On the Taketa bound for normally monomial p–groups of maximal class

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Abstract

A longstanding problem in the representation theory of finite solvable groups, sometimes called the Taketa problem, is to find strong bounds for the derived length $\mathrm{dl}(G)$ in terms of the number $|\mathrm{cd}(G)|$ of irreducible character degrees of the group G. For p–groups an old result of Taketa implies that $\mathrm{dl}(G) \leq |\mathrm{cd}(G)|$, and while it is conjectured that the true bound is much smaller (in fact, logarithmic) for large $\mathrm{dl}(G)$, it turns out to be extremely difficult to improve on Taketa's bound at all. Here, therefore, we suggest to first study the problem for a restricted class of p–groups, namely normally monomial p–groups of maximal class. We exhibit some structural features of these groups and show that if G is such a group, then $\mathrm{dl}(G) \leq \frac{1}{2}|\mathrm{cd}(G)| + \frac{11}{2}$.

1. Introduction

In 1930, K. Taketa proved that finite monomial groups are solvable, and his proof implied that the derived length $\mathrm{dl}(G)$ of the group G is bounded by the number $|\mathrm{cd}(G)|$ of irreducible complex character degrees of G, i.e.,

$$\mathrm{dl}(G) \leq |\mathrm{cd}(G)|.$$

Since p–groups are monomial, this bound in particular holds for p–groups.
In the 1970's Isaacs picked up the problem and asked for more general and better bounds. Since then, numerous people have worked on this problem, which we shall call Taketa problem henceforth (see e.g. [8] for a more detailed account of the history of the problem), with the result that today it is known that $\mathrm{dl}(G) \leq 2|\mathrm{cd}(G)|$ in general, and it is believed that there are universal constants C_1, C_2 such that for any finite solvable group we have

$$(*) \quad \mathrm{dl}(G) \leq C_1 \log|\mathrm{cd}(G)| + C_2.$$

This conjecture has first been stated early on by Isaacs for p–groups only, but today this is more or less the only case for which it has not been proved yet, i.e., proving $(*)$ for p–groups essentially will imply $(*)$ for arbitrary solvable groups (see [7]). So curiously enough, while establishing a linear bound for p–groups is quite easy and the conjectural logarithmic bound arose from p–groups, it turned out that the p–group case is the core problem in proving $(*)$ for arbitrary solvable groups. In fact, it seems as if even the slightest improvement of Taketa's inequality (such as $\mathrm{dl}(G) \leq |\mathrm{cd}(G)| - 1$ for large $\mathrm{dl}(G)$) is almost out of reach of today's techniques. This is evidenced by the fact that, to the authors' knowledge, there is only one result in this direction, namely Slattery's result [17] that if G is a p–group and $\mathrm{cd}(G) = \{1, p, p^2, \ldots, p^n\}$, then $\mathrm{dl}(G) \leq n - 1$, which is an improvement in a very specialized situation. Recently A. Moretó could prove a logarithmic bound in the situation of Slattery's theorem. (His result is even more general, see [15, Theorem D].)
Moreover, the evidence supporting $(*)$ for p–groups stems from studying quite specific families of p–groups, mostly Sylow subgroups of some classical groups, for which the character degrees are sufficiently well–known. Given the abundance of p–groups, this evidence is not too strong, and there might as well be a corner in the universe of p–groups where a counterexample is hiding.

In view of all these difficulties, in this paper we propose to start modestly on the problem by considering it for a very restricted class $\mathcal{C}$ of p–groups only that still in many ways catches the typical behavior of p–groups and thus could provide essential insight for further progress on the problem. In particular, it could be one of the places where a counterexample to $(*)$ is hiding, if one exists. Otherwise, it should be possible to prove $(*)$ for groups in $\mathcal{C}$ (although we have not been able to do so).

The class $\mathcal{C}$ we have in mind is the set of all normally monomial p–groups of maximal class. In Section 2 we will collect some basic structural properties of the groups in $\mathcal{C}$, and we will also see that for these groups $|\mathrm{cd}(G)|$ can be determined from the group structure without using character theory. In Section 3 we will – via the Lazard correspondence – translate the Taketa problem to a problem in Lie algebras which are easier to study.
Finally we will use our results to prove that

$$\mathrm{dl}(G) \leq \frac{1}{2}|\mathrm{cd}(G)| + \frac{11}{2}$$

for all groups in $\mathcal{C}$, thus giving a first improvement on Taketa's bound (see Corollary 3.12(c) below).

2. Normally monomially p–groups of maximal class

In this section, we will investigate the structure of the groups in $\mathcal{C}$ and collect some useful information on these groups. It is well–known that p–groups are monomial, i.e., every irreducible complex character χ is induced from a linear character of some subgroup U_χ (depending on χ). If in an M–group G for every $\chi \in \mathrm{Irr}(G)$ the subgroup U_χ can be chosen to be normal in G, then (following [10]) we call G *normally monomial* and also say that G is an nM–group. With this notation the class $\mathcal{C}$ of groups we are interested in here is

$$\mathcal{C} = \{G \mid G \text{ is a normally monomial } p\text{–group of maximal class}\}.$$

For our study of this class of groups we make use of a characterization of nM–groups due to G. A. How [2, 3] which immediately implies the following:

Theorem 2.1. *Let G be a finite group with a unique minimal normal subgroup N, and let A be an abelian normal subgroup of maximal order in G.*

Then G is an nM–group if and only if the following hold:

(1) G/N is an nM–group and

(2) $N \leq [A, g] = \langle [a, g] \mid a \in A\rangle$ for all $g \in G - A$.

Proof. This follows immediately from [2]. $\diamond$

Note that while the groups in $\mathcal{C}$ have quite a restricted structure, still $\mathcal{C}$ contains groups of arbitrary derived length, as was shown in [10] and [16]. Hence it does make sense to study the Taketa problem for the groups in $\mathcal{C}$.

