u,m)}{\partial\mu^n}\right|_{\mu=0} \\
&+t^n \times \Pr_{0,x\in(l,u)}\left(\sup_{0\le s\le \tau} m*s+W\left(s\right)<u;\ \inf_{0\le s\le t} m*s+W\left(s\right)>l\right)
\end{aligned}
\tag{4.1}
$$

Theorem 1 gives us an expression for the Laplace transform respect to time $t$ of the function:

$$
\Omega\left(t,\mu,x;l,u,m\right) = e^{-mz-\frac{m^2}{2}\tau}\mathcal{L}^{-1}\left[\omega\left(\gamma,\mu,x;l,u,m\right);\gamma\rightarrow t\right]
$$

and the n-th order derivatives $\Omega^{(n)}$and $\omega^{(n)}$ can be computed using the Cauchy integral formula using the expression for $\omega$. We have indeed:

**Theorem 4.1.** *(**Cauchy integral formula**) Let $\omega$ be analytic everywhere within and on a simple closed contour $C$, taken in the positive sense. If $\mu_0$ is any point interior to $C$, then:*

$$
\omega\left(\gamma,\mu_0,x;l,u,m\right) = \frac{1}{2\pi i}\int_C \frac{\omega\left(\gamma,\mu,x;l,u,m\right)}{\mu-\mu_0}d\mu
$$

*and in general the n-th order derivative with respect to $\mu$ is given by:*

$$
\omega^{(n)}\left(\gamma,\mu_0,x;l,u,m\right) = \frac{n!}{2\pi i}\int_C \frac{\omega\left(\gamma,\mu,x;l,u,m\right)}{\left(\mu-\mu_0\right)^{n+1}}d\mu\blacklozenge
$$

Using this result and a numerical procedure for inverting the univariate Laplace transform, we can compute all the moments of the random variable occupation time. In order to calculate the value of the contour integral, we can choose as path the unitary circle with center at the origin. Moreover, we have $\mu_0 = 0$. So we obtain:

$$\begin{aligned} \omega^{(n)}(\gamma, 0, x; l, u, m) &= \frac{n!}{2\pi i}\int_C \frac{\omega(\gamma, \mu, x; l, u, m)}{\mu^{n+1}} d\mu \qquad (4.2) \\ &= \frac{n!}{2\pi i}\int_0^{2\pi} \frac{\omega(\gamma, e^{i\theta}, x; l, u, m)}{e^{(n+1)i\theta}} ie^{i\theta} d\theta \\ &= \frac{n!}{2\pi}\int_0^{2\pi} \omega\left(\gamma, e^{i\theta}, x; l, u, m\right) e^{-ni\theta} d\theta \end{aligned}$$

Then the strategy for computing the moments is the following:

1. numerically compute the integral (4.2);

2. numerically compute the inverse respect to $\gamma$ of the function $\varpi^{(n)}$and obtain $\Omega^{(n)}$,

3. then compute the nth-moment using formula 4.1.

We have performed the first step using a Legendre quadrature formula with 100 nodes, Press et al. [23] page 150 and following, whilst the second step has been done using two different numerical procedures: the Euler algorithm as in Abate and Whitt [1] and the univariate version of the Padé approximant method as described in Singhal and Vlach [24]. The results of the two methods agree up to the seventh digit and the computation of each moment requires less than one second. Takacs [26] has obtained, using a combinatorial approach, a recurrence relationship for the moments of the occupation time of the Brownian motion without drift in terms of the moments of the local time of the BM. An advantage of the method discussed in the present context is that we can compute separately the different moments and this fact can reduce enormously the computational time.

Figures 3-5 illustrate the behavior of expected value and standard deviation of the occupation time. With regard to the behavior of the expected occupation time respect to the starting level of the Brownian Motion, Figure 3, we observe the bell-shaped form: the expected value is at the highest when we are inside the barriers and as we move far from them, it decreases. In Figure 4, we have the expected fraction of time spent inside the barriers that is a decreasing function of time if the index starts inside the barriers (as time passes it is more likely to move out), otherwise it is increasing and then decreasing (as time passes it is more likely to move in and then out again). In the case of zero drift the shape is symmetric around the mid-point of the monitoring interval. In Figure 5 we represent the standard deviation always respect to the starting point of the Brownian Motion: the particular shape is due to the fact that when the index starts exactly on the barriers it continues in moving in and out of the interval: so the standard deviation is at a maximum when we are on the barriers.

## 5. Conclusions

In this chapter we have studied a generalization of the Lévy arc-sine law providing an expression for the Laplace Transform of the characteristic function. We have used the Feynman-Kac equation and solved the related PDE. A possible extension is related to the derivation of the joint density of the occupation time and the final level of the Brownian Motion and it can be done easily following the same procedure. We have also discussed how to invert numerically the double transform in order to obtain the density function and we have shown that using standard numerical procedures, in particular the Fast Fourier Transform, the task can be accomplished without difficulty. We have also shown how to compute the moments using the Cauchy integral formula and the expression for the characteristic function. In next chapter we will discuss in more detail the application of the present results to the pricing of occupation time derivatives.

**Figure 3: Expected value of the occupation time of the Brownian motion with drift (l=1, u=1, t=1)**

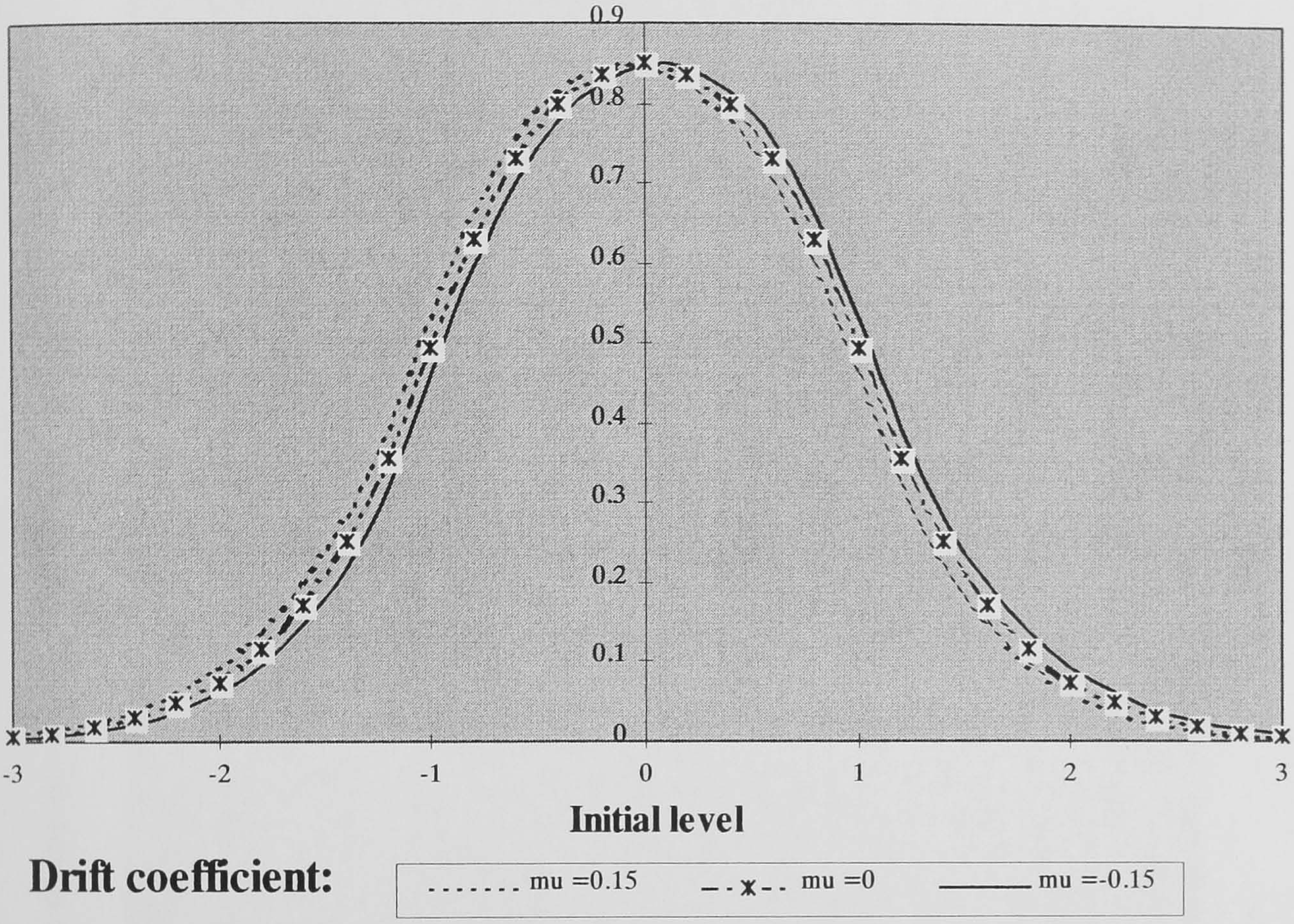



**Figure 4: Fraction of expected time spent inside the barriers by a BM with drift (l=1, u=1, drift =0.15)**

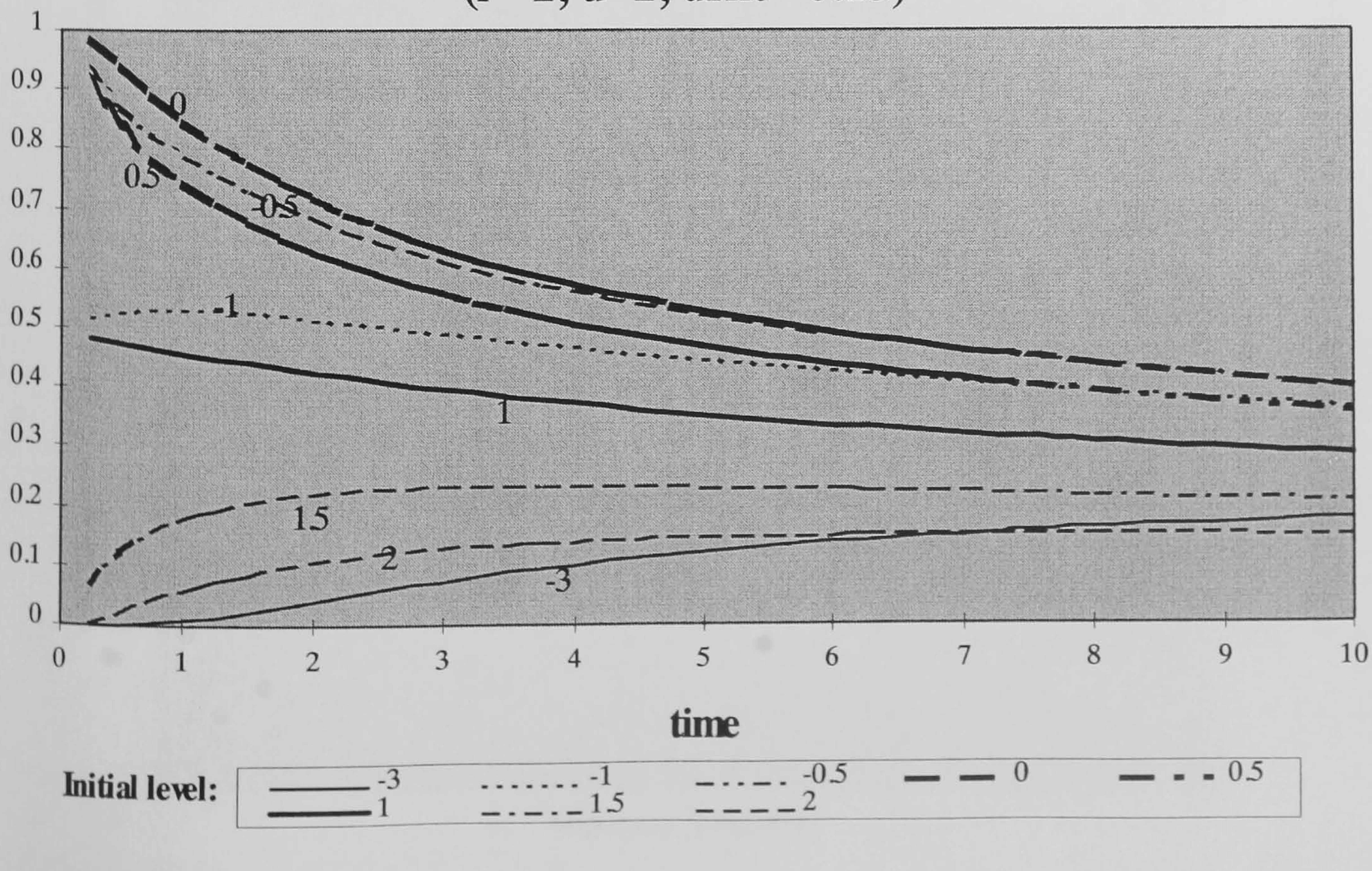
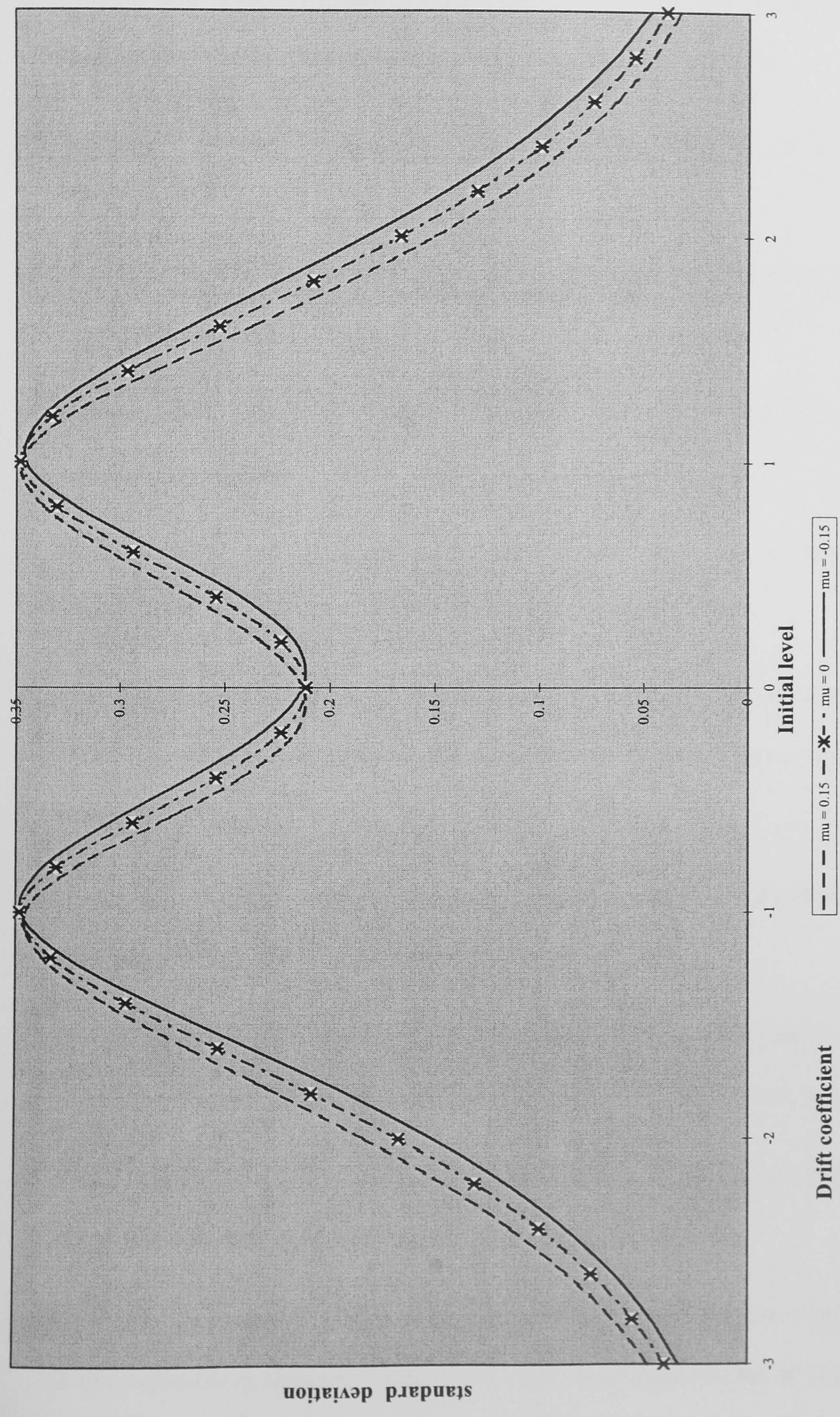


**Figure 5: Standard deviation of the occupation time of the Brownian motion with drift (l=-1, u=1, t=1)**



## References

[1] Abate, J. and Whitt, W. (1992), The Fourier-series method for inverting transforms of probability distributions, *Queueing Systems Theory Appl.* 10 5-88.

[2] Abramowitz, M., and Stegun, I.A.(1972). *Mathematical Functions*, Dover Publications, Inc., New York.

[3] Akahori, J. (1995). Some formulae for a new type of path-dependent options, *Ann. Appl. Probab.* 5, 383-388.

[4] Borodin, A. N. , Salminen P. (1996). *Handbook of Brownian Motion - Facts and Formulae*, Birkhauser.

[5] Boyle, P. (1977). Options: A MonteCarlo approach, *Journal of Financial Economics* 4, 323-338.

[6] Choudhury G. L., Lucantoni D. M, and Whitt W. (1994), Multidimensional Transform Inversion with applications to the transient M/G/1 Queue, *Ann. Appl. Probab.* 5, 389-398.

[7] Crump, K. (1976), Numerical Inversion of Laplace transform using Fourier Series approximation, *J. Assoc. Comp. Mach.* 23, 89-96.

[8] Dassios, A. (1995). The distribution of the quantile of a Brownian motion with drift and the pricing of related path-dependent options, *Ann. Appl. Probab.* 4, 719-740.

[9] Davies, B. and Martin B. (1979), Numerical Inversion of the Laplace transform: a survey and comparison of methods, *J. Comp. Phys.* 33, 1-32.

[10] Embrechts, P., Rogers, L. C. G. and Yor, M. (1995). A proof of Dassios' representation of the $\alpha$-quantile of Brownian motion with drift, *Ann. Appl. Probab.* 5, 757-767.

[11] Fusai, G. and Tagliani A. (1999), Numerical Methods for Occupation Time Derivatives, *Proceedings of the Conference on Numerical methods in Finance*, Venice, May. The revised version is the 3rd chapter of the present thesis and has been accepted for publication in *Journal of Computational Finance.*

[12] Gradshteyn I. S., Ryzhik I. M. (1980). *Table of Integrals, Series and Products,* Academic Press.

[13] Harrison J. M., Kreps D. (1979). Martingales and Arbitrage in Multiperiod Securities Markets, *Journal of Economic Theory*, 20 July, 381-408.

[14] Hull, J. (1997), *Options, Futures, and Other Derivatives.* Englewood Cliffs, NJ: Prenctice Hall.

[15] Jansons K. M. (1997). The Distribution of Time Spent by a Standard Excursion above a given Level, with Applications to Ring Polymers Near a Discontinuity in Potential, *Electronic Comm. in Prob.*, 2, 53-58.

[16] Karatzas I. and Shreve S. E. (1991), *Brownian Motion and Stochastic Calculus,* Springer-Verlag, New York.

[17] Kevorkian J. (1989), *Partial Differential Equations - Analytical Solution Techniques*, Chapman & Hall, New York.

[18] Lévy, P. (1939), Sur certain processus stochastiques homogènes, *Compositio Math.*, Vol. 7, pages 283-339.

[19] Linetski, V. (1996), Step options: The Feynman-Kac Approach to occupation time derivatives. University of Michigan Dept. of Industrial and Operations Engineering Technical Report 96-18.

[20] Navatte P. and F. Quittard-Pinon (1999), The valuation of interest rate digital options and range notes revisited, *European Financial Management*, Vol. 5, no. 3, pages 425-440.

[21] Necati Özişik M. (1989), *Boundary Value Problems of Heat Conduction*, Dover.

[22] Pechtl, A. (1995), Classified Information, in *Over the Rainbow*, ed by Robert Jarrow, Risk Publications, pag. 71-74.

[23] Press, W. H., Teukolsky S. A., Vetterling W. T. and Flannery B. P. (1997). *Numerical Recipes in C*, Cambridge University Press, version 2.08

[24] Singhal, K., Vlach, J. and Vlach M. (1975), Numerical inversion of multidimensional Laplace transforms, *Proc. IEEE* 63, 1627-1628.

[25] Takàcs, L. (1996). On a generalization of the arc-sine law, *Ann. Appl. Probab.* 6, 1035-1040.

[26] Takàcs (1998), Sojourn Times for the Brownian Motion, *Journal of Applied Mathematics and Stochastic Analysis*, Vol. 11, no. 3, 231-246.

[27] Taleb, N. (1997), *Dynamic Hedging*, John Wiley.

[28] Tucker, A. L. and Wei J. Z. (1997), The Latest Range, *Advances in Futures and Options Research*, Vol. 9, pages 287-296.

[29] Turnbull, S. M. (1995), Interest Rate Digital Options and Range Notes, *Journal of Derivatives* 3, 92-101.

[30] Zauderer, E. (1989). *Partial Differential Equations of Applied Mathematics*, Wiley & Sons, New York.

## A. Solution of the PDE

In order to solve the PDE (3.2):

$$-\frac{\partial v(t,x,\mu)}{\partial t}+\frac{1}{2}\frac{\partial^2 v(t,x,\mu)}{\partial^2 x}+m\frac{\partial v(t,x,\mu)}{\partial x}+i\mu\mathbf{1}_{(l<x<u)}v(t,x,\mu)=0 \tag{A.1}$$

with initial condition:

$$v(0,x)=1,\ \forall x\in(-\infty;+\infty) \tag{A.2}$$

and boundary conditions:

$$v(t,\pm\infty)=1,\ \forall t>0 \tag{A.3}$$

we consider the following transformation:

$$v(t,x,\mu)=e^{\alpha x+\beta t}h(t,x,\mu)$$

and setting $\alpha=-m$ and $\beta=-\frac{m^2}{2}$, we get the following PDE for the function $h(t,x)$ :

$$-\frac{\partial h(t,x,\mu)}{\partial t}+\frac{1}{2}\frac{\partial^2 h(t,x,\mu)}{\partial^2 x}+i\mu\mathbf{1}_{(l<x<u)}h(t,x,\mu)=0 \tag{A.4}$$

with initial condition $h(0,x,\mu)=e^{-\alpha x}$. We can make a second transformation defining $z=(x-l)/(u-l)$ and we introduce a new function:

$$y(t,z,\mu)=h(t,(u-l)z+l,\mu)$$

In the following we suppress the dependence of $y(t,z,\mu)$ on $\mu$ and we write $y(t,z) \equiv y(t,z,\mu)$. We get the following PDE for the function $y(t,z)$:

$$-\frac{\partial y(t,z)}{\partial t} + c^2 \frac{\partial^2 y(t,z)}{\partial^2 z} + i\mu \mathbf{1}_{(0<z<1)} y(t,z) = 0 \tag{A.5}$$

where $c^2 = \frac{1}{2(u-l)^2}$, and $y(0,z) = e^{-\alpha((u-l)z+l)}$.

### A.1. Solution of the PDE with Neumann boundary conditions

In order to solve (A.5) we can distinguish three cases, $z < 0$, $0 < z < 1$, and $z>1$ and require a continuous and differentiable solution at these boundary points. As consequence, we can guarantee that the characteristic function as well is continuous and differentiable at $x = l$ and at $x = u$. So we can solve the PDE (A.5) imposing Neumann boundary conditions at $z = 0$ and at $z = 1$:

$$\begin{aligned} \left.\frac{\partial y(t,z)}{\partial z}\right|_{z=0^-} &= L(t) = \left.\frac{\partial y(t,z)}{\partial z}\right|_{z=0^+} \\ \left.\frac{\partial y(t,z)}{\partial z}\right|_{z=1^-} &= U(t) = \left.\frac{\partial y(t,z)}{\partial z}\right|_{z=1^+} \end{aligned} \tag{A.6}$$

where $L(t)$ and $U(t)$ are unknown functions to be chosen in order to have continuity of the solution at $z = 0$ and at $z = 1$,i.e.:

$$\begin{aligned} y(t,z)|_{z=0^-} &= y(t,z)|_{z=0^+} \\ y(t,z)|_{z=1^-} &= y(t,z)|_{z=1^+} \end{aligned} \tag{A.7}$$

In the case $z < 0$ and $z > 1$, to solve (A.5) with boundary conditions (A.6) amounts to solve the heat equation in a semi-infinite region with Neumann boundary condition. The solution can be found in Zauderer [30] (page 268, eq. 5.121). In the case $0 < z < 1$, considering the new transformation $g(t,z) = e^{-i\mu t} y(t,z)$, we get that $g(t,z)$ satisfies the heat equation in a finite strip with Neumann boundary $e^{-i\mu t} L(t)$ and $e^{-i\mu t} U(t)$ at $z = 0$

and $z=1$, respectively. The solution in this case can be found in Necati[21] (page 62, eq.2-73a). In conclusion the function $y(t,z)$ can be expressed in terms of the unknown functions $L(t)$ and $U(t)$ in the following way:

$$y(t,z) = \begin{cases} 1_{(z>1)} \begin{pmatrix} e^{-\alpha u} \displaystyle\int_0^{+\infty} \left[G(z-1-\xi,t)+G(z-1+\xi,t)\right] e^{-\alpha(u-l)\xi} d\xi \\ -2c^2 \displaystyle\int_0^t G(z-1,t-\theta)\,U(\theta)\,d\theta \end{pmatrix} \\ 1_{(0<z<1)} \begin{pmatrix} e^{i\mu t} \frac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)} + c^2 \int_0^t e^{i\mu(t-\theta)} \left(U(\theta)-L(\theta)\right) d\theta \\ +2c^2 \displaystyle\sum_{n=1}^{+\infty} y_n(t) \cos n\pi z + \sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t} \phi_n \cos n\pi z \end{pmatrix} \\ 1_{(z<0)} \begin{pmatrix} e^{-\alpha l} \displaystyle\int_0^{+\infty} \left[G(-z-\xi,t)+G(-z+\xi,t)\right] e^{\alpha(u-l)\xi} d\xi \\ +2c^2 \displaystyle\int_0^t G(-z,t-\theta)\,L(\theta)\,d\theta \end{pmatrix} \end{cases} \tag{A.8}$$

where:

$$\begin{aligned} G(x,t) &= \frac{e^{-\frac{x^2}{4c^2 t}}}{\sqrt{4\pi c^2 t}} \\ \phi_n &= 2\int_0^1 e^{-\alpha((u-l)\xi+l)} \cos(n\pi\xi)\,d\xi \\ y_n(t) &= \int_0^t e^{(i\mu-\lambda_n)(t-\theta)} \left((-1)^n U(\theta)-L(\theta)\right) \\ \lambda_n &= (n\pi c)^2 \end{aligned}$$

In order to find the unknown functions $U(t)$ and $L(t)$ we now require the continuity of the function $y(t,z)$ at $z=0$ and $z=1$, i.e. we impose conditions (A.7). However it is more convenient to transform the continuity conditions in the following way:

$$\begin{aligned} \lim_{z\to1^+} y(t,z) + \lim_{z\to0^-} y(t,z) &= \lim_{z\to1^-} y(t,z) + \lim_{z\to0^+} y(t,z) \\ \lim_{z\to1^+} y(t,z) - \lim_{z\to0^-} y(t,z) &= \lim_{z\to1^-} y(t,z) - \lim_{z\to0^+} y(t,z) \end{aligned}$$

and introducing the functions $D(t) := U(t) - L(t)$ and $S(t) := U(t) + L(t)$, we obtain:

$$\lim_{z\to 1^+} y(t,z) + \lim_{z\to 0^-} y(t,z)$$
$$= e^{-\alpha u+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(\alpha\tfrac{\sqrt{t}}{\sqrt{2}}\right) + e^{-\alpha l+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(-\tfrac{\alpha\sqrt{t}}{\sqrt{2}}\right) - 2c^2 \int_0^t \tfrac{1}{\sqrt{4\pi c^2(t-\theta)}} D(\theta)\,d\theta$$

$$\lim_{z\to 1^+} y(t,z) - \lim_{z\to 0^-} y(t,z)$$
$$= e^{-\alpha u+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(\alpha\tfrac{\sqrt{t}}{\sqrt{2}}\right) - e^{-\alpha l+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(-\tfrac{\alpha\sqrt{t}}{\sqrt{2}}\right) - 2c^2 \int_0^t \tfrac{1}{\sqrt{4\pi c^2(t-\theta)}} S(\theta)\,d\theta$$

$$\lim_{z\to 1^-} y(t,z) + \lim_{z\to 0^+} y(t,z)$$
$$= 2e^{i\mu t}\tfrac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)} + 2c^2\int_0^t e^{i\mu(t-\theta)} D(\theta)\,d\theta + \sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t}\phi_n\left(1+(-1)^n\right)$$
$$+2c^2\sum_{n=1}^{+\infty}\left(1+(-1)^n\right)\int_0^t e^{(i\mu-\lambda_n)(t-\theta)} D(\theta)\,d\theta$$

$$\lim_{z\to 1^-} y(t,z) - \lim_{z\to 0^+} y(t,z)$$
$$= 2c^2\sum_{n=1}^{+\infty}\left(1-(-1)^n\right)\int_0^t e^{(i\mu-\lambda_n)(t-\theta)} S(\theta)\,d\theta + \sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t}\phi_n\left((-1)^n-1\right)$$

The determination of the functions $D(t)$ and $S(t)$ requires to solve respect to them the following integral equations:

$$\left\{\begin{array}{l} e^{-\alpha u+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(\tfrac{\alpha\sqrt{t}}{\sqrt{2}}\right) + e^{-\alpha l+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(-\tfrac{\alpha\sqrt{t}}{\sqrt{2}}\right) - 2c^2 \int_0^t \tfrac{1}{\sqrt{4\pi c^2(t-\theta)}} D(\theta)\,d\theta \\ = 2e^{i\mu t}\tfrac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)} + 2c^2\int_0^t e^{i\mu(t-\theta)} D(\theta)\,d\theta + \sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t}\phi_n\left(1+(-1)^n\right) \\ +2c^2\sum_{n=1}^{+\infty}\left(1+(-1)^n\right)\int_0^t e^{(i\mu-\lambda_n)(t-\theta)} D(\theta)\,d\theta \end{array}\right. \tag{A.9}$$

$$\left\{\begin{array}{l} e^{-\alpha u+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(\tfrac{\alpha\sqrt{t}}{\sqrt{2}}\right) - e^{-\alpha l+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(-\tfrac{\alpha\sqrt{t}}{\sqrt{2}}\right) - 2c^2 \int_0^t \tfrac{1}{\sqrt{4\pi c^2(t-\theta)}} S(\theta)\,d\theta \\ = 2c^2\sum_{n=1}^{+\infty}\left(1-(-1)^n\right)\int_0^t e^{(i\mu-\lambda_n)(t-\theta)} S(\theta)\,d\theta + \sum_{n=1}^{+\infty} e^{(i\mu-\lambda_n)t}\phi_n\left((-1)^n-1\right) \end{array}\right. \tag{A.10}$$

We remark that these integral equations involve separately the functions $S(t)$ and $D(t)$. We solve them using the Laplace transform. We call $s(\gamma)$ and $d(\gamma)$ the Laplace transforms, with respect to the time variable $t$, of the functions $S(t)$ and $D(t)$:

$$s(\gamma) \; : \; = \mathcal{L}[S(t); t \to \gamma] = \int_0^t e^{-\gamma t} S(t)\,dt$$
$$d(\gamma) \; : \; = \mathcal{L}[D(t); t \to \gamma] = \int_0^t e^{-\gamma t} D(t)\,dt$$

Laplace transforming (A.9) and (A.10), we obtain two linear equations to be solved separately for each function $s(\gamma)$ and $d(\gamma)$:

$$\left\{ \begin{array}{l} \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} + \frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)} - \frac{2c^2}{\sqrt{4c^2\gamma}} d(\gamma) \\ = 2\frac{1}{\gamma - i\mu}\frac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)} + 2c^2 \frac{1}{\gamma - i\mu} d(\gamma) + \frac{1}{\pi^2 c^2}\sum_{n=1}^{+\infty} \frac{(1+(-1)^n)}{n^2+a^2}\phi_n \\ +d(\gamma)\frac{2c^2}{\pi^2c^2}\sum_{n=1}^{+\infty}\frac{(1+(-1)^n)}{n^2+a^2} \end{array} \right. \tag{A.11}$$

$$\left\{ \begin{array}{l} \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} - \frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)} - \frac{2c^2}{\sqrt{4c^2\gamma}} s(\gamma) \\ = s(\gamma)\frac{2c^2}{\pi^2c^2}\sum_{n=1}^{+\infty}\frac{(1-(-1)^n)}{n^2+a^2} + \frac{1}{\pi^2c^2}\sum_{n=1}^{+\infty}\frac{((-1)^n-1)}{n^2+a^2} \end{array} \right.$$

Using the following summation series formula in Gradshteyn and Ryzhik[12], page 40 as formula 1.445.1:

$$\sum_{n=1}^{\infty} \frac{n\sin(nx)}{n^2+a^2} = \frac{\pi}{2}\frac{\sinh[a(\pi-x)]}{\sinh[a\pi]}; 0 < x < 2\pi \tag{A.12}$$

formula 1.445.2:

$$\sum_{n=1}^{\infty} \frac{\cos(nx)}{n^2+a^2} = \frac{\pi}{2a}\frac{\cosh[a(\pi-x)]}{\sinh[a\pi]} - \frac{1}{2a^2}; 0 < x \leq 2\pi \tag{A.13}$$

formula 1.445.3:

$$\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2+a^2} = \frac{\pi}{2a} \frac{\cosh[ax]}{\sinh[a\pi]} - \frac{1}{2a^2}; -\pi \leq x \leq \pi \tag{A.14}$$

and formula 1.445.4:

$$\sum_{n=1}^{\infty} \frac{(-1)^n n \sin(nx)}{n^2+a^2} = -\frac{\pi}{2} \frac{\sinh[ax]}{\sinh[a\pi]}; -\pi < x < \pi \tag{A.15}$$

where $a\pi = \sqrt{(\gamma - i\mu)/c^2}$, we obtain:

$$\begin{cases} \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} + \frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)} - \frac{1}{c}\int_0^1 \frac{\cosh(a\pi(1-x))+\cosh(a\pi x)}{\sqrt{\gamma-i\mu}\sinh(a\pi)} e^{-\alpha((u-l)x+l)} dx \\ = c\left(\frac{1}{\sqrt{\gamma}} + \frac{1}{\sqrt{\gamma-i\mu}} \frac{\cosh(a\pi)+1}{\sinh(a\pi)}\right) d(\gamma) \end{cases}$$
$$\begin{cases} \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} - \frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)} - \frac{1}{c}\int_0^1 \frac{\cosh(a\pi x)-\cosh(a\pi(1-x))}{\sqrt{\gamma-i\mu}\sinh(a\pi)} e^{-\alpha((u-l)x+l)} dx \\ = c\left(\frac{1}{\sqrt{\gamma}} + \frac{1}{\sqrt{\gamma-i\mu}} \frac{\cosh(a\pi)-1}{\sinh(a\pi)}\right) s(\gamma) \end{cases} \tag{A.16}$$

We observe:

$$\begin{aligned} &\frac{1}{c}\int_0^1 \frac{\cosh(a\pi(1-x))+\cosh(a\pi x)}{\sqrt{\gamma-i\mu}\sinh(a\pi)} e^{-\alpha((u-l)x+l)} dx \\ &= \frac{\left(\sqrt{\gamma-i\mu}\left(e^{-\alpha u}+e^{-\alpha l}\right)\sinh(a\pi)+\frac{\alpha}{\sqrt{2}}\left(e^{-\alpha u}-e^{-\alpha l}\right)(\cosh(a\pi)+1)\right)}{\sqrt{\gamma-i\mu}\sinh(a\pi)\left(\gamma-i\mu-\frac{\alpha^2}{2}\right)} \end{aligned}$$

$$\begin{aligned} &\frac{1}{c}\int_0^1 \frac{\cosh(a\pi x)-\cosh(a\pi(1-x))}{\sqrt{\gamma-i\mu}\sinh(a\pi)} e^{-\alpha((u-l)x+l)} dx \\ &= \frac{\left(\sqrt{\gamma-i\mu}\left(e^{-\alpha u}-e^{-\alpha l}\right)\sinh(a\pi)+\frac{\alpha}{\sqrt{2}}\left(e^{-\alpha u}+e^{-\alpha l}\right)(\cosh(a\pi)-1)\right)}{\sqrt{\gamma-i\mu}\sinh(a\pi)\left(\gamma-i\mu-\frac{\alpha^2}{2}\right)} \end{aligned}$$

so substituting these expressions in (A.16) and solving respect to the quantities $cd(\gamma)/\sqrt{\gamma}$ and $cs(\gamma)/\sqrt{\gamma}$ we obtain (3.6) in Theorem 1.

From equation (A.8), we get as well the Laplace transform of the function $y(t,z)$ when

$z = 0$ and when $z = 1$. Then the expressions for the characteristic function when $x = l$ and when $x = u$, are:

$$\begin{aligned} v(t, u, \mu) &= e^{-mu-\frac{m^2}{2}t} y(t, 1) \\ v(t, l, \mu) &= e^{-ml-\frac{m^2}{2}t} y(t, 0) \end{aligned} \tag{A.17}$$

## A.2. Solution of the PDE with Dirichlet boundary conditions

In order to find the expression for the function $v(t, x, \mu)$ for a generic value of $x$, we can now solve the PDE (3.2) in three different regions ($x < l$, $l < x < u$ and $u < x$) using as Dirichlet boundary conditions at $x = l$ and $x = u$ the known values in (A.17). This, after the same transformation as before, amounts to solve:

$$-\frac{\partial y(t, z)}{\partial t} + c^2 \frac{\partial^2 y(t, z)}{\partial^2 z} + i\mu \mathbf{1}_{(0<z<1)} y(t, z) = 0 \tag{A.18}$$

where $c^2 = \frac{1}{2(u-l)^2}$, and $y(0, z) = e^{-\alpha((u-l)z+l)}$, in three different regions using Dirichlet boundary conditions at $z = 0$ and at $z = 1$ the known values of $y(t, 0)$ and $y(t, 1)$.

### A.2.1. Case $x < l$ and $x > u$

In this case, we have the heat equation with Dirichlet boundary conditions and the solution can be found in Zauderer [30], page. 265, eq. 5.105:

$$y(t,x) = \begin{cases} \mathbf{1}_{(x>u)} \left( \begin{array}{l} \int_0^{+\infty} \left[ \frac{e^{-\frac{(x-u-\zeta)^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(x-u+\zeta)^2}{2t}}}{\sqrt{2\pi t}} \right] e^{-\alpha(\zeta+u)} d\zeta + \\ + \int_0^t \frac{(x-u)}{\sqrt{2\pi(t-\theta)^3}} e^{-\frac{(x-u)^2}{2(t-\theta)}} e^{-\alpha u - \beta\theta} q(\theta)\, d\theta \end{array} \right) \\ \\ \mathbf{1}_{(x<l)} \left( \begin{array}{l} \int_0^{+\infty} \left[ \frac{e^{-\frac{(l-x-\zeta)^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(l-x+\zeta)^2}{2t}}}{\sqrt{2\pi t}} \right] e^{-\alpha(l-\zeta)} d\zeta + \\ + \int_0^t \frac{(l-x) e^{-\frac{(l-x)^2}{2(t-\theta)}}}{\sqrt{2\pi(t-\theta)^3}} e^{-\alpha l - \beta\theta} p(\theta)\, d\theta \end{array} \right) \end{cases}$$

and then:

$$v(t,x,\mu) = e^{\alpha x + \beta t} y(t,x) = \begin{cases} \mathbf{1}_{(x>u)} \left( \begin{array}{l} e^{\alpha x + \beta t} \int_0^{+\infty} \left[ \frac{e^{-\frac{(x-u-\zeta)^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(x-u+\zeta)^2}{2t}}}{\sqrt{2\pi t}} \right] e^{-\alpha(u+\zeta)} d\zeta \\ + e^{\alpha(x-u)} \int_0^t \frac{(x-u) e^{-\frac{(x-u)^2}{2(t-\theta)}}}{\sqrt{2\pi(t-\theta)^3}} e^{\beta(t-\theta)} q(\theta)\, d\theta \end{array} \right) \\ \\ \mathbf{1}_{(x<l)} \left( \begin{array}{l} e^{\alpha x + \beta t} \int_0^{+\infty} \left[ \frac{e^{-\frac{(l-x-\zeta)^2}{2t}}}{\sqrt{2\pi t}} - \frac{e^{-\frac{(l-x+\zeta)^2}{2t}}}{\sqrt{2\pi t}} \right] e^{-\alpha(l-\zeta)} d\zeta \\ + e^{-\alpha(l-x)} \int_0^t \frac{(l-x) e^{-\frac{(l-x)^2}{2(t-\theta)}}}{\sqrt{2\pi(t-\theta)^3}} e^{\beta(t-\theta)} p(\theta)\, d\theta \end{array} \right) \end{cases} \tag{A.19}$$

where $q(t)$ and $p(t)$ are the known values of the characteristic function at $x = u$ and $x = l$: $q(t) := v(t,u)$ and $q(t) := v(t,l)$.

Using some algebra and comparing with BS [4] (formula 1.2.4 page 198), we can remark that when $x > u$:

$$\begin{aligned}
&e^{\beta t}\int_0^{+\infty}\left(\frac{e^{-\frac{(x-u-\zeta)^2}{2t}}}{\sqrt{2\pi t}}-\frac{e^{-\frac{(x-u+\zeta)^2}{2t}}}{\sqrt{2\pi t}}\right)e^{\alpha(x-u-\zeta)}d\zeta\\
&=1-\left[\tfrac{1}{2}\operatorname{Erfc}\left(\tfrac{x-u+mt}{\sqrt{2t}}\right)+\tfrac{e^{-2m(x-u)}}{2}\operatorname{Erfc}\left(\tfrac{x-u-mt}{\sqrt{2t}}\right)\right]\\
&=\Pr_{0,x\in(u,+\infty)}\left(\inf_{0\le s\le t} ms+W(s)>u\right)
\end{aligned}\tag{A.20}$$

where we have used the fact that $\beta = -m^2/2$ and $\alpha = -m$. Similarly, when $x < l$ it can be shown with some algebra and using formula 1.1.4 page 197 in BS [4] that:

$$\begin{aligned}
&e^{\alpha x+\beta t}\int_0^{+\infty}\left[\frac{e^{-\frac{(l-x-\zeta)^2}{2t}}}{\sqrt{2\pi t}}-\frac{e^{-\frac{(l-x+\zeta)^2}{2t}}}{\sqrt{2\pi t}}\right]e^{-\alpha(l-\zeta)}d\zeta\\
&=1-\left[\tfrac{1}{2}\operatorname{Erfc}\left(\tfrac{l-x-mt}{\sqrt{2t}}\right)+\tfrac{e^{2m(l-x)}}{2}\operatorname{Erfc}\left(\tfrac{l-x+mt}{\sqrt{2t}}\right)\right]\\
&=\Pr_{0,x\in(-\infty,l)}\left(\sup_{0\le s\le t} ms+W(s)<l\right)
\end{aligned}\tag{A.21}$$

In order to find the expression for $\Omega(t,\mu,x;u,l,m)$ in Theorem 1, we can use equation (A.17) and substituting it in (A.19), we obtain:

$$\Omega(t,\mu,x;u,l,m)=\begin{cases}
\mathbf{1}_{(x>u)}e^{\alpha x+\beta t}\int_0^t\frac{(x-u)e^{-\frac{(x-u)^2}{2(t-\theta)}}}{\sqrt{2\pi(t-\theta)^3}}y(\theta,1)\,d\theta\\
\\
\mathbf{1}_{(x<l)}e^{\alpha x+\beta t}\int_0^t\frac{(l-x)e^{-\frac{(l-x)^2}{2(t-\theta)}}}{\sqrt{2\pi(t-\theta)^3}}y(\theta,0)\,d\theta
\end{cases}$$

and then if we consider the Laplace transform of the integrals we obtain the expression for $\omega(\gamma,\mu,x;l,u,m)$ in Theorem 1.

