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Dividends and Volatility: A
Nonparametric Investigation**

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American Options with Stochastic Dividends and Volatility: A Nonparametric Investigation[*]

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Résumé / Abstract

Cet article examine les contrats optionnels de type américain lorsque l'actif sous-jacent paie des dividendes et a une volatilité stochastiques. Nous présentons une discussion complète des fondations théoriques de l'évaluation des options américaines et de leurs frontières d'exercice. Nous démontrons leur dépendance par rapport aux diverses sources d'incertitude qui déterminent le taux de dividendes et la volatilité, et dérivons les prix d'équilibre des actifs, titres dérivés ainsi que les politiques optimales d'exercice dans un modèle d'équilibre général. Les modèles théoriques conduisent à des expressions complexes qui sont difficiles à estimer. C'est pourquoi nous adoptons une approche non-paramétrique qui permet d'examiner des formes réduites. Nous utilisons des méthodes non-paramétriques pour estimer les prix d'options à l'achat et les frontières d'exercice conditionnelles aux dividendes et à la volatilité. Puisque cette dernière est un processus latent nous proposons plusieurs approches, fondées en particulier sur des estimateurs-filtres EGARCH, des volatilités implicites et historiques. L'approche non-paramétrique nous permet de tester si les prix d'options et les décisions d'exercice sont principalement déterminés par les dividendes, comme suggéré par Harvey et

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Whaley (1992a, b) et Fleming et Whaley (1994) pour le contrat OEX, ou si la volatilité stochastique complémente l'incertitude sur les dividendes. Nous établissons que les dividendes seuls ne rendent pas compte de tous les aspects de l'évaluation de ces options et des décisions d'exercice, ce qui suggère la nécessité d'inclure la volatilité stochastique.

In this paper, we consider American option contracts when the underlying asset has stochastic dividends and stochastic volatility. We provide a full discussion of the theoretical foundations of American option valuation and exercise boundaries. We show how they depend on the various sources of uncertainty which drive dividend rates and volatility, and derive equilibrium asset prices, derivative prices and optimal exercise boundaries in a general equilibrium model. The theoretical models yield fairly complex expressions which are difficult to estimate. We therefore adopt a nonparametric approach which enables us to investigate reduced forms. Indeed, we use nonparametric methods to estimate call prices and exercise boundaries conditional on dividends and volatility. Since the latter is a latent process, we propose several approaches, notably using EGARCH filtered estimates, implied and historical volatilities. The nonparametric approach allows us to test whether call prices and exercise decisions are primarily driven by dividends, as has been advocated by Harvey and Whaley (1992a,b) and Fleming and Whaley (1994) for the OEX contract, or whether stochastic volatility complements dividend uncertainty. We find that dividends alone do not account for all aspects of call option pricing and exercise decisions, suggesting a need to include stochastic volatility.

1 Introduction

The early exercise feature of American option contracts considerably complicates their valuation. Even the relatively simple case of an underlying asset with a Geometric Brownian Motion (GBM) price process and constant dividend rate requires numerical algorithms to value the option and determine the optimal exercise policy.[1]

Two critical assumptions, namely (1) a constant dividend rate and (2) constant volatility, are often cited as restrictive and counter-factual. For the OEX contract, the most widely traded American-type option written on the S&P100 Stock Index, Harvey and Whaley (1992a, b) and Fleming and Whaley (1994) underline the importance of the amount and the timing of dividends. To account for discrete dividend payments on the S&P100 index portfolio they use a modification of the Cox, Ross and Rubinstein (1979) binomial method which reduces the index level by the discounted flow of dividends during the lifetime of the option. Using this approach they show that ignoring dividends has a significant impact on pricing errors. The computations are already so demanding in this simple case with constant volatility that the stochastic volatility case appears to be beyond reach. It is interesting to note that for European-type options, like the SPX contract on the S&P500 Stock Index, there has been far more interest in studying the stochastic volatility case.[2] One may therefore wonder whether it is either stochastic volatility, or stochastic dividends, *or both*, which determine American as well as European options. The purpose of our paper is to address this question. We study the case of American options as it is considerably more difficult than the European-type contract. The approach we take, however,

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[1] A whole range of numerical procedures have been proposed, including finite differences, binomial, multinomial, quasi-analytical, quadratic methods as well as the method of lines and Richardson extrapolations. A partial list of contributions includes Brennan and Schwartz (1977), Cox, Ross and Rubinstein (1979), Geske (1979), Whaley (1981), Geske and Johnson (1984), Barone-Adesi and Whaley (1987), Boyle (1988), Breen (1991), Yu (1993), Broadie and Detemple (1996) and Carr and Faguet (1994), among others. For a review of these procedures, see Broadie and Detemple (1996).

[2] See for instance Hull and White (1987), Johnson and Shanno (1987), Scott (1987), Wiggins (1987), Chesney and Scott (1989), Stein and Stein (1991), Heston (1993), among others.

readily applies to the SPX or similar contracts. For European options with stochastic volatility we could recover pricing formula such as those derived by Hull and White (1987) and others, except that we do not necessarily impose the specific volatility processes they assumed.

We first study the theoretical foundations of American option pricing and characterize the exercise boundary assuming stochastic volatility *and* stochastic dividend rate. Then, we test the models empirically and test whether it is dividends, volatility or both which affect the OEX contract. American option pricing models with stochastic dividends and volatility are prohibitively complex to conduct a structural econometric analysis. Fortunately, in testing the impact of dividends and volatility we do not have to handle a fully specified structural model. Instead, we follow a different approach which uses *market* data, both on exercise decisions and option prices, and relies on *nonparametric* statistical techniques. Let us illustrate this intuitively for the case of the exercise boundary. Suppose that we have observations on the exercise decisions of investors who own American options, along with the features of the contracts being exercised.[3] The idea is that with enough data, such as ten years of daily observations, we should be able to gather information about investors' perceptions of the exercise boundary and their response to volatility and dividends.[4] The computation of exercise boundaries, and in particular the inclusion of stochastic volatility and dividends in the analysis will be discussed in detail in the paper. The approach just described for the exercise boundaries can also be applied to the pricing of the option, again assuming that we have data on call and put contracts and their attributes. As noted before, the latter could apply to American as well as European contracts.

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[3]Such data are available for the S&P100 Index option or OEX contract, as they are collected by the Option Clearing Corporation (OCC). Option exercise data have been used in a number of studies, including Ingersoll (1977), Bodurtha and Courtadon (1986), Overdahl (1988), Dunn and Eades (1989), Gay, Kolb and Yung (1989), Zivney(1991), French and Maberly (1992) and Diz and Finucane (1993).

[4]Questions as to whether market participants exercise "optimally", regardless of what the model or assumptions might be, will not be the main focus of our paper although several procedures that we suggest would create a natural framework to address some of these issues. For the most recent work on testing market rationality using option exercise data and for a review of the related literature, see Diz and Finucane (1993).

The idea of applying nonparametric methods to option pricing has been suggested recently in a number of paper, *e.g.*, Abken *et. al.* (1996), Aït-Sahalia (1996), Aït-Sahalia and Lo (1995), Gouriéroux, Monfort and Tenreiro (1994), Hutchinson, Lo and Poggio (1994), Jackwerth and Rubinstein (1996), Madan and Milne (1994) and Stutzer (1995). As there are a multitude of nonparametric methods it is no surprise that the aforementioned papers use different methods. Moreover, they do not address the same topics either. Indeed, some aim for nonparametric corrections of standard (say Black-Scholes) option pricing formula, others estimate risk-neutral densities, *etc.* So far this literature has focused exclusively on European type options. By studying American options, our paper models both pricing and exercise strategies via nonparametric methods. In addition, our analysis features a combination of volatility filtering based on EGARCH models and nonparametric analysis hitherto not explored in the literature.[5] This combination has several advantages as it helps to reduce the high dimensionality of nonparametric methods and is a relatively simple way to introduce conditional volatility.

In section 2 of the paper, we provide a rigourous theoretical treatment of American option pricing with stochastic volatility and stochastic dividend. We show how option values and exercise boundaries depend on the various sources of uncertainty in the model. Section 3 is devoted to the nonparametric estimation of American options with stochastic dividends and/or volatility. Formal tests for the impact of random volatility are presented. We use data on call prices as well as exercise decisions and study the pricing of options and exercise decisions assuming random dividends and volatility. Section 4 concludes the paper. In Appendix A we examine the relationship between the aggregate dividend process and the equilibrium index value, its volatility, the endogenous dividend rate and equilibrium interest rate in a general framework with state dependent utility.

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[5]In addition to the EGARCH filtered volatilities we will also consider implied volatilities and historical volatilities.

2 American option valuation with stochastic dividends and volatility

Much has been written on the valuation of American options. The earliest analysis of the subject by McKean (1965) and Van Moerbeke (1976) formulates the pricing problem as a free boundary problem. A formal justification based on no-arbitrage arguments for the valuation of an American contingent claim is provided by Bensoussan (1984) and Karatzas (1988) in the context of a general market model, in which the the underlying asset price follows an Itô process. It should not come as a surprise that the distributional properties of the underlying asset price determine those of the exercise boundary. However, in such a general context, analytical closed-form solutions are typically not available. The standard approach then specifies a process for the underlying asset price, generally a Geometric Brownian Motion (GBM), and searches for numerically efficient algorithms to compute the pricing formula and the exercise boundary. This particular case is now well understood and its theoretical properties have been extensively studied by Kim (1990), Jacka (1991), Carr, Jarrow and Myneni (1992), Myneni (1992) and Broadie and Detemple (1996).

In this section, we study American options in a more general setting which allows both for a stochastic dividend yield and stochastic volatility. We consider a financial market in which the stock price S satisfies

$$\mathrm{d}S_t = S_t[\mu(Y_t, Z_t, t) - \delta(Y_t, Z_t, t)]\mathrm{d}t + S_t[\sigma_1(Y_t, Z_t, t)\mathrm{d}W_{1t}$$

$$+\sigma_2(Y_t, Z_t, t)\mathrm{d}W_{2t} + \sigma_3(Y_t, Z_t, t)\mathrm{d}W_{3t}], \tag{2.1}$$

$$\mathrm{d}Y_t = \mu^Y(Y_t, t)\mathrm{d}t + \sigma_1^Y(Y_t, t)\mathrm{d}W_{1t} + \sigma_2^Y(Y_t, t)\mathrm{d}W_{2t}, \tag{2.2}$$

$$\mathrm{d}Z_t = \mu^Z(Z_t, t)\mathrm{d}t + \sigma_1^Z(Z_t, t)\mathrm{d}W_{1t} + \sigma_3^Z(Z_t, t)\mathrm{d}W_{3t}, \tag{2.3}$$

for $t \in [0, T]$ and where S_0, Y_0 and Z_0 are given. Here $\mu(Y_t, Z_t, t)$, $\sigma_1(Y_t, Z_t, t)$, $\sigma_2(Y_t, Z_t, t)$ and $\sigma_3(Y_t, Z_t, t)$ represent the drift and the volatility coefficients of the stock price process and $\delta(Y_t, Z_t, t)$ is the dividend rate on the stock. These coefficients depend on time and on the

