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Integral Equation Methods for Ocean Acoustics with Depth-Dependent Background Sound Speed

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Abstract

Integral Equation Methods for Ocean Acoustics with Depth-Dependent Background Sound Speed.

Time-harmonic acoustic wave propagation in an ocean with depth-dependent background sound speed can be described by the Helmholtz equation in an infinite, two- or three-dimensional waveguide of finite height. A crucial subproblem for the anayltic and numeric treatment of associated wave propagation problems is a Liouville eigenvalue problem that involves the depth-dependent contrast. For different types of background sound speed profiles, we discuss discretization schemes for the Liouville eigenvalue problem arising in the vertical variable. Due to variational theory in Sobolev spaces, we then show well-posedness of weak solutions to the corresponding scattering problem from a bounded inhomogeneity inside such an ocean: We introduce an exterior Dirichlet-to-Neumann operator for depth-dependent sound speed and prove boundedness, coercivity, and holomorphic dependence of this operator in suitable function spaces adapted to our weak solution theory. Analytic Fredholm theory then implies existence and uniqueness of solution for the scattering problem for all but a countable sequence of frequencies. Introducing the Green's function of the waveguide, we prove equivalence of the source problem for the Helmholtz equation with depth-dependent sound speed profile, Neumann boundary condition on the bottom and Dirichlet boundary condition on the top surface, to the Lippmann-Schwinger integral equation in dimensions two and three. Next, we periodize the Lippmann-Schwinger integral equation in dimensions two and three. The periodized version of the Lippmann-Schwinger integral equation and an interpolation projection onto a space spanned by finitely many eigenfunctions in the vertical variable and trigonometric polynomials in the horizontal variables, two different collocation schemes are derived. A result of Sloan [J.Approx Theory, 39:97-117,1983] on non-polynomial interpolation yields both converge and algebraic convergence rates depending on the smoothness of the inhomogeneity and the source of both schemes. Using one collocation scheme we present numerical results in dimension two. We further present an optimization technique of the vertical transform process, when the height of the obstacle is small compared to the finite height of the ocean, which makes computation in dimension three possible. If several scatters are present in the waveguide, this discretization technique leads to one computational domain containing all scatterers. For a three dimensional waveguide, we reformulate the Lippmann-Schwinger integral equation as a coupled system in an union of several boxes, each containing one part of the scatter.

Résumé

Equations intégrales volumiques pour l'acoustique sous marine avec vitesse dépendant de la profondeur.

On s'intéresse à la propagation d'une onde acoustique en régime harmonique dans un milieu océanique où la vitesse du son dépend de la profondeur et qui peut être modélisée par l'équation de Helmholtz dans un milieu infini en dimensions transverses mais à hauteur finie. Dans un premier temps, à fin de construire l'opérateur Dirichlet-Neumann permettant de borner le domaine de calcul, nous étudions le problème aux valeurs propres de Liouville qui intervient sur l'axe vertical et sa discrétisation par des approches éléments finis ou spectrales. Nous étudions ensuite à l'aide de méthodes variationnelles le problème de diffraction par une inhomogénéité à support borné incluse dans l'océan en utilisant l'opérateur Dirichlet-Neumann extérieur. Nous démontrons que dans des espaces de fonctions adaptés, l'opérateur de diffraction est borné, coercif et dépend holomorphiquement de la fréquence. La théorie de Fredholm analytique implique alors l'existence et l'unicité de la solution du problème de diffraction pour toutes les fréquences sauf pour une suite discrète de valeurs ayant $+\infty$ comme seul point d'accumulation. Après l'introduction de la fonction de Green du guide d'onde océanique, nous reformulons le problème de diffraction à l'aide d'une équation intégrale de Lippmann-Schwinger. Cette équation est ensuite périodisée dans les directions transverses. En considérant l'espace d'approximation engendré par les vecteurs propres (du problème de Liouville) suivant l'axe vertical et des polynômes trigonométriques suivant les directions transverses, nous obtenons deux schémas de collocation différents de l'équation volumique (suivant la manière de projeter l'opérateur intégral). Le taux de convergence algébrique qui dépend de la régularité d'inhomogénéité et de la source, ainsi qu'un résultat d'interpolation non-polynomial de Sloan [J.Approx Theory, 39:97-117,1983], nous permettent de démontrer la convergence (et sa vitesse) pour les deux schémas de collocation. La méthode est ensuite testée et validée numériquement sur des exemples synthétiques. La dernière partie s'intéresse au cas d'un guide contenant plusieurs objets diffractants très espacés. Dans le cas de dimension trois, nous pouvons reformuler l'équation intégrale de Lippmann-Schwinger en un système couplé de plusieurs équations posées sur des (petits) domaines contenant chacun un seul objet. Cette technique permet de réduire considerablement le coût numérique.

Zusammenfassung

Volumenintegralgleichungsmethoden für Ozeane mit tiefenabhängiger Schallausbreitungsgeschwindigkeit.

In dieser Arbeit gehen wir auf zeitharmonische Wellenausbreitung in einem Ozean mit tiefenabhängiger Schallausbreitungsgeschwindigkeit ein, die für einen in horizontaler Ebene unendlichen Ozean mit endlicher Tiefe durch die Helmholtz-Gleichung modelliert wird. Für unterschiedliche Schallprofile betrachten wir für das Liouville-Eigenwertproblem, das in der vertikalen Variable auftritt, verschiedene Approximationstechniken, wie z.B. eine Finite-Element Methode und eine spektrale Methode. Mit Hilfe der Theorie von variationellen Formulierungen in Sobolevräumen, zeigen wir dann, dass für eine beschränkte Inhomogenität im Ozean das Streuproblem gut gestellt ist. Weiter führen wir den Dirichlet-Neumann Operator für tiefenabhängige Schallausbreitungsgeschwindigkeit ein und zeigen in einem an das Model angepassten Funktionenraum, dass dieser Operator beschränkt, koerziv und holomorph abhängig von der Frequenz ist. Existenz und Eindeutigkeit einer schwachen Lösung des Streuproblems kann dann mit Hilfe von analytischer Fredholmtheorie für alle Frequenzen bis auf eine abzählbare Menge ohne endlichen Häufungspunkt, gezeigt werden. Mit Hilfe der Greenschen Funktion wird dann Äquivalenz zwischen dem Streuproblem, welches durch die Helmholtz Gleichung mit Dirichlet Randbedingungen an der Oberfläche und Neumann Randbedingungen am Meeresgrund beschrieben wird und der Lippmann-Schwinger Integralgleichung für Dimension zwei und drei, gezeigt. Die eingeführte Greensche Funktion wird dann in horizontaler Variable periodisiert. Mit Hilfe einer Interpolationsprojektion auf einem endlichdimensionalen Raum, der in horizontaler Variable durch trigonometrische Funktionen und in vertikaler Variable durch Eigenfunktionen des Liouville-Eigenwertproblems aufgespannt wird, erhalten wir aus der periodisierten Integralgleichung zwei verschiedene Kollokationsmethoden.

Die algebraische Konvergenzrate der Kollokationsmethoden folgt aus der Glattheit der Inhomogenität und der Quelle, sowie einem Resultat für nicht-polynomiale Interpolation von Sloan [J.Approx Theory, 39:97-117,1983]. Für Dimension zwei und drei führen wir numerische Experimente für eine Kollokationsmethode durch. Weiter analysieren wir eine Methode, bei der die Inhomogenität im Verhältnis zu Ozeantiefe klein ist. Diese Methode spart Speicherplatz und Rechenzeit. Wenn im dreidimensionalen Fall mehrere Inhomogenitäten im Ozean vorhanden sind, können wir die Lippmann-Schwinger Integralgleichung zu einem gekoppelten System, von mehreren Gleichungen auf kleinen Gebieten umschreiben. Diese Technik reduziert die numerischen Kosten drastisch.

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Outline of the Thesis

To briefly indicate the structure of this thesis, Chapter 2 introduces the wave propagation problem and the Liouville eigenvalue problem for a two- or three-dimensional ocean with depth-dependency of the sound speed. Using different background sound speed profiles, we discuss discretization schemes for the Liouville eigenvalue problem arising in the vertical variable.

In Chapter 3, due to variational theory in Sobolev spaces, we first show well-posedness of the corresponding scattering problem from a bounded inhomogeneity inside such an ocean. We further introduce an exterior Dirichlet-to-Neumann operator for depth-dependent background sound speed and prove boundedness, coercivity, and holomorphic dependence of this operator in function spaces adapted to our weak solution theory. Then, analytic Fredholm theory implies existence and uniqueness of solution for the scattering problem for all but a countable sequence of frequencies.

Chapter 4 studies the Green's function for an ocean of dimensions two and three with background sound speed dependency, Neumann boundary condition on the bottom and Dirichlet boundary on the top. We further prove equivalence of the source problem using the Helmholtz equation, to the Lippmann-Schwinger integral equation in dimensions two and three. We moreover periodize the Lippmann-Schwinger integral equation.

Based on the periodized version of the Lippmann-Schwinger integral equation and an interpolation projection onto a space spanned by finitely many eigenfunctions in the vertical variable and trigonometric polynomials in the horizontal variables two different collocation schemes are derived, in Chapter 5. A algebraic convergence rate depending on the smoothness of the inhomogeneity and the source and a result of Sloan [J.Approx Theory, 39:97-117,1983] on non-polynomial interpolation prove convergence of both schemes, as well as convergence rates.

Using one collocation scheme, we present in Chapter 6 numerical results in dimension two and three. We further present an optimization technique of the vertical transform process, when the height of the obstacle is small compared to the finite height of the ocean, which makes computation in dimension three possible.

If several scatters are present in the waveguide, the discretization technique from Chapter 6 leads to one computational domain containing all scatterers and hence to a large linear system. In Chapter 7, we reformulate the Lippmann-Schwinger integral equation for a three dimensional waveguide as a coupled system with an union of several boxes, each containing one part of the scatter.

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Chapter 1

Introduction

1.1 Motivation for Sound Propagation in an Ocean

The research of sound wave propagation inside an ocean is an active research topic in applied mathematics and engineering at least since the mid-20th century for its crucial importance for techniques for oil exploration or like SONAR (Sound Navigation And Ranging), see, e.g., [Buc92] or the introduction of [BGWX04]. In the beginning of the 21st century exact simulations for sound propagation in an ocean became even more important since the living environment and the communication of marine mammals is crucial affected by increasing influences of man-made ocean noise creation, e.g. construction and the operation of offshore wind farms or shipping. Legal thresholds for emitted sound energies requires precise models and quantitatively exact simulations of sonic intensities for ocean explorations, e.g. for acoustic pulses produced by firing air guns. This requirement to a model can be described by scattering of time-harmonic acoustic waves in the ocean by different elliptic and parabolic equation approximations. (e.g. Ames and Lee give in [AL87] a survey of the history and the beginning of researches of ocean acoustic propagation.) Accurately describing acoustic waves with small amplitude, the Helmholtz equation is an attractive model for time-harmonic wave propagation in an ocean of finite height, where already various established discretization schemes for the approximation of its solutions, see, e.g. [AK77], [Buc92], exist. Well-known techniques of approximation are for example finite elements or boundary elements methods, however, these memory intensive techniques are limited to small oceans and shallow water, see, e.g. [BGT85]. Furthermore, another common simple model to compute wave propagation in ocean acoustic researches is the near field Lloyd mirror pattern with convergence-zone propagation modification discussed in [Jen11][Chapter 1.4].

Liouville Eigenvalue Problem. An alternative approximation using a spectral volumetric equation method for constant background sound speed for a 3D waveguide with finite height is presented in [LN12], however, reasonable models for sound propagation over large distances and depth oceans imperatively require a depth-dependent background sound speed. We point out that the background sound speed in the ocean depends, e.g. on the salinity, pressure (via depth) or the temperature of the ocean, whereas it is well-known that the temperature can fluctuate during the year. Figure 1.1 shows the sound speed fluctuation, depending on the season, for a depth ocean.

In this thesis we chose the approach of time-harmonic acoustic wave propagation in an ocean with depth-dependent background sound speed by the Helmholtz equation in an infinite, two- or three-dimensional waveguide of finite height.

Existence and Uniqueness of Solution to the Scattering Problem. Using a variational approach for wave scattering in this ocean with depth-dependent background sound speed, we introduce the theory for weak solutions. [SS10] implies that the existence of Gårding inequalities is required to obtain convergence of numerical approximations, using variation theory of weak solutions in Sobolev spaces for the Helmholtz equation. For an ocean with constant background sound speed, where eigenvectors and eigenvalues have an explicit representation, [AGL08], [AGL11]

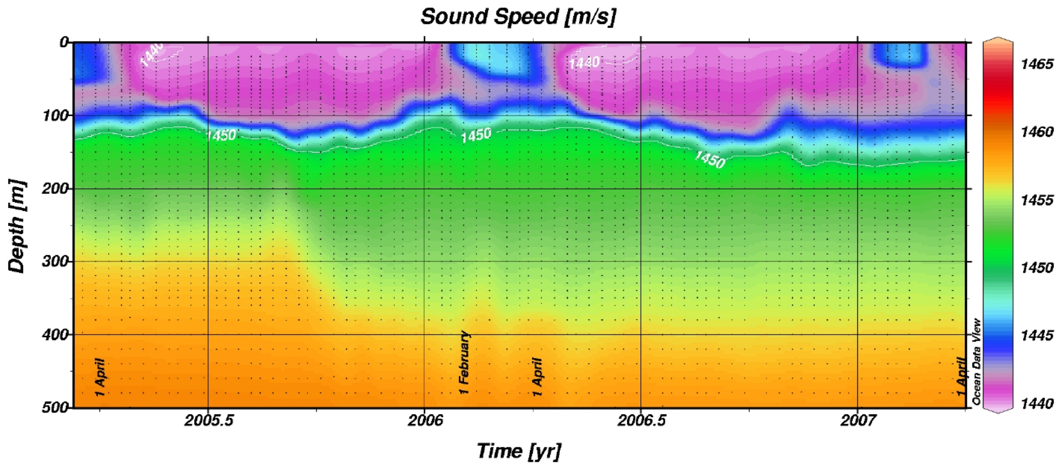


Figure 1.1: Sound speed (Evolution of time, Source: Alfred-Wegener-Institut Bremerhaven)

present the theory for weak solutions using the variational approach for wave scattering using the Helmholtz equation. The non-trivial difficulty compared to known results for constant sound speed, is that the eigenvalues and eigenvectors have no explicitly representation. In consequence, we prove holomorphic dependence of the eigenvalues on the frequency via abstract perturbation theory for an ocean with depth-dependent background sound speed using tools from [AGL08]. We further introduce an exterior Dirichlet-to-Neumann operator for depth-dependent sound speed and prove boundedness, coercivity, and holomorphic dependence of this operator in function spaces adapted to our weak solution theory. Then, analytic Fredholm theory implies existence and uniqueness of solution for the scattering problem for all but possibly a countable sequence of frequencies with accumulation point $+\infty$.

Lippmann-Schwinger Integral Equation. Solving the Helmholtz equation with corresponding boundary conditions and radiation conditions is equivalent to solving the Lippmann-Schwinger integral equation. For constant background sound speed this is well-known by [CK13]: if the background sound speed depends on the depth of the ocean the proof is more challenging. Using the eigenvectors and eigenvalues, we first introduce the Green's function for dimension two and three and we further introduce the volumetric integral operator $\mathcal{V}$ using the Green's function G. Existence of classical (i.e., twice differentiable) solutions to the Helmholtz equation for constant and depth-dependent sound speed has been shown via integral equation techniques by Xu and Gilbert in a series of papers in [GX89, Xu92, Xu97, GL97, BGWX04], too. Xu and Gilbert used the fact that the fundamental solution G can be separated into a free space Green's function and a part correcting the boundary conditions and taking into account the depth-dependent sound speed, thus using well-known volume integral equation tools from [CK13]. We present here, however, an alternative technique to obtain the required boundedness of the volumetric integral operator, that $\mathcal{V}$ is bounded from $L^2(\Lambda_\rho)$ into $H^2(\Lambda_\rho)$. Then, we can proove that the Helmholtz equation with corresponding boundary and radiation conditions is equivalent to the Lippmann-Schwinger equation, for the volumetric integral operator. Furthermore, we periodize the Green's function in dimension two and three in the hoirzontal component and we moreover introduce the periodized Lippmann-Schwinger equation. In dimension three, we discover that the decay rate of the Fourier coefficients of the periodized Green's function is not sufficient to prove convergence for the discretized integral equation we use for the numerical approach, later on. This differs from well-known convergence theory for constant background-speed, which is presented in [LN12]. Using a cut-off function, we improve this decay rate in dimension three.

Numerical Approximation of the Periodized Scattering Problem. For the numerical approximation we follow a technique from Vainikko, see [Vai00], [SV02]. The crucial advantage of this technique is that we can solve the discrete system on the periodized domain by an iterative method, like GMRES, and avoid the need to evaluate the volumetric integral equation $\mathcal{V}$ by integration of the involved Green's function of the problem. We consider an approximation space, spanned by finitely many eigenfunctions of the periodized volume integral operator that are composed by eigenfunctions of the Liouville eigenvalue problem in the vertical variable and trigonometric polynomials in the horizontal variables. This differs again from well-known numerical approximation for constant background-speed, i.e. see [LN12], where the approximation space, in the vertical axis is spanned by finitely many trigonometric functions. Depending on whether the interpolation projection onto this approximation space is applied to the integral operator or to the contrast times the integral operator, we obtain two different collocation schemes.

The error estimates for the horizontal part of the interpolation projection are well-known, i.e. see [SV02], however, the error estimates for the vertical part of the interpolation projection differ from well-known theory. Using an approximation result of Sloan, see [Slo83], we bound the interpolation error of an eigenfunction interpolation in L^2, by the distance in the maximum norm of the interpolated function to the approximation space. Then, an embedding result allows to transfer this error estimate to an estimate in Sobolev spaces. Coupling the estimates for the horizontal and vertical part, we obtain convergence rates for the two collocation schemes and the non-standard interpolation operator.

Numerical Computations using the Collocation Method. The fast Fourier transform and the inverse fast Fourier transform can be used to implement the horizontal part of the transform, however, a corresponding fast discrete transform for the vertical part does in general not exist. The classical article [CT65] for the fast Fourier transform, together with [SV02] and the fact that we use matrix vector multiplications in the horizontal axis, imply that the transfer of grid values to Fourier coefficients costs in general for dimension two least $O(N_1 N_2^2 \log(N_1))$ operations, where N_j denotes the number of collocation points in direction x_j for $j = 1, 2$ and in dimension three $O(N_1 N_2 N_3^2 \log(N_1 N_2))$ operations, where N_j denotes the number of collocation points in direction x_j for $j = 1, 2, 3$

Once the Liouville eigenvalue problem is approximated in advance, the fully discrete collocation scheme can be implemented numerically. Evaluating the fully discrete operator requires to transfer point values at the grid points into Fourier coefficients. Due to the non-constant sound speed, in the vertical component this transform consists of matrix-vector multiplications with a matrix containing the approximated eigenvectors and its inverse matrix. Here, iterative algorithms, see e.g. [SM03] are used to reduce a considerable amount of memory compared to standard MATLAB routines for matrix inversion.

As the discrete system can be restricted to the support of the inhomogeneity, an optimized scheme depending on the support of the inhomogeneity allows to economize memory and computation time compared to other volumetric methods such as finite-element techniques involving exterior Dirichlet-to-Neumann operators, see [BGT85]. This is in particular attractive if the height of the inhomogeneity is small compared to the height of the ocean, which is frequently the case in ocean acoustics.

Combined Spectral/Multipole Method. If several scatters are present in the waveguide, the discretization technique already presented leads to one computational domain containing all scatterers. For a three dimensional waveguide, we reformulate the Lippmann-Schwinger integral equation as a coupled system with an union of several boxes, each containing one part of the scatter. [LN12] and [GR87] discovered a combined spectral/multipole method applied to an ocean with constant background sound speed. We apply this idea to non-constant background speed to make computations for several distributed inhomogeneities and sources placed over large distances in the ocean possible.

Chapter 2

Liouville Eigenvalue Problem

We are interested in solutions to the Helmholtz equation in a waveguide with depth-dependent sound speed and Dirichlet and Neumann boundary conditions on the two boundaries respectively. We first expand u by separation of variables acting on the horizontal variable $\tilde{x}$ and acting on the vertical variable x_m. In consequence, we obtain a Liouville eigenvalue problem. We analyze approximation techniques for different types of background sound speed that we exploit later for numerical computations of the solution to the wave guide problem.

2.1 Sound Propagation Model in an Ocean

This section deals with the introduction of the mathematical setting of waves traveling in an ocean with depth-dependent background sound speed.

First, we consider some model assumptions. The domain of interest is a waveguide $\Omega = \mathbb{R}^{m-1} \times (0, H)$, where $H > 0$ is the constant depth and $m = 2, 3$ the dimension of the ocean. We point out that an ocean with dimension $m > 3$ has no physical interest and is not considered in this work. By the fact that the vertical coordinate axis is singled out in the definition of the waveguide Ω, we establish for a point x in the waveguide Ω the notation

$$x = (x_1, x_2)^T = (\tilde{x}, x_m)^T \qquad \text{for } m = 2 \text{ and}$$
$$x = (x_1, x_2, x_3)^T = (\tilde{x}, x_m)^T \qquad \text{for } m = 3.$$

In this work, the propagation of time-harmonic waves at angular frequency $\omega > 0$ with time-dependence $\exp(-i\omega t)$ in the waveguide Ω is formally modeled by the Helmholtz equation

$$\Delta u(x) + \frac{\omega^2}{c^2(x_m)} n^2(x) u(x) = 0 \qquad \text{in } \Omega, \tag{2.1}$$

where n denotes the refractive index, $c \in L^\infty(0, H)$ the background sound speed depending on the depth of the ocean. Furthermore, to model a local perturbation of the sound speed inside the inhomogeneous waveguide Ω, we suppose for $n^2 : \Omega \to \mathbb{C}$ that $n = 1$ outside some bounded and open set $D \subset \overline{\Omega}$. In particular, we define the contrast by

$$q(x) := n^2(x) - 1 \qquad \text{for } x \in \Omega,$$

and note that $\operatorname{supp}(q) \subset \overline{D}$. We rewrite (2.1) as

$$\Delta u(x) + \frac{\omega^2}{c^2(x_m)} (1 + q(x)) u(x) = 0 \qquad \text{in } \Omega. \tag{2.2}$$

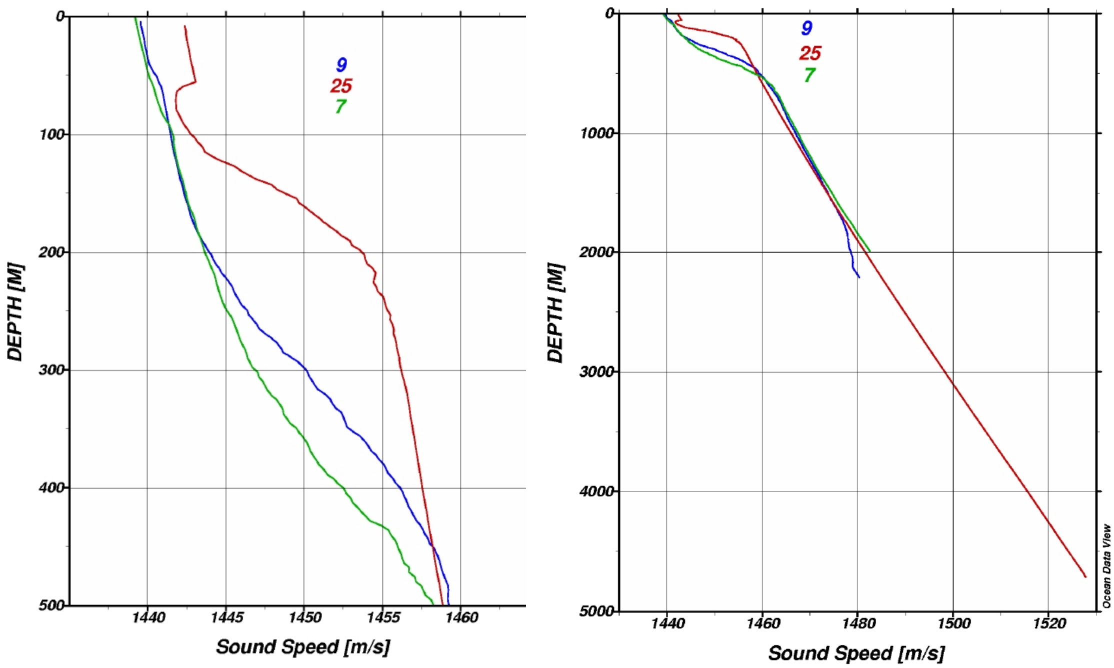
We now give some details on the depth-dependent background sound speed. Figure 1.1 and 2.1 give us an idea for reasonable bounds the sound speed c. We point out that Figure 2.1 gives a physical motivation for our assumptions on c. In a nutshell, the background sound speed lies between $1440m/s$ and $1540m/s$. In the sequel, we consider that the background sound speed c depending on the depth of the ocean is be bounded away from zero by

$$0 < c_- \leq c(x_m) \leq c_+, \qquad \text{for almost all } x_m \in [0, H]. \tag{2.3}$$

Hence, we have

$$0 < \frac{\omega}{c_+} \leq \frac{\omega}{c(x_m)} \leq \frac{\omega}{c_-} \qquad \text{for almost all } x_m \in [0, H]. \tag{2.4}$$

Next, we model the free surface of the ocean by sound soft boundary conditions



(a) (b)

Figure 2.1: Sound speed profile depending on the depth of the ocean. Profile 25: deep ocean, profile 7: shelf, profile 9: shelf in winter. (Source: Alfred-Wegener-Institut Bremerhaven)

$$u = 0 \text{ on } \Gamma_0 := \{x \in \mathbb{R}^m : x_m = 0\}, \tag{2.5}$$

and the seabed of the ocean by sound hard boundary conditions

$$\frac{\partial u}{\partial x_m} = 0 \text{ on } \Gamma_H := \{x \in \mathbb{R}^m : x_m = H\}. \tag{2.6}$$

Other boundary conditions modeling for instance interaction of elastic seabed with the ocean (e.g. [GL97]), are possible. However, for simplicity, we assume Neumann boundary condition (2.6) on the bottom of ocean describing a perfectly reflecting bottom.

