On Polytopes Arising in Cluster Algebras & Finite Frames

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Universität Bremen

On Polytopes Arising in Cluster Algebras & Finite Frames

Dissertation

zur Erlangung des akademischen Grades

Doktor der Naturwissenschaften
(Dr. rer. nat.)

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vorgelegt am 29.09.2015

Tag des Kolloquiums:
08. Februar 2016

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Zusammenfassung

Polytope tauchen in vielen Zusammenhängen auf, zwei davon sind Cluster-Algebren und endliche Frames. Im ersten Teil untersuchen wir graphentheoretische Eigenschaften von Polytopen, die im Zusammenhang mit Cluster-Algebren von endlichem Typ auftauchen. Wir führen die grundlegenden Begriffe und Konstruktionen für Cluster-Algebren von endlichem Typ ein, betrachten dann deren Austauschgraphen und formulieren eine Vermutung über die Hamiltonizität dieser Graphen. Im zweiten Teil untersuchen wir Polytope, welche in der Konstruktion von endlichen Frames mit gegebenen Längen der Framevektoren und gegebenem Spektrum des Frameoperators auftreten. Nach einer Einführung in endliche Frames geben wir für straffe Frames konstanter Norm eine nicht redundante Beschreibung der Polytope durch Gleichungen und Ungleichungen an. Daraus leiten wir die Dimension und die Anzahl der Facetten der Polytope ab. Dabei erhalten wir kombinatorisch zwei Isomorphismen zwischen Polytopen, die zu Frames gehören. Anschließend diskutieren wir, wie diese Isomorphismen durch die Umkehrung der Reihenfolge der Framevektoren und das Naimark-Komplement beschrieben werden.

Abstract

Polytopes appear in many contexts, two being cluster algebras and finite frames. At first we study graph theoretic properties of polytopes arising in the context of cluster algebras of finite type. We introduce the basic terms and constructions for cluster algebras of finite type, then we consider their exchange graphs and give a conjecture about the Hamiltonicity of the exchange graphs. Then we study polytopes, which arise in the construction of finite frames with given lengths of frame vectors and given spectrum of the frame operator. After an introduction to finite frames, we give a non-redundant description of those polytopes for equal norm tight frames in terms of equations and inequalities. From this, we derive the dimension and number of facets of the polytopes. In this process we combinatorially obtain two isomorphisms between polytopes associated to frames. Afterwards we discuss how these isomorphisms are described by reversing the order of frame vectors and taking Naimark complements.

Acknowledgements

Graduating and writing a thesis is not a task one can do without the support and guidance of others. Therefore, I want to express my appreciation and thanks to all those people, without whom this work would not have been possible.

At first I want to thank my advisor Eva-Maria Feichtner for her continuous support through the past years and the opportunity to learn interesting and beautiful mathematics.

Many thanks to all my colleagues at **ALTA**, past and present, for a fruitful atmosphere, insights in many different topics of mathematics and the countless hours of discussions about topics besides math.

I thank my coauthor Christoph Pegel with whom I wrote the article *Polytopes of Eigensteps of Equal Norm Tight Frames*.

I also want to appreciate the work of those who proofread the manuscript of this thesis. It is not an easy task to read all the formulas and the text on a topic you are not into. Thanks, Mareike Bötcher, Jan-Philipp Litza, Christoph Pegel and Ingo Sieverdingbeck.

The participants of the workshop *Frames and Algebraic & Combinatorial Geometry* deserve my thanks for their feedback to the work presented in the fourth chapter, as they gave me more insight into the connections between frames and geometry.

My family, in particular my parents, deserve my thanks for their support through all the years. Also I am grateful to Anneke Bötcher, whose love not only brightened my life but gave me the strength to go through this project.

Contents

1 Introduction

From elementary school to latest research, polytopes arise as fundamental geometric objects, which have been studied since ancient times. In many different branches of mathematics, from linear programming and combinatorial optimization to algebraic topology and algebraic geometry, polytopes play an important role. They were first studied as polygons in two-dimensional space and later as polyhedra in three dimensions. Geometry in higher dimensions was not studied until the 19th century. Then Ludwig Schläfli introduced so called *polyschemes*, now called polytopes, in his book *Theorie der vielfachen Kontinuität* [Sch01, §10]. The book was not published until 1901. Since then, polytopes have been studied intensively and are now a rich and active field of study not only in pure mathematics but with many connections to applied fields.

For deep insight into the world of polytopes we refer to the classic book of Branko Grünbaum [Grü03], the text book of Günter M. Ziegler [Zie95] and the book of Alexander Barvinok [Bar02], the latter taking a broader approach by also discussing non-polyhedral convex sets.

The versatility of polytopes can be seen in the fact that they appear both in cluster algebras and finite frames. These are two young fields of study, which have nothing in common, the former coming from Lie theory and representation theory, while the latter has its origins in functional analysis and signal processing.

This thesis consists of five chapters, including this introduction—the first chapter. The second chapter deals with polytopes associated to cluster algebras. After laying the foundations by introducing the basic terms and construction methods, we present a conjecture about graph properties of the 1-skeleton of polytopes associated to cluster algebras of finite type.

The third and fourth chapter deal with a different topic, namely finite frames. In Chapter 3 we give a brief introduction of finite frames and discuss properties important later in the thesis. Chapter 4 is devoted to the study of polytopes arising from spaces of frames with certain properties. We give a non-redundant description of these polytopes in terms of equations and inequalities. From this we derive the dimension and number of facets of the polytopes. Further we obtain two affine isomorphisms between the polytopes and discuss their properties combinatorially and from the frames perspective. We show how these isomorphisms are described by reversing the order of frame vectors and taking Naimark complements. The final chapter is designated to study the connections of the polytopes from chapter 4 with similar constructions by Stanley and Gelfand & Tsetlin.

We fix the following notations. The set of natural numbers $\mathbb{N}$ always contains the zero, and with $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ we denote the ring of integers, the field of fractions, the field of real numbers and the field of complex numbers, respectively. Although non-convex polytopes are also interesting, all polytopes that appear in this thesis are convex, that is for any two points contained in the polytope, the straight line segment between these points is also contained in the polytope. A convex polytope is either defined to be the convex hull of finitely many points in a Euclidean space or a bounded intersection of finitely many halfspaces in the said space. Both definitions are equivalent, which is stated by the *main theorem* of polytope theory. When looking at the faces of a polytope of dimension d, the faces of dimension $0, 1, d-1$ are called *vertices*, *edges* and *facets*, respectively. Two special faces of a polytope are that of dimension -1 and d, which are the empty face, denoted $\emptyset$, and the polytope itself. The term *proper face* refers to all faces of the polytope except the polytope itself. The number of faces of each dimension of the polytope is encoded in the *f-vector*, $f = (f_{-1}, f_0, f_1, \dots, f_d) \in \mathbb{N}^{d+2}$, where f_{-1} is by convention equal to 1. The set of all faces of a polytope, ordered by inclusion, forms the *face-lattice*.

2 Cluster Algebras

Cluster algebras (fr. Algèbres amassées) are a certain class of commutative algebras which have been introduced in the early 2000s by Sergey Fomin and Andrej Zelevinsky as a tool to study total positivity and dual canonical bases in Lie theory. They presented their results in a series of four papers [FZ02; FZ03a; BFZ05; FZ07]. Since then cluster algebras established themselves as a field of study on their own and have been proven to be a useful tool in various fields of mathematics, namely tropical geometry, Teichmüller theory and Poisson geometry among many others. Loosely speaking, cluster algebras are commutative rings with a distinguished family of generators called *cluster variables* which are grouped into overlapping sets called *clusters*.

This Chapter is organized as follows. In Section 2.1 we give the definition of cluster algebras and some basic results so we can state the famous classification of cluster algebras of finite type in the following section. Section 2.3 is devoted to explain how to construct cluster algebras from quivers. In Section 2.4 we construct cluster algebras from triangulations. In the final section we discuss graph properties of the exchange graph of cluster algebras of finite type.

We follow [Wil14] and [Mar13] for Section 2.1 to Section 2.3. Our reference for Section 2.4 is [FZ03a] and [DRS10].

2.1 Basic Definitions

Although we are focussing on cluster algebras of geometric type in this chapter, we define cluster algebras in a more generalized form for better understanding.

We start with the definition of a semifield.

Definition 2.1. (*Semifield*): A *semifield* $(\mathbb{P}, \oplus, \cdot)$ is an abelian multiplicative group $(\mathbb{P}, \cdot)$, with an auxiliary addition $\oplus$, which is associative, commutative and distributive with respect to the multiplication.

Example 2.2: The positive real numbers $(\mathbb{R}_{>0}, \oplus, \cdot)$ are a semifield with $\oplus$ being the usual addition. A more important example of a semifield is the *tropical semifield*, given in Definition 2.4.

An interesting property of semifields is the following.

Lemma 2.3:
Every semifield is torsion-free as a multiplicative group.

Proof. Let $p \in \mathbb{P}$, $m > 1$ and $p^m = 1$. Then,

$$\begin{aligned} p &= \frac{p^m \oplus p^{m-1} \oplus \cdots \oplus p}{p^{m-1} \oplus p^{m-2} \oplus \cdots \oplus 1} \\ &= \frac{p^{m-1} \oplus p^{m-2} \oplus \cdots \oplus 1}{p^{m-1} \oplus p^{m-2} \oplus \cdots \oplus 1} \\ &= 1. \end{aligned}$$

□

Among semifields the class of *tropical semifields* is of particular interest.

Definition 2.4. (*Tropical semifield*): Let I be an index set. A *tropical semifield* $\mathrm{Trop}(p_i \mid i \in I)$ is an abelian multiplicative group freely generated by the p_i with an auxiliary addition $\oplus$ defined by

$$\prod_{i \in I} p_i^{a_i} \oplus \prod_{i \in I} p_i^{b_i} = \prod_{i \in I} p_i^{\min(a_i, b_i)}.$$

The most famous example of a tropical semifield is the field of real numbers with addition replaced by taking the minimum and multiplication by taking the usual addition: $(\mathbb{R}, \min, +)$. This has been studied extensively over the last two decades.

Notation 2.5: We denote with $\mathcal{F}$ the field of rational functions in n independent variables with coefficients in the group ring $\mathbb{QP}$.

Definition 2.6. (*Free generating set*): A tuple $(x_1, \ldots, x_n)$ of elements of the field $\mathcal{F}$ is a *free generating set*, if the elements $x_1, \ldots, x_n$ are algebraically independent over the group ring $\mathbb{QP}$ and $\mathcal{F} = \mathbb{QP}(x_1, \ldots, x_n)$.

In the construction of cluster algebras skew-symmetrizable matrices play an important role.

Definition 2.7. (*Skew-symmetrizable*): An $n \times n$ matrix B is *skew-symmetrizable*, if there exists a diagonal matrix D with positive diagonal entries such that DB is skew-symmetric. An $m \times n$ matrix $\widetilde{B}$ of rank n is called *skew-symmetrizable*, if the *principal part* $B = (b_{ij})_{1 \le i,j \le n}$ is skew-symmetrizable.

Skew-symmetrizable matrices look like a generalization of skew-symmetric matrices, but as the following proposition states, they are in some sense skew-symmetric.

Proposition 2.8. (*[FZ02, Section 4]*):
Any skew-symmetrizable matrix is sign-skew-symmetric, that is, the matrix of signs is skew-symmetric.

Note, that this implies, that every diagonal entry of a skew-symmetrizable matrices equals zero. We now define a key ingredient in the construction of a cluster algebra.

Definition 2.9. (*Labeled seed*): A *labeled seed* Σ in the field $\mathcal{F}$ is a triple $\Sigma = (X, Y, B)$, where

- $X = (x_1, \ldots, x_n)$ is an n-tuple of elements of $\mathcal{F}$ forming a free generating set;
- $Y = (y_1, \ldots, y_n)$ is an n-tuple of elements of $\mathbb{P}$;
- $B = (b_{ij})$ is a skew-symmetrizable $n \times n$ integer matrix, called the *exchange matrix*.

For labeled seeds, we define the following equivalence relation. Two labeled seeds $\Sigma = (X, Y, B)$ and $\Sigma' = (X', Y', B')$ are said to be equivalent, in symbols $\Sigma \sim \Sigma'$, if and only if there exists a permutation π on the set $\{1, \ldots, n\}$ such that for all $1 \le i, j \le n$

$$x'_i = x_{\pi(i)},$$
$$y'_i = y_{\pi(i)} \quad \text{and}$$
$$b'_{ij} = b_{\pi(i),\pi(j)}.$$

The equivalence classes of this relation are called *unlabeled seeds*, or just *seeds*.

Given a labeled seed, one can construct other labeled seeds as follows.

Definition 2.10. (*Seed mutation*): Let (X, Y, B) be a labeled seed in $\mathcal{F}$, as above, and let $k \in \{1, \ldots, n\}$. The *seed mutation* μ_k in direction k transforms (X, Y, B) into the labeled seed $\mu_k(X, Y, B) = (X', Y', B')$ as follows. $X' = (x'_1, \ldots, x'_n)$ is given by $x'_i = x_i$ for all $i \neq k$ and

$$x'_k = \frac{y_k \prod\limits_{b_{ik}>0} x_i^{b_{ik}} + \prod\limits_{b_{ik}<0} x_i^{-b_{ik}}}{(y_k \oplus 1) x_k}. \tag{2.1}$$

$Y' = (y'_1, \ldots, y'_n)$ is given by

$$y'_j = \begin{cases} y_k^{-1}, & \text{if } j = k; \\ y_j y_k^{\max(b_{kj},0)} (y_k \oplus 1)^{-b_{kj}}, & \text{if } j \neq k. \end{cases} \tag{2.2}$$

$B' = (b'_{ij})$ is given by

$$b'_{ij} = \begin{cases} -b_{ij}, & \text{if } i = k \text{ or } j = k; \\ b_{ij} + \operatorname{sgn}(b_{ik}) \max(b_{ik} b_{kj}, 0), & \text{otherwise.} \end{cases} \tag{2.3}$$

Remark 2.11: It is easy to see that B' is also a skew-symmetrizable matrix.

Remark 2.12: One easily checks that μ_k is an involution.

Definition 2.13. (*Cluster pattern*): We consider the n-regular tree $\mathbb{T}_n$ whose edges are labeled by the numbers $1, \ldots, n$, such that the n edges at each vertex receive different labels. A *cluster pattern* is an assignment of a labeled seed $\Sigma_t = (X_t, Y_t, B_t)$ to every vertex $t \in \mathbb{T}_n$, such that the

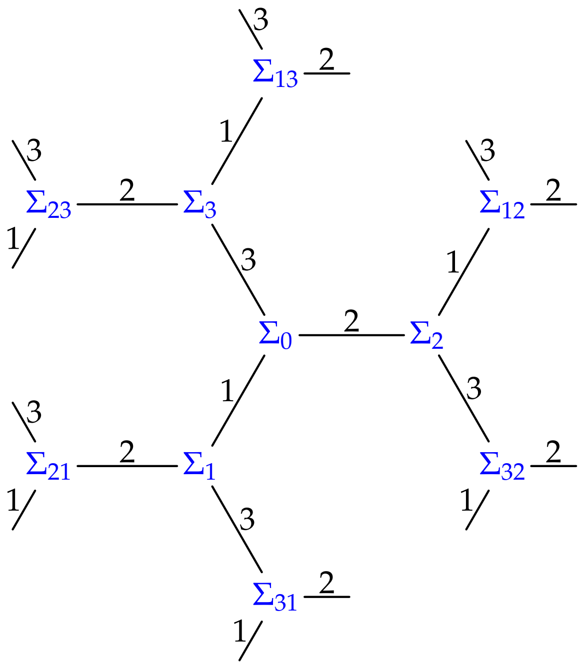


Figure 2.1: A cluster pattern of rank 3. Here we use non-repeating sequences of numbers $1, 2, 3$ for indexing the clusters, where Σ_0 is the initial cluster.

seeds assigned to the endpoints of any edge $t \overset{k}{\text{———}} t'$ are obtained from each other by the seed mutation in direction k. The components of Σ_t are written as:

$$X_t = (x_{1;t}, \dots, x_{n;t}), \qquad Y_t = (y_{1;t}, \dots, y_{n;t}), \qquad B_t = (b^t_{ij}).$$

We elucidate the above definition with Figure 2.1 which shows a cluster pattern of rank 3.

Notation 2.14: The X_t are called *clusters*, the elements $x_{i;t}$ are called *cluster variables* and we denote with $\mathcal{X}$ the union of all clusters, that is, the set of all cluster variables:

$$\mathcal{X} = \bigcup_{t \in \mathbb{T}_n} X_t = \{x_{i;t} \mid t \in \mathbb{T}_n, 1 \leq i \leq n\}.$$

Now, we have all ingredients to define a cluster algebra in the most general way.

Definition 2.15. (***Cluster algebra***)**:** The *cluster algebra* $\mathcal{A}$ associated to a given cluster pattern is the $\mathbb{ZP}$-subalgebra of the ambient field $\mathcal{F}$ generated by all the cluster variables: $\mathcal{A} = \mathbb{ZP}[\mathcal{X}]$. We choose an arbitrary seed $\Sigma = (X, Y, B)$ from the underlying cluster pattern and write $\mathcal{A}(X, Y, B)$ for the cluster algebra.

All clusters of a cluster algebra contain the same number of cluster variables. The number of cluster variables in each cluster is called the *rank* of the cluster algebra. We see from the definition above that a rank 1 cluster algebra contains only two clusters, since a 1-regular tree contains only two vertices.

Later, we will need the following definition, which associate to each cluster algebra a certain graph.

Definition 2.16. (***Exchange graph***)**:** Let $\mathcal{A}$ be a cluster algebra. The *exchange graph* of $\mathcal{A}$ is the undirected graph, whose vertices are the unlabeled seeds of $\mathcal{A}$ and an egde is drawn between two vertices if they are connected by a mutation.

We say that a cluster algebra is of *finite type* if the number of cluster variables in $\mathcal{X}$ is finite, or equivalently, if the number of seeds is finite. One of the main observations is the fact that all cluster variables are Laurent polynomials in the variables of an arbitrary cluster. This property is called the *Laurent phenomenon*.

Theorem 2.17. (***[FZ02, Thm. 3.1]***)**:**
In a cluster algebra, any cluster variable is expressed in terms of any given cluster as a Laurent polynomial with coefficients in the group ring $\mathbb{ZP}$.

As Fomin and Zelevinsky proved the above theorem, they conjectured an even stronger result.

Conjecture 2.18. (***[FZ02, Section 3]***)**:**
In a cluster algebra, any cluster variable is expressed in terms of any given cluster as a Laurent polynomial with all coefficients being non-negative *integer linear combination of elements of the semifield* $\mathbb{P}$.

This conjecture is known as the *positivity conjecture*. Over the years it was proven that the conjecture holds in some special cases. In 2013, Kyung-yong Lee and Ralf Schiffler proved in [LS15] that the conjecture holds for all skew-symmetric cluster algebras, which are all cluster algebras, whose exchange matrices are skew-symmetric.

Definition 2.19. (*Geometric type*): A cluster algebra $\mathcal{A} = \mathbb{ZP}[\mathcal{X}]$ is said to be of *geometric type* if the semifield $\mathbb{P}$ is a tropical semifield.