Now let G be a p–group of maximal class, that is, if $|G| = p^n$ and $n \geq 2$, then $\mathrm{cl}(G) = n - 1$. A lot of structural information is available on p–groups of maximal class (see e.g. [4, III, §14]), and we will freely use the basic well–known facts on these groups. We introduce some more notation that we will use for the rest of the paper.

Let G be a p–group of maximal class with $|G| = p^n$ and $n \geq 4$. Let $G = \gamma_1(G) > \gamma_2(G) > \ldots > \gamma_{n-1}(G)$ be the lower central series of G, and let $G_1 = C_G(\gamma_2(G)/\gamma_4(G))$. Then $|G : G_1| = p$ and G_1 is characteristic in G. Any maximal subgroup of G except for G_1 is of maximal class.
We define $G_i = G_i(G) = \gamma_i(G)$ for $i \geq 2$ and $G_0 = G$ so that altogether $G_0 > G_1 > \ldots > G_n = 1$ is a characteristic series of G with $G' = G_2$. Also, if $i \geq 2$, then G_i is the only normal subgroup of G of index p^i in G.
Moreover, as in [4, III, Def. 14.5 and Hauptsatz 14.7] we say that G is *non–exceptional* or *of non–exceptional type* if and only if $[G_i, G_1] \leq G_{i+2}$ for $i \in \mathbb{N}$ or equivalently if and only if $[G_i, G_j] \leq G_{i+j+1}$ for all i, j with $i + j > 2$. If G is not non–exceptional, we say that G is exceptional or of exceptional type. The main results about exceptionality are due to Blackburn and can also be found in [4, III, §14]. In particular, as G/G_{n-1} is always non–exceptional, for studying the Taketa problem on $\mathcal{C}$, one often can assume that G is non–exceptional.
Let $G = \langle G_1, s\rangle$ and $G_1 = \langle G_2, s_1\rangle$ for suitable $s, s_1 \in G$. If we define recursively $s_{i+1} = [s, s_i]$ for $i \geq 1$, then it is well–known that if G is non–exceptional, then $G_i = \langle s_i, G_{i+1}\rangle$ for $i \in \mathbb{N}$.

Non–exceptionality has good hereditary properties, as the following lemma shows.

Lemma 2.2. *Let G be a non–exceptional p–group of maximal class of order p^n with $n \geq 3$ and let H be a maximal subgroup of G with $H \neq G_1$. Then H is a non–exceptional p–group of maximal class of order p^{n-1}, and we have*

$$G_j(H) = G_{j+1} \text{ for } j = 1, \ldots, n-2.$$

Proof. By [4, III, Satz 14.22] we know that H is a p–group of maximal class. Now $H \trianglelefteq G$ and $G_j(H)$ is characteristic in H of index p^j for $j = 1, \ldots, n-2$. Therefore $G_j(H)$ is normal in G of index p^{j+1} $(j = 1, \ldots, n-2)$, and so by [4, III, Hilfsatz 14.2b)] we have $G_j(H) = G_{j+1}$ for $j = 1, \ldots, n-2$. In particular, $G_1(H) = G_2 = G'$, and therefore $[G_1(H), G_i(H)] = [G_2, G_{i+1}] \leq G_{i+3} = G_{i+2}(H)$ for all $i \in \mathbb{N}$. This shows that H is non–exceptional, and the lemma is proved. $\diamond$

Now we can study the groups in $\mathcal{C}$. First we prove an important hereditary property of belonging to $\mathcal{C}$, showing that the groups in $\mathcal{C}$ are rather well–behaved.

Lemma 2.3. *Let $G \in \mathcal{C}$ be non–exceptional and let $H \leq G$ be a maximal subgroup of G with $G_1 \neq H$. Then $H \in \mathcal{C}$ and H is non–exceptional.*

Proof. By Lemma 2.2 we know that H is a non–exceptional p–group of maximal class. It remains to show that H is an nM–group. We prove this by induction on n, where $|G| = p^n$. If $n \leq 3$, then H is abelian and everything is trivial. So let $n \geq 4$. By Lemma 2.2 we have $G_j(H) = G_{j+1}$ for $j = 1, \ldots, n-2$. Now by induction, applied to G/G_{n-1} and its maximal subgroup H/G_{n-1}, we see that $H/G_{n-1} \in \mathcal{C}$. Thus by Theorem 2.1 we have to show that if $A \trianglelefteq H$ is a maximal abelian normal subgroup of G and $g \in H - A$, then

$$[\langle g\rangle, A] \geq G_{n-2}(H) = G_{n-1} \quad (+).$$

As $n \geq 4$, H is not abelian. Let A be a maximal abelian normal subgroup of H. If $|H : A| \geq p^2$, then obviously $n \geq 5$ and again by [4, III, Hilfssatz 14.2b)] clearly $A = G_j(H) = G_{j+1}$ for some $j \in \{2, \ldots, n-3\}$ (which is nonempty). But then A is obviously also the maximal abelian normal subgroup of G, and since $G \in \mathcal{C}$ and thus satisfies the conclusion of Theorem 2.1, obviously this implies (+). Therefore as H is nonabelian, it remains to consider the case $|H : A| = p$. So if $g \in H - A$, then $H = \langle g, A\rangle$ and thus $[\langle g\rangle, A] = H'$. Since $|H/H'| = p^2$, we have $H' = G_j(H) = G_{j+1} \geq G_{n-1}$ for some $j \in \{2, \ldots, n-2\}$ and so altogether we get $[\langle g\rangle, A] \geq G_{n-1}$, as wanted. This finishes the proof of the lemma. $\diamond$

We now take a closer look at the metabelian sections of the groups in $\mathcal{C}$ of exponent p.