2.2 The Liouville Eigenvalue Problem

Assume that $q = 0$. Using separation of variables, we assume the series expansion of the solution as

$$u(\tilde{x}, x_m) = \sum_{j \in \mathbb{N}} \alpha(j) w_j(\tilde{x}) u_j(x_m), \qquad \text{with coefficients } \alpha(j) \in \mathbb{C}.$$

By the definition of the Laplace operator we obtain that

$$\Delta(w_j u_j) = u_j \frac{\partial^2 w_j}{\partial x_1^2} + w_j \frac{\partial^2 u_j}{\partial x_2^2} \qquad \text{for } m = 2 \text{ and}$$

$$\Delta(w_j u_j) = u_j \frac{\partial^2 w_j}{\partial x_1^2} + u_j \frac{\partial^2 w_j}{\partial x_2^2} + w_j \frac{\partial^2 u_j}{\partial x_3^2} \qquad \text{for } m = 3.$$

For simplicity, we write in the following for the Laplacian acting on the horizontal variables

$$\frac{\partial^2 w_j}{\partial x_1^2} = \Delta_{\tilde{x}} w_j \quad (m=2) \qquad \text{and} \qquad \frac{\partial^2 w_j}{\partial x_1^2} + \frac{\partial^2 w_j}{\partial x_2^2} = \Delta_{\tilde{x}} w_j \quad (m=3).$$

If an arbitrary function ϕ depends only on one variable we abbreviate its first derivative by one prime and obviously its second derivative by two primes e.g. $\phi' := \partial\phi/\partial x_m$ and $\phi'' := \partial\phi^2/\partial x_m^2$. We can now replace in the Helmholtz equation (2.1) the term $u(\tilde{x}, x_m)$ to find for one series term that

$$u_j \Delta_{\tilde{x}} w_j + w_j \frac{\partial^2 u_j}{\partial x_m^2} + \frac{\omega^2}{c^2(x_m)} w_j u_j = 0.$$

Next, for $w_j(\tilde{x}), u_j(x_m) \neq 0$ we compute

$$\frac{\Delta_{\tilde{x}} w_j(\tilde{x})}{w_j(\tilde{x})} = -\frac{1}{u_j(x_m)} \frac{\partial^2 u_j(x_m)}{\partial x_m^2} - \frac{\omega^2}{c^2(x_m)}. \tag{2.7}$$

The left-hand side of the last equation (2.7) depends only on the horizontal variables $\tilde{x}$ and the right-hand side of (2.7) depends only on the vertical variable x_m. This only holds if each side is constant, say

$$\frac{\Delta_{\tilde{x}} w_j(\tilde{x})}{w_j(\tilde{x})} = -\frac{1}{u_j(x_m)} \frac{\partial^2 u_j(x_m)}{\partial x_m^2} - \frac{\omega^2}{c^2(x_m)} := -\mu_j \qquad \text{in } \Omega, \tag{2.8}$$

where $\mu_j \in \mathbb{C}$. We point out that this definition of the eigenvalue problem is not the common way to define it. We choose this sign to work later on with positive wave numbers for propagating modes, following the usual convention of scattering problems, where we want that the (square) root of negative μ_j tends to $+i\infty$ for $j \to \infty$. Note that this choice of the sign of μ_j influences the definition of the radiation conditions later on. Now, we see

$$u_j'' + \frac{\omega^2}{c^2(x_m)} u_j = \mu_j u_j \qquad \text{in } (0, H), \tag{2.9}$$

and

$$\Delta_{\tilde{x}} w_j + \mu_j w_j = 0 \qquad \text{in } \mathbb{R}^{m-1}. \tag{2.10}$$

Next, we investigate equation (2.9) with the corresponding boundary conditions (2.5) and (2.6). We say μ is an eigenvalue of the operator $(\partial^2/\partial x_m^2 + \omega^2/c^2(x_m))$ on $[0, H]$ if there exists a function ϕ, not identically equal zero, solving the Liouville-type eigenvalue problem

$$\begin{aligned} \phi''(x_m) + \frac{\omega^2}{c^2(x_m)} \phi(x_m) - \mu\, \phi(x_m) = 0 \qquad & \text{in } [0, H], \\ \phi(0) = 0 \qquad \text{and} \qquad & \phi'(H) = 0. \end{aligned} \tag{2.11}$$

Then, function ϕ is a corresponding eigenfunction to eigenvalue μ.

We introduce the space of ℓ-times differentiable functions by

$$C_W^\ell(0, H) := \{u \in C^\ell(0, H) : u(0) = 0\},$$

and the Sobolev space

$$H_W^1([0, H]) := \{u \in H^1(0, H) : u(0) = 0\}.$$

The weak formulation of (2.11) is obtained by multiplying the first equation of (2.11) with a test function $v \in H^1_W([0,H])$, formally integrating by parts, plugging in the boundary condition from line two of (2.11) and multiplying the equation by -1. As a result the weak formulation is

$$\int_0^H \phi' \, \overline{v'} \, dx_m - \int_0^H \frac{\omega^2}{c^2(x_m)} \phi \, \overline{v} \, dx_m = -\int_0^H \mu \phi \overline{v} \, dx_m \qquad \text{for all } v \in H^1_W([0,H]). \tag{2.12}$$

The solution to (2.12) satisfies

$$\phi'' + \frac{\omega^2}{c^2(x_m)} \phi - \mu\phi = 0, \qquad \text{in } L^2([0,H]),$$

which implies that $\phi \in H^2([0,H])$. We now define

$$a(\phi, v) := \int_0^H \phi' \, \overline{v'} \, dx_m - \int_0^H \frac{\omega^2}{c^2(x_m)} \phi \overline{v} \, dx_m,$$

and we arrive at the following variational formulation: Seek the eigenpair $(\phi, \mu) \in H^1_W([0,H]) \times \mathbb{C}$ such that

$$a(\phi, v) = -\mu \int_0^H \phi \overline{v} \, dx_m \qquad \text{for all } v \in H^1_W([0,H]).$$

By using a compact perturbation we now show that the sesquilinear form a is coercive. We add a term equal zero to the sesquilinear form a and we compute

$$\begin{aligned} a(\phi, \phi) &= \int_0^H \phi' \, \overline{\phi'} \, dx_m - \int_0^H \frac{\omega^2}{c^2(x_m)} \phi \, \overline{\phi} \, dx_m + \int_0^H \phi \, \overline{\phi} \, dx_m - \int_0^H \phi \, \overline{\phi} \, dx_m \\ &\geq \|\phi\|^2_{H^1([0,H])} - \left(\frac{\omega^2}{c_+^2} \|\phi\|^2_{L^2([0,H])} + \|\phi\|^2_{L^2([0,H])} \right). \end{aligned}$$

We now see by [McL00, pg.43-44] that a is coercive, with respect to the pivot space $L^2([0,H])$.

Next, since $\omega / c(x_m)$ is real valued, we obtain that a is self-adjoint. Therefore, well-known eigenvalues theory (see e.g. [McL00]) implies existence of a sequence of real eigenvalues $\{\mu_j : j \in \mathbb{N}\} \subset \mathbb{R}$ such that $\mu_j \to -\infty$ as j tends to infinity with corresponding eigenfunctions $\{\phi_j : j \in \mathbb{N}\}$ in $H^1_W([0,H])$, such that orthogonality conditions hold. We consider that the eigenvalues are ordered as

$$\mu_1 \geq \mu_2 \geq \mu_3 \geq \dots > -\infty.$$

We furthermore define their square roots λ_j as

$$\lambda_j := \begin{cases} \sqrt{\mu_j} & \text{for } \mu_j \geq 0, \\ i\sqrt{|\mu_j|} & \text{for } \mu_j \leq 0, \end{cases} \qquad \text{for all } j \in \mathbb{N}. \tag{2.13}$$

More precisely, we have extended the root function from the real axis to a holomorphic function in the complex plane with branch cut along the negative imaginary axis. Positive eigenvalues μ_j creates propagating modes ϕ_j, which transmit all of the energy, negative eigenvalues μ_j lead to imaginary λ_j, creating evanescent modes.

Plugging all together, the eigenpair (ϕ_j, λ_j^2), where $\phi_j \in H^2([0,H])$ denotes the j-th eigenfunction and $\lambda_j^2 \in \mathbb{R}$ the j-th eigenvalue, solves

$$\phi_j'' + \frac{\omega^2}{c^2(x_m)} \phi_j - \lambda_j^2 \phi_j = 0 \qquad \text{almost everywhere in } (0,H),$$

and since $H^2([0,H]) \subset C^{1,1/2}([0,H])$ it holds for all $\phi_j \in C^{1,1/2}([0,H])$ the boundary conditions

$$\phi_j(0) = 0 \qquad \text{and} \qquad \phi_j'(H) = 0,$$

in the classical sense of point wise evaluation. Additionally, we assume that the eigenfunctions $\phi_j \in H^1_W([0,H])$ are normed by

$$\int_0^H \phi_j(x_m)\overline{\phi_j(x_m)}\,dx_m = 1 \qquad \text{for all } j \in \mathbb{N}.$$

Plugging all together gives us the following corollary.

Corollary 2.2.1. *The eigenpair (ϕ_j, λ_j^2), where $\phi_j \in H^2([0,H])$ denotes the j-th eigenfunction and $\lambda_j^2 \in \mathbb{R}$ the j-th eigenvalue, solves*

$$\phi_j'' + \frac{\omega^2}{c^2(x_m)}\phi_j - \lambda_j^2\phi_j = 0 \qquad \textit{almost everywhere in } (0,H), \tag{2.14}$$
$$\phi_j(0) = 0 \qquad \textit{and} \qquad \phi_j'(H) = 0,$$

and it holds the variational formulation

$$\int_0^H \left[\phi_j' \cdot v' + \left(\lambda_j^2 - \frac{\omega^2}{c^2(x_m)}\right)\phi_j\overline{v}\right] dx_m = 0 \qquad \textit{for all } v \in H^1_W([0,H]). \tag{2.15}$$

Remark 2.2.2. *For an ocean with constant background sound speeds c the eigenvalues can be explicitly computed by*

$$\lambda_j^2 := \frac{\omega^2}{c^2} - \left(\frac{(2j-1)\pi}{2H}\right)^2 \qquad \textit{for } j \in \mathbb{N},$$

and the eigenvectors are given by

$$\phi_j(x_m) = \sin\left(\frac{(2j-1)\pi}{2H}\,x_m\right), \qquad x_m \in [0,H].$$

Theorem 2.2.3 (Number of Propagating Modes)**.** *Assume that the sound speed $x_m \mapsto c(x_m)$ is bounded as (2.3) and assume that $\omega > 0$ is fixed. We further consider that for all $j \in \mathbb{N}$ the pair (ϕ_j, λ_j^2) solves (2.14) and the pair $(\phi^\pm, \lambda_\pm^2)$ solves the corresponding eigenvalue problem for constant background sound speed $c_\mp$. Then, it holds for $(\lambda_j^+)^2$ and $(\lambda_j^-)^2$ as defined in Remark 2.2.2 that*

$$(\lambda_j^+)^2 \le \lambda_j^2 \le (\lambda_j^-)^2 \qquad \textit{for all } j \in \mathbb{N}.$$

Proof. Let V_j for $j \in \mathbb{N}$ be any j-dimensional subspace of $H^1_W([0,H])$. By Min-Max Theorem (or also called Courant-Fischer Theorem from [Yse10, Theorem 5.9]), we obtain that

$$\begin{aligned}
\lambda_j^2 &= \min_{V_j} \max_{\phi_j \in V_j, \|\phi_j\|_{H^1(0,H)}=1} a(\phi_j,\phi_j) \\
&= \min_{V_j} \max_{\phi_j \in V_j, \|\phi_j\|_{H^1(0,H)}=1} \int_0^H \left(|\phi_j'(x_m)|^2 - \frac{\omega^2}{c(x_m)^2}|\phi_j(x_m)|^2\right) dx_m \\
&\lesseqgtr \min_{V_j} \max_{\phi_j^\pm \in V_j, \|\phi_j^\pm\|_{H^1(0,H)}=1} \int_0^H \left(|\phi_j'^\pm(x_m)|^2 - \frac{\omega^2}{c_\mp^2}|\phi_j^\pm(x_m)|^2\right) dx_m = (\lambda_j^\pm)^2 \qquad \text{for all } j \in \mathbb{N}.
\end{aligned}$$

This finishes the proof. □

Lemma 2.2.4. *a) There exists a constant C such that $|\lambda_j^2| > Cj^2$ fur sufficient large j.*

b) There are constants $0 < c_0 < C$ such that $c_0 j \le \|\phi_j'\|_{L^2(0,H)} \le Cj$, $\forall j \in \mathbb{N}$. It further holds that $\|\phi_j'\|_{L^2(0,H)} \le C(1+|\lambda_j|^2)^{1/2}$ for all $j \in \mathbb{N}$.

Proof. a) Due to Theorem 2.2.3 we know that

$$(\lambda_j^+)^2 \leq \lambda_j^2 \leq (\lambda_j^-)^2 \qquad \text{for all } j \in \mathbb{N},$$

and by Remark 2.2.2 we have

$$(\lambda_j^\pm)^2 = \frac{\omega^2}{c_\mp^2} - \left(\frac{(2j-1)\pi}{2H}\right)^2 \qquad \text{for all } j \in \mathbb{N}.$$

Then, we obtain for j large enough that

$$\lambda_j^2 \lesseqgtr (\lambda_j^\pm)^2 = \frac{w^2}{c_\mp^2} - \frac{(4j^2-2j+1)\pi^2}{4H^2} \lesseqgtr C^\pm j^2,$$

where $C^\pm > 0$ is independent of j.

b) Obviously, we have

$$\|\phi_j^{'}(x_m)\|^2_{L^2(0,H)} = \int_0^H |\phi_j^{'}(x_m)|^2 dx_m \qquad \text{for all } j \in \mathbb{N}.$$

From (2.14) and (2.15) we have

$$\|\phi_j^{'}(x_m)\|^2_{L^2(0,H)} = \int_0^H |\phi_j'|^2 dx_m = \int_0^H \left[\frac{\omega^2}{c^2(x_m)} - \lambda_j^2\right] |\phi_j|^2 \, dx_m.$$

If we assume that $\|\phi_j'\|^2_{L^2(0,H)} = 0$ we obtain that the eigenfunction is constant, however, this will be a contradiction to the Dirichlet boundary condition holding for ϕ_j. In consequence, a) yields that

$$\max\left(\frac{\omega^2}{c_+^2} + \lambda_j^2, 0\right) < \|\phi_j^{'}\|^2_{L^2(0,H)} \leq \lambda_j^2 + \frac{\omega^2}{c_-^2} \leq \frac{\pi^2(2j-1)^2}{4H^2} + \omega^2 \frac{c_+^2 - c_-^2}{c_+^2 c_-^2}. \tag{2.16}$$

The estimate $\|\phi_j^{'}\|^2_{L^2(0,H)} < Cj^2$ follows directly from the right-hand side of (2.16). As $\|\phi_j'\|_{L^2(0,H)}$ cannot vanish, and as the left-hand side of (2.16) grows quadratically, there is $c_0 > 0$ such that $0 < c_0 j^2 \leq \|\phi_j^{'}\|^2_{L^2(0,H)}$. Finally, $\|\phi_j^{'}\|_{L^2(0,H)} \leq C(1+\lambda_j|^2)^{1/2}$ for all $j \in \mathbb{N}$ follows again by exploiting part a).

□

Lemma 2.2.5. *For the eigenvector $\phi_j \in H^1([0,H])$ it holds that*

$$\max_{x_m \in [0,H]} \phi_j(x_m) \leq C(H) \qquad \text{for } j \in \mathbb{N},$$

where $C(H) > 0$ depends on H but not on $j \in \mathbb{N}$.

Proof. The proof is rather technical and uses separation of variables. For simplicity we set $\phi := \phi_j$ and $\lambda := \lambda_j$. By the definition of the eigenvalue problem with corresponding boundary conditions given in (2.11) we have

$$\phi''(x_m) - \lambda^2 \phi(x_m) = \underbrace{-\frac{\omega^2}{c^2(x_m)}\phi(x_m)}_{=:f(x_m)}, \tag{2.17}$$

with boundary conditions $\phi'(H) = 0$ and $\phi(0) = 0$. Now, we choose the following ansatz function as the general solution for the eigenvalue problem

$$\phi(x_m) = \alpha(x_m)\exp(-\lambda x_m).$$

Furthermore, by differentiating the ansatz function we see that

$$\phi'(x_m) = \alpha'(x_m)\exp(-\lambda x_m) - \lambda\alpha(x_m)\exp(-\lambda x_m),$$
$$\phi''(x_m) = \alpha''(x_m)\exp(-\lambda x_m) - 2\lambda\alpha'(x_m)\exp(-\lambda x_m) + \lambda^2\phi(x_m).$$

Next, we insert $\phi''(x_m)$ in (2.17), multiply with $\exp(-\lambda x_m)$ and we obtain

$$(\alpha''(x_m) - 2i\lambda\alpha'(x_m))\exp(-2\lambda x_m) = f(x_m)\exp(-\lambda x_m).$$

Furthermore, we deduce

$$(\alpha'(x_m)\exp(-2\lambda x_m))' = f(x_m)\exp(-\lambda x_m).$$

By integration it follows that

$$\alpha'(x_m)\exp(-\lambda x_m) = \alpha'(H)\exp(-\lambda H) + \int_H^{x_m} f(t)\exp(-\lambda t)\,dt.$$

We moreover see

$$\alpha'(x_m) = \alpha'(H) + \exp(\lambda x_m)\int_H^{x_m} f(t)\exp(-\lambda t)\,dt.$$

Then, a second integration gives

$$\alpha(x_m) = \alpha(0) + \alpha'(H)x_m + \int_0^{x_m}\exp(\lambda s)\int_H^s f(t)\exp(-\lambda t)\,dt\,ds. \tag{2.18}$$

Since the Dirichlet boundary condition implies that $\alpha(0) = 0$, we focus on $\alpha'(H)$. Plugging $x_m = H$ in the last equation shows

$$\alpha(H) = \alpha'(H)H + \int_0^{H}\exp(\lambda s)\int_H^s f(t)\exp(-\lambda t)\,dt\,ds. \tag{2.19}$$

Now, it holds for the ansatz function that

$$\alpha(x_m) = \phi(x_m)\exp(\lambda x_m).$$

Next, by differentiation we see that

$$\alpha'(x_m) = \phi'(x_m)\exp(\lambda x_m) + \lambda\exp(\lambda x_m)\phi(x_m).$$

Then, we use $x_m = H$ in the last equation to write

$$\alpha'(H) = \lambda\underbrace{\exp(\lambda H)\phi(H)}_{=\alpha(H)}. \tag{2.20}$$

Furthermore, it holds that the integral term in (2.19) is constant. Consequently, equation (2.19) and (2.20) satisfying

$$\begin{pmatrix} 1 & -H \\ -\lambda & 1 \end{pmatrix}\begin{pmatrix} \alpha(H) \\ \alpha'(H) \end{pmatrix} = \begin{pmatrix} C \\ 0 \end{pmatrix},$$

where C is a constant. Therefore, we obtain the boundary term

$$\alpha'(H) = C\frac{\lambda}{1-\lambda H}.$$

For equation (2.18) it follows that

$$\alpha(x_m) = C\frac{\lambda}{1-\lambda H}x_m + \int_0^{x_m}\exp(\lambda s)\int_H^s f(t)\exp(-\lambda t)\,dt\,ds.$$

Plugging the last equation in the definition of the ansatz function and then applying Cauchy-Schwarz inequality finish the proof (since $\|f\|_{L^2} \leq \omega/c$). □

Remark 2.2.6. *a) Consider that the sound speed $x_m \mapsto c(x_m)$ is bounded by constants given in (2.3) and for all $j \in \mathbb{N}$ the pair (ϕ_j, λ_j^2) solving (2.14). Then there exists a finite number $J \in \mathbb{N}$ of propagating eigenmodes, where the number $J \in \mathbb{N}$ depends on the parameters ω, c and H. We call this assemblage of the parameters ω, c and H* ***ocean configuration****.*

b) Due to the ocean configuration, there exists for $0 \leq j \leq J(\omega, c, H)$ real-valued λ_j and for $j > J(\omega, c, H)$ purely imaginary ones. For a special choice of ω, c or H, also $\lambda_j = \lambda_j^2 = 0$ could be possible for some $j \in \mathbb{N}$. We call this choice ***exceptional ocean configuration****.*

c) For fixed depth H and given background sound speed, we obtain the frequency dependency of the eigenvalues $\lambda^2 = \lambda^2(\omega)$ and we call the frequencies resulting vanishing eigenvalues $\lambda^2(\omega) = 0$ ***exceptional frequencies****.*

In this work we exclude these exceptional frequencies. Nevertheless, if we exclude these exceptional frequencies the analysis, however, will be influenced. We investigate the case of these exceptional frequencies later on. We further want extend $\omega \mapsto \lambda_j^2(\omega)$ as a holomorphic function into a complex open neighborhood of $\mathbb{R}_{>0}$ in $\mathbb{C}$.

Lemma 2.2.7. *For all $\omega_* > 0$ there exists an open neighborhood $U(\omega_*) \subset \mathbb{C}$ of $\mathbb{R}_{>0}$ and index functions $\ell_j : U(\omega_*) \to \mathbb{N}$ that satisfy $\cup_{j \in \mathbb{N}} \ell_j(\omega) = \mathbb{N}$ and $\ell_j(\omega) \neq \ell'_j(\omega)$ for $j \neq j' \in \mathbb{N}$ and all $\omega \in U$, such that the eigenvalue curves $\omega \mapsto \lambda^2_{\ell_j(\omega)}(\omega)$ are real-analytic functions in $U(\omega_*) \cap \mathbb{R}$ and extend to holomorphic functions in $U(\omega_*)$ for all $j \in \mathbb{N}$.*

Proof. In the following we use results on holomorphic families of operators from [Kat95, Chapter VII, §2 and §4]. We first choose some $\omega_* > 0$. The differential operators $L(\omega)\, u = u'' + (\omega_*^2/c^2)u$ on $(0, H)$ with boundary conditions $u(0) = 0$ and $u'(H) = 0$ yield a selfadjoint holomorphic family of type (A) since $u \mapsto (\omega_*^2/c^2)u$ is bounded on $L^2(0, H)$, $\omega_* \mapsto (\omega_*^2/c^2)u$ is holomorphic in $\omega_* \in \mathbb{C}$, and the domain $\{u \in H^2(0, H),\, v(0) = 0\}$ of $L(\Omega)$ is independent of $\omega_* \in \mathbb{C}$, compare [Kat95, Ch. VII, §1.1, §2.1, Th. 2.6]. These differential operators also form of a holomorphic family of type (B) since the associated sesquilinear form a from (2.12) is bounded. We now differ to two cases, if λ_j^2 is multiple eigenvalue and if λ_j^2 has multiplicity one.

If $\lambda_j^2(\omega_*)$ is a multiple eigenvalue, then the function $\omega \mapsto \lambda_j^2(\omega)$ is in general not differentiable at ω^*, such that the eigenvalue index needs to be re-ordered to obtain smooth eigenvalue curves, compare [Kat95, Ch. VII, §3.1, Ch. 2, Th. 6.1]. Indeed, the latter reference shows that if $\lambda_j^2(\omega_*)$ is a multiple eigenvalue then it has finite multiplicity and there exists a complex neighborhood U_j of ω_* and an index function $\ell_j : U_j \cap \mathbb{R} \to \mathbb{N}$ such that $\omega \mapsto \lambda^2_{\ell_j(\omega)}(\omega)$ can be extended holomorphically from $U_j \cap \mathbb{R}$ into U_j.

If eigenvalue $\lambda_j^2(\omega_*)$ has multiplicity one, then [Kat95, Ch. VII, §3.1, Example 4.23] gives for $j \in \mathbb{N}$ that for each eigenvalue $\lambda_j^2(\omega_*)$, there is a complex neighborhood U_j of ω_* such that $\omega \mapsto \lambda_j^2(\omega)$ can be extended from $U_j \cap \mathbb{R}$ as a holomorphic function of ω into U_j. In consequence, while the curves $\omega \mapsto \lambda_j^2(\omega)$ are merely piece-wise analytic for real $\omega > 0$ and the corresponding eigenvalue sheets are piece-wise holomorphic in a complex neighborhood of $\mathbb{R}_{>0}$, analyticity can be obtained by re-ordering indices via the index functions ℓ_j.

It remains to see that the intersection of the neighborhoods $\cap_{j \in \mathbb{N}} U_j$ is non-empty. For $N \in \mathbb{N}$ this holds for any finite union $\cup_{j=1,\dots,N} U_j$.

Due to the eigenvalue estimates in Lemma 2.2.4 we further know that the eigenvalues multiplicity one holds for $j > j_0$, where j_0

$$j_0 := \left\lceil \frac{H^2}{2\pi^2} \left(\omega_*^2 \left[\frac{1}{c_-^2} - \frac{1}{c_+^2} \right] + 1 \right) \right\rceil.$$

Then, the distance

$$d_j = \begin{cases} \lambda_j^2(\omega) - \lambda_{j-1}^2(\omega) & j \geq 2, \\ \lambda_2^2(\omega) - \lambda_1^2(\omega) & j = 1, \end{cases}$$

of $\lambda_j^2(\omega_*)$ to the rest of the spectrum

$$\{\lambda_\ell^2(\omega_*) \text{ where } \lambda_\ell^2(\omega_*) \neq \lambda_j^2(\omega_*) \text{ for } j \neq l\},$$

is bounded by one. Applying Theorem 4.8 in [Kat95, Ch. VII], compare also (4.45) in the same chapter, we deduce that for all $j > j_0$ the holomorphic extension of $\lambda_j^2(\omega_*)$ has a convergence radius of at least $(1+1/c_-^2)^{-1}$. (Set $\varepsilon = 1$, $a = 1$, $b = 0$, and $c = 1/c_-^2 \geq \|1/c^2\|_{L^\infty(0,H)}$ in (4.45).) Further, for $j \leq j_0$ all eigenvalues $\lambda_j^2(\omega_*)$, can be extend to a holomorphic functions in

$$U(\omega_*) := \cap_{j=1}^{j_0} U_j \cap B(\omega_*, 1).$$

This finishes the proof.

□

Theorem 2.2.8. *For $j \in \mathbb{N}$ there exists a complex neighborhood U of $\mathbb{R}_{>0}$ and index functions $\ell_j : U \to \mathbb{N}$ such that the eigenvalue curves $\lambda_{\ell_j(\omega)}^2(\omega)$ are real-analytic curves that extend to holomorphic functions in U. For each compact subset W of U, the set of frequencies where some eigenvalue vanishes*

$$K_0 = \{\omega \in W : \text{ there is } j \in \mathbb{N} \text{ such that } \lambda_j^2(\omega) = 0\},$$

is composed by a finite number of discrete values.