We now show how to find the density function of the occupation time given in equation (3.8) in Theorem 1. Using in (A.19) the fact that:

$$\begin{aligned}
q(t)&=v(t,u;l,u,m)=\int_0^t e^{i\mu\theta}f_\tau(\theta,t,u)\,d\theta\\
p(t)&=v(t,l;l,u,m)=\int_0^t e^{i\mu\theta}f_\tau(\theta,t,l)\,d\theta
\end{aligned}\tag{A.22}$$

we can observe that, for $x > u$, we have:

$$
\begin{aligned}
& e^{\alpha(x-u)} \int_0^t \frac{(x-u)e^{-\frac{(x-u)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} e^{\beta(t-\tau)} q(\tau)\, d\tau \\
& = e^{\alpha(x-u)} \int_0^t \frac{(x-u)e^{-\frac{(x-u)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} e^{\beta(t-\tau)} \left( \int_0^\tau e^{i\mu\theta} f_\tau(\theta, \tau, u)\, d\theta \right) d\tau \\
& = e^{\alpha(x-u)} \int_0^t e^{i\mu\theta} \left( \int_\theta^t \frac{(x-u)e^{-\frac{(x-u)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} e^{\beta(t-\tau)} f_\tau(\theta, \tau, u)\, d\tau \right) d\theta
\end{aligned}
$$

and then:

$$
\begin{aligned}
& v(t, x, \mu; l, u, m) \\
& = 1 \times \Pr_{0, x\in(u,+\infty)} \left( \inf_{0\le s\le t} ms + W(s) > u \right) \\
& + e^{\alpha(x-u)} \int_0^t e^{i\mu\theta} \left( \int_\theta^t \frac{(x-u)e^{-\frac{(x-u)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} e^{\beta(t-\tau)} f_\tau(\theta, \tau, u)\, d\tau \right) d\theta
\end{aligned}
$$

and so comparing with (3.1) the density function of the occupation when $x > u$ can be expressed in terms of the density function when $x = u$. Similarly, for $x < l$, we obtain:

$$
\begin{aligned}
& v(t, x, \mu; l, u, m) \\
& = 1 \times \Pr_{0, x\in(-\infty,l)} \left( \sup_{0\le s\le t} ms + W(s) < l \right) \\
& + e^{-\alpha(l-x)} \int_0^t e^{i\mu\theta} \left( \int_\theta^t \frac{(l-x)e^{-\frac{(l-x)^2}{2(t-\tau)}}}{\sqrt{2\pi(t-\tau)^3}} e^{\beta(t-\tau)} f_\tau(\theta, \tau, l)\, d\tau \right) d\theta
\end{aligned}
$$

Comparing these expressions with (3.1) the density function of the occupation when $x < l$ can be expressed in terms of the density function when $x = l$.

### A.2.2. Case $l < x < u$

In this case the PDE (3.2) becomes:

$$
-\frac{\partial v(t, x, \mu)}{\partial t} + \frac{1}{2} \frac{\partial^2 v(t, x, \mu)}{\partial^2 x} + m \frac{\partial v(t, x, \mu)}{\partial x} + i\mu v(t, x, \mu) = 0 \qquad \text{(A.23)}
$$

and has to be solved in the finite region $l < x < u$, with boundary conditions:

$$\begin{aligned} v(t,u) &= q(t) \\ v(t,l) &= p(t) \end{aligned}$$

and initial condition:

$$v(0,x) = 1$$

If we consider the transformation $y(t,z) = e^{-\alpha((u-l)z+l)-kt}v(t,(u-l)z+l)$, we get for the function $y(t,z)$ the heat equation in a finite region $0 < z < 1$,:

$$-\frac{\partial y(t,z)}{\partial t} + c^2 \frac{\partial^2 y(t,z)}{\partial^2 z} = 0 \tag{A.24}$$

with boundary conditions:

$$y(t,1) = e^{-\alpha u - kt}q(t)\,;\ y(t,0) = e^{-\alpha l - kt}p(t)$$

and initial condition $y(0,z) = e^{-\alpha((u-l)z+l)}$. The solution can be found in Necati [21] (page 62, eq.2-73a). Then the expression for characteristic function is given by:

$$\begin{aligned} &v(t,x,\mu) = e^{\alpha x + kt}\sum_{n=1}^{\infty} w_n(t)\sin\left(n\pi\left(\tfrac{x-l}{u-l}\right)\right) \\ &+e^{i\mu t}2e^{\alpha x - \frac{\alpha^2}{2}t}\int_0^1 \left[\sum_{n=1}^{\infty} e^{-(cn\pi)^2 t}\sin(n\pi z)\sin(n\pi\xi)\right] e^{-\alpha(\xi(u-l)+l)}d\xi \end{aligned}$$

where:

$$w_n(t) = 2n\pi c^2 \int_0^t e^{-\lambda_n(t-s)}\left[e^{-\alpha l - ks}p(s) - (-1)^n e^{-\alpha u - ks}q(s)\right] ds \tag{A.25}$$

Using the properties of the theta function, compare Kevorkian [17] at page 25-26, and comparing with the expression in BS [4] (formula 1.15.4, page. 211) we obtain:

$$
\begin{aligned}
&2e^{-mx-\frac{\alpha^2}{2}t}\int_0^1\left[\sum_{n=1}^{\infty}e^{-(cn\pi)^2t}\sin(n\pi z)\sin(n\pi\xi)\right]e^{m(\xi(u-l)+l)}d\xi \\
&=\Pr_{0,x\in(l,u)}\left(\sup_{0\le s\le t}ms+W(s)<u;\ \inf_{0\le s\le t}ms+W(s)>l\right)
\end{aligned}
\tag{A.26}
$$

In order to find the expression for $\Omega(t,x;u,l,m)$ when $x\in(l,u)$ in Theorem 1, we can use equation (A.17) and substituting it in (A.25), we obtain:

$$
w_n(t)=2n\pi c^2\int_0^t e^{-\lambda_n(t-s)-i\mu s}\left[y(s,0)-(-1)^n y(s,1)\right]ds
$$

and then:

$$
\begin{aligned}
&\Omega(t,\mu,x;u,l,m)\\
&=e^{\alpha x+\beta t}\sum_{n=1}^{\infty}2n\pi c^2\sin\left(n\pi\left(\tfrac{x-l}{u-l}\right)\right)\textstyle\int_0^t e^{-(\lambda_n-i\mu)(t-s)}\left(y(s,0)-(-1)^n y(s,1)\right)ds
\end{aligned}
$$

If we consider the Laplace transform of the series we get:

$$
\begin{aligned}
&\omega(\gamma,\mu,x;l,u,m)\\
&=\tfrac{1}{\pi^2c^2}\sum_{n=1}^{\infty}\tfrac{2n\pi c^2}{n^2+\frac{\gamma-i\mu}{\pi^2c^2}}\sin\left(n\pi\left(\tfrac{x-l}{u-l}\right)\right)\left(\mathcal{L}\left[y(t,0);t\to\gamma\right]-(-1)^n\mathcal{L}\left[y(t,1);t\to\gamma\right]\right)ds
\end{aligned}
$$

and using the summation formulas (A.12) and (A.13) we obtain the expression in Theorem 1.

We now show how to find the density function of the occupation time given in equation (3.8) in Theorem 1. Substituting in expression (A.25) the functions $q(t)$ and $p(t)$ as given in (A.22), we have:

$$
\begin{aligned}
& e^{\alpha x+kt} w_n(t) \\
& = -2n\pi c^2 \int_0^t e^{\left(k-(n\pi c)^2\right)(t-s)} \\
& \times \left[ (-1)^n e^{-\alpha(u-x)} \int_0^s e^{i\mu\theta} f_\tau(\theta, s, u) - e^{\alpha(x-l)} \int_0^s e^{i\mu\theta} f_\tau(\theta, s, l)\, d\theta \right] ds
\end{aligned}
$$

With a change of variable, $(\xi = t - s, \nu = t + \tau - s = \xi + \tau)$, and using the fact that $k = -\alpha^2/2 + i\mu$, we get:

$$
\begin{aligned}
& = -2n\pi c^2 \int_0^t e^{-\left(\frac{\alpha^2}{2}+(n\pi c)^2\right)\xi} \\
& \times \int_\xi^t \left[ (-1)^n e^{-\alpha(u-x)} e^{i\mu\theta} f_\tau(\theta-\xi, t-\xi, u) - e^{\alpha(x-l)} e^{i\mu\theta} f_\tau(\theta-\xi, t-\xi, l) \right] d\theta d\xi \\
& = 2n\pi c^2 \int_0^t e^{i\mu\theta} \\
& \times \textstyle\int_0^\theta e^{-\left(\frac{\alpha^2}{2}+(n\pi c)^2\right)\xi} \left( e^{\alpha(x-l)} f_\tau(\theta-\xi, t-\xi, l) - (-1)^n e^{-\alpha(u-x)} f_\tau(\theta-\xi, t-\xi, u) \right) d\xi d\theta
\end{aligned}
$$

So for a generic starting point $x \in (l, u)$ we have:

$$
\begin{aligned}
& v(t, x, \mu; l, u, m) \\
& = e^{i\mu t} \Pr_{0, x\in(l,u)} \left( \sup_{0\le s\le t} ms + W(s) < u; \inf_{0\le s\le t} ms + W(s) > l \right) \\
& + \int_0^t e^{i\mu\theta} \left[ \sum_{n=1}^{\infty} 2n\pi c^2 \sin\left( n\pi \left( \tfrac{x-l}{u-l} \right) \right) \int_0^\theta e^{-\left(\frac{\alpha^2}{2}+n^2\pi^2c^2\right)\xi} \right. \\
& \left. \times \left( e^{\alpha(x-l)} f_\tau(\theta-\xi, t-\xi, l) - (-1)^n e^{-\alpha(u-x)} f_\tau(\theta-\xi, t-\xi, u) \right) d\xi \right] d\theta
\end{aligned}
$$

and so we recognize in the square brackets inside the integral the density function of the occupation when $l < x < u$, as shown in Theorem 1.

## B. The expressions for the two steps univariate inversion

For the fact that standard numerical Laplace inversion packages perform unidimensional inversion, we need an expression for the Laplace transform of the real part and the imaginary part of the functions $D(t)$ and $S(t)$ and not just the Laplace transform of the functions $D(t)$ and $S(t)$ as described in the previous section. In this section we provide the formula to be used in this case.

### B.1. Computation of the functions $D(t)$ and $S(t)$

We need to remark that the functions $D(t)$ and $S(t)$ are complex functions, so in the above integral equations we need to separate real and complex part. We write

$$\begin{aligned} D(t) &= D_1(t) + iD_2(t) \\ S(t) &= S_1(t) + iS_2(t) \end{aligned}$$

We have to solve two integral equations respect to the complex functions $D(t) = D_1(t) + iD_2(t)$ and $S(t) = S_1(t) + iS_2(t)$. The first one is given by:

$$\left\{ \begin{array}{l} e^{-\alpha u + \frac{1}{2}\alpha^2 t} \operatorname{Erfc}\left(\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) + e^{-\alpha l + \frac{1}{2}\alpha^2 t} \operatorname{Erfc}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) \\ -2c^2 \int_0^t \frac{1}{\sqrt{4\pi c^2 (t-\theta)}} D(\theta)\, d\theta \\ = 2e^{i\mu t} \frac{e^{-\alpha l} - e^{-\alpha u}}{\alpha(u-l)} + 2c^2 \int_0^t e^{i\mu(t-\theta)} D(\theta)\, d\theta + \sum\limits_{n=1}^{+\infty} e^{(i\mu - \lambda_n)t} \phi_n \left(1 + (-1)^n\right) \\ +2c^2 \sum\limits_{n=1}^{+\infty} \left(1 + (-1)^n\right) \int_0^t e^{(i\mu - \lambda_n)(t-\theta)} D(\theta)\, d\theta \end{array} \right. \tag{B.1}$$

while the second one is:

$$\left\{ \begin{array}{l} e^{-\alpha u + \frac{1}{2}\alpha^2 t} \operatorname{Erfc}\left(\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) - e^{-\alpha l + \frac{1}{2}\alpha^2 t} \operatorname{Erfc}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) \\ -2c^2 \int_0^t \frac{1}{\sqrt{4\pi c^2 (t-\theta)}} S(\theta)\, d\theta \\ = 2c^2 \sum\limits_{n=1}^{+\infty} \left(1 - (-1)^n\right) \int_0^t e^{(i\mu - \lambda_n)(t-\theta)} S(\theta)\, d\theta + \sum\limits_{n=1}^{+\infty} e^{(i\mu - \lambda_n)t} \phi_n \left((-1)^n - 1\right) \end{array} \right. \tag{B.2}$$

Using the Euler formula, $e^{i\mu t} = \cos(\mu t) + i \sin(\mu t)$, we substitute in the equation (B.1) and we obtain a system of two integral equations for the real part and for the imaginary part of the function $D(t)$:

real part (B.3)

$$\begin{cases} e^{-\alpha u+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(\frac{\alpha\sqrt{t}}{\sqrt{2}}\right)+e^{-\alpha l+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) \\ -2c^2\int_0^t \frac{1}{\sqrt{4\pi c^2(t-\theta)}}D_1(\theta)\,d\theta \\ =2\cos(\mu t)\,\frac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)}+\sum\limits_{n=1}^{+\infty}\cos(\mu t)\,e^{-\lambda_n t}\phi_n\left(1+(-1)^n\right) \\ +2c^2\int_0^t\cos(\mu(t-\theta))\,D_1(\theta)\,d\theta-2c^2\int_0^t\sin(\mu(t-\theta))\,D_2(\theta)\,d\theta \\ +2c^2\sum\limits_{n=1}^{+\infty}\left(1+(-1)^n\right)\int_0^t e^{-\lambda_n(t-\theta)}\cos(\mu(t-\theta))\,D_1(\theta)\,d\theta \\ -2c^2\sum\limits_{n=1}^{+\infty}\left(1+(-1)^n\right)\int_0^t e^{-\lambda_n(t-\theta)}\sin(\mu(t-\theta))\,D_2(\theta)\,d\theta \end{cases} \tag{B.4}$$

imaginary part

$$\begin{cases} -2c^2\int_0^t \frac{1}{\sqrt{4\pi c^2(t-\theta)}}D_2(\theta)\,d\theta \\ =2\sin(\mu t)\,\frac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)} \\ +\sum\limits_{n=1}^{+\infty}\sin(\mu t)\,e^{-\lambda_n t}\phi_n\left(1+(-1)^n\right) \\ +2c^2\int_0^t\sin(\mu(t-\theta))\,D_1(\theta)\,d\theta+2c^2\int_0^t\cos(\mu(t-\theta))\,D_2(\theta)\,d\theta \\ +2c^2\sum\limits_{n=1}^{+\infty}\left(1+(-1)^n\right)\int_0^t e^{-\lambda_n(t-\theta)}\sin(\mu(t-\theta))\,D_1(\theta)\,d\theta \\ +2c^2\sum\limits_{n=1}^{+\infty}\left(1+(-1)^n\right)\int_0^t e^{-\lambda_n(t-\theta)}\cos(\mu(t-\theta))\,D_2(\theta)\,d\theta \end{cases}$$

Similarly, using in equation (B.2) the Euler formula and $S(t)=S_1(t)+iS_2(t)$, we obtain a system of two integral equations for the real part and for the imaginary part of the function $S(t)$:

real part (B.5)

$$\begin{cases} e^{-\alpha u+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(\frac{\alpha\sqrt{t}}{\sqrt{2}}\right)-e^{-\alpha l+\frac{1}{2}\alpha^2 t}\,\mathrm{Erfc}\left(-\frac{\alpha\sqrt{t}}{\sqrt{2}}\right) \\ -2c^2\int_0^t \frac{1}{\sqrt{4\pi c^2(t-\theta)}}S_1(\theta)\,d\theta \\ =2c^2\sum\limits_{n=1}^{+\infty}\left(1-(-1)^n\right)\int_0^t e^{-\lambda_n(t-\theta)}\cos\left(\mu\left(t-\theta\right)\right)S_1(\theta)\,d\theta \\ -2c^2\sum\limits_{n=1}^{+\infty}\left(1-(-1)^n\right)\int_0^t e^{-\lambda_n(t-\theta)}\sin\left(\mu\left(t-\theta\right)\right)S_2(\theta)\,d\theta \\ +\sum\limits_{n=1}^{+\infty}e^{-\lambda_n t}\cos(\mu t)\,\phi_n\left((-1)^n-1\right) \end{cases} \tag{B.6}$$

imaginary part

$$\begin{cases} -2c^2\int_0^t \frac{1}{\sqrt{4\pi c^2(t-\theta)}}S_2(\theta)\,d\theta \\ =2c^2\sum\limits_{n=1}^{+\infty}\left(1-(-1)^n\right)\int_0^t e^{-\lambda_n(t-\theta)}\sin\left(\mu\left(t-\theta\right)\right)S_1(\theta)\,d\theta \\ +2c^2\sum\limits_{n=1}^{+\infty}\left(1-(-1)^n\right)\int_0^t e^{-\lambda_n(t-\theta)}\cos\left(\mu\left(t-\theta\right)\right)S_2(\theta)\,d\theta \\ +\sum\limits_{n=1}^{+\infty}e^{-\lambda_n t}\sin(\mu t)\,\phi_n\left((-1)^n-1\right) \end{cases}$$

So, using the convolution property of the Laplace transform, we get two systems of linear equations in the functions $s_j(\gamma)$ and $d_j(\gamma)$, $j=1,2$:

$$\begin{cases} \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}+\frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)}-\frac{2c^2}{\sqrt{4c^2\gamma}}d_1(\gamma) \\ =2\frac{\gamma}{\gamma^2+\mu^2}\frac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)}+\frac{1}{2}\sum\limits_{n=1}^{+\infty}\left(\frac{1}{\gamma+\lambda_n-i\mu}+\frac{1}{\gamma+\lambda_n+i\mu}\right)\phi_n\left(1+(-1)^n\right) \\ +2c^2\frac{\gamma}{\gamma^2+\mu^2}d_1(\gamma)-2c^2\frac{\mu}{\gamma^2+\mu^2}d_2(\gamma) \\ +c^2\sum\limits_{n=1}^{+\infty}\left(1+(-1)^n\right)\left(\frac{1}{\gamma+\lambda_n-i\mu}+\frac{1}{\gamma+\lambda_n+i\mu}\right)d_1(\gamma) \\ -\frac{c^2}{i}\sum\limits_{n=1}^{+\infty}\left(1+(-1)^n\right)\left(\frac{1}{\gamma+\lambda_n-i\mu}-\frac{1}{\gamma+\lambda_n+i\mu}\right)d_2(\gamma) \end{cases}$$

$$\begin{cases} -\frac{2c^2}{\sqrt{4c^2\gamma}} d_2(\gamma) = 2\frac{\mu}{\gamma^2+\mu^2}\frac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)} \\ +\frac{1}{2i}\sum_{n=1}^{+\infty}\left(\frac{1}{\gamma+\lambda_n-i\mu}-\frac{1}{\gamma+\lambda_n+i\mu}\right)\phi_n\left(1+(-1)^n\right) \\ +2c^2\frac{\mu}{\gamma^2+\mu^2}d_1(\gamma)+2c^2\frac{\gamma}{\gamma^2+\mu^2}d_2(\gamma) \\ +\frac{c^2}{i}\sum_{n=1}^{+\infty}\left(1+(-1)^n\right)\left(\frac{1}{\gamma+\lambda_n-i\mu}-\frac{1}{\gamma+\lambda_n+i\mu}\right)d_1(\gamma) \\ +c^2\sum_{n=1}^{+\infty}\left(1+(-1)^n\right)\left(\frac{1}{\gamma+\lambda_n-i\mu}+\frac{1}{\gamma+\lambda_n+i\mu}\right)d_2(\gamma) \end{cases}$$

and then:

$$\begin{cases} \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}+\frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)}-2\frac{\gamma}{\gamma^2+\mu^2}\frac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)} \\ -\frac{1}{2}\sum_{n=1}^{+\infty}\left(\frac{1}{\gamma+\lambda_n-i\mu}+\frac{1}{\gamma+\lambda_n+i\mu}\right)\phi_n\left(1+(-1)^n\right) \\ =2c^2\left(\frac{1}{\sqrt{4c^2\gamma}}+\frac{\gamma}{\gamma^2+\mu^2}+\frac{1}{2}\sum_{n=1}^{+\infty}\left(1+(-1)^n\right)\left(\frac{1}{\gamma+\lambda_n-i\mu}+\frac{1}{\gamma+\lambda_n+i\mu}\right)\right)d_1(\gamma) \\ -2c^2\left(\frac{\mu}{\gamma^2+\mu^2}+\frac{1}{2i}\sum_{n=1}^{+\infty}\left(1+(-1)^n\right)\left(\frac{1}{\gamma+\lambda_n-i\mu}-\frac{1}{\gamma+\lambda_n+i\mu}\right)\right)d_2(\gamma) \end{cases} \tag{B.7}$$

$$\begin{cases} -2\frac{\mu}{\gamma^2+\mu^2}\frac{e^{-\alpha l}-e^{-\alpha u}}{\alpha(u-l)}-\frac{1}{2i}\sum_{n=1}^{+\infty}\left(\frac{1}{\gamma+\lambda_n-i\mu}-\frac{1}{\gamma+\lambda_n+i\mu}\right)\phi_n\left(1+(-1)^n\right) \\ =2c^2\left(\frac{\mu}{\gamma^2+\mu^2}+\frac{1}{2i}\sum_{n=1}^{+\infty}\left(1+(-1)^n\right)\left(\frac{1}{\gamma+\lambda_n-i\mu}-\frac{1}{\gamma+\lambda_n+i\mu}\right)\right)d_1(\gamma) \\ +2c^2\left(\frac{1}{\sqrt{4c^2\gamma}}+\frac{\gamma}{\gamma^2+\mu^2}+\frac{1}{2}\sum_{n=1}^{+\infty}\left(1+(-1)^n\right)\left(\frac{1}{\gamma+\lambda_n-i\mu}+\frac{1}{\gamma+\lambda_n+i\mu}\right)\right)d_2(\gamma) \end{cases}$$

Using (A.13) and (A.14), we can write :

$$\begin{aligned} C(\gamma,\mu) &= cA(\gamma,\mu)\,d_1(\gamma)-cB(\gamma,\mu)\,d_2(\gamma) \\ E(\gamma,\mu) &= cB(\gamma,\mu)\,d_1(\gamma)+cA(\gamma,\mu)\,d_2(\gamma) \end{aligned}$$

where we have defined the following (complex) functions:

$$A(\gamma,\mu) = \left(\frac{1}{\sqrt{\gamma}} + \frac{\cosh[a\pi]+1}{2\sqrt{\gamma - i\mu}\sinh[a\pi]} + \frac{\cosh[b\pi]+1}{2\sqrt{\gamma + i\mu}\sinh[b\pi]}\right)$$
$$B(\gamma,\mu) = \frac{1}{i}\left(\frac{\cosh[a\pi]+1}{2\sqrt{\gamma - i\mu}\sinh[a\pi]} - \frac{\cosh[b\pi]+1}{2\sqrt{\gamma + i\mu}\sinh[b\pi]}\right)$$
$$C(\gamma,\mu) = \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} + \frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)} - \frac{1}{2}w(\gamma,\mu)$$
$$E(\gamma,\mu) = -\frac{1}{2i}z(\gamma,\mu)$$
$$F(\gamma,\mu) = A^2(\gamma,\mu) + B^2(\gamma,\mu)$$

$$\begin{aligned} &w(\gamma,\mu) \\ &= \frac{1}{c}\int_0^1 \left(\frac{\cosh[a\pi(1-\xi)]+\cosh[a\pi\xi]}{\sqrt{\gamma - i\mu}\sinh[a\pi]} + \frac{\cosh[b\pi(1-\xi)]+\cosh[b\pi\xi]}{\sqrt{\gamma + i\mu}\sinh[b\pi]}\right) e^{-\alpha((u-l)\xi+l)}d\xi \\ &= w_1(\gamma,\mu) + w_2(\gamma,\mu) \end{aligned}$$

$$\begin{aligned} &z(\gamma,\mu) \\ &= \frac{1}{c}\int_0^1 \left(\frac{\cosh[a\pi(1-\xi)]+\cosh[a\pi\xi]}{\sqrt{\gamma - i\mu}\sinh[a\pi]} - \frac{\cosh[b\pi(1-\xi)]+\cosh[b\pi\xi]}{\sqrt{\gamma + i\mu}\sinh[b\pi]}\right) e^{-\alpha((u-l)\xi+l)}d\xi \\ &= w_1(\gamma,\mu) - w_2(\gamma,\mu) \end{aligned}$$

$$w_1(\gamma,\mu) = \frac{1}{\sqrt{\gamma - i\mu}\sinh[a\pi]}\left(\frac{\sqrt{\gamma - i\mu}\left(e^{-\alpha l}+e^{-\alpha u}\right)\sinh[a\pi] - \frac{\alpha}{\sqrt{2}}\left(e^{-\alpha l}-e^{-\alpha u}\right)(\cosh[a\pi]+1)}{\gamma - i\mu - \frac{\alpha^2}{2}}\right)$$
$$w_2(\gamma,\mu) = \frac{1}{\sqrt{\gamma + i\mu}\sinh[b\pi]}\left(\frac{\sqrt{\gamma + i\mu}\left(e^{-\alpha l}+e^{-\alpha u}\right)\sinh[b\pi] - \frac{\alpha}{\sqrt{2}}\left(e^{-\alpha l}-e^{-\alpha u}\right)(\cosh[b\pi]+1)}{\gamma + i\mu - \frac{\alpha^2}{2}}\right)$$
$$a\pi = \sqrt{\frac{\gamma - i\mu}{c^2}};\ b\pi = \sqrt{\frac{\gamma + i\mu}{c^2}}$$

then the functions $d_1(\gamma)$ and $d_2(\gamma)$ are given by:

$$\begin{aligned} cd_1(\gamma) &= \frac{A(\gamma,\mu)C(\gamma,\mu) + B(\gamma,\mu)E(\gamma,\mu)}{F(\gamma,\mu)} \\ cd_2(\gamma) &= \frac{A(\gamma,\mu)E(\gamma,\mu) - B(\gamma,\mu)C(\gamma,\mu)}{F(\gamma,\mu)} \end{aligned}$$

Using a similar procedure and Laplace transforming the system of integral equations

(B.5) and substituting the expressions for the series at last we obtain the following linear system:

$$
\begin{aligned}
G(\gamma,\mu) &= M(\gamma,\mu)\, s_1(\gamma) - N(\gamma,\mu)\, s_2(\gamma) \\
H(\gamma,\mu) &= N(\gamma,\mu)\, s_1(\gamma) + M(\gamma,\mu)\, s_2(\gamma)
\end{aligned}
$$

where we have defined the following functions:

$$G(\gamma,\mu) = \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} - \frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)} - \frac{1}{2}k(\gamma,\mu)$$

$$H(\gamma,\mu) = -\frac{1}{2i}j(\gamma,\mu)$$

$$M(\gamma,\mu) = \left(\frac{1}{\sqrt{\gamma}} + \frac{\cosh[a\pi]-1}{2\sqrt{\gamma-i\mu}\sinh[a\pi]} + \frac{\cosh[b\pi]-1}{2\sqrt{\gamma+i\mu}\sinh[b\pi]}\right)$$

$$N(\gamma,\mu) = \frac{1}{i}\left(\frac{\cosh[a\pi]-1}{2\sqrt{\gamma-i\mu}\sinh[a\pi]} - \frac{\cosh[b\pi]-1}{2\sqrt{\gamma+i\mu}\sinh[b\pi]}\right)$$

$$P(\gamma,\mu) = M^2(\gamma,\mu) + N^2(\gamma,\mu)$$

$$
\begin{aligned}
&k(\gamma,\mu) \\
&= \frac{1}{c}\int_0^1 \left(\frac{\cosh[a\pi\xi]-\cosh[a\pi(1-\xi)]}{\sqrt{\gamma-i\mu}\sinh[a\pi]} + \frac{\cosh[b\pi\xi]-\cosh[b\pi(1-\xi)]}{\sqrt{\gamma+i\mu}\sinh[b\pi]}\right) e^{-\alpha((u-l)\xi+l)}d\xi \\
&= k_1(\gamma,\mu) + k_2(\gamma,\mu)
\end{aligned}
$$

$$
\begin{aligned}
&j(\gamma,\mu) \\
&= \frac{1}{c}\int_0^1 \left(\frac{\cosh[a\pi\xi]-\cosh[a\pi(1-\xi)]}{\sqrt{\gamma-i\mu}\sinh[a\pi]} - \frac{\cosh[b\pi\xi]-\cosh[b\pi(1-\xi)]}{\sqrt{\gamma+i\mu}\sinh[b\pi]}\right) e^{-\alpha((u-l)\xi+l)}d\xi \\
&= k_1(\gamma,\mu) - k_2(\gamma,\mu)
\end{aligned}
$$

$$k_1(\gamma,\mu) = \frac{1}{\sqrt{\gamma-i\mu}\sinh[a\pi]}\left(\frac{\sqrt{\gamma-i\mu}\left(e^{-\alpha u}-e^{-\alpha l}\right)\sinh[a\pi]+\frac{\alpha}{\sqrt{2}}\left(e^{-\alpha u}+e^{-\alpha l}\right)(\cosh[a\pi]-1)}{\gamma-i\mu-\frac{\alpha^2}{2}}\right)$$

$$k_2(\gamma,\mu) = \frac{1}{\sqrt{\gamma+i\mu}\sinh[b\pi]}\left(\frac{\sqrt{\gamma+i\mu}\left(e^{-\alpha u}-e^{-\alpha l}\right)\sinh[b\pi]+\frac{\alpha}{\sqrt{2}}\left(e^{-\alpha u}+e^{-\alpha l}\right)(\cosh[b\pi]-1)}{\gamma+i\mu-\frac{\alpha^2}{2}}\right)$$

and the functions $s_1(\gamma)$ and $s_2(\gamma)$ are given by:

$$\begin{aligned} cs_1(\gamma) &= \frac{M(\gamma,\mu)\,G(\gamma,\mu)+N(\gamma,\mu)\,H(\gamma,\mu)}{P(\gamma,\mu)} \\ cs_2(\gamma) &= \frac{M(\gamma,\mu)\,H(\gamma,\mu)-N(\gamma,\mu)\,G(\gamma,\mu)}{P(\gamma,\mu)} \end{aligned}$$

## B.2. Computation of the function $f_\tau(\tau,t,x;l,u)$

Now we want compute the characteristic function $v(t,x,\mu)$:

$$v(t,x,\mu) := v(t,x,\mu;u,l,m) = \mathbf{E}_{0,x}\left[e^{i\mu\tau(t,x;u,l,m)}\right]$$
$$= \int_0^t e^{i\mu s} f_\tau(s,t,x;l,u,m)\,ds + 1\,\mathrm{Pr}_{0,x}[\tau(t,x;u,l,m)=0] + e^{i\mu t}\,\mathrm{Pr}_{0,x}[\tau(t,x;u,l,m)=t] \tag{B.8}$$

and the function $f_\tau(\tau,t,x;l,u,m)$ for $0<\tau<t$. We describe below the procedure that we can follow. We distinguish the case where $x<l$ or $x>u$ from the case $l<x<u$.

### B.2.1. Case $x<l$ or $x>u$

In the precedent section we have shown that the characteristic function is given by:

$$v(t,x,\mu) = \begin{cases} \mathbf{1}_{(x>u)} \begin{pmatrix} 1\times \mathrm{Pr}_{0,x>u}(\tau(t,x;u,l,m)=0)+ \\ +e^{\alpha x+\beta t}\int_0^t \frac{(x-u)}{\sqrt{2\pi(t-\theta)^3}} e^{-\frac{(x-u)^2}{2(t-\theta)}} y(\theta,1)\,d\theta \end{pmatrix} \\ \\ \mathbf{1}_{(x<l)} \begin{pmatrix} 1\times \mathrm{Pr}_{0,x>l}(\tau(t,x;u,l,m)=0)+ \\ +e^{\alpha x+\beta t}\int_0^t \frac{(l-x)}{\sqrt{2\pi(t-\theta)^3}} e^{-\frac{(l-x)^2}{2(t-\theta)}} y(\theta,0)\,d\theta \end{pmatrix} \end{cases} \tag{B.9}$$

and $\Pr_{0,x}[\tau(t,x;u,l,m)=t]=0$ if $x<l$ or $x>u$. Comparing (B.8) and (B.9), we have that:

$$\Omega(t,\mu,x) := \Omega(t,\mu,x;l,u,m) := \int_0^t e^{i\mu s} f_\tau(s,t,x;l,u,m)\,ds$$
$$= \begin{cases} \mathbf{1}_{(x>u)} e^{\alpha x+\beta t} \int_0^t \frac{(x-u)}{\sqrt{2\pi(t-\theta)^3}} e^{-\frac{(x-u)^2}{2(t-\theta)}} y(\theta,1)\,d\theta \\ \\ \mathbf{1}_{(x<l)} e^{\alpha x+\beta t} \int_0^t \frac{(l-x)}{\sqrt{2\pi(t-\theta)^3}} e^{-\frac{(l-x)^2}{2(t-\theta)}} y(\theta,0)\,d\theta \end{cases} \tag{B.10}$$

where the Laplace transform of the real and imaginary part of the functions $y(t,1)$ and $y(t,0)$ are given by:

$$\begin{aligned} \mathcal{L}[\mathrm{Re}(y(t,1));t\to\gamma] &= \frac{e^{-\alpha u}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)} - \frac{c}{2\sqrt{\gamma}}(s_1(\gamma)+d_1(\gamma)) \\ \mathcal{L}[\mathrm{Im}(y(t,1));t\to\gamma] &= -\frac{c}{2\sqrt{\gamma}}(s_2(\gamma)+d_2(\gamma)) \\ \mathcal{L}[\mathrm{Re}(y(t,0));t\to\gamma] &= \frac{e^{-\alpha l}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)} + \frac{c}{2\sqrt{\gamma}}(s_1(\gamma)-d_1(\gamma)) \\ \mathcal{L}[\mathrm{Im}(y(t,0));t\to\gamma] &= \frac{c}{2\sqrt{\gamma}}(s_2(\gamma)-d_2(\gamma)) \end{aligned} \tag{B.11}$$

In order to find the function $\Omega(t,\mu,x)$ is not convenient to invert the above expressions for the Laplace transforms of $y(t,0)$ and $y(t,1)$ and then numerically integrate them in (B.10), but to find the Laplace transform of the function $\Omega(t,\mu,x)$ using the convolution property. In this Laplace transform there will appear the quantities above. Then we can numerically invert it. In this way we can compute just one Laplace inversion and avoid the computation of the integrals in (B.10). Once we have found the function $\Omega(t,\mu,x)$ we can compute the function $f(\tau,t,x;l,u)$ through a new numerical inversion.

We can use the following Laplace transform formula, compare AS [2]:

$$\mathcal{L}\left[\frac{ae^{-\frac{a^2}{2t}}}{\sqrt{2\pi t^3}};t\to\gamma\right] = e^{-\sqrt{2}a\sqrt{\gamma}}$$

So using the convolution and the translation property of the Laplace transform, we have:

$$\mathcal{L}\left[\operatorname{Re}\left(\Omega\left(t,\mu,x\right)\right);t\rightarrow\gamma\right]$$
$$=\begin{cases}\mathbf{1}_{(x>u)}e^{\alpha x}e^{-\sqrt{2}(x-u)\sqrt{\gamma-\beta}}\left(\frac{e^{-\alpha u}}{\sqrt{\gamma-\beta}\left(\sqrt{\gamma-\beta}+\frac{\alpha}{\sqrt{2}}\right)}-\frac{c}{2\sqrt{\gamma-\beta}}\left(s_1\left(\gamma-\beta\right)+d_1\left(\gamma-\beta\right)\right)\right)\\ \\ \mathbf{1}_{(x<l)}e^{\alpha x}e^{-\sqrt{2}(l-x)\sqrt{\gamma-\beta}}\left(\frac{e^{-\alpha l}}{\sqrt{\gamma-\beta}\left(\sqrt{\gamma-\beta}-\frac{\alpha}{\sqrt{2}}\right)}+\frac{c}{2\sqrt{\gamma-\beta}}\left(s_1\left(\gamma-\beta\right)-d_1\left(\gamma-\beta\right)\right)\right)\end{cases} \tag{B.12}$$

whilst the Laplace transform of the imaginary part of the function $y\left(t,z\right)$ is given by:

$$\mathcal{L}\left[\operatorname{Im}\left(\Omega\left(t,\mu,x\right)\right);t\rightarrow\gamma\right]$$
$$=\begin{cases}-\mathbf{1}_{(x>u)}e^{\alpha x}\frac{e^{-\sqrt{2}(x-u)\sqrt{\gamma-\beta}}}{\sqrt{\gamma-\beta}}\frac{c}{2}\left(s_2\left(\gamma-\beta\right)+d_2\left(\gamma-\beta\right)\right)\\ \\ \mathbf{1}_{(x<l)}e^{\alpha x}\frac{e^{-\sqrt{2}(l-x)\sqrt{\gamma-\beta}}}{\sqrt{\gamma-\beta}}\frac{c}{2}\left(s_2\left(\gamma-\beta\right)-d_2\left(\gamma-\beta\right)\right)\end{cases} \tag{B.13}$$

So we obtain:

$$\operatorname{Re}\left(\Omega\left(t,\mu,x\right)\right)$$
$$=\begin{cases}\mathbf{1}_{(x>u)}e^{\alpha x+\beta t}\mathcal{L}^{-1}\left[\frac{e^{-\sqrt{2}(x-u)\sqrt{\gamma}}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}e^{-\alpha u}-\frac{e^{-\sqrt{2}(x-u)\sqrt{\gamma}}}{\sqrt{\gamma}}\frac{c}{2}\left(s_1\left(\gamma\right)+d_1\left(\gamma\right)\right);t\right]\\ \\ \mathbf{1}_{(x<l)}e^{\alpha x+\beta t}\mathcal{L}^{-1}\left[\frac{e^{-\sqrt{2}(l-x)\sqrt{\gamma}}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)}e^{-\alpha l}+\frac{e^{-\sqrt{2}(l-x)\sqrt{\gamma}}}{\sqrt{\gamma}}\frac{c}{2}\left(s_1\left(\gamma\right)-d_1\left(\gamma\right)\right);t\right]\end{cases} \tag{B.14}$$

and

$$\operatorname{Im}\left(\Omega\left(t,\mu,x\right)\right)$$
$$=\begin{cases}-\mathbf{1}_{(x>u)}e^{\alpha x+\beta t}\mathcal{L}^{-1}\left[\frac{e^{-\sqrt{2}(x-u)\sqrt{\gamma}}}{\sqrt{\gamma}}\frac{c}{2}\left(s_2\left(\gamma\right)+d_2\left(\gamma\right)\right);t\right]\\ \\ \mathbf{1}_{(x<l)}e^{\alpha x+\beta t}\mathcal{L}^{-1}\left[\frac{e^{-\sqrt{2}(l-x)\sqrt{\gamma}}}{\sqrt{\gamma}}\frac{c}{2}\left(s_2\left(\gamma\right)-d_2\left(\gamma\right)\right);t\right]\end{cases} \tag{B.15}$$

where the notation $\mathcal{L}^{-1}\left[f\left(\gamma\right);t\right]$ is used for the inverse Laplace transform of the function

$f(\gamma)$.