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current values of the state variables Y and Z which satisfy the stochastic differential equations (2.2)-(2.3). Two state variables are required to model a stochastic dividend yield which is imperfectly correlated with the volatility coefficients of the stock price process. We suppose that the coefficients $\mu^Y(Y_t, t), \sigma_1^Y(Y_t, t)$ and $\sigma_2^Y(Y_t, t)$, which are functions of (Y_t, t), and $\mu^Z(Z_t, t), \sigma_1^Z(Z_t, t), \sigma_3^Z(Z_t, t)$ which are functions of (Z_t, t), satisfy standard Lipschitz and growth conditions: this ensures the existence of a unique solution to (2.2)-(2.3). The processes W_1, W_2 and W_3 are independent Brownian motion processes which represent the uncertainty in the economy. We also suppose that the interest rate r is constant. As shown in Appendix A, these assumptions can be supported as the equilibrium outcome in a general economy with *stochastic* dividend (level) process and representative agent with state dependent utility function. In the remainder of this section, we operate in the context of this model. In this general economy, the equilibrium market prices of W_1-, W_2- and W_3-risks are functions of both state variables Y and Z

$$
\begin{align}
\theta_{1t} &= \theta_1(Y_t, Z_t, t), \tag{2.4} \\
\theta_{2t} &= 0, \tag{2.5} \\
\theta_{3t} &= \theta_3(Y_t, Z_t, t), \tag{2.6}
\end{align}
$$

which are explicitly related to the characteristics of the underlying dividend process (see Theorem A.1, Corollary A.1, and model 1 in Appendix A). In this economy W_1-risk is priced since it affects the change in the dividend (level) process, W_2-risk has market price 0 since it is unrelated to dividend (level) risk, and W_3-risk is priced since it affects marginal utility. The risk neutralized processes for the stock price and the volatility and dividend rate state variables are given by $\mathrm{d}S_t = S_t[r - \delta(Y_t, Z_t, t)]\mathrm{d}t + S_t[\sigma_1(Y_t, Z_t, t)\mathrm{d}W_{1t}^*$

$$
+\sigma_2(Y_t, Z_t, t)\mathrm{d}W_{2t}^* + \sigma_3(Y_t, Z_t, t)\mathrm{d}W_{3t}^*], \tag{2.7}
$$

$\mathrm{d}Y_t = [\mu^Y(Y_t, t) - \theta_1(Y_t, Z_t, t)\,\sigma_1^Y(Y_t, t)]\mathrm{d}t$

$$
+\sigma_1^Y(Y_t, t)\mathrm{d}W_{1t}^* + \sigma_2^Y(Y_t, t)\mathrm{d}W_{2t}^*, \tag{2.8}
$$

$\mathrm{d}Z_t = [\mu^Z(Z_t, t) - \theta_1(Y_t, Z_t, t)\,\sigma_1^Z(Z_t, t) - \theta_3(Y_t, Z_t, t)\,\sigma_3^Z(Z_t, t)]\mathrm{d}t$

$$
+\sigma_1^Z(Z_t, t)\mathrm{d}W_{1t}^* + \sigma_3^Z(Z_t, t)\mathrm{d}W_{3t}^*, \tag{2.9}
$$

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for $t \in [0, T]$, where S_0, Y_0 and Z_0 are given, and where W_1^*, W_2^* and W_3^* are Brownian motion processes relative to the equivalent martingale measure Q (*i.e.*, the risk-neutral measure).

The stock price model with stochastic volatility (2.7)-(2.8) is fairly general since it allows for arbitrary correlation between the volatility process and the stock price process as well as for a fairly general structure of the drift and volatility coefficients of the state variable processes Y and Z. We note in particular that the volatility and the dividend innovations need not be spanned by the stock price innovations, *i.e.*, the basic model is one in which volatility risk and dividend risk cannot be hedged away by trading the other securities in the model (the stock and the bond). In order to price zero net supply contingent claims, we take a general equilibrium approach [see *e.g.*, Cox, Ingersoll and Ross (1985)] in which the financial market is effectively complete. In this context, the value of any contingent claim is simply given by its shadow price, *i.e.*, the price at which the representative agent is content to forgo holding the asset. The equilibrium risk premium on this claim is therefore the sum of the market prices of W_1- and W_3-risks, each multiplied by the sensitivity of the claim to W_1- and W_3-risk (see Theorem A.2 in appendix A).

Consider now an American call option contract with maturity date T and payoff $(S - K)^+$ at the exercise time. Let $\mathcal{S}_{[t,T]}$ denote the class of stopping times taking values in the interval $[t, T]$. In our representative agent economy, the value of this contract C_t is the maximum present value that can be achieved over this set of stopping times,

$$C_t = \sup_{\tau \in \mathcal{S}_{[t,T]}} E_t^Q \left[e^{-r(\tau - t)} (S_\tau - K)^+ \right], \quad t \in [0, T], \qquad (2.10)$$

where E^Q denotes the expectation under the equivalent martingale measure Q. Standard transformations also yield the early exercise premium representation for the American call option:

$$C_t = C_t^E + E_t^Q \left[\int_t^T e^{-r(s-t)} (\delta_s S_s - rK) \mathbb{I}_{\{s = \tau(s)\}} \mathrm{d}s \right], \qquad (2.11)$$

where $\mathbb{I}_A$ denotes the indicator function of the set A, C_t^E is the value of a European call and $\tau(t)$ is the optimal stopping time in $\mathcal{S}_{[t,T]}$ (*i.e.*, the

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optimal exercise time) defined by

$$\tau(t) \equiv \inf\{\nu \in [t,T] : C_\nu = (S_\nu - K)^+\}. \tag{2.12}$$

Since the economy under consideration is fully described by the pair of processes (Y, Z), the optimal exercise time $\tau(t)$ can also be defined by

$$\tau(t) \equiv \inf\{\nu \in [t,T] : S_\nu \geq B(Y_\nu, Z_\nu, \nu)\}, \tag{2.13}$$

where $B(Y_t, Z_t, t)$ represents the optimal exercise boundary, *i.e.*, the boundary of the immediate exercise region. The event $\{s = \tau(s)\}$ can then be written as $\{S_\nu \geq B(Y_\nu, Z_\nu, \nu)\}$. Summarizing, we have the following result:

Theorem 2.1 *Consider the financial market in which the stock price process is given by (2.1)-(2.3) and the interest rate is constant. In this economy, the price at date $t \in [0,T]$ of an American call option is given by:*

$$C(S_t, Y_t, Z_t, t) = C^E(S_t, Y_t, Z_t, t) + \Pi\big(S_t, Y_t, Z_t, t, B(\,\cdot\,)\big), \tag{2.14}$$

where C^E denotes the value of a European option with maturity date T and $\Pi\big(S_t, Y_t, Z_t, t, B(\,\cdot\,)\big)$ denotes the early exercise premium,

$$\Pi\big(S_t, Y_t, Z_t, t, B(\,\cdot\,)\big) \equiv$$

$$E_t^Q\left[\int_t^T e^{-r(s-t)}\big[\delta(Y_s, Z_s, s)S_s - rK\big]\mathbb{I}_{\{S_s \geq B(Y_s, Z_s s)\}}\mathrm{d}s\right]. \tag{2.15}$$

The optimal exercise boundary B satifies the recursive integral equation

$$B(Y_t, Z_t, t) - K = C_t^E\big(B(Y_t, Z_t, t), Y_t, Z_t, t\big)$$

$$+\Pi\big(B(Y_t, Z_t, t), Y_t, Z_t, t, B(\,\cdot\,)\big), \quad t \in [0,T], \tag{2.16}$$

$$B(Y_T, Z_T, T) = \max\left\{\frac{r}{\delta(Y_T, Z_T, T)}K, K\right\}. \tag{2.17}$$

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It should parenthetically be noted that we supressed K in (2.14) and all subsequent formulas to simplify the notation. A useful property of the American option price is given next,

Corollary 2.1 *Consider the financial market model with stochastic volatility of Theorem 2.1. The American option valuation formula is homogeneous of degree one in the triple (S,B,K),*

$$C_t(S, B, K) = K C_t(S/K, B/K, 1), \qquad (2.18)$$

for all $t \in [0, T]$ and $S \in \mathbb{R}^+$.

This property is important for the econometric evaluation of the model discussed in Section 3. The property states that the ratio of the option price over the exercise price is independent of the absolute level of the stock price (equivalently of the absolute level of the exercise price).

Formulas (2.14)-(2.17) for the American option price can be written more explicitly using the structure of the underlying asset price processes S and Y. Solving equation (2.7) for the stock price gives

$$S_\nu = S_t \exp\left[\int_t^\nu \left[r - \delta(Y_s, Z_s, s) - \tfrac{1}{2}\sigma_1(Y_s, Z_s, s)^2 - \tfrac{1}{2}\sigma_2(Y_s, Z_s, s)^2\right.\right.$$
$$\left.-\tfrac{1}{2}\sigma_3(Y_s, Z_s, s)^2\right]\mathrm{d}s + \int_t^\nu \left[\sigma_1(Y_s, Z_s, s)\,\mathrm{d}W_{1s}^* + \sigma_2(Y_s, Z_s, s)\,\mathrm{d}W_{2s}\right.$$
$$\left.\left.+\sigma_3(Y_s, Z_s, s)\,\mathrm{d}W_{3s}^*\right]\right], \ (2.19)$$

for $\nu > t$. Substituting this into (2.14)-(2.17) produces a valuation formula for the American option for a fairly general class of diffusion volatility processes. Once the optimal exercise boundary has been determined this formula can be computed by simulating the paths of the Brownian motion processes W_1^*, W_2^* and W_3^*.

More can be said for the following model with a single state variable Y,

$$\mathrm{d}S_t = S_t\left[(r-\delta)\mathrm{d}t + \sigma_1(Y_t, t)\left(\rho\mathrm{d}W_t^* + \sqrt{1-\rho^2}\,\mathrm{d}B_t^*\right)\right] \quad (2.20)$$
$$\mathrm{d}Y_t = \left[\mu^Y(Y_t, t) - \sigma^Y(Y_t, t)\rho\theta_1(Y_t, t)\right]\mathrm{d}t + \sigma^Y(Y_t, t)\mathrm{d}W_t^* \ ,(2.21)$$

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where δ and $\rho \in (-1, 1)$ are constants, and W^* and B^* are independent Brownian motion processes under Q.[6] Let

$$a_{t,v} \equiv \left[\int_t^v \sigma_1(Y, u)^2 \mathrm{d}u \right]^{1/2} \qquad (2.22)$$

$$w_{t,v}^1 \equiv (a_{t,v})^{-1} \left[\int_t^v \sigma_1(Y, u) \mathrm{d}W_u^* \right] \qquad (2.23)$$

and

$$d_0(S_t, B_v, a_{t,v}) \equiv \left[\log\left(\frac{S_t}{B_v} \right) + (r - \delta)(v - t) + \frac{1}{2} a_{t,v}^2 \right] \frac{1}{a_{t,v}} \qquad (2.24)$$

$$d(S_t, B_v, a_{t,v}, \rho, w_{t,v}^1) \equiv \frac{1}{\sqrt{1 - \rho^2}} d_0(S_t, B_v, a_{t,v}) + \frac{1}{\sqrt{1 - \rho^2}} \left[\rho w_{t,v}^1 - \rho^2 a_{t,v} \right].$$
$$(2.25)$$

