Statistics of rare events in infinite ergodic theory

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THERE IS NO BRANCH OF MATHEMATICS, HOWEVER ABSTRACT, WHICH MAY NOT SOME DAY BE APPLIED TO PHENOMENA OF THE REAL WORLD.

– Nikolai Ivanovich Lobachevsky

German summary

Diese Dissertationsschrift beschäftigt sich mit der zeitlichen Entwicklung von Wahrscheinlichkeitsdichten seltener Ereignisse in einem dynamischen System, dem ein σ-endliches, invariantes Maß zugrunde liegt. Bekannte distributionale Konvergenzsätze werden erweitert und es werden Observablen betrachtet, für die keine distributionale Konvergenz hin zu einem Gleichgewichtszustand gilt. Ein Kernwerkzeug dieser Untersuchungen ist der Transferoperator, der die Entwicklung von Wahrscheinlichkeitsdichten bezüglich eines dynamischen Systems beschreibt.
Es wird eine Familie von Markov-Intervall-Abbildungen untersucht, die zwischen der Zeltabbildung und der Farey Abbildung interpoliert. Hierfür wird distributionale Konvergenz für Wahrscheinlichkeitsdichten mit Singularitäten betrachtet. Es kann gezeigt werden, dass unter gewissen Voraussetzungen auch hierfür Grenzwertsätze gelten. Ein besonderes Augenmerk wird hierbei auf die Farey Abbildung gelegt, da in diesem Fall ein Wechselspiel von chaotischer und regulärer Dynamik auftritt, das durch einen indifferenten Fixpunkt im Ursprung erzeugt wird. In der theoretischen Physik ist dieses Phänomen auch bekannt als *Intermittency*. Außerdem kann gezeigt werden, dass das Grenzwertverhalten entlang der ω-Limesmenge der Singularität von den diophantischen Eigenschaften der Singularität abhängt. Dieser Teil ist teilweise in [KKS16] veröffentlicht.
Im letzten Teil der Arbeit wird untersucht, inwieweit die Voraussetzungen der bekannten Konvergenzresultate erweitert werden können. Es zeigt sich, dass es hier natürliche Grenzen gibt und selbst für verhältnismäßig reguläre Beobachtungsgrößen keine distributionale Konvergenz zur Gleichverteilung zu erreichen ist. Dieser Teil legt die Familie der α-Farey Abbildungen zugrunde, eine Familie stückweise linearer Markov Intervall Abbildungen, die es ermöglicht, verschiedene Systeme mit einem instabilen, indifferenten Fixpunkt im Ursprung und verschiedenen regulär variierenden Wanderraten zu erzeugen. Der dritte Teil ist in [KKSS15] veröffentlicht.
Dieser Arbeit liegt die Idee der Erneuerungstheorie für Operatoren zugrunde; eine Idee, die in [Sar02] entwickelt wurde und durch [MT12] weiter vertieft wurde. Sie generalisiert Ideen der klassischen Erneuerungstheorie für Operatoren und ermöglicht so Aussagen zur distributionalen Konvergenz, indem zuerst die Konvergenz des Transferoperators des induzierten dynamischen Systems gezeigt wird und diese dann für den Transferoperator des eigentlichen dynamischen Systems zurückgefolgert wird.

Contents

Chapter 1

Introduction

1.1 Background

Rare events are events for which we expect to wait infinitely long, but the probability that they happen is equal to one. Naturally, these events have a certain impact on a system and what comes to mind are catastrophes like earthquakes, nuclear accidents, volcano eruptions or tsunamis. After such a catastrophe a system can become disordered or chaotic for some time, after which it returns to normal. It could even end up in chaos and never return to normal. Thus, an important question to ask is which one of the two scenarios really happens. In mathematics and physics this phenomena is known as intermittency.
One way of examining this kind of phenomena is to look at distributional convergence. Distributional convergence is a vital area of research in ergodic theory and dynamical systems and is concerned with the evolution in time of probability densities imposed on the system. Models to examine such phenomena are given by maps of the unit interval.

Expanding maps of the unit interval have been widely studied in the last decades. However, in recent years an increasing amount of interest has aroused in maps exhibiting indifferent fixed points. That is, maps which are expanding everywhere, except at unstable fixed points. Around this point trajectories are considerably slowed down and cause the interplay of regular and chaotic dynamics. From a measure theoretical point of view, this might lead to the invariant measure having infinite mass. The first models representing intermittency were the so-called Pomeau-Manneville maps [PM80]. Further models for intermittent maps are given by Markov interval maps with indifferent fixed points.
Methods from finite ergodic theory are not applicable in this setting because they do not yield meaningful information about the system. For instance, it is known that Birkhoff's ergodic theorem breaks down [Aar97, Theorem 2.4.2], see Chapter 6 for further details.

The main object of consideration in this thesis are non-singular, conservative and ergodic dynamical systems, that is a quadruplet $(X, \mathfrak{B}, \mu, T)$, where $(X, \mathfrak{B}, \mu)$ is a σ-finite measure space and where $T\colon X \to X$ is an ergodic transformation. For formal definitions and basic results deeper introduction to this field and the nomenclature the reader is referred to Chapter 3.

For such a system a vital tool to explore its statistical behaviour is the transfer operator [Bal00]. The transfer operator is the linear operator $\widehat{T}\colon L^1_\mu \to L^1_\mu$, uniquely defined, for $f \in L^1_\mu$ and $g \in L^\infty_\mu$, via the dual relation

$$\int_X g \cdot \widehat{T}(f) \mathrm{d}\mu = \int_X g \circ T \cdot f \mathrm{d}\mu.$$

This operator plays an important role in finding invariant measures for a system, since the constant density $\mathbb{1}$ is a fixed point of the transfer operator, meaning $\widehat{T}(\mathbb{1}) = \mathbb{1}$, whenever T is μ-invariant, see for instance [LY73]. With the transfer operator at hand, one can, instead of looking at ergodic sums, look at dual ergodic sums, $\sum_{k=0}^m \widehat{T}^k(f)$ and ask whether these converge, or at what rate they converge or diverge. This leads to the notion of pointwise dual ergodicity. Further results on the asymptotic behaviour of those dual ergodic sums have been achieved, under certain conditions, for instance by [CF90, Tha95, Zwe98, Zwe00]. Having statements about the dual ergodic sums, a good question to ask is, whether we can determine the asymptotics of the individual iterates of the transfer operator. This question turns out to be considerably more delicate and is at the heart of this thesis. For a deeper account on the topic of distributional convergence and about the starting point for the research carried out for this thesis, the reader is referred to Section 6.2.

In the beginning of the 21st century a new method evolved to discern distributional convergence results, namely operator renewal theory. For this method arguments and techniques from classical renewal theory are lifted to an operator setting, see [Sar02, MT12], which plays a crucial role in this thesis. With the help of these techniques it is possible to obtain convergence for individual iterates of the transfer operator, by exploiting convergence results for the so-called return time operator.

The starting point is to understand the previously known convergence results. To elaborate how far the theory reaches, there are two natural ways to modify the setting. One of them is to adjust the transformation itself, in particular to change the wandering rate of the transformation, which is done in the third part of this thesis. Another option is to extend the class of observables that is considered, which is done in Part II. The second part considers distributional convergence of observables which are integrable, but posses singularities. It turns out that the limiting behaviour on the ω-limit set of the pole depends on the diophantine properties of the pole. Yet, distributional convergence can still be obtained on compact subsets, that do not intersect the ω-limit set of the pole and that are bounded away from the indifferent fixed point, as Theorems 8.1, 8.2 and 8.6 show.

Part III focuses on how to modify the transformation in general and its wandering rate in particular. It turns out that additional assumptions may be required, if the wandering rate is no longer slowly varying but regularly varying, as Theorem 12.3 shows. However, under additional assumptions on the wandering rate, see Theorem 12.2, or on the observable, see Theorem 12.1, convergence to equilibrium can still be obtained.

Before stating the main results of this thesis in Section 1.2, we give an overview of the structure of the thesis. The thesis splits up in three parts.

Part I gives an introduction to the underlying theory of this thesis. It will unify notation and introduce necessary definitions and statements that are needed in Part II and Part III.

Chapter 2 introduces the notation that is used throughout this thesis. It gives an overview of the key variables and also introduces the maps and transformations central to this thesis. A brief introduction to dynamical systems and ergodic theory is given in Chapter 3, followed by a short introduction to regular varying functions in Chapter 4. There are two main examples which are introduced in Chapter 5 and which elucidate the theory. These two main examples will also play a key role in the main results of this thesis. We will work with the first example in Part II and focus on the second in Part III.

As infinite ergodic theory and the transfer operator is at the core of this thesis, the first part concludes with Chapter 6 on transfer operator methods. It consists of three sections. Section 6.1 introduces the transfer operator and the necessary relations, Section 6.2 gives an overview of the state of the art in distributional convergence and finally, Section 6.3 introduces operator renewal theory.

The second part has partly been published in [KKS16]. The nomenclature as well as several passages, including the central definitions and the main results, are adopted from there.

Before the main results of Part II are stated in Chapter 8, important notation and some definitions, as used in [KKS16], are needed and hence introduced in Chapter 7. After the main results are stated, pictures and heuristics are given to elucidate the theory in Chapter 9. After which complete proofs of Theorems 8.1, 8.2 and 8.6 are given in Chapter 10.

Part III has partly been published in [KKSS15]. It is structured similar to Part II. Chapter 11 introduces the necessary definitions and Chapter 12 states the main results of Part III. After giving complete proofs of the main results in Chapter 13, this thesis ends with Chapter 14 which comments on how the results complement and extend the previously known results in infinite ergodic theory. In particular this Chapter comments on how the current results can be seen in the light of [MT15].

1.2 Statement of main results

This section states the main results. The necessary notion is introduced briefly; nevertheless, for thorough introduction to the notion and further details, the reader is referred to Part II and Part III respectively.

1.2.1 Main results of Part II

For $r \in [0,1]$, the map $T_r \colon [0,1] \to [0,1]$ is defined by

$$T_r(x) := \begin{cases} \dfrac{(2-r)\cdot x}{1-r\cdot x} & \text{if } 0 \le x \le 1/2, \\ \dfrac{(2-r)\cdot(1-x)}{1-r+r\cdot x} & \text{if } 1/2 < x \le 1. \end{cases}$$

Let $\mathcal{P}_r$ denote the Perron-Frobenius operator for T_r, see Definition 6.1, page 31. Further $\omega_r(\beta)$ denotes the ω-limit set of β, see Equation (7.1), page 64, and $\mathfrak{U}_{\beta,a}$ is a class of functions that are integrable and have a pole at β of order α, see Definition 7.2, page 63, for further properties of $\mathfrak{U}_{\beta,a}$. The term intermediate a-type is defined in Definition 7.3, page 64. Heuristically, we can say that, if a number is of *intermediate a-type*, we have some kind of control over the growth rate of its continued fraction entries. For $r \in [0,1]$, we let h_r denote the invariant density of T_r, absolutely continuous with respect to the Lebesgue measure. Given these definitions, we can state the main results of Part II. The first theorem is a statement on convergence to equilibrium of unbounded observables for $r \in [0,1)$.

Theorem (Theorem 8.1)**.** *For $r \in [0,1)$, if $a \in (0,1)$ and $\beta \in [0,1]$, then, for each $v \in \mathfrak{U}_{\beta,a}$, we have that*

$$\lim_{n\to\infty} \mathcal{P}_r^n(v) = \int v \, d\lambda \cdot h_r, \tag{1.1}$$

uniformly on compact subsets of $[0,1] \setminus \omega_r(\beta)$ and pointwise outside a set with Hausdorff dimension equal to zero.
If $\beta \in [0,1]$ is pre-periodic with respect to T_r and has period length strictly greater than one, then on the finite set $\omega_r(\beta)$ we have that

$$\liminf_{n\to+\infty} \mathcal{P}_r^n(v) = \int v \, d\lambda \cdot h_r \quad \textit{and} \quad \limsup_{n\to+\infty} \mathcal{P}_r^n(v) = +\infty.$$

In the case that $\beta \in [0,1]$ is pre-periodic with respect to T_r and has period length equal to one, then on the singleton $\omega_r(\beta)$, the limit in (1.1) is equal to $+\infty$.

The next theorem deals with the case $r = 1$.

Theorem (Theorem 8.2)**.** *If $a \in (0,1)$ and if $\beta \in (0,1]$ is either rational or irrational of intermediate a-type, then, for each $v \in \mathfrak{U}_{\beta,a}$, we have that*

$$\lim_{n\to\infty} \ln(n) \cdot \mathcal{P}_1^n(v) = \int_{[0,1]} v \, d\lambda \cdot h_1, \tag{1.2}$$

uniformly on compact subsets of $(0,1] \setminus \omega_1(\beta)$ and pointwise outside a set with Hausdorff dimension equal to zero. If $\beta \in (0,1]$ is pre-periodic with respect to T_1

and has period length strictly greater than one, then on the finite set $\omega_1(\beta)$ we have that

$$\liminf_{n\to+\infty} \ln(n)\cdot \mathcal{P}_1^n(v) = \int v \,\mathrm{d}\lambda \cdot h_1 \quad \text{and} \quad \limsup_{n\to+\infty} \ln(n)\cdot \mathcal{P}_1^n(v) = +\infty.$$

In the case that $\beta \in (0,1]$ is pre-periodic with respect to T_1 and has period length equal to one, then on the singleton $\omega_1(\beta)$, the limit in (1.2) is equal to $+\infty$.

For $\beta \in [0,1]\setminus\mathbb{Q}$, we let $[a_1, a_2, \dots]$ denote the continued fraction expansion of β. In the following theorem, for the observable $v_{\beta,a}(x) = |\beta - x|^{-a}$ and a non-periodic β, we demonstrate that on the ω-limit set, the values of the limit inferior and limit superior depend on the diophantine properties of β.
For $n\in\mathbb{N}$, let $p_n = p_n(\beta)$ and $q_n = q_n(\beta)$ denote the unique integers, such that $\gcd(p_n, q_n) = 1$ and

$$\frac{p_n}{q_n} = [a_1, a_2, \dots, a_n].$$

Theorem (Theorem 8.6)**.** *1. There exist non-periodic β and $\varrho \in (0,1]$ both with bounded continued fraction entries but such that, on the one hand, if $a \in (0,1)$, then on $\omega_1(\beta)$, we have that*

$$\lim_{n\to+\infty} \ln(n)\cdot \mathcal{P}_1^n(v_{\beta,a}) = \int v_{\beta,a}\,\mathrm{d}\lambda\cdot h_1.$$

On the other hand, if $a \in (0,1/2)$, then on $\omega_1(\varrho)$, we have that

$$\lim_{n\to\infty} \ln(n)\cdot \mathcal{P}_1^n(v_{\varrho,a}) = \int v_{\varrho,a}\,\mathrm{d}\lambda\cdot h_1;$$

otherwise, if $a\in(1/2,1)$, then on $\omega_1(\varrho)$

$$\liminf_{n\to+\infty} \ln(n)\cdot \mathcal{P}_1^n(v_{\varrho,a}) = \int v_{\varrho,a}\,\mathrm{d}\lambda\cdot h_1$$
$$\text{and} \quad \limsup_{n\to+\infty} \ln(n)\cdot \mathcal{P}_1^n(v_{\varrho,a}) = +\infty.$$

2. Let $a \in (0,1)$ and let $\beta = [0; a_1, a_2, \dots] \in (0,1]$ be of intermediate a-type such that

$$\lim_{n\to+\infty} a_n = +\infty,$$

which implies that $\omega_1(\beta) = \{1/n\colon n\in\mathbb{N}\}\cup\{0\}$.
Fix $k\in\mathbb{N}$ and let $l(k) := \min\{i\in\mathbb{N}\colon a_m \ge k \text{ for all } m \ge i\}$. For all $j \ge l(k)$, set $n_{k,j}\in\mathbb{N}$ to be the unique integer satisfying $T_1^{n_{k,j}}(\beta) = [0; k, a_{j+1}, a_{j+2}, \dots]$ and set

$$\mathscr{S}_{k,j} := \frac{(a_{j+1})^a \cdot \ln\left(n_{k,j}\right)}{(q_j)^{2\cdot(1-a)}},$$

where q_n is as defined in (5.1)*. If* $\limsup_{j\to\infty} \mathscr{S}_{k,j} = 0$*, then*

$$\lim_{n\to+\infty} \ln(n) \cdot \mathcal{P}_1^n(v_{\beta,a}) \left(\frac{1}{k}\right) = \int v_{\beta,a} \,\mathrm{d}\lambda \cdot h_1 ;$$

otherwise,

$$\liminf_{n\to+\infty} \ln(n) \cdot \mathcal{P}_1^n(v_{\beta,a}) \left(\frac{1}{k}\right) = \int v_{\beta,a} \,\mathrm{d}\lambda \cdot h_1$$

$$\text{and} \quad \limsup_{n\to+\infty} \ln(n) \cdot \mathcal{P}_1^n(v_{\beta,a}) \left(\frac{1}{k}\right) > \int v_{\beta,a} \,\mathrm{d}\lambda \cdot h_1 .$$

1.2.2 Main results of Part III

Let F_α denote the α-Farey map, given by Equation (5.11). Further, let $\widehat{F_\alpha}$ denote the transfer operator with respect to the invariant measure μ_α and let h_α denote the density of the invariant measure with respect to the Lebesgue measure. We have the following statements on distributional convergence.

Theorem (Theorem 12.1)**.** *For* $\delta \in (1/2, 1]$*, let* $([0,1], \mathscr{B}, \mu_\alpha, F_\alpha)$ *denote a* δ*-expansive* α*-Farey system. If* $v \in \mathcal{L}^1_\lambda([0,1])$ *and if*

$$D_\alpha \left(\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1} \left(\frac{v}{h_\alpha} \cdot \mathbb{1}_{A_n} \right) \right) = \mathfrak{O}(1)$$

$$\text{and} \quad \left\| v \cdot \mathbb{1}_{A_n} \right\|_\infty = \mathfrak{o}\left(\frac{1}{t_n}\right),$$

then uniformly on compact subsets of $(0,1]$*,*

$$\lim_{n\to\infty} w_n \cdot \widehat{F}_\alpha^n \left(\frac{v}{h_\alpha}\right) = \frac{1}{\Gamma(1+\delta)\cdot\Gamma(2-\delta)} \cdot \int v \,\mathrm{d}\lambda .$$

For the next theorem we need the space of functions $\mathcal{A}_\alpha$. It is given by

$$\mathcal{A}_\alpha := \left\{ v \in \mathcal{L}^1_{\mu_\alpha}([0,1]) \colon \begin{array}{l} \|v\|_\infty < \infty \ , \sum_{k=1}^\infty \left\| \widehat{F}_\alpha^{k-1}(v \cdot \mathbb{1}_{A_k}) \right\|_\infty < +\infty \\ \text{and} \quad \widehat{F}_\alpha^{n-1}(v \cdot \mathbb{1}_{A_n}) \in \mathcal{B}_\alpha \text{ for all } n \in \mathbb{N} \end{array} \right\}.$$

The term *moderately increasing* is an additional assumption on slowly varying functions, defined in Definition 11.1. Theorem 12.2 shows that we can weaken the conditions on the observable compared to Theorem 12.1, if we have more information on the wandering rate.

Theorem (Theorem 12.2)**.** *Let* $([0,1], \mathscr{B}, \mu_\alpha, F_\alpha)$ *denote a* 1*-expansive* α*-Farey system and assume that the wandering rate is moderately increasing. If* $v \in \mathcal{A}_\alpha$*, then, uniformly on compact subsets of* $(0,1]$*,*

$$\lim_{n\to\infty} w_n \cdot \widehat{F}_\alpha^n(v) = \int v \,\mathrm{d}\mu_\alpha .$$

Nevertheless, Theorem 12.3 shows, that precaution is needed if the conditions on the wandering rate are weakened. In the next theorem Γ_δ is a constant given in Equation (5.14).

Theorem (Theorem 12.3)**.** *Let $\delta \in (1/2, 1)$ and let $([0,1], \mathscr{B}, \mu_\alpha, F_\alpha)$ denote a δ-expansive α-Farey system. There exists a positive, locally constant, Riemann integrable function $v \in \mathcal{A}_\alpha$ of bounded variation, such that, for all $x \in \overline{A}_1$,*

$$\liminf_{n\to\infty} w_n \cdot \widehat{F}_\alpha^n(v)(x) = \Gamma_\delta \int v \, \mathrm{d}\mu_\alpha$$
$$\text{and} \quad \limsup_{n\to\infty} w_n \cdot \widehat{F}_\alpha^n(v)(x) = +\infty.$$

Part I

Theoretical background and preliminaries

Chapter 2

Notation

This chapter begins with an overview of the main variables and maps used throughout the thesis. The following notation is used.

- λ denotes the Lebesgue measure.
- T_r, $r \in [0, 1]$ denotes a family of intermittent interval transformations. It is given in Equation (5.2). If $r = 1$, T_1 is known as the Farey transformation. This family of transformations is crucial in Part II.
- μ_r is the invariant measure of T_r absolutely continuous with respect to λ. It is unique up to multiplication with a constant.
- h_r is the density of μ_r with respect to λ. That is $h_r := \mathrm{d}\mu_r/\mathrm{d}\lambda$. In particular h_1 is the density of the invariant measure of the Farey map with respect to the Lebesgue measure. That is, $h_1 = 1/x$. The density h_r is given in Equation (5.4).
- F_α is the family of α-Farey maps, introduced in (5.11)
- μ_α is the invariant measure of F_α absolutely continuous with respect to λ.
- h_α is the invariant density of μ_α with respect to λ. That is $h_\alpha := \mathrm{d}\mu_\alpha/\mathrm{d}\lambda$. It is given by Equation (5.15).

Furthermore, for a measure space $(X, \mathfrak{B}, \mu)$, we let $\mathcal{L}^1_\mu(X)$ denote the space of functions, for which

$$\int_X |f| \, \mathrm{d}\mu < +\infty.$$

By $L^1_\mu(X)$ we denote the Banach space of equivalence classes $[f]_\mu$ of functions, where for each representative $f\colon [0, 1] \to \mathbb{C}$ of $[f]_\mu$,

$$\|f\|_{\mu,1} := \int_X |f| \, \mathrm{d}\mu < +\infty,$$

and where f, g belong to the same equivalence class, if and only if, $\|f - g\|_{\mu,1} = 0$. Throughout, following convention, we write $f \in L^1_\mu(X)$ to mean a function $f\colon X \to \mathbb{C}$ which belongs to the equivalence class $[f]_\mu$ of $\mathcal{L}^1_\mu(X)$. A similar construction is done for the spaces $\mathcal{L}^\infty_\mu$ and L^∞_μ.
To simplify notation, the index μ_r is replaced by r in Part II, for instance, we write $\mathcal{L}^1_r(X) \coloneqq \mathcal{L}^1_{\mu_r}(X)$.

We use the Landau notation $\mathfrak{o}(\cdot)$ and $\mathfrak{O}(\cdot)$ as well as $\sim$ and $\ll$.
The symbol $\sim$ between the elements of two sequences of real numbers $(b_n)_{n\in\mathbb{N}}$ and $(c_n)_{n\in\mathbb{N}}$ means that the sequences are asymptotically equivalent, namely that $\lim_{n\to+\infty} b_n/c_n = 1$.
We use the Landau notation $b_n = \mathfrak{o}(c_n)$, if $\lim_{n\to+\infty} b_n/c_n = 0$. The notions $b_n = \mathfrak{O}(c_n)$ and $b_b \ll c_n$ are used interchangeably, if $\lim_{n\to+\infty} b_n/c_n < C < \infty$. The same notation is used between two real-valued function f and g, defined on the set of real numbers $\mathbb{R}$.
These variables are the crucial ones throughout this thesis. The rest of the necessary notation is introduced along the way and we turn towards the introduction of dynamical systems and ergodic theory in Chapter 3.

Chapter 3

Basics of dynamical systems and ergodic theory

For a reader with a background in dynamical systems and ergodic theory this chapter is a recapitulation. It is included to recall and introduce central definitions that are needed for this thesis. The purpose is to motivate the questions being asked in Part II and Part III on the one hand. On the other hand this chapter gives a brief introduction to the matter for readers of different backgrounds and it unifies notations, which differ in literature. For a thorough account on the subject of dynamical systems and ergodic theory the reader is referred to standard references such as [Aar97, Den05, Wal82]. Additionally, a variety of lecture notes can be found online.

Dynamical comes from the ancient greek word 'δύναμις' (*'dynamis'*), which means *force*. In physics, it is the study of force and its impact on mass, hence the study of motion. In mathematics, studying *dynamical systems* means studying, how a system changes. Usually, the term *system* in general means something isolated. In reality this can for instance be a bowl of dough or a population on a planet; in mathematics, a system is usually a space X on which further properties are imposed.

It is a dynamical system if a force is applied to the system, for instance the dough is kneaded, the population is, for example due to births and deaths, changed, or from the mathematical point of view, if a map T maps the space X into itself.

The term *ergodic* comes from the two ancient greek words 'ἔργον' and 'ὁδός' ('*ergon*' and '*hodos*'), meaning *work* and *path*. This term dates back to the 1930s and the ergodic hypothesis by Boltzmann. As it turned out this hypothesis was wrong in its original form, so that additional assumptions on the system were required; but nevertheless, it was the starting point of ergodic theory in its current form.

This thesis, in particular, is dealing with measure theoretical dynamical systems. A measure theoretical dynamical system is a quadruplet $(X, \mathfrak{B}, \mu, T)$, where $(X, \mathfrak{B}, \mu)$

is a standard measure space. That means X is a complete and separable metric space, equipped with the Borel σ-algebra $\mathfrak{B}$ and a not necessarily finite but σ-finite measure μ. A σ-algebra is called the Borel σ-algebra, if it is generated by the collection of open subsets of X (compare [Aar97, §1.0]). Here and throughout this thesis, the Borel σ-algebra of a space X is denoted by $\mathfrak{B}_X$. If it is clear from the context to which space X we refer to, the subscript is omitted and simply $\mathfrak{B}$ is written.
Moreover, the transformation T is a measurable map that maps X into itself. If it is implicitly clear, to which system it is referred to, the phrases *system* and *transformation* are used interchangeably, both meaning the quadruplet $(X, \mathfrak{B}, \mu, T)$. We recall a few definitions from measure theory, see for instance [Aar97, Chapter 1].

Definition 3.1 (**measure/probability preserving, absolutely continuous, non-singular, ergodic**)**.** Let $(X, \mathfrak{B}, \mu, T)$ denote a measure theoretical dynamical system and let $T^{-1}(A)$ denote the preimage of A under T.
We call a transformation (or a system) *measure preserving*, if μ is T-invariant. That is, for every Borel set A we have that $\mu(A) = \mu(T^{-1}(A))$. If μ is a probability measure, we call T *probability preserving*.
A transformation is said to be *absolutely continuous*, if preimages of Borel sets of measure zero have zero measure.
The system is said to be *non-singular*, if B is a set of zero measure is equivalent to $T^{-1}(B)$ is a set of zero measure.
Finally, we call the system *ergodic*, if every invariant set has measure zero or its complement has measure zero.

Each finite measure μ can be normalised by dividing by $\mu(X)$. So the case that the invariant measure is finite but not a probability measure is neglected, as it is common in literature. *Ergodicity* means that a system can not be decomposed into subsystems acting independently of each other.
This thesis studies long term behaviour of a system. That is, the multiple iteration of the map T. Thus, we consider n-fold iterations T^n, meaning for $x \in X$ and $n \in \mathbb{N}$,

$$T^0(x) := x \text{ and } T^n(x) := T^{n-1}(T(x)).$$

We focus on conservative dynamical systems, which is defined after introducing the notion of wandering sets.

Definition 3.2 (**Wandering set** [Aar97, §1.1])**.** Let $(X, \mathfrak{B}, \mu, T)$ denote a non-singular dynamical system. A set $W \in \mathfrak{B}$ is called a *wandering set for* T, if the sets $\{T^{-n}(W)\}_{n=0}^{\infty}$ are disjoint almost everywhere.

Definition 3.3 (**Conservative**)**.** A non-singular dynamical system $(X, \mathfrak{B}, \mu, T)$ is called *conservative*, if each wandering set has measure zero.

A useful parameter to partition a system is the first return time. Linked hereto are the level sets of the first return time and the induced transformation.

Definition 3.4 (**First return time, induced transormation**). Let $(X, \mathfrak{B}, \mu, T)$ denote a conservative and ergodic dynamical system. Let $A \subset X$. We define the *first return time* by $\phi_A\colon A \to \mathbb{N} \cup \{+\infty\}$, by

$$\phi_A(x) := \inf\{n \in \mathbb{N} \colon T^n(x) \in A\}$$

and call the collection of sets $\{\phi_A = n\}_{n\in\mathbb{N}} := \{y \in A \colon \phi_A(y) = n\}_{n\in\mathbb{N}}$ the *level sets of the first return time*.

As common, we define the infimum over the empty set to be $+\infty$. Since the system is conservative, the set of points for which the return time is $+\infty$ has zero measure. The notion of the first return time leads straight to the *induced transformation*. The induced transformation, with respect to a set A, with finite and positive measure, is a way to look at a dynamical system with a possibly infinite invariant measure through "finite measure glasses", by cutting out the excursions between two visits to the set A. It is given by $T_A\colon A \to A$, with

$$T_A(x) := \begin{cases} T^{\phi_A(x)}(x) & \text{if } \phi_A(x) < +\infty, \\ x & \text{else.} \end{cases} \tag{3.1}$$

The first return time and the induced transformation is not to be confused with the first entry time and the jump transformation, defined next.

Definition 3.5 (**First entry time, Jump transformation**). Adopt the setting of Definition 3.4. The *first entry time*, $e_A(x)\colon X \to \mathbb{N} \cup \infty$ is given by

$$e_A(x) := \inf\{n \in \mathbb{N} \colon T^{n-1}(x) \in A\}.$$

The *jump transformation*, $T_{\text{jump},A}(x)\colon X \to X$ is given by

$$T_{\text{jump},A}(x) := \begin{cases} T^{e_A(x)}(x) & \text{if } e_A(x) < +\infty, \\ x & \text{else.} \end{cases}$$

In a conservative dynamical system and for a set of positive measure A, we can define the level sets of the first entry time by

$$A_n := \{e_A = n\} := \{x \in X \colon e_A(x) = n\}, \tag{3.2}$$

which gives a partition of X by $\{e_A = n\}_{n\in\mathbb{N}}$.

As we can see, the jump transformation and the induced transformation are similar, but not the same. It turns out, under certain circumstances these modifications of the same transformation are isomorphic. This is explained more thoroughly in [Kau11, Section 3.2].
Yet another important characteristic of a dynamical system is the wandering rate. It characterises how fast a set is spread under the dynamics of a system.

Definition 3.6 (**Wandering rate**)**.** Let $(X, \mathfrak{B}, \mu, T)$ denote a measure theoretical dynamical system, and let $A \in \mathfrak{B}$ with $0 < \mu(A) < \infty$. The *wandering rate* w_n is given by

$$w_n := \mu\left(\bigcup_{k=0}^{n-1} T^{-k}(A)\right).$$

As we will see later, the wandering rate is under certain assumptions on the system independent of the set A up to asymptotic equivalence.
A first step to determine the long term behaviour of a system is to look at ergodic sums, which leads straight to Birkhoff's Ergodic Theorem and the questions that arise naturally for infinite invariant measures.

Definition 3.7 (**Ergodic sum**)**.** Let $(X, \mathfrak{B}, \mu, T)$ denote a measure theoretical dynamical system. We call $S_n f := \sum_{k=0}^{n-1} f \circ T$ the *ergodic sum* for a measurable, complex valued function f.

If f is a characteristic function of a measurable set of finite and positive measure, A, that is $f = \mathbb{1}_A$, we call $S_n(\mathbb{1}_A)$ the *soujourn time* of A. This describes the spent time in A.
A central theorem in ergodic theory is Birkhoff's ergodic theorem. For the purpose of this thesis, we look at a slightly less general version than the one used elsewhere, compare for instance [Aar97, 2.2.6].
Heuristically, it states that in the long run the average over time tends to the average over space.

Theorem 3.8 (**Birkhoff's ergodic theorem**)**.** *Suppose that $(X, \mathfrak{B}, \mu, T)$ is an ergodic, probability preserving system. That is, μ is a T invariant probability measure. Then we have for all $f \in L^1_\mu$,*

$$\lim_{n\to\infty} \frac{S_n f}{n} = \int_X f \,\mathrm{d}\mu \text{ for } \mu\text{-almost every } x \in X.$$

If $(X, \mathfrak{B}, \mu, T)$ is a conservative, ergodic, measure preserving, system with a σ-finite, infinite measure, then we have for all $f \in L^1_\mu$,

$$\lim_{n\to\infty} \frac{S_n f}{n} = 0 \text{ for } \mu\text{-almost every } x \in X.$$

The second part of this theorem gives rise to further questions, one being, whether there is a better normalizing sequence than $1/n$. It is not possible to answer this question affirmatively, as Aaronson proves with his theorem.

Theorem 3.9 ([Aar97, Theorem 2.4.2.])**.** *Suppose T is a conservative, ergodic, measure preserving transformation of the σ-finite, infinite measure space $(X, \mathfrak{B}, \mu)$, and let $(a_n)_{n\in\mathbb{N}} > 0$, then for all non-negative $f \in L^1_\mu$,*

$$\liminf_{n\to\infty} \frac{S_n f}{a_n} = 0 \text{ almost everywhere}$$

or for all non-negative $f \in L^1_\mu$ there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ tending to infinity, such that

$$\lim_{k\to\infty} \frac{S_{n_k} f}{a_{n_k}} = \infty \text{ almost everywhere.}$$

This theorem shows, that in infinite ergodic theory the ergodic sum is underestimated or overestimated infinitely often. So the question that naturally arises is whether we can do better.
Sure, we can do better, as it was shown by Hopf's ratio ergodic theorem [Hop37]. More explicitly, Hopf's ratio ergodic theorem states that for two L^1_μ functions f and g, with $g \geq 0$ and $\int g \, \mathrm{d}\mu > 0$, the quotient of the ergodic sums $S_n f / S_n g$ converges almost surely to $\int f \, \mathrm{d}\mu / \int g \, \mathrm{d}\mu$.
Besides the ratio ergodic theorem by Hopf, this question leads to the transfer operator and the transfer operator method. When using this method ergodic theorists are interested in the long term behaviour of densities and not just single points. In other words distributional convergence is investigated. Before we turn towards the transfer operator method, two further chapters are included. Chapter 4 deals with regular varying functions and because number theory offers models to learn, understand and apply ergodic theory, Chapter 5 introduces relevant topics of number theory. After these two sections we return to the current questions in infinite ergodic theory, discuss distributional convergence, give a justification, why the term *distributional convergence* is used and give an overview of the state of the art.

Chapter 4

Regular variation

In this chapter we state several theorems, propositions and lemmata on regular varying functions that are needed in the sequel. The proofs of the following statements are omitted and can be found in the given sources or various other standard literature.

Definition 4.1 (**Slowly varying function, regular varying function**)**.** Let $a \in \mathbb{R}^+$. We call a function $\ell\colon [a, \infty) \to \overline{\mathbb{R}}$ *slowly varying*, if it is measurable, locally Riemann integrable and for each $\eta > 0$, we have that

$$\lim_{x \to \infty} \frac{\ell(\eta x)}{\ell(x)} = 1.$$

A function $r\colon [a, \infty) \to \overline{\mathbb{R}}$ is called *regular varying of order* δ, if there exists a slowly varying function $\ell\colon [a, \infty) \to \overline{\mathbb{R}}$, such that $r(x)$ can be written as

$$r(x) = x^{\delta} \cdot \ell(x).$$

We also say $\ell(x)$ varies slowly, respectively regularly, at infinity.

The next lemma states several properties of slowly varying functions.

Lemma 4.2 ([KKSS15, Lemma 2.6])**.** *Let $a \in \mathbb{N}$ and let $L\colon [a, +\infty) \to \mathbb{R}$ denote a positive slowly varying function.*

(i) [Sen76, page 2] *For a compact interval $I \subset \mathbb{R}^+$ we have that*

$$\lim_{x \to +\infty} \frac{L(p \cdot x)}{L(x)} = 1$$

holds uniformly with respect to $p \in I$, and hence, for a fixed $b \in \mathbb{R}^+$,

$$\lim_{x \to +\infty} \frac{L(x - b)}{L(x)} = 1.$$

(ii) [Sen76, page 18] *For a fixed $b \in \mathbb{R}^+$ we have that*

$$\lim_{x\to+\infty} \frac{L(x)}{x^b} = 0 \quad \text{and} \quad \lim_{x\to+\infty} L(x)\cdot x^b = +\infty.$$

(iii) [Sen76, page 41] *If L is continuous and strictly increasing, we denote the inverse function of L by L^{-1}. If we further have*

$$\lim_{x\to+\infty} L(x) = +\infty,$$

then, for a fixed $c \in (0,1)$,

$$\lim_{x\to+\infty} \frac{L^{-1}(c\cdot x)}{L^{-1}(x)} = 0.$$

(iv) [Sen76, page 50] *If $M\colon [a+1,+\infty) \to \mathbb{R}$ is defined to be the linear interpolation of the function*

$$n \mapsto \sum_{k=a+1}^{n} \frac{L(k)}{k},$$

then M is a slowly varying function and

$$\lim_{x\to\infty} \frac{L(x)}{M(x)} = 0.$$

The next theorem states, how asymptotics of summands imply the asymptotic behaviour of a sum and vice versa. It is needed in the proof of the main results. We let $\Gamma(\cdot)$ denote the Γ-function.

Theorem 4.3 (**Karamata's Tauberian theorem**). *Let, for $n \in \mathbb{N}$, $\mathfrak{q}_n \geq 0$ and let $0 < \rho < \infty$. Suppose that $L\colon \mathbb{N} \to \mathbb{R}$ varies slowly at infinity. If*

$$\mathfrak{q}_n \sim \frac{n^{\rho-1}\cdot L(n)}{\Gamma(\rho)} \quad \text{, as } n \to \infty, \tag{4.1}$$

then we have that

$$\sum_{k=0}^{n-1} \mathfrak{q}_k \sim \frac{n^{\rho}\cdot L(n)}{\Gamma(\rho+1)} \quad \text{, as } n \to \infty. \tag{4.2}$$

If the sequence $\{\mathfrak{q}_n\}_{n\in\mathbb{N}}$ is eventually monotone (4.2) *implies* (4.1)*.*

The first part of the theorem, also known as an *Abelian theorem*, is a consequence of [Fel71, Chapter VIII.9, Theorem 1]. The second part, the *Tauberian theorem*, follows from [Fel71, Chapter XIII, Theorem 5].
With this theorem we conclude this chapter. For further details on functions of slow and regular variation, the reader is, for instance, referred to [BGT87, Sen76].

Chapter 5

Number theory - the two examples

5.1 The Farey map and a family of interval maps

This section introduces the notation of [KKS16], so parts of it are published therein.

5.1.1 Continued fractions

> "When Huygens set about constructing a model of the solar system by using toothed wheels, he was confronted with the problem of determining what numbers of teeth for the wheels would give a ratio for two interconnected wheels (equal to the ratio of their periods of rotation) that would be as close as possible to the ratio α of the periods of revolution of the corresponding planets. At the same time the number of teeth obviously could not, for technical reasons, be too high. Thus, Huygens's problem was to find a rational number with numerator and denominator not exceeding a certain bound that would still be as close as possible to the given number α." [Khi97, page 28]

Continued fractions give means to solve this problem, as they yield 'best approximations' to a given real number. For an explicit definition of the term 'best approximation' the reader is referred to [Khi97, Section 2.6]. So continued fractions have an application in both engineering and diophantine approximation. Part II will rely heavily on continued fractions, so let us turn to continued fractions in general and to the techniques we need in this thesis in particular.
Every number $x \in \mathbb{R} \setminus \mathbb{Q}$ has a unique continued fraction expansion (see for

example [Khi97]), given by

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \dots}}},$$

where the so-called continued fraction entries a_i are natural numbers for $i \in \mathbb{N}$ and $a_0 \in \mathbb{N}_0$. The number a_0 is the integer part $\lfloor x \rfloor$ of the number x, that is the biggest integer not exceeding x. The other entries are generated by the Gauß map $G : [0,1] \longrightarrow [0,1]$,

$$G(x) := \frac{1}{x} - \left\lfloor \frac{1}{x} \right\rfloor$$

via the following algorithm,

$$a_i = \left\lfloor \frac{1}{G^{i-1}(x)} \right\rfloor.$$

Here and throughout this thesis, we restrict ourselves to the unit interval, that is in this thesis we always assume $a_0 = 0$. Furthermore, we denote the continued fraction expansion of an irrational $\beta \in [0,1]$ by $\beta := [a_1, a_2, \dots] := [0; a_1, a_2, \dots]$, where $a_n \in \mathbb{N}$, for all $n \in \mathbb{N}$. Continued fraction expansions are as well declared for rational numbers $\beta \in [0,1]$. In this case the continued fraction expansion is finite and since

$$\beta = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\ddots}{\cfrac{1}{a_k}}}} = \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{\ddots}{\cfrac{1}{a_k - 1 + \cfrac{1}{1}}}}},$$

this continued fraction expansion is no longer unique. Hence, if $\beta \in [0,1] \cap \mathbb{Q} \setminus \{1\}$, we set $\beta := [a_1, a_2, \dots a_k]$ and assume without loss of generality, that the last continued fraction entry, a_k, is greater than one.

If there exists an $M \in \mathbb{N}_0$ and $n \in \mathbb{N}$, such that for all $m > M$, $a_m = a_{m+n}$, then we say that β is pre-periodic with period length n or that β has period length n and write $\beta = [a_1, a_2, \dots, a_M, \overline{a_{M+1}, a_{M+2}, \dots, a_{M+n}}]$.

Moreover, for $\beta \in [0,1]$, we define $p_n = p_n(\beta)$ and $q_n = q_n(\beta)$ recursively by

$$\begin{aligned} p_{-1} := 1, \quad q_{-1} := 0, \quad p_0 := 0, \quad q_0 := 1, \\ p_n := a_n \cdot p_{n-1} + p_{n-2}, \quad \text{and} \quad q_n := a_n \cdot q_{n-1} + q_{n-2}. \end{aligned} \tag{5.1}$$

For $n \in \mathbb{N}$, we have that

$$\frac{p_n}{q_n} = [a_1, a_2, \ldots, a_n]$$

and $p_{n-1} \cdot q_n - p_n \cdot q_{n-1} = 1$. Closely related to continued fractions is the Farey Map, which is part of a family of Markov interval maps. This family is introduced in the next section.

5.1.2 A family of interval maps on the threshold to intermittency

In this paragraph, a family $\{T_r \colon [0,1] \to [0,1]\}_{r \in [0,1]}$ of Markov interval maps interpolating between the Tent map T_0 and the Farey map T_1 is considered. This family of maps was considered in [DEIK07, GI05, KS08], recently its spectrum was studied numerically in [BABCI15]. The current notation is the one used in [KKS16]. For $r \in [0,1]$, the map $T_r \colon [0,1] \to [0,1]$ is given by

$$T_r(x) := \begin{cases} \dfrac{(2-r)\cdot x}{1 - r\cdot x} & \text{if } 0 \leq x \leq 1/2, \\ \dfrac{(2-r)\cdot(1-x)}{1 - r + r\cdot x} & \text{if } 1/2 < x \leq 1. \end{cases} \tag{5.2}$$

The jump transformation with respect to $[1/2, 1]$ of the Farey map T_1 is the *Gauß map* that encodes the continued fraction expansion algorithm, see Section 5.1.1. For $r \in (0,1]$, the map T_r has two fixed points, one at zero and one at $1 - (3 - \sqrt{9 - 4\cdot r})/(2\cdot r)$. If $r = 0$, the map has as well two fixed points, one at zero, the other at $2/3$. The inverse branches $f_{r,0}, f_{r,1} \colon [0,1] \to [0,1]$ of T_r are given by

$$\begin{aligned} f_{r,0}(x) &:= \left(T_{r|[0,1/2]}\right)^{-1}(x) = \frac{x}{2 - r + r\cdot x} \\ \text{and} \quad f_{r,1}(x) &:= \left(T_{r|[1/2,1]}\right)^{-1}(x) = \frac{1 + (1-r)\cdot(1-x)}{2 - r + r\cdot x}. \end{aligned} \tag{5.3}$$

For a picture of the Farey map the reader is referred to Figure 5.1.
It was shown, for $r \in [0,1)$ in [GI05] and for $r = 1$ in [KS08], that the absolutely continuous invariant measure μ_r of T_r is given by

$$h_r(x) := \frac{\mathrm{d}\mu_r}{\mathrm{d}\lambda}(x) = \begin{cases} 1 & \text{if } r = 0, \\ \dfrac{-r}{\ln(1-r)} \dfrac{1}{1 - r + r\cdot x} & \text{if } r \in (0,1), \\ \dfrac{1}{x} & \text{if } r = 1. \end{cases} \tag{5.4}$$

For $r \in [0,1)$, the density with respect to the Lebesgue measure, h_r gives rise to a probability measure, whereas h_1 is the density of an infinite and σ-finite measure.

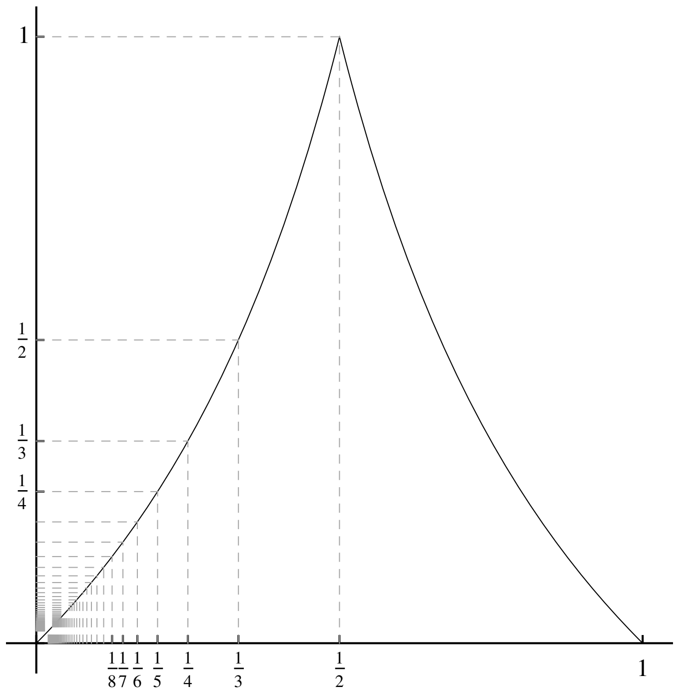


Figure 5.1: The Farey map, $T_1(x)$

5.1.3 The *r*-coding and the Farey coding

Similar to other codings known in number theory such as codings for continued fractions and β-transformations, the family of maps T_r, with $r \in [0,1]$, gives rise to a coding for numbers in the unit interval, see [KKS16, Section 4]. To this end we let $\Sigma := \{0,1\}$, $\Sigma^n := \{0,1\}^n$, for $n \in \mathbb{N}$, and let $\Sigma^{\mathbb{N}}$ denote the set of all infinite words over the alphabet Σ. For $\beta \in [0,1]$ we let $\vartheta_r(\beta)$ denote the infinite word $(\vartheta_{r,1}(\beta), \vartheta_{r,2}(\beta) \dots) \in \Sigma^{\mathbb{N}}$, where we define

$$\vartheta_{r,n}(\beta) := \begin{cases} 0 & \text{if } T_r^{n-1}(\beta) \le 1/2, \\ 1 & \text{otherwise.} \end{cases}$$

We define the *r*-coding of β by $\beta := [\vartheta_{r,1}(\beta), \vartheta_{r,2}(\beta), \dots]_r$. A similar coding is used for α-Farey systems lateron, see Section 6.1.2.
For $n \in \mathbb{N}$ and for $\vartheta = (\vartheta_1, \vartheta_2, \dots) \in \Sigma^{\mathbb{N}}$, we define $\vartheta|_n := (\vartheta_1, \dots, \vartheta_n) \in \Sigma^n$ and, for $\varphi = (\varphi_1, \varphi_2, \dots, \varphi_n) \in \Sigma^n$, we set

$$\begin{aligned} f_{r,\varphi} &:= f_{r,\varphi_1} \circ \cdots \circ f_{r,\varphi_n} \\ \text{and} \quad [\varphi]_r &= [(\varphi_1, \varphi_2, \dots, \varphi_n)]_r := f_{r,\varphi}([0,1]). \end{aligned} \tag{5.5}$$

The set $[\varphi]_r$ is referred to as a *cylinder set of length n* with respect to T_r.
In the proof of the main results we need the adjacent cylinder sets of a given cylinder set. Hence, for later purpose, we let $\vartheta_r^{\pm}(\beta)|_n \in \Sigma^n$ denote unique finite

words such that

$$[\vartheta_r^+(\beta)|_n]_r \cap [\vartheta_r(\beta)|_n]_r \neq \emptyset, \quad [\vartheta_r^-(\beta)|_n]_r \cap [\vartheta_r(\beta)|_n]_r \neq \emptyset \tag{5.6}$$

and such that, for all $x \in (0,1)$, either one of the following sets of inequalities hold.

$$f_{\vartheta_r^-(\beta)|_n}(x) \leq f_{\vartheta_r(\beta)|_n}(x) < f_{\vartheta_r^+(\beta)|_n}(x) \quad \text{or} \quad f_{\vartheta_r^-(\beta)|_n}(x) < f_{\vartheta_r(\beta)|_n}(x) \leq f_{\vartheta_r^+(\beta)|_n}(x).$$

In the case when there exists $\vartheta \in \Sigma^m$, for an $m \in \mathbb{N}$, such that either $f_{r,\vartheta}(0) = \beta$ or $f_{r,\vartheta}(1) = \beta$, then it can occur that $\vartheta_r^+(\beta)|_m = \vartheta_r(\beta)|_m$ or that $\vartheta_r^-(\beta)|_m = \vartheta_r(\beta)|_m$. We call such points *r-rationals*. If β is an r-rational, it is mapped to zero under the iteration of T_r eventually. If $r = 1$, the set of r-rationals is precisely the set of rational numbers in the closed unit interval $[0,1]$ and all rational numbers are mapped to zero under the action of the Farey map eventually.
For ease of notation, we set

$$\begin{aligned} \mathfrak{W}_{r,n}(\beta) &:= \{\vartheta_r^-(\beta)|_n, \vartheta_r(\beta)|_n, \vartheta_r^+(\beta)|_n\} \\ \text{and} \quad [W_{r,n}(\beta)] &:= [\vartheta_r^-(\beta)|_n]_r \cup [\vartheta_r(\beta)|_n]_r \cup [\vartheta_r^+(\beta)|_n]_r. \end{aligned} \tag{5.7}$$

$\mathfrak{W}_{r,n}(\beta)$ refers to a collection of words, whereas $[W_{r,n}(\beta)]$ refers to a interval consisting of the cylinder set of length n and its adjacent cylinder set or sets respectively. If we look at two adjacent cylinders or more exactly at their coding, we observe that it differs in exactly one letter, as described in the following lemma. This observation is needed in the proofs of the main theorems in Part II.