### B.2.2. Case $l < x < u$

In this case we have shown that the characteristic function is given by:

$$v(t,x,\mu) \\ = 1_{(l<x<u)}\left(e^{i\mu t}\,\mathrm{Pr}_{0,x}\left[\tau(t,x;u,l,m)=t\right]+e^{\alpha x+\beta t}\sum_{n=1}^{\infty}\varpi_n(t)\sin\left(n\pi\left(\tfrac{x-l}{u-l}\right)\right)\right) \tag{B.16}$$

where:

$$\begin{aligned}\varpi_n(t) &= 2n\pi c^2\int_0^t\left(\cos\mu(t-s)+i\sin\mu(t-s)\right)\\ &\quad\times e^{-n\pi c^2(t-s)}\left[y(s,0)-(-1)^n y(s,1)\right]ds\end{aligned}$$

Moreover $\mathrm{Pr}_{0,x}\left[\tau(t,x;u,l,m)=0\right]=0$ if $l<x<u$. Comparing (B.8) and (B.16), we have that:

$$\begin{aligned}&\Omega(t,\mu,x) := \textstyle\int_0^t e^{i\mu s}f_\tau(s,t,x;l,u)\,ds\\ &= \mathbf{1}_{(l<x<u)}e^{\alpha x+\beta t}\sum_{n=1}^{\infty}\varpi_n(t)\sin\left(n\pi\left(\tfrac{x-l}{u-l}\right)\right)\end{aligned} \tag{B.17}$$

Then if we consider the Laplace transform we have:

$$\mathcal{L}\left[\varpi_n(t);t\to\gamma\right] \\ = 2n\pi c^2\mathcal{L}\left[\left(\cos\mu t+i\sin\mu t\right)e^{-\lambda_n t};t\to\gamma\right]\mathcal{L}\left[y(t,0)-(-1)^n y(t,1);t\to\gamma\right]$$

and we obtain:

$$\mathcal{L}\left[\operatorname{Re}\left(\varpi_n\left(t\right)\right);t\to\gamma\right]$$

$$=2n\pi c^2\mathcal{L}\left[e^{-\lambda_n t}\cos\mu t;t\to\gamma\right]\left(\mathcal{L}\left[\operatorname{Re}\left(y\left(t,0\right)\right)-\left(-1\right)^n\operatorname{Re}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)$$

$$-2n\pi c^2\mathcal{L}\left[e^{-\lambda_n t}\sin\mu t;t\to\gamma\right]\left(\mathcal{L}\left[\operatorname{Im}\left(y\left(t,0\right)\right)-\left(-1\right)^n\operatorname{Im}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)$$

$$=n\pi c^2\left(\tfrac{1}{\lambda_n+\gamma-i\mu}+\tfrac{1}{\lambda_n+\gamma+i\mu}\right)\left(\mathcal{L}\left[\operatorname{Re}\left(y\left(t,0\right)\right)-\left(-1\right)^n\operatorname{Re}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)$$

$$-\tfrac{n\pi c^2}{i}\left(\tfrac{1}{\lambda_n+\gamma-i\mu}-\tfrac{1}{\lambda_n+\gamma+i\mu}\right)\left(\mathcal{L}\left[\operatorname{Im}\left(y\left(t,0\right)\right)-\left(-1\right)^n\operatorname{Im}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)$$

$$=\tfrac{n\pi c^2}{\lambda_n+\gamma-i\mu}\left(\mathcal{L}\left[\operatorname{Re}\left(y\left(t,0\right)\right);t\to\gamma\right]-\tfrac{1}{i}\mathcal{L}\left[\operatorname{Im}\left(y\left(t,0\right)\right);t\to\gamma\right]\right)$$

$$-\tfrac{(-1)^n n\pi c^2}{\lambda_n+\gamma-i\mu}\left(\mathcal{L}\left[\operatorname{Re}\left(y\left(t,1\right)\right);t\to\gamma\right]-\tfrac{1}{i}\mathcal{L}\left[\operatorname{Im}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)$$

$$+\tfrac{n\pi c^2}{\lambda_n+\gamma+i\mu}\left(\mathcal{L}\left[\operatorname{Re}\left(y\left(t,0\right)\right);t\to\gamma\right]+\tfrac{1}{i}\mathcal{L}\left[\operatorname{Im}\left(y\left(t,0\right)\right);t\to\gamma\right]\right)$$

$$-\tfrac{(-1)^n n\pi c^2}{\lambda_n+\gamma+i\mu}\left(\mathcal{L}\left[\operatorname{Re}\left(y\left(t,1\right)\right);t\to\gamma\right]+\tfrac{1}{i}\mathcal{L}\left[\operatorname{Im}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)$$

and:

$$\mathcal{L}\left[\operatorname{Im}\left(\varpi_n\left(t\right)\right);t\to\gamma\right]$$

$$=2n\pi c^2\mathcal{L}\left[e^{-\lambda_n t}\sin\mu t;t\to\gamma\right]\left(\mathcal{L}\left[\operatorname{Re}\left(y\left(t,0\right)\right)-\left(-1\right)^n\operatorname{Re}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)$$

$$+2n\pi c^2\mathcal{L}\left[e^{-\lambda_n t}\cos\mu t;t\to\gamma\right]\left(\mathcal{L}\left[\operatorname{Im}\left(y\left(t,0\right)\right)-\left(-1\right)^n\operatorname{Im}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)$$

$$=\tfrac{n\pi c^2}{i}\left(\tfrac{1}{\lambda_n+\gamma-i\mu}-\tfrac{1}{\lambda_n+\gamma+i\mu}\right)\left(\mathcal{L}\left[\operatorname{Re}\left(y\left(t,0\right)\right)-\left(-1\right)^n\operatorname{Re}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)$$

$$+n\pi c^2\left(\tfrac{1}{\lambda_n+\gamma-i\mu}+\tfrac{1}{\lambda_n+\gamma+i\mu}\right)\left(\mathcal{L}\left[\operatorname{Im}\left(y\left(t,0\right)\right)-\left(-1\right)^n\operatorname{Im}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)$$

$$=\tfrac{n\pi c^2}{\lambda_n+\gamma-i\mu}\left(\mathcal{L}\left[\operatorname{Im}\left(y\left(t,0\right)\right);t\to\gamma\right]+\tfrac{1}{i}\mathcal{L}\left[\operatorname{Re}\left(y\left(t,0\right)\right);t\to\gamma\right]\right)$$

$$-\tfrac{(-1)^n n\pi c^2}{\lambda_n+\gamma-i\mu}\left(\mathcal{L}\left[\operatorname{Im}\left(y\left(t,1\right)\right);t\to\gamma\right]+\tfrac{1}{i}\mathcal{L}\left[\operatorname{Re}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)$$

$$+\tfrac{n\pi c^2}{\lambda_n+\gamma+i\mu}\left(\mathcal{L}\left[\operatorname{Im}\left(y\left(t,0\right)\right);t\to\gamma\right]-\tfrac{1}{i}\mathcal{L}\left[\operatorname{Re}\left(y\left(t,0\right)\right);t\to\gamma\right]\right)$$

$$-\tfrac{(-1)^n n\pi c^2}{\lambda_n+\gamma+i\mu}\left(\mathcal{L}\left[\operatorname{Im}\left(y\left(t,1\right)\right);t\to\gamma\right]-\tfrac{1}{i}\mathcal{L}\left[\operatorname{Re}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)$$

So we have:

$$
\begin{aligned}
&\mathcal{L}\left[\operatorname{Re}\left(\Omega\left(t,\mu,x\right)\right);t\to\gamma\right]\\
&=\mathbf{1}_{(l<x<u)}e^{\alpha x}\sum_{n=1}^{\infty}\mathcal{L}\left[\operatorname{Re}\left(\varpi_n\left(t\right)\right);t\to\gamma-\beta\right]\sin\left(n\pi\left(\tfrac{x-l}{u-l}\right)\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathcal{L}\left[\operatorname{Im}\left(\Omega\left(t,\mu,x\right)\right);t\to\gamma\right]\\
&=\mathbf{1}_{(l<x<u)}e^{\alpha x}\sum_{n=1}^{\infty}\mathcal{L}\left[\operatorname{Im}\left(\varpi_n\left(t\right)\right);t\to\gamma-\beta\right]\sin\left(n\pi\left(\tfrac{x-l}{u-l}\right)\right)
\end{aligned}
$$

We can use the fact that, compare [12], page.. 40 as formula 1.445.1:

$$
\sum_{n=1}^{\infty}\frac{n\sin\left(nx\right)}{n^2+a^2}=\frac{\pi}{2}\frac{\sinh\left[a\left(\pi-x\right)\right]}{\sinh\left[a\pi\right]};0<x\leq 2\pi \tag{B.18}
$$

and formula 1.445.4:

$$
\sum_{n=1}^{\infty}\frac{\left(-1\right)^n n\sin\left(nx\right)}{n^2+a^2}=-\frac{\pi}{2}\frac{\sinh\left[ax\right]}{\sinh\left[a\pi\right]};-\pi\leq x\leq\pi \tag{B.19}
$$

If we define:

$$
\begin{aligned}
&\Psi_1\left(\gamma\right):=\sum_{n=1}^{\infty}\mathcal{L}\left[\operatorname{Re}\left(\varpi_n\left(t\right)\right);t\to\gamma\right]\sin\left(n\pi\left(\tfrac{x-l}{u-l}\right)\right)\\
&=\tfrac{\sinh\left[a\pi\left(\frac{u-x}{u-l}\right)\right]}{2\sinh\left[a\pi\right]}\left(\mathcal{L}\left[\operatorname{Re}\left(y\left(t,0\right)\right);t\to\gamma\right]-\tfrac{1}{i}\mathcal{L}\left[\operatorname{Im}\left(y\left(t,0\right)\right);t\to\gamma\right]\right)\\
&+\tfrac{\sinh\left[a\pi\left(\frac{x-l}{u-l}\right)\right]}{2\sinh\left[a\pi\right]}\left(\mathcal{L}\left[\operatorname{Re}\left(y\left(t,1\right)\right);t\to\gamma\right]-\tfrac{1}{i}\mathcal{L}\left[\operatorname{Im}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)\\
&+\tfrac{\sinh\left[b\pi\left(\frac{u-x}{u-l}\right)\right]}{2\sinh\left[b\pi\right]}\left(\mathcal{L}\left[\operatorname{Re}\left(y\left(t,0\right)\right);t\to\gamma\right]+\tfrac{1}{i}\mathcal{L}\left[\operatorname{Im}\left(y\left(t,0\right)\right);t\to\gamma\right]\right)\\
&+\tfrac{\sinh\left[b\pi\left(\frac{x-l}{u-l}\right)\right]}{2\sinh\left[b\pi\right]}\left(\mathcal{L}\left[\operatorname{Re}\left(y\left(t,1\right)\right);t\to\gamma\right]+\tfrac{1}{i}\mathcal{L}\left[\operatorname{Im}\left(y\left(t,1\right)\right);t\to\gamma\right]\right)
\end{aligned}
\tag{B.20}
$$

$$
a\pi=\sqrt{\tfrac{\gamma-i\mu}{c^2}};\,b\pi=\sqrt{\tfrac{\gamma+i\mu}{c^2}}
$$

and:

$$\begin{aligned}
\Psi_2(\gamma) &:= \sum_{n=1}^{\infty} \mathcal{L}\left[\mathrm{Im}\left(\varpi_n(t)\right); t \to \gamma\right] \sin\left(n\pi\left(\tfrac{x-l}{u-l}\right)\right) \\
&= \tfrac{\sinh\left[a\pi\left(\frac{u-x}{u-l}\right)\right]}{2\sinh[a\pi]} \left(\mathcal{L}\left[\mathrm{Im}\left(y(t,0)\right); t \to \gamma\right] + \tfrac{1}{i}\mathcal{L}\left[\mathrm{Re}\left(y(t,0)\right); t \to \gamma\right]\right) \\
&+ \tfrac{\sinh\left[a\pi\left(\frac{x-l}{u-l}\right)\right]}{2\sinh[a\pi]} \left(\mathcal{L}\left[\mathrm{Im}\left(y(t,1)\right); t \to \gamma\right] + \tfrac{1}{i}\mathcal{L}\left[\mathrm{Re}\left(y(t,1)\right); t \to \gamma\right]\right) \\
&+ \tfrac{\sinh\left[b\pi\left(\frac{u-x}{u-l}\right)\right]}{2\sinh[b\pi]} \left(\mathcal{L}\left[\mathrm{Im}\left(y(t,0)\right); t \to \gamma\right] - \tfrac{1}{i}\mathcal{L}\left[\mathrm{Re}\left(y(t,0)\right); t \to \gamma\right]\right) \\
&+ \tfrac{\sinh\left[b\pi\left(\frac{x-l}{u-l}\right)\right]}{2\sinh[b\pi]} \left(\mathcal{L}\left[\mathrm{Im}\left(y(t,1)\right); t \to \gamma\right] - \tfrac{1}{i}\mathcal{L}\left[\mathrm{Re}\left(y(t,1)\right); t \to \gamma\right]\right)
\end{aligned} \tag{B.21}$$

The Laplace transforms of the imaginary and real part of the function $y(t,0)$ and $y(t,1)$ are given in equation (B.11).

In conclusion, we have:

$$\Omega(t,\mu,x) := \int_0^t e^{i\mu s} f_\tau(s,t,x;l,u)\,ds$$

$$\mathrm{Re}\left(\Omega(t,\mu,x)\right) = \mathbf{1}_{(l<x<u)} e^{\alpha x+\beta t}\mathcal{L}^{-1}\left[\Psi_1(\gamma); t\right] \tag{B.22}$$

$$\mathrm{Im}\left(\Omega(t,\mu,x)\right) = \mathbf{1}_{(l<x<u)} e^{\alpha x+\beta t}\mathcal{L}^{-1}\left[\Psi_2(\gamma); t\right]$$

Inverting these two quantities we can obtain the function $f_\tau(\tau,t,x;l,u)$ when $l<x<u$.

# Chapter 3. Pricing of occupation time derivatives: continuous and discrete monitoring

## 1. Introduction

In the present work we examine the pricing problem for occupation time derivatives comparing the case of discrete and continuous time monitoring. The payoff of these contracts depends on the time spent by an index below a given level (hurdle or switch derivatives) or inside a band (corridor and Parisian derivatives and range notes). In particular, we examine the case of the corridor bond, bond where the coupon is proportional to the time spent inside a given band, and the corridor option that guarantees a minimum coupon. The structure of their payoff is similar to FX range floaters boost and step structures as described in Hull [17][1], Linetsky [20], Pechtl [24], Tucker et al.[30], Turnbull[31], Davydov and Linetsky [11] and Bregagnolio[6]. Chacko and Das [9] have examined the case of the Asian corridor bond.

The focus of the chapter is then to examine the differences between the price of the contract assuming continuous or, the more realistic, discrete time monitoring. Indeed the difference between the two prices can be relevant mainly when the residual life of the option is short or when the monitoring frequency is low (e.g. monthly). Moreover, as observed in Broadie and Glasserman [7] for barrier options, in Heynen and Kat [16]

[1] The bibliography is after the conclusions in the present chapter.

for lookback options and in several articles on *Derivative Week,* a sizable portion of real contracts specify fixed times for monitoring the asset and this can introduce substantial differences between discrete and continuous monitoring.

Assuming continuous time monitoring and describing the evolution of the index by a Geometric Brownian Motion, we can obtain a closed form solution for the density of the occupation time below a given level and the double Laplace transform for the density function of the time spent inside a band. We propose two numerical methods for this inversion. We remark that this is the first work in finance where the numerical inversion of a bidimensional Laplace transform is discussed. We can show that this inversion, at least in the case studied in this paper, can be performed very quickly and with a great accuracy. Moreover the numerical methods adopted are remarkably easy to understand and perform.

In the case of discrete time monitoring we could give the option price as an iterate integral (one integral for every monitoring dates), but this method, adopted for example in [16] for studying lookback options, can be very expensive. So we follow a Partial Differential Equation approach à la Black and Scholes along the lines described in Wilmott et al. [12] and [32]. In order to numerically solve the PDE, taking into account the actual performance of high-speed computers, we draw our attention toward a proper finite difference scheme that has to satisfy the following requirements: the numerical solution a) exists; b) is positive; c) converges to the exact one as the discretization steps tend to zero; d) is free of unwanted oscillations arising at the monitoring dates, due to the discontinuity introduced by the influx of new information. We believe that these are minimal properties because they respect the physical nature (i.e. financial) of the problem. For this reason, in the present work, we resort to an upwind finite-difference scheme for the first spatial derivative and we show through a numerical analysis that it guarantees all of these requirements independently from the discretization step and the parameter values. i.e. all properties are satisfied unconditionally. Finally, with numerical examples we confirm the significant price difference between the contracts that are monitored continuously and those that are monitored discretely.

In the second section we describe the main contracts. In the third section we show how to compute analytically the price for the corridor bond. In the fourth section we examine the pricing problem for the corridor option and we show the convenience of computing the density function of the occupation time assuming continuous time monitoring, whilst with discrete monitoring we prefer adopt a PDE approach. In the final section we compare the different methods. As Appendices we give the main results concerning the double Laplace transform of the density function and its inversion and the numerical analysis of the adopted scheme for the PDE. We discuss also, Appendix D, the so called digital corridor option, where at the expiry the holder of the option receives a fixed amount if the occupation time of the interval has been greater than a prefixed level.

## 2. The contracts

There are several forms of occupation time derivatives. The more common are the so called hurdle (or switch or range) and corridor derivatives, Hull [17], Pechtl [24], Tucker and Wei [30], Turnbull[31], Bregagnolio and Iori [6], Linetski [20]. Miura [21] and Akahori [2] introduce the quantile option. Davydov and Linetsky [11] in a recent and independent work extends the single barrier case to the double barrier step options.

Let us define $x$ be the index level at the current time $t$, and $Y(x, T, t; u, l)$ the time spent by the index inside the band $[l, u]$, in the time interval $[t; T]$. With continuous time monitoring, we can write:

$$Y(x, T, t; u, l) = \int_t^T 1_{(l<X_s<u)} ds$$

where $1_{(l<X_s<u)}$ stands for the indicator function of the set $[l, u]$. If $l = 0$, we are considering the time spent below the level $u$. Assuming discrete time monitoring, with $n$ monitoring dates $t_1$,..., $t_n$, where $t = t_0 < t_1 < ... < t_n = T$, we have:

$$Y(x, T, t; u, l) = \sum_{i=1}^{n} 1_{(l<X_{t_i}<u)} (t_i - t_{i-1})$$

A corridor bond pays at time $T$ the amount:

$$N * \frac{Y(x, T, t; u, l)}{T - t}$$

where $N$ is the nominal value of the bond. The corridor option guarantees a minimum amount $N * mc$ at the expiry, so the payoff is given by:

$$N * \max \left[ \frac{Y(x, T, t; u, l)}{T - t}; mc \right]$$

In the case of the hurdle derivative, we set $l = 0$.

In Appendix D we discuss the evaluation of a digital corridor option that pays at time $T$ the amount:

$$N * 1_{Y(x,T,t;u,l)>K}$$

i.e. a fixed amount $N$ if the occupation time is greater than $K$.

In all previous cases, if the lower barrier goes to zero we have hurdle derivatives.

## 3. The corridor bond

In order to price the corridor bond, we can use the well known fact that in absence of arbitrage opportunities the price is given by the expected value under the risk neutral measure of the discounted payoff, [15]. We can easily obtain the price of the contract either with continuous time monitoring either with discrete time monitoring.

### 3.1. Continuous and discrete time monitoring

The price is given by the expected value of the discounted payoff. Assuming a constant risk free interest rate $r$,we obtain:

$$\begin{aligned} E_{t,x}\left[e^{-r(T-t)} * N * \tfrac{Y(x,T,t;u,l)}{T-t}\right] &= e^{-r(T-t)} * \tfrac{N}{T-t} E_{t,x}\left[\int_t^T 1_{(l<X_s<u)} ds\right] \\ &= e^{-r(T-t)} * \tfrac{N}{T-t} \int_t^T E_{t,x}\left(1_{(l<X_s<u)}\right) ds \\ &= e^{-r(T-t)} * \tfrac{N}{T-t} \int_t^T \mathrm{Pr}_{t,x}\left(l < X_s < u\right) ds \end{aligned} \tag{3.1}$$

where in the second line we have used the Fubini's theorem for changing expectation and integral. The same result can be found in [24].

Assuming that the dynamic of the underlying price is described by a GBM, $dX = rXdt + \sigma XdW_t$, we obtain, taking the log of the prices:

$$\mathrm{Pr}_{t,x}\left(l < X_s < u\right) = \Phi\left(h\left(x,u,s-t\right)\right) - \Phi\left(h\left(x,l,s-t\right)\right)$$

where $\Phi\left(x\right) = \int_{-\infty}^{x} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw$ and $h\left(x,l,\tau\right) = \frac{1}{\sigma\sqrt{\tau}}\left(\ln\frac{l}{x} - \left(r - \frac{\sigma^2}{2}\right)\tau\right)$.

In the case of discrete monitoring, similarly to the previous case, setting $\Delta = t_{i+1} - t_i$, i.e. we have a constant distance between monitoring dates, we obtain:

$$\begin{aligned} &E_{t,x}\left[e^{-r(T-t)} * N * \tfrac{Y_d(x,T,t;u,l)}{T-t}\right] \\ &= e^{-r(T-t)} * \tfrac{N}{T-t} \sum_{i=1}^{n} \left(\Phi\left(h\left(x,u,t_i-t\right)\right) - \Phi\left(h\left(x,l,t_i-t\right)\right)\right)\Delta \end{aligned}$$

**Remark**: Recalling the fact that the local time at a level $x$ for a Standard Brownian Motion is the amount of time spent by the Brownian path in the vicinity of the point $x$, compare Karatzas and Shreve (KS henceforth) [18] pag. 203, we can obtain a representation of the occupation time in terms of integral of the local time respect to the space variable instead as integral of the indicator function respect to time. Using the representation of the semimartingale local time for a continuous semimartingale given in KS, pag. 218 formula (7.3), we have for every Borel measurable function $k : \Re \to [0, \infty)$:

$$\int_t^T k\left(X_s\right) d < X >_s = 2 \int_0^{+\infty} k(\xi) L_t(\xi) d\xi$$

where $< X >_s$ denotes the quadratic variation of the price process. In the case of GBM d$< X >_s = \sigma^2 X^2 ds$ and then choosing $k\left(X\right) = 1_{(l<X<u)}/\sigma^2 X^2$, we obtain:

$$\int_t^T 1_{(l<X_s<u)} ds = 2 \int\limits_0^{+\infty} \frac{1_{(l<\xi<u)}}{\sigma^2 \xi^2} L_t(\xi) d\xi = 2 \int\limits_l^u \frac{L_t(\xi)}{\sigma^2 \xi^2} d\xi$$

Unfortunately this representation does not seem to be useful for obtaining the price of the corridor option, because we should know the joint density law of the local times $L_t\left(\xi\right)$, for $l < \xi < u$. Obviously, for pricing the corridor bond, we need just the expected value of $L_t\left(\xi\right)$. Some more results, but with reference to the Brownian motion with no drift, can be found in Takacs [28]. Important results have been found in Carr and Jarrow [8] in the discussion of the stop loss strategy.

## 4. The corridor option

In order to compute the price for the options we can follow different ways:

1. find the density function of the r.v. $Y\left(x, T, t; u, l\right)$ and compute the expected value of the discounted payoff;

2. write and numerically solve the PDE describing the price of the contingent claim;

3. run a Monte Carlo simulation for the dynamics of the underlying asset and compute for every possible path the final value of the occupation time and the payoff of the contingent claim and then average the results of different paths.

The first approach results useful assuming continuous time monitoring. Unfortunately, there is no closed form expression for the density function, but we are able to obtain its double Laplace transform and in next section, we will see that from this double transform we can compute the double Laplace transform of the corridor option price. The second approach becomes preferable in the case of discrete time monitoring. The Monte Carlo simulation could be preferred in the case of discrete monitoring, because in the simulation the underlying can be checked only at discrete, albeit small, intervals.

### 4.1. Continuous time monitoring

Assuming continuous time monitoring can be realistic when the residual life of the contract is quite long or when the monitoring frequency is high (e.g. daily). In this case, it is easier to find the density function of the occupation time, rather than numerically solve a PDE with two state variables (asset price and occupation time) or to run a Monte Carlo simulation. The problem with the PDE is that we need to augment the state space introducing a new variable, locally risk-free, that takes into account the time spent inside the band. This fact makes the PDE a degenerate parabolic equation (the covariance matrix is singular) and the numerical method can suffer. The Monte Carlo method has a limit in the intrinsic discreteness of the simulation so we do not know if the process has crossed or not the barriers and we cannot compute the time spent inside the barriers during each step of the simulation[2]. For this reason it should be preferable to use the Monte Carlo

[2]For a related problem of pricing double barrier options, Baldi et al. [5] illustrate how to improve the Montecarlo method providing approximations for the exit probability from an interval. Other important

method in presence of discrete monitoring.

We observe that to find the time spent inside the band $[l, u]$ by the GBM $X_t$, $X_0 \equiv x$, is equivalent to find the time spent inside the band $[L \equiv \ln(l)/\sigma, U \equiv \ln(u)/\sigma]$ by the Arithmetic Brownian Motion (ABM), starting at $z = \ln(x)/\sigma$, with drift $m = \left(r - \sigma^2/2\right)/\sigma$ and unitary diffusion coefficient. In the following, we set $\tau = T - t$.

Using the Feynman-Kac formula, [18] chapter 4.4, Fusai [13] has obtained an expression for the Laplace transform with respect $\tau$ of a function appearing in the expression of the moments generating function $v(z, \tau, \mu)$ of the r.v. $Y$:

$$\begin{aligned}
v(z,\tau,\mu) &= \mathbf{E}_{t,z}\left[e^{-\mu Y(z,t+\tau,t;U,L)}\right] \qquad (4.1)\\
&= \int_t^{t+\tau} e^{-\mu y}\,\mathrm{Pr}_{t,z}\left(Y(z,t+\tau,t;U,L) \in dy\right) + 1 \times \mathrm{Pr}_{t,z}\left[Y(z,T,t;U,L) = 0\right]\\
&+ e^{-\mu(T-t)} \times \mathrm{Pr}_{t,z}\left[Y(z,T,t;U,L) = T-t\right]
\end{aligned}$$

where we have taken into account that the time spent inside the band can be equal to $T-t$ or equal to 0. In Theorem 1 in Appendix A we give the analytical expression for the second and third terms in the expression above[3] and the Laplace transform of the first term:

$$\omega(\gamma,\mu,z) \equiv \omega(\gamma,\mu,z;L,U,m) = \mathcal{L}\left[\Omega(z,\tau);\tau \to \gamma\right] = \int_0^{+\infty} e^{-\gamma\tau}\Omega(z,\tau)\,d\tau$$

where:

$$\Omega(z,\tau) \equiv \Omega(z,\tau;L,U,m) = \int_t^{t+\tau} e^{-\mu y}\,\mathrm{Pr}_{t,z}\left(Y(z,t+\tau,t;U,L) \in dy\right)$$

We can use then the expression for $\omega(\gamma,\mu,z)$ for obtaining the double Laplace trans-

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suggestions can be found in Andersen and Brotherton-Ratcliffe [4]. We do not have investigated, if the suggested methods can be used in the present problem.

[3] In the case of only one barrier, $L = -\infty$ ($l = 0$), the density function admits a closed form expression very easy to compute, compare Takacs [29] and Fusai [13] that simplify the results in Akahori [2].

form of the corridor option price. Indeed, we observe that we can write the undiscounted price of the corridor option with strike $K$ and residual live $\tau$ as:

$$C(\tau, K) \equiv C(\tau, K; x, t, l, u)$$
$$= \int_K^\tau (y - K) \Pr_{t,x}[Y(t+\tau, x; u, l) \in dy] + (\tau - K)^+ \times \Pr_{t,x\in(l,u)}[Y(t+\tau, x; u, l) = \tau]$$

and with some algebra, we obtain:

$$C(\tau, K)$$
$$= E_{t,x}[Y(t+\tau, x; u, l)] - K(1 - \Pr_{t,x}[Y(t+\tau, x; u, l) = 0])$$
$$+ \int_0^K (K - y) \Pr_{t,z}(Y(t+\tau, z; U, L) \in dy)$$

where $E_{t,x}[Y(t+\tau, x; u, l)]$, the expected value of the r.v. $Y(t+\tau, x; u, l)$, can be obtained by (3.1).

If we consider now the Laplace transform with respect to $K$ of the third term and we exploit the convolution property of the Laplace transform, we obtain:

$$\mathcal{L}\left[\int_0^K (K - y) \Pr_{t,z}(Y(t+\tau, z; U, L) \in dy); K \to \mu\right]$$
$$= \mathcal{L}[K; K \to \mu] \mathcal{L}[\Pr_{t,z}(Y(t+\tau, z; U, L) \in dK); K \to \mu]$$
$$= \frac{1}{\mu^2}\Omega(t, \mu, x; l, u, m)$$

and then $C(\tau, K)$ is given by:

$$C(\tau, K)$$
$$= E_{t,x}[Y(t+\tau, x; u, l)] - K(1 - \Pr_{t,x}[Y(t+\tau, x; u, l) = 0])$$
$$+\mathcal{L}^{-1}\left[\frac{\Omega(t,\mu,x;l,u,m)}{\mu^2}; \mu \to K\right]$$
$$= E_{t,x}[Y(t+\tau, x; u, l)] - K(1 - \Pr_{t,x}[Y(t+\tau, x; u, l) = 0])$$
$$+\mathcal{L}^{-1}\left[\frac{\omega(\gamma,\mu,x;l,u,m)}{\mu^2}; \mu \to K, \gamma \to t\right]$$

So the pricing of the corridor option requires the inversion of the double Laplace transform[4] of the quantity $\omega(\gamma, \mu, x; l, u, m) / \mu^2$. We can remark that in order to calculate the Greeks of the contract we can simply calculate the derivatives of the double Laplace transform and invert them.

As suggested in Abate and Whitt [1], for numerically computing the inverse of the above quantity, we have considered two different methods[5] in order to have a cross check on the results. The two methods are: a) the Fourier-series method firstly introduced for multidimensional transform inversion by Choudhury et al. [10], and b) the Padé approximation as suggested in Singhal et al. [25].

We describe the two inversion methods in Appendix B, whilst Tables 1-5 compare the inversion for different strikes, index, volatility and time to maturity. We can see that the two methods give results quite similar, sometimes up to the seventh digit. Using the Padé inversion technique, the inversion is performed very quickly, requiring less than one second. As described in Appendix B, the Fourier-series method, although a little slower, has the advantage of permitting a control of the different type of errors (aliasing, truncation and roundoff) that can arise in the numerical inversion. In every case, both inversion techniques appear very accurate.

We remark that we believe that this is the first paper in finance to use the numerical inversion of a multidimensional Laplace transform. Moreover with success[6]. The very good performance and the easy implementation of the proposed methods[7] contradicts the

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[4]For numerical purposes, it is convenient to divide in (8) and (9) the numerator and the denominator by $\sinh(a\pi)$ and to use the fact that $\tanh(a\pi/2) = (\cosh(a\pi) - 1) / \sinh(a\pi) = \sinh(a\pi) / (\cosh(a\pi) + 1)$

[5]We have also tried a third method presented in Moorthy [22] that makes use of the Fourier series representation, but the inversion was quite sensible to the choice of the parameters and so we do not report the results here.

[6]At the moment this paper was completed and submitted, we learned that the same multidimensional inversion algorithm proposed by CLW has been used by Davydov and Linetski [11] for pricing double step options.

[7]All calculations have been done on a Compaq Presario Notebook P233MMX. The code for the numerical inversion has been written in C using Microsoft Visual C++ 5.0. In the Padé inversion the poles and the residues have been previously computed using Mathematica 3.0.

common idea that to perform the numerical inversion of a Laplace transform is difficult to implement and represents "a well known ill-conditioned problem". This common misconcept is due, as well documented in Abate and Whitt [1], to one of the first tentatives of inversion that reduced the numerical inversion to the solution of an ill-conditioned linear system and so the diffusion of the idea that all available inversion techniques are unstable.

Finally, in table 6 we compare the prices of the corridor option, assuming continuous monitoring, obtained by the Laplace inversion (Padé inversion method) and MC simulation. From the we observe that the Monte Carlo simulation, without AV, is usually accurate to the third or fourth digit. Moreover the use of the antithetic variate does not seem to improve substantially the estimate[8]. Obviously the average computing time has been greatest for the MC simulation[9].

[8] We encountered a similar problem when we have used as control variate the price of the corridor bond. The standard error in this case increased.

[9] It took around 42' to run the 50000 simulations with five different starting points for the index, and considering simultaneously different monitoring frequencies (step by step, daily, weekly and monthly). We used 1200 steps for year.

# Price of the corridor option in the Choudhury et al. inversion method

## Lower Barrier =100, Upper Barrier =110, interest rate =0.05, volatility =0.2, time to maturity = 1yr

| n= | 20 | 20 | 38 | 50 | 50 |
|---|---|---|---|---|---|
| n+m= | 30 | 40 | 49 | 60 | 70 |
| K=0.2 | | | | | |
| Index Level | | | | | |
| 90 | 0.0463060 | 0.0463038 | 0.0463037 | 0.0463038 | 0.0463038 |
| 95 | 0.0792317 | 0.0792444 | 0.0792466 | 0.0792449 | 0.0792444 |
| 100 | 0.1246993 | 0.1247227 | 0.1246598 | 0.1247147 | 0.1247227 |
| 105 | 0.1469007 | 0.1469239 | 0.1469240 | 0.1469234 | 0.1469239 |
| 110 | 0.1161024 | 0.1161262 | 0.1160632 | 0.1161182 | 0.1161262 |
| 115 | 0.0735381 | 0.0735554 | 0.0735559 | 0.0735551 | 0.0735554 |
| 120 | 0.0457274 | 0.0457253 | 0.0457256 | 0.0457254 | 0.0457253 |
| Avg. CPU | 2"29/100 | 4"43/100 | 7"14/100 | 8"43/100 | 12"15/100 |
| K=0.6 | | | | | |
| Index Level | | | | | |
| 90 | 0.0009085 | 0.0009014 | 0.0009012 | 0.0009013 | 0.0009014 |
| 95 | 0.0026546 | 0.0026893 | 0.0026967 | 0.0026910 | 0.0026893 |
| 100 | 0.0066776 | 0.0067873 | 0.0065826 | 0.0067598 | 0.0067874 |
| 105 | 0.0093788 | 0.0094617 | 0.0094691 | 0.0094623 | 0.0094618 |
| 110 | 0.0062090 | 0.0063189 | 0.0061142 | 0.0062915 | 0.0063191 |
| 115 | 0.0026166 | 0.0026664 | 0.0026691 | 0.0026661 | 0.0026664 |
| 120 | 0.0010898 | 0.0010822 | 0.0010831 | 0.0010825 | 0.0010822 |
| Avg. CPU | 2"29/100 | 4" | 6" | 8"57/100 | 11"57/100 |

| n= | 20 | 20 | 38 | 50 | 50 |
|---|---|---|---|---|---|
| n+m= | 30 | 40 | 49 | 60 | 70 |
| K=0.4 | | | | | |
| Index Level | | | | | |
| 90 | 0.0101504 | 0.0101457 | 0.0101456 | 0.0101456 | 0.0101457 |
| 95 | 0.0213119 | 0.0213357 | 0.0213406 | 0.0213368 | 0.0213358 |
| 100 | 0.0399710 | 0.0400375 | 0.0399036 | 0.0400197 | 0.0400376 |
| 105 | 0.0502940 | 0.0503482 | 0.0503518 | 0.0503482 | 0.0503483 |
| 110 | 0.0372085 | 0.0372753 | 0.0371414 | 0.0372576 | 0.0372754 |
| 115 | 0.0202611 | 0.0202948 | 0.0202964 | 0.0202945 | 0.0202948 |
| 120 | 0.0107746 | 0.0107697 | 0.0107704 | 0.0107699 | 0.0107697 |
| Avg. CPU | 2"43/100 | 4"14/100 | 5"57/100 | 8"43/100 | 11"43/100 |
| K=0.8 | | | | | |
| Index Level | | | | | |
| 90 | 0.0000173 | 0.0000080 | 0.0000076 | 0.0000078 | 0.0000079 |
| 95 | 0.0000064 | 0.0000518 | 0.0000619 | 0.0000541 | 0.0000519 |
| 100 | 0.0000969 | 0.0002497 | -0.0000257 | 0.0002126 | 0.0002499 |
| 105 | 0.0003132 | 0.0004241 | 0.0004354 | 0.0004253 | 0.0004243 |
| 110 | 0.0000793 | 0.0002324 | -0.0000431 | 0.0001952 | 0.0002326 |
| 115 | -0.0000094 | 0.0000563 | 0.0000601 | 0.0000559 | 0.0000564 |
| 120 | 0.0000230 | 0.0000126 | 0.0000138 | 0.0000130 | 0.0000126 |
| Avg. CPU | 2"71/100 | 4" | 5"86/100 | 8"43/100 | 13"29/100 |

**Table 1**: In the first two lines n and m are the numbers of terms used in the Euler summation technique.
The inversion has been performed with A1=A2=20 and l1=l2=2.
In the table under the column "Price" there are the prices when the inversion is done setting n=50 and m+n=70 whilst in the other places appear the percentage difference

# Price of the corridor option with the Padé inversion method

## Lower Barrier =100, Upper Barrier =110, interest rate =0.05, volatility =0.2, time to maturity = 1yr

| **Num. Degree** | 1 | 2 | 3 | 3 | 4 |
|---|---|---|---|---|---|
| **Den. Degree** | 20 | 19 | 18 | 19 | 18 |
| K=0.2 | | | | | |
| Index Level | | | | | |
| 90 | 0.0463038 | 0.0463038 | 0.0463038 | 0.0463038 | 0.0463039 |
| 95 | 0.0792444 | 0.0792445 | 0.0792443 | 0.0792445 | 0.0792445 |
| 100 | 0.1247228 | 0.1247228 | 0.1247224 | 0.1247230 | 0.1247226 |
| 105 | 0.1469239 | 0.1469239 | 0.1469237 | 0.1469240 | 0.1469239 |
| 110 | 0.1161262 | 0.1161263 | 0.1161259 | 0.1161265 | 0.1161261 |
| 115 | 0.0735554 | 0.0735554 | 0.0735552 | 0.0735555 | 0.0735555 |
| 120 | 0.0457253 | 0.0457253 | 0.0457253 | 0.0457253 | 0.0457254 |
| Avg. CPU | 14/100 | 14/100 | 14/100 | 29/100 | 14/100 |

| **Num. Degree** | 1 | 2 | 3 | 3 | 4 |
|---|---|---|---|---|---|
| **Den. Degree** | 20 | 19 | 18 | 19 | 18 |
| K=0.4 | | | | | |
| Index Level | | | | | |
| 90 | 0.0101457 | 0.0101457 | 0.0101456 | 0.0101458 | 0.0101459 |
| 95 | 0.0213358 | 0.0213358 | 0.0213355 | 0.0213359 | 0.0213361 |
| 100 | 0.0400376 | 0.0400377 | 0.0400368 | 0.0400381 | 0.0400375 |
| 105 | 0.0503483 | 0.0503483 | 0.0503478 | 0.0503485 | 0.0503485 |
| 110 | 0.0372754 | 0.0372755 | 0.0372746 | 0.0372759 | 0.0372753 |
| 115 | 0.0202948 | 0.0202949 | 0.0202945 | 0.0202950 | 0.0202951 |
| 120 | 0.0107697 | 0.0107697 | 0.0107696 | 0.0107698 | 0.0107699 |
| Avg. CPU | 29/100 | 14/100 | 14/100 | 29/100 | 29/100 |

| **Num. Degree** | 1 | 2 | 3 | 3 | 4 |
|---|---|---|---|---|---|
| **Den. Degree** | 20 | 19 | 18 | 19 | 18 |
| K=0.6 | | | | | |
| Index Level | | | | | |
| 90 | 0.0009020 | 0.0009020 | 0.0009014 | 0.0009014 | 0.0009019 |
| 95 | 0.0026920 | 0.0026900 | 0.0026887 | 0.0026890 | 0.0026899 |
| 100 | 0.0067888 | 0.0067860 | 0.0067854 | 0.0067882 | 0.0067871 |
| 105 | 0.0094607 | 0.0094586 | 0.0094601 | 0.0094629 | 0.0094619 |
| 110 | 0.0063204 | 0.0063177 | 0.0063171 | 0.0063199 | 0.0063188 |
| 115 | 0.0026689 | 0.0026669 | 0.0026657 | 0.0026662 | 0.0026670 |
| 120 | 0.0010832 | 0.0010828 | 0.0010821 | 0.0010821 | 0.0010826 |
| Avg. CPU | 14/100 | 14/100 | 14/100 | 29/100 | 14/100 |

| **Num. Degree** | 1 | 2 | 3 | 3 | 4 |
|---|---|---|---|---|---|
| **Den. Degree** | 20 | 19 | 18 | 19 | 18 |
| K=0.8 | | | | | |
| Index Level | | | | | |
| 90 | 0.0000000 | 0.0000064 | 0.0000066 | 0.0000073 | 0.0000076 |
| 95 | 0.0000502 | 0.0000528 | 0.0000510 | 0.0000533 | 0.0000518 |
| 100 | 0.0002647 | 0.0002622 | 0.0002545 | 0.0002584 | 0.0002535 |
| 105 | 0.0004509 | 0.0004433 | 0.0004338 | 0.0004353 | 0.0004321 |
| 110 | 0.0002463 | 0.0002440 | 0.0002367 | 0.0002405 | 0.0002359 |
| 115 | 0.0000558 | 0.0000580 | 0.0000559 | 0.0000582 | 0.0000565 |
| 120 | 0.0000094 | 0.0000114 | 0.0000113 | 0.0000123 | 0.0000122 |
| Avg. CPU | 29/100 | 29/100 | 14/100 | 29/100 | 14/100 |

**Table 2**: In the first two lines num and den degree are the numerator and denominator degrees for the Padé approximants
In the table are given the prices when the inversion is done setting num=4 and den=18 whilst
in the other places appear 100 x the percentage difference

# Comparison between Padé and CLW (Fourier) Algorithms

## Lower Barrier =100, Upper Barrier =110, interest rate =0.05, volatility =0.5

| | Fourier -Series Inversion | | Padé inversion | Fourier -Series Inversion | | Padé inversion | Fourier -Series Inversion | | Padé inversion |
|---|---|---|---|---|---|---|---|---|---|
| | n=20 n+m=40 | n=50 n+m=70 | num=4 den=18 | n=20 n+m=40 | n=50 n+m=70 | n=4 m=18 | n=20 n+m=40 | n=50 n+m=70 | num=4 den=18 |
| Index Level | K = 0.2, T-t =1yr | | | K = 0.4, T-t =1yr | | | K = 0.6, T-t =1yr | | |
| 90 | 0.010357 | 0.010357 | 0.010357 | 0.000205 | 0.000205 | 0.000206 | 0.000000 | 0.000000 | 0.000001 |
| 95 | 0.013600 | 0.013600 | 0.013600 | 0.000319 | 0.000319 | 0.000319 | 0.000000 | 0.000000 | 0.000000 |
| 100 | 0.017445 | 0.017445 | 0.017445 | 0.000479 | 0.000479 | 0.000479 | 0.000000 | 0.000000 | 0.000001 |
| 105 | 0.019611 | 0.019611 | 0.019611 | 0.000575 | 0.000575 | 0.000575 | 0.000001 | 0.000001 | 0.000001 |
| 110 | 0.017951 | 0.017951 | 0.017951 | 0.000493 | 0.000493 | 0.000493 | 0.000000 | 0.000001 | 0.000001 |
| 115 | 0.014867 | 0.014867 | 0.014867 | 0.000357 | 0.000357 | 0.000357 | 0.000000 | 0.000001 | 0.000001 |
| 120 | 0.012330 | 0.012330 | 0.012331 | 0.000259 | 0.000259 | 0.000259 | 0.000000 | 0.000000 | 0.000001 |
| Avg. CPU | 3"86/100 | 11"43/100 | 29/100 | 4" | 11"29/100 | 14/100 | 3"86/100 | 11"29/100 | 14/100 |

| Index Level | K = 0.2, T-t =2 yr | | | K = 0.5, T-t =2 yr | | | K =0.8, T-t =2 yr | | |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.037567 | 0.037567 | 0.037567 | 0.001656 | 0.001656 | 0.001656 | 0.000008 | 0.000009 | 0.000009 |
| 95 | 0.044291 | 0.044291 | 0.044291 | 0.002146 | 0.002146 | 0.002146 | 0.000016 | 0.000016 | 0.000016 |
| 100 | 0.051560 | 0.051560 | 0.051560 | 0.002721 | 0.002721 | 0.002721 | 0.000018 | 0.000018 | 0.000018 |
| 105 | 0.055611 | 0.055611 | 0.055611 | 0.003041 | 0.003041 | 0.003041 | 0.000014 | 0.000014 | 0.000015 |
| 110 | 0.053056 | 0.053056 | 0.053056 | 0.002800 | 0.002800 | 0.002800 | 0.000019 | 0.000019 | 0.000019 |
| 115 | 0.047777 | 0.047777 | 0.047777 | 0.002341 | 0.002341 | 0.002342 | 0.000017 | 0.000017 | 0.000018 |
| 120 | 0.043089 | 0.043090 | 0.043090 | 0.001960 | 0.001960 | 0.001960 | 0.000010 | 0.000010 | 0.000011 |
| Avg. CPU | 3"86/100 | 11"43/100 | 14/100 | 3"86/100 | 11"29/100 | 14/100 | 3"86/100 | 11"43/100 | 14/100 |

**Table 3**: In the first two columns of each block there are the prices when the inversion is done with the CLW algorithm with setting a) n=20 and n+m=40 b) n=50 and m+n=70 respectively. In the third column using the Padé algorithm with num=4, den=18

# Comparison between Padé and CLW (Fourier) Algorithms

## Lower Barrier =100, Upper Barrier =110, interest rate =0.05, volatility =0.3152

| | Fourier -Series Inversion | | Padé inversion | Fourier -Series Inversion | | Padé inversion | Fourier -Series Inversion | | Padé inversion |
|---|---|---|---|---|---|---|---|---|---|
| | n=20 n+m=40 | n=50 n+m=70 | num=4 den=18 | n=20 n+m=40 | n=50 n+m=70 | n=4 m=18 | n=20 n+m=40 | n=50 n+m=70 | num=4 den=18 |
| Index Level | K = 0.2, T-t =1yr | | | K = 0.4, T-t =1 yr | | | K = 0.6, T-t =1 yr | | |
| 90 | 0.028317 | 0.028317 | 0.028317 | 0.002997 | 0.002997 | 0.002998 | 0.000065 | 0.000065 | 0.000066 |
| 95 | 0.040976 | 0.040976 | 0.040976 | 0.005179 | 0.005179 | 0.005180 | 0.000153 | 0.000153 | 0.000153 |
| 100 | 0.056857 | 0.056857 | 0.056857 | 0.008431 | 0.008431 | 0.008431 | 0.000326 | 0.000326 | 0.000326 |
| 105 | 0.065363 | 0.065363 | 0.065363 | 0.010332 | 0.010333 | 0.010333 | 0.000442 | 0.000443 | 0.000443 |
| 110 | 0.056839 | 0.056839 | 0.056839 | 0.008429 | 0.008429 | 0.008429 | 0.000326 | 0.000326 | 0.000326 |
| 115 | 0.042835 | 0.042835 | 0.042835 | 0.005534 | 0.005534 | 0.005534 | 0.000169 | 0.000169 | 0.000170 |
| 120 | 0.032163 | 0.032163 | 0.032163 | 0.003620 | 0.003620 | 0.003620 | 0.000088 | 0.000088 | 0.000088 |
| Avg. CPU | 3"86/100 | 11"29/100 | 14/100 | 3"86/100 | 11"29/100 | 14/100 | 3"86/100 | 11"29/100 | 14/100 |

| Index Level | K = 0.2, T-t =2 yr | | | K = 0.5, T-t =2 yr | | | K =0.8, T-t =2 yr | | |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.084109 | 0.084109 | 0.084109 | 0.014306 | 0.014306 | 0.014307 | 0.000967 | 0.000968 | 0.000968 |
| 95 | 0.104684 | 0.104684 | 0.104684 | 0.019569 | 0.019569 | 0.019569 | 0.001513 | 0.001513 | 0.001514 |
| 100 | 0.127560 | 0.127560 | 0.127560 | 0.026005 | 0.026005 | 0.026005 | 0.002278 | 0.002278 | 0.002278 |
| 105 | 0.139097 | 0.139097 | 0.139096 | 0.029420 | 0.029420 | 0.029420 | 0.002713 | 0.002713 | 0.002714 |
| 110 | 0.127521 | 0.127521 | 0.127521 | 0.025997 | 0.025997 | 0.025997 | 0.002277 | 0.002277 | 0.002277 |
| 115 | 0.107476 | 0.107476 | 0.107476 | 0.020328 | 0.020328 | 0.020328 | 0.001599 | 0.001599 | 0.001599 |
| 120 | 0.090625 | 0.090625 | 0.090625 | 0.015922 | 0.015922 | 0.015923 | 0.001127 | 0.001128 | 0.001128 |
| Avg. CPU | 3"86/100 | 11"29/100 | 14/100 | 3"71/100 | 11"14/100 | 14/100 | 3"86/100 | 11"57/100 | 14/100 |

**Table 4**: In the first two columns of each block there are the prices when the inversion is done with the CLW algorithm with setting a) n=20 and n+m=40 b) n=50 and m+n=70 respectively. In the third column using the Padé algorithm with num=4, den=18

# Comparison between Padé and CLW (Fourier) Algorithms

Lower Barrier =100, Upper Barrier =110, interest rate =0.05, volatility =0.2

| | Fourier -Series Inversion | | Padé inversion | Fourier -Series Inversion | | Padé inversion | Fourier -Series Inversion | | Padé inversion |
|---|---|---|---|---|---|---|---|---|---|
| | n=20 n+m=40 | n=50 n+m=70 | num=4 den=18 | n=20 n+m=40 | n=50 n+m=70 | n=4 m=18 | n=20 n+m=40 | n=50 n+m=70 | num=4 den=18 |
| Index Level | K = 0.2, T-t =1yr | | | K = 0.4, T-t = 1 yr | | | K = 0.6, T-t =1 yr | | |
| 90 | 0.046304 | 0.046304 | 0.046304 | 0.010146 | 0.010146 | 0.010146 | 0.000901 | 0.000901 | 0.000902 |
| 95 | 0.079244 | 0.079244 | 0.079245 | 0.021336 | 0.021336 | 0.021336 | 0.002689 | 0.002689 | 0.002690 |
| 100 | 0.124723 | 0.124723 | 0.124723 | 0.040037 | 0.040038 | 0.040037 | 0.006787 | 0.006787 | 0.006787 |
| 105 | 0.146924 | 0.146924 | 0.146924 | 0.050348 | 0.050348 | 0.050349 | 0.009462 | 0.009462 | 0.009462 |
| 110 | 0.116126 | 0.116126 | 0.116126 | 0.037275 | 0.037275 | 0.037275 | 0.006319 | 0.006319 | 0.006319 |
| 115 | 0.073555 | 0.073555 | 0.073555 | 0.020295 | 0.020295 | 0.020295 | 0.002666 | 0.002666 | 0.002667 |
| 120 | 0.045725 | 0.045725 | 0.045725 | 0.010770 | 0.010770 | 0.010770 | 0.001082 | 0.001082 | 0.001083 |
| Avg. CPU | 3"86/100 | 11"43/100 | 14/100 | 3"86/100 | 13"29/100 | 29/100 | 4" | 11"29/100 | 14/100 |

| Index Level | K = 0.2, T-t =2 yr | | | K = 0.5, T-t =2 yr | | | K =0.8, T-t =2 yr | | |
|---|---|---|---|---|---|---|---|---|---|
| 90 | 0.140240 | 0.140240 | 0.140240 | 0.044762 | 0.044762 | 0.044762 | 0.009078 | 0.009078 | 0.009078 |
| 95 | 0.189989 | 0.189990 | 0.189990 | 0.067194 | 0.067194 | 0.067194 | 0.015731 | 0.015731 | 0.015732 |
| 100 | 0.247316 | 0.247316 | 0.247316 | 0.095816 | 0.095816 | 0.095816 | 0.025495 | 0.025495 | 0.025495 |
| 105 | 0.271130 | 0.271130 | 0.271130 | 0.108983 | 0.108983 | 0.108983 | 0.030494 | 0.030494 | 0.030495 |
| 110 | 0.230267 | 0.230267 | 0.230267 | 0.089206 | 0.089206 | 0.089206 | 0.023736 | 0.023736 | 0.023736 |
| 115 | 0.171638 | 0.171638 | 0.171638 | 0.061464 | 0.061464 | 0.061464 | 0.014644 | 0.014644 | 0.014645 |
| 120 | 0.127418 | 0.127418 | 0.127418 | 0.042140 | 0.042140 | 0.042140 | 0.008982 | 0.008982 | 0.008982 |
| Avg. CPU | 3"86/100 | 11"29/100 | 14/100 | 4" | 11"43/100 | 14/100 | 3"86/100 | 12" | 14/100 |

**Table 5**: In the first two columns of each block there are the prices when the inversion is done with the CLW algorithm with setting a) n=20 and n+m=40 b) n=50 and m+n=70 respectively. In the third column using the Padé algorithm with num=4, den=18

**Table 6. Price for the corridor option**

Parameters setting: L=100, U=110, r=0.05, $\sigma$ =0.2, $\tau$=1yr, K=0.2 ; MC: 50000 simulations, 1200 steps

| *Index level* | 90 | 100 | 105 | 110 | 120 |
|---|---|---|---|---|---|
| Padé (n=4, m=18) | 0.04630 | 0.12472 | 0.14692 | 0.11613 | 0.04573 |
| Monte Carlo | 0.04635 | 0.12474 | 0.14751 | 0.11642 | 0.04607 |
| std. error*1000 | 0.427 | 0.670 | 0.711 | 0.660 | 0.435 |
| Monte Carlo +AV | 0.04661 | 0.12495 | 0.14728 | 0.11583 | 0.04607 |
| std. error*1000 | 0.265 | 0.518 | 0.665 | 0.358 | 0.271 |

### 4.2. Discrete time monitoring

In the case of discrete time monitoring, we could try to use the pricing formula for the continuous time case in order to approximate the solution in the discrete time case, but this approximation could work well only if the monitoring frequency is high or if the option has a long time to expiry. So we prefer to discuss alternative approaches: a PDE method and a Monte Carlo simulation. Another possibility is to express the option price as a multiple iterate integral as done in Heynen and Kat [16] for lookback options: one integral for each monitoring date, so that with monthly monitoring, we should iterate over twelve integrals and the computational effort and the time required can be quite high. A referee signalled that recently Aitsahlia and Lai [3] have considerably improved this recursive numerical integration procedure exploiting the duality property of random walks.