Proof. Using Lemma 2.2.7 we cover the positive reals $(0, \infty)$ with the neighborhoods $U(\omega)$ of $\omega > 0$. Then for each compact interval $[1/\ell, \ell]$ with $\ell \in \mathbb{N}$ there exists a finite sub cover, which for $j \in \mathbb{N}$ allows to holomorphically continue all eigenvalue functions $\omega \mapsto \lambda_{\ell_j(\omega)}^2(\omega)$, into a complex neighborhood U_ℓ of $[1/\ell, \ell]$. If $\ell \in \mathbb{N}$ is chosen arbitrary, this holds for the claimed open set $U = \cup_{\ell \in \mathbb{N}} U_\ell$ in $\mathbb{C}$ containing $\mathbb{R}_{>0}$.

It remains to prove the last claim of theorem. Due to Theorems 1.9 and 1.10 in [Kat95, Ch. VII, §1.3] we obtain that for each compact subset W of U the set

$$K_0 = \{\omega \in W : \text{there is } j \in \mathbb{N} \text{ such that } \lambda_j^2(\omega) = 0\} \subset W,$$

is either finite (for each $\omega \in W$ there exists some $j = j(\omega) \in \mathbb{N}$ such that $\lambda_j^2(\omega) = 0$) or equals W (the number of frequencies ω in W such that there is $j = j(\omega) \in \mathbb{N}$ such that $\lambda_j^2(\omega) = 0$ is finite). Obviously, the set K_0 is finite since Lemma 2.2.4 excludes the first case. □

2.3 Discretization of the Liouville Eigenvalue Problem

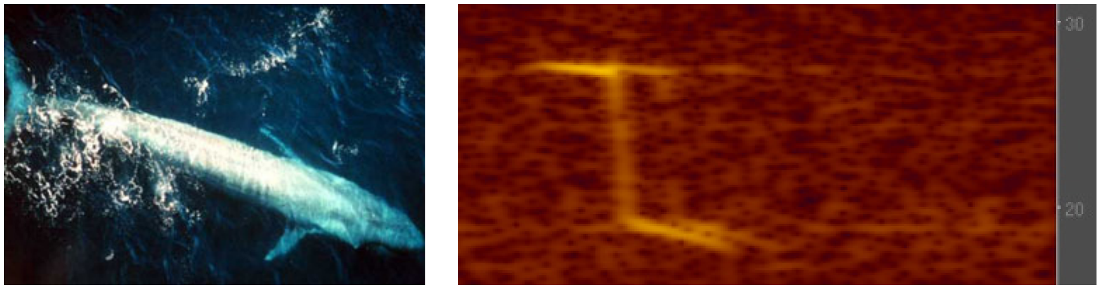
In this section we will discuss the numerical discretization of the Liouville eigenvalue problem (2.14). More precisely, we present different schemes to approximate the eigenpair (ϕ_j, λ_j^2), where $\phi_j \in H^2([0, H])$ denotes the j-th eigenfunction and $\lambda_j^2 \in \mathbb{R}$ the j-th eigenvalue solving the Liouville eigenvalue problem (2.14). Due to Remark 2.2.6 the eigenvalues $\{\lambda_j^2\}_{j\in\mathbb{N}}$ and the eigenvectors $\{\phi_j\}_{j\in\mathbb{N}}$ depend on the ocean configuration, e.g. the sound speed c, the frequency ω and the height H. A crucial point is the representation of the sound speed c, which influences the computation technique of eigenvalues and eigenfunctions. Thus, we discuss in the following different models. We recall that, if we assume that the background speed is constant, then the eigenfunctions and eigenvalues have the well-known representation,

$$\lambda_j^2 = \frac{\omega^2}{c^2} - \left(\frac{(2j-1)\pi}{2H}\right)^2 \quad \text{and} \quad \phi_j(x_m) = \sin\left(\frac{(2j-1)\pi}{2H}\, x_m\right) \quad \text{for all } j \in \mathbb{N}, x_m \in [0, H].$$

First bounds of c are motivated by Figure 2.1, where roughly speaking the background sound speed lies between $1440m/s$ and $1540m/s$.

In ocean acoustics the communication of blue whales is an important research topic. For example Figure 2.2 b) shows the signal of a Z Call of a blue whale a). Thus, Figure 2.2 b) gives us a first reasonable value for the frequencies for the computations later on.

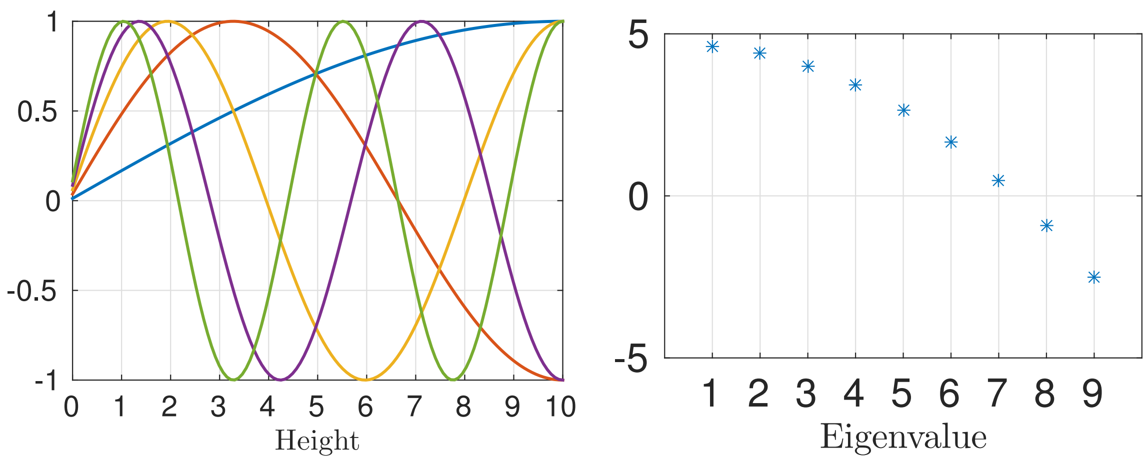
A first motivation for the approximation of eigenvalues and eigenvectors are given in Figure 2.3, where we compute the eigenvectors $\{\phi_j\}_{j=1,2...,5}$ and the eigenvalues $\{\lambda_j^2\}_{j=1,2...,9}$ for constant



a) b)

Figure 2.2: Blue whale Z call: 10s. 28Hz, a downsweep of 1s from 28Hz to 19Hz, 10s. 19Hz. (Source: Alfred-Wegener-Institut Bremerhaven)

background sound speed $c = 1460$, height $H = 10$ and frequency $\omega = 500$. We use this technique later on to evaluate the approximation error of eigenvalues and eigenvectors for different techniques, as a first indicator. We point out that Figure 2.3 (b) shows that eigenvalues $\{\lambda_j^2\}_{j=1,2\ldots,7}$ are propagating ones.



(a) (b)

Figure 2.3: Constant background sound speed $c = 1460$, height $H = 10$ and frequency $\omega = 500$. (a) Eigenvectors $\{\phi_j\}_{j=1,2\ldots,5}$ (b) Eigenvalues $\{\lambda_j^2\}_{j=1,2\ldots,9}$

Due to the fact that we consider an inhomogeneous ocean, the non constant background speed yields different computation techniques of the eigenfunction $\phi_j(x_m)$ and the eigenvalue λ_j^2: If we consider that we have an n-layered ocean with piecewise constant background speed on each layer and using transmission conditions on each one, then we have a semi explicit representation of eigenvectors and eigenvalues. If we suppose continuous background sound speed, which is sufficiently smooth, then we can compute eigenvectors and eigenvalues by a spectral method. If the non-constant background speed is arbitrary, but still sufficiently smooth, we can apply a finite element method (FEM).

For completeness, we first introduce common models for the background sound speed in the ocean. For a realistic model the background sound speed depends on the vertical axis x_m and respects various parameters, e.g. temperature, depth of the ocean, material decomposition. [KPL12, Chapter 2] gives different approaches for background sound speed models, depending on seasonal variability or on storm surges. For more basic sound velocity models we refer the reader to [Lur10]. We first present the well-known and quite simple model of Medwin developed in 1975, see [MC97], for depth oceans with depth of maximum one kilometer. The sound speed in m/s, depending on the temperature in Celsius T, the salinity S in PSU (Practical salinity units), and the depth of the ocean H, is denoted by

$$c(x_m) = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + (1.34 - 0.01T)(S - 35) + 0.016x_m, \quad x_m \in [0, H].$$

Here, x_m is the actual depth in meters. We note that [MC97] gives also possible reasonable values of the salinity and temperatures.

A different approach to compute the background sound speed in the ocean is introduced by Chen and Milero in [Lur10, Section 2.6]. This model is widely used and confirmed by UNESCO but needs more efforts to compute since ones needs the salinity S in PSU, the temperature in Celsius T and the hydrostatic pressure in bars P computed by Leroy formula

$$P(x_m, \varphi) = 10052.405(1 + 5.28 * 10^{-3}\sin^2\varphi)x_m + 2.36\,10^{-6}x_m^2 + 10.196,$$

where φ is the latitude in meter. Using the following model parameters depending on the temperature,

$$\begin{aligned}
c_0 =& 1402.388 + 5.03711\,T - \frac{5.80852}{10^2}T^2 + \frac{3.3420}{10^4}T^3 - \frac{1.47800}{10^6}T^4 + \frac{3.1464}{10^9}T^5,\\
c_1 =& 0.153563 + \frac{6.8982}{10^4}T - \frac{8.1788}{10^6}T^2 + \frac{1.3621}{10^7}T^3 - \frac{6.1185}{10^{10}}T^4,\\
c_2 =& \frac{3.1260}{10^5} - \frac{1.7107}{10^6}T + \frac{2.5974}{10^8}T^2 - \frac{2.5335}{10^{10}}T^3 + \frac{1.0405}{10^{12}}T^4,\\
c_3 =& \frac{9.7729}{10^9} - \frac{3.8504}{10^{10}}T - \frac{2.3643}{10^{12}}T^2,
\end{aligned}$$

and the parameters depending on the temperature and the hydrostatic pressure,

$$\begin{aligned}
A =& A_0 + A_1P + A_2P^2 + A_3P^3,\\
A_0 =& 1.389 - \frac{1.262}{10^2}T + \frac{7.164}{10^5}T^2 + \frac{2.006}{10^6}T^3 - \frac{3.21}{10^8}T^4,\\
A_1 =& \frac{9.4742}{10^5} - \frac{1.258}{10^5}T - \frac{6.4885}{10^{-8}}T^2 + \frac{1.0507}{10^8}T^3 - \frac{2.0122}{10^{10}}T^4,\\
A_2 =& -\frac{3.9064}{10^7} + \frac{9.1041}{10^9}T - \frac{1.6002}{10^{10}}T^2 + \frac{7.988}{10^{12}} * T^3,\\
A_3 =& \frac{11}{10^{10}} + \frac{6.649}{10^{12}}T - \frac{3.389}{10^{13}}T^2,\\
B =& \frac{1.922}{10^2} - \frac{4.42}{10^5}T + \left(\frac{7.3637}{10^3} + \frac{1.7945}{10^7}T\right)P,\\
C =& \frac{1.727}{10^3} - \frac{7.9836}{10^6}P,
\end{aligned}$$

we can further compute the sound speed in meter per second by

$$c(x_m) = c_0 + c_1P + c_2P^2 + c_3P^3 + AS + BS^{3/2} + CS^2.$$

For non constant background sound speed $c(x_m)$ we have no explicit representation of the eigenvectors $\{\phi_j\}_{j\in\mathbb{N}}$, in a form containing sine or cosine. Thus, we now present the well-known Ritz-Galerkin approximation applied to the Liouville eigenvalue problem acting on the vertical variable (2.14). We recall the variational formulation of the Liouville eigenvalue problem given by equation (2.15),

$$\int_0^H \left[\phi_j' \cdot \overline{v}' + \left(\lambda_j^2 - \frac{\omega^2}{c^2(x_m)}\right)\phi_j\overline{v}\right] dx_m = 0 \qquad \text{for all } v \in H^1_W([0, H]).$$

Let $N \in \mathbb{N}$ denote the discretization parameter and define the grid $\mathbb{N}_N = \{j \in \mathbb{N}, 1 \leq j \leq N\}$, and

$$\left\{x_j^{(N)} := \left(H\frac{j}{N}\right) : j \in \mathbb{N}_N\right\} \subset [0, H]. \tag{2.21}$$

Using piece-wise linear Ritz-Galerkin approximation to approximate the latter variational formulation, we moreover represent $j, N \in \mathbb{N}$ the approximated eigenfunction $\phi_j^{(N)}$ as a linear combination of basis functions ψ_k by

$$\phi_j^{(N)} = \sum_{k=1}^{N} \xi_i \psi_k(x_m),$$

where

$$\psi_j(x_m) = \begin{cases} (x_m - x_{j-1}^{(N)})/(x_j^{(N)} - x_{j-1}^{(N)}), & \text{if } x_{j-1}^{(N)} \leq x_m \leq x_j^{(N)}, \\ (x_{j+1}^{(N)} - x_m)/(x_{j+1}^{(N)} - x_j^{(N)}), & \text{if } x_j^{(N)} \leq x_m \leq x_{j+1}^{(N)}, \\ 0, & \text{otherwise }, \end{cases}$$

and its derivative

$$\psi_j'(x_m) = \begin{cases} 1/(x_j^{(N)} - x_{j-1}^{(N)}), & \text{if } x_{j-1}^{(N)} \leq x_m \leq x_j^{(N)}, \\ -1/(x_{j+1}^{(N)} - x_j^{(N)}), & \text{if } x_j^{(N)} \leq x_m \leq x_{j+1}^{(N)}, \\ 0, & \text{otherwise }, \end{cases}$$

where $x_0^{(N)} = 0$. The fact that $x_0^{(N)} = 0$ yields the Dirichlet boundary conditions $x_m = 0$. For $j = N$, we obtain

$$\psi_N(x_m) = \begin{cases} 0, & \text{if } 0 \leq x_m \leq x_{N-1}^{(N)}, \\ (x_m - x_{N-1}^{(N)})/(H - x_{N-1}^{(N)}), & \text{if } x_{N-1}^{(N)} \leq x_m \leq H, \end{cases}$$

and its derivative

$$\psi_N'(x_m) = \begin{cases} 0, & \text{if } 0 \leq x_m \leq x_{N-1}^{(N)}, \\ 1/(H - x_{N-1}^{(N)}), & \text{if } x_{N-1}^{(N)} \leq x_m \leq H. \end{cases}$$

We replace the eigenfunction ϕ_j in the latter variational formulation by its approximated one $\phi_j^{(N)}$ and we denote the approximated eigenvalues by $\lambda_{j,h}^2$ and we deduce for all $j \in \mathbb{N}$ that the eigenpair $(\phi_j^{(N)}, \lambda_{j,h}^2)$ solves

$$\int_0^H \left[\sum_{k=1}^{N} \xi_k \psi_k'(x_m)\psi_j'(x_m) - \frac{\omega^2}{c(x_m)^2}\sum_{k=1}^{N} \xi_k \psi_k(x_m)\psi_j(x_m) + \lambda_{j,h}^2 \sum_{k=1}^{N} \xi_k \psi_i(x_m)\psi_j(x_m)\right] dx_m = 0.$$

Consequently, we obtain for all $j \in \mathbb{N}$ the Raleigh-Ritz-Galerkin equation

$$\sum_{k=1}^{N} \left(\int_{x_{j-1}^{(N)}}^{x_{j+1}^{(N)}} \left[\psi_k'(x_m)\psi_j'(x_m) - \frac{\omega^2}{c(x_m)^2}\psi_k(x_m)\psi_j(x_m) + \lambda_{j,h}^2 \psi_k(x_m)\psi_j(x_m)\, dx_m\right] dx_m\right) \xi_k = 0.$$

In matrix notation we have

$$(S - M)\xi = -\lambda_h^2 B\xi, \tag{2.22}$$

where the entries of the stiffness matrix S and the mass matrix M are defined by

$$s_{kj} = \int_{x_{j-1}^{(N)}}^{x_{j+1}^{(N)}} \psi_k'\psi_j'\, dx_m, \qquad m_{kj,c} = \int_{x_{j-1}^{(N)}}^{x_{j+1}^{(N)}} \frac{\omega^2}{c(x_m)^2}\psi_k\psi_j\, dx_m,$$

and the entries of the mass matrix B are defined by

$$b_{kj} = \int_{x_{j-1}^{(N)}}^{x_{j+1}^{(N)}} \psi_k \psi_j \, dx_m.$$

For completeness, the mass matrix S can be computed explicitly for $j = 1, 2, ..., N-1$,

$$\begin{aligned}
s_{jj} &= \int_{x_{j-1}^{(N)}}^{x_j^{(N)}} \frac{1}{(x_j^{(N)} - x_{j-1}^{(N)})^2} \, dx_m + \int_{x_j^{(N)}}^{x_{j+1}^{(N)}} \frac{1}{(x_{j+1}^{(N)} - x_j^{(N)})^2} \, dx_m = \frac{1}{x_j^{(N)} - x_{j-1}^{(N)}} + \frac{1}{x_{j+1}^{(N)} - x_j^{(N)}}, \\
s_{j,j+1} &= -\int_{x_j^{(N)}}^{x_{j+1}^{(N)}} \frac{1}{(x_{j+1}^{(N)} - x_j^{(N)})^2} \, dx_m = -\frac{1}{x_{j+1}^{(N)} - x_j^{(N)}}, \\
s_{NN} &= \int_{x_{N-1}^{(N)}}^{H} \frac{1}{(H - x_{N-1}^{(N)})^2} \, dx_m = \frac{1}{H - x_{N-1}^{(N)}}.
\end{aligned}$$

We further know that $s_{j+1,j} = s_{j,j+1}$. To calculate the mass matrix M a numerical quadrature will be necessary to evaluate

$$\begin{aligned}
m_{jj} &= -\int_{x_{j-1}^{(N)}}^{x_j^{(N)}} \frac{\omega^2}{c(x_m)^2} \frac{(x_m - x_{j-1})^2}{(x_j - x_{j-1})^2} \, dx_m - \int_{x_j^{(N)}}^{x_{j+1}^{(N)}} \frac{\omega^2}{c(x_m)^2} \frac{(x_{j+1}^{(N)} - x_m)^2}{(x_{j+1}^{(N)} - x_j^{(N)})^2} \, dx_m, \\
m_{j,j+1} &= \int_{x_j^{(N)}}^{x_{j+1}^{(N)}} \frac{\omega^2}{c(x_m)^2} \frac{(x_{j+1}^{(N)} - x_m)(x_m - x_j^{(N)})}{(x_{j+1}^{(N)} - x_j^{(N)})^2} \, dx_m, \\
m_{NN} &= \int_{x_{N-1}^{(N)}}^{H} \frac{\omega^2}{c(x_m)^2} \frac{(x_m - x_{N-1}^{(N)})^2}{(H - x_{N-1}^{(N)})^2} \, dx_m.
\end{aligned}$$

Again, the entries of the mass matrix B can be computed explicitly,

$$\begin{aligned}
b_{jj} &= \int_{x_{j-1}^{(N)}}^{x_j^{(N)}} \frac{(x_m - x_{j-1})^2}{(x_j - x_{j-1})^2} \, dx_m + \int_{x_j^{(N)}}^{x_{j+1}^{(N)}} \frac{(x_{j+1}^{(N)} - x_m)^2}{(x_{j+1}^{(N)} - x_j^{(N)})^2} \, dx_m, \\
&= \frac{1}{3}\left(x_j^{(N)} - x_{j-1}^{(N)}\right) + \frac{1}{3}\left(x_{j+1}^{(N)} - x_j^{(N)}\right), \\
b_{j,j+1} &= \int_{x_j^{(N)}}^{x_{j+1}^{(N)}} \frac{(x_{j+1}^{(N)} - x_m)(x_m - x_j^{(N)})}{(x_{j+1}^{(N)} - x_j^{(N)})^2} \, dx_m = \frac{1}{6}\left(x_{j+1}^{(N)} - x_j^{(N)}\right), \\
b_{NN} &= \int_{x_{N-1}^{(N)}}^{H} \frac{(x_m - x_{N-1}^{(N)})^2}{(H - x_{N-1}^{(N)})^2} \, dx_m = \frac{1}{3}\left(H - x_{N-1}^{(N)}\right).
\end{aligned}$$

It is well-known that by the definition of the hat functions ψ_i stiffness matrix and mass matrix are sparse.

Remark 2.3.1. *Let $\tilde{M}$ denotes the mass matrix using constant background-sound speed c_{max} instead of depth-dependent sound speed. If we replace now the left-hand side in (2.22) by $(S - M - \tilde{M})\xi$ we can compute the eigenvalues $\tilde{\lambda}_h^2 := \lambda_h^2 + \lambda_{max,h}^2$ on the right-hand side in (2.22). Since $\lambda_{max,h}^2$ can be computed analytically, we deduce λ_h^2. Using this technique for background sound speed, roughly speaking similar, to the constant one c_{max}, we see a significant speed up in the routine computing the eigenvalues. However, if the the non constant back ground speed differs too much from the constant one the rapid convergence is not ensured.*

It is worth to see that the approximation error of the eigenvector $\phi_j(x_m)$ depends on the number of computed eigenvalues. Motivated by Figure 2.3 (a) for constant sound speed, we see that for large eigenvalue the oscillation of the eigenvector increases and hence the choice of the mesh has to be adapted. For non-constant sound speed this also holds.

To obtain error estimates of the approximated eigenvalues and eigenvectors to the exact ones using the Ritz-Galerkin methods, we look at the variational formulation.

$$\int_0^H \left[\phi_j' \cdot \overline{v}' + \left(\lambda_j^2 - \frac{\omega^2}{c^2(x_m)}\right)\phi_j \overline{v}\right] dx_m = 0 \qquad \text{for all } v \in H^1_W([0,H]).$$

We recall again the variational formulation of the Liouville eigenvalue problem given by equation (2.15), and denote for simplicity $(\phi, \lambda^2) := (\phi_j, \lambda_j^2)$,

$$\int_0^H \phi' \, \overline{v}' \, dx_m - \int_0^H \frac{\omega^2}{c^2(x_m)} \phi \overline{v} \, dx_m = -\int_0^H \lambda^2 \phi \overline{v} \, dx_m \qquad \text{for all } v \in H^1_W([0,H]).$$

We now rewrite for all $v \in H^1_W([0,H])$ the last equation as

$$\int_0^H \phi' \, \overline{v}' \, dx_m - \int_0^H \frac{\omega^2}{c^2(x_m)} \phi \overline{v} \, dx_m + 2\frac{\omega^2}{c_-^2} \int_0^H \phi \overline{v} \, dx_m = \left(2\frac{\omega^2}{c_-^2} - \lambda^2\right) \int_0^H \phi \overline{v} \, dx_m.$$

Next, we define the operator $K : L^2([0,H]) \to L^2([0,H])$ by requiring that if $f \in L^2([0,H])$ then $Kf \in H^1_W([0,H])$ satisfies for all $v \in H^1_W([0,H])$ that

$$\int_0^H (Kf)' \, \overline{v}' \, dx_m - \int_0^H \frac{\omega^2}{c^2(x_m)} Kf \overline{v} \, dx_m + 2\frac{\omega^2}{c_-^2} \int_0^H Kf \overline{v} \, dx_m = 2\frac{\omega^2}{c_-^2} \int_0^H f \overline{v} \, dx_m.$$

Since the sesquilinear form on the left of this variational formulation is bounded from below, Lax-Milgrams lemma ensures that a solution to this problem exists, such that K is well-defined. Then, we have

$$Kv = \frac{2\omega^2}{c_-^2}\Big(\frac{2\omega^2}{c_-^2} - \lambda^2\Big)^{-1} v.$$

Theorem 2.3.2. *The operator K is a bounded and compact map from $L^2([0,H])$ into $L^2([0,H])$.*

The proof follows since K maps f boundedly to ϕ, $\phi \in H^1_W([0,H])$ and $H^1_W([0,H])$ is compactly embedded in $L^2([0,H])$.

Now, we denote the discrete variant of operator K by K_N, i.e. $K_N f = v_N$ is of the form $v_N = \sum_{j=1}^N \xi_j \psi_j$ and solves

$$\int_0^H v_N' \overline{\psi_k}' dx_m - \int_0^H \left(\frac{\omega^2}{c(x_m)^2} + \frac{2\omega^2}{c_-^2}\right) v_N \overline{\psi_k} dx_m = 2\frac{\omega^2}{c_-^2} \int_0^H f \overline{\psi_k} dx_m, \qquad k = 1, ..., N.$$

Then, the convergence of the approximated eigenvalues of the compact, self-adjoint operator K to the exact one is given by [Osb75, Theorem 3] and applied to a self-adjoint operator by [Mon03, Theorem 2.52].

Lemma 2.3.3. *For the operator K_N it holds that $K_N f \to Kf$ in $L^2([0,H])$ for N tends to infinity and the set $\mathcal{K} = \{K_N : L^2([0,H]) \to L^2([0,H])\}$ is collectively compact.*

Proof. We first consider

$$X_N = \operatorname{span}\{\psi_j, j = 1, ..., N\}.$$

For $f \in L^2([0,H])$ we have that $u_N := K_N f$ solves

$$\int_0^H \left[u_N' \overline{v_N}' + \left(\frac{2\omega^2}{c_-^2} - \frac{\omega^2}{c^2(x_m)}\right) u_N \overline{v_N}'\right] dx_m = \frac{2\omega^2}{c_-^2} \int_0^H f \overline{v_N} \, dx_m \qquad \text{for all } v_N \in X_N.$$

We point out that the test function v_N in the last equation can equivalently replaced by all ψ_k, where $k = 1, ..., N$. We further see the corresponding continuous problem: Find $u := Kf \in H^1_W([0,H])$, such that

$$\int_0^H \left[u'\overline{v}' + \left(\frac{2\omega^2}{c_-^2} - \frac{\omega^2}{c^2(x_m)} \right) u\overline{v}' \right] dx_m = \frac{2\omega^2}{c_-^2} \int_0^H f\overline{v}\, dx_m \qquad \text{for all } v_N \in X_N.$$

Due to Lax-Milgrams lemma we know that a unique solution exists, since it holds

$$\|u\|_{H^1_W([0,H])} = \|Kf\|_{H^1_W([0,H])} \leq C\|f\|_{H^1_W([0,H])}.$$

Since a unique solution exists, we know by Céa's Lemma that

$$\|u - u_N\|_{H^1_W([0,H])} \leq \inf_{v_N \in X_N} \|u - v_N\| \leq C\frac{1}{N}\|u\|_{H^2([0,H])} \to 0 \qquad \text{for } N \to \infty. \tag{2.23}$$

Thus,

$$\|Kf - K_N f\|_{H^1_W([0,H])} \to 0 \qquad \text{for } N \to \infty \text{ and for all } f \in L^2([0,H]).$$

This ends the first part of the proof.