For cluster algebras of geometric type we can simplify the exchange relations. In order to do that, let $x_{n+1}, \ldots, x_m$ be the generators of the tropical semifield $\mathbb{P} = \operatorname{Trop}(x_{n+1}, \ldots, x_m)$. Then, we can write the coefficients $y_{i;t}$ at the seed $\Sigma_t = (X_t, Y_t, B_t)$ as Laurent monomials in $x_{n+1}, \ldots, x_m$, so we can define the integers b^t_{ij} for $j \in \{1, \ldots, n\}$ and $n < i \leq m$ as the coefficients of the monomial representation of the $y_{i;t}$:

$$y_{j;t} = \prod_{i=n+1}^{m} x_i^{b^t_{ij}}.$$

This gives a natural way of including the exchange matrix B_t as the principal $n \times n$ submatrix in the larger $m \times n$ matrix $\widetilde{B}_t = (b^t_{ij})$, where $1 \leq i \leq m$ and $1 \leq j \leq n$, whose matrix elements b^t_{ij} with $i > n$ encode the coefficients $y_{j;t}$. We then have the following lemma.

Lemma 2.20:
Let $\mathcal{A} \subset \mathbb{QP}(x_1, \ldots, x_n)$ be a cluster algebra of geometric type and let the tropical semifield $\mathbb{P} = \operatorname{Trop}(x_{n+1}, \ldots, x_m)$ have generators $x_{n+1}, \ldots, x_m$. Then the exchange relation (2.1) *reduces to:*

$$x'_k x_k = \prod_{b_{ik}>0} x_i^{b_{ik}} + \prod_{b_{ik}<0} x_i^{-b_{ik}}.$$

and the mutation rule (2.3) *is also valid for the $m \times n$ case and includes the exchange relation* (2.2).

Proof. We write

$$y_j = \prod_{i=n+1}^{m} x_i^{b_{ij}} = \left(\prod_{\substack{i=n+1 \\ b_{ij}>0}}^{m} x_i^{b_{ij}} \right) \left(\prod_{\substack{i=n+1 \\ b_{ij}<0}}^{m} x_i^{b_{ij}} \right)$$

and obtain

$$y_j \oplus 1 = \prod_{\substack{i=n+1 \\ b_{ij}<0}}^{m} x_i^{b_{ij}}.$$

The exchange relation (2.1) then becomes

$$\begin{aligned}
x_k' x_k &= \left(\left(\prod_{\substack{i=n+1 \\ b_{ik}>0}}^{m} x_i^{b_{ik}} \right) \left(\prod_{\substack{i=n+1 \\ b_{ik}<0}}^{m} x_i^{b_{ik}} \right) \prod_{\substack{i=1 \\ b_{ik}>0}}^{n} x_i^{b_{ik}} + \prod_{\substack{i=1 \\ b_{ik}<0}}^{n} x_i^{-b_{ik}} \right) \prod_{\substack{i=n+1 \\ b_{ik}<0}}^{m} x_i^{-b_{ik}} \\
&= \left(\prod_{\substack{i=n+1 \\ b_{ik}>0}}^{m} x_i^{b_{ik}} \right) \prod_{\substack{i=1 \\ b_{ik}>0}}^{n} x_i^{b_{ik}} + \left(\prod_{\substack{i=n+1 \\ b_{ik}<0}}^{m} x_i^{-b_{ik}} \right) \prod_{\substack{i=1 \\ b_{ik}<0}}^{n} x_i^{-b_{ik}} \\
&= \prod_{\substack{i=1 \\ b_{ik}>0}}^{m} x_i^{b_{ik}} + \prod_{\substack{i=1 \\ b_{ik}<0}}^{m} x_i^{-b_{ik}}.
\end{aligned}$$

For the exchange relation (2.2) we obtain the following. The first identity $y_j' = y_k^{-1}$ if $j = k$ becomes

$$y_k^{-1} = \prod_{i=n+1}^{m} x_i^{-b_{ik}}.$$

The second identity $y_j' = y_j y_k^{\max(b_{kj},0)} (y_k \oplus 1)^{-b_{kj}}$ if $j \neq k$ becomes

$$\begin{aligned}
y_j' &= \left(\prod_{i=n+1}^{m} x_i^{b_{ij}} \right) \left(\prod_{i=n+1}^{m} x_i^{b_{ik}} \right)^{\max(b_{kj},0)} \left(\prod_{\substack{i=n+1 \\ b_{ik<0}}}^{m} x_i^{b_{ik}} \right)^{-b_{kj}} \\
&= \prod_{i=n+1}^{m} x_i^{b_{ij} + b_{ik} \max(b_{kj},0) - \max(-b_{ik},0) b_{kj}} \\
&= \prod_{i=n+1}^{m} x_i^{b_{ij} + \operatorname{sgn}(b_{ik}) \max(b_{ik} b_{kj},0)},
\end{aligned}$$

and we see that the exponents are encoded in the exchange relation (2.3).

□

The above lemma allows us to skip the coefficients $Y = (y_1, \ldots, y_n)$ in the definition of cluster algebras of geometric type and take the generators $x_{n+1}, \ldots, x_m$ of the tropical semifield instead. In this context, the coefficients $x_{n+1}, \ldots x_m$ are called stable or *frozen variables*. The union $\widetilde{X} = X \cup \{x_{n+1}, \ldots, x_m\}$ is then called the *extended cluster* of a seed $(\widetilde{X}, \widetilde{B})$. For cluster algebras of geometric type we now write $\mathcal{A} = \mathcal{A}(\widetilde{X}, \widetilde{B})$.

Cluster algebras of geometric type appear in various contexts. For example, Joshua S. Scott showed in [Sco06] that the homogeneous coordinate ring of the Grassmannian $Gr_{k,n}$ is a cluster algebra of geometric type. In particular the homogeneous coordinate ring of the Grassmannian $Gr_{2,n+3}$ is a cluster algebra of type A_n. Other important examples of Grassmannians whose homogeneous coordinate rings are cluster algebras of finite type, are $Gr_{3,6}$, $Gr_{3,7}$ and $Gr_{3,8}$, which are of type D_4, E_6 and E_8, respectively. In the next section we explain the meaning of the different types of cluster algebras.

2.2 Classification of cluster algebras of finite type

The classification of cluster algebras of finite type was an astonishing and important result in the development of cluster algebra theory. At first we present a condition which guarantees that a cluster algebra is of finite type.

Theorem 2.21. (*[FZ03a, Thm. 1.8]*):
As in Section 2.1, let $X = (x_1, \ldots, x_n)$ be a free generating set of the field $\mathcal{F}$. Let $Y = (y_1, \ldots, y_n)$ be an n-tuple of elements of the semiring $\mathbb{P}$ and let $B = (b_{ij})$ be an skew-symmetrizable $n \times n$ integer matrix. The cluster algebra $\mathcal{A} = \mathcal{A}(X, Y, B)$ is of finite type if and only if for every seed (X', Y', B') in $\mathcal{A}$ the entries of the matrix $B' = (b'_{ij})$ satisfy the inequalities $|b'_{ij} b'_{ji}| \leq 3$, for all $1 \leq i, j \leq n$.

In order to formulate the classification results, we need some terminology from Lie theory.

Definition 2.22. (*Generalized Cartan matrix*): Let $A = (a_{ij})$ be an $n \times n$

integer matrix satisfying

$$a_{ii} = 2, \qquad \text{for all } 1 \le i \le n; \tag{2.4}$$
$$a_{ij} \le 0, \qquad \text{for all } i \ne j; \tag{2.5}$$
$$a_{ij} = 0, \text{if and only if } a_{ji} = 0. \tag{2.6}$$

Then the matrix A is called a *generalized Cartan matrix*.

Note, that every skew-symmetrizable matrix is sign-skew-symmetric, that is, the matrix of signs is skew-symmetric. This allows us to assign a generalized Cartan matrix to every skew-symmetrizable matrix in the following way.

Definition 2.23. (***Cartan Counterpart***)**:** Let $B = \big(b_{ij}\big)$ be a skew-symmetrizable integer square matrix. The *Cartan counterpart* of B is the generalized Cartan matrix $A = \big(a_{ij}\big)$ given by

$$a_{ij} = \begin{cases} 2, & \text{if } i = j; \\ -\big|b_{ij}\big|, & \text{if } i \ne j. \end{cases}$$

If B is not a square matrix, then we define the Cartan counterpart of B as the Cartan counterpart of the principal part of B.

Definition 2.24. (***Strongly isomorphic***)**:** Let $\mathcal{F}$ and $\mathcal{F}'$ be two field over the same semifield $\mathcal{P}$. Let $\mathcal{A}(X, Y, B) \subset \mathcal{F}$ and $\mathcal{A}(X', Y', B') \subset \mathcal{F}'$ be two cluster algebras. We say that $\mathcal{A}(X, Y, B)$ and $\mathcal{A}\left(X', Y', B'\right)$ are *strongly isomorphic* (or isomorphic as cluster algebras) if there is a field isomorphism $\phi\colon \mathcal{F} \to \mathcal{F}'$ such that $\big(\phi(X), Y, B\big)$ is a seed in $\mathcal{A}(X', Y', B')$; in other words, if and only if there is a seed in $\mathcal{A}(X', Y', B')$ whose coefficients and exchange matrix are Y and B, respectively.

The exchange graphs of two strongly isomorphic cluster algebras are isomorphic and determined by the choice of Y and B. This justifies to write $\mathcal{A}(Y, B)$ instead of $\mathcal{A}(X, Y, B)$.

Moreover, every cluster algebra over a fixed semifield $\mathbb{P}$ belongs to a series $\mathcal{A}(-, B)$ of all cluster algebras having a seed with exchange matrix B. If the choice of Y does not matter, we write $\mathcal{A}(B)$ for the cluster algebra.

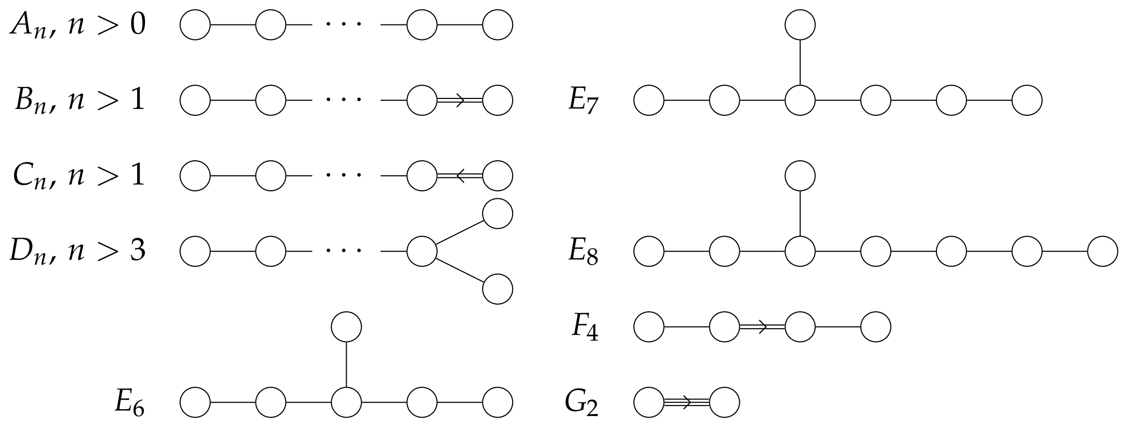


Figure 2.2: The finite Dynkin diagrams.

Theorem 2.25. (*[FZ03a, Theorem 1.4]*):
All cluster algebras in a family $\mathcal{A}(-, B)$ are simultaneously of finite or infinite type. A cluster algebra is of finite type if and only if the Cartan counterpart of the principal part of the exchange matrix of one of its seeds is a Cartan matrix of finite type, that is a Cartan matrix for a root system of type A_n, B_n, C_n, D_n, E_6, E_7, E_8, F_4 or G_2.

To every Cartan matrix one can assign a directed graph, called *Dynkin diagram*. Dynkin diagrams are used to classify semisimple Lie algebras. The finite Dynkin diagrams are depicted in Figure 2.2, see the book of James E. Humphreys [Hum72] for insight.

This theorem gives rise to a bijection between strong isomorphism classes of cluster algebras of finite type and finite Dynkin diagrams. In particular, the Dynkin type of a cluster algebra does not depend on the choice of the coefficient vector Y. Therefore, for the case of cluster algebras of geometric type, we can waive the frozen variables.

2.3 Quivers

In this section, we construct cluster algebras from quivers. We are using quivers to develope a deeper understanding of cluster algebras of finite type. Therefore, let from now on all cluster algebras be of finite type.

Definition 2.26. (*Quiver*): A *quiver* Q is a directed graph $Q = (V, E, s, t)$ with vertex set V, arrow set E and two maps $s : E \to V$ and $t : E \to V$ taking an arrow to its starting point and target point, respectively. A quiver is called *finite* if both V and E are finite sets. In this case we set $V = \{1, \dots, n\}$. A loop is an arrow α whose starting point and target point coincide. A 2-cycle is a pair of distinct arrows β and γ such that $s(\beta) = t(\gamma)$ and $t(\beta) = s(\gamma)$. The *underlying graph* of Q is the undirected graph obtained from Q by replacing all arrows with undirected edges. A vertex which is not a target point of any arrow is called a *source*, a vertex which is not a starting point of any arrow is called a *sink*.

In all cases we consider, it is clear what the starting and target point of an arrow is. Hence, we omit the maps s and t and write $Q = (V, E)$. From now on, let $Q = (V, E)$ be a finite quiver without loops or 2-cycles. We label the vertices of Q with numbers $\{1, \dots, n\}$.

We encode the combinatorial data of the quiver in a skew-symmetric *exchange matrix* $B = (b_{ij})$, where

$$b_{ij} = -b_{ji} = \ell,$$

whenever there are ℓ arrows from vertex i to vertex j. The loop-freeness of Q and the non-existence of oriented 2-cycles guarantee that B is skew-symmetric and well-defined.

Vice versa we can associate a quiver $Q(B)$ to every skew-symmetric matrix B. This construction can be extended to skew-symmetric $m \times n$ matrices $\widetilde{B}$. For that, we construct a quiver $\widetilde{Q} = Q(\widetilde{B})$ with vertices $1, \dots, m$, having $Q(B)$ as a full subquiver on the vertices $1, \dots, n$. The vertices $n + 1, \dots, m$ are called *frozen vertices*.

The matrix mutation $\mu_k(B)$ from Equation (2.3) on page 6 induces a mutation of the quiver in the following way.

Lemma 2.27:
Let $Q = (V, E)$ be a finite quiver with associated exchange matrix $B = (b_{ij})$ and $1 \leq k \leq n$. A mutation of B in direction k (or at a vertex k) results in a new quiver $Q' = \mu_k(Q)$ which one obtains from Q through the following three steps:

1. *For each pair of arrows $i \to k \to j$: introduce a new arrow $i \to j$;*

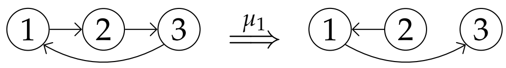


Figure 2.3: A quiver mutation in direction 1.

2. *Reverse direction of all arrows incident to k;*

3. *Remove all oriented 2-cycles.*

Proof. The statement follows directly from the definition of the exchange matrix and Equation (2.3) on page 6. □

Note that for quivers with frozen vertices, one can only perform a mutation at the non-frozen ones. The following example elucidates the quiver mutation.

Example 2.28: Figure 2.3 shows the quivers for the matrices

$$B = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} \text{ and } \mu_1(B) = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Definition 2.29. (*Mutation equivalence*): Two quivers Q and Q' are *mutation equivalent* if one can get from Q to Q' by a sequence of mutations.

For example, the two quivers in Figure 2.3 are mutation equivalent, since they are connected by a mutation. They are mutation equivalent to a quiver of type A_3.

Remark 2.30: A mutation at a vertex $v \in V$ which is a sink or a source just reverses the orientation of all arrows having v as target point or starting point, respectively.

As in Section 2.1, let $\mathcal{F}$ be the field of rational functions in n independent variables over $\mathbb{Q}$. Note, that this is equivalent to Notation 2.5 on page 4 by choosing $\mathbb{P}$ to be the trivial semifield (which is tropical). Also as before, we let $X = \{x_1, \dots, x_n\}$ be a *free generating set* such that $\mathcal{F} = \mathbb{Q}(x_1, \dots, x_n)$. We label the vertices of the quiver with $x_1, \dots, x_n$ and analogue to Notation 2.14 on page 7 we call the pair (Q, X) a *seed* and X a *cluster*.

A *seed mutation* at a mutable vertex x_k replaces Q by $\mu_k(Q)$ and x_k by the new cluster variable x'_k which fulfills the exchange relation

$$x_k x'_k = \prod_{x_k \leftarrow x_i} x_i + \prod_{z \rightarrow x_i} x_i.$$

Definition 2.31. (*Cluster algebra associated to a quiver*): The *cluster algebra* $\mathcal{A}(Q)$ associated to a quiver Q is the subring of the ambient field $\mathcal{F}$ generated by all elements of all clusters from all possible mutations of the seed (Q, X).

Since to every finite quiver Q, there is an associated skew-symmetric matrix (the exchange matrix B of Q), the cluster algebra $\mathcal{A}(Q)$ is called a *skew-symmetric cluster algebra* of geometric type. As in the previous section we say that a cluster algebra is of *finite type* if there are only finitely many distinct clusters.

The following theorem gives a classification of all cluster algebras of finite type associated to a quiver.

Theorem 2.32. (*[FZ03a, Thm. 1.8]*):
The cluster Algebra $\mathcal{A}(Q)$ associated to a quiver Q is of finite type if and only if the underlying graph of Q is a simply laced Dynkin diagram, that is, a Dynkin diagram of type A, D, or E. See Figure 2.2 on page 13 for reference.

Theorem 2.32 states that there are two infinite series of strong isomorphism classes of skew-symmetric cluster algebras (type A_n and D_n) and three exceptional types (E_6, E_7 and E_8). The other types of cluster algebras of finite type (B_n, C_n, F_4 and G_2) can not be realized from quivers.

2.4 Triangulations

In this section we discuss the natural relationship between certain cluster algebras of geometric type and triangulations. We start with the triangulation of convex polygons and discuss generalizations of this approach.

For insight into the world of triangulations we refer to the book of Jesús A. De Loera, Jörg Rambau and Francisco Santos [DRS10] from which we took the main definitions.

Definition 2.33. (*Triangulation*): A *triangulation* of a point configuration $A = \{a_1, \ldots, a_n\} \in \mathbb{R}^d$ is a collection of d-simplices with vertices in A, that satisfies the following two properties:

1. The union of all simplices equals the convex hull of A. (*Union Property*)

2. Any pair of simplices intersects in a common face (possibly empty). (*Intersection Property*)

To describe a triangulation we will label the points from 1 to n and give the list of vertex sets of the d-simplices in the triangulation. By a triangulation of a polytope we mean a triangulation of the point configuration given by the vertices of the polytope.