#### 4.2.1. The PDE approach

Along the lines in Wilmott et al. [32] and [12], we can show that the price $V(x, \tau, y)$ of the corridor option satisfies the following system of PDE's:

$$-\frac{\partial V(x,\tau,y)}{\partial \tau}+rx\frac{\partial V(x,\tau,y)}{\partial x}+\frac{\sigma^2}{2}x^2\frac{\partial^2 V(x,\tau,y)}{\partial x^2}=rV(x,\tau,y) \tag{4.2}$$

where $\tau \geq 0, 0 < x < +\infty$ $0 \leq y \leq n$, where we have redefined the time $t$ in time to expiry of the option, $\tau = T - t$ [10] and $n$ gives the number of monitoring dates, $n = [\tau/\Delta]$, with $\Delta$ the time distance between two consecutive monitoring dates. The initial condition to be satisfied is[11]:

$$V(x,0,y)=\max(y-K;0)*\Delta \tag{4.3}$$

In the PDE's above $y$ is an integer number representing the number of times the index has spent inside the band at the monitoring dates. We treat (4.2) as a system of PDE's indexed on $y$: between monitoring dates $y$ is fixed, and, for each fixed $y$, $V(x,\tau,y)$ evolves according to (4.2). So we could solve different PDE simultaneously, one for each possible value of $y$, until the next monitoring date. An arbitrage argument, Wilmott et al. [32], requires that the option price has to be continuous across the monitoring dates i.e.:

$$V\left(x,\tau_j^+,y^+\right)=V\left(x,\tau_j^-,y^-\right)$$

or if $y^+ = y$:

$$V\left(x,\tau_j^+,y\right)=V\left(x,\tau_j^-,y+1_{(l<x<u)}\right) \tag{4.4}$$

The "jump condition" (4.4) links at the monitoring dates PDE's for different values of

[10] As a consequence, if the times $t_1$, ..., $t_n$ are the monitoring dates, now they correspond to $\tau_i = T - t_i$. So $\tau_0 = 0$, and when the time to expiry of the option is $\tau_{i-1} \leq \tau < \tau_i$ it means that we have again $i$ monitoring dates and the residual life of the contract is $i * \Delta$.

[11] The initial condition depends on the fact that we can write:

$$N*\max\left[\frac{y}{(T-t)};mc\right]*\Delta=\frac{N}{T-t}*(\max[y-K;0]+K)*\Delta$$

where $K = (T-t)\,mc$. So the payoff can be viewed as $N/(T-t)$ call options on $y$ plus the same quantity of bonds of nominal value $K$. In the following we will concentrate the attention on the quantity $\max[y-K;0]*\Delta$.

$y$: depending on the position of the index respect to the band, there will be an exchange of information between the different PDE's for adjacent values of $y$, as illustrated in Figure 1.

Figure 2 illustrates the updating process that occurs after one third of year and after four monitoring dates, assuming $K = 2.4$. The PDE indexed with $y = 0$ at the monitoring date receives new information from the PDE indexed with $y = 1$ if the index is inside the barriers, otherwise updates itself. Similarly, the PDE indexed with $y = 1$ ($y = 2$) receives new information from the PDE indexed with $y = 2$ ($y = 3$) if the index is inside the barriers, otherwise updates itself. The PDE with $y = [K] + 1$ does not need an updating because at the expiry the option will be surely exercised ($y > K$), so we have an analytical expression for it (compare eq. 4.7 below). The updated values for each PDE can be then used to start to solve again separately the different PDE's between monitoring dates.

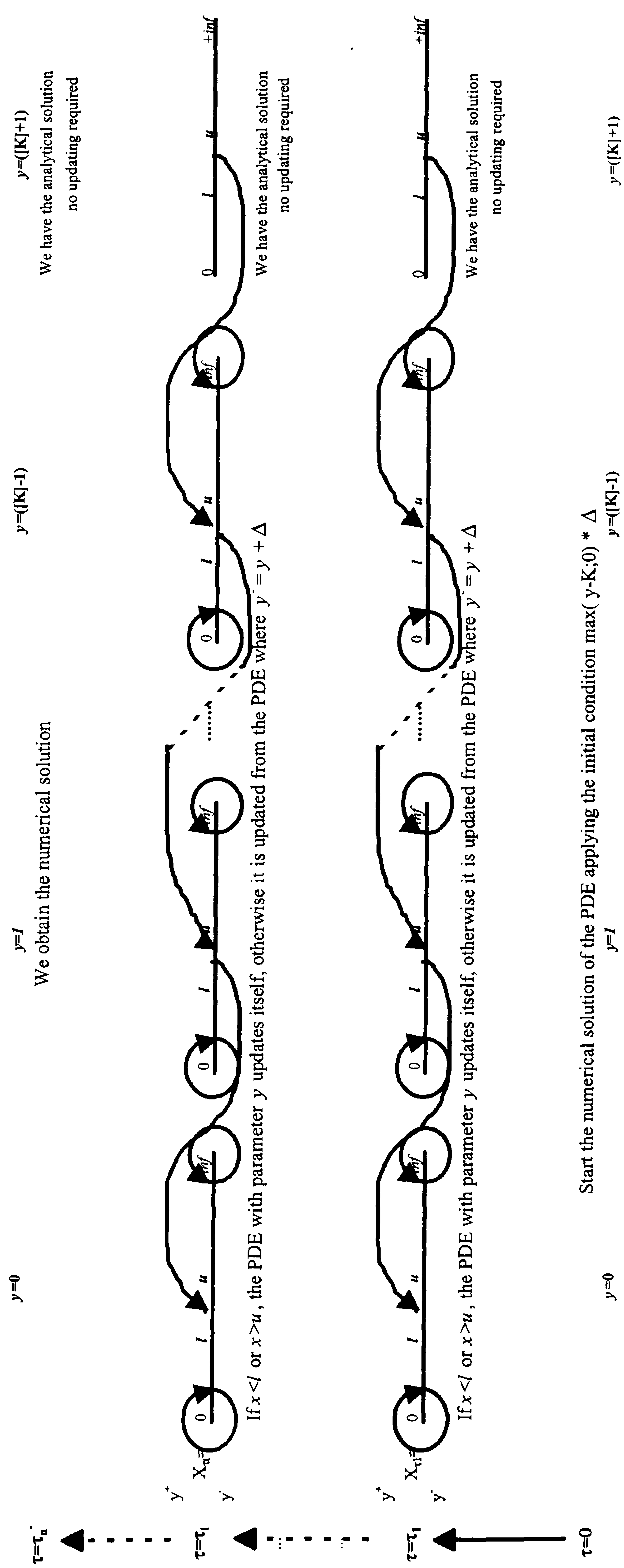



**Figure 1**: Updating process for the Corridor Option with Discrete Time Monitoring

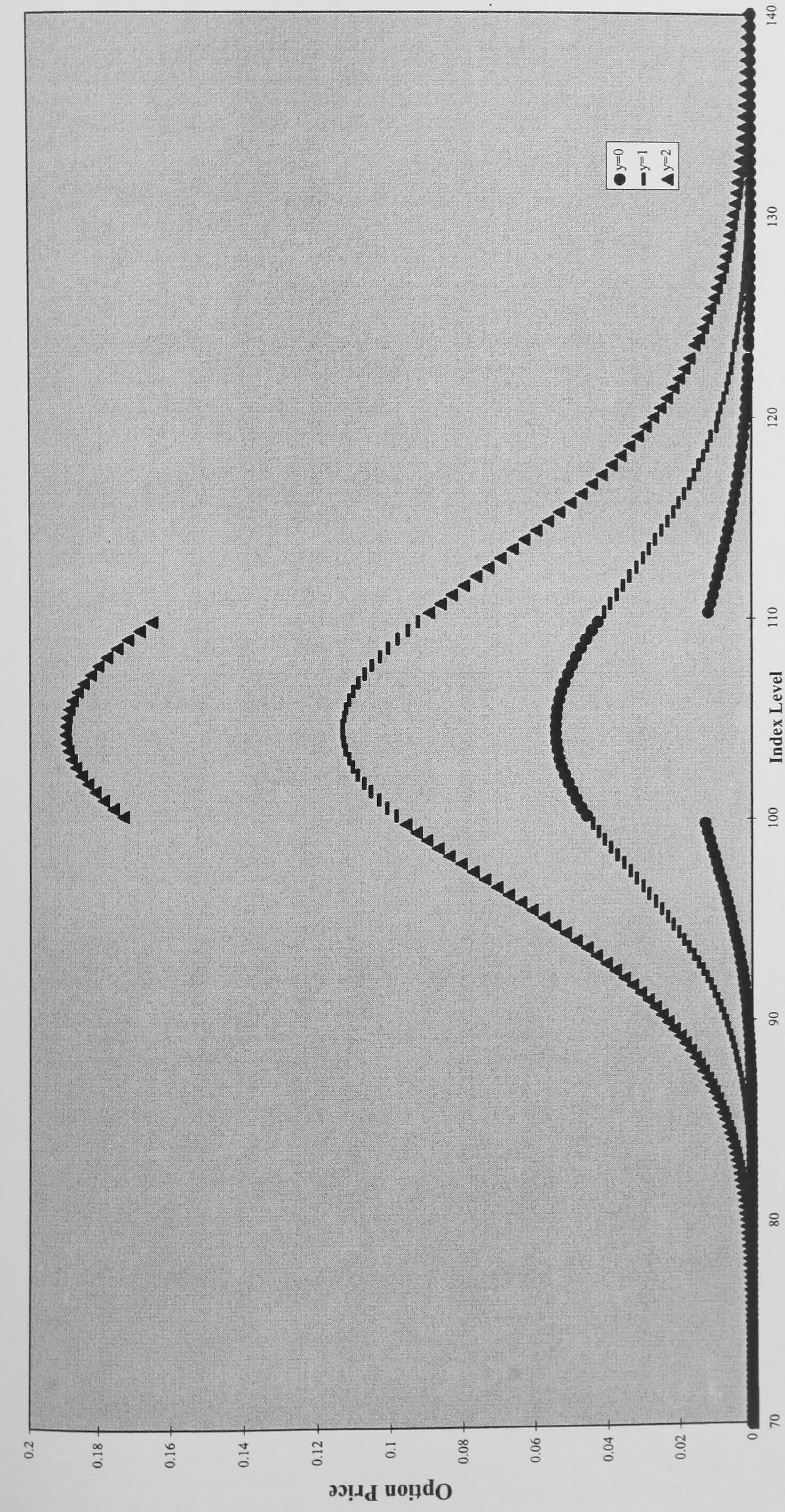



**Figure 2**: The jump condition for the updating process after four monitoring dates and for different values of the parameter $y$ ($r$=0.05, σ=0.2, K=2.4, $l$=100, $u$=110, $\tau$ = 1/3 years)

Regarding the boundary conditions, we observe that for a GBM process, if $x = 0$ the price will be forever equal to 0, and then the stock cannot spend any more time inside the interval and the final occupation time will be equal to $y$. So we know with certainty the final payoff and we obtain the condition:

$$V(0, \tau, y) = e^{-r\tau} \max(y - K; 0) * \Delta \tag{4.5}$$

Similarly, for the case $x \to +\infty$, we will get:

$$V(+\infty, \tau, y) = e^{-r\tau} \max(y - K; 0) * \Delta \tag{4.6}$$

In conclusion, we have to solve the PDE (4.2), with initial condition (4.3) and boundary conditions (4.5) and (4.6) and continuity condition (4.4) at times $\tau_i$. The computational domain is $\tau > 0, 0 < x < +\infty$.

Let us observe that when $y \geq [K] + 1 \geq K$, where $[K]$ is the greatest integer equal or less than $K$, and when the time to maturity is $\tau$ the option will be surely exercised, it is like a corridor bond, and we have an analytical solution given by:

$$V(x, \tau, y) = e^{-r\tau} \left( (y - K) \Delta + \sum_{i=1}^{n} \left( \Phi(h(x, u, i * \Delta)) - \Phi(h(x, l, i * \Delta)) \right) \Delta \right) \tag{4.7}$$

where $\Phi(x) = \int_{-\infty}^{x} \frac{e^{-\frac{w^2}{2}}}{\sqrt{2\pi}} dw$ and $h(x, l, \tau) = \frac{1}{\sigma\sqrt{\tau}} \left( \ln(l/x) - \left( r - \frac{\sigma^2}{2} \right) \tau \right)$. This remark allows us to solve only a restricted number of PDE's instead of having to solve a PDE for each possible value of $y$. For example with monthly monitoring and with a strike equal to 2.4, we need to solve just three equations (corresponding to $y = 0, 1$and 2) instead of thirteen and the computational time is not prohibitive[12].

[12]To solve numerically the PDE with monthly monitoring took around 1'05" for 1000 different index levels and 300 discretization steps respect to $\tau$ and K=12*0.2=2.4. If we increase the strike price, we need

At the monitoring dates $\tau_i$, taking into account the known solution for $y > K$, the updating condition can be written:

$$V\left(x,\tau_j^+,y\right) = \begin{cases} V\left(x,\tau_j^-,y\right) & x \notin [l,u] \\ \\ V\left(x,\tau_j^-,y+1\right) & x \in [l,u] \end{cases}$$
$$\text{when } y = 0,1,...,[K]-1,\ j=1,...,n$$

and:

$$V\left(x,\tau_j^+,[K]\right) = \begin{cases} V\left(x,\tau_j^-,[K]\right) & x \notin [l,u] \\ \\ V\left(x,\tau_j^-,[K]+1\right) & x \in [l,u] \end{cases}$$

with $j = 1,...,n$.

When the time to maturity is $\tau$, i.e. there are again $n$ monitoring times, the expression $V\left(x,\tau_j^-,[K]+1\right)$ is given by (4.7). When we have again $j$ monitoring dates, $j = n-1$, $n-2$, ..., 2, 1, the analytical solution is always given by (4.7), once we have set $\tau = \tau_{j,}y = [K]+1$, $n = j$. When $\tau_0 = 0$ the option is expired and we apply the initial condition $\max(y-k;0)*\Delta$.

We remark that the change of information between PDE's implies that, for the nature of the updating of the occupation time, we are introducing a "discontinuity" in the new initial condition to be used at every monitoring date as well illustrated in figure (2). This fact could generate unwanted spurious oscillations in the numerical solution if a scheme is adopted without paying attention to the nature of the problem. For example a Crank-Nicholson scheme is usually the preferred one, just because it is reputed unconditionally

solve more PDE: e.g. if K=12*0.4=4.8, the time required becomes 1'50". If we increase the monitoring frequency, also the computational time increases: e.g. with daily monitoring, K=0.2 and 150 discretization steps respect to $\tau$, it took more than 2h30'. But in this case it is more convenient to approximate the solution using the continuous time formula. The code for the numerical solution of the PDE has been written in Fortran.

stable and its local precision is maximal for PDE's of the parabolic type. But if adopted in the present context, this natural choice forgets the nature of the problem. Indeed as shown in Smith [26], pag. 122-124, "numerical studies indicate that very slowly decaying finite oscillations can occur with the Crank-Nicholson method in the neighborhood of discontinuities in the initial values...". So for our problem it is necessary to devote some attention in choosing the numerical scheme because to adopt a Crank-Nicholson scheme without a preliminary analysis could be very dangerous[13]. In effect doing a numerical analysis similar to that one in Appendix C, it is easily proved that the absence of spurious obscillations in the Crank-Nicholson scheme requires the following relationship between spatial and time step:

$$\frac{\Delta\tau}{(\Delta x)^2} \leq \frac{16}{\sigma^2}$$

and depending on the values of the volatility, this restriction can become very demanding. For example if we choose $\Delta x$ =0.01 (i.e. 100 state steps) and $\sigma = 0.2$, we have that $\Delta\tau \leq$0.04 (i.e. 25 time steps between two monitoring dates), whilst if $\Delta x$ =0.001 (i.e. 1000 state steps) then we need to require $\Delta\tau \leq$0.0004 (i.e. 2500 time steps) and higher accuracy can be obtained only with a very small time step.

For this reason, we would like to use a proper numerical method that guarantees the respect of some minimal conditions very natural from the financial point of view: a) the solution exists, b) it is positive, c) it converges to the exact one as the discretization steps tend to zero, d) it is free of unwanted oscillations arising at the prefixed updating dates, due to exchange of information between PDE's. In particular in Appendix C we describe a proper numerical scheme that can be used in the present case. Through a detailed numerical analysis, we can show that all above requirements are satisfied unconditionally and

[13] We know of only a paper by Zvan and al. [33] documenting the limits of a Crank-Nicholson scheme when applied to the pricing of barrier options.

in particular the oscillations due to discontinuities in the updating process are eliminated very quickly and do not perturb the numerical solution. The cost to be paid is a lower accuracy than in the Crank-Nicholson: for the proposed scheme the order of convergence is $o(\Delta\tau, \Delta x)$.

### 4.2.2. The Monte Carlo simulation

The advantage of using Monte Carlo simulation is its practicality, in the sense that is easy to understand and applicable without effort to different problems and assuming more general processes for the underlying. Moreover in the case of discrete monitoring the bias due to the underestimation of the maximum and overestimation of the minimum does not appear, because the simulation can be devised to match the actual observation dates. But as remarked in Andersen and Brotherton-Ratcliffe [4] simulation could become prohibitively lengthy if there are too many monitoring dates, although this problem is common to the PDE approach as well. Moreover, when the dimensionality of the problem (number of state variables) is low as in the present case, a finite-difference method can be preferable because more accurate. But the real advantage of using a PDE approach is that we can solve simultaneously for different initial index levels (1000 points in the numerical examples) and we can calculate quite easily and with great accuracy the Greeks of the contract. In particular the numerical solution can be of support in calculating an hedging strategy that in the case of exotic options appear very important, whilst in the case of the Monte Carlo simulation this is not possible. For example in the present case, particular care has to be devoted to the calculation of the gamma near the barriers, because it presents a jump.

Table 7 compares the results from the Monte Carlo simulation (with and without control variate) and the numerical solution of the PDE, assuming monthly monitoring. The

MC estimate has a higher standard error in this case than it did for continuous monitoring (Table 6), even though the latter could not match the observation dates. Moreover also in this case, the use of the Antithetic Variate reduces the standard error, but it appears to give a more biased estimate respect to the numerical solution of the PDE. Also the use of the corridor bond as Control Variate was not useful in reducing the standard error and we do not report the relative results.

**Table 7: Price for the corridor option**

Parameters setting: K=0.2 and L=100, U=110, r=0.05, $\sigma$ =0.2, $\tau$=1yr.

PDE: $(x, \tau)$-grid=1000x300; MC: 50000 simulations, steps matching the monitoring dates

| *index level* | PDE (monthly) | MC; (1000*s.e.) | MC+AV; (1000*s.e.) |
|---|---|---|---|
| 90 | 0.05235 | 0.05328;(0.482) | 0.05355;(0.298) |
| 100 | 0.12120 | 0.12161;(0.697 ) | 0.12207;(0.541) |
| 105 | 0.13281 | 0.13404;( 0.726) | 0.13351;(0.659) |
| 110 | 0.11328 | 0.11387;(0.687) | 0.11316;(0.387) |
| 120 | 0.05160 | 0.05218;(0.487) | 0.05234;( 0.304) |

## 5. A comparison between discrete and continuous monitoring

In this section we compare the results coming from the inversion of the Laplace transform, with the numerical solution of the PDE.

From table 8 and figure 3 we can appreciate the difference between the price with continuous (inverse Laplace) and discrete time monitoring (PDE). If we consider the index level varying in the interval 85-125, we can see that the percentage difference between continuous and monthly monitoring can go from +10% to -11%, depending on the position of the index respect to the barriers. This difference is greatly reduced if we compare daily and continuous monitoring.

In every case the price of the contract with continuous time monitoring is the highest (lowest) when the index is inside (outside) the band[14]. This fact is due to the nature of the contract: if the index is inside the band, and we assume continuous time monitoring, then the passage of the time will increase the value of the contract until the moment in which the index crosses the barriers. Instead if we assume discrete time monitoring, we cannot exploit completely the passage of time: we register the position of the index only at discrete dates and if they are quite distant (e.g. a month), it is possible that the index in the meantime has moved outside the band and so the occupation time cannot increase. In this case the time between two monitoring dates can be entirely lost if the index moves outside the band. Viceversa if we are outside the band, and between two monitoring dates the index moves inside the band the occupation time increases by the distance $\Delta$ : the contract earns the entire time distance between monitoring dates. Instead with continuous time monitoring we lose every instant until the process crosses the barriers. So the continuous time formula will overvalue (undervalue) the discrete time formula when the index is inside (outside) the band. Then it will be important to distinguish between discrete and continuous monitoring time.

In figure 4 and table 9 we illustrate the effect of the monitoring frequency on the delta of the contract: in this case we can see that higher the monitoring frequency, higher the absolute value of the delta. So the monitoring frequency assumes importance for replicating the contract, mainly when the index level is near the barriers. We also investigated if it is useful to shift the barriers in the continuous time formula in order to get an accurate price for the case of discrete monitoring. This is for example the idea in Broadie et al. [7]: they examine the pricing problem for several kinds of barrier and lookback options and they show that shifting in an opportune way the barrier in the continuous formula

[14]The result depends also on the value of the drift of the process, so that the overvaluation (undervaluation) does not occur exactly at the extremes of the band

they can get an accurate price for the case of discrete time monitoring. Basically the idea we pursued was to find the lower barrier that equates the price of the corridor bond with discrete and continuous time monitoring and then use it as input in the double Laplace transform for obtaining an approximate price for the discrete corridor option. The percentage difference between the numerical solution of the PDE and the price obtained by this procedure was reduced for example from 11% to around the 3% when the index level was around 90, but in general this procedure was not completely satisfactory. So more work has to be done in this case.

## 6. Conclusion

In the present paper we have examined the effect of the monitoring frequency on the price and the delta of corridor derivatives, using either a numerical solution of a PDE either the double inverse Laplace transform. We have shown that the two methods can be fruitfully coupled. Indeed as the monitoring frequency increases, it can be too expensive to solve the PDE and so it becomes more convenient to approximate the solution using the inverse Laplace transform: the difference between the prices and deltas with daily and continuous monitoring is very small. Viceversa for low monitoring frequency we can use the numerical solution of the PDE. In both cases, as check test we can always use Monte Carlo simulation, but it is much more expensive relatively to the time required and the standard variance reduction techniques do not seem work very well. It remains to be investigated if we can approximate the discrete time formula using the much more efficient continuous time formula using a shift argument similar to that used in Broadie et al.[7].

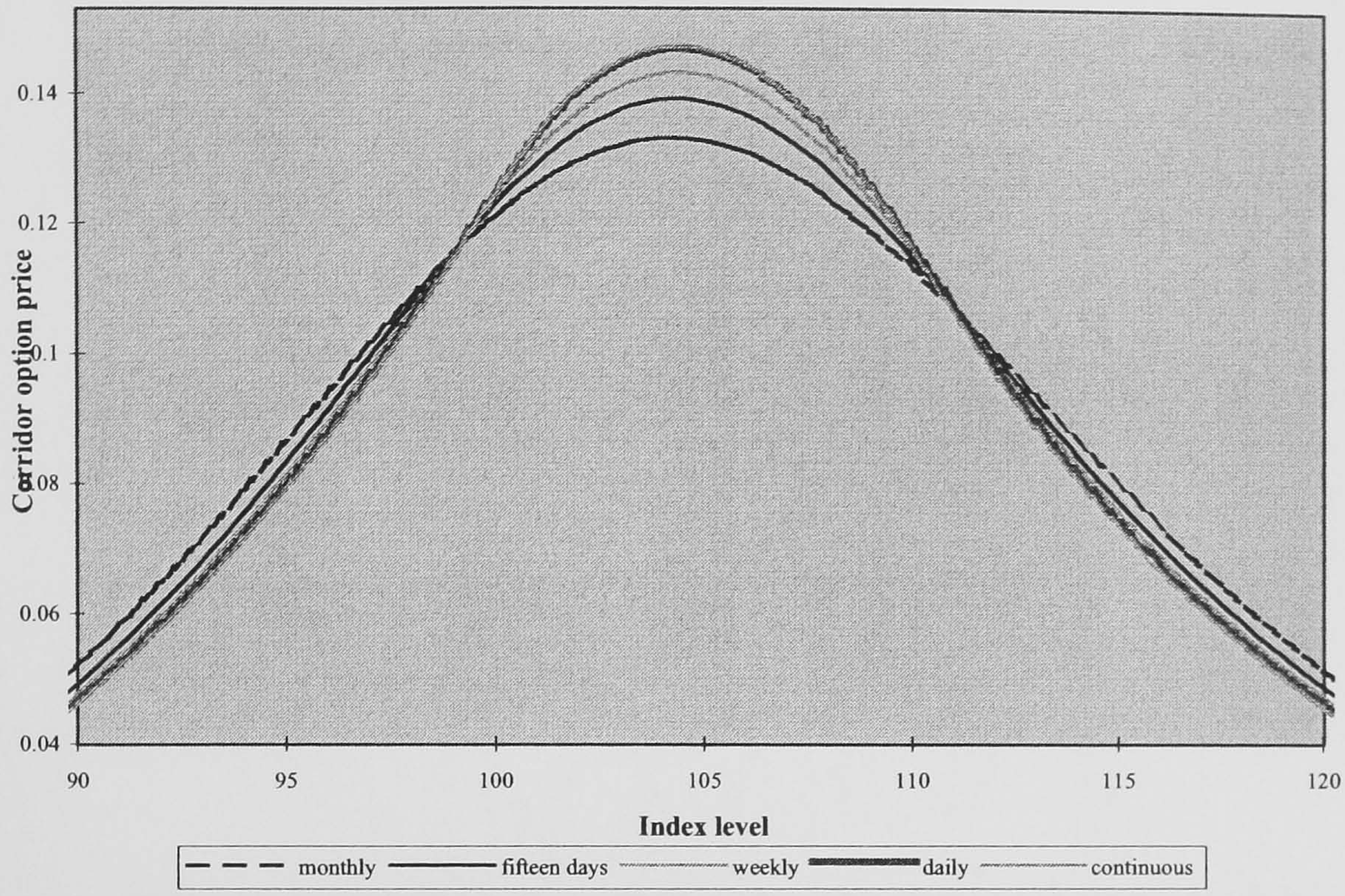



**Figure 3:** Price of the corridor option and monitoring frequency
(l=100, u=110, K*mon. freq=0.2, $\tau$=1, r=5%, $\sigma$=0.2)

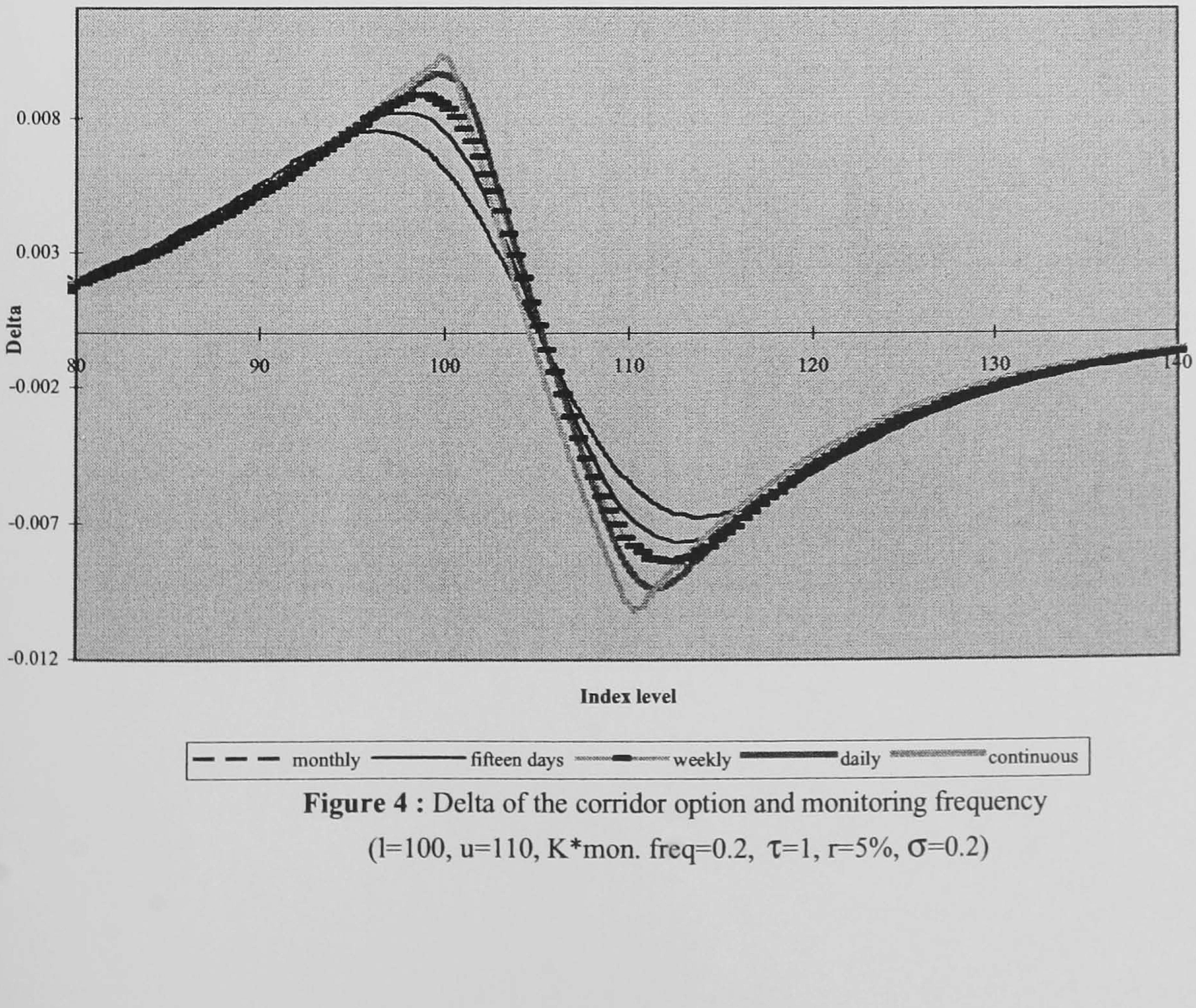



**Figure 4 :** Delta of the corridor option and monitoring frequency
(l=100, u=110, K*mon. freq=0.2, $\tau$=1, r=5%, $\sigma$=0.2)

**Table 8. Effect of Monitoring Frequency on the Prices**

Parameters setting: L=100, U=110, $\sigma$=0.2, r=0.05, $\tau$=1yr, K*mon.frequency=0.2.

| | Monitoring Frequency | | | | |
|---|---|---|---|---|---|
| | % difference respect to cont. monitoring | | | | Price |
| Index Level | Monthly | Beweekly | Weekly | Daily | Continuos |
| 80 | 19.52% | 10.81% | 5.58% | 1.33% | 0.0115 |
| 85 | 15.96% | 8.23% | 4.15% | 1.29% | 0.0245 |
| 90 | 13.05% | 6.53% | 3.29% | 0.92% | 0.0463 |
| 95 | 9.28% | 4.95% | 2.47% | 0.70% | 0.0793 |
| 100 | -2.81% | -1.45% | -0.64% | 0.04% | 0.1247 |
| 102 | -7.74% | -4.66% | -2.57% | -0.37% | 0.1406 |
| 105 | -9.58% | -5.54% | -2.77% | -0.25% | 0.1469 |
| 108 | -7.31% | -4.25% | -2.17% | -0.02% | 0.1338 |
| 110 | -2.45% | -1.04% | -0.21% | 0.59% | 0.1161 |
| 115 | 9.09% | 5.45% | 3.14% | 1.37% | 0.0736 |
| 120 | 12.85% | 6.72% | 3.66% | 1.69% | 0.0457 |
| 125 | 14.95% | 8.15% | 4.57% | 1.75% | 0.0279 |

**Table 9. Effect of Monitoring Frequency on the Delta**

Parameters setting: L=100, U=110, $\sigma$=0.2, r=0.05, $\tau$=1yr, K*mon.frequency=0.2.

| | Monitoring Frequency | | | | |
|---|---|---|---|---|---|
| | % difference respect to cont. monitoring | | | | Delta |
| Index Level | Monthly | Beweekly | Weekly | Daily | Continuos |
| 80 | 10.34% | 10.34% | 10.34% | -3.71% | 0.0019 |
| 85 | 11.62% | 2.98% | 2.98% | 2.98% | 0.0034 |
| 90 | 7.58% | 2.46% | -1.81% | -1.81% | 0.0054 |
| 95 | -3.65% | -0.29% | -1.35% | -2.50% | 0.0078 |
| 100 | -45.94% | -32.73% | -21.92% | -9.90% | 0.0104 |
| 102 | -39.72% | -20.00% | -4.33% | 6.91% | 0.0056 |
| 105 | -42.81% | -29.84% | -42.81% | -42.81% | -0.0013 |
| 108 | -38.10% | -25.22% | -15.57% | -6.74% | -0.0072 |
| 110 | -43.82% | -31.45% | -21.25% | -10.32% | -0.0104 |
| 115 | -3.20% | 1.28% | 3.13% | 3.13% | -0.0069 |
| 120 | 12.77% | 8.07% | 8.07% | 3.37% | -0.0044 |
| 125 | 13.20% | 6.90% | 6.90% | 6.07% | -0.0028 |

## References

[1] Abate J. and W. Whitt (1992), The Fourier-series method for inverting transforms of probability distributions, *Queueing Systems* 10, pp. 5-88.

[2] Akahori, J. (1995). Some formulae for a new type of path-dependent options, *Ann. Appl. Probab.* 5, 383-388.

[3] Aitsahlia F. and T. L. Lai (1998), Random walk Duality and the Valuation of Discrete Lookback Options, *Applied Mathematical Finance*, vol. 5, n. 3, 227-240.

[4] Andersen L. and R. Brotherton-Ratcliffe (1996), Exact Exotics, *Risk* Vol.9, No. 10, October.

[5] Baldi, P., L. Caramellino and M. G. Iovino (1998), Pricing General Barrier Options: a Numerical Approach using Sharp Large Deviations, *Mathematical Finance*, vol. 4, no. 4, 293-321.

[6] Bregagnolio T., Iori L. (1995), Their name is Bond, *Investire*, pp. 88-92, January (in Italian).

[7] Broadie M., Glasserman P. and Kou S. (1997), A Continuity Correction for Discrete Barrier Options, *Mathematical Finance*, Vol. 7, no. 4, October.

[8] Carr, P. and R. A. Jarrow (1990), The Stop-Loss Start-Gain Paradox: A new Decomposition into Intrinsec and Time Value, *The Review of Financial Studies*, Vol. 3, no. 3, pagg. 469-492.

[9] Chacko G. and S. R. Das (1997), Average Interest, w.p. Harvard Business School, May.

[10] Choudhury G. L., Lucantoni D. M, and Whitt W. (1994), Multidimensional Transform Inversion with Applications to the Transient M/G/1 Queue, *Ann. Appl. Probab.* 5, 389-398.

[11] Davydov D. and V. Linetsky (1999), Structuring, pricing and hedging double barrier step options, w.p. University of Michigan, November.

[12] Dewynne, J.N. and Wilmott, P. (1995), A note on average rate options with discrete sampling, *SIAM J. Appl. Math.* Vol. 55, No. 1, pp. 267-276, February.

[13] Fusai, G. (2000), Corridor Options and Arc-Sine Law, *Annals of Applied Probability,* vol. 10, n.2, may 2000, 634-663.

[14] Fusai, G. (2000), Applications of Laplace Transform for Evaluating Occupation Time Options and other Derivatives, PH.D. Thesis, Warwick University.

[15] Harrison, J. M. and Pliska, S. R. (1981), Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Process. Appl.*, Vol. 15, 214-260.

[16] Heynen, R. C. and H. M. Kat (1995), Lookback options with discrete and partial monitoring of the underlying price, *Applied Mathematical Finance*, Vol. 2, No. 4, December, pp. 273-284.

[17] Hull, J. (1997), *Options, Futures, and Other Derivatives.* Englewood Cliffs, NJ: Prenctice Hall.

[18] Karatzas I, and S.E. Shreve (1997), Brownian Motion and Stochastic Calculus, Springer, 2nd ed.

[19] Lévy, P. (1939), Sur certain processus stochastiques homogènes, Compositio Math., Vol. 7, pages 283-339.

[20] Linetski, V. (1996), Step options: The Feynman-Kac Approach to occupation time derivatives. University of Michigan Dept. of Industrial and Operations Engineering Technical Report 96-18.

[21] Miura, R. (1992), A Note on Look-Back Option based on Order Statistics, *Hitotsubashi Journal of Commerce and Management*, Vol. 27, pp. 15-28.

[22] Moorthy, M. V. (1995), Numerical Inversion of two-dimensional Laplace transforms - Fourier series representation, *Applied Numerical Mathematics*, Vol. 17, pp. 119-127.

[23] Ortega, J.M. (1998), *Matrix Theory*, Plenum Press, New York.

[24] Pechtl, A. (1995), Classified Information, in *Over the Rainbow*, ed by Robert Jarrow, Risk Publications, pag. 71-74.

[25] Singhal, K., Vlach, J. and Vlach M. (1975), Numerical inversion of multidimensional Laplace transforms, *Proc. IEEE* 63, 1627-1628.

[26] Smith, G. D. (1985), *Numerical solution of partial differential equations: finite difference methods*, Oxford University Press.

[27] Stoer, J. and R. Bulirsch (1980), *Introduction to Numerical Analysis*, Springer Verlag, New York.

[28] Takacs, L. (1995), On the local time of the Brownian motion, *The Annals of Applied Probability*, vol. 5, No. 3, pagg. 741-756.

[29] Takacs, L. (1996) On a generalization of the Arc-Sine Law, *The Annals of Applied Probability*, Vol. 6, No. 3, pagg. 1035-1040.

[30] Tucker, A. L. and Wei J. Z. (1997), The Latest Range, *Advances in Futures and Options Research*, Vol. 9, pages 287-296.

[31] Turnbull, S. M. (1995), Interest Rate Digital Options and Range Notes, *Journal of Derivatives* 3, 92-101.

[32] Wilmott, P. Dewynne, J.N. and Howison, S. (1993), *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press

[33] Zvan, R., K. R. Vetzal and P. A. Forsyth (1997), PDE methods for barrier options, University of Waterloo Department of Computer Science Technical Report CS-97-27.

## A. The double Laplace transform of the density law

If we define the moment generating function (mgf) $v(z,\tau,\mu)$ of the r.v. $Y$

$$\begin{aligned}
&v(z,\tau,\mu) \equiv v(z,\tau,\mu;m,U,L) = \mathbf{E}_{t,z}\left[e^{-\mu Y(z,t+\tau,t;U,L)}\right] \qquad \text{(A.1)}\\
&= \int_t^{t+\tau} e^{-\mu y}\,\mathrm{Pr}_{t,z}\left(Y(z,t+\tau,t;U,L) \in dy\right) + 1 \times \mathrm{Pr}_{t,z}\left[Y(z,T,t;U,L) = 0\right]\\
&+ e^{-\mu(T-t)} \times \mathrm{Pr}_{t,z}\left[Y(z,T,t;U,L) = T-t\right]
\end{aligned}$$

it can be shown that, using the Feynman-Kac formula (Karatzas and Shreve [18] chapter 4.4), the function $v(z,\tau,\mu)$ satisfies the following PDE:

$$-\frac{\partial v(t,z)}{\partial t} + \frac{1}{2}\frac{\partial^2 v(t,z)}{\partial^2 z} + m\frac{\partial v(t,z)}{\partial z} - \mu \mathbf{1}_{(L<z<U)} v(t,z) = 0 \qquad \text{(A.2)}$$

with initial condition:

$$v(z,0) = 1, \forall z \in (-\infty;+\infty) \qquad \text{(A.3)}$$

and boundary conditions:

$$v(\pm\infty,\tau) = 1, \forall \tau > 0 \qquad \text{(A.4)}$$

The above PDE can be solved taking the Laplace transform with respect to time and solving the second differential equation requiring continuity and differentiability of the solution at the points $L$ and $U$ and boundedness of the solution at $\pm\infty$. This requires to solve a linear system with four equations and four unknowns. A more efficient way for obtaininig the solution is presented in Fusai [13].