We next show, that for every bounded set $U \subset L^2([0,H])$ is $\mathcal{K}(U) = \{K_N f : f \in U, N \in \mathbb{N}\}$ a relatively compact set in $L^2([0,H])$. Due to (2.23) and since $f \in U$ and U is bounded, we discover that

$$\|u - K_N f\|_{H^1_W([0,H])} \leq \frac{C}{N}\|u\|_{H^2([0,H])} \leq \frac{C}{N}\|f\|_{L^2([0,H])} \leq \frac{C}{N}C_u.$$

In consequence, the reverse triangle inequality $\|K_N f\| - \|u\| \leq \|u - K_N f\|$ yields

$$\|K_N f\|_{H^1_W([0,H])} \leq \frac{C}{N}C_u \qquad \text{for all } f \in U \text{ and for all } N \in \mathbb{N}.$$

In particular, $K(U)$ is bounded. As the embedding of $H^1_W([0,H])$ in $L^2([0,H])$ is compact, every sequence in $K(U)$ contains a convergent subsequence. Then, $K(U)$ is relatively compact and the set $\mathcal{K} = \{K_N : L^2([0,H]) \to L^2([0,H])\}$ is collectively compact. This finishes the proof. $\square$

We have now collected all requirements to apply the convergence result from [Mon03, Theorem 2.52] to our setting.

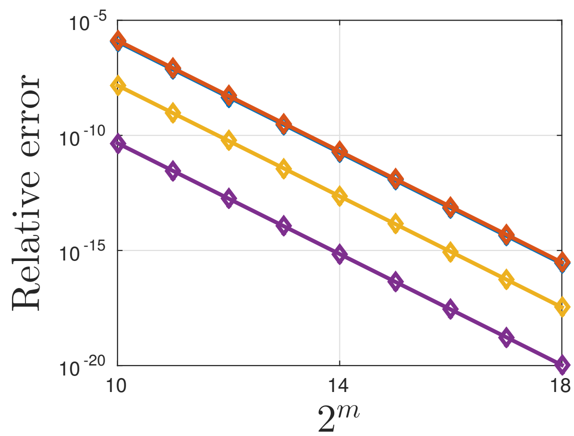


Figure 2.4: Relative L^2-error of the finite element method, using a uniform mesh with discretization 2^m, where $m = 10, ..., 18$, for height $h = 200$, $c = 1400$, frequency $f = 27$ (red), $f = 50$ (blue), $f = 500$ (yellow) and $f = 2000$ (magenta) compared to the exact analytic solution.

Theorem 2.3.4. *Assume $\varepsilon > 0$ such that the circle of radius ε about λ^2 encloses no other eigenvalues of operator K. Then for h small enough the circle of radius ε centered at λ^2 encloses precisely N eigenvalues of the discrete problem $\lambda_{h,j}^2$, $j = 1, ...N$. Moreover, there holds*

$$\dim \Phi(\lambda^2) = \dim \oplus_{j=1}^{N} \Phi(\lambda_{h,j}^2),$$

where $\Phi(\lambda^2)$ denotes the eigenspace corresponding to the eigenvalue λ^2 and $\Phi(\lambda_{h,j}^2)$ denotes the eigenspace corresponding to the eigenvalue $\lambda_{h,j}^2$. Further, there exists a constant $C > 0$ such that there holds

$$|\lambda^2 - \lambda_{h,j}^2| \leq C\Big[\sum_{j,k=1} |((K - K_h)\phi_j, \phi_k)_{L^2([0,H])}| + \|(K - K_h)|_{\Phi(\lambda^2)}\|_{L^2([0,H])}^2\Big], \tag{2.24}$$

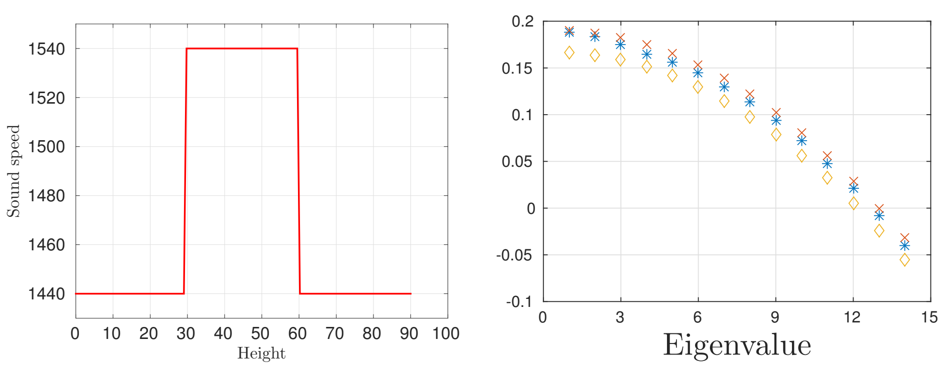
where $(K - K_N)|_{\Phi(\lambda^2)}$ is the restriction of $\Phi(\lambda^2)$.

Furthermore, we obtain the approximation error of the eigenvectors to the exact ones by [Osb75, Theorem 5].

Theorem 2.3.5. *Let $\lambda_{h,j}^2$ be an approximated eigenvalue of operator K such that $\lim_{h\to 0} \lambda_{h,j}^2 = \lambda_j^2$ and let α the smallest integer such that the nullspace of $(\lambda^2 - K)^\alpha$ equals the nullspace $(\lambda^2 - K)^{\alpha+1}$. Assume for each j that $\phi_j^{(N)}$ is a unit vector satisfying $(\lambda_{h,j}^2 - T_h)^k \phi_j^{(N)} = 0$ for some positive integer $k \leq \alpha$. Then, for any integer l such that $k \leq l \leq \alpha$, there is a vector $\phi_j \in \Phi(\lambda^2)$ such that $(\lambda_j^2 - K)^l \phi_j = 0$ and for $C > 0$ holds*

$$\|\phi_j - \phi_j^{(N)}\|_{L^2([0,H])} \leq C\|(K - K_h)|_{\Phi(\lambda^2)}\|_{L^2([0,H])}^{(l-k+1)/\alpha}.$$

We want now to consider numerical examples for the introduced Ritz-Galerkin method. For the first example, we consider the height of the ocean $h = 200$, the constant sound speed $c = 1400$. We look now to the relative error of the computation using the introduced Ritz-Galerkin method to the the exact analytic solution. Figure 2.4 shows the relative error of the solution for a uniform mesh with discretization parameter 2^m, where $m = 10, ..., 18$ for different frequencies $f = 27$ (red), $f = 50$ (blue), $f = 500$ (yellow) and $f = 2000$ (magenta). For the second example, we consider a three layered ocean with height 90 with height of each layer of 30.



(a) (b)

Figure 2.5: Three-layered ocean with height $h = 90$ and constant sound speed on each layer, $c_1 = c_3 = 1440$ and $c_2 = 1540$. (a) Sound speed profile (b) Analytic computed eigenvalues for frequency $f = 100Hz$ and sound speed $c_1 = 1440$ (yellow, diamond) and $c_2 = 1540$ (red, cross). Eigenvalues λ_j^2 computed on an uniform mesh with 2^{14} grid points by finite elements method (blue, star)

Furthermore, let the constant sound speed on layer one and three be $c_1 = c_3 = 1440$ and on layer two $c_2 = 1540$, see 2.5 a). By Theorem 2.2.3 and the Remark 2.2.2 we can give for each eigenvalue an upper and a lower estimate. Note that using the upper and lower estimates of the eigenvalues in Theorem 2.2.3, we obtain an estimate of the number of positive eigenvalues before the evaluation of the eigenvalues of the non-constant sound speed with the Ritz-Galerkin method. Those eigenvalues $(\lambda_j^-)^2$ and $(\lambda_j^+)^2$ of constant sound speed c_+ and c_- can be computed in an analytic way. Then, we use the eigenvalues for constant sound speed c_+ or c_- as initialization parameter for the iterative eigenvalue solver of MATLAB to deduce better approximation of the eigenvalues of non constant background sound speed. Figure 2.5 shows the eigenvalues $\{\lambda_j^2\}_{j=1,...,15}$, computed by finite element methods for an 3-layered ocean, using as initialization eigenvalues of constant sound speed $(c_- + c_+)/2$. We point out that the computing environment, here MATLAB, for high frequency sources ($> 500Hz$) and deeper oceans $h > 200$ does not find all positive eigenvalues. Note that the jump of the sound speed on the layer connection provokes this effect. A possibility to fix this effect is using a continuous background sound speed, which we discuss later on.

A Multi Layered Ocean. We approximate an inhomogeneous ocean by a multi-layered homogeneous ocean. In consequence, we introduce a model of an n-layered ocean, with constant sound speed on each layer and using transmission conditions to connect each one. [ZAM00] presents the geometry of an upper and a lower layer. In this work, we expand this geometry to a multi-layered homogeneous ocean geometry. We assume a n-layered homogeneous ocean and the depth of layer k is denoted by

$$d_k = \sum_{j=1}^{k} s_j,$$

where $k = 1, ..., n$ and s_k is the corresponding size of layer k. We set $d_0 = 0$ and, of course, we have $d_n = H$. We denote in the following c_k as the constant background sound speed of layer k and we assume that the waves are modeled by the Helmholtz equation. We further assume that for the sound speed of each layer k holds $0 < c_- \le c_k \le c_+$. Note that Theorem 2.2.3 gives an estimate of the number of positive eigenvalues. Similar, like in Chapter 1, we obtain by separation of variables the eigenpair (ϕ_j^k, λ_j^2) of layer k solving

$$\frac{\partial^2 \phi_j^{(k)}}{\partial x_m^2} + \frac{\omega^2}{c_k^2}\phi_j^{(k)} - \lambda_j^2 \phi_j^{(k)} = 0 \qquad \text{almost everywhere on layer } k, \tag{2.25}$$

and corresponding boundary condition on layer k, which we will discuss in the following. The boundary conditions of the our waveguide the ocean gives us the boundary condition

$$\phi_j^{(1)}(0) = 0, \qquad \text{and} \qquad \frac{\partial \phi_j^{(n)}}{\partial x_m}(H) = 0. \tag{2.26}$$

Furthermore, we consider transmission conditions on the interfaces. For $k = 1, ..., n-1$ we have continuity conditions and continuity of the derivative with respect to x_m at each layer,

$$\phi_j^{(k)}(d_k) = \phi_j^{(k+1)}(d_k), \qquad \text{and} \qquad \frac{\partial \phi_j^{(k)}}{\partial x_m}(d_k) = \frac{\partial \phi_j^{(k+1)}}{\partial x_m}(d_k). \tag{2.27}$$

For simplicity arguments for $k = 1, ..., n$ we set $\alpha_k = \sqrt{\omega^2/c_k^2 - \lambda_j^2}$. Then, the general solution of equation (2.25) is given by

$$\phi_j^{(k)} = A_k \sin(\alpha_k x_m) + B_k \cos(\alpha_k x_m), \qquad d_{k-1} \le x_m \le d_k, \tag{2.28}$$

where $k = 1, ..., n$. Further, by the boundary condition (2.26), we deduce $B_1 = 0$ and

$$\alpha_n A_n \cos(\alpha_n H) - \alpha_n B_n \sin(\alpha_n H) = 0.$$

We moreover compute by continuity condition (2.27) for $k = 2, ..., n-1$ that

$$
\begin{aligned}
A_1 \sin(\alpha_1 d_1) - A_2 \sin(\alpha_2 d_1) - B_2 \cos(\alpha_2 d_1) &= 0,\\
A_k \sin(\alpha_k d_k) + B_k \cos(\alpha_k d_k) - A_{k+1} \sin(\alpha_{k+1} d_k) - B_{k+1} \cos(\alpha_{k+1} d_k) &= 0,\\
\alpha_1 A_1 \cos(\alpha_1 d_1) - \alpha_2 A_2 \cos(\alpha_2 d_1) + \alpha_2 B_2 \sin(\alpha_2 d_1) &= 0,\\
\alpha_k A_k \cos(\alpha_k d_k) - \alpha_k B_k \sin(\alpha_k d_k) - \alpha_{k+1} A_{k+1} \cos(\alpha_{k+1} d_k) + \alpha_{k+1} B_{k+1} \sin(\alpha_{k+1} d_k) &= 0.
\end{aligned}
$$

This gives us by Dirichlet boundary conditions at the free surface a system of $(2n-1) \times (2n-1)$ unknowns. Of course, the evolution of the eigenvectors requires the normalization condition with ρ_k as the normalization constant of the eigenfunction of the corresponding layer,

$$
\frac{1}{\rho} \int_0^H (\phi_j)^2 dx_m = \sum_{k=0}^{n} \frac{1}{\rho_k} \int_{d_k}^{d_{k+1}} (\phi_j^{(k+1)})^2 dx_m = 1.
$$

For completeness, the integrals are given by

$$
\begin{aligned}
\int_0^{d_1} (\phi_j^{(1)})^2 dx_m =& \frac{{A_1}^2 \left(-\cos(\alpha_1\, d_1) \sin(\alpha_1\, d_1) + \alpha_1\, d_1\right)}{2\alpha_1}, \qquad (2.29)\\
\int_{d_k}^{d_{k+1}} (\phi_j^{(k)})^2 dx_m =& \frac{{A_{k+1}}^2 \cos(\alpha_{k+1}\, d_k) \sin(\alpha_{k+1}\, d_k)}{2\alpha_k} + \frac{A_{k+1} B_{k+1} \left(\cos(\alpha_{k+1}\, d_k)\right)^2}{\alpha_{k+1}}\\
&- \frac{{B_{k+1}}^2 \cos(\alpha_{k+1}\, d_k) \sin(\alpha_k\, d_k)}{2\alpha_{k+1}} - \frac{{A_{k+1}}^2 \cos(\alpha_{k+1}\, d_{k+1}) \sin(\alpha_{k+1}\, d_{k+1})}{2\alpha_{k+1}}\\
&+ \frac{{B_{k+1}}^2 [\alpha_{k+1}\, d_{k+1} - \alpha_{k+1}\, d_k]}{2\alpha_{k+1}} + \frac{{A_{k+1}}^2 [\alpha_{k+1}\, d_{k+1} - \alpha_{k+1}\, d_k]}{2\alpha_{k+1}}\\
&- \frac{A_{k+1} B_{k+1} \left(\cos(\alpha_{k+1}\, d_{k+1})\right)^2}{\alpha_{k+1}} + \frac{{B_{k+1}}^2 \cos(\alpha_{k+1}\, d_{k+1}) \sin(\alpha_{k+1}\, d_{k+1})}{2\alpha_{k+1}}.
\end{aligned}
$$

To obtain now the eigenvalues λ_j^2 for an n-layered ocean we solve the equation $\det(\mathcal{A}_n(\lambda_j)) = 0$. For example, if $n = 2$ we compute for one eigenvalue λ_j^2 that $\det(\mathcal{A}_2(\lambda_j)) = 0$, where

$$
\mathcal{A}_2(\lambda_j) := \begin{pmatrix} \sin(\alpha_1 d_1) & -\sin(\alpha_2 d_1) & -\cos(\alpha_2 d_1) \\ \alpha_1 \cos(\alpha_1 d_1) & -\alpha_2 \cos(\alpha_2 d_1) & \alpha_2 \sin(\alpha_2 d_1) \\ 0 & \alpha_2 \cos(\alpha_2 d_2) & -\alpha_2 \sin(\alpha_2 d_2) \end{pmatrix}.
$$

If $n = 3$ then $\det(\mathcal{A}_3(\lambda_j)) = 0$, where

$$
\mathcal{A}_3(\lambda_j) := \begin{pmatrix} \sin(\alpha_1 d_1) & -\sin(\alpha_2 d_1) & 0 & -\cos(\alpha_2 d_1) & 0 \\ 0 & \sin(\alpha_2 d_2) & -\sin(\alpha_3 d_2) & \cos(\alpha_2 d_2) & -\cos(\alpha_3 d_2) \\ \alpha_1 \cos(\alpha_1 d_1) & -\alpha_2 \cos(\alpha_2 d_1) & 0 & \alpha_2 \sin(\alpha_2 d_1) & 0 \\ 0 & \alpha_2 \cos(\alpha_2 d_2) & -\alpha_3 \cos(\alpha_3 d_2) & -\alpha_2 \sin(\alpha_2 d_2) & \alpha_3 \sin(\alpha_3 d_2) \\ 0 & 0 & \alpha_3 \cos(\alpha_3 d_3) & 0 & -\alpha_3 \sin(\alpha_3 d_3) \end{pmatrix}.
$$

To reduce now the system of unknowns we chose an alternative ansatz. For layer $k = 1, ..., n-1$ the general solution of equation (2.28) holds, however, for the last layer n we assume

$$
\phi_j^{(n)} = A_n \sin(\alpha_n (x_m - H)) + B_n \cos(\alpha_n (x_m - H)), \qquad \text{for } x_m \in [d_{n-1}, H].
$$

Consequently, we deduce

$$
\frac{\partial \phi_j^{(n)}}{\partial x_m} = \alpha_n A_n \sin(\alpha_n (x_m - H)) + \alpha_n B_n \cos(\alpha_n (x_m - H)) \qquad \text{for } x_m \in [d_{n-1}, H].
$$

From the Dirichlet boundary condition in (2.26) we obtain that $B_1 = 0$ and for the Neumann boundary condition in (2.26) we see that $A_n = 0$. Thus,

$$
\begin{aligned}
A_1 \sin(\alpha_1 d_1) - A_2 \sin(\alpha_2 d_1) - B_2 \cos(\alpha_2 d_1) &= 0,\\
\alpha_1 A_1 \cos(\alpha_1 d_1) - \alpha_2 A_2 \cos(\alpha_2 d_1) + \alpha_2 B_2 \sin(\alpha_2 d_1) &= 0.
\end{aligned}
$$

Further, for each layer $k = 2, ..., n-1$ we have

$$A_k \sin(\alpha_k d_k) + B_k \cos(\alpha_k d_k) - A_{k+1} \sin(\alpha_{k+1} d_k) - B_{k+1} \cos(\alpha_{k+1} d_k) = 0,$$
$$\alpha_k A_k \cos(\alpha_k d_k) - \alpha_k B_k \sin(\alpha_k d_k) - \alpha_{k+1} A_{k+1} \cos(\alpha_{k+1} d_k) + \alpha_{k+1} B_{k+1} \sin(\alpha_{k+1} d_k) = 0.$$

To this end, for layer n we compute

$$A_{n-1} \sin(\alpha_{n-1} d_{n-1}) + B_{n-1} \cos(\alpha_{n-1} d_{n-1}) - B_n \cos(\alpha_n (d_{n-1} - H)) = 0,$$
$$\alpha_{n-1} A_{n-1} \cos(\alpha_{n-1} d_{n-1}) - \alpha_{n-1} B_{n-1} \sin(\alpha_{n-1} d_{n-1}) + \alpha_n B_n \sin(\alpha_n (d_{n-1} - H)) = 0.$$

Once more the evolution of the eigenvectors requires the normalization condition. Thus, we denote the normalization constant β and it holds

$$\int_0^H |\beta \phi_j|^2 dx_m = 1.$$

Now, to obtain the eigenvalues λ_j^2 we solve $\det(\mathcal{A}_n(\lambda_j)) = 0$. Plugging all together, we have to solve a system of $(2n-2) \times (2n-2)$ unknowns. For example, if $n = 2$, then $\mathcal{A}_2(\lambda)$ is given by

$$\mathcal{A}_2(\lambda_j) := \begin{pmatrix} \sin(\alpha_1 d_1) & -\cos(\alpha_2 (d_1 - H)) \\ \alpha_1 \cos(\alpha_1 d_1) & \alpha_2 \sin(\alpha_2 (d_1 - H)) \end{pmatrix}.$$

If $n = 3$, then

$$\mathcal{A}_3(\lambda) := \begin{pmatrix} \sin(\alpha_1 d_1) & -\sin(\alpha_2 d_1) & -\cos(\alpha_2 d_1) & 0 \\ \alpha_1 \cos(\alpha_1 d_1) & -\alpha_2 \cos(\alpha_2 d_1) & \alpha_2 \sin(\alpha_2 d_1) & 0 \\ 0 & \sin(\alpha_2 d_2) & \cos(\alpha_2 d_2) & -\cos(\alpha_3 (d_2 - H)) \\ 0 & \alpha_2 \cos(\alpha_2 d_2) & -\alpha_2 \sin(\alpha_2 d_2) & \alpha_3 \sin(\alpha_3 (d_2 - H)) \end{pmatrix}.$$

We point out that solving the non-linear equation $\det(\mathcal{A}_n(\lambda_j)) = 0$, numerical computing environments use finite difference methods to compute the Jacobian of $\det(\mathcal{A}_n(\lambda_j))$, this may can leads for high frequencies to an error. To minimize this non negligible error, one can compute the Jacobian in an analytic way using well-known derivation techniques for determinants. Then, we first have

$$\alpha_k'(\lambda_j) = -2\lambda_j (\omega^2 / c_k^2 - \lambda_j^2)^{-1/2} = -\frac{2\lambda_j}{\alpha_k}.$$

To obtain the derivative of $\det(\mathcal{A}_2(\lambda_j))$ with respect to λ_j we need

$$\sin(\alpha_k d_k)' = d_k \alpha_k' \cos(\alpha_k d_k), \qquad \text{and} \qquad \cos(\alpha_k d_k)' = -d_k \alpha_k' \sin(\alpha_k d_k).$$

Moreover, we have

$$(\alpha_k \cos(\alpha_k d_k))' = \alpha_k' \cos(\alpha_k d_k) - \alpha_k d_k \alpha_k' \sin(\alpha_k d_k),$$
$$(\alpha_k \sin(\alpha_k d_k))' = \alpha_k' \sin(\alpha_k d_k) + \alpha_k d_k \alpha_k' \cos(\alpha_k d_k).$$

By well-known derivation techniques for determinants, it holds

$$\frac{\partial \det \mathcal{A}_2(\lambda)}{\partial \lambda} = \operatorname{tr}\left(\operatorname{adj}\left(\mathcal{A}_2(\lambda) \frac{\partial \mathcal{A}_2}{\partial \lambda} \right) \right) = \operatorname{tr}\left[\begin{pmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{pmatrix} \begin{pmatrix} a_{11}' & a_{12}' \\ a_{21}' & a_{22}' \end{pmatrix} \right],$$

where tr denotes the trace and adj the adjoined of a matrix. In consequence, we have

$$\frac{\partial \det \mathcal{A}_2(\lambda)}{\partial \lambda} = a_{22} a_{11}' - a_{12} a_{21}' - a_{21} a_{12}' + a_{11} a_{22}',$$

Frequency	Method	Error of ϕ_5
27Hz	FEM (uniform mesh)	1.1×10^{-8}
27Hz	Multi-layer approach	8.9×10^{-11}
500Hz	FEM (uniform mesh)	1.1×10^{-8}
500Hz	Multi-layer approach	1.8×10^{-10}
1 Khz	FEM (uniform mesh)	1.1×10^{-8}
1 Khz	Multi-layer approach	9.1×10^{-10}

Table 2.1: Error estimates for a multi layered ocean using FEM and a multi layered approach scheme.

where $a_{22}, a_{12}, a_{21}, a_{11}$ denote the entries of $\mathcal{A}_2(\lambda_j)$ and

$$\begin{aligned} a'_{11} &= d_1\alpha'_1\cos(\alpha_1 d_1), \\ a'_{21} &= \alpha'_1\cos(\alpha_k d_1) - \alpha_1 d_1\alpha'_1\sin(\alpha_1 d_1), \\ a'_{12} &= (d_1 - H)\alpha'_2\sin(\alpha_2(d_1 - H)), \\ a'_{22} &= \alpha'_2\sin(\alpha_2(d_1 - H)) + \alpha_2(d_1 - H)\alpha'_2\cos(\alpha_2(d_1 - H)). \end{aligned}$$

By rigorously computations one can obtain the Jacobian for $n = 3$, too.

To evaluate this algorithm we assume for simplicity $n = 3$, which restrict us to a three layered ocean. We further use MATLAB to solve the non-linear equation $\det(\mathcal{A}_n(\lambda_j)) = 0$. To evaluate this method and to compare it with the Ritz-Galerkin approach, we use the same sound speed on each layer and compare the solution of the approach of one eigenvector with the solution of an homogeneous ocean, computed by analytic techniques. Table 2.3 gives us a first motivation for fixed height $H = 250$, $d_1 = 100$, $d_2 = 200$, constant background sound speed on each layer $c1 = c2 = c3 = 1440$ and different frequencies ω. For the finite element method (FEM) mesh we used an uniform mesh of 2^{17} points.

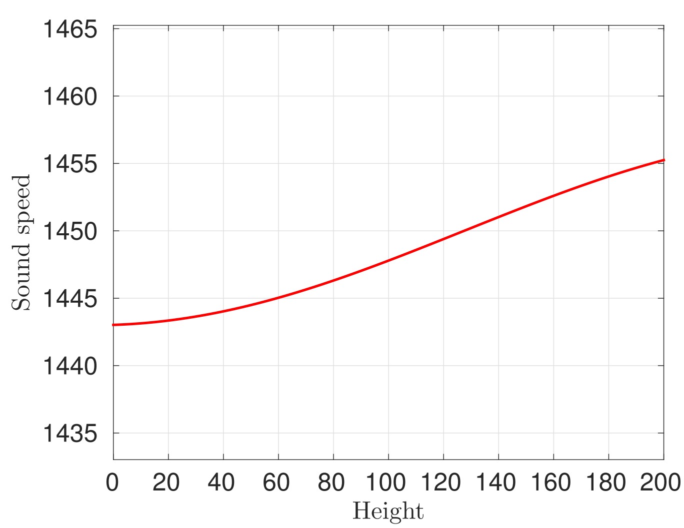


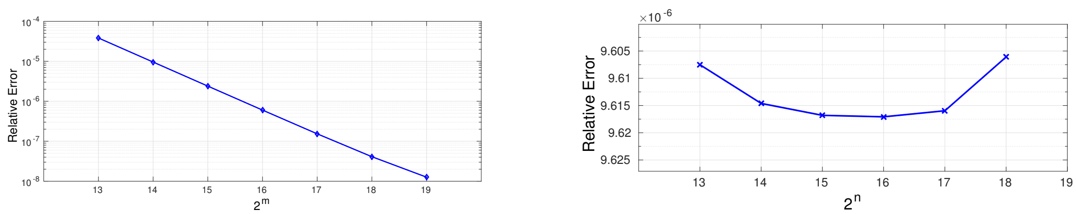
Figure 2.6: Continuous background sound speed approximation of sound speed profile 7 for height $H = 200$.

For a height $H = 200$ this approximation is given by Figure 2.6.

