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RÉSUMÉ

Une des propriétés désirables d'une procédure de vote est qu'elle doit être exempte de retrait stratégique d'un candidat à l'élection. Duttu, Jackson et Le Breton (*Econometrica*, 2001) ont établi des théorèmes démontrant que cette propriété est incompatible avec certaines propriétés désirables de procédures de vote. Cet article montre que la généralisation non binaire du théorème d'Arrow par Grether et Plott peut être utilisée pour faire une démonstration assez simple de deux de ces théorèmes d'impossibilité.

Mots clés : axiome de choix social, stabilité du candidat, économie politique, élection

ABSTRACT

A desirable property of a voting procedure is that it be immune to the strategic withdrawal of a candidate for election. Dutta, Jackson, and Le Breton (*Econometrica*, 2001) have established a number of theorems that demonstrate that this condition is incompatible with some other desirable properties of voting procedures. This article shows that Grether and Plott's nonbinary generalization of Arrow's Theorem can be used to provide simple proofs of two of these impossibility theorems.

Key words : axiomatic social choice, candidate stability, political economy, voting

1. Introduction

A desirable property of a voting procedure is that it be immune to the strategic withdrawal of a candidate for election. Dutta, Jackson, and Le Breton [3] have recently established a number of theorems which demonstrate that this condition is incompatible with some other desirable properties of voting procedures. In this article, we show that Grether and Plott's [5] nonbinary generalization of Arrow's [1] Theorem can be used to provide relatively simple proofs of two of Dutta, Jackson, and Le Breton's impossibility theorems.

For any profile of linear voter preferences, a voting rule determines a winning candidate from any subset of candidates drawn from a fixed list of potential candidates based on the preferences of the voters over the candidates running for office. A voting procedure is *candidate stable* if no candidate would prefer to withdraw from an election when all of the other potential candidates enter. In other words, it is a Nash equilibrium for all potential candidates to stand for election. When there is no overlap between the candidates and voters, Dutta, Jackson, and Le Breton restrict attention to voting rules that respect *unanimity* in the sense that a candidate who is ranked first among the candidates on the ballot by all of the voters is elected. Their candidate stability theorem for the no overlap case shows that unanimity and candidate stability jointly imply that a single voter determines the outcome in any election in which all or all but one of the potential candidates is on the ballot.[1]

Dutta, Jackson, and Le Breton also consider a stronger version of their candidate stability axiom. A voting procedure is *strongly candidate stable* if the election outcome is unaffected when a candidate withdraws who would lose if every candidate enters the election. When there is no overlap between candidates and voters, the two candidate stability conditions are equivalent, at least when candidates rank themselves first, which is what Dutta, Jackson, and Le Breton assume. In their strong candidate stability theorem, candidates can also be voters. Because candidates are assumed to rank themselves first, the unanimity condition is strengthened to require that a candidate is chosen if he or she is the top-ranked candidate on the ballot by all voters once the self-preferences of non-elected voting candidates is ignored. Dutta, Jack-

[1]When candidates are permitted to vote, Dutta, Jackson, and Le Breton have shown that candidate stability is incompatible with a weak unanimity condition that is consistent with a candidate ranking him- or herself first and a monotonicity condition that is satisfied by many common voting procedures. We do not consider this result here.

son, and Le Breton's strong candidate stability theorem shows that when this unanimity condition is combined with strong candidate stability, the voting rule must be dictatorial in the sense described above.

Dutta, Jackson, and Le Breton do not prove their candidate stability theorem for the no overlap case directly; they instead show that it is a corollary to their strong candidate stability theorem. Their proof of the latter result is quite long and complicated. Although, as Dutta, Jackson, and Le Breton [3, p. 1021] note, 'the logic of Arrow's theorem cannot be directly applied' to prove their results, they nevertheless are able to use Wilson's [7] generalization of Arrow's Theorem at a key step in their argument. The proofs provided here demonstrate that the incompatibility of Dutta, Jackson, and Le Breton's axioms in the two theorems described above follows fairly directly from a restricted-domain version of Arrow's impossibility theorem.

In Section 2, we present Grether and Plott's Theorem. Dutta, Jackson, and Le Breton's theorems are described in Section 3. Our proofs of the candidate stability and strong candidate stability theorems follow in Sections 4 and 5, respectively. In Section 6, we relate our analysis to the recent articles on multivalued voting procedures by Eraslan and McLennan [4] and Rodríguez-Álvarez [6].

2. The Grether-Plott theorem

Let $\mathcal{N} = \{1, \ldots, n\}$ with $n \geq 2$ be a finite set of individuals and $X = \{x_1, \ldots, x_m\}$ with $m \geq 3$ be a finite set of alternatives. An *agenda* is a nonempty subset of X. An *ordering* of X is a reflexive, complete, and transitive binary relation on X. The corresponding strict preference relation P and indifference relation I are defined as follows: For all $x, y \in X$, (a) $xPy \leftrightarrow xRy$ and $\neg(yRx)$ and (b) $xIy \leftrightarrow xRy$ and yRx. A *linear ordering* of X is an antisymmetric ordering; i.e., an ordering for which no two distinct alternatives are indifferent. Let $\mathcal{R}$ denote the *set of all orderings* and $\mathcal{L}$ denote the *set of all linear orderings* of X.