Lemma 5.1 ([KKS16, Lemma 4.1]). *Let $r \in [0,1]$ and $n \in \mathbb{N}$ be fixed. If $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_n)$ and $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ denote two distinct, yet adjacent, elements of Σ^n. That is, we have that $[\vartheta]_r \neq [\nu]_r$ and $[\vartheta]_r \cap [\nu]_r \neq \emptyset$, then there exists a unique $i \in \{1,2,\dots,n\}$ such that $\vartheta_i \neq \nu_i$ and $\vartheta_j = \nu_j$ for all $j \in \{1,2,\dots,n\} \setminus \{i\}$.*

Proof of Lemma 5.1. For $n = 1$, we have that $[(0)]_r = f_{r,0}([0,1]) = [0,1/2]$ and $[(1)]_r = f_{r,1}([0,1]) = [1/2,1]$ and we proceed by induction on n. So, suppose the statement is true for $n \in \mathbb{N}$. Let $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_{n+1})$ and $\nu = (\nu_1, \nu_2, \dots, \nu_{n+1})$ denote two distinct elements of Σ^{n+1}, with $[\vartheta]_r \cap [\nu]_r \neq \emptyset$. We have two cases to consider, namely, if there exists a word $\xi \in \Sigma^n$ such that $[\vartheta]_r \cup [\nu]_r = [\xi]_r$, or not. In the case that there exists a word $\xi = (\xi_1, \dots, \xi_n) \in \Sigma^n$ with $[\vartheta]_r \cup [\nu]_r = [\xi]_r$, then, by construction, either $\vartheta = (\xi_1, \xi_2, \dots, \xi_n, 0)$ and $\nu = (\xi_1, \xi_2, \dots, \xi_n, 1)$, or $\vartheta = (\xi_1, \xi_2, \dots, \xi_n, 1)$ and $\nu = (\xi_1, \xi_2, \dots, \xi_n, 0)$, in which case the result follows.
In the case that there does not exist a word $\xi \in \Sigma^n$ with $[\vartheta]_r \cup [\nu]_r = [\xi]_r$, then, by construction, there exist $\xi = (\xi_1, \xi_2, \dots, \xi_n), \eta = (\eta_1, \eta_2, \dots, \eta_n) \in \Sigma^n$ such that $[\xi]_r \cap [\eta]_r \neq \emptyset$, $[\vartheta]_r \subset [\xi]_r$ and $[\nu]_r \subset [\eta]_r$. Since $f_{r,1}$ is monotonically decreasing, an odd number of applications of $f_{r,1}$ is order reversing and an even number or no applications of $f_{r,1}$ is order preserving. Therefore, by the inductive hypothesis, we have that either $f_{r,\xi}$ is order preserving and $f_{r,\eta}$ is order reversing, or $f_{r,\xi}$ is order reversing and $f_{r,\eta}$ is order preserving. Assuming the former of these two cases, by construction we have that $\vartheta = (\xi_1, \dots, \xi_n, 1)$ and $\nu = (\eta_1, \dots, \eta_n, 1)$, in which

case the result follows. In the remaining case, namely that $f_{r,\xi}$ is order reversing and $f_{r,\eta}$ is order preserving, by construction we have that $\vartheta = (\xi_1, \dots, \xi_n, 0)$ and $\nu = (\eta_1, \dots, \eta_n, 0)$, which concludes the proof.

□

For the case $r = 1$, the Farey-coding and the Continued Fraction coding is linked, as the Gauß transformation is the jump transformation with respect to $[1/2, 1]$ of the Farey map. The number of consecutive zeroes in the Farey coding determines the continued fraction entry and vice versa. If for two natural numbers k, m a number has a block of $(k-1)$ consecutive zeroes in the Farey-coding, and this block is followed by the m^{th} one, the m^{th} continued fraction entry is $a_m = k$. For example,

$$x := [0,0,0,1, \quad 0,0,1, \quad 0,0,0,0,0,0,1, \dots]_1 = [4,3,7,\dots] = \cfrac{1}{4 + \cfrac{1}{3 + \cfrac{1}{7 + \dots}}}.$$

Considering the Farey coding we observe two further technical results, Lemmata 5.2 and 5.3. To state and prove the lemma we introduce further variables.
For $n \in \mathbb{N}$ and $\beta \in (0, 1]$, We recall that $p_n = p_n(\beta)$ and $q_n = q_n(\beta)$ are as defined in (5.1), and define $k(n) = k(n,\beta)$, $m(n) = m(n,\beta)$ and $r(n) = r(n,\beta)$ by

$$\begin{aligned} k(n) &:= \max\{k \in \{1,2,\dots,n\} \colon \vartheta_{1,k}(\beta) = 1\}, \\ m(n) &:= \#\{\ell \in \{1,2,\dots,n\} \colon \vartheta_{1,\ell}(\beta) = 1\} \\ \text{and} \quad r(n) &:= n - k(n). \end{aligned} \tag{5.8}$$

The first lemma is a list of properties that can be discerned from the given definitions.

Lemma 5.2 ([KKS16]). *For $k(n)$, $m(n)$ and $r(n)$, given in* (5.8)*, and the Farey map T_1, we have the following properties.*

1. *If $k(n) = n$, then $a_{m(n)} = n - k(n-1)$.*
2. *If $(b_m)_{m\in\mathbb{N}}$ is a sequence of positive real numbers, then we have that*
$$T_1([0; b_1, b_2, b_3 \dots]) = \begin{cases} [0; b_1 - 1, b_2, b_3 \dots] & \text{if } b_1 > 1, \\ [0; b_2, b_3 \dots] & \text{if } b_1 = 1. \end{cases}$$
3. *For $n \in \mathbb{N}$, we have that*
$$f_{1,\vartheta_1(\beta)|_n}(0) = \frac{p_{m(n)}}{q_{m(n)}} = [0; a_1, a_2, \dots, a_{m(n)}]$$
and
$$f_{1,\vartheta_1(\beta)|_n}(1) = \frac{(r(n)+1)\cdot p_{m(n)} + p_{m(n)-1}}{(r(n)+1)\cdot q_{m(n)} + q_{m(n)-1}} = [0; a_1, a_2, \dots, a_{m(n)}, r(n)+1].$$

Proof of Lemma 5.2. 1. The first and the third statement follow by definition.

2. The second statement follows by the definition of the Farey map and the fact, that the Gauß map is the jump transformation of the Farey map. This property is very important in the proof of the main result in Part II. In fact, it states that for each natural number $n > 1$, we have that $T_1\left([1/(n+1), 1/n]\right) = [1/n, 1/(n-1)]$ and $T_1\left([1/2, 1]\right) = [0, 1]$. This can be seen in Figure 5.1.

□

Lemma 5.3 ([KKS16, Lemma 4.11]). *For $n \in \mathbb{N}$ and $\beta \in (0, 1]$, we have that*

$$f_{1,\vartheta_1(\beta)|_n}(x) = \frac{(r(n) \cdot p_{m(n)} + p_{m(n)-1}) \cdot x + p_{m(n)}}{(r(n) \cdot q_{m(n)} + q_{m(n)-1}) \cdot x + q_{m(n)}}, \tag{5.9}$$

where $p_n = p_n(\beta)$ and $q_n = q_n(\beta)$ are as defined in (5.1)*.*

Proof of Lemma 5.3. The function $f_{1,\vartheta_1(\beta)|_n}$ is a Möbius transformation and as such it is uniquely determined by its values at three distinct points. Let us consider the case when $\vartheta_{1,n}(\beta) = 1$. By definition we have that $r(n) = 0$ and so the function of (5.9) becomes

$$x \mapsto \frac{p_{m(n)-1} \cdot x + p_{m(n)}}{q_{m(n)-1} \cdot x + q_{m(n)}}. \tag{5.10}$$

By Lemma 5.2.3. given above,

$$0 \mapsto \frac{p_{m(n)}}{q_{m(n)}} = f_{\vartheta_1(\beta)|_n}(0) \quad \text{and} \quad 1 \mapsto \frac{p_{m(n)-1} + p_{m(n)}}{q_{m(n)-1} + q_{m(n)}} = f_{\vartheta_1(\beta)|_n}(1).$$

Since $f_{1,\vartheta_1(\beta)|_n}$ is a contraction, by Banach's fixed point theorem, there exists a unique $x \in [0, 1]$ such that $f_{1,\vartheta_1(\beta)|_n}(x) = x$. By Lemma 5.2.1. and 2. given above the pre-periodic point

$$[0; \overline{a_1, \dots, a_{m(n)}}] := [0; a_1, \dots, a_{m(n)}, a_1, \dots, a_{m(n)}, a_1, \dots, a_{m(n)}, \dots, a_1, \dots, a_{m(n)}, \dots]$$

is a fixed point of $f_{1,\vartheta_1(\beta)|_n}$. Further, by [DK02, Exercise 1.3.10] it follows that the point $[0; \overline{a_1, \dots, a_{m(n)}}]$ is a fixed point of the map given in (5.10). This completes the proof of the result for $\vartheta_n(\beta) = 1$.
The result for the case when $\vartheta_{1,n}(\beta) = 0$, follows from the definition of $r(n)$ and the case when $\vartheta_{1,n}(\beta) = 1$, together with the observation that we have for $n \in \mathbb{N}$ and all $x \in [0, 1]$, that $f_{1,0}^n(x) = x/(1 + n \cdot x)$.

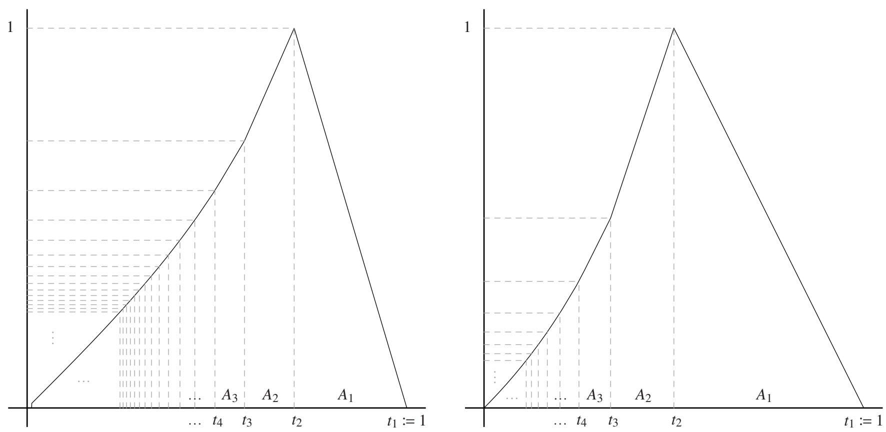
□

5.2 The family of α-Farey systems

Another family of systems that is relevant in this thesis is given by the family of α-*Farey maps*. This family is important in Part III. These maps are interesting from a number-theoretical point of view as well as from a dynamical viewpoint, since they offer piecewise linear versions of transformations with infinite (and finite) measures having different wandering rates. To introduce these maps, this thesis follows the notation used in [KKSS15], hence, parts of this section are published in [KKSS15]. For further details about the α-Farey system, see [KMS12, Mun11].
For the definition of the α-Farey transformation, we define a countable infinite partition $\alpha := \{A_n : n \in \mathbb{N}\}$ of $(0,1)$ by non-empty, right-open and left-closed intervals A_n. It is assumed throughout that the atoms of α are ordered from right to left, starting with A_1, and that these atoms only accumulate at zero. We let a_n denote the Lebesgue measure $\lambda(A_n)$ of the atom $A_n \in \alpha$. Furthermore, $t_n := \sum_{k=n}^{\infty} a_k$ denotes the Lebesgue measure of the n-th tail of α, namely $\lambda\left(\bigcup_{k=n}^{\infty} A_k\right)$.
The α-Farey map $F_\alpha \colon [0,1] \to [0,1]$ is defined by

$$F_\alpha(x) := \begin{cases} \dfrac{1-x}{a_1} & \text{if } x \in \overline{A}_1 := A_1 \cup \{1\}, \\ \dfrac{a_{n-1} \cdot (x - t_{n+1})}{a_n} + t_n & \text{if } x \in A_n, \text{ for } n \geq 2, \\ 0 & \text{if } x = 0. \end{cases} \tag{5.11}$$

For a picture of two α-Farey maps with respect to different partitions see Figure 5.2.



(a) $\delta = 65/128$. (b) $\delta = 1$.

Figure 5.2: The α-Farey map, where $t_n = n^{-\delta}$, for all $n \in \mathbb{N}$.

Throughout, we will assume that the partition α satisfies the condition that the sequence $(t_n)_{n\in\mathbb{N}}$ is not summable. For $\delta \in (0,1]$, an α-Farey map F_α is said to

be δ-*expansive* if the sequence $(a_n)_{n\in\mathbb{N}}$ is regularly varying of order $-(1+\delta)$, that is, if there exists a slowly varying function $\ell\colon \mathbb{R}\to\mathbb{R}$ such that, for all $n\in\mathbb{N}$, $a_n = \delta\cdot\ell(n)\cdot n^{-(1+\delta)}$.
In this situation, the Abelian part of 4.3 implies that

$$\lim_{n\to\infty}\frac{\ell(n)\cdot n^{-\delta}}{t_n} = \lim_{n\to\infty}\frac{\ell(n)\cdot n^{-\delta}}{\sum_{k=n}^{\infty} a_k} = \lim_{n\to\infty}\frac{\ell(n)\cdot n^{-\delta}}{\sum_{k=n}^{\infty}\delta\cdot\ell(n)\cdot n^{-(1+\delta)}} = 1. \tag{5.12}$$

By definition we have for all $n\in\mathbb{N}$, that $t_{n+1}<t_n$. This in combination with (5.12) implies by the Tauberian part of Theorem 4.3, for $\delta\in(0,1)$, that

$$w_n \sim \overline{\Gamma}_\delta\cdot n^{1-\delta}\cdot\ell(n), \tag{5.13}$$

where

$$\overline{\Gamma}_\delta := \frac{\Gamma(1-\delta)}{\Gamma(2-\delta)}. \tag{5.14}$$

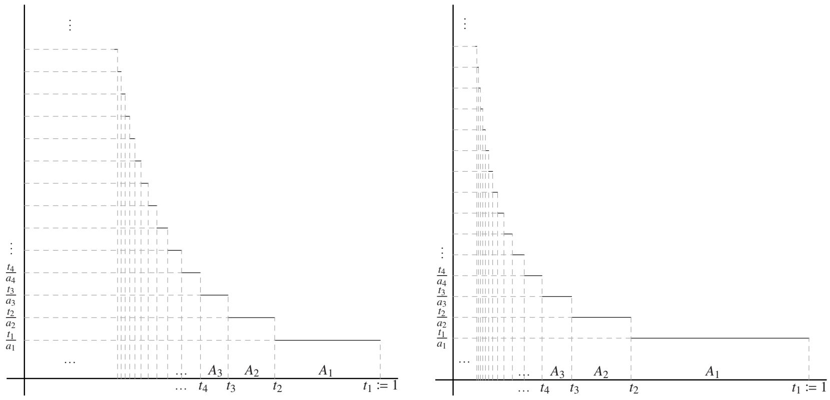
Therefore, the Lebesgue measure of the n-th tail of α is asymptotic to a regularly varying function of order $-\delta$, which is called expansive of order δ in [KMS12]. Thus, δ-expansive implies expansive of order δ in the sense of [KMS12]. However, an expansive α-Farey map of order δ is not necessarily δ-expansive.
This can be seen by observing that the Abelian part of Theorem 4.3 requires less assumptions than the Tauberian part.
For the *dyadic partition*, namely $\{[1/2^n, 1/2^{n+1}) : n\in\mathbb{N}\}\cup\{[1/2,1]\}$, the α-Farey map coincides with the tent map T_0, introduced in Subsection 5.1.2, see [KMS12]. In this case $t_n\sim 2^{-n}$, and hence the sequence $(t_n)_{n\in\mathbb{N}}$ would be summable which contradicts our assumption for this part of the thesis. This partition would give rise to an invariant probability measure, namely the Lebesgue measure. Hence, in that case we could apply tools from finite ergodic theory.
By [KMS12, Lemma 2.5], α-Farey transformations give rise to an invariant measure μ_α whose density with respect to the Lebesgue measure is given by

$$h_\alpha := \frac{\mathrm{d}\mu_\alpha}{\mathrm{d}\lambda} = \sum_{n\in\mathbb{N}}\frac{t_n}{a_n}\mathbb{1}_{A_n}. \tag{5.15}$$

See [KMS12] for a proof of this statement and see Figure 5.3 for a plot of the invariant density.
As in the previous section, we define the inverse branches of the α-Farey map by

$$f_{\alpha,0}(x) := (F_\alpha|_{[0,t_2]})^{-1}(x) \quad\text{and}\quad f_{\alpha,1}(x) := (F_\alpha|_{[t_2,1]})^{-1}(x). \tag{5.16}$$

These inverse branches are needed in the pointwise representation of the transfer operator. Note, that in the last formula the index α refers to a partition, whereas the index r in (5.3) is an element in $[0,1]$. In the sequel it will be clear from the context, to which system we refer to, so no confusion will appear.



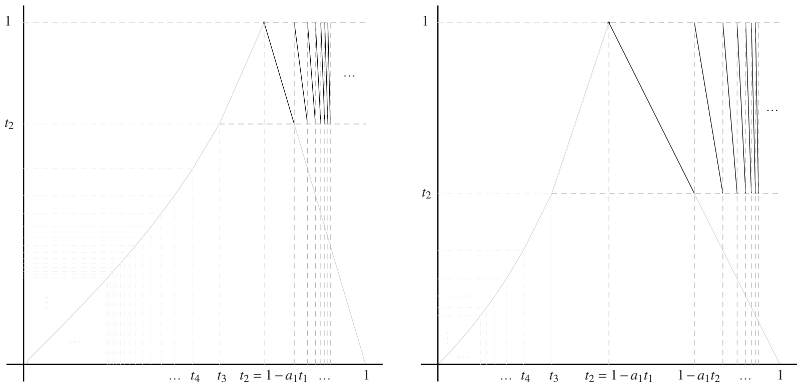
(a) $\delta = 65/128$. (b) $\delta = 1$.

Figure 5.3: Plot of the density function h_α for the α-Farey map, where $t_n = n^{-\delta}$ for all $n \in \mathbb{N}$.

We denote the induced transformation, as defined in Definition 3.4, of the α-Farey map for the set A_1 by $F_{\alpha A_1}$. It is given by $F_{\alpha A_1} : \overline{A}_1 \to \overline{A}_1$,

$$F_{\alpha A_1}(x) := \begin{cases} F_\alpha^{\phi_{A_1}(x)}(x) & \text{if } x \in A_1, \\ t_2 & \text{if } x = 1. \end{cases} \tag{5.17}$$

See Figure 5.4 for two examples of the induced transformation with respect to the same partitions as in Figure 5.2.



(a) $\delta = 65/128$. (b) $\delta = 1$.

Figure 5.4: Plot of $F_{\alpha A_1}$, where $t_n = n^{-\delta}$ for all $n \in \mathbb{N}$.

Chapter 6

The transfer operator method

After introducing some topics of number theory, we return to the questions being asked at the end of Chapter 3.

The first section of this chapter is split up into two parts. First we introduce the general theory and the necessary relations in Subsection 6.1.1, afterwards, in Subsection 6.1.2, we apply these tools to the example systems given in Chapter 5. The second section, Section 6.2, gives an overview on the state of the art of distributional convergence. Finally, Section 6.3 introduces operator renewal theory, by first giving a short introduction to classical renewal theory and secondly by introducing renewal equations for operators. We conclude this section, and hence this chapter, with Subsection 6.3.3 in which two example systems are given that give rise to Banach spaces for which operator renewal theory can be applied.

6.1 The transfer operator

6.1.1 The theory behind the transfer operator - defining relations

As mentioned before, an important tool to examine limiting behaviour of dynamical systems is the transfer operator, whose job it is to describe the iterations of densities and hence distributions over time. The transfer operator plays the crucial role in this thesis and therefore a short introduction to this operator is given. For a thorough introduction to this operator the reader is referred to [LM94, Wal82], or a variety of lecture notes.

This chapter gives the definition used in [LM94], with a slight adjustment of the notation to be in accordance with the rest of this thesis.

Definition 6.1 (**Transfer operator** [LM94, Definition 3.2.3])**.** Let $(X, \mathfrak{B}, \mu)$ denote a measure space and $f \in L^1_\mu(X)$. If $T\colon X \to X$ is a non-singular transformation, the unique operator $\widehat{T}\colon L^1_\mu(X) \to L^1_\mu(X)$ that satisfies for each $A \in \mathfrak{B}$, with $\mu(A) < \infty$, that

$$\int_A \widehat{T}(f(x))\mathrm{d}\mu(x) = \int_{T^{-1}(A)} f(x)\mathrm{d}\mu(x),$$

is called the *transfer operator* with respect to μ.

The uniqueness of the operator follows by the Radon-Nykodym theorem (compare [LM94, Theorem 2.2.1]). Approximation arguments from measure theory yield the more common dual relation of the transfer operator, namely for $g \in L^\infty_\mu(X)$ and each $f \in L^1_\mu(X)$, we have that

$$\int_X g \cdot \widehat{T}(f(x))\mathrm{d}\mu(x) = \int_X (g \circ T)(x) \cdot f(x)\mathrm{d}\mu(x). \tag{6.1}$$

The operator $U\colon L^\infty_\mu(X) \to L^\infty_\mu(X)$, given by $U(g(x)) := (g \circ T)(x)$, is known as the Koopman operator.
From a probabilistic point of view, this operator can be described in the following way. Let Z denote a random variable on the space X, whose density on X with respect to the invariant measure μ is given by f. Then, the random variable $T^n \circ Z$ has the density $\widehat{T^n} f$.

In literature the Perron-Frobenius operator is often defined by the following relation: Let f denote the invariant density of T with respect to the Lebesgue measure. For all Borel sets $A \subset [0, 1]$, let $\nu_f(A) := \int \mathbb{1}_A \cdot f \,\mathrm{d}\lambda$. We have that

$$\mathcal{P}(f) = \frac{\mathrm{d}\nu_f \circ T^{-1}}{\mathrm{d}\lambda}. \tag{6.2}$$

The name of the transfer operator varies in literature. Throughout this thesis, we call the transfer operator with respect to the invariant measure for a transformation T, *transfer operator* and denote it by $\widehat{T}$. The transfer operator with respect to the Lebesgue measure will be called *Perron-Frobenius operator* and is denoted by $\mathcal{P}$. There is a relation between these two operators, which is given in the next lemma, Lemma 6.2.

Lemma 6.2. *Let $(X, \mathfrak{B}, T, \mu)$ denote a measure preserving, ergodic dynamical system. Let μ be absolutey continuous with respect to the Lebesgue measure λ and let h denote the density of μ with respect to λ. For $f \in L^1_\mu$, we have that*

$$\widehat{T}(f) = \frac{\mathcal{P}(f \cdot h)}{h}.$$

Proof of Lemma 6.2. Let $g \in L^\infty_\mu$ and $f \in L^1_\mu$. We have that

$$\begin{aligned}
\int g \cdot \widehat{T}(f) d\mu &= \int (g \circ T) \cdot f \cdot h \,\mathrm{d}\lambda \\
&= \int g \cdot \mathcal{P}(f \cdot h) \, d\lambda \\
&= \int g \cdot \mathcal{P}(f \cdot h) \cdot \frac{1}{h} \,\mathrm{d}\mu.
\end{aligned}$$

Since $g \in L^\infty_\mu$ and $f \in L^1_\mu$ were chosen arbitrarily, the lemma is proven.

□

Working with the transfer operator is often easier with a pointwise defined version of it. By *a pointwise version* of the transfer operator we understand a function in $\mathcal{L}^1_\mu$ that satisfies the dual relation given in (6.1). Its equivalence class is an element in L^1_μ. Such a version for the Perron-Frobenius operator is given by

$$\mathcal{P}(f)(x) = \sum_{y \in T^{-1}(x)} \left| \frac{1}{T'(y)} \right| \cdot f(y). \tag{6.3}$$

This formula together with Lemma 6.2 allows us to calculate explicit pointwise versions of the transfer operator, which is done for the two main examples in Subsection 6.1.2.
Afterwards, with the transfer operator at hand, we give an overview of the state of the art of distributional convergence.

6.1.2 The two examples - Part 2

To calculate explicit pointwise versions of the transfer operator, recall the general Definition 6.1, the defining relations of Chapter 5 and in particular Lemma 6.2 and Equation (6.3).

The transfer operator of the family T_r

Let the derivatives of the contractions $f_{r,0}$ and $f_{r,1}$ be denoted by $f'_{r,0}$ and $f'_{r,1}$ respectively, see (5.3).
For $r \in [0,1]$, a pointwise version of the *Perron-Frobenius operator*

$$\mathcal{P}_r \colon \mathcal{L}^1_\lambda([0,1]) \to \mathcal{L}^1_\lambda([0,1])$$

of T_r can be obtained by (6.3). For $f \in \mathcal{L}^1_\lambda([0,1])$, we have that

$$\mathcal{P}_r(f) := \left| f'_{r,0} \right| \cdot f \circ f_{r,0} + \left| f'_{r,1} \right| \cdot f \circ f_{r,1}. \tag{6.4}$$

This representation coincides with (6.2). The definition of $\mathcal{P}_r$ can be extended to well-defined $\mathbb{C}$-valued or $\overline{\mathbb{R}}$-valued functions, which will be of use in Part II. To apply Lemma 6.2 we need to recall the invariant density of T_r from Equation (5.4). Let us now focus on the case $r = 1$. This is also the case we require most in Part **II**. The proofs of the statements for the cases $r \in [0,1)$ can be done by considering the Perron-Frobenius operator only.

It turns out, see for instance [KS08, Kau11], that the operator $\widehat{T}_1$ can be written in terms of the inverse branches of T_1. We have,

$$\begin{aligned}\widehat{T}_1(f)(x) &= \frac{1}{h_1}\cdot \mathcal{P}_1\left(h_1\cdot f\right)\\ &= x\cdot\left(\frac{1+x}{x}\cdot\frac{f\left(\frac{x}{1+x}\right)}{(1+x)^2}+\frac{f\left(\frac{1}{1+x}\right)}{1+x}\right)\\ &= \frac{1}{(1+x)}\cdot f\left(\frac{x}{1+x}\right)+\frac{x}{1+x}\cdot f\left(\frac{1}{1+x}\right)\\ &= f_{1,0}(x)\cdot f\circ f_{1,1}(x)+f_{1,1}(x)\cdot f\circ f_{1,0}(x).\end{aligned} \tag{6.5}$$

This representation and the "symmetry" in the formula comes in very handy in the sequel. The results presented later do not require this symmetry, but calculations become considerably easier with formula (6.5).

The α-Farey transfer operator

By (6.1), the transfer operator for the α-Farey map is given for all $v \in \mathcal{L}^1_{\mu_\alpha}([0,1])$ and all measurable functions w with $\|w\|_\infty < \infty$ by the defining dual relation

$$\int \widehat{F}_\alpha(v)\cdot w\,\mathrm{d}\mu_\alpha = \int v\cdot w\circ F_\alpha\,\mathrm{d}\mu_\alpha. \tag{6.6}$$

In a similar manner as in the previous example we can calculate a pointwise defined version of this operator which satisfies (6.6). To distinguish between the pointwise version and the $L^1_{\mu_\alpha}$-version of the transfer operator, in [KKSS15] the pointwise version is called the *α-Farey transfer operator*. It is given by the positive linear operator $\widehat{F}_\alpha\colon \mathcal{L}^1_{\mu_\alpha}([0,1]) \to \mathcal{L}^1_{\mu_\alpha}([0,1])$, with

$$\widehat{F}_\alpha(v) := \sum_{n\in\mathbb{N}}\left(\frac{t_{n+1}}{t_n}v\circ f_{\alpha,0}+\left(1-\frac{t_{n+1}}{t_n}\right)v\circ f_{\alpha,1}\right)\cdot \mathbb{1}_{A_n}, \tag{6.7}$$

where $f_{\alpha,0}$ and $f_{\alpha,1}$ refer to the inverse branches of F_α, see (5.16).
Equation (6.7) is a consequence of Lemma 6.2 and Equation (5.15). The explicit calculation follows the same path as in the other example in the previous subsection. It is explicitly caried out in [Kau11, Section 3.3.5].
Later in this thesis we want to look at the individual iterates of the transfer operator. Thus, we have to calculate pointwise versions of it as well. To this end, let $\vartheta \in \Sigma^k$ denote a word of length k over the alphabet $\Sigma := \{0,1\}$ and define the following set of constants recursively by

$$\begin{aligned} c_{n,(0)} &:= \frac{t_{n+1}}{t_n}, & c_{n,(\vartheta_1,\dots,\vartheta_k,0)} &:= c_{n,(0)}\cdot c_{n+1,\vartheta},\\ c_{n,(1)} &:= 1-\frac{t_{n+1}}{t_n}, & c_{n,(\vartheta_1,\dots,\vartheta_k,1)} &:= c_{n+1,(1)}\cdot c_{1,\vartheta}.\end{aligned} \tag{6.8}$$

In particular, letting $0_k := (\underbrace{0, 0, \dots, 0}_{k\text{-times}})$, we have for each $k \in \mathbb{N}$ that

$$c_{1,0_k} = t_{k+1}.$$

As in (5.5), we define for $n \in \mathbb{N}$ and each word $\varphi \in \Sigma^n$,

$$f_{\alpha,\varphi} := f_{\alpha,\varphi_1} \circ \cdots \circ f_{\alpha,\varphi_n}.$$

Pointwise versions of the iterations of the transfer operator of the α-Farey map are given by the following lemma. The proof of it is technical but straight forward.

Lemma 6.3 ([KKSS15, Lemma 2.2])**.** *Let $F_\alpha \colon [0,1] \to [0,1]$ denote an arbitrary α-Farey map and let $u \in \mathcal{L}^1_{\mu_\alpha}$. For each $k \in \mathbb{N}$, we have that*

$$\widehat{F}_\alpha^k(u) = \sum_{n=1}^{\infty} \sum_{\vartheta \in \Sigma^k} c_{n,\vartheta} \cdot u \circ f_{\alpha,\vartheta} \cdot \mathbb{1}_{A_n}.$$

Proof of Lemma 6.3. We proceed by induction on k. The start of the induction is an immediate consequence of (6.7). If we suppose that the statement is true for a natural number $k \in \mathbb{N}$, we have that

$$\begin{aligned}
\widehat{F}_\alpha^{k+1}(u) &= \widehat{F}_\alpha\left(\widehat{F}_\alpha^k(u)\right) = \widehat{F}_\alpha\left(\sum_{n\in\mathbb{N}} \sum_{\vartheta\in\Sigma^k} c_{n,\vartheta} u \circ f_{\alpha,\vartheta} \cdot \mathbb{1}_{A_n}\right) \\
&= \sum_{m=1}^{\infty}\left(\sum_{n\in\mathbb{N}} \sum_{\vartheta\in\Sigma^k} \frac{t_{m+1}}{t_m} c_{n,\vartheta} u \circ f_{\alpha,\vartheta} \circ f_{\alpha,0} \cdot \mathbb{1}_{A_n} \circ f_{\alpha,0}\right. \\
&\qquad \left. + \left(1 - \frac{t_{m+1}}{t_m}\right) c_{n,\vartheta} u \circ f_{\alpha,\vartheta} \circ f_{\alpha,1} \cdot \mathbb{1}_{A_n} \circ f_{\alpha,1}\right) \cdot \mathbb{1}_{A_m} \\
&= \sum_{m=1}^{\infty}\left(\sum_{\vartheta\in\Sigma^k} \frac{t_{m+1}}{t_m} c_{m+1,\vartheta} u \circ f_{\alpha,\vartheta} \circ f_{\alpha,0} + \left(1 - \frac{t_{m+1}}{t_m}\right) c_{1,\vartheta} u \circ f_{\alpha,\vartheta} \circ f_{\alpha,1}\right) \cdot \mathbb{1}_{A_m}.
\end{aligned}$$

Using the defining relations given in (6.8), this completes the proof of Lemma 6.3.

□

We state yet another technical lemma that is needed in the sequel.

Lemma 6.4 ([KKSS15, Lemma 2.5])**.** *For each $n \in \mathbb{N}$, we define*

$$10_n := (1, \underbrace{0, 0, \dots, 0}_{n\text{-times}}).$$

We have for each α-Farey map F_α, that

$$c_{1,10_{n-1}} = \mu_\alpha(\{\phi_{A_1} = n\}) = a_n = t_n - t_{n+1},$$

Proof of Lemma 6.4. By construction of the α-Farey map, we have that

$$\{\phi_{A_1} = 1\} = [1 - a_1 t_1, 1 - a_1 t_2]$$

and, for all integers $n > 1$, we have that

$$\{\phi_{A_1} = n\} = (1 - a_1 t_n, 1 - a_1 t_{n+1}].$$

Thus,

$$\mu_\alpha(\{\phi_{A_1} = n\}) = \int \mathbb{1}_{\{\phi_{A_1}=n\}} \cdot \frac{\mathrm{d}\mu_\alpha}{\mathrm{d}\lambda}\,\mathrm{d}\lambda = \frac{a_1}{\lambda(\{\phi_{A_1} = n\})} = t_n - t_{n+1}. \tag{6.9}$$

We show by induction on n that we have for each $k \in \mathbb{N}$, that

$$c_{k,10_{n-1}} = \frac{t_{k+n-1} - t_{k+n}}{t_k}. \tag{6.10}$$

By (6.7), we have for each $k \in \mathbb{N}$, that $c_{k,(1)} = 1 - t_{k+1}/t_k = (t_k - t_{k+1})/t_k$, which is the start of the induction. Hence, suppose that the statement in (6.10) is true for an $n \in \mathbb{N}$. From (6.8), we have that

$$c_{k,10_n} := \left(\frac{t_{k+1}}{t_k}\right) \cdot c_{k+1,10_{n-1}},$$

for each $k \in \mathbb{N}$, which gives

$$c_{k,10_n} = \frac{t_{k+1}}{t_k} \cdot \frac{t_{k+n} - t_{k+n+1}}{t_{k+1}} = \frac{t_{k+n} - t_{k+n+1}}{t_k}.$$

This completes the proof of the statement in (6.10).
Setting $k = 1$ in (6.10), we obtain that $c_{1,10_{n-1}} = t_n - t_{n+1}$, for all $n \in \mathbb{N}$. Combining this with (6.9), completes the proof.

□

6.2 Distributional convergence - state of the art

This paragraph gives an overview of known convergence results for the transfer operator. In particular it will point out, why the problems turn out to be considerably more delicate in infinite ergodic theory.

Theorem 3.9 shows that it is not possible to find a normalising sequence for the ergodic sum for a dynamical system with an infinite invariant measure. Can we ask a more suitable question? A good starting point is to look at the evolution of densities instead of the pointwise behaviour of a dynamical system. For this procedure we need the transfer operator introduced in Chapter 6.1. That means that instead of considering ergodic sums we look at the dual analogs, namely we try to find suitable constants (r_n), with $n \in \mathbb{N}$, and look at

$$\frac{1}{r_n} \cdot \sum_{k=0}^{n-1} \widehat{T}(f). \tag{6.11}$$

Furthermore, the transfer operator is an important tool to prove the existence of invariant measures itself, because we have that if a transformation T is invariant with respect to μ, the constant one function $\mathbb{1}$ is a fixed point of the transfer operator, that is, $\widehat{T}(\mathbb{1}) = \mathbb{1}$. Lasota and Yorke used this approach in [LY73] for differentiable, uniformly hyperbolic interval maps. Additionally they point out, for the example of the Pomeau-Manneville map, which challenges might occur if one considers maps with indifferent fixed points.
To find analogue statements for maps with indifferent fixed points, we need the notion of *pointwise dual ergodicity* introduced by Aaronson.
For the next two definitions let $(X, \mathfrak{B}, \mu, T)$ denote a conservative, ergodic, measure preserving system.

Definition 6.5 (**Pointwise dual ergodic** [Aar97, § 3.7]). T is called *pointwise dual ergodic* if there are constants r_n such that for all $f \in L^1_\mu$

$$\frac{1}{r_n} \cdot \sum_{k=0}^{n-1} \widehat{T}^k(f) \to \int_X f \,\mathrm{d}\mu \quad \text{almost everywhere as } n \to \infty.$$

The sequence $(r_n)_{n\in\mathbb{N}}$ is called *return sequence*.

The wandering rate of a pointwise dual ergodic system is independent of the set A up to asymptotic equivalence, if A has positive and finite measure and A is a so-called uniform set. For further details and the definition of a *uniform* set, see [Aar97, Section 3.8]. Closely linked to pointwise dual ergodicity is the notion of *Darling-Kac sets*.

Definition 6.6 (**Darling-Kac set** [Aar97, § 3.7]). A set $A \in \mathfrak{B}$, with $0 < \mu(A) < \infty$ is called a *Darling-Kac set*, if there are constants $r_n > 0$ such that

$$\frac{1}{r_n} \cdot \sum_{k=0}^{n-1} \widehat{T}^k \mathbb{1}_A \to \mu(A) \quad \text{almost everywhere uniformly on A as } n \to \infty.$$

The wandering rate and the return sequence $(r_n)_{n\in\mathbb{N}}$ are linked via the relation, see [Aar97, Proposition 3.8.7],

$$r_n \sim \frac{n}{\Gamma(1+\delta)\cdot\Gamma(2-\delta)\cdot w_n}, \tag{6.12}$$

where we assume that the wandering rate is regularly varying with index δ. The notion of Darling-Kac sets is important, because under certain circumstances the existence of such a set implies pointwise dual ergodicity as the next proposition shows.

Proposition 6.7 ([Aar97, Proposition 3.7.5]). *Suppose T is a conservative, ergodic, measure preserving transformation of $(X, \mathfrak{B}, \mu)$. If T has a Darling-Kac set, then T is pointwise dual ergodic.*

Aaronson, Denker and Urbański show in [ADU93] that Gibbs-Markov maps are pointwise dual ergodic. Furthermore, exactness of such maps is shown and sufficient conditions are given, to see whether the invariant measure is finite or infinite. In the infinite measure case they prove the existence of Darling-Kac sets and investigate return times and the asymptotics of the return sequences.
Further investigations of the limiting behaviour of (6.11) have been made by Collet and Ferrero in [CF90], in which they examine this limiting behaviour for maps of the form $T\colon [0,1] \to [0,1]$ with $T(x) := x + a \cdot x^2 + \mathfrak{o}\left(x^2\right)$, in a neighbourhood of zero. Thaler proves a limit theorem for a class of functions, nowadays known as Thaler maps in [Tha95] and Zweimüller extends in [Zwe98, Zwe00] this class of transformations to so-called AFN maps and gives further sufficient conditions for the existence of Darling-Kac maps. An AFN map with full branches is a Thaler map.
When investigating the asymptotic behaviour of (6.11), it seems natural to ask, what can be said about the individual iterates of the transfer operator. Of course, these kind of questions have first been asked in finite ergodic theory. These results translated to our setting read as follows. That is, we consider the Perron Frobenius operator $\mathcal{P}_r$ of T_r for $r \in [0,1)$.

Theorem 6.8 ([Bal00, Col96, Kel84, Ryc83]). *For $r \in [0,1)$ there exist constants $M = M(r) > 0$ and $p = p(r) \in (0,1)$ such that*

$$\left\| \mathcal{P}_r^n(f) - \int f \, \mathrm{d}\lambda \cdot h_r \right\|_{\mathrm{BV}} \leq M \cdot p^n \cdot \|f\|_{\mathrm{BV}}.$$

This result is called exponential decay of correlation and is needed in the proof of Theorem 8.1. However, we can not always expect exponential decay of correlations, as Gouëzel shows in [Gou04]. In his paper, systems for which we have polynomial decay of correlations are considered.
In the infinite ergodic theory setting, first investigations in that direction date back to Thaler in [Tha00] in which, once more, Thaler maps are considered. Thaler discerns statements about the limiting behaviour of the individual iterates of the transfer operator. Since the Farey map, T_1, is not a Thaler map, further investigations were necessary. Kesseböhmer and Slassi show in [KS08] that for each

$$f \in \mathcal{D} := \left(f \in \mathcal{L}^1_\mu \colon f \in C^2((0,1)) \text{ with } f' > 0 \text{ and } f'' \leq 0 \right), \tag{6.13}$$

$w_n \cdot \widehat{T}^n(f)$ converges to $\int f \, d\mu_1$ almmost everywhere uniformly on $(\sqrt{(2)} - 1, 1]$.
The results of Theorem 6.8 rely on the fact that in finite ergodic theory, the Perron-Frobenius operator has a spectral gap, a property that the transfer operator in infinite ergodic theory lacks in general. This fact is the starting point of operator renewal theory, in which ideas from classical renewal theory are transferred to

the operator setting. With the help of these ideas, it is possible to obtain further convergence results for the individual iterates of the transfer operator. This is the topic of the next section, Section 6.3.

6.3 Operator renewal theory

6.3.1 Classical renewal theory

This section begins with a subsection about classical renewal theory. It is included to give a short overview of the main results in renewal theory. Afterwards it is explained how these methods can be used to obtain results in infinite ergodic theory.

Classical renewal theory is a part of probability theory and has its origin in the twentieth century. The first landmark result in this area was achieved by Erdős, Feller and Pollard in [EFP49]. Originally it arose from questions of self renewing aggregates. As an example, it is good to have the replacement of light bulbs in mind.

For an introduction to elementary renewal theory, the reader is referred to [Fel68, Fel71]. In this subsection we stick with Feller's notation. We let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space and we let $\mathcal{E}$ denote an event, for instance, we say a certain *success* occurs, whatever *success* might mean in a specific setting. For instance, it could mean the replacement, the *renewal*, of a light bulb. The two non-negative sequences of real numbers $(f_i)_{i\in\mathbb{N}_0}$ and $(u_i)_{i\in\mathbb{N}_0}$ are defined by $f_0 := 0$, $u_0 := 1$ and for $n \in \mathbb{N}$ by

$$\begin{aligned} u_n &:= \mathbb{P}\left(\mathcal{E} \text{ occurs at the } n\text{-th trial}\right), \\ f_n &:= \mathbb{P}\left(\mathcal{E} \text{ occurs for the first time at the } n\text{-th trial}\right). \end{aligned}$$

This setting yields $\sum_{i=0}^{\infty} f_i \leq 1$, however, we assume that $\sum_{i=0}^{\infty} f_i = 1$. This property is, in the language of probability theory, known as the Event $\mathcal{E}$ being *persistent*. Furthermore, we assume, without loss of generality, that the greatest common divisor of all the indices $i \in \mathbb{N}$ for wich $u_i > 0$ is one. That is, $\mathcal{E}$ is non-periodic. The periodic case can be traced back to the non-periodic case.

The probability of the event '$\mathcal{E}$ occurs for the first time at trial k, $k \in \mathbb{N}$, and then again at trial number $n \in \mathbb{N}$, with $n > k$' is given by $f_k \cdot u_{n-k}$. These events are mutually exclusive for different k and hence we have for $n \geq 1$, that (compare [Fel68, Section XIII.3])

$$u_n = f_1 \cdot u_{n-1} + f_2 \cdot u_{n-1} + \cdots + f_n \cdot u_0.$$

By [Fel68, Section XIII.3, Theorem 3], we have that

$$\lim_{n\to\infty} u_n = \frac{1}{\sum_{j=1}^{\infty} j \cdot f_j} \tag{6.14}$$

and interpret the limit as zero, if $\sum_{j=1}^{\infty} j \cdot f_j = \infty$.
In the case of an infinite mean and under additional assumptions, namely that the tail of the random variable is regularly varying, Garsia and Lamperti in [GL62] as well as Erickson in [Eri70] and Doney in [Don97] prove statements about the exact asymptotic behaviour, that is how fast the limit in (6.14) tends to zero. How this probability theoretic results help to answer the questions that are asked in this thesis is explained in the next subsection.

6.3.2 The theory of operator renewal theory

In [Sar02] Sarig generalised the previously known results from renewal theory by combining renewal theoretical arguments with operator theory; a new approach was born. To describe the general setting, let $(X, \mathfrak{B}, \mu, T)$ denote a conservative and non-singular dynamical system and let $\mathbb{D}$ denote the open unit ball in $\mathbb{C}$ and $\overline{\mathbb{D}}$ its closure. Further, let $A \in \mathfrak{B}$ be such that $0 < \mu(A) < \infty$ and let $T_A(x)$ denote the induced transformation with respect to A, defined in (3.1). Sarig showed the following.

Proposition 6.9 ([Sar02, Proposition 1]). *For $n \in \mathbb{N}_0$, we define the return time operator*

$$T_n(f) := \mathbb{1}_A \cdot \widehat{T}^n(f \cdot \mathbb{1}_A)$$

and the first return time operator

$$R_n(f) := \mathbb{1}_A \cdot \widehat{T}^n(f \cdot \mathbb{1}_{\{\phi_A = n\}}).$$

Furthermore, we denote the identity operator by I and define $R(z) := \sum_{n=1}^{\infty} z^n \cdot R_n$ and $T(z) := I + \sum_{n=1}^{\infty} z^n \cdot T_n$.
Then for all $z \in \overline{\mathbb{D}}$, we have that

$$T(z) = (I - R(z))^{-1}. \tag{6.15}$$

Furthermore, $R(1) = \sum_{n=1}^{\infty} R_n$ is the transfer operator of the induced transformation T_A.

Besides Proposition 6.9, [Sar02] proves lower bounds on the decay of correlations. In the case of finite ergodic theory, [Gou04] generalizes Sarigs results. A general proof of this proposition can be found in [Sar02]. For the case, that T is the Farey map, a down to earth proof, which follows Sarigs arguments but gives more details, can be found in [Kau11]. It has also been shown in [Kau11] that these operator renewal equations yield the classical renewal relation for the so-called sum level sets found for the α-Farey map in [KMS12].
Gouëzel generalised Sarigs methods [Gou04, Gou05] and combining this approach, with classical results from probability theory by Garsia and Lamperti [GL62] as well as Erickson [Eri70], Melbourne and Terhesiu [MT12] proved a landmark result on the asymptotic rate of convergence of the return time operator T_n. We adopt their

setting which is as follows.
Let $(X, \mathfrak{B}, T, \mu)$ denote an infinite, σ-finite measure preserving system. We fix a set $Y \in \mathfrak{B}$ with $0 < \mu(Y) < \infty$. Without loss of generality we can rescale μ such that $\mu(Y) = 1$. Furthermore, let ϕ_Y denote the first return time with respect to Y, see Definition 3.4. We assume that the tail probabilities $\mu\,(y \in Y\colon \phi_Y(y) > n)$ are regularly varying of order $\delta \in (1/2, 1]$, that is, there exists a slowly varying function $\ell\colon \mathbb{N} \to \mathbb{R}$, such that

$$\mu(y \in Y\colon \phi_Y(y) > n) = \frac{\ell(n)}{n^{\delta}}.$$

We impose further functional analytic conditions on the first return map, namely we assume the existence of a function space $\mathcal{B} \subset \mathcal{L}^{\infty}$ that contains the constant functions and satisfies the following conditions:

(R1) If $f \in \mathcal{B}$, then $f \in \mathcal{L}^{\infty}([0, 1])$ and $R(1)(f) \in \mathcal{B}$.