Indeed the error of the approximated eigenvectors for different frequencies are nearly the same using the finite element method, since MATLAB uses a quadrature technique to approximate the integral in the FEM. For a three-layered ocean, where the sound speed on layer one is equal to the sound speed of layer three and different to layer two, like in Figure 2.5 (a), this method has difficulties to find a high number of eigenvalues. As the iterative eigenvalue solver uses an initialization values to find eigenvalues, the finite element can be provided the start values to compute these eigenvalues. If the eigenvalue acting as start value computed by the finite element method, however, differs to strong from the exactly value, the iterative solver, is not able to find a better approximated eigenvalue by the three-layered ocean model. If one computes only a couple of eigenvalues, like to evaluate the truncated Green's function, this approach gives better approximations of eigenvalues and eigenvectors as using the Ritz-Galerkin approach. For the collocation method we introduce later on, this method does not yield a sufficiently large number of eigenvalues. We point out that to find eigenvalues by approximation in MATLAB with, roughly speaking good initialization values, are fast operations, however, computing the initialization values by Ritz-Galerkin methods influences significantly the runtime of this algorithm.

A Ritz-Galerkin Approximation applied to Continuous Background Sound Speed. We want now consider a continuous background sound speed. We first approximate the sound speed profile 7 in Figure 2.1 in Chapter 1 by a continuous function.

Figure 2.7 shows then the convergence of the eigenvalues (a) and eigenvectors (b) of the finite element method for continuous sound speed in Figure 2.6 on $[0, H]$, with height $H = 200$ and frequency $\omega = 27$.



(a) (b)

Figure 2.7: Ocean configuration: height $H = 200$, frequency $\omega = 27$ and continuous approximation of sound speed profile. a) Relative L^2-error of the eigenvalues computed by FEM on uniform grid $N = 2^m$ where $m = 13, ..., 19$. b) Relative L^2-error of the eigenvectors computed by FEM on uniform grid $N = 2^m$ where $m = 13, ..., 18$.

To compare the convergence of different FEM grids of the interval $[0, H]$ we project functions in the finite-dimensional space defined by the N discretized points $x_j^{(N)}$ from (2.21) onto a uniform grid of 2^{14} discretization points.

Since we use an uniform discretization of the interval $[0, H]$ of 2^m points, where $m = 14, ...19$, an interpolation between the grids is not necessary. We point out that the projection to the reference grids does not influence the error. Furthermore, we fix the number of eigenvalues to compute to $\mathcal{J} = 2^6$. Note that the MATLAB routine **eigs** restricts $\mathcal{J} \leq N$. The reference solution with uniform discretization of the interval $[0, H]$ in 2^{21} points has been computed in nine hours on a eight processors workstation, where the solution took nearly five percent of the memory of 128 GB RAM. Note that for frequencies larger than 50, continuous sound speed given in Figure 2.7 a) and $H = 200$, the MATLAB routine **eigs** does not find the correct eigenvectors anymore. We further see in Figure 2.7 b) that the relative error increases for a certain choice of the mesh size. This effect follows form the quadrature approximation of the integral in the finite element method.

A Spectral Method. We present now a spectral method to compute the eigenvalues and the eigenfunctions in the vertical direction for smooth background sound speed. In particular, this spectral method is a Ritz-Galerkin method with trigonometric basis functions. We recall our 1D-eigenvalue problem from (2.14) in the vertical direction and for simplicity we denote the eigenpair $(\phi, \lambda) := (\phi_j, \lambda_j^2)$ which solves

$$\phi'' + \frac{\omega^2}{c(x_m)^2}\phi - \lambda^2\phi = 0, \qquad \text{almost everywhere in } (0, H),$$

and corresponding boundary conditions $\phi(0) = 0$ and $\phi'(H) = 0$. We further denote for $\alpha_j := (2j-1)\pi/(2H)$ a basis

$$\varphi_j(x_m) = \sin(\alpha_j x_m) \qquad \text{for } j \in \mathbb{N}, x_m \in [0, H].$$

Then, for the vectors $\varphi_j(x_m)$ holds

$$\varphi_j(0) = 0, \qquad \text{and} \qquad \varphi_j'(H) = \alpha_j \cos\left(\frac{\pi(2j-1)}{2}\right) = 0.$$

Next, due to the variational formulation (2.15),

$$\int_0^H \left(\phi'\overline{v}' - \frac{\omega^2}{c(x_m)^2}\phi\overline{v}\right) dx_m - \phi(0)\overline{v}(0) + \phi'(H)\overline{v}(H) = -\lambda^2 \int_0^H \phi\overline{v}\, dx_m. \tag{2.30}$$

We further know that all eigenvalues λ^2 are real. If the background speed c is constant, then it holds for the eigenvalues $\lambda_j^2 = \omega^2/c^2 - \alpha_j^2$ and

$$\varphi_j'' + \frac{\omega^2}{c^2}\varphi_j = \left(\frac{\omega^2}{c^2} - \alpha_j^2\right)\varphi_j + \varphi_j(0) + \varphi_j''(H).$$

Therefore, we recall the Sobolev space

$$X := \left\{\phi \in H^2([0, H]) : \phi(0) = 0 \text{ and } \phi'(H) = 0\right\},$$

with norm

$$\|\phi_j\|^2_{H^2([0,H])} = \|\phi_j\|^2_{L^2([0,H])} + \|\phi_j'\|^2_{L^2([0,H])} + \|\phi_j''\|^2_{L^2([0,H])},$$

such that

$$\|\phi_j\|^2_{H^2([0,H])} = \|\phi_j\|^2_{L^2([0,H])} + |\alpha_j|^2\|\phi_j\|^2_{L^2([0,H])} + |\alpha_j|^4\|\phi_j\|^2_{L^2([0,H])}.$$

The basis function $\varphi_j \in X([0, H])$ are orthogonal in $L^2([0, H])$. Next, for $N \in \mathbb{N}$ the Ritz-Galerkin approximation of (2.30) using $X_N = \text{span}\{\varphi_j, j = 1, ...N\}$ gives us for $u^N \in X_N$ that

$$\int_0^H \left[(u^N)'(v^N)' - \frac{\omega^2}{c^2(x_m)}u^N v^N\right] dx_m = -\lambda^2 \int_0^H u^N v^N dx_m,$$

where

$$u^N = \sum_{k=1}^N c_k\varphi_k \qquad \text{and} \qquad v^N = \sum_{\ell=1}^N b_\ell\varphi_\ell, \tag{2.31}$$

and for its derivative, it holds

$$(u^N)' = \sum_{k=1}^N c_k\alpha_k\varphi_k' \qquad \text{and} \qquad (v^N)' = \sum_{\ell=1}^N b_\ell\alpha_\ell\varphi_\ell'.$$

Then, we obtain for the variational formulation

$$\int_0^H \left[\sum_{k=1}^N c_k\alpha_k\varphi_k' \sum_{\ell=1}^N b_\ell\alpha_\ell\varphi_\ell' - \frac{\omega^2}{c^2(x_m)}\sum_{k=1}^N c_k\varphi_k \sum_{\ell=1}^N b_\ell\varphi_\ell\right] dx_m = -\lambda_j^2 \sum_{k=1}^N c_k\varphi_k \sum_{\ell=1}^N b_\ell\varphi_\ell dx_m.$$

Due to the definition of φ_j and its orthogonality, we have

$$\sum_{k=1}^{N} c_k b_k \int_0^H \alpha_k^2 \cos^2(\alpha_k x_m) dx - \sum_{k=1}^{N}\sum_{\ell=1}^{N} c_k b_\ell \int_0^H \frac{\omega^2}{c^2(x_m)} \sin(\alpha_k x_m)\sin(\alpha_\ell x_m) dx_m$$
$$= -\lambda_j^2 \sum_{k=1}^{N} c_k b_k \int_0^H \sin^2(\alpha_k x_m) dx_m.$$

By partial integration and the fact that $\sin(2\alpha_k H) = \sin(\pi(2k-1)) = 0$ for $k \in \mathbb{N}$, we obtain

$$\int_0^H \sin^2(\alpha_k x_m) dx_m = \left[\frac{x_m}{2} - \frac{\sin(2\alpha_k x_m)}{4\alpha_k}\right]_0^H = \frac{H}{2} - \frac{\sin(2\alpha_k H)}{4\alpha_k} = \frac{H}{2},$$
$$\int_0^H \cos^2(\alpha_k x_m) dx_m = \left[\frac{x_m}{2} + \frac{\sin(2\alpha_k x_m)}{4\alpha_k}\right]_0^H = \frac{H}{2} + \frac{\sin(2\alpha_k H)}{4\alpha_k} = \frac{H}{2},$$

and further by changing the order of the unknown variables,

$$\frac{H}{2}\sum_{k=1}^{N} \alpha_k^2 b_k c_k - \sum_{k=1}^{N}\sum_{\ell=1}^{N} \int_0^H \frac{\omega^2}{c^2(x_m)} \sin(\alpha_k x_m)\sin(\alpha_\ell x_m) dx_m\, b_\ell c_k = -\frac{H}{2}\lambda^2 \sum_{k=1}^{N} b_k c_k.$$

By multiplication with factor $2/H$ we deduce in matrix notation

$$\left(A - \frac{2}{H}B\right)\xi = -\lambda^2 I \xi, \tag{2.32}$$

where

$$A = \operatorname{diag}(\alpha_k^2)_{k=1}^N, \qquad \xi = \begin{pmatrix}\xi_1\\ \vdots \\ \xi_N\end{pmatrix}, \qquad B_{kl} = \int_0^H \frac{\omega^2}{c^2(x_m)}\varphi_k(x)\varphi_\ell(x) dx_m.$$

Next, by well-known trigonometric functions identities, we see that

$$\varphi_k(x_m)\varphi_\ell(x_m) = \sin(\alpha_k x_m)\sin(\alpha_\ell x_m) = \frac{1}{2}[\cos((\alpha_k - \alpha_\ell)x_m) - \cos(\alpha_k + \alpha_j)x_m)].$$

In consequence, we obtain

$$\begin{aligned} B_{kl} &= \int_0^H \frac{\omega^2}{c^2(x_m)}\varphi_k(x_m)\varphi_\ell(x_m) dx_m \\ &= \frac{1}{2}\int_0^H \frac{\omega^2}{c^2(x_m)} \cos((\alpha_k - \alpha_\ell)x_m) dx_m - \frac{1}{2}\int_0^H \frac{\omega^2}{c^2(x_m)}\cos((\alpha_k + \alpha_\ell)x_m) dx_m. \end{aligned} \tag{2.33}$$

Now, we assume that the background sound speed can be extended to a $2H$-periodic continuous function. Then, the trigonometric basis for continuous function of $L^2([0, 2H])$ is given by

$$\phi_\ell(x_m) = \frac{1}{\sqrt{2H}} \exp\left(i\ell\pi \frac{x_m}{H}\right) \qquad \text{for } \ell \in \mathbb{Z}.$$

Furthermore, for simplicity we set $a(x_m) := \omega^2/c^2(x_m)$, and we can represent the $2H$-periodic function $a \in C([0, 2H])$ as a Fourier series,

$$a(x_m) = \frac{1}{\sqrt{2H}} \sum_{\ell\in\mathbb{Z}} \hat{a}(\ell) \exp\left(i\ell\pi \frac{x_m}{H}\right) \qquad \text{for } \ell \in \mathbb{Z},$$

with Fourier coefficients defined by

$$\hat{a}(\ell) = \frac{1}{\sqrt{2H}} \int_0^{2H} a(x_m) \exp\left(-i\ell\pi \frac{x_m}{H}\right) dx_m \qquad \text{for } \ell \in \mathbb{Z}.$$

Note that since $a(x_m)$ is real, there holds

$$\hat{a}(-\ell) = \frac{1}{\sqrt{2H}} \int_0^{2H} a(x_m) \exp\left(i\ell\pi \frac{x_m}{H}\right) dx_m = \overline{\hat{a}(\ell)} \qquad \text{for } \ell \in \mathbb{Z}.$$

We moreover have the orthogonality

$$\frac{1}{2H} \int_0^{2H} \exp\left(i\ell\pi \frac{x_m}{H}\right) \exp\left(-ik\pi \frac{x_m}{H}\right) dx_m = \delta_{\ell,k} \qquad \text{for } \ell, k \in \mathbb{Z}.$$

For $M \in \mathbb{N}$ we consider

$$\mathbb{Z}_{M,2H} := \left\{\ell \in \mathbb{Z} : -\frac{M}{2} < \ell \leq \frac{M}{2}\right\},$$

and we set the polynomial space of $2H$-periodic continuous functions by

$$\mathcal{T}_{M,2H} = \operatorname{span}\left\{\exp\left(i\ell\pi \frac{x_m}{H}\right) : x_m \in [0, 2H], \ell \in \mathbb{Z}_{M,2H}\right\}.$$

We moreover define for a $2H$-periodic function the interpolation projection $Q_M \in \mathcal{T}_{M,2H}$ by

$$(Q_M a)\left(\frac{2Hj}{M}\right) = a\left(\frac{2Hj}{M}\right), \qquad \text{where } j = 0, 1, ..., M-1.$$

Therefore, we have for $j = 0, 1, ..., M-1$ the representation for the nodal values by

$$(Q_M a)(x_m) := a_M(x_m) = \sum_{j=0}^{M-1} a\left(\frac{2Hj}{M}\right) \varphi_{M,j}(x_m),$$
$$\varphi_{M,j}(x_m) = \frac{1}{M} \sum_{\ell \in \mathbb{Z}_{M,2H}} \exp\left(2i\pi\ell\left(\frac{x_m}{2H} - \frac{j}{M}\right)\right).$$

Here, the basis functions $\varphi_{M,j} \in \mathcal{T}_{M,2H}$ satisfy

$$\varphi_{M,j}\left(\frac{2Hk}{M}\right) = \delta_{jk} \qquad \text{for } j, k = 0, 1, ..., M-1,$$
$$\int_0^{2H} \varphi_{M,j}(x_m)\overline{\varphi_{M,k}(x_m)} dx_m = \frac{1}{M}\delta_{jk} \qquad \text{for } j, k = 0, 1, ..., M-1.$$

For $\ell \in \mathbb{Z}_{M,2H}$ and given $Q_M a \in \mathcal{T}_{M,2H}$ its Fourier coefficients are computed by

$$\hat{a}_M(\ell) = \frac{1}{\sqrt{2H}} \int_0^{2H} a_M(x_m) \exp\left(-i\ell\pi \frac{x_m}{H}\right) dx_m = \frac{\sqrt{2H}}{M} \sum_{j=0}^{M-1} a_M\left(\frac{2Hj}{M}\right) \exp\left(-i2\pi l \frac{j}{M}\right), \tag{2.34}$$

or shortly

$$\hat{a}_M\left(k + \left\lfloor -\frac{M}{2} \right\rfloor + 1\right) = \frac{\sqrt{2H}}{M} \mathcal{F}_M(a_M)(k), \qquad k = 0, ..., M-1,$$

where $\mathcal{F}_M$ is the one-dimensional discrete Fourier transform (1D-FFT) defined by

$$\mathcal{F}_M(k) = \frac{1}{\sqrt{M}} \sum_{j=0}^{M-1} a_M(k) \exp\left(\frac{-2\pi i\ell k}{M}\right),$$

where $k = 0, 1, ..., M-1$ and a_M denotes the vector of nodal values $a(2Hk/M)$ for $k = 0, ..., M-1$.

For $M \in \mathbb{N}$ the orthogonal projection in $L^2([0, 2H])$ to $\mathcal{T}_{M,2H}$ is given by

$$a_M(x_m) = \frac{1}{\sqrt{2H}} \sum_{\ell \in \mathbb{Z}_{M,2H}} \hat{a}_M(\ell) \exp\left(i\pi\ell \frac{x_m}{H}\right). \tag{2.35}$$

Note that the computation of $a_M(x_m)$ by the orthogonal projection has high numerical costs, since for the computation of $\hat{a}_M(\ell)$ in (2.34) the integrals have to be numerically approximated. Furthermore, for given Fourier coefficients we compute the nodal values $a_M \in \mathcal{T}_{M,2H}$ by

$$a_M\left(\frac{2Hj}{M}\right) = \frac{\sqrt{M}}{\sqrt{2H}} \sum_{k=0}^{M-1} \hat{a}_M(j) \exp\left(-2i\pi \left(k + \left\lfloor -\frac{M}{2} \right\rfloor + 1\right) \frac{j}{M}\right) \qquad j = 0, 1, ..., M-1,$$

or shortly

$$a_M(k) = \frac{M}{\sqrt{2H}} \mathcal{F}_M^{-1} \hat{a_M} \left(k + \left\lfloor -\frac{M}{2} \right\rfloor + 1\right) \qquad \text{for } k = 0, ..., M-1,$$

where $\mathcal{F}_M^{-1}$ is the one-dimensional inverse discrete Fourier transform (1D-IFFT), defined by

$$\mathcal{F}_M(k) = \sqrt{M} \sum_{j=0}^{M-1} \hat{a_M}(k) \exp\left(\frac{2\pi i\ell k}{M}\right) \qquad \text{for } k = 0, 1, ...M-1.$$

Now by the assumption that the sound speed c is $2H$-periodic continuous function, we use the approximation of $\omega^2/c^2(x_m)$ by $a_M(x_m)$, where $x_m \in [0, H]$, and we obtain for the matrix element B_{kj} in (2.32) that

$$B_{kj} = \frac{1}{2}\left[\int_0^H a_M(x_m) \cos((\alpha_k - \alpha_j)x_m)\, dx_m - \int_0^H a_M(x_m) \cos((\alpha_k + \alpha_j)x_m)\, dx_m\right].$$

Using the definition of $\alpha_j = \pi(2j-1)/(2H)$ and the definition of the orthogonal projection (2.35), we have for each variable of $B_{k\ell}$ that

$$\int_0^H a_M(x_m) \cos((\alpha_k - \alpha_j)x_m) dx_m$$
$$= \frac{1}{\sqrt{2H}} \sum_{\ell \in \mathbb{Z}_{M,2H}} \hat{a}_M(\ell) \int_0^H \exp\left(\frac{i\ell\pi}{H} x_m\right) \cos\left(\frac{\pi(k-j)}{H} x_m\right) dx_m,$$

and

$$\int_0^H a_M(x_m) \cos((\alpha_k + \alpha_j)x_m) dx_m$$
$$= \frac{1}{\sqrt{2H}} \sum_{\ell \in \mathbb{Z}_{M,2H}} \hat{a}_M(\ell) \int_0^H \exp\left(\frac{i\ell\pi}{H} x_m\right) \cos\left(\frac{\pi(k+j-1)}{H} x_m\right) dx_m.$$

Furthermore, with the change of variables $m = k - j$ and by the definition of the cosines, we deduce

$$\int_0^H \exp\left(\frac{i\pi\ell}{H} x_m\right) \cos\left(\frac{\pi(k-j)}{H} x_m\right) dx_m$$
$$= \frac{1}{2} \int_0^H \exp\left(\frac{i\pi\ell}{H} x_m\right) \left[\exp\left(\frac{i\pi m}{H} x_m\right) + \exp(-\frac{i\pi m}{H} x_m)\right] dx_m$$
$$= \frac{1}{2} \int_0^H \left[\exp\left(\frac{i\pi(\ell+m)}{H} x_m\right) + \exp\left(\frac{i\pi(\ell-m)}{H} x_m\right)\right] dx_m. \tag{2.36}$$

Moreover, by integration we obtain for $\ell \neq |m|$ that

$$\int_0^H \exp\left(\frac{i\pi\ell}{H}x_m\right)\cos\left(\frac{\pi m}{H}x_m\right)dx_m$$
$$= -\frac{ih}{2\pi}\left[\frac{1}{(\ell+m)}\exp\left(\frac{i\pi(\ell+m)}{H}x_m\right)+\frac{1}{(\ell-m)}\exp\left(\frac{i\pi(\ell-m)}{H}x_m\right)\right]_0^H.$$

Next, the binomial formula implies

$$\int_0^H \exp\left(\frac{i\pi\ell}{H}x_m\right)\cos\left(\frac{\pi m}{H}x_m\right)dx_m$$
$$= -\frac{ih}{2\pi(\ell^2-m^2)}\left[(\ell-m)\exp\left(\frac{i\pi(\ell+m)}{H}x_m\right)+(\ell+m)\exp\left(\frac{i\pi(\ell-m)}{H}x_m\right)\right]_0^H.$$

Now, we reduce the last equation to the form

$$\begin{aligned}\int_0^H \exp\left(\frac{i\pi\ell}{H}x_m\right)\cos\left(\frac{\pi m x_m}{H}\right)dx_m &\\ = -&\frac{ih}{2\pi(\ell^2-m^2)}\left[(\ell-m)\exp(i\pi(\ell+m))+(\ell+m)\exp(i\pi(\ell-m))-2\ell\right]\\ =&\frac{(-1)ih}{2\pi(\ell^2-m^2)}\Big[\ell\exp(i\pi\ell)\left[\exp(i\pi m)+\exp(-i\pi m)\right]\\ &\qquad -m\exp(i\pi\ell)\left[\exp(i\pi m)-\exp(-i\pi m)\right]-2\ell\Big].\end{aligned}$$

Since $m \in \mathbb{Z}$ we have

$$\int_0^H \exp\left(\frac{i\pi\ell}{H}x_m\right)\cos\left(\frac{\pi m}{H}x_m\right)dx_m$$
$$= \frac{(-1)ih}{2\pi(\ell^2-m^2)}\left[\exp(i\pi\ell)[2\ell\cos(m\pi)-\underbrace{2im\sin(\pi m)}_{=0}]-2\ell\right].$$

Depending on $m \in \mathbb{Z}$ the cosines is one or minus one in the sense that

$$\int_0^H \exp\left(\frac{i\pi\ell}{H}x_m\right)\cos\left(\frac{\pi m}{H}x_m\right)dx_m = \frac{ih\ell}{\pi(\ell^2-m^2)}\left[1-(-1)^{m+\ell}\right].$$

Let us now derive the form if $\ell = m$ and $\ell \neq 0$ in (2.36). In particular, we have

$$\int_0^H \exp\left(\frac{i\pi(\ell+m)}{H}x_m\right)+\exp\left(\frac{i\pi(\ell-m)}{H}x_m\right)dx_m = \int_0^H\left(\exp\left(\frac{2i\pi\ell}{H}x_m\right)+1\right)dx_m,$$

and by integration we arrive at

$$\int_0^H\left(\exp\left(\frac{2i\pi\ell}{H}x_m\right)+1\right)dx_m = -\frac{ih}{2\pi\ell}\left(\exp(2i\pi\ell)-1\right)+H = \frac{ih}{2\pi\ell}\left(1-\exp(2i\pi\ell)\right)+H.$$

For $\ell \in \mathbb{Z}$ we have

$$\int_0^H \exp\left(\frac{i\pi\ell}{H}x_m\right)\cos\left(\frac{\pi m}{H}x_m\right)dx_m = \frac{H}{2}.$$

Similar, for $\ell = -m$ and $\ell \in \mathbb{Z}$ we deduce for (2.36) that

$$\int_0^H \exp\left(\frac{i\pi\ell}{H}x_m\right)\cos\left(\frac{\pi m}{H}x_m\right)dx_m = \frac{H}{2}.$$

Next, for $|\ell| = m$ and $\ell = 0$ we deduce

$$\int_0^H \exp\left(\frac{i\pi(\ell+m)}{H}x_m\right) + \exp\left(\frac{i\pi(\ell-m)}{H}x_m\right)\,dx_m = \int_0^H 2\,dx_m = 2H.$$

Consequently, we have for $m = k - j$ that

$$\int_0^H a(x_m)\cos((\alpha_k - \alpha_j)x_m)dx_m$$
$$= \frac{1}{\sqrt{2H}} \sum_{\ell\in\mathbb{Z}_{M,2H}} \hat{a}_M(\ell) \left(\left[\frac{ih\ell\left(1-(-1)^{k-j+\ell}\right)}{\pi(\ell^2-(k-j)^2)}\right]_{|\ell|\neq k-j} + \left[\frac{H}{2}\right]_{|\ell|=k-j,\ell\neq 0} + [H]_{|\ell|=k-j,\ell=0}\right).$$

Similar, we have for $m = k + j - 1$ that

$$\int_0^H a(x_m)\cos((\alpha_k + \alpha_j)x_m)dx_m$$
$$= \frac{1}{\sqrt{2H}} \sum_{\ell\in\mathbb{Z}_{M,2H}} \hat{a}_M(\ell) \Bigg(\left[\frac{ih\ell(1-(-1)^{k+j-1+\ell})}{\pi(\ell^2-(k+j-1)^2)}\right]_{|\ell|\neq k+j-1}$$
$$+ \left[\frac{H}{2}\right]_{|\ell|=k+j-1,\ell\neq 0} + [H]_{|\ell|=k+j-1,\ell=0}\Bigg).$$

To this end, for given $\hat{a}(\ell)$ we deduce

$$B_{kj} = \frac{1}{2\sqrt{2H}} \sum_{\ell\in\mathbb{Z}_{M,2H}} \hat{a}_M(\ell) \Bigg(\left[\frac{ih\ell(1-(-1)^{k-j+\ell})}{\pi(\ell^2-(k-j)^2)}\right]_{|\ell|\neq k-j} + \left[\frac{H}{2}\right]_{|\ell|=k-j,\ell\neq 0} + [H]_{k=j}$$
$$- \left[\frac{ih\ell(1-(-1)^{k+j-1+\ell})}{\pi(\ell^2-(k+j-1)^2)}\right]_{|\ell|\neq k+j-1} - \left[\frac{H}{2}\right]_{|\ell|=k+j-1,\ell\neq 0} - [H]_{|\ell|=k+j-1,\ell=0}\Bigg).$$

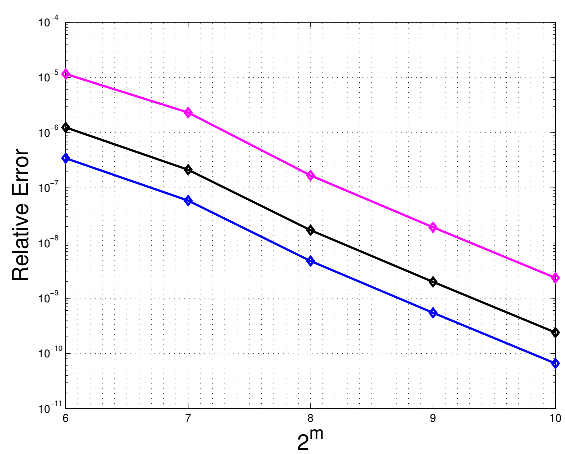
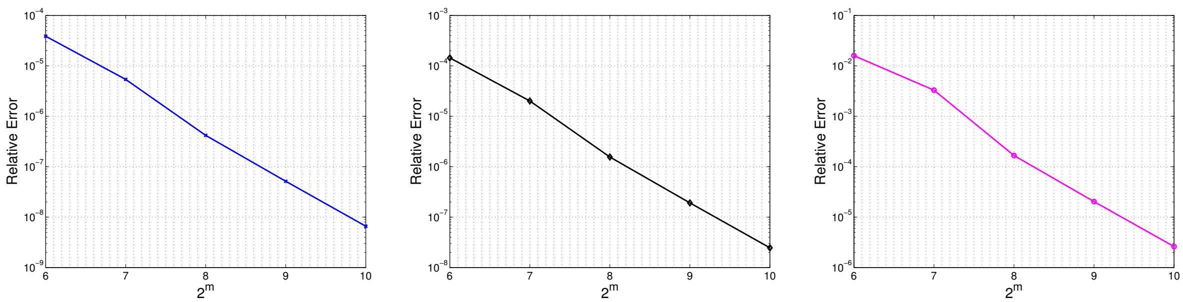


Figure 2.8: Relative L^2-error of the eigenvalues computed by the spectral method for frequency $\omega = 27$ (blue), $\omega = 50$ (black), $\omega = 500$ (magenta) for $N = 2^6$ eigenvalues, discretization of the interval $[0, H]$ of 2^{14} points and $M = 2^m$ where $m = 6, ..., 10$.