Each individual $i \in \mathcal{N}$ has a preference ordering $R_i \in \mathcal{R}$ of X. A *preference profile* $\mathbf{R} = (R_1, \ldots, R_n)$ is an n-tuple of individual preference orderings. Two preferences $R^1, R^2 \in \mathcal{R}$ coincide on $A \subseteq X$ if for all $x, y \in A$, $xR^1y \leftrightarrow xR^2y$. Two profiles $\mathbf{R}^1, \mathbf{R}^2 \in \mathcal{R}^n$ coincide on $A \subseteq X$ if R_i^1 and R_i^2 coincide on A for all $i \in \mathcal{N}$.

The set of admissible profiles and/or the set of admissible agendas may be restricted *a priori*. The *preference domain* is $\mathcal{D}$, a nonempty subset of $\mathcal{R}^n$. The *agenda domain* is $\mathcal{A}$, a collection of nonempty subsets of X.

A *social choice correspondence* $C\colon \mathcal{A}\times\mathcal{D}\to X$ is a mapping that assigns a nonempty subset of the agenda to each admissible agenda and admissible profile. The set $C(A,\mathbf{R})$ is the *choice set*. If for all $A\in\mathcal{A}$ and all $\mathbf{R}\in\mathcal{D}$, $C(A,\mathbf{R})$ contains a single alternative, then C is a *social choice function*. In this case, we write x instead of $\{x\}$ when $\{x\}$ is the choice set.

In its choice-theoretic formulation, Arrow's [1] Theorem demonstrates that the four Arrow social choice correspondence axioms are incompatible when the preference domain is unrestricted (i.e., $\mathcal{D}=\mathcal{R}^n$) and every nonempty subset of X is an admissible agenda. The Arrow axioms are Arrow's Choice Axiom, Independence of Infeasible Alternatives, Weak Pareto, and Nondictatorship.

Arrow's Choice Axiom places restrictions on how choices are made out of different agendas for a fixed preference profile.

Arrow's Choice Axiom. For all $A^1,A^2\in\mathcal{A}$ and all $\mathbf{R}\in\mathcal{D}$, if $A^1\subseteq A^2$ and $C(A^2,\mathbf{R})\cap A^1\neq\emptyset$, then $C(A^1,\mathbf{R})=C(A^2,\mathbf{R})\cap A^1$.

Informally, for a given profile $\mathbf{R}$, if the agenda A^1 is a subset of the agenda A^2 and the choice sets for these two agendas have at least one alternative in common, then the choice set for the smaller agenda consists of that part of the choice set for the larger agenda that is contained in the smaller agenda.

Independence of Infeasible Alternatives requires the choice set to be independent of preferences over alternatives not in the agenda.

Independence of Infeasible Alternatives. For all $A\in\mathcal{A}$ and all $\mathbf{R}^1,\mathbf{R}^2\in\mathcal{D}$, if $\mathbf{R}^1$ and $\mathbf{R}^2$ coincide on A, then $C(A,\mathbf{R}^1)=C(A,\mathbf{R}^2)$.

For all $A\in\mathcal{A}$ and all $\mathbf{R}\in\mathcal{D}$, the *weak Pareto set* is

$$\mathcal{P}(A,\mathbf{R})=\{x\in A\mid \nexists y\in A \text{ such that } yP_ix \text{ for all } i\in\mathcal{N}\}.$$

The Weak Pareto axiom requires the choice set to be a subset of the weak Pareto set.

Weak Pareto. For all $A\in\mathcal{A}$ and all $\mathbf{R}\in\mathcal{D}$, $C(A,\mathbf{R})\subseteq\mathcal{P}(A,\mathbf{R})$.

For a nonempty set $A\subseteq X$ and an ordering $R\in\mathcal{R}$, the *set of best alternatives* in A according to R is

$$B(A,R)=\{x\in A\mid xRy \text{ for all } y\in A\}.$$

An individual $d \in \mathcal{N}$ is a *dictator* for the social choice correspondence $C\colon \mathcal{A} \times \mathcal{D} \to X$ if $C(A, \mathbf{R}) \subseteq B(A, R_d)$ for all $A \in \mathcal{A}$ and all $\mathbf{R} \in \mathcal{D}$. That is, d is a dictator if the choice set is always a subset of d's best alternatives in the agenda. Nondictatorship is the requirement that there be no dictator.

Nondictatorship. There is no dictator.

Grether and Plott [5] investigated the consistency of the Arrow axioms when the only admissible agendas are those subsets of X containing at least k alternatives, where $k < |X|$.

k-Set Feasibility. There exists a positive integer $k < |X|$ such that $A \in \mathcal{A}$ if and only if $|A| \geq k$.

Grether and Plott assumed that the preference domain is unrestricted. Their theorem is also valid for the domain of linear preference profiles, and it is this version of their theorem that is relevant here.

Unrestricted Linear Preference Domain. $\mathcal{D} = \mathcal{L}^n$.

The Grether-Plott Theorem shows that the Arrow axioms are inconsistent with an unrestricted linear preference domain when the agenda domain satisfies k-Set Feasibility.