Proposition 2.4. Let G be a p–group of maximal class and of exponent p. Suppose that there are $1 < i < j$ such that $\gamma_{i-1}(G)' > \gamma_i(G)' = \ldots = \gamma_j(G)' > \gamma_{j+1}(G)'$. By [4, III, Hilfssatz 14.2b)] there is a $k > j$ such that $\gamma_i(G)' = \gamma_k(G) > 1$. For $U \leq G$ write $\overline{U} = U\gamma_{k+1}(G)/\gamma_{k+1}(G)$, and put $N = \overline{\gamma_i(G)}$. Then the following hold:

(a) $N = E \times A$ for an extraspecial group E with $\overline{\gamma_k(G)} = Z(E)$ and an elementary abelian group A.

(b) $\overline{\gamma_k(G)} \times A = Z(\overline{\gamma_i(G)}) \trianglelefteq \overline{G}$, in particular, there is an l such that $j < l \leq k$ and $\overline{\gamma_l(G)} = \overline{\gamma_k(G)} \times A$, and $l - i$ is even. For $s \in \{0, \ldots, \frac{l-i}{2}\}$ we have $C_N(\overline{\gamma_{l-s}(G)}) = \overline{\gamma_{i+s}(G)} = E_s \times A_s$ with $E_s \leq E$ extraspecial of order $\frac{|E|}{p^{2s}}$ and $A \leq A_s \leq Z(\overline{\gamma_{i+s}(G)})$ elementary abelian, $d(A_s) = d(A) + s$ and $\overline{\gamma_{l-s}(G)} = A_s \times \overline{\gamma_k(G)}$. Moreover, $C_N(\gamma_{l-s}(G)) = C_G(\gamma_{l-s}(G))$ for $s = 1, \ldots, \frac{l-i}{2}$, and $\overline{\gamma_{\frac{l+i}{2}}(G)}$ is the maximal abelian normal subgroup of $\overline{\gamma_i(G)}$ (and thus of $\overline{G}$).

Proof. Clearly we may assume that $\gamma_{k+1}(G) = 1$. Now $N/N' = \gamma_i(G)/\gamma_k(G)$ is elementary abelian and N' has order p and $N' \leq Z(N)$. As $\exp(\overline{G}) = p$ we can write $Z(N) = N' \times A$ for an elementary abelian subgroup A. So now it is easy to see that for $\tilde{N} := N/A$ we have $Z(\tilde{N}) = Z(N)/A = N'A/A \cong N'$: Namely clearly $Z(N)/A \leq Z(\tilde{N})$, and if $\tilde{x} = xA \in Z(\tilde{N})$, then $[\langle x\rangle, N] \leq A$ and thus $[\langle x\rangle, N] \leq N' \cap A = 1$, so $x \in Z(N)$ and $\tilde{x} \in Z(N)/A$.
Consequently $Z(\tilde{N}) = N'A/A = (N/A)' = \tilde{N}' \cong N'$ is cyclic of order p, i. e., $\tilde{N}$ is extraspecial of exponent p. Thus we conclude that $N = E \times A$ where $E \cong \tilde{N}$ is extraspecial with $Z(E) = \gamma_k(G)$. So (a) is shown.
By [4, III, Hilfssatz 14.2b)] there is an $l \in \mathbb{N}$ such that $\gamma_l(G) = Z(N) = \gamma_k(G) \times A$, and clearly $j < l \leq k$. Moreover $N/Z(N) \cong E/Z(E)$ and so $l - i = d(N/Z(N))$ is even. We prove the statement on $C_N(\gamma_{l-s}(G))$ in (b) by induction on s. For $s = 0$ the assertion is trivial. So let $s \geq 1$. By induction we see that $C_N(\gamma_{l-s}(G)) \leq C_N(\gamma_{l-(s-1)}(G)) = \gamma_{i+s-1}(G) = E_{s-1} \times A_{s-1}$ with E_{s-1} extraspecial, $|E_{s-1}| = \frac{|E|}{p^{2(s-1)}}$, A_{s-1} elementary abelian, $A \leq A_{s-1} \leq Z(\gamma_{i+(s-1)}(G))$, and $\gamma_{l-(s-1)}(G) = A_{s-1} \times \gamma_k(G)$ is abelian. Hence $A_{s-1} \leq C_N(\gamma_{l-s}(G)) \trianglelefteq G$ and so $\gamma_{l-(s-1)}(G) \leq C_N(\gamma_{l-s}(G))$. As $\gamma_{l-s}(G)/A_{s-1}$ is of order p^2 and a normal subgroup of the extraspecial group $\gamma_{i+s-1}/A_{s-1} \cong E_{s-1}$, from the well–known structure of extraspecial groups we conclude that $C_N(\gamma_{l-s}(G)) = E_{s-2} \times \langle x\rangle \times A_{s-1}$ for a suitable $x \in E_{s-1} \cap \gamma_{l-s}(G)$ and $E_{s-2} \leq E_{s-1}$ is extraspecial of order $\frac{|E|}{p^s}$. Moreover since $|C_N(\gamma_{l-(s-1)}(G))/C_N(\gamma_{l-s}(G))| = p$, we have $C_N(\gamma_{l-s}(G)) = \gamma_{i+s}(G)$. If we put $A_s = \langle x\rangle \times A_{s-1}$, then obviously $A \leq A_s \leq Z(\gamma_{i+s}(G))$, A is elementary abelian,

$d(A_s) = d(A) + s$ and $\gamma_{l-s}(G) = A_s \times \gamma_k(G)$.
Finally observe that as $C_N(\gamma_{l-s}(G)) \trianglelefteq G$ and $C_N(\gamma_{l-s}(G)) < N$ for $s \geq 1$, we have $C_G(\gamma_{l-s}(G)) = C_N(\gamma_{l-s}(G))$ for $s = 1, \ldots, \frac{l-i}{2}$. Also from the above it is clear that for $s = \frac{l-i}{2}$ we see that $\gamma_{l-s}(G) = \gamma_{\frac{l+i}{2}}(G)$ is the maximal abelian normal subgroup of G. So also (b) is proved. $\diamond$

Next we turn to the irreducible characters of the groups in $\mathcal{C}$. We give a purely group theoretic description of $|\mathrm{cd}(G)|$.
As usual, we denote by $\mathrm{Irr}(G)$ the set of ordinary irreducible characters of G and write $\mathrm{cd}(G) = \{\chi(1) \mid \chi \in \mathrm{Irr}(G)\}$ for the set of the degrees of these characters.