Now, we use MATLAB to compute the eigenvalues λ^2 in (2.32) and the coefficients c_k in (2.31). For more details on solving the corresponding eigenvalue equations with MATLAB we refer the reader to [Tre00, Chapter 9]. The first test problem that we consider is to compute the eigenvalues and eigenvectors acting on the vertical axis by the spectral method for different ocean configurations $J(\omega, c, H)$. Once more, we approximate the sound speed profile 7 in Figure 2.1 by a continuous function and its approximation is given by Figure 2.6. For simplicity, we moreover fix the number of eigenvalues to compute to $\mathcal{J} = 2^6$ and the number of discretization points to $N = 2^{14}$ for the uniform mesh on the interval $[0, H]$. The choice of this discretization parameter is an arbitrary assumption. We now analysis the error of the eigenvalues and eigenvectors for different frequencies and discretization parameter M. Note that the eigenvalue computing package of MATLAB requires $M \geq \mathcal{J}$.

We first see that for discretization points $M = 2^m$, where $m = 6, ..., 10$, the L^2-error of the eigenvalues for the frequencies $\omega = 27$ (blue), $\omega = 50$ (black) and $\omega = 500$ (magenta) are given by Figure 2.8. The computation of the reference solution $M = 2^{12}$ was computed in reasonable time of 29 hours on a 8 processor workstation with 128 GB RAM and used memory of two percent.



(a) (b) (c)

Figure 2.9: Relative L^2-error of the eigenvectors computed by the spectral method for $H = 200$, c continuous, $N = 2^6$ eigenvalues, discretization of the interval $[0, H]$ is 2^{14} points, $M = 2^m$ and for frequencies a) $\omega = 27$ (b), $\omega = 50$ (c), $\omega = 500$.

Next, the L^2-error of the eigenvectors are given in Figure 2.9. Here the L^2-error of the approximated eigenvectors are given by (a) for frequency $\omega = 27$, by (b) for $\omega = 50$, and by (c) for $\omega = 500$. Note that the memory efficient of the spectral method compared to the finite element method is higher, since for the reconstruction of the eigenfunctions only $\mathcal{J} \leq M$ values have to been saved. We have to point out that the eigenvectors after saving have to be computed by the formula in (2.31), however, in general disk space is cheaper than computing time.

Chapter 3

Existence and Uniqueness of Solution to the Scattering Problem

In this section we want to rigorously set up a radiation condition for solutions to the Helmholtz equation for the acoustic scattering ocean model introduced above. We first introduce the scattering problem in the wave guide. Then, we set up spectral characterizations of Sobolev-type function spaces to analyze the Dirichlet-Neumann operator in dimensions two and three for the waveguide operator scattering problem. Our aim is to derive existence and uniqueness result by using Fredholm theory and a Garding inequality.

3.1 Scattering in the Waveguide and Radiation Condition

We recall the eigenfunctions $\phi_j \in H^1_W([0,H]) \cap H^2([0,H])$ to the eigenvalue problem (2.14), solving

$$\phi_j'' + \frac{\omega^2}{c^2(x_m)}\phi_j - \lambda_j^2 \phi_j = 0 \qquad \text{in } L^2([0,H]),$$

as well as the corresponding eigenvalues λ_j^2. Based on the eigenpairs $(\phi_j, \lambda_j^2)_{j\in\mathbb{N}}$ solving (2.14), we recall the construction of solutions to the Helmholtz equation by series expansion

$$u(\tilde{x}, x_m) = \sum_{j\in\mathbb{N}} \alpha(j) w_j(\tilde{x}) \phi_j(x_m), \qquad \text{with coefficients } \alpha(j) \in \mathbb{C},$$

where w_j solves the Helmholtz equation in (2.10),

$$\Delta_{\tilde{x}} w_j + \lambda_j^2 w_j = 0 \qquad \text{in } \mathbb{R}^{m-1}.$$

Together with the equations for ϕ_j, we obtain that incident fields represented by plane waves

$$x \mapsto \phi_j(x_m)\exp(i\lambda_j\, \tilde{x}\cdot\theta) \qquad \text{for } x \in \Omega, j \in \mathbb{N}, \tag{3.1}$$

with direction of propagation $\theta \in \mathbb{R}^{m-1}$, $|\theta|_2 = 1$, solve the unperturbed Helmholtz equation,

$$\Delta u + \frac{\omega^2}{c^2(x_m)} u = 0 \qquad \text{in } \Omega. \tag{3.2}$$

Due to the boundary conditions in (2.14), for ϕ_j, these waveguide modes satisfy the waveguide boundary conditions

$$u = 0 \text{ on } \Gamma_0 \qquad \text{and} \qquad \frac{\partial u}{\partial x_m} = 0 \text{ on } \Gamma_H.$$

If $\lambda_j^2 > 0$ these plane waves correspond to propagating waveguide modes and obviously for $\lambda_j^2 < 0$ they are associated with evanescent ones. Recall the idea of exceptional frequencies in Remark 2.2.6, that for fixed H and c for a special choice of ω we obtain eigenvalues $\lambda_j^2 = 0$. For this choice the waveguide modes are constant in the horizontal variable $\tilde{x}$. In the following we exclude this ocean configurations by the following assumption.

Assumption 3.1.1. *We assume from now that the frequency $\omega > 0$ is chosen such that $\lambda_j \neq 0$ for all $j \in \mathbb{N}$.*

Indeed, this assumption makes sense since we showed in Theorem 2.2.8 that the case of exceptional ocean configurations only holds for at a most countable set of frequencies without finite accumulation point in $(0, \infty)$ and due to Lemma 2.2.4 we have for sufficiently small frequency $\omega > 0$ that all eigenvalues λ_j^2 are positive and therefore the only possible accumulation point of exceptional ocean configurations is $+\infty$. We point out that [AGL08] treats the case of exceptional ocean configurations for constant background speed.

To get uniqueness for the Helmholtz equation (and for physical aspects), the series representation of the solution of the Helmholtz equation for the horizontal variables (2.10) are required to be bounded for $|\tilde{x}|$ sufficiently large enough. In consequence, we need similarly to the case where the background sound speed is constant (see e.g. [AGL08]) a kind of Sommerfeld's radiation conditions. We see that they differ for propagating and evanescent modes. Let J_0 be the first index of λ_j (counting multiplicity) such that $\lambda_j^2 < 0$. Thus, to get now uniqueness for the Helmholtz equation for the horizontal variables (2.10) function $w_j \in C^\infty(|\tilde{x}| > \rho)$ is required to satisfy

$$\begin{cases} \lim_{|\tilde{x}|\to\infty} \sqrt{|\tilde{x}|} \left(\frac{\partial w_j}{\partial |\tilde{x}|} - i|\lambda_j| w_j \right) = 0, \text{ uniformly in } \frac{\tilde{x}}{|\tilde{x}|}, & \text{if } j < J_0, \\ w_j(\tilde{x}) \text{ is bounded in } |\tilde{x}| > \rho & \text{if } j \geq J_0, \end{cases} \quad \text{for all } j \in \mathbb{N}. \tag{3.3}$$

We call (3.3) radiation conditions and any solution to the Helmholtz equation solving (3.3) in the domain $(|\tilde{x}| > \rho) \times [0, H]$ is called a radiating solution.

We have now the tools to analyze the scattering problem in an inhomogeneous ocean. We first introduce for $k = 1, 2$ the space of functions

$$H^k_{W,\mathrm{loc}}(\Omega) = \{u : \Omega \to \mathbb{C} : u|_B \in H^k(B \cap \Omega) \text{ for every ball } B = B(0, R), R > 0, \text{ and } u|_{\Gamma_0} = 0\}.$$

Then, we consider an incident field u^i, satisfying the unperturbed Helmholtz equation

$$\Delta u^i + \frac{\omega^2}{c^2(x_m)} u^i = 0 \text{ in } \Omega, \tag{3.4}$$

and boundary conditions

$$u^i = 0 \text{ on } \Gamma_0 := \{x \in \mathbb{R}^m : x_m = 0\} \quad \text{and} \quad \frac{\partial u^i}{\partial x_m} = 0 \text{ on } \Gamma_H := \{x \in \mathbb{R}^m : x_m = H\}. \tag{3.5}$$

In consequence, by multiplication (3.4) with a test function $v \in H^1(\Omega)$, partial integration and plugging in the boundary conditions (3.5), we deduce the weak formulation: The incident field $u^i \in H^1_{W,\mathrm{loc}}(\Omega)$ solves (3.4) and (3.5) weakly in Ω,

$$\int_\Omega \left(\nabla u^i \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)} u^i \, \overline{v} \right) dx = 0 \qquad \text{for all } v \in H^1_W(\Omega) \text{ with compact support.} \tag{3.6}$$

The incident field u^i can be e.g. a plane wave of direction $\theta \in \mathbb{R}^{m-1}, |\theta| = 1$ (see (3.1)) having the form

$$x \mapsto u^i(x) = \phi_j(x_m) \exp(i\lambda_j \, \tilde{x} \cdot \theta) \qquad \text{for } x \in \Omega,$$

for fixed $j \in \mathbb{N}$. In the presence of a scatterer $D \subset \Omega$, where

$$q(x) = n^2(x) - 1 \qquad \text{for } x \in \Omega, \tag{3.7}$$

this object creates a scattered field u^s, which provides the total field

$$u(x) = u^i(x) + u^s(x) \qquad \text{for } x \in \Omega, \tag{3.8}$$

where u solves the Helmholtz equation

$$\Delta u(x) + \frac{\omega^2}{c^2(x_m)} n^2(x) u(x) = 0 \qquad \text{for } x \in \Omega, \tag{3.9}$$

and boundary conditions

$$u = 0 \text{ on } \Gamma_0 := \{x \in \mathbb{R}^m : x_m = 0\} \quad \text{and} \quad \frac{\partial u}{\partial x_m} = 0 \text{ on } \Gamma_H := \{x \in \mathbb{R}^m : x_m = H\}.$$

Plugging all together, we have the following wave propagation problem: Find the total field $u \in H^1_{W,\mathrm{loc}}(\Omega)$ solving

$$\int_\Omega \left(\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)} n^2\, u \overline{v} \right) = 0, \qquad \text{for all } v \in H^1_W(\Omega) \text{ with compact support.} \tag{3.10}$$

We moreover require that the scattered fields satisfies the radiation conditions (3.3): More precisely, u^s can be represented as

$$u^s(x) = \sum_{j \in \mathbb{N}} \alpha(j) w_j(\tilde{x}) \phi_j(x_m), \tag{3.11}$$

for $|\tilde{x}| > \rho$, where ρ is such that $D \subset B(0, \rho)$ and where w_j satisfies (3.3).

.

Remark 3.1.2. *Any solution solving (3.10) can be represented in series form as in (3.11) since the eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}} \subset H^1_W([0,H])$ are an orthonormal basis of $L^2(0,H)$. In consequence, the assumption on the scattered field u^s simply requires radiation conditions (3.3) to be satisfied.*

3.2 Spectral Characterizations and Periodic Function Spaces

We first start by introducing classical Fourier theory, which is adapted to the Helmholtz problem (2.1) with corresponding boundary conditions (2.5) and (2.6). Essentially for the analysis and, later on, for computations, we restrict ourselves to the rectangular domain

$$\Lambda_\rho := \{x \in \Omega : |\tilde{x}|_\infty < \rho\}, \qquad \rho > 0.$$

Linked to the notation for points x, we introduce

$$\tilde{\Lambda}_\rho := \{\tilde{x} \in \mathbb{R}^{m-1} : |\tilde{x}|_\infty < \rho\}, \qquad \rho > 0.$$

Additionally, we denote for $\rho > 0$ the cylindrical domain

$$M_\rho = \{x \in \mathbb{R}^{m-1} \times [0,H] : |\tilde{x}|_2 < \rho\}, \qquad \rho > 0.$$

We point out that ρ is chosen to be large enough, such that $\overline{D} \subset M_\rho \subset \Omega$ holds. Next, for dimension two we define the boundaries of the cylindrical domain M_ρ by

$$\Gamma_{0,\rho} := \{x \in \mathbb{R}^2 : -\rho < x_1 < \rho, x_2 = 0\} \quad \text{and} \quad \Gamma_{H,\rho} := \{x \in \mathbb{R}^2 : -\rho < x_1 < \rho, x_2 = H\},$$

thus $\Gamma_{0,\rho}$ and $\Gamma_{H,\rho}$ are lines fixed in $x_2 = 0$ and $x_2 = H$. Further, let

$$\Sigma_{\pm\rho} : = \{x \in \Omega : x_1 = \pm\rho, x_2 \in (0,H)\} \quad \text{and} \quad \Sigma_\rho := \Sigma_{-\rho} \cup \Sigma_{+\rho}.$$

Similar, we consider for dimension three

$$\Gamma_{0,\rho} := \{x \in \mathbb{R}^3 : |\tilde{x}|_2 < \rho, x_3 = 0\} \quad \text{and} \quad \Gamma_{H,\rho} := \{x \in \mathbb{R}^3 : |\tilde{x}|_2 < \rho, x_3 = H\}.$$

Then, we define the boundary of the cylindrical domain M_ρ that is contained in Ω by

$$\Sigma_\rho : = \{x \in \Omega : |\tilde{x}|_2 = \rho, x_3 \in (0,H)\}.$$

For dimension three we further know that this part of the boundary of M_ρ can be represented as

$$\Sigma_\rho = \left\{x = (\rho\cos\varphi, \rho\sin\varphi, x_3)^\top : \varphi \in (0,\pi),\, x_3 \in (0,H)\right\}. \tag{3.12}$$

Furthermore, we define

$$\mathbb{Z}^m_+ := \{\boldsymbol{n} \in \mathbb{Z}^m : n_m > 0\}.$$

With respect to our notation of points we write for $\boldsymbol{n} \in \mathbb{Z}^2_+$ the vector $\boldsymbol{n} = (n_1, n_2)$ and for $\boldsymbol{n} \in \mathbb{Z}^3_+$ that $\boldsymbol{n} = (n_1, n_2, n_3) = (\tilde{\boldsymbol{n}}, n_3)$. We point out that vectors are distinguished from scalars by the use of bold typeface, however, for simplicity we use this convention not for points and vectors in Ω. Next, we establish orthonormal basis functions in $L^2(\Lambda_\rho)$. Consider the

$$v_{\tilde{\boldsymbol{n}}}(\tilde{x}) : = \frac{1}{\sqrt{2\rho}} \exp\left(i\frac{\pi}{\rho} n_1 x_1\right) \qquad \text{for } m = 2, n_1 \in \mathbb{Z}, x_1 \in [-\rho,\rho] \text{ and}$$

$$v_{\tilde{\boldsymbol{n}}}(\tilde{x}) : = \frac{1}{2\rho} \exp\left(i\frac{\pi}{\rho}\ \tilde{\boldsymbol{n}} \cdot \tilde{x}\right) \qquad \text{for } m = 3, \tilde{\boldsymbol{n}} \in \mathbb{Z}^2, \tilde{x} \in \tilde{\Lambda}_\rho.$$

Corollary 3.2.1. *For the vector $v_{\tilde{\boldsymbol{n}}} \in C^\infty(\tilde{\Lambda}_\rho)$ holds*

$$\max_{\tilde{x}\in\tilde{\Lambda}_\rho} |v_{\tilde{\boldsymbol{n}}}(\tilde{x})| \le C(\rho), \qquad \tilde{\boldsymbol{n}} \in \mathbb{Z}^{m-1},$$

where $C(\rho)$ depends on the domain $\tilde{\Lambda}_\rho$.

The proof follows directly by rigorous computations.

Since for $n_m \in \mathbb{N}$ the eigenfunctions $\phi_{n_m}(x_m)$ of different eigenvalues are orthonormal and complete in $L^2(0,H)$, the composition with the orthonormal functions $v_{\tilde{\boldsymbol{n}}}(\tilde{x})$, where $\tilde{\boldsymbol{n}} \in \mathbb{Z}^{m-1}$ forms a complete orthonormal system basis in $L^2(\Lambda_\rho) = L^2(\tilde{\Lambda}_\rho \times (0,H))$, for $m = 2,3$. More precisely, the $\varphi_{\boldsymbol{n}}$ defined by

$$\varphi_{\boldsymbol{n}}(x) := \frac{1}{(2\rho)^{(m-1)/2}} \phi_{n_m}(x_m) \exp\left(i\frac{\pi}{\rho}\ \tilde{\boldsymbol{n}} \cdot \tilde{x}\right), \qquad \text{where } \tilde{\boldsymbol{n}} \in \mathbb{Z}^{m-1}, n_m \in \mathbb{N}, x \in \Lambda_\rho, \tag{3.13}$$

forms an orthonormal basis in $L^2(\Lambda_\rho)$. The proof follows from [BB93] by using theory of tensor products in Hilbert spaces.

Due to completeness of the orthonormal eigenfunctions $\{\varphi_{\boldsymbol{k}}\}_{\boldsymbol{k}\in\mathbb{Z}^m_+} \subset L^2(\Lambda_\rho)$, every function $u \in L^2(\Lambda_\rho)$ can be represented as its Fourier series,

$$u(x) = \sum_{\boldsymbol{k}\in\mathbb{Z}^m_+} \hat{u}(\boldsymbol{k})\varphi_{\boldsymbol{k}}(x), \qquad \text{with } \hat{u}(\boldsymbol{k}) = (u, \varphi_{\boldsymbol{k}})_{L^2(\Lambda_\rho)} = \int_{\Lambda_\rho} u\overline{\varphi_{\boldsymbol{k}}}\, dx, \qquad \boldsymbol{k} \in \mathbb{Z}^m_+, x \in \Lambda_\rho, \tag{3.14}$$

where the series converges in $L^2(\Lambda_\rho)$. As usual, we call the coefficients $\hat{u}(\boldsymbol{k})$ the Fourier coefficients of u. Furthermore, the integral exists by the fact that $\phi_{k_m}(x_m)$ is in $H^1_W([0,H])$. With respect to the separation of horizontal and vertical variables we denote the Fourier coefficient of $u \in L^2(\Lambda_\rho)$, still depending on the horizontal variables, by

$$\hat{u}(k_m, \tilde{x}) := (u, \phi_{k_m})_{L^2(0,H)} = \int_0^H u(\tilde{x}, x_m)\, \phi_{k_m}(x_m)\, dx_m, \qquad k_m \in \mathbb{N}, x \in \Lambda_\rho,$$

Indeed, there is no complex-conjugation of the eigenvectors $\{\phi_{k_m}(x_m)\}_{k_m\in\mathbb{N}}$ since they are real-valued. Furthermore, we set the Fourier coefficients of $u \in L^2(\tilde{\Lambda}_\rho)$, which still depend on the vertical axis, by

$$\hat{u}(\tilde{\boldsymbol{k}}, x_m) := (u, v_{\tilde{\boldsymbol{k}}})_{L^2(\tilde{\Lambda}_\rho)} = \int_{\tilde{\Lambda}_\rho} u(\tilde{x}, x_m)\, \overline{v_{\tilde{\boldsymbol{k}}}(\tilde{x})}\, d\tilde{x}, \qquad \tilde{\boldsymbol{k}} \in \mathbb{Z}^{m-1}, x \in \Lambda_\rho.$$

Every element $u \in L^2(\Lambda_\rho)$ can hence be expanded into Fourier series of the form

$$u(x) = \sum_{k_m\in\mathbb{N}} \hat{u}(k_m, \tilde{x})\phi_{n_j}(x_m), \qquad x \in \Lambda_\rho, j \in \mathbb{N}, \tag{3.15}$$

and

$$u(x) = \sum_{\tilde{\boldsymbol{k}}\in\mathbb{N}^{m-1}} \hat{u}(\tilde{\boldsymbol{k}}, x_m)v_{\tilde{\boldsymbol{k}}}(\tilde{x}), \qquad x \in \Lambda_\rho. \tag{3.16}$$

With reference to the introduced Fourier series we next define periodic Sobolev space on Λ_ρ by

$$H^s(\Lambda_\rho) = \left\{ u = \sum_{\boldsymbol{k}\in\mathbb{Z}^m_+} \hat{u}(\boldsymbol{k})\varphi_{\boldsymbol{k}}, \sum_{\boldsymbol{k}\in\mathbb{Z}^m_+} (1 + |\tilde{\boldsymbol{k}}|^2 + |\lambda_{k_m}|^2)^s |\hat{u}(\boldsymbol{k})|^2 < \infty \right\}, \qquad s \in \mathbb{R}, \tag{3.17}$$

with (squared) norm

$$\|u\|^2_{H^s(\Lambda_\rho)} = \sum_{\boldsymbol{k}\in\mathbb{Z}^m_+} (1 + |\tilde{\boldsymbol{k}}|^2 + |\lambda_{k_m}|^2)^s |\hat{u}(\boldsymbol{k})|^2.$$

Analogously, we define Sobolev space with index $s \in \mathbb{R}$ on $\tilde{\Lambda}_\rho = (-\rho, \rho)^{m-1}$ and on $(0, H)$ by

$$H^s(\tilde{\Lambda}_\rho) = \left\{ v = \sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}^{m-1}} \hat{v}(\tilde{\boldsymbol{k}})v(\tilde{\boldsymbol{k}}) : \sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}^{m-1}} (1 + |\tilde{\boldsymbol{k}}|^2)^s \left|\hat{v}(\tilde{\boldsymbol{k}}, x_m)\right|^2 < \infty \right\}, \tag{3.18}$$

where

$$\hat{v}(\tilde{\boldsymbol{k}}, x_m) = (v, v_{\tilde{\boldsymbol{k}}})_{L^2(\tilde{\Lambda}_\rho)}, = \int_{\tilde{\Lambda}_\rho} v\, \overline{v_{\tilde{k}}}\, d\tilde{x},$$

with (squared) norm

$$\|v\|^2_{H^s(\tilde{\Lambda}_\rho)} = \sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}^{m-1}} (1 + |\tilde{\boldsymbol{k}}|^2|)^s |\hat{v}(\tilde{\boldsymbol{k}}, x_m)|^2.$$

Furthermore, let

$$H^s(0, H) = \left\{ w = \sum_{k_m\in\mathbb{N}} \hat{w}(k_m, \tilde{x})\phi_{k_m} : \sum_{k_m\in\mathbb{N}} (1 + |\lambda_{k_m}|^2)^s \left|\hat{w}(k_m, \tilde{x})\right|^2 < \infty \right\} \quad s \in \mathbb{R}, \tag{3.19}$$

where

$$\hat{w}(k_m, \tilde{x}) = (w, \phi_{k_m})_{L^2(\tilde{\Lambda}_\rho)} = \int_0^H w\, \phi_{k_m}\, dx_m,$$

and with (squared) norm

$$\|w\|^2_{H^s(0,H)} = \sum_{k_m\in\mathbb{N}} (1 + |\lambda_{k_m}|^2)^s |\hat{w}(k_m, \tilde{x})|^2.$$

For the special choice $s = 0$ we recover in (3.17) the $L^2(\Lambda_\rho)$ space by $H^0(\Lambda_\rho) = L^2(\Lambda_\rho)$. Due to [AH09, Example 7.4.2] or [SV02, Lemma 5.3.2] we prove later on that for $s > m/2$ the space $H^s(\Lambda_\rho)$ is continuously embedded in $C(\overline{\Lambda_\rho})$.

As for any complete orthonormal system, due to Parseval's identity the following lemma holds.