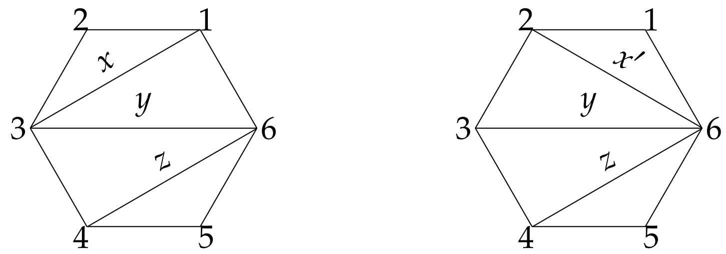
Example 2.34: The *zig-zag triangulation* of the regular hexagon depicted in Figure 2.4a on the next page is written as:

$$\{\{1,2,3\},\{1,3,6\},\{3,4,6\},\{4,5,6\}\},$$

whereas the 6th *standard triangulation* depicted in Figure 2.4b on the following page would be written as

$$\{\{1,2,6\},\{2,3,6\},\{3,4,6\},\{4,5,6\}\}.$$

We denote with C_n the convex polygon with n vertices, labeled counterclockwise from 1 to n. Since the number of triangulations does not depend on the coordinates of the vertices of the polygon, we denote this number with t_n. A triangulation of the n-gon consists of $n-3$ noncrossing diagonals. Two diagonals cross if and only if they involve four vertices in an alternating way. That is, if $1 \leq i < j < k < l \leq n$, then the only two diagonals crossing each other involving the four points i, j, k and l are $\{i,k\}$ and $\{j,l\}$. A *diameter* or long diagonal of a polygon with $2n$ vertices is a diagonal between the vertices i and $i+n$ (mod $2n$).



(a) Zig-zag triangulation.

(b) 6th standard triangulation.

Figure 2.4: Two triangulation of a hexagon.

Lemma 2.35. (*[DRS10, Theorem 1.1.2]*):
The number t_n of triangulations of a convex n-gon is

$$t_n = \frac{1}{n-1}\binom{2n-4}{n-2}.$$

The sequence of numbers t_n is known as the *Catalan numbers* and is an important integer sequence in combinatorics. Catalan numbers appear in various contexts such as

1. the numbers of binary trees with $n-2$ nodes;
2. bracketings of products of $n-1$ factors.

Definition 2.36. (*Diagonal flip*): Let $\{i,j\}$ be a diagonal in the triangulation of an n-gon and $\{i,j,k\}$ and $\{i,l,j\}$ the triangles sharing $\{i,j\}$ as a common edge. Then $\{i,j\}$ is a diagonal in the quadrilateral $\{i,l,j,k\}$. Replacing the diagonal $\{i,j\}$ with the other diagonal $\{k,l\}$ in this quadrilateral results in a different triangulation of the n-gon. This operation is called *diagonal flip*.

Figure 2.4 shows the flip of the diagonal $\{1,3\}$ with label x to the diagonal $\{2,6\}$ with label x'.

Definition 2.36 allows us to look at the set of triangulations of a regular n-gon as vertices of a graph (the *graph of flips*) where an edge between two vertices exists if and only if there exists a diagonal flip turning one triangulation to the other. This graph is regular of degree $n-3$, since we can perform a flip for any of the $n-3$ diagonals in a triangulation of the n-gon.

Proposition 2.37. (*[DRS10, p. 7]*):
The graph of flips of an n-gon is connected.

One of the properties of the flip graph which is not easily seen is that it is Hamiltonian. This was first proven by JOAN M. LUCAS in 1987 [Luc87]. In 1999 FERRAN HURTADO and MARC NOY gave a more simple proof [HN99]. The Hamiltonicity of the flip graph was the inspiration for Conjecture 2.44 on page 25, therefore we state it as the following Lemma:

Lemma 2.38. (*[HN99, Theorem 4.1]*):
The graph of flips of an n-gon is a Hamiltionian graph for $n \geq 5$.

As our interest is not as much in the combinatorics of the polygon as it is in that of the triangulations, from now on, we switch to a concise labeling of the diagonals instead of the vertices. To every triangulation T of an n-gon we assign an $(n-3) \times (n-3)$ matrix $B = B(T) = (b_{ij})$ as follows. Label the diagonals of the triangulations from 1 to $n-3$. Set $b_{ij} = 1$, if i, j are two diagonals of a triangle in the triangulation and j follows i in clockwise order. Similarly let $b_{ij} = -1$, if i, j are two diagonals of a triangle in the triangulation and i follows j in clockwise order. Otherwise, set $b_{ij} = 0$. This makes B a skew-symmetric matrix. Mutations of B corresponds to flips inside the triangulation and therefore we write $T' = \mu_k(T)$ if T' is a triangulation obtained from T by flipping the diagonal with label k. More precisely, we have that $\mu_k(B(T)) = B(\mu_k(T))$ for all $1 \leq k \leq n-3$. If we also label the boundary edges of the polygon from $n-2$ to $2n-3$, we get a $(2n-3) \times (n-3)$-matrix $\widetilde{B}$, whose principal part is B.

From triangulations of an n-gon we can construct a cluster algebra, which is explained in the following.

We recall *Ptolemy's theorem*, which states that in an inscribed quadrilateral the sum of the products of the lengths of the two pairs of opposite sides equals the product of the lengths of the two diagonals.

Given a triangulation of an n-gon with associated matrix B and a free generating set $(x_1, \ldots, x_{n-3})$ of $\mathcal{F} = \mathbb{Q}(x_1, \ldots, x_{n-3})$. We take the latter as our initial cluster. We assign to every diagonal of the triangulation an element of the free generating set, that is, label the diagonals with $x_1, \dots, x_{n-3}$. Then we perform all possible flips on the diagonals and obtain in each step a new cluster of variables in the following way. A flip of the diagonal x_i yields a new diagonal x_i' such that the Ptolemy relation

$$x_i x_i' = x_a x_c + x_b x_d$$

holds, where x_a, x_b, x_c, x_d are the edges of the corresponding quadrilateral in clockwise order with diagonals x_i and x_i'. If one of the edges belongs to the boundary of the n-gon, we set the corresponding variable equal to 1. The *cluster algebra associated to an n-gon* is the subalgebra of $\mathcal{F}$ generated by all cluster variables obtained in this process. Again, we obtain n frozen variables if we label the boundary edges of the n-gon with variables x_{n-2} to x_{2n-3} instead of ones.

Since to every triangulation we assign a skew-symmetric matrix with all entries either being -1, 0 or 1, cluster algebras associated to an n-gon are of type A_{n-3}. In other words, the graph of flips of a triangulation of a regular n-gon is isomorphic to the exchange graph of the cluster algebra of type A_{n-3}.

As we have seen, one can construct cluster algebras of type A_n from quivers and from triangulations of the n-gon. These two construction methods can be interchanged in the following way. Let T be a triangulation of an n-gon and label the diagonals from 1 to $n-3$. Let $Q(T)$ be the quiver with vertex set $1, \dots, n-3$ and arrows as follows. Draw an arrow from vertex i to vertex j whenever the diagonals i and j are two sides of a triangle and j follows i in the order induced by anticlockwise orientation of the boundary of the triangle. By this, flips of diagonals correspond to quiver mutations.

The construction of cluster algebras from triangulations is not restricted to cluster algebras of type A_n. It is possible to obtain cluster algebras of other types by special constructions of triangulations. In the following we explain the construction for cluster algebras of type B_n and C_n.

Definition 2.39. (*Centrally-symmetric triangulation*): Let P be a regular polygon with $2n$ vertices. Let $\mathbb{Z}/2\mathbb{Z}$ operate on P by rotation of P by 180°.

A *centrally-symmetric triangulation* of P is a maximal subset of noncrossing $\mathbb{Z}/2\mathbb{Z}$-orbits of diagonals of P.

Note, that every centrally-symmetric triangulation contains exactly one diameter. Performing a flip of a centrally-symmetric triangulation is either flipping the diameter or flipping both diagonals of a $\mathbb{Z}/2\mathbb{Z}$-orbit. Figure 2.5 on page 23 shows the graph of flips for the centrally-symmetric triangulation of the hexagon.

Now, we want to construct cluster algebras from centrally-symmetric triangulations. Let P be a regular n-gon on $2n+2$ vertices. The number of $\mathbb{Z}/2\mathbb{Z}$-orbits of diagonals in a centrally-symmetric triangulation of P equals n. Again, we take $x_1, \ldots, x_n$ as our initial cluster. We assign each x_i to a $\mathbb{Z}/2\mathbb{Z}$-orbit. The exchange relations corresponding to flips are slightly more complicated in this case than for type A_n. Let x_i be a diagonal of the quadrilateral consisting of the four edges x_a, x_b, x_c, x_d in clockwise order. As in the type A_n case, we set them equal to 1, if it is a edge of the boundary of the $(2n+2)$-gon. The flipped diagonal of x_i is x_i' and the exchange relations are as follows. If none of the edges x_a, x_b, x_c, x_d or the diagonal x_i is a diameter of the $(2n+2)$-gon, we have

$$x_i x_i' = x_a x_c + x_b x_d. \tag{2.7}$$

If one of the edges x_a, x_b, x_c, x_d is a diameter of the $(2n+2)$-gon, say x_a, we have

$$x_i x_i' = \begin{cases} x_a^2 x_c + x_b x_d, & \text{for type } B_n, \\ x_a x_c + x_b x_d, & \text{for type } C_n. \end{cases} \tag{2.8}$$

If x_i is a diameter of the $(2n+2)$-gon, we have

$$x_i x_i' = \begin{cases} x_a + x_b, & \text{for type } B_n, \\ x_a^2 + x_b^2, & \text{for type } C_n. \end{cases} \tag{2.9}$$

This definition gives rise to a bijection between the set of clusters of a cluster algebra of type B_n or C_n and the set of centrally-symmetric triangulations of a $(2n+2)$-gon.

Example 2.40: Consider the centrally-symmetric triangulation depicted in the upper left of Figure 2.5 on the next page. Performing the flips as depicted, we get the following cluster variables for the cluster algebra of type C_2

- x,
- y,
- $x' = \frac{1+y}{x}$,
- $y' = \frac{1+x'^2}{y} = \frac{1+x^2+y^2+2y}{x^2y}$,
- $x'' = \frac{1+y'}{x'} = \frac{1+x^2+y}{xy}$,
- $y'' = \frac{1+x''^2}{y'} = \frac{1+x^2}{y}$,
- $x''' = \frac{1+y''}{x''} = x$.

Further mutations do not yield new cluster variables. The cluster algebra of type C_2 is then generated by the six Laurent polynomials in x and y listed above. It is clear that this cluster algebra could not be associated to a quiver as the accompanying exchange matrix is not skew-symmetric since it is mutation-equivalent to the matrix

$$B = \begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}.$$

The resulting cluster algebra is strongly isomorphic to a cluster algebra of type B_2 by swapping the variables x and y.

With the construction of cluster algebras from quivers and triangulations we now have explicit construction methods for the cluster algebras of type A_n, B_n, C_n, D_n, E_6, E_7 and E_8.

Remark 2.41: Thinking of generalizations of the construction of cluster algebras from triangulations, one way would be to consider triangulations of higher dimensional polytopes instead of 2-dimensional polygons. Steffen Oppermann and Hugh Thomas showed in [OT12] that the process of flipping diagonals could be generalized to triangulations of the cyclic polytope in even dimensions. In this case the role of the diagonals would be adopted by the $d/2$-dimensional faces of the d-simplices of the

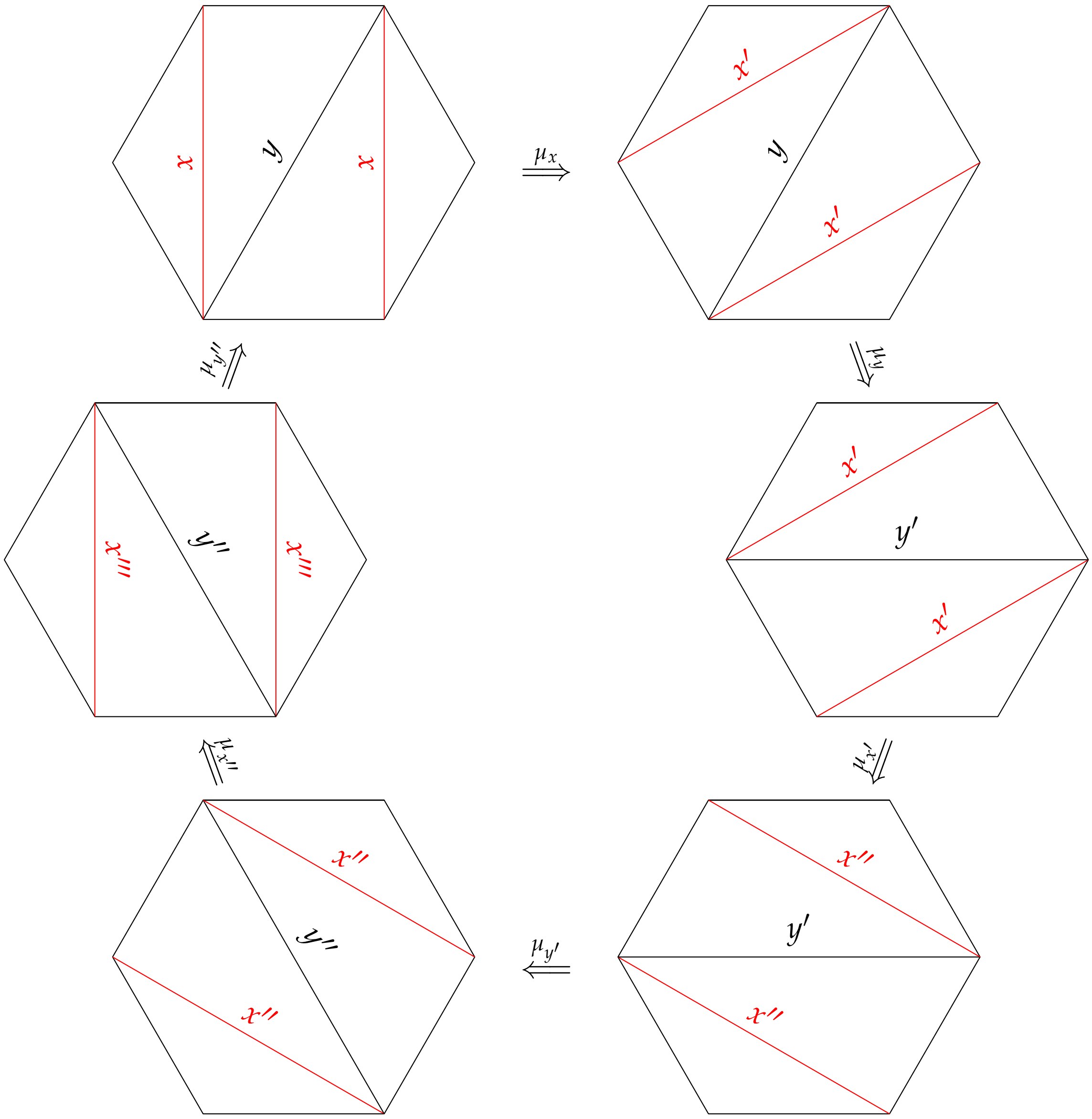


Figure 2.5: Flips of a centrally-symmetric triangulation of a hexagon. It turns out that $x''' = x$ and $y''' = y$.

triangulation which intersect with the interior of the cyclic polytope. It is not understood how to generalize this construction to odd dimensions, since in this case the number of simplices one needs to triangulate a polytope is not determined by the number of vertices. For example the 3-cube can be triangulated using either five or six 3-simplices. This would require to think of a totally different concept to replace the diagonal flips. Until now it is not known how this could be achieved.

2.5 Hamiltonicity of the Exchange Graph

Although the exchange graph of a cluster algebra is an n-regular tree, for cluster algebras of finite type it can be realized as the 1-skeleton of a convex polytope of dimension n.

Theorem 2.42. (*[FZ03b, Theorem 5.10]*):
The exchange graph of a cluster algebra of finite type is the 1*-skeleton of a convex polytope.*

This polytope is called the *generalized associahedron* of the corresponding type. The vertices of the generalized associahedron correspond to clusters and the facets correspond to cluster variables. This implies that the generalized associahedron is a simple polytope and therefore it is completely described by its 1-skeleton, as the following theorem states.

Theorem 2.43. (*[Zie95, §3.4]*):
If P is a simple polytope, then the 1*-skeleton of P determines the entire combinatorial structure of P. In other words, if two simple polytopes have isomorphic* 1*-skeleta, then their face lattices are also isomorphic.*

For cluster algebras of type A_n, the exchange graph is the 1-skeleton of the *associahedron*, often called *Stasheff polytope*. For type B_n cluster algebras, the corresponding polytope is the *cyclohedron* or *Bott-Taubes polytope*. Since the exchange graphs of cluster algebras of type B_n and C_n are isomorphic, the latter are also associated to the cyclohedron.

The 1-skeleton of a generalized associahedron is an n-regular graph, where n is the dimension of the generalized associahedron (which is

	A_n	B_n/C_n	D_n	E_6	E_7	E_8	F_4	G_2
f_0	$\frac{1}{n+2}\binom{2n+2}{n+1}$	$\binom{2n}{n}$	$\frac{3n-2}{n}\binom{2n-2}{n-1}$	833	4160	25080	105	8
f_1	$\frac{n}{2(n+2)}\binom{2n+2}{n+1}$	$\frac{n}{2}\binom{2n}{n}$	$\frac{3n-2}{2}\binom{2n-2}{n-1}$	2499	14560	100320	210	8

Table 2.1: The number of vertices and edges of generalized associahedra of different types.

the rank of the cluster algebra). Therefore, the number of edges f_1 of the graph is equal to $\frac{n}{2}f_0$, where f_0 is the number of vertices of the generalized associahedron (which is equal to the number of clusters).

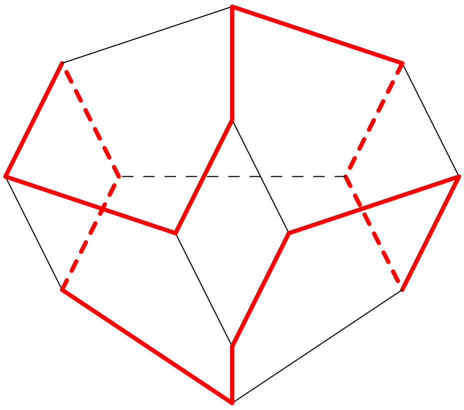
The number of vertices and edges of the generalized associahedra are given in Table 2.1.

We conjecture, that the 1-skeleton of the generalized associahedron has the following nice property.

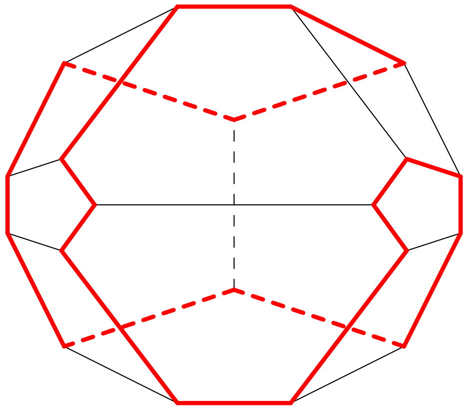
Conjecture 2.44:
The 1-skeleton of the generalized associahedron of a cluster algebra of finite type is a Hamiltonian graph, that is, there exists a closed path visiting each vertex exactly once.