With this notation we have

Theorem 2.2 *Consider the financial market model (2.20)-(2.21). The price at date $t \in [0, T]$ of an American call option is:*

$$C_t(S_t, Y_t, t) = E_t \left[C_Y^E(S_t, K, a_{t,T}, \rho, w_{t,T}^1) \right] + E_t \left[\int_t^T G_Y(S_t, B_v, a_{t,v}, \rho, w_{t,v}^1) \mathrm{d}v \right],$$
$$(2.26)$$

where

$$C_Y^E(S_t, K, a_{t,T}, \rho, w_{t,T}^1) \equiv S_t \exp\left[-\delta(T - t) - \frac{1}{2} \rho^2 a_{t,T}^2 + \rho a_{t,T} w_{t,T}^1 \right]$$

$$\times N\left(d(S_t, K, a_{t,T}, \rho, w_{t,T}^1) - K \exp\left[-r(T - t) \right] \right.$$

$$N\left(d(S_t, K, a_{t,T}, \rho, w_{t,T}^1) - \sqrt{1 - \rho^2}\, a_{t,T} \right) \qquad (2.27)$$

and

$$G_Y(S_t, B_v, a_{t,v}, \rho, w_{t,v}^1) \equiv \delta S_t \exp\left[-\delta(v - t) - \frac{1}{2} \rho^2 a_{t,v}^2 + \rho a_{t,v} w_{t,v}^1 \right]$$

$$\times N\left(d(S_t, B_v, a_{t,v}, \rho, w_{t,v}^1) \right) - r K \exp\left[-r(v - t) \right]$$

$$N\left(d(S_t, B_v, a_{t,v}, \rho, w_{t,v}^1) - \sqrt{1 - \rho^2}\, a_{t,v} \right). \qquad (2.28)$$

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[6] The model (2.20)-(2.21) can be obtained from (2.7)-(2.9) by taking $\sigma_1^Y = b_1 \sigma^Y$, $\sigma_2^Y = b_2 \sigma^Y$, $\mathrm{d}W_t^* \equiv (b_1^2 + b_2^2)^{-1/2}[b_1 \mathrm{d}W_{1t}^* + b_2 \mathrm{d}W_{2t}^*]$ and $\rho \equiv (b_1^2 + b_2^2)^{-1/2} b_1$, and eliminating the state variable Z and the Brownian motion W_3^*.

The optimal exercise boundary satisfies the recursive integral equation

$$B(Y_t, t) - K = C_t\big(B(Y_t, t), Y_t, t\big), \qquad (2.29)$$

$$B(Y_T, T) = \max\big\{\frac{r}{\delta}K, K\big\} \qquad (2.30)$$

subject to the relevant boundary conditions implied by the limiting behavior of the state variable process Y.

Expressions (2.26)-(2.28) for the early exercise premium and the value of the American option are not closed form expressions. One expectation with respect to the trajectories of Y (equivalently, with respect to the trajectories of W^*) remains to be taken. If the optimal exerise surface $B(\cdot, \cdot)$ has been identified, explicit computation of the option value can be performed by simulating the path of Y. Such calculations are standard for pricing European-type contracts, *i.e.*, computing the formula for $E_t(C_Y^E)$ where C_Y^E is given in (2.27) (see references appearing in the Introduction on this subject). The determination of the excercise boundary, however, is a nontrivial step in this computation. As (2.29) reveals it involves solving a recursive integral equation in two dimensions. This difficult step is bypassed in the nonparametric approach developed in the next section.

3 Nonparametric methods for American option pricing with stochastic volatility and dividends

The results in section 2 showed that the reduced forms for equilibrium American option prices and exercise decisions depend in a nontrivial way on two latent state processes Y and Z (see also Appendix A). They also established that the call price is homogeneous of degree one in (S, K), the strike, under relatively mild regularity conditions (see Corollary 2.1). The main obstacle is that call prices as well as exercise boundaries under stochastic volatility and random dividends become fairly complex functions of these state processes. Indeed, considering a fully specified parametric framework would require the computation of intricate expressions involving conditional expectations and identifying the exercise boundary which solves a recursive integral equation. It is the main reason why no attempts were made to compute prices and excercise decisions under such general conditions. Fortunately the task of determining whether both stochastic volatility and dividends affect the valuation of the OEX contract can be accomplished by using nonparametric methods. Moreover, these also yield a method for pricing calls and exercising contracts conditional on volatility and dividends. In a first subsection, we describe the generic specification of the model used in the nonparametric approach. Some of the technical issues regarding the nonobservability of the volatility are discussed in a second subsection. The third subsection presents the estimation techniques and results while the final one is devoted to testing the effect of volatility and dividends on option valuation.

3.1 The generic reduced form specification

In the economy of section 2 and appendix A two state variables Y and Z affect the equilibrium call prices and exercise decisions. Therefore $T - t$ periods before maturity we have the following relations

$$(C/K)_t = \breve{g}_C\big((S/K)_t, T - t, Y_t, Z_t\big), \quad (B/K)_t = \breve{g}_B\big(T - t, Y_t, Z_t\big). \quad (3.1)$$

The functions $\breve{g}_B$ and $\breve{g}_C$ are viewed as the reduced forms of the general equilibrium specification discussed in the previous section and in appendix A. We deleted on purpose all the parameters which help to determine the relations appearing in (3.1). Indeed, one of the advantages of the nonparametric approach is that we will not (have to) specify the preference parameters or the stochastic process for the underlying asset.

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Since the reduced forms (3.1) involve two undefined and unobservable state variables, they are of no interest for the econometrician and the practitioner. Both would prefer a relation which expresses C/K and B/K as functions of variables having an economic interpretation. Observing the model derived in section 2 in its more general formulation, we see that there are mainly two chanels through which Y and Z affect the call price and exercise decisions, namely (1) the dividend rate δ and (2) the volatility of the underlying asset price σ (see Theorem 2.1 and Corollary A.1). Therefore we will be interested in estimating the relationships appearing in the model of section 2:

$$(C/K)_t = g_C\big((S/K)_t, T-t, \delta_t, \sigma_t\big), \qquad (B/K)_t = g_B\big(T-t, \delta_t, \sigma_t\big),$$
$$\text{(3.2)}$$

where $\delta_t = \delta(Y_t, Z_t, t)$ and $\sigma_t = \sigma(Y_t, Z_t, t)$. Relationship (3.2) is what one could call an "empirical reduced form" of the option pricing model developed in section 2.

The idea is that with enough observations on call prices, exercise decisions, dividends and volatility, we should be able to recover the reduced forms from the data. However, by being nonparametric in both the formulation of the theoretical model and its econometric treatment, there are issues we cannot address.[7] Nevertheless, the nonparametric approach does achieve the main goal of our econometric anaylsis, namely to determine whether the volatility and/or the dividend rate affect the valuation of the contract and the exercise policy. The models studied so far in the empirical finance literature on American options have concentrated almost exclusively on the effect of the dividends and implicitly assume that there is only one state variable acting through the dividend rate, see *e.g.*, Harvey and Whaley (1992a, b), Fleming and Whaley (1994). They used a modified Cox-Ross-Rubinstein algorithm, yielding:

$$(C/K)_t = \tilde{g}_C\big((S/K)_t, T-t, \delta(Y_t, t)\big), \qquad (B/K)_t = \tilde{g}_B\big(T-t, \delta(Y_t, t)\big),$$
$$\text{(3.3)}$$

where $\tilde{g}_C$ and $\tilde{g}_B$ are *specific* functions related to the GBM specification. Even within the Harvey and Whaley and Fleming and Whaley framework of a single state variable and time-varying dividends, our nonparametric approach does not necessarily assume a GBM process.

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[7] For instance, suppose that in estimating nonparametrically the relations in (3.2) we find that both σ and δ affect B/K and C/K. Then from appendix A we can note that model 1 and model 3 are possible candidates for the true underlying model. Indeed, model 1 is the most general one which yields σ and δ as functions of Y and Z. Model 3 is more restrictive in the sense that the underlying economic model restricts δ to be a function of Z. Such issues can only be addressed via a fully specified structural model.

Moreoever, it is also worth noting that the nonparametric methods not only allow us to price contracts, similar to Harvey and Whaley (1992a,b) and Fleming and Whaley (1994), but also to compute excercise boundaries conditional on dividends. Finally, within this framework we can also cover models with stochastic volatility but a single state variable:

$$(C/K)_t = g_C\big((S/K)_t, T - t, \delta(Y_t, t), \sigma(Y_t, t)\big),$$

$$(B/K)_t = g_B\big(T - t, \delta(Y_t, t), \sigma(Y_t, t)\big), \qquad (3.4)$$

such as model 2 in appendix A and Theorem 2.2 (which includes the implied binomial tree models of Rubinstein (1994)). In the next subsection, we devote our attention to the specification of the latent volatility variable process and the estimation issues associated with it.

3.2 Volatility measurement and estimation issues

We noted in the Introduction that models often encountered in the literature on European options feature stochastic volatility, see Hull and White (1987), Johnson and Shanno (1987), Scott (1987), Wiggins (1987), Chesney and Scott (1989), Stein and Stein (1991), Heston (1993), among others. The results obtained for European options, and those for American options with stochastic volatility discussed in section 2, show that in order to price a call one has to integrate over a path of future volatilities for the remaining lifetime of the contract.[8] The first step will consist of estimating the current state. Since it is a latent process we need to extract it from the (return) data. Once those estimated volatilities are obtained we will estimate nonparametrically their relationship with the call prices which are assumed to be functions of the expected value of future volatilities, given current values of the state variables. Obviously, even with an explicit model for volatility, the computation of this expectation for European and certainly American type contracts is extremely challenging. It is this difficult step which is bypassed here via the use of market data and nonparametric methods.

In principle, one could filter σ_t from the data using a sample of observations on the series S_t. We obviously need a parametric model if we were to do this in an explicit and optimal way. This however would be incompatible with a nonparametric approach. Hence, we need to proceed somehow without violating the main results of section 2 and at the

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[8] This distinction between the current state and its future path over the remaining term of the contract was also important in the case of dividend series. Indeed Harvey and Whaley (1992a, b) and Fleming and Whaley (1994) reduce the index by the discounted flow of dividends during the lifetime of the option.

same time without making specific parametric assumptions. One could consider a nonparametric fit between σ_t and $(S/K)_t$ and past squared returns $(\log S_{t-j} - \log S_{t-j-1})^2$, $j = 1, \ldots, L$, for some finite lag L, resulting in the following $L + 2$-dimensional nonparametric fit:

$$(C/K)_t = \ddot{g}_C \left[(S/K)_t, T - t, \delta_t, (\log S_{t-j} - \log S_{t-j-1})^2, j = 1, 2, \ldots, L \right], \quad (3.5)$$

$$(B/K)_t = \ddot{g}_B \left[T - t, \delta_t, (\log S_{t-j} - \log S_{t-j-1})^2, j = 1, 2, \ldots, L \right], (3.6)$$

considered for instance by Pagan and Schwert (1990). It is clear that this approach is rather unappealing as it would typically require a large number of lags, say $L = 20$ with daily observations. Hence, we face the typical curse of dimensionality problem often encountered in nonparametric analysis.[9] A more appealing way to proceed is to summarize the information contained in past squared returns (possibly the infinite past). We will consider three different strategies: (*a*) historical volatilities, (*b*) EGARCH volatilities and (*c*) implied volatilities. Each are discussed in detail in a first subsection. The final subsection elaborates on nonparametric estimation issues.