Theorem 1. (Grether-Plott [5]) *There is no social choice correspondence with an unrestricted linear preference domain that satisfies k-Set Feasibility, Arrow's Choice Axiom, Independence of Infeasible Alternatives, Weak Pareto, and Nondictatorship.*

3. Strategic candidacy

The framework introduced in the previous section needs to be modified somewhat in order to describe the Dutta-Jackson-Le Breton [3] model of strategic candidacy. Let $\mathcal{C} = \{1, \ldots, m\}$ be the set of potential *candidates*, $\mathcal{V}$ be the set of *voters*, and $\mathcal{N} = \mathcal{C} \cup \mathcal{V}$, where $|\mathcal{N}| = n$. The candidate set $\mathcal{C}$ corresponds to the set of alternatives X in the preceding section. We assume that $m \geq 3$ and $|\mathcal{V}| \geq 2$. We consider both the case in which some candidates are voters and the case in which they are not. The set $\mathcal{C}$ is assumed to be ordered in such a way that the nonvoting candidates (if any) appear first. Let $\mathcal{C}_1$ be the set of *nonvoting candidates* and $\mathcal{C}_2$ be the set of *voting candidates*, where $|\mathcal{C}_1| = m_1$ and $|\mathcal{C}_2| = m_2$. Note that $\mathcal{V} = \mathcal{C}_2 \cup (\mathcal{N} \backslash \mathcal{C})$ and $m_1 + m_2 = m$.

Both candidates and voters have preferences over candidates. For $i \in \mathcal{C}$, let

$$\mathcal{L}_i = \{R \in \mathcal{L} \mid B(\mathcal{C}, R) = \{i\}\}.$$

$\mathcal{L}_i$ is the set of linear preferences on $\mathcal{C}$ that rank candidate i first. Candidate i is assumed to have a preference in $\mathcal{L}_i$. Let $\mathcal{L}_{\mathcal{C}_1} = \prod_{i \in \mathcal{C}_1} \mathcal{L}_i$, $\mathcal{L}_{\mathcal{C}_2} = \prod_{i \in \mathcal{C}_2} \mathcal{L}_i$, and $\mathcal{L}_{\mathcal{C}} = \mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}_{\mathcal{C}_2}$. Voters who are not candidates can have any preference in $\mathcal{L}$. A *preference profile* is now a vector $\mathbf{R} = (\mathbf{R}_{\mathcal{C}_1}, \mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N} \backslash \mathcal{C}})$, where $\mathbf{R}_{\mathcal{C}_1}$ is the subprofile of nonvoting candidates' preferences, $\mathbf{R}_{\mathcal{C}_2}$ is the subprofile of voting candidates' preferences, and $\mathbf{R}_{\mathcal{N} \backslash \mathcal{C}}$ is the subprofile of noncandidates' preferences. The set of *admissible preference profiles* is $\mathcal{L}^* = \mathcal{L}_{\mathcal{C}} \times \mathcal{L}^{n-m}$.

Any subset of the set of potential candidates may stand for election, but only one candidate is elected. Let $\mathcal{X}$ denote the set of all nonempty subsets of $\mathcal{C}$. A *voting function* is a social choice function $V \colon \mathcal{X} \times \mathcal{L}^* \to \mathcal{C}$.

Dutta, Jackson, and Le Breton require a voting function to satisfy Independence of Infeasible Alternatives, modified in the obvious way to apply to preference profiles in $\mathcal{L}^*$. In this context, Independence of Infeasible Alternatives requires the election outcome only to depend on the preferences over candidates who enter the election.

Independence of Nonvoters' Preferences requires the outcome of an election to only depend on the voters' preferences.

Independence of Nonvoters' Preferences. For all $A \in \mathcal{X}$, all $\mathbf{R}^1_{\mathcal{C}_1}, \mathbf{R}^2_{\mathcal{C}_1} \in \mathcal{L}_{\mathcal{C}_1}$, and all $(\mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N} \backslash \mathcal{C}}) \in \mathcal{L}_{\mathcal{C}_2} \times \mathcal{L}^{n-m}$, $V(A, (\mathbf{R}^1_{\mathcal{C}_1}, \mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N} \backslash \mathcal{C}})) = V(A, (\mathbf{R}^2_{\mathcal{C}_1}, \mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N} \backslash \mathcal{C}}))$.

A voting function satisfies Unanimity if candidate j is chosen when all of the voters agree that j is the best candidate running for office.

Unanimity. For all $A \in \mathcal{X}$ and all $\mathbf{R} \in \mathcal{L}^*$, if $B(A, R_i) = \{j\}$ for all $i \in \mathcal{V}$, then $V(A, \mathbf{R}) = j$.

Because of the restriction on candidates' preferences, Unanimity is vacuous if there is more than one voting candidate. When this is the case, it is more appropriate to use Strong Unanimity instead.

Strong Unanimity. For all $A \in \mathcal{X}$ and all $\mathbf{R} \in \mathcal{L}^*$, if $B(A, R_i) = \{j\}$ for all $i \in \mathcal{N} \backslash \mathcal{C}$ and $B(A \backslash \{i\}, R_i) = \{j\}$ for all $i \in \mathcal{C}_2 \backslash \{j\}$, then $V(A, \mathbf{R}) = j$.

Strong Unanimity requires candidate j to be chosen if j is unanimously preferred by all voters to all of the other candidates in the election once each voting candidate's self preference (other than that of candidate j if j is a voting candidate) is ignored. If there are no voting candidates, Unanimity and Strong Unanimity are equivalent.

A voting function is Candidate Stable if each candidate prefers the outcome when all candidates are on the ballot to the outcome that would obtain if he or she withdrew from the election.

Candidate Stability. For all $i \in \mathcal{C}$ and all $\mathbf{R} \in \mathcal{L}^*$, $V(\mathcal{C}, \mathbf{R}) R_i V(\mathcal{C} \backslash \{i\}, \mathbf{R})$.

Strong Candidate Stability requires the election outcome to be unaffected if a candidate withdraws who would lose if every candidate stood for office.