We give some evidence why this conjecture might be true. The graph for the A_n is Hamiltonian, this is a known result, see Lemma 2.38 on page 19. Figure 2.6a on the following page shows a Hamiltonian path for the case A_3. For finite type cluster algebras of rank two it is clear that they satisfy Conjecture 2.44, since the cluster pattern is a 2-regular tree, hence the generalized associahedron is a 2-dimensional polytope and the 1-skeleton of it is already a cycle. For rank three, one has that $A_3 = D_3$ and therefore the graph of the generalized associahedron of type D_3 is also Hamiltonian. The graph for the type B_3/C_3 cluster algebras with marked Hamiltonian path is depicted in Figure 2.6b on the following page.

For rank four we get the exceptional type F_4, which has a generalized associahedron of dimension four with 105 vertices. We computed a Hamiltonian path using the computer algebra system `sage`. For the exceptional types E_6, E_7 and E_8 it is unfeasable to check Hamiltonicity by hand since they have 833, 4160 and 25080 vertices, respectively. Using



(a) Type A_3.



(b) Type B_3/C_3.

Figure 2.6: Hamiltonian cycles through the generalized associahedra of types A_3 and B_3/C_3.

`sage` we were able to compute Hamiltonian cycles for these three associahedra. Also we used a computer to check for cluster algebras of ranks 4 to 8 that Conjecture 2.44 on page 25 holds.

In the final preparation of this thesis we learned that Thibault Manneville and Vincent Pilaud in [MP14] have proven Hamiltonicity for the case of B_n/C_n. They used a graph-theoretic approach and showed Hamiltonicity for all graph associahedra, a certain class of associahedra which includes the generalized associahedra of type B_n/C_n.

In total we could turn Conjecture 2.44 on page 25 into a theorem for most of the cases:

Theorem 2.45:
If $\mathcal{A}$ is a cluster algebra of type A_n, B_n, C_n, E_6, E_7, E_8, F_4 or G_2, the graph of the generalized associahedron of $\mathcal{A}$ is Hamiltonian.

The only remaining cases are the type D_n cluster algebras for which we believe one can construct a similar argument as in [MP14], but we were not able to do this yet.

3 Finite Frames

Frames were first introduced in 1952 by Richard J. Duffin and Albert C. Schaeffer in an article on non-harmonic Fourier series [DS52]. In the 1980's Ingrid Daubechies, Alex Grossmann and Yves Meyer picked up on this and showed that frames have great significance in signal processing [DGM86]. Since then, frame theory has evolved to be useful for many applications whenever one needs redundant and stable representations of data. Coming from Fourier series and functional analysis, frames were first studied for infinite-dimensional vector spaces, such as $L^2(\mathbb{R})$ and others. Theory of frames for finite-dimensional vector spaces is a much younger subject which has been fueled by the necessity for real world applications and implementations in computer systems. Applications range from problems in engineering and computer sciences to applied and pure mathematics.

We focus on finite frames and organize this chapter as follows. In the first section we give the definitions of frames and some of their basic properties. The second section is dedicated to several operators that come with a frame. We discuss properties of them and in particular eigenvalues of the frame operator and the Gramian operator. This will come in handy for Chapter 4. In Section 3.3 we explain how to construct new frames from old ones and how to obtain frames with certain desirable properties.

3.1 Definitions and Basic Properties

In this section we define frames and discuss their main properties. We follow the book of Ole Christensen [Chr03] and the first chapter of the book edited by Peter G. Casazza and Gitta Kutyniok [CK13], the latter giving a detailed introduction and collection of recent results in finite frame theory.

Throughout the rest of this thesis let $\mathcal{H}$ denote a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and let $\mathbb{F}$ be either $\mathbb{R}$ or $\mathbb{C}$. Since every finite-dimensional Hilbert space is either isomorphic to $\mathbb{R}^d$ or $\mathbb{C}^d$ by a choice of an orthonormal basis, we assume in this case $\mathcal{H} = \mathbb{F}^d$ and use coordinates with respect to the standard basis.

A well known result in Hilbert space theory is the following proposition, first formulated by Marc-Antoine Parseval.

Proposition 3.1. (*Parseval's identity*):
Let $(e_i)_{i\in I}$ be an orthonormal basis for a Hilbert space $\mathcal{H}$. Then for every $x \in \mathcal{H}$, we have

$$\|x\|^2 = \sum_{i\in I} \left|\langle x, e_i\rangle\right|^2.$$

As an implication of Parseval's identity one can reconstruct a vector x from the *frame coefficients* $\left(\langle x, e_i\rangle\right)_{i\in I}$, namely for every $x \in \mathcal{H}$, we have

$$x = \sum_{i\in I} \langle x, e_i\rangle e_i. \tag{3.1}$$

Equation (3.1) is called the perfect *reconstruction formula*. In fact, other families of vectors besides orthonormal bases fulfill Parseval's identity. For example, it is satisfied by all families of vectors with norm $\sqrt{2/N}$ in $\mathbb{R}^2$ which form a regular N-gon.

Definition 3.2. (*Bessel sequence*): For a vector space $\mathcal{H}$ with inner product $\langle\cdot,\cdot\rangle$, a family of vectors $\{f_i\}_{i\in I}$ is a *B-Bessel sequence* for a constant $B > 0$, called *Bessel bound*, if for every $x \in \mathcal{H}$ we have

$$\sum_{i\in I} \left|\langle x, f_i\rangle\right|^2 \leq B\|x\|^2.$$

For finite families of vectors we have the following result.

Proposition 3.3:
Every finite family of vectors $(f_n)_{n=1}^N$ in a d-dimensional Hilbert space $\mathcal{H}$ is a Bessel-sequence.

Proof. From the Cauchy-Schwarz inequality we have for all $x \in \mathcal{H}$

$$\sum_{n=1}^{N} |\langle x, f_n \rangle|^2 \leq \sum_{n=1}^{N} \|f_n\|^2 \|x\|^2,$$

therefore, $\sum_{n=1}^{N} \|f_n\|^2$ is a Bessel bound for $(f_n)_{n=1}^{N}$. $\square$

Definition 3.4. (*Frame*): For a vector space $\mathcal{H}$ with inner product $\langle \cdot, \cdot \rangle$, a family of vectors $\{f_i\}_{i \in I}$ is a *frame* if there exist constants $A, B > 0$ such that for every $f \in \mathcal{H}$

$$A\|f\|^2 \leq \sum_{i \in I} |\langle f, f_i \rangle|^2 \leq B\|f\|^2. \tag{3.2}$$

The constants A and B are called *frame bounds*. These are not unique and for every frame there exist *optimal frame bounds* which are the supremum of all lower frame bounds and the infimum of all upper frame bounds.

Although one can do differently, for finite-dimensional vector spaces, we will only consider frames which consist of finitely many vectors.

Definition 3.5. (*Types of frames*): A frame is called

- *tight* if one can choose $A = B$ in Equation (3.2) and *Parseval* if $A = B = 1$ is possible;
- *equal norm* if all f_i have the same norm, in particular, it is called *unit norm* if $\|f_i\| = 1$ for all $i \in I$;
- *equiangular* if it is unit norm and there exists a constant c such that $|\langle f_i, f_j \rangle| = c$ for all $i \neq j$.

A finite unit norm tight frame is usually abbreviated *FUNTF* and an equiangular tight frame is called *ETF*. The term *Parseval frame* comes from the fact, that those frames satisfy Parseval's identity and the reconstruction property (Equation (3.1) on page 28).

We elucidate the above definition with some examples.

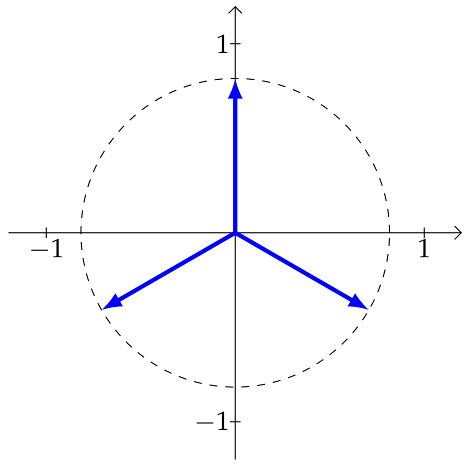


Figure 3.1: The Mercedes-Benz frame

Example 3.6: The *Mercedes-Benz frame* for $\mathbb{R}^2$ is given by

$$\left(\sqrt{\frac{2}{3}}\begin{pmatrix}0\\1\end{pmatrix}, \sqrt{\frac{2}{3}}\begin{pmatrix}\frac{\sqrt{3}}{2}\\-\frac{1}{2}\end{pmatrix}, \sqrt{\frac{2}{3}}\begin{pmatrix}-\frac{\sqrt{3}}{2}\\-\frac{1}{2}\end{pmatrix}\right).$$

This is an ETF. Figure 3.1 explains its name.

Example 3.7: Let $(e_n)_{n=1}^N$ be an orthonormal basis for a d-dimensional Hilbert space $\mathcal{H}$. Then the following holds.

(i) $(e_1, e_1, e_2, e_3, \ldots, e_N)$ is a frame for $\mathcal{H}$ with optimal frame bounds $A = 1$ and $B = 2$;

(ii) $(e_1, e_1, e_2, e_2, \ldots, e_N, e_N)$ is a tight frame for $\mathcal{H}$ with frame bounds $A = B = 2$;

(iii) $\left(e_1, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{2}}e_2, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \frac{1}{\sqrt{3}}e_3, \ldots\right)$, where each vector $\frac{1}{\sqrt{n}}e_n$ is repeated n times, is a Parseval frame.

We note the following property of finite frames:

Proposition 3.8. (*[CK13, Lemma 1.2(ii)]*):
Let $(f_n)_{n=1}^N$ be a family of vectors in a d-dimensional Hilbert space $\mathcal{H}$. Then, $(f_n)_{n=1}^N$ is a frame for $\mathcal{H}$ if and only if $(f_n)_{n=1}^N$ is a spanning set for $\mathcal{H}$.

Proof. Suppose that $\operatorname{span}\big((f_n)_{n=1}^N\big) \neq \mathcal{H}$, then there exists an $f \in \mathcal{H}$, where $f \neq 0$ such that $\langle f, f_n \rangle = 0$ for all $1 \leq n \leq N$. Therefore $(f_n)_{n=1}^N$ cannot be a frame. Conversely, assume that $(f_n)_{n=1}^N$ is not a frame, then there exists a sequence $(x_i)_{i=1}^\infty$ of normalized vectors in $\mathcal{H}$ such that $\sum_{n=1}^N \big|\langle x_i, f_n \rangle\big|^2 \leq 1/n$ for all $i \in \mathbb{N}$. Hence the limit x of a convergent subsequence of $(x_i)_{i=1}^\infty$ satisfies $\langle x, f_n \rangle = 0$ for all $1 \leq n \leq N$. Since $\|x\| = 1$, it follows that $(f_n)_{n=1}^N$ is not a spanning set. □

Further more, we have the following properties for Parseval frames:

Proposition 3.9. (*[CK13, Lemma 1.2(i),(iii)]*):
Let $\mathcal{H}$ be a d-dimensional Hilbert space. Then the following holds.

(i) Every orthonormal basis of $\mathcal{H}$ is a Parseval frame.

(ii) Every unit norm Parseval frame for $\mathcal{H}$ is an orthonormal basis of $\mathcal{H}$.

Proof. (i) Follows directly from Parseval's identity.

(ii) Let $(f_n)_{n=1}^N$ be a unit norm Parseval frame. Then for every index $m \in \{1, \ldots, N\}$, we have

$$\|f_m\|^2 = \sum_{n=1}^N \big|\langle f_m, f_n \rangle\big|^2 = \|f_m\|^4 + \sum_{\substack{n=1 \\ n \neq m}}^N \big|\langle f_m, f_n \rangle\big|^2.$$

Since $\|f_n\| = 1$ for every $n \in \{1, \ldots, N\}$, we have

$$\sum_{\substack{n=1 \\ n \neq m}}^N \big|\langle f_m, f_n \rangle\big|^2 = 0, \qquad \text{for all } m \in \{1, \ldots, N\}.$$

Hence, $\langle f_n, f_k \rangle = 0$ for all $n \neq k$. Since every finite frame for $\mathcal{H}$ is also a spanning set for $\mathcal{H}$, we have $(f_n)_{n=1}^N$ being an orthonormal basis. □

We close this section with the definition of the *spark* of a frame.

Definition 3.10. (*Spark of a frame*): Let $F = (f_n)_{n=1}^N$ be a frame for a d-dimensional Hilbert space $\mathcal{H}$. The *spark* of F is the cardinality of the smallest linearly dependent subset of the frame. A frame is *full spark* if every d-element subset of F is linearly independent, so it has spark $d+1$.

Example 3.11: Consider the following two frames for $\mathbb{R}^2$, each consisting of two orthonormal bases.

$$F = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix} \right)$$
$$G = \left(\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{pmatrix}, \begin{pmatrix} -\frac{\sqrt{3}}{2} \\ \frac{1}{2} \end{pmatrix} \right)$$

Then F is a frame with spark two and G is a frame having spark three. Hence G has full spark, since every subset of two frame vectors forms a basis of $\mathbb{R}$, which is not the case for F.

In many applications full spark frames are desirable because they are maximally robust against erasure of frame vectors. That is, one can remove $N - d$ vectors of the frame and is still left with a spanning set for the vector space. Jameson Cahill showed in his master thesis that full spark frames are dense in the space of frames and even that full spark Parseval frames are dense in the space of Parseval frames [Cah09]. Cahill, Dustin G. Mixon and Nate Strawn proved that this is also true for FUNTFs [CMS13].

3.2 Finite Frames and Operators

We now look at the operators associated to a frame, which encode crucial properties of the frame. Every finite frame comes with two linear operators, the *synthesis operator* T and its adjoint, the *analysis operator* T^*. They are given by

$$\begin{aligned} T\colon \mathbb{F}^N &\longrightarrow \mathcal{H} \\ (c_n)_{n=1}^N &\longmapsto \sum_{n=1}^N c_n f_n \end{aligned}$$

and

$$\begin{aligned} T^*\colon \mathcal{H} &\longrightarrow \mathbb{F}^N \\ x &\longmapsto \big(\langle x, f_n \rangle\big)_{n=1}^N. \end{aligned}$$

Strictly speaking, we can assign these two operators to any sequence of vectors in $\mathcal{H}$ whether it forms a frame or not.

We now show some properties of these two operators.

Lemma 3.12. (*[CK13, §1.4.1]*):
Let $(f_n)_{n=1}^N$ be a sequence of vectors in a d-dimensional Hilbert space $\mathcal{H}$ with associated synthesis operator T and analysis operator T^*.

(i) We have

$$\|T^*x\|^2 = \sum_{n=1}^{N} \left|\langle x, f_n\rangle\right|^2 \quad \textit{for all } x \in \mathcal{H}.$$

Hence, $(f_n)_{n=1}^N$ is a frame for $\mathcal{H}$ if and only if T^* is injective.

(ii) $(f_n)_{n=1}^N$ is a frame for $\mathcal{H}$ if and only if T is surjective.

Proof. (i) Follows directly from the definitions of the analysis operator and frames.

(ii) Follows from $(\operatorname{im} T)^\perp = (\ker T^*)$ and (i). □

Remark 3.13: The matrix structure of the analysis and synthesis operator is quite simple. Let $(f_n)_{n=1}^N$ be a frame for a d-dimensional Hilbert space $\mathcal{H}$, then the associated synthesis operator T is

$$T = \begin{pmatrix} | & | & \cdots & | \\ f_1 & f_2 & \cdots & f_N \\ | & | & \cdots & | \end{pmatrix},$$

that is, writing the frame vectors as columns of a matrix. Similarly one writes the analysis operator T^* as a matrix whose rows are the frame vectors.

To every finite frame we also associate the so called *frame operator* which encodes important properties of the frame and is useful in many ways.

Definition 3.14. (*Frame operator*): Let $(f_n)_{n=1}^N$ be a sequence of vectors in a d-dimensional Hilbert space $\mathcal{H}$ with associated synthesis operator T.

Then the associated *frame operator* $S\colon \mathcal{H} \longrightarrow \mathcal{H}$ is given by:

$$Sx = TT^*x = \sum_{n=1}^{N} \langle x, f_n \rangle f_n, \quad x \in \mathcal{H}.$$

Note, we are again stretching the terms and call S a *frame operator* whether the sequence $(f_n)_{n=1}^N$ forms a frame or not.

Proposition 3.15:
Let $(f_n)_{n=1}^N$ be a sequence of N vectors in a d-dimensional Hilbert space with associated synthesis operator T and frame operator S. Then S is the sum of N operators of rank less or equal to one, namely

$$S = TT^* = f_1 f_1^* + f_2 f_2^* + \cdots + f_N f_N^*.$$

Proof. This follows immediately from the definition of the frame operator. □

Clearly, S is a self-adjoint, invertible operator and has only real eigenvalues.

Lemma 3.16. (*[CK13, §1.4.2]*):
Let $(f_n)_{n=1}^N$ be a sequence of vectors in a d-dimensional Hilbert space $\mathcal{H}$ with associated frame operator S. Then, for all $x \in \mathcal{H}$,

$$\langle Sx, x \rangle = \sum_{n=1}^{N} \left| \langle x, f_n \rangle \right|^2.$$

Proof. Since $\langle Sx, x \rangle = \langle TT^*x, x \rangle = \|T^*x\|^2$, the statement follows from Lemma 3.12 on page 33. □

Theorem 3.17. (*[CK13, §1.4.2]*):
Let $(f_n)_{n=1}^N$ be a frame for a d-dimensional Hilbert space $\mathcal{H}$ with frame operator S having eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$. Then λ_1 and λ_d are the optimal upper and lower frame bound, respectively.

Proof. Let $(e_i)_{i=1}^d$ denote the normalized eigenvectors of the frame operator S forming an orthonormal basis of $\mathcal{H}$, with respective eigenvalues

$(\lambda_i)_{i=1}^d$ written in non-increasing order. Let $x \in \mathcal{H}$. Since $x = \sum_{i=1}^d \langle x, e_i \rangle e_i$, we obtain

$$Sx = \sum_{i=1}^{d} \lambda_i \langle x, e_i \rangle e_i.$$

By Lemma 3.16 on page 34, we have

$$\begin{aligned} \sum_{n=1}^{N} \left| \langle x, f_n \rangle \right|^2 = \langle Sx, x \rangle &= \left\langle \sum_{i=1}^{d} \lambda_i \langle x, e_i \rangle e_i, \sum_{i=1}^{d} \langle x, e_i \rangle e_i \right\rangle \\ &= \sum_{i=1}^{d} \lambda_i \left| \langle x, e_i \rangle \right|^2 \leq \lambda_1 \sum_{i=1}^{d} \left| \langle x, e_i \rangle \right|^2 = \lambda_1 \|x\|^2. \end{aligned}$$

Thus the optimal upper frame bound of the frame $(f_n)_{n=1}^N$ is less or equal to λ_1. Equality follows from

$$\sum_{n=1}^{N} \left| \langle e_1, f_n \rangle \right|^2 = \langle Se_1, e_1 \rangle = \langle \lambda_1 e_1, e_1 \rangle = \lambda_1.$$

In a similar way one proves the statement for the lower frame bound. □

We state the following theorem about self-adjoint operators by Roger A. Horn and Charles R. Johnson [HJ85, §4.3], but formulate it in frame theory language.