Proposition 2.5. Let $G \in \mathcal{C}$.
(a) Let $i \in \{0, 1, \ldots, n-1\}$ and suppose that there is a $1 \neq \lambda \in \mathrm{Irr}(G_i')$ with $\lambda(1) = 1$ and $G_{i+1}' \leq \ker(\lambda)$. If $\mu \in \mathrm{Irr}(G_{i+1})$ is an extension of λ (which clearly exists), then $\mu^G \in \mathrm{Irr}(G)$.

(b) Let $\chi \in \mathrm{Irr}(G)$ with $\chi(1) \neq 1$. Then there is an $i \in \{0, \ldots, n-1\}$ and a linear $\mu \in \mathrm{Irr}(G_{i+1})$ with $\mu^G = \chi$ and such that for $\lambda = \mu|_{G_i'}$ we have $\lambda \neq 1$ and $G_{i+1}' \leq \ker(\lambda)$.

Proof. (a) Note that as $\lambda \neq 1$, we have $n \geq 3$. Clearly we may assume that $G_{i+1}' = 1$, so that G_{i+1} is the unique maximal abelian normal subgroup of G. So there is an extension μ of λ to G_{i+1}. Now by [5, Proposition 19.12] λ cannot be extended to G_i, as by hypothesis $G_i' \not\leq \ker(\lambda)$. Hence as G is an nM–group, it follows easily that μ^G must be irreducible.
(b) As G is an nM–group, and $\chi(1) \neq 1$, χ is induced from a linear character μ of some G_{i+1} for a suitable $i \in \{0, \ldots, n-2\}$. So clearly $G_{i+1}' \leq \ker(\lambda)$. Now if even $G_i' \leq \ker(\lambda)$, then $G_i' \leq \ker(\mu)$ and thus $G_i' \leq \ker(\chi)$, so that χ can be seen as an element of $\mathrm{Irr}(G/G_i')$. Hence G_i/G_i' is an abelian normal subgroup of G/G_i' and thus by Ito $|G : G_{i+1}| = \mu^G(1) = \chi(1)$ divides $|G/G_i|$ which is a contradiction. Therefore $G_i' \not\leq \ker(\lambda)$ and we are done. $\diamond$

Corollary 2.6. *Let $G \in C$. Define $s(G) = |\{G_i' \mid i = 0, \ldots, n\}|$.*
Then $|\mathrm{cd}(G)| = s(G)$.

Proof. By Proposition 2.5 we conclude that $\mathrm{cd}(G) - \{1\} = \{|G : G_{i+1}| \mid i \in \{0, \ldots, n-1\}$ such that there is a $1 \neq \mu \in \mathrm{Irr}(G_{i+1}/G_{i+1}')$ with $\mu|_{G_i'} \neq 1\}$. Now since for any $i \in \{0, 1, \ldots, n-1\}$ there exists such a $1 \neq \mu \in \mathrm{Irr}(G_{i+1}/G_{i+1}')$ with $\mu|_{G_i'} \neq 1$ if and only if $G_i' > G_{i+1}'$, we conclude that

$$|\mathrm{cd}(G)| - 1 = |\{i \mid i \in \{0, 1, \ldots, n-1\} \text{ with } G_i' > G_{i+1}'\}|$$

which immediately implies $|\mathrm{cd}(G)| = s(G)$, as wanted. $\diamond$

3. Exploring $\mathcal{C}$ via Lie algebras

An important and rather well–known tool we have to use is a strong structure–preserving correspondence between certain p–groups and certain Lie rings, which was discovered by W. Magnus (see [13]) and later independently by M. Lazard (see [11]). It runs as follows:

Theorem 3.1. *Let p be a prime, let Γ_p denote the set of finite p–groups $(P, \cdot_P)$ (where $\cdot_P$ denotes the group multiplication) whose nilpotency class is less than p, and let Λ_p denote the set of finite nilpo-*

tent Lie rings $(L, [\cdot,\cdot]_L)$ (where $[\cdot,\cdot]_L$ denotes the Lie bracket) whose order is a power of p and whose nilpotency class is less than p. Then there exists a bijection $\phi : \Gamma_p \to \Lambda_p$ with $\phi : (P, \cdot_P) \mapsto (P, [\cdot,\cdot]_P)$ for $(P, \cdot_P) \in \Gamma_p$ (i.e., the set P remains the same under ϕ, so the set P carries a group structure and a Lie ring structure at the same time), such that the following holds:

(1) If $U \subseteq P$, then $(U, \cdot_P)$ is a subgroup of $(P, \cdot_P)$ if and only if $(U, [\cdot,\cdot]_P)$ is a Lie–subring of $(P, [\cdot,\cdot]_P)$, and $(U, \cdot_P)$ is a normal subgroup of $(P, \cdot_P)$ if and only if $(U, [\cdot,\cdot]_P)$ is an ideal of $(P, [\cdot,\cdot]_P)$.