3.2.1 Volatility measurement

(*a*) **Historical volatilities.** Practitioners regulary use the most recent past of the quadratic variation of S to extract volatility. Typically, these estimates amount to a 20 or 30 days average of past squared returns. Such a statistic is obviously easy to compute, does not involve any parameters and solves in a rather simple way the curse of dimensionality problem alluded to before. In using historical volatilities, we replace $\sigma(Y_t, Z_t, t)^2$ by $L^{-1} \sum_{j=0}^{L-1} (\log S_{t-j} - \log S_{t-j-1})^2$ and obtain a nonparametric estimation problem similar to that involving dividends. A slightly more complicated scheme, notably appearing in RiskMetrics$^{\text{TM}}$, is to use the infinite past through an exponentially weighted moving average specification. This amounts to

$$\hat{\sigma}_t = \lambda \hat{\sigma}_{t-1} + \lambda(1 - \lambda)(\log S_t - \bar{r}_{t-1})^2,$$

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[9]The nonparametric estimators of regression functions $Y = f(X)$, where X is a vector of dimension d, are *local* smoothers, in the sense that the estimate of f at some point x depends only on the observations (X_i, Y_i) with X_i in a neighborhood $\mathcal{N}(x)$ of x. The so-called curse of dimensionality captures the fact that, if we measure the degree of localness of a smoother by the proportion of observations (X_i, Y_i) for which X_i is in $\mathcal{N}(x)$, then the smoother becomes less local when d increases, in the sense that the $\mathcal{N}(x)$ corresponding to a fixed degree of localness loses its neighbouring property as the dimension of X increases. A consequence of this is that unless the sample size increases drastically, the precision of the estimate deteriorates as we add regressors in f. For more details on the curse of dimensionality and how to deal with it, see Hastie and Tibshirani (1990), Scott (1992, chap. 7) and Silverman (1990, p. 91 – 94). We propose here a different approach.

where $\bar{r}_t = (1-\lambda)\log S_t + \lambda \bar{r}_{t-1}$. Obviously, such a specification involves parameter estimation. One can fix λ at some value, not necessarily obtained via formal statistical estimation.[10] The empirical quantiles of the filtered volatilities are given in Table 2.

(b) **EGARCH volatilities.** The ARCH class of models could be viewed as filters to extract the (continuous time) conditional variance process from discrete time data. Several papers were devoted to the subject, namely Nelson (1990, 1991, 1992, 1996a,b) and Nelson and Foster (1994, 1995), which brought together two approaches, ARCH and continuous time SV, for modelling time-varying volatility in financial markets. Nelson's first contribution in his 1990 paper was to show that ARCH models, which model volatility as functions of past (squared) returns, converge weakly to a diffusion process, either a diffusion for $\log \sigma_t^2$ or a Constant Elasticity of Variance (CEV) process. In particular, it was shown that a GARCH(1,1) model observed at finer and finer time intervals $\Delta t = h$ with conditional variance parameters $\omega_h = h\omega, \alpha_h = \alpha\left(h/2\right)^{\frac{1}{2}}$ and $\beta_h = 1 - \alpha\left(h/2\right)^{\frac{1}{2}} - \theta h$ and conditional mean $\mu_h = hc\sigma_t^2$ converges to a diffusion limit

$$\begin{aligned}
\mathrm{d}\log S_t &= c\,\sigma_t^2 \mathrm{d}t + \sigma_t \mathrm{d}W_t & (3.7) \\
\mathrm{d}\sigma_t^2 &= (\omega - \theta\sigma_t^2)\mathrm{d}t + \sigma_t^2 \mathrm{d}W_t^\sigma. & (3.8)
\end{aligned}$$

Similarly, it was also shown that a sequence of AR(1)-EGARCH(1,1) models converges weakly to an Ornstein-Uhlenbeck diffusion for $\ln \sigma_t^2$

$$\mathrm{d}\ln \sigma_t^2 = \alpha(\beta - \ln \sigma_t^2)\mathrm{d}t + \mathrm{d}W_t^\sigma. \qquad (3.9)$$

These basic insights show that the continuous time stochastic difference equations emerging as diffusion limits of ARCH models were no longer ARCH but instead SV models. Moreover, following Nelson (1992), even when misspecified, ARCH models still keep desirable properties regarding extracting the continuous time volatility. The argument is that for a wide variety of misspecified ARCH models the difference between the (EG)ARCH volatility estimates and the true underlying diffusion volatilities converges to zero in probability as the length of the sampling time interval goes to zero at an appropriate rate. This powerful argument allows us to use the EGARCH model as filter which is not necessarily incompatible with the underlying (unspecified) structural model. Indeed, it is worth noting that setting $c = 1$ in (3.7) and using (3.9) yields

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[10]In the case of RiskMetrics™ for daily data, one sets $\lambda = .94$, a value which we retained for our computations.

Table 1: *EGARCH(1,1) filtering of the volatility*

Parameter	Estimate	Estimated standard error
μ	0.00043	0.00025
ω	-0.93279	0.10280
β	0.89609	0.01133
γ	-0.11361	0.01045
θ	0.22466	0.02026

a stochastic volatility model which falls within the class of processes described by the equilibrium equation for S in appendix A.

Volatilities are extracted using the following AR(1)-EGARCH(1,1) specification:

$$
\begin{aligned}
\ln S_t &= \mu + \ln S_{t-1} + e_t, \\
\ln \sigma_t^2 &= \omega + \beta \sigma_{t-1}^2 + \gamma \frac{e_{t-1}}{\sigma_t} + \theta \left[\frac{|e_{t-1}|}{\sigma_{t-1}} - \sqrt{\frac{2}{\pi}} \right].
\end{aligned}
$$

The estimation from S&P 100 data is summarized in Table 1, while Table 2 provides a summary of the distribution of extracted volatilities.

(c)   Implied volatilities.   Last but not least, we can look through the window of a (modified) Black-Scholes economy pricing formula and compute the implied volatilities from call prices which are quoted on the market. The computation of implied volatilities is discussed in Harvey and Whaley (1992a) and Fleming and Whaley (1994). They do take into account the dividend process. Indeed, they compute the present value of the dividend stream during the life of the option to adjust the index and subsequently apply the (constant volatility) Cox-Ross-Rubinstein algorithm. If there are two state variables, and hence both $\sigma(Y_t, Z_t, t)$ and $\delta(Y_t, Z_t, t)$, we expect that implied volatilities paired with the observed dividend series δ reflect the joint process (Y_t, Z_t). The empirical quantiles of implied volatilities are given in Table 2.

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Table 2: *Empirical quantiles of filtered conditional variances* ($\hat{\sigma}^2$)

Quantiles	Type of volatility extraction		
	$\hat{\sigma}^2_{\text{EGARCH}}$	$\hat{\sigma}^2_{\text{RiskMetrics™}}$	$\hat{\sigma}^2_{\text{Implied}}$
min	0.00005	0.00003	0.00002
5%	0.00006	0.00004	0.00006
25%	0.00008	0.00006	0.00009
50%	0.00010	0.00007	0.00012
75%	0.00013	0.00010	0.00017
95%	0.00023	0.00033	0.00039
max	0.00917	0.00326	0.00364

3.2.2 Estimation issues

The purpose of this section is to point out several issues regarding the nonparametric estimation of

$$(C/K)_t = g_C\big((S/K)_t, T-t, \delta_t, \hat{\sigma}_t\big) \text{ and } (B/K)_t = g_B\big((S/K)_t, T-t, \delta_t, \hat{\sigma}_t\big), \tag{3.10}$$

where σ_t is now replaced by $\hat{\sigma}_t$ which represents any of the volatility estimations discussed in the previous section. It is beyond the scope and purpose of this paper to provide all the technical details. Instead, we will briefly touch on the issues and provide the relevant references to the literature. The purpose of applying nonparametric statistical estimation is to recover g_C or g_B from the data. This estimation method can only be justified if it applies to a situation where the regularity conditions for such techniques are satisfied. To discuss this let us briefly review the context of nonparametric estimation. In general it deals with the estimation of relations such as

$$Y_i = g(Z_i) + u_i, \quad i = 1, \ldots, n, \tag{3.11}$$

where, in the simplest case, $((Y_i, Z_i), i = 1, \ldots, n)$ is a family of i.i.d. pairs of random variables, and $E(u|Z) = 0$, so that $g(z) = E(Y|Z = z)$. The error terms $u_i, i = 1, \ldots, n$, are assumed to be independent, while g is a function with certain smoothness properties which is to be estimated from the data. Several estimation techniques exist, including

kernel-based methods, smoothing splines, orthogonal series estimators such as Fourier series, Hermite polynomials and neural networks, among many others. Most of the applications involving options data cited in the Introduction involve the use of kernel-based methods. Kernel smoothers produce an estimate of g at $Z = z$ by giving more weight to observations (Y_i, Z_i) with Z_i "close" to z. More precisely, the technique introduces a *kernel function*, K, which acts as a weighting scheme (it is usually a probability density function, see Silverman (1986, p. 38)) and a *smoothing parameter* λ which defines the degree of "closeness" or neighborhood. The most widely used kernel estimator of g in (3.11) is the Nadaraya-Watson estimator defined by

$$\hat{g}_\lambda(z) = \frac{\sum_{i=1}^n K\left(\frac{Z_i - z}{\lambda}\right) Y_i}{\sum_{i=1}^n K\left(\frac{Z_i - z}{\lambda}\right)}, \tag{3.12}$$

so that $\left(\hat{g}_\lambda(Z_1), \ldots, \hat{g}_\lambda(Z_n)\right)' = W_n^K(\lambda) Y$, where $Y = (Y_1, \ldots, Y_n)'$ and W_n^K is a $n \times n$ matrix with its (i, j)-th element equal to $K\left(\frac{Z_j - Z_i}{\lambda}\right) / \sum_{k=1}^n K\left(\frac{Z_k - Z_i}{\lambda}\right)$. W_n^K is called the *influence matrix* associated with the kernel K. The parameter λ controls the level of neighboring in the following way. For a given kernel function K and a fixed z, observations (Y_i, Z_i) with Z_i far from z are given more weight as λ increases; this implies that the larger we choose λ, the less $\hat{g}_\lambda(z)$ is changing with z. In other words, the degree of smoothness of $\hat{g}_\lambda$ increases with λ. As in parametric estimation techniques, the issue here is to choose K and λ in order to obtain the best possible fit. Nonparametric estimation becomes more complicated when the errors are not i.i.d. Under general conditions, the kernel estimator remains convergent and asymptotically normal. Only the asymptotic variance is affected by the correlation of the error terms (see for instance Aït-Sahalia (1996) on this matter). It is still not clear in the literature what should be done in this case to avoid over- or undersmoothing.[11] The characterization of the correlation in the data may be problematic in option price applications, however. The

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[11]When the observed pairs of (Y, Z) are drawn from a stationary dynamic bivariate process, Robinson (1983) provides conditions under which kernel estimators of regression functions are consistent. He also gives some central limit theorems which ensure the asymptotic normality of the estimators. The conditions under which these results are obtained have been weakened by Singh and Ullah (1985). These are mixing conditions on the bivariate process (Y, Z). For a detailed treatment, see Györfy *et al.* (1989). This reference (chap. 6) also discusses the choice of the smoothing parameter in the context of nonparametric estimation from time series observations. When the autocorrelation function of u is unknown, one has to make the transformation from sample estimates obtained from a first step smoothing. Altman (1987, 1990) presents some simulations results which show that in some situations, this so called whitening method seems to work relatively well. However there is no general

relevant time scale for the estimation of g is not calendar time, as in a standard time series context, but rather the time to expiration of the contracts which are sampled sequentially through the cycle of emissions. It becomes even more difficult once it is realized that at each time t several contracts are listed simultaneously and trading may take place only in a subset of contracts. To choose the bandwith parameter we followed a procedure called *generalized cross-validation*, described in Craven and Wahba (1979) and used in the context of option pricing in Broadie *et. al.* (1995).