Theorem 3.18. (*Horn-Johnson*):
Let $(f_n)_{n=1}^N$ be a sequence of vectors in a d-dimensional Hilbert space $\mathcal{H}$ with frame operator S and let $f \in \mathcal{H}$ be a given vector. If the eigenvalues $\big(\lambda_i(S)\big)_{i=1}^d$ of S and $\big(\lambda_i(S + ff^*)\big)_{i=1}^d$ of $S + ff^*$ are arranged in non-increasing order, we have for all $2 \leq i \leq d$,

$$\lambda_i(S) \leq \lambda_i(S + ff^*) \leq \lambda_{i-1}(S). \tag{3.3}$$

The property stated in (3.18) is called *interlacing*. In other words, the Horn-Johnson Theorem 3.18 states that the spectra of the frame operators S and $S + ff^*$ interlace.

The following theorem gives the relation between frame vectors and the eigenvalues of the associated frame operator.

Theorem 3.19. (*[CK13, §1.4.2]*):
Let $(f_n)_{n=1}^N$ be a frame for a d-dimensional Hilbert space $\mathcal{H}$ with frame operator S having normalized eigenvectors $(e_i)_{i=1}^d$ and respective eigenvalues $(\lambda_i)_{i=1}^d$. Then for all $1 \leq i \leq d$ we have

$$\lambda_i = \sum_{n=1}^N \left|\langle e_i, f_n\rangle\right|^2.$$

Moreover,

$$\operatorname{Tr} S = \sum_{i=1}^d \lambda_i = \sum_{n=1}^N \|f_n\|^2.$$

Proof. Using Lemma 3.16 on page 34 we obtain for all $1 \leq i \leq d$

$$\lambda_i = \langle Se_i, e_i\rangle = \sum_{n=1}^N \left|\langle e_i, f_n\rangle\right|^2.$$

Furthermore, we have

$$\begin{aligned}\sum_{i=1}^d \lambda_i &= \sum_{i=1}^d \sum_{n=1}^N \left|\langle e_i, f_n\rangle\right|^2 \\ &= \sum_{n=1}^N \sum_{i=1}^d \left|\langle e_i, f_n\rangle\right|^2 = \sum_{n=1}^N \|f_n\|^2.\end{aligned}$$ □

The frame operator S is fundamental for the reconstruction of a vector (or a signal) from the frame coefficients. If one has an orthonormal basis the reconstruction is fairly simple as we have seen in Section 3.1. The following Theorem gives a similarly simple reconstruction formula for frames.

Theorem 3.20. (*[CK13, §1.5.1]*):
Let $(f_n)_{n=1}^N$ be a frame for a d-dimensional Hilbert space $\mathcal{H}$ with frame operator S. Then, for every $x \in \mathcal{H}$, we have

$$x = \sum_{n=1}^N \langle x, f_n\rangle S^{-1} f_n = \sum_{n=1}^N \left\langle x, S^{-1} f_n\right\rangle f_n.$$

Proof. We write x as $x = S^{-1}Sx$ and $x = SS^{-1}x$ and obtain the statement directly from Definition 3.14 on page 33. $\square$

This theorem also explains why tight frames are desirable. In case of tight frames the frame operator is a positive multiple of the identity and inverting is a trivial task and also numerically optimally stable. This is important for applications. From Theorem 3.20 we obtain as a corollary the reconstruction for Parseval frames.

Corollary 3.21. (*[CK13, §1.5.1]*):
Let $(f_n)_{n=1}^N$ be a Parseval frame for a d-dimensional Hilbert space $\mathcal{H}$. Then for every $x \in \mathcal{H}$, we have

$$x = \sum_{n=1}^{N} \langle x, f_n \rangle f_n.$$

Definition 3.22. (*Gramian operator*): Let $(f_n)_{n=1}^N$ be a frame for a d-dimensional Hilbert space $\mathcal{H}$ with synthesis operator T. The *Gramian operator* of $(f_n)_{n=1}^N$ is the operator $G\colon \mathbb{F}^N \to \mathbb{F}^N$ given by

$$G(a_n)_{n=1}^N = T^*T(a_n)_{n=1}^N = \left(\sum_{n=1}^{N} a_n \langle f_n, f_k \rangle \right)_{k=1}^{N} = \sum_{n=1}^{N} a_n \big(\langle f_n, f_k \rangle \big)_{k=1}^{N}.$$

We can represent the Gramian operator as a matrix (called *Gramian matrix*):

$$G = \begin{pmatrix} \|f_1\|^2 & \langle f_2, f_1 \rangle & \cdots & \langle f_N, f_1 \rangle \\ \langle f_1, f_2 \rangle & \|f_2\|^2 & \cdots & \langle f_N, f_2 \rangle \\ \cdots & \cdots & \ddots & \cdots \\ \langle f_1, f_N \rangle & \langle f_2, f_N \rangle & \cdots & \|f_N\|^2 \end{pmatrix}$$

From the matrix representation we can derive the following properties of the Gramian operator:

Proposition 3.23. (*[CK13, Theorem 1.7]*):
Let $(f_n)_{n=1}^N$ be a frame for a d-dimensional Hilbert space $\mathcal{H}$ with associated synthesis operator T, frame operator S and Gramian operator G. Then the following statements hold:

(i) The nonzero eigenvalues of G and S coincide.

(ii) The operator G is invertible if and only if $N = d$.

(iii) The sequence $(f_n)_{n=1}^N$ is a Parseval frame if and only if G is an orthogonal projection of rank d.

(iv) An operator $U\colon \mathcal{H} \to \mathcal{H}$ is unitary if and only if the Gramian operator of $(Uf_n)_{n=1}^N$ is equal to G.

Proof. (i) We use singular value decomposition to write $T = U\Sigma V^*$ were U is a $d \times d$ unitary matrix, Σ is a $d \times N$ rectangular diagonal matrix and V^* is an $N \times N$ unitary matrix. Hence we have $TT^* = U\Sigma\Sigma^* U^*$ where $\Sigma\Sigma^*$ is a $d \times d$ diagonal matrix whose diagonal entries are the eigenvalues of TT^*. Therefore the diagonal entries of Σ and the diagonal entries of Σ^* are the square roots of the eigenvalues of TT^* and of the nonzero eigenvalues of T^*T.

(ii) follows directly from (i).

(iii) G is self-adjoint and has rank d and $G^2 = (T^*T)(T^*T) = T^*(TT^*)T$. Since T^* is injective and T is surjective it follows that G is an orthogonal projection if and only if $TT^* = I_d$ which is the case if and only if $(f_n)_{n=1}^N$ is Parseval.

(iv) Let U be a unitary operator on $\mathcal{H}$. The entries of the Gramian operator of $(Uf_n)_{n=1}^N$ are $\langle Uf_n, Uf_m \rangle = \langle f_n, f_m \rangle$. Conversely, we have $\langle f_n, f_m \rangle = \langle Uf_n, Uf_m \rangle = \langle UU^* f_n, f_m \rangle$. Since $\operatorname{span}\big((f_n)_{n=1}^N\big) = \mathcal{H}$, we have that UU^* must be the identity on $\mathcal{H}$. □

3.3 Construction of Finite Frames

Often one is in need of frames with certain properties, such as tightness for easy reconstruction of a signal from frame coefficients. In this section we show how to construct tight frames from arbitrary sequences of vectors and other frames.

To simplify the notation from now on, we identify a finite frame $(f_i)_{i=1}^N$ with the matrix $F = (f_1|f_2|\cdots|f_N)$ which is also the synthesis operator as we learned in the previous section. Hence it is useful to also denote the *synthesis operator* F and the *analysis operator* F^* instead of T and T^*, respectively. The frame operator S is then written as $S = FF^*$ and the Gramian operator G as $G = F^*F$. We note at this point, that some authors interchange the notation for the synthesis and analysis operator. So they write F^*F for the frame operator and FF^* for the Gramian operator.

The following Lemma which gives a *construction of tight frames* was first proven by Peter G. Casazza and Nicole Leonhard [CL08].

Lemma 3.24. (*[CK13, §1.6.1]*)**:**
Let $F = (f_n)_{n=1}^N$ be a family of vectors in a d-dimensional Hilbert space $\mathcal{H}$ and let the frame operator FF^* having normalized eigenvectors $(e_i)_{i=1}^d$ and respective eigenvalues $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_d$. Let $1 \leq k \leq d$ be such that $\lambda_1 = \lambda_2 = \cdots = \lambda_k > \lambda_{k+1}$. Then

$$(f_n)_{n=1}^N \cup \left(\sqrt{\lambda_1 - \lambda_i}\, e_i\right)_{i=k+1}^d$$

is a tight frame for $\mathcal{H}$.

In particular, $d - k$ is the least number of vectors which can be added to $(f_n)_{n=1}^N$ to obtain a tight frame.

It is indeed an interesting result that one can construct tight frames from any given finite sequence of vectors.

Proof. Let $h_i = \sqrt{\lambda_1 - \lambda_i}\, e_i$ for $k+1 \leq i \leq d$. The frame operator S for the family $(f_n)_{n=1}^N \cup (h_i)_{i=k+1}^d$ is given by

$$Sf = FF^*f + \sum_{i=k+1}^{d} \langle f, h_i \rangle h_i.$$

By Proposition 3.1 on page 28, we have

$$f = \sum_{i=1}^{d} \langle f, e_i \rangle e_i.$$

for an arbitrary $f \in \mathcal{H}$. Therefore,

$$FF^*f = \sum_{i=1}^{d} \langle f, e_i \rangle FF^* e_i = \sum_{i=1}^{d} \lambda_i \langle f, e_i \rangle e_i.$$

Since $\lambda_1 = \lambda_2 = \cdots = \lambda_k$, we have

$$\begin{aligned} Sf &= \sum_{i=1}^{d} \lambda_i \langle f, e_i \rangle e_i + \sum_{i=k+1}^{d} (\lambda_1 - \lambda_i) \langle f, e_i \rangle e_i \\ &= \sum_{i=1}^{k} \lambda_i \langle f, e_i \rangle e_i + \sum_{i=k+1}^{d} \lambda_i \langle f, e_i \rangle e_i + \lambda_1 \sum_{i=k+1}^{d} \langle f, e_i \rangle e_i - \sum_{i=k+1}^{d} \lambda_i \langle f, e_i \rangle e_i \\ &= \lambda_1 \sum_{i=1}^{d} \langle f, e_i \rangle e_i \\ &= \lambda_1 f. \end{aligned}$$

Hence, $(f_n)_{n=1}^N \cup (h_i)_{i=k+1}^d$ is a tight frame for $\mathcal{H}$.

Now assume that there exists a family of vectors $(g_m)_{m \in I}$ with frame operator S such, that $(f_n)_{n=1}^N \cup (g_m)_{m \in I}$ is an A-tight frame. Hence by Theorem 3.19 on page 36 we have $A \geq \lambda_1$. Define $\widetilde{S}$ to be the operator on $\mathcal{H}$ given by

$$\widetilde{S} e_i = \begin{cases} 0, & 1 \leq i \leq k \\ (\lambda_1 - \lambda_i) e_i, & k+1 \leq i \leq d. \end{cases}$$

Then, we have $A \cdot I_d = FF^* + S$ and

$$\begin{aligned} \langle Sx, x \rangle &= \big\langle (A \cdot I_d - FF^*) x, x \big\rangle \\ &\geq \big\langle (\lambda_1 I_d - FF^*) x, x \big\rangle = \langle \widetilde{S} x, x \rangle, \quad \text{for all } x \in \mathcal{H}. \end{aligned}$$

Since $\widetilde{S}$ has $d-k$ non-zero eigenvalues, this is also true for S. Hence $|I| \geq d-k$, which proves the second statement. □

If one has a finite frame for a finite-dimensional Hilbert space $\mathcal{H}$ with frame bounds A and B then one can easily construct a frame for every subspace of $\mathcal{H}$ with the same frame bounds by the following Proposition.

Proposition 3.25. (*[CK13, §1.6.1]*):
Let $(f_n)_{n=1}^N$ be a frame for a d-dimensional Hilbert space $\mathcal{H}$ with frame bounds A and B. Let P be an orthogonal projection of $\mathcal{H}$ onto a subspace W. Then $(Pf_n)_{n=1}^N$ is a frame for W with frame bounds A and B.

Proof. For any $w \in W$, we have

$$A\|w\|^2 = A\|Pw\|^2 \le \sum_{n=1}^{N} \left|\langle Pw, f_n\rangle\right|^2$$
$$= \sum_{n=1}^{N} \left|\langle w, Pf_n\rangle\right|^2 \le B\|Pw\|^2 = B\|w\|^2.$$

This proves the claim. □

It follows directly that, if $(f_n)_{n=1}^N$ is a Parseval frame, this also holds for $(Pf_n)_{n=1}^N$.

One of the most useful methods for constructing new frames from existing ones which carries over many important properties is given by the following theorem [Cas+13].

Theorem 3.26. (*Naimark's Theorem*):
A family of vectors $F = (f_n)_{n=1}^N$ is a Parseval frame for a d-dimensional Hilbert space $\mathcal{H}$ if and only if there is an N-dimensional Hilbert space $\mathcal{H}' \supset \mathcal{H}$ with an orthonormal basis $(e_n)_{n=1}^N$ so that the orthogonal projection $P\colon \mathcal{H}' \to \mathcal{H}$ satisfies $Pe_n = F^* f_n$ for all $1 \le n \le N$. Moreover, $\big((I-P)e_n\big)_{n=1}^N$ is a Parseval frame for an $(N-d)$-dimensional Hilbert space. Such a frame is called a Naimark complement *of F.*

Proof. Since F is a Parseval frame, the analysis operator F^* is an isometry. Let P be the orthogonal projection of $\mathbb{F}^N$ onto $F^*(\mathcal{H})$. Then for any F^*f we have

$$\langle F^*f, Pe_n\rangle = \langle F^*f, e_n\rangle = \langle f, Fe_n\rangle = \langle f, f_n\rangle = \langle F^*f, F^*f_n\rangle.$$

Hence, we have $Pe_n = F^*f_n$. □

Peter G. Casazza, Matthew Fickus, Dustin G. Mixon, Jesse Peterson and Ihar Smalyanau extended the construction of Naimark complements to arbitrary finite frames [Cas+13].

Definition 3.27. (*General Naimark complement*): Let $(f_n)_{n=1}^N$ be a finite sequence of vectors in a d-dimensional Hilbert space $\mathcal{H}$ with Bessel bound B. Suppose, the corresponding frame operator have eigenvalues

$B = \lambda_1 = \cdots = \lambda_k > \lambda_{k+1} \geq \cdots \geq \lambda_d$. Let $(f_n)_{n=1}^N \cup (h_i)_{i=k+1}^d$ be the completion to a B-tight frame from Lemma 3.24 on page 39. Further, let $F^*\colon \mathcal{H} \to \mathbb{F}^{d+N-k}$ be the analysis operator for the Parseval frame

$$\left(\frac{1}{\sqrt{B}} f_n\right)_{n=1}^N \cup \left(\frac{1}{\sqrt{B}} h_i\right)_{i=k+1}^d .$$

F^* is an isometry and by Naimark's theorem, there exists an orthonormal basis $(e_n)_{n=1}^{d+N-k}$ for $\mathbb{F}^{d+N-k}$ such that $\sqrt{B}Pe_n = F^* f_n$ for all $n = 1, \ldots, N$ where P is the orthogonal projection onto $F^*(\mathcal{H})$. Then

$$(g_n)_{n=1}^N := \left(\sqrt{B}(I-P)e_n\right)_{n=1}^N$$

is a *general Naimark* complement of $(f_n)_{n=1}^N$.

If we restrict ourselves to tight frames, the above definition gets much simpler. Let $F = (f_n)_{n=1}^N$ be a B-tight frame for a d-dimensional Hilbert space. A general Naimark complement for F is a frame $G = (g_n)_{n=1}^N$ for a $(N-d)$-dimensional Hilbert space if and only if

$$\frac{1}{\sqrt{B}} \begin{pmatrix} F \\ G \end{pmatrix}$$

is a unitary matrix. In particular, we have that $F^*F + G^*G = B \cdot I_N$.

Another method to construct finite frames is to build Horn matrices whose existence is guaranteed by the Schur-Horn Theorem which we explain in the following.

Definition 3.28. (*Majorizing sequences*): Let $(a_n)_{n=1}^N$ and $(b_n)_{n=1}^N$ be two nonnegative nonincreasing sequences. We say $(a_n)_{n=1}^N$ *majorizes* $(b_n)_{n=1}^N$, denoted $(a_n)_{n=1}^N \succeq (b_n)_{n=1}^N$, if

$$\begin{aligned} \sum_{n=1}^m a_n &\geq \sum_{b=1}^m b_n, \quad \text{for all } m = 1, \ldots, N-1 \\ \sum_{n=1}^N a_n &= \sum_{k=1}^N b_n. \end{aligned}$$

A result of Issai Schur states that the spectrum of a self-adjoint matrix majorizes its diagonal entries [Sch23]. Alfred Horn later proved the converse result, if $(\lambda_n)_{n=1}^N \succeq (\mu_n)_{n=1}^N$, then there exists a self-adjoint matrix with spectrum $(\lambda_n)_{n=1}^N$ and $(\mu_n)_{n=1}^N$ as diagonal entries [Hor54]. Together these two results are known as the *Schur-Horn Theorem*.

Theorem 3.29. (*Schur-Horn Theorem*):
There exists a self-adjoint matrix with spectrum $(\lambda_n)_{n=1}^N$ and diagonal entries $(\mu_n)_{n=1}^N$ if and only if $(\lambda_n)_{n=1}^N \succeq (\mu_n)_{n=1}^N$.

Since the proof of Horn was non-constructive, only few methods were known to construct Horn matrices. In 2013 Jameson Cahill, Matthew Fickus, Dustin G. Mixon, Miriam J. Poteet and Nathanial K. Strawn gave an explicit construction method using *eigensteps*, which is a sequence of interlacing spectra [Cah+13]. Eigensteps will be discussed in the next chapter.

4 Polytopes of Eigensteps

Eigensteps have been introduced by Jameson Cahill, Matthew Fickus, Dustin G. Mixon, Miriam J. Poteet and Nathanial K. Strawn in [Cah+13] to construct all finite frames for a given spectrum and set of lengths. The results have been adopted in [Fic+13] to obtain an algorithm to construct all self-adjoint matrices with prescribed spectrum and diagonal. The existence of such matrices is given by the Schur-Horn theorem (Theorem 3.29 on page 43). The fact that eigensteps form a polytope, and therefore a path-connected set, has been used in [CMS13] to obtain connectivity and irreducibility results for algebraic varieties of finite unit norm tight frames in real and complex space. Parametrizing this polytope is crucial to apply the algorithms described in [Cah+13] and [Fic+13].

In this chapter, we consider the case of equal norm tight frames, where the describing equations and inequalities of the polytope of eigensteps can be drastically simplified. To be precise, we obtain a description of the polytope where the remaining inequalities are in one-to-one correspondence with the facets of the polytope.