(2) If $(H, \cdot_P)$ and $(K, \cdot_P)$ are normal subgroups of $(P, \cdot_P)$, then the set of elements of the commutator group $[(H, \cdot_P), (K, \cdot_P)] = \langle [x, y] \mid x \in (H, \cdot_P), y \in (K, \cdot_P)\rangle$ coincides with the set of elements of the ideal $\langle [x, y]_P \mid x \in (H, [\cdot,\cdot]_P),\ y \in (K, [\cdot,\cdot]_P)\rangle$ of $(P, [\cdot,\cdot]_P)$ generated by all Lie brackets of elements of $(H, [\cdot,\cdot]_P)$ with elements of $(K, [\cdot,\cdot]_P)$.
In particular, if $(P^{(i)}, \cdot_P)$ for $i \in \mathbb{N}$ are the subgroups of the derived series of $(P, \cdot_P)$, then the $(P^{(i)}, [\cdot,\cdot]_P)$ are the ideals of the derived series of $(P, [\cdot,\cdot]_P)$; so the derived lengths of $(P, \cdot_P)$ and $(P, [\cdot,\cdot]_P)$ coincide.
Likewise, if $(\gamma_i(P), \cdot_P)$ for $i \in \mathbb{N}$ are the subgroups of the lower central series of $(P, \cdot_P)$, then the $(\gamma_i(P), [\cdot,\cdot]_P)$ are the ideals of the lower central series of $(P, [\cdot,\cdot]_P)$; so the nilpotency classes of $(P, \cdot_P)$ and $(P, [\cdot,\cdot]_P)$ coincide.

Proof. This follows from the work Magnus [13] and Lazard [11]. (Alternatively, see [9, Example 10.24 and the comments preceeding it]). $\diamond$

Since Lie rings are often easier to handle than p–groups (as becomes obvious by looking at the corresponding Jacobi identities, for example), we will often work in the setting of Lie rings and then via Theorem 3.1 translate the obtained results into statements about p–groups.

So to study the groups in $\mathcal{C}$ via Lie algebras, we will restrict ourselves to the class

$$\mathcal{C}_p := \{G \mid G \in \mathcal{C},\ G \text{ has nilpotency class less than } p \text{ and } G \text{ is non–exceptional of exponent } p\}.$$

This, however, is not a great restriction for studying the Taketa problem on $\mathcal{C}$ because if G is a p–group of maximal class with $\mathrm{dl}(G) \geq 4$, by [12, Corollary 2.7] the class of G is at most $9p - 40$, and thus $\mathrm{dl}(G_p) \leq 4$. Hence if G has class $\geq p$, then by [4, III, Hauptsatz 14.6 and Hilfssatz 14.14])

$$\mathrm{dl}(G) \leq \mathrm{dl}(G/G_p) + 4 \text{ and } G/G_p \in \mathcal{C}_p.$$

We next reword the Taketa problem on $\mathcal{C}_p$ in terms of Lie algebras.

For any Lie algebra L, we define the derived Lie subalgebra by

$$L' = [L, L] = \langle [u, v] \mid u, v \in L\rangle.$$

In general, for Lie algebras U, V we let $[U, V] = \langle [u, v] \mid u \in U, v \in V\rangle$.

Definition 3.2. Let p be a prime. We define a class $\mathcal{L}_p$ of Lie algebras in the following way:
$L \in \mathcal{L}_p$ if and only if the following hold:

(a) L is a Lie algebra over $\mathrm{GF}(p)$.

(b) $n := \dim_{\mathrm{GF}(p)} L \leq p$

(c) There are $e_1, \ldots, e_n \in L$ such that $L = \langle e_1, \ldots, e_n \rangle$, and if we put $L_i = \langle e_i, e_{i+1}, \ldots, e_n \rangle$ for $i \in \mathbb{N}$ (so that $L_1 = L$ and $L_m = 0$ for $m > n$), then

$$[L, L_i] = [e_1, L_i] = L_{i+1} \text{ for } i \geq 2,$$

where $[e_1, U] = \langle [e_1, u] \mid u \in U \rangle$ (generated as $\mathrm{GF}(p)$–vector space) for any $U \subseteq L$.

Using the fact that $[[L_i, L_j], L_k] \subseteq [[L_j, L_k], L_i] + [[L_k, L_i], L_j]$, one can easily see (by induction) that $[L_i, L_j] \subseteq L_{i+j}$ for all i, j. By [4, III, Hauptsatz 14.7] this shows that the (via Lazard) corresponding group of any $L \in \mathcal{L}_p$ is non–exceptional. Hence altogether the Lazard–correspondence induces a bijection between $\mathcal{C}_p$ and $\mathcal{L}_p$.

Remark 3.3. Let $L \in \mathcal{L}_p$. Then the following hold.

(a) $L_i' = [L_i, L_{i+1}]$ for all i

(b) $[L_i, L_j] \subseteq L_{i+j}$ for all i, j, more precisely $[L_i, L_j] = L_k$ for all i, j and some $k \geq i + j$.

Proof. (a) is trivial.
(b) follows immediately from the corresponding facts for the groups in $\mathcal{C}_p$, but can also be shown directly from the definition. $\diamond$

Example 3.4. Let p be a prime, $n \leq p$ and $L = \langle e_1, \ldots, e_n \rangle$ a $\mathrm{GF}(p)$–vector space. Define the bilinear form $[\cdot, \cdot]$ on L by putting $[e_i, e_j] = (i - j)e_{i+j}$ and linearly extending this to arbitrary elements of L. With this, L becomes a Lie algebra and even $L \in \mathcal{L}_p$ (see e.g. [10]).
At this point, except for trivial variations this seems to be the only known family of Lie algebras in $\mathcal{L}_p$ with increasing derived length. We will refer to this family as the "standard example".

Definition 3.5. For $L \in \mathcal{L}_p$, define $s(L) = |\{L_i' \mid i = 1, \ldots, n\}|$. Also, for $k \in \mathbb{N}$ define the function $f : \mathbb{N} \to \mathbb{N}$ by

$$f(k) = \min\{s(L) \mid L \in \mathcal{L}_p \text{ and } \mathrm{dl}(L) \leq k\}$$

With this, the Taketa problem for $\mathcal{L}_p$ is to find good upper bounds for $\mathrm{dl}(L)$ in terms of $s(L)$, and any result on this is, via the Lazard correspondence, a result on the Taketa problem for $\mathcal{C}_p$.