In Theorem 1 below, given a function $G(\tau)$, we denote with $g(\gamma)$ its Laplace transform with respect to the variable $\tau = T - t$:

$$g(\gamma) \equiv \mathcal{L}[G(\tau); \tau \to \gamma] = \int_0^T e^{-\gamma\tau} G(\tau)\, d\tau$$

and with $\mathcal{L}^{-1}[g(\gamma); \gamma \to \tau]$ its inverse Laplace transform. We have:

**Theorem A.1.** *:The Laplace transform (moment generating function) of the density law of the occupation time of the interval $[L, U]$ by the ABM has the following representation:*

$$v(z,\tau,\mu) = \Omega(z,\tau; L, U, m) \\ + \begin{cases} 1 \times \Pr_{0,z\in[U,+\infty)}\left(\inf\limits_{0\le s\le\tau} m * s + W(s) > U\right) \\ \\ e^{-\mu\tau} \times \Pr_{0,z\in(L,U)}\left(\sup\limits_{0\le s\le\tau} m * s + W(s) < U; \inf\limits_{0\le s\le t} m * s + W(s) > L\right) \\ \\ 1 \times \Pr_{0,z\in(-\infty,L]}\left(\sup\limits_{0\le s\le\tau} m * s + W(s) < L\right) \end{cases} \tag{A.5}$$

*where:*

$$\Pr_{0,z\in[U,+\infty)}\left(\inf_{0\le s\le\tau} m * s + W(s) > U\right) \\ = 1 - \left[\tfrac{1}{2}\operatorname{Erfc}\left(\tfrac{z-U+m\tau}{\sqrt{2\tau}}\right) + \tfrac{e^{-2m(z-U)}}{2}\operatorname{Erfc}\left(\tfrac{z-U-m\tau}{\sqrt{2\tau}}\right)\right];$$

$$\Pr_{0,z\in(L,U)}\left(\sup_{0\le s\le\tau} m * s + W(s) < U; \inf_{0\le s\le t} m * s + W(s) > L\right) \\ = 2e^{-mz-\frac{m^2}{2}\tau} \int_0^1 \left[\sum_{n=1}^{\infty} e^{-(cn\pi)^2\tau} \sin(n\pi z)\sin(n\pi\xi)\right] e^{m(\xi(U-L)+L)} d\xi;$$

$$\Pr_{0,z\in(-\infty,L]}\left(\sup_{0\le s\le\tau} m*s+W(s)<L\right)$$
$$=1-\left[\frac{1}{2}\,\mathrm{Erfc}\left(\frac{L-z-m\tau}{\sqrt{2\tau}}\right)+\frac{e^{2m(L-z)}}{2}\,\mathrm{Erfc}\left(\frac{L-z+m\tau}{\sqrt{2\tau}}\right)\right]$$

*and:*

$$\begin{aligned}\Omega(z,\tau;L,U,m) &\equiv \int_0^\tau e^{-\mu y}\Pr_{t,z}\left(y(z,\tau,0;U,L)\in dy\right)\\ &= e^{-mz-\frac{m^2}{2}\tau}\mathcal{L}^{-1}\left[\varpi(\gamma,z;L,U,m);\tau\right]\end{aligned}$$

$$\varpi(\gamma,z;L,U,m)$$
$$=\begin{cases}\mathbf{1}_{(z\ge U)}e^{-\sqrt{2}(z-U)\sqrt{\gamma}}\mathcal{L}\left[A(\tau);\gamma\right]\\[2ex] \mathbf{1}_{(L<z<U)}\frac{\mathcal{L}[A(\tau);\gamma]\sinh\left(a\pi\left(\frac{U-z}{U-L}\right)\right)+\mathcal{L}[B(\tau);\gamma]\sinh\left(a\pi\left(\frac{z-L}{U-L}\right)\right)}{\sinh(a\pi)}\\[2ex] \mathbf{1}_{(z\le L)}e^{-\sqrt{2}(L-z)\sqrt{\gamma}}\mathcal{L}\left[B(\tau);\gamma\right]\end{cases}$$

$$\mathcal{L}\left[A(\tau);\tau\to\gamma\right]=\frac{e^{-\alpha U}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}-\frac{c}{2\sqrt{\gamma}}\left(s(\gamma)+d(\gamma)\right)$$
$$\mathcal{L}\left[B(\tau);\tau\to\gamma\right]=\frac{e^{-\alpha L}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)}+\frac{c}{2\sqrt{\gamma}}\left(s(\gamma)-d(\gamma)\right)$$

*with:*

$$\frac{c}{\sqrt{\gamma}}d(\gamma)=\frac{\sqrt{\gamma+\mu}\sinh(a\pi)}{\left(\sqrt{\gamma+\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)+1)\right)}\left(\frac{e^{-\alpha U}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}+\frac{e^{-\alpha L}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)}+\frac{\left(\frac{\alpha}{\sqrt{2}}\left(e^{-\alpha L}-e^{-\alpha U}\right)(\cosh(a\pi)+1)-\sqrt{\gamma+\mu}\left(e^{-\alpha U}+e^{-\alpha L}\right)\sinh(a\pi)\right)}{\sqrt{\gamma+\mu}\sinh(a\pi)\left(\gamma+\mu-\frac{\alpha^2}{2}\right)}\right)$$

$$\frac{c}{\sqrt{\gamma}}s(\gamma)=\frac{\sqrt{\gamma+\mu}\sinh(a\pi)}{\sqrt{\gamma+\mu}\sinh(a\pi)+\sqrt{\gamma}(\cosh(a\pi)-1)}\left(\frac{e^{-\alpha U}}{\sqrt{\gamma}\left(\sqrt{\gamma}+\frac{\alpha}{\sqrt{2}}\right)}-\frac{e^{-\alpha L}}{\sqrt{\gamma}\left(\sqrt{\gamma}-\frac{\alpha}{\sqrt{2}}\right)}-\frac{\left(\frac{\alpha}{\sqrt{2}}\left(e^{-\alpha U}+e^{-\alpha L}\right)(\cosh(a\pi)-1)+\sqrt{\gamma+\mu}\left(e^{-\alpha U}-e^{-\alpha L}\right)\sinh(a\pi)\right)}{\sqrt{\gamma+\mu}\sinh(a\pi)\left(\gamma+\mu-\frac{\alpha^2}{2}\right)}\right) \tag{A.6}$$

$$a\pi=\sqrt{\frac{\gamma+\mu}{c^2}};\quad \alpha=-m;\quad \beta=-\frac{m^2}{2};\quad c^2=\frac{1}{2(U-L)^2}\ \blacklozenge \tag{A.7}$$

## B. The numerical inversion

In this section, we describe the idea underlying the two algorithms adopted for the numerical inversion of the double Laplace transform. As remarked in Choudhury et al. [10] little attention has been given to inversion of multidimensional transforms and moreover nothing at all regard their application in finance.

### B.1. The Padé inversion method

This method, proposed by Singhal et al. (henceforth SVV) [25], seems to have been for some time the only known inversion technique for the numerical inversion of multidimensional Laplace transforms. Fortunately its implementation is very easy and moreover the computational time required for the inversion has resulted to be less than 1". Its success depends strongly on the smoothness property of the original function.

This technique starts from the inversion formula in the complex plane. If $\tilde{f}\,(s_1, s_2)$ is the double Laplace transform, the inverse Laplace transform $f\,(t_1, t_2)$ is obtained applying the Bromwich inversion formula in two variables:

$$f\,(t_1, t_2) = \left(\frac{1}{2\pi i}\right)^2 \int_{c_1 - i\infty}^{c_1 + i\infty} \int_{c_2 - i\infty}^{c_2 + i\infty} e^{s_1 t_1} e^{s_2 t_2}\, \tilde{f}\,(s_1, s_2)\, ds_1 ds_2$$

where $c_1$ and $c_2$ are the right-most singularities. Substituting $s_k t_k = w_k$, $k = 1, 2$, we have:

$$f\,(t_1, t_2) = \frac{1}{t_1 t_2} \left(\frac{1}{2\pi i}\right)^2 \int_{c_1' - j\infty}^{c_1' + j\infty} \int_{c_2' - j\infty}^{c_2' + j\infty} e^{w_1} e^{w_2}\, \tilde{f}\left(\frac{w_1}{t_1}, \frac{w_2}{t_2}\right) dw_1 dw_2 \qquad \text{(B.1)}$$

Then SVV approximate the exponential function $e^{z_k}$ by a Padé rational function:

$$e^{w_k} \simeq \psi_{n_k,m_k}(w_k) = \frac{\sum_{j=0}^{n_k} (n_k + m_k - j)! \binom{n_k}{j} w_k^j}{\sum_{j=0}^{m_k} (-1)^j (n_k + m_k - j)! \binom{m_k}{j} w_k^j}$$

where $n_k < m_k$ in order that the function:

$$\tilde{f}\left(\frac{w_1}{t_1}, \frac{w_2}{t_2}\right) \psi_{n_1,m_1}(w_1)\, \psi_{n_2,m_2}(w_2)$$

has two more poles than zeros in each variable. Then

$$\psi_{n_k,m_k}(w_k) = \sum_{j=1}^{m_k} \frac{r_{kj}}{w_k - w_{kj}}$$

where $w_{kj}$ are the poles of the approximation and $r_{kj}$ are the corresponding residues. Substituting $\psi_{n_k,m_k}(w_k)$ for $e^{w_k}$ in (B.1), we get an approximation $\bar{f}(t_1, t_2)$ to $f(t_1, t_2)$:

$$\bar{f}(t_1, t_2) = \frac{1}{t_1 t_2} \left(\frac{1}{2\pi i}\right)^2 \int_{c_1'-i\infty}^{c_1'+i\infty} \int_{c_2'-i\infty}^{c_2'+i\infty} \tilde{f}\left(\frac{w_1}{t_1}, \frac{w_2}{t_2}\right) \sum_{j=1}^{m_1} \frac{r_{1j}}{w_1 - w_{1j}} \sum_{j=1}^{m_2} \frac{r_{2j}}{w_2 - w_{2j}} dw_1 dw_2$$

Interchanging the sums with the integrals, using the residue calculus and closing the path in the right half plane we get the final formula:

$$\bar{f}(t_1, t_2) = \frac{1}{t_1 t_2} \sum_{j=1}^{m_1} \sum_{j=1}^{m_2} r_{1j} r_{2j}\, \tilde{f}\left(\frac{w_{1j}}{t_1}, \frac{w_{2j}}{t_2}\right)$$

The inversion reduces to a double summation and requires just $m_1 \times m_2$ function evaluations. The Padé inversion formula can be easily programmed once we have calculated the poles and the residues of the Padé approximant. This can be done very easily in Mathematica version 3.0 using the function Pade[] for computing the approximant, the function NSolve[] to find the poles and the function NResidue[] to find the corresponding residues, as illustrated here below, where we give the Mathematica 3.0 code for computing the poles and the residues of the Padé approximant. It requires the use of the packages NResidue and Pade.

```
<< NumericalMath`NResidue`;
<< Calculus`Pade`;
poles[nk_, mk_] := x/.NSolve[Denominator[Pade[Exp[x], {x, 0, nk, mk}]] == 0, x]
res[nk_, mk_, i_] := NResidue[Pade[Exp[x], {x, 0, nk, mk}], {x, poles[nk, mk][[i]]}]
```

The computation of the poles and residues can be quite time consuming especially if the degree of the numerator and denominator is high. However, it needs to be done only once and then we can store the calculated poles and residues and perform the numerical inversion very quickly[15]. Regarding the choice of the degree of the numerator and denominator, we have seen that if $m_k > 20$ we can incur in roundoff errors because the residues and the poles can assume very large values. A check test is to verify that the sum of the residues is zero. For our problem a good choice was to set $n_k$ equal to 4 and $m_k = 18$.

### B.2. The Fourier series method

The inversion formula proposed in the Choudhury et al. (henceforth CLW) [10] is a multidimensional version of the algorithm in Abate and Whitt [1], with an enhancement in order to control simultaneously the aliasing and the round-off errors. The authors damp the given function multiplying by a two dimensional decaying exponential function and then approximate the damped function by a periodic function constructed by aliasing. The two exponential parameters allow to control the aliasing error in the approximation by the periodic function. The inversion formula is then the two dimensional Fourier series of the periodic function. Moreover expressing the two dimensional series as an alternating

[15] However, the computation of poles and residues can be done easily in C or Fortran as well, using double precision arithmetic.

series nested within a second alternating series CLW can apply the Euler transformation to compute the infinite series from finitely many terms. If $\tilde{f}(s_1, s_2)$ is the double Laplace transform of the function $f(t_1, t_2)$, their inversion formula is given by:

$$
\begin{aligned}
\bar{f}(t_1,t_2) &= \frac{\exp\left(\frac{A_1}{2l_1}+\frac{A_2}{2l_2}\right)}{4t_1t_2l_1l_2} \\
&\times \left\{ \tilde{f}\left(\frac{A_1}{2l_1t_1}, \frac{A_2}{2l_2t_2}\right) + 2\sum_{k_1=1}^{l_2}\sum_{k=0}^{+\infty}(-1)^k \operatorname{Re}\left[e^{-\frac{ik_1\pi}{l_2}}\tilde{f}\left(\frac{A_1}{2l_1t_1}, \frac{A_2}{2l_2t_2} - \frac{ik_1\pi}{t_2l_2} - \frac{ik\pi}{t_2}\right)\right] \right. \\
&+2\sum_{j_1=1}^{l_1}\sum_{j=0}^{+\infty}(-1)^j \operatorname{Re}\left[\sum_{k_1=1}^{l_2}\sum_{k=0}^{+\infty}(-1)^k e^{-\left(\frac{ij_1\pi}{l_1}+\frac{ik_1\pi}{l_2}\right)}\tilde{f}\left(\frac{A_1}{2l_1t_1} - \frac{ij_1\pi}{l_1t_1} - \frac{ij\pi}{t_1}, \frac{A_2}{2l_2t_2} - \frac{ik_1\pi}{t_2l_2} - \frac{ik\pi}{t_2}\right)\right] \\
&+2\sum_{j_1=1}^{l_1}\sum_{j=0}^{+\infty}(-1)^j \operatorname{Re}\left[e^{-\left(\frac{ij_1\pi}{l_1}\right)}\tilde{f}\left(\frac{A_1}{2l_1t_1} - \frac{ij_1\pi}{l_1t_1} - \frac{ij\pi}{t_1}, \frac{A_2}{2l_2t_2}\right)\right. \\
&\left.\left. +\sum_{k_1=1}^{l_2}\sum_{k=0}^{+\infty}(-1)^k e^{-\left(\frac{ij_1\pi}{l_1}-\frac{ik_1\pi}{l_2}\right)}\tilde{f}\left(\frac{A_1}{2l_1t_1} - \frac{ij_1\pi}{l_1t_1} - \frac{ij\pi}{t_1}, \frac{A_2}{2l_2t_2} + \frac{ik_1\pi}{t_2l_2} + \frac{ik\pi}{t_2}\right)\right]\right\}
\end{aligned}
$$

and

$$f(t_1,t_2) = \bar{f}(t_1,t_2) + \bar{e}$$

CLW are able to show that, if the $|f(t_1,t_2)| \leq C$ for some constant $C$ and for all $t_1$and $t_2$, then the aliasing error $\bar{e}$ can be bounded as

$$|\bar{e}| \leq C\left(e^{-A_1} + e^{-A_2}\right)$$

and then with $A_1$ and $A_2$ we can control the aliasing error. This is en effect our case, because it is often easy to bound the price of a financial contract and then to find the constant $C$. In the numerical examples we have set $A_1 = A_2 = 20$.

In the inversion formula above we can remark the presence of infinite sums of the form $\sum_{k=0}^{\infty}(-1)^k a_k$, where $a_k$ is real or complex. In this case we can apply the Euler transformation and approximate the infinite sum by:

$$
\begin{aligned}
E(m,n) &= S_n + \sum_{k=0}^{m} \binom{m}{k} 2^{-m} S_{n+k} \\
S_n &= \sum_{k=0}^{n} (-1)^k a_n
\end{aligned}
$$

So the Euler transformation requires the computation of just $m+n+1$ terms in the sum and then the final sum is extrapolated by the $m+1$ extraterms $n, ..., n+m$. CLW state that, in their experience, using the Euler summation technique with $n = 38$ and $m = 11$ can reduce to $10^{-13}$ or lower the truncation error coming from using a finite number of terms. In the inversion, we have verified that we can obtain an accurate answer with $n = 20$ and $m = 20$ terms, i.e. a total of just 41 terms. Although we have implemented the inversion in C using Microsoft Visual C++, we remark that the Euler algorithm can be found in Mathematica version 3.0 in the package NumericalMath'NLimit'.

The parameters $l_1$ and $l_2$ can be used in order to control the roundoff error, due to multiplying large numbers by small ones. In particular the quantity

$$
\exp\left(A_1/2l_1 + A_2/2l_2\right) / \left(4t_1t_2l_1l_2\right)
$$

can be large and we can decrease it increasing $l_1$and $l_2$, although this fact increases the computational time that is proportional to the product $l_1l_2$. The authors suggest that the roundoff and the aliasing errors can be set about of the same order of magnitude and that for two-dimensional inversion $l_1 = l_2 = 2$ is adequate. In effect this choice worked well in our problem.

## C. The finite difference scheme

Our main system of PDE's (4.2)

$$-V_\tau + rxV_x + \frac{\sigma^2}{2}x^2V_{xx} = rV \tag{C.1}$$

with continuity condition across the sampling dates $\tau_j$,

$$V\left(x, \tau_j^+, y\right) = V\left(x, \tau_j^-, y + 1_{(l<x<u)}\right) \tag{C.2}$$

and initial condition$V(x,0,y) = \max(y-K;0) * \Delta$ is solved by a proper finite difference scheme in order to satisfy the requirements specified in the main text, i.e.: a) existence, b) positivity, c) convergence to the exact one and d) absence of spurious oscillations. From a computational point of view the change of variable

$$X = \frac{1}{1+x}$$

which maps the $x$-interval $[0,+\infty)$ onto the $X$-interval $[0,1]$ avoids assuming a finite arbitrarily large computational domain. We get the more convenient boundary value and initial conditions problem:

$$\begin{aligned} &-\tfrac{\partial V}{\partial \tau} + \left[r\left(X^2 - X\right) + \sigma^2 X\left(1-X\right)^2\right]\tfrac{\partial V}{\partial X} + \tfrac{\sigma^2}{2}X^2(1-X)^2\tfrac{\partial^2 V}{\partial X^2} - rV = 0 \\ &V(\tau,0) = V(\tau,1) = e^{-r\tau}\max(y-K,0), \qquad V(0,X) = \max(y-K,0) \end{aligned} \tag{C.3}$$

Equally spaced points $\tau_n = n\Delta\tau$, $n = 1,...,M_\tau$, $X_j = j\Delta X$, $j = 1,...,M_X$ have been chosen, with steps size $\Delta\tau$, $\Delta X$ respectively. $M_\tau \cdot M_X$ grid points are so obtained.

An implicit scheme for the derivatives $\partial V/\partial x$ and $\partial^2 V/\partial x^2$ is used:

- the first derivative $\frac{\partial V}{\partial \tau}$ is discretized by a first order backward-difference scheme:

$$\frac{\partial V}{\partial \tau} \simeq \frac{V_j^{n+1} - V_j^n}{\Delta \tau}$$

- the first derivative $\frac{\partial V}{\partial X}$ has been treated by a *first-order upwind scheme*, as the convection term coefficient can take both negative and positive values:

$$\frac{\partial V}{\partial x} \simeq \begin{cases} \frac{V_{j+1}^{n+1} - V_j^{n+1}}{\Delta x} & \text{if } \; r\left(X^2 - X\right) + \sigma^2 X \left(1 - X\right)^2 > 0 \\ \\ \frac{V_j^{n+1} - V_{j-1}^{n+1}}{\Delta x} & \text{if } \; r\left(X^2 - X\right) + \sigma^2 X \left(1 - X\right)^2 < 0 \end{cases}$$

- for the second derivative $\frac{\partial^2 V}{\partial X^2}$ a centered-difference formula has been chosen:

$$\frac{\partial^2 V}{\partial x^2} \simeq \frac{V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}}{\Delta x^2}$$

a) The difference equation is as follows.

If $r(X_j^2 - X_j) + \sigma^2 X_j (1 - X_j)^2 > 0$, then:

$$\begin{aligned} &-\frac{V_j^{n+1} - V_j^n}{\Delta \tau} + \left[r(X_j^2 - X_j) + \sigma^2 X_j (1 - X_j)^2\right] \frac{V_{j+1}^{n+1} - V_j^{n+1}}{\Delta X} \\ &+\frac{1}{2}\sigma^2 X_j^2 (1 - X_j^2)^2 \frac{V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}}{(\Delta X)^2} - rV_j^{n+1} = 0 \end{aligned} \tag{C.4}$$

otherwise if $r(X_j^2 - X_j) + \sigma^2 X_j (1 - X_j)^2 < 0$:

$$\begin{aligned} &-\frac{V_j^{n+1} - V_j^n}{\Delta \tau} + [r(X_j^2 - X_j) + \sigma^2 X_j (1 - X_j)^2] \frac{V_j^{n+1} - V_{j-1}^{n+1}}{\Delta X} \\ &+\frac{1}{2}\sigma^2 X_j^2 (1 - X_j)^2 \frac{V_{j+1}^{n+1} - 2V_j^{n+1} + V_{j-1}^{n+1}}{(\Delta X)^2} - rV_j^{n+1} = 0 \end{aligned} \tag{C.5}$$

Here $j = 1, ..., M_X$, $X_j = X(j\Delta X)$, $V_j^n = V(n\Delta\tau, j\Delta X)$, represents the approximate solution at the point $(n\Delta\tau, j\Delta X)$, $V_0^n = V_{M_X+1}^n = e^{-rn\Delta\tau}\max(y - K, 0)$.

A set of $M_\tau$ uncoupled $M_X$-order linear systems are so obtained. Each system $AV_{n+1} = b_n$, $n = 0, ..., M_\tau$ contains the grid points having a $\tau = \tau_{n+1}$ constant value. Some interesting features of each $A$ matrix are the following:

- $A$ is tridiagonal (with diagonal positive components and off diagonal non positive components), nonsymmetric, irreducible (its associated graph is strongly connected), strictly diagonally dominant;
- $b_n$ is non negative and contains both the approximate values of the solution previously calculated at the time $\tau_n$ and the boundary conditions at the time $\tau_{n+1}$. Therefore:
- $A$ is nonsingular having eigenvalues with positive real parts (from Gerschgorin's Theorem);
- $A$ is an $M$-matrix (compare Ortega [23], p. 110);
- $A^{-1} > 0$ strictly.

b) Then $V_{n+1} = A^{-1}b_n$ is positive. Thus the numerical solution in all grid points is positive.

c) For what concerns the convergence we resort to the following *Lax's equivalence theorem* (compare Smith [26]): Given a properly posed linear initial-value problem and a linear finite difference approximation to it that satisfies the consistency condition, stability is the necessary and sufficient condition for convergence.

1. Consistency

Now we rewrite (C.3) into operational form $L(V) = 0$, where $V$ indicates the exact solution. Let $V_j^n$ the approximate solution at the point $(n\Delta\tau, j\Delta X)$ and $T_j^n$ the local truncation error. Expand each term $V_{j+1}^{n+1}$, $V_j^{n+1}$, $V_{j-1}^{n+1}$ by Taylor's series about $(n\Delta\tau, j\Delta X)$.

The principal part of the local truncation error

$$
\begin{aligned}
T_j^n &= L\left(V\left(n\Delta\tau, j\Delta X\right)\right) \\
&+\Delta\tau\left[-r\frac{\partial V}{\partial\tau} - \frac{1}{2}\frac{\partial^2 V}{\partial\tau^2} + \left(r(X_j^2 - X_j) + \sigma^2 X_j(1-X_j)^2\right)\frac{\partial^2 V}{\partial\tau\partial X}\right. \\
&\left.+\frac{1}{2}\sigma^2 X_j^2(1-X_j)^2\frac{\partial^3 V}{\partial\tau\partial X^2}\right] + \Delta X\frac{\left(r(X_j^2-X_j)+\sigma^2 X_j(1-X_j)^2\right)}{2}\frac{\partial^2 V}{\partial X^2}
\end{aligned}
\tag{C.6}
$$

is obtained. Being $V$ solution of the differential equation, then $L(V_j^n) = 0$ holds. In the hypothesis that $\frac{\partial^2 V}{\partial\tau^2}, \frac{\partial^2 V}{\partial\tau\partial X}, \frac{\partial^2 V}{\partial X^2}, \frac{\partial^3 V}{\partial\tau\partial X^2}$ are bounded then as $(\Delta\tau, \Delta X) \to 0$ we have $T_j^n \to 0$, so that the numerical scheme given by (C.4) and (C.5) is consistent with (C.3).

2. Stability

Writing (C.4) and (C.5) into matrix form at the value $\tau = \tau_{n+1} = (n+1)\Delta\tau$ we obtain:

$$
AV_{n+1} = V_n + C \tag{C.7}
$$

where $A = [a_{ij}]$ is a tridiagonal diagonally dominant matrix, with $a_{ii} - \sum_{j\neq i}^{M_x} \mid a_{ij} \mid = 1 + r\Delta\tau$, $i = 1, ..., M_x$, whilst the vector $C$ includes the boundary conditions. From Gerschgorin theorem the eigenvalues $\lambda_i(A)$ satisfy $\lambda_i(A) \geq 1 + r\Delta\tau$ and then

$$
0 < \lambda_i(A^{-1}) \leq \frac{1}{1 + r\Delta\tau} < 1 \tag{C.8}
$$

Thus the scheme is unconditionally stable and then the convergence, via Lax's equivalence theorem, is proved.

d) The matrix $A$ is tridiagonal, diagonally dominant, with all its off-diagonal entries strictly negative. Then $A$ is similar to a real symmetric tridiagonal matrix $DAD^{-1}$ with non-zero off-diagonal entries (Ortega [23], p. 113), so that all the eigenvalues $\lambda_i(A)$ are real. Here $D$ is a diagonal matrix.

The characteristic polynomial $P_{M_x}(\lambda)$ associated to $DAD^{-1}$ admits $M_x$ real and distinct zeros (Stoer [27]) so that $A$ admits $M_x$ real distinct eigenvalues. Hence the $M_x$ eigenvectors $v_s$ of $A$ are linearly independent and can be used as a basis for the $M_x$ dimensional space of the vector $V_0$ of initial values. In other words, $V_0$ can be expressed as $V_0 = \sum_{j=1}^{M_x} c_j v_j$ where the $c_j$ are constant. From (C.7) we have

$$\begin{aligned} V_n &= \left(A^{-1}\right) V_{n-1} + A^{-1}C = \ldots = \left(A^{-1}\right)^n V_0 + A^{-1}C \sum_{j=0}^{n-1} \left(A^{-1}\right)^j \\ &= \left(A^{-1}\right)^n \sum_{j=1}^{M_x} c_j v_j + A^{-1}C \sum_{j=0}^{n-1} \left(A^{-1}\right)^j = \sum_{j=1}^{M_x} c_j \left(A^{-1}\right)^n v_j + A^{-1}C \sum_{j=0}^{n-1} \left(A^{-1}\right)^j \\ &= \sum_{j=1}^{M_x} c_j \lambda_j^n \left(A^{-1}\right) v_j + A^{-1}C \sum_{j=0}^{n-1} \left(A^{-1}\right)^j \end{aligned} \tag{C.9}$$

Taking into account (C.8) then the numerical scheme (C.4),(C.5) is $L_0$-stable (Smith [26], p.121) and unwanted finite oscillations in the numerical solution are rapidly dampened.

In summarizing, the scheme satisfies all the requirements $\forall(\Delta\tau, \Delta x)$.

## D. The digital corridor option

In this section we discuss the pricing of the digital corridor option that pays a cash amount if the occupation time at the expiry of the option is greater than a fixed level $K$ :

$$payoff_{t+\tau} = 1_{(Y(t+\tau,x;u,l)>K)}$$

We remark on the importance of this contract as basic element in the construction of the corridor option, in exactly the same manner as the digital option for the plain vanilla options. Indeed the corridor option with residual life $\tau$, can be expressed as sum of corridor digital options with ascending strikes:

$$corropt_t\left(K\right) = \sum_{n=0}^{\left[(\tau-\bar{K})/\Delta K\right]} digcorropt_t\left(\bar{K} + n\Delta K\right)\Delta K$$

where [] stands for the integer part. In the limit, we have:

$$corropt_t\left(\bar{K}\right) = \int_{\bar{K}}^{\tau} digcorropt_t\left(K\right)dK$$

In order to price this contract we need to discount the following quantity[16]:

$$\begin{aligned}
&E_{t,x}\left[1_{(Y(t+\tau,x;u,l)>K)}\right] = \Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right)>K\right] \\
&= \textstyle\int_K^{t+\tau}\Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right)\in dy\right] + \Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right)=\tau\right] \\
&= 1-\Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right)=0\right] - \textstyle\int_0^K\Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right)\in dy\right]
\end{aligned}$$

In this expression, the only problem can come from the integral term, because all the other quantities have a known closed form expression. If we consider the Laplace transform with respect of $K$ of this quantity, this transform is given by the Laplace transform of $\Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right)\in dy\right]$ divided by $\mu$ and then the undiscounted price is given by:

$$\begin{aligned}
&= 1-\Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right)=0\right] - \mathcal{L}^{-1}\left[\tfrac{\Omega(t,\mu,x;l,u,m)}{\mu};\mu\to K\right] \\
&= 1-\Pr_{t,x}\left[Y\left(t+\tau,x;u,l\right)=0\right] - \mathcal{L}^{-1}\left[\tfrac{\omega(\gamma,\mu,x;l,u,m)}{\mu};\mu\to K,\gamma\to t\right]
\end{aligned}$$

so in order to price this option we need to compute the double Laplace inverse of the quantity $\omega\left(\gamma,\mu,x;l,u,m\right)/\mu$. The numerical routines adopted for the inversion in the case of the corridor option can then be modified very quickly to cope with this pricing problem. In Table 10 we compare the prices obtained using the two different numerical procedures and we can verify that the accuracy is very high and the computational time very low.

[16]In the following we use the fact that $1=\Pr_{t,x}\left[Y_{t+\tau}=0\right]+\Pr_{t,x}\left[Y_{t+\tau}=\tau\right]+\int_K^{t+\tau}\Pr_{t,x}\left[Y_{t+\tau}\in dy\right]+\int_0^K\Pr_{t,x}\left[Y_{t+\tau}\in dy\right]$.

**Table 10. Prices of digital corridor options**

Parameters setting: U=110, L=100, r=0.05, $\sigma$ = 0.2, $\tau$ =1yr.

| Index Level | Fourier | Padé |
|---|---|---|
| 90 | 0.286586 | 0.286588 |
| 95 | 0.425404 | 0.425388 |
| 100 | 0.582029 | 0.582005 |
| 105 | 0.646513 | 0.646487 |
| 110 | 0.5421136 | 0.542088 |
| 115 | 0.387798 | 0.387776 |
| 120 | 0.269909 | 0.269911 |
| Average CPU | 3"85/100 | 25/100 |

Regarding the discrete time monitoring case we remark that we can apply the same procedure adopted for the corridor option, i.e. solving the PDE (4.2), but now the initial condition is given by:

$$V(x,0,y) = 1_{(y>K)} \tag{D.1}$$

and the boundary conditions:

$$V(0,\tau,y) = e^{-r\tau}1_{(y>K)} \tag{D.2}$$

$$V(+\infty,\tau,y) = e^{-r\tau}1_{(y>K)} \tag{D.3}$$

In this case $y$ is an integer number representing the number of times the index has spent inside the band at the monitoring dates. We observe that when $y \geq [K]+1 \geq K$, where $[K]$ is the greatest integer strictly smaller than $K$, and when the time to maturity is $\tau$ the option will be surely exercised, and we have an analytical solution given by $V(x,\tau,y) = e^{-r\tau}$.

We remark that for this kind of contract the discontinuities are introduced not only at the monitoring dates, but at the initial date as well and the numerical scheme described above is then a correct one for the problem at hand.

In Table 11 we report the price of this contract in presence of continuous and discrete time monitoring. In the discrete time case, the value of $K_d$ has been set equal to $K_d = [K_c * n]$ where [] is the integer part, $n$ the number of monitoring dates and $K_c$ the fixed level (as ratio to the residual life) used in the continuous time formula. In this way if $K_c = 0.2$ and with monthly monitoring, we have $K_d = [0.2 * 12] = [2.4] = 2$. The remaining parameters are the same as in the previous table.

**Table 11. Discrete and Continuos Monitoring**

**in pricing Digital Corridor Options**

Parameters setting: U=110, L=100, r=0.05, $\sigma = 0.2$, $\tau$ =1yr, K=0.2.

| Index Level | Monthly Monitoring | Continuous Monitoring | %Difference |
|---|---|---|---|
| 90 | 0.267553 | 0.286588 | -6.64% |
| 95 | 0.400571 | 0.425388 | -5.83% |
| 100 | 0.519848 | 0.582005 | -10.68% |
| 105 | 0.553547 | 0.646487 | -14.38% |
| 110 | 0.481742 | 0.542088 | -11.13% |
| 115 | 0.361531 | 0.387776 | -6.77% |
| 120 | 0.250799 | 0.269911 | -7.08% |

In this case the price of the option in the discrete time monitoring appears to be systematically lower than in the corresponding continuous case. The likely reason is in the discrete distribution of the occupation time in the case of discrete monitoring, which concentrates the masses of probability in 0, $\Delta$,$2\Delta$.

# Chapter 4. Valuation of Asian Options: new insights

## 1. Introduction

Asian options, options for which the payoff depends on the arithmetic average value of the asset price over some time period, have had a very large success in the last years, because they reduce the possibility of market manipulation near the expiry date and offer a better hedge to firms having a stream of positions.

Actually there is no closed form solution for the price of this option because in the usual Black-Scholes framework, the arithmetic average is a sum of correlated lognormal distributions for which there is no recognizable density function. Several tentatives have been done. There are techniques that provide the Laplace transform with respect to time of the option price as in Geman and Yor [24][1], or methods assuming some form for the density of the average in order to obtain closed form solution, Turnbull and Wakeman [51], Levy [32], Vorst [53] and the references therein, Milevsky and Posner [39]. Another approach is to include the path-dependent variables into the state space and cast the pricing problem into the Black-Scholes framework, and to solve an augmented and degenerate parabolic PDE, as in Barraquand and Pudet [6] or in Zvan et al. [58], or to scale the problem and reduce the PDE to a one-dimensional problem as in Rogers and Shi [43]. Other possibilities

[1] The bibliography is after the conclusions in the present chapter.

consist in Monte Carlo simulation with variance reduction techniques, e.g. Clewlow and Carverhill [12], and binomial approximations, Hull and White [26]. Finally Rogers and Shi [43] obtain very tight upper and lower bounds for the price.

In the present paper we show that the density law of the logarithm of the arithmetic average (henceforth logav) is uniquely determined by its moments. On the basis of this result and using the logarithmic moments instead of the algebraic moments, we try to recover completely the true density using the maximum entropy approach. The maximum entropy principle, Jaynes [28] and Golan et al. [25], is the most popular one in choosing the approximating distribution and is well known and widely diffused in Statistics. In Finance has been already used for estimating the implied risk neutral distribution from the option prices, Buchen and Kelly [10], Avellaneda [5]. Assuming the given moments as known information, the maximum entropy (ME) principle chooses, out of the distributions consistent with the given partial information, the one having maximum entropy, or equivalently the most uncertain, accomplishing a principle of scientific honesty.

For a given number of moments, the approximated distribution will be: a) always positive, b) uniquely determined by the infinite sequence of logarithmic moments. Moreover c) the sequence of approximants is converging to the true distribution in some norm, d) a bound to the error made in pricing Asian options is provided. We remark that the above properties a), c) and d) are not satisfied when we approximate the true density with an Edgeworth series using the algebraic moments.

We compare the proposed method with:

a) the lognormal density approximation in Turnbull and Wakeman [51] and in Levy [32];

b) the Edgeworth series approximation in Turnbull and Wakeman [51] and in Ritchken et al. [42];

c) the reciprocal gamma density approximation as suggested in Milevsky and Posner [39];

d) the Generalized Beta density of the second type with the same first four moments as the average;

e) the Edgeworth series approximation around a normal, using the logarithmic moments instead of the algebraic.

Finally, in order to have a reference point for comparing the different approximations we use:

1. the lower and upper bounds given in Rogers and Shi [43] and in Thompson [49].

2. the numerical inversion of the Laplace transform given in Geman and Yor, [24].

We do not consider Monte Carlo simulation because, although there are a number of techniques for variance reduction that allow to limit the number of simulations, it is still quite time consuming requiring in every case several thousands of simulations. Either we do not consider partial differential equation solution methods because respect to the above methods their implementation is slightly more involved.

In section 1 we describe the Black-Scholes framework underlying the Asian option problem. In section 2 we discuss the Laplace transform technique and we present three very simple to implement inversion techniques (the Dubner Abate method as discussed in Abate and Whitt, [1]). In section 3 we discuss the general moment problem: i.e. we recall some basic fact relevant when we want approximate a distribution using its moments. Then we describe different approximations based on the moments of the arithmetic average: the lognormal and the Edgeworth series, the reciprocal gamma and the GB2 density approximation. Then we show that the density of the logav is uniquely determined by its moments and on the basis of this result we discuss how to approximate the density of the logav. Finally in section 6, we compare the different approximations and then we draw some conclusion. In Appendix A we give some more details about the derivation of the Geman-Yor formula. In Appendix B we discuss a linear approximation to the Asian option price and in Appendix C we describe the use of lower and upper bounds to the Asian price.

## 2. The general framework

We assume that under the martingale measure the underlying asset price evolves according to a Geometric Brownian Motion:

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t \\ S_t &= S_0 e^{\left(r-\frac{\sigma^2}{2}\right)t+\sigma W_t} \end{aligned}$$

where $r$ is the instantaneous risk free rate, $\sigma$ is the instantaneous volatility and $W_t$ is a Wiener process. Assuming that the average is computed continuously in time in the period $(0,T)$, we define the price average in the following way:

$$A_t = \frac{\int_0^t S_w dw}{t} = \frac{S_0 \int_0^t e^{\left((r-\sigma^2/2)w+\sigma W_w\right)} dw}{t}$$

and the payoff at time $T$ of a Asian option is given by max$[A_T - K$,0] for a call option and by max$[K - A_T$,0] for a put option.

It is convenient to redefine the option payoff in the following way:

$$\max\left[A_T - K, 0\right] = S_0 \max\left[\frac{\bar{A}_T}{T} - \bar{K}, 0\right]$$

where $\bar{K}=K/S_0$ and $\bar{A_T} = \int_0^T e^{\left((r-\sigma^2/2)w+\sigma W_w\right)} dw$. If the option is into the averaging period $A_T$ refers to the average over the remaining time to expiry, but the strike price $\bar{K}$ must be replaced by $K^*$ and the option value must be multiplied by $(T-t)/T$, with:

$$K^* = \frac{T}{T-t}\bar{K} - \frac{t}{T-t}\bar{A}^*$$

and where $A^*$ is the realized average during the period $(0,t)$. If the asset pays a continuous dividend yield $q$ the asset price has to be corrected in $Se^{-qT}$ and the risk free rate becomes $r-q$.

## 3. Analytical results: The Geman-Yor Laplace transform

Using the fact that a Geometric Brownian Motion is a time-changed squared Bessel process and the stability by additivity of this process, Geman and Yor [24] show that the Asian option price can be expressed as[2]:

$$c_{gy} = \frac{e^{-r(T-t)}}{T}\frac{4S}{\sigma^2}c(h,q) \tag{3.1}$$

and they are able to obtain the Laplace transform wrt $h = \sigma^2 (T-t)/4$ of the function $c(h,q)$:

$$\begin{aligned} C(\lambda,q) &= \int_0^{+\infty} e^{-\lambda h} c(h,q)\,dh \\ &= \frac{\int_0^{1/(2q)} e^{-x} x^{\frac{1}{2}(\mu-v)-2}(1-2qx)^{\frac{1}{2}(\mu+v)+1}dx}{\lambda(\lambda-2-2v)\Gamma\left(\frac{1}{2}(\mu-v)-1\right)} \end{aligned} \tag{3.2}$$

where:

$$\begin{aligned} v &= \frac{2r}{\sigma^2}-1; q = \frac{\sigma^2}{4S}(K*T-t*a) \\ \mu &= \sqrt{2\lambda+v^2} \end{aligned} \tag{3.3}$$

[2] In Appendix A, we report a simple proof taken from Yor [55].

where $S$ is the current stock price, $t$ the current date, $T$ the expiry date of the option, $K$ the strike price, $r$ the interest rate, $\sigma$ the volatility and $a$ the current average. A similar result has been obtained recently by Lipton [35] reducing the augmented Black-Scholes PDE through the use of a non-dimensional variable and then solving it using the Laplace transform, compare Chapter 1, Section 3.3.2 of the present thesis.

In order to consider a constant integration domain, we can make a change of integrating variable $z = 2qx$ and the expression for the integral in (3.2) becomes:

$$\frac{1}{(2q)^{1+\delta}} \int_0^1 e^{-\frac{z}{2q}} z^{\frac{1}{2}(\mu-v)-2} (1-z)^{\frac{1}{2}(\mu+v)+1} \, dz \tag{3.4}$$

where $\delta = \frac{1}{2}(\mu - v) - 2$. The above integral exists if the exponents of $z$ and $(1-z)$ are greater than -1. This amounts to require that $\lambda > 2 + 2v$, so $2 + 2v$ represents a pole of the Laplace transform[3].

Geman and Eydeland [22] suggest to replace $e^{-z/2q}$ in (3.4) using a series expansion and the relationship between beta and gamma functions. Then they obtain the following approximate expression for the Laplace transform[4]:

$$\frac{1}{\lambda(\lambda - 2v - 2)} \sum_{i=0}^{M} \frac{1}{i!} \left(-\frac{1}{2q}\right)^{1+\delta+i} \frac{\Gamma((\mu - v)/2 - 1 + i)\, \Gamma((\mu + v)/2 + 2)}{\Gamma(\mu + 1 + i)\, \Gamma((\mu - v)/2 - 1)}$$

where $M$ is the number of terms used in the series expansion. For the numerical inversion they apply the Fast Fourier Transform. However, we have applied to this expression the different algorithms for the numerical inversion described below and the inversion was accurate if $\sigma\sqrt{T} > 0.3$ with $M$=40. If $\sigma\sqrt{T} < 0.3$ the numerical inversion results to be

[3]When these exponents are in the interval $(-1, 0)$, although the integral exists, we can have numerical problems, so when $2 + 2v < \lambda < 8 + 4v$, it is convenient to operate a substitution making the change of variable $w = z^{1+\frac{1}{2}(\mu-v)-2}$.

[4]We need to remark that in the Geman and Eydeland paper there are several typographical errors, so some attention has to be taken.

completely inaccurate, independently on the number of terms in the series expansion.

### 3.1. The density law of the average

The expressions in (3.1) and (3.2) for the price of the Asian option gives us the possibility of computing the Laplace transform of the density of the average. Indeed, we can express the implied distribution and the implied density function using the first and the second derivatives respect to the strike price of the call option price. Applying this result to the present case, we can obtain an expression for the Laplace transform of the distribution function and of the density of the average and then invert it. It is easy to show that:

$$\begin{aligned}\int_0^K f_{gy}(y)\,dy &= e^{r(T-t)}\frac{\partial c_{gy}}{\partial K}+1\\ f_{gy}(K) &= e^{r(T-t)}\frac{\partial^2 c_{gy}}{\partial K^2}\end{aligned}$$

In particular, deriving respect to $K$ the equation (3.2), we obtain:

$$\begin{aligned}\frac{\partial c_{gy}}{\partial K} &= \frac{e^{-r(T-t)}}{T}\frac{4S}{\sigma^2}\frac{\partial c(h,q)}{\partial K}\\ &= \frac{e^{-r(T-t)}}{T}\frac{4S}{\sigma^2}\mathcal{L}^{-1}\left[\frac{\partial C(\lambda,q)}{\partial q}\frac{\partial q}{\partial K}\right]\\ &= e^{-r(T-t)}\mathcal{L}^{-1}\left[\frac{\partial C(\lambda,q)}{\partial q}\right]\end{aligned}$$

and:

$$\begin{aligned}\frac{\partial^2 c_{gy}}{\partial K^2} &= e^{-r(T-t)}\mathcal{L}^{-1}\left[\frac{\partial^2 C(\lambda,q)}{\partial q^2}\frac{\partial q}{\partial K}\right]\\ &= e^{-r(T-t)}\frac{\sigma^2 T}{4S}\mathcal{L}^{-1}\left[\frac{\partial^2 C(\lambda,q)}{\partial q^2}\right]\end{aligned}$$

where:

$$\frac{\partial C(\lambda,q)}{\partial q} = -\frac{(\mu+v+2)\int_0^{1/(2q)} e^{-x}x^{\frac{1}{2}(\mu-v)-1}(1-2qx)^{\frac{1}{2}(\mu+v)}\,dx}{\lambda(\lambda-2-2v)\,\Gamma\left(\frac{1}{2}(\mu-v)-1\right)}$$

and:

$$\frac{\partial^2 C(\lambda, q)}{\partial q^2} = \frac{(\mu + v + 2)(\mu + v) \int_0^{1/(2q)} e^{-x} x^{\frac{1}{2}(\mu - v)} (1 - 2qx)^{\frac{1}{2}(\mu + v - 2)} dx}{\lambda (\lambda - 2 - 2v) \Gamma\left(\frac{1}{2}(\mu - v) - 1\right)}$$

So with very little changes to the routine for the determination of the Asian option price, we can compute the density law of the arithmetic average. As alternative, we could use the result in Yor [56] that represents the density law of the arithmetic average taken at an independent exponential time in terms of the ratio of a beta and a gamma independent random variables.

### 3.2. The numerical inversion

Although there are numerous and well understood techniques for the Laplace inversion, in Finance this technique has received a very limited attention, mainly because it is reputed, against the evidence, to be a method slow and difficult to implement accurately. In the case of Asian option we are aware only of a work by Fu et al. [21], and among the different routines tested we agree with them in using the Abate-Whitt algorithm for the inversion.