This chapter is based on the paper the author of this thesis has co-authored with Christoph Pegel [HP15]. It is organized as follows. We start with the necessary preliminaries in Section 4.1 in order to study the polytope of eigensteps in a purely combinatoric manner in Sections 4.2 and 4.3. We give formulae for the dimension of the polytope and its number of facets. In Section 4.4 we come back to frame theory and describe how the affine isomorphisms of polytopes we obtained combinatorially are described by reversing the order of frame vectors and taking Naimark complements. We end with Section 4.5, where we discuss our results and some open questions.

4.1 Preliminaries

Let F_n denote the truncation of the frame F to the first n frame vectors and $(\lambda_{i,n})_{i=1}^{d}$ the spectrum of the frame operator $F_n F_n^*$. Then, we have that $F_0 F_0^*$ is a $d \times d$ zero matrix, thus having all eigenvalues equal to zero: $(\lambda_{i,0})_{i=1}^{d} = 0$. For $n = N$, we have that $F_N = F$.

The problem discussed in [Cah+13] is the following: given a non-increasing sequence of norm-squares $(\mu_n)_{n=1}^{N}$ and a non-increasing, non-negative spectrum $(\lambda_i)_{i=1}^{d}$, find all matrices $F = (f_n)_{n=1}^{N}$ such that $\|f_n\|^2 = \mu_n$ for all n and the non-increasing spectrum $\sigma(FF^*)$ of the frame operator FF^* is $(\lambda_i)_{i=1}^{d}$. To achieve this, the authors of [Fic+13] divide the task into two steps. First, find all possible sequences of spectra $((\lambda_{i,n})_{i=1}^{d})_{n=0}^{N}$, such that there exists an F with $\|f_n\|^2 = \mu_n$ and $\sigma(F_n F_n^*) = (\lambda_{i,n})_{i=1}^{d}$ for all n.

Any such sequence of spectra is called a *valid sequence of eigensteps* for the given input data $(\mu_n)_{n=1}^{N}$ and $(\lambda_i)_{i=1}^{d}$. Then, for a given valid sequence of eigensteps, find all F such that $\|f_n\|^2 = \mu_n$ and $\sigma(F_n F_n^*) = (\lambda_{i,n})_{i=1}^{d}$ for all n by iteratively adding frame vectors following an elaborate algorithm.

By Theorem 2 in [Cah+13], the *interlacing conditions* of Theorem 3.18 on page 35 and the *trace condition* of Theorem 3.19 on page 36 together with $\lambda_{i,0} = 0$ and $\lambda_{i,N} = \lambda_i$ for all i completely characterize the valid sequences of eigensteps. Since all conditions are linear equations or linear inequalities, the valid sequences of eigensteps form a polytope $\Lambda((\mu_n)_{n=1}^{N}, (\lambda_i)_{i=1}^{d})$ in $\mathbb{R}^{d\times(N+1)}$.

In this chapter, we consider the case of equal norm tight frames, where the lengths and spectra are constant, that is, we have $\mu_n = \mu$ for all n and $\lambda_n = \frac{N\mu}{d}$ for all n. This implies that we have the following identity for the frame operator: $FF^* = \frac{N\mu}{d} \cdot I_d$. In particular, this covers equal norm Parseval frames for $\mu = \frac{d}{N}$ and FUNTFs for $\mu = 1$. Besides these two, further important classes of finite equal norm tight frames are *Gabor frames*, which play an important role in time-frequency analysis, and *Grassmannian frames*, which are useful in communication and coding theory, see [CL08; SH03] for reference. To avoid fractions and increase readability, we discuss equal norm tight frames with $\mu = d$, hence $FF^* = N \cdot I_d$. By scaling, the results can of course be transferred to arbitrary finite equal norm tight frames.

Let $\Lambda_{N,d} := \Lambda((d)_{n=1}^N, (N)_{i=1}^d)$ denote the polytope of finite equal norm tight frames with norm-squares $\mu = d$. We arrive at the following combinatorial definition of the polytope of eigensteps:

Definition 4.1. (*Polytopes of eigensteps*): For integers $0 \leq d \leq N$, we define the *polytope of eigensteps of equal norm tight frames* $\Lambda_{N,d}$ as the set of all matrices

$$\lambda = \left(\lambda_{i,n}\right)_{\substack{1\leq i\leq d,\\ 0\leq n\leq N}} \in \mathbb{R}^{d\times(N+1)}$$

satisfying the following conditions:

$$\lambda_{i,0} = 0 \qquad \text{for } 1 \leq i \leq d, \tag{4.1}$$

$$\lambda_{i,N} = N \qquad \text{for } 1 \leq i \leq d, \tag{4.2}$$

$$\sum_{i=1}^{d} \lambda_{i,n} = dn \qquad \text{for } 0 \leq n \leq N, \tag{4.3}$$

$$\lambda_{i,n} \leq \lambda_{i,n+1} \qquad \text{for } 1 \leq i \leq d \text{ and } 0 \leq n < N, \tag{4.4}$$

$$\lambda_{i,n} \leq \lambda_{i-1,n-1} \qquad \text{for } 1 < i \leq d \text{ and } 0 < n \leq N. \tag{4.5}$$

We will refer to (4.1) and (4.2) as the *first* and *last column conditions*, respectively. The equations in (4.3) are *column sum conditions*, while (4.4) and (4.5) will be referred to as the *horizontal* and *diagonal inequalities*, respectively, for reasons obvious from Figure 4.1.

4.2 The Dimension of Polytopes of Finite Equal Norm Tight Frames

In this section we determine the dimension of $\Lambda_{N,d}$. The dimension of the solution set of a system of linear equations and inequalities can be computed from the number of variables and the number of linearly independent equations, including those arising from inequalities that are always satisfied with equality. Thus, the first step is to remove

$$\begin{array}{cccccccc}
0 = & \lambda_{d,0} & \lambda_{d,1} & \lambda_{d,2} & \cdots & \lambda_{d,N-1} & \lambda_{d,N} & = N \\
0 = & \lambda_{d-1,0} & \lambda_{d-1,1} & \lambda_{d-1,2} & \cdots & \lambda_{d-1,N-1} & \lambda_{d-1,N} & = N \\
0 = & \lambda_{d-2,0} & \lambda_{d-2,1} & \lambda_{d-2,2} & \cdots & \lambda_{d-2,N-1} & \lambda_{d-2,N} & = N \\
 & \vdots & \vdots & \vdots & & \vdots & \vdots & \\
0 = & \lambda_{1,0} & \lambda_{1,1} & \lambda_{1,2} & \cdots & \lambda_{1,N-1} & \lambda_{1,N} & = N \\
\hline
\Sigma & 0 & d & 2d & \cdots & (N-1)d & Nd &
\end{array}$$

Figure 4.1: The conditions for a valid sequence of eigensteps for equal norm tight frames with $\mu = d$. A wedge $\lambda_{i,j}$ ◀ $\lambda_{k,l}$ denotes an inequality $\lambda_{i,j} \leq \lambda_{k,l}$.

redundant equalities and recognize inequalities that are always satisfied with equality.

Proposition 4.2:
A matrix $(\lambda_{i,n}) \in \mathbb{R}^{d\times(N+1)}$ is a point of $\Lambda_{N,d}$ if and only if the following conditions are satisfied:

$$\lambda_{i,n} = 0 \qquad \textit{for} \quad i > n, \tag{4.6}$$

$$\lambda_{i,n} = N \qquad \textit{for} \quad i < n + d - N + 1, \tag{4.7}$$

$$\sum_{i=1}^{d} \lambda_{i,n} = dn \qquad \textit{for} \quad 0 < n < N, \tag{4.8}$$

$$\lambda_{i,n} \leq \lambda_{i,n+1} \qquad \textit{for} \quad 1 \leq i \leq d, \quad i \leq n < N - d + i - 1, \tag{4.9}$$

$$\lambda_{i,n} \leq \lambda_{i-1,n-1} \qquad \textit{for} \quad 1 < i \leq d, \quad i \leq n < N - d + i, \tag{4.10}$$

$$\lambda_{d,d} \geq 0, \tag{4.11}$$

$$\lambda_{1,N-d} \leq N. \tag{4.12}$$

Proof. The idea behind the proof is to use the first and last column conditions together with the horizontal and diagonal inequalities to obtain triangles in the eigenstep tableaux that consist of fixed 0- or N-entries.

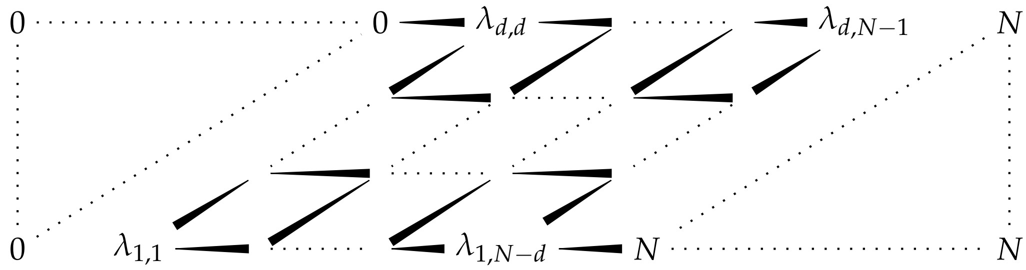


Figure 4.2: The modified conditions for a valid sequence of eigensteps for equal norm tight frames with only the inequalities required by Proposition 4.2.

Using those fixed triangles we can drop many of the now redundant inequalities from the system in Definition 4.1. The remaining inequalities form a parallelogram with two legs as depicted in Figure 4.2.

We first prove the necessity of the modified conditions. The triangles described by (4.6) and (4.7) (from now on referred to as the two *triangle conditions*, see Figure 4.2 for reference) are an immediate consequence of the first and last column conditions together with the horizontal and diagonal inequalities. The remaining equations and inequalities already appear as part of the definition of $\Lambda_{N,d}$.

To prove sufficiency, we first see that the first and last column conditions are implied by the triangle conditions. The first and last column are always fixed, so the column sum conditions can be weakened to (4.8). Condition (4.11) together with the weakened horizontal and diagonal inequalities (4.9) and (4.10) is enough to guarantee that all $\lambda_{i,n}$ are non-negative. Thus, we will refer to (4.11) as the *lower bound condition*. Similarly (4.12) guarantees $\lambda_{i,n} \leq N$ for all entries and will be referred to as the *upper bound condition*. Hence, from the original horizontal and diagonal inequalities (4.4) and (4.5) we only need those involving solely entries outside of the 0- and N-triangles. □

The remaining inequalities required by Proposition 4.2 on page 48 are depicted in Figure 4.2. Note that Proposition 4.2 on page 48 holds only for equal norm tight frames, in particular (4.7) is false for frames which are not tight.

With the modified conditions from Proposition 4.2 we are now able to compute the dimension of $\Lambda_{N,d}$.

Theorem 4.3:
The dimension of the polytope $\Lambda_{N,d}$ is 0 for $d = 0$ and $d = N$, otherwise

$$\dim(\Lambda_{N,d}) = (d-1)(N-d-1).$$

Proof. For $d = 0$ the only point in $\Lambda_{N,d}$ is the empty $0 \times (N+1)$ matrix, hence $\dim(\Lambda_{N,0}) = 0$. For $d = N$, the 0- and N-triangles fill up the whole matrix. Thus, $\Lambda_{N,N}$ also consists of a single point and $\dim(\Lambda_{N,N}) = 0$.

Otherwise, the triangle and sum conditions given by (4.6), (4.7) and (4.8) are linearly independent. Thus, by counting the equations, we obtain

$$\begin{aligned}\dim(\Lambda_{N,d}) &\leq d(N+1) - 2 \cdot \frac{d(d+1)}{2} - (N-1) \\ &= (d-1)(N-d-1).\end{aligned}$$

To verify $\dim(\Lambda_{N,d}) \geq (d-1)(N-d-1)$, we show that $\Lambda_{N,d}$ contains a *special point* $\widehat{\lambda}$ that satisfies all the inequalities (4.9) to (4.12) *strictly*, with the difference between the left and right hand sides of each inequality being equal to 1. The entries of $\widehat{\lambda}$ that are not fixed by the triangle conditions are given by

$$\widehat{\lambda}_{i,n} := d + n - 2i + 1 \quad \text{for} \quad i \leq n \leq N - d + i - 1. \tag{4.13}$$

See Example 4.4 for reference. The smallest value in (4.13) is $\widehat{\lambda}_{d,d} = 1$, the largest is $\widehat{\lambda}_{1,N-d} = N - 1$, so the lower and upper bound conditions are strictly satisfied. The horizontal and diagonal inequalities hold strictly as well, since

$$\begin{aligned}\widehat{\lambda}_{i,n} &= d + n - 2i + 1 \\ &< d + (n+1) - 2i + 1 = \widehat{\lambda}_{i,n+1}\end{aligned}$$

and

$$\begin{aligned}\widehat{\lambda}_{i,n} &= d + n - 2i + 1 \\ &< d + (n-1) - 2(i-1) + 1 = \widehat{\lambda}_{i-1,n-1}.\end{aligned}$$

It remains to verify the column sum conditions (4.8). For that, letting $i_0 := \max\{0, n + d - N\}$ and $i_1 := \min\{d, n\}$ we have

$$\begin{aligned}
\sum_{i=1}^{d} \widehat{\lambda}_{i,n} &= \sum_{i=1}^{i_0} N + \sum_{i=i_0+1}^{i_1} \widehat{\lambda}_{i,n} + \sum_{i=i_1+1}^{d} 0 \\
&= i_0 N + \sum_{i=i_0+1}^{i_1} (d+n-2i+1) \\
&= i_0 N + (i_1 - i_0)(d+n-i_1-i_0).
\end{aligned}$$

In all four cases, this expression evaluates to dn, which completes the proof. □

We elucidate the above result with an example.

Example 4.4: For $N = 6$, $d = 4$, the dimension of the polytope of eigensteps of equal norm tight frames $\Lambda_{6,4}$ is $\dim(\Lambda_{6,4}) = 3$ and we obtain the special point of $\Lambda_{6,4}$ as

$$\widehat{\lambda} = \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 2 & 6 \\ 0 & 0 & 0 & 2 & 3 & 6 & 6 \\ 0 & 0 & 3 & 4 & 6 & 6 & 6 \\ 0 & 4 & 5 & 6 & 6 & 6 & 6 \end{pmatrix}.$$

This tableau satisfies all inequalities in Proposition 4.2 strictly while also satisfying the column sum and triangle conditions.

4.3 The Facets of Polytopes of Eigensteps of Finite Equal Norm Tight Frames

In this section we investigate which of the remaining inequalities describing $\Lambda_{N,d}$ are necessary. In other words, we find the facet-describing inequalities of $\Lambda_{N,d}$. In particular, we obtain a formula for the number of facets.

To reduce the number of inequalities we need to consider separately, we use two kinds of dualities. One is an affine isomorphism between $\Lambda_{N,d}$ and $\Lambda_{N,N-d}$ that translates horizontal to diagonal inequalities and vice versa. The other is an affine involution on $\Lambda_{N,d}$, reversing the order of rows and columns of the eigenstep tableaux. We will see in Section 4.4 how these dualities correspond to certain operations on equal norm tight frames.

From the proof of Theorem 4.3 on page 50 we know that the affine hull $\operatorname{aff}(\Lambda_{N,d})$ is the affine subspace of $\mathbb{R}^{d\times(N+1)}$ defined by the triangle and sum conditions (Equations (4.6) to (4.8) on page 48).

Proposition 4.5:
There is an affine isomorphism $\Psi_{N,d}\colon \operatorname{aff}(\Lambda_{N,d}) \longrightarrow \operatorname{aff}(\Lambda_{N,N-d})$ *given by*

$$(\Psi_{N,d}(\lambda))_{i,n} = \begin{cases} \lambda_{d+i-n,N-n}, & \text{for } i \le n \le d+i-1, \\ 0, & \text{for } n < i, \\ N, & \text{for } n > d+i-1, \end{cases}$$

that restricts to an affine isomorphism $\Lambda_{N,d} \to \Lambda_{N,N-d}$.

As a map of eigenstep tableaux, $\Psi_{N,d}$ can be understood as interchanging rows and diagonals in the parallelogram of non-fixed entries while adjusting the sizes of 0- and N-triangles. For $N = 5$ and $d = 3$ the map $\Psi_{5,3}\colon \operatorname{aff}(\Lambda_{5,3}) \to \operatorname{aff}(\Lambda_{5,2})$ is given by

$$\Psi_{5,3}\begin{pmatrix} 0 & 0 & 0 & \lambda_{3,3} & \lambda_{3,4} & 5 \\ 0 & 0 & \lambda_{2,2} & \lambda_{2,3} & 5 & 5 \\ 0 & \lambda_{1,1} & \lambda_{1,2} & 5 & 5 & 5 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \lambda_{3,3} & \lambda_{2,2} & \lambda_{1,1} & 5 \\ 0 & \lambda_{3,4} & \lambda_{2,3} & \lambda_{1,2} & 5 & 5 \end{pmatrix}.$$

Note, that the entries outside the 0- and N-triangles that are in the same column, will be mapped in the same column under Ψ.

In Figure 4.3 on the facing page we illustrate the general structure of an image of an eigenstep tableau under $\Psi_{N,d}$.

Proof. We first need to verify that $\lambda' := \Psi_{N,d}(\lambda)$ is a point in $\operatorname{aff}(\Lambda_{N,N-d})$ for $\lambda \in \operatorname{aff}(\Lambda_{N,d})$. The triangle conditions are satisfied by the definition of $\Psi_{N,d}$. To verify the sum conditions for $\Lambda_{N,N-d}$, let $1 \le n \le N-1$ and $m := N - n$, then we have the following identity.