Strong Candidate Stability. For all $i \in \mathcal{C}$ and all $\mathbf{R} \in \mathcal{L}^*$, if $V(\mathcal{C}, \mathbf{R}) \neq i$, then $V(\mathcal{C}, \mathbf{R}) = V(\mathcal{C} \backslash \{i\}, \mathbf{R})$.

This axiom is a strengthening of Candidate Stability if the set of voters and candidates overlap. However, when no voter is a candidate, these conditions are equivalent for a voting function that satisfies Independence of Nonvoters' Preferences.

Lemma 1. (Dutta-Jackson-Le Breton [3]) *If a voting function satisfies Independence of Nonvoters' Preferences and $\mathcal{C}_2 = \emptyset$, then it satisfies Candidate Stability if and only if it satisfies Strong Candidate Stability.*[2]

Dutta, Jackson, and Le Breton use a strengthened version of the nondictatorship axiom introduced in the preceding section. Let

$$\mathcal{A}^{m-1} = \{A \in \mathcal{X} \mid |A| \geq m - 1\}.$$

A voter $d \in \mathcal{V}$ is a *dictator for large elections* for the voting function $V : \mathcal{X} \times \mathcal{L}^* \to \mathcal{C}$ if $V(A, \mathbf{R}) = B(A, R_d)$ for all $A \in \mathcal{A}^{m-1}$ and all $\mathbf{R} \in \mathcal{L}^*$. Informally, an individual is a dictator for large elections if this individual's most-preferred candidate is elected whenever all or all but one of the candidates run for office.

[2]We have only stated the part of Lemma 2 in Dutta, Jackson, and Le Breton [3] that deals with Candidate Stability. Their lemma also assumes that the voting function satisfies Independence of Infeasible Alternatives and Unanimity, but these assumptions are not needed to show the equivalence of Candidate Stability and Strong Candidate Stability.

Strong Nondictatorship. There is no dictator for large elections.

Theorem 2 is the Dutta-Jackson-Le Breton candidate stability theorem for the case in which there is no overlap between candidates and voters.

Theorem 2. (Dutta-Jackson-Le Breton [3]) *If $\mathcal{C}_2 = \emptyset$, there is no voting function that satisfies Independence of Nonvoters' Preferences, Independence of Infeasible Alternatives, Unanimity, Candidate Stability, and Strong Nondictatorship.*

When candidates can also be voters, Dutta, Jackson, and Le Breton have established the following theorem about voting functions that satisfy Strong Candidate Stability.

Theorem 3. (Dutta-Jackson-Le Breton [3]) *If a voting function satisfies Independence of Nonvoters' Preferences, Independence of Infeasible Alternatives, Strong Unanimity, and Strong Candidate Stability, then there is a dictator for large elections and the dictator is in $\mathcal{V}\backslash\mathcal{C}$.*

In view of Lemma 1 and the equivalence between Unanimity and Strong Unanimity when $\mathcal{C}_2 = \emptyset$, Theorem 2 is a simple corollary to Theorem 3. Note that when $\mathcal{V} = \mathcal{C}$, there does not exist a voting function satisfying the properties in Theorem 3.

4. A proof of the candidate stability theorem

In this section, we use the Grether-Plott Theorem to help prove Theorem 2. Throughout this section, we assume that $\mathcal{C}_2 = \emptyset$; i.e., there are no candidates who are permitted to vote.

We first establish a preliminary result concerning social choice correspondences whose agenda domain is $\mathcal{A}^{m-1}$ and whose preference domain is $\mathcal{L}^n$, where $\mathcal{A}^{m-1}$ is now the set of subsets of X of cardinality at least $m-1$. Let $C^*\colon \mathcal{A}^{m-1} \times \mathcal{L}^n \to X$ be such a social choice correspondence.

An analogue to Unanimity for C^* is Unanimity*, which requires x to be chosen if it is everyone's best choice in the agenda.

Unanimity*. For all $A \in \mathcal{A}^{m-1}$ and all $\mathbf{R} \in \mathcal{L}^n$, if $B(A, R_i) = \{x\}$ for all $i \in \mathcal{N}$, then $C^*(A, \mathbf{R}) = \{x\}$.

The following lemma shows that Weak Pareto is implied by Arrow's Choice Axiom, Independence of Infeasible Alternatives, and Unanimity* for a social choice correspondence whose domain is $\mathcal{A}^{m-1} \times \mathcal{L}^n$.

Lemma 2. *If a social choice correspondence $C^*\colon \mathcal{A}^{m-1} \times \mathcal{L}^n \to X$ satisfies Arrow's Choice Axiom, Independence of Infeasible Alternatives, and Unanimity*, then it also satisfies Weak Pareto.*

Proof. Let $C^*\colon \mathcal{A}^{m-1} \times \mathcal{L}^n \to X$ be a social choice correspondence that satisfies Arrow's Choice Axiom, Independence of Infeasible Alternatives, and Unanimity*. We have two cases to consider.

Case 1. We first show that for any $\mathbf{R} \in \mathcal{L}^n$, $C^*(X, \mathbf{R})$ is contained in the weak Pareto set. On the contrary, suppose that there exists an $\mathbf{R} \in \mathcal{L}^n$ and $x, y \in X$ such that xP_iy for all $i \in \mathcal{N}$, but $y \in C^*(X, \mathbf{R})$. Without loss of generality, we suppose that $x = x_1$ and $y = x_2$.