(R2) The inequality $\|f\|_{\mathcal{L}^{\infty}} \leq \|f\|_{\mathcal{B}}$ holds for all $f \in \mathcal{B}$.

(R3) For all $n \in \mathbb{N}$, the operator $R_n|_{\mathcal{B}}$ is bounded and linear. Moreover, there exists a constant $C > 0$, such that

$$\|R_n\| \leq C \cdot \mu(\{y = n\}).$$

(R4) *Spectral Gap*: The operator $R(1)$ restricted to $\mathcal{B}$ has a simple and isolated eigenvalue at 1.

(R5) *Aperiodocity*: For $z \in \overline{\mathbb{D}} \setminus \{1\}$, the value 1 is not in the spectrum of $R(z)$.

As before, let $\Gamma(\cdot)$ denote the Γ-function and define

$$\Gamma_{\delta} := \frac{1}{\Gamma(1+\delta) \cdot \Gamma(2-\delta)}. \tag{6.16}$$

For $\delta \in (1/2, 1]$, Melbourne and Terhesiu obtain the following theorem, which is stated without a proof here.

Theorem 6.10 ([MT12, Theorem 2.1])**.** *Given the above setting, for $\delta \in (1/2, 1]$, we have that*

$$\lim_{n\to\infty} \sup_{v\in\mathcal{B}\colon \|v\|_{\mathcal{B}}=1} \left\| \mathbb{1}_Y \cdot w_n \cdot T_n(v) - \Gamma_{\delta} \cdot \int v \mathrm{d}\mu \right\|_{\mathcal{B}} = 0.$$

Gouëzel shows in [Gou11] that the result holds true for $\delta \in (0, 1/2]$ as well if we impose further assumptions on the tails of ϕ_Y.
This theorem relies on the fact, that although actual transfer operator does not have a spectral gap, the transfer operator of the induced map, $R(1)$, has one. Using operator renewal theory, the convergence of the induced transfer operator can, under certain circumstances, be extended to the convergence of the actual transfer operator on Y. This is done in detail in Part II and Part III.

If we have the convergence of the transfer operator on a set of positive and finite measure, here Y, it is possible to extend this convergence result to each set of positive and finite measure. In [MT12] this is done using a method involving a Young tower construction. An extension theorem that uses more down to earth calculations was given in [Kau11]. To keep this thesis as self-contained as possible a slightly generalised version of this theorem and a proof of it is included. The theorem is stated and proven for an α-Farey system. This version of the theorem and the proof was given in [KKSS15]. The statement can be transfered to the Farey system. Since the proof follows the exact same route, with slightly adapted calculations, it will be omitted in the latter case.

Theorem 6.11 ([KKSS15, Theorem 1.1]). *Assume that the wandering rate of an α-Farey system $([0,1],\mathscr{B},\mu_\alpha,F_\alpha)$ satisfies $\lim_{n\to\infty} w_n/w_{n+1} = 1$. We have that, if $v \in \mathcal{L}^1_{\mu_\alpha}([0,1])$ satisfies*

$$\lim_{n\to+\infty} w_n \cdot \widehat{F}^n_\alpha(v) = \Gamma_\delta \cdot \int v \, \mathrm{d}\mu_\alpha$$

uniformly on $\overline{A}_1$, then the same holds on any compact subset of $(0,1]$. The same statement holds when replacing uniform convergence by almost everywhere uniform convergence.

For $\delta \in (0,1]$, the wandering rate of a δ-expansive α-Farey system satisfies $\lim_{n\to\infty} w_n/w_{n+1} = 1$.

Proof of Theorem 6.11. Let us first recall that, for $x \in (0,1]$ and $n \in \mathbb{N}$,

$$\begin{aligned}\left(\mathcal{P}^{n+1}_\alpha(h_\alpha \cdot v)\right)(x) &= \mathcal{P}_\alpha\left(\left(\mathcal{P}^n_\alpha(h_\alpha \cdot v)\right)(x)\right)\\ &= \left(\mathcal{P}^n_\alpha(h_\alpha \cdot v)\right)(f_{\alpha,0}(x)) \cdot \left|f'_{\alpha,0}(x)\right|\\ &\quad + \left(\mathcal{P}^n_\alpha(h_\alpha \cdot v)\right)(f_{\alpha,0}(x)) \cdot \left|f'_{\alpha,1}(x)\right|,\end{aligned}$$

which gives

$$\left(\mathcal{P}^n_\alpha(h_\alpha \cdot v)\right)(f_{\alpha,0}(x)) = \frac{\left(\mathcal{P}^{n+1}_\alpha(h_\alpha \cdot v)\right)(x) - \left(\mathcal{P}^n_\alpha(h_\alpha \cdot v)\right)(f_{\alpha,0}(x)) \cdot \left|f'_{\alpha,1}(x)\right|}{\left|f'_{\alpha,0}(x)\right|}. \tag{6.17}$$

We proceed by induction on n as follows. The start of the induction is given by the assumption in the theorem. That is, the convergence holds on the first partition element A_1. For the inductive step, assume that the statement holds for $\bigcup_{i=1}^k A_i$, for some $k \in \mathbb{N}$. Then consider an arbitrary $y \in A_{k+1}$, and let x denote the unique element in A_k such that $f_{\alpha,0}(x) = y$. Using (6.17), the fact that $\widehat{F}_\alpha = h_\alpha^{-1} \cdot \mathcal{P}_\alpha(h_\alpha \cdot v)$ and the inductive hypothesis in tandem with the assumption that $\lim w_n/w_{n+1} = 1$, we obtain that

$$\begin{aligned}
w_n \cdot \left(\widehat{F}_\alpha^n(v)\right)(y) &= w_n \cdot \left(\widehat{F}_\alpha^n(v)\right)(f_{\alpha,0}(x)) \\
&= \frac{w_n \cdot (\mathcal{P}^n (h_\alpha \cdot v))(f_{\alpha,0}(x))}{h_\alpha (f_{\alpha,0}(x))} \\
&= \frac{w_n \cdot \left(\mathcal{P}_\alpha^{n+1} (h_\alpha \cdot v)\right)(x) - \left|f'_{\alpha,1}(x)\right| \cdot w_n \cdot (\mathcal{P}_\alpha^n (h_\alpha \cdot v))(f_{\alpha,0}(x))}{h_\alpha (f_{\alpha,0}(x)) \cdot \left|f'_{\alpha,0}(x)\right|} \\
&\sim \frac{h_\alpha(x) - h_\alpha (f_{\alpha,0}(x)) \cdot \left|f'_{\alpha,1}(x)\right|}{h_\alpha (f_{\alpha,0}(x)) \cdot \left|f'_{\alpha,0}(x)\right|} \cdot \Gamma_\delta \cdot \int v \, \mathrm{d}\mu_\alpha \\
&= \Gamma_\delta \cdot \int v \, \mathrm{d}\mu_\alpha .
\end{aligned}$$

The last equality is a consequence of the eigenequation

$$\mathcal{P}_\alpha (h_\alpha(x)) = h_\alpha(x) = h_\alpha (f_{\alpha,0}(x)) \cdot \left|f'_{\alpha,1}(x)\right| + h_\alpha (f_{\alpha,0}(x)) \cdot \left|f'_{\alpha,0}(x)\right| .$$

□

The analogous statement for the Farey system reads as follows.

Theorem 6.12 ([KKS16, Theorem 4.10]). *If $f \in \mathcal{L}^1_\mu([0,1])$ satisfies*

$$\ln(n) \cdot \widehat{T}_1^n(f) \to \int f \, \mathrm{d}\mu_1$$

uniformly on Y, then the same convergence holds on any compact subset of $(0,1]$.

The same statement holds when replacing uniform convergence by almost everywhere uniform convergence.

Proof of Theorem 6.12. Since the wandering rate of the Farey map is asymptotic to $\ln(n)$, which satisfies $\ln(n)/\ln(n+1) \sim 1$, the proof follows in the same way as the proof of Theorem 6.11.

□

By similar proofs to the ones given above, we can obtain the result of Theorem 6.11 for further interval maps, such as Gibbs-Markov maps, Thaler maps and Pomeau-Manneville maps.
In the next subsection we give two examples of Banach spaces which satisfy conditions *(R1)-(R5)*.

6.3.3 The two examples - Part 3

As in Chapter 5 and Section 6.1, we conclude this chapter with two examples elucidating the theory. That is, two examples of Banach spaces are given, where conditions *(R1)-(R5)* are satisfied. As before, the first one is published in [KKS16], the second one in [KKSS15]. The fact that these Banach spaces satisfy these conditions is widely considered as folklore, but thorough proofs are included here. In both cases condition *(R4)* relies on the notion of quasi-compactness of an operator and on a theorem which is known as the theorem on the difference of two norms. The first version of this theorem is due to Doeblin and Fortet [DF37]. The generalisations are due to Ionescu-Tulcea and Marinescu [ITM50] and due to Hennion and Hervé [HH01]. The theorem is used in each of the two examples and hence needed for the main results in Parts II and III. We thus state the version of [HH01] slightly adapted to fit our notation before giving the two examples explicitly. We start by introducing the notion of quasi-compactness. For a bounded linear operator L on a Banach space $\mathfrak{L}$, we let $\rho(L)$ denote its spectral radius.

Definition 6.13 (**Quasi-compact**). A bounded linear operator L on a Banach space $\mathfrak{L}$ is called *quasi-compact* if there is a direct sum decomposition $\mathfrak{L} = \mathfrak{F} \oplus \mathfrak{H}$ and $0 < \rho < \rho(L)$ where

1. $\mathfrak{F}, \mathfrak{H}$ are closed and L-invariant, that is, $L(\mathfrak{H}) \subseteq \mathfrak{H}$ and $L(\mathfrak{F}) \subset \mathfrak{F}$,
2. $\mathfrak{F}$ is finite dimensional and all eigenvalues of $L|_{\mathfrak{F}} \colon \mathfrak{F} \to \mathfrak{F}$ have modulus larger than ρ and
3. the spectral radius of $L|_{\mathfrak{H}} \colon \mathfrak{H} \to \mathfrak{H}$ is smaller than ρ.

To show the validity of condition *(R4)*, and hence the existence of a spectral gap, it suffices to show, that the operator $R(1)$ is quasi-compact, which is done in the next theorem.

Theorem 6.14 ([HH01, Theorem XIV.3]). *Suppose that $(\mathfrak{L}, \|\cdot\|_{\mathfrak{L}})$ is a Banach space and $L \colon \mathfrak{L} \to \mathfrak{L}$ is a bounded linear operator with spectral radius $\rho(L)$. Assume that there exists a semi-norm $\|\cdot\|'_{\mathfrak{L}}$ with the following properties.*

Continuity *The semi-norm $\|\cdot\|'_{\mathfrak{L}}$ is continuous on $\mathfrak{L}$.*

Pre-compactness *For a sequence $(f_n)_{n\in\mathbb{N}}$ in $\mathfrak{L}$, if $\sup_{n\in\mathbb{N}}\|f_n\|_{\mathfrak{L}} < +\infty$, there exists a subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ and $g \in \mathfrak{L}$ such that*

$$\lim_{k\to+\infty} \|L(f_{n_k}) - g\|'_{\mathfrak{L}} = 0.$$

Boundedness *There exists $M > 0$ such that $\|L(f)\|'_{\mathfrak{L}} \le M \cdot \|f\|'_{\mathfrak{L}}$, for all $f \in \mathfrak{L}$.*

Doeblin-Fortet Inequality *There exist $k \in \mathbb{N}$, $r \in (0, \rho(L))$ and $R \ge 0$ such that, for all $f \in \mathfrak{L}$,*

$$\|L^k(f)\|_{\mathfrak{L}} \le r^k \cdot \|f\|_{\mathfrak{L}} + R \cdot \|f\|'_{\mathfrak{L}}.$$

Under these conditions the operator $L\colon \mathfrak{L} \to \mathfrak{L}$ is quasi-compact.

The Banach space of functions of bounded variation

The class of functions that is used in Part II of this thesis is the class of functions of bounded variation. Firstly, a short recapitulation of functions of bounded variation is given, including various properties that are required in the sequel. Secondly, the Banach space is defined and it is shown that this Banach space satisfies conditions *(R1)-(R5)*. For a more thorough account on functions of bounded variation the reader is referred to [BG97, Fre03], for example. We begin with the terms variation and bounded variation.

Definition 6.15 (**Varation, bounded variation**)**.** Let $[c, d]$ denote a compact interval in $\mathbb{R}$ and let f denote a function such that $f\colon [c, d] \to \mathbb{C}$. The *variation* is defined by

$$V_{[c,d]}(f) := \sup_P \left\{ \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| \right\}.$$

We take the supremum over all finite partitions $P := \{I_i = [x_{i-1}, x_i] \colon i \in \{1, 2, \dots, n\}\}$, for which $c := x_0 < x_1 < \cdots < x_{n-1} < x_n =: d$, is a chain of points belonging to $[c, d]$, for an $n \in \mathbb{N}$.
We say f is of *bounded variation*, if and only if $V_{[c,d]}(f)$ is finite.

The following two propositions state various properties of functions of bounded variation that are used in the sequel.
Proposition 6.16 is concerned with $\mathbb{R}$-valued functions and Proposition 6.17 is concerned with $\mathbb{C}$-valued functions.

Proposition 6.16 ([BG97, Chapter 2])**.** *Let $f, g \in \mathcal{L}^1_\lambda([a, b])$ denote two $\mathbb{R}$-valued functions of bounded variation.*

1. *The supremum norm $\|f\|_\infty$ of f is finite.*
2. *For $x \in [a, b]$ we have that*
$$|f(x)| \leq V_{[a,b]}(f) + \frac{\|f\|_1}{b-a}.$$
3. *The sum, difference and product of two functions of bounded variation are of bounded variation, and moreover,*
$$\begin{aligned} &V_{[a,b]}(f \pm g) \leq V_{[a,b]}(f) + V_{[a,b]}(g) \\ \text{and} \quad &V_{[a,b]}(f \cdot g) \leq V_{[a,b]}(g) \cdot \|f\|_\infty + V_{[a,b]}(f) \cdot \|g\|_\infty. \end{aligned}$$
4. *If $c \in (a, b)$, then f is of bounded variation on the intervals $[a, c]$ and $[c, d]$ and moreover, $V_{[a,b]}(f) = V_{[a,c]}(f) + V_{[c,b]}(f)$.*

5. *The function f has a representation as the difference of two non-decreasing functions.*

6. *A function of bounded variation is differentiable Lebesgue almost everywhere.*

7. *For a set $U \subseteq Y$, let $C^1(U)$ denote the differentiable real-valued functions defined on U. Letting*

$$\Psi_{[a,b]} := \{\psi \in C^1([a,b])\colon \|\psi\|_\infty \leq 1 \text{ and } \psi(a) = \psi(b) = 0\},$$

we have that

$$V_{[a,b]}(f) = \sup_{\psi \in \Psi_{[a,b]}} \int f \cdot \psi' \, \mathrm{d}\lambda.$$

Proposition 6.17 ([Fre03, page 74 f.])**.** *Let $f, g \in \mathcal{L}^1_\lambda([a,b])$ denote two $\mathbb{C}$-valued functions of bounded variation.*

1. *The supremum norm $\|f\|_\infty$ of f is finite.*

2. *The sum, difference and product of two functions of bounded variation are of bounded variation.*

3. *A $\mathbb{C}$-valued function is of bounded variation, if and only if its real and imaginary parts are of bounded variation. In particular, if $f = \mathfrak{Re}(f) + i \cdot \mathfrak{Im}(f)$, then*

$$\max\{V_{[a,b]}(\mathfrak{Re}(f)), V_{[a,b]}(\mathfrak{Im}(f))\} \leq V_{[a,b]}(f) \leq V_{[a,b]}(\mathfrak{Re}(f)) + V_{[a,b]}(\mathfrak{Im}(f)).$$

In particular, by Proposition 6.17.3. and additional regularity conditions, for instance the linearity of the transfer operator, we can simplify the matter and restrict our thoughts without loss of generality to positive real-valued functions.
The class of functions of bounded variation that vanish on the complement of a certain set leads to our example, which is in line with the first main example in this thesis.

Proposition 6.18 ([KKS16, Proposition 4.8])**.** *Let $Y = [1/2, 1]$ and let $\mathrm{BV}(Y)$ denote the space of $\mathbb{C}$-valued right-continuous functions with domain $[0,1]$ that vanish on the complement of Y and which are of bounded variation. We define, for all $f \in \mathrm{BV}(Y)$, the norm $\|f\|_{\mathrm{BV}} := \|f\|_\infty + V_Y(f)$. The space $\mathrm{BV}(Y)$ is a Banach space and satisfies conditions* (R1) *to* (R5)*.*

Proof of Proposition 6.18. A short sketch of the proof of this proposition can be found in [KKS16], though more details are given here. Since Y is compact, each $f \in \mathrm{BV}(Y)$ is an $\mathcal{L}^1_{\mu_1}(Y)$ function. Furthermore, by [Fre03, p.74] we have that $(\mathrm{BV}(Y), \|\cdot\|_{\mathrm{BV}})$ is a Banach space.
We start with showing conditions *(R2)* and *(R1)*.

(R2) The inequality $\|f\|_{\mathcal{L}^\infty} \leq \|f\|_{\mathcal{B}}$ holds for all $f \in \mathcal{B}$.

By the definition of the $\mathrm{BV}(Y)$-norm, we have for $f \in \mathrm{BV}(Y)$ that

$$\|f\|_{\mathrm{BV}} := \|f\|_\infty + V_Y(f) \geq \|f\|_\infty$$

and hence condition *(R2)* follows from the definition of the BV-norm. This observation turns out to be useful in the proof that condition *(R1)* is satisfied for $\mathrm{BV}(Y)$, which follows next.

(R1) If $f \in \mathcal{B}$, then $f \in \mathcal{L}^\infty([0,1])$ and $R(1)(f) \in \mathcal{B}$.

There are two possible ways of showing condition *(R1)*. One of them, certainly the faster one, is to deduct *(R1)* from condition *(R3)* as a corollary, because $R(1) = \sum_{n\geq 1} R_n$. The other one is more straight forward and since the calculations done there are also needed in showing *(R4)*, we follow this route.
As seen in the proof of *(R2)*, we have that $f \in \mathrm{BV}(Y)$ implies $f \in \mathcal{L}^\infty$. Thus, it remains to show that for all $f \in \mathrm{BV}(Y)$, we have that $R(1)(f) \in \mathrm{BV(Y)}$. To do so, we first recall the Proposition by Sarig, Proposition 6.9, that states that $R(1)$ is the transfer operator of the on Y induced transformation. That is, for all $w \in \mathcal{L}^1_{\mu_1}(Y)$ and $u \in \mathcal{L}^\infty(Y)$, we have that

$$\int_Y R(1)(w) \cdot u \,\mathrm{d}\mu_1 = \int_Y w \cdot u \circ T_1^{\phi_Y(x)} \,\mathrm{d}\mu_1, \tag{6.18}$$

where ϕ_Y is the first return time of $y \in Y$. This observation in combination with Proposition 6.16.2. leads to

$$\begin{aligned}
\|R(1)(f)(x)\|_{\mathrm{BV}} &= \|R(1)(f)(x)\|_\infty + V_Y(R(1)(f)) \\
&= \frac{1}{1-\frac{1}{2}} \cdot \int R(1)f \,\mathrm{d}\lambda + V_Y(R(1)(f)) + V_Y(R(1)(f)) \\
&\leq 2 \cdot \left(\int R(1)f \,\mathrm{d}\mu_1 + V_Y(R(1)(f)) \right) \qquad (6.19) \\
&= 2\left((V_Y(R(1)(f)) + \int f \,\mathrm{d}\mu_1 \right) \\
&= 2\left(V_Y(R(1)(f)) + \|f\|_{1,1} \right).
\end{aligned}$$

Hence, it suffices to show that $V_Y(R(1)(f))$ is bounded, which will be done in the next lemma.

Lemma 6.19. *For $f \in \mathrm{BV(Y)}$, we have that $R(1)(f)$ is of bounded variation.*

Proof of Lemma 6.19. The proof is technical and has its key in Proposition 6.16.7. Furthermore, by Proposition 6.17.3., we can assume, without loss of generality, that f is real-valued. Let further, for an interval $[c,d)$

$$\Psi_{[c,d)} := \Psi_{[c,d]} := \left\{ \psi \in \mathrm{C}^1([c,d]) \colon \|\psi\|_\infty < \infty,\ \psi(c) = \psi(d) = 0 \right\},$$

and let U_k denote the level sets of the first return times to Y. That is

$$U_k := \{y \in Y : \phi_Y(y) = k\} = \left[\frac{k}{k+1}, \frac{k+1}{k+2}\right). \tag{6.20}$$

It follows, that

$$V_Y(R(1)(f)) = \sup_{\psi \in \Psi_Y} \int_Y R(1)(f)(x) \cdot \psi'(x)\, \mathrm{d}\lambda.$$

Furthermore, we have that

$$\begin{aligned}
\int_Y R(1)(f)(x) \cdot \psi'(x)\, \mathrm{d}\lambda &= \int_Y R(1)(f)(x) \cdot \psi'(x) \cdot \frac{1}{h_1(x)}\, \mathrm{d}\mu_1 \\
&= \int_Y f(x) \cdot \psi'\left(T_1^{\phi_y}(x)\right) \cdot \frac{1}{h_1\left(T_1^{\phi_y}(x)\right)}\, \mathrm{d}\mu_1 \\
&= \sum_{k=1}^{\infty} \int \mathbb{1}_{U_k}(x) \cdot f(x) \cdot \psi'\left(T_1^k(x)\right) \cdot \left(T_1^k(x)\right) \mathrm{d}\mu_1 \\
&= \sum_{k=1}^{\infty} \int \mathbb{1}_{U_k}(x) \cdot f(x) \cdot \psi'\left(T_1^k(x)\right) \cdot \left(T_1^k(x)\right) \cdot h(x)\, \mathrm{d}\lambda.
\end{aligned}$$

For each $\psi \in \Psi_Y$ and each $k \in \mathbb{N}$, we define

$$\psi_k(x) := \begin{cases} \psi \circ T_1^k(x) & \text{if } x \in U_k \setminus \partial U_k, \\ 0 & \text{otherwise.} \end{cases}$$

We can conclude, for $k \in \mathbb{N}$, that $\psi_k \in \Psi_{U_k}$ and by the chain rule, we have for x lying in the interior of U_k and for each $k \in \mathbb{N}$, that

$$\psi_k'(x) = \left(\psi\left(T_1^k(x)\right)\right)' = \psi'\left(T_1^k(x)\right) \cdot \left(T_1^k\right)'(x).$$

If we furthermore define on the interior of U_k and for each $k \in \mathbb{N}$,

$$g_k(x) := \frac{T_1^k(x)}{\left(T_1^k\right)'(x)} \cdot h(x),$$

an application of Proposition 6.16.3., 4. and 7. yields for all $f \in \mathrm{BV}(Y)$, that

$$\begin{aligned} V_Y(R(1)(f)) &= \sum_{k=1}^{\infty} \sup_{\psi \in \Psi_{U_k}} \int \mathbb{1}_{U_k}(x) \cdot f(x) \cdot \frac{\psi_k'(x)}{\left(T_1^k\right)'(x)} \cdot T_1^k(x) \cdot h(x)\, \mathrm{d}\lambda \\ &= \sum_{k=1}^{\infty} \sup_{\psi \in \Psi_{U_k}} \int \mathbb{1}_{U_k}(x) \cdot f(x) \cdot g_k(x) \cdot \psi_k'(x)\, \mathrm{d}\lambda \\ &= \sum_{k=1}^{\infty} V_{U_k}(f \cdot g_k) \\ &\le \sum_{k=1}^{\infty} V_{U_k}(g_k) \cdot \|f\|_\infty + V_{U_k}(f) \cdot \|g_k\|_\infty \\ &< 0.461 \cdot \|f\|_\infty + \frac{V_Y(f)}{2}, \end{aligned} \tag{6.21}$$

which implies that $R(1)(f)$ is of bounded variation. To see that (6.21) holds we have to look at T_1^k and its properties. We have that,

$$\begin{aligned} \mathbb{1}_{U_k} T_1^k(x) &= \frac{1-x}{k \cdot x - (k-1)}, \\ \left(T_1^k\right)'(x) &= \frac{-1}{(k \cdot x - (k-1))^2}, \end{aligned}$$

and hence

$$g_k(x) = \begin{cases} k \cdot x - 2 \cdot k + 1 + \dfrac{(k-1)}{x} = \dfrac{(x-1) \cdot (k \cdot x - (k-1))}{x} & \text{if } x \in U_k, \\ 0 & \text{otherwise.} \end{cases}$$

We note that g_k has no roots in U_k and is negative on each U_k. The roots would be at $x_1 = (k-1)/k$ and $x_2 = 1$, neither of them lies inside U_k. Further calculations yield on the interior of each U_k, that

$$\begin{aligned} \frac{\partial}{\partial x} g_k(x) &= k - \frac{k-1}{x^2}, \\ \frac{\partial^2}{\partial x^2} g_k(x) &= 2 \cdot \frac{k-1}{x^3}. \end{aligned}$$

Hence, $g_1(x)$ is negative and linearly increasing on U_1. Likewise, $g_2(x)$ is negative on U_2 and attains its minimum at $x = \left(2 + \sqrt{5}\right)/2 \in U_2$. For $k \ge 3$ we have that $g_k(x)$ is negative and monotonically decreasing, hence it attains its maximal absolute value on each U_k at the right edge. These observations yield, that

$$\left\| g_k|_{U_k} \right\|_\infty = \begin{cases} \frac{1}{2} & \text{if } k = 1, \\ 3 - 2 \cdot \sqrt{2} & \text{if } k = 2, \\ \frac{2}{(k+1)\cdot(k+2)} & \text{otherwise.} \end{cases}$$

Hence, we have that

$$V_{U_k}(g_k) = \begin{cases} \frac{1}{2} - \frac{1}{3} = \frac{1}{6} & \text{if } k = 1, \\ 2 \cdot \left| \frac{1}{6} - \left(3 - 2 \cdot \sqrt{2}\right) \right| = \frac{17}{3} - 4 \cdot \sqrt{2} & \text{if } k = 2, \\ \left| g_k\left(\frac{k}{k+1}\right) - g_k\left(\frac{k+1}{k+2}\right) \right| = \frac{k-2}{k \cdot (k+1) \cdot (k+2)} & \text{otherwise.} \end{cases}$$

In particular, we have that $\|g_k\|_\infty < 1$ and that

$$\begin{aligned} \sum_{k=1}^{\infty} V_{U_k}(g_k) &\leq \frac{1}{6} + \frac{17}{3} - 4 \cdot \sqrt{2} + \sum_{k=3}^{\infty} \frac{k-2}{k \cdot (k+1) \cdot (k+2)} \\ &\leq \frac{1}{6} + \frac{17}{3} - 4 \cdot \sqrt{2} + \sum_{k=3}^{\infty} \frac{1}{(k+1)^2} \\ &= \frac{1}{6} + \frac{17}{3} - 4 \cdot \sqrt{2} + \sum_{k=1}^{\infty} \frac{1}{(k)^2} - 1 - \frac{1}{4} - \frac{1}{9} \\ &= \frac{1}{6} + \frac{17}{3} - 4 \cdot \sqrt{2} + \frac{\pi^2}{6} - 1 - \frac{1}{4} - \frac{1}{9} \\ &< 0.461, \end{aligned} \tag{6.22}$$

which proves the assertion of the lemma and hence condition *(R2)* is satisfied.

□

Note that (6.21) in particular implies that we have

$$V_Y(R(1))(f) \leq \frac{\|f\|_\infty + V_Y(f)}{2} = \frac{\|f\|_{\mathrm{BV}(Y)}}{2}. \tag{6.23}$$

This observation is useful in the proof of the Doeblin-Fortet inequality, which is needed in the proof of condition *(R4)*. Yet, let us first focus on condition *(R3)*.

(R3) For all $n \in \mathbb{N}$, the operator $R_n|_{\mathcal{B}}$ is bounded and linear. Moreover, there exists a constant $C > 0$, such that $\|R_n\| \leq C \cdot \mu_1(\{y \in Y\colon \phi_Y(y) = n\})$.

The linearity of powers of $\widehat{T}_1$ is inherited by the linearity of the operator $\widehat{T}_1$ and so, for all $n \in \mathbb{N}$, we have that R_n is a linear operator. The next aim is to show that there exists $C < \infty$ such that the operator norm of $R_n|_{\mathrm{BV}(Y)} \leq C \cdot \mu_1(\{y \in Y\colon \phi_Y(y) = n\})$. First, we prove the result for $n \in \{1, 2\}$ by an explicit calculation. Afterwards, the result is proven for integers $n \geq 3$. To this end, observe that for $x \in [0, 1]$, we have that $\mathbb{1}_{U_1} \circ f_{1,0}(x) = 0$ and $\mathbb{1}_{U_2} \circ f_{1,1} \circ f_{1,1}(x) = 0$. Hence, we discern,

$$\widehat{T}_1(\mathbb{1}_{U_1} \cdot f)(x) = f_{1,0}(x) \cdot \mathbb{1}_{U_1} \circ f_{1,1}(x) \cdot f \circ f_{1,1}(x) \text{ and} \tag{6.24}$$

$$\widehat{T}_1^2(\mathbb{1}_{U_2} \cdot f)(x) = f_{1,1}(x) \cdot f_{1,0} \circ f_{1,0}(x) \cdot f \circ f_{1,1} \circ f_{1,0}(x) \cdot \mathbb{1}_{U_2} \circ f_{1,1} \circ f_{1,0}(x). \tag{6.25}$$

(6.24) in tandem with the definition of the $\mathrm{BV}(Y)$-norm and Proposition 6.16.3. implies, for a real-valued function $f \in \mathrm{BV}(Y)$, that

$$\begin{aligned}
\|R_1(f)\|_{\mathrm{BV}(Y)} &= \|\mathbb{1}_Y \widehat{T}_1(\mathbb{1}_{U_1} \cdot f)\|_{\mathrm{BV}(Y)} = \|\mathbb{1}_Y \cdot \widehat{T}_1(\mathbb{1}_{U_1} \cdot f)\|_\infty + V_Y\left(\mathbb{1}_Y \cdot \widehat{T}_1(\mathbb{1}_{U_1} \cdot f)\right) \\
&\leq \left\|\mathbb{1}_Y \cdot f_{1,0} \cdot \mathbb{1}_{U_1} \circ f_{1,1}\right\|_\infty \left\|f \circ f_{1,1}\right\|_\infty \\
&\quad + V_Y\left(\mathbb{1}_Y \cdot f_{1,0} \cdot \mathbb{1}_{U_1} \circ f_{1,1}\right) \cdot \left\|f \circ f_{1,1}\right\|_\infty \\
&\quad + \left\|\mathbb{1}_Y \cdot f_{1,0} \cdot \mathbb{1}_{U_1} \circ f_{1,1}\right\|_\infty \cdot V_Y\left(f \circ f_{1,1}\right) \\
&\leq \left\|f_{1,0}\right\|_\infty \cdot \left\|\mathbb{1}_{U_1} \circ f_{1,1}\right\|_\infty \|f\|_\infty \\
&\quad + V_Y\left(\mathbb{1}_Y \cdot f_{1,0}\right) \cdot V_Y\left(\mathbb{1}_{U_1} \circ f_{1,1}\right) \cdot \|f\|_\infty \\
&\quad + \left\|f_{1,0}\right\|_\infty \cdot \left\|\mathbb{1}_{U_1} \circ f_{1,1}\right\|_\infty \cdot V_Y(f) \\
&\leq \|f\|_\infty + 2 \cdot \|f\|_\infty + V_Y(f) \\
&\leq 3\|f\|_{\mathrm{BY}_Y}.
\end{aligned}$$

The same follows for $n = 2$, by using (6.25).
Let us now consider the case $n \geq 3$. First, we observe that

$$f_{1,0} \colon [0,1] \to \left[0, \frac{1}{2}\right] \quad \text{and} \quad f_{1,1} \colon [0,1] \to \left[\frac{1}{2}, 1\right],$$

which implies for $n \in \mathbb{N}$, that

$$U_n := \{y \in Y \colon \phi_Y(y) = n\} = f_{1,1} \circ f_{1,0}^{n-1}([0,1)).$$

Together with the representation of $\widehat{T}_1$ given in (6.5) and an inductive argument, this yields for $f \in \mathrm{BV}(Y)$ that

$$\widehat{T}_1^n(\mathbb{1}_{U_n} \cdot f) = f_{1,0}^n \cdot \prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k \cdot \mathbb{1}_{[1/2,1)} \cdot \left(f \circ f_{1,1} \circ f_{1,0}^{n-1}\right).$$

Furthermore, for $k \in \mathbb{N}$ and $x \in [0,1]$, we have that

$$f_{1,0}^k(x) = \frac{x}{1 + k \cdot x} \quad \text{and} \quad f_{1,1} \circ f_{1,0}^k(x) = \frac{1 + k \cdot x}{1 + (k+1) \cdot x}. \tag{6.26}$$

Hence, since $f_{1,0}^k$ is a positive monotonically increasing function and since $f_{1,1} \circ f_{1,0}^k$ is a positive monotonic decreasing contracting C^1-function, it follows, that

$$\left\|\mathbb{1}_{[1/2,1)} \cdot f_{1,0}^k\right\|_\infty = \frac{1}{1+k} \quad \text{and} \quad \left\|\mathbb{1}_{[1/2,1)} \cdot f_{1,1} \circ f_{1,0}^k\right\|_\infty = \frac{2+k}{2+(k+1)}.$$

Hence, it follows that

$$\left\|\mathbb{1}_{[1/2,1)} \cdot f_{1,0}^n \cdot \prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k\right\|_\infty \leq \frac{1}{1+n} \cdot \prod_{k=0}^{n-2} \frac{2+k}{2+(k+1)} = \frac{2}{(n+1)^2}.$$

Moreover, we discern

$$V_Y(f \circ f_{1,1} \circ f_{1,0}^{n-1}) \leq V_Y(f)$$
$$\text{and} \quad V_Y\left(\prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k\right) \leq \left\|\prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k\right\|_\infty \leq \frac{2}{n+1}.$$

This in tandem with Proposition 6.16.3. implies, for a $\mathbb{R}$-valued function $f \in \mathrm{BV}(Y)$, that

$$\begin{aligned}
\|R_n(f)\|_{\mathrm{BV}(Y)} &= \|\mathbb{1}_Y \cdot \widehat{T}_1^n(\mathbb{1}_{U_n} \cdot f)\|_{\mathrm{BV}(Y)} \\
&= \left\|\mathbb{1}_Y \cdot \widehat{T}_1^n(\mathbb{1}_{U_n} \cdot f)\right\|_\infty + V_Y\left(\mathbb{1}_Y \cdot \widehat{T}_1^n(\mathbb{1}_{U_n} \cdot f)\right) \\
&= \left\|\mathbb{1}_{[1/2,1)} \cdot f_{1,0}^n \cdot \prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k \cdot \left(f \circ f_{1,1} \circ f_{1,0}^{n-1}\right)\right\|_\infty \\
&\quad + V_Y\left(\mathbb{1}_{[1/2,1)} \cdot f_{1,0}^n \cdot \prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k \cdot \left(f \circ f_{1,1} \circ f_{1,0}^{n-1}\right)\right) \\
&\leq \left\|\mathbb{1}_{[1/2,1)} \cdot f_{1,0}^n\right\|_\infty \cdot \left\|\mathbb{1}_{[1/2,1)} \cdot \prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k\right\|_\infty \cdot \left\|\left(f \circ f_{1,1} \circ f_{1,0}^{n-1}\right)\right\|_\infty \\
&\quad + V_Y\left(\mathbb{1}_{[1/2,1)} \cdot f_{1,0}^n\right) \cdot \left\|\prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k \cdot \left(f \circ f_{1,1} \circ f_{1,0}^{n-1}\right)\right\|_\infty \\
&\quad + \left\|\mathbb{1}_{[1/2,1)} \cdot f_{1,0}^n\right\|_\infty \cdot V_Y\left(\prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k \cdot \left(f \circ f_{1,1} \circ f_{1,0}^{n-1}\right)\right) \\
&\leq \left\|\mathbb{1}_{[1/2,1)} \cdot f_{1,0}^n\right\|_\infty \cdot \left\|\mathbb{1}_{[1/2,1)} \cdot \prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k\right\|_\infty \cdot \|f\|_\infty \\
&\quad + \left\|\mathbb{1}_{[1/2,1)} \cdot f_{1,0}^n\right\|_\infty \cdot \left\|\mathbb{1}_{[1/2,1)} \cdot \prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k\right\|_\infty \cdot \|f\|_\infty \\
&\quad + \left\|\mathbb{1}_{[1/2,1)} \cdot f_{1,0}^n\right\|_\infty \cdot \left\|\mathbb{1}_{[1/2,1)} \cdot \prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k\right\|_\infty \cdot V_Y(f) \\
&\leq \left\|\mathbb{1}_{[1/2,1)} \cdot f_{1,0}^n\right\|_\infty \cdot \left\|\mathbb{1}_{[1/2,1)} \cdot \prod_{k=0}^{n-2} f_{1,1} \circ f_{1,0}^k\right\|_\infty \cdot (2 \cdot \|f\|_\infty + V_Y(f)) \\
&\leq \frac{1}{n+1} \cdot \frac{2}{(n+1)^2} \cdot 2 \cdot \|f\|_{\mathrm{BV}(Y)} \\
&= \frac{4}{(n+1)^3} \cdot \|f\|_{\mathrm{BV}(Y)}.
\end{aligned}$$

By linearity of the operators R_n, with $n \in \mathbb{N}$, the triangle inequality and Proposition 6.17.3., it follows for all $f \in \mathrm{BV}(Y)$ that

$$\|R_n f\|_{\mathrm{BV}_Y} \leq \frac{8 \cdot \|f\|_{\mathrm{BV}(Y)}}{(n+1)^3}.$$

Finally, observing

$$\begin{aligned}\mu(U_n) &= \int \mathbb{1}_{U_n}(x) \cdot \frac{1}{x} \cdot \mathrm{d}\lambda = \ln\left(1 + \frac{1}{n \cdot (n+2)}\right) \\ &\geq \frac{1}{n \cdot (n+1)} - \frac{1}{2} \cdot \left(\frac{1}{n \cdot (n+1)}\right)^2 \\ &\geq \frac{1}{(n+1)^3},\end{aligned}$$

yields the required result, which shows that condition *(R3)* is satisfied.

(R4) Spectral Gap: The operator $R(1)$ restricted to $\mathcal{B}$ has a simple isolated eigenvalue at 1.

We start with showing that 1 is an eigenvalue of $R(1)$. Recall that $h_1(x) = 1/x$ and that, for all $k \in \mathbb{N}$,

$$U_k := \overline{\{y \in Y \colon \phi_Y(y) = k\}} = \left[\frac{k}{k+1}, \frac{k+1}{k+2}\right].$$

Utilising (6.26), we conclude that for $x \in [0,1]$

$$\begin{aligned}R(1)(\mathbb{1}_Y)(x) &= \lim_{m\to+\infty} \sum_{k=1}^{m} \mathbb{1}_Y(x) \cdot \widehat{T}_1^k(\mathbb{1}_{U_k})(x) \\ &= \lim_{m\to+\infty} \sum_{k=1}^{m} \mathbb{1}_Y(x) \cdot x \cdot \mathcal{P}_1^k(\mathbb{1}_{U_k} \cdot h_1)(x) \\ &= \lim_{m\to+\infty} \sum_{k=1}^{m} \mathbb{1}_Y(x) \cdot \frac{x}{(1+(k-1)\cdot x)\cdot(1+k\cdot x)} \\ &= \mathbb{1}_Y(x) \cdot x \cdot \lim_{m\to+\infty} \sum_{k=1}^{m} \left(\frac{1-k}{(1+(k-1)\cdot x)} + \frac{k}{(1+k\cdot x)}\right) \\ &= \mathbb{1}_Y(x).\end{aligned}$$

Hence, the function $\mathbb{1}_Y$ is an eigenfunction of the operator $R(1)$ with eigenvalue one and therefore the spectral radius $\rho(R(1)|_{\mathrm{BV}(Y)})$ of $R(1)$ restricted to the Banach space $\mathrm{BV}(Y)$ is equal to one. In order to show that 1 is an isolated eigenvalue it is sufficient to show that $R(1)$ is quasi-compact. By Theorem 6.14, this follows from the three properties, continuity, pre-compactness and boundedness and the Doeblin-Fortet inequality.
We let the semi-norm of Theorem 6.14 be $\|\cdot\|_{1,1}$.

Continuity Let $(f_n)_{n\in\mathbb{N}}$ denote a convergent sequence in $\mathrm{BV}(Y)$ and denote its limit by $f \in \mathrm{BV}(Y)$. By the definition of $\|\cdot\|_{\mathrm{BV}(Y)}$, we have that

$$\lim_{n\to+\infty} \|f_n - f\|_\infty = 0,$$

and hence

$$\lim_{n\to+\infty} \|f_n - f\|_{1,1} \leq \lim_{n\to+\infty} \int \|f_n - f\|_\infty \ \mathrm{d}\mu_1 = \lim_{n\to+\infty} \ln(2) \cdot \|f_n - f\|_\infty = 0.$$

Pre-compactness From (6.18) one can deduce that

$$\|R(1)(f)\|_{\mathcal{L}^1_1(Y)} = \|f\|_{\mathcal{L}^1_1(Y)}.$$

Therefore, by linearity of the operator $R(1)$, Egorov's theorem [Bog07, Theorem 2.2.1], Proposition 6.16.5. and Proposition 6.17.2. and 3., it is sufficient to show the following. Given a sequence $(f_n\colon Y \to \mathbb{R})_{n\in\mathbb{N}}$ of non-decreasing, non-negative functions which are bounded everywhere such that there exists a constant $M \in \mathbb{R}$ with $\|f_n\|_{\mathrm{BV}(Y)} = 2 \cdot \|f_n\|_\infty \leq 2 \cdot M$, then there exists a monotonic subsequence $(n_k)_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that the sequence $(f_{n_k})_{n_k\in\mathbb{N}}$ converges to a function f, with finite $\mathrm{BV}(Y)$-norm, pointwise almost everywhere. Recall that, by the definition of $\mathrm{BV}(Y)$, the functions f_n and f are right-continuous. To this end, let R denote a countable dense subset of Y and let $\{r_k\}_{k\in\mathbb{N}}$ be an enumeration of R. Since the sequence $\{f_n(r_1)\}_{n\in\mathbb{N}}$ is a bounded subsequence, by the Bolzano-Weierstraß theorem, there exists an accumulation point $j_1 \in [0, M]$ and a monotonic sequence of natural numbers $(n_k^{(1)})_{k\in\mathbb{N}}$ such that $\lim_{k\to+\infty} f_{n_k^{(1)}}(r_1) = j_1$. The same argument applied to the sequence $\left(f_{n_k^{(1)}}(r_2)\right)_{k\in\mathbb{N}}$ produces an accumulation point $j_2 \in [0, M]$ and a monotonic sequence $\left(n_k^{(2)}\right)_{k\in\mathbb{N}}$ of natural numbers such that $\lim_{k\to+\infty} f_{n_k^{(2)}}(r_2) = j_2$. Continuing this procedure ad infinitum leads to a sequence of points $(j_k)_{k\in\mathbb{N}}$, which belongs to the interval $[0, M]$, and therefore it leads to a nested sequence of monotonic subsequences $\left(\left(n_k^{(m)}\right)_{k\in\mathbb{N}}\right)_{m\in\mathbb{N}}$ of the natural numbers such that for all $m \in \mathbb{N}$,

$$\lim_{k\to+\infty} f_{n_k^{(m)}}(r_i) = j_i,$$

for all $i \in \{1, 2, 3, \dots, m\}$. We will show that there exists a positive function $f\colon Y \to \mathbb{R}$ with $\|f\|_{\mathrm{BV}(Y)} \leq 2 \cdot M$ which is the almost everywhere pointwise limit of the sequence of functions $\left(f_{n_k^{(k)}}\right)_{k\in\mathbb{N}}$. Define

$$f(x) := \begin{cases} \lim\limits_{k\to+\infty} f_{n_k^k}(x) & \text{if } x \in R \\ \lim\limits_{r\downarrow x;\ r\in R} f(r) & \text{if } x \in Y \setminus R. \end{cases}$$

This is well defined since, for all $k \in \mathbb{N}$, the function $f_{n_k^{(k)}}$ is right-continuous, non-decreasing, non-negative and bounded above by M everywhere, and so, on R the function f is right-continuous, non-decreasing, non-negative and bounded above by M. Therefore, we have that

$$\|f\|_{BV} = 2 \cdot \|f\|_\infty \leq 2 \cdot M.$$

In particular we have that f is of bounded variation and so differentiable almost everywhere, and hence continuous almost everywhere. Let U denote the set of points where f is discontinuous. If $x \in R \setminus U$, then the pointwise convergence follows by construction. If $x \in Y \setminus (R \cup U)$, then since f is continuous on this set, we have that

$$f(x) = \lim_{\substack{y \uparrow x;\\ y \in Y \setminus (U \cup R)}} f(y) = \lim_{\substack{y \uparrow x;\\ y \in Y \setminus (U \cup R)}} \lim_{\substack{r \downarrow y;\\ r \in R}} \liminf_{k \to +\infty} f_{n_k^{(k)}}(r) \leq \liminf_{k \to +\infty} f_{n_k^{(k)}}(x)$$

and that

$$\begin{aligned} f(x) &= \lim_{\substack{y \downarrow x;\\ y \in Y \setminus (U \cup R)}} f(y) \\ &= \lim_{\substack{y \downarrow x;\\ y \in Y \setminus (U \cup R)}} \left(\lim_{\substack{r \downarrow y;\\ r \in R}} \left(\limsup_{k \to +\infty} f_{n_k^{(k)}}(r) \right) \right) \\ &\geq \limsup_{k \to +\infty} f_{n_k^{(k)}}(x). \end{aligned}$$

Thus, the limit $\lim_{k \to +\infty} f_{n_k^{(k)}}(x)$ exists and equals $f(x)$ for all $x \in Y \setminus U$, which yields pre-compactness.

Boundedness Indeed, as mentioned above, from (6.18) we can deduce that $\|R(1)\|_{\mathcal{L}_1^1(Y)} = 1$.

Doeblin-Fortet Inequality By Proposition 6.16.2., (6.19), (6.21) and (6.23) we have for an $\mathbb{R}$-valued $f \in \mathrm{BV}(Y)$, that

$$\begin{aligned} \|R(1)^2(f)\|_{\mathrm{BV}(Y)} &\leq 2 \cdot \|f\|_{1,1} + 2 \cdot V_Y(R(1)^2(f)) \\ &\leq 2 \cdot \|f\|_{1,1} + V_Y(R(1)(f)) + 2 \cdot 0.461 \cdot \|R(1)(f)\|_\infty \\ &\leq 2 \cdot \|f\|_{1,1} + V_Y(R(1)(f)) \\ &\quad + 0.922 \cdot 2 \cdot \|f\|_{1,1} + 0.922 \cdot V_Y(R(1)(f)) \\ &\leq 2 \cdot (1.922) \cdot \|f\|_{1,1} + 1.922 \cdot V_Y(R(1)(f)) \\ &\leq 4 \cdot \|f\|_{1,1} + \frac{1.922}{2} \cdot \|f\|_{\mathrm{BV}(Y)}. \end{aligned}$$

Hence,

$$\begin{aligned} \|R(1)^4(f)\|_{\mathrm{BV}(Y)} &\leq 4 \cdot \|f\|_{1,1} + 4 \cdot \|f\|_{1,1} + \left(\frac{1.922}{2}\right)^2 \cdot \|f\|_{\mathrm{BV}(Y)} \\ &= 8 \cdot \|f\|_{1,1} + (0.961)^2 \cdot \|f\|_{\mathrm{BV}(Y)}. \end{aligned}$$

Inductively, we discern for a real-valued $f \in \mathrm{BV}(Y)$

$$\|R(1)^k(f)\|_{\mathrm{BV}(Y)} \leq 2 \cdot k \cdot \|f\|_{1,1} + (0.961)^{\frac{k}{2}} \cdot \|f\|_{\mathrm{BV}(Y)}.$$

For a complex-valued $f \in \mathrm{BV}(Y)$, this yields

$$\|R(1)^k(f)\|_{\mathrm{BV}(Y)} \leq 4k \cdot \|f\|_{1,1} + 2 \cdot (0.961)^{\frac{k}{2}} \cdot \|f\|_{\mathrm{BV}(Y)}.$$

In particular we have that $0.961 < 1$. Thus, choosing k sufficiently large, here $k \geq 36$, we have that

$$2 \cdot (0.961)^{\frac{k}{2}} < 1,$$

hence the Doeblin-Fortet inequality is fulfilled.