Lemma 3.2.2. *For $u \in L^2(M_\rho)$ we have that the Fourier coefficients $\hat{u}(j,\cdot)$ belongs to $L^2(\{|\tilde{x}| < \rho\})$. Further, it holds*

$$\|u\|^2_{L^2(M_\rho)} = \sum_{k_m \in \mathbb{N}} \|\hat{u}(k_m,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})} \qquad \textit{and} \qquad \|u\|^2_{L^2(\Lambda_\rho)} = \sum_{\boldsymbol{k}\in\mathbb{Z}^m_+} |\hat{u}(\boldsymbol{k})|^2.$$

Proof. By the definition of the L^2-norm on the cylindrical domain M_ρ it follows that

$$\|u\|^2_{L^2(M_\rho)} = \int_{M_\rho} |u(\tilde{x}, x_m)|^2 \, dx.$$

Next, we obtain for fixed $\tilde{x}$-variable by Parseval's identity that for any continuous function $u \in C(\overline{M}_\rho)$ there holds

$$\int_0^H |u(\cdot, x_m)|^2 \, dx_m = \|u(\cdot, x_m)\|^2_{L^2(0,H)} = \sum_{j=1}^{\infty} |u(j,\cdot)|^2.$$

In consequence, we see

$$\|u\|^2_{L^2(M_\rho)} = \sum_{j=1}^{\infty} \|\hat{u}(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})}.$$

By the definition of continuous functions in $L^2(M_\rho)$, the latter equality holds for all $u \in L^2(M_\rho)$. □

Lemma 3.2.3. *a) For $u \in H^1_W(M_\rho)$ we have that the Fourier coefficients $\hat{u}(j,\cdot)$ belong to $H^1(\{|\tilde{x}| < \rho\})$ and*

$$\left\|\frac{\partial u}{\partial x_i}\right\|^2_{L^2(M_\rho)} = \sum_{j=1}^{\infty} \left\|\frac{\partial}{\partial x_i}\hat{u}(j,\cdot)\right\|^2_{L^2(\{|\tilde{x}|<\rho\})}, \qquad \textit{where } i = 1, ..., m-1.$$

b) If $u \in C^2(\overline{M_\rho})$ then the series expansion

$$u(x) = \sum_{k_m \in \mathbb{N}} \hat{u}(k_m, \tilde{x})\phi_{n_j}(x_m), \qquad x \in \Lambda_\rho,$$

converges uniformly and absolutely. Further the derivative (term by term) of this series expansion with respect to the vertical axis exists and converges uniformly and absolutely,

$$\frac{\partial u}{\partial x_m}(x) = \sum_{j\in\mathbb{Z}} \hat{u}(j,\tilde{x})\frac{\partial \phi_j}{\partial x_m}(x_m) \qquad \textit{for } x \in \overline{M_\rho}.$$

Proof. a) Consider in this proof $i = 1, ..., m-1$. If $u \in H^1(M_\rho)$ then it holds for the weak derivatives $\partial u/\partial x_i \in L^2(M_\rho)$. Conversely, if the distribution derivatives $\partial u/\partial x_i \in L^2(M_\rho)$ then we know that $u \in H^1(M_\rho)$. Consequently, since

$$\left\|u \mapsto \frac{\partial u}{\partial x_i}\right\|_{L^2(M_\rho)} \le C\|u\|_{H^1(M_\rho)},$$

where $C > 0$, we can interchange sum and derivation with respect to the variable x_i. Furthermore, due to the fact that $\{\phi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis, we obtain

$$\frac{\partial}{\partial x_i}\int_0^H u(\tilde{x}, x_m)\phi_k(x_m)\, dx_m = \frac{\partial}{\partial x_i}\int_0^H \sum_{j=1}^{\infty} \hat{u}(j,\tilde{x})\phi_j(x_m)\phi_k(x_m)\, dx_m = \frac{\partial}{\partial x_i}\hat{u}(k,\tilde{x}). \tag{3.20}$$

We moreover see by Cauchy-Schwarz inequality that

$$\left\| \int_0^H \frac{\partial u}{\partial x_i}(x)\overline{\phi_k(x_m)}\,dx_m \right\|^2_{L^2(\{|\tilde{x}|<\rho\})} \leq \left\| \left\| \frac{\partial u}{\partial x_i}(\tilde{x},\cdot) \right\|_{L^2([0,H])} \|\phi_k\|_{L^2([0,H])} \right\|^2_{L^2(\{|\tilde{x}|<\rho\})} = \left\| \frac{\partial u}{\partial x_i} \right\|^2_{L^2(\Lambda_\rho)}.$$

In consequence, we see that the operator of all partial derivation $x_i \mapsto \partial\hat{u}(k,x_i)/\partial x_i$ is square-integrable and $\hat{u}(k,\tilde{x}) \in H^1(M_\rho)$. To this end, Parseval's identity and (3.20) imply

$$\left\| \frac{\partial u}{\partial x_i} \right\|^2_{L^2(M_\rho)} = \sum_{j=1}^{\infty} \left\| \frac{\partial}{\partial x_i}\hat{u}(j,\cdot) \right\|_{L^2(\{|\tilde{x}|<\rho\})}.$$

b) Due to [LS60, Chapter 2, §4-6] we obtain that for an expansion in terms of Sturm-Liouville eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}}$ we can interchange sum and derivative in x_m direction. □

For simplicity we set for dimension two $\hat{u}'(j,x_1) := \partial\hat{u}(j,x_1)/\partial x_1$. To denote in the following lemma the existence of $C>0$ independent of u such that it holds

$$C^{-1}\|u\|_A^2 \leq \|u\|_B^2 \leq C\|u\|_A^2,$$

we indicate by $\|u\|_A^2 \simeq \|u\|_B^2$ the equivalence of two norms $\|\cdot\|_{A,B}$.

Lemma 3.2.4. *For $u \in H^1(M_\rho)$ it holds for $m=2$ that*

$$\|u\|^2_{H^1(M_\rho)} \simeq \int_{-\rho}^{\rho} \sum_{j=1}^{\infty} |\hat{u}'(j,x_1)|^2\,dx_1 + \int_{-\rho}^{\rho} \sum_{j=1}^{\infty} (1+|\lambda_j|^2)|\hat{u}(j,x_1)|^2\,dx_1. \tag{3.21}$$

The equivalence constants depend on ω and J. However, they can be chosen uniformly for frequencies ω in any compact subset of $\mathbb{R}_{>0}$.

Proof. Due to the fact that $H^1_W(M_\rho) \cap C^2(\overline{M_\rho})$ is a dense subset of $H^1_W(M_\rho)$ we show the result for $u \in H^1_W(M_\rho) \cap C^2(\overline{M_\rho})$. Lemma 3.2.3 b) states that we have for $u \in H^1_W(M_\rho) \cap C^2(\overline{M_\rho})$ the derivative (term by term) of its series expansion with respect to x_2 axis converges uniformly and absolutely and

$$\frac{\partial u}{\partial x_2}(x) = \sum_{j\in\mathbb{Z}} \hat{u}(j,x_1)\frac{\partial \phi_j}{\partial x_2}(x_2) \qquad \text{for } x \in \overline{M_\rho}.$$

Using the expression of the Fourier series in (3.15), we consider the truncated series

$$u_N(x) = \sum_{j=1}^{N} \hat{u}(j,x_1)\phi_j(x_2) \qquad \text{for } N \in \mathbb{N}.$$

We note that $u_N \to u$ as $N \to \infty$ in $H^1(M_\rho)$ since due to Lemma 3.2.3 it holds that u is twice differentiable. We furthermore obtain

$$\|u_N\|^2_{L^2(M_\rho)} = \int_{-\rho}^{\rho}\int_0^H \sum_{j,j'=1}^{N} \hat{u}(j,x_1)\phi_j(x_2)\overline{u(j',x_1)}\phi_{j'}(x_2)\,dx_2dx_1. \tag{3.22}$$

By the orthogonality of the eigenfunction $\{\phi_j\}_{j\in\mathbb{N}}$ we observe that

$$\|u_N\|^2_{L^2(M_\rho)} = \int_{-\rho}^{\rho} \sum_{j=1}^{N} |\hat{u}(j,x_1)|^2\,dx_1 \leq \|u\|^2_{L^2(M_\rho)}.$$

Next, we deduce for the gradient for $m = 2$ that

$$\nabla u_N = \frac{\partial u_N}{\partial x_1} e_{x_1} + \frac{\partial u_N}{\partial x_2} e_{x_2}, \qquad \text{where } e_{x_1} = (1,0)^T \text{ and } e_{x_2} = (0,1). \tag{3.23}$$

We point out that by Lemma 3.2.3 we can then interchange sum and derivative and by the continuity of the norm and Lemma 3.2.3 this interchange is still valid for if N tends to infinity. Once more exploiting Lemma 3.2.3 for the derivative in x_1 direction, there holds

$$\left\| \frac{\partial u_N}{\partial x_1} \right\|^2_{L^2(M_\rho)} = \int_{-\rho}^{\rho} \sum_{j=1}^{N} |\hat{u}'(j, x_1)|^2 \, dx_1.$$

By taking the limit $N \to \infty$, we see

$$\|u\|^2_{L^2(M_\rho)} = \int_{-\rho}^{\rho} \sum_{j=1}^{\infty} |\hat{u}(j, x_1)|^2 \, dx_1 \qquad \text{and} \qquad \left\| \frac{\partial u}{\partial x_1} \right\|^2_{L^2(M_\rho)} = \int_{-\rho}^{\rho} \sum_{j=1}^{\infty} |\hat{u}'(j, x_1)|^2 \, dx_1. \tag{3.24}$$

Again by Lemma 3.2.3 we can interchange sum and derivative and by the continuity of the norm and Lemma 3.2.3 this interchange is still valid if N tends to infinity. In consequence, we treat directly this case, where N tends to infinity. In particular, for the derivation in x_2 direction we compute that

$$\left\| \frac{\partial u}{\partial x_2} \right\|^2_{L^2(M_\rho)} = \int_{-\rho}^{\rho} \sum_{j,j'=1}^{\infty} \hat{u}(j', x_1) \overline{\hat{u}(j', x_1)} \, dx_1 \int_0^H \phi_j'(x_2) \phi_{j'}'(x_2) \, dx_2.$$

Using the variational formulation of the eigenvalue problem for the j-th eigenpair (λ_j^2, ϕ_j) in (2.15) we obtain

$$\left\| \frac{\partial u}{\partial x_2} \right\|^2_{L^2(M_\rho)} = \int_{-\rho}^{\rho} \sum_{j,j'=1}^{\infty} \hat{u}(j, x_1) \overline{\hat{u}(j', x_1)} \, dx_1 \int_0^H \left(\frac{\omega^2}{c^2(x_2)} - \lambda_j^2 \right) \phi_j(x_2) \phi_{j'}(x_2) \, dx_2. \tag{3.25}$$

Furthermore, we deduce

$$- \int_{-\rho}^{\rho} \sum_{j,j'=1}^{\infty} \hat{u}(j, x_1) \overline{\hat{u}(j', x_1)} \, dx_1 \lambda_j^2 \int_0^H \phi_j(x_2) \phi_{j'}(x_2) \, dx_2 = - \sum_{j=1}^{\infty} \lambda_j^2 \|u_j\|^2_{L^2([-\rho,\rho])}, \tag{3.26}$$

and

$$\int_{-\rho}^{\rho} \sum_{j,j'=1}^{\infty} \hat{u}(j, x_1) \overline{\hat{u}(j', x_1)} \, dx_1 \int_0^H \frac{\omega^2}{c^2(x_2)} \phi_j(x_2) \phi_{j'}(x_2) \, dx_2$$
$$= \int_{-\rho}^{\rho} \int_0^H \frac{\omega^2}{c^2(x_2)} \left| \sum_{j=1}^{\infty} \hat{u}(j, x_1) \phi_j(x_2) \right|^2 dx_2 \, dx_1. \tag{3.27}$$

Due to the estimate of the background sound speed in (2.4), we obtain

$$\begin{aligned} \left\| \frac{\partial u}{\partial x_2} \right\|^2_{L^2(M_\rho)} &\leqslant \frac{\omega^2}{c_\mp^2} \|u\|^2_{L^2(M_\rho)} - \sum_{j=1}^{\infty} \lambda_j^2 \|u_j\|^2_{L^2([-\rho,\rho])} \\ &= \sum_{j=1}^{\infty} \left(\frac{\omega^2}{c_\mp^2} - \lambda_j^2 \right) \|u_j\|^2_{L^2([-\rho,\rho])} \\ &= \sum_{j=1}^{\infty} \left(\frac{\omega^2}{c_\mp^2} - \lambda_j^2 \right) \int_{-\rho}^{\rho} |\hat{u}(j, x_1)|^2 dx_1. \end{aligned} \tag{3.28}$$

Next, we know by Corollary 2.2.6 and Remark 2.2.2 that there is a finite number $1 \leq j \leq J(\omega, c, H)$ of positive eigenvalues λ_j^2 to control this inequality. For this finite number of eigenvalues we now show by contradiction that there exists $c > 0$ such that

$$\left\| \frac{\partial u}{\partial x_2} \right\|_{L^2(M_\rho)}^2 \geq c \sum_{j=1}^{\infty} (1 + |\lambda_j|^2) \int_{-\rho}^{\rho} |\hat{u}(j, x_1)|^2 \, dx_1 \qquad \text{for all } u \in H_W^1(M_\rho). \tag{3.29}$$

We assume that the last inequality does not hold uniformly for all $u \in H_W^1(M_\rho)$. Then, there is a sequence $\{u^{(k)}\}_{k \in \mathbb{N}} \subset H_W^1(M_\rho)$ such that

$$\left\| \frac{\partial u^{(k)}}{\partial x_2} \right\|_{L^2(M_\rho)}^2 \longrightarrow 0 \text{ as } k \to \infty,$$

while

$$\sum_{j=1}^{\infty} (1 + |\lambda_j|^2) \int_{-\rho}^{\rho} |\hat{u}^{(k)}(j, x_1)|^2 \, dx_1 = 1 \qquad \text{for all } k \in \mathbb{N}. \tag{3.30}$$

To obtain a contradiction we prove that

$$\|v\|_{L^2(M_\rho)} \leq C \left\| \frac{\partial v}{\partial x_2}(x_1, \cdot) \right\|_{L^2(M_\rho)} \qquad \text{for all } v \in H_W^1(M_\rho).$$

We start off by Poincaré's inequality to see

$$\|v(x_1, \cdot)\|_{L^2([0,H])}^2 \leq \frac{H^2}{2} \left\| \frac{\partial v}{\partial x_2}(x_1, \cdot) \right\|_{L^2([0,H])}^2 \qquad \text{for all } v \in H_W^1(M_\rho) \cap C^2(\overline{M_\rho}).$$

Furthermore, by integration in horizontal variable we find

$$\|v\|_{L^2(M_\rho)}^2 \leq \frac{H^2}{2} \left\| \frac{\partial v}{\partial x_2}(x_1, \cdot) \right\|_{L^2(M_\rho)}^2 \qquad \text{for all } v \in H_W^1(M_\rho) \cap C^2(\overline{M_\rho}).$$

Since $H_W^1(M_\rho) \cap C^2(\overline{M_\rho}) \subset H_W^1(M_\rho)$ is dense in $H_W^1(M_\rho)$, the latter estimate holds for all $v \in H_W^1(M_\rho)$. In consequence, we have for $C = \max_{1 \leq j \leq J}(1 + |\lambda_j|^2)$ that

$$\begin{aligned} \sum_{j=1}^{J} (1 + |\lambda_j|^2) \int_{-\rho}^{\rho} |\hat{u}^{(k)}(j, x_1)|^2 \, dx_1 &\leq C \|u^{(k)}\|_{L^2(M_\rho)}^2 \\ &\leq CH \left\| \frac{\partial u^{(k)}}{\partial x_2} \right\|_{L^2(M_\rho)}^2 \longrightarrow 0 \qquad \text{as } k \to \infty. \end{aligned} \tag{3.31}$$

Together with (3.28), the last inequality shows

$$\begin{aligned} &\sum_{j=J+1}^{\infty} (1 + |\lambda_j|^2) \int_{-\rho}^{\rho} |\hat{u}^{(k)}(j, x_1)|^2 \, dx_1 \\ &\qquad \leq C \left\| \frac{\partial u^{(k)}}{\partial x_2} \right\|_{L^2(M_\rho)}^2 + \underbrace{\sum_{j=1}^{J} (|\lambda_j|^2 - C) \|\hat{u}^{(k)}(j, x_1)\|_{L^2([-\rho,\rho])}^2}_{\to 0 \text{ by } (3.31)} \longrightarrow 0 \qquad \text{as } k \to \infty. \end{aligned}$$

In particular, we deduce

$$\sum_{j=1}^{\infty} (1 + |\lambda_j|^2) \int_{-\rho}^{\rho} |\hat{u}^{(k)}(j, x_1)|^2 \, dx_1 \longrightarrow 0 \qquad \text{as } k \to \infty,$$

which contradicts our assumption (3.30) and proves the estimate (3.29). Plugging this together with estimate (3.24), we deduce the norm equivalence of $\| \cdot \|_{H^1(M_\rho)}$.

□

Remark 3.2.5. *Using an inner product with weight $\omega^2/c^2(x_m)$ defined by*

$$(\phi_j, \phi_k)_{L^2_\diamond([0,H])} = \int_0^H \frac{\omega^2}{c^2(x_m)} \phi_j(x_m)\overline{\phi_k(x_m)}\, dx_m,$$

the norm equivalence can also be shown. Using this inner product, we can define a second eigenvalue problem and there hold orthogonality conditions, too. Then, techniques from [GK69][Chapter VI, Theorem 2.1.2] show that equivalence to the standard L^2-norm holds, such that there exists a constant $C > 0$ satisfying

$$\frac{1}{C}(\phi,\phi)_{L^2_\diamond([0,H])} \le \|\phi\|^2_{L^2([0,H])} \le C(\phi,\phi)_{L^2_\diamond([0,H])} \qquad \text{for all } \phi \in L^2([0,H]).$$

Now, due to the cylindrical form of M_ρ in dimension three, we use cylinder coordinates

$$x = (r\cos\varphi, r\sin\varphi, x_3) \in M_\rho \qquad \text{for } r \in (0,\rho], \varphi \in [0,2\pi] \text{ and } x_3 \in [0,H].$$

Moreover, since $\exp(in\varphi)$ represents an (non-normalized) orthogonal basis of $L^2(\Sigma_\rho)$, we can expand a function $u \in L^2(M_\rho)$ into the Fourier series,

$$u(x) = \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} \hat{u}(n,j,r) \exp(in\varphi)\phi_j(x_3), \tag{3.32}$$

where $\hat{u}(n,j,r)$ denotes the Fourier coefficient in the three-dimensional case with r dependency,

$$\hat{u}(n,j,r) = \frac{1}{2\pi r} \int_0^{2\pi} \int_0^H u(r,\varphi,x_3)\overline{\exp(in\varphi)}\phi_j(x_3)\, dx_3\, d\varphi, \qquad n \in \mathbb{Z}, j \in \mathbb{N}, 0 < r < \rho.$$

Once more for simplicity we denote the derivative with respect to the radial variable r as

$$\hat{u}'(n,j,r) := \frac{\partial \hat{u}}{\partial r}(n,j,r).$$

Lemma 3.2.6. *For $m = 3$ it holds for $u \in H^1(M_\rho)$ that*

$$\|u\|^2_{H^1(M_\rho)} \simeq \int_0^\rho \sum_{n\in\mathbb{Z}} \sum_{j=1}^{\infty} |\hat{u}'(n,j,r)|^2 r dr + \int_0^\rho \sum_{n\in\mathbb{Z}} \sum_{j=1}^{\infty} (1 + |\lambda_j|^2 + \frac{n^2}{r^2})|\hat{u}(n,j,r)|^2\, r\, dr.$$

The equivalence constants depend on ω and J. However, they can be chosen uniformly for frequencies ω in any compact subset of $\mathbb{R}_{>0}$.

Proof. Again like in dimension two, due to the fact that $H^1_W(M_\rho) \cap C^2(\overline{M_\rho})$ is a dense subset of $H^1_W(M_\rho)$ we claim the result for $u \in H^1_W(M_\rho) \cap C^2(\overline{M_\rho})$ and Lemma 3.2.3 b) states that we have for $u \in H^1_W(M_\rho) \cap C^2(\overline{M_\rho})$ the derivative (term by term) of its series expansion with respect to the vertical axis

$$\frac{\partial u}{\partial x_3}(x) = \sum_{j\in\mathbb{Z}} \hat{u}(j,\tilde{x}) \frac{\partial \phi_j}{\partial x_3}(x_3) \qquad \text{for } x \in \overline{M_\rho},$$

and converges uniformly and absolutely. Next, we use the Fourier series in (3.32), apply Parseval's identity and exploit that $\{\phi_j\}_{j\in\mathbb{N}}$ forms an orthonormal basis to see that

$$\|u\|^2_{L^2(M_\rho)} = \sum_{j=1}^{\infty} \|u(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})} = \sum_{n\in\mathbb{Z}} \sum_{j=1}^{\infty} \int_0^\rho |\hat{u}(n,j,r)|^2 r\, dr.$$

Next, we use the representation of the gradient in cylinder coordinates given by

$$\nabla u = \frac{\partial u}{\partial r} e_r + \frac{1}{r}\frac{\partial u}{\partial \varphi} e_\varphi + \frac{\partial u}{\partial x_3} e_{x_3}, \qquad \text{with } e_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{pmatrix}, e_\varphi = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{pmatrix}, e_{x_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{3.33}$$

Due to Lemma 3.2.3 we can interchange sum and derivative to see for $\nabla_{\tilde{x}} u \in H^1(\{|\tilde{x}| < \rho\})$ that

$$\|\nabla_{\tilde{x}} u\|^2_{L^2(\{|\tilde{x}|<\rho\})} = \sum_{j=1}^{\infty} \|\nabla_{\tilde{x}} u(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})}.$$

Consequently, the transformation to cylinder coordinates in (3.33) and a straightforward computation shows us like in [Kir11, A.35] that

$$\begin{aligned}\|\nabla_{\tilde{x}} u(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})} &= \left\| \frac{\partial u}{\partial r} e_r + \frac{1}{r}\frac{\partial u}{\partial \varphi} e_\varphi \right\|^2_{L^2(\{|\tilde{x}|<\rho\})} \\ &= 2\pi \sum_{n\in\mathbb{Z}} \sum_{j=1}^{\infty} \int_0^\rho \left[|\hat{u}'(n,j,r)|^2 + \left(1 + \frac{n^2}{r^2}\right) |\hat{u}(n,j,r)|^2 \right] r\, dr.\end{aligned}$$

Next we look at the derivative in the vertical variable

$$\left\| \frac{\partial u}{\partial x_3} \right\|^2_{L^2(M_\rho)} = \int_{\{|\tilde{x}|<\rho\}} \sum_{n\in\mathbb{Z}} \sum_{j,j'=1}^{\infty} \hat{u}(n,j,r)\overline{\hat{u}(n,j',r)} \int_0^H \phi_j'(x_3)\phi_{j'}'(x_3)\, dx_3\, dr.$$

Analogously, like in dimension two the variational formulation of the eigenvalue problem for the j-th eigenpair (λ_j^2, ϕ_j) in (2.15) shows like in the computation of equations (3.25-3.28) that

$$\begin{aligned}\left\| \frac{\partial u}{\partial x_3} \right\|^2_{L^2(M_\rho)} &= \int_{\{|\tilde{x}|<\rho\}} \sum_{j,j'=1}^{\infty} u(j,\tilde{x})\overline{u(j',\tilde{x})}\, d\tilde{x} \int_0^H \left(\frac{\omega^2}{c^2(x_3)} - \lambda_j^2 \right) \phi_j(x_3)\phi_{j'}(x_3)\, dx_3 \\ &= \sum_{j,j'=1}^{\infty} \int_{\{|\tilde{x}|<\rho\}} u(j,\tilde{x})\overline{u(j',\tilde{x})}\, d\tilde{x} \left[\int_0^H \frac{\omega^2}{c^2(x_3)} \phi_j \phi_{j'}\, dx_3 - \lambda_j^2 \int_0^H \phi_j \overline{\phi_{j'}}\, dx_3 \right] \\ &= \int_{\{|\tilde{x}|<\rho\}} \int_0^H \frac{\omega^2}{c^2(x_3)} \left| \sum_{j=1}^{\infty} u(j,\tilde{x})\phi_j(x_3) \right|^2 dx_3\, d\tilde{x} - \sum_{j=1}^{\infty} \lambda_j^2 \|u(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})} \\ &\leqslant \frac{\omega^2}{c_\mp^2} \|u\|^2_{L^2(M_\rho)} + \sum_{j=1}^{\infty} \lambda_j^2 \|u(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})} \\ &= \sum_{j=1}^{\infty} \left[\frac{\omega^2}{c_\mp^2} - \lambda_j^2 \right] \|u(j,\cdot)\|^2_{L^2(\{|\tilde{x}|<\rho\})} = \sum_{j=1}^{\infty} \left[\frac{\omega^2}{c_\mp^2} - \lambda_j^2 \right] \sum_{n\in\mathbb{Z}} \int_0^\rho |\hat{u}(j,n,r)|^2\, r\, dr.\end{aligned}$$

We see again by Corollary 2.2.6 and Remark 2.2.2 that the first J terms on the right-hand side might be negative. These finitely many terms can be estimated by the L^2-Norm of u, like in the two-dimensional case in the proof of Lemma 3.2.4.

Then, plugging all the results together we observe that

$$\|u\|^2_{H^1(M_\rho)} \simeq \int_0^\rho \sum_{n\in\mathbb{Z}} \sum_{j=1}^{\infty} |\hat{u}'(n,j,r)|^2 r dr + \int_0^\rho \sum_{n\in\mathbb{Z}} \sum_{j=1}^{\infty} (1 + |\lambda_j|^2 + \frac{n^2}{r^2}) |\hat{u}(n,j,r)|^2\, r\, dr.$$

□

Note that the statement of Remark 3.2.5 on norm equivalences holds for $m = 3$, too.

We next define the trace operator for continuous functions $u \in C(\overline{\Sigma_\rho})$ by $T : u \mapsto u|_{\Sigma_\rho}$. Due to [McL00] we know the trace operator T can be extended to a bounded linear operator from $H^1(M_\rho)$ into $H^{1/2}(M_\rho) \subset L^2(M_\rho)$. Using now the definition of the Fourier series, we introduce a special subspace of this trace space for functions in M_ρ. For $x = (\pm\rho, x_2) \in \mathbb{R}^2$ this special trace space adapted to $H^1_W(M_\rho)$ is denoted by

$$V_2 = \left\{ \psi \in L^2(\Sigma_\rho) : \psi|_{\Sigma_{\pm\rho}} = \sum_{j=1}^{\infty} \hat{\psi}_\pm(j)\phi_j(x_2), x_2 \in [0, H] : \sum_{j=1}^{\infty}(1 + |\lambda_j|^2)^{1/2}|\hat{\psi}_\pm(j)|^2 < \infty \right\},$$

and it holds $V_2 \subset L^2(\Sigma_\rho)$. Note that, the notation for Fourier coefficients $\hat{\psi}_\pm(j)$ indicate to which part of the boundary these coefficients are associated.