By Arrow's Choice Axiom, $x_2 \in C^*(X\backslash\{x_3\}, \mathbf{R})$. Consider a profile of preferences $\mathbf{R}^1 \in \mathcal{L}^n$ that coincides with $\mathbf{R}$ on $X\backslash\{x_3\}$ and has x_3 ranked last by all $i \in \mathcal{N}$. By Independence of Infeasible Alternatives, $x_2 \in C^*(X\backslash\{x_3\}, \mathbf{R}^1)$. It then follows from Arrow's Choice Axiom that $C^*(X, \mathbf{R}^1) \cap \{x_2, x_3\} \neq \emptyset$.

Arrow's Choice Axiom now implies that $C^*(X\backslash\{x_4\}, \mathbf{R}^1) \cap \{x_2, x_3\} \neq \emptyset$. Consider a profile $\mathbf{R}^2 \in \mathcal{L}^n$ that coincides with $\mathbf{R}^1$ on $X\backslash\{x_4\}$ and has x_4 ranked last by all $i \in \mathcal{N}$. By Independence of Infeasible Alternatives, $C^*(X\backslash\{x_4\}, \mathbf{R}^2) = C^*(X\backslash\{x_4\}, \mathbf{R}^1)$ and, hence, $C^*(X\backslash\{x_4\}, \mathbf{R}^2) \cap \{x_2, x_3\} \neq \emptyset$. By Arrow's Choice Axiom, we then have $C^*(X, \mathbf{R}^2) \cap \{x_2, x_3, x_4\} \neq \emptyset$.

Repeated use of this argument leads to the conclusion that $C^*(X, \mathbf{R}^{m-2}) \cap \{x_2, \ldots, x_m\} \neq \emptyset$ for some profile $\mathbf{R}^{m-2} \in \mathcal{L}^n$ that coincides with $\mathbf{R}$ on $\{x_1, x_2\}$ and has $x_1P_i^{m-2}x_3$, $x_2P_i^{m-2}x_3$, and $x_3P_i^{m-2}x_4P_i^{m-2} \cdots P_i^{m-2}x_m$ for all $i \in \mathcal{N}$. Because $x_1P_ix_2$ for all $i \in \mathcal{N}$, we therefore have $x_1P_i^{m-2}x_j$ for all $i \in \mathcal{N}$ and all $j = 2, \ldots, m$. Hence, Unanimity* is violated.

Case 2. We now show that for all $x \in X$ and all $\mathbf{R} \in \mathcal{L}^n$, $C^*(X\backslash\{x\}, \mathbf{R})$ is contained in the weak Pareto set. On the contrary, suppose that there exist distinct $x, y, z \in X$ and $\mathbf{R} \in \mathcal{L}^n$ such that yP_iz for all $i \in \mathcal{N}$, but $z \in C^*(X\backslash\{x\}, \mathbf{R})$. Consider a profile $\mathbf{R}^1 \in \mathcal{L}^n$ that coincides with $\mathbf{R}$ on $X\backslash\{x\}$ and has x ranked last by all $i \in \mathcal{N}$. By Independence of Infeasible Alternatives, $z \in C^*(X\backslash\{x\}, \mathbf{R}^1)$. By Arrow's Choice Axiom, $C^*(X, \mathbf{R}^1) \cap \{x, z\} \neq \emptyset$, which contradicts what was established in Case 1 because y Pareto dominates both x and z in $\mathbf{R}^1$. □

When $\mathcal{C}_2 = \emptyset$, if a voting function V satisfies Independence of Nonvoters'

Preferences, one can identify V with a social choice function C_V with domain $\mathcal{X} \times \mathcal{L}^{n-m}$ by setting, for all $A \in \mathcal{X}$ and all $\mathbf{R}_{\mathcal{V}} \in \mathcal{L}^{n-m}$,

$$C_V(A, \mathbf{R}_{\mathcal{V}}) = V(A, (\bar{\mathbf{R}}_{\mathcal{C}_1}, \mathbf{R}_{\mathcal{V}})),$$

where $\bar{\mathbf{R}}_{\mathcal{C}_1}$ is an arbitrary subprofile in $\mathcal{L}_{\mathcal{C}_1}$.[3] Let $\bar{C}_V \colon \mathcal{A}^{m-1} \times \mathcal{L}^{n-m} \to \mathcal{C}$ be the restriction of C_V to $\mathcal{A}^{m-1} \times \mathcal{L}^{n-m}$.

We now prove Theorem 2.

Proof of Theorem 2.[4] Suppose that $\mathcal{C}_2 = \emptyset$ and the voting function $V \colon \mathcal{X} \times \mathcal{L}_{\mathcal{C}_1} \times \mathcal{L}^{n-m} \to \mathcal{C}$ satisfies all the assumptions of Theorem 2 except Strong Nondictatorship. Let $\bar{C}_V$ be the social choice function defined above. Because V satisfies Independence of Infeasible Alternatives and Unanimity, $\bar{C}_V$ satisfies Independence of Infeasible Alternatives and Unanimity*. By Lemma 1, V satisfies Strong Candidate Stability. Because $\bar{C}_V$ is a function and V satisfies Independence of Nonvoters' Preferences, the satisfaction of Strong Candidate Stability by V is equivalent to the satisfaction of Arrow's Choice Axiom by $\bar{C}_V$. Hence, by Lemma 2, $\bar{C}_V$ satisfies Weak Pareto. $\bar{C}_V$ satisfies k-Set Feasibility with $k = m - 1$. Therefore, by the Grether-Plott Theorem (Theorem 1), $\bar{C}_V$ is dictatorial. But $\bar{C}_V$ being dictatorial is equivalent to V having a dictator for large elections. Thus, V does not satisfy Strong Nondictatorship. $\square$

5. A proof of the strong candidate stability theorem

We now consider the possibility that some candidates vote. As in the preceding section, we identify a voting function V with a social choice function C_V whose preference domain only includes preferences of voters. Because candidates who vote rank themselves first, the preference domain of C_V is restricted. We show that if V satisfies the assumptions of Theorem 3, then, restricted to agendas in $\mathcal{A}^{m-1}$, C_V can be extended to the unrestricted linear preference domain in such a way that all of the assumptions of Lemma 2 are satisfied. The conclusions of Theorem 3 follow relatively straightforwardly from this observation.