In order to compute the inverse Laplace transform the above integrals have been computed using Gaussian quadrature (Legendre) with 100 nodes. Moreover we have tried an inversion method, due to Dubner and Abate [17] and described in detail in Abate and Whitt [1], without a complete error analysis, but using as test the bounds given in Rogers and Shi [43] and in Thompson [49]. The inversion formula is given by:

$$c_{da}(h, q) \approx \sum_{k=0}^{m} \binom{m}{k} 2^{-m} s_{n+k}(h)$$

where:

$$s_n(h) = \frac{e^{A/2}}{2h}\operatorname{Re}\left(C\left(\frac{A}{2h},q\right)\right) + \frac{e^{A/2}}{h}\sum_{k=1}^{n}(-1)^k\operatorname{Re}\left(C\left(\frac{A+2k\pi i}{2h},q\right)\right) \tag{3.5}$$

The algorithm requires the specification of three parameters: $A$, $n$ and $m$ and Fu et al. [21] have shown that a good choice results to be $A = 18.4$, m= 15, n= 11, except when $\sigma = 0.1$ where we have used m= 35, n= 15. We agree with them in this choice. The algorithm has been implemented in Microsoft Visual C++ Version 5.0 and all the calculations have been done on a Compaq Presario with Pentium 133 processor.

The inversion did not present problems except for low volatility levels ($\sigma\sqrt{T} < 0.08$) when we had numerical problems in the computation of the integral in (3.2), due to finite precision of computers (double precision provides 16 decimal places). In Table 1 we report the result of the inversion and the lower and upper bound described in Appendix C. More results can be found in Tables 6a-6c.

We have used also a different algorithm due to Crump, [13] and it gave results identical to those obtained with the AW algorithm: the two algorithms are indeed based on the same idea of approximating the integral in the inversion formula with a trapezium rule. We tried a third method based on the Padé rational approximant, [46], but the results were not very satisfactory, likely because the Laplace transform is not well described by a series expansion in the $\gamma$ variable.

| Table1: Laplace Inversion and Bounds on the Asian Option Price | | | | | |
|---|---|---|---|---|---|
| $\sigma\sqrt{T}$ | $rT$ | $K$ | Lower Bound | AW | Upper Bound |
| 0.1 | 0.09 | 100 | 4.91508 | 4.91512 | 4.91541 |
| 0.2 | 0.09 | 100 | 6.77700 | 6.77735 | 6.77866 |
| 0.3 | 0.09 | 100 | 8.82755 | 8.82876 | 8.83329 |
| 0.4 | 0.09 | 100 | 10.92090 | 10.92377 | 10.93596 |
| 0.5 | 0.09 | 100 | 13.02253 | 13.02816 | 13.05680 |

## 4. The moment problem

Before describing in detail the different approaches that make use of the moments, we review some questions related to the determination of the distribution of the average using the moments technique.

1. There is some initial reason to think that matching moments will give a good approximation. For example, Lindsay and Roeder [34] show that if two distributions functions have the same first $n$ moments, then they must cross each other at least $n$ times. Moreover, Akhiezer [2], page 66, shows that the possible error can be bounded, and establishes the relevant relationships of the difference between two distributions that share the same $2m$ moments. However more recently, Lindsay and Basak [33] show that this bound results to be quite strict only in the tails of the distribution.

2. When the random variable assumes only positive values, the distribution recovering from its moments (the so called *Stieltjes moment problem*) arises both existence and determinacy questions. We consider uniquely the latter one. In Stieltjes moment problem an infinite sequence of moments determines a unique distribution iff this distribution decays

asymptotically as an exponential $\exp(-\alpha x^{\lambda})$, $\alpha > 0$, $\lambda \geq 1/2$. For example, this does not happen in the case of the lognormal density, that is, there are multiple distributions with exactly the same moments[5]. This distribution has indeed tails decaying slower than $\exp(-\alpha x^{\lambda})$.

3. In the case the given information is represented uniquely by the sequence of moments, there are sufficient conditions for the moment problem to be determinate or indeterminate. The following two sufficient criteria can be useful

a) the popular Carleman's criterion gives a sufficient condition that can be used to verify if the infinite sequence of moments determines uniquely a distribution. The criterion requires that:

$$\sum_{n=1}^{\infty} \frac{1}{\mu_n^{\frac{1}{2n}}} = +\infty \tag{4.1}$$

b) the Krein condition, compare Akhiezer [2], requires the distribution function to be absolutely continuous with density $f(x) > 0$, $x > 0$ (and $f(x) = 0$, $x \leq 0$) and the moments of all order exist. If :

$$\int_0^{\infty} \frac{-\ln f(x)}{1+x^2} dx < \infty$$

then the distribution function is not determined by its moments.

c) In Shohat-Tamarkin, [44] it is possible to find a necessary and sufficient condition for the determinacy problem, but their condition seems too complicated to be very useful.

In next subsections, we review the different moment based approximations that have been proposed in the literature for solving the Asian option pricing problem. Then we propose first a new approximation based on the GB2 distribution that, for its great flexibility, results to be very accurate. But this approximation shares with the previous the

[5]For a proof compare Stoyanov [47], pagg.102-4, where there are also examples of distributions with the same moments as the lognormal density.

limit that there is no guarantee that using the moments of the arithmetic average we can recover the true density law (determinacy problem). Then we show that a more natural approximation requires the moments of $\ln \bar{A}_T$. In fact, we are able to show that the density of this random variable is uniquely determined by its moments. On the basis of this result we propose three more approximations: i) lognormal, ii) Edgeworth approximation around a normal distribution and iii) ME distribution.

### 4.1. Approximations based on the moments of $\bar{A}_T$

On a practical side, the most used approximations in pricing Asian options are those based on the tentative of recovering the density of the average from its moments. In order to use this method, we need an expression for the moments of the arithmetic average. They can be found in Geman and Yor [24], page 359, and are given by the following expression:

$$\mu_n := E\left[\bar{A}_T{}^n\right] = \frac{n!}{\lambda^{2n}}\left\{\sum_{j=0}^{n} d_j^{(v/\lambda)} \exp\left[\left(\frac{\lambda^2 j^2}{2} + \lambda j v\right) t\right]\right\} \tag{4.2}$$

where:

$$\begin{aligned} d_j^{(\beta)} &= 2^n \prod_{\substack{i\neq j \\ 0\leq i\leq n}} \left[(\beta+j)^2 - (\beta+i)^2\right]^{-1} \\ \lambda &= \sigma; v = \left(r - \sigma^2/2\right)/\sigma \end{aligned}$$

and in particular for $n = 1$, we have:

$$\mu_1 = \frac{e^{2(v+1)t} - 1}{2(v+1)}$$

For the fact that the moment problem for the lognormal density is undeterminate, in the case of the arithmetic average one might expect that approximations based on a sequence of algebraic moments might either fail to converge, or converge to another distribution with the same moments. A difficult proof would no doubt be required and maybe a similar approximation could be numerically very unstable for more than a few moments. On the other hand, in the case of the arithmetic average we have a convolution of an infinite number of lognormal distributions and this convolution can change the distribution from a heavy tail density (the lognormal) to a light-tail density (the arithmetic average) and so there could a (little) possibility that the moments determine uniquely the distribution. But we need to remark that the effect of averaging can just reduce the value of the constant $\alpha$, so that at finite values of $x$ the tails seem decay fast, but for very large values of $x$ the term $x^\lambda$ in the product $\alpha x^\lambda$ becomes again important. Geman and Yor [24], page 359, have shown that the Carleman's criterion is not satisfied in the case of the average of the geometric Brownian motion[6] and so nothing can be said about the determinacy problem.

Although the determinacy problem in this case is unsolved, there are several approximations based on trying to recover the density of $\bar{A}_T$ using the information content of its algebraic moments. For these approximations there exists no analytical bound for the error term as a function of the number of moments included in the approximation. Then for a fixed number of moments, the relative size of the error needs to be examined using numerical analysis and so nothing can be said about convergence.

### 4.1.1. The lognormal approximation

To use a lognormal density as approximating density of the average is a very common approximation among the practitioners and it has been suggested in Turnbull and Wakeman [51] and in Levy [32].

[6]However, Geman and Yor use a criterion that has to be used for the so called Hamburger's moment problem, i.e. when the r.v. is defined over the entire real axis, whilst in the present case the r.v. is defined over the positive axis. This implies that in (4.1) we need to use $\mu_n$ and not $\mu_{2n}$ as they do.

If we assume that $\ln \bar{A}_T$ has a normal distribution with mean $m$ and variance $v^2$, then, for any integer $n$ :

$$E_{\log}\left[\bar{A_T}^n\right] = e^{nm+0.5n^2v^2}$$

and where $E_{\log}$ stands for expected value assuming a (risk-neutral) lognormal distribution for the average. The idea is to choose $m$ and $v$ to fit the true mean and variance of $\bar{A}_T\ /T$. We obtain:

$$\begin{aligned} m &= 2\log\mu_1 - \frac{1}{2}\log\mu_2 - \log T \\ v^2 &= \log\mu_2 - 2\log\mu_1 \end{aligned} \tag{4.3}$$

and the price of the Asian call option is given by a modified Black-Scholes formula:

$$c_{log} = S_0 e^{m+v^2/2-rT} N(d_1) - e^{-rT} K N(d_2) \tag{4.4}$$

where $d_1 = (\ln S_0/K + m + v^2)/v$, $d_2 = d_1 - v$.

### 4.1.2. The Edgeworth series approximation

Using a two-parameters density we can capture the mean and the variance of the distribution, but we could have important differences between the fitted and the true skewness and kurtosis as we increase the volatility $\sigma$ or the time to maturity of the option. For this reason Turnbull and Wakeman [51] and Ritchken et al. [42] propose to use a fourth-order Edgeworth series expansion. If $f_{\log}\left(y; m, v^2\right)$ is the lognormal density with parameters $m$ and $v^2$, then the Edgeworth approximation $f_{edg}\left(y; m, v^2\right)$ to the true density is given by:

$$f_{edg}\left(y; m, v^2\right) = f_{\log}\left(y; m, v^2\right) + \sum_{i=1}^{4} \frac{k_i}{i!} \frac{\partial^i f_{\log}\left(y; m, v^2\right)}{\partial y^i} + e(y) \tag{4.5}$$

where $e(y)$ is the residual error term and $k_i$ is the difference between the i-th cumulant of the exact distribution and the approximate distribution:

$$k_i = \chi_i(f) - \chi_i(l)$$

and:

$$\begin{aligned}
\chi_1(f) &= \mu_1 \\
\chi_2(f) &= E\left(\bar{A}_T - \mu_1\right)^2 \\
\chi_3(f) &= E\left(\bar{A}_T - \mu_1\right)^3 \\
\chi_4(f) &= E\left(\bar{A}_T - \mu_1\right)^4 - 3\chi_2(f)
\end{aligned}$$

The first cumulant is the mean, the second the variance, the third a measure of skewness and the fourth a measure of kurtosis.

Unfortunately there exists no analytical bound for the error term $e(y)$ as a function of the number of terms included in the approximation. If all moments exist, it can be shown that $e(y)$ goes uniformly to zero in $y$ as the number of moments included in the approximation increases. For a fixed number of moments, the relative size of the error needs to be examined using numerical analysis and so nothing can be said about convergence. Moreover, if the error between the lognormal model and the true one is small, then the approximating technique on average adds noise, although not large in an absolute sense. For details compare Jarrow and Rudd [27].

Once the parameters $m$ and $v^2$ have been set according to (4.3) and then $k_1 = k_2 = 0$ in the expansion above, the option price formula is given by, compare Jarrow and Rudd

[27][7] page 355:

$$c_{edg} = c_{log} + e^{-rT}\frac{S_0}{T}\left[ -\frac{k_3}{6}\left.\frac{\partial f_{\log}\left(y;m,v^2\right)}{\partial y}\right|_{y=TK/S_0} + \frac{k_4}{24}\left.\frac{\partial^2 f_{\log}\left(y;m,v^2\right)}{\partial y^2}\right|_{y=TK/S_0}\right] + \epsilon\left(K\right) \tag{4.6}$$

where $c_{log}$ is given by (4.4).

In Figure 1, for the set of parameters $T$=1, $r$=0.09, $S$=1, $\sigma = 0.4$, we compare the transition densities: a) of the underlying asset (i.e. the lognormal density), b) of the arithmetic average obtained with the numerical Laplace inversion, c) the lognormal approximation to the average, fitting the first two moments of the arithmetic average and d) the Edgeworth approximations, fitting the first four moments of the arithmetic average. Note that using the Edgeworth approximation, for high volatility levels $\left(\sigma\sqrt{T} > 0.3\right)$the density becomes bimodal and besides it can assume negative values.

### 4.1.3. The reciprocal gamma approximation

Milevsky and Posner [39] have proved that the stationary density for the arithmetic average of a geometric Brownian motion is given by a reciprocal gamma density, i.e. the reciprocal of the average has a gamma density. The same result, using different techniques, have been also obtained by Dufresne [18], De Schepper et al. [16], Yor [54], page 525.

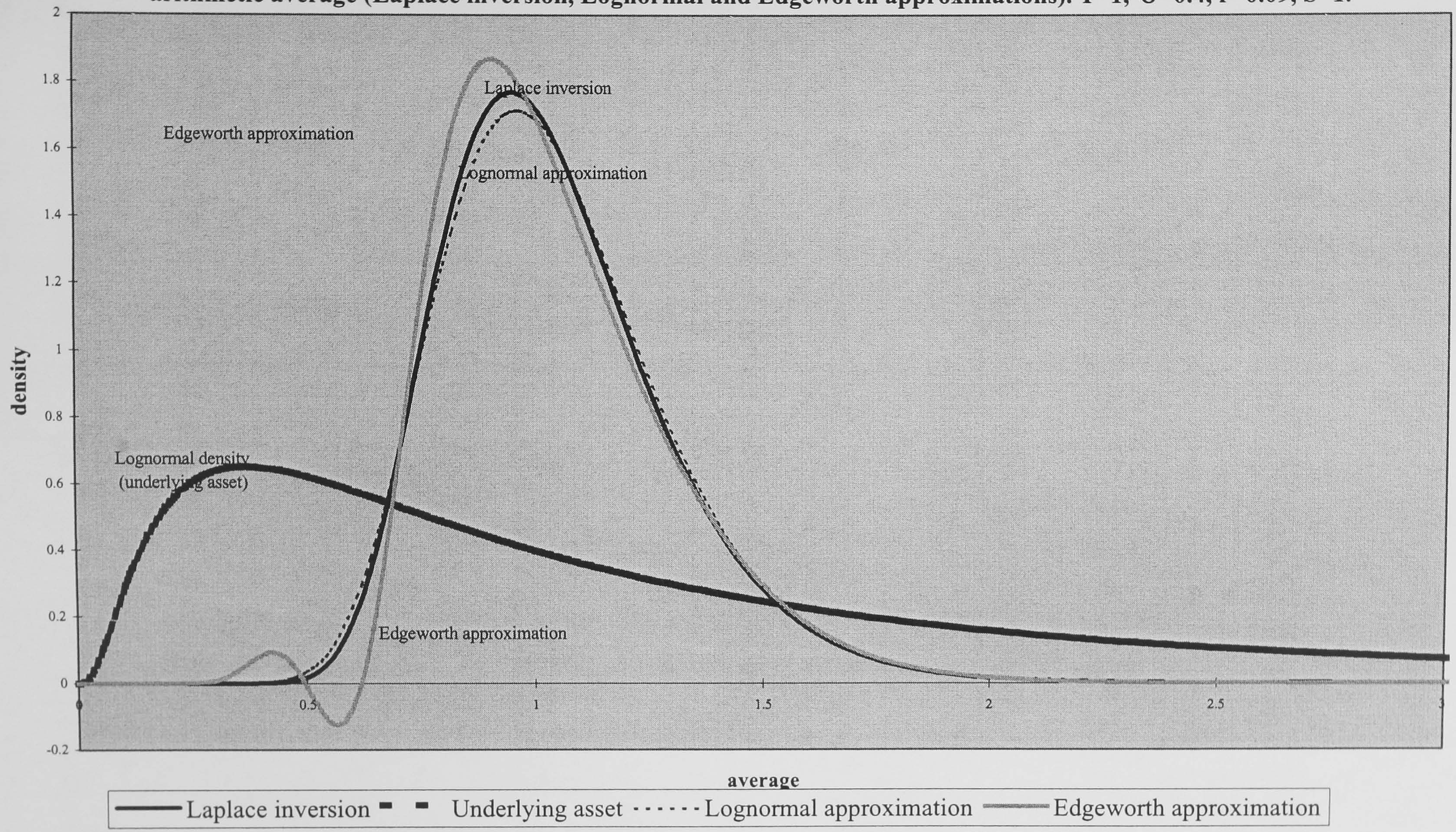
In particular they have shown that, if $r - \sigma^2/2 < 0$, $\bar{A}_\infty = \lim_{t\to\infty} \bar{A}_t$has a density given by:

$$f_{rg}\left(x;\alpha,\beta\right) = \frac{\beta^{-\alpha}x^{-1-\alpha}}{\Gamma\left(\alpha\right)}\exp\left(-1/(x\beta)\right)$$

where $\alpha = 1\text{-}2r/\sigma^2$and $\beta = \sigma^2/2$. It is then easy to verify that $1/\bar{A}_\infty$ has a gamma density.

[7]Respect to the formula in Jarrow and Rudd in the present formula does not appear the term involving the difference between the second-order cumulants because, by construction, it is equal to zero.

**Figure 1: Density for the underlying asset in the lognormal case and different approximations to the arithmetic average (Laplace inversion, Lognormal and Edgeworth approximations). T=1, $\sigma$=0.4, r=0.09, S=1.**



Milevsky and Posner [39] state also that, according to the Kolmogorov-Smirnov criterion, this density is a better fit to the true density than the lognormal, where the true density has been estimated using Monte Carlo simulation.

The idea in Milevsky and Posner [39] is to approximate the density of the average using the stationary density given above, but where the parameters $\alpha$ and $\beta$ are chosen in order to match the first two moments of the average. The moments of the reciprocal gamma distribution are:

$$E_{rg}\left[\left(\frac{1}{A_\infty}\right)^n\right] = \frac{1}{\beta\,(\alpha-1)\,(\alpha-2)\ldots(\alpha-n)}$$

and are positive, provided that $\alpha > n$. Fitting the first two moments to the corresponding moments of the arithmetic average $S_0\,\bar{A}_T\,/T$, the parameters $\alpha$ and $\beta$ are given by:

$$\alpha = \frac{S_0}{T}\frac{2\mu_2-\mu_1^2}{\mu_2-\mu_1^2}, \beta = \left(\frac{S_0}{T}\right)^2\frac{\mu_2-\mu_1^2}{\mu_2\mu_1}$$

and, assuming continuous averaging, the price $c_{rg}$ of the Asian option with residual life equal to $T$ is given by:

$$c_{rg} = \frac{1-e^{-rT}}{rT}S_0 G\left(\frac{1}{K};\alpha-1,\beta\right) - e^{-rT}KG\left(\frac{1}{K};\alpha,\beta\right) \tag{4.7}$$

where $G\,(x;\alpha,\beta)$ is the cumulative density function of the gamma density function and can be expressed in terms of the incomplete Gamma function, Press et al. [40], page 216.

### 4.1.4. The GB2 approximation

We propose the generalized beta of the second kind (GB2) for describing the density of the average for the fact that particular cases of this density are the lognormal, i.e. the distribution of the underlying asset, and the reciprocal gamma density, i.e. the stationary density law of the average. There are several application of the GB2 distribution in economics and Finance, compare McDonald [37], Bookstaber and McDonald [7], Aparicio and Hodges [4], Rebonato [41] and McDonald [38] and the references therein. Unfortunately, at the same manner as other approximations used by practitioners, the use of a GB2 density does not permit to give a bound to the committed error, although its use is more theoretically founded.

The GB2 density depends on four parameters $a$, $b$, $p$, $q$ and is given by:

$$f_{gb2}(x; a, b, p, q) = \begin{cases} \frac{a x^{ap-1}}{b^{ap} B(p,q)(1+(x/b)^a)^{p+q}}, & x \geq 0 \\ 0 & elsewhere \end{cases}$$

with $B(p, q)$ the beta function:

$$B(p, q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$$

and $\Gamma(.)$ is the gamma function.

The $nth$ order moments are given by:

$$E_{gb2}[A_t^n] = \frac{b^n B(p + n/a; q - n/a)}{B(p; q)}, -p < n/a < q$$

The parameters $a$, $b$, $p$, $q$ determine the shape and location of the density in a complex manner. The parameter $b$ is a scale parameter and depends on the unit of measure. To larger values of $a$ or $q$ correspond thinner tails of the distribution. Indeed if $a$ assumes large values the density function is largely concentrated around the value of the parameter $b$: this can be verified noting that for large $a$ the mean is near $b$ and the variance tends to zero. The parameters $p$ and $q$ determine the value of skewness that can be positive or negative. Moreover to high values for $p$ and $q$ correspond lighter tails in the distribution.

Practically, the problem reduces to the choice of three parameters, because setting $n = 1$ we can choose $b$ in such a way to equate the first moment of the true density with the first moment of the approximating GB2 density:

$$b = E[A_t] \frac{B(p;q)}{B(p+1/a;q-1/a)}$$

In order to choose the parameters $a$, $p$ and $q$ we minimize respect to them the following objective function:

$$\min_{a,p,q} \left( \sum_{n=2}^{4} \left( E_{gb2}[A_t^n] - E[A_t^n] \right)^2 \right)^{1/2} \tag{4.8}$$

i.e. we minimize the root of the squared errors using the second, third and fourth moments. In the objective function we do not include the first moment, because by construction it is equal in the two distributions. The minimization results to be difficult as the volatility level increases, mainly because we need add constraints requiring that the arguments of the beta function stay positive. In Table 2, we report the estimated parameters varying the interest rate and the volatility level.

Using the estimated parameters, we can then compute the Asian option price integrating the fitted density over the option payoff. We can follow two ways. One possibility is to use the closed form expression for the truncated moments[8] for the GB2 density given in [7], whilst the second one is to do a numerical integration. We have preferred the second way and in order to avoid the problem of integrating over an unbounded domain, we have numerically integrated the density function over the payoff of the put option:

$$p_{gb2} = e^{-rT} \int_0^{KT/S_0} (K - xS_0/T) f_{gb2}(x; a, b, p, q) \, dx \tag{4.9}$$

[8] In particular, the truncated moments can be expressed in terms of the hypergeometric function ${}_2F_1$, but the numerical computation of this function is quite cumbersome, compare Press and al., pag. 271.

The price of the call option can bė then obtained exploiting the put-call parity for Asian options:

$$c_{gb2} = p_{gb2} + e^{-rT}(E\left[\bar{A}_T\right] S_0/T - K) \tag{4.10}$$

where we have used the fact that, by construction, $E_{gb2}\left[\bar{A}_T\right] = E\left[\bar{A}_T\right]$. We have computed the integral in (4.9) using a Gaussian Legendre quadrature with 100 nodes, whilst the parameters in the GB2 density are the solutions to (4.8).

### 4.2. Approximations based on the moments of $1/\bar{A}_T$

Recently, Dufresne [19] has proved the following important and surprising result:

**Theorem 4.1.** *(Dufresne) The distribution of* $1/\bar{A}_T$ *is determined by its moments.*

Exploiting this result, it could be reasonable to recover the density of $1/\bar{A}_T$ and then the density of $\bar{A}_T$. Indeed, for the computation of the price of the Asian option Dufresne uses Laguerre expansions of the price in terms of the moments of $1/\bar{A}_T$. Unfortunately, the computation of the moments requires to solve numerically a differential-difference equation for the $k$-th moment of $1/\bar{A}_T$ and this fact may arise problems in the estimation of higher order moments. In order to obtain a good approximation, Dufresne has to use at least the first 11 moments for volatility levels around 0.3, and the first 5-6 moments for higher volatility levels, confirming the fact that a high volatility increases the accuracy of the approximation methods (Laplace transform, numerical solution of the PDE). The main advantage of this approach is the convergence of the approximated price to the true one as we increase the number of considered moments. The main disadvantages are: a) no analytical bound on the error introduced by truncating the series after $n$ terms is available, b) it is not guaranteed that the density assumes always positive values, c) the approximation works well only for $\sigma\sqrt{T} > 0.28$.

| Table 2: Parameters for the GB2 density (T=1yr) | | | | | | |
|---|---|---|---|---|---|---|
| $r$ | $\sigma$ | $a$ | $b$ | $p$ | $q$ | Ob. Func. (4.8) |
| 0.05 | 0.05 | 7.206668 | 1.024784 | 46.14809 | 46.08233 | 4.5E-07 |
| 0.09 | 0.05 | 20.4018 | 1.023616 | 6.083099 | 6.159717 | 3.45E-07 |
| 0.15 | 0.05 | 15.53325 | 1.021687 | 10.32477 | 9.841517 | 1.85E-08 |
| 0.05 | 0.1 | 4.266127 | 0.967796 | 37.49543 | 29.61429 | 3.33E-07 |
| 0.09 | 0.1 | 4.253543 | 0.988102 | 37.28448 | 29.53633 | 3.82E-07 |
| 0.15 | 0.1 | 4.234851 | 1.019615 | 36.97273 | 29.4205 | 4.64E-07 |
| 0.05 | 0.2 | 2.506444 | 0.868733 | 29.95544 | 20.26853 | 3.71E-06 |
| 0.09 | 0.2 | 2.49837 | 0.888634 | 29.71946 | 20.25761 | 4.96E-06 |
| 0.15 | 0.2 | 2.489439 | 0.916865 | 29.44206 | 20.1353 | 4.49E-06 |
| 0.05 | 0.3 | 1.793849 | 0.739806 | 28.83111 | 16.70598 | 9.7E-06 |
| 0.09 | 0.3 | 1.793375 | 0.757319 | 28.45935 | 16.59103 | 9.21E-06 |
| 0.15 | 0.3 | 1.780583 | 0.781104 | 28.36091 | 16.61345 | 9.53E-06 |
| 0.05 | 0.4 | 1.417093 | 0.609365 | 28.65367 | 14.48636 | 3.19E-05 |
| 0.09 | 0.4 | 1.414075 | 0.625293 | 28.31111 | 14.45149 | 3.44E-05 |
| 0.15 | 0.4 | 1.400483 | 0.644665 | 28.30516 | 14.54752 | 3.79E-05 |
| 0.05 | 0.5 | 1.16881 | 0.483159 | 29.64121 | 13.19732 | 0.000102 |
| 0.09 | 0.5 | 1.154146 | 0.486798 | 30.17376 | 13.38046 | 0.000106 |
| 0.15 | 0.5 | 1.129694 | 0.49015 | 31.20385 | 13.71576 | 0.000113 |

### 4.3. Approximations based on the moments of $\ln \bar{A}_T$

In this section we show that:

a) the distribution of the logarithm of the average is determined by its moments;

b) we then show how to calculate the logarithmic moments;

c) we discuss three possibilities for recovering the density using the information content of the logarithmic moments.

We have the following result:

**Theorem 4.2.** *The distribution of the logarithm of the average is determined by its moments.*

**Proof**: The proof is quite simple once we register the result in Dufresne [19] that $E\left[\bar{A_T}^{\gamma}\right]$ exists finite for every real value of $\gamma$[9]. This fact implies that the moment generating function of $\ln \bar{A_T}$:

$$E\left[e^{\gamma \ln \bar{A_T}}\right] = \int_{-\infty}^{+\infty} e^{\gamma x} f_{\ln \bar{A_T}}(x)\, dx = \int_{0}^{+\infty} e^{\gamma \ln x} f_{\bar{A_T}}(x)\, dx = E\left[\bar{A_T}^{\gamma}\right]$$

exists finite for every real value of $\gamma$. Then the density function to be recovered has an exponentially decaying behavior[10] and this is a sufficient condition for moment problem determinacy♦.

This result appears natural if we recall that the logarithm of a lognormal distribution has a normal distribution that is determined in a unique way from the sequence of its moments. The distribution of the average should be not very different from a lognormal distribution and so it is natural the result that the logarithm of the average has a distribution determined entirely from its moments.

[9] It is important to remember that the existence of all (power) moments is a necessary condition for the distribution to be determined by its moments. If the moments are not finite it means that the tails do not have an exponential decaying behavior.

[10] If there exists $\int_0^{+\infty} e^{\gamma x} f_{\ln A}(x)\, dx$ then $f_{\ln A}(x)$ has an asymptotic behavior given by $e^{-\alpha x}$, $|\alpha| > |\gamma|$.

#### 4.3.1. The moments of $\ln \bar{A}_T$

In order to calculate the moments of $\ln \bar{A}_T$ we use the expression given[11] in Geman and Yor [23] for the moments of $\bar{A}_T^{\,n}$, $n$ real.

If we define:

$$D_h = \int_0^h e^{2(W_s+vs)} ds$$

we have, by the scaling property of BM:

$$A_t = \frac{4}{\sigma^2} D_{\frac{\sigma^2 t}{4}}$$

For every $n \geq 0$ ($n$ a real number), the Laplace transform of $E\left(D_h^n\right)$ with respect to $h$, compare Geman and Yor[23], is given by:

$$\begin{aligned} \mathcal{L}_h\left(E\left(D_h^n\right)\right) &:= \textstyle\int_0^{+\infty} e^{-\lambda h} E\left(D_h^n\right) dh \\ &= \frac{1}{\lambda} \frac{\Gamma(n+1)\Gamma((\mu+v)/2+1)\Gamma((\mu-v)/2-n)}{2^n \Gamma((\mu-v)/2)\Gamma(n+1+(\mu+v)/2)} \end{aligned} \tag{4.11}$$

so we have that:

$$\mathcal{L}_h\left(E\left(A_t^n\right)\right) = \mathcal{L}_h\left(E\left(\frac{4}{\sigma^2} D_h\right)^n\right) = \left(\frac{4}{\sigma^2}\right)^n \mathcal{L}_h\left(E\left(D_h\right)^n\right)$$

If we differentiate $m$ times the above expression with respect to $n$ and then we set $n = 0$, we obtain:

$$\left.\frac{\partial^m \mathcal{L}_h\left(E\left(A_t^n\right)\right)}{\partial^m n}\right|_{n=0} = \mathcal{L}_h\left(E\left(\ln^m A_t\right)\right)$$

i.e. the Laplace transform of the desired moments. Then the moments are obtained by inversion. The procedure is the following:

[11] Alternatively, the formula below can be obtained using the same approach described in Appendix A for the determination of the Laplace transform of the Asian option price.

1) For every fixed $\lambda$, we compute the $m$-order derivative of (4.11) with respect to $n$.

2) We operate the numerical inversion choosing the values of $T$, $\sigma$ and $r$ and so we fix the corresponding value of $h = \sigma^2 T/4$.

The differentiation can be done analytically using programs like Maple or Mathematica or numerically applying the Cauchy-Goursat integral theorem[12], compare Churchill and Brown [11]. Only for higher order moments ($m \geq 8$), we started to observe differences between the two results. We used the second method for the higher speed of calculus. The inversion has been done using the Abate-Whitt algorithm with the same parameter settings as for the inversion of the Laplace transform of the Asian option price. Respect to the inversion of the Asian option price, we did not have problems for low volatility levels. Moreover, the computation of the Laplace transform in this case appears much less difficult than respect to the Geman-Yor formula.

### 4.3.2. The Asian option prices

Once we have calculated the moments of ln $\bar{A}_T$ we have to choose how many moments consider for recovering the density of ln $\bar{A}_T$ . We now discuss three possible choices: a) the normal distribution, b) an Edgeworth expansion around a normal, c) a maximum entropy approximation.

**The normal approximation** A simple choice is to approximate the density of ln $\bar{A}_T$ with a normal density with the same two moments. This implies that the distribution of $\bar{A}_T$ is lognormal and we can use the Black-Scholes model. In particular, in this case the

[12] In applying the Cauchy integral theorem, note that the real part of the singularities for $n$ are given by -1, Re-$(1+(\mu+v)/2)=r_0$ and by Re$((\mu+v)/2)=r_1>0$. We can verify that $|r_1| < |r_0|$, so that as contour in the Cauchy integral we have chosen the circle centered at the origin with radius $R <$min[1, Re$((\mu+v)/2)$]. The numerical computation of the Cauchy integral is obtained trough the Lyness algorithm that reduces the computation of the integral along a closed path in the complex plane to the computation of a real integral over $[-\pi, \pi]$ by the trapezoidal rule.

Asian option price is given by:

$$c_{en2} = Se^{m_1+v^2/2-rT}N(d_1) - e^{-rT}KN(d_2) \tag{4.12}$$

where:

$$\begin{aligned}
m_1 &= E\left(\ln \bar{A_T}\right) \\
v^2 &= E\left(\ln^2 \bar{A_T}\right) - m_1^2 \\
d_1 &= \frac{\ln(S/K) + m_1 + v^2}{v} \\
d_2 &= d_1 - v
\end{aligned}$$

The formula is very similar to (4.4), but with the difference that in (4.4) we have fitted the first two moments of $\ln \bar{A_T}$, whilst in (4.4) we have fitted the first two moments of $\bar{A_T}$. This fact implies that the expected value of $\bar{A_T}$ implied by using the normal approximation is different from the true expected values. As consequence we can expect a bias as we decrease the strike price or as we increase the interest rate.

Finally, we remark that if we want to approximate the density of $\bar{A_T}$ using a lognormal distribution, in the context of a ME approach such a distribution is obtained resorting to $E\left[\ln \bar{A_T}\right]$ and $E\left[\ln^2 \bar{A_T}\right]$, and not to $E\left[\bar{A_T}\right]$ and $E\left[\bar{A_T}^2\right]$ as usually is done.

**The Edgeworth approximation** A second possibility is to use as density for $\ln \bar{A_T}$ an Edgeworth series approximation around a normal distribution and not, as usually it is done, to approximate the density of $\bar{A_T}$ using an Edgeworth series around a lognormal distribution.

In the present case the approximating density for $\ln \bar{A_T}$ is given by:

$$\begin{aligned} f_{edgnorm}(y) &= f_{norm}\left(y; m, v^2\right) + \sum_{i=1}^{4} \frac{k_i}{i!} \frac{\partial^i f_{norm}\left(y; m, v^2\right)}{\partial y^i} + e(y) && (4.13) \\ m &= E\left(\ln \bar{A_T}\right) && (4.14) \\ v^2 &= E\left(\ln^2 \bar{A_T}\right) - m^2 && (4.15) \end{aligned}$$

where $f_{norm}(y)$ is the normal density with mean $m$ and variance $v^2$, $e(y)$ is the residual error term and $k_i$ is the difference between the i-th cumulant of the exact distribution and the approximate distribution:

$$k_i = \chi_i(f) - \chi_i(l)$$

and:

$$\begin{aligned} \chi_1(f) &= \mu_1 = E\left(\ln \bar{A_T}\right) \\ \chi_2(f) &= E\left(\ln \bar{A_T} - \mu_1\right)^2 \\ \chi_3(f) &= E\left(\ln \bar{A_T} - \mu_1\right)^3 \\ \chi_4(f) &= E\left(\ln \bar{A_T} - \mu_1\right)^4 - 3\chi_2(f) \end{aligned}$$

The approximating density for $\bar{A_T}$ is given by the following expression:

$$f_{\bar{A_T}}(y) = \frac{f_{ednorm}\left(\ln y; m, v^2\right)}{y}$$

Using this density we have computed the price of the Asian call option exploiting the fact that:

$$\begin{aligned} \int_0^{+\infty} \left(y\tfrac{S}{T} - K\right)^+ f_{\bar{A_T}}(y)\, dy &= \tfrac{S}{T} \int_{KT/S}^{+\infty} \left(y - K\tfrac{T}{S}\right)^+ f_{\bar{A_T}}(y)\, dy \\ &= \tfrac{S}{T} \int_0^1 \left(\tfrac{KS}{Tw} - K\tfrac{T}{S}\right)^+ f_{\bar{A_T}}\left(\tfrac{KT}{Sw}\right) \tfrac{KS}{Tw^2} dw \end{aligned}$$

so that the infinite integration range has been reduced to a finite one.

Obviously these two approximations have the limit that there is no expression for bounding the error, but respect to the approximations based on the lognormal distribution are much more theorethically founded.

**The Maximum Entropy (ME) approximation** The third possibility is to consider the ME distribution, method pioneered in statistics by Jaynes [28]. Obviously the ME approach can appear more time-consuming and cumbersome than the other empirical methods, and there is little hope to use it for real-time trading, but it can be very useful where the computational time is not an issue. However the computation of the ME approximation did not require more than eight seconds. Moreover the ME approach can be helpful for having a better understanding of the correct number of moments that we should consider in the problem at hand. Indeed we are able to give a theoretical bound to the error committed as functions of the number of considered moments. Finally the ME distribution is very flexible in the sense that, given the information content in the moments, it is able to reproduce it.

Given the first $2M$ characterising moments, in our case the logarithmic moments, we can try reconstruct the true but unknown density with the ME density:

$$E_0(\ln^j x) = \int_0^{\infty} \ln^j x f_{true}(x)dx = \int_0^{\infty} \ln^j x f_{2M}(x)dx \qquad j = 0, ..., 2M \tag{4.16}$$

where the ME distribution $f_{2M}(x)$, obtained as solution of a variational problem, is given by, compare Kesavan [30]:

$$f_{2M}(x) = \exp\left(-\sum_{j=0}^{2M} \lambda_j \ln^j x\right) \tag{4.17}$$

and where $(\lambda_1, ..., \lambda_{2M})$ are Lagrange's multipliers that solve the minimization problem:

$$\min_{\lambda_1,...,\lambda_{2M}} \Gamma(\lambda_1, ..., \lambda_{2M}) = \sum_{j=1}^{2M} \lambda_j E_0\left(\ln^j x\right) + \ln\left[\int_0^{+\infty} \exp\left(-\sum_{j=1}^{2M} \lambda_j \ln^j(x)\right) dx\right] \quad (4.18)$$

whilst $\lambda_0$, the normalization constant, is obtained by (4.16) with $j$=0.

The unicity of the ME solution, if it exists, is guaranteed by the convexity of the potential function $\Gamma(\lambda_1, ..., \lambda_{2M})$, Kesavan [30]. The above approximant (4.17) is flexible and assumes different shapes: for $M = 1$ we obtain a lognormal distribution.

The choice of the ME approximant is also due to the fact that the ME density $f_{2M}(x)$ converges to the true density in entropy, compare Tagliani [48], i.e.:

$$\lim_{M\to\infty} -\int_0^{+\infty} f_{2M}(x) \ln f_{2M}(x)\, dx = -\int_0^{+\infty} f_{true}(x) \ln f_{true}(x)\, dx =: H[f_{true}]$$

and in the $L_1$ norm, exploiting the fact that, Kullback [31]:

$$\int_0^{+\infty} |f_{2M}(x) - f_{true}(x)|\, dx \leq \sqrt{2\left(H[f_{2M}] - H[f_{true}]\right)}$$

As more moments are added, then the entropies satisfy the inequalities:

$$H[f_2] \geq H[f_4] \geq ... \geq H[f_{2M}] \geq ... \quad (4.19)$$

In general the sequence of differences $H[f_2]-H[f_4]$, $H[f_4]-H[f_2]$,...,$H[f_{2M-2}]-H[f_{2M}]$ decays to zero more or less fast. In the former case it means the distribution information content is represented by few first moments, so that the approximant having the given first moments well fits the underlying unknown distribution.

These results are particularly important because, if we are interested in the Asian option price we need to compute, exploiting the put-call parity, the following quantity:

$$p_{true} = \int_0^K (K-a)\, f_{true}(a)\, da$$

where now $f_{true}$ stands for the true but unknown density of $A_T$. Exploiting the convergence in $L_1$ we can then find a bound to the difference between the true price $p_{true}$ and the approximated price $p_{2M}$. Indeed, considering an even number $2M$ of moments, we have :

$$\begin{aligned}
|p_{true} - p_{2M}| &= \left|\int_0^{+\infty} (K - aS_0/t)^+ f_{true}(a)\, da - \int_0^{+\infty} (K - aS_0/t)^+ f_{2M}(a)\, da\right| \\
&= \left|\int_0^{+\infty} (K - aS_0/t)^+ (f_{true}(a) - f_{2M}(a))\, da\right| \\
&\le \int_0^{+\infty} |K - aS_0/t|\, |f_{true}(a) - f_{2M}(a)|\, da \\
&\le \int_0^{+\infty} K\, |f_{true}(a) - f_{2M}(a)|\, da \\
&\le K \int_0^{\infty} |f_{true}(a) - f_{2M}(a)|\, da
\end{aligned}$$

and so we obtain an explicit bound for the error:

$$|p_{true} - p_{2M}| \le K\sqrt{2\left(H[f_{2M}] - H[f_{true}]\right)}, \tag{4.20}$$

This expression turns out to depend on $H[f_{true}]$ that is not known, but we can estimate it as follows.

For every distribution, unlike the ME ones with assigned moments, the content of information is shared among the infinite moments. Any additional moment causes a decrease in entropy. Nevertheless for the distributions usually met in the applications, the information content is shared among the moments so that the first moments generate a greater decrease in entropy compared to the higher order moments. Consequently it's reasonable to expect that the sequence of points $(2j, H[f_{2j}])$, $j = 1, 2, \ldots$ lies on a convex curve. Therefore the quantity $\Delta^2 H[f_{2j}] =: H[f_{2(j+1)}] - 2H[f_{2j}] + H[f_{2(j-1)}]$, obtained up to a multiplicative factor by means of finite difference discretization, may be considered as positive. Then the new accelerated sequence

$$H^{acc}[f_{2j}] = H[f_{2j}] - \frac{(H[f_{2(j+1)}] - H[f_{2j}])^2}{\Delta^2 H[f_{2j}]}$$

obtained by means Aitken $\Delta^2$−process is such that $H^{acc}[f_{2j}] < H[f_{2j}]$ and is converging faster to $H[f]$. It is reasonable assume that $H[f] \simeq H^{acc}[f_{2(M-2)}]$ so that an upper bound for the approximation error may be obtained uniquely in terms of calculated quantities:

$$\begin{aligned} \mid p_{true} - p_{2(M-2)} \mid \; &\leq K\sqrt{2(H[f_{2(M-2)}] - H[f_{true}])} \\ &\simeq K\sqrt{2(H[f_{2(M-2)}] - H^{acc}[f_{2(M-2)}])} \\ &= K\sqrt{2\frac{(H[f_{2(M-1)}]-H[f_{2(M-2)}])^2}{(H[f_{2M}]-2H[f_{2(M-1)}]+H[f_{2(M-2)}])}} \end{aligned} \qquad (4.21)$$

In this way, given a desired prefixed error, we have a theoretical criteria for choosing the optimal number of moments in constructing the approximating density.

In Table 3 we report the value of the minimized function[13] in (4.18) for different parameter values (volatility varying between 5% and 100%, interest rate equal to 5%, 9% and 15%) and for a different number of moments (2, 4, 6). From the different Figures we can verify that the most important decrease in the entropy occurs when we increase the number of moments from two to four. The information content of adding the fifth and the sixth moment is very low. For low volatility levels the value of the potential function has a minimum when we consider just the first two moments. This is due to the fact that once we have fitted the first two moments the function to be minimized is very flat.