Note that we clearly have $1 \leq s(L) \leq n$ for $L \in \mathcal{L}_p$ with $|L| = p^n$.
Furthermore, by Remark 3.3(b) the terms of the derived series of L are just a subset of $\{L_1', \ldots, L_n'\}$

which is why $\mathrm{dl}(L) \leq s(L)$ for any $L \in \mathcal{L}_p$. Hence $f(k) \geq k$ for all $k \in \mathbb{N}$.
Moreover, it can easily be checked that the standard example yields $f(k) \leq 2^{k-1}$ for all $k \in \mathbb{N}$. So altogether we have

$$(**) \quad k \leq f(k) \leq 2^{k-1}.$$

So is the true upper bound exponential or polynomial in k?
The conjecture is that it is exponential, and this conjecture is equivalent to Conjecture $(*)$ in the introduction for the groups in $\mathcal{C}_p$. As to the small values of k, by $(**)$ we obtain $f(1) = 1,\ f(2) = 2$ and $3 \leq f(3) \leq 4$ and $4 \leq f(4) \leq 8$. We will show below that $f(3) = 3$ and $f(4) \geq 5$. More generally we will see that $f(k) \geq 2k - 3$ for all $k \in \mathbb{N}$.

The following settings will be used throughout the remainder of this paper.

Definition 3.6. (a) Let $L \in \mathcal{L}_p$ with $|L| = p^n$ and let e_1 be as in Definition 3.2, and let $e_2 \in L_2 - L_3$. It is then easy to see that if we successively define $e_{i+1} = [e_1, e_i]$ $(i \in \mathbb{N})$, then $e_i \in L_i - L_{i+1}$ for all $i = 1, \ldots, n$ and $e_k = 0$ for $k \geq n+1$; hence $L = \langle e_1, \ldots, e_n \rangle$.

(b) Now we introduce a "differential operator" d on L as follows:
For $v \in L$ define $d(v) = [e_1, v]$. Repeated operations of d are written as powers of d, and we put $d^0(v) = v$ and $d^j(v) = 0$ whenever $j < 0$.

(c) We also introduce the structural coefficients $C^k_{i,j} \in \mathrm{GF}(p)$ with respect to our specially chosen basis $\{e_1, \ldots, e_n\}$, that is, we define them such that

$$[e_i, e_j] = \sum_{k=1}^{n} C^k_{i,j} e_k \text{ for } i, j \in \{1, \ldots, n\}.$$

Next we will see that once $[e_2, e_3], [e_3, e_4], \ldots, [e_l, e_{l+1}]$ are given (i.e., the $C^k_{i,i+1}$ are given for $i = 2, \ldots, l$ and $k = 1, \ldots, n$), where $l = \frac{n}{2} - 1$ if n is even and $l = \frac{n-1}{2}$ if n is odd, then this already determines L completely (note that $L'_{l+1} = 0$), that is, all the other structural coefficients are determined. This can be useful for constructing examples.

Lemma 3.7. *Let $L \in L_p$. Then for all $i > 1$ and $j \geq 0$ we have*

$$[e_i, e_{i+j+1}] = d([e_i, e_{i+j}]) - [e_{i+1}, e_{i+j}]$$

and

$$[e_i, e_{i+j+1}] = \sum_{k=0}^{n-i} (-1)^k \binom{j-k}{k} d^{j-2k}([e_{i+k}, e_{i+k+1}]).$$

Proof. The first equation follows immediately from the Jacobi identity for e_1, e_i and e_j. The second equation follows by a routine induction on j. $\diamond$

The reader may compare the second formula in Lemma 3.7 to [1, Theorem 4.5].

Example 3.8. (a) We now can show that $f(3) = 3$. For this, we only have to construct an example L with $\mathrm{dl}(L) = s(L) = 3$.

The easiest such example seems to be the following:
Let p be a prime with $p > 8$, and put $L = \langle e_1, \ldots, e_8 \rangle$ with the e_i as in Definition 3.6 and $[e_2, e_3] = e_7$, $[e_3, e_4] = e_7$. (By Lemma 3.7 this fully defines L.)
The only other nonzero $[e_i, e_j]$ not involving e_1 then are

$$[e_3, e_5] = [e_2, e_4] = e_8, \ [e_2, e_5] = -e_7 \text{ and } [e_2, e_6] = -2e_8.$$

It is easy to check that L has the claimed properties.
Note that this L really does not depend on p and also works over $\mathbb{Z}$.
The next example will be different in this respect.

(b) Let F be a field and L an F–vector space with F–basis $\{e_1, \ldots, e_{13}\}$. We try to turn this into a Lie algebra by defining

$[e_1, e_i] = e_{i+1}$ for $i = 2, \ldots, 12$,
$[e_2, e_3] = 37e_7$, $[e_3, e_4] = -74e_9$, $[e_4, e_5] = -111e_{11}$, $[e_5, e_6] = -37e_{11}$ and $[e_6, e_7] = e_{13}$.

Then we have the following:

- α) If $F = \mathbb{Q}$, then this does not lead to a Lie algebra; some Jacobi identity will be violated.
- β) If $F = \mathrm{GF}(37)$, then $L \in \mathcal{L}_{37}$, and $\mathrm{dl}(L) = s(L) = 3$, so here we have another example showing that $f(3) = 3$.
- γ) If $F = \mathrm{GF}(223)$, then $L \in \mathcal{L}_{223}$, and $\mathrm{dl}(L) = 3$ and $s(L) = 6$.