Thus, the operator $R(1)$ is quasi-compact, in particular we have shown that R(1) has a spectral gap, which finishes the proof of condition *(R4)* and only condition *(R5)* is left to be shown.

(R5) Aperiodocity: For $z \in \overline{\mathbb{D}} \setminus \{1\}$, the value 1 is not in the spectrum of $R(z)$.

For $z \in \mathbb{D}$, we define the operator $T(z) \colon \mathcal{L}^1_{\mu_1}(Y) \to \mathcal{L}^1_{\mu_1}(Y)$ by

$$T(z)(f) := \sum_{n=1}^{+\infty} z^n \cdot \mathbb{1}_Y \cdot \widehat{T}^n(\mathbb{1}_Y \cdot f).$$

By Proposition 6.9 we have that

$$R(z) \circ T(z)(f) = T(z)(f) - f = T(z) \circ R(z)(f).$$

This implies that 1 does not belong to the spectrum of the operator $R(z)$. If we assume it would be an eigenvector, we would immediately get $T(z)(f) = T(z)(f) - f$, which is equivalent to $f = 0$ and thus a contradiction. Hence, it is sufficient to show the result for $z \in \mathbb{S} \setminus \{1\}$. For this, we will follow the arguments given in the proof of [Gou04, Lemma 6.7]. To this end let $t \in (0, 2\pi)$ and let $z = \mathrm{e}^{i \cdot t}$ be fixed. Suppose, by way of contradiction, that $R(z)(f) = f$ for a non-zero $f \in \mathrm{BV}(Y)$. Let $\mathcal{L}^2_1(Y)$ denote the space of $\mathbb{C}$-valued square integrable functions with respect to the measure μ_1 that have domain $[0, 1]$ and are supported on Y. Further, let $\langle \cdot, \cdot \rangle$ denote the associated bilinear form and define the operator $W \colon \mathcal{L}^\infty_{\mu_1}(Y) \to \mathcal{L}^\infty_{\mu_1}(Y)$, by

$$W(u) := \mathrm{e}^{-i \cdot t \cdot \phi_Y} \cdot u \circ T_1^{\phi_Y}$$

for $u \in \mathcal{L}^{\infty}(Y)$. We observe that

$$\begin{aligned} R(z)(v) &= \sum_{n=0}^{\infty} z^n \cdot R_n(v) \\ &= \sum_{n=0}^{\infty} z^n \cdot \mathbb{1}_Y \cdot \widehat{T}^n\left(v \cdot \mathbb{1}_{\{\phi_Y = n\}}\right) \\ &= \sum_{n=0}^{\infty} e^{i \cdot t \cdot n} \cdot \mathbb{1}_Y \cdot \widehat{T}^n\left(v \cdot \mathbb{1}_{\{\phi_Y = n\}}\right) \\ &= \sum_{n=0}^{\infty} \mathbb{1}_Y \cdot \widehat{T}^n\left(v \cdot \mathbb{1}_{\{\phi_Y = n\}} \cdot e^{i \cdot t \cdot n}\right) \\ &= R(1)\left(e^{i \cdot t \cdot \phi_Y} \cdot v\right). \end{aligned}$$

Combining this observation with the dual relation given in (6.18), we have for all $v \in \mathrm{BV}(Y)$ and $u \in \mathcal{L}^{\infty}_{\mu_1}(Y)$, that

$$\begin{aligned} \langle u, R(z)(v)\rangle = \int \overline{u} \cdot R(z)(v)\, \mathrm{d}\mu_1 &= \int \overline{u} \cdot R(1)(\mathrm{e}^{i \cdot t \cdot \phi_Y} \cdot v)\, \mathrm{d}\mu_1 \\ &= \int \overline{u} \circ T_1^{\phi_Y} \cdot \mathrm{e}^{i \cdot t \cdot \phi_Y} \cdot v\, \mathrm{d}\mu_1 = \langle W(u), v\rangle, \end{aligned}$$

and thus,

$$\begin{aligned} \|W(f) - f\|^2_{\mathcal{L}^2_1(Y)} &= \|W(f)\|^2_{\mathcal{L}^2_1(Y)} - 2 \cdot \Re\mathrm{e}\, \langle W(f), f\rangle + \|f\|^2_{\mathcal{L}^2_1(Y)} \\ &= \|W(f)\|^2_{\mathcal{L}^2_1(Y)} - 2 \cdot \Re\mathrm{e}\, \langle f, R(z)(f)\rangle + \|f\|^2_{\mathcal{L}^2_1(Y)} \\ &= \|W(f)\|^2_{\mathcal{L}^2_1(Y)} - 2 \cdot \Re\mathrm{e}\, \langle f, f\rangle + \|f\|_2{}^2 \\ &= \|W(f)\|^2_{\mathcal{L}^2_1(Y)} - \|f\|^2_{\mathcal{L}^2_1(Y)}, \end{aligned} \tag{6.27}$$

By another application of (6.18), we also have that

$$\|W(f)\|^2_{\mathcal{L}^2_1(Y)} = \int |f|^2 \circ T_1^{\phi_Y}\, \mathrm{d}\mu_1 = \int |f|^2\, \mathrm{d}\mu_1 = \|f\|^2_{\mathcal{L}^2_1(Y)}. \tag{6.28}$$

From (6.27) and (6.28), we obtain that $W(f) - f$ is zero μ_1-almost everywhere. Since by definition of $\mathrm{BV}(Y)$, we have that f is right-continuous, $W(f)$ is right-continuous, and so the function $W(f) - f$ is zero everywhere.

We now have a right-continuous function f so that $\mathrm{e}^{-i \cdot t \cdot \phi_Y} \cdot f \circ T_1^{\phi_Y} = f$. Since T_1 is ergodic with respect to μ_1 by [Aar97, Proposition 1.4.8, 1.5.1 and 1.5.3] we have that $T_1^{\phi_Y}$ is ergodic with respect to μ_1. Thus, by [Wal82, Theorem 1.6], we obtain that $|f|$ is constant everywhere. As f is non-zero, this constant is non-zero, and so, we obtain that $\mathrm{e}^{-i \cdot t \cdot \phi_Y} = f/(f \circ T_1^{\phi_Y})$. However, since for each $n \in \mathbb{N}$, there exists an $x \in Y$ such that $T_1^{\phi_Y}(x) = x$ and such that $\phi_Y(x) = n$. Hence, we have that $\mathrm{e}^{-i \cdot t \cdot n} = 1$ for all $n \in \mathbb{N}$. This contradicts the choice of t, namely that t belongs to the open interval $(0, 2 \cdot \pi)$.

This finishes the proof of condition *R(5)* and hence proves Proposition 6.18.

□

The Banach space of piece-wise Lipschitz continuous functions

The second example is published in [KKSS15]. Its underlying dynamical system is a δ-expansive α-Farey system, so recall the definitions of this system from Section 5.2.Further, note that conditions *(R1)-(R5)* coincide with the conditions called (H1) and (H2) in [KKSS15].
To define the Banach space and its norm we use the level sets of first return time to A_1 given in Definition 3.4. These level sets of the first return times define a countable-infinite partition of A_1, via $\{\{\phi_{A_1} = n\} : n \in \mathbb{N}\}$ which we denote by β_α. Furthermore, let

$$D_\alpha(f) := \sup_{a \in \beta_\alpha} \sup_{x \neq y \in a} \frac{|f(x) - f(y)|}{|x - y|},$$

and define

$$\|\cdot\|_{\mathcal{B}_\alpha} := \|\cdot\|_\infty + D_\alpha(\cdot). \tag{6.29}$$

Let $\mathcal{B}_\alpha$ denote the set of functions with domain $[0, 1]$ that vanish on the complement of $\overline{A}_1$ and which have finite $\|\cdot\|_{\mathcal{B}_\alpha}$-norm. In particular, if $f \in \mathcal{B}_\alpha$, then f is Lipschitz continuous on each atom of β_α, zero outside of $\overline{A}_1$ and bounded everywhere.
We show that for every δ-expansive α-Farey system, the Banach space $(\mathcal{B}_\alpha, \|\cdot\|_{\mathcal{B}_\alpha})$ satisfies conditions *(R1)-(R5)*. The proof follows the arguments of [KKSS15].

Proposition 6.20 ([KKSS15, Proposition 2.4]). *For a δ-expansive α-Farey system, the pair $(\mathcal{B}_\alpha, \|\cdot\|_{\mathcal{B}_\alpha})$ forms a Banach space and conditions* (R1)-(R5) *are satisfied.*

A very similar space was previously considered in [AD01], the only difference being that in this thesis we consider functions with finite $\|\cdot\|_\infty$-norm whereas Aaronson and Denker consider functions that are bounded almost everywhere. That is, this thesis distinguishes between functions that differ on a set of measure zero, [AD01] does not. In [AD01, Section 1] it is also shown, that the pair $(\mathcal{B}_\alpha, \|\cdot\|_{\mathcal{B}_\alpha})$ forms a Banach space. The slight differences between the Banach space considered in [AD01] and the one considered here do not change any of the calculations considerably.

Proof of Proposition 6.20. Let us start with the first three conditions.

(R1) If $f \in \mathcal{B}_\alpha$, then $f \in \mathcal{L}^\infty([0, 1])$ and $R_\alpha(1)(f) \in \mathcal{B}_\alpha$.

(R2) The inequality $\|f\|_{\mathcal{L}^\infty} \leq \|f\|_{\mathcal{B}_\alpha}$ holds for all $f \in \mathcal{B}_\alpha$.

(R3) For all $n \in \mathbb{N}$, the operator $R_{\alpha,n}|_{\mathcal{B}_\alpha}$ is bounded and linear. Moreover, there exists a constant $C > 0$, such that

$$\|R_{\alpha,n}\| \leq C \cdot \mu_\alpha(\{\phi_{A_1} = n\}).$$

As in the previous example, condition *R(2)* is satisfied by the definition of the norm. Furthermore, by the definition of the current Banach space itself and the definition of its norm, we have that $f \in \mathcal{L}^\infty([0,1])$, whenever $f \in \mathcal{B}_\alpha$. So it is left to show that $R_\alpha(1)$ maps $\mathcal{B}_\alpha$ into itself. To show this, we make use of Lemmata 6.3 and 6.4. We let $f \in \mathcal{B}_\alpha$ and fix $k \in \mathbb{N}$. We have that

$$R_{\alpha,k}(f) = \mathbb{1}_{\overline{A}_1} \cdot \widehat{F}_\alpha^k(\mathbb{1}_{\{\phi_{A_1}=k\}} \cdot f) = \mathbb{1}_{\overline{A}_1} \cdot \mu_\alpha(\{\phi_{A_1} = k\}) \cdot f \circ f_{\alpha,10_{k-1}}.$$

This in combination with the definition of the partition β_α yields that

$$\|R_{\alpha,k}(f)\|_{\mathcal{B}_\alpha} \leq \mu_\alpha(\{\phi_{A_1} = k\}) \cdot \|f\|_{\mathcal{B}_\alpha}. \tag{6.30}$$

Hence, for each $k \in \mathbb{N}$ the operator $R_{\alpha,k}$ maps $\mathcal{B}_\alpha$ into itself and by the definition of $R_\alpha(1)$ and since $\sum_{n\in\mathbb{N}} \mu_\alpha(\{\phi_{A_1} = n\}) = \mu_\alpha(A_1) = 1$, we have that

$$\|R(1)f\|_{\mathcal{B}_\alpha} \leq \sum_{n\in\mathbb{N}} \|R_{\alpha,n}(f)\|_{\mathcal{B}_\alpha} \leq \sum_{n\in\mathbb{N}} \mu_\alpha(\{\phi_{A_1} = n\}) \cdot \|f\|_{\mathcal{B}_\alpha} = \|f\|_{\mathcal{B}_\alpha}.$$

That shows, that $R_\alpha(1)$ maps $\mathcal{B}_\alpha$ into itself. Hence, conditions *R(1)* and *R(2)* are satisfied and we can turn towards condition *R(3)*.
For all $n \in \mathbb{N}$, the linearity of $R_{\alpha,n}$ follows from the linearity of the operator $\widehat{F}_\alpha$. Furthermore, as seen in (6.30), we have that

$$\sup_{f\in\mathcal{B}_\alpha :\, \|f\|_{\mathcal{B}_\alpha}=1} \|R_{\alpha,n}(f)\|_{\mathcal{B}_\alpha} \leq \mu_\alpha(\phi = n).$$

Hence, condition *(R3)* holds true as well and we can focus on the last two conditions.

(R4) *Spectral Gap*: The operator $R(1)$ restricted to $\mathcal{B}$ has a simple isolated eigenvalue at 1.

To proof this condition we make use of [AD01, Theorem 1.6], which is based on Theorem 6.14. We state this theorem slightly adapted to fit our notation and omit the proof.
Heuristically, this theorem states that the transfer operator restricted to the Banach space $\mathcal{B}_\alpha$ can be decomposed into a projection, namely to the integral with respect to the invariant measure and another operator Q, which is orthogonal in the space of linear operators of the current Banach space to the projection and has spectral radius less than one. The projection has eigenvalue one. That means that the iterated application of $\widehat{F}_\alpha$ has a smoothing property on functions of $\mathcal{B}_\alpha$.

Theorem 6.21 ([AD01, Theorem 1.6])**.** *Adopt the setting of Proposition 6.20. If $F_{\alpha A_1}$ is a piecewise linear expansive Markov map, we have for $f \in \mathcal{B}_\alpha$ that*

$$R_\alpha(1)f = \int f \mathrm{d}\mu_\alpha + Q(f),$$

with $\rho(Q) < 1$ and for all for $g \in \mathcal{B}_\alpha$, we have that

$$\int Q(g)\mathrm{d}\lambda \cdot h_\alpha = Q\left(\int (g)\mathrm{d}\lambda \cdot h_\alpha\right) = 0.$$

This theorem yields condition *(R4)*. To apply it we need to show that the induced map is a piecewise linear expansive Markov map. That means that the induced map $F_{\alpha\, A_1}(x)$ is expansive on each partition element $\{\phi = n\}$, which follows from the following observations.

Firstly, we observe that on the set $\{\phi = n\}$ the absolute value of the derivative of $F_{\alpha\, A_1}(x)$ is constant and equal to $1/(t_n - t_{n-1})$. Since, by definition, $(t_n)_{n\in\mathbb{N}}$ is a monotonically decreasing sequence which is bounded above by one, it follows that there exists a constant $c > 0$, such that for all $n \in \mathbb{N}$, we have that $1/(t_n - t_{n-1}) > c > 1$. Hence, the induced map is expansive.
Secondly, the partition β_α is a countable partition of A_1 and we have for all $n \in \mathbb{N}$, that $\mu_\alpha\left(F_{\alpha\, A_1}(x)(\{\phi = n\})\right) = 1$.
Moreover, the σ-algebra generated by $\{F^n_{\alpha\, A_1}(\{\phi = n\})\colon\, n, m \in \mathbb{N}\}$ is equal to the Borel σ-algebra on $\overline{A_1}$.
Lastly, for each $n \in \mathbb{N}$ and the word $\psi := (1, \underbrace{0, 0, \dots, 0}_{(n-1)-\text{times}}, 1)$, we have that,

$$f_{\alpha,\psi}([0, 1]) = \overline{\{\phi = n\}}.$$

Given these properties condition *(R4)* is a consequence of Theorem 6.21. Hence, condition *R(5)* is left to be shown.

(R5) Aperiodocity: For $z \in \overline{\mathbb{D}} \setminus \{1\}$, the value 1 is not in the spectrum of $R(z)$.

As in the previous example we let $\mathbb{S} := \{z \in \mathbb{C}\colon |z| = 1\}$. To show condition *(R5)* we distinguish between the cases $z \in \mathbb{D}$ and $z \in \mathbb{S} \setminus \{1\}$
Similar as in the other example, we obtain that 1 is not an eigenvalue for $z \in \mathbb{D}$. Since if we assume that there exists an eigenfunction $w \in \mathcal{B}_\alpha$, with $w \neq 0$, such that $R(z)w = w$, we obtain a contradiction if we substitute this into the formula $(T(z) \circ R(z))(w) = T(z)(w) - w$.
The remaining case is $z \in \mathbb{S} \setminus \{1\}$ and follows the same calculations as in the previous example, with minor technical differences. Thus, conditions *(R1)–(R5)* are satisfied.

□

This concludes the first part of this thesis and we turn towards the second main part which deals with the convergence to equilibrium of unbounded observables.

Part II

Convergence to equilibrium of unbounded observables in infinite ergodic theory

Chapter 7

Central definitions I

Before the main results of this part are stated in Chapter 8, important notation and some definitions, as used in [KKS16], are needed and hence introduced in the current chapter. After the main results are stated, pictures and heuristics are given to elucidate the theory, see Chapter 9, followed by thorough proofs of Theorems 8.1, 8.2 and 8.6 in Chapter 10.
This part works with the family of maps T_r, which was introduced in Section 5.1, so recall the definitions and formulas given therein, in particular Equation (5.2).
Two important function spaces which we will use are defined below.

Definition 7.1 ($\mathbf{BV}(A)$). Let $A \subseteq [0,1]$. We define the space $\mathrm{BV}(A)$ to be the set of right-continuous functions $f\colon [0,1] \to \mathbb{C}$ such that the norm $\|f\|_{\mathrm{BV}} := V_{[A]}(f) + \|f\|_\infty$ is finite.

Definition 7.2 ($\mathfrak{U}_{\beta,a}$). Let $a \in (0,1)$ and $\beta \in [0,1]$. We define the space $\mathfrak{U}_{\beta,a}$ to be the set of functions $v\colon [0,1] \to \overline{\mathbb{R}}$ such that

1. $\lim_{x\uparrow\beta} v(x) = \lim_{x\downarrow\beta} v(x) = +\infty$,

2. for each compact subset $K \subset [0,1] \setminus \{\beta\}$, we have that $v \cdot \mathbb{1}_K \in \mathrm{BV}(0,1)$.

3. there exists a connected open neighbourhood $U \subset [0,1]$ of β, under the Euclidean subspace topology, and two constants C_1, C_2 such that for all $x \in U$, with $x \neq \beta$
$$\frac{C_1}{|\beta - x|^a} \leq v(x) \leq \frac{C_2}{|\beta - x|^a}.$$

Conditions (b) and (c) immediately imply that if $v \in \mathfrak{U}_{\beta,a}$, then v belongs to the so-called improper Riemann integrable functions. Moreover, without loss of generality, throughout we assume that v is positive. Note that, by the linearity of the transfer operator, the crucial condition is $C_1 \cdot |\beta - x|^{-a} \leq v(x) \leq C_2 \cdot |\beta - x|^{-a}$. So for a simplification of the matter the reader might think, without loss of generality, of the observable being $v(x) = |\beta - x|^{-a}$.

As it is common in literature, we define the *ω-limit set of $\beta \in [0,1]$* with respect to T_r to be the set of accumulation points of the orbit $(T_r^n(\beta))_{n\in\mathbb{N}_0}$ and denote it by

$$\omega_r(\beta) := \bigcap_{k\in\mathbb{N}_0} \overline{\{T_r^\ell(\beta) : \ell \geq k\}}. \tag{7.1}$$

As in the continued fraction setting, we call a point $\beta \in (0,1]$ *pre-periodic with respect to T_r*, if there exists an $M \in \mathbb{N}_0$ and $n \in \mathbb{N}$, such that for all $m > M$,

$$b_m = b_{m+n}. \tag{7.2}$$

For a given pre-periodic point β with respect to T_r, we define the *period length* of β with respect to T_r to be the minimal n such that the equality in (7.2) holds. We write $\beta = [b_1, b_2, \ldots, b_M, \overline{b_{M+1}, b_{M+2}, \ldots, b_{M+n}}]_r$.
Indeed, for $r \in (0,1]$, we have that $1-(3-\sqrt{9-4\cdot r})/(2\cdot r)$ is periodic and hence pre-periodic with respect to T_r. For $r = 1$, an example for a pre-periodic number is $\sqrt{2}/(3\cdot\sqrt{2}+1) = [3,\overline{1}] = [0,0,\overline{1}]_1$.

Before we turn towards the main results we need to introduce yet another definition. Namely, the main results of this part for the case $r = 1$ require the notion of *intermediate a-type* introduced in [KKS16].

Definition 7.3 ([KKS16], **Intermediate *a*-type**)**.** Let β be an irrational number. For $\beta = [a_1, a_2, \ldots] \in [0,1]$, we let $s_{n,j}/t_{n,j} = [a_1, \ldots, a_{n-1}, j]$, with $s_{n,j}, t_{n,j} \in \mathbb{N}$ co-prime. Using the terminology from continued fraction expansion one refers to $s_{n,j}/t_{n,j}$ as an *intermediate approximant to β*.
Given an $a \in (0,1)$ we say that β is of *intermediate a-type* if and only if there exists an $\epsilon > 0$, such that

$$\sum_{n=1}^{+\infty}\sum_{k=1}^{a_n} (t_{n,j})^{-2\cdot(1-a)+\epsilon} < +\infty.$$

Remark 7.4. A closer look at the term *intermediate a-type* shows the following.

1. If $a < 1/2$, every irrational β, is of intermediate a-type.
2. If β is pre-periodic, or more generally, if the continued fraction entries a_i of β are bounded, then β is of intermediate a-type, for all $a \in (0,1)$.
3. If for $\beta = [a_1, a_2, \ldots]$, there exists a fixed $K \in \mathbb{N}$, such that the sequence of continued fraction entries of β satisfies $a_n = \mathfrak{o}(n^K)$, then β is of intermediate a-type.
4. It follows from the results of [KS07] that

$$\dim_{\mathcal{H}}(\{\beta \in [0,1] : \beta \text{ is of intermediate } a\text{-type for all } a \in (0,1)\}) = 1.$$

Here and throughout we will denote the Hausdorff dimension of a set $A \subset \mathbb{R}$ by $\dim_{\mathcal{H}}(A)$, see for instance [Fal14] for the definition and further details on the Hausdorff dimension of a set.
Heuristically we can say that, if a number is of *intermediate a-type*, we have some kind of control over the growth rate of its continued fraction entries.

Chapter 8

Convergence to equilibrium - Statement of the main results

With the definitions of the previous section at hand and the notion given in Section 5.1, we are in the position to state the main results of Part II.
The first theorem deals with the case $r \in [0,1)$. That means, we have a finite invariant measure, the transformation is piecewise expanding and the transformation is of bounded distortion, which makes the proofs considerably easier.
The second and the third theorem deal with the case $r = 1$. So more sophisticated methods involving methods from infinite ergodic theory are applied in the proofs of Theorem 8.2 and Theorem 8.6. Yet, many of the ideas behind the proofs are similar.

Theorem 8.1 ([KKS16, Theorem 3.1]). *For $r \in [0,1)$, if $a \in (0,1)$ and $\beta \in [0,1]$, then, for each $v \in \mathfrak{U}_{\beta,a}$, we have that*

$$\lim_{n\to\infty} \mathcal{P}_r^n(v) = \int v \, \mathrm{d}\lambda \cdot h_r, \tag{8.1}$$

uniformly on compact subsets of $[0,1] \setminus \omega_r(\beta)$ and pointwise outside a set with Hausdorff dimension equal to zero.
If $\beta \in [0,1]$ is pre-periodic with respect to T_r and has period length strictly greater than one, then on the finite set $\omega_r(\beta)$ we have that

$$\liminf_{n\to+\infty} \mathcal{P}_r^n(v) = \int v \, \mathrm{d}\lambda \cdot h_r \quad \textit{and} \quad \limsup_{n\to+\infty} \mathcal{P}_r^n(v) = +\infty.$$

In the case that $\beta \in [0,1]$ is pre-periodic with respect to T_r and has period length equal to one then on the singleton $\omega_r(\beta)$, the limit in (8.1) *is equal to $+\infty$.*

Theorem 8.2 ([KKS16, Theorem 3.2]). *If $a \in (0,1)$ and if $\beta \in (0,1]$ is either rational or irrational of intermediate a-type, then, for each $v \in \mathfrak{U}_{\beta,a}$, we have that*

$$\lim_{n\to\infty} \ln(n) \cdot \mathcal{P}_1^n(v) = \int_{[0,1]} v \, \mathrm{d}\lambda \cdot h_1, \tag{8.2}$$

uniformly on compact subsets of $(0,1] \setminus \omega_1(\beta)$ *and pointwise outside a set with Hausdorff dimension equal to zero. If* $\beta \in (0,1]$ *is pre-periodic with respect to* T_1 *and has period length strictly greater than one, then on the finite set* $\omega_1(\beta)$ *we have that*

$$\liminf_{n\to+\infty} \ln(n)\cdot \mathcal{P}_1^n(v) = \int v \,\mathrm{d}\lambda \cdot h_1 \quad \textit{and} \quad \limsup_{n\to+\infty} \ln(n)\cdot \mathcal{P}_1^n(v) = +\infty. \tag{8.3}$$

In the case that $\beta \in (0,1]$ *is pre-periodic with respect to* T_1 *and has period length equal to one, then on the singleton* $\omega_1(\beta)$*, the limit in* (8.2) *is equal to* $+\infty$*.*

Before we state a theorem about convergence on the set of exceptional points, that is the ω-limit set of the pole, we make a few remarks on the statements of the first two theorems.

Remark 8.3 ([KKS16, Remark 2])**.** The $\ln(n)$-term in (8.2) and (8.3) is the *wandering rate* of the Farey map T_1, introduced in Definition 3.6. The wandering rate for $r \in [0,1)$ is asymptotic to a constant.

Remark 8.4 ([KKS16, Remark 3])**.** We highlight an interesting difference between Theorems 8.1 and 8.2, which is a result of the Farey map having an indifference fixed point at zero. In the case that $r \in [0,1)$, $a \in (0,1)$, if β is an r-rational (see Subsection 5.1.3) and $v \in \mathfrak{U}_{\beta,a}$, we have that

$$\lim_{n\to\infty} \mathcal{P}_r^n(v)(0) = +\infty.$$

For $r = 1$, $a \in (0,1)$, if β is a rational number and $v \in \mathfrak{U}_{\beta,a}$, we have that

$$\lim_{n\to\infty} \ln(n)\cdot \mathcal{P}_1^n(v)(0) = 0.$$

The points $0, 1/2$ and 1 are r-rationals for all $r \in [0,1]$.

Remark 8.5 ([KKS16, Remark 4])**.** In the case that one replaces the norm $\|\cdot\|_\infty$ by the essential supremum norm in the definition of $\mathrm{BV}(0,1)$, and hence in the definition of $\mathfrak{U}_{\beta,a}$, the limit in (8.2) holds uniformly Lebesgue almost everywhere on compact subsets of $(0,1) \setminus \omega_1(\beta)$ and pointwise Lebesgue almost everywhere on $(0,1]$.

In the following theorem, for the observable $v_{\beta,a}(x) = |\beta - x|^{-a}$ and a non-periodic β, we demonstrate that on the ω-limit set, the values of the limit inferior and limit superior depend on the diophantine properties of β. We let q_n be as defined in (5.1).

Theorem 8.6 ([KKS16, Theorem 3.3])**.** *We have the following.*

1. *There exist non-periodic* β *and* $\varrho \in (0,1]$ *both with bounded continued fraction entries but such that, on the one hand, if* $a \in (0,1)$*, then on* $\omega_1(\beta)$*, we have that*

$$\lim_{n\to+\infty} \ln(n)\cdot \mathcal{P}_1^n(v_{\beta,a}) = \int v_{\beta,a} \,\mathrm{d}\lambda \cdot h_1.$$

On the other hand, if $a \in (0, 1/2)$, then on $\omega_1(\varrho)$, we have that

$$\lim_{n\to\infty} \ln(n) \cdot \mathcal{P}_1^n(v_{\varrho,a}) = \int v_{\varrho,a} \,\mathrm{d}\lambda \cdot h_1;$$

otherwise, if $a \in [1/2, 1)$, then on $\omega_1(\varrho)$

$$\liminf_{n\to+\infty} \ln(n) \cdot \mathcal{P}_1^n(v_{\varrho,a}) = \int v_{\varrho,a} \,\mathrm{d}\lambda \cdot h_1$$
$$\text{and} \quad \limsup_{n\to+\infty} \ln(n) \cdot \mathcal{P}_1^n(v_{\varrho,a}) = +\infty.$$

2. *Let $a \in (0, 1)$ and let $\beta = [0; a_1, a_2, \dots] \in (0, 1]$ be of intermediate a-type such that*

$$\lim_{n\to+\infty} a_n = +\infty,$$

which implies that $\omega_1(\beta) = \{1/n \colon n \in \mathbb{N}\} \cup \{0\}$.
Fix $k \in \mathbb{N}$ and let $l(k) := \min\{i \in \mathbb{N} \colon a_m \geq k \text{ for all } m \geq i\}$. For all $j \geq l(k)$, set $n_{k,j} \in \mathbb{N}$ to be the unique integer satisfying $T_1^{n_{k,j}}(\beta) = [0; k, a_{j+1}, a_{j+2}, \dots]$ and set

$$\mathscr{S}_{k,j} := \frac{(a_{j+1})^a \cdot \ln\left(n_{k,j}\right)}{(q_j)^{2\cdot(1-a)}},$$

where q_n is as defined in (5.1). *If $\limsup_{j\to\infty} \mathscr{S}_{k,j} = 0$, then*

$$\lim_{n\to+\infty} \ln(n) \cdot \mathcal{P}_1^n(v_{\beta,a})\left(\frac{1}{k}\right) = \int v_{\beta,a} \,\mathrm{d}\lambda \cdot h_1;$$

otherwise,

$$\liminf_{n\to+\infty} \ln(n) \cdot \mathcal{P}_1^n(v_{\beta,a})\left(\frac{1}{k}\right) = \int v_{\beta,a} \,\mathrm{d}\lambda \cdot h_1$$
$$\text{and} \quad \limsup_{n\to+\infty} \ln(n) \cdot \mathcal{P}_1^n(v_{\beta,a})\left(\frac{1}{k}\right) > \int v_{\beta,a} \,\mathrm{d}\lambda \cdot h_1.$$

Chapter 9

Heuristics behind Theorem 8.2

The aim of the current chapter is to elucidate the ideas behind Theorem 8.2. The pictures illustrate various examples of the stated convergence results. Throughout this chapter we set $r = 1$.

The pictures in the current chapter deal with the individual iterates of the transfer operator, whereas Theorem 8.2 deals with the individual iterates of the Perron-Frobenius operator. By Lemma 6.2 however, we can change between these two operators back and forth.

We begin with an example for which convergence results of the individual iterates of the transfer operator were previously known. We let $g_1 : [0, 1] \to \mathbb{R}$ be given by

$$g_1(x) := \frac{\sqrt{x}}{2}.$$

Figure 9.1 shows the observable g_1 together with a horizontal line at the value

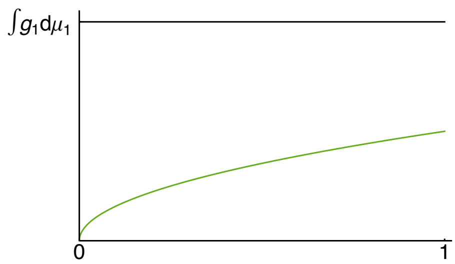
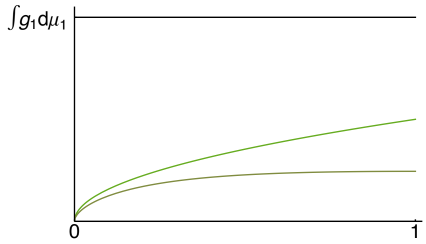


Figure 9.1: Graph of the observable g_1 and of the function $\int g_1 \mathrm{d}\mu_1 \cdot \mathbb{1}_{[0,1]} = \mathbb{1}_{[0,1]}$.

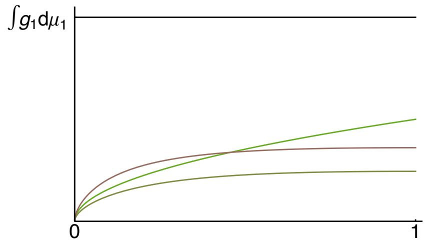
of its μ_1-integral. This observable is in many ways nice, in particular it is twice differentiable and concave on $(0, 1)$. Furthermore, it belongs to the class $\mathcal{D}$ considered in [KS08], see (6.13). These facts together with the extension theorem, Theorem 6.11, yield uniformly on compact subsets of $(0, 1]$, that

$$\lim_{n\to\infty} \ln(n)\widehat{T}_1^n(g_1) = \int g_1 \mathrm{d}\mu_1 = 1.$$

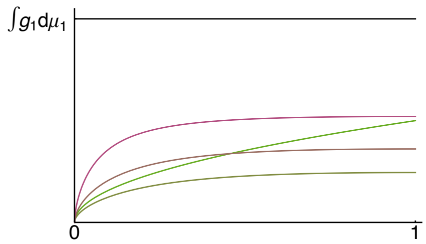
This convergence result is also known as the smoothing property of the transfer operator and can be seen in Figure 9.2.



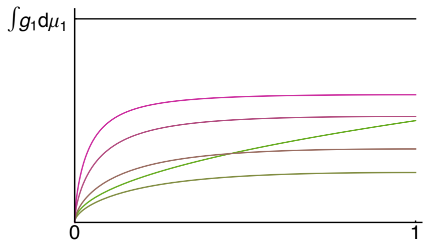
(a) $\ln(k+1) \cdot \widehat{T_1}^k(g_1)$, for $k \in \{0, 1\}$



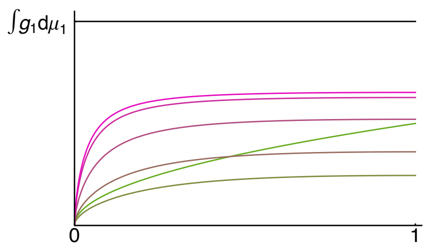
(b) $\ln(k+1) \cdot \widehat{T_1}^k(g_1)$, for $k \in \{0, 1, 2\}$.



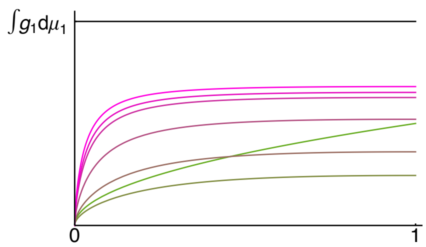
(c) $\ln(k+1) \cdot \widehat{T_1}^k(g_1)$, for $k \in \{0, 1, 3, 5\}$



(d) $\ln(k+1) \cdot \widehat{T_1}^k(g_1)$, for $k \in \{0, 1, 2, 5, 10\}$.



(e) $\ln(k+1) \cdot \widehat{T_1}^k(g_1)$, for $k \in \{0, 1, 2, 5, 10, 12\}$



(f) $\ln(k+1) \cdot \widehat{T_1}^k(g_1)$, for $k \in \{0, 1, 2, 5, 10, 12, 15\}$.

Figure 9.2: Iterates of the transfer operator applied to g_1 normalised with the wandering rate.

The question that naturally arises is, what can we do to extend these well-known convergence results. This was the starting point for the research that led to the results of Part II. So far it had always been assumed that the considered observables were bounded. However, since the definition of the transfer operator can be extended to $\mathcal{L}^1_\mu$-functions, this restriction is not necessary. So what happens, if we look at the individual iterates of the transfer operator applied to an unbounded and integrable observable?
To this end, let $a \in (0,1)$ and $\beta \in [0,1]$. We consider observables of the form $g\colon [0,1] \to \mathbb{R}$ with $g(x) = x/|x-\beta|^a$. These observables are μ_1-integrable and have a pole of order a at β. As it turns out, the position of the pole is crucial for the behaviour of the iterates of the transfer operator.
To get a first impression, we start with a pole at a rational number, namely $\beta = 1/3$ and choose $a := 1/3$. We let $g_2\colon [0,1] \to \mathbb{R}$ be given by

$$g_2(x) := \frac{x}{\left|x - \frac{1}{3}\right|^{\frac{1}{3}}}.$$

See Figure 9.3 for the graph of g_2 and a constant function of its μ_1-integral. Taking

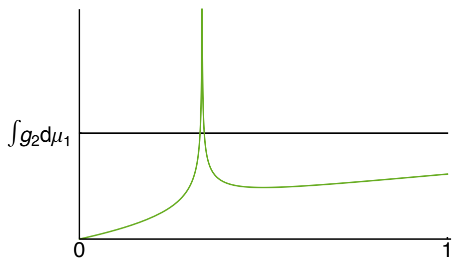


Figure 9.3: Graph of the observable g_2 and of the function $\int g_2 \mathrm{d}\mu_1 \cdot \mathbb{1}_{[0,1]}$.

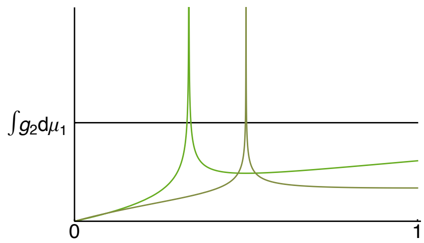
the representation of the transfer operator (6.5), see page 34, into account, we observe two facts. The first is, that $T_1(1/3) = 1/2$. Since the transfer operator can be written via preimages of the transformation, we have that if g_2 has a pole at $1/3$, the first iteration, $\widehat{T}_1(g_2)$ has a pole at $1/2$. The second observation is, that the first iteration splits into a sum of two summands. One of them is an observable for which convergence results are known and the other summand has a pole at $1/2$. Applying the transfer operator once more, we see that we have four summands. For three of them, the convergence is known and one having pole at 1. After the third iteration, the pole vanishes. This has to happen, since $T_1(1) = 0$ and μ_1 has infinite mass at zero. Hence, an observable which is not zero for $x = 0$, or even unbounded at zero, is not μ_1 integrable, but we have for all $v \in \mathcal{L}^1_{\mu_1}$, that

$$\int \widehat{T}_1(v) \mathrm{d}\mu_1 = \int v \mathrm{d}\mu_1.$$

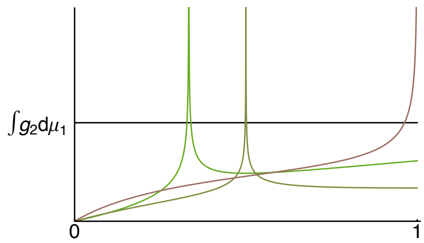
So, after the third iteration of the transfer operator applied to our observable, we are in a class of observables for which the convergence of the individual iterates of the transfer operator is known. Hence, we get the convergence,

$$\lim_{n\to\infty} \ln(n)\widehat{T}_1^n(g_2(x)) = \int g_2 \mathrm{d}\mu_1.$$

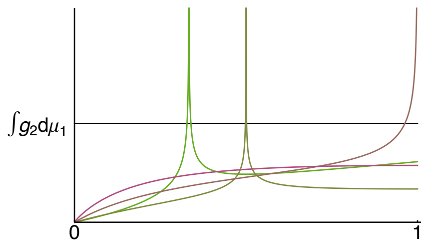
This pattern holds for all rational numbers, since each rational number is mapped



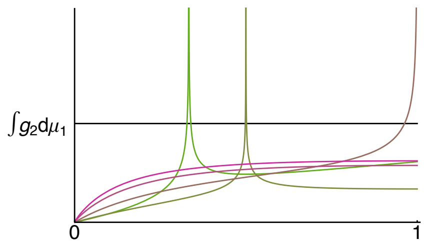
(a) $\ln(k+1)\cdot\widehat{T_1}^k(g_2)$, for $k \in \{0, 1\}$



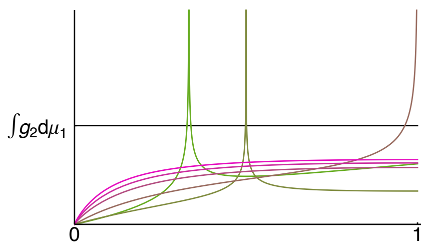
(b) $\ln(k+1)\cdot\widehat{T_1}^k(g_2)$, for $k \in \{0, 1, 2\}$.



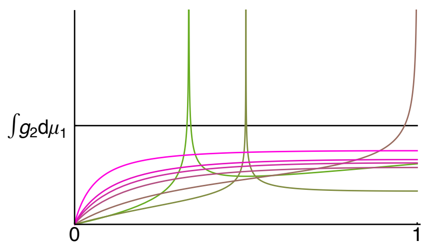
(c) $\ln(k+1)\cdot\widehat{T_1}^k(g_2)$, for $k \in \{0, 1, 2, 3\}$



(d) $\ln(k+1)\cdot\widehat{T_1}^k(g_2)$, for $k \in \{0, 1, 2, 3, 4\}$.



(e) $\ln(k+1)\cdot\widehat{T_1}^k(g_2)$, for $k \in \{0, 1, 2, 3, 4, 5\}$



(f) $\ln(k+1)\cdot\widehat{T_1}^k(g_2)$, for $k \in \{0, 1, 2, 3, 4, 5, 10\}$.

Figure 9.4: Iterates of the transfer operator applied to the observable g_2 normalised with the wandering rate.

to zero under the Farey map after a finite number of iterations. This behaviour can be seen in Figure 9.4 and is an important difference to the finite measure case, as explained in Remark 8.4.

What happens if the pole is not mapped to zero eventually? One example that comes to mind is the fixed point of the Farey map at $\gamma := (\sqrt{5}-1)/2$.
So let us have a look at the observable $g_3\colon [0,1] \to \mathbb{R}$,

$$g_3(x) := \frac{x}{|x-\gamma|^{\frac{1}{3}}}.$$

A picture of the observable g_3 is displayed in Figure 9.5 together with a graph of the constant function with the value $\int_{[0,1]} g_3 \mathrm{d}\mu_1$. We observe, that the pole does not

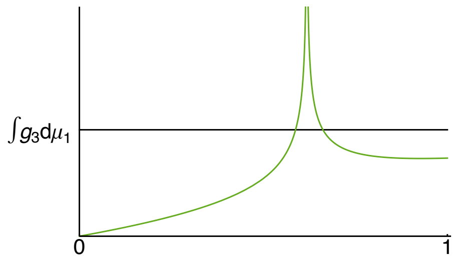
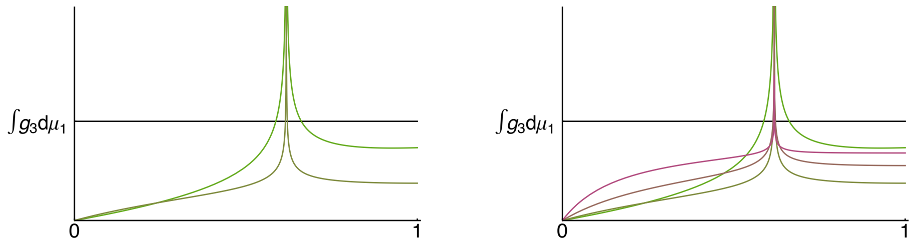


Figure 9.5: Graph of the observable g_3 and of the function $\int g_3 \mathrm{d}\mu_1 \cdot \mathbb{1}_{[0,1]}$.

move, if we apply the transfer operator to the observable. Nevertheless, we observe, that the pointwise version of the first iteration of the transfer operator splits up into two summands. One of them is without the pole and convergence for it is known, the other summand has a pole. Inductively we can see, that the n-th iteration of the transfer operator consists of a sum of 2^n summands. One of them has a pole and for the remaining $2^n - 1$ summands without a pole the convergence is known. What essentially happens is, that the part containing the pole gets less important in the long run. It still has the pole, but the diameter of the cusp gets thinner, in the sense that for a constant $C > \int g_3 d\mu_1$, we have that $\mu_1(\{x\colon \widehat{T}(g_3)(x) > C\})$ gets smaller. The sum of the remaining $2^n - 1$ parts converges to the integral on each compact subset not containing γ. This can be seen in Figure 9.6.



(a) $\ln(k+1) \cdot \widehat{T_1}^k(g_3)$, for $k \in \{0, 1\}$

(b) $\ln(k+1) \cdot \widehat{T_1}^k(g_3)$, for $k \in \{0, 1, 2, 4\}$.

Figure 9.6: Iterates of the transfer operator applied to the observable g_3 normalised with the wandering rate.

Having seen the case in which the pole does not move under the iterations of the transfer operator, a natural question to ask is what happens if it does, without being a rational number. A natural point to look at, is a periodic point. Such a point is for instance $1/\sqrt{2} = [0, 1, \overline{2}] = [0, \overline{1, 0}]_1$, which is pre-periodic with respect to the Gauß map and purely periodic under the Farey map. Thus, we look at the observable g_4, displayed in Figure 9.7, which is given by $g_4 \colon [0, 1] \to \mathbb{R}$,

$$g_4(x) := \frac{x}{\left|x - \frac{1}{\sqrt{2}}\right|^{\frac{1}{3}}}.$$

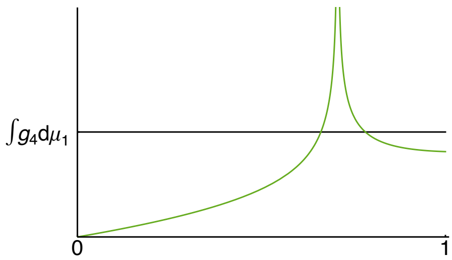
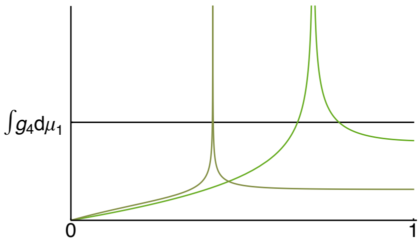
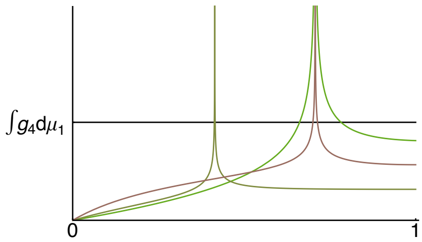


Figure 9.7: Graph of the observable g_4 and of the function $\int g_4 \mathrm{d}\mu_1 \cdot \mathbb{1}_{[0,1]}$.

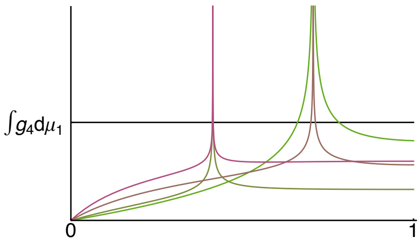
In this case the pole moves along its orbit and we observe, that the cusp get thinner. Furthermore, on each compact subset outside the orbit, the individual iterates of the transfer operator normalised by the wandering rate, converge to the constant function with the value $\int g_4 \mathrm{d}\mu_1$. On the orbit we observe that the limit inferior is as well equal to $\int g_4 \mathrm{d}\mu_1$, but each time the pole comes back to the orbit point, the convergence can of course not hold. This scenario can be seen in Figure 9.8.



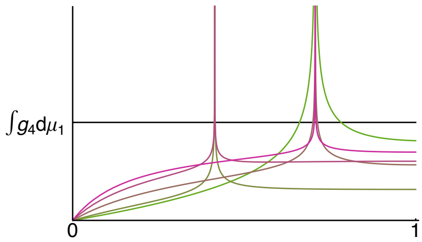
(a) $\ln(k+1) \cdot \widehat{T_1}^{k}(g_4)$, for $k \in \{0, 1\}$



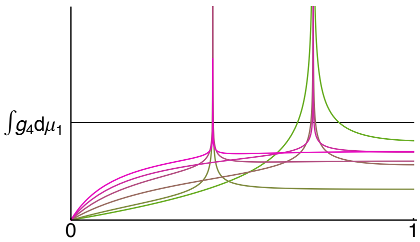
(b) $\ln(k+1) \cdot \widehat{T_1}^{k}(g_4)$, for $k \in \{0, 1, 2\}$.



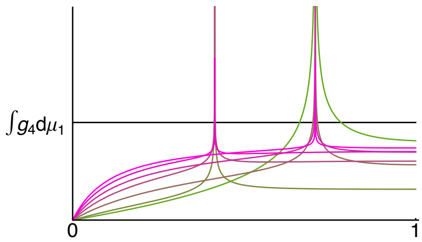
(c) $\ln(k+1) \cdot \widehat{T_1}^{k}(g_4)$, for $k \in \{0, 1, 2, 3\}$



(d) $\ln(k+1) \cdot \widehat{T_1}^{k}(g_4)$, for $k \in \{0, 1, 2, 3, 4\}$.



(e) $\ln(k+1) \cdot \widehat{T_1}^{k}(g_4)$, for $k \in \{0, 1, 2, 3, 4, 5\}$



(f) $\ln(k+1) \cdot \widehat{T_1}^{k}(g_4)$, for $k \in \{0, 1, 2, 3, 4, 5, 6\}$.