Another technical matter to deal with is the estimation of reduced forms using implied volatilities, historical volatilities or EGARCH volatilities which all amount to different filtering devices to surpass the complicated multidimensional nonparametric fit involving past squared returns. However, choosing and working with a measurement of the latent volatility variable raises a more serious problem of errors in the variables generated by using filtered volatility. There are different ways of dealing with this issue. Some amount to kernel regression estimation procedures proposed by Muus (1994) involving kernels based on a characteristic function specification. As these procedures are rather complicated we will refrain from applying them. More interestingly, Rilstone (1996) studies the generic problem of generated regressors, which is a regressor like $\hat{\sigma}_t$, in a standard kernel-based regression model and shows how it affects the convergence rates of the estimators while maintaining their properties of consistency and asymptotic normality.

3.3   Estimation results

We focus our attention on the OEX contract which was also studied by Harvey and Whaley (1992a, b) and Fleming (1994). The empirical investigation rests on a combination of five different data sets. They are: (1) time series data of the daily closure of the S&P100 Index, (2) data on daily call option prices at the market closure obtained from the Chicago Board Option Exchange (CBOE), (3) observations on the daily exercises of the OEX contract as recorded by the Option Clearing Corporation (OCC), (4) dividend series of the companies listed in the S&P100 Index and (5) series of filtered volatilities described in section 3.2.1.[12] The sample we consider runs from January 3rd 1984 to March

result on the efficiency of the procedure. See also Härdle and Linton (1994, section 5.2) and Andrews (1991, section 6).

[12]The implied volatilities series for the OEX contract is that calculated by Fleming and Whaley (1994). The data refered to in (3) is described in Diz and Finucane (1993), while the dividend series are those calculated by Harvey and Whaley (1992b).

30$^{\text{th}}$ 1990.

We consider first call prices normalized by the strike price K. The degree of moneyness is measured via the ratio S/K. The empirical results are reported in two sets of six figures. To summarize the results we classify the options in three classes of maturity (see *e.g.*, Rubinstein (1985)): (1) very short maturities which are less than one month, denoted TTM1 in the figures, (2) maturities between one and two months, denoted TTM2 and finally (3) maturities between two and three months denoted TTM3. Regarding volatility we classified the data according to the empirical quartiles of the volatility distribution appearing in Table 2. The same strategy is applied to the dividend rate process, except that we took a roughly 50-50 percent cut of the distribution which conveniently was separated as $\delta_t = 0$ versus $\delta_t > 0$, where δ_t denotes observations of the dividend rate. Figure 1 consists of six of graphs. It can be interpreted as a 3×2 matrix, the rows corresponding to the three time-to-maturity classes, TTM1 (top) to TTM3 (bottom), and columns to the two classes of observed dividend rates, $\delta_t = 0$ (left) and $\delta_t > 0$ (right). Each graph contains four curves representing the quartiles of the volatility distribution. Figure 1 covers the case of EGARCH volatilities.

The first thing to note is that the cases $\delta_t = 0$ and $\delta_t > 0$ look quite similar across the different maturities. As time to maturity increases, there is a larger impact of volatility. This is obviously not surprising as the option price is more sensitive to changes in volatility and to the volatility level itself over longer time horizons. What is more surprising perhaps is that, particularly with TTM3, there is a distinct pattern emerging for the fourth volatility quartile while the first three seem to be lumped together. For at-the-money options the difference is roughly a two to three percent upward shift in the price ratio C/K. In section 3.4 we will actually discuss how this translates into actual option prices. For smaller maturities this difference disappears, as expected. The results so far seem to suggest two things: (1) conditioning on δ_t does not displace pricing of options and (2) the volatility effect seems to be present only for large (fourth quartile) volatilities. We also report results using implied volatilities rather than EGARCH ones. These appear in Figure 2 and show that the results are robust with regard to the specification of volatility.
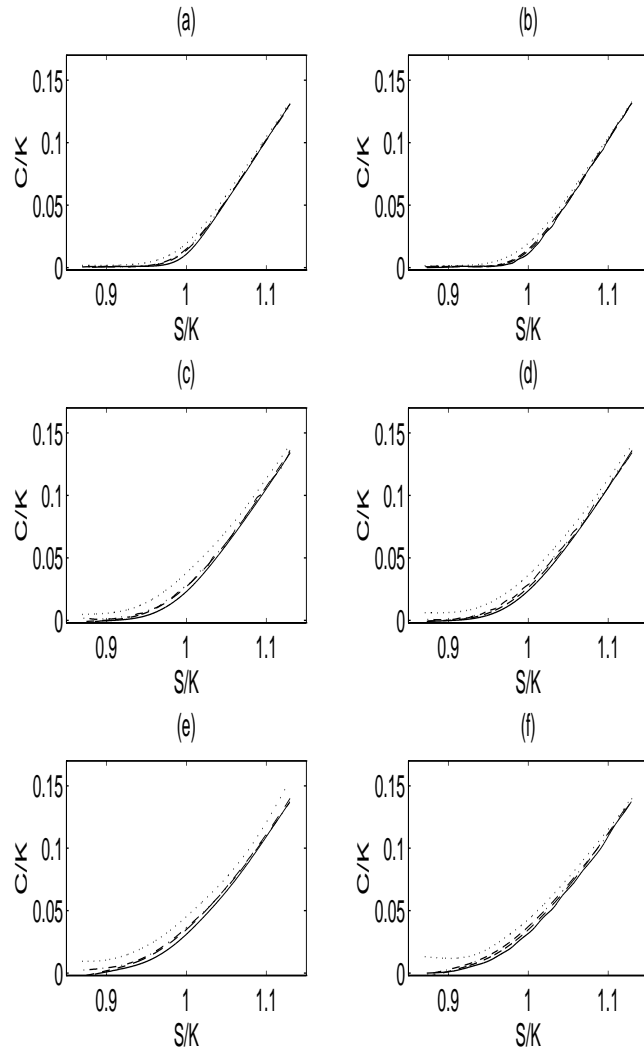
Figure 1: *Estimated call prices conditional on dividend and EGARCH volatility quartiles.* — : *first quartile;* − − : *second quartile;* − · − : *third quartile;* · · · : *fourth quartile.*
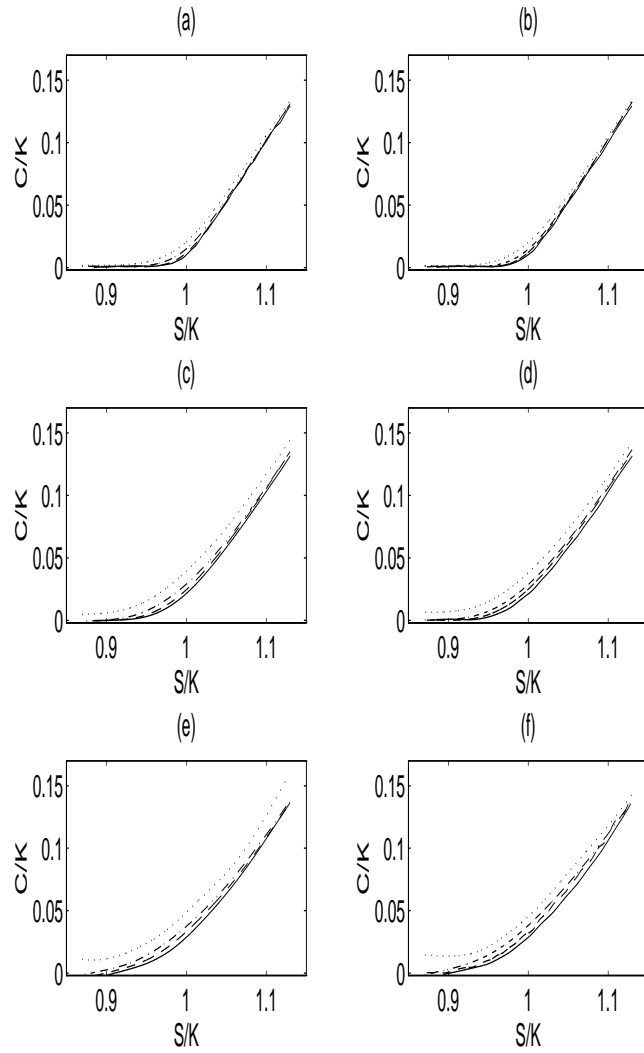
Figure 2: *Estimated call prices conditional on dividend and implied volatility quartiles.* — : *first quartile;* − − : *second quartile;* − · − : *third quartile;* · · · : *fourth quartile.*

Since graphical appearances may be decieving we must rely on explicit statistical testing to find out whether volatility and/or dividends matter in pricing OEX index options. Indeed, the graphs only make the distinction, adopted for convenience, $\delta_t = 0$ versus $\delta_t > 0$. We therefore consider now a formal procedure for testing whether volatility and/or dividends should be included in relations (3.10). Aït-Sahalia, Bickel and Stoker (1994) proposed a test for the exclusion of variables in a regression function estimated by kernel methods. If we consider a relationship like g_C in (3.10), we may wish to test whether the dividend rate δ is a variable which contributes to the variation of (normalized) call prices. We are therefore considering the test of $H_0(\delta) : C/K = g_C^{0\delta}\big((S/K), \tau, \sigma\big)$ against $H_1(\delta) : C/K = g_C\big((S/K), \tau, \sigma, \delta\big)$. Alternatively, we may also test for the presence of an impact of volatility on call prices by considering a test of $H_0(\sigma)$ against $H_1(\sigma)$, where these hypotheses are defined in a similar way reversing the role of δ and σ.

The test statistic proposed by Aït-Sahalia, Bickel and Stoker (1994) is based on the mean square difference of prediction errors by the two competing models $g_C^{0\beta}$ and g_C, $\beta = \delta$ or σ. It is shown that a normalized version of the test statistic is asymptotically normally distributed, under some regularity conditions bearing mainly on the kernel function, the convergence of the bandwidth and the joint distribution of the variables involved in the relation defined by $H_1(\beta)$. The test results appear in Table 3 where t_β represents the statistic used for testing $H_0(\beta)$ against $H_1(\beta)$, where β stands for σ and δ. [13] Since t_β is asymptotically $N(0,1)$ under $H_0(\beta)$ we find a rejection of the null hypothesis in all cases. In other words, neither the volatility nor the dividend rate can be omitted from the relationship g_C in (3.10). Hence, based on this evidence we have to conclude that the emphasis on dividends alone in the pricing of OEX options, as articulated in Harvey and Whaley (1992a, b) and Fleming and Whaley (1994), is not enough to characterize option pricing in this market. [14]

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[13]One regularity condition for applying the tests deserves some attention. Namely, if we consider a test of $H_0(\delta)$ against $H_1(\delta)$, it is clear that the condition that the density of $(C/K, S/K, \sigma, \delta)$ is r (where r is the order of the kernel used in the estimation) times continuously differentiable for some $r \geq 2$ is not met as δ is a random variable for which the value 0 is a mass point. We therefore report in Table 3 two statistics for each test, one based the entire sample and one based on the observation points with $\delta_t > 0$. The latter should in principle not suffer from a mass point accumulation in the data. Fortunately the results are invariant to this issue as can be noted from the table.