[13] The minimization has been done in C with the multidimensional conjugate gradient method using the routine frprmn in Press and al. [40] pag. 423. The gradient of the function is obtained by differencing with respect to $\lambda_j$ the function (4.18) and is given by the difference $E_M[\ln^j x] - E_{true}[\ln^j x]$.

| Table 3: The potential function in (4.18) and the bound in the ME approach | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | *Volatility level* | | | | | | | |
| r=0.05, K=1 | 0.05 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.8 | 1 |
| 2M=2 | -2.09532 | -1.40389 | -0.71754 | -0.32336 | -0.0514 | 0.151684 | 0.536641 | 0.683853 |
| 2M=4 | -2.09531 | -1.40396 | -0.71793 | -0.32425 | -0.05298 | 0.149212 | 0.530328 | 0.674049 |
| 2M=6 | -2.09523 | -1.40403 | -0.71794 | -0.32425 | -0.05298 | 0.14921 | 0.530316 | 0.674011 |
| bound | 0.0022 | 0.0289 | 0.0282 | 0.0422 | 0.0563 | 0.0703 | 0.1125 | 0.1403 |
| r=0.09, K=1 | | | | | | | | |
| 2M=2 | -2.07014 | -1.37871 | -0.69242 | -0.29833 | -0.02649 | 0.176444 | 0.560749 | 0.70738 |
| 2M=4 | -2.0701 | -1.37879 | -0.69281 | -0.29921 | -0.02805 | 0.173995 | 0.554487 | 0.697645 |
| 2M=6 | -2.06988 | -1.37882 | -0.69281 | -0.29921 | -0.02805 | 0.173994 | 0.554475 | 0.697607 |
| bound | 0.0040 | 0.0161 | 0.0280 | 0.0419 | 0.0560 | 0.0700 | 0.1120 | 0.1398 |
| r=0.15, K=1 | | | | | | | | |
| 2M=2 | -2.03215 | -1.34075 | -0.65453 | -0.26057 | 0.011088 | 0.213787 | 0.597113 | 0.742873 |
| 2M=4 | -2.03207 | -1.34081 | -0.65492 | -0.26144 | 0.009546 | 0.211374 | 0.59093 | 0.733245 |
| 2M=6 | n.a. | -1.34081 | -0.65492 | -0.26144 | 0.009545 | 0.211374 | 0.590918 | 0.733241 |
| bound | n.a. | 0.0116 | 0.0277 | 0.0416 | 0.0556 | 0.0695 | 0.1113 | 0.1388 |

From Table 3 it is evident that using the first four moments can be quite enough. Moreover in their computation the numerical errors were negligible and also the ME estimate is not time consuming: the estimation procedure requires less than 8 seconds and for a given set of parameters the computation of the unknowns $\lambda_j$ is done only once. In the Table we report also the bound given in equation (4.21), that refers to an approximation with the first two moments. The approximation is very good for low volatility levels, but as it increases the bound becomes less tight and more moments are required. As we will illustrate in the next section the first four moments will be sufficient, compare Table 5.

| Moments | Determinacy | Approximant | Convergence | Notes |
|---|---|---|---|---|
| $E_0\left(A_t^n\right)$ | Not known | Edgeworth | Not known | very simple; negative densities; no error bound. |
| $E_0\left(\left(\frac{1}{A_t}\right)^n\right)$ | Yes | Laguerre | Yes | not very fast; numerical problems; no error bound. |
| $E_0\left(\ln A_t^n\right)$ | Yes | Edgeworth | Yes | simple; good accuracy; no error bound. |
| $E_0\left(\ln A_t^n\right)$ | Yes | ME | Yes | not very fast, very accurate; error bound. |

The scheme above compares the different methods that exploit the information content of a sequence of moments and the properties of the approximation used for recovering the density of the average and the option price. Regard to the Edgeworth approximation using the algebraic moments, the convergence to the true density is not known, because the determinacy problem is unsolved. Instead using the Edgeworth approximation with logarithm moments the determinacy problem is positively solved, so that it makes sense to investigate the convergence of the approximation that is uniform as the number of moments considered increase.

## 5. A comparison between the different approximations

The different approximations can be compared in several ways:

1. evaluating with the Kolmogorov-Smirnov criterion the distance between the approximating distribution and the distribution obtained by the Laplace inversion; unfortunately, as we will this measure does not seem to be useful once we consider the pricing problem.

2. compare the different approximations with the lower and the upper bound to the option price as given in Rogers and Shi [43] and in Thompson [49]. We describe these bounds in Appendix C. The lower bound appears to be very strict, but from this bound we cannot recover easily information regard to the density function of the average or the Greeks of the contract;

3. evaluate the difference (mean, absolute, percentage) between the option price coming from the different approximations and the one obtained by the Laplace transform inversion.

As we can see from Table 4 where we report the Kolmogorov Smirnov measure between the density obtained with the numerical inversion and the different approximations, we can draw the following conclusions:

a) for low volatility levels ($\sigma < 0.2$), the different approximations perform equally well and the first two moments seems to be sufficient to recover the density of the average. The lognormal approximation based on the logarithmic moments does slightly better than the lognormal approximation based on the power moments. The reciprocal gamma behaves better than the lognormal. However as we increase the volatility level, it is important to include higher order moments.

b) if we restrict our attention to approximations based on the moments of the average the GB2 approximation dominates the Edgeworth approximation. Moreover using the Edgeworth approximation, for high volatility levels we obtain a bimodal density. Besides the density can assume negative values. This fact should discourage from using an Edgeworth approximation when the distribution is not uniquely determined by its moments.

c) In every case the approximations based on the moments of the logarithm of the average perform better and in this sense the Edgeworth approximation around a normal and the ME with four moments are very good.

d) with regard to the ME approximation, to use the first six moments instead that just the first four does not seem to improve significantly the results.

If we compare the different approximations when we have to price Asian options, we can see a synthesis of the results in Table 5 and in more detail in Tables 6 and 6a-6f. In Table 5, using as benchmark the result of the Laplace inversion, we report the average of the mean squared error (MSE), the average mean absolute error (MAE) and the absolute maximum error (MAXA), where the average is taken considering the above errors in Tables 6a-6f, when we vary the volatility level ($\sigma$ =0.05, 0.1, 0.2, 0.3, 0.4, 0.5). In the case $\sigma$ = 0.05 the column with the Laplace inversion in Table 6 has been substituted by the average of the low and the upper bound, because in this case we had numerical problems in the computation of the Laplace transform. In Table 5 we report also the percentage of times the price of a given approximation stays inside the lower and upper bound given in Rogers and Shi [43] and in Thompson [49].

**Table 4: Kolmogorov-Smirnov distance between the true distribution and the different approximations**

Parameters setting: r=0.05, T=1, $S_0$=1

| Volatility | Fitlog | EdgeLog | RGamma | GB2 | ME2 | EdgeNorm | ME4 | ME6 |
|---|---|---|---|---|---|---|---|---|
| 0.1 | 0.03055 | 0.03301 | 0.03441 | 0.03248 | 0.03053 | 0.03282 | 0.03199 | 0.03168 |
| 0.2 | 0.01327 | 0.01821 | 0.01961 | 0.01569 | 0.01246 | 0.01661 | 0.01656 | 0.01647 |
| 0.3 | 0.01071 | 0.01725 | 0.01549 | 0.00990 | 0.00833 | 0.01123 | 0.01142 | 0.01132 |
| 0.4 | 0.01200 | 0.02530 | 0.01397 | 0.00673 | 0.00801 | 0.00851 | 0.00889 | 0.00886 |
| 0.5 | 0.01470 | 0.06956 | 0.01344 | 0.00617 | 0.00859 | 0.00687 | 0.00746 | 0.00740 |
| Maximum | 0.03055 | 0.06956 | 0.03441 | 0.03248 | 0.03053 | 0.03282 | 0.03199 | 0.03168 |
| Average | 0.01624 | 0.03266 | 0.01938 | 0.01420 | 0.01358 | 0.01521 | 0.01527 | 0.01514 |

Legend. Fitlog: Lognormal Approximation using the first two moments of the average,

EdgeLog: Edgeworth expansion around a Lognormal using the first four moments,

RGamma: Reciprocal Gamma approximation, GB2: GB2 approximation

ME2 (ME4, ME6): ME approximation using the first two (four, six) logarithmic moments

EdgeNorm: Edgeworth expansion around a Normal using the first four logarithmic moments,

**Table 5: Evaluation of different approximations**

| | MSE | MAE | MAXA | %Inside |
|---|---|---|---|---|
| AW | - | - | - | 96.67% |
| ME4 | 0.00028 | 0.00091 | 0.00380 | 64.44% |
| Lower Bound | 0.00049 | 0.00183 | 0.00647 | - |
| EdgeNorm | 0.00083 | 0.00304 | 0.01279 | 31.11% |
| GB2 | 0.00203 | 0.00682 | 0.03363 | 31.11% |
| Upper Bound | 0.00235 | 0.00888 | 0.03631 | - |
| ME2 | 0.01179 | 0.03858 | 0.24215 | 3.33% |
| RGamma | 0.01362 | 0.05009 | 0.19608 | 2.22% |
| FitLog | 0.01661 | 0.05933 | 0.24819 | 8.89% |
| EdgeLog | 0.01928 | 0.06085 | 0.56647 | 31.11% |

Legend.

AW: numerical inversion of the Laplace transform;

Lower Bound: lower bound in [43] and in [49];

Upper Bound: upper bound in [49];

MSE: mean squared error, MAE: mean absolute error;

MAXA: maximum absolute error; others as in table 4.

We can draw the following conclusions:

a) Comparing Tables 6a-6f, we can see the very good results that we can obtain with the numerical Laplace inversion. Indeed also with a volatility level at 10% when the bounds to the option price are very tight, the price obtained stays inside this small range: e.g. when $\sigma = 0.1$, $r = 0.05$, $T = 1$, $S_0 = 100$ and $K = 100$ the lower bound is 3.64134 and the upper bound is 3.64157 and the Laplace inversion[14] gives 3.64139.

b) A second aspect are the accurate results obtained with the maximum entropy estimate based on four moments. The fact that the percentage of times this estimate stays inside the bound is only 64.4% is due to the difficulties in estimating the ME distribution for low volatility levels ($\sigma = 5\%$). However, as we increase the volatility, also the ME estimate stays inside the bounds and the improvement respect to the lognormal approximation is around 65 times. The Edgeworth expansion around the normal distribution confirms the fact that using the first four logarithmic moments is the correct choice, although we get price just below the lower bound. The maximum error committed with a ME approach with four moments is equal at 0.0032 when $\sigma$=10% and reduces to 0.0015 as $\sigma$ increases at 50%. For this reason the bounds given in Table 3 and relative at an approximation with the first two moments can appear too conservative.

c) For low volatility levels the bounds appear very tight and in general the lower bound is very near the price given by the numerical Laplace inversion.

d) Among the approximations that use the power moments, the Edgeworth expansion around a lognormal distribution performs very badly, mainly as we increase the volatility level. The use of the Reciprocal Gamma approximation does not seem to improve significantly the results respect to the lognormal approximation. Another interesting point is that also with very low volatility levels the prices obtained with the approximations based on the power moments stay very often outside the bounds. However the maximum absolute difference between the lognormal approximation and the Laplace answer is 0.008 for volatility at the 10% level and increases at 0.248 for a volatility at 50%. This non-enormous absolute error can explain the success of the lognormal approximation among practitioners.

[14] In this case the Abate-Whitt algorithm has been used with a total of 50 (35+15) terms, instead than the usual 26 (15+11) as in the other tables.

**Table 6a: A Comparison of Asian Option Pricing Models (volatility 5%, S=100)**

| r | K | Lower Bound | FitLog | EdgeLog | RGamma | GB2 | ME2 | EdgeNorm | ME4 | p B -Low.B | Upper Bound |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 90 | 11.93050 | 11.93050 | 11.93050 | 11.93050 | 11.93051 | 11.93050 | 11.93051 | 11.93063 | 11.93050 | 11.93051 |
| 0.05 | 95 | 7.17773 | 7.17799 | 7.17773 | 7.17756 | 7.17818 | 7.17797 | 7.17786 | 7.17813 | 7.17778 | 7.17783 |
| 0.05 | 100 | 2.71617 | 2.71818 | 2.71615 | 2.71478 | 2.71808 | 2.71799 | 2.71596 | 2.71845 | 2.71620 | 2.71622 |
| 0.05 | 105 | 0.33723 | 0.33545 | 0.33729 | 0.33849 | 0.33533 | 0.33525 | 0.33718 | 0.33490 | 0.33729 | 0.33736 |
| 0.05 | 110 | 0.00802 | 0.00759 | 0.00802 | 0.00834 | 0.00789 | 0.00758 | 0.00826 | 0.00746 | 0.00810 | 0.00819 |
| 0.09 | 90 | 13.37821 | 13.37821 | 13.37821 | 13.37821 | 13.37821 | 13.37820 | 13.37821 | 13.37840 | 13.37821 | 13.37821 |
| 0.09 | 95 | 8.80884 | 8.80888 | 8.80884 | 8.80881 | 8.80892 | 8.80887 | 8.80890 | 8.80908 | 8.80886 | 8.80887 |
| 0.09 | 100 | 4.30823 | 4.30972 | 4.30823 | 4.30720 | 4.31021 | 4.30964 | 4.30877 | 4.31051 | 4.30830 | 4.30837 |
| 0.09 | 105 | 0.95833 | 0.95815 | 0.95838 | 0.95851 | 0.95729 | 0.95788 | 0.95694 | 0.95781 | 0.95841 | 0.95849 |
| 0.09 | 110 | 0.05210 | 0.05088 | 0.05214 | 0.05303 | 0.05137 | 0.05081 | 0.05292 | 0.05007 | 0.05223 | 0.05236 |
| 0.15 | 90 | 15.39763 | 15.39763 | 15.39763 | 15.39763 | 15.39763 | 15.39763 | 15.39763 | 15.39790 | 15.39763 | 15.39763 |
| 0.15 | 95 | 11.09409 | 11.09410 | 11.09409 | 11.09409 | 11.09410 | 11.09409 | 11.09410 | 11.09436 | 11.09410 | 11.09410 |
| 0.15 | 100 | 6.79435 | 6.79463 | 6.79436 | 6.79417 | 6.79485 | 6.79462 | 6.79464 | 6.79515 | 6.79441 | 6.79447 |
| 0.15 | 105 | 2.74441 | 2.74638 | 2.74443 | 2.74303 | 2.74647 | 2.74621 | 2.74418 | 2.74830 | 2.74449 | 2.74458 |
| 0.15 | 110 | 0.42374 | 0.42233 | 0.42389 | 0.42497 | 0.42192 | 0.42212 | 0.42330 | 0.42041 | 0.42393 | 0.42413 |
| | MSE | 2.03E-05 | 2.81E-04 | 1.24E-05 | 1.96E-04 | 3.08E-04 | 2.83E-04 | 1.25E-04 | 4.66E-04 | - | 2.03E-05 |
| | MAE | 5.76E-05 | 7.54E-04 | 3.66E-05 | 5.23E-04 | 8.56E-04 | 7.74E-04 | 2.96E-04 | 1.32E-03 | - | 5.76E-05 |
| | MAPE | 9.07E-02 | 6.69E-01 | 8.31E-02 | 3.46E-01 | 3.77E-01 | 6.98E-01 | 2.43E-01 | 9.35E-01 | - | 9.07E-02 |
| | MAXA | 1.99E-04 | 1.98E-03 | 8.91E-05 | 1.46E-03 | 2.01E-03 | 2.04E-03 | 1.47E-03 | 3.80E-03 | - | 1.99E-04 |

**Table 6b: A Comparison of Asian Option Pricing Models (volatility 10%, S=100)**

| r | K | Lower Bound | FitLog | EdgeLog | RGamma | GB2 | ME2 | EdgeNorm | ME4 | AW | Upper Bound |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 90 | 11.95108 | 11.95333 | 11.95127 | 11.94968 | 11.95222 | 11.95305 | 11.95113 | 11.95019 | 11.95107 | 11.95223 |
| 0.05 | 95 | 7.40774 | 7.41517 | 7.40754 | 7.40275 | 7.40957 | 7.41415 | 7.40785 | 7.40472 | 7.40778 | 7.40860 |
| 0.05 | 100 | 3.64134 | 3.64748 | 3.64091 | 3.63698 | 3.64007 | 3.64538 | 3.64122 | 3.63998 | 3.64139 | 3.64157 |
| 0.05 | 105 | 1.31057 | 1.30684 | 1.31097 | 1.31307 | 1.30800 | 1.30471 | 1.31054 | 1.31346 | 1.31068 | 1.31098 |
| 0.05 | 110 | 0.33107 | 0.32399 | 0.33155 | 0.33616 | 0.33098 | 0.32285 | 0.33133 | 0.33419 | 0.33121 | 0.33224 |
| 0.09 | 90 | 13.38519 | 13.38629 | 13.38535 | 13.38452 | 13.38584 | 13.38611 | 13.38518 | 13.38510 | 13.38537 | 13.38603 |
| 0.09 | 95 | 8.91183 | 8.91721 | 8.91185 | 8.90820 | 8.91374 | 8.91657 | 8.91188 | 8.91301 | 8.91185 | 8.91296 |
| 0.09 | 100 | 4.91508 | 4.92310 | 4.91459 | 4.90938 | 4.91536 | 4.92141 | 4.91511 | 4.91826 | 4.91512 | 4.91541 |
| 0.09 | 105 | 2.06993 | 2.07045 | 2.07002 | 2.06952 | 2.06731 | 2.06818 | 2.06999 | 2.07137 | 2.07007 | 2.07038 |
| 0.09 | 110 | 0.63006 | 0.62338 | 0.63072 | 0.63501 | 0.62887 | 0.62178 | 0.63033 | 0.62773 | 0.63027 | 0.63102 |
| 0.15 | 90 | 15.39877 | 15.39906 | 15.39885 | 15.39859 | 15.39898 | 15.39894 | 15.39877 | 15.39850 | 15.39715 | 15.39915 |
| 0.15 | 95 | 11.11948 | 11.12196 | 11.11966 | 11.11781 | 11.12073 | 11.12167 | 11.11956 | 11.11985 | 11.11923 | 11.12061 |
| 0.15 | 100 | 7.02768 | 7.03485 | 7.02746 | 7.02254 | 7.02948 | 7.03384 | 7.02780 | 7.03026 | 7.02774 | 7.02845 |
| 0.15 | 105 | 3.61249 | 3.61869 | 3.61216 | 3.60801 | 3.61154 | 3.61669 | 3.61242 | 3.61592 | 3.61263 | 3.61301 |
| 0.15 | 110 | 1.41329 | 1.41080 | 1.41384 | 1.41542 | 1.41089 | 1.40869 | 1.41339 | 1.41279 | 1.41362 | 1.41431 |
| | MSE | 1.14E-04 | 1.35E-03 | 1.40E-04 | 9.49E-04 | 4.43E-04 | 1.32E-03 | 1.15E-04 | 5.51E-04 | - | 2.37E-04 |
| | MAE | 2.22E-04 | 4.60E-03 | 3.79E-04 | 3.21E-03 | 1.52E-03 | 4.54E-03 | 2.26E-04 | 1.88E-03 | - | 7.84E-04 |
| | MAPE | 8.86E-03 | 3.07E-01 | 1.84E-02 | 2.12E-01 | 6.68E-02 | 3.60E-01 | 7.12E-03 | 1.30E-01 | - | 4.09E-02 |
| | MAXA | 1.62E-03 | 7.98E-03 | 1.70E-03 | 5.74E-03 | 2.75E-03 | 8.49E-03 | 1.62E-03 | 3.29E-03 | - | 2.00E-03 |

**Table 6c: A Comparison of Asian Option Pricing Models (volatility 20%, S=100)**

| r | K | Lower Bound | FitLog | EdgeLog | RGamma | GB2 | ME2 | EdgeNorm | ME4 | AW | Upper Bound |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 90 | 12.59560 | 12.62976 | 12.58815 | 12.57293 | 12.60201 | 12.61924 | 12.59635 | 12.59449 | 12.59599 | 12.60071 |
| 0.05 | 95 | 8.81839 | 8.85250 | 8.80916 | 8.79446 | 8.82021 | 8.83719 | 8.81851 | 8.81728 | 8.81876 | 8.82073 |
| 0.05 | 100 | 5.76271 | 5.78284 | 5.75849 | 5.74752 | 5.75866 | 5.76438 | 5.76225 | 5.76222 | 5.76309 | 5.76443 |
| 0.05 | 105 | 3.50684 | 3.50607 | 3.50989 | 3.50603 | 3.49965 | 3.48743 | 3.50658 | 3.50740 | 3.50733 | 3.50890 |
| 0.05 | 110 | 1.98924 | 1.97107 | 1.99646 | 2.00119 | 1.98290 | 1.95501 | 1.98978 | 1.99076 | 1.98990 | 1.99272 |
| 0.09 | 90 | 13.83122 | 13.86167 | 13.82573 | 13.81078 | 13.83835 | 13.85294 | 13.83193 | 13.83200 | 13.83150 | 13.83721 |
| 0.09 | 95 | 9.99536 | 10.03043 | 9.98596 | 9.97052 | 9.99947 | 10.01702 | 9.99567 | 9.99627 | 9.99565 | 9.99807 |
| 0.09 | 100 | 6.77700 | 6.80355 | 6.77025 | 6.75716 | 6.77542 | 6.78628 | 6.77670 | 6.77776 | 6.77735 | 6.77866 |
| 0.09 | 105 | 4.29594 | 4.30408 | 4.29618 | 4.28890 | 4.28988 | 4.28541 | 4.29560 | 4.29643 | 4.29647 | 4.29798 |
| 0.09 | 110 | 2.54546 | 2.53453 | 2.55152 | 2.55250 | 2.53857 | 2.51730 | 2.54580 | 2.54574 | 2.54622 | 2.54854 |
| 0.15 | 90 | 15.64160 | 15.66533 | 15.63944 | 15.62548 | 15.64827 | 15.65889 | 15.64217 | 15.64203 | 15.64176 | 15.64890 |
| 0.15 | 95 | 11.79357 | 11.82644 | 11.78570 | 11.76994 | 11.79946 | 11.81588 | 11.79408 | 11.79435 | 11.79376 | 11.79747 |
| 0.15 | 100 | 8.40852 | 8.44079 | 8.39956 | 8.38433 | 8.41016 | 8.42593 | 8.40858 | 8.40937 | 8.40884 | 8.41041 |
| 0.15 | 105 | 5.64129 | 5.66133 | 5.63718 | 5.62561 | 5.63775 | 5.64361 | 5.64108 | 5.64191 | 5.64186 | 5.64336 |
| 0.15 | 110 | 3.55469 | 3.55639 | 3.55754 | 3.55282 | 3.54826 | 3.53832 | 3.55486 | 3.55494 | 3.55561 | 3.55777 |
| | MSE | 1.26E-04 | 6.34E-03 | 1.65E-03 | 4.60E-03 | 1.46E-03 | 5.09E-03 | 1.42E-04 | 1.89E-04 | - | 8.44E-04 |
| | MAE | 4.45E-04 | 2.17E-02 | 5.70E-03 | 1.59E-02 | 5.23E-03 | 1.77E-02 | 4.82E-04 | 5.93E-04 | - | 2.78E-03 |
| | MAPE | 1.04E-02 | 3.16E-01 | 9.62E-02 | 2.28E-01 | 1.11E-01 | 3.77E-01 | 9.37E-03 | 1.04E-02 | - | 4.43E-02 |
| | MAXA | 9.20E-04 | 3.48E-02 | 9.69E-03 | 2.51E-02 | 7.68E-03 | 3.49E-02 | 8.63E-04 | 1.49E-03 | - | 7.13E-03 |

**Legend**: MSE:mean squared error, MAE:mean absolute error, MAPE:mean absolute percentage error, MAXA: maximum absolute error. The names of columns are as in Table 5.

**Table 6d: A Comparison of Asian Option Pricing Models (volatility 30%, S=100)**

| r | K | Lower Bound | FitLog | EdgeLog | RGamma | GB2 | ME2 | EdgeNorm | ME4 | AW | Upper Bound |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 90 | 13.95242 | 14.03812 | 13.89562 | 13.89339 | 13.96194 | 13.98819 | 13.95336 | 13.95423 | 13.95383 | 13.96202 |
| 0.05 | 95 | 10.67537 | 10.74879 | 10.63393 | 10.62185 | 10.67723 | 10.68962 | 10.67462 | 10.67700 | 10.67664 | 10.68218 |
| 0.05 | 100 | 7.94436 | 7.99251 | 7.92948 | 7.90645 | 7.93883 | 7.92806 | 7.94274 | 7.94595 | 7.94563 | 7.95053 |
| 0.05 | 105 | 5.75726 | 5.77417 | 5.76852 | 5.74051 | 5.74682 | 5.70916 | 5.75607 | 5.75898 | 5.75868 | 5.76383 |
| 0.05 | 110 | 4.07011 | 4.05724 | 4.09909 | 4.07482 | 4.05808 | 3.99604 | 4.07041 | 4.07206 | 4.07180 | 4.07864 |
| 0.09 | 90 | 14.98279 | 15.06704 | 14.92543 | 14.92399 | 14.99389 | 15.02159 | 14.98403 | 14.98304 | 14.98396 | 14.99285 |
| 0.09 | 95 | 11.65475 | 11.73287 | 11.60605 | 11.59733 | 11.65910 | 11.67758 | 11.65443 | 11.65486 | 11.65589 | 11.66128 |
| 0.09 | 100 | 8.82755 | 8.88576 | 8.80190 | 8.78216 | 8.82447 | 8.82377 | 8.82615 | 8.82782 | 8.82876 | 8.83329 |
| 0.09 | 105 | 6.51635 | 6.54628 | 6.51749 | 6.49026 | 6.50755 | 6.48187 | 6.51500 | 6.51712 | 6.51779 | 6.52257 |
| 0.09 | 110 | 4.69491 | 4.69511 | 4.71737 | 4.69045 | 4.68338 | 4.63266 | 4.69475 | 4.69639 | 4.69671 | 4.70265 |
| 0.15 | 90 | 16.51203 | 16.59082 | 16.45856 | 16.45608 | 16.52476 | 16.55195 | 16.51362 | 16.51251 | 16.51291 | 16.52372 |
| 0.15 | 95 | 13.14575 | 13.22661 | 13.09056 | 13.08547 | 13.15352 | 13.17760 | 13.14612 | 13.14635 | 13.14669 | 13.15258 |
| 0.15 | 100 | 10.20872 | 10.27814 | 10.16897 | 10.15454 | 10.20970 | 10.22093 | 10.20789 | 10.20971 | 10.20983 | 10.21407 |
| 0.15 | 105 | 7.73683 | 7.78408 | 7.72184 | 7.69780 | 7.73146 | 7.72207 | 7.73560 | 7.73848 | 7.73830 | 7.74274 |
| 0.15 | 110 | 5.72816 | 5.74800 | 5.73774 | 5.70942 | 5.71858 | 5.68520 | 5.72761 | 5.73062 | 5.73012 | 5.73547 |
| | MSE | 3.56E-04 | 1.51E-02 | 9.77E-03 | 1.17E-02 | 2.33E-03 | 9.90E-03 | 5.17E-04 | 1.41E-04 | - | 1.63E-03 |
| | MAE | 1.35E-03 | 5.14E-02 | 3.24E-02 | 4.06E-02 | 7.95E-03 | 3.31E-02 | 1.78E-03 | 4.71E-04 | - | 6.06E-03 |
| | MAPE | 1.82E-02 | 4.99E-01 | 3.29E-01 | 4.03E-01 | 1.13E-01 | 4.86E-01 | 2.48E-02 | 5.46E-03 | - | 7.26E-02 |
| | MAXA | 1.96E-03 | 8.43E-02 | 5.85E-02 | 6.12E-02 | 1.37E-02 | 7.58E-02 | 2.89E-03 | 1.02E-03 | - | 1.08E-02 |

**Table 6e: A Comparison of Asian Option Pricing Models (volatility 40%, S=100)**

| r | K | Lower Bound | FitLog | EdgeLog | RGamma | GB2 | ME2 | EdgeNorm | ME4 | AW | Upper Bound |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 90 | 15.62084 | 15.77682 | 15.43893 | 15.50994 | 15.63718 | 15.64171 | 15.62057 | 15.62355 | 15.62417 | 15.64085 |
| 0.05 | 95 | 12.66307 | 12.79563 | 12.55138 | 12.56333 | 12.66936 | 12.64673 | 12.66028 | 12.66568 | 12.66614 | 12.68038 |
| 0.05 | 100 | 10.13297 | 10.23032 | 10.09832 | 10.05435 | 10.12982 | 10.07371 | 10.12921 | 10.13581 | 10.13597 | 10.14937 |
| 0.05 | 105 | 8.01277 | 8.06917 | 8.04507 | 7.96123 | 8.00232 | 7.91132 | 8.00984 | 8.01613 | 8.01592 | 8.02963 |
| 0.05 | 110 | 6.26877 | 6.28428 | 6.34866 | 6.24609 | 6.25387 | 6.13109 | 6.26815 | 6.27280 | 6.27226 | 6.28824 |
| 0.09 | 90 | 16.49702 | 16.65395 | 16.29773 | 16.38398 | 16.51620 | 16.52672 | 16.49741 | 16.49932 | 16.49997 | 16.51601 |
| 0.09 | 95 | 13.50789 | 13.64791 | 13.36949 | 13.40168 | 13.51777 | 13.50557 | 13.50568 | 13.51018 | 13.51071 | 13.52377 |
| 0.09 | 100 | 10.92090 | 11.03114 | 10.85701 | 10.83224 | 10.92136 | 10.87906 | 10.91734 | 10.92351 | 10.92377 | 10.93596 |
| 0.09 | 105 | 8.72680 | 8.79965 | 8.73326 | 8.66299 | 8.71938 | 8.64385 | 8.72355 | 8.73002 | 8.72994 | 8.74234 |
| 0.09 | 110 | 6.89990 | 6.93321 | 6.96109 | 6.86426 | 6.88709 | 6.77955 | 6.89850 | 6.90392 | 6.90349 | 6.91747 |
| 0.15 | 90 | 17.79766 | 17.95199 | 17.58134 | 17.68445 | 17.82007 | 17.83676 | 17.79902 | 17.79946 | 17.80010 | 17.81632 |
| 0.15 | 95 | 14.78330 | 14.92999 | 14.61047 | 14.67046 | 14.79799 | 14.79840 | 14.78215 | 14.78515 | 14.78577 | 14.79776 |
| 0.15 | 100 | 12.13053 | 12.25606 | 12.02394 | 12.02929 | 12.13642 | 12.11226 | 12.12765 | 12.13277 | 12.13320 | 12.14398 |
| 0.15 | 105 | 9.84134 | 9.93595 | 9.80601 | 9.76058 | 9.83903 | 9.78516 | 9.83808 | 9.84428 | 9.84443 | 9.85542 |
| 0.15 | 110 | 7.90130 | 7.96003 | 7.92858 | 7.84648 | 7.89259 | 7.80772 | 7.89918 | 7.90519 | 7.90501 | 7.91712 |
| | MSE | 7.94E-04 | 2.85E-02 | 3.10E-02 | 2.32E-02 | 3.11E-03 | 1.84E-02 | 1.34E-03 | 1.14E-04 | - | 3.54E-03 |
| | MAE | 3.05E-03 | 1.00E-01 | 9.86E-02 | 8.53E-02 | 1.07E-02 | 5.72E-02 | 4.95E-03 | 3.96E-04 | - | 1.36E-02 |
| | MAPE | 3.03E-02 | 8.28E-01 | 7.87E-01 | 7.34E-01 | 1.06E-01 | 6.75E-01 | 4.99E-02 | 3.52E-03 | - | 1.30E-01 |
| | MAXA | 3.70E-03 | 1.54E-01 | 2.19E-01 | 1.16E-01 | 2.00E-02 | 1.41E-01 | 6.76E-03 | 6.44E-04 | - | 1.67E-02 |

**Table 6f: A Comparison of Asian Option Pricing Models (volatility 50%, S=100)**

| r | K | Lower Bound | FitLog | EdgeLog | RGamma | GB2 | ME2 | EdgeNorm | ME4 | AW | Upper Bound |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.05 | 90 | 17.42759 | 17.67918 | 17.00382 | 17.24381 | 17.46095 | 17.39602 | 17.42452 | 17.43280 | 17.43406 | 17.46883 |
| 0.05 | 95 | 14.69709 | 14.91574 | 14.46257 | 14.52848 | 14.71794 | 14.61395 | 14.69100 | 14.70235 | 14.70315 | 14.73576 |
| 0.05 | 100 | 12.31491 | 12.48954 | 12.25646 | 12.17159 | 12.32366 | 12.17722 | 12.30800 | 12.32062 | 12.32079 | 12.35269 |
| 0.05 | 105 | 10.26058 | 10.38535 | 10.34604 | 10.14915 | 10.25893 | 10.07064 | 10.25511 | 10.26709 | 10.26657 | 10.29936 |
| 0.05 | 110 | 8.50694 | 8.58075 | 8.69620 | 8.43044 | 8.49725 | 8.27112 | 8.50475 | 8.51444 | 8.51326 | 8.54958 |
| 0.09 | 90 | 18.18295 | 18.43698 | 17.69986 | 17.99498 | 18.21877 | 18.16603 | 18.18078 | 18.18746 | 18.18884 | 18.22077 |
| 0.09 | 95 | 15.43707 | 15.66486 | 15.13557 | 15.25983 | 15.46142 | 15.37335 | 15.43160 | 15.44169 | 15.44272 | 15.47216 |
| 0.09 | 100 | 13.02253 | 13.21198 | 12.89937 | 12.86687 | 13.03528 | 12.90732 | 13.01569 | 13.02769 | 13.02816 | 13.05680 |
| 0.09 | 105 | 10.92375 | 11.06751 | 10.95396 | 10.79735 | 10.92609 | 10.75741 | 10.91769 | 10.92981 | 10.92963 | 10.95880 |
| 0.09 | 110 | 9.11795 | 9.21323 | 9.26553 | 9.02510 | 9.11188 | 8.90499 | 9.11458 | 9.12515 | 9.12431 | 9.15600 |
| 0.15 | 90 | 19.30062 | 19.55391 | 18.73925 | 19.10964 | 19.33935 | 19.30180 | 19.29993 | 19.30425 | 19.30572 | 19.33469 |
| 0.15 | 95 | 16.54546 | 16.78229 | 16.14761 | 16.35843 | 16.57452 | 16.50743 | 16.54126 | 16.54927 | 16.55052 | 16.57621 |
| 0.15 | 100 | 14.09498 | 14.30234 | 13.87252 | 13.92346 | 14.11353 | 14.01101 | 14.08877 | 14.09944 | 14.10024 | 14.12496 |
| 0.15 | 105 | 11.94030 | 12.10904 | 11.88030 | 11.79330 | 11.94878 | 11.80815 | 11.93397 | 11.94578 | 11.94600 | 11.97103 |
| 0.15 | 110 | 10.06490 | 10.18997 | 10.13926 | 9.94841 | 10.06471 | 9.88644 | 10.06033 | 10.07171 | 10.07128 | 10.09793 |
| | MSE | 1.51E-03 | 4.81E-02 | 7.31E-02 | 4.10E-02 | 4.55E-03 | 3.58E-02 | 2.75E-03 | 2.35E-04 | - | 7.80E-03 |
| | MAE | 5.84E-03 | 1.77E-01 | 2.28E-01 | 1.55E-01 | 1.47E-02 | 1.18E-01 | 1.05E-02 | 7.99E-04 | - | 3.00E-02 |
| | MAPE | 4.66E-02 | 1.29E+00 | 1.55E+00 | 1.15E+00 | 1.03E-01 | 1.08E+00 | 8.38E-02 | 5.90E-03 | - | 2.39E-01 |
| | MAXA | 6.47E-03 | 2.48E-01 | 5.66E-01 | 1.96E-01 | 3.36E-02 | 2.42E-01 | 1.28E-02 | 1.47E-03 | - | 3.63E-02 |

**Legend**: MSE:mean squared error, MAE:mean absolute error, MAPE:mean absolute percentage error, MAXA: maximum absolute error. The names of columns are as in Table 5.

## 6. Conclusion

In the present chapter we have given new insights regard the problem of pricing Asian options, showing how approximations based on the moments of the logarithm of the arithmetic average are more theoretically founded than approximations based on the algebraic moments of the arithmetic average. Then we have discussed how to recover the unknown density on the basis of this information. Another possible application of the results in the present paper is to the pricing of options with stochastic volatility as in the Hull and White model where the pricing problem reduces to the determination of the density of the average variance over the residual life of the option. An important extension left for future work is to examine if approximations based on the logarithmic moments are possible also in the case of discrete monitoring of the average and in the case of floating strike Asian options.

## References

[1] Abate, J. and Whitt, W. (1992), The Fourier-series method for inverting transforms of probability distributions, *Queueing Systems Theory Appl.* 10 5-88.

[2] Akhiezer N. I. (1965), The Classical Moment Problem and some Related Questions in Analysis, Oliver & Boyd, London.

[3] Andreasen J. (1998), The pricing of discretely sampled Asian and lookback options: a change of numeraire approach, *The Journal of Computational Finance*. Vol. 2, No. 1, 5-23.

[4] Aparicio S. D. and S. Hodges (1998), Implied Risk-Neutral Distribution: A Comparison of Estimation Methods, FORC Preprint 98/95, University of Warwick.

[5] Avellaneda M. (1999), Minimum Entropy Calibration of Asset Pricing Models, to appear in International *Journal of Theoretical and Applied Finance.*

[6] Barraquand, J. and Pudet T. (1996), Pricing of American path-dependent contingent claims, *Mathematical Finance*, Vol. 6, 17-51.

[7] Bookstaber R. M. and J. B. McDonald (1987), A general distribution for describing security price returns, J. Business vol. 60, 401-424.

[8] Borwein J. M. and A.S. Lewis, Convergence of best entropy estimates, SIAM J. Optimization, **1**, 191-205, (1991).

[9] Bouaziz L., E. Briys and M. Crouhy (1994), The pricing of forward starting Asian options, Journal of Banking and Finance, vol. 18, 823-839.

[10] Buchen P. W. and M. Kelly (1996), The Maximum Entropy Distribution of an Asset Inferred from Option Prices, Journal of Financial and Quantitative Analysis, vol. 31, no.1, 143-159, march.

[11] Churchill R. V. and J. W. Brown (1990), Complex Variables and Applications, Fifth edition, Mc-Graw-Hill International Editions.

[12] Clewlow L. J. and A. P. Carverhill (1994), Quicker on the Curves, *Risk*, Vol. 7 No. 5, 63-64.

[13] Crump, K. (1976), Numerical Inversion of Laplace transform using Fourier Series approximation, *J. Assoc. Comp. Mach.* 23, 89-96.

[14] Curran, M. (1992), Beyond average intelligence, Risk, Vol. 5, No. 10, pag. 60.

[15] Curran, M. (1992), Valuing Asian and Portfolio Options by Conditioning on the Geometric Mean Price, Management Science, Vol. 40, No. 12, pagg.1705-1711.

[16] De Schepper A., M. Teunen and M. Goovaerts (1994), An Analytical Inversion of a Laplace transform related to annuities certain, Insurance: Math. Econonom., 14, 33-37.

[17] Dubner H. and Abate J. (1968), Numerical Inversion of Laplace transforms by relating them to the finite Fourier cosine transform, J. ACM 15, 1 January, 115-123.

[18] Dufresne D. (1989), Weak Convergence of Random Growth Processes with Applications to Insurance, Insurance: Math. Econonom., 8, 187-201.

[19] Dufresne D. (1999), Laguerre Series for Asian and other options, w.p. University of Melbourne, February.

[20] Frontini M. and A. Tagliani (1997) Entropy-convergence in Stieltjes and Hamburger moment problem, Applied Math. and Computation, **88**, 39-51.

[21] Fu M. C., D. Madan and T. Wang (1998), Pricing Continuous Asian Options: a comparison of Monte Carlo and Laplace transform inversion methods, *The Journal of Computational Finance,* Vol. 2, No. 1, 49-74.

[22] Geman H. and A. Eydeland (1995), Domino Effect, Risk, Vol. 8, n. 4, april, 65-67.

[23] Geman H. and M. Yor (1992), Quelques relations entre processus de Bessel, options asiatique et fonctions confluentes hypergéométriques, C.R. Acad. Sci. Paris, t. 314, Série I, p. 471-474.

[24] Geman H. and M. Yor (1993), Bessel Processes, Asian Options and Perpetuities, Mathematical Finance, vol. 3, no. 4, pagg. 349-375.

[25] Golan A., G Judge, D. Miller (1996), Maximum entropy econometrics: robust estimation with limited data", J. Wiley & Sons.

[26] Hull, J. and A. White (1993), Efficient Procedures for Valuing European and American Path Dependent Options, *Journal of Derivatives*, Vol. 1, 21-31.

[27] Jarrow R. and A. Rudd (1982), Approximate Option Valuation for Arbitrary Stochastic Processes, Journal of Financial Economics, vol. 10, pp. 347-369.

[28] Jaynes E.T. (1978), Where do we stand on maximum entropy, in R.D. Levine and M. Tribus Eds., *The maximum entropy formalism*, MIT Press Cambridge MA, 15-118.

[29] Karlin S. and L.S. Shapley (1953), *Geometry of moment spaces*, AMS Memoirs **12**, Providence RI.

[30] Kesavan, H.K., Kapur, J.N. (1992) *Entropy Optimization Principles with Applications*, Academic Press.

[31] Kullback, S., A lower bound for discrimination information in terms of variation, IEEE Trans. on Information Theory (Corresp.) , IT-13, 126-127, (1967).

[32] Levy E. (1992), Pricing European Average Rate Currency Options, Journal of International Money and Finance, vol. 11, 474-491.

[33] Lindsay, B. G. and P. Basak (1995), Moments determine the tail of a distribution (but not much else), Technical report 95-7, Center for likelihood Studies, Dept. of Statistics, The Pennsylvania State University, University Park, PA.

[34] Lindsay, B. G. and K. Roeder (1997), Moment-based oscillation properties of mixture models, Ann. Statist., 25, pagg. 378-386.

[35] Lipton A. (1999), Similarities via self-similarities, Risk, vol. 12, no. 9, 101-105. September.

[36] MacDonald J. R. (1964), Accelerated convergence, divergence, iteration, extrapolation and curve fitting, J. Appl. Phys. 10, 3034-3041.

[37] McDonald J. B. (1984), Some generalized functions for the size distribution of income, Econometrica, vol. 52, 647-663.

[38] McDonald J. B. (1996), Probability distributions for financial models, Handbook of Statistics G.S. Maddala and C. R. Rao eds, vol. 14, chapt. 14, 427-461.

[39] Milevsky M. A. and S. E. Posner (1998), Asian Options, the sum of lognormals and the reciprocal gamma distribution, *Journal of Financial and Quantitative Analysis*, vol. 33, no. 3, pagg. 409-422, Sept..

[40] Press, W. H., Teukolsky S. A., Vetterling W. T. and Flannery B. P. (1997). *Numerical Recipes in C*, Cambridge University Press, version 2.08

[41] Rebonato R. (1999), Volatility and Correlation in the Pricing of Equity, FX and Interest-Rate Options, chpt. 7, 190-214, Wiley Financial Engineering Series.

[42] Ritchken P., L. Sankarasubramanian and A. M. Vijh (1990), The Valuation of Path-Dependent contracts on the average, Management Science,

[43] Rogers L. C. G. and Shi Z. (1992), The Value of an Asian option, J. of Applied Probability, Vol. 32, pagg. 1077-1088.

[44] Shohat J. A. and J.D. Tamarkin (1943), *The problem of moments*, AMS Mathematical Survey **1**, Providence RI, (1943).

[45] Simon R. M., Stroot M. T. and G. H. Weiss (1972), Numerical inversion of Laplace transforms with applications to the percentage labeled experiments, Comput. Biomed. Res. 6, pagg. 596-607.

[46] Singhal, K., Vlach, J. and M. Vlach (1975), Numerical inversion of multidimensional Laplace transforms, *Proc. IEEE* 63, 1627-1628.

[47] Stoyanov J. (1997), Counterexamples in Probability, 2nd edition, Wiley Series in Probability and Statistics.

[48] Tagliani A. (2000), Recovering a probability density function from its Mellin transform, to appear in Applied Mathematics and Computation.

[49] Thompson G. W. P. (1999), Topics in Mathematical Finance, PH.D. Thesis, Queens' College, January.

[50] Thompson G. W. P. (1999), Fast narrow bounds on the Value of Asian options, w.p. Judge Institute of Management Studies, U. of Cambridge.

[51] Turnbull S. and L. Wakeman (1991), A Quick Algorithm for Pricing European Average Options, Journal of Financial and Quantitative Analysis, vol. 26, pagg. 377-389.

[52] Vlach J. and K. Singhal (19 ), Computer Methods for Circuit Analysis and Design, Van Nostrand Reinhold Company, New York.

[53] Vorst, T. (1996), Averaging Options, in I. Nelken, *The Handbook of Exotic Options*, Chpt. 6, pp. 175-199, IRWIN.

[54] Yor M. (1992), On some exponential functionals of Brownian Motion, Adv. Appl. Prob. 24, pagg. 509-531.

[55] Yor M. (1992), Some Aspects of Brownian Motion. Part I: Some Special Functionals. Birkhauser Verlag. Basel - Boston - Berlin.

[56] Yor M. (1992), Sur les lois des fonctionnelles exponentielles du mouvement brownien, considérées en certains instants aléatoires, C. R. Acad. Sci Paris Série I, 314, 951-956.

[57] Wilmott, P. Dewynne, J.N. and Howison, S. (1993), *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press.

[58] Zvan R., P. A. Forsyth and K. R. Vetzal (1997), Robust Numerical methods for PDE models of Asian options, *The Journal of Computational Finance*, Vol. 1, N. 2, 39-77.

## A. The derivation of the Geman-Yor formula

In this Appendix we report a derivation of the Laplace transform of the Asian option price found in Yor [56]. The result is based on a a remarkable expression of the law of the arithmetic average taken at an indipendent exponential time. We report the result because we liked the immediate and simple derivation.

We set (assuming for simplicitiy that $t = 0$ and $a = 0$ in (3.3)):

$$v = \frac{2r}{\sigma^2} - 1; q = \frac{\sigma^2}{4S}KT; \mu = \sqrt{2\lambda + v^2}$$

and define:

$$D_h = \int_0^h e^{2(W_s + vs)} ds$$

Yor [55] shows that $D_h$ taken at an exponentially distributed random time $T_\lambda$, independent of $W$, with mean $1/\lambda$, is characterized by:

$$2D_{T_\lambda} \stackrel{\mathcal{L}}{=} \frac{B_{1,\alpha}}{G_\beta}$$

where $\alpha = (\mu + v)/2$, $\beta = (\mu - v)/2$, and the random variable $B_{1,\alpha} \sim Beta(1, \alpha)$ and $G_\beta \sim Gamma(\beta, 1)$ are independent.