Figure 9.8: Iterates of the transfer operator applied to the observable g_4 normalised with the wandering rate.

Finally, we have a look at a pole that is neither rational nor periodic. We choose $\beta = 2^{(-1/3)}$ and $a = 9/20$. Still, we have that $a < 1/2$, so the pole is of intermediate a-type, see Remark 7.4, and hence we fall into the case, considered in Theorem 8.2. Thus, let $g_5 \colon [0,1] \to \mathbb{R}$ be given by

$$g_5(x) := \frac{x}{\left|x - \frac{1}{\sqrt[3]{2}}\right|^{\frac{9}{20}}}.$$

See Figure 9.9 for a picture of the observable g_5.

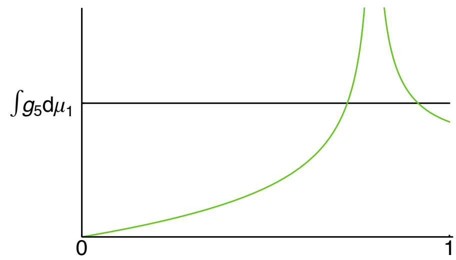
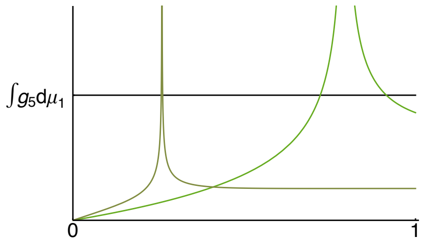
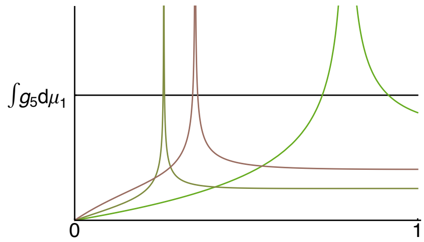


Figure 9.9: Graph of the observable g_5 and of the function $\int g_5 \mathrm{d}\mu_1 \cdot \mathbb{1}_{[0,1]}$.

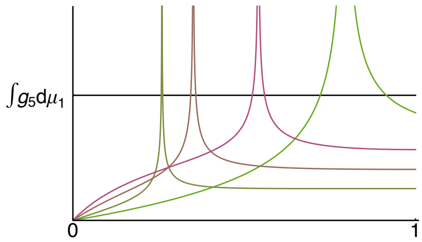
We observe that the diameter of the cusp gets smaller with each iteration of the transfer operator normalised by the wandering rate. Although the pole can get "close" to where it has been before, after the pole has been mapped to a certain point, we discern convergence at this point. This behaviour is described in Theorem 8.2 and can be seen in Figure 9.10.



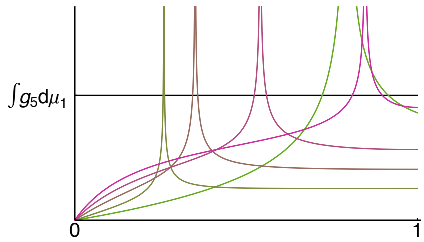
(a) $\ln(k+1)\cdot\widehat{T_1}^{\,k}(g_5)$, for $k \in \{0, 1\}$



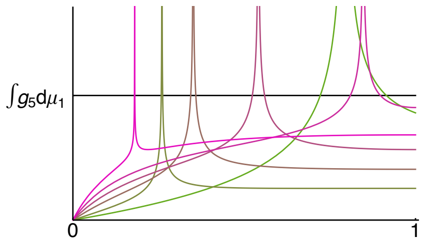
(b) $\ln(k+1)\cdot\widehat{T_1}^{\,k}(g_5)$, for $k \in \{0, 1, 2\}$.



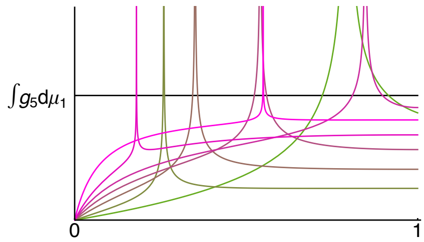
(c) $\ln(k+1)\cdot\widehat{T_1}^{\,k}(g_5)$, for $k \in \{0, 1, 2, 3\}$



(d) $\ln(k+1)\cdot\widehat{T_1}^{\,k}(g_5)$, for $k \in \{0, 1, 2, 3, 4\}$.



(e) $\ln(k+1)\cdot\widehat{T_1}^{\,k}(g_5)$, for $k \in \{0, 1, 2, 3, 4, 5\}$



(f) $\ln(k+1)\cdot\widehat{T_1}^{\,k}(g_5)$, for $k \in \{0, 1, 2, 3, 4, 5, 10\}$.

Figure 9.10: Iterates of the transfer operator applied to an observable with a non-periodic pole normalised with the wandering rate

Chapter 10

Proofs of the main results

10.1 The case $r \in [0, 1)$

Before we start with the proof, recall the definition of $[W_{r,n}(\beta)]$ as defined in (5.7). The definition of this set is crucial, since by the linearity of the Perron-Frobenius operator we have that

$$\mathcal{P}_r^n(v) = \mathcal{P}_r^n(v \cdot \mathbb{1}_{[0,1]\setminus[W_{r,n}(\beta)]}) + \mathcal{P}_r^n(v \cdot \mathbb{1}_{[W_{r,n}(\beta)]}).$$

Having this equality in mind, the proof splits up into three parts. The first part is to show that we have $\lim_{n\to\infty} \mathcal{P}_r^n(v \cdot \mathbb{1}_{[0,1]\setminus[W_{r,n}(\beta)]}) = \int v d\lambda \cdot h_r$. The second part is to show that $\lim_{n\to\infty} \mathcal{P}_r^n(v \cdot \mathbb{1}_{[W_{r,n}(\beta)]})$ is equal to zero outside a set of Hausforff dimension equal to zero. These two first parts yield the first statement of the theorem. Finally, we have to consider pre-periodic points to get the final statement of Theorem 8.1.

A sketch of the proof is published in [KKS16], though more details are given here. Before we start with the first part of the proof, we state a lemma from [KK12] without a proof and its corollary about bounded distortion, which helps throughout the proof.

Lemma 10.1 ([KK12, Lemma 3.2]). *Let $r \in [0, 1)$ be fixed. There is a sequence $(\varrho_m)_{m\in\mathbb{N}_0}$, dependent on r, with $\varrho_m > 0$ for each $m \in \mathbb{N}_0$ and $\lim_{m\to+\infty} \varrho_m = 1$, such that, for all $m, n \in \mathbb{N}_0$, $\vartheta \in \Sigma^m$, $\varphi \in \Sigma^n$ and $x, y \in [\vartheta]_r$, we have that*

$$\varrho_m^{-1} \leq \left| \frac{f'_{r,\varphi}(x)}{f'_{r,\varphi}(y)} \right| \leq \varrho_m.$$

Here Σ^0 denotes the set containing the empty set and $f_{r,\emptyset}$ denotes the identity function $[0, 1] \ni x \mapsto x$.

Corollary 10.2 ([KKS16, Lemma 4.3]). *Let $n \in \mathbb{N}$ be fixed. If $\vartheta = (\vartheta_1, \vartheta_2, \dots, \vartheta_n)$ and $\nu = (\nu_1, \nu_2, \dots, \nu_n)$ denote two distinct, yet adjacent elements of Σ^n. That is,*

$[\vartheta] \cap [\nu] \neq \emptyset$*, then there exists a positive constant* K *such that, for all* $x, y \in [0,1]$*,*

$$K^{-1} \leq \left| \frac{f'_{r,\vartheta}(x)}{f'_{r,\nu}(y)} \right| \leq K.$$

Proof of Corollary 10.2. This corollary follows by combining the chain rule with Lemma 5.1 and Lemma 10.1.

□

Proof of Theorem 8.1. The first part of the proof is dealt with in the following lemma and follows from results on distributional convergence in finite ergodic theory stated in Theorem 6.8.

Lemma 10.3 ([KKS16, Lemma 4.5])**.** *For* $r \in [0,1)$*,* $a \in (0,1)$*,* $\beta \in [0,1]$ *and* $v \in \mathfrak{U}_{\beta,a}$*, we have that*

$$\lim_{n\to\infty} \mathcal{P}_r^n \left(v \cdot \mathbb{1}_{[0,1]\setminus[W_{r,n}(\beta)]} \right) = \int v \, \mathrm{d}\lambda \cdot h_r,$$

uniformly on $[0,1]$*.*

Proof of Lemma 10.3. Let $N \in \mathbb{N}$ be fixed. Since $v \cdot \mathbb{1}_{[0,1]\setminus[W_r(\beta)|_N]} \in \mathrm{BV}(0,1)$, we have by Theorem 6.8, that

$$\lim_{n\to+\infty} \mathcal{P}_r^n (v \cdot \mathbb{1}_{[0,1]\setminus[W_r(\beta)|_N]}) = \int v \cdot \mathbb{1}_{[0,1]\setminus[W_r(\beta)|_N]} \, \mathrm{d}\lambda \cdot h_r$$

uniformly on $[0,1]$. We will shortly show, with the aid of Lemmata 5.1 and Corollary 10.2, that, uniformly on $[0,1]$, there exists a positive constant $K \in \mathbb{R}$ such that for all $x \in [0,1]$

$$\lim_{n\to+\infty} \mathcal{P}_r^n (v \cdot \mathbb{1}_{[W_r(\beta)|_N]\setminus[W_r(\beta)|_n]})(x) \leq K \cdot \sum_{k=N}^{+\infty} \frac{1}{(2-r)^{k\cdot(1-a)}}. \tag{10.1}$$

As v is improper Riemann integrable and as $\lim_{N\to+\infty} \lambda([\vartheta_r(\beta)|_N]) = 0$, we have that

$$\lim_{N\to+\infty} \int v \cdot \mathbb{1}_{[0,1]\setminus[W_r(\beta)|_N]} \, \mathrm{d}\lambda = \int v \, \mathrm{d}\lambda$$

and by the properties of a geometric series we have that

$$\lim_{N\to+\infty} \sum_{k=N}^{+\infty} \frac{1}{(2-r)^{k\cdot(1-a)}} = 0.$$

Thus, assuming the inequality given in (10.1), since $\mathcal{P}_r$ is a positive linear operator and since N was chosen arbitrarily, the result follows. Hence, all that is left to show, is the inequality stated in (10.1). To this end, let $U \subset [0,1]$ denote an open set and let C_2 be a constant such that Condition (c) in the definition of $\mathfrak{U}_{\beta,a}$, see page 63, is satisfied. Let $n > N \geq 2$ with $[W_r(\beta)|_N] \subseteq U$ be fixed. For all $x \in [0,1]$, we have that

$$\begin{aligned} \frac{2-r}{4} &\leq \left|f'_{r,0}(x)\right| \leq \frac{1}{2-r} \\ \text{and} \quad \frac{2-r}{4} &\leq \left|f'_{r,1}(x)\right| \leq \frac{1}{2-r}. \end{aligned} \tag{10.2}$$

This in tandem with Lemma 10.1 and corrolary 10.2 and the mean value theorem, gives that there exists a positive constant $\varrho \in \mathbb{R}$ such that the following chain of inequalities hold, for all $x \in [0,1]$. In the last equality we set

$$K := \frac{\varrho^2 \cdot C_2 \cdot 4^{1+a}}{(2-r)^{2\cdot a}}.$$

$$\begin{aligned}
&\mathcal{P}_r^n(v \cdot \mathbb{1}_{[W_r(\beta)|_N]\setminus[W_r(\beta)|_n]})(x) \\
&= \sum_{\substack{\vartheta\in\Sigma^n\setminus\mathfrak{W}_{r,n}(\beta) \\ [\vartheta]\subseteq[W_r(\beta)|_N]}} \left|f'_{r,\vartheta}(x)\right| \cdot v \circ f_{r,\vartheta}(x) \\
&\leq \sum_{\substack{\vartheta\in\Sigma^n\setminus\mathfrak{W}_{r,n}(\beta) \\ [\vartheta]\subseteq[W_r(\beta)|_N]}} \varrho \cdot \lambda([\vartheta]) \cdot \sup\{v(y) \colon y \in [\vartheta]\} \\
&\leq \sum_{k=N+1}^{n} \sum_{\substack{\vartheta\in\Sigma^k\setminus\mathfrak{W}_{r,k}(\beta) \\ [\vartheta]\subseteq[W_r(\beta)|_{k-1}]}} \varrho \cdot \lambda([\vartheta]) \cdot \sup\{v(y) \colon y \in [\vartheta]\} \\
&\leq \sum_{k=N+1}^{n} \sum_{\substack{\vartheta\in\Sigma^k\setminus\mathfrak{W}_{r,k}(\beta) \\ [\vartheta]\subseteq[W_r(\beta)|_{k-1}]}} \varrho \cdot C_2 \cdot \lambda([\vartheta]) \cdot \sup\{|y-\beta|^{-a} \colon y \in [\vartheta]\} \\
&\leq \sum_{k=N+1}^{n} 2 \cdot \varrho^2 \cdot C_2 \cdot \left(\frac{4^a \cdot \lambda([\vartheta_r^-(\beta)|_{k-1}])^{1-a}}{(2-r)^{1+a}} + \frac{4^a \cdot \lambda([\vartheta_r^+(\beta)|_{k-1}])^{1-a}}{(2-r)^{1+a}} \right) \\
&\leq \sum_{k=N+1}^{n} \varrho^2 \cdot C_2 \cdot 4^{1+a} \cdot (2-r)^{-(1+a)-(k-1)\cdot(1-a)} \\
&= K \cdot \sum_{k=N+1}^{n} (2-r)^{-k\cdot(1-a)}
\end{aligned}$$

Hence, the proof is completed. □

The second part of this proof considers the part of the observable, that contains the pole. To this end we define the tail of an observable.

Definition 10.4 (***r*-tail of the observable**, [KKS16, Definition 4.1])**.** Let $r \in [0,1)$, $a \in (0,1)$ and $\beta \in [0,1]$ and let $v_{\beta,a}(x) = |x-\beta|^{-a}$. We define the *r-tail of the observable* $v_{\beta,a}$ by

$$v^{n,r} := v_{\beta,a,n,r} := \mathcal{P}_r^n(v_{\beta,a} \cdot \mathbb{1}_{[W_{r,n}(\beta)]}) = \sum_{\vartheta \in \mathfrak{W}_{r,n}(\beta)} \left|f'_{r,\vartheta}(x)\right| \cdot v_{\beta,a} \circ f_{r,\vartheta}. \tag{10.3}$$

For $\eta > 0$, we further define

$$A_{n,r,\eta} := \{x \in [0,1] \colon v^{n,r}(x) > \eta\}.$$

Since $\mathcal{P}_r$ is a positive linear operator and v is non-negative we discern that

$$0 \le \lim_{n\to\infty} \mathcal{P}_r^n(v \cdot \mathbb{1}_{[W_{r,n}(\beta)]}) \le C_2 \cdot \lim_{n\to\infty} \mathcal{P}_r^n(v^{n,r}).$$

We show in the next lemma that the latter limit is equal to zero outside a set of Hausdorff dimension zero.
For $s > 0$ and $\delta > 0$, we let $\mathcal{H}^s_\delta$ denote the δ-approximation to the s-dimensional Hausdorff measure and we let $\mathcal{H}^s$ denote the s-dimensional Hausdorff measure.

Lemma 10.5 ([KKS16, Lemma 4.6])**.** *For $r \in [0,1)$, $a \in (0,1)$, $\beta \in [0,1]$, $n \in \mathbb{N}$ and $\eta > 0$, we have that*

$$\dim_{\mathcal{H}}\left(\limsup_{n\to+\infty} A_{n,r,\eta}\right) = 0.$$

Proof of Lemma 10.5. Set $z = T_r^n(\beta)$ and observe that z is the unique real number in $[0,1]$ with $f_{r,\vartheta_r(\beta)|_n}(z) = \beta$. By the mean value theorem there exists $u \in (0,1)$ such that

$$\begin{aligned} \left|\beta - f_{r,\vartheta_r(\beta)|_n}(x)\right| &= \left|f_{r,\vartheta_r(\beta)|_n}(z) - f_{r,\vartheta_r(\beta)|_n}(x)\right| \\ &= |x-z| \cdot \left|f'_{r,\vartheta_r(\beta)|_n}(u)\right| \\ &= \left|x - T_r^n(\beta)\right| \cdot \left|f'_{r,\vartheta_r(\beta)|_n}(u)\right|. \end{aligned}$$

By construction, we have that $\left|\beta - f_{r,\vartheta_r^\pm(\beta)|_n}(x)\right| \ge \left|\beta - f_{r,\vartheta_r(\beta)|_n}(x)\right|$. Recall, that $\vartheta_r^\pm$ are the unique words that code the adjacent cylinders of the cylinder coded by $\vartheta_r(\beta)|_n$, see (5.6). This in tandem with (10.2) and Lemma 10.1 and Corollary 10.2, yields the following set inclusions. We let $B(y,\rho)$ denote the open Euclidean ball of radius ρ centred at y.

$$\begin{aligned} A_{n,r,\eta} &= \{x \in [0,1] \colon v^{n,r}(x) > \eta\} \\ &= \left\{x \in [0,1] \colon \textstyle\sum_{\vartheta \in \mathfrak{W}_{r,n}(\beta)} \left|f'_{r,\vartheta}(x)\right| \cdot v_{\beta,a} \circ f_{r,\vartheta} > \eta\right\} \\ &= \left\{x \in [0,1] \colon \textstyle\sum_{\vartheta \in \mathfrak{W}_{r,n}(\beta)} \left|f'_{r,\vartheta}(x)\right| \cdot \left|x - T_r^n(\beta)\right|^{-a} \cdot \left|f'_{r,\vartheta_r(\beta)|_n}(u)\right|^{-a} > \eta\right\} \\ &\subseteq \left\{x \in [0,1] \colon \left|x - T_r^n(\beta)\right| < (2-r)^{(1-1/a)\cdot n} \cdot (3 \cdot \eta \cdot K)^{1/a}\right\} \\ &= B\left(T_r^n(\beta), (2-r)^{(1-1/a)\cdot n} \cdot (3 \cdot \eta \cdot K)^{1/a}\right). \end{aligned}$$

Hence, given $\delta > 0$, there exists a natural number $M = M(\delta) \in \mathbb{N}$ such that

$$\left\{B\left(T_r^n(\beta), (2-r)^{(1-1/a)\cdot n} \cdot (3\cdot\eta\cdot K)^{1/a}\right) : n \geq M \text{ and } n \in \mathbb{N}\right\}$$

is an open δ-cover of $\limsup_{n\to+\infty} A_{n,r,\eta}$. We have that

$$\begin{aligned}
\mathcal{H}^s_\delta\left(\limsup_{n\to+\infty} A_{n,r,\eta}\right) &\leq \sum_{n=M}^{+\infty} \lambda\left(B\left(T_r^n(\beta), (2-r)^{(1-1/a)\cdot n} \cdot (3\cdot\eta\cdot K)^{1/a}\right)\right)^s \\
&\leq \sum_{n=M}^{+\infty} (2-r)^{(1-1/a)\cdot s\cdot n} \cdot (3\cdot\eta\cdot K)^{s/a} \\
&= \frac{(3\cdot\eta\cdot K)^{s/a}\cdot(2-r)^{(1-1/a)\cdot s\cdot M}}{1-(2-r)^{(1-1/a)\cdot s}}.
\end{aligned}$$

Since $a \in (0,1)$, this last quantity is finite for all $s > 0$ and $\delta > 0$, and so $\mathcal{H}^s(\limsup_{n\to+\infty} A_{n,r,\eta})$ is finite for all $s > 0$, which implies

$$\dim_{\mathcal{H}}(\limsup_{n\to+\infty} A_{n,r,\eta}) = 0,$$

as required and the first two parts of Theorem 8.1 are proven.
All that remains to show is the third part of the theorem, namely that if $\beta \in [0,1]$ is pre-periodic with respect to T_r and has period length strictly greater than one, then on $\omega_r(\beta)$ we have that

$$\liminf_{n\to+\infty} \mathcal{P}_r^n(v) = \int v \,\mathrm{d}\lambda \cdot h_r \quad \text{and} \quad \limsup_{n\to+\infty} \mathcal{P}_r^n(v); = +\infty;$$

and in the case that $\beta \in [0,1]$ is pre-periodic with respect to T_r and has period length equal to one then on the singleton $\omega_r(\beta)$ we have that the limit in (8.1) is equal to $+\infty$.
By linearity of $\mathcal{P}_r^n$ and Lemma 10.3, it suffices to show, if $\beta \in [0,1]$ is pre-periodic with respect to T_r and has period length strictly greater than one, then on $\omega_r(\beta)$

$$\liminf_{n\to+\infty} v^{n,r} = 0 \quad \text{and} \quad \limsup_{n\to+\infty} v^{n,r} = +\infty;$$

and in the case that $\beta \in [0,1]$ is pre-periodic with respect to T_r and has period length equal to one, then on the singleton $\omega_r(\beta)$

$$\lim_{n\to+\infty} v^{n,r} = +\infty.$$

Indeed if β is pre-periodic with respect to T_r and has period length $m \geq 1$, then letting $n \in \mathbb{N}_0$, be the minimal integer so that $T_r^{n+k}(\beta) = T_r^{n+k+m}(\beta)$, for all $k \in \mathbb{N}_0$, we have that

$$f_{r,(\vartheta_{r,n+j+1}(\beta),\ldots,\vartheta_{r,n+j+m}(\beta))}\left(T_r^{n+j}(\beta)\right) = T_r^{n+j}(\beta),$$

for all $j \in \{0, 1, \dots, m-1\}$. Further, $\omega_r(\beta) = \left\{T_r^n(\beta), \dots, T_r^{n+m-1}(\beta)\right\}$, and hence, for $j \in \{0, 1, \dots, m-1\}$, it follows for all $k \in \mathbb{N}_0$ that

$$v^{n+j+k\cdot m,r}\left(T_r^{n+j}(\beta)\right) = +\infty.$$

To complete the proof we will show, for $m > 1$ and $i, j \in \{0, 1, \dots, m-1\}$ with $i \neq j$, that

$$\lim_{k\to+\infty} v^{n+j+k\cdot m,r}\left(T_r^{n+i}(\beta)\right) = 0.$$

To this end set $L := \min\{|T_r^{n+j}(\beta) - T_r^{n+i}(\beta)\colon i, j \in \{0, 1, \dots, m-1\} \text{ and } i \neq j\}$. By (10.2) and Corollary 10.2, there exists a positive constant $\varrho \in \mathbb{R}$ such that the following chain of inequalities hold. Here, we apply the mean value theorem in a similar manner as in the proof of Lemma 10.5.

$$\begin{aligned}
&\lim_{k\to+\infty} v^{n+j+k\cdot m,r}\left(T_r^{n+i}(\beta)\right) \\
&= \lim_{k\to+\infty} \sum_{\vartheta\in\mathfrak{W}_{r,n+j+k\cdot m}(\beta)} \left|f'_{r,\vartheta}\left(T_r^{n+i}(\beta)\right)\right| \cdot \left|\beta - f_{r,\vartheta}\left(T_r^{n+i}(\beta)\right)\right|^{-\alpha} \\
&\leq \lim_{k\to+\infty} 3\cdot\varrho\cdot\left|f'_{r,\vartheta_1(\beta)|_{n+j+k\cdot m}}\left(T_r^{n+i}(\beta)\right)\right| \cdot \left|\beta - f_{r,\vartheta_1(\beta)|_{n+j+k\cdot m}}\left(T_r^{n+i}(\beta)\right)\right|^{-\alpha} \\
&\leq \lim_{k\to+\infty} 3\cdot\varrho\cdot\left|f'_{r,\vartheta_1(\beta)|_{n+j+k\cdot m}}\left(T_r^{n+i}(\beta)\right)\right| \\
&\qquad\cdot\left|f_{r,\vartheta_1(\beta)|_{n+j+k\cdot m}}\left(T_r^{n+j+k\cdot m}(\beta)\right) - f_{r,\vartheta_1(\beta)|_{n+j+k\cdot m}}\left(T_r^{n+i}(\beta)\right)\right|^{-\alpha} \\
&\leq \lim_{k\to+\infty} 3\cdot\varrho^{1+a}\cdot\left|f'_{r,\vartheta_1(\beta)|_{n+j+k\cdot m}}\left(T_r^{n+i}(\beta)\right)\right|^{1-a} \cdot \left|T_r^{n+j+k\cdot m}(\beta) - T_r^{n+i}(\beta)\right|^{-a} \\
&= 3\cdot\varrho^{1+a}\cdot\left|T_r^{n+j}(\beta) - T_r^{n+i}(\beta)\right|^{-a} \lim_{k\to\infty}(2-r)^{(a-1)\cdot(n+j+k\cdot m)} \\
&= 3\cdot\varrho^{1+a}\cdot L^a \lim_{k\to\infty}(2-r)^{(a-1)\cdot(n+j+k\cdot m)} \\
&= 0.
\end{aligned}$$

This completes the proof of Theorem 8.1.

□□

10.2 The case $r = 1$

In the previous proof, the bounded distortion property of the family of maps T_r, with $r \in [0, 1)$, played a crucial role. Unfortunately, T_1 does no longer have this property, since the absolut value of the derivative of $T_1(x)$ equals one for $x = 0$ and $x = 1$. So more sophisticated methods are required in the proof of Theorem 8.2.
In addition, we have to divide the proof of Theorem 8.2 into two cases. Firstly, the case, when β is a rational number is conidered. Secondly, the case when β is an

irrational number of intermediate a-type is considered. In this case similar methods to the ones used in Theorem 8.1 are applied. These methods are not applicable for the rational case and hence this case is considered separately.
This part is published as [KKS16, Subsection 5.2], though more details are given here.
Before we start with the proof of the theorem we have to discuss, how the convergence of the induced transfer operator stated in Theorem 6.10 can be extended to convergence of the actual transfer operator. In [KKSS15], it has been shown, that we have to be careful for regular varying wandering rates. Yet, $\ln(\cdot)$ is slowly varying and even moderately increasing, see Part III for further details, so no problems arise in this case. For the next theorem, we recall the definition of the level sets of the first entry time Y_k given in (3.2) on page 15.The next theorem was first published in [MT12], however, in the current form it was used in [KKS16, Theorem 4.9], where the notation is slightly adapted.

Theorem 10.6 ([MT12, Theorem 10.4]). *Let $f \in \mathrm{BV}([0,1])$ be such that $\|f\|_\infty < +\infty$ and let*

$$\widetilde{f_k} := \mathbb{1}_{Y_k} \cdot f.$$

If

$$\sum_{k=0}^{+\infty} \left\| \widehat{T}_1^k(\widetilde{f_k}) \right\|_\infty < +\infty, \tag{10.4}$$

then on Y

$$\lim_{n\to+\infty} \ln(n) \cdot \widehat{T}_1^n(f) = \int f \, \mathrm{d}\mu_1.$$

Before we prove this theorem we give a lemma, that shows that the class of observables that satisfies the conditions of Theorem 10.6 is not vain.

Lemma 10.7 ([KKS16, Remark 8]). *If $f \in \mathrm{BV}(0,1)$, then f/h_1 satisfies the conditions of Theorem 10.6.*

Proof of Lemma 10.7. Observe that, by the pointwise version of the transfer operator given in (6.5), we have that

$$\widehat{T}_1^n\left(\frac{f}{h_1} \cdot \mathbb{1}_{Y_n}\right) = \left(\prod_{k=0}^{n-1} f_{1,1} \circ f_{1,0}^k\right) \cdot \frac{f \circ f_{1,0}^n}{h_1 \circ f_{1,0}^n} \cdot \mathbb{1}_Y.$$

Therefore, since f, $f_{1,0}$ and $f_{1,1}$ are of bounded variation and the composition and product of functions of bounded is again of bounded variation, see Lemma 6.16.3., it follows that $\widehat{T}^n(f/h_1 \cdot \mathbb{1}_{Y_n}) \in \mathrm{BV}(Y)$. Further, we observe that

$$\begin{aligned} f_{1,0}^k(x) &= \frac{x}{1+k\cdot x}, \\ f_{1,1} \circ f_{1,0}^k(x) &= \frac{1+k\cdot x}{1+(k+1)\cdot x}, \\ \prod_{k=0}^{n-1} f_{1,1} \circ f_{1,0}^k(x) &= \frac{1}{1+nx}, \\ \frac{1}{h_1 \circ f_{1,0}^k(x)} \cdot \mathbb{1}_Y(x) &= \frac{x}{1+k\cdot x} \cdot \mathbb{1}_Y(x). \end{aligned} \tag{10.5}$$

Combining (10.5) with the fact that a function of bounded variation has finite supremum norm (see Lemma 6.16.1.), we have that

$$\sum_{k=0}^{+\infty} \left\| \widehat{T}_1^k \left(\frac{f}{h_1} \cdot \mathbb{1}_{Y_k} \right) \right\|_\infty \leq \sum_{k=0}^{+\infty} \frac{\|f\|_\infty}{(k+1)^2} < +\infty.$$

□

Proof of Theorem 10.6. In this proof we use the Landau notation, little $\mathfrak{o}(\cdot)$. The first part of this proof is inspired by the first paragraph of the proof of [MT12, Theorem 10.4]. However, to keep this thesis as self contained as possible, a thorough proof is given here.

By Theorem 6.10 and Proposition 6.18, we have, for each $n \in \mathbb{N}_0$, that there exist $\theta_n \colon [0,1] \to \mathbb{C}$ supported on a subset of Y with $\|\theta_n\|_\infty = \mathfrak{o}\left((\ln(n+2))^{-1}\right)$ and

$$\mathbb{1}_Y \cdot \widehat{T}_1^n(\mathbb{1}_Y \cdot f) = \frac{1}{\ln(n+2)} \cdot \int f \, \mathrm{d}\mu_1 \cdot \mathbb{1}_Y + \theta_n \cdot f.$$

As before, we define $\widetilde{f_j} := \mathbb{1}_{Y_j} \cdot f$ and observe that we have on Y,

$$\widehat{T}_1^n(f) = \widehat{T}_1^{n-j}\left(\widehat{T}_1^j(f)\right) = \sum_{j=1}^{n} \mathbb{1}_Y \cdot \widehat{T}_1^{n-j}\left(\widehat{T}_1^j\left(\widetilde{f_j}\right) \cdot \mathbb{1}_Y\right)$$

Thus, for all natural numbers $n > 1$, we have on Y, that

$$\begin{aligned}
&\left|\ln(n) \cdot \widehat{T}_1^n(f) - \int f \, \mathrm{d}\mu_1\right| \\
&= \left|\ln(n) \cdot \sum_{j=0}^{n} \widehat{T}_1^{n-j}\left(\mathbb{1}_Y \cdot \widehat{T}_1^j\left(\widetilde{f}_j\right)\right) - \int f \, \mathrm{d}\mu_1\right| \\
&\leq \left|\ln(n) \cdot \sum_{j=0}^{n} \frac{1}{\ln(n-j+2)} \int \widehat{T}_1^j\left(\widetilde{f}_j\right) \mathrm{d}\mu_1 - \int f \, \mathrm{d}\mu_1\right| \\
&\quad + \ln(n) \cdot \sum_{j=0}^{n} \left\|\theta_{n-j}\right\|_\infty \cdot \left\|\mathbb{1}_Y \cdot \widehat{T}_1^j\left(\widetilde{f}_j\right)\right\|_\infty \\
&\leq \sum_{j=0}^{n} \left(\frac{\ln(n)}{\ln(n-j+2)} - 1\right) \cdot \int \left|\widetilde{f}_k\right| \mathrm{d}\mu_1 &(10.6)\\
&\quad + \ln(n) \cdot \sum_{j=0}^{n} \left\|\theta_{n-j}\right\|_\infty \cdot \left\|\mathbb{1}_Y \cdot \widehat{T}_1^j\left(\widetilde{f}_j\right)\right\|_\infty &(10.7)\\
&\quad + \sum_{j=n+1}^{+\infty} \int \left|\widetilde{f}_j\right| \mathrm{d}\mu_1. &(10.8)
\end{aligned}$$

We now proceed by showing that the three terms (10.6), (10.7) and (10.8) converge to zero as n tends to infinity, for all $x \in Y$.

(a) Since

$$\mu_1(Y_j) = \ln\left(1 + \frac{1}{j+1}\right) \sim \frac{1}{j+1}$$

and since $f \in \mathcal{L}^\infty([0,1])$, there exists a constant $c > 0$ such that for all $j \in \mathbb{N}_0$,

$$\left\|\widetilde{f}_j\right\|_{1,1} \leq \frac{c}{j+1}.$$

For $\epsilon > 0$, if $0 \leq j \leq n - n^{(1/(1+\epsilon))} + 2$, then for all $n \in \mathbb{N}$, we have that

$$\frac{ln(n)}{\ln(n-j+2)} \leq 1 + \epsilon.$$

For a real number x we let $\lceil x \rceil$ denote the smallest integer greater or equal to x. Thus, for a given $\epsilon > 0$, we have that

$$\begin{aligned}
&\lim_{n\to+\infty}\sum_{j=0}^{n}\frac{\ln(n)}{\ln(n-j+2)}\cdot\int\left|\widetilde{f_j}\right|\,\mathrm{d}\mu_1\\
&\leq \lim_{n\to+\infty}\sum_{j=0}^{n-\left\lceil n^{\frac{1}{1+\epsilon}}\right\rceil+1}(1+\epsilon)\cdot\int\left|\widetilde{f_j}\right|\,\mathrm{d}\mu_1+\lim_{n\to+\infty}\sum_{j=n-\left\lceil n^{\frac{1}{1+\epsilon}}\right\rceil+2}^{n}\frac{c\cdot\ln(n)}{j\cdot\ln(n-j+2)}\\
&\leq (1+\epsilon)\cdot\int|f|\,\mathrm{d}\mu_1+\frac{c}{\ln(2)}\cdot\lim_{n\to+\infty}\frac{\left(\left\lceil n^{\frac{1}{1+\epsilon}}\right\rceil+2\right)\cdot\ln(n)}{\left(n-n^{\frac{1}{1+\epsilon}}+2\right)}\\
&= (1+\epsilon)\cdot\int|f|\,\mathrm{d}\mu_1.
\end{aligned}$$

Moreover, since for all integers $n > 1$ and for $j \in \{0, 1, 2, \dots, n\}$, we have that

$$\frac{\ln(n)}{\ln(n-j+2)} \geq 1,$$

it follows that,

$$\lim_{n\to+\infty}\sum_{j=0}^{n}\frac{\ln(n)}{\ln(n-j+2)}\cdot\int\left|\widetilde{f_j}\right|\,\mathrm{d}\mu_1 \geq \lim_{n\to+\infty}\sum_{j=0}^{n}\int\left|\widetilde{f_j}\right|\,\mathrm{d}\mu_1=\int|f|\,\mathrm{d}\mu_1.$$

Hence, (10.6) tends to zero.

(b) For $j \in \mathbb{N}_0$, the map $f_{1,1}\circ f_{1,0}^{j}$ is order reversing and an inductive argument can be used to show that

$$f_{1,1}\circ f_{1,0}^{j}(x)=\frac{1+j\cdot x}{1+(j+1)\cdot x}.$$

Using the fact that $Y_k \subseteq f_{1,0}^{k}\circ f_{1,1}((0,1])$, for $k\in\mathbb{N}$, and the representation of $\widehat{T_1}$ given in (6.5), an inductive argument yields, for all $j\in\mathbb{N}$, that

$$\widehat{T}_1^{j}(\widetilde{f_j})(x)=\left(\prod_{k=0}^{j-1}f_{1,1}\circ f_{1,0}^{k}(x)\right)\cdot\widetilde{f_j}\circ f_{1,0}^{j}(x),$$

and thus, that

$$\begin{aligned}
\left\|\mathbb{1}_Y\cdot\widehat{T}^{j}(\widetilde{f_j})\right\|_\infty &\leq \left(\prod_{k=0}^{j-1}\frac{1+\frac{k}{2}}{1+\frac{k+1}{2}}\right)\cdot\left\|\widetilde{f_j}\right\|_\infty\\
&\leq \frac{2}{j+2}\cdot\left\|\widetilde{f_j}\right\|_\infty\\
&\leq \frac{2}{j+2}\cdot\|f\|_\infty.
\end{aligned}\tag{10.9}$$

Since $\|\theta_n\|_\infty = \mathfrak{o}(1/\ln(n+2))$, given an $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $\|\theta_m\|_\infty \leq 2 \cdot \epsilon / \ln(m)$, for all $m \geq N_\epsilon$. Moreover, the value

$$\Theta := \sup\{\|\theta_n\|_\infty : n \in \mathbb{N}_0\}$$

is positive and finite. Combining these statements, we have the following inequality.

$$\begin{aligned}\ln(n) \cdot \sum_{j=0}^{n} \|\theta_{n-j}\|_\infty \cdot \|\widehat{T}_1^j(\widetilde{f_j})\|_\infty \leq 2 \cdot \epsilon \cdot \sum_{j=0}^{n-N_\epsilon} \frac{\ln(n)}{\ln(n-j)} \cdot \left\|\widehat{T}_1^j\left(\widetilde{f_j}\right)\right\|_\infty \\ + 2 \cdot \Theta \cdot \|f\|_\infty \cdot \ln(n) \cdot \sum_{j=n-N_\epsilon+1}^{n} \frac{1}{j}.\end{aligned}$$

Using (10.4) and (10.9) a similar argument to that given in (a) yields that

$$\lim_{n\to+\infty} 2 \cdot \epsilon \cdot \sum_{j=0}^{n-N_\epsilon} \frac{\ln(n)}{\ln(n-j)} \cdot \left\|\widehat{T}_1^j\left(\widetilde{f_j}\right)\right\|_\infty \leq 2 \cdot \epsilon \cdot (1+\epsilon) \cdot \sum_{k=0}^{+\infty} \left\|\widehat{T}_1^k\left(\widetilde{f_k}\right)\right\|_\infty.$$

Thus, for a given $\epsilon > 0$, we have that

$$\begin{aligned}&\lim_{n\to+\infty} \ln(n) \cdot \sum_{j=0}^{n} \left\|\theta_{n-j}\right\|_\infty \cdot \left\|\widehat{T}_1^j\left(\widetilde{f_j}\right)\right\|_\infty \\ &\leq 2 \cdot \epsilon \cdot (1+\epsilon) \cdot \sum_{j=0}^{+\infty} \left\|\widehat{T}_1^j\left(\widetilde{f_j}\right)\right\|_\infty + \lim_{n\to+\infty} 2 \cdot \Theta \cdot \|f\|_\infty \cdot \ln(n) \cdot \sum_{j=n-N_\epsilon+1}^{n} \frac{1}{j} \\ &\leq 2 \cdot \epsilon \cdot (1+\epsilon) \cdot \sum_{j=0}^{+\infty} \left\|\widehat{T}_1^j\left(\widetilde{f_j}\right)\right\|_\infty + 2 \cdot \Theta \cdot \|f\|_\infty \lim_{n\to+\infty} \ln(n) \cdot \ln\left(\frac{n}{n-N_\epsilon}\right).\end{aligned}$$

Finally, an application of L'Hôpital's rule yields that

$$\lim_{n\to\infty} \ln(n) \cdot \ln\left(\frac{n}{n-N_\epsilon}\right) = 0,$$

which in turn implies, that

$$\lim_{n\to+\infty} \ln(n) \cdot \sum_{j=0}^{n} \left\|\theta_{n-j}\right\|_\infty \cdot \left\|\widehat{T}_1^j\left(\widetilde{f_j}\right)\right\|_\infty \leq 2 \cdot \epsilon \cdot (1+\epsilon) \cdot \sum_{j=0}^{\infty} \left\|\widehat{T}_1^j\left(\widetilde{f_j}\right)\right\|_\infty.$$

Since ϵ was arbitrarily chosen, and taking (10.4) into account, we have that (10.7) tends to zero

(c) Since $f \in \mathcal{L}_1^1([0,1])$, using the definition of $\widetilde{f_j}$, we obtain that (10.8) converges to zero and the proof is complete.

□

Now we are in the position to give a proof of Theorem 8.2. We start with the case, where β is a rational number.

Proof of Theorem 8.2 for β rational. Let $a \in (0,1)$, and let $\beta \in (0,1]$ be a rational number. As before, let $v \in \mathfrak{U}_{\beta,a}$. As β is a rational number, there exists a minimal $n \in \mathbb{N}$ such that $T^n(\beta) = 0$, let n be fixed as such. Further, we have that $\omega_1(\beta) = \{0\}$. Since the case $\beta = 1$ is a simplification of the matter, we can, without loss of generality, assume that $\beta \neq 1$. By the definition of the Farey map, there exist exactly two finite words $\eta, \eta' \in \Sigma^n$ such that

(a) $f_{1,\eta}(0) = \beta = f_{1,\eta'}(0)$,

(b) $f_{1,\eta}(x) < \beta < f_{1,\eta'}(x)$, for all $x \in (0,1]$, and

(c) $f_{1,\xi}(x) \neq \beta$, for all words $\xi \in \Sigma^n \setminus \{\eta, \eta'\}$ and all $x \in [0,1]$.

The pointwise version of the Perron-Frobenius operator yields, for $k \in \mathbb{N}$, that

$$\mathcal{P}_1^k(v)(x) = \sum_{\xi \in \Sigma^k} \left| f'_{1,\xi} \right| \cdot v \circ f_{1,\xi}.$$

Hence, by linearity of the operator $\mathcal{P}_1$, we have, for all natural numbers $k > n$, that

$$\begin{aligned}
\mathcal{P}_1^k(v) &= \mathcal{P}_1^{k-n}\left(\mathcal{P}_1^n(v)\right) \\
&= \mathcal{P}_1^{k-n}\left(\mathcal{P}_1^n\left(v \cdot \mathbb{1}_{[0,1]\setminus[\eta]\cap[0,1]\setminus[\eta']}\right)\right) + \mathcal{P}_1^{k-n}\left(\mathcal{P}_1^n\left(v \cdot \mathbb{1}_{[\eta]\cup[\eta']}\right)\right) \\
&= \mathcal{P}_1^{k-n}\left(\sum_{\xi \in \Sigma^k \setminus \{\eta,\eta'\}} \left| f'_{1,\xi} \right| \cdot v \circ f_{1,\xi}\right) + \mathcal{P}_1^{k-n}\left(\mathcal{P}_1^n\left(v \cdot \mathbb{1}_{[\eta]\cup[\eta']}\right)\right).
\end{aligned}$$

If $\xi \in \{0,1\}^{n-1} \setminus \{\eta, \eta'\}$, then since $\beta \notin f_{1,\xi}([0,1])$, since the functions $f_{1,\xi}$, $f'_{1,\xi}$, $1/h_1$ are all of bounded variation, since $v \in \mathfrak{U}_{\beta,a}$ and since $[\xi]$ is a compact interval bounded away from β, by Proposition 6.16, it follows that for $x \in [0,1]$, the function

$$x \mapsto \frac{1}{h_1(x)} \sum_{\xi \in \Sigma^k \setminus \{\eta,\eta'\}} \left| f'_{1,\xi}(x) \right| \cdot v \circ f_{1,\xi}(x)$$

is of bounded variation. Hence, by Proposition 6.18 and Theorems 10.6 and 6.12 together with Lemma 10.7, we have that

$$\begin{aligned}
\lim_{k\to\infty} \ln(k) \cdot \mathcal{P}_1^k\left(v \cdot \mathbb{1}_{[0,1]\setminus[\eta]\cap[0,1]\setminus[\eta']}\right) &= \int \mathcal{P}_1^n\left(v \cdot \mathbb{1}_{[0,1]\setminus[\eta]\cap[0,1]\setminus[\eta']}\right) \mathrm{d}\lambda \cdot h_1 \\
&= \int v \cdot \mathbb{1}_{[0,1]\setminus[\eta]\cap[0,1]\setminus[\eta']} \mathrm{d}\lambda \cdot h_1.
\end{aligned}$$

Therefore, to complete the proof we need to show that

$$\lim_{k\to+\infty} \ln(k) \cdot \mathcal{P}_1^k\left(v \cdot \mathbb{1}_{[\eta]\cup[\eta']}\right) = \int v \cdot \mathbb{1}_{[\eta]\cup[\eta']} \mathrm{d}\lambda \cdot h_1.$$

To this end let $m > n$ be a fixed natural number which satisfies for all $\xi \in \Sigma^m$ that

$$\lambda([\xi]) \leq \min\{|a-\beta|, |b-\beta|,\}$$

where $U = (a, b)$ is the open connected set such that there exist constants C_1, C_2, so that $C_1 \cdot v_{\beta,a} \leq v \leq C_2 \cdot v_{\beta,a}$ on U, as in condition (c) in the description of $\mathfrak{U}_{a,\beta}$. Let $\nu, \nu' \in \Sigma^m$ be the unique words satisfying

$$[\nu] \cap [\nu'] = \{\beta\}, \quad [\nu] \subset [\eta] \quad \text{and} \quad [\nu'] \subset [\eta'] .$$

Indeed, we necessarily have that $f_{1,\nu}(0) = \beta = f_{1,\nu'}(0)$. Using identical arguments to those above, we can conclude that

$$\lim_{k\to+\infty} \ln(k) \cdot \mathcal{P}_1^k \left(v \cdot \mathbb{1}_{[\eta]\setminus[\nu]\cup[\eta']\setminus[\nu']}\right) = \int v \cdot \mathbb{1}_{[\eta]\setminus[\nu]\cup[\eta']\setminus[\nu']} \,\mathrm{d}\lambda \cdot h_1 .$$

Moreover, by positivity of the operator $\mathcal{P}_1$ we have that

$$C_1 \cdot \mathcal{P}_1^k \left(v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']}\right) \leq \mathcal{P}_1^k \left(v \cdot \mathbb{1}_{[\nu]\cup[\nu']}\right) \leq C_2 \cdot \mathcal{P}_1^k \left(v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']}\right) .$$

We claim, and will shortly prove, that

$$\lim_{k\to+\infty} \ln(k) \cdot \mathcal{P}_1^k \left(v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']}\right) = \int v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']} \,\mathrm{d}\lambda \cdot h_1 . \tag{10.10}$$

Assuming this, we conclude, for all $m \in \mathbb{N}$, that

$$\begin{aligned} \liminf_{k\to+\infty} \ln(k) \cdot \mathcal{P}_1^k(v) \geq C_1 \cdot & \int v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']} \,\mathrm{d}\lambda \cdot h_1 \\ & + \int v \cdot \mathbb{1}_{[0,1]\setminus[\nu]\cap[0,1]\setminus[\nu']} \,\mathrm{d}\lambda \cdot h_1 \end{aligned} \tag{10.11}$$

and

$$\begin{aligned} \limsup_{k\to+\infty} \ln(k) \cdot \mathcal{P}_1^k(v) \leq C_2 \cdot & \int v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']} \,\mathrm{d}\lambda \cdot h_1 \\ & + \int v \cdot \mathbb{1}_{[0,1]\setminus[\nu]\cap[0,1]\setminus[\nu']} \,\mathrm{d}\lambda \cdot h_1 . \end{aligned} \tag{10.12}$$

The words ν, ν' are dependent on m. Since the left hand side of (10.11) and (10.12) are independent of m and since $\lambda(\nu), \lambda(\nu')$ both converge to zero as n tends to infinity in combination with the fact that $v_{\beta,a}$ is improper Riemann integrable, the statement follows. All that is left to show is the euqality given in (10.10).
By Proposition 6.18, Theorem 10.6 and Theorem 6.12 together with Lemma 10.7 it is sufficient to show that, for $x \in [0, 1]$, the function

$$x \mapsto \widehat{T}_1^m \left(\frac{v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']}}{h_1}\right)(x)$$

is of bounded variation. In order to show this, recall that $f_{1,\nu}$ and $f_{1,\nu'}$ are Möbius transformations and can, for $i \in \{1,2\}$, with $a_i, b_i, c_i, d_i \in \mathbb{Z}$, be written as

$$f_\nu(x) = \frac{a_1 \cdot x + b_1}{c_1 \cdot x + d_1} \quad \text{and} \quad f_{\nu'}(x) = \frac{a_2 \cdot x + b_2}{c_2 \cdot x + d_2}.$$

We observe that

$$\widehat{T}_1^m\left(\frac{v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']}}{h_1}\right)(x) = \sum_{i=1}^{2} \frac{x}{(c_i \cdot x + d_i)^2} \cdot \left(\frac{(-1)^{i+1}}{\beta - \frac{a_i \cdot x + b_i}{c_i \cdot x + d_i}}\right)^a.$$

The desired conclusion, namely that $\widehat{T}_1^m\left(v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']}/h_1\right)$ is of bounded variation follows from the following four observations.