[14]One important comment needs to be made to understand the comparison with the Fleming-Harvey-Whaley findings. Namely, there is a difference between our state variable specification and theirs. Indeed, we use concurrent δ_t instead of the future flow of dividend over the lifetime of the option. The Fleming-Harvey-Whaley ap-

Table 3: *Goodness of fit test statistics.*

	t_σ	t_δ
Full sample	675.8855	73.2789
Obs. with $\delta > 0$	378.5861	82.2617

To conclude this section we turn our attention to the data on exercise decisions. Broadie *et. al.* (1995) describe in detail how to extract from the data set observations on excercise decisions. These observations are used to derive a kernel estimate g_B in (3.10). The resulting surface is shown in Figure 3 for different filtered volatilities, taking the implied volatility as a representative example here. [15] We also found, but do not report here for the purpose of streamlining the presentation, that both dividends and volatility again play a significant role (in a statistical sense). It is interesting to study the surface plotted in Figure 3. We notice that the surface is relatively insensitive with respect to volatility, except at the high end scale of volatility. This evidence is in line with the call price functionals which showed an upward shift only for the upper quartile of the volatility distribution. It is important to note that the evidence reported here comes from a very different and separate data set involving observations regarding excercise decisions rather than call prices.

3.4   Nonparametric pricing of American call options

In addition to the statistical issues involved in the specification of an option pricing functional we must also assess option pricing errors. In Table 4 we report the results of numerical computations which compare the pricing of an OEX call using (1) the binomial tree approach, (2) the algorithm for American option pricing developed Broadie and De-temple (1996) and last but not least (3) the nonparametric functionals retrieved from the data. These are respectively denoted Bin, B-D and

proach assumes future dividends to be known to compute their implied volatilities. In practice they have to be predicted. When the autocorrelation function of δ_t is computed we find strong and cyclical autocorrelations. This means that δ_t contains a fair amount of information regarding future dividend payments. This makes our approach a reasonable proxy without having to model explicitly the prediction model for future dividends.

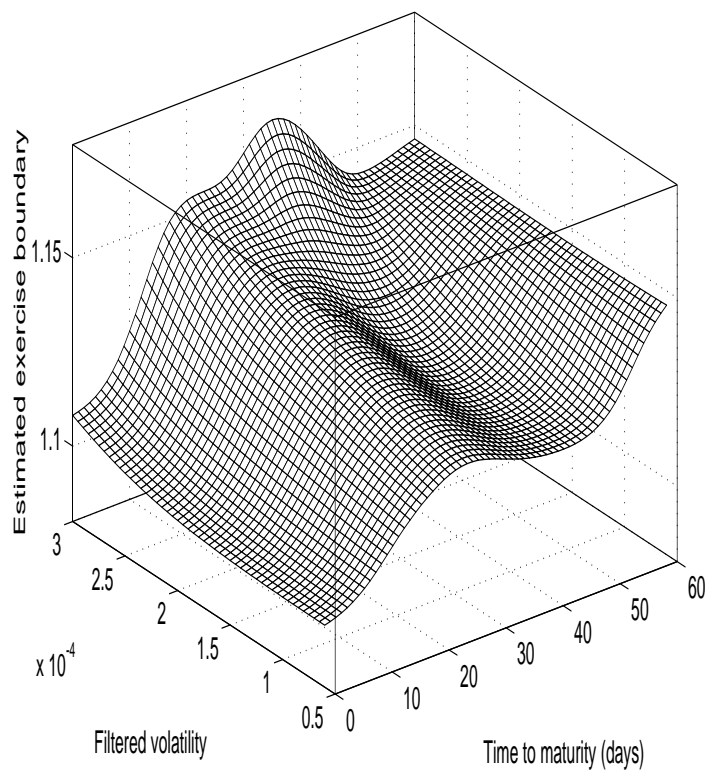[15]Figure 3 does not involve conditioning on values of δ_t.

Figure 3: *Estimated exercise boundary as a function of implied volatility.*

Nonparametric in Table 4. A number of hypothetical situations were postulated for these calculations. First, we examined prices quoted on nondividend paying days. Hence, δ_t is assumed zero and we therefore compare a nonparametric pricing functional which explicitly conditions on this event while the parametric approaches do not. [16] The B-D algorithm for instance assumes that the S&P100 index follows a geometric Brownian Motion with constant volatility and constant dividend flow δ.[17] To deal with volatility we compared two extremes, namely volatility days which reside in the first and fourth quartile of the distribution. [18] From the results in the previous section we know that this amounts to comparing two typical situations which can be characterized as low and high volatilities since the first three quartiles appear to be lumped together (*cfr.* Figures 1 and 2). Moreover, we examined three maturities, namely 28, 56 and 84 days. These are hypothetical TTM specifications falling in the three broad categories we studied. The particular choice of days in inconsequential for our results. The options priced are either at-the-money or else 5 percent in- and out-of-the-money. For the nonparametric pricing scheme we computed the average price over the entire range of the *low* and *high* volatility quartiles while the parametric pricing schemes were computed for the upper and lower limits of the empirical quartile ranges. This provides a pricing bracket which we can compare with the nonparametric results. All the results in Table 4 refer to the ratio C/K and can be easily interpreted in a dollar sense by picking K = 100 for instance.

The results in Table 4 show that parametric models consistently misprice the OEX option. In particular, the average nonparametric price in general fails to belong to the interval prescribed by the parametric models. Moreover, this conclusion is uniform across the parametric models. In addition, for low volatilities we note underpricing by the parametric model for nearly all maturities. [19] In contrast, for high volatility we note that the nonparametric pricing schemes belong to the parametric range for medium maturities (56 days and 84 days) while the parametric models overprice for short maturities out- or at-the-money options. The magnitude of the errors can be considerable. Taking $K = 100$ we note that they may be 20 cents or more per contract. In percentage terms the

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[16]Results pertaining to $\delta_t > 0$ are not reported but yield to conclusions similar to those we report for $\delta_t = 0$.

[17]The dividend rate was set equal to the sample average of the S&P100 dividend series contructed by Harvey and Whaley (see also Broadie *et. al.* (1995) for more details).

[18]All calculations in Table 4 are made with the EGARCH volatility estimates.

[19]An exception are the out-of-the money short maturity options.

Table 4: *American call option normalized prices (C/K).*

(a) Time to maturity = 28 days

Volatility		Low volatilities			High volatilities		
Moneyness		out	at	in	out	at	in
Nonparametric		0.00114480	0.01666828	0.05533171	0.00285930	0.02091207	0.05696213
Bin	lower	0.00096633	0.01371195	0.05142301	0.00389952	0.02006459	0.05465180
	upper	0.00174318	0.01579428	0.05228001	0.00824606	0.02660485	0.05931876
B-D	lower	0.00097033	0.01372321	0.05142182	0.00390517	0.02008115	0.05464477
	upper	0.00174216	0.01580728	0.05228078	0.00823746	0.02662683	0.05930739

(b) Time to maturity = 56 days

Volatility		Low volatilities			High volatilities		
Moneyness		out	at	in	out	at	in
Nonparametric		0.00574492	0.02590474	0.06231479	0.01194974	0.03423728	0.06610578
Bin	lower	0.00351903	0.01945440	0.05436309	0.00955299	0.02839967	0.06074075
	upper	0.00528273	0.02238628	0.05626256	0.01698366	0.03761091	0.06863641
B-D	lower	0.00351797	0.01947021	0.05436360	0.00954500	0.02842289	0.06075945
	upper	0.00529623	0.02240454	0.05625452	0.01697063	0.03764160	0.06861070

(c) Time to maturity = 84 days

Volatility		Low volatilities			High volatilities		
Moneyness		out	at	in	out	at	in
Nonparametric		0.01017253	0.03236089	0.0684330	0.02033059	0.04318956	0.07282010
Bin	lower	0.00625150	0.02386238	0.05735078	0.01458586	0.03477430	0.06610690
	upper	0.00881129	0.02743838	0.06003424	0.02425917	0.04601170	0.07630737
B-D	lower	0.00624208	0.02388162	0.05735069	0.01459030	0.03480239	0.06611287
	upper	0.00879815	0.02746057	0.06003006	0.02426214	0.04604874	0.07628135

pricing errors sometimes exceed 30 to 40 percent of the price. Needless to say that such differences are very significant in the pricing of these options.

4   Conclusion

We considered American option contracts when the underlying asset or index has stochastic dividends and stochastic volatility. This situation is quite common in financial markets and generalizes many cases studied in the literature so far. The theoretical models which were derived in section 2 yield fairly complex expressions which are difficult to compute. It motivated us to adopt a nonparametric approach to estimate call prices and exercise boundaries conditionnal on dividends and volatility. Using data from the OEX contract we find that dividend payments are important, confirming earlier results of Harvey and Whaley (1992a,b) and Fleming and Whaley (1994), but also uncover a significant volatility effect hitherto ignored in the literature on American options. In that respect our results join the extensive efforts undertaken in the case of

European style options. Yet, the nonparametric approach we present is more flexible since it does not require the specification of an explicit model for the underlying index. This flexibility inherent in the nonparametric approach applies to American and European contracts, or even more exotic option designs provided a sufficiently active market yields enough data to compute the estimates. It allowed us to uncover a rather interesting effect of volatility on option pricing in the case of the OEX contract. Indeed, it appears that OEX option prices are relatively insensitive to volatility movements except when the latter starts to behave in the extreme upper end of the distribution. Our approach also joins the recent efforts of applying nonparametric methods to option pricing. Yet the analysis in this paper is novel since it extends the domain of application of the nonparametric approach to stochastic volatility and to a class of contracts which involve both exercise timing decisions and pricing determination. The method proposed in this paper has also substantial practical applications for users of OEX options. In that regard knowledge of the empirical exercise boundary and the pricing function can help in decisions involving the purchase of the OEX contract or its exercise prior to maturity.

A Stock, bond and contingent claim valuation with stochastic volatility

In this appendix, we develop a consistent model of contingent claim valuation with stochastic volatility of the underlying stock price and a stochastic dividend rate. Our general equilibirum approach endogenizes the equilibrium value of the stock, its dividend rate and volatility coefficients, the interest rate and the values of contingent claims. Equilibrium values are expressed as functions of the primitives of the economy: the dividend process on the stock and the preferences of the representative agent. The model is sufficiently general to deliver the basic stylized facts which characterize for instance the S&P100 Index process: stochastic volatility and stochastic, imperfectly correlated, dividend rate. The economy is described in the first subsection; the second subsection covers the equilibrium properties of stocks and derivative securities.

A.1 The Economy

We consider a continous time pure exchange economy with a representative agent and a finite time period $[0, T]$. The uncertainty is represented by a three-dimensional Brownian Motion process $W \equiv (W_1, W_2, W_3)$ defined on a probability space $(\Omega, \mathcal{F}, P)$. The information structure of the representative agent is the filtration generated by W. The economy has a financial market with two primitive assets, a risky asset (stock) and an instantaneously riskless bond. The risky stock is in unit supply. It generates a flow of dividend payments D which satisfies the stochastic differential equation

$$\mathrm{d}D_t = D_t[\mu^D(Y_t, Z_t, t)\mathrm{d}t + \rho^D(Y_t, Z_t, t)\mathrm{d}W_{1t}], \quad t \in [0, T], \qquad (A.1)$$

$$\mathrm{d}Y_t = \mu^Y(Y_t, t)\mathrm{d}t + \sigma_1^Y(Y_t, t)\mathrm{d}W_{1t} + \sigma_2^Y(Y_t, t)\mathrm{d}W_{2t}, \quad t \in [0, T], \qquad (A.2)$$

$$\mathrm{d}Z_t = \mu^Z(Z_t, t)\mathrm{d}t + \sigma_1^Z(Z_t, t)\mathrm{d}W_{1t} + \sigma_3^Z(Z_t, t)\mathrm{d}W_{3t}, \quad t \in [0, T], \qquad (A.3)$$

where D_0, Y_0 and Z_0 are given. The variables Y and Z are state variables which captures the stochastic fluctuations in the volatility coefficient of the dividend process. The drift is also affected by Y and Z. The price of the stock, S, satisfies the stochastic differential equation

$$\mathrm{d}S_t + \mathrm{d}D_t = S_t[\mu_t\mathrm{d}t + \sigma_{1t}\mathrm{d}W_{1t} + \sigma_{2t}\mathrm{d}W_{2t} + \sigma_{3t}\mathrm{d}W_{3t}], \quad t \in [0, T], \; (A.4)$$

and has an initial value S_0. The initial value S_0 and the coefficients $(\mu, \sigma_1, \sigma_2, \sigma_3)$ which appear in (A.4) are determined in equilibrium.