(c) The following is an example $L \in \mathcal{L}_{103}$ with $\mathrm{dl}(L) = 3$ and $s(L) = 7$:
Let the e_i be as in Definition 3.6 and put
$[e_2, e_3] = e_7$, $[e_3, e_4] = 2e_9$, $[e_4, e_5] = 28e_{11}$, $[e_5, e_6] = 107e_{13}$, $[e_6, e_7] = -7e_{15}$, $[e_7, e_8] = 2e_{15}$.
(This system is not consistent over $\mathbb{Z}$, though.)

Example 3.8(b) and (c) were found and verified with the help of a program written in MATHEMATICA that takes as input the structural coefficients for $[e_i, e_{i+1}]$ for $i \geq 2$ and checks whether this leads to a Lie algebra (i.e., it checks whether all the Jacobi identities are valid) for some prime and if yes, it determines $\mathrm{dl}(L)$ and $s(L)$. Despite this useful tool as of yet we have been unable to produce a new interesting example (other than the standard example) of derived length 4.

We now start working towards our main results. The following result is the crucial step.

Theorem 3.9. *Let $L \in L_p$. If $L_2' = L_3' = L_m$ for some $m \in \mathbb{N}$, then $L_4' \geq L_{2m-5}$.*

Proof. If $L_m = 0$, then there is nothing to do. So let $L_m \neq 0$, then $m \geq 7$. Clearly we may assume that $n = 2m - 5$. Working towards a contradiction, we assume that $L_4' = 0$.
First, observe that as $[e_2, e_3] \in L_m \subseteq L_7$ and $L_4' = 0$, we have $[e_2, e_3, e_4] = 0$, where we use the

usual convention that $[u, v, w] := [[u, v], w]$ for $u, v, w \in L$.
Second, observe that $[e_2, e_4] = d([e_2, e_3])$ by Lemma 3.7 and thus

$$\begin{aligned}
[e_4, e_2, e_3] &= [e_3, [e_2, e_4]] = [e_3, d([e_2, e_3])] \\
&= \sum_{k=m}^{n} C_{2,3}^{k} [e_3, e_{k+1}] \\
&= \sum_{k=m}^{n} C_{2,3}^{k} \sum_{j=0}^{n-3} (-1)^j \binom{k-3-j}{j} d^{k-3-2j}([e_{3+j}, e_{4+j}]).
\end{aligned}$$

As $L'_4 = 0$ it further follows that $[e_{3+j}, e_{3+j+1}] = 0$ for $j \geq 1$, so $[e_4, e_2, e_3] = \sum_{k=m}^{n} C_{2,3}^{k} d^{k-3}[e_3, e_4]$, and as $[e_3, e_4] \in L_m$, for $k \geq m$ we have $d^{k-3}[e_3, e_4] \in L_{m+k-3} \subseteq L_{2m-3} = 0$, as $n = 2m - 5$.
Thus $[e_4, e_2, e_3] = 0$.
Thirdly, consider

$$\begin{aligned}
[e_3, e_4, e_2] &= \sum_{k=m}^{n} C_{3,4}^{k} [e_k, e_2] \\
&= -\sum_{k=m}^{n} C_{3,4}^{k} [e_2, e_k] \\
&= -\sum_{k=m}^{n} C_{3,4}^{k} \sum_{j=0}^{n-2} (-1)^j \binom{k-3-j}{j} d^{k-3-2j}([e_{2+j}, e_{3+j}]) \\
&= -\sum_{k=m}^{n} C_{3,4}^{k} (d^{k-3}([e_2, e_3]) - (k-4) d^{k-5}([e_3, e_4])).
\end{aligned}$$

Now as for $k \geq m$ clearly $d^{k-3}([e_2, e_3]) \in L_{m+k-3} \subseteq L_{2m-3} = 0$ and for $k \geq m+1$ also $d^{k-5}([e_3, e_4]) \in L_{m+k-5} \subseteq L_{2m-4} = 0$, we obtain

$$\begin{aligned}
[e_3, e_4, e_2] &= C_{3,4}^{m} (m-4) d^{m-5}([e_3, e_4]) \\
&= (m-4) C_{3,4}^{m} \sum_{l=m}^{n} C_{3,4}^{l} e_{l+m-5} \\
&= (m-4) (C_{3,4}^{m})^2 e_{2m-5}.
\end{aligned}$$

Now as $m - 4 \neq 0$ and by the Jacobi identity we have

$$0 = [e_2, e_3, e_4] + [e_4, e_2, e_3] + [e_3, e_4, e_2] = 0 + 0 + (m-4)(C_{3,4}^{m})^2 e_{2m-5},$$

it follows that $C_{3,4}^{m} = 0$.
But on the other hand, as $L'_4 = 0$, we have

$$L_m = L'_3 = \langle [e_i, e_j] \mid i, j \geq 3 \rangle = \langle [e_3, e_j] \mid j \geq 4 \rangle = \langle d^j([e_3, e_4]) \mid j \geq 4 \rangle,$$

and this obviously forces $C_{3,4}^{m} \neq 0$. This contradiction proves the theorem. $\diamond$

Corollary 3.10. *Let $L \in \mathcal{L}_p$. Suppose that for some $i \geq 0$ we have $L'_{2+i} = L'_{3+i} = L_{m+i}$. Then $L'_{4+i} \supseteq L_{2m-5+i}$.*

Proof. Clearly we may assume that $|L| = p^{2m-5+i}$. Now consider $M = \langle e_1, e_2', e_3', \ldots, e_{2m-5}' \rangle \subseteq L$, where $e_j' = e_{j+i}$ for $j = 2, \ldots, 2m-5$, and put $M_1 = M$ and $M_j = \langle e_j', \ldots, e_{2m-5}' \rangle = L_{j+i}$. Then also $M \in \mathcal{L}_p$ and $M_2' = M_3' = M_m$. Thus by Theorem 3.9 we have $L_{4+i}' = M_4' \supseteq M_{2m-5} = L_{2m-5+i}$, as wanted. $\diamond$

Note that Corollary 3.10 is more interesting than it seems at first sight. Naively, one might think that generalizing Theorem 3.9 appropriately to the situation in Corollary 3.10 would yield only the conclusion $L_{4+i}' \supseteq L_{2(m+i)-5} = L_{2m+2i-5}$. Instead, we get the above stronger conclusion, so the result gets more powerful as i increases.