[3]When $\mathcal{C}_2 = \emptyset$, we write $\mathbf{R}_{\mathcal{V}}$ instead of $\mathbf{R}_{\mathcal{N} \setminus \mathcal{C}}$.

[4]Although Theorem 2 is a corollary to Theorem 3, we provide a separate proof of Theorem 2 because the direct proof is quite short.

When $\mathcal{C}_2$ is not necessarily empty, the procedure used in Section 4 to associate a social choice function with a voting function needs to be generalized. Consider an arbitrary $\bar{\mathbf{R}}_{\mathcal{C}_1} \in \mathcal{L}_{\mathcal{C}_1}$. If a voting function V satisfies Independence of Nonvoters' Preferences, one can identify V with a social choice function C_V with domain $\mathcal{X} \times \mathcal{L}_{\mathcal{C}_2} \times \mathcal{L}^{n-m}$ by setting, for all $A \in \mathcal{X}$ and all $(\mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N}\setminus\mathcal{C}}) \in \mathcal{L}_{\mathcal{C}_2} \times \mathcal{L}^{n-m}$,

$$C_V(A, (\mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N}\setminus\mathcal{C}})) = V(A, (\bar{\mathbf{R}}_{\mathcal{C}_1}, \mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N}\setminus\mathcal{C}})).$$

Let $\bar{C}_V \colon \mathcal{A}^{m-1} \times \mathcal{L}_{\mathcal{C}_2} \times \mathcal{L}^{n-m} \to \mathcal{C}$ be the restriction of C_V to $\mathcal{A}^{m-1} \times \mathcal{L}_{\mathcal{C}_2} \times \mathcal{L}^{n-m}$.

We want to extend $\bar{C}_V$ to a social choice function $\tilde{C}_V$ whose domain is $\mathcal{A}^{m-1} \times \mathcal{L}^{n-m_1}$; i.e., to a social choice function with an unrestricted linear preference domain. For any $(\mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N}\setminus\mathcal{C}}) \in \mathcal{L}^{n-m_1}$, define $(\mathbf{R}^\circ_{\mathcal{C}_2}, \mathbf{R}^\circ_{\mathcal{N}\setminus\mathcal{C}}) \in \mathcal{L}_{\mathcal{C}_2} \times \mathcal{L}^{n-m}$ as follows:

(i) if $i \notin \mathcal{C}_2$, then $R^\circ_i = R_i$, and

(ii) if $i \in \mathcal{C}_2$, then $R^\circ_i \in \mathcal{L}_i$ is such that R°_i coincides with R_i on $\mathcal{C}\setminus\{i\}$.

Note that $(\mathbf{R}^\circ_{\mathcal{C}_2}, \mathbf{R}^\circ_{\mathcal{N}\setminus\mathcal{C}})$ is uniquely defined. The social choice function $\tilde{C}_V$ is defined by setting, for all $A \in \mathcal{A}^{m-1}$ and all $(\mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N}\setminus\mathcal{C}}) \in \mathcal{L}^{n-m_1}$,

$$\tilde{C}_V(A, (\mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N}\setminus\mathcal{C}})) = \bar{C}_V(A, (\mathbf{R}^\circ_{\mathcal{C}_2}, \mathbf{R}^\circ_{\mathcal{N}\setminus\mathcal{C}})).$$

Lemma 3 demonstrates that if a voting function V satisfies the assumptions of Theorem 3, then the social choice function $\tilde{C}_V$ satisfies the assumptions of Lemma 2.

Lemma 3. *If a voting function V satisfies Independence of Nonvoters' Preferences, Independence of Infeasible Alternatives, Strong Unanimity, and Strong Candidate Stability, then the social choice function $\tilde{C}_V$ satisfies Arrow's Choice Axiom, Independence of Infeasible Alternatives, and Unanimity*.*