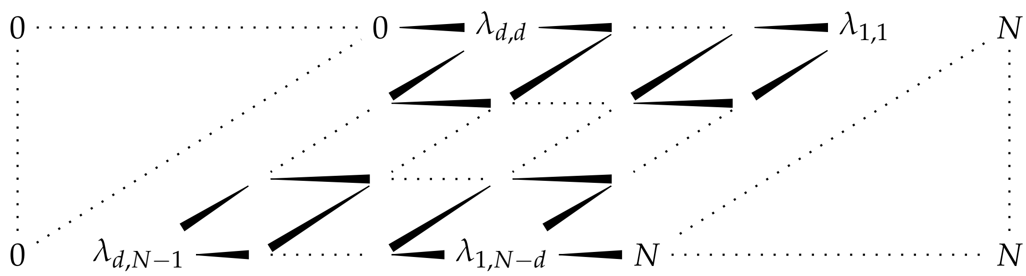


Figure 4.3: The image of a sequence of eigensteps λ as in Figure 4.2 under the affine isomorphism $\Psi_{N,d}\colon \Lambda_{N,d} \to \Lambda_{N,N-d}$ from Proposition 4.5.

$$\begin{aligned}
\sum_{i=1}^{N-d} \lambda'_{i,n} &= \sum_{i=1}^{\max\{0,n-d\}} N \quad + \sum_{i=\max\{0,n-d\}+1}^{\min\{N-d,n\}} \lambda_{d+i-n,N-n} \\
&= \max\{0,n-d\}N + \sum_{j=\max\{0,m+d-N\}+1}^{\min\{d,m\}} \lambda_{j,m} \\
&= \max\{0,n-d\}N + dm - \max\{0,m+d-N\}N \\
&= \max\{0,n-d\}N - \max\{0,d-n\}N + d(N-n) \\
&= (n-d)N + d(N-n) = (N-d)n.
\end{aligned}$$

To see that $\Psi_{N,d}$ restricts to an affine map $\Lambda_{N,d} \to \Lambda_{N,N-d}$ we need to consider all inequalities. Let $\lambda \in \Lambda_{N,d}$ and $\lambda' = \Psi_{N,d}(\lambda)$. The lower and upper bound conditions are satisfied, since $\lambda'_{N-d,N-d} = \lambda_{d,d} \geq 0$ and $\lambda'_{1,d} = \lambda_{1,N-d} \leq N$. The remaining horizontal and diagonal inequalities (4.9) and (4.10) interchange under $\Psi_{N,d}$. Let $j := d+i-n$ and $m := N-n$, then we have

$$\begin{aligned}
& \lambda'_{i,n} \leq \lambda'_{i,n+1} && \text{for} \quad 1 \leq i \leq N-d, i \leq n < d+i-1 \\
\Leftrightarrow \quad & \lambda_{j,m} \leq \lambda_{j-1,m-1} && \text{for} \quad 1 < j \leq d, j \leq m < N-d+j
\end{aligned}$$

and

$$\begin{aligned}
& \lambda'_{i,n} \leq \lambda'_{i-1,n-1} && \text{for} \quad 1 < i \leq N-d, i \leq n < d+i \\
\Leftrightarrow \quad & \lambda_{j,m} \leq \lambda_{j,m+1} && \text{for} \quad 1 \leq j \leq d, j \leq m < N-d+j-1.
\end{aligned}$$

Hence, $\Psi_{N,d}$ restricts to an affine map $\Lambda_{N,d} \longrightarrow \Lambda_{N,N-d}$. It is an isomor-

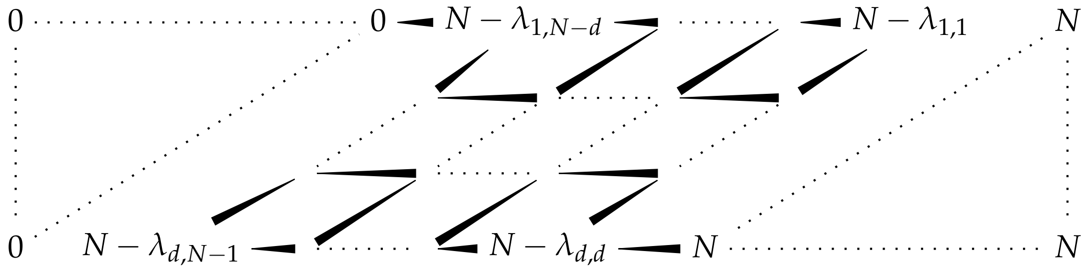


Figure 4.4: The image of a sequence of eigensteps λ as in Figure 4.2 under the involution $\Phi_{N,d}$ from Proposition 4.6.

phism on both the affine hulls and the polytopes themselves, since $\Psi_{N,d}$ and $\Psi_{N,N-d}$ are mutually inverse. This needs to be checked only for the non-fixed entries:

$$\begin{aligned}
(\Psi_{N,N-d}(\Psi_{N,d}(\lambda)))_{i,n} &= (\Psi_{N,d}(\lambda))_{N-d+i-n,N-n} \\
&= \lambda_{d+N-d+i-n-N+n,N-N+n} = \lambda_{i,n}, \\
(\Psi_{N,d}(\Psi_{N,N-d}(\lambda)))_{i,n} &= (\Psi_{N,N-d}(\lambda))_{d+i-n,N-n} \\
&= \lambda_{N-d+d+i-n-N+n,N-N+n} = \lambda_{i,n}. \qquad \square
\end{aligned}$$

Proposition 4.6:
There is an affine involution $\Phi_{N,d} \colon \mathbb{R}^{d\times(N+1)} \longrightarrow \mathbb{R}^{d\times(N+1)}$ *given by*

$$\Phi(\lambda)_{i,n} = N - \lambda_{d-i+1,N-n},$$

that restricts to an affine involution $\Lambda_{N,d} \to \Lambda_{N,d}$.

The involution $\Phi_{N,d}$ can be understood as rotating the whole eigenstep tableau by $180°$ and subtracting every entry from N, as depicted in Figure 4.4.

Proof. It is clear that $\Phi_{N,d}$ is an affine map $\mathbb{R}^{d\times(N+1)} \to \mathbb{R}^{d\times(N+1)}$. We use the original system of equations and inequalities given in Definition 4.1 on page 47 to verify $\Phi(\lambda) \in \Lambda_{N,d}$ when $\lambda \in \Lambda_{N,d}$. For $n = 0, N$ we obtain

$$\begin{aligned}
\Phi(\lambda)_{i,0} &= N - \lambda_{d-i+1,N} = N - N = 0, \\
\Phi(\lambda)_{i,N} &= N - \lambda_{d-i+1,0} = N - 0 = N.
\end{aligned}$$

Hence, the first and last column sum condition Equations (4.1) and (4.2) are satisfied by $\Phi(\lambda)$. The column sum conditions (4.3) are satisfied, since

$$\begin{aligned}\sum_{i=1}^{d} \Phi(\lambda)_{i,n} &= \sum_{i=1}^{d} \left(N - \lambda_{d-i+1,N-n}\right) \\ &= \sum_{j=1}^{d} \left(N - \lambda_{j,N-n}\right) \\ &= dN - d(N-n) = dn.\end{aligned}$$

For the horizontal and diagonal inequalities, we observe that $\lambda_{i,n} \leq \lambda_{i',n'}$ is equivalent to $N - \lambda_{i',n'} \leq N - \lambda_{i,n}$.

Finally, $\Phi_{N,d}$ is an involution on both $\mathbb{R}^{d\times(N+1)}$ and $\Lambda_{N,d}$, since

$$\begin{aligned}\left((\Phi_{N,d} \circ \Phi_{N,d})(\lambda)\right)_{i,n} &= N - \left(\Phi_{N,d}(\lambda)\right)_{d-i+1,N-n} \\ &= N - (N - \lambda_{i,n}) = \lambda_{i,n}.\end{aligned}$$ □

The results noted in the following remark are easily verified by direct computation.

Remark 4.7: The special point $\widehat{\lambda}$ of $\Lambda_{N,d}$ is fixed under $\Phi_{N,d}$ and mapped to the special point of $\Lambda_{N,N-d}$ by $\Psi_{N,d}$. Furthermore, Φ and Ψ commute. To be precise:

$$\Phi_{N,N-d} \circ \Psi_{N,d} = \Psi_{N,d} \circ \Phi_{N,d}.$$

Using the dualities given by Φ and Ψ, we now construct points that witness the necessity of most of the inequalities in Proposition 4.2 on page 48.

Lemma 4.8:
Let $N \geq 5$ and $2 \leq d \leq N-2$. Consider one of the inequalities in (4.9) to (4.12) which is not $\lambda_{2,2} \leq \lambda_{1,1}$, $\lambda_{1,1} \leq \lambda_{1,2}$, $\lambda_{d,N-2} \leq \lambda_{d,N-1}$ or $\lambda_{d,N-1} \leq \lambda_{d-1,N-2}$. Then there is a point in $\mathbb{R}^{d\times(N+1)}$ satisfying all conditions of Proposition 4.2 except the considered inequality.

Proof. The idea behind the proof is to start with the special point $\widehat{\lambda} \in \Lambda_{N,d}$ and locally change entries such that just one of the inequalities fails, while

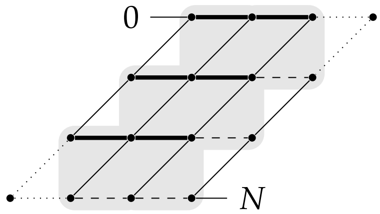
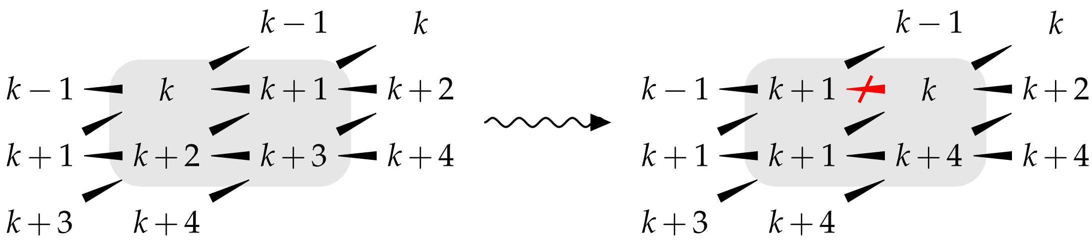


Figure 4.5: The horizontal inequalities treated by the modification (4.14) shown in bold.

preserving all other conditions. Since $\Psi_{N,d}$ translates horizontal (4.9) to diagonal inequalities (4.10) and vice versa, it is enough to consider only horizontal inequalities. Also, since $\Phi_{N,d}$ maps the top row ($i = d$) to the bottom row ($i = 1$), the inequalities in the bottom row do not need to be considered either. Since $\Phi_{N,d}$ maps the first diagonal ($i = n$) to the last ($i = n + d - N + 1$) and vice versa, we do not need to consider the last horizontal inequality in each row. The remaining horizontal inequalities are treated with the following modification of $\widehat{\lambda}$:

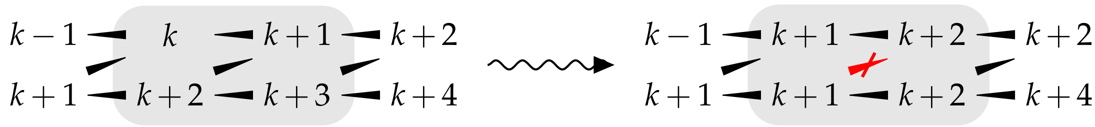


(4.14)

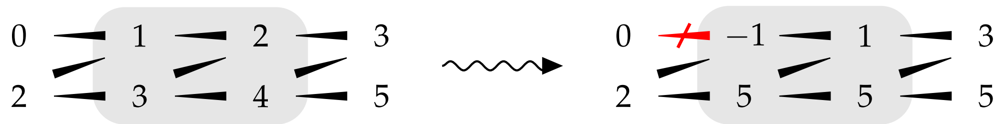
Note that this modification of $\widehat{\lambda}$ does not alter the column sums and causes only the marked red inequality in (4.14) to fail. If $2 \leq d < N - 2$ the square of modified entries in (4.14) fits into the parallelogram of non-fixed entries. In Figure 4.5 we demonstrate how this modification can be used to obtain points that let each of the bold inequalities fail individually. The dashed inequalities are covered by the above argument using $\Phi_{N,d}$, while the dotted inequalities are the four exceptions mentioned in Lemma 4.8 on page 55.

If $d = N - 2$, the parallelogram of non-fixed entries becomes too thin to fit the squares of (4.14), so this case has to be treated separately. Instead of considering the horizontal inequalities for $d = N - 2$, we can use the duality given by $\Psi_{N,d}$ and consider the diagonal inequalities for $d = 2$.

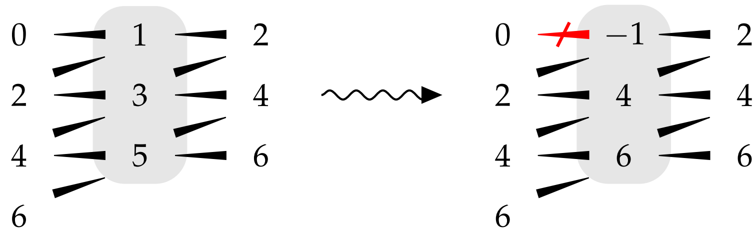
We use the following modification of $\widehat{\lambda}$:



The only inequality that remains to be treated is the lower bound condition $\lambda_{d,d} \geq 0$. The upper bound condition then follows from the duality given by $\Phi_{N,d}$. Here we use a modification of $\widehat{\lambda}$ to construct a point that causes only the lower bound condition to fail. We first do this for $d = 2$:



This also covers the case $d = N - 2$ by dualizing using $\Psi_{N,2}$. For the case $2 < d < N - 2$ we use a different modification of $\widehat{\lambda}$:



□

The following example shows the witnessing points in two dimensions.

Example 4.9: We construct the points given by Lemma 4.8 explicitly for $N = 5$, $d = 2$. The special point of $\Lambda_{5,2}$ is

$$\widehat{\lambda} = \begin{pmatrix} 0 & 0 & 1 & 2 & 3 & 5 \\ 0 & 2 & 3 & 4 & 5 & 5 \end{pmatrix}.$$

The half-spaces described by the non-exceptional inequalities are

$$H_1\colon \lambda_{2,2} \geq 0, \qquad H_2\colon \lambda_{2,2} \leq \lambda_{2,3},$$
$$H_3\colon \lambda_{2,3} \leq \lambda_{1,2}, \qquad H_4\colon \lambda_{1,2} \leq \lambda_{1,3} \quad \text{and}$$
$$H_5\colon \lambda_{1,3} \leq 5.$$

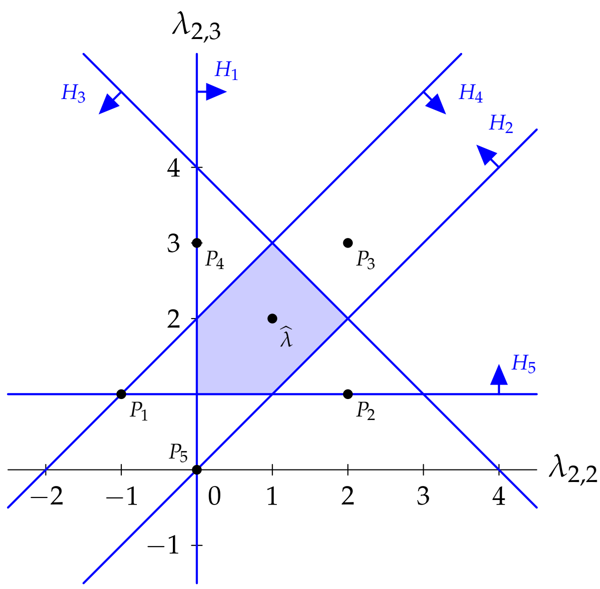


Figure 4.6: For $\Lambda_{5,2}$ we have five necessary inequalities. The points P_i which satisfy all conditions but the defining inequality for the half-space H_i are constructed in Example 4.9 on page 57.

Applying Lemma 4.8 on page 55 yields the following five points P_i, each satisfying all conditions except lying in the half-space H_i:

$$P_1 = \begin{pmatrix} 0 & 0 & -1 & 1 & 3 & 5 \\ 0 & 2 & 5 & 5 & 5 & 5 \end{pmatrix},$$

$$P_2 = \begin{pmatrix} 0 & 0 & 2 & 1 & 3 & 5 \\ 0 & 2 & 2 & 5 & 5 & 5 \end{pmatrix},$$

$$P_3 = \begin{pmatrix} 0 & 0 & 2 & 3 & 3 & 5 \\ 0 & 2 & 2 & 3 & 5 & 5 \end{pmatrix},$$

$$P_4 = \Phi_{5,2}(P_2) = \begin{pmatrix} 0 & 0 & 0 & 3 & 3 & 5 \\ 0 & 2 & 4 & 3 & 5 & 5 \end{pmatrix},$$

$$P_5 = \Phi_{5,2}(P_1) = \begin{pmatrix} 0 & 0 & 0 & 0 & 3 & 5 \\ 0 & 2 & 4 & 6 & 5 & 5 \end{pmatrix}.$$

The two variables $\lambda_{2,2}$ and $\lambda_{2,3}$ completely parametrize the polytope, since $\lambda_{1,1} = 2$, $\lambda_{1,2} = 4 - \lambda_{2,2}$, $\lambda_{1,3} = 6 - \lambda_{2,3}$ and $\lambda_{2,4} = 3$ by the column sum conditions. Hence, we can illustrate the situation in the plane, as done in Figure 4.6.

Using Lemma 4.8 on page 55, we prove the following theorem, giving the number of facets of $\Lambda_{N,d}$.

Theorem 4.10:
For $2 \leq d \leq N-2$ *the number of facets of* $\Lambda_{N,d}$ *is*

$$d(N-d-1)+(N-d)(d-1)-2. \tag{4.15}$$

Proof. In a first step, we show that for the case of $N \geq 5$ the amount of $d(N-d-1)+(N-d)(d-1)-2$ inequalities are sufficient to describe $\Lambda_{N,d}$ in its affine hull.

Let $N \geq 5$, and $2 \leq d \leq N-2$. Counting the horizontal and diagonal inequalities (4.9) and (4.10) yields $d(N-d-1)+(d-1)(N-d)$ inequalities.

We now show that the four inequalities between non-fixed entries that are already mentioned in Lemma 4.8 on page 55 are in fact not necessary. Recall that these are

$$\lambda_{2,2} \leq \lambda_{1,1}, \tag{4.16}$$
$$\lambda_{1,1} \leq \lambda_{1,2}, \tag{4.17}$$
$$\lambda_{d,N-2} \leq \lambda_{d,N-1}, \tag{4.18}$$
$$\lambda_{d,N-1} \leq \lambda_{d-1,N-2}, \tag{4.19}$$

From the column sum and triangle conditions it follows that $\lambda_{1,1} = d$ and $\lambda_{1,2} + \lambda_{2,2} = 2d$.

Thus (4.16) and (4.17) are both equivalent to $\lambda_{2,2} \leq d$, which is already implied by $\lambda_{2,2} \leq \lambda_{2,3} \leq \lambda_{1,2} = 2d - \lambda_{2,2}$, when $d \leq N-2$. Therefore (4.16) and (4.17) are superfluous.

Again, by the column sum and triangle conditions, we have the equalities $\lambda_{d,N-1} = N-d$ and $\lambda_{d,N-2} + \lambda_{d-1,N-2} = 2(N-d)$. Thus (4.18) and (4.19) are both equivalent to $\lambda_{d-1,N-2} \geq N-d$, which is already implied for $d \leq N-2$ by

$$\lambda_{d-1,N-2} \geq \lambda_{d-1,N-3} \geq \lambda_{d,N-2} = 2(N-d) - \lambda_{d-1,N-2}.$$

However the two arguments are independent only when $N \geq 5$, since for $N = 4$ and $d = 2$ we have $\lambda_{2,2} = \lambda_{d,N-2}$ and $\lambda_{1,2} = \lambda_{d-1,N-2}$.

Counting all inequalities, including the lower and upper bound conditions, excluding the four superfluous inequalities, we have

$$\begin{aligned} d(N-d-1) + (d-1)(N-d) + 2 - 4 \\ = d(N-d-1) + (d-1)(N-d) - 2 \end{aligned} \tag{4.20}$$

inequalities that are sufficient to describe $\Lambda_{N,d}$. From Lemma 4.8 on page 55 we know that all these inequalities are actually necessary, hence we obtain the desired number of facets.

For the case $N = 4$, $d = 2$, we have $\dim(\Lambda_{4,2}) = (2-1)(4-2-1) = 1$. The only polytope of dimension 1 is a line segment, the two endpoints being its facets. Thus, $\Lambda_{4,2}$ has two facets, as given by Equation (4.15) on page 59 for $N = 4$, $d = 2$. $\square$

From Lemma 4.8 on page 55 and Theorem 4.10 on page 59 we conclude that removing the four exceptional inequalities from the description of $\Lambda_{N,d}$ in Proposition 4.2 on page 48 yields a non-redundant system of equations and inequalities.