The riskless bond with instantaneous maturity is in zero net supply. It pays an interest rate r per unit time which is also determined in equilibrium.

The representative agent has preferences represented by the von Neumann-Morgenstern index

$$U(c) = E\left[\int_0^T u(Z_t, c_t, t)\mathrm{d}t\right],\qquad(A.5)$$

where $u(Z, c, t)$ is a state-dependent instantaneous utility function and Z represents a utility shock. The function $u(\,\cdot\,)$ satisfies standard conditions: it is twice continuously differentiable with respect to c, strictly concave and increasing with respect to c and has the limiting values $\lim_{c\uparrow\infty} u'(Z, c, t) = 0$ and $\lim_{c\downarrow 0} u'(Z, c, t) = \infty$, for all $Z \in \mathbb{R}^+$ and $t \in [0, T]$.

The preference model (A.3),(A.5) is fairly general. It includes, in particular, the standard model with constant subjective discount rate $u(c, t) = e^{-\beta t} u(c)$ which is obtained by setting $\mu^Z = -Z_t\beta$, $\sigma_1^Z = \sigma_3^Z = 0$ and $u(Z, c, t) = Zu(c)$. It also includes models with stochastic discount rate obtained for $u(Z, c, t) = e^{\int_0^t Z_\nu \mathrm{d}\nu} u(c)$ where Z follows the stochastic process (A.3). The consideration of state-dependent utility functions gives us the additional degree of freedom required to model equilibrium dividend rate processes ($\delta_t \equiv D_t/S_t$) which are stochastic and partially correlated with the price volatility process.

The representative agent consumes and invests in the stock and the riskless asset. A consumption policy is a progressively measurable process c such that $\int_0^T c_v \mathrm{d}v < \infty$, ($P$-a.s.). An investment policy is a progressively measurable process π such that $\int_0^T \pi_v^2(\sigma_{1v}^2 + \sigma_{2v}^2)\mathrm{d}v < \infty$, ($P$-a.s.). Here π represents the investment in the stock. The investment in the bond is $X - \pi$ where X denotes the wealth of the agent. A consumption-investment policy (c, π) generates the wealth process

$$\mathrm{d}X_t = [r_t X_t - c_t]\mathrm{d}t + \pi_t\left[(\mu_t - r_t)\mathrm{d}t + \sigma_{1t}\mathrm{d}W_{1t} + \sigma_{2t}\mathrm{d}W_{2t} + \sigma_{3t}\mathrm{d}W_{3t}\right], \quad t \in [0, T],$$
$$(A.6)$$

where X_0 is given. An investment policy is admissible if the associated wealth process satisfies

$$X_t \geq 0, \quad t \in [0, T].\qquad(A.7)$$

A consumption-investment policy (c, π) is optimal for the preferences $U(c)$ if it cannot be dominated by any alternative admissible consumption-investment policy. A collection of processes $((S, r), c, \pi)$ is an equilibrium if (i) taking prices as given the policy (c, π) is optimal for the agent and (ii) markets clear: $c = D$ and $X - \pi = 0$.

A.2 The equilibrium values of the stock, the interest rate and contingent claims

In this subsection we first provide equilibrium valuation formulas for the stock, the bond and zero net supply contingent claims for the general economy described above (Theorems A.1 and A.2 and corollary A.1). We then examine conditions on the primitives of the model under which the stock price has various stochastic volatility structures and dividend rate processes while the equilibrium interest rate is constant. These conditions lead to the canonic market model which serves as our starting point in Section 2 for the valuation of American options and to the various reduced forms which are tested in Section 3

Our first two Theorems state standard pricing results which hold in pure exchange economies [see *e.g.* Lucas (1978), Duffie and Zame (1989), Karatzas, Lehoczky and Shreve (1990), Detemple and Zapatero (1991); see also Cox, Ingersoll and Ross (1985) for production economies]. Let $b_t \equiv \exp\left(-\int_0^t r_\nu \mathrm{d}\nu\right)$ denote the discount factor for date t cash flows. Similarly, let $b_{t,\nu} \equiv b_\nu/b_t$ denote the discount factor at date t for cash flows received at date ν. We also use the notation $u_1 \equiv \partial u/\partial Z$, $u_2 \equiv \partial u/\partial c$, $u_{11} \equiv \partial^2 u/\partial Z^2$, *etc*, where ∂u denotes the partial derivatives of the utility function. We have,

Theorem A.1 *Consider the economy with stochastic dividend process (A.1)-(A.3) described above. The equilibrium interest rate is given by*

$$r_t = -\frac{u_{21}}{u_2}\mu^Z(Z_t, t) - \frac{u_{22}}{u_2}D_t\mu^D(Y_t, Z_t, t) - \frac{1}{2}\frac{u_{211}}{u_2}\left[\sigma^Z(Z_t, t)\right]^2$$

$$-\frac{1}{2}\frac{u_{222}}{u_2}\left[D_t\rho^D(Y_t, Z_t,, t)\right]^2 - \frac{u_{221}}{u_2}\sigma_1^Z(Z_t, t)\rho^D(Y_t, Z_t, t)D_t, \quad \text{(A.8)}$$

where $\left[\sigma^Z(Z_t, t)\right]^2 \equiv \sigma_1^Z(Z_t, t)^2 + \sigma_3^Z(Z_t, t)^2$. *The price of the dividend paying asset is given by*

$$S_t = E_t^Q\left[\int_t^T b_{t,\nu}D_\nu \mathrm{d}\nu\right], \quad \text{(A.9)}$$

for $t \in [0, T]$. *The expectation in (A.9) is taken relative to the equilibrium equivalent martingale measure based on the equilibrium market prices of risk*

$$\theta_{1t} = -\frac{u_{22}}{u_2}D_t\rho^D(Y_t, Z_t, t) - \frac{u_{21}}{u_2}\sigma_1^Z(Z_t, t), \quad \text{(A.10)}$$

$$\theta_{2t} = 0, \quad \text{(A.11)}$$

$$\theta_{3t} = -\frac{u_{21}}{u_2}\sigma_3^Z(Z_t, t), \quad \text{(A.12)}$$

for all $t \in [0, T]$. The equilibrium risk premium on the stock is given by

$$\mu_t - r_t = \sum_{i=1}^{3} \theta_{it}\sigma_{it}, \quad t \in [0, T]. \tag{A.13}$$

In the competitive equilibrium of Theorem A.1 zero net supply contingent claims can be easily valued. Suppose that we add to the basic economic model of Section A.1 a zero net supply claim with maturity date T_1, terminal cash-flow B and flow of payments $\mathrm{d}f_\nu$, $\nu \in [0, T_1]$. For this asset it is easy to show that,

Theorem A.2 *Consider the economy with stochastic dividend process (A.1)-(A.3) and suppose that a zero net supply contingent claim with characteristics (f, B, T_1) is marketed. The equilibrium value V of the contingent claim is given by*

$$V_t = E_t^Q \left[\int_t^{T_1} b_{t,v} \,\mathrm{d}f_v + b_{t,T_1} B \right], \tag{A.14}$$

for $t \in [0, T]$. The expectation in (A.14) is taken relative to the equilibrium equivalent martingale measure based on the equilibrium market prices of risk (A.10)-(A.12) and the discount factor is based on the equilibrium interest rate (A.8). The equilibrium risk premium on a zero net supply contingent claim with volatility coefficients ρ_1, ρ_2 and ρ_3 is

$$\alpha_t - r_t = \sum_{i=1}^{3} \theta_{it}\rho_{it}, \quad t \in [0, T], \tag{A.15}$$

where α represents the drift of the contingent claim price.

Let us consider the stock in the economy of Theorem A.1. Applying the representation formula of Theorem A.2 and simplifying leads to the following expressions for the equilibrium stock price

$$S_t = E_t^Q \left[\int_t^{T} b_{t,v} D_v \,\mathrm{d}v \right] \tag{A.16}$$

$$= E_t^Q \left[\int_t^{T} \exp\left(-\int_t^{v} r_s \,\mathrm{d}s \right) D_t \exp\left[\int_t^{v} \left(\mu_s^D - \frac{1}{2}\rho_s^{D\,2} \right) \mathrm{d}s + \int_t^{v} \rho_s^D \,\mathrm{d}W_{1s} \right] \mathrm{d}v \right] \tag{A.17}$$

(by definition (A.1))

$$= D_t E_t^Q \left[\int_t^{T} \exp\left[\int_t^{v} -\left(r_s - \mu_s^D + \theta_{1s}\rho_s^D + \frac{1}{2}\rho_s^{D\,2} \right) \mathrm{d}s + \int_t^{v} \rho_s^D \,\mathrm{d}W_{1s}^* \right] \mathrm{d}v \right] \tag{A.18}$$

where $\mathrm{d}W_{1t}^* \equiv \mathrm{d}W_{1t} + \theta_{1t}\mathrm{d}t$

$$\equiv D_t W(D_t, Y_t, Z_t, t), \qquad (\text{A.19})$$

where $W(D, Y, Z, t)$ denotes the conditional expectation appearing in the previous line. Note that this function depends on the level of the dividend payment, D, because the equilibrium interest rate in (A.8) depends on D for a sufficiently general specification of preferences. The equilibrium dividend rate δ is given by

$$\delta_t = \delta(D_t, Y_t, Z_t, t) \equiv \frac{D_t}{S_t} = W(D_t, Y_t, Z_t, t)^{-1}. \qquad (\text{A.20})$$

Summarizing, we have

Corollary A.1 *In the equilibrium of Theorem A.1, the stock price is*

$$S_t = D_t W(D_t, Y_t, Z_t, t) \qquad (\text{A.21})$$

where

$$W(D_t, Y_t, Z_t, t)$$

$$\equiv E_t^Q \left[\int_t^T \exp\left[\int_t^v -\left(r_s - \mu_s^D + \theta_{1s}\rho_s^D + \frac{1}{2}\rho_s^{D\,2}\right)\mathrm{d}s + \int_t^v \rho_s^D \mathrm{d}W_{1s}^* \right] \mathrm{d}v \right] \quad (\text{A.22})$$

where r and θ_1 are given in (A.8) and (A.10) respectively. The stock price satisfies the following stochastic differential equation

$$\mathrm{d}S_t = S_t \left[\left(r_t - \delta(D_t, Y_t, Z_t, t)\right)\mathrm{d}t + \sigma_{1t}\mathrm{d}W_{1t}^* + \sigma_{2t}\mathrm{d}W_{2t}^* + \sigma_{3t}\mathrm{d}W_{3t}^* \right], \qquad (\text{A.23})$$

where the volatility coefficients can be written as

$$\sigma_{1t} = \left(1 + \frac{W_D}{W}\right)\rho^D(Y_t, Z_t, t) + \frac{W_Y}{W}\sigma_1^Y(Y_t, t) + \frac{W_Z}{W}\sigma_1^Z(Z_t, t), \quad (\text{A.24})$$

$$\sigma_{2t} = \frac{W_Y}{W}\sigma_2^Y(Y_t, t), \qquad (\text{A.25})$$

$$\sigma_{3t} = \frac{W_Z}{W}\sigma_3^Z(Y_t, t). \qquad (\text{A.26})$$

For economies in which the interest rate is independent of the dividend level, the equilibrium dividend rate becomes

$$\delta_t = \delta(Y_t, Z_t, t) = W(Y_t, Z_t, t)^{-1} \qquad (\text{A.27})$$

a function of (Y, Z) solely. The term W_D in the volatility expression (A.24) is then equal to zero. This property of the equilibrium holds,

for instance, when preferences are of the power form with multiplicative state variable effect: $u(Z, c, t) = v(Z)\frac{1}{\gamma}c^\gamma$.