Proof. First, we show that $\tilde{C}_V$ satisfies Arrow's Choice Axiom. Let $(\mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N}\setminus\mathcal{C}}) \in \mathcal{L}^{n-m_1}$ and $c \in \mathcal{C}$ be such that $\tilde{C}_V(\mathcal{C}, (\mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N}\setminus\mathcal{C}})) \neq c$. By definition, $\tilde{C}_V(\mathcal{C}, (\mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N}\setminus\mathcal{C}})) = \bar{C}_V(\mathcal{C}, (\mathbf{R}^\circ_{\mathcal{C}_2}, \mathbf{R}^\circ_{\mathcal{N}\setminus\mathcal{C}})) = V(\mathcal{C}, (\bar{\mathbf{R}}_{\mathcal{C}_1}, \mathbf{R}^\circ_{\mathcal{C}_2}, \mathbf{R}^\circ_{\mathcal{N}\setminus\mathcal{C}}))$. Thus, $V(\mathcal{C}, (\bar{\mathbf{R}}_{\mathcal{C}_1}, \mathbf{R}^\circ_{\mathcal{C}_2}, \mathbf{R}^\circ_{\mathcal{N}\setminus\mathcal{C}})) \neq c$. Strong Candidate Stability then implies that $V(\mathcal{C}\setminus\{c\}, (\bar{\mathbf{R}}_{\mathcal{C}_1}, \mathbf{R}^\circ_{\mathcal{C}_2}, \mathbf{R}^\circ_{\mathcal{N}\setminus\mathcal{C}})) = V(\mathcal{C}, (\bar{\mathbf{R}}_{\mathcal{C}_1}, \mathbf{R}^\circ_{\mathcal{C}_2}, \mathbf{R}^\circ_{\mathcal{N}\setminus\mathcal{C}}))$. By definition, we also have $\tilde{C}_V(\mathcal{C}\setminus\{c\}, (\mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N}\setminus\mathcal{C}})) = \bar{C}_V(\mathcal{C}\setminus\{c\}, (\mathbf{R}^\circ_{\mathcal{C}_2}, \mathbf{R}^\circ_{\mathcal{N}\setminus\mathcal{C}})) = V(\mathcal{C}\setminus\{c\}, (\bar{\mathbf{R}}_{\mathcal{C}_1}, \mathbf{R}^\circ_{\mathcal{C}_2}, \mathbf{R}^\circ_{\mathcal{N}\setminus\mathcal{C}}))$.

It then follows that $\tilde{C}_V(\mathcal{C}\backslash\{c\},(\mathbf{R}_{\mathcal{C}_2},\mathbf{R}_{\mathcal{N}\backslash\mathcal{C}})) = \tilde{C}_V(\mathcal{C},(\mathbf{R}_{\mathcal{C}_2},\mathbf{R}_{\mathcal{N}\backslash\mathcal{C}}))$. Thus, $\tilde{C}_V$ satisfies Arrow's Choice Axiom.

Second, we show that $\tilde{C}_V$ satisfies Independence of Infeasible Alternatives. Consider any $A \in \mathcal{A}^{m-1}$ and let $(\mathbf{R}_{\mathcal{C}_2},\mathbf{R}_{\mathcal{N}\backslash\mathcal{C}}),(\bar{\mathbf{R}}_{\mathcal{C}_2},\bar{\mathbf{R}}_{\mathcal{N}\backslash\mathcal{C}}) \in \mathcal{L}^{n-m_1}$ coincide on A. By construction, $(\mathbf{R}^\circ_{\mathcal{C}_2},\mathbf{R}^\circ_{\mathcal{N}\backslash\mathcal{C}})$ coincides with $(\bar{\mathbf{R}}^\circ_{\mathcal{C}_2},\bar{\mathbf{R}}^\circ_{\mathcal{N}\backslash\mathcal{C}})$ on A. Because V satisfies Independence of Infeasible Alternatives, we then have $V(A,(\bar{\mathbf{R}}_{\mathcal{C}_1},\mathbf{R}^\circ_{\mathcal{C}_2},\mathbf{R}^\circ_{\mathcal{N}\backslash\mathcal{C}})) = V(A,(\bar{\mathbf{R}}_{\mathcal{C}_1},\bar{\mathbf{R}}^\circ_{\mathcal{C}_2},\bar{\mathbf{R}}^\circ_{\mathcal{N}\backslash\mathcal{C}}))$. Using the definitions of C_V and $\tilde{C}_V$, it now follows that $\tilde{C}_V(A,(\mathbf{R}_{\mathcal{C}_2},\mathbf{R}_{\mathcal{N}\backslash\mathcal{C}})) = \tilde{C}_V(A,(\bar{\mathbf{R}}_{\mathcal{C}_2},\bar{\mathbf{R}}_{\mathcal{N}\backslash\mathcal{C}}))$, and, hence, that $\tilde{C}_V$ satisfies Independence of Infeasible Alternatives.

Third, we show that $\tilde{C}_V$ satisfies Unanimity*. Consider any $A \in \mathcal{A}^{m-1}$. Let $(\mathbf{R}_{\mathcal{C}_2},\mathbf{R}_{\mathcal{N}\backslash\mathcal{C}}) \in \mathcal{L}^{n-m_1}$ and $c \in \mathcal{C}$ be such that for all $i \in \mathcal{V}$, $B(A,R_i) = \{c\}$. Note that $B(A,R^\circ_i) = \{c\}$ for all $i \in \mathcal{N}\backslash\mathcal{C}$ and $B(A\backslash\{i\},R^\circ_i) = \{c\}$ for all $i \in \mathcal{C}_2\backslash\{c\}$. Because V satisfies Strong Unanimity, $V(A,(\bar{\mathbf{R}}_{\mathcal{C}_1},\mathbf{R}^\circ_{\mathcal{C}_2},\mathbf{R}^\circ_{\mathcal{N}\backslash\mathcal{C}})) = c$. Using the definitions of C_V and $\tilde{C}_V$, we conclude that $\tilde{C}_V(A,(\mathbf{R}_{\mathcal{C}_2},\mathbf{R}_{\mathcal{N}\backslash\mathcal{C}})) = c$. Thus, $\tilde{C}_V$ satisfies Unanimity*. □

Lemmas 2 and 3 are now used to prove Theorem 3.