For pricing Asian options we are interested in the following quantity:

$$\bar{A}_t = \int_0^t e^{\sigma W_s + \left(r - \frac{\sigma^2}{2}\right)s} ds$$

that, by the time-scaling property of the Brownian motion, is related to the expression of

$\dot{D_h}$ by:

$$\bar{A_t} = \frac{4}{\sigma^2} D_{\frac{\sigma^2 t}{4}}$$

In particular we need to compute:

$$\begin{aligned} e^{-rT} E\left(\tfrac{S_0}{T}\ \bar{A}_t - K\right)^+ &= e^{-rT} E\left(\tfrac{4}{\sigma^2}\tfrac{S_0}{T} D_{\frac{\sigma^2 T}{4}} - K\right)^+ \\ &= e^{-rT}\tfrac{4}{\sigma^2}\tfrac{S_0}{T} E\left(D_{\frac{\sigma^2 T}{4}} - \tfrac{\sigma^2 T}{4 S_0} K\right)^+ \\ &= e^{-rT}\tfrac{4}{\sigma^2}\tfrac{S_0}{T} E\left(D_h - q\right)^+ \end{aligned}$$

and then if we compute the Laplace transform with respect to $h$ of the above quantity, exploiting the relationship between Laplace transform and expectation wrt to an exponential distributed random variable, we obtain:

$$\int_0^{+\infty} e^{-\lambda h} E\left(D_h - q\right)^+ dh = \frac{1}{2\lambda} E\left(2D_{T_\lambda} - 2q\right)^+$$

Now using the above result due to Yor, we have:

$$\begin{aligned} E\left(2D_{T_\lambda} - 2q\right)^+ &= E\left(\tfrac{B_{1,\alpha}}{G_\beta} - 2q\right)^+ \\ &= \int_0^1 \int_0^{+\infty} \left(\tfrac{u}{x} - 2q\right)^+ \tfrac{x^{\beta-1}}{\Gamma(\beta)} e^{-x} dx \alpha \left(1-u\right)^{\alpha-1} du \\ &= \int_0^1 \int_0^{u/2q} \left(\tfrac{u}{x} - 2q\right) \tfrac{x^{\beta-1}}{\Gamma(\beta)} e^{-x} dx \alpha \left(1-u\right)^{\alpha-1} du \\ &= \int_0^{1/2q} \tfrac{x^{\beta-1}}{\Gamma(\beta)} e^{-x} dx \int_{2qx}^1 \left(\tfrac{u}{x} - 2q\right) \alpha \left(1-u\right)^{\alpha-1} du \\ &= \tfrac{1}{1+\alpha} \int_0^{1/2q} \tfrac{x^{\beta-2}}{\Gamma(\beta)} e^{-x} \left(1-2qx\right)^{\alpha+1} dx \end{aligned}$$

where we have used:

$$\int_{2qx}^1 \left(\frac{u}{x} - 2q\right) \alpha \left(1-u\right)^{\alpha-1} du = \frac{\left(1-2qx\right)^{\alpha+1}}{\left(1+\alpha\right) x}$$

So using $\alpha = (\mu + v)/2$, $\beta = \alpha - v = (\mu - v)/2$, and $\Gamma(\beta) = (\beta - 1)\Gamma(\beta - 1)$ and

$(1+\alpha)(\beta-\dot{1}) = (\lambda - 2 - 2v)/2$ we finally obtain

$$\begin{aligned}\int_0^{+\infty} e^{-\lambda h} E\left(D_h - q\right)^+ dh &= \tfrac{1}{2\lambda}\tfrac{1}{(1+\alpha)(\beta-1)}\int_0^{1/2q} \tfrac{x^{\beta-2}}{\Gamma(\beta-1)} e^{-x}\left(1-2qx\right)^{\alpha+1} dx \\ &= \tfrac{1}{\lambda(\lambda-2-4v)}\int_0^{1/2q} \tfrac{x^{(\mu-v)/2-2}}{\Gamma((\mu-v)/2-1)} e^{-x}\left(1-2qx\right)^{(\mu+v)/2+1} dx\end{aligned}$$

i.e.the expression (3.1) given in the main text.

## B. A linear approximation for Asian Options

Bouaziz et al. [9] have proposed for pricing Asian option with floating strike a linear approximation to the exponential function defining the arithmetic average. In this Appendix, we have investigated if it can be usuful to exploit a similar approximation also for Asian options with fixed strike. Respect to the approximation in [9], we have used a first order Taylor approximation only for the term involving the Brownian motion, so that we have:

$$\begin{aligned}\int_0^T e^{ms+\sigma W_s} ds &\approx \int_0^T e^{ms}\left(1+\sigma W_s\right) ds \\ &= \tfrac{e^{mT}}{m} + \sigma \int_0^T e^{ms} W_s ds\end{aligned}$$

where $m = r - \sigma^2/2$. We observe that $\int_0^T e^{ms} W_s ds$ has normal distribution characterised by:

$$\begin{aligned}E\left(\int_0^T e^{ms} W_s ds\right) &= 0 \\ E\left(\int_0^T e^{ms} W_s ds\right)^2 &= E\left(\int_0^T e^{ms} W_s ds\right)\left(\int_0^T e^{mr} W_r dr\right) \\ &= E \int_0^T \int_0^T e^{ms} e^{mr} W_s W_r ds dr \\ &= \int_0^T \int_0^T e^{ms} e^{mr} E\left(W_s W_r\right) ds dr \\ &= \int_0^T \int_0^T e^{ms} e^{mr} \min\left(s,r\right) ds dr \\ &= \int_0^T e^{mr}\left(\int_0^r e^{ms} s ds + \int_r^T e^{ms} r ds\right) dr\end{aligned}$$

and with some standard integration, we obtain:

$$v^2 := E\left(\int_0^T e^{ms} W_s ds\right)^2 = \frac{4e^{mT} - 1 - 3e^{2mT} + 2e^{2mT} mT}{2m^3}.$$

We conclude that:

$$\int_0^T e^{ms} W_s ds \sim N\left(0, v^2\right)$$

For pricing Asian options we need to compute:

$$E\left[\frac{S}{T}\left(\frac{e^{mT}}{m} + \sigma \int_0^T e^{ms} W_s ds\right) - K\right]^+ = \sigma \frac{S}{T} E\left[\int_0^T e^{ms} W_s ds - \hat{K}\right]^+$$

where $\hat{K} = \left(KT/S - e^{mT}/m\right)/\sigma$. Computing the expected value, we obtain the linear approximation to the Asian option price:

$$c_{lin} = \sigma \frac{S}{T} e^{-rT}\left(\frac{e^{-\frac{\hat{K}^2}{2v^2}}\sqrt{v^2}}{\sqrt{2\pi}} - \hat{K}\,\Phi\left(-\frac{\hat{K}}{v^2}\right)\right)$$

In the Table below we have compared the linear approximation with the lower and upper bound described in Appendix C, when $\sigma = 5\%$ and $10\%$, $T = 1$, $r = 0.05$, $S = 100$. As we can see, the linear approximation underestimates the true price and the error can be acceptable only a volatility level at 5% or below. We do not report here the results for higher volatility levels, where the difference affects the first digit as well.

| Table 7: The Linear Approximation | | | | | | |
|---|---|---|---|---|---|---|
| | $\sigma = 0.05$ | | | $\sigma = 0.1$ | | |
| $K$ | Lower | Linear | Upper | Lower | Linear | Upper |
| 95 | 7.1777 | 7.1184 | 7.1778 | 7.4077 | 7.2289 | 7.4086 |
| 100 | 2.7162 | 2.6781 | 2.71622 | 3.6412 | 3.5047 | 3.6416 |
| 105 | 0.3372 | 0.3128 | 0.3374 | 1.3106 | 1.1936 | 1.3110 |

An improvement in the approximation could be to use a second order Taylor expansion, but this requires the determination of the distribution of the sum of correlated normal (the terms $W_s$ in the expansion) and non-central chi-squared variables (the terms $W_s^2$ in the expansion).

## C. The upper and lower bounds for the price

A different approach to the problem of Asian option price is to find a lower and upper bound for the price and practically they result to be surprisingly accurate, i.e. the size of the interval results to be small and in general the Asian option price will be near the lower bound. The idea of finding the bounds is due to Curran [14], [15] and to Rogers and Shi [43]. These bounds have been improved by Thompson [49], [50] that was able to simplify the computation of the lower bound given in Rogers and Shi. Moreover, Thompson gives an upper bound for the option price, much more stronger than the one obtained by Rogers and Shi. Using a lower and upper bound can give us some better feeling in choosing the different methods described above and then understand the limits of the different approaches.

The Asian option price is obtained through the computation of the quantity

$$E\left(\frac{S}{T}\int_0^T e^{\left(r-\sigma^2/2\right)s+\sigma W_s}ds - K\right)^+ .$$

Making the substitution $h = s/T$ and using the scaling property of the BM:

$$\begin{aligned}\frac{1}{T}\int_0^T e^{\left(r-\sigma^2/2\right)s+\sigma W_s}ds &= \int_0^1 e^{\left(r-\sigma^2/2\right)Th+\sigma W_{Th}}dh \\ &= \int_0^1 e^{\left(r-\sigma^2/2\right)Th+\sigma\sqrt{T}W_h}dh\end{aligned}$$

we can always consider the case with time to expiry equal to 1 and set the interest rate equal to $rT$ and the volatility equal to $\sigma\sqrt{T}$. So in the following we will assume that the residual life of the option is equal to 1 and the risk-free rate $r$ stands for $rT$ and the volatility $\sigma$ stands for $\sigma\sqrt{T}$. Moreover we set:

$$X = \int_0^1 e^{\left(r-\sigma^2/2\right)Th+\sigma\sqrt{T}W_h}dh - K$$

and the option price is given by $EX^+$.

Using the iterated conditional expectation and the fact that $X^+ \geq X$ and that $X^+$ is a positive quantity, then:

$$E\left(X^+\right) = E\left[E\left(X^+\left|Z\right.\right)\right] \geq E\left[E\left(X\left|Z\right.\right)^+\right]$$

and the accuracy of the lower bound can be estimated from the quite general bounds:

$$0 \leq E\left(X^+\right) - E\left[E\left(X\left|Z\right.\right)^+\right] \leq \frac{1}{2}E\left[\sqrt{Var\left(X\left|Z\right.\right)}\right]$$

The method can be very successfully if the conditioning variable $Z$ is chosen to make the conditional variance small. Rogers and Shi realize that a good choice results to be $Z = \int_0^T W_s ds$ and then they are able to give an expression to $E\left[E\left(X\left|Z\right.\right)^+\right]$ as a double integral:

$$c \geq c_{low} \equiv e^{-r} \int_{-\infty}^{+\infty} \sqrt{3}\phi\left(\sqrt{3}z\right) \left[\int_0^1 S_0 e^{3\sigma t(1-t/2)z+\alpha t+\frac{1}{2}\sigma^2\left(t-3t^2(1-t/2)^2\right)} dt - K\right]^+ dz$$

where $\alpha = \left(r - \sigma^2/2\right)$, and where $\phi(x)$ is the normal density.

Thompson gives a simpler version of the above formula, trying to approximate the event that the option makes a positive payout with a simpler event. If $\mathcal{A} = \{\int_0^1 S_t dt > K\}$, we have for any event $\mathcal{A}^*$:

$$E\left[\int_0^1 S_t dt - K\right]^+ \geq E\left[\left(\int_0^1 S_t dt - K\right) 1_{\mathcal{A}^*}\right] = \int_0^T E\left(S_u - K; \mathcal{A}^*\right) du \qquad \text{(C.1)}$$

and a natural choice is $\mathcal{A}^* = \{\int_0^T W_u du > \gamma\}$ where $\gamma$ is chosen in an optimal way maximinizing the rhs in (C.1). The optimal value of $\gamma$, $\gamma^*$, results to be the unique solution to:

$$\int_0^1 S_0 \exp\left(3\gamma^* \sigma t\,(1-t/2) + \alpha t + \frac{1}{2}\sigma^2\left(t - 3t^2\,(1-t/2)^2\right)\right) dt = K \qquad \text{(C.2)}$$

and the lower bound is:

$$e^{-r} \int_0^1 E\left(S_u - K; 1_{\left(\int_0^1 W_u du > \gamma^*\right)}\right) dt$$

i.e.:

$$c_{low} = e^{-r} \int_0^1 S_0 e^{\alpha t + \frac{1}{2}\sigma^2 t} \Phi\left(\frac{-\gamma^* + \sigma t\,(1-t/2)}{1/\sqrt{3}}\right) dt - K\Phi\left(\frac{-\gamma^*}{1/\sqrt{3}}\right) \qquad \text{(C.3)}$$

where $\Phi$ is the normal distribution function. The bound given in Curran is for the case of discrete averaging and is obtained solving in an approximate way (C.2) and can provide a valid alternative to the numerical solution of (C.2). Finally, Thompson shows also the lower bound is the same as the one given in Rogers and Shi with the advantage of being expressed as a single integral. The computation of the optimal value of $\gamma$ can be done very quickly with standard optimization routines, such as bisection or Newton search.

The upper bound given in Thompson is:

$$c \leq c_{up} \equiv e^{-r} \int_0^1 \int_{-\infty}^{+\infty} 2v\varphi(w) \left[ a(t,x)\,\Phi\left(\frac{a(t,x)}{b(t,x)}\right) + b(t,x)\,\varphi\left(\frac{a(t,x)}{b(t,x)}\right)\right] dw\,dv$$

where $v = \sqrt{t}$, $w = x/\sqrt{t}$,

$$\begin{aligned}
a(t,x) &= S_0 \exp(\sigma x + \alpha t) - K(\mu_t + \sigma x) + K\sigma(1 - t/2)\,x \\
b(t,x) &= K\sigma\sqrt{\frac{1}{3} - t(1 - t/2)^2} \\
\mu_t &= \frac{1}{K}\left(S_0 \exp(\alpha t) + \gamma\sqrt{v_t}\right) \\
v_t &= c_t^2 t + 2(K\sigma)\,c_t t(1 - t/2) + (K\sigma)^2/3 \\
c_t &= S_0 \exp(\alpha t)\,\sigma - K\sigma \\
\gamma &= (K - S_0(e^{\alpha} - 1)/\alpha) \Big/ \int_0^1 \sqrt{v_t}\,dt
\end{aligned}$$

This upper bound results to be tighter than the one given in Rogers and Shi [43].

# Conclusions and Further Research

In the first chapter we showed how the application of the Laplace transform technique can allow to solve with great accuracy many familiar pricing problems. Unfortunately in some cases, as for Asian option with floating strike or for double barrier options in the square-root model, also the computation of the Laplace transform can be cumbersome. A possible way for coping with this problem is to use a so called hybrid method, compare Zinober et al. [6]. The method consists in the transformation of the original PDE into the Laplace space through elimination of the time dependency and subsequent numerical solution of the resulting steady-state second order differential equation in the complex space. The exploitation of the Laplace transform to solve the time dependency restricts the application to linear parabolic PDE's, with the possible inclusion of a jump process. Some preliminary tests conducted in Fusai and Tagliani [3] appear promising and very competitive, either for the computational time and the accuracy, respect to the numerical solution of the PDE. The proposed approach has also the advantage that it is easily parallelizable, whilst the finite difference schemes most often employed, e.g. Crank-Nicolson techniques, are inherently sequential.

Another important area that future research should concentrate on is the application of the Laplace transform technique to the pricing of discrete monitored contracts and American options. As illustrated in the third chapter the price differences between continuous and discrete monitoring of the underlying can be very relevant. It would then be very important to understand how the Laplace technique can cope with this kind of problems. At this regard, we have started some promising work related to the pricing of discrete monitored barrier options, Fusai et al.[4]. Regard American options, a possible idea is to explore the so called randomization used in Carr [1], that seems to present an analogy with the Laplace transform technique.

In the second and third chapter, we have discussed the pricing problem for occupation time derivatives showing that multidimensional Laplace transforms can be inverted with great accuracy, but very few things have been said about the hedging problem. We believe that more research has to be done regard the comparison of dynamic and static hedging of this kind of contracts. We have conducted some Monte Carlo simulation comparing a delta-hedging dynamic strategy, with different rebalancing periods, versus a mean-variance static strategy, but the results of the static strategy were not very satisfactory. So it could be interesting to investigate some hybrid strategies, as in Carr et al. [2], where a standard dynamic trading strategy in the underlying asset is enhanced by static positions in options.

Finally, in last chapter we have proposed a new method for evaluating fixed strike Asian options, resorting to the maximum entropy density approximation and exploiting the logarithmic moments. Then we have showed, using the first four moments, we can estimate with high accuracy the Asian option price. It is our intention to try to apply this approach to binomial trees, along the lines in Rubinstein who develops a simple technique for valuing European and American derivatives with underlying asset risk-neutral returns that depart from lognormal in terms of higher order moments (skewness and kurtosis).

## References

[1] Carr P. (1998), Randomization and the American Put, *Review of Financial Studies.*

[2] Carr P., A. Lipton and D. Madan (2000), Going with the flow, *Risk*, August, 85-89 .

[3] Fusai G. and A. Tagliani (2000), The use of the Laplace transform for the numerical solution of PDE in Finance, *w.p.* presented at XXIV Convegno Amases, Padenghe sul Garda, Italy, September.

[4] Fusai G. and M.C. Recchioni (2000), Quasi-analytical valuation of discrete monitored barrier options, *w.p.* in preparation.

[5] Rubinstein M. (1998), Edgeworth Binomial Trees, *Journal of Derivatives*, Spring, 20-27.

[6] Zinober A. S. I. and E. Huntley (1980) The numerical solution of linear time-dependent partial differential equations by the Laplace transform and finite difference approximations, *J. of Computational and Applied Mathematics*, vol. 6, no. 4, pagg. 253-8.

# Bibliography

This bibliography has been constructed using the complete set of references that already appear at the end of the different chapters of the Thesis. Then the numbering of the items listed below does not correspond to the number of the references used in each chapter.

## References


[1] Abate, J. G. L. Choudhury and W. Whitt (1996), On the Laguerre method for numerically inverting Laplace transforms, *Informs . comput.*, vol. 8, 413-27.

[2] Abate, J. G. L. Choudhury and W. Whitt (1998), Numerical Inversion of multidimensional Laplace transforms by the Laguerre method, *Performance Evaluation*, vol. 31, 229-43.

[3] Abate, J. and Whitt, W. (1992), The Fourier-series method for inverting transforms of probability distributions, *Queueing Systems Theory Appl.* 10 5-88.

[4] Abramowitz, M., and Stegun, I.A.(1972). *Mathematical Functions*, Dover Publications, Inc., New York.

[5] Akahori, J. (1995). Some formulae for a new type of path-dependent options. *Ann. Appl. Probab. 5*, 383-388.

[6] Akhiezer N. I. (1965), The Classical Moment Problem and some Related Questions in Analysis, Oliver & Boyd, London.

[7] Aitsahlia F. and T. L. Lai (1998), Random walk Duality and the Valuation of Discrete Lookback Options, *Applied Mathematical Finance*, vol. 5, n. 3, 227-240.

[8] Andersen L. and R. Brotherton-Ratcliffe (1996), Exact Exotics, *Risk* Vol.9, No. 10, October.

[9] Andreasen J. (1998), The pricing of discretely sampled Asian and lookback options: a change of numeraire approach, *The Journal of Computational Finance,* Vol. 2, No. 1, 5-23.

[10] Aparicio S. D. and S. Hodges (1998), Implied Risk-Neutral Distribution: A Comparison of Estimation Methods, FORC Preprint 98/95, University of Warwick.

[11] Avellaneda M. (1999), Minimum Entropy Calibration of Asset Pricing Models, to appear in International *Journal of Theoretical and Applied Finance.*

[12] Bakshi G. and D. Madan (1999), Spanning and Derivative-Security Valuation, forthcoming in *Journal of Financial Economics.*

[13] Baldi, P., L. Caramellino and M. G. Iovino (1998), Pricing General Barrier Options: a Numerical Approach using Sharp Large Deviations, *Mathematical Finance*, vol. 4, no. 4, 293-321.

[14] Barraquand, J. and Pudet T. (1996), Pricing of American path-dependent contingent claims, *Mathematical Finance*, Vol. 6, 17-51.

[15] Bellman R., R. E. Kalaba and J. Lockett (1966), Numerical Inversion of the Laplace Transform. Application to Biology, Economics, Engineering and Physics, American Elsevier, New York.

[16] Black F. and M. Scholes (1973), The Pricing of Options and Corporate Liabilities, *Journal of Political Economy*, vol. 81, pagg. 637-654.

[17] Bookstaber R. M. and J. B. McDonald (1987), A general distribution for describing security price returns, J. Business vol. 60, 401-424.

[18] Borodin A. N. and P. Salminen (1996), Handbook of Brownian Motion - Facts and Formulae, Birkhauser.

[19] Borwein J. M. and A.S. Lewis, Convergence of best entropy estimates, SIAM J. Optimization, **1**, 191-205, (1991).

[20] Bouaziz L., E. Briys and M. Crouhy (1994), The pricing of forward starting Asian options, Journal of Banking and Finance, vol. 18, 823-839.

[21] Boyle, P. (1977). Options: A MonteCarlo approach, *Journal of Financial Economics* 4, 323-338.

[22] Bregagnolio T., Iori L. (1995), Il loro nome è bond, *Investire*, pp. 88-92, January.

[23] Broadie M., Glasserman P. and Kou S. (1997), A Continuity Correction for Discrete Barrier Options, *Mathematical Finance*, Vol. 7, no. 4, October.

[24] Buchen P. W. and M. Kelly (1996), The Maximum Entropy Distribution of an Asset Inferred from Option Prices, Journal of Financial and Quantitative Analysis, vol. 31, no.1, 143-159, march.

[25] Buser S. A. (1986), Laplace Transforms as Present Value Rules: A Note, *Journal of Finance*, Vol. XLI, no. 1, march, pagg 243-7.

[26] Carr, P. and R. A. Jarrow (1990), The Stop-Loss Start-Gain Paradox: A new Decomposition into Intrinsec and Time Value, *The Review of Financial Studies*, Vol. 3, no. 3, pagg. 469-492.

[27] Cathcart L. (1998), The Pricing of Floating Rate Instruments, *Journal of Computational Finance*, Vol. 1, no. 4, summer, pagg. 31-51.

[28] Chacko G. and S. R. Das (1997), Average Interest, w.p. Harvard Business School, may.

[29] Chen R. and L. Scott (1995), Interest Rate Options in Multifactor Cox-Ingersoll-Ross Models of the Term Structure, *Journal of Derivatives*, vol. 3, pagg. 53-72.

[30] Cheng A. H.-D, Sidauruk P. and Abousleiman Y. (1994), Approximate Inversion of the Laplace Transform, *Mathematica Journal*, Vol.4, No. 2.

[31] Chesney M., M. Jeanblanc-Picqué and M. Yor (1995), Brownian excursion and Parisian barrier options, Advances in Applied Probability,

[32] Chesney M., J. Cornwall, M. Jeanblanc-Picqué, G. Kentwell and M. Yor (1997), Parisian Pricing, Risk, Vol. 10, no. 1, january, pagg. 77-80.

[33] Choudhury G. L., Lucantoni D. M, and Whitt W. (1994), Multidimensional Transform Inversion with applications to the transient M/G/1 Queue, Ann. Appl. Probab. 5, 389-398.

[34] Churchill R. V. and J. W. Brown (1990), Complex Variables and Applications, Fifth edition, Mc-Graw-Hill International Editions.

[35] Clewlow L. J. and A. P. Carverhill (1994), Quicker on the Curves, *Risk*, Vol. 7 No. 5, 63-64.

[36] Cox J. C. (1996), The Constant Elasticity of Variance Option Pricing Model, *The Journal of Portfolio Management*, Special Issue, pagg.15-17.

[37] Cox J. C., J. E. Ingersoll and S. A. Ross (1985), A Theory of the Term Structure of Interest Rates, *Econometrica*, vol. 53, pagg. 363-84.

[38] Cox J. C. and S. Ross (1976), The Valuation of Options for Alternative Stochastic Processes, *Journal of Financial Economics*, pagg. 145-66.

[39] Craig I. and A. M. Thompson (1994), Why Laplace Transforms are Difficult to Invert Numerically, *Computers in Physics*, vol. 8, no. 6 nov-december.

[40] Crump, K. (1976), Numerical Inversion of Laplace Transform using Fourier Series Approximation, *J. Assoc. Comp. Mach.* 23, 89-96.

[41] Curran, M. (1992), Beyond average intelligence, Risk, Vol. 5, No. 10, pag. 60.

[42] Curran, M. (1992), Valuing Asian and Portfolio Options by Conditioning on the Geometric Mean Price, Management Science, Vol. 40, No. 12, pagg.1705-1711.

[43] Dassios, A. (1995). The Distribution of the Quantile of a Brownian Motion with Drift and the Pricing of Related Path-Dependent Options, ***Ann. Appl. Probab.*** **4,** 719-740.

[44] Davies B. and B. L. Martin (1970), Numerical Inversion of Laplace Transforms: a Critical Evaluation and Review of Methods, *J. Comp. Physics*, vol. 33, pagg. 1-32.

[45] Davydov D. and V. Linetsky, (1999), Structuring, Pricing and Hedging Double Barrier Step Options, w.p. University of Michigan, november 1999.

[46] De Schepper A., M. Teunen and M. Goovaerts (1994), An Analytical Inversion of a Laplace transform related to annuities certain, *Insurance: Math. Econonom.*, 14, 33-37.

[47] Dewynne, J.N. and Wilmott, P. (1995), A note on average rate options with discrete sampling, *SIAM J. Appl. Math.* Vol. 55, No. 1, pp. 267-276, February.

[48] Doetsch G (1970), Introduction to the Theory and Application of the Laplace Transformation, Springer-Verlag.

[49] Dubner H. and Abate J. (1968), Numerical Inversion of Laplace Transforms by Relating them to the Finite Fourier Cosine Transform, *J. ACM* 15, 1 January, 115-123.

[50] Duffie D., J. Pan and K. Singleton (1999), Transform Analysis and Option Pricing for Affine Jump-Diffusions, w.p. Stanford University, February.

[51] Duffy D. A. (1993), On the Numerical Inversion of Laplace Transforms: Comparison of three new Methods on Characteristic Problems from Applications, *ACM Trans. on Math. Soft.*, vol. 19, no. 3, Sept., pagg. 333-59.

[52] Dufresne D. (1989), Weak Convergence of Random Growth Processes with Applications to Insurance, *Insurance: Math. Econonom.*, 8, 187-201.

[53] Dufresne D. (1999), Laguerre Series for Asian and other options, w.p. University of Melbourne, February.

[54] Dyke P. P. G. (1999), An Introduction to Laplace Transforms and Fourier Series, Springer.

[55] Embrechts, P., Rogers, L. C. G. and Yor, M. (1995). A proof of Dassios' representation of the $\alpha$-quantile of Brownian motion with drift, Ann. Appl. Probab. 5, 757-767.

[56] Feller W. (1951), Two Singular Diffusion Problems, *Annals of Mathematics*, vol. 54, No.1, pagg. 173-82.

[57] Feller W. (1971), An Introduction to Probability Theory and its Applications, vol. II, 2nd ed., Wiley, New York.

[58] Frontini M. and A. Tagliani (1997) Entropy-convergence in Stieltjes and Hamburger moment problem, Applied Math. and Computation, **88**, 39-51.

[59] Fu M. C., D. Madan and T. Wang (1998), Pricing Continuous Asian Options: a comparison of MonteCarlo and Laplace Transform Inversion Methods, *The Journal of Computational Finance,* Vol. 2, No. 1, 49-74.

[60] Fusai, G. (2000), Corridor Options and Arc-Sine Law, *Annals of Applied Probability,* vol. 10, n.2, may 2000, 634-663.

[61] Fusai, G. and Tagliani A. (1999), Numerical Methods for Occupation Time Derivatives, *Proceedings of the Conference on Numerical methods in Finance,* Venice, May. The revised version is the 3rd chapter of the present thesis and has been accepted for publication in *Journal of Computational Finance.*

[62] Garbow B. S., G. Giunta, J. N. Lyness and A. Murli (1988), Software for an implementation of Weeks' method for the inverse Laplace transform problem, *ACM Trans. Math. Software,* Vol. 14, 163-70.

[63] Garbow B. S., G. Giunta, J. N. Lyness and A. Murli (1988), Algorithm 662: A FORTRAN software package for numerical inversion of the Laplace transform based on Weeks' method, ACM *Trans. Math. Software,* Vol. 14, 171-6.

[64] Geman H. and A. Eydeland (1995), Domino Effect, *Risk,* Vol. 8, n. 4, april, 65-67.

[65] Geman H. and M. Yor (1992), Quelques relations entre processus de Bessel, options asiatique et fonctions confluentes hypergéométriques, C.R. Acad. Sci. Paris, t. 314, Série I, p. 471-474.

[66] Geman H. and M. Yor (1993), Bessel Processes, Asian Options and Perpetuities, *Mathematical Finance*, vol. 3, no. 4, pagg. 349-375.

[67] Geman H. and M. Yor (1996), Pricing and hedging double barrier options: a probabilistic approach, *Mathematical Finance*, 6, n. 4, pagg. 365-78.

[68] Golan A., G Judge, D. Miller (1996), Maximum entropy econometrics: robust estimation with limited data", J. Wiley & Sons.

[69] Gradshteyn I. S., Ryzhik I. M. (1980). *Table of Integrals, Series and Products*, Academic Press.

[70] Gustafsson B., H-O. Kreiss and J. Oliger (1996) Time Dependent Problems and Difference Methods, Wiley and Sons Inc.

[71] Harrison J. M., Kreps D. (1979). Martingales and Arbitrage in Multiperiod Securities Markets, *Journal of Economic Theory*, 20 July, 381-408.

[72] Harrison, J. M. and Pliska, S. R. (1981), Martingales and stochastic integrals in the theory of continuous trading, *Stochastic Process. Appl.*, Vol. 15, 214-260.

[73] He H., Keirstead W. P. and Rebholz J. (1998), Double Lookbacks, *Mathematical Finance*, vol. 8, no. 3, July.

[74] Heynen, R. C. and H. M. Kat (1995), Lookback options with discrete and partial monitoring of the underlying price, *Applied Mathematical Finance*, Vol. 2, No. 4, December, pp. 273-284.

[75] Heynen R. C. and H. M. Kat (1997), Barrier options, in Exotic Options: the State of the Art, ed. Clelow and Strickland, International Thomson Business Press.

[76] Hochstadt H. (1986), The Functions of Mathematical Physics, Dover Publications, New York.

[77] Hugonnier J. N. (1999), The Feynman-Kac Formula and Pricing of Occupation Time Derivatives, *International Journal of Theoretical and Applied Finance*, vol. 2, no. 2, pagg. 153-78.

[78] Hui C. H. (1996), One-Touch Double Barrier Binary Option Values, *Appl. Financial Econ.* vol. 6, 343-6.

[79] Hui C. H., C. F. Lo and P.H. Yuen (2000), Comment on 'Pricing Double Barrier Options using Laplace Transforms' by Antoon Pelsser, *Finance and Stochastic* vol. 4, pagg. 105-7.

[80] Hull, J. (1997), Options, Futures, and Other Derivatives. Englewood Cliffs, NJ: Prenctice Hall.

[81] Hull, J. and A. White (1993), Efficient Procedures for Valuing European and American Path Dependent Options, *Journal of Derivatives*, Vol. 1, 21-31.

[82] Ikeda N. and S. Watanabe (1981), Stochastic Differential Equations and Diffusion Processes, North-Holland.

[83] Ince (1964), Ordinary Differential Equations, Dover Publications, Inc., New York.

[84] Ingersoll J. E. (1986), Theory of Financial Decisions Making,

[85] Ito K. and McKean Jr. H. P. (1974), Diffusion Processes and Their Sample Paths, Springer Verlag.

[86] Jamshidian F. (1996), Pricing of Contingent Claims in the One-Factor Term Structure Model, republished in Vasicek and Beyond, edited by Lane Hughston, Risk Publications, London. Original w.p. dated 1987.

[87] Jamshidian F. (1997), A Note on Analytical Valuation of Double Barrier Options, w.p. Sakura Global Capital, February.

[88] Jamshidian F. (1997), A note on analytical valuation of double barrier options, w.p. Sakura Global Capital.

[89] Jansons K. M. (1997). The Distribution of Time Spent by a Standard Excursion above a given Level, with Applications to Ring Polymers Near a Discontinuity in Potential, *Electronic Comm. in Prob.*, 2, 53-58.

[90] Jarrow R. and A. Rudd (1982), Approximate Option Valuation for Arbitrary Stochastic Processes, Journal of Financial Economics, vol. 10, pp. 347-369.

[91] Jaynes E.T. (1978), Where do we stand on maximum entropy, in R.D. Levine and M. Tribus Eds., *The maximum entropy formalism*, MIT Press Cambridge MA, 15-118.

[92] Johnson N. L. and S. Kotz (1970), Distributions in Statistics: Continuous Univariate Distributions, vol. 2, Houghton Miffin Company, Boston.

[93] Karatzas I. and S. E. Shreve (1991), Brownian Motion and Stochastic Calculus. 2nd ed., Springer.

[94] Karlin S. and L.S. Shapley (1953), *Geometry of moment spaces*. AMS Memoirs **12**, Providence RI.

[95] Karlin S. and H. M. Taylor (1981), A Second Course in Stochastic Processes. Academic Press.

[96] Kesavan, H.K., Kapur, J.N. (1992) *Entropy Optimization Principles with Applications*, Academic Press.

[97] Kevorkian J. (1989), *Partial Differential Equations - Analytical Solution Techniques*, Chapman & Hall, New York.

[98] Kullback, S., A lower bound for discrimination information in terms of variation, IEEE Trans. on Information Theory (Corresp.) , IT-13, 126-127, (1967).

[99] Kunitomo N. and M. Ikeda (1992), Pricing options with curved boundaries, *Mathematical Finance* 2, pagg. 275-98.

[100] Kwok Y. K. and D. Barthez (1989), An Algorithm for the Numerical Inversion of Laplace Transforms, *Inverse Problems* vol. 5, pagg. 1089-95.

[101] Lamberton D. and B. Lapeyre (1996), Introduction to Stochastic Calculus Applied to Finance, Chapman and Hall.

[102] Lebedev N. N. (1972), Special functions and their applications, Dover Publications, New York.

[103] Leblanc B. and O. Scaillet (1998), Path dependent options on yields in the affine term structure model, *Finance and Stochastics*, vol. 2, pagg. 349-67.

[104] Lévy, P. (1939), Sur certain processus stochastiques homogènes. *Compositio Math.*, Vol. 7, pages 283-339.

[105] Levy E. (1992), Pricing European Average Rate Currency Options, Journal of International Money and Finance, vol. 11, 474-491.

[106] Lindsay, B. G. and P. Basak (1995), Moments determine the tail of a distribution (but not much else), Technical report 95-7, Center for likelihood Studies, Dept. of Statistics, The Pennsylvania State University, University Park, PA.

[107] Lindsay, B. G. and K. Roeder (1997), Moment-based oscillation properties of mixture models, Ann. Statist., 25, pagg. 378-386.

[108] Linetski, V. (1996), Step options: The Feynman-Kac Approach to occupation time derivatives. University of Michigan Dept. of Industrial and Operations Engineering Technical Report 96-18.

[109] Linetski, V. (1999), Step options, *Mathematical Finance*, vol. 9, no. 1, January, pagg. 55-96.

[110] Lipton A. (1999), Similarities via self-similarities, *Risk*, september, pagg. 101-5.

[111] Longman I. M. (1975), Application of best Rational Function Approximation for Laplace Transform Inversion, *J. Comp. Appl. Math.*, vol. 1, pagg. 17-23.

[112] Longstaff F. A. (1990), The valuation of options on yields, *Journal of Financial Economics*, vol. 26, pagg. 97-121.

[113] Longstaff F. A. (1995), Hedging interest rate risk with options on average interest rates, *The Journal of Fixed Income*, vol. 4, no. 4, march, pagg. 37-45.

[114] MacDonald J. R. (1964), Accelerated convergence, divergence, iteration, extrapolation and curve fitting, *J. Appl. Phys.* 10, 3034-3041.

[115] Maghsoodi Y. (1996), Solution of the Extended CIR Term Structure and Bond Option Valuation, *Mathematical Finance*, vol. 6, no. 1, jan., pagg. 89-109.

[116] McDonald J. B. (1984), Some generalized functions for the size distribution of income, Econometrica, vol. 52, 647-663.

[117] McDonald J. B. (1996), Probability distributions for financial models, Handbook of Statistics G.S. Maddala and C. R. Rao eds, vol. 14, chapt. 14, 427-461.

[118] Milevsky M. A. and S. E. Posner (1998), Asian Options, the Sum of Lognormals and the Reciprocal Gamma Distribution, *Journal of Financial and Quantitative Analysis,* vol. 33, no. 3, pagg. 409-22, September.

[119] Miura, R. (1992), A Note on Look-Back Option based on Order Statistics, *Hitotsubashi Journal of Commerce and Management*, Vol. 27, pp. 15-28.

[120] Moorthy, M. V. (1995), Numerical Inversion of two-dimensional Laplace transforms - Fourier series representation, *Applied Numerical Mathematics*, Vol. 17, pp. 119-127.

[121] Moorthy, M. V. (1995), Inversion of the multi-dimensional Laplace transform-expansion by Laguerre series, *Z angew Math Phys*, 46, 793-806.

[122] Navatte P. and F. Quittard-Pinon (1999), The valuation of interest rate digital options and range notes revisited, *European Financial Management*, Vol. 5, no. 3, pages 425-440.

[123] Necati Özişik M. (1989), Boundary Value Problems of Heat Conduction, Dover.

[124] Ortega, J.M. (1998), *Matrix Theory*, Plenum Press, New York.

[125] Pechtl, A. (1995), Classified Information, in Over the Rainbow, ed by Robert Jarrow, Risk Publications, pag. 71-74

[126] Pelsser A. (2000), Pricing double barrier options using Laplace transforms, *Finance Stochastics,* vol. 4, pagg. 95-104.

[127] Piessens R. (1969), New quadrature formulas for the numerical inversion of the Laplace transform, *BIT*, vol. 9, pagg. 351-61.

[128] Platzman L. K., J. C. Ammons and J. J. Bartholdi (1988), A Simple and Efficient Algorithm to Compute Tail Probabilities from Transforms, *Oper. Res.*, vol. 26, pagg. 137-44.

[129] Press, W. H., Teukolsky S. A., Vetterling W. T. and Flannery B. P. (1997). *Numerical Recipes in C*, Cambridge University Press, version 2.08

[130] Rebonato R. (1999), Volatility and Correlation in the Pricing of Equity, FX and Interest-Rate Options, chpt. 7, 190-214, Wiley Financial Engineering Series.

[131] Ritchken P., L. Sankarasubramanian and A. M. Vijh (1990), The Valuation of Path-Dependent contracts on the average, Management Science,

[132] Rogers L. C. G. and Shi Z. (1992), The Value of an Asian option, *J. of Applied Probability*, Vol. 32, pagg. 1077-88.

[133] Sankaran M., 1963, Approximations to the Non Central Chi-Square Distribution, *Biometrika* 50, 199-204.

[134] Sbuelz A. (1999), A general treatment of barrier options and semi-static hedges of double barrier options, w.p. London Business School.

[135] Schroder, M. (1989), Computing the CEV option pricing formula, *Journal of Finance* vol. 44, pagg. 211-219.

[136] Selby, M. J. P. (1983), The Application of Option Theory to the Evaluation of Risky Debt, PH.D. Thesis, London Business School, May.

[137] Shiga, T. and S. Watanabe (1973), Bessel diffusions as a One-Parameter Family of Diffusion Processes, *Z. Wahrsch.*, 27, pagg. 37-46.

[138] Shimko, D. (199), Finance in continuous time: A Primer, Kolb Publishing Company.

[139] Shohat J. A. and J.D. Tamarkin (1943), *The problem of moments*, AMS Mathematical Survey **1**, Providence RI, (1943).

[140] Simon R. M., Stroot M. T. and G. H. Weiss (1972), Numerical Inversion of Laplace Transforms with Applications to the Percentage Labeled Experiments, *Comput. Biomed. Res.* 6, pagg. 596-607.

[141] Singhal, K., Vlach, J. (1975), Computation of Time Domain Response by Numerical Inversion of the Laplace transform, Journal of the Franklin Institute, no. 2, pagg.110-27.

[142] Singhal, K., Vlach, J. and M. Vlach (1975), Numerical inversion of multidimensional Laplace transforms, *Proc. IEEE* 63, 1627-1628.

[143] Smith, G. D. (1985), *Numerical solution of partial differential equations: finite difference methods*, Oxford University Press.

[144] Stoer, J. and R. Bulirsch (1980), *Introduction to Numerical Analysis*, Springer Verlag, New York.

[145] Stoyanov J. (1997), Counterexamples in Probability, 2nd edition, Wiley Series in Probability and Statistics.

[146] Strauss, W.A. (1992), Partial Differential Equations: An Introduction, Wiley and Sons,

[147] Tagliani A. (2000), Recovering a probability density function from its Mellin transform, to appear in Applied Mathematics and Computation.

[148] Takàcs, L. (1995), On the local time of the Brownian motion, *The Annals of Applied Probability*, vol. 5, No. 3, pagg. 741-756.

[149] Takàcs, L. (1996), On a generalization of the Arc-Sine Law, *The Annals of Applied Probability*, Vol. 6, No. 3, pagg. 1035-1040.

[150] Takàcs (1998), Sojourn Times for the Brownian Motion, *Journal of Applied Mathematics and Stochastic Analysis*, Vol. 11, no. 3, 231-246.

[151] Talbot, A. (1979), The Accurate Numerical Inversion of Laplace Transforms, *J. Inst. Math. Appl.* vol. 23, n. 1, pagg. 97-120.

[152] Taleb, N. (1997), *Dynamic Hedging*, John Wiley.

[153] Thompson G. W. P. (1999), Topics in Mathematical Finance, PH.D. Thesis, Queens' College, January.

[154] Thompson G. W. P. (1999), Fast narrow bounds on the Value of Asian options, w.p. Judge Institute of Management Studies, U. of Cambridge.

[155] Tucker, A. L. and Wei J. Z. (1997), The Latest Range, Advances in Futures and Options Research, Vol. 9, pages 287-296.

[156] Turnbull, S. M. (1995), Interest Rate Digital Options and Range Notes, *Journal of Derivatives* 3, 92-101.

[157] Turnbull S. and L. Wakeman (1991), A Quick Algorithm for Pricing European Average Options, Journal of Financial and Quantitative Analysis, vol. 26, pagg. 377-389.

[158] Vlach J. and K. Singhal (1993), Computer Methods for Circuit Analysis and Design, Van Nostrand Reinhold Company, New York, 2nd edition.

[159] Vorst, T. (1996), Averaging Options, in I. Nelken, *The Handbook of Exotic Options*, Chpt. 6, pp. 175-199, IRWIN.

[160] Yor M. (1992), On some exponential functionals of Brownian Motion, *Adv. Appl. Prob.* 24, pagg. 509-531.

[161] Yor M. (1992), Some Aspects of Brownian Motion. Part I: Some Special Functionals. Birkhauser Verlag. Basel - Boston - Berlin.

[162] Yor M. (1992), Sur les lois des fonctionnelles exponentielles du mouvement brownien, considérées en certains instants aléatoires, C. R. Acad. Sci Paris Série I, 314, 951-956.

[163] Weeks W. T. (1966), Numerical inversion of Laplace transforms using Laguerre functions, *ACM* 13, 3, july, pagg. 419-29.

[164] Weideman J. A. C. (1999), Algorithms for Parameter Selection in the Weeks Method for Inverting the Laplace Transform, *SIAM J. Sci. Comput.*, vol. 21, no. 1, pagg. 111-28.

[165] Wellekens C. J. (1970), Generalization of Vlach's Method for the Numerical Inversion of the Laplace Transform, *Electronics Letter*, vol. 6, n. 23, November, pagg. 742–4.

[166] Wilmott, P. Dewynne, J.N. and Howison, S. (1993), *Option Pricing: Mathematical Models and Computation*, Oxford Financial Press.

[167] Wynn P. (1961), The $\epsilon$-algorithm and operational formulas of numerical analysis, *Math. Comp.* vol. 15, pagg. 151-8.

[168] Zakian V. (1970), Optimisation of Numerical Inversion of Laplace Transforms, *Electronics Letter*, vol. 6, n. 21, October, pagg. 677-9.

[169] Zauderer, E. (1989), *Partial Differential Equations of Applied Mathematics*, John Wiley & Sons, 2nd ed..

[170] Zvan R., P. A. Forsyth and K. R. Vetzal (1997), Robust Numerical methods for PDE models of Asian options, *The Journal of Computational Finance,* Vol. 1, N. 2, 39-77.

[171] Zvan, R., K. R. Vetzal and P. A. Forsyth (1997), PDE methods for barrier options, University of Waterloo Department of Computer Science Technical Report CS-97-27.