4.4 Connections between Frame and Eigenstep Operations

Until now we focussed on the combinatorics of sequences of eigensteps. In this section, we give descriptions of the affine isomorphisms $\Phi_{N,d}$ and $\Psi_{N,d}$ in terms of the underlying frames.

For this section, we fix the following notations: given a frame $F = (f_n)_{n=1}^{N}$, let λ_F denote the sequence of eigensteps associated to an equal norm tight frame F, that is $\lambda_F := (\sigma(F_n F_n^*))_{n=1}^{N}$ and let $\widetilde{F} := (f_{N-n})_{n=0}^{N-1}$ denote the frame with reversed order of frame vectors.

We obtain the following result:

Proposition 4.11:
Let $F = (f_n)_{n=1}^{N}$ be an equal norm tight frame in $\mathbb{F}^d$ with $\|f_n\|^2 = d$, then

$$\Phi_{N,d}(\lambda_F) = \lambda_{\widetilde{F}}.$$

Proof. Decomposing the frame operator of F we have

$$N \cdot I_d = FF^* = \sum_{k=1}^{N} f_k f_k^* = \sum_{k=1}^{n} f_k f_k^* + \sum_{k=n+1}^{N} f_k f_k^* = F_n F_n^* + \widetilde{F}_{N-n}\widetilde{F}_{N-n}^*.$$

Thus, if $v \in \mathbb{F}^d$ is an eigenvector of $F_n F_n^*$ with eigenvalue γ, we obtain

$$\widetilde{F}_{N-n}\widetilde{F}_{N-n}^* v = (N \cdot I_d - F_n F_n^*)v = (N-\gamma)v.$$

So v is an eigenvector of partial frame operator $\widetilde{F}_{N-n}\widetilde{F}_{N-n}^*$ with eigenvalue $N-\gamma$ and $\lambda_{\widetilde{F}} = \Phi_{N,d}(\lambda_F)$. □

Recall that by Definition 3.27 on page 41, for a given equal norm tight frame $F = (f_n)_{n=1}^N$ in $\mathbb{F}^d$ with $\|f_n\|^2 = d$, a frame $G = (g_n)_{n=1}^N$ in $\mathbb{F}^{N-d}$ satisfying $F^*F + G^*G = N \cdot I_N$ is called a *Naimark complement* of F. The following proposition shows how the duality described by $\Psi_{N,d}$ corresponds to taking a Naimark complement and reversing the order of frame vectors.

Proposition 4.12:
Let $F = (f_n)_{n=1}^N$ be an equal norm tight frame in $\mathbb{F}^d$ with norms $\|f_n\|^2 = d$ and $G = (g_n)_{n=1}^N$ a Naimark complement of F, then

$$\Psi_{N,d}(\lambda_F) = \lambda_{\widetilde{G}}.$$

Proof. Since $\Psi_{N,d}(\lambda_F) = \lambda_{\widetilde{G}}$ is equivalent to $\lambda_F = \Psi_{N,N-d}(\lambda_{\widetilde{G}})$, we only need to consider the case $N \geq 2d$. We first consider the columns of F with indices $n < d$. Since F_n is an $d \times n$ matrix, $F_n F_n^*$ has at most n non-zero eigenvalues. To be precise, the spectrum of the frame operator of F_n is

$$\sigma(F_n F_n^*) = (\lambda_{1,n}, \ldots, \lambda_{n,n}, \underbrace{0, \ldots, 0}_{d-n}).$$

In order to obtain the eigensteps of G, we switch to Gram matrices. The Gram matrix of F_n is the $n \times n$ matrix $F_n^* F_n$, with spectrum

$$\sigma(F_n^* F_n) = (\lambda_{1,n}, \ldots, \lambda_{n,n}),$$

which is obtained by considering the singular value decomposition of F_n.

Since G is a Naimark complement of F, we have $F^*F + G^*G = N \cdot I_N$. In particular,

$$\begin{aligned} N \cdot I_N = F^*F + G^*G &= \begin{pmatrix} F^* & G^* \end{pmatrix} \begin{pmatrix} F \\ G \end{pmatrix} \\ &= \begin{pmatrix} F_n^* & G_n^* \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} F_n & \cdots \\ G_n & \cdots \end{pmatrix} = \begin{pmatrix} F_n^*F_n + G_n^*G_n & \cdots \\ \vdots & \ddots \end{pmatrix}. \end{aligned}$$

The first n rows and columns of this identity yield $F_n^*F_n + G_n^*G_n = N \cdot I_n$. Therefore

$$\sigma(G_n^*G_n) = (N - \lambda_{n,n}, \dots, N - \lambda_{1,n}).$$

Going back to the frame operator of G_n, which is the $(N-d) \times (N-d)$ matrix $G_nG_n^*$, we have

$$\sigma(G_nG_n^*) = (N - \lambda_{n,n}, \dots, N - \lambda_{1,n}, \underbrace{0, \dots, 0}_{N-d-n}).$$

Finally, using $\widetilde{G}_{N-n}\widetilde{G}_{N-n}^* + G_nG_n^* = GG^* = N \cdot I_{N-d}$, we obtain

$$\sigma(\widetilde{G}_{N-n}\widetilde{G}_{N-n}^*) = (\underbrace{N, \dots, N}_{N-d-n}, \lambda_{n,n}, \dots, \lambda_{1,n}),$$

which shows that the $(N-n)$-th column of the image of λ_F under Ψ is equal to the $(N-n)$-th column of $\lambda_{\widetilde{G}}$ for $n < d$.

For $n > N-d$, let $m := N-n$ so that $m < d$. Hence, we have that the $(N-m)$-th column of $\Psi(\lambda_{\widetilde{F}})$ is the $(N-m)$-th column of λ_G by the previous argument. Since $\lambda_{\widetilde{F}} = \Phi_{N,d}(\lambda_F)$ and $\lambda_G = \Phi_{N,N-d}(\lambda_{\widetilde{G}})$, we know that $\Psi_{N,d}(\Phi_{N,d}(\lambda_F))$ and $\Phi_{N,N-d}(\lambda_{\widetilde{G}})$ agree in the n-th column. Using $\Psi_{N,d} \circ \Phi_{N,d} = \Phi_{N,N-d} \circ \Psi_{N,d}$ and the fact that $\Phi_{N,N-d}$ reverses the column order, we conclude that $\Psi_{N,d}(\lambda_F)$ and $\lambda_{\widetilde{G}}$ agree in the $(N-n)$-th column as desired.

We now consider $d \leq n \leq N-d$. By the same arguments as before, we have

$$\sigma(F_nF_n^*) = (\lambda_{1,n}, \dots, \lambda_{d,n}),$$

$$\sigma(F_n^* F_n) = (\lambda_{1,n}, \ldots, \lambda_{d,n}, \underbrace{0, \ldots, 0}_{n-d}), \quad \text{and}$$
$$\sigma(G_n^* G_n) = (\underbrace{N, \ldots, N}_{n-d}, N - \lambda_{d,n}, \ldots, N - \lambda_{1,n}).$$

Since G_n is an $(N-d) \times n$ matrix, with $N - d \geq n$, the spectrum of the frame operator of G_n is

$$\sigma(G_n G_n^*) = (\underbrace{N, \ldots, N}_{n-d}, N - \lambda_{d,n}, \ldots, N - \lambda_{1,n}, \underbrace{0, \ldots, 0}_{N-d-n}),$$

thus

$$\sigma(\widetilde{G}_{N-n} \widetilde{G}_{N-n}^*) = (\underbrace{N, \ldots, N}_{N-d-n}, \lambda_{1,n}, \ldots, \lambda_{d,n}, \underbrace{0, \ldots, 0}_{n-d}),$$

which shows that the $(N-n)$-th column of $\Psi(\lambda_F)$ is equal to the $(N-n)$-th column of $\lambda_{\widetilde{G}}$ for $d \leq n \leq N - d$. $\square$

4.5 Conclusion and Open Problems

As we have seen, in the special case of equal norm tight frames we are able to obtain a general non-redundant description of the polytope of eigensteps in terms of equations and inequalities. However, this description does not generalize to non-tight frames, where we lose the N-triangle in the eigenstep tableau. Hence, even the dimension of $\Lambda((\mu_n)_{n=1}^N, (\lambda_i)_{i=1}^d)$ will depend on the multiplicities of eigenvalues in the spectrum that cause smaller triangles of fixed entries in the eigenstep tableaux.

On the frame theoretic end, it might be interesting to study properties of frames F corresponding to certain points of the polytope. For example, interesting classes of equal norm tight frames might be the frames F such that λ_F is the special point $\widehat{\lambda}$, a boundary point of $\Lambda_{N,d}$ or a vertex of $\Lambda_{N,d}$.

From a discrete geometers point of view, it might be interesting to find a description of polytopes of eigensteps in terms of vertices. However, even restricting to equal norm tight frames, we were not able to calculate the number of vertices of $\Lambda_{N,d}$ in general, let alone find a description of the polytope as a convex hull of vertices.

5 Related Work

In discrete geometry there are some classes of polytopes that have a construction similar to polytopes of eigensteps. In this chapter we give an outline of two of them, namely Stanley's order polytopes and Gelfand-Tsetlin polytopes, and show which similarities they have to polytopes of eigensteps.

In Section 5.1 we give the definition of Stanley's order polytope and discuss the relation to polytopes of eigensteps. In Section 5.2 we do the same with the Gelfand-Tsetlin polytope.

5.1 Order Polytopes

Richard P. Stanley introduced two kinds of polytope associated to a poset in 1986 [Sta86], the order polytope and the chain polytope. The construction of order polytopes is similar to that of polytopes of eigensteps. In fact it can be viewed as a generalization of polytopes of eigensteps. We follow Stanley's paper to define order polytopes and show some useful properties, then we investigate the relation between order polytopes and polytopes of eigensteps.

Let $P = \{x_1, \dots, x_n\}$ be a poset and V the vector space of all functions $f\colon P \to \mathbb{R}$. The vector space V is n-dimensional and comes with an inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle f, g \rangle = \sum_{x \in P} f(x) g(x).$$

Hence, V is a Euclidean space.

Definition 5.1. (*Order polytope*): The *order polytope* $\mathcal{O}(P)$ of the poset P is the subset of V defined by the conditions

$$0 \le f(x) \le 1 \qquad \text{for all } x \in P, \tag{5.1}$$

$$f(x) \le f(y) \qquad \text{if } x \le y \text{ in } P. \tag{5.2}$$

Since the order polytope $\mathcal{O}(P)$ is defined by linear inequalities and (5.1) gives a bound, $\mathcal{O}(P)$ is indeed a convex polytope. One can show that the dimension of $\mathcal{O}(P)$ is the cardinality n of the poset P.

It is convenient to extend the poset P to a poset $\widehat{P}$ by adjoining a minimum element $\widehat{0}$ and a maximum element $\widehat{1}$ to P. Let $\widehat{V}$ be the vector space of all functions from $f\colon \widehat{P} \to \mathbb{R}$. The polytope $\widehat{\mathcal{O}}(P)$ is the set of all elements $g \in \widehat{V}$ such that,

$$g(\widehat{0}) = 0, \qquad g(\widehat{1}) = 1, \tag{5.3}$$

$$g(x) \le g(y) \qquad \text{if } x \le y \text{ in } \widehat{P}. \tag{5.4}$$

Let $\varphi\colon \widehat{\mathcal{O}}(P) \to \mathcal{O}(P)$ be the map obtained by restriction to P. This is a linear map and a bijection and hence it defines a combinatorial equivalence of polytopes.

Each of the inequalities in (5.4) defines a facet of the order polytope. More precisely a facet of $\mathcal{O}(P)$ consists of all $f \in \mathcal{O}(P)$ such that $f(x) = f(y)$ for some fixed pair (x, y) for which y covers x in $\widehat{P}$. Therefore the number of facets of $\mathcal{O}(P)$ is the number of cover relations in $\widehat{P}$.

As usual when studying polytopes one is interested in the entire facial structure of the polytope. For the order polytope this is a relatively simple task, as one can describe the faces of the polytopes via partitions of the poset $\widehat{P}$.

Every face of a polytope is an intersection of facets, hence faces F_π of $\mathcal{O}(P)$ corresponds to certain partitions $\pi = \{B_1, \dots, B_k\}$ of $\widehat{P}$ into nonempty pairwise disjoint blocks as follows:

$$F_\pi = \left\{ f \in \widehat{\mathcal{O}}(P) : f \text{ is constant on the blocks } B_i \text{ of } \pi \right\}. \tag{5.5}$$

If F_π is a face of $\widehat{P}$, we call π a *face partition*. If every block B_i of the partition π is connected as a subposet of $\widehat{P}$, we call the partition π

connected. A partition $\pi = \{B_1, \dots, B_k\}$ is called *closed* if for any $i \neq j$ there exists an $f \in F_\pi$ such that $f(B_i) \neq f(B_j)$. Let $\leq_\pi$ be a binary relation on π by setting $B_i \leq_\pi B_j$ if $x \leq y$ for some $x \in B_i$ and $y \in B_j$. We call π *compatible* if the transitive closure of $\leq_\pi$ is a partial order.

The following theorem characterizes the faces of the order polytope via partitions of the underlying poset.

Theorem 5.2. (*[Sta86, Theorem 1.2]*):
A partition of P is a closed face partition if and only if it is connected and compatible.

When looking on the eigenstep tableau λ of a finite equal norm tight frame consisting of N vectors of a d-dimensional Hilbert space, we can interpret the non-fixed part of it as a Hasse diagram of a poset P. We get $\widehat{P}$ by adding the lower bound condition (4.11) and the upper bound condition (4.12) to P, that is adding a minimal element 0 and a maximal element N to the poset.

Besides the fact that one can construct the order polytope for every poset, while the polytope of eigensteps is only defined for posets arising from eigenstep tableaux, there is one key difference between these two polytopes. For the construction of the polytope of eigensteps one has to consider the sum conditions for the columns of the poset, which has the following geometric interpretation. The polytope of eigensteps is the polytope resulting from intersecting the order polytope with $N-3$ linear equations, namely the sum conditions for the columns $n = 2, \dots, N-2$ of the eigenstep tableau. This implies that Theorem 5.2 does not hold for polytopes of eigensteps, since there are non-connected partitions, which define a face of the polytope. Figure 5.1a on the next page depicts a non-connected partition which corresponds to a face (a vertex) of the polytope of eigensteps $\Lambda_{8,4}$. Unlike the order polytope, where one can read the dimension of a face from the number of components in the defining partition, it is impossible in the case of polytopes of eigensteps. The partition depicted in Figure 5.1a on the following page has three connected components, but there are also vertices of $\Lambda_{8,4}$ which are defined by eigenstep tableaux having up to seven components. One of these tableaux is shown in Figure 5.1b on the next page.

Figure 5.1: Two partitions, each defining a vertex of $\Lambda_{8,4}$.

5.2 Gelfand-Tsetlin Polytopes

The second kind of polytope whose construction is similar to that of the polytope of eigensteps is the Gelfand-Tsetlin polytope, named after Israel M. Gelfand and Michael L. Tsetlin. Gelfand-Tsetlin polytopes arise in the representation theory of the general linear group $\mathrm{GL}_n(\mathbb{C})$ and were introduced in 1950 [GT50]. We follow the paper of Jesús A. De Loera and Tyrrell B. McAllister [DM04] for the definitions.

We start with the definition of a *Gelfand-Tsetlin pattern*.

Definition 5.3. (*Gelfand-Tsetlin pattern*): A triangular shaped double-indexed sequence of sequences $(\lambda_{i,n})_{1\leq i\leq n\leq d}$ given by

$$\lambda_{i,n} \geq 0, \qquad \text{for } 1 \leq i \leq n \leq d, \tag{5.6}$$
$$\lambda_{i,n+1} \geq \lambda_{i,n} \geq \lambda_{i+1,n+1}, \qquad \text{for } 1 \leq i \leq n \leq d-1. \tag{5.7}$$

is called a *Gelfand-Tsetlin pattern*.

Note that the inequalities in (5.7) are interlacing conditions for the sequences $(\lambda_{i,n})$ and $(\lambda_{i,n+1})$.

Definition 5.4. (*Gelfand-Tsetlin polytope*): Let $\lambda = (\lambda_1, \ldots, \lambda_d)$ and $\mu = (\mu_1, \ldots, \mu_d)$ be tuples of non-negative integers. The *Gelfand-Tsetlin polytope* $\mathrm{GT}(\lambda, \mu) \subset \mathbb{R}^{d(d+1)/2}$ is the convex polytope of Gelfand-Tsetlin patterns

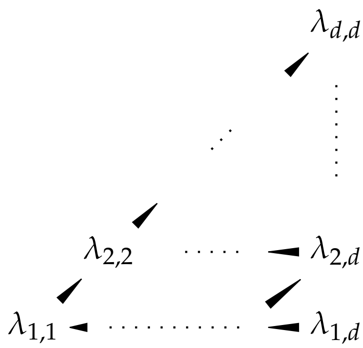


Figure 5.2: A Gelfand-Tsetlin pattern.

which satisfy

$$\lambda_{i,d} = \lambda_i, \qquad \text{for } 1 \leq i \leq d \tag{5.8}$$

$$\sum_{i=1}^{n} \lambda_{i,n} = \sum_{i=1}^{n} \mu_i, \qquad \text{for } 1 \leq n \leq d. \tag{5.9}$$

Clearly, in order to fulfill (5.7) to (5.9), the sequence λ has to be non-increasing and must majorize the sequence μ. We see immediately the similarity to the definition of eigenstep polytopes. This will be even more clear if we depict the sequence $(\lambda_{i,n})_{1\leq i \leq n \leq d}$ as shown in Figure 5.2. In fact, the Gelfand-Tsetlin pattern looks like a part of an eigenstep tableau.

The key difference between a Gelfand-Tsetlin pattern and an eigenstep tableau is the triangular shape of the Gelfand-Tsetlin pattern. In fact, for a tuple $\lambda = (N, \ldots, N)$ for a constant N the only Gelfand-Tsetlin pattern satisfying eqs. (5.8) and (5.9) is the constant N-triangle in an eigenstep tableau in the polytope $\Lambda_{N,d}$ for $N > d$. In general, for the polytope $\Lambda\Big((\mu_n)_{n=1}^{N}, (\lambda_i)_{i=1}^{d} \Big)$ the lower-right triangle of an eigenstep tableau can be interpreted as a Gelfand-Tsetlin pattern. Since there is not a unique way to restrict the sum conditions onto the triangle, we can not interpret the Gelfand-Tsetlin polytope as a subpolytope of the polytope of eigensteps.

From a general point of view it might be interesting to have a concept unifying polytopes of eigensteps, order polytopes and Gelfand-Tsetlin polytopes. We leave this to future research.

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Curriculum Vitae

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1988–2001	Primary and secondary school in Bremerhaven
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2011	Master of Education in Mathematics and Physics (*Thesis: Ein Schülerseminar zur tropischen Mathematik*) Advisors: Prof. Dr. Angelika Bikner-Ahsbahs and Prof. Dr. Eva-Maria Feichtner
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