Corollary 3.11. *Let $L \in \mathcal{L}_p$ with $\mathrm{dl}(L) = k$. Then $s(L) \geq 2k-3$. Thus also $f(k) \geq 2k-3$. In particular, $f(4) \geq 5$.*

Proof. We may assume that $k \geq 3$. Define $m_i \in \mathbb{N}$ $(i = 0, \ldots, k)$ such that $L^{(i)} = L_{m_i}$ for $i = 1, \ldots, k$, so $L_{m_i}' = L_{m_{i+1}}$ for $i = 1, \ldots, k-1$ and $L_{m_k} = 0$. So we have $m_0 = 1$, $m_1 = 3$ and $m_{i+1} \geq 2m_i + 1$ and hence $m_1 > m_2 > \ldots > m_k$ with $m_{i+1} - m_i \geq 2$ for $i = 0, \ldots, k-1$. Also $L_{m_0}' > L_{m_1}' > \ldots > L_{m_k}' = 0$. Next we establish the following:
Claim: For each $i \in \{1, \ldots, k-2\}$ we have one of the following:
(1) $L_{m_{i-1}}' > L_{m_i-1}' > L_{m_i}'$ or
(2) $L_{m_i}' > L_r' > L_{m_{i+1}-1}'$ for a suitable r or
(3) $L_{m_i}' > L_r' = 0$ for some r.
To establish the claim, assume that (1) does not hold. Now as $L_{m_{i-1}}' = L_{m_i}$ and $L_{m_i-1}' \leq L_{2m_i-1}$ and $2m_i - 1 > m_i$, we have $L_{m_{i-1}}' > L_{m_i-1}'$, and so (1) not holding means that $L_{m_i-1}' = L_{m_i}'$. So let $t \in \mathbb{Z}$ be maximal such that $L_{m_i-1}' = L_{m_i}' = L_t' = L_{m_{i+1}}$ (possibly $t = m_i$) and note that then $m_{i+1} \geq 2t+1$. Now as $L_{t-1}' = L_t'$, we may apply Corollary 3.10 which yields that $L_{t+1}' \supseteq L_{2(m_{i+1}-t+3)-5+t-3} = L_{2(m_{i+1}-1)-t} \supseteq L_{2(m_{i+1}-1)}$, and as $L_{m_{i+1}-1}' \subseteq L_{2(m_{i+1}-1)+1}$, we conclude that if we put $r = t+1$, then $L_{m_i}' = L_t' > L_r' > L_{m_{i+1}-1}'$ if $L_r' > 0$ which is (2), or we get (3) if $L_r' = 0$. This proves the claim.
Now by the claim we have $|\{L_2', L_3', \ldots, L_{m_k-1}'\}| \geq 2(k-2)$, as careful counting shows. Thus including $L_1' = L_3$, we find $s(L) \geq 2k-3$, as wanted. This proves the corollary. $\diamond$

The fact that $f(4) \geq 5$ can also be seen directly quite easily, using Theorem 3.9. It shows that also in $\mathcal{C}_p$ there is no p–group G with $\mathrm{dl}(G) = |\mathrm{cd}(G)| = 4$. It is not known whether such a group can exist.

We finally translate our results on Lie algebras back to groups. This gives us our main results.

Corollary 3.12. *(a) If $G \in \mathcal{C}_p$ and $\mathrm{dl}(G) = 4$, then $|\mathrm{cd}(G)| \geq 5$.*

(b) If $G \in \mathcal{C}_p$, then $\mathrm{dl}(G) \leq \frac{1}{2}|\mathrm{cd}(G)| + \frac{3}{2}$.

(c) If $G \in \mathcal{C}$, then $\mathrm{dl}(G) \leq \frac{1}{2}|\mathrm{cd}(G)| + \frac{11}{2}$.

Proof. (a) and (b) are immediate consequences of Corollary 3.11.
(c) Let $G \in \mathcal{C}$ of order p^n. If G has class less than p, then G/G_{n-1} is non–exceptional of exponent p and so $G/G_{n-1} \in \mathcal{C}_p$, so that by (b)

$$\mathrm{dl}(G) \leq \mathrm{dl}(G/G_{n-1}) + 1 \leq \frac{1}{2}|\mathrm{cd}(G/G_{n-1})| + \frac{3}{2} + 1 \leq \frac{1}{2}|\mathrm{cd}(G)| + \frac{5}{2}.$$

So we may assume that G has class $\geq p$. Recall from the beginning of this section that then $\mathrm{dl}(G) \leq \mathrm{dl}(G/G_p) + 4$ and $G/G_p \in \mathcal{C}_p$. Hence by (b) we have

$$\mathrm{dl}(G) \leq \frac{1}{2}|\mathrm{cd}(G/G_p)| + \frac{3}{2} + 4 \leq \frac{1}{2}|\mathrm{cd}(G)| + \frac{11}{2},$$

and so we are done. ◇

Note that in the proof of Corollary 3.12(c) by a more careful argument, applying Corollary 3.12(b) also to groups $\langle e_1, e_p, \ldots, e_{2p-2}\rangle \in \mathcal{C}_p$ etc. (where $G = \langle e_1, \ldots, e_n\rangle$, and $n \geq 2p-2$ in this case), one could slightly improve the constant $\frac{11}{2}$ in that result, but we did not deem this worth the effort.

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