1. For all $t \in (0,1]$, we have that

$$V_{[t,1]}\left(\widehat{T}_1^m\left(\frac{v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']}}{h_1}\right)\right) < +\infty.$$

2. For $i \in \{1,2\}$, by L'Hôpital's rule we have that

$$\lim_{x\to 0} \frac{(-1)^{i+1} \cdot x}{\beta - \frac{a_i \cdot x + b_i}{c_i \cdot x + d_i}} = d_i^2.$$

3. By L'Hôpital's rule, we have that

$$\lim_{x\to 0} \widehat{T}_1^m\left(\frac{v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']}}{h_1}\right)(x) = \sum_{i=1}^{2} \lim_{x\to 0} \frac{x}{(c_i \cdot x + d_i)^2} \cdot \left(\frac{(-1)^{i+1}}{\beta - \frac{a_i \cdot x + b_i}{c_i \cdot x + d_i}}\right)^a = 0.$$

4. We have that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{dx}}\left(\widehat{T}_1^m\left(\frac{v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']}}{h_1}\right)(x)\right) \\
&= \sum_{i=1}^{2} \frac{\mathrm{d}}{\mathrm{dx}}\left(\frac{x}{(c_i \cdot x + d_i)^2} \cdot \left(\frac{(-1)^{i+1}}{\beta - \frac{a_i \cdot x + b_i}{c_i \cdot x + d_i}}\right)^a\right) \\
&= \sum_{i=1}^{2} \frac{-c_i \cdot x + d_i}{(c_i \cdot x + d_i)^3} \cdot \left(\frac{(-1)^{i+1}}{\beta - \frac{a_i \cdot x + b_i}{c_i \cdot x + d_i}}\right)^a - \frac{(-1)^{i+1} \cdot a \cdot x}{(c_i \cdot x + d_i)^4} \cdot \left(\frac{(-1)^{i+1}}{\beta - \frac{a_i \cdot x + b_i}{c_i \cdot x + d_i}}\right)^{a+1},
\end{aligned}$$

   which is non-negative on an open neighbourhood of zero.

Hence, $\widehat{T}_1^m\left(v_{\beta,a} \cdot \mathbb{1}_{[\nu]\cup[\nu']}/h_1\right)$ is of bounded variation and the statement of the theorem follows.

□

To finish the proof of Theorem 8.2 we have to consider the case when β is an irrational number of intermediate a-type

Proof of Theorem 8.2 for β irrational of intermediate a-type. As before, we observe that, by the linearity of the Perron-Frobenius operator, we have that

$$\ln(n) \cdot \mathcal{P}_1^n(v) = \ln(n) \cdot \mathcal{P}_1^n(v \cdot \mathbb{1}_{[0,1]\setminus[\vartheta_1(\beta)|_n]}) + \ln(n) \cdot \mathcal{P}_1^n(v \cdot \mathbb{1}_{[\vartheta_1(\beta)|_n]}).$$

Since $h_1 \cdot \widehat{T}_1(f) = \mathcal{P}_1(f \cdot h_1)$, it is sufficient to show two statements. The first statement is

$$\lim_{n\to\infty} \ln(n) \cdot \widehat{T}_1^n \left(\frac{v \cdot \mathbb{1}_{[0,1]\setminus[\vartheta_1(\beta)|_n]}}{h_1} \right) = \int v \, \mathrm{d}\lambda$$

uniformly on compact subsets of $(0, 1]$. This is done in Lemma 10.9. For the second statement, we have to introduce, similar to the case $r \in [0, 1)$, see Equation (10.3), the tail of an observable.

Definition 10.8 (**1-tail of the observable**, [KKS16, Definition 4.1])**.** For $a \in (0, 1)$, $\beta \in [0, 1]$ and $n \in \mathbb{N}$, we define the 1*-tail of the observable* $v_{\beta,a}\colon x \mapsto |x - \beta|^{-a}$ by

$$v^{n,1} := v_{\beta,a,n,1} := \mathcal{P}_r^n(v_{\beta,a} \cdot \mathbb{1}_{[W_{r,n}(\beta)]}) = \left| f'_{1,\vartheta_1(\beta)|_n}(x) \right| \cdot v_{\beta,a} \circ f_{1,\vartheta_1(\beta)|_n}. \tag{10.13}$$

Further, for $\eta > 0$, set

$$A_{n,1,\eta} := \left\{ x \in [0, 1]\colon \ln(n) \cdot v^{n,1}(x) > \eta \right\}.$$

Further, we observe that since $v \in \mathfrak{U}_{\beta,a}$ is non-negative and $\mathcal{P}_1$ is a positive linear operator, that there exists a positive constant C with

$$\begin{aligned} 0 &\le \lim_{n\to\infty} \ln(n) \cdot \mathcal{P}_1^n \left(v \cdot \mathbb{1}_{[\vartheta_1(\beta)|_n]} \right) \\ &\le \lim_{n\to\infty} \ln(n) \cdot C \cdot \mathcal{P}_r^n \left(v \cdot \mathbb{1}_{[\vartheta_1(\beta)|_n]} \right) \\ &= \lim_{n\to\infty} \ln(n) \cdot C \cdot v^{n,1}. \end{aligned}$$

Hence, the second part to show is, similar to the case $r \in [0, 1)$, that the last limit is equal to zero outside a set of Hausdorff dimension zero. This is done in Lemma 10.10.

Lemma 10.9 ([KKS16, Lemma 4.12])**.** *For $a \in (0, 1)$, $\beta \in (0, 1]$ of intermediate a-type and $v \in \mathfrak{U}_{\beta,a}$, we have that*

$$\lim_{n\to\infty} \ln(n) \cdot \widehat{T}_1^n \left(\frac{v \cdot \mathbb{1}_{[0,1]\setminus[\vartheta_1(\beta)|_n]}}{h_1} \right) = \int v \, \mathrm{d}\lambda$$

uniformly on compact subsets of $(0, 1]$.

Proof of Lemma 10.9. Recall the definition of $k(n)$, $m(n)$ and $r(n)$ given in (5.8). Let K denote a compact subset of $(0,1]$ and let $c, b \in (0,1]$ be such that $K \subseteq [c,b]$. Let $N \in \mathbb{N}$ be fixed. By Proposition 6.18, Theorem 10.6 and Theorem 6.12 together with Lemma 10.7, since the function $v \cdot \mathbb{1}_{[0,1]\setminus[\vartheta_1(\beta)|_N]}$ is of bounded variation, it follows that

$$\lim_{n\to\infty} \ln(n) \cdot \widehat{T}_1^n\left(\frac{v \cdot \mathbb{1}_{[0,1]\setminus[\vartheta_1(\beta)|_N]}}{h_1}\right) = \int v \cdot \mathbb{1}_{[0,1]\setminus[\vartheta_1(\beta)|_N]} \, \mathrm{d}\lambda \text{ uniformly on K.}$$

Let $t_{n,j}$ be as in Definition 7.3, page 64 and let $\widetilde{N}$ denote the unique integer such that $a_1 + a_2 + ... a_{\widetilde{N}} \le N < a_1 + a_2 + ... a_{\widetilde{N}+1}$.
Then, by linearity and positivity of the transfer operator $\widehat{T}_1$, since

$$\lim_{k\to+\infty} \lambda([\vartheta_1(\beta)|_k]) = 0,$$

since the observable v is Lebesgue integrable and since β is of intermediate a-type, it suffices to show that there exists a positive constant C such that

$$\lim_{n\to+\infty} \ln(n) \cdot \widehat{T}_1^n\left(\frac{v_{\beta,a} \cdot \mathbb{1}_{[\vartheta_1(\beta)|_N]\setminus[\vartheta_1(\beta)|_n]}}{h_1}\right) \le C \cdot \sum_{k=\widetilde{N}}^{+\infty} \sum_{j=1}^{a_k} t_{k,j}^{-2\cdot(1-a)+\epsilon},$$

for an $\epsilon \in (0, 2\cdot(1-a))$.
To this end, for each integer $k > 1$, let $\overline{\vartheta_1(\beta)|_k} \in \Sigma^k$ denote the unique word of length k such that $[\vartheta_1(\beta)|_{k-1}] = [\vartheta_1(\beta)|_k] \cup [\overline{\vartheta_1(\beta)|_k}]$. By Lemma 5.3 we have for all $x \in K$, that

1. $\left|f'_{\overline{\vartheta_1(\beta)|_k}}(x)\right| \le \dfrac{1}{(c \cdot ((r(k)+1)\cdot q_{m(k)} + q_{m(k)-1})^2},$

2. if $r(k)+1 \ne a_{m(k)}$, then

$$\begin{aligned}
&\left|\beta - f_{\overline{\vartheta_1(\beta)|_k}}(x)\right| \\
&\ge \left|\frac{(r(k)+2)\cdot p_{m(k)} + p_{m(k)-1}}{(r(k)+2)\cdot q_{m(k)} + q_{m(k)-1}} - \frac{(r(k)+1)\cdot p_{m(k)} + p_{m(k)-1}}{(r(k)+1)\cdot q_{m(k)} + q_{m(k)-1}}\right| \\
&\ge \frac{1}{2\cdot((r(k)+1)\cdot q_{m(k)} + q_{m(k)-1})^2},
\end{aligned}$$

3. if $r(k)+1 = a_{m(k)}$, letting

$$z_k = \begin{cases} b & \text{if } m(k) \text{ is even,} \\ c & \text{if } m(k) \text{ is odd,} \end{cases}$$

then

$$\begin{aligned}
&\left|\beta - f_{\overline{\vartheta_1(\beta)|_k}}(x)\right| \\
&\geq \left|\frac{(r(k)+1)\cdot p_{m(k)} + p_{m(k)-1}}{(r(k)+1)\cdot q_{m(k)} + q_{m(k)-1}} - \frac{\left(r(k)\cdot p_{m(k)} + p_{m(k)-1}\right)\cdot x + p_{m(k)}}{\left(r(k)\cdot q_{m(k)} + q_{m(k)-1}\right)\cdot x + q_{m(k)}}\right| \\
&\geq \frac{1 - z_k}{\left((r(k)+1)\cdot q_{m(k)} + q_{m(k)-1}\right)^2}.
\end{aligned}$$

Since constant functions are of bounded variation, we have by Proposition 6.18, Theorem 10.6 and Proposition 6.12 together with Lemma 10.7, that there exists a positive constant C', so that for all $k \in \mathbb{N}$ and $x \in K$

$$\widehat{T}_1^k\left(\frac{1}{h_1}\right)(x) \leq \frac{C'}{\ln(k+1)}.$$

Noting that $t_{m(k),r(k)+1} = (r(k)+1)\cdot q_{m(k)} + q_{m(k)-1}$ and, letting ϵ be such that

$$\sum_{n=1}^{+\infty}\sum_{k=1}^{a_n} t_{n,j}^{-2\cdot(1-a)+\epsilon} < +\infty,$$

we have that

$$\begin{aligned}
&\lim_{n\to+\infty} \ln(n)\cdot \widehat{T}_1^n\left(\frac{v_{\beta,a}\cdot \mathbb{1}_{[\vartheta_1(\beta)|_N]\setminus[\vartheta_1(\beta)|_n]}}{h_1}\right) \\
&= \lim_{n\to+\infty} \ln(n) \sum_{k=N+1}^{n-1} \widehat{T}_1^{n-k}\left(\widehat{T}_1^k\left(\frac{v_{\beta,a}\cdot \mathbb{1}_{[\overline{\vartheta_1(\beta)|_k}]}}{h_1}\right)\right) \\
&= \lim_{n\to+\infty} \ln(n) \sum_{k=N+1}^{n-1} \widehat{T}_1^{n-k}\left(\left|f'_{\overline{\vartheta_1(\beta)|_k}}\right| \cdot \frac{1}{\left|\beta - f_{\overline{\vartheta_1(\beta)|_k}}\right|^a}\cdot\frac{1}{h_1}\right) \\
&\leq \lim_{n\to+\infty} \frac{C'}{2\cdot a^2\cdot(1-z_k)} \sum_{k=N+1}^{n-1} \frac{\ln(n)}{\ln(n-k+1)}\cdot\frac{1}{\left((r(k)+1)\cdot q_{m(k)} + q_{m(k)-1}\right)^{2\cdot(1-a)}} \\
&\leq \lim_{n\to+\infty} \frac{C'}{2\cdot a^2\cdot(1-z_k)} \sum_{k=N+1}^{\lfloor n/2\rfloor} \frac{\ln(n)}{\ln(n/2)}\cdot\frac{1}{\left((r(k)+1)\cdot q_{m(k)} + q_{m(k)-1}\right)^{2\cdot(1-a)-\epsilon}} \\
&\quad + \lim_{n\to+\infty} \frac{C'}{2\cdot a^2\cdot(1-z_k)} \sum_{k=\lfloor n/2\rfloor+1}^{n-1} \frac{2\cdot\ln(n)}{n^\epsilon}\cdot\frac{1}{\left((r(k)+1)\cdot q_{m(k)} + q_{m(k)-1}\right)^{2\cdot(1-a)-\epsilon}} \\
&\leq \frac{C'}{a^2\cdot(1-z_k)} \sum_{k=N+1}^{+\infty} \frac{1}{\left((r(k)+1)\cdot q_{m(k)} + q_{m(k)-1}\right)^{2\cdot(1-a)-\epsilon}} \\
&\leq \frac{C'}{a^2\cdot(1-z_k)} \sum_{k=\widetilde{N}}^{+\infty}\sum_{j=1}^{a_k} t_{k,j}^{-2\cdot(a-1)+\epsilon}.
\end{aligned}$$

This completes the proof of Lemma 10.9.

□

The aim of the next lemma is to provide an analogous result for $r = 1$ of Lemma 10.5. The idea behind the proofs of Lemmata 10.5 and 10.10 are similar, however, in the case that $r = 1$, several technical difficulties arise and thus need to be taken care of.

Lemma 10.10 ([KKS16, Lemma 4.13])**.** *For $a \in (0,1)$, $\beta \in [0,1]$ irrational and of intermediate a-type, $n \in \mathbb{N}$ and $\eta > 0$, we have that*

$$\dim_{\mathcal{H}}\left(\limsup_{n\to+\infty} A_{n,1,\eta}\right) = 0.$$

Proof of Lemma 10.10. It is sufficient to prove, for all $k \in \mathbb{N}$, $\eta > 0$ and $\epsilon \in (0,(2\cdot k\cdot(k+1))^{-1})$, that

$$\dim_{\mathcal{H}}\left(\limsup_{n\to+\infty} A_{n,1,\eta} \cap \left(\frac{1}{k+1}+\epsilon, \frac{1}{k}-\epsilon\right)\right) = 0.$$

To this end, for $n \in \mathbb{N}$, observe that $T_1^n(\beta)$ is the unique real number in $[0,1]$ such that $f_{1,\vartheta_1(\beta)|_n}\left(T_1^n(\beta)\right) = \beta$. In the sequel we distinguish between the two cases

$$T_1^n(\beta) \in \left(\frac{1}{k}, \frac{1}{k+1}\right) \quad \text{and} \quad T_1^n(\beta) \notin \left(\frac{1}{k}, \frac{1}{k+1}\right).$$

Since β is irrational, the case that there exists a $k \in \mathbb{N}$, such that $T_1^n(\beta) = 1/k$ can not occur.

If $T_1^n(\beta) \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$, then, for all $x \in (1/(k+1)+\epsilon, 1/k-\epsilon)$, by the mean value theorem and Lemma 5.3, there exists $u \in \left(\frac{1}{k+1}, \frac{1}{k}\right)$ such that

$$\begin{aligned}
\left|\beta - f_{1,\vartheta_1(\beta)|_n}(x)\right| &= \left|f_{1,\vartheta_1(\beta)|_n}\left(T_1^n(\beta)\right) - f_{1,\vartheta_1(\beta)|_n}(x)\right| \\
&= \left|x - T_1^n(\beta)\right| \cdot \left|f'_{1,\vartheta_1(\beta)|_n}(u)\right| \\
&= \frac{\left|x - T_1^n(\beta)\right|}{\left|(r(n)u+1)\cdot q_{m(n)} + q_{m(n)-1}\cdot u\right|^2} \\
&\geq \frac{k^2}{\left|x - T_1^n(\beta)\right| \cdot \left|(r(n)+k)\cdot q_{m(n)} + q_{m(n)-1}\right|^2}.
\end{aligned}$$

If $T_1^n(\beta) \notin \left(\frac{1}{k+1}, \frac{1}{k}\right)$, then, for all $x \in \left(\frac{1}{k+1}+\epsilon, \frac{1}{k}-\epsilon\right)$, since $f_{1,\vartheta_1(\beta)|_n}$ is order preserving or order reversing, we have that

$$\begin{aligned}
&\left|\beta - f_{1,\vartheta_1(\beta)|_n}(x)\right| = \left|f_{1,\vartheta_1(\beta)|_n}\left(T_1^n(\beta)\right) - f_{1,\vartheta_1(\beta)|_n}(x)\right| \\
&\quad\geq \min\left\{\left|f_{1,\vartheta_1(\beta)|_n}\left(\frac{1}{k}\right) - f_{1,\vartheta_1(\beta)|_n}(x)\right|, \left|f_{1,\vartheta_1(\beta)|_n}\left(\frac{1}{k+1}\right) - f_{1,\vartheta_1(\beta)|_n}(x)\right|\right\}
\end{aligned}$$

and so by the mean value theorem and Lemma 5.3, there exists $u \in (1/(k+1), 1/k)$ such that

$$\left|\beta - f_{1,\vartheta_1(\beta)|_n}(x)\right| \geq \epsilon \cdot \left|f'_{1,\vartheta_1(\beta)|_n}(u)\right| = \frac{\epsilon}{\left|(r(n) \cdot u + 1) \cdot q_{m(n)} + q_{m(n)-1} \cdot u\right|^2} \geq \frac{\epsilon \cdot k^2}{\left|(r(n) + k) \cdot q_{m(n)} + q_{m(n)-1}\right|^2}.$$

Hence, for $x \in (1(k+1) + \epsilon, 1/k - \epsilon)$, we have that

$$\ln(n) \cdot v^{n,1}(x) = \frac{\ln(n)}{((r(n) \cdot x + 1) \cdot q_{m(n)} + q_{m(n)-1} \cdot x)^2} \cdot \frac{1}{\left|\beta - f_{1,\vartheta_1(\beta)|_n}(x)\right|^a}$$

$$\leq \begin{cases} \dfrac{(k+1)^2 \cdot \ln(n)}{\left|T_1^n(\beta) - x\right|^a \cdot k^{2 \cdot a} \cdot ((r(n) + k) \cdot q_{m(n)} + q_{m(n)-1})^{2 \cdot (1-a)}} & \text{if } T_1^n(\beta) \in \left(\frac{1}{k+1}, \frac{1}{k}\right), \\[2ex] \dfrac{(k+1)^2 \cdot \ln(n)}{\epsilon^a \cdot k^{2 \cdot a} \cdot ((r(n) + k) \cdot q_{m(n)} + q_{m(n)-1})^{2 \cdot (1-a)}} & \text{if } T_1^n(\beta) \notin \left(\frac{1}{k+1}, \frac{1}{k}\right). \end{cases}$$

Since,

$$\lim_{n \to +\infty} \frac{(k+1)^2 \cdot \ln(n)}{\epsilon^a \cdot k^{2 \cdot a} \cdot ((r(n) + k) \cdot q_{m(n)} + q_{m(n)-1})^{2 \cdot (1-a)}} \leq \lim_{n \to +\infty} \frac{(k+1)^2 \cdot \ln((r(n) + k) \cdot q_{m(n)} + q_{m(n)-1})}{\epsilon^a \cdot k^{2 \cdot a} \cdot ((r(n) + k) \cdot q_{m(n)} + q_{m(n)-1})^{2 \cdot (1-a)}} = 0,$$

there exists $M \in \mathbb{N}$ such that, for all $x \in (1/(k+1) + \epsilon, 1/k - \epsilon)$ and $n \geq M$, if we have $T_1^n(\beta) \notin (1/(k+1), 1/k)$, then we have that $\ln(n) \cdot v^{n,1}(x) < \eta$. Therefore, for all $n \geq M$, we have that

$$A_{n,1,\eta} \cap \left(\frac{1}{k+1} + \epsilon, \frac{1}{k} - \epsilon\right) = \emptyset;$$

otherwise, if $T_1^n(\beta) \in (1/(k+1), 1/k)$, then

$$\begin{aligned} &A_{n,1,\eta} \cap \left(\frac{1}{k+1} + \epsilon, \frac{1}{k} - \epsilon\right) \\ &= \left\{x \in \left(\frac{1}{k+1} + \epsilon, \frac{1}{k} - \epsilon\right) : \ln(n) \cdot v^{n,1}(x) \geq \eta\right\} \\ &\subseteq \left\{x \in \left(\frac{1}{k+1} + \epsilon, \frac{1}{k} - \epsilon\right) : \right. \\ &\qquad \left. \frac{(k+1)^2 \cdot \ln(n)}{\left|T_1^n(\beta) - x\right|^a \cdot k^{2 \cdot a} \cdot ((r(n) + k) \cdot q_{m(n)} + q_{m(n)-1})^{2 \cdot (1-a)}} \geq \eta\right\} \\ &\subseteq B\left(T_1^n(\beta), \frac{(k+1)^{2/a} \cdot \ln(n)^{1/a}}{\eta^{1/a} \cdot k^2 \cdot ((r(n) + k) \cdot q_{m(n)} + q_{m(n)-1})^{2 \cdot (1/a-1)}}\right) \cap \left(\frac{1}{k+1} + \epsilon, \frac{1}{k} - \epsilon\right). \end{aligned}$$

Hence, given $\delta > 0$, there exists a natural number $K = K(\delta) \geq M$ such that

$$\left\{ B\left(T_1^n(\beta), \frac{(k+1)^{2/a} \cdot \ln(n)^{1/a}}{\eta^{1/a} \cdot k^2 \cdot ((r(n)+k) \cdot q_{m(n)} + q_{m(n)-1})^{2\cdot(1/a-1)}} \right) : \\ n \geq K \text{ and } \exists\, l \in \mathbb{N} \text{ so that } n = -k + \sum_{i=1}^{l} a_i \right\}$$

is an open δ-cover of $\limsup_{n\to+\infty} A_{n,1,\eta} \cap (1/(k+1)+\epsilon, 1/k-\epsilon)$. Therefore, for $s > 0$ and $\delta > 0$, we have that

$$\begin{aligned}
&\mathcal{H}_\delta^s\left(\limsup_{n\to+\infty} A_{\eta,n} \cap \left(\frac{1}{k+1} + \epsilon, \frac{1}{k} - \epsilon \right) \right) \\
&\leq \sum_{n=M}^{+\infty} \lambda \left(B\left(T_1^n(\beta), \frac{2^{2\cdot(1/a-1)} \cdot (k+1)^{2/a} \cdot \ln(n)^{1/a}}{\eta^{1/a} \cdot k^2 \cdot ((r(n)+k+1) \cdot q_{m(n)} + q_{m(n)-1})^{2\cdot(1/a-1)}} \right) \right. \\
&\qquad\qquad \left. \cap \left(\frac{1}{k+1} + \epsilon, \frac{1}{k} - \epsilon \right) \right)^s \\
&\leq \frac{2^{s+2\cdot(1/a-1)} \cdot (k+1)^{2\cdot s/a}}{\eta^{s/a} \cdot k^{2\cdot s}} \sum_{m=m(K)}^{+\infty} \frac{\ln\left(\sum_{\ell=1}^{m+1} a_\ell \right)^{s/a}}{(q_{m+1})^{2\cdot s\cdot(1/a-1)}} \\
&\leq \frac{2^{s+2\cdot(1/a-1)} \cdot (k+1)^{2\cdot s/a}}{\eta^{s/a} \cdot k^{2\cdot s}} \sum_{m=m(K)}^{+\infty} \frac{\ln(q_{m+1})^{s/a}}{(q_{m+1})^{2\cdot s\cdot(1/a-1)}}.
\end{aligned}$$

In the above we have used that if $y \in [1/(\ell+2), 1/(\ell+1)]$, for an $\ell \in \mathbb{N}$, then $T_1(y) \in [1/(\ell+1), 1/\ell]$.
The last infinite sum is finite for all $s > 0$ and $\delta > 0$ since, by the recursive definition of q_n, we have that q_n grows at least at an exponential rate as n tends to infinity. Thus, $\mathcal{H}^s(\limsup_{n\to+\infty} A_{n,1,\eta})$ is finite for all $s > 0$. This yields that $\dim_{\mathcal{H}}(\limsup_{n\to+\infty} A_{n,1,\eta}) = 0$ as required.

□

Thus, all that remains to show is that if $\beta \in (0,1]$ is irrational, pre-periodic with respect to T_1 and has period length strictly greater than one, then on $\omega_1(\beta)$ we have that

$$\liminf_{n\to+\infty} \ln(n) \cdot \mathcal{P}_1^n(v) = \int v \, d\lambda \cdot h_1 \quad \text{and} \quad \limsup_{n\to+\infty} \ln(n) \cdot \mathcal{P}_1^n(v) = +\infty;$$

and in the case that $\beta \in (0,1]$ is pre-periodic with respect to T_1 and has period length equal to one then on the singleton $\omega_1(\beta)$ we have that the limit in (8.1) is equal to $+\infty$.
By positivity and linearity of $\mathcal{P}_1^n$ and Lemma 10.9, it suffices to show, if $\beta \in (0,1]$ is irrational, pre-periodic with respect to T_1 and has period length strictly greater than one, then on $\omega_1(\beta)$,

$$\liminf_{n\to+\infty} \ln(n) \cdot v^{n,1} = 0 \quad \text{and} \quad \limsup_{n\to+\infty} \ln(n) \cdot v^{n,1} = +\infty;$$

and in the case that $\beta \in (0,1]$ is pre-periodic with respect to T_1 and has period length equal to one, then on the singleton $\omega_1(\beta)$,

$$\lim_{n\to+\infty} \ln(n) \cdot v^{n,1} = +\infty.$$

Indeed if β is pre-periodic with respect to T_1 and has period length $l \geq 1$, then letting $n \in \mathbb{N}_0$, be the minimal integer so that $T_1^{n+k}(\beta) = T_1^{n+k+l}(\beta)$, for all $k \in \mathbb{N}_0$, we have that

$$f_{1,(\vartheta_{1,n+j+1}(\beta),\dots,\vartheta_{1,n+j+l}(\beta))}\left(T_1^{n+j}(\beta)\right) = T_1^{n+j}(\beta),$$

for all $j \in \{0,1,\dots,l-1\}$. Further, $\omega_1(\beta) = \left\{T_1^n(\beta),\dots,T_1^{n+l-1}(\beta)\right\}$, and hence, for $j \in \{0,1,\dots,l-1\}$, it follows, for all $k \in \mathbb{N}_0$, that

$$v^{n+j+k\cdot l,1}\left(T_1^{n+j}(\beta)\right) = +\infty.$$

To complete the proof we will show, for $l > 1$ and $i,j \in \{0,1,\dots,l-1\}$ with $i \neq j$, that

$$\lim_{k\to+\infty} v^{n+j+k\cdot l,1}\left(T_1^{n+i}(\beta)\right) = 0.$$

To this end set $L := \min\left\{\left|T_1^{n+j}(\beta) - T_1^{n+i}(\beta)\right| : i,j \in \{0,1,\dots,l-1\} \text{ and } i \neq j\right\}$ and set

$$\begin{aligned} c &:= \min\left\{T_1^{n+j}(\beta) \colon j \in \{0,1,\dots,l-1\}\right\} \\ \text{and} \quad b &:= \max\left\{T_1^{n+j}(\beta) \colon j \in \{0,1,\dots,l-1\}\right\}. \end{aligned}$$

Since β is irrational and pre-periodic with period $m > 1$, it follows that $0 < c < b < 1$ and therefore, we have for all $i,j \in \{0,1,\dots,l-1\}$, with $i \neq j$ and $k \in \mathbb{N}$, that

$$\left|f'_{1,\vartheta_1(\beta)|_{n+j+k\cdot l}}\left(T_1^{n+i}(\beta)\right)\right| \leq \frac{1}{c^2 \cdot \left((r(n+j+k\cdot l)+1)\cdot q_{m(n+j+k\cdot l)} + q_{m(n+j+k\cdot l)-1}\right)^2}.$$

Further, we have, for all $i,j \in \{0,1,\dots,l-1\}$ with $i \neq j$ and $k \in \mathbb{N}$, that

$$\begin{aligned} \left|\beta - f_{1,\vartheta_1(\beta)|_{n+j+k\cdot l}}\left(T_1^{n+i}(\beta)\right)\right| &\geq \left|f_{1,\vartheta_1(\beta)|_{n+j+k\cdot l}}\left(T_1^{n+j+k\cdot l}(\beta)\right) - f_{1,\vartheta_1(\beta)|_{n+j+k\cdot l}}\left(T_1^{n+i}(\beta)\right)\right| \\ &\geq \inf_{u\in[c,b]} \left|f'_{1,\vartheta_1(\beta)|_{n+j+k\cdot l}}(u)\right| \cdot \left|T_1^{n+j}(\beta) - T_1^{n+i}(\beta)\right| \\ &\geq \frac{L}{\left((r(n+j+k\cdot l)+1)\cdot q_{m(n+j+k\cdot l)} + q_{m(n+j+k\cdot l)-1}\right)^2}. \end{aligned}$$

Hence, for all $i, j \in \{0, 1, \dots, l-1\}$ with $i \neq j$, we have

$$\begin{aligned}
0 &\leq \lim_{l\to+\infty} v^{n+j+l\cdot m,1}\left(T_1^{n+i}(\beta)\right)\\
&\leq \lim_{l\to+\infty} \frac{\left|f'_{1,\vartheta_1(\beta)|_{n+j+l\cdot m}}\left(T_1^{n+i}(\beta)\right)\right|}{\left|\beta - f_{1,\vartheta_1(\beta)|_{n+j+l\cdot m}}\left(T_1^{n+i}(\beta)\right)\right|^a}\\
&\leq \lim_{l\to+\infty} \frac{1}{c^2\cdot L^a\cdot\left((r\,(n+j+k\cdot l)+1)\cdot q_{m(n+j+k\cdot l)}+q_{m(n+j+k\cdot l)-1}\right)^{2\cdot(1-a)}}\\
&= 0.
\end{aligned}$$

This completes the proof of Theorem 8.2 and we begin with the proof of Theorem 8.6.

□□

Proof of Theorem 8.6.1. This proof is a constructive proof. Within this proof set

$$\beta := [0; \underbrace{1,1,}_{2\cdot 1}2, \underbrace{1,1,1,1,}_{2\cdot 2}2, \underbrace{1,1,1,1,1,1,}_{2\cdot 3}2, \dots]$$
$$\text{and} \quad \kappa := [0; \underbrace{1,1,}_{2^1}2, \underbrace{1,1,1,1,}_{2^2}2, \underbrace{1,1,\dots,1,}_{2^3}2, \dots].$$

Furthermore, for $n \in \mathbb{N}$, set

$$\Lambda(n,\beta) := n\cdot(n+2) \quad \text{and} \quad \Lambda(n,\kappa) := 2^n + n - 2.$$

We observe that $\beta, \kappa \in [1/2, 1]$. As before, we let $a_n(\beta)$ and $a_n(\kappa)$ denote the n-th continued fraction entry of β and κ respectively. Hence, a calculation yields that $a_{\Lambda(n,\beta)-1}(\beta) = a_{\Lambda(n,\kappa)-1}(\kappa) = 2$. Further, we can show that

$$\omega_1(\beta) = \omega_1(\kappa) = \left\{[0; \underbrace{1,1,\dots,1,}_{k}2,\overline{1}] : k \in \mathbb{N}_0\right\} \cup \left\{[0;\overline{1}]\right\}.$$

We have that $\gamma := (\sqrt{5}-1)/2 = [0;\overline{1}]$. By (10.13), we have for each $n \in \mathbb{N}$, that

$$v_{\tau,a,n,1} = \frac{\left|f'_{1,\vartheta_1(\tau)|_n}\right|}{\left|\tau - f_{1,\vartheta_1(\tau)|_n}\right|^a}.$$

Following the same arguments as in beginning of the proof of Theorem 8.2, it is sufficient to show, on $\omega_1(\beta) = \omega_1(\kappa)$, that

$$\limsup_{n\to+\infty} \ln(n)\cdot v_{\beta,a,n,1} = 0$$
$$\text{and} \quad \limsup_{n\to+\infty} \ln(n)\cdot v_{\kappa,a,n,1} = \begin{cases} 0 & \text{if } a \in \left(0,\frac{1}{2}\right), \\ +\infty & \text{if } a \in \left[\frac{1}{2},1\right). \end{cases} \tag{10.14}$$

To this end fix $k \in \mathbb{N}_0$ and set

$$\zeta_k := [0; \underbrace{1, 1, \dots, 1}_{k}, 2, \overline{1}] \in \left[\frac{1}{3}, 1\right].$$

We will show that the equalities given in (10.14) hold for ζ_k, the result for γ is a simplification of this case.
To this end let $\tau \in \{\beta, \kappa\}$. By the mean value theorem, for each $n \in \mathbb{N}$, there exists $u_n(\tau) \in (1/3, 1)$ such that

$$\begin{aligned}
|\tau - f_{1,\vartheta_1(\tau)|_n}(\zeta_k)| &= \left| f_{1,\vartheta_1(\tau)|_n}\left(T_1^n(\tau)\right) - f_{1,\vartheta_1(\tau)|_n}(\zeta_k)\right| \\
&= \left|T_1^n(\tau) - \zeta_k\right| \cdot \left|f'_{1,\vartheta_1(\tau)|_n}(u_n(\tau))\right| \\
&= \frac{\left|T_1^n(\tau) - \zeta_k\right|}{\left((r(n,\tau)u_n(\tau) + 1) \cdot q_{m(n,\tau)}(\tau) + q_{m(n,\tau)-1}(\tau) \cdot u_n(\tau)\right)^2} \\
&\begin{cases} \geq \dfrac{\left|T_1^n(\tau) - \zeta_k\right|}{5^2 \cdot (q_{m(n,\tau)}(\tau))^2}, \\[2ex] \leq \dfrac{\left|T_1^n(\tau) - \zeta_k\right|}{\left(q_{m(n,\tau)}(\tau)\right)^2}. \end{cases}
\end{aligned}$$

See definition (5.8), page 26 for $m(n,\tau)$ and $r(n,\tau)$. For $l \in \mathbb{N}_0$, the integers $p_l(\tau)$ and $q_l(\tau)$ are as defined in (5.1), page 22.
Thus, for $\tau \in \{\beta, \kappa\}$ and $k \in \mathbb{N}_0$, we have that

$$\begin{aligned}
&\limsup_{n\to\infty} \ln(n) \cdot v_{\tau,a,n,1}(\zeta_k) \\
&= \limsup_{n\to\infty} \frac{\ln(n)}{\left((r(n,\tau)\cdot \zeta_k + 1) \cdot q_{m(n,\tau)}(\tau) + q_{m(n,\tau)-1} \cdot \zeta_k\right)^2} \frac{1}{\left|\tau - f_{1,\vartheta_1(\tau)|_n}(\zeta_k)\right|^a} \\
&\begin{cases} \geq \limsup\limits_{n\to\infty} \dfrac{\ln(n)}{5^2 \cdot \left(q_{m(n,\tau)}(\tau)\right)^{2\cdot(1-a)} \cdot \left|T_1^n(\tau) - \zeta_k\right|^a}, \\[2ex] \leq \limsup\limits_{n\to\infty} \dfrac{5^{2\cdot a} \cdot \ln(n)}{\left(q_{m(n,\tau)}(\tau)\right)^{2\cdot(1-a)} \cdot \left|T_1^n(\tau) - \zeta_k\right|^a} \end{cases} \\
&\begin{cases} \geq \limsup\limits_{n\to\infty} \dfrac{\ln(n)}{5^2 \cdot \left(q_{m(n,\tau)}(\tau)\right)^{2\cdot(1-a)} \cdot \left|T_1^{n-(k+1)}(\tau) - \gamma\right|^a \cdot \left|\left(f_{1,1}^k \circ f_{1,0} \circ f_{1,1}\right)'(0)\right|^a} \\[2ex] \leq \limsup\limits_{n\to\infty} \dfrac{5^{2\cdot a} \cdot \ln(n)}{\left(q_{m(n,\tau)}(\tau)\right)^{2\cdot(1-a)} \cdot \left|T_1^{n-(k+1)}(\tau) - \gamma\right|^a \cdot \left|\left(f_{1,1}^k \circ f_{1,0} \circ f_{1,1}\right)'(1)\right|^a}. \end{cases}
\end{aligned}$$

Since $\left|\left(f_{1,1}^{k} \circ f_{1,0} \circ f_{1,1}\right)'(x)\right|^{a}$ is bounded and bounded away from zero, it suffices to show, for $a \in (0,1)$, that

$$\limsup_{n\to+\infty} \frac{\ln(n)}{\left(q_{m(n,\beta)}(\beta)\right)^{2\cdot(1-a)} \cdot \left|T_1^{n-(k+1)}(\beta) - \gamma\right|^{a}} = 0 \tag{10.15}$$

and

$$\limsup_{n\to+\infty} \frac{\ln(n)}{\left(q_{m(n,\kappa)}(\kappa)\right)^{2\cdot(1-a)} \cdot \left|T_1^{n-(k+1)}(\kappa) - \gamma\right|^{a}} = \begin{cases} 0 & \text{if } a \in \left(0, \frac{1}{2}\right), \\ +\infty & \text{if } a \in \left[\frac{1}{2}, 1\right). \end{cases} \tag{10.16}$$

We will first show the equality given in (10.15) after which we will show the equality given in (10.16). We observe, for $l \in \mathbb{N}$, that if $n-(k+1) = \Lambda(l,\beta) + (l-1)$, for an $l \in \mathbb{N}$, then

$$T_1^{n-(k+1)}(\beta) = [0; 2, \underbrace{1, 1, \dots, 1}_{2\cdot(l+1)}, 2, \underbrace{1, 1, \dots, 1}_{2\cdot(l+2)}, 2, \underbrace{1, 1, \dots, 1}_{2\cdot(l+3)}, 2, \dots] \in \left[\frac{1}{3}, \frac{1}{2}\right],$$

and hence,

$$\begin{aligned} \frac{\ln(n)}{\left(q_{m(n,\beta)}(\beta)\right)^{2\cdot(1-a)} \cdot \left|T_1^{n-(k+1)}(\beta) - \gamma\right|^{a}} &\le \frac{\ln\left(\Lambda(l,\beta) + (l-1) + (k+1)\right)}{\left(q_{\Lambda(l,\beta)}(\beta)\right)^{2\cdot(1-a)} \cdot \left|\frac{1}{2} - \gamma\right|^{a}} \\ &\sim \frac{2 \cdot \ln(l)}{\left(q_{l\cdot(l+2)}(\beta)\right)^{2\cdot(1-a)} \cdot \left|\frac{1}{2} - \gamma\right|^{a}}. \end{aligned} \tag{10.17}$$

Since the sequence $\left(q_j\right)_{j\in\mathbb{N}}$ grows exponentially, the last term converges to zero as $l \to \infty$. In the first inequality of (10.17), we have used the fact that

$$n - (k+1) = \Lambda(l,\beta) + (l-1).$$

In the case that $n-(k+1) \notin \{\Lambda(j,\beta) + (j-1) \colon j \in \mathbb{N}\}$, set $l = l(n) \in \mathbb{N}$ to be the maximal integer such that $n-(k+1) > \Lambda(l,\beta) + (l-1)$, in which case we observe, that $n-(k+1) - \Lambda(l,\beta) \ge l$. That implies

$$\begin{aligned} 2\cdot(l+1) + 1 &= 3\cdot(l+1) - l \\ &\ge 3\cdot(l+1) - (n-(k+1) - \Lambda(l,\beta)) \\ &= 3\cdot(l+1) - n + (k+1) + \Lambda(l,\beta), \end{aligned}$$

and hence, we have that

$$T_1^{n-(k+1)}(\beta) = [0; \underbrace{1, 1, \quad \dots \quad, 1}_{\substack{3\cdot(l+1)+(k+1)+\Lambda(l,\beta)-n \\ \le 2\cdot(l+1)+1}}, 2, \underbrace{1, 1, \dots, 1}_{2\cdot(l+2)}, 2, \underbrace{1, 1, \dots, 1}_{2\cdot(l+3)}, 2, \dots].$$

By Lemma 5.2, the at most $2 \cdot (l+1)+1$ first ones come from at most $2 \cdot (l+1)$ ones in the first place and one additional one that has been a two one iteration earlier.
This yields

$$\begin{aligned}
&\frac{\ln(n)}{\left(q_{m(n,\beta)}(\beta)\right)^{2\cdot(1-a)} \cdot \left|T_1^{n-(k+1)}(\beta)-\gamma\right|^a} \\
&= \frac{\ln(n)}{\left(q_{m(n,\beta)}(\beta)\right)^{2\cdot(1-a)} \cdot \left|f_{1,1}^{3\cdot(l+1)+(k+1)+\Lambda(l,\beta)-n}\left(T_1^{\Lambda(l+1,\beta)+l}(\beta)\right) - f_{1,1}^{3\cdot(l+1)+(k+1)+\Lambda(l,\beta)-n}(\gamma)\right|^a} \\
&\leq \frac{\ln\left((l+2)\cdot(l+5)\right)}{\left(q_{l\cdot(l+2)}(\beta)\right)^{2\cdot(1-a)} \cdot \inf_{u\in[0,1]}\left|\left(f_{1,1}^{3\cdot(l+1)+(k+1)+\Lambda(l,\beta)-n}\right)'(u)\right|^a \cdot \left|\frac{1}{2}-\gamma\right|^a} \\
&= \frac{\ln\left((l+2)\cdot(l+5)\right)\cdot\left(q_{3\cdot(l+1)+(k+1)+\Lambda(l,\beta)-n}(\gamma)\right)^a}{\left(q_{l\cdot(l+2)}(\beta)\right)^{2\cdot(1-a)} \cdot \left|\frac{1}{2}-\gamma\right|^a} \\
&= \frac{\ln\left((l+2)\cdot(l+5)\right)\cdot\left(q_{2\cdot(l+1)+1}(\beta)\right)^a}{\left(q_{l\cdot(l+2)}(\beta)\right)^{2\cdot(1-a)} \cdot \left|\frac{1}{2}-\gamma\right|^a}.
\end{aligned} \tag{10.18}$$

Since the sequence $(q_j(\beta))_{j\in\mathbb{N}}$ grows exponentially, (10.18) converges to zero as $l = l(n) \to \infty$. The equality stated in (10.15) follows from (10.17) and (10.18) and we can turn our attention towards the case $\tau = \kappa$.
We will prove the equality given in (10.16). The result for, $a \in (0, 1/2)$, follows in a similar manner to the previous case. Indeed, observe that if

$$n-(k+1) = \Lambda(l,\kappa) + (l-1),$$

for an $l \in \mathbb{N}$, then

$$T_1^{n-(k+1)}(\kappa) = [0; 2, \underbrace{1,1,\dots,1}_{2^{l+1}}, 2, \underbrace{1,1,\dots,1}_{2^{l+2}}, 2, \underbrace{1,1,\dots,1}_{2^{l+3}}, 2, \dots] \in \left[\frac{1}{3}, \frac{1}{2}\right], \tag{10.19}$$

and hence, for n sufficiently large,

$$\frac{\ln(n)}{\left(q_{m(n,\kappa)}(\kappa)\right)^{2\cdot(1-a)} \cdot \left|T_1^{n-(k+1)}(\kappa)-\gamma\right|^a} \leq \frac{(l+1)\cdot\ln(2)}{\left(q_{2^l}(\kappa)\right)^{2\cdot(1-a)} \cdot \left|\frac{1}{2}-\gamma\right|^a}. \tag{10.20}$$

The sequence $\left(q_j(\kappa)\right)_{j\in\mathbb{N}}$ grows exponentially, in particular there exists a positive constant c so that we have for $j \in \mathbb{N}$, that $1/(c\cdot\kappa^j) \leq q_j(\kappa) \leq c/\kappa^j$. Therefore, the latter term in (10.20) converges to zero as $l \to \infty$.
In the case that $n-(k+1) \notin \{\Lambda(j,\kappa)+(j-1) \colon j \in \mathbb{N}\}$, set $l = l(n) \in \mathbb{N}$ to be the maximal integer such that $n-(k+1) > \Lambda(l,\kappa)+(l-1)$, in which case

$$T_1^{n-(k+1)}(\kappa) = [0; \underbrace{1,1,\quad\dots\quad,1}_{\substack{2^{l+1}+(l+1)+(k+1)+\Lambda(l,\kappa)-n \\ \leq 2^{l+1}+1}}, 2, \underbrace{1,1,\dots,1}_{2^{l+2}}, 2, \underbrace{1,1,\dots,1}_{2^{l+3}}, 2, \dots].$$

We also observe that $q_i(\gamma) \leq q_i(\kappa)$, for all $i \in \mathbb{N}_0$. Therefore, it follows that

$$\begin{aligned}\frac{\ln(n)}{(q_{m(n,\kappa)}(\kappa))^{2\cdot(1-a)}} \cdot \frac{1}{|T_1^{n-(k+1)}(\kappa) - \gamma|^a} &\leq \frac{(l+2)\cdot\ln(2)}{(q_{2^l}(\kappa))^{2\cdot(1-a)}} \cdot (q_{2\cdot 2^l+2}(\gamma))^a \\ &\leq \frac{(l+2)\cdot\ln(2)}{(q_{2^l}(\gamma))^{2\cdot(1-a)} \cdot (q_{2\cdot 2^l+2}(\gamma))^{-a}}.\end{aligned} \tag{10.21}$$

Thus, if $a \in (0, 1/2)$, the last term of (10.21) converges to zero as $l = l(n) \to \infty$. The equality in (10.16) for $a \in (0, 1/2)$ follows from (10.20) and (10.21).
Let us now examine the case, where $a \in [1/2, 1)$.
It follows from an inductive argument that we have for all $n \in \mathbb{N}$, that $q_l(\kappa) \leq 2^n \cdot q_l(\gamma)$ for all integers $l \in [\Lambda(n,\kappa), \Lambda(n+1,\kappa))$. Further, by (10.19) we have for all $n \in \mathbb{N}$, that

1. $$\begin{aligned}\left|\gamma - T_1^{\Lambda(n,\kappa)+n-1}(\kappa)\right| &= |\gamma - [0; 2, \underbrace{1, \dots, 1}_{2^{n+1}}, 2, \underbrace{1, \dots, 1}_{2^{n+2}}, 2 \dots]| \\ &\geq \left|\gamma - \frac{1}{2}\right| \quad \text{and}\end{aligned}$$

2. $$\begin{aligned}\left|\gamma - T_1^{\Lambda(n,\kappa)+n+1}(\kappa)\right| &= |\gamma - [0; \underbrace{1, \dots, 1}_{2^n}, 2, \underbrace{1, \dots, 1}_{2^{n+1}}, 2, \dots]| \\ &\leq \left|\gamma - \frac{p_{2^n}(\gamma)}{q_{2^n}(\gamma)}\right| \\ &\leq \frac{1}{(q_{2^n}(\gamma))^2}\end{aligned}$$

Therefore, if $a \in [1/2, 1)$, since there exists a positive constant $\mathfrak{c}$ such that

$$\frac{1}{\gamma^n \cdot \mathfrak{c}} \leq q_n(\gamma) \leq \frac{\mathfrak{c}}{\gamma^n},$$

for all $n \in \mathbb{N}$, since $2^{2\cdot n\cdot(1-a)} \leq \gamma^{-2\cdot n\cdot(1-a)}$ and since

$$\begin{aligned}&2^{n+1}a - 2\cdot n\cdot(1-a) - 2\cdot(1-a)\cdot(2^n + n - 2) \\ &= 2^{n+1}\cdot(2\cdot a - 1) - 4\cdot(n-1)\cdot(1-a),\end{aligned}$$

we have that

$$\begin{aligned}&\limsup_{n\to+\infty} \frac{\ln(\Lambda(n,\kappa)+n+1)}{\left(q_{\Lambda(n,\kappa)}(\kappa)\right)^{2\cdot(1-a)} \cdot \left|T_1^{\Lambda(n,\kappa)+n+1}(\kappa) - \gamma\right|^a} \\ &\geq \limsup_{n\to+\infty} \frac{n\cdot\ln(2)\cdot(q_{2^n}(\gamma))^{2\cdot a}}{2^{2\cdot n\cdot(1-a)}\cdot(q_{2^n+n-2}(\gamma))^{2\cdot(1-a)}} \\ &\geq \limsup_{n\to+\infty} \frac{n\cdot\ln(2)}{\mathfrak{c}^2\cdot\gamma^{2^{n+1}\cdot(2\cdot a-1)+4\cdot(n-1)\cdot(1-a)}} \\ &= +\infty.\end{aligned}$$

Moreover, since the sequence $(q_j(\kappa))_{j\in\mathbb{N}}$ grows exponentially, it follows that

$$\begin{aligned}
&\liminf_{n\to+\infty} \frac{\ln(\Lambda(n,\kappa)+n-1)}{\left(q_{\Lambda(n,\kappa)-1}(\kappa)\right)^{2\cdot(1-a)} \cdot \left|T_1^{\Lambda(n,\kappa)+n-1}(\kappa)-\gamma\right|^a} \\
&\leq \liminf_{n\to+\infty} \frac{\ln(\Lambda(n,\kappa)+n-1)}{\left(q_{\Lambda(n,\kappa)-1}(\kappa)\right)^{2\cdot(1-a)} \cdot \left|\gamma-\frac{1}{2}\right|^a} = 0,
\end{aligned}$$

which proofs the assertion. The case $a = 1/2$ is included here which has not been included in [KKS16].