Canonic option pricing models assume that the interest rate is constant. In the economic context above, this amounts to the further restriction

$$\frac{u_{21}}{u_2}\mu^Z - \frac{u_{22}}{u_2}D\mu^D - \frac{1}{2}\frac{u_{211}}{u_2}\sigma^{Z2} - \frac{1}{2}\frac{u_{222}}{u_2}D^2\rho^{D2} - \frac{u_{221}}{u_2}\sigma_1^Z\rho^D D = \lambda \quad (A.28)$$

for some constant λ. This restriction is a joint condition on the preferences of the representative agent and on the structure of the dividend process (A.1)-(A.3). If (A.28) holds, we obtain the following model for our primary securities (under the pricing measure Q) :

model 1:

$$dS_t = S_t\left[\left(r - \delta(Y_t, Z_t, t)\right)dt + \sigma_1(Y_t, Z_t, t)dW_{1t}^* + \sigma_2(Y_t, Z_t, t)dW_{2t}^*\right.$$
$$\left. + \sigma_3(Y_t, Z_t, t)dW_{3t}^*\right],$$
$$dY_t = \left(\mu^Y(Y_t, t) - \theta_{1t}\sigma_1^Y(Y_t, t)\right)dt + \sigma_1^Y(Y_t, t)dW_{1t}^* + \sigma_2^Y(Y_t, t)dW_{2t}^*,$$
$$dZ_t = \left(\mu^Z(Z_t, t) - \theta_{1t}\sigma_1^Z(Z_t, t) - \theta_{3t}\sigma_3^Z(Z_t, t)\right)dt + \sigma_1^Z(Z_t, t)dW_{1t}^*$$
$$+ \sigma_3^Z(Z_t, t)dW_{3t}^*,$$

where $(\theta_1, \theta_2, \theta_3)$ are given in (A.10)-(A.12), $\delta(Y, Z, t) \equiv W(Y, Z, t)^{-1}$ and r is constant. The volatility coefficients of the stock are

$$\sigma_{1t} = \rho^D(Y_t, Z_t, t) + \frac{W_Y}{W}\sigma_1^Y(Y_t, t) + \frac{W_Z}{W}\sigma_1^Z(Z_t, t),$$

$$\sigma_{2t} = \frac{W_Y}{W}\sigma_2^Y(Y_t, t),$$

$$\sigma_{3t} = \frac{W_z}{W}\sigma_3^Z(Y_t, t).$$

Model 1 is fairly general to the extent that both the dividend rate and the volatility coefficients of the stock price depend on Y and Z.

This is the general model which underlies our treatment of American options in Section 2 and our econometric investigation in Section 3.

In Section 3 we are led to consider various reduced forms which are subcases of model 1. In the remainder of this Appendix we explore conditions on the structure of the economy which give rise to those special cases.

Suppose that in addition to (A.28) we also require

$$\mu_t^D - \theta_{1t}\rho_t^D = \phi(Y_t, t), \qquad (A.29)$$

$$\rho_t^D = \rho^D(Y_t, t), \qquad (A.30)$$

$$\theta_{1t} = \psi(Y_t, t), \qquad (A.31)$$

where ϕ and ψ are functions of the state variables Y but not Z. Then, it can be verified from (A.22) that $W = W(Y_t, t)$ and that the volatility coefficients (σ_1, σ_2) are functions of Y alone while $\sigma_3 = 0$. Thus, our first subcase is

model 2:

$$\begin{aligned}
\mathrm{d}S_t &= S_t\left[\left(r - \delta(Y_t, t)\right)\mathrm{d}t + \sigma_1(Y_t, t)\mathrm{d}W_{1t}^* + \sigma_2(Y_t, t)\mathrm{d}W_{2t}^*\right], \\
\mathrm{d}Y_t &= \left(\mu^Y(Y_t, t) - \theta_{1t}\sigma_1^Y(Y_t, t)\right)\mathrm{d}t + \sigma_1^Y(Y_t, t)\mathrm{d}W_{1t}^* + \sigma_2^Y(Y_t, t)\mathrm{d}W_{2t}^*,
\end{aligned}$$

where θ_1 is given by (A.31), $\delta(Y, t) \equiv W(Y, t)^{-1}$ and r is constant. The volatility coefficients of the stock are

$$\begin{aligned}
\sigma_{1t} &= \rho^D(Y_t, t) + \frac{W_Y}{W}\sigma_1^Y(Y_t, t), \\
\sigma_{2t} &= \frac{W_Y}{W}\sigma_2^Y(Y_t, t), \\
\sigma_{3t} &= 0.
\end{aligned}$$

This model underlies the reduced form specification (3.4) which is discussed in Section 3.

An alternative case of interest is when (assuming that (A.28) also holds)

$$\mu_t^D - \theta_{1t}\rho_t^D = \phi(Z_t, t), \qquad (A.32)$$

$$\rho_t^D = \rho^D(Y_t, t), \qquad (A.33)$$

$$\theta_{1t} = \psi(Y_t, t), \qquad (A.34)$$

$$\sigma_{1t}^Z = 0, \qquad (A.35)$$

Condition (A.32) is satisfied for the multiplicative power utility $u(Z, c) = v(Z)\frac{1}{\gamma}c^\gamma$ provided that the drift of the dividend process $\mu^D(Y, Z, t)$ has the appropriate structure. Condition (A.35) implies that Y and Z are independent processes under the P-measure. Independence under P combined with multiplicative power utility function ensures that the market price of risk θ_1 satisfies (A.34). Note that this preference structure also implies that θ_3 is a function of Z alone: the processes Y and Z are then also independent under the pricing measure Q.

From (A.32)-(A.35) we can write

$$W(Y_t, Z_t, t)$$

$$= E_t^Q \left[\int_t^T \exp\left(-r(v-t) - \int_t^v \phi(Z_s, s)\mathrm{d}s - \frac{1}{2}\int_t^v \rho_s^{D\,2}\mathrm{d}s + \int_t^v \rho_s^D \mathrm{d}W_{1s}^* \right) \mathrm{d}v \right]$$

$$= E_t^Q \left[\int_t^T \exp\left(-r(v-t) - \int_t^v \phi(Z_s, s)\mathrm{d}s \right) E_t^Q \left[\exp\left(-\frac{1}{2}\int_t^v \rho_s^{D\,2}\mathrm{d}s \right. \right. \right.$$

$$\left. \left. \left. + \int_t^v \rho_s^D \mathrm{d}W_{1s}^* \right) |\mathcal{F}_v^Z \right] \mathrm{d}v \right]$$

$$= E_t^Q \left[\int_t^T \exp\left(-r(v-t) - \int_t^v \phi(Z_s, s)\mathrm{d}s \right) \mathrm{d}v \right]$$

$$= W(Z_t, t).$$

In the equality above we used the measurability of the first exponential with respect to $\mathcal{F}_v^Z$. The third equality follows from the Q independence of Y and Z and the martingale property of the exponential in question. Our model 3 then reads

model 3:

$$\begin{aligned}
\mathrm{d}S_t &= S_t \left[\left(r - \delta(Z_t, t) \right)\mathrm{d}t + \sigma_1(Y_t, t)\mathrm{d}W_{1t}^* + \sigma_3(Z_t, t)\mathrm{d}W_{3t}^* \right], \\
\mathrm{d}Y_t &= \left(\mu^Y(Y_t, t) - \theta_{1t}\sigma_1^Y(Y_t, t) \right)\mathrm{d}t + \sigma_1^Y(Y_t, t)\mathrm{d}W_{1t}^* + \sigma_2^Y(Y_t, t)\mathrm{d}W_{2t}^*, \\
\mathrm{d}Z_t &= \left(\mu^Z(Z_t, t) - \theta_{3t}\sigma_3^Z(Z_t, t) \right)\mathrm{d}t + \sigma_3^Z(Z_t, t)\mathrm{d}W_{3t}^*,
\end{aligned}$$

where θ_1 is given in (A.34) and θ_3 in (A.12), $\delta(Z, t) \equiv W(Z, t)^{-1}$ and r is constant. The volatility coefficients of the stock are

$$\begin{aligned}
\sigma_{1t} &= \rho^D(Y_t, t), \\
\sigma_{2t} &= 0, \\
\sigma_{3t} &= \frac{W_Z}{W}\sigma_3^Z(Z_t, t).
\end{aligned}$$

In this model the dividend rate is stochastic and depends on Z alone while the volatility of the stock depends both on Y and Z. A subcase of this model is when $W = W(t)$ is independent of both Y and Z. Then the dividend rate is a function of time alone and $\sigma_3 = 0$. This subcase is the model with pure volatility risk (and no dividend yield risk).

model 4:

$$\begin{aligned}
\mathrm{d}S_t &= S_t \left[\left(r - \delta(t) \right)\mathrm{d}t + \sigma_1(Y_t, t)\mathrm{d}W_{1t}^* \right], \\
\mathrm{d}Y_t &= \left(\mu^Y(Y_t, t) - \theta_{1t}\sigma_1^Y(Y_t, t) \right)\mathrm{d}t + \sigma_1^Y(Y_t, t)\mathrm{d}W_{1t}^* + \sigma_2^Y(Y_t, t)\mathrm{d}W_{2t}^*,
\end{aligned}$$

where θ_1 is given in (A.34), $\delta(t) = W(t)^{-1}$ and r is constant. The volatility coefficients of the stock are

$$\sigma_1(Y_t, 1) = \rho^D(Y_t, t),$$

$$\sigma_2 = 0 \text{ and } \sigma_3 = 0.$$

Proof of Theorem 2.2: Define $w_{t,v}^2 \equiv (a_{t,v})^{-1} \left[\int_t^v \sigma_1(Y_u, u) \mathrm{d}B_u^* \right]$. Using (2.22) - (2.23) and the definition of $w_{t,v}^2$ enables us to write the solution of (2.20) as:

$$S_v = S_t \exp \left[(r - \delta)(v - t) - \frac{1}{2} a_{t,v}^2 + \rho a_{t,v} w_{t,v}^1 + \sqrt{1 - \rho^2} \, a_{t,v} w_{t,v}^2 \right].$$

Note that the event $\left\{ S_v \geq B_v(Y_v, v) \right\}$ is equivalent to $\left\{ w_{t,v}^2 \geq -d(S_t, B_v, a_{t,v}, \rho, w_{t,v}^1) + \sqrt{1 - \rho^2} \, a_{t,v} \right\}$, where the function $d(\cdot)$ is defined in (2.24)-(2.25). Since $w_{t,v}^2$ has a standard normal distribution conditional on the trajectories of Y we can first integrate the representations (2.14) and (2.15) with respect to $w_{t,v}^2$ conditionally on $\{Y_s : s \in [t, v]\}$, and then integrate over the trajectories of Y. This leads to the expressions in the Theorem.

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