Proof of Theorem 3. Suppose that the voting function V satisfies the assumptions of Theorem 3. By Lemmas 2 and 3, the social choice function $\tilde{C}_V$ satisfies Arrow's Choice Axiom, Independence of Infeasible Alternatives, and Weak Pareto. Because $\tilde{C}_V$ satisfies k-Set Feasibility with $k = m-1$ and has an unrestricted linear preference domain, $\tilde{C}_V$ is dictatorial by the Grether-Plott Theorem. Because $\tilde{C}_V$ coincides with $\bar{C}_V$ on the domain of $\bar{C}_V$, $\bar{C}_V$ is also dictatorial. Hence, there is a dictator for large elections for V. Strong Unanimity implies that this dictator must belong to $\mathcal{V}\backslash\mathcal{C}$. □

6. Multivalued voting procedures

In this section, we relate our analysis to the recent articles by Eraslan and McLennan [4] and Rodríguez-Álvarez [6] on multivalued voting procedures.[5]

A *voting correspondence* is a social choice correspondence $V\colon \mathcal{X}\times\mathcal{L}^* \to \mathcal{C}$. Eraslan and McLennan [4] consider the following candidate stability axiom.

Strong Candidate Stability*. For all $i \in \mathcal{C}$ and all $\mathbf{R} \in \mathcal{L}^*$, either $V(\mathcal{C},\mathbf{R}) = \{i\}$ or $V(\mathcal{C}\backslash\{i\},\mathbf{R}) = V(\mathcal{C},\mathbf{R})\backslash\{i\}$.

[5] Dutta, Jackson, and Le Breton [3] only briefly consider multivalued voting procedures.

For voting functions, Strong Candidate Stability* is equivalent to Strong Candidate Stability. Restricted to agendas in $\mathcal{A}^{m-1}$, Strong Candidate Stability* is equivalent to Arrow's Choice Axiom.

For a voting correspondence V, we can construct a social choice correspondence $\tilde{C}_V$ with domain $\mathcal{A}^{m-1} \times \mathcal{L}^{n-m_1}$ as in the preceding section. With only trivial changes to the proof, Lemma 3 and, hence, Theorem 3 are valid for voting correspondences if Strong Candidate Stability* is substituted for Strong Candidate Stability.[6] Eraslan and McLennan have shown that this theorem can be further strengthened. Their main result shows that, in the voting correspondence version of Theorem 3, V is characterized by a serial dictatorship, and this is true even if individuals are permitted to have weak preference orderings (subject to the proviso that each candidate ranks him- or herself strictly above the other candidates) and the unanimity condition is weakened somewhat.[7]

For the Candidate Stability axiom to apply to voting correspondences, preferences over candidates need to be extended to preferences over subsets of candidates. Rodríguez-Álvarez [6] considers two such extensions based on the work of Barberà, Dutta, and Sen [2]. In the first of these extensions, what Rodríguez-Álvarez calls BDS1, sets of candidates are ordered by their conditional expected utilities using an initial probability distribution over the set $\mathcal{C}$ and a von Neumann-Morgenstern utility function representing the preference on $\mathcal{C}$. The corresponding multivalued version of Candidate Stability requires that for any preference profile $\mathbf{R} \in \mathcal{L}^*$, no candidate i would prefer to withdraw when the set of candidates is $\mathcal{C}$ using any preference over subsets of candidates that is obtainable from R_i with the BDS1 extension procedure. Lemma 2 in Rodríguez-Álvarez [6] demonstrates that if a voting correspondence V satisfies Independence of Nonvoters' Preferences and Inde-

[6]We only need to replace $\tilde{C}_V(\mathcal{C}, (\mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N}\backslash\mathcal{C}})) \neq c$ and $V(\mathcal{C}, (\bar{\mathbf{R}}_{\mathcal{C}_1}, \mathbf{R}^\circ_{\mathcal{C}_2}, \mathbf{R}^\circ_{\mathcal{N}\backslash\mathcal{C}})) \neq c$ with $\tilde{C}_V(\mathcal{C}\backslash\{c\}, (\mathbf{R}_{\mathcal{C}_2}, \mathbf{R}_{\mathcal{N}\backslash\mathcal{C}})) \neq \emptyset$ and $V(\mathcal{C}\backslash\{c\}, (\bar{\mathbf{R}}_{\mathcal{C}_1}, \mathbf{R}^\circ_{\mathcal{C}_2}, \mathbf{R}^\circ_{\mathcal{N}\backslash\mathcal{C}})) \neq \emptyset$, respectively, in the proof that $\tilde{C}_V$ satisfies Arrow's Choice Axiom.

[7]Our proof can be easily adapted to establish the voting correspondence version of Theorem 3 (using Strong Candidate Stability*) when individuals can have any weak preference subject to the restriction on candidates' preferences described above. To apply our proof strategy, the social choice correspondence $\tilde{C}_V$ would need to be defined for the domain $\mathcal{A}^{m-1} \times \mathcal{R}^{n-m_1}$. For $i \in \mathcal{C}_2$, the preference R_i° associated with R_i would then be required to have i as the uniquely best candidate, but would preserve any indifferences among the other candidates. Note that both the Grether-Plott Theorem and Lemma 2 are valid for the preference domain $\mathcal{R}^n$.

pendence of Infeasible Alternatives and there are no voting candidates, then V satisfies the multivalued version of Candidate Stability using the BDS1 extension procedure if and only if it satisfies Strong Candidate Stability*. Rodríguez-Álvarez has also shown that Theorem 2 holds for voting correspondences when Candidate Stability is applied using the BDS1 preference extension procedure. Using his Lemma 2 instead of Lemma 1, our proof of Theorem 2 provides an alternative way of establishing his theorem.

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