□

Proof of Theorem 8.6.2. Since $\lim_{n\to+\infty} a_n = +\infty$, we have that the ω-Limit set of β is given by

$$\omega_1(\beta) = \left\{\frac{1}{k} : k \in \mathbb{N}\right\} \cup \{0\}.$$

Following the same arguments as in beginning of the proof of Theorem 8.2, it is sufficient to show, for a fixed $k \in \mathbb{N}$, that

$$\limsup_{n\to+\infty} \ln(n) \cdot v_{\beta,a,n,1}\left(\frac{1}{k}\right) \begin{cases} = 0 & \text{if } \limsup\limits_{j\to\infty} \mathscr{S}_{k,j} = 0, \\ > 0 & \text{if } \limsup\limits_{j\to\infty} \mathscr{S}_{k,j} > 0. \end{cases}$$

and

$$\liminf_{n\to+\infty} \ln(n) \cdot v_{\beta,a,n,1}\left(\frac{1}{k}\right) = 0.$$

For $n \in \mathbb{N}$, $T_1^n(\beta)$ is the unique real number in $[0,1]$ such that $f_{1,\vartheta_1(\beta)|_n}\left(T_1^n(\beta)\right) = \beta$. We fix $k \in \mathbb{N}$. It is sufficient to show the validity of the limes inferior statement for the case $T_1^n(\beta) \notin (1/(k+1), 1/k)$ and the limes superior statement for the case $T_1^n(\beta) \in (1/(k+1), 1/k)$. If $T_1^n(\beta) \in (1/(k+1), 1/k)$, we first observe, that, by the mean value theorem, there exists $u := u(n) \in (1/(k+1), 1/k)$, such that

$$\begin{aligned}
\left|\beta - f_{1,\vartheta_1(\beta)|_n}\left(\frac{1}{k}\right)\right| &= \left|f_{1,\vartheta_1(\beta)|_n}\left(T_1^n(\beta)\right) - f_{1,\vartheta_1(\beta)|_n}\left(\frac{1}{k}\right)\right| \\
&= \left|\frac{1}{k} - T_1^n(\beta)\right| \cdot \left|f'_{1,\vartheta_1(\beta)|_n}(u)\right| \\
&= \frac{\left|\frac{1}{k} - T_1^n(\beta)\right|}{\left|(r(n)\cdot u+1)\cdot q_{m(n)} + q_{m(n)-1}\cdot u\right|^2} \\
&\begin{cases} \geq \frac{k^2\cdot\left|\frac{1}{k}-T_1^n(\beta)\right|}{\left|(r(n)+k)\cdot q_{m(n)}+q_{m(n)-1}\right|^2}, \\ \\ \leq \frac{(k+1)^2\cdot\left|\frac{1}{k}-T_1^n(\beta)\right|}{\left|(r(n)+k+1)\cdot q_{m(n)}+q_{m(n)-1}\right|^2}. \end{cases}
\end{aligned}$$

Furthermore, we have that $T_1^n(\beta) = [0; k, a_{m(n)}, a_{m(n)+1}, \dots]$; that is $n = n_{k,m(n)}$ which in turn implies, that

$$\begin{aligned}
&\limsup_{j\to+\infty} \ln\left(n_{k,j}\right)\cdot v_{\beta,a,n_{k,j},1}\left(\frac{1}{k}\right)\\
&= \limsup_{j\to+\infty} \frac{k^2\cdot\ln\left(n_{k,j}\right)}{\left(\left(r(n_{k,j})+k\right)\cdot q_{m(n_{k,j})} + q_{m(n_{k,j})-1}\right)^2\cdot\left|\beta - f_{1,\vartheta_1(\beta)|_{n_{k,j}}}\left(\frac{1}{k}\right)\right|^a}\\
&\begin{cases}
\le \limsup\limits_{j\to+\infty} \dfrac{k^{2\cdot(1-a)}\cdot\ln\left(n_{k,j}\right)}{\left|\frac{1}{k} - T_1^{n_{k,j}}(\beta)\right|^a\cdot\left((r(n_{k,j})+k)\cdot q_{m(n_{k,j})} + q_{m(n_{k,j})-1}\right)^{2\cdot(1-a)}}\\[2ex]
\ge \limsup\limits_{j\to+\infty} \dfrac{k^{2\cdot(1+a)}\cdot\ln\left(n_{k,j}\right)}{2^{2\cdot a}\cdot\left|\frac{1}{k} - T_1^{n_{k,j}}(\beta)\right|^a\cdot\left((r(n_{k,j})+k)\cdot q_{m(n_{k,j})} + q_{m(n_{k,j})-1}\right)^{2\cdot(1-a)}}
\end{cases}\\
&\begin{cases}
\le \limsup\limits_{j\to+\infty} \dfrac{k^2\cdot\left(a_{j+1}+1\right)^a\cdot\ln\left(n_{k,j}\right)}{\left(q_j\right)^{2\cdot(1-a)}}\\[2ex]
\ge \limsup\limits_{j\to+\infty} \dfrac{k^{2\cdot(1+2\cdot a)}\cdot\left(a_{j+1}\right)^a\cdot\ln\left(n_{k,j}\right)}{2^{2\cdot a}\cdot\left(q_j\right)^{2\cdot(1-a)}}
\end{cases}\\
&\begin{cases}
= \limsup\limits_{j\to+\infty} k^2\cdot\mathscr{S}_{k,j}\\
= \limsup\limits_{j\to+\infty} k^{2\cdot(1+2\cdot a)}\cdot 4^{-a}\cdot\mathscr{S}_{k,j},
\end{cases}
\end{aligned}$$

and the first statement follows. Hence, the case $T_1^n(\beta) \notin (1/(k+1), 1/k)$ is left to be considered. Since $f_{1,\vartheta_1(\beta)|_n}$ is either order preserving or order reversing, we have for $n \in \mathbb{N}$ sufficiently large that

$$\begin{aligned}
&\left|\beta - f_{1,\vartheta_1(\beta)|_n}\left(\frac{1}{k}\right)\right|\\
&= \left|f_{1,\vartheta_1(\beta)|_n}(T_1^n(\beta)) - f_{1,\vartheta_1(\beta)|_n}\left(\frac{1}{k}\right)\right|\\
&\ge \begin{cases}
\left|f_{1,\vartheta_1(\beta)|_n}\left(\frac{1}{2}\right) - f_{1,\vartheta_1(\beta)|_n}(1)\right| & \text{if } k = 1,\\[2ex]
\min\Big\{\left|f_{1,\vartheta_1(\beta)|_n}\left(\frac{1}{k+1}\right) - f_{1,\vartheta_1(\beta)|_n}\left(\frac{1}{k}\right)\right|, & \\
\qquad\left|f_{1,\vartheta_1(\beta)|_n}\left(\frac{2\cdot k-1}{2\cdot k\cdot(k-1)}\right) - f_{1,\vartheta_1(\beta)|_n}\left(\frac{1}{k}\right)\right|\Big\} & \text{otherwise.}
\end{cases}
\end{aligned}$$

By the mean value theorem there exists $u \in (1/(k+1), (2\cdot k-1)/(2\cdot k\cdot(k-1)))$ if $k \neq 1$ and $u \in (1/2, 1)$ if $k = 1$ such that

$$\begin{aligned}\left|\beta - f_{1,\vartheta_1(\beta)|_n}\left(\frac{1}{k}\right)\right| &\geq \frac{\left|f'_{1,\vartheta_1(\beta)|_n}(u)\right|}{2\cdot k\cdot(k+1)}\\ &= \frac{1}{(2\cdot k\cdot(k+1))\cdot\left|(r(n)\cdot u+1)\cdot q_{m(n)} + q_{m(n)-1}\cdot u\right|^2}\\ &\geq \frac{1}{3\cdot 2\cdot k\cdot\left|(r(n)+\max\{k-1,1\})\cdot q_{m(n)} + q_{m(n)-1}\right|^2}.\end{aligned}$$

Hence, we have that

$$\begin{aligned}0 &\leq \ln(n)\cdot v_{\beta,a,n,1}\left(\frac{1}{k}\right)\\ &= \frac{\ln(n)}{\left(\left(\frac{r(n)}{k}+1\right)\cdot q_{m(n)} + \frac{q_{m(n)-1}}{k}\right)^2\cdot\left|\beta - f_{1,\vartheta_1(\beta)|_n}\left(\frac{1}{k}\right)\right|^a}\\ &\leq \frac{6^{2\cdot a}\cdot k^{2\cdot(1-a)}\cdot\ln(n)}{((r(n)+1)\cdot q_{m(n)} + q_{m(n)-1})^{2\cdot(1-a)}}.\end{aligned}$$

Since $(r(n)+1)\cdot q_{m(n)} + q_{m(n)-1} > n$, for all $n \in \mathbb{N}$, it follows that

$$\liminf_{n\to+\infty} \ln(n)\cdot v_{\beta,a,n,1}\left(\frac{1}{k}\right) = 0,$$

which completes the proof of Theorem 8.6.

□□

Part III

The asymptotics of the α-Farey transfer operator

Chapter 11

Central definitions II

This chapter introduces the necessary definitions that are needed in this part of the thesis and begins with the notion of *moderately increasing*. This notion classifies different slowly varying functions and shows that there is a variety in the class of slowly varying functions. The major parts of this part are published in [KKSS15].

Definition 11.1 (**Moderately increasing**)**.** A slowly varying function $w_n \colon \mathbb{N} \to \mathbb{R}$ is called moderately increasing, if

$$\left(\frac{w_n}{w_{\lceil n \cdot w_n^{-2} \rceil}}\right)_{n \in \mathbb{N}}$$

is a bounded sequence.

Note that the constant function and for instance the slowly varying functions $\ln(n)$, $e^{\sqrt{\ln(n)}}$ and $e^{\frac{\ln(n)}{\ln(\ln(n))}}$ are moderately increasing. Yet, we have that

$$\begin{aligned} &\lim_{n\to\infty} \frac{\ln(n)}{e^{\sqrt{\ln(n)}}} = 0, \\ &\lim_{n\to\infty} \frac{\ln(n)}{e^{\frac{\ln(n)}{\ln(\ln(n))}}} = 0, \\ \text{and} \quad &\lim_{n\to\infty} \frac{e^{\sqrt{\ln(n)}}}{e^{\frac{\ln(n)}{\ln(\ln(n))}}} = 0, \end{aligned} \tag{11.1}$$

which demonstrates that three moderately increasing functions, although they are all slowly varying, lie in different asymptotic classes.
The fact that *slowly varying* does not imply *moderately increasing* can be seen in the following example.

Example 11.2. This is an example of a slowly varying function that is not moderately increasing. Given a function $m \colon [0, \infty) \to \mathbb{R}$, we let $l(x) := \exp(m(x) \cdot (\log x))$. By the definition of m and l, we observe the following two equivalences.

1. $\lim_{x\to\infty} l(c \cdot x)/l(x) = 1$, for all $c \geq 1$, if and only if

$$\lim_{x\to\infty} m(x+d) - m(x) = 0, \text{ for all } d \geq 0. \tag{11.2}$$

2. $l(x)/l\left(x/l(x)^2\right)$ is bounded, if and only if

$$m(x) - m(x - 2 \cdot m(x)) \text{ is bounded from above.} \tag{11.3}$$

Thus, a function m that satisfies (11.2) but not (11.3) would give a counterexample. To this end, we let $(c_k)_{k\in\mathbb{N}}$ denote a decreasing sequence of positive numbers that converges to zero. Let $x_{k+1} := k/2 \cdot c_k^2$. Further, let $b_1 = 0$ and for $k \geq 2$ let

$$b_{k+1} := (c_k - c_{k+1}) \cdot x_{k+1} + b_k.$$

Define m to be the continuous piecewise linear function given by

$$m(x) := c_k \cdot x + b_k, \text{ for } x \in [x_k, x_{k+1}].$$

Since $c_k \to 0$, for $k \to \infty$, we have that m satisfies (11.2). On the other hand, we have that

$$\begin{aligned} m(x_{k+1}) - m(x_{k+1} - 2 \cdot m(x_{k+1})) &\geq c_k \cdot x_{k+1} + b_k - (c_k \cdot (x_{k+1} - 2 \cdot m(x_{k+1})) + b_k) \\ &= 2 \cdot c_k \cdot m(x_{k+1}) \; = \; 2 \cdot c_k^2 \cdot \frac{k}{2 \cdot c_k^2} + 2 \cdot c_{k+1} \cdot b_{k+1} \\ &\geq k. \end{aligned}$$

Hence, (11.3) is violated and m is a slowly varying function that is not moderately increasing.
I am grateful to F. Ekström and T. Samuel for providing this example.

We continue with the introduction of the function space for which convergence results are going to be discussed in the sequel.
Recall from Paragraph 6.3.3 that if $f \in \mathcal{B}_\alpha$, then f is Lipschitz continuous on each atom of β_α, zero outside of $\overline{A}_1$ and bounded everywhere. We define

$$\mathcal{A}_\alpha := \left\{ v \in \mathcal{L}^1_{\mu_\alpha}([0,1]) \colon \begin{array}{l} \|v\|_\infty < \infty \,, \sum_{k=1}^\infty \left\| \widehat{F}_\alpha^{k-1}(v \cdot \mathbb{1}_{A_k}) \right\|_\infty < +\infty \\ \text{and} \quad \widehat{F}_\alpha^{n-1}(v \cdot \mathbb{1}_{A_n}) \in \mathcal{B}_\alpha \text{ for all } n \in \mathbb{N} \end{array} \right\}. \tag{11.4}$$

For examples of observables belonging to $\mathcal{A}_\alpha$, the reader is referred to Example 11.3 and the discussion succeeding Theorem 12.3. We call the condition

$$\sum_{k=1}^\infty \left\| \widehat{F}_\alpha^{k-1}(v \cdot \mathbb{1}_{A_k}) \right\|_\infty < +\infty, \tag{11.5}$$

the *summability condition*.
The next example shows that the definition of $\mathcal{A}_\alpha$ is not vain.

Example 11.3. We assume that $([0,1], \mathscr{B}, \mu_\alpha, F_\alpha)$ is a 1-expansive α-Farey system with a moderately increasing wandering rate. Let $\mathscr{D}_{\mu_\alpha}$, be given by

$$\mathscr{D}_{\mu_\alpha} := \left\{ f \colon f \in \mathcal{L}^1_{\mu_\alpha}([0,1]) \text{ and } f \in C^2((0,1)) \text{ with } f' > 0 \text{ and } f'' \leq 0 \right\},$$

and for $f \in \mathscr{D}_{\mu_\alpha}$, set $u := f/h_\alpha$. We claim that $u \in \mathcal{A}_\alpha$.

To proof this claim we have a look at a technical Lemma, which is also crucial in the proof of the main results.

Lemma 11.4 ([KKSS15, Lemma 2.3])**.** *For each $n \in \mathbb{N}$, we have that*

$$\widehat{F}_\alpha^{n-1}(\mathbb{1}_{A_n}) = t_n \cdot \mathbb{1}_{A_1} \tag{11.6}$$

and hence, by the definition of the norm, $\left\| \widehat{F}_\alpha^{n-1}(\mathbb{1}_{A_n}) \right\|_{\mathcal{B}_\alpha} = \left\| \widehat{F}_\alpha^{n-1}(\mathbb{1}_{A_n}) \right\|_\infty = t_n$.

Proof of Lemma 11.4. For $n = 1$ the result follows, since $t_1 = 1$. By Lemma 6.3, we have for $n \neq 1$, on $[0,1)$, that

$$\widehat{F}_\alpha^{n-1}(\mathbb{1}_{A_n}) = \sum_{k=1}^{\infty} c_{k,0_{n-1}} \cdot \mathbb{1}_{A_n} \circ f_{\alpha,0_{n-1}} \cdot \mathbb{1}_{A_k} = \sum_{k=1}^{\infty} c_{k,0_{n-1}} \cdot \mathbb{1}_{A_1} \cdot \mathbb{1}_{A_k} = t_n \cdot \mathbb{1}_{A_1}.$$

This completes the proof and we can show the claim of Example 11.3.

□

We are required to show that $u \in \mathcal{L}^1_{\mu_\alpha}([0,1])$, that $\|u\|_\infty < +\infty$, that for all $j \in \mathbb{N}$, $\widehat{F}_\alpha^{j-1}(u \cdot \mathbb{1}_{A_j}) \in \mathcal{B}_\alpha$, and that the summability condition (11.5) holds.
By definition, each function belonging to $\mathscr{D}_{\mu_\alpha}$ is convex and continuous on $(0,1)$, twice differentiable and μ_α-integrable. Thus, $f \in \mathcal{L}^1_{\mu_\alpha}([0,1])$ and $\|u\|_\infty < +\infty$. Combining this with the fact that $1/h_\alpha$ is μ_α integrable, non-negative and bounded, we have that $u \in \mathcal{L}^1_{\mu_\alpha}([0,1])$ and $\|u\|_\infty < +\infty$. Let us now turn to the second assertion, namely that $\widehat{F}_\alpha^{n-1}(u \cdot \mathbb{1}_{A_n}) \in \mathcal{B}_\alpha$, for all $n \in \mathbb{N}$. We immediately have that $\widehat{F}_\alpha^{0}(u \cdot \mathbb{1}_{A_1}) = u \cdot \mathbb{1}_{A_1} \in \mathcal{B}_\alpha$. For $n \geq 2$, note that, if g is a differentiable Lipschitz function on $\overline{A}_1$, then $D_\alpha(g) = \sup\{|g'| : x \in \overline{A}_1\}$. Thus, by Lemma 11.4 and the chain rule, we have that, for each integer $n \geq 2$,

$$\begin{aligned} \left\| \widehat{F}_\alpha^{n-1}(u \cdot \mathbb{1}_{A_n}) \right\|_{\mathcal{B}_\alpha} &= \left\| c_{1,0_{n-1}} \cdot \left(\left(\frac{f}{h_\alpha} \right) \circ f_{\alpha,0_{n-1}} \right) \right\|_{\mathcal{B}_\alpha} \\ &= \left\| a_n \cdot f \circ f_{\alpha,0_{n-1}} \right\|_{\mathcal{B}_\alpha} \\ &= \left\| a_n \cdot f \circ f_{\alpha,0_{n-1}} \right\|_\infty + D_\alpha(a_n f \circ f_{\alpha,0_{n-1}}) \\ &\leq a_n \cdot \left(\|f\|_\infty + \left\| f' \cdot \mathbb{1}_{A_n} \right\|_\infty \right). \end{aligned} \tag{11.7}$$

Since $f \in \mathscr{D}_{\mu_\alpha}$, we have that $\|f\|_\infty < +\infty$ and that $0 \leq f'(x) \leq (f(t_{n+1}) - f(t_{n+2}))/a_{n+1}$, for all $x \in A_n$. That is, the derivative of f on each level set of the first return time is less than or equal to the slope of the straight line through the endpoints of the level set of the first return time on the right side of it. Therefore, since $a_n = \ell(n) \cdot n^{-2}$, it follows, that

$$\sum_{n=2}^{\infty} a_n \cdot \left\| f' \cdot \mathbb{1}_{A_n} \right\|_\infty \leq \sum_{n=2}^{\infty} a_n \cdot f'\left(\xi_{n+1}\right) = \sum_{n=2}^{\infty} \frac{a_n \cdot \left(f(t_{n+1}) - f(t_{n+2})\right)}{a_{n+1}} \leq \frac{a_2 \cdot f\left(t_3\right)}{a_3}.$$

Combining this with (11.7) and using the facts that the sequence $(a_n)_{n\in\mathbb{N}}$ is summable and that $\|f\|_\infty$ and $\|u \cdot \mathbb{1}_{A_1}\|_{\mathcal{B}_\alpha}$ are finite, the summability condition in (11.5) follows. Hence, it follows that $u \in \mathcal{A}_\alpha$.

Chapter 12

Convergence and non-convergence – main results of Part III

Using the definitions of the previous chapter and the notion given in Section 5.2, we are in the position to state the main results of this part. The first theorem is a generalisation of [KKSS15, Theorem 1.3 (ii)], inspired by recent developments published in [MT15]. The second theorem is an improvement in a certain situation. Finally, the third theorem shows that convergence results do not hold in full generality.
In Chapter 14 and after the statement of the theorems, we discuss how the results of Theorem 12.1, Theorem 12.2 and Theorem 12.3 complement, extend and follow from the results of [MT15] and [KKSS15] and other previously known results.

Theorem 12.1. *Let $\delta \in (1/2, 1]$ and let $([0,1], \mathscr{B}, \mu_\alpha, F_\alpha)$ denote a δ-expansive α-Farey system. If $v \in \mathcal{L}^1_\lambda([0,1])$ and if*

$$D_\alpha\left(\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1}\left(\frac{v}{h_\alpha} \cdot \mathbb{1}_{A_n}\right)\right) = \mathfrak{O}(1) \tag{12.1}$$

$$\text{and} \quad \left\|v \cdot \mathbb{1}_{A_n}\right\|_\infty = \mathfrak{o}\left(\frac{1}{t_n}\right), \tag{12.2}$$

then uniformly on compact subsets of $(0,1]$,

$$\lim_{n\to\infty} w_n \cdot \widehat{F}_\alpha^n\left(\frac{v}{h_\alpha}\right) = \Gamma_\delta \cdot \int \frac{v}{h_\alpha}\, \mathrm{d}\mu_\alpha. \tag{12.3}$$

The constant Γ_δ is given by (6.16).

Conditions (12.1) and (12.2) are quite restrictive. We can do better, if we have more information on the system as Theorem 12.2 shows.

Theorem 12.2 ([KKSS15, Theorem 1.3 (i)]). *Let* $([0,1],\mathscr{B},\mu_\alpha,F_\alpha)$ *denote a* 1-*expansive* α*-Farey system and assume that the wandering rate is moderately increasing. If* $v \in \mathcal{A}_\alpha$*, then we have, uniformly on compact subsets of* $(0,1]$ *that*

$$\lim_{n\to\infty} w_n \cdot \widehat{F}_\alpha^n(v) = \int v \,\mathrm{d}\mu_\alpha. \tag{12.4}$$

However, we have to be pre-cautious, if the assumption become too weak, as the next theorem shows.

Theorem 12.3 ([KKSS15, Theorem 1.3 (iii)]). *For* $\delta \in (1/2,1)$*, let* $([0,1],\mathscr{B},\mu_\alpha,F_\alpha)$ *denote a* δ*-expansive* α*-Farey system. There exists a positive, locally constant, Riemann integrable function* $v \in \mathcal{A}_\alpha$ *of bounded variation, such that, for all* $x \in \overline{A}_1$*,*

$$\begin{aligned} &\liminf_{n\to\infty} w_n \cdot \widehat{F}_\alpha^n(v)(x) = \Gamma_\delta \cdot \int v\,\mathrm{d}\mu_\alpha \\ \textit{and}\quad &\limsup_{n\to\infty} w_n \cdot \widehat{F}_\alpha^n(v)(x) = +\infty. \end{aligned} \tag{12.5}$$

The limes inferior in (12.5) *can be replaced by a limes, if we exclude a set of integers having asymptotic upper density zero.*

The asymptotic upper density $\overline{d}(\cdot)$ of a subset L of the natural numbers is defined to be

$$\overline{d}(L) := \limsup_{n\to\infty} \frac{\#\{k \in L\colon k < n\}}{n}. \tag{12.6}$$

Theorem 12.1 resembles [KKSS15, Theorem 1.3 (ii)]. In the latter, the authors assumed as well two properties, namely that

1. the sequence $(D_\alpha(\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1}(v \cdot \mathbb{1}_{A_n})))_{n\in\mathbb{N}}$ is bounded and
2. there exist constants $c \in \mathbb{R}$ and $\eta \in (0,\delta)$ with $\left\|v \cdot \mathbb{1}_{A_n}\right\|_\infty \le c \cdot n^\eta$, for all $n \in \mathbb{N}$.

We observe that, by the definition of h_α, the first condition implies (12.1), since $D_\alpha(\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1}(v/h_\alpha \cdot \mathbb{1}_{A_n})) = 1/n \cdot D_\alpha(\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1}(v \cdot \mathbb{1}_{A_n})$. The second condition implies (12.2), because we have that

$$\lim_{n\to\infty} t_n \cdot \left\|v \cdot \mathbb{1}_{A_n}\right\|_\infty \le \lim_{n\to\infty} c \cdot \frac{\ell(n)}{n^\delta} \cdot n^\eta = 0.$$

We end this chapter with a series of examples and remarks which indicate how the current results can be extended.

Example 12.4 ([KKSS15, Remark 1.4]). It is immediate that if $a_n = n^{-1} \cdot (n+1)^{-1}$, then $t_n = n^{-1}$ and $w_n \sim \ln(n)$, and that these parameters give rise to an example of an α-Farey system which satisfies the conditions of Theorem 12.2. Indeed there exist many examples of α-Farey systems for which the conditions of Theorem 12.2 are satisfied, but where the wandering rate behaves very differently to the function

$n \mapsto \ln(n)$. Letting $\delta = 1$, as we see in Lemma 4.2 (iv), the sequence $(w_n)_{n\in\mathbb{N}}$ is slowly varying and $\lim_{n\to\infty} n \cdot t_n/w_n = 0$. We also have that

$$w_n = \sum_{j=1}^{n} t_j = n \cdot \sum_{j=n+1}^{\infty} a_j + \sum_{j=1}^{n} j \cdot a_j = n \cdot t_{n+1} + \sum_{j=1}^{n} j \cdot a_j.$$

Using this we deduce the following two examples.

Example 12.5. If $a_n = \frac{e^{\sqrt{\ln(n)}}}{n^2 \cdot \sqrt{\ln(n)}}$, then $t_n \sim \frac{e^{\sqrt{\ln(n)}}}{n \cdot \sqrt{\ln(n)}}$ and $w_n \sim e^{\sqrt{\ln(n)}}$.

Example 12.6. Let $\kappa(n) := \frac{1}{(\ln(\ln(n))-1)\cdot(\ln(\ln(n)))^2}$.

If $a_{n-16} = \frac{\kappa(n)}{n^2} \cdot e^{\frac{\ln(n)}{\ln(\ln(n))}}$, then $t_n \sim \frac{\kappa(n)}{n} \cdot e^{\frac{\ln(n)}{\ln(\ln(n))}}$ and $w_n \sim e^{\frac{ln(n)}{\ln(\ln(n))}}$.

Indeed the above two sets of parameters give rise to examples of 1-expansive α-Farey systems whose wandering rate is moderately increasing. Equation (11.1) demonstrates the difference between these wandering rates.

Remark 12.7 ([KKSS15, Remark 1.6])**.** If in the definition of the norm $\|\cdot\|_{\mathcal{B}_\alpha}$, one replaces the norm $\|\cdot\|_\infty$ by the *essential supremum norm* $\|\cdot\|_{\text{ess sup}}$, then by appropriately adapting the proofs given in the sequel, one can obtain a proof of Theorem 12.1 where the uniform convergence on compact subsets of $(0, 1]$ is replaced by uniform convergence almost everywhere on compact subsets of $(0, 1]$.

Remark 12.8. As the proof of Theorem 10.6, parts of the proof of Theorem 12.1 and Theorem 12.2 are inspired by the proof of [MT15, Theorem 10.4].

Remark 12.9 ([KKSS15, Remark 1.8])**.** Thaler [Tha00] discerned the precise asymptotic behaviour of iterates of the associated Perron-Frobenius operator $\mathcal{P}$ for certain interval maps $T\colon [0, 1] \to [0, 1]$ with two monotonically increasing, differentiable branches whose invariant measure has infinite mass and whose tail probabilities are regularly varying with exponent $-\delta \in [-1, 0)$. He proved that one has for all Riemann integrable functions u with domain $[0, 1]$ that

$$\lim_{n\to+\infty} w_n(T) \cdot \mathcal{P}^n(u) = \Gamma_\delta \cdot \left(\int u \, \mathrm{d}\lambda \right) \cdot h \tag{12.7}$$

uniformly almost everywhere on compact subsets of $(0, 1]$. Here, h denotes the associated invariant density. However, α-Farey maps do not fall into this class of interval maps. Using the relationship between the transfer and the Perron-Frobenius operator, Theorem 12.2 together with the assumption that the Banach space of functions of bounded variation with the norm $\|\cdot\|_{\text{ess sup}} + \mathrm{Var}(\cdot)$ satisfies the functional analytic conditions *(R1) -(R5)*, given in Section 6.3.2, shows that Thaler's result can be extended to δ-expansive α-Farey maps. Results of this form have also been obtained in [TZ06] for AFN maps. Yet, an α-Farey map is also not an AFN map.

Remark 12.10. In the last remark of this chapter, we rewrite our results in terms of the maps F_α and in the context of probability theory. To this end, we fix a $[0,1]$-valued random variable X_0 with distribution $\mathbb{P}_{X_0} := \mathbb{P} \circ {X_0}^{-1}$ absolutely continuous with respect to μ_α and with probability density $v \in \mathcal{L}^1_{\mu_\alpha}([0,1])$, namely $\mathrm{d}\mathbb{P}_{X_0} = v\,\mathrm{d}\mu_\alpha$. We now consider the process $(X_n)_{n\in\mathbb{N}}$ with $X_n := F_\alpha^n \circ X_0$. The convergence results in the above theorem means that the scaled distribution $w_n \cdot \mathcal{D}(X_n)$ of X_n converges weakly to $\mu_\alpha \cdot \Gamma_\delta \int v\,\mathrm{d}\mu_\alpha$. This follows since for all compact intervals $A \subset (0,1]$ we have for n tending to infinity, that

$$\begin{aligned} w_n \cdot \int \mathbb{1}_A \circ X_n \,\mathrm{d}\mathbb{P} &= w_n \cdot \int \mathbb{1}_A \circ F_\alpha^n \circ X_0 \,\mathrm{d}\mathbb{P} \\ &= w_n \cdot \int \mathbb{1}_A \circ F_\alpha^n \cdot v\,\mathrm{d}\mu_\alpha \\ &= \int \mathbb{1}_A \cdot \widehat{F}_\alpha^n(v)\,\mathrm{d}\mu_\alpha \\ &\to \mu_\alpha(A) \cdot \Gamma_\delta \int v\,\mathrm{d}\mu_\alpha. \end{aligned}$$

In the case of an ergodic probability preserving dynamical system $(X, \mathfrak{B}, \mathbb{P}, T)$, estimates on the rate of mixing of the system have been well studied in [Gou04], in particular the rate of convergence of

$$\int v \cdot w \circ T^n \,\mathrm{d}\mathbb{P} - \int v\,\mathrm{d}\mathbb{P} \cdot \int w\,\mathrm{d}\mathbb{P},$$

where $v, w \in \mathcal{L}^1_{\mathbb{P}}(X)$. Indeed the current results imply the following

$$\lim_{n\to\infty} w_n \cdot \int v \cdot w \circ {F_\alpha}^n \cdot \mathbb{1}_{{F_\alpha}^{-n}(A)}\,\mathrm{d}\mu_\alpha - \Gamma_\delta \cdot \int v\,\mathrm{d}\mu_\alpha \cdot \int w \cdot \mathbb{1}_A\,\mathrm{d}\mu_\alpha = 0,$$

where $w \in \mathcal{L}^\infty_{\mu_\alpha}[0,1]$, v satisfies the conditions of Theorem12.1 or 12.2 depending on δ and A is a compact subset of $(0,1]$.

Chapter 13

Proof of Theorems 12.1 - 12.3

13.1 Asymptotics of the α-Farey transfer operator for $\delta \in (1/2, 1]$

Proof of Theorem 12.1. To apply Theorem 6.10 to the observable v/h_α, we need to show, that $\left\| \mathbb{1}_{A_1} \cdot \widehat{F}^{n-1} \left(v/h_\alpha \cdot \mathbb{1}_{A_n} \right) \right\|_{\mathcal{B}_\alpha}$ is bounded. By Conditions (12.1) and (12.2), it is bounded, since by (5.15) and (11.6), we have, for all $n \in \mathbb{N}$, that

$$\begin{aligned}
&\left\| \mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1} \left(\frac{v}{h_\alpha} \cdot \mathbb{1}_{A_n} \right) \right\|_{\mathcal{B}_\alpha} \\
&= \left\| \mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1} \left(\frac{v}{h_\alpha} \cdot \mathbb{1}_{A_n} \right) \right\|_\infty + D_\alpha \left(\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1} \left(\frac{v}{h_\alpha} \cdot \mathbb{1}_{A_n} \right) \right) \\
&= a_n \cdot \left\| v \cdot \mathbb{1}_{A_n} \right\|_\infty + D_\alpha \left(\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1} \left(\frac{v}{h_\alpha} \cdot \mathbb{1}_{A_n} \right) \right).
\end{aligned}$$

We also discern that $\left\| \mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-1} \left(v/h_\alpha \cdot \mathbb{1}_{A_n} \right) \right\|_\infty$ is summable for $\delta \in (1/2, 1]$, as by condition (12.2), we have that

$$\sum_{k=0}^{\infty} a_k \left\| v \cdot \mathbb{1}_{A_k} \right\|_\infty \ll \sum_{k=0}^{\infty} a_k \cdot t_k \sim \sum_{k=0}^{\infty} \frac{(\ell(k))^2}{k^{1+2\cdot\delta}}. \tag{13.1}$$

For the proof of Theorem 12.1, we make use of Theorem 6.10 and Proposition 6.20. We have that there is a $\theta_n : [0, 1] \to \mathbb{R}$, such that $\sup\{|\theta_n(x)| : x \in \overline{A}_1\} = \mathfrak{o}(1/w_n)$ and, for each $n \in \mathbb{N}_0$, we have that

$$\widehat{F}_\alpha^n \left(\frac{v}{h_\alpha} \right) \cdot \mathbb{1}_{\overline{A}_1} = \frac{\Gamma_\delta}{w_n} \cdot \int v \, \mathrm{d}\lambda \cdot \mathbb{1}_{\overline{A}_1} + \theta_n \cdot \frac{v}{h_\alpha} \cdot \mathbb{1}_{\overline{A}_1}.$$

This in combination with the observation that, for $g \in L^1_{\mu_\alpha}$, we have

$$\int g \mathrm{d}\lambda = \sum_{j=0}^{\infty} \int \frac{g}{h_\alpha} \cdot \mathbb{1}_{A_{j+1}} \mathrm{d}\mu_\alpha = \sum_{j=0}^{\infty} \int \widehat{F}_\alpha^j \left(\frac{g}{h_\alpha} \cdot \mathbb{1}_{A_{j+1}} \right) \mathrm{d}\mu_\alpha,$$

yields, on $\overline{A}_1$, that

$$\begin{aligned}
&\left| w_n \cdot \widehat{F}_\alpha^n\left(\frac{v}{h_\alpha}\right) - \Gamma_\delta \cdot \int v \, \mathrm{d}\lambda \right| \\
&= \left| w_n \cdot \sum_{j=0}^{n} \mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-j}\left(\widehat{F}_\alpha^j\left(\frac{v}{h_\alpha} \cdot \mathbb{1}_{A_{j+1}}\right)\right) - \Gamma_\delta \cdot \int v \, \mathrm{d}\lambda \right| \\
&= \Bigg| \Gamma_\delta \cdot \sum_{j=0}^{n} \frac{w_n}{w_{n-j}} \int \widehat{F}_\alpha^j\left(\frac{v}{h_\alpha} \cdot \mathbb{1}_{A_{j+1}}\right) \mathrm{d}\mu_\alpha \\
&\qquad - \Gamma_\delta \cdot \int v \, \mathrm{d}\lambda + w_n \cdot \sum_{j=0}^{n} \theta_{n-j} \cdot \widehat{F}_\alpha^j\left(\frac{v}{h_\alpha} \cdot \mathbb{1}_{A_{j+1}}\right) \Bigg| \\
&\leq \Gamma_\delta \cdot \left(\sum_{j=0}^{n} \left(\frac{w_n}{w_{n-j}} - 1 \right) \cdot \int \left| v \cdot \mathbb{1}_{A_{j+1}} \right| \mathrm{d}\lambda \right. && (13.2) \\
&\quad + w_n \cdot \sum_{j=0}^{n} \left\| \theta_{n-j} \right\|_\infty \cdot \left\| \widehat{F}_\alpha^j\left(\frac{v}{h_\alpha} \cdot \mathbb{1}_{A_{j+1}}\right) \right\|_\infty && (13.3) \\
&\quad + \Gamma_\delta \cdot \sum_{j=n+1}^{\infty} \int \left| v \cdot \mathbb{1}_{A_{j+1}} \right| \mathrm{d}\lambda. && (13.4)
\end{aligned}$$

To prove the theorem, we need to show, that each of the summands (13.2), (13.3) and (13.4) tends to zero as n tends to infinity.

To see that (13.2) converges to zero, we split the sum in it up into two parts. To this end, fix an ϵ, with $0 < \epsilon < 1$ and write

$$\begin{aligned}
\sum_{j=0}^{n} \left(\frac{w_n}{w_{n-j}} - 1 \right) \cdot \int \left| v \cdot \mathbb{1}_{A_{j+1}} \right| \mathrm{d}\lambda &= \sum_{j=0}^{\lfloor \epsilon \cdot n \rfloor} \left(\frac{w_n}{w_{n-j}} - 1 \right) \cdot \int \left| v \cdot \mathbb{1}_{A_{j+1}} \right| \mathrm{d}\lambda \\
&\quad + \sum_{j=\lfloor \epsilon \cdot n \rfloor + 1}^{n} \left(\frac{w_n}{w_{n-j}} - 1 \right) \cdot \int \left| v \cdot \mathbb{1}_{A_{j+1}} \right| \mathrm{d}\lambda.
\end{aligned}$$

We observe that

$$\begin{aligned}
\sum_{j=0}^{\epsilon \cdot n} \left(\frac{w_n}{w_{n-j}} - 1 \right) \cdot \int \left| v \cdot \mathbb{1}_{A_{j+1}} \right| \mathrm{d}\lambda &\ll \int |v| \, \mathrm{d}\lambda \cdot \left(\frac{w_n}{w_{(1-\epsilon)n}} - 1 \right) \\
&\sim \int |v| \, \mathrm{d}\lambda \cdot \left(\frac{n^{1-\delta} \cdot \ell(n)}{(1-\epsilon)^{1-\delta} \cdot n^{1-\delta} \cdot \ell((1-\epsilon) \cdot n)} - 1 \right) \\
&\sim \int |v| \, \mathrm{d}\lambda \cdot \left(\frac{1}{(1-\epsilon)^{1-\delta}} - 1 \right).
\end{aligned}$$

Furthermore, we observe, using Theorem 4.3, that

$$
\begin{aligned}
&\sum_{j=\lfloor \epsilon\cdot n\rfloor+1}^{n} \left(\frac{w_n}{w_{n-j}} - 1\right) \cdot \int \left|v \cdot \mathbb{1}_{A_{j+1}}\right| \mathrm{d}\lambda \\
&\leq \frac{w_n}{n} \cdot \sum_{j=\lfloor \epsilon\cdot n\rfloor}^{n} \frac{1}{w_{n-j}} \cdot j \cdot \int \left|v \cdot \mathbb{1}_{A_{j+1}}\right| \mathrm{d}\lambda \\
&\leq \frac{w_n}{n} \cdot \sum_{j=\lfloor \epsilon\cdot n\rfloor}^{n} \frac{1}{w_{n-j}} \cdot (j+1) \cdot a_{j+1} \cdot \left\|v \cdot \mathbb{1}_{A_{j+1}}\right\|_\infty \\
&\ll \frac{w_n}{n} \cdot \left(\sum_{j=\lfloor \epsilon\cdot n\rfloor}^{n} \frac{1}{w_{n-j}}\right) \cdot \max_{0\leq i\leq \epsilon\cdot n} t_i \cdot \left\|v \cdot \mathbb{1}_{A_{i+1}}\right\|_\infty \\
&= \frac{w_n}{n} \cdot \left(\sum_{j=0}^{\lceil (1-\epsilon)\cdot n\rceil} \frac{1}{w_j}\right) \cdot \max_{0\leq i\leq \epsilon\cdot n} t_i \cdot \left\|v \cdot \mathbb{1}_{A_{i+1}}\right\|_\infty \\
&\sim \frac{w_n}{n} \cdot \frac{(1-\epsilon)\cdot n}{w_{(1-\epsilon)\cdot n}} \cdot \max_{0\leq i\leq \epsilon\cdot n} t_i \cdot \left\|v \cdot \mathbb{1}_{A_{i+1}}\right\|_\infty \\
&= (1-\epsilon)^\delta \cdot \frac{\ell(n)}{\ell((1-\epsilon)\cdot n)} \cdot \max_{0\leq i\leq \epsilon\cdot n} t_i \cdot \left\|v \cdot \mathbb{1}_{A_{i+1}}\right\|_\infty .
\end{aligned}
$$

By Condition (12.2), the last term tends to zero as n tends to infinity, and since ϵ was chosen arbitrarily, we obtain that (13.2) tends so zero.
The next aim is to show that (13.3) tends to zero.
By (5.13) and the fact that $\sup\{|\theta_n(x)| : x \in \overline{A}_1\} = \mathfrak{o}(1/w_n)$, given $\xi > 0$ there exists $M(\xi) \in \mathbb{N}$ such that, for all $m \geq M(\xi)$,

$$
\frac{\overline{\Gamma}_\delta \cdot l(m) \cdot m^{1-\delta}}{e^\xi} \leq w_m \leq \overline{\Gamma}_\delta \cdot e^\xi \cdot l(m) \cdot m^{1-\delta} \quad \text{and} \quad ||\theta_m||_\infty \leq \frac{\xi}{w_m}.
$$

Moreover, there exist constants $c_1, c_2 \in \mathbb{R}$ such that, for all $n \in \mathbb{N}_0$,

$$
w_n \geq \overline{\Gamma}_\delta \cdot \left(\frac{l(n) \cdot n^{1-\delta}}{e^\xi} + c_1\right) \quad \text{and} \quad ||\theta_n||_\infty \leq \frac{c_2}{w_n}.
$$

Furthermore, since F_α is δ-expansive, by (13.1), we have that the sequence $\left(a_n \cdot \left\|v \cdot \mathbb{1}_{A_n}\right\|_\infty\right)_{n\in\mathbb{N}}$ is summable and

$$
a_n \cdot w_{n-1} \sim \frac{\overline{\Gamma}_\delta \cdot \delta \cdot (l(n))^2}{n^{2\cdot\delta}}.
$$

These properties together with Lemma 11.4, imply the existence of a constant

$c_3 > 0$ such that

$$
\begin{aligned}
0 &\leq \limsup_{n\to+\infty} w_n \cdot \sum_{j=0}^{n} \left\|\theta_{n-j}\right\|_\infty \cdot \left\|\widehat{F}_\alpha^j\left(v \cdot \frac{\mathbb{1}_{A_{j+1}}}{h_\alpha}\right)\right\|_\infty \\
&\leq \limsup_{n\to+\infty} \xi \cdot e^{2\cdot\xi} \cdot \sum_{j=0}^{n-M(\xi)} \frac{\ell(n)\cdot n^{1-\delta}}{\ell(n-j)\cdot(n-j)^{1-\delta}} \cdot \left\|v\cdot \mathbb{1}_{A_{j+1}}\right\|_\infty \cdot a_{j+1} \\
&\quad + \limsup_{n\to+\infty} \frac{c_2\cdot w_n \cdot a_{n+1}}{w_0} \\
&\quad + \limsup_{n\to+\infty} c_3 \cdot e^{2\cdot\xi} \cdot \sum_{j=n-M(\xi)+1}^{n-1} \frac{l(n)\cdot n^{1-\delta}}{l(n-j)\cdot(n-j)^{1-\delta} + c_1\cdot e^{\xi}} \cdot a_{j+1} \cdot \left\|v\cdot \mathbb{1}_{A_{j+1}}\right\|_\infty \\
&= \xi \cdot e^{2\cdot\xi} \cdot \sum_{j=0}^{\infty} a_{j+1} \cdot \left\|v\cdot \mathbb{1}_{A_{j+1}}\right\|_\infty .
\end{aligned}
$$

Since $\xi > 0$ was chosen arbitrarily, the result for (13.3) follows.
Finally, since $v \in \mathcal{L}_\lambda^1([0,1])$, we discern, that (13.4) tends to zero. Since the arguments given above are independent of a given point in $\overline{A}_1$, an application of Theorem 6.11 finishes the proof of Theorem 12.1.

□□

13.2 Asymptotics of the α-Farey transfer operator for $\delta = 1$

Throughout this section, we let $([0,1], \mathscr{B}, \mu_\alpha, F_\alpha)$ denote a 1-expansive α-Farey system with wandering rate w_n. In order to prove Theorem 12.2, we will use the auxiliary results, Lemmata 13.1 and 13.2. Before which we require the linear interpolation of the wandering rate, which we denote by $\mathfrak{w}(\cdot)$. That is, we define the function $\mathfrak{w}\colon [0,\infty) \to \mathbb{R}$ by

$$
\mathfrak{w}(x) := \begin{cases} \frac{x}{2} + \frac{1}{2} & \text{if } x \in [0,1], \\ \\ t_{n+1}\cdot(x-n) + w_n & \text{if } x \in [n, n+1], \text{ for } n \in \mathbb{N}. \end{cases} \tag{13.5}
$$

Further, for $\sigma \in \mathbb{R}^+$, we define for all $x \geq \mathfrak{w}^{-1}((1+\sigma)/2)$,

$$
j_\sigma(x) := x - \mathfrak{w}^{-1}\left(\frac{\mathfrak{w}(x)}{1+\sigma}\right).
$$

Lemma 13.1 ([KKSS15, Lemma 4.1])**.** *For a given $\sigma \in \mathbb{R}^+$, we have that $j_\sigma(x) \sim x$.*

Proof of Lemma 13.1. For $\sigma \in \mathbb{R}^+$, we have that

$$\lim_{x\to\infty} \frac{\mathfrak{w}^{-1}\left(\frac{\mathfrak{w}(x)}{1+\sigma}\right)}{x} = \lim_{x\to\infty} \frac{\mathfrak{w}^{-1}\left(\frac{\mathfrak{w}(x)}{1+\sigma}\right)}{\mathfrak{w}^{-1}(\mathfrak{w}(x))} = 0,$$

and the result follows. The second equality follows from the fact that $\mathfrak{w}$ is a positive, strictly monotonically increasing function and Lemma 4.2 (iii).

□

Lemma 13.2 ([KKSS15, Lemma 4.2]). *Let $(\delta_j)_{j\in\mathbb{N}}$ denote a sequence of positive real numbers such that $\sum_{j=1}^{\infty} \delta_j \cdot t_j < \infty$. If the wandering rate is moderately increasing, then*

$$\lim_{n\to\infty} \sum_{j=0}^{n} \frac{w_n}{w_{n-j}} \cdot \delta_{j+1} \cdot t_{j+1} = \sum_{j=1}^{\infty} \delta_j \cdot t_j.$$

Proof of Lemma 13.2. We assume that $\sup\{\delta_j : j \in \mathbb{N}\} = 1$, without loss of generality. Furthermore, as we will see in (13.6), we may assume, without loss of generality, that $\lfloor j_\sigma(n)\rfloor + 1 \le \lfloor n - n\cdot(\mathfrak{w}(n))^{-2}\rfloor$. If this would not be the case, we would split the following sum only into two parts, leaving out the second summand. Let $\sigma \in \mathbb{R}^+$ be fixed. By definition of $\mathfrak{w}$, we have, for $n \ge \mathfrak{w}^{-1}((1+\sigma)/2)$, that

$$\begin{aligned}
&\sum_{j=0}^{n} \frac{w_n}{w_{n-j}} \cdot \delta_{j+1} \cdot t_{j+1} \\
&\le \frac{\mathfrak{w}(n)}{\mathfrak{w}(n - j_\sigma(n))} \cdot \sum_{j=0}^{\lfloor j_\sigma(n)\rfloor} \delta_{j+1} \cdot t_{j+1} + \frac{\mathfrak{w}(n)}{\mathfrak{w}\left(\frac{n}{(\mathfrak{w}(n))^2}\right)} \cdot \sum_{j=\lfloor j_\sigma(n)\rfloor+1}^{\left\lfloor n - \frac{n}{(\mathfrak{w}(n))^2}\right\rfloor} \delta_{j+1} \cdot t_{j+1} \\
&\quad + 2\mathfrak{w}(n) \cdot \sum_{j=\left\lfloor n - \frac{n}{(\mathfrak{w}(n))^2}\right\rfloor+1}^{n} t_{j+1}.
\end{aligned}$$

By Lemma 4.2 (iv) and since $(t_j)_{j\in\mathbb{N}}$ is a regularly varying sequence of order -1, we have that,

$$\lim_{n\to\infty} 2\cdot\mathfrak{w}(n) \sum_{j=\lfloor n-n(\mathfrak{w}(n))^{-2}\rfloor+1}^{n} t_{j+1} \le \lim_{n\to\infty} \frac{2\cdot t_{n+1}\cdot n}{\mathfrak{w}(n)} = 0. \tag{13.6}$$

Further, since $\mathfrak{w}$ is an unbounded monotonically increasing function we have that $n - n\cdot(\mathfrak{w}(n))^{-2} \sim n$ and, by Lemma 13.1, we have that $j_\sigma(n) \sim n$. These statements in tandem with the assumptions that $\sum_{j=1}^{\infty} \delta_j \cdot t_j < \infty$ and the assumption that the wandering rate is moderately increasing, yield the following:

$$\lim_{n\to\infty} \frac{\mathfrak{w}(n)}{\mathfrak{w}\left(\frac{n}{(\mathfrak{w}(n))^2}\right)} \cdot \sum_{j=\lfloor j_\sigma(n)\rfloor+1}^{\left\lfloor n-\frac{n}{(\mathfrak{w}(n))^2}\right\rfloor} \delta_{j+1} \cdot t_{j+1} = 0.$$

Finally, observing that

$$\frac{\mathfrak{w}(n)}{\mathfrak{w}(n - j_\sigma(n))} = 1 + \sigma,$$

finishes the proof, as σ was chosen arbitrarily.

□

After these two auxiliary results, we prove the main result.

Proof of Theorem 12.2. Following the same arguments as in the proof of Theorem 12.1, we have by Theorem 6.10 and Proposition 6.20, that for each $n \in \mathbb{N}$, there exists $\theta_n : (0,1) \to \mathbb{R}$ such that $\sup\{|\theta_n(x)| : x \in \overline{A}_1\} = \mathfrak{o}(1/w_n)$ and

$$\widehat{F}_\alpha^n(\mathbb{1}_{\overline{A}_1} \cdot v) \cdot \mathbb{1}_{\overline{A}_1} = \frac{1}{w_n} \cdot \int \mathbb{1}_{\overline{A}_1} \cdot v \, \mathrm{d}\mu_\alpha \cdot \mathbb{1}_{\overline{A}_1} + \theta_n \cdot v \cdot \mathbb{1}_{\overline{A}_1}. \tag{13.7}$$

By (13.7), we have on $\overline{A}_1$ that

$$\begin{aligned}
&\left| w_n \cdot \widehat{F}_\alpha^n(v) - \int v \, \mathrm{d}\mu_\alpha \right| \\
&= \left| w_n \cdot \sum_{j=0}^{n} \mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^{n-j}\left(\mathbb{1}_{A_1} \cdot \widehat{F}_\alpha^j\left(v \cdot \mathbb{1}_{A_{j+1}}\right)\right) - \int v \, \mathrm{d}\mu_\alpha \right| \\
&= \left| w_n \cdot \sum_{j=0}^{n} \frac{1}{w_{n-j}} \int \widehat{F}_\alpha^j\left(v \cdot \mathbb{1}_{A_{j+1}}\right) \mathrm{d}\mu_\alpha - \int v \, \mathrm{d}\mu_\alpha + w_n \cdot \sum_{j=0}^{n} \theta_{n-j} \cdot \widehat{F}_\alpha^j(v \cdot \mathbb{1}_{A_{j+1}}) \right| \\
&\leq \sum_{j=0}^{n} \left(\frac{w_n}{w_{n-j}} - 1 \right) \cdot \int \left| v \cdot \mathbb{1}_{A_{j+1}} \right| \mathrm{d}\mu_\alpha && (13.8) \\
&\quad + w_n \cdot \sum_{j=0}^{n} \left\| \theta_{n-j} \right\|_\infty \cdot \left\| \widehat{F}_\alpha^j\left(v \cdot \mathbb{1}_{A_{j+1}}\right) \right\|_\infty && (13.9) \\
&\quad + \sum_{j=n+1}^{\infty} \int |v \cdot \mathbb{1}_{A_{j+1}}| \, \mathrm{d}\mu_\alpha. && (13.10)
\end{aligned}$$

Since $v \in \mathcal{A}_\alpha \subseteq \mathcal{L}^1_{\mu_\alpha}([0,1])$, it follows that (13.10) converges to zero. To see that (13.8) and (13.9) converge to zero, observe that

(i) Since $v \in \mathcal{A}_\alpha$, we have that $v \in \mathcal{L}^1_{\mu_\alpha}([0,1])$ and, moreover, we have that

$$\int \left| v \cdot \mathbb{1}_{A_j} \right| \mathrm{d}\mu_\alpha = \frac{t_j}{a_j} \cdot \int v \cdot \mathbb{1}_{A_j} \, \mathrm{d}\lambda.$$

(ii) Since $v \in \mathcal{A}_\alpha$ we have that $||v||_\infty$ is finite, and so the sequence

$$\left(\frac{1}{a_j} \cdot \int v \cdot \mathbb{1}_{A_j} \, \mathrm{d}\lambda \right)_{j \in \mathbb{N}}$$

is a bounded sequence.

(iii) Using Lemma 11.4 together with the fact that $\widehat{F}_\alpha$ is positive and linear and the fact that $v \in \mathcal{A}_\alpha$, we have that $\left|\widehat{F}_\alpha^{j-1}(v \cdot \mathbb{1}_{A_j})(x)\right| \leq \|v\|_\infty \cdot t_j$.

(iv) Given $\epsilon > 0$, there exists $N_\epsilon \in \mathbb{N}$ such that $\|\theta_m\|_\infty \leq \epsilon/\mathfrak{w}(m)$, for all $m \geq N_\epsilon$.

Combining these observations with Lemma 13.2 and (11.5), we have that (13.8) and (13.9) converge to zero. Since the arguments given above are independent of a given point in $\overline{A}_1$, an application of Theorem 6.11 now finishes the proof.

□□

In the proof of Theorem 12.1 and Theorem 12.2 we have not used the specific structure of $\mathcal{B}_\alpha$. We only used that $\mathcal{B}_\alpha$ is a Banach space which satisfies conditions *(R1)* to *(R5)*. Thus, we may replace $\mathcal{B}_\alpha$ by an arbitrary Banach space which satisfies conditions *(R1)* to *(R5)*. For such alternative Banach spaces see Remark 12.9 or Part II. In doing such a substitution one may change uniform convergence to almost everywhere uniform convergence.

13.3 Non-convergence for $\delta \in (1/2, 1)$

This section gives a constructive proof of Theorem 12.3.

Proof of Theorem 12.3. The proof is divided into several parts. First, we define a class of observables $\mathcal{V}$. Second, in Proposition 13.7 we will show that if $v \in \mathcal{V}$, then v is bounded, of bounded variation, Riemann integrable and belongs to $\mathcal{L}^1_{\mu_\alpha}([0,1])$. Third, in Proposition 13.8 we will show that if $v \in \mathcal{V}$, then it belongs to the space $\mathcal{A}_\alpha$, in particular we will show, that the summability condition given in (11.5) is satisfied for all $v \in \mathcal{V}$. Finally, in Proposition 13.10 we will show that, if $v \in \mathcal{V}$, then

$$\liminf_{n\to+\infty} w_n \cdot \widehat{F}_\alpha^n(v)(x) = \Gamma_\delta \cdot \int v \mathrm{d}\mu_\alpha \quad \text{and} \quad \limsup_{n\to+\infty} w_n \cdot \widehat{F}_\alpha^n(v)(x) = +\infty.$$

Combing these results yields a proof of Theorem 12.3. To define the set $\mathcal{V}$, we let g_1, g_2 and g_3 denote three positive constants, depending on δ, such that

(C1) $g_1 > \frac{1}{1-\delta}$,

(C2) $\delta g_1 > g_2$,

(C3) there exists $\epsilon \in (0, \delta - 1/2)$, such that $g_2 \cdot (\delta - \epsilon) > (2\cdot\delta + 2\cdot\epsilon - 1)\cdot g_1 + g_3$.

These constants give rise to three sequences, $(N_k)_{k\in\mathbb{N}}$, $(n_k)_{k\in\mathbb{N}}$ and $(s_k)_{k\in\mathbb{N}}$, whose elements are given by

$$N_k := \left\lceil 2^{g_1\cdot k}\right\rceil, n_k := \left\lfloor 2^{g_2\cdot k}\right\rfloor \quad \text{and} \quad s_k := \frac{1}{2^{g_3\cdot k}}.$$

Finally, we let $\mathcal{V}$ denote the class of observables $v : [0,1] \to \mathbb{R}$ which are of the form:

$$v := \sum_{k=1}^{\infty} s_k \cdot \sum_{j=N_k-n_k}^{N_k} \mathbb{1}_{A_j}. \tag{13.11}$$

The next two examples show, that the class of observables is not vain.

Example 13.3. For $\delta \in (1/2, 1)$, choose

$$g_1 := \frac{1+\epsilon}{1-\delta},$$
$$g_2 := \frac{\delta}{1-\delta},$$

Then g_1 and g_2 satisfy the conditions **(C1)** and **(C2)**. With these choices we can verify that **(C3)** is equivalent to

$$g_3 < (1-\delta) - \frac{3\cdot\delta + 1 + 2\cdot\epsilon}{1-\delta} \cdot \epsilon.$$

Hence, by choosing $\epsilon > 0$ sufficiently small, it follows that the conditions **(C1)**, **(C2)** and **(C3)** can be satisfied simultaneously.

Example 13.4. For $\delta \in (1/2, 1)$, set

$$g_1 := \frac{1}{(1-\delta)^2},$$
$$g_2 := \frac{\delta^2 + 2\cdot\delta - 1}{2\cdot\delta\cdot(1-\delta)^2},$$
$$g_3 := \frac{1}{8},$$
$$\text{and} \quad \epsilon := \frac{\delta\cdot(1-\delta)^2}{2\cdot\delta^2 + 12\cdot\delta - 2}.$$

For each $\delta \in (1/2, 1)$, these values are positive and satisfy conditions **(C1)**, **(C2)** and **(C3)**.
For instance, if $\delta := 3/4$, then we have $g_1 = 16$, $g_2 = 34/3$, $g_3 = 1/8$ and $\epsilon = 3/520$.

The main reason why we require the sequence $(s_k)_{k\in\mathbb{N}}$, is to ensure that v is of bounded variation. Further, condition **(C3)** is only required in the proof of the second statement of Proposition 13.10, specifically when Lemma 13.9 is used.
Before we begin with Proposition 13.7, we give two technical lemmata which we will use in its proof.

Lemma 13.5 ([KKSS15, Lemma 4.7])**.** *If $s \in (0,1)$ and $1 < b < a^s$, then*

$$\sum_{k=1}^{\infty} a^{(1-s)\cdot k} - (a^k - b^k)^{1-s} \le \sum_{k=1}^{\infty} \left(\frac{b}{a^s}\right)^k < +\infty.$$

Proof of Lemma 13.5. By assumption, we have that $a/b > 1$ and so

$$1 - \left(\frac{b}{a}\right)^k \leq \left(1 - \left(\frac{b}{a}\right)^k\right)^{1-s},$$

which implies for each $k \in \mathbb{N}$, that

$$\left(\frac{a}{b}\right)^k - \left(\frac{a}{b}\right)^k \cdot \left(1 - \left(\frac{b}{a}\right)^k\right)^{1-s} \leq 1.$$

Hence, we have that

$$\begin{aligned}
\sum_{k=1}^{\infty} a^{(1-s)\cdot k} - (a^k - b^k)^{1-s} &= \sum_{k=1}^{\infty} \left(\frac{b}{a^s}\right)^k \cdot \left(\left(\frac{a}{b}\right)^k - \left(\frac{a}{b}\right)^k \cdot \left(1 - \left(\frac{b}{a}\right)^k\right)^{1-s}\right) \\
&\leq \sum_{k=1}^{\infty} \left(\frac{b}{a^s}\right)^k \\
&< +\infty.
\end{aligned}$$

□

The next lemma shows that the sum in the definition of the observable v, given in (13.11) is reasonable.

Lemma 13.6 ([KKSS15, Lemma 4.8]). *For $k \in \mathbb{N}$, we have that $N_{k+1} - n_{k+1} > N_k$.*

Proof of Lemma13.6. We have, for all $k \in \mathbb{N}$, that

$$\begin{aligned}
\frac{N_{k+1} - n_{k+1}}{N_k} &\geq \frac{2^{g_1\cdot(k+1)} - 2^{g_2\cdot(k+1)} - 1}{2^{g_1\cdot k} + 1} \\
&= \frac{2^{g_1\cdot(k+1)}\left(1 - 2^{(g_2-g_1)\cdot(k+1)} - 2^{-g_1\cdot(k+1)}\right)}{2^{g_1\cdot k}\cdot(1 + 2^{-g_1\cdot k})} \\
&= \frac{2^{g_1}\cdot\left(1 - 2^{(g_2-g_1)\cdot(k+1)} - 2^{-g_1\cdot(k+1)}\right)}{2^{-g_1\cdot k} + 1} \\
&\geq \frac{2^{g_1}\cdot\left(1 - 2^{2\cdot(g_2-g_1)} - 2^{-2g_1}\right)}{2^{-g_1} + 1}.
\end{aligned}$$

By **(C1)** and **(C2)**, we have that $g_1 > (1-\delta)^{-1}$ and $g_2 - g_1 < 0$ and hence, the last term is strictly greater than one.

□

Proposition 13.7. *An observable v defined as in* (13.11) *is bounded, of bounded variation, Riemann integrable and belongs to the space $\mathcal{L}^1_{\mu_\alpha}([0,1])$.*

Proof of Proposition 13.7. By its construction, the observable v is Riemann integrable. Moreover, v is measurable, as each of the atoms of α is measurable and v is the sum of indicator functions of atoms of α. Further, the range of v is equal to $\{0\} \cup \{s_k : k \in \mathbb{N}\}$, and thus, $\|v\|_\infty = s_1$. By Lemma 13.6, we have that $N_{k+1} - n_{k+1} > N_k$, and so the variation of v is equal to $2 \cdot \sum_{k=1}^{\infty} s_k$, which is finite, as $s_k := 2^{-g_3 \cdot k}$ and as g_3 is positive. This shows that v is of bounded variation. It remains to show that v is μ_α-integrable. For this recall that $\mu_\alpha(A_k) = t_k$, for each $k \in \mathbb{N}$. Choose a positive constant $\eta < \min\{\delta, g_3/g_1\}$ and recall that $t_n \sim l(n) \cdot n^{-\delta}$. By Lemma 4.2 (ii), there exists a constant $c > 0$ such that $t_n \leq c \cdot l(n) \cdot n^{-\delta} \leq c \cdot n^{\eta-\delta}$, for each $n \in \mathbb{N}$. Therefore, by Lemma 13.5 and Lemma 13.6, we have that

$$\begin{aligned}
\int |v| \, \mathrm{d}\mu_\alpha &= \sum_{k=1}^{\infty} s_k \cdot \sum_{j=N_k-n_k}^{N_k} t_j \\
&\leq \sum_{k=1}^{\infty} 2^{-g_3 \cdot k} \cdot \sum_{j=N_k-n_k}^{N_k} \frac{c}{j^{\delta-\eta}} \\
&\leq \frac{c}{1-\delta+\eta} \cdot \sum_{k=1}^{\infty} \left(\frac{2^{g_1 \cdot k \cdot (1-\delta+\eta)} - \left(2^{g_1 \cdot k} - 2^{g_2 \cdot k}\right)^{1-\delta+\eta}}{2^{-g_3 \cdot k}} + \frac{N_k^{\eta-\delta}}{2^{g_3 \cdot k}} \right) \\
&\leq \frac{c}{1-\delta+\eta} \cdot \sum_{k=1}^{\infty} \left(2^{(g_2 - \delta g_1 + \eta g_1 - g_3) \cdot k} + \frac{N_k^{\eta-\delta}}{2^{-g_3 \cdot k}} \right).
\end{aligned}$$

The last series converges, since $\eta < \min\{\delta, g_3/g_1\}$, $g_2 < \delta \cdot g_1$ and $N_k > 1$, for all $k \in \mathbb{N}$.

$\square$

Our next aim is to show that v belongs to $\mathcal{A}_\alpha$, in particular it satisfies the summability condition given in (11.5).

Proposition 13.8 ([KKSS15, Propositions 4.10 and 4.11])**.** *An observable v defined as in* (13.11) *belongs to $\mathcal{A}_\alpha$.*

Proof of Proposition 13.8. By Proposition 13.7, we have that $v \in \mathcal{L}^1_{\mu_\alpha}([0,1])$ and that $\|v\|_\infty = 1$. Moreover, by Lemma 11.4, we have on $[0,1]$, that, for each $j \in \mathbb{N}$, there exists a natural number k, such that

$$\widehat{F}_\alpha^{j-1}\left(v \cdot \mathbb{1}_{A_j}\right)(x) = \begin{cases} t_j \cdot s_k & \text{if } N_k - n_k \leq j \leq N_k \text{ and if } x \in A_1, \\ 0 & \text{otherwise.} \end{cases} \tag{13.12}$$

Therefore, $\widehat{F}_\alpha^{j-1}\left(v \cdot \mathbb{1}_{A_j}\right) \in \mathcal{B}_\alpha$, for all $j \in \mathbb{N}$, and hence, it is left to show, that an observable v defined as in (13.11) satisfies the summability condition given in (11.5).

Lemma 11.4 and (13.12) together imply that

$$\sum_{k=0}^{\infty} \left\| \widehat{F}_{\alpha}^{k} \left(v \cdot \mathbb{1}_{A_{k+1}} \right) \right\|_{\infty} = \sum_{k=1}^{\infty} s_k \cdot \sum_{j=N_k-n_k}^{N_k} t_j = \sum_{k=1}^{\infty} s_k \cdot \sum_{j=N_k-n_k}^{N_k} \mu_\alpha(A_j) = \int |v| \, \mathrm{d}\mu_\alpha.$$

The last term is finite, since, by Proposition 13.7, we have that $v \in \mathcal{L}^1_{\mu_\alpha}([0,1])$.

□

In the proof of Proposition 13.10, we will require the following auxiliary result.

Lemma 13.9 ([KKSS15, Lemma 4.12])**.** *For each $N \in \mathbb{N}$, the sequence*

$$\left(s_k \sum_{j=N_k-n_k}^{N_k-N} \frac{N_k^{1-\delta} \cdot l(N_k) \cdot l(j)}{(N_k - j)^{1-\delta} \cdot l(N_k - j) \cdot j^{\delta}} \right)_{k \in \mathbb{N}}$$

diverges to infinity.

Proof of Lemma 13.9. The result follows from combining the following three observations.

(i) Using the facts that $\delta \in (1/2, 1)$ and $\epsilon > 0$, that the sequence $(N_k)_{k\in\mathbb{N}}$ is not bounded above and is strictly monotonically increasing, that $s_k := 2^{-g_3 \cdot k}$ and that N is a fixed natural number, we have that

$$\lim_{k\to+\infty} s_k \cdot \left(\frac{N_k}{N_k - N} \right)^{\delta} \cdot {N_k}^{1-2\cdot\delta-2\cdot\epsilon} \cdot N^{\delta-\epsilon} = 0.$$

(ii) For each $k \in \mathbb{N}$, we have that

$$\begin{aligned} s_k \cdot N_k^{1-2\cdot\delta-2\cdot\epsilon} n_k^{\delta-\epsilon} &\geq 2^{-g_3 \cdot k} \cdot 2^{g_1 \cdot (1-2\cdot\delta-2\cdot\epsilon)\cdot k} \cdot \left(2^{g_2 \cdot (\delta-\epsilon)} - 1 \right) \\ &= 2^{(g_1\cdot(1-2\cdot\delta-2\cdot\epsilon)+g_2\cdot(\delta-\epsilon)-g_3)\cdot k} - 2^{(g_1\cdot(1-2\cdot\delta-2\cdot\epsilon)-g_3)\cdot k}. \end{aligned}$$

Using condition **(C3)** with the facts that $\delta \in (1/2, 1)$, $\epsilon > 0$ and that g_1, g_2 and g_3 are positive, it follows that

$$\lim_{k\in\mathbb{N}} s_k \cdot {N_k}^{1-2\cdot\delta-2\cdot\epsilon} \cdot {n_k}^{\delta-\epsilon} = +\infty.$$

(iii) There exist constants $\kappa, \xi > 0$ such that, for all $k \in \mathbb{N}$ sufficiently large,

$$\begin{aligned} &\sum_{j=N_k-n_k}^{N_k-N} \frac{{N_k}^{1-\delta} \cdot l(N_k) \cdot l(j)}{(N_k - j)^{1-\delta} \cdot l(N_k - j) \cdot j^{\delta}} \\ &\geq \left(\frac{1}{N_k - N} \right)^{\delta} \cdot \frac{l(N_k - N) \cdot l(N_k) \cdot N_k^{1-\delta}}{e^{\xi}} \cdot \sum_{j=N_k-n_k}^{N_k-N} \frac{1}{(N_k - j)^{1-\delta} \cdot l(N_k - j)} \\ &\geq \frac{\kappa}{e^{\xi}} \cdot \left(\frac{N_k}{N_k - N} \right)^{\delta} \cdot N_k^{1-2\cdot\delta-2\cdot\epsilon} \cdot \left(n_k^{\delta-\epsilon} - N^{\delta-\epsilon} \right). \end{aligned}$$

Here, the first inequality follows from the facts that $l(\cdot)$ is a slowly varying function and that $\lim_{k\to\infty}(N_k - n_k)/N_k = 1$ together with Lemma 4.2 (i). The second inequality follows from Lemma 4.2 (ii), which guarantees the existence of the constant $\kappa > 0$ such that, we have for all $n \in \mathbb{N}$,

$$\frac{n^{\epsilon}}{\kappa} \geq l(n) \geq \frac{\kappa}{n^{\epsilon}}$$

□

Proposition 13.10 ([KKSS15, Proposition 4.13]). *For an observable v as defined in* (13.11)*, we have that, on $\overline{A}_1$,*

$$\liminf_{n\to+\infty} w_n \cdot \widehat{F}_{\alpha}^{n}(v) = \Gamma_{\delta} \cdot \int v \, \mathrm{d}\mu_{\alpha} \quad \textit{and} \quad \limsup_{n\to+\infty} w_n \cdot \widehat{F}_{\alpha}^{n}(v) = +\infty. \tag{13.13}$$

Proof of Proposition 13.10. By Theorem 6.10 and Proposition 6.20, we have uniformly on $\overline{A}_1$ that

$$\lim_{n\to+\infty} l(n) \cdot n^{1-\delta} \cdot \mathbb{1}_{\overline{A}_1} \cdot \widehat{F}_{\alpha}^{n}(\mathbb{1}_{A_1}) = \frac{\Gamma_{\delta}}{\overline{\Gamma}_{\delta}} \cdot \mu_{\alpha}(A_1) \cdot \mathbb{1}_{\overline{A}_1} = \frac{\Gamma_{\delta}}{\overline{\Gamma}_{\delta}} \cdot \mathbb{1}_{\overline{A}_1}.$$

Thus, given $\xi > 0$, there exists $N(\xi) \in \mathbb{N}$ such that, for all $n \geq N(\xi)$ on $\overline{A}_1$,

$$\frac{e^{\xi} \cdot \Gamma_{\delta} \cdot n^{\delta-1}}{\overline{\Gamma}_{\delta} \cdot l(n)} \geq \widehat{F}_{\alpha}^{n}(\mathbb{1}_{A_1}) \geq \frac{e^{-\xi} \cdot \Gamma_{\delta} \cdot n^{\delta-1}}{\overline{\Gamma}_{\delta} \cdot l(n)}. \tag{13.14}$$

We will first show the second statement in (13.13). To this end, observe that by (5.13) it is sufficient to show that, on A_1,

$$\limsup_{k\to+\infty} l(N_k) \cdot {N_k}^{1-\delta} \cdot \widehat{F}_{\alpha}^{N_k}(v)(x) = +\infty. \tag{13.15}$$

In order to see this, let $\xi > 0$ be fixed and let $p(\xi) \in \mathbb{N}$ denote the smallest integer for which $n_{p(\xi)} > N(\xi)$. Since $\widehat{F}_{\alpha}$ is a positive linear operator, we have, for all $k > p(\xi)$, that

$$l(N_k) \cdot {N_k}^{1-\delta} \cdot \widehat{F}_{\alpha}^{N_k}(v) \geq s_k \cdot l(N_k) \cdot {N_k}^{1-\delta} \cdot \sum_{j=N_k-n_k}^{N_k-N(\xi)} \widehat{F}_{\alpha}^{N_k}(\mathbb{1}_{A_j}). \tag{13.16}$$

Now, Lemma 4.2 (i) implies that

$$\lim_{n\to+\infty} \frac{n^{1-\delta} \cdot l(n)}{(n+1)^{1-\delta} \cdot l(n+1)} = 1.$$

As the sequence $(a_n)_{n\in\mathbb{N}}$ is positive and since $a_n = \delta \cdot l(n)/n^{1+\delta}$ the value

$$r := \inf\left\{\frac{n^{1-\delta} \cdot l(n)}{(n+1)^{1-\delta} \cdot l(n+1)}\right\}$$

is finite and strictly greater than zero. Hence, by (11.6), (13.14) and (13.16) and the fact that $t_n \sim l(n)/n^{\delta}$, we have on $\overline{A}_1$ that, for each $k \in \mathbb{N}$ sufficiently large,

$$
\begin{aligned}
& l(N_k) \cdot {N_k}^{1-\delta} \cdot \widehat{F}_{\alpha}^{N_k}(v) \\
& \geq \frac{\Gamma_\delta \cdot s_k}{\overline{\Gamma}_\delta \cdot e^{\xi}} \cdot \sum_{j=N_k-n_k}^{N_k-N(\xi)} \frac{(N_k-j)^{1-\delta} \cdot l(N_k-j)}{(N_k-j+1)^{1-\delta} \cdot l(N_k-j+1)} \cdot \frac{N_k^{1-\delta} \cdot l(N_k) \cdot t_j}{(N_k-j)^{1-\delta} \cdot l(N_k-j)} \\
& \geq \frac{\Gamma_\delta \cdot r \cdot s_k}{\overline{\Gamma}_\delta \cdot e^{2\cdot\xi}} \cdot \sum_{j=N_k-n_k}^{N_k-N(\xi)} \frac{N_k^{1-\delta} \cdot l(N_k) \cdot l(j)}{(N_k-j)^{1-\delta} \cdot l(N_k-j) \cdot j^{\delta}}.
\end{aligned}
$$

By Lemma 13.9, the last term diverges.
All that remains to show is that the first statement of (13.13) holds. For this, observe that, by positivity and linearity of $\widehat{F}$, Theorem 6.10, Proposition 6.20 and (11.6), we have on $\overline{A}_1$ that, for each $k \in \mathbb{N}$,

$$
\begin{aligned}
\Gamma_\delta \cdot \sum_{l=1}^{N_k} \int v \cdot \mathbb{1}_{A_l} \, d\mu_\alpha &= \Gamma_\delta \cdot \sum_{m=1}^{k} s_m \cdot \sum_{j=N_m-n_m}^{N_m} t_j \\
&= \sum_{m=1}^{k} s_m \cdot \sum_{j=N_m-n_m}^{N_m} \liminf_{n\to+\infty} w_n \cdot \widehat{F}^{n-j+1}\left(\widehat{F}_{\alpha}^{j+1}\left(\mathbb{1}_{A_j}\right)\right) \\
&\leq \liminf_{n\to+\infty} w_n \cdot \widehat{F}^{n}\left(\sum_{m=1}^{k} s_m \cdot \sum_{j=N_m-n_m}^{N_m} \mathbb{1}_{A_j}\right) \\
&\leq \liminf_{n\to+\infty} w_n \cdot \widehat{F}^{n}(v).
\end{aligned}
$$

Since $k \in \mathbb{N}$ was arbitrary, the above inequalities imply that on $\overline{A}_1$,

$$
\liminf_{n\to+\infty} w_n \cdot \widehat{F}^{n}(v) \geq \Gamma_\delta \cdot \int v \, d\mu_\alpha.
$$

Suppose that this inequality is strict, namely, suppose that there exists a constant $c > 0$ such that on $\overline{A}_1$,

$$
\liminf_{n\to+\infty} w_n \cdot \widehat{F}^{n}(v) \geq c > \Gamma_\delta \cdot \int v \, d\mu_\alpha.
$$

This assumption together with (5.13) implies that, given $\xi > 0$, there exists $M(\xi) \in \mathbb{N}$ such that, for all $n \geq M(\xi)$ and $x \in \overline{A}_1$,

$$
\widehat{F}^{n}(v)(x) \geq \frac{c \cdot n^{\delta-1}}{e^{\xi} \cdot \overline{\Gamma}_\delta \cdot l(n)}.
$$

The constant $\overline{\Gamma}_\delta$ is given in (5.14). Thus, by Karamata's Tauberian theorem, Theorem 4.3, it follows that, for all $n \geq M(\xi)$ and $x \in \overline{A}_1$,

$$\sum_{k=1}^{n} \widehat{F}^k(v)(x) \geq \sum_{k=1}^{M(\xi)} \widehat{F}^k(v)(x) + \sum_{k=M(\xi)+1}^{n} \frac{c \cdot k^{\delta-1}}{e^{\xi} \cdot \overline{\Gamma}_{\delta} \cdot l(k)}$$
$$\geq \sum_{k=1}^{M(\xi)} \widehat{F}^k(v)(x) + \frac{\overline{\Gamma}_{1-\delta} \cdot c \cdot n^{\delta}}{e^{\xi} \cdot \overline{\Gamma}_{\delta} \cdot l(n)}.$$

Hence,

$$\liminf_{n\to+\infty} \frac{w_n}{n} \cdot \sum_{k=1}^{n} \widehat{F}^n(v)(x) = \liminf_{n\to+\infty} \frac{l(n) \cdot \overline{\Gamma}_{\delta}}{n^{\delta}} \cdot \sum_{k=1}^{n} \widehat{F}^n(v)(x)$$
$$\geq \overline{\Gamma}_{1-\delta} \cdot c > \overline{\Gamma}_{1-\delta} \cdot \Gamma_{\delta} \cdot \int v \, \mathrm{d}\mu_{\alpha}.$$

This is a contradiction, since by (5.13) and by combining Theorem 6.10 with Theorem 4.3, we have that the set $\overline{A}_1$ is a Darling-Kac set and therefore, by Proposition 6.7, the α-Farey system is pointwise dual ergodic, meaning that, for μ_{α}-almost every $x \in [0, 1]$, we have that

$$\lim_{n\to+\infty} \frac{w_n}{n} \cdot \sum_{k=1}^{n} \widehat{F}^n(v)(x) = \overline{\Gamma}_{1-\delta} \cdot \Gamma_{\delta} \cdot \int v \, \mathrm{d}\mu_{\alpha}.$$

□

All that is left to show is the final statement of Theorem 12.3. We observe, that the divergence in (13.15) occurs only along a zero density subsequence. That is a subsequence n_k of the natural numbers, such that $\lim_{n\to\infty} \#\{n_k : n_k < n\}/n = 0$. Following similar arguments as in [GL62, p. 226], we have the following lemma. Recall the definition of the asymptotic upper density given in (12.6).

Lemma 13.11. *In a pointwise dual ergodic system $(X, \mathfrak{B}, \mu, T)$, with a wandering rate that varies at ∞ regularly with index δ, the limes inferior in* (12.5) *can be replaced by a limes, if we exclude a set of integers having asymptotic upper density zero.*

Proof. We let $(b_k)_{k\in\mathbb{N}}$ denote a sequence of real numbers and let $B \in \mathbb{R}$. To prove the lemma we claim the following implication. If we have for all subsets $K \subseteq \mathbb{N}$ with positive asymptotic upper density that

$$\liminf_{k\to\infty,\, k\in K} b_k = B, \tag{13.17}$$

we have that there exists a subset $\widetilde{K} \subseteq \mathbb{N}$ with asymptotic upper density equal to zero such that

$$\limsup_{k\to\infty,\, k\notin\widetilde{K}} b_k = B.$$

To prove this claim, we let $\mathfrak{N}_k := \{n \in \mathbb{N} : b_n \geq B + 1/k\}$. Then there exists an $\mathfrak{n}_k$, such that we have for all $n \geq \mathfrak{n}_k$ that $\#(\mathfrak{N}_k \cap [1, n])/n \leq 1/k$. By construction, we have $\mathfrak{N}_k \subseteq \mathfrak{N}_{k+1}$.
With

$$\mathfrak{B} := \bigcup_{i=1}^{\infty} (\mathfrak{N}_i \cap [\mathfrak{n}_i, \mathfrak{n}_{i+1}]) ,$$

we have for n, with $n_k < n \leq n_{k+1}$, that $\#(\mathfrak{B} \cap [1, n])/n \leq \#(\mathfrak{N}_k \cap [1, n])/n \leq 1/k$. Hence, $\mathfrak{B}$ is a set of asymptotic upper density equal to zero and by its construction we have

$$\limsup_{k \to \infty,\, k \notin \mathfrak{B}} b_k = B.$$

This proves the claim. To prove the lemma, it hence suffices to prove (13.17) for $b_k := w_k \cdot T^k(f)$. We prove this by way of contradiction, namely, we assume the existence of a subset of the natural numbers $K \subset \mathbb{N}$ with positive asymptotic upper density $\overline{d}(K) = \rho > 0$ such that there exists a constant $J > \int_X f \,\mathrm{d}\mu$ with

$$\liminf_{k \to \infty,\, k \in K} w_k \widehat{T}^k(f) \geq \Gamma_\delta \cdot J.$$

Exploiting pointwise dual ergodicity, this leads to a contradiction, since

$$\begin{aligned}
\overline{\Gamma}_{1-\delta} \cdot \Gamma_\delta \cdot \int_X f \,\mathrm{d}\mu &= \lim_{n \to \infty} \frac{w_n}{n} \cdot \sum_{k=0}^{n-1} \widehat{T}^k(f) \\
&= \lim_{n \to \infty} \left(\frac{w_n}{n} \cdot \sum_{k=0,\, k \notin K}^{n-1} \widehat{T}^k(f) + \frac{w_n}{n} \cdot \sum_{k=0,\, k \in K}^{n-1} \widehat{T}^k(f) \right) \\
&= \liminf_{n \to \infty} \left(\frac{1}{n} \cdot \sum_{k=0,\, k \notin K}^{n-1} w_k \cdot \widehat{T}^k(f) \right) + \limsup_{n \to \infty} \left(\frac{1}{n} \cdot \sum_{k=0,\, k \in K}^{n-1} w_k \cdot \widehat{T}^k(f) \right) \\
&\geq (1 - \rho) \cdot \overline{\Gamma}_{1-\delta} \cdot \Gamma_\delta \cdot \int_X f \,\mathrm{d}\mu + \rho \cdot \overline{\Gamma}_{1-\delta} \cdot \Gamma_\delta \cdot J \\
&> \overline{\Gamma}_{1-\delta} \cdot \Gamma_\delta \cdot \int_X f \,\mathrm{d}\mu.
\end{aligned}$$

This finishes the proof of Lemma 13.11. I am grateful to Ian Melbourne for posing this question.

□

Finally, the proof of Theorem 12.3 is a consequence of Propositions 13.7, 13.8 and 13.10 and Lemma 13.11.

□□

Chapter 14

Sufficient conditions for convergence

Theorem 12.3 shows that we have to be careful when trying to obtain distributional convergence. Melbourne and Terhesiu impose additional assumptions on the observable in [MT15], such that convergence of the individual iterates of the transfer operator can be obtained. In this chapter we discuss, how the results in [MT15] extend and complement the results of Theorems 12.2 and 12.3.
We first state the version of their theorem [MT15, Theorem 10.4], slightly adapted to fit our notation. It is assumed, that the tail probabilities are regularly varying. That is, we assume that there exists $\delta \in (1/2, 1]$ and a slowly varying function $\ell(\cdot)$, such that

$$\mu_\alpha\left(y \in Y : \phi_Y(y) > n\right) = t_n \sim \frac{\ell(n)}{n^\delta}.$$

Furthermore, we define

$$\mathcal{M}(n) := \begin{cases} \ell(n) & \text{if } \delta \in \left(\frac{1}{2}, 1\right), \\ \sum_{j=1}^{n} \frac{\ell(j)}{j} & \text{if } \delta = 1. \end{cases}$$

We state the theorem here for the Banach Space $\mathcal{B}_\alpha$. Yet, the result in [MT15] is stated for a general Banach Space that satisfies conditions *(R1)-(R5)*, and hence one could replace $\mathcal{B}_\alpha$ by another suitable Banach space.

Theorem 14.1 ([MT15, Theorem 10.4])**.** *Suppose that $v \in \mathcal{A}_\alpha$. Furthermore, suppose that*

$$\left\| \widehat{F}_\alpha^{k-1}(v \cdot \mathbb{1}_{A_k}) \right\|_{\mathcal{B}_\alpha} = \mathfrak{o}\left(\frac{1}{n}\right) \tag{14.1}$$

$$\text{or} \quad \sum_{k=n}^{\infty} \left\| \widehat{F}_\alpha^{k-1}(v \cdot \mathbb{1}_{A_k}) \right\|_{\mathcal{B}_\alpha} = \mathfrak{o}\left(\frac{1}{\mathcal{M}(n) \cdot n^{1-\delta}}\right), \tag{14.2}$$

then $\lim_{n\to\infty} \mathcal{M}(n) \cdot n^{1-\beta} \cdot \widehat{F}_\alpha^n(v) = \Gamma_\delta \cdot \int_X v \, \mathrm{d}\mu_\alpha$ *uniformly on Y and pointwise on X.*

As mentioned before, the *'pointwise on X'* statement relies on a Young-tower construction in [MT12], see [MT12, Section 10.1]. In this thesis it is done via the extension Theorem 6.11.
Conditions (12.1) and (12.2) together for an observable v are similar to condition (14.1) of Theorem 14.1 for the observable v/h_α.
The minor differences are, that on the one hand [MT15] considers a general Banach space not a particular one, as this thesis does. On the other hand, it assumes that $\left\|\widehat{F}_\alpha^{k-1}(v \cdot \mathbb{1}_{A_k})\right\|_{\mathcal{B}_\alpha} = \mathfrak{o}\left(n^{-1}\right)$, this thesis requires this regularity assumption only in the supremum part of the norm, see (12.2). The assumption on the other part of the norm is weaker, see (12.1). Yet, for a δ-expansive α-Farey system, we have that $\left\|\widehat{F}_\alpha^n(v/h_\alpha \cdot \mathbb{1}_{A_{n+1}})\right\|_\infty = \mathfrak{o}\left(n^{-1}\right)$ if and only if $\left\|v \cdot \mathbb{1}_{A_n}\right\|_\infty = \mathfrak{o}\left(t_n^{-1}\right)$. This can be seen, since

$$\begin{aligned}
n \cdot \left\|\widehat{F}_\alpha^n\left(\frac{v}{h_\alpha} \cdot \mathbb{1}_{A_{n+1}}\right)\right\|_\infty &= n \cdot \frac{a_n}{t_n} \cdot \left\|\widehat{F}_\alpha^n(v \cdot \mathbb{1}_{A_{n+1}})\right\|_\infty \\
&= n \cdot a_n \left\|v \cdot \mathbb{1}_{A_{n+1}}\right\|_\infty \\
&\sim t_n \cdot \left\|v \cdot \mathbb{1}_{A_n}\right\|_\infty .
\end{aligned}$$

However, our observable, constructed in the proof of Theorem 12.3, does neither satisfy (14.1) nor (14.2). Furthermore, neither (14.1) nor (14.2) implies the conditions of Theorem 12.2 and vice versa.
First, we show that an observable $v \in \mathcal{V}$ does not satisfy (14.1), for which it is sufficient, to show that $\limsup_{k\to\infty} k \cdot \widehat{F}_\alpha^{k-1}(v \cdot \mathbb{1}_{A_k}) > 0$. As in Theorem 12.3, we consider only the case $\delta \in (1/2, 1)$ here.
By the definition of the $\mathcal{B}_\alpha$-norm, given in (6.29), and by the definition of the class of functions $\mathcal{V}$, we observe, that if $v \in \mathcal{V}$, we have $\|v\|_{\mathcal{B}_\alpha} = \|v\|_\infty$. Moreover, v and $\widehat{F}_\alpha^{k-1}(v \cdot \mathbb{1}_{A_k})$ are non-negative. Furthermore, by (13.12) and Lemma 13.6, we have for $N_k - n_k \le j_k \le N_k$, that

$$\begin{aligned}
\limsup_{k\to\infty} k \cdot \widehat{F}_\alpha^{k-1}(v \cdot \mathbb{1}_{A_k}) &\ge \limsup_{k\to\infty} j_k \cdot t_{j_k} \cdot s_k \\
&\ge \limsup_{k\to\infty} (N_k - n_k)^{1-\delta} \cdot \ell(N_k) \cdot s_k \\
&\ge \limsup_{k\to\infty} N_{k-1}^{1-\delta} \cdot \ell(N_k) \cdot s_k \\
&\ge \limsup_{k\to\infty} \frac{1}{2^{g_1\cdot(1-\delta)}} \cdot 2^{g_1\cdot(1-\delta)\cdot k} \cdot 2^{-g_3\cdot k} \cdot \ell\left(2^{g_1\cdot(1-\delta)\cdot k}\right).
\end{aligned}$$

The last term diverges, since, $\ell(\cdot)$ is slowly varying and by conditions **(C1)**-**(C3)** of $\mathcal{V}$ and the fact that $1/2 < \delta < 1$, we have that

$$\begin{aligned}
(1-\delta)\cdot g_1 - g_3 &\ge (1-\delta)\cdot g_1 + (2\cdot\delta + 2\cdot\epsilon - 1)\cdot g_1 - (\delta-\epsilon)\cdot g_2 \\
&\ge (1-\delta)\cdot g_1 + (2\cdot\delta + 2\cdot\epsilon - 1)\cdot g_1 - (\delta-\epsilon)\cdot\delta\cdot g_1 \\
&= (\delta - \delta^2 + \epsilon\cdot\delta + 2\cdot\epsilon)\cdot g_1 \\
&> 0.
\end{aligned}$$

The next aim is to show that an observable $v \in \mathcal{V}$ does not satisfy (14.2) either. Since $\ell(\cdot)$ is slowly varying, there exists a constant $C \in \mathbb{R}$ and a sufficiently small $\eta > 0$ such that $0 < 1 - 2 \cdot \delta - 2 \cdot \eta < 1$ and such that

$$\begin{aligned}
&\limsup_{k\to\infty} \ell(k) \cdot k^{1-\delta} \cdot \sum_{i=k}^{\infty} \left\| \widehat{F}_{\alpha}^{k-1}(v \cdot \mathbb{1}_{A_k}) \right\|_{\infty} \\
&\geq \limsup_{k\to\infty} \ell(k) \cdot k^{1-\delta} \cdot \sum_{i=k}^{\infty} s_i \cdot \sum_{j=N_i-n_i}^{N_i} \frac{\ell(j)}{j^{\delta}} \\
&\geq \limsup_{k\to\infty} C \cdot N_k^{1-\delta-\eta} \cdot \sum_{i=k}^{\infty} s_i \cdot \frac{n_i}{N_i^{\delta+\eta}} \\
&\geq \limsup_{k\to\infty} C \cdot 2^{g_1 \cdot (1-\delta-\eta)\cdot k} \cdot \sum_{i=k}^{\infty} 2^{(g_2-g_3-g_1\cdot(\delta+\eta))\cdot i} \\
&= \limsup_{k\to\infty} C \cdot \frac{2^{g_1\cdot(1-\delta-\eta)\cdot k} \cdot 2^{(g_2-g_3-g_1\cdot(\delta+\eta))\cdot k}}{1 - 2^{(g_2-g_3-g_1\cdot(\delta+\eta))}} \\
&= \limsup_{k\to\infty} C \cdot \frac{2^{((1-2\cdot\delta-2\cdot\eta)\cdot g_1+g_2-g_3)\cdot k}}{1 - 2^{(g_2-g_3-g_1\cdot(\delta+\eta))}}
\end{aligned} \tag{14.3}$$

By condition **(C3)**, there is an $\epsilon \in (0, \delta - 1/2)$, such that

$$g_3 < g_2 \cdot (\delta - \epsilon) - g_1 \cdot (2 \cdot \delta + 2 \cdot \epsilon - 1).$$

Combining this with condition **(C2)** yields, that

$$g_3 < \left(\frac{\delta^2 - \epsilon \cdot \delta - 2 \cdot \delta - 2 \cdot \epsilon + 1}{\delta} \right) \cdot g_2 \leq g_2.$$

Hence, we have that $g_2 - g_3 > 0$. Combining this with the fact that $1-2\cdot\delta-2\cdot\eta > 0$ yields that the last term of (14.3) diverges and hence condition (14.2) is not satisfied. This thesis concludes with a set of examples that show that neither (14.1) nor (14.2) implies the conditions of Theorem 12.2 and vice versa. In examples 14.2-14.5, the underlying system is a one expansive α-Farey system with a slowly varying wandering rate.

Example 14.2. The first example satisfies both (14.1) and (14.2), and belongs to the system $([0, 1], \mathfrak{B}, F_\alpha, \mu_\alpha)$, with a wandering rate w_n, that is slowly varying but not moderately increasing. That is, $t_n \sim \ell(n)/n$ and $\ell(\cdot)$ is slowly varying but not moderately increasing. The reader is referred to Example 11.2 for such a wandering rate. Moreover we define the observable $\phi_0 \colon [0, 1] \to \mathbb{R}$ by $\phi_0(0) := 0$ and

$$\phi_0(x) := \sum_{k=1}^{\infty} \frac{1}{k^3} \cdot \mathbb{1}_{A_k}(x).$$

We have on A_1, that

$$\widehat{F}_\alpha^{n-1}(v \cdot \mathbb{1}_{A_n}) = \frac{\ell(n)}{n^4}$$

Hence, (14.1) and (14.2) are satisfied simultaneously, but the wandering rate is not moderately increasing. Yet, the summability condition given in (11.5) is still satisfied.

Example 14.3. The next example deals with a 1-expansive α-Farey system, with a moderately increasing wandering rate and an observable that does not satisfy (14.1). We assume that the wandering rate w_n is asymptotic to a slowly varying, moderately increasing and increasing function $\ell(\cdot)$.
For $k \in \mathbb{N}$, we define the observables $\phi_{1,k} \colon [0,1] \to \mathbb{R}$ and the observable $\phi_1 \colon [0,1] \to \mathbb{R}$ by

$$\phi_{1,k}(x) := \mathbb{1}_{A_{k^3}}(x)$$
$$\text{and} \quad \phi_1(x) := \sum_{k=1}^{\infty} \phi_{1,k}(x)$$

We observe on A_1, that

$$n \cdot \widehat{F}_\alpha^{n-1}(v \cdot \mathbb{1}_{A_n}) = \begin{cases} \ell(j^3), & \text{if } x \in A_{j^3}, j \in \mathbb{N} \\ 0, & \text{otherwise.} \end{cases}$$

Hence, (14.1) is not satisfied.

Example 14.4. This example is an example of a 1-expansive α-Farey system with a moderately increasing wandering rate, such that (14.2) is not satisfied. For $n \in \mathbb{N}$, we define

$$a_n := \frac{\ln(n)}{n^2},$$
$$s_n := \frac{1}{(\ln(n+1))^3},$$
$$\text{and} \quad \phi_3(x) := \sum_{n=1}^{\infty} s_n \cdot \mathbb{1}_{A_n}(x)$$

Hence, $t_n \sim \ln(n)/n$ and we have on A_1, that

$$\widehat{F}_\alpha^{n}(\phi_3 \cdot \mathbb{1}_{A_{n-1}}) \sim t_n \cdot s_n = \frac{1}{n \cdot (\ln(n))^2}.$$

In particular, the summability condition (11.5) is satisfied. Furthermore, we note that $\mathcal{M}(n) \sim (\ln(n))^2/2$, which implies that

$$\mathcal{M}(n) \cdot \sum_{k=n}^{\infty} \widehat{F}_\alpha^{n}(\phi_3 \cdot \mathbb{1}_{A_{n-1}}) \sim \frac{1}{2} \cdot \ln(n).$$

Thus, (14.2) is not satisfied.

Examples 14.3 and 14.4 only failed to satisfy one of the conditions (14.1) and (14.2). Example 14.3 still satisfies (14.2) and Example 14.4 satisfies (14.1). We want to find an example that neither satisfies (14.1) nor (14.2) but still satisfies the conditions of Theorem 12.2. A combination of the two examples 14.3 and 14.4 yields an example we are looking for.

Example 14.5. We adopt the settings of the previous examples and define the observable $\phi_4\colon [0,1] \to \mathbb{R}$ by

$$\phi_4(x) := \phi_2(x) + \phi_3(x).$$

This example satisfies the conditions of Theorem 12.2. Yet, neither (14.1) nor (14.2) are satisfied.

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