The inner product for the Hilbert space V_2 is defined by

$$(\theta, \psi)_{V_2} := \sum_{j=1}^{\infty}(1 + |\lambda_j|^2)^{1/2} \left(\hat{\theta}_+(j)\overline{\hat{\psi}_+(j)} + \hat{\theta}_-(j)\overline{\hat{\psi}_-(j)} \right) \qquad \text{for } \theta, \psi \in V_2.$$

The dual of V_2 with pivot space $L^2(\Sigma_\rho)$ is denoted as V_2' and is a Hilbert space. Its inner product is defined by

$$(\theta, \psi)_{V_2'} = \sum_{j=1}^{\infty}(1 + |\lambda_j|^2)^{-1/2} \left(\hat{\theta}_+(j)\overline{\hat{\psi}_+(j)} + \hat{\theta}_-(j)\overline{\hat{\psi}_-(j)} \right) \qquad \text{for } \theta, \psi \in V_2'.$$

Owing to the definition of the Fourier series in (3.32) in dimension three, we have for $x \in \mathbb{R}^3$ and $x = (\rho\cos\varphi, \rho\sin\varphi, x_3)$ the space

$$V_3 = \left\{ \psi \in L^2(\Sigma_\rho) : \psi(x) = \sum_{j=1}^{\infty}\sum_{n\in\mathbb{Z}} \hat{\psi}(n, j)\exp(in\varphi)\phi_j(x_3) \text{ for } (\varphi, x_3) \in (0, 2\pi) \times [0, H] : \right.$$
$$\left. \sum_{j=1}^{\infty}\sum_{n\in\mathbb{Z}}(1 + |n|^2 + |\lambda_j|^2)^{1/2}|\hat{\psi}(n, j)|^2 < \infty\} \subset L^2(\Sigma_\rho) \right\}.$$

The inner product for this Hilbert space V_3 is given by

$$(\theta, \psi)_{V_3} := 2\pi\rho \sum_{j=1}^{\infty}\sum_{n\in\mathbb{Z}}(1 + |n|^2 + |\lambda_j|^2)^{1/2}\hat{\theta}(n, j)\overline{\hat{\psi}(n, j)} \qquad \text{for } \theta, \psi \in V_3,$$

We call V_3' the dual space of V_3 with respect to $L^2(\Sigma_\rho)$ and for its inner product we write

$$(\theta, \psi)_{V_3'} = 2\pi\rho \sum_{j=1}^{\infty}\sum_{n\in\mathbb{Z}}(1 + |n|^2 + |\lambda_j|^2)^{-1/2}\hat{\theta}(n, j)\overline{\hat{\psi}(n, j)} \qquad \text{for } \theta, \psi \in V_3'.$$

Theorem 3.2.7. *The trace operator $T : H^1(M_\rho) \to V_m$, where $m = 2, 3$, is continuous and is onto.*

Proof. Again, we prove the result for smooth functions u in the dense set $H^1_W(M_\rho) \cap C^1(\overline{\Omega})$. We first treat the proof in dimension two. By Lemma 3.2.4 we obtain

$$\|u\|^2_{H^1(M_\rho)} \simeq \int_0^\rho \sum_{j=1}^{\infty} |\hat{u}'(j, x_1)|^2 \, dx_1 + \int_0^\rho \sum_{j=1}^{\infty}(1 + |\lambda_j|^2)|\hat{u}(j, x_1)|^2 \, dx_1.$$

Note that we apply techniques used in [Kir11, Appendix A.5]. The Cauchy-Schwarz inequality in

$L^2(-\rho,\rho)$ gives

$$\begin{aligned}(2\rho)^2|\hat{u}(j,\rho)|^2 =&\int_{-\rho}^{\rho}\frac{d}{dx_1}\left((x_1+\rho)^2|\hat{u}(j,x_1)|^2\right)\,dx_1\\ =&2\int_{-\rho}^{\rho}(x_1+\rho)|\hat{u}(j,x_1)|^2dx_1+2\int_{-\rho}^{\rho}(x_1+\rho)^2\,\mathrm{Re}\left[\hat{u}(n,\rho)\overline{\hat{u}'(n,\rho)}\right]dx_1\\ \le&4\rho\int_{-\rho}^{\rho}|\hat{u}(j,x_1)|^2\,dx_1\\ &+2\left(\int_{-\rho}^{\rho}(x_1+\rho)^2|\hat{u}(j,x_1)|^2\,dx_1\int_{-\rho}^{\rho}(x_1+\rho)|\hat{u}'(j,x_1)|^2\,dx_1\right)^{1/2}.\end{aligned}$$

Now, we use

$$2ab\le a^2+b^2,(x_1+\rho)^2\le 4\rho^2 \quad\text{ and }\quad (1+|\lambda_j^2|)^{1/2}\le 1+|\lambda_j|^2 \text{ for } |x_1|<\rho,$$

to obtain

$$(1+|\lambda_j|^2)^{1/2}|u(j,\rho)|^2\le C(\rho)(1+|\lambda_j|^2)\int_{-\rho}^{\rho}|\hat{u}(j,x_1)|^2\,dx_1+\int_{-\rho}^{\rho}|\hat{u}'(j,x_1)|^2\,dx_1.$$

Repeating the same computation for $-\rho$ instead of ρ and summing over $j\in\mathbb{N}$ shows that

$$\begin{aligned}\|Tu\|_{V_2}=\|u|_{\Sigma_\rho}\|_{V_2}^2=&\sum_{j\in\mathbb{N}}(1+|\lambda_j|^2)^{1/2}[|\hat{u}(j,\rho)|^2+|\hat{u}(j,-\rho)|^2]\\ =&C(\rho)\sum_{j\in\mathbb{N}}\int_{-\rho}^{\rho}(1+|\lambda_j|^2)^{1/2}[|\hat{u}(j,\rho)|^2+|\hat{u}(j,-\rho)|^2]\,dx_1\le C\|u\|_{H^1(M_\rho)},\end{aligned}$$

where the last inequality uses the norm equivalence in dimension two from Lemma 3.2.4.

Our task is now to prove the three-dimensional case. The Cauchy-Schwarz inequality in $L^2([0,\rho])$ allows us to estimate

$$\begin{aligned}\rho^2|\hat{u}(n,j,\rho)|^2=&\int_0^{\rho}\frac{d}{dr}(r^2|\hat{u}(n,j,\rho)|^2)\,dr\\ =&2\int_0^{\rho}r|\hat{u}(n,j,\rho)|^2\,dr+2\,\mathrm{Re}\int_0^{\rho}\hat{u}(n,j,\rho)\overline{u'(n,j,\rho)}r^2\,dr\\ \le&2\int_0^{\rho}r|\hat{u}(n,j,\rho)|^2\,dr+2\rho\left(\int_0^{\rho}|\hat{u}(n,j,\rho)|^2r\,dr\int_0^{\rho}|\hat{u}'(n,j,\rho)|^2r\,dr\right)^{1/2}.\end{aligned}$$

Similarly to dimension two, we now obtain

$$\begin{aligned}(1+|n|^2+|\lambda_j|^2)^{1/2}|\hat{u}(n,j,\rho)|^2\le& C(1+|n|^2+|\lambda_j|^2)\int_0^{\rho}\left[|\hat{u}(j,n,r)|^2+|\hat{u}'(j,n,r)|^2\right]r\,dr\\ \le& C\max(1,\rho^2)\int_0^{\rho}\left[\left(2+\frac{|n|^2}{r^2}+|\lambda_j|^2\right)|\hat{u}(j,n,r)|^2+|\hat{u}'(j,n,r)|^2\right]r\,dr.\end{aligned}$$

Norm equivalence of Lemma 3.2.6, summation over $j\in\mathbb{N}$ and $n\in\mathbb{Z}$ yield

$$\|Tu\|_{V_3}\le C\|u\|_{H^1(M_\rho)}.$$

Due to Lemma 3.2.4 for $m=2$ and Lemma 3.2.6 for $m=3$ and the definition of V_m we further know, that T is onto. This completes the proof. □

Owing to the fact that V_m' is the dual space to V_m for the pivot space $L^2(\Sigma_\rho)$ we write equivalently for

$$(v,u)_{V_m'\times V_m}=(v,u)_{L^2(\Sigma_\rho)}\qquad\text{for } u\in V_m \text{ and } v\in V_m', m=2,3.$$

Due to the fact that $\{\phi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis,

$$(v,u)_{L^2(\Sigma_\rho)} = \int_{\Sigma_\rho} \overline{u} v \, ds$$
$$= \int_0^H \sum_{j=1}^\infty \sum_{j'=1}^\infty \hat{v}(j,x_1)\phi_j(x_2)\overline{\hat{u}(j',x_1)}\phi_{j'}(x_2)\,dx_2 = \sum_{j=1}^\infty \hat{v}(j,x_1)\overline{\hat{u}(j,x_1)}. \tag{3.34}$$

Additionally, we use for dimension three the orthogonality of the basis $\{\exp(in\cdot)\}_{n\in\mathbb{Z}}$. Then, analogously like in dimension three we can compute that

$$(v,u)_{L^2(\Sigma_\rho)} = \int_{\Sigma_\rho} \overline{u} v \, ds$$
$$= \int_0^{2\pi}\int_0^H \sum_{n,n'\in\mathbb{Z}} \sum_{j,j'=1}^\infty \hat{v}(n,j,r)\phi_j(x_3)\exp(in\varphi)\overline{\hat{u}(n',j',r)}\phi_{j'}(x_3)\exp(-in'\varphi)\,dx_3\,d\varphi$$
$$= 2\pi\rho \sum_{n\in\mathbb{Z}}\sum_{j=1}^\infty \overline{\hat{u}(n,j,r)}\hat{v}(n,j,r). \tag{3.35}$$

This implies that

$$|(v,u)| \le \|v\|_{V_m'}\|u\|_{V_m} \qquad \text{for all } v\in V_m' \text{ and } u\in V_m, m=2,3.$$

3.3 The Exterior Dirichlet-to-Neumann Operator

In this section, we construct an exterior Dirichlet-to-Neumann map on the surface Σ_ρ that maps Dirichlet boundary values on Σ_ρ to the Neumann data on Σ_ρ of the (unique) radiating solution in $\Omega\setminus M_\rho$ to the Helmholtz equation (2.1). To analyze this exterior Dirichlet-to-Neumann map we require that Assumption 3.1.1 holds, i.e. no eigenvalue $\lambda_j^2\in\mathbb{R}$ vanishes. In consequence, we can denote a formal solution to the Helmholtz (2.1) on $\Omega\setminus M_\rho$ that satisfies the boundary conditions (2.5) and (2.6) of the form

$$u(x) = \sum_{j\in\mathbb{N}} \begin{cases} \hat{u}(j)\exp(i\lambda_j|x_1|)\phi_j(x_2) & x=\binom{x_1}{x_2}\in\Omega\setminus M_\rho, m=2,\\ \sum_{n\in\mathbb{Z}}\hat{u}(n,j)H_n^{(1)}(\lambda_j r)\exp(in\varphi)\phi_j(x_3), & x=\begin{pmatrix} r\cos\varphi\\ r\sin\varphi\\ x_3\end{pmatrix}\in\Omega\setminus M_\rho, m=3,\end{cases} \tag{3.36}$$

where $H_n^{(1)}$ denotes the Hankel function of the first kind and order n. Indeed, in dimension two the mode

$$x_1 \mapsto w_j(\tilde{x}) = \exp(i\lambda_j|x_1|),$$

satisfies the one-dimensional Helmholtz equation in $(\partial^2/\partial x_1^2+\lambda_j^2)v_j = 0$ in $\mathbb{R}\setminus\{0\}$. If $\lambda_j^2>0$ (i.e. $\lambda_j\in\mathbb{R}_{>0}$) and if $\lambda_j^2<0$ (i.e. $\lambda_j\in i\mathbb{R}_{>0}$) the corresponding modes satisfies the radiation conditions (3.3). Obviously, in dimension three the Hankel function $H_n^{(1)}$ of the first kind and order n satisfies Bessel's differential equation such that

$$\tilde{x}\mapsto w_{j,n}(\tilde{x}) = H_n^{(1)}(i\lambda_j r)\exp(in\varphi), \qquad \tilde{x}=\binom{r\cos\varphi}{r\sin\varphi},\ r>0,$$

satisfies the two-dimensional Helmholtz equation in $(\Delta_{\tilde{x}}+\lambda_j^2)w_{n,j}=0$ in $\mathbb{R}^2\setminus\{0\}$, see [CK13]. We further know that the asymptotic expansion of the Hankel function for large arguments shows that each term of u satisfies the radiation conditions (3.3) such that (3.36) defines a radiating solution.

Next, we formally obtain the normal derivative on Σ_ρ by

$$\frac{\partial u}{\partial r}(x) = i\sum_{j\in\mathbb{N}}^\infty \begin{cases} \hat{u}(j)\lambda_j\exp(i\lambda_j|x_1|)\phi_j(x_2), & x=\binom{x_1}{x_2}\in\Sigma_\rho, m=2\\ \sum_{n\in\mathbb{Z}}\lambda_j\hat{u}(n,j)H_n^{(1)'}(\lambda_j\rho)\exp(in\varphi)\phi_j(x_3), & x=\begin{pmatrix}\rho\cos\varphi\\ \rho\sin\varphi\\ x_3\end{pmatrix}\in\Sigma_\rho, m=3.\end{cases}$$

Note that in dimension two it holds that $\partial u/\partial r(x) = \pm\partial u/\partial x_1(x)$, where $x \in \Sigma_\rho$. This formula motivates the following definition.

Definition 3.3.1. *For $\psi \in V_m$ with corresponding Fourier coefficients $\{\hat{\psi}_\pm(j)\}_{j\in\mathbb{N}}$ for $m = 2$ and $\{\hat{\psi}(j,n)\}_{j\in\mathbb{N},n\in\mathbb{Z}}$ for $m = 3$ the Dirichlet-to-Neumann operator $\mathbf{\Lambda}$ is denoted by*

$$\mathbf{\Lambda} : V_m \to V_m', \psi \mapsto \sum_{j\in\mathbb{N}} \begin{cases} i\lambda_j \hat{\psi}_\pm(j)\phi_j(x_2) & x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \in \Sigma_\rho, m = 2, \\ \sum_{n\in\mathbb{Z}} \lambda_j \frac{H_n^{(1)'}(\lambda_j\rho)}{H_n^{(1)}(\lambda_j\rho)} \hat{\psi}(n,j) e^{in\varphi}\phi_j(x_3), & x = \left(\begin{smallmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ x_3 \end{smallmatrix}\right) \in \Sigma_\rho, m = 3. \end{cases} \tag{3.37}$$

The following result shows that the Dirichlet-to-Neumann operator $\mathbf{\Lambda}$ is well-defined and bounded from V_m into V_m'.

Theorem 3.3.2. *The Dirichlet-to-Neumann operator $\mathbf{\Lambda}$ from (3.37) is well-defined and bounded from V_m into V_m',*

$$\|\mathbf{\Lambda}\psi\|_{V_m'} \le C\|\psi\|_{V_m} \qquad \text{for all } \psi \in V_m,$$

where $C > 0$ is a constant.

Proof. (1) Since all eigenvalues λ_j^2 are non-zero by Assumption 3.1.1, then in dimension three each term of the series of the Dirichlet-to-Neumann map is well-defined. In the two-dimensional case this holds even in the case if $\lambda_j^2 = 0$ for some $j \in \mathbb{N}$.

(2) By the definition of the Dirichlet-to-Neumann map for dimension two, computation (3.34) and the fact $\lambda_j^2 \le Cj^2$ (see Lemma 2.2.4), we find

$$\|\mathbf{\Lambda}\psi\|_{V_2'}^2 = \sum_{j=1}^\infty (1+|\lambda_j|^2)^{-1/2} \left|\widehat{\mathbf{\Lambda}\psi}(j)\right|^2 \le \sum_{j=1}^\infty (1+|\lambda_j|^2)^{1/2} \left|\hat{\psi}_+(j) + \hat{\psi}_-(j)\right|^2 = \|\psi\|_{V_2}.$$

Now, we treat the three-dimensional case. Due to [AS64, Equation 9.1.5] we know that for $z \in \mathbb{C}, z \ne 0$ and the Hankel function $H_n^1(z)$ of the first kind and order n, that the relation $H_{-n}^1(z) = (-1)^n H_n^1(z)$ holds. In consequence, we compute

$$\frac{H_n^{(1)'}(\lambda_j\rho)}{H_n^{(1)}(\lambda_j\rho)} = \frac{H_{-n}^{(1)'}(\lambda_j\rho)}{H_{-n}^{(1)}(\lambda_j\rho)}.$$

For simplicity, we further introduce the Fourier coefficients

$$\hat{w}(n,j) := \begin{cases} \hat{\psi}(n,j), & n = 0 \\ (|\hat{\psi}(n,j)|^2 + |\hat{\psi}(-n,j)|^2)^{1/2}, & n \ne 0. \end{cases} \tag{3.38}$$

Together with the definition of the Dirichlet-to-Neumann operator, the assumption that $\psi \in V_3$ is given by $\psi = \sum_{n\in\mathbb{Z},j\in\mathbb{N}} \hat{\psi}(n,j)\exp(in\varphi)\phi_j(x_3)$ and computation (3.35) we deduce

$$\begin{aligned} \|\mathbf{\Lambda}\psi\|_{V_3'}^2 &= 2\pi\rho \sum_{j=1}^\infty \sum_{n\in\mathbb{Z}} (1+|n|^2+|\lambda_j|^2)^{-1/2} \left|\lambda_j\hat{\psi}(n,j)\frac{H_n^{(1)'}(\lambda_j\rho)}{H_n^{(1)}(\lambda_j\rho)}\right|^2 \\ &= 2\pi\rho \sum_{j=1}^\infty \sum_{n=0}^\infty (1+|n|^2+|\lambda_j|^2)^{-1/2} \hat{w}(n,j)^2 \left|\lambda_j \frac{H_n^{(1)'}(\lambda_j\rho)}{H_n^{(1)}(\lambda_j\rho)}\right|^2. \end{aligned}$$

Note again that all terms are well-defined since $\lambda_j^2 \ne 0$, for all $j \in \mathbb{N}$ by the Assumption 3.1.1. We point out that $|H_n^{(1)}(z)|$ is strictly positive for $z \in \mathbb{C}, z > 0$. Indeed, $z > 0$ holds by the definition of λ_j^2 in (2.13). Next, from the Appendix of [AGL08] we have the relation

$$\frac{H_n^{(1)'}(z)}{H_n^{(1)}(z)} = \frac{H_{|n|-1}^{(1)}(z)}{H_|n|^{(1)}(z)} - \frac{|n|}{z} \qquad \text{for } z \in \mathbb{C}, arg(z) \in (-\pi/2, \pi/2). \tag{3.39}$$

This implies

$$\|\mathbf{\Lambda}\psi\|^2_{V_3'} \le C(\rho)\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}(1+|n|^2+|\lambda_j|^2)^{-1/2}|\lambda_j|^2|\hat{w}(n,j)|^2\left(\left|\frac{H^{(1)}_{n-1}(\lambda_j\rho)}{H^{(1)}_n(\lambda_j\rho)}\right|^2+\frac{n^2}{|\lambda_j|^2\rho^2}\right).$$

We separately estimate each part of the sum on the right-hand side of the last equation. By the definition of the Fourier coefficients $\hat{w}(n,j)$ in (3.38) we obtain

$$\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}(1+|n|^2+|\lambda_j|^2)^{-1/2}|\lambda_j|^2|\hat{w}(n,j)|^2\frac{n^2}{|\lambda_j|^2\rho^2}$$
$$\le \frac{1}{\rho^2}\sum_{j=1}^{\infty}\sum_{n\in\mathbb{Z}}(1+|n|^2+|\lambda_j|^2)^{-1/2}n^2|\hat{\psi}(n,j)|^2 \le \frac{1}{\rho^2}\|\psi\|^2_{V_3}.$$

Next, by Lemma A.2 in the Appendix of [AGL08] we know that it holds

$$\left|\frac{H^{(1)}_{n-1}(z)}{H^{(1)}_n(z)}\right| \le C(\rho), \qquad \text{for } |z|\ge\rho>0, arg(z)\in(-\pi/2,\pi/2], n\in\mathbb{N}. \tag{3.40}$$

In particular, due to fact that $\lambda_j^2 \le Cj^2$ (see Lemma 2.2.4) holds, we find that

$$\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}(1+|n|^2+|\lambda_j|^2)^{-1/2}|\lambda_j|^2|\hat{w}(n,j)|^2\left|\frac{H^{(1)}_{n-1}(\lambda_j\rho)}{H^{(1)}_n(\lambda_j\rho)}\right|^2$$
$$\le C(\rho)\sum_{j=1}^{\infty}\sum_{n=0}^{\infty}(1+|n|^2+|\lambda_j|^2)^{-1/2}\lambda_j|^2|\hat{w}(n,j)|^2| \le C(\rho)\|\psi\|^2_{V_3}.$$

□

Lemma 3.3.3. *For $\psi\in V_m$, the function*

$$u(x)=\begin{cases}\sum_{j=1}^{\infty}\hat{\psi}_+(j)\frac{\exp(i\lambda_j x_1)}{\exp(i\lambda_j\rho)}\phi_j(x_2) & x_1>\rho,\\ \sum_{j=1}^{\infty}\hat{\psi}_-(j)\frac{\exp(-i\lambda_j x_1)}{\exp(-i\lambda_j\rho)}\phi_j(x_2) & x_1<-\rho,\end{cases} \qquad x=\binom{x_1}{x_2}\in\Omega\setminus M_\rho, m=2 \tag{3.41}$$

and

$$u(x)=\sum_{j=1}^{\infty}\sum_{n\in\mathbb{Z}}\hat{\psi}(n,j)\frac{H^{(1)}_n(\lambda_j r)}{H^{(1)}_n(\lambda_j\rho)}\exp(in\varphi)\phi_j(x_3), \qquad x=\begin{pmatrix}r\cos\varphi\\ r\sin\varphi\\ x_3\end{pmatrix}\in\Omega\setminus\overline{M_\rho}, m=3 \tag{3.42}$$

is the unique weak solution in $H^1_{W,\mathrm{loc}}(\Omega\setminus M_\rho)$ to the Helmholtz equation

$$\Delta u+\frac{\omega^2}{c^2(x_m)}u=0 \quad \text{in } \Omega\setminus\overline{M_\rho} \tag{3.43}$$

that satisfies boundary values for $u|_{\Sigma_\rho}=\psi$ and the waveguide boundary condition on (2.5) and (2.6) and the radiation conditions (3.3). Further u belongs to $H^1_{\mathrm{loc}}(\Omega\setminus\overline{M_\rho})$ and there is a constant $C=C(R)>0$ independent of ψ such that

$$\|u\|_{H^1_{\mathrm{loc}}(M_R\setminus\overline{M_\rho})}\le C\|\psi\|_{V_m} \qquad \text{for all } R>\rho.$$

Proof. (1) We first proof that the function u belongs to $H^1_{W,\mathrm{loc}}(\Omega\setminus M_\rho)$ for dimension two. Due to (3.41) we know that u can be written by

$$u(x)=\sum_{j\in\mathbb{N}}\hat{u}_\pm(j)\frac{\exp(i\lambda_j|x_1|)}{\exp(i\lambda_j\rho)}, \qquad \pm x_1\gtrless\pm\rho, x_2\in[0,H].$$

Next, we show that the latter series converges in $H^1(M_R \setminus M_\rho)$ for arbitrary $R > \rho$, such that $u \in H^1_{\mathrm{loc}}(\Omega \setminus \overline{M_\rho})$. By the proof of norm equivalence in dimension two of Theorem 3.2.4 we have

$$\begin{aligned}
\|u\|^2_{H^1_{W,\mathrm{loc}}(M_R\setminus M_\rho)} &\simeq \sum_{j\in\mathbb{N}} \int_{\rho<|x_1|<R} (1+|\lambda_j|^2) \left[|\hat{u}_-(j,x_1)|^2 + |\hat{u}_+(j,x_1)|^2 \right] \, dx_1 \\
&\le \sum_{j\in\mathbb{N}} (1+|\lambda_j|^2) \left| \int_{\rho<|x_1|<R} \exp(-i\lambda_j(|x_1|-\rho)) \, dx_1 \right| \left[\left|\hat{\psi}_-(j)|^2 + |\hat{\psi}_+(j)\right|^2 \right] \\
&= 2\sum_{j\in\mathbb{N}} (1+|\lambda_j|^2) \left| \frac{\exp(-2i\lambda_j\rho) - \exp(-i\lambda_j(R+\rho))}{\lambda_j} \right| \left[\left|\hat{\psi}_-(j)|^2 + |\hat{\psi}_+(j)\right|^2 \right] \\
&\le C \sum_{j\in\mathbb{N}} (1+|\lambda_j|^2)^{1/2} \left[\left|\hat{\psi}_-(j)|^2 + |\hat{\psi}_+(j)\right|^2 \right] = C\|\psi\|^2_{V_2}.
\end{aligned}$$

This finishes the proof for dimension two. Next we treat the three-dimensional case. In this part we abbreviate the domain $M_R \setminus \overline{M_\rho}$ for $R > \rho$ as $M_{\rho,R}$ and the corresponding two-dimensional domain $\{\rho < |\tilde{x}| < R\} \subset \mathbb{R}^2$ by $\tilde{M}_{\rho,R}$. For $|\tilde{x}| > \rho$, the function u from (3.42) can be written as

$$u(x) = \sum_{j\in\mathbb{N}} \hat{u}(j,\tilde{x})\phi_j(x_3) \quad \text{with } \hat{u}(j,\tilde{x}) = \sum_{n\in\mathbb{Z}} \hat{\psi}(j,n) \frac{H_n^{(1)}(\lambda_j r)}{H_n^{(1)}(\lambda_j \rho)} \exp(in\varphi), \quad \tilde{x} = \left(\begin{smallmatrix} r\cos\varphi \\ r\sin\varphi \end{smallmatrix} \right).$$

We first show that the latter series converges in $H^1(M_{\rho,R})$ for arbitrary $R > \rho$, such that $u \in H^1_{\mathrm{loc}}(\Omega \setminus \overline{M_\rho})$. For $\xi \in \mathbb{R}$ and a parameter $k^2 = 1 + \max_{j\in\mathbb{N}}(\lambda_j^2) < \infty$ we set

$$\alpha(\xi) = \begin{cases} \sqrt{k^2-\xi^2} & \text{if } k^2 \ge \xi^2, \\ i\sqrt{\xi^2-k^2} & \text{if } k^2 \le \xi^2. \end{cases}$$

Then we use the latter function to write

$$\tilde{v}_{n,\xi}(\tilde{x}) := \frac{H_n^{(1)}(r\alpha(\xi))}{H_n^{(1)}(\rho\alpha(\xi))} \exp(in\varphi) \quad \text{for } \tilde{x} = r \left(\begin{smallmatrix} \cos\varphi \\ \sin\varphi \end{smallmatrix} \right) \in \tilde{M}_{\rho,R} \text{ and } n \in \mathbb{Z}.$$

Of course, the smooth function $\tilde{v}_{n,\xi}$ belongs to $H^1(\tilde{M}_{\rho,R})$. Next, [CH07, Lemma A6] states that

$$\|\tilde{v}_{n,\xi}\|^2_{H^1(\tilde{M}_{\rho,R})} \le C(\rho,R)\,(1+n^2+\xi^2)^{1/2}, \qquad \text{for } \xi \in \mathbb{R}, n \in \mathbb{Z},$$

where $C > 0$ is independent of ξ, n. Due to $k^2 = 1 + \max_{j\in\mathbb{N}}(\lambda_j^2)$ we know that $k^2 - \lambda_j^2 > 1$ for all $j \in \mathbb{N}$. Then there is a unique positive solution $\xi_j > 0$ to $\xi_j^2 = k^2 - \lambda_j^2$. We point put that the latter equation implies

$$\alpha(\xi_j) = i\lambda_j \qquad \text{and} \qquad \xi_j^2 \le C(k)(1+|\lambda_j|^2).$$

In consequence, we estimate

$$\begin{aligned}
\|\tilde{v}_{n,\xi_j}\|^2_{H^1(\tilde{M_{\rho,R}})} &\le C(\rho,R)\,(1+n^2+\xi_j^2)^{1/2} \\
&= C(\rho,R)\,(1+n^2+k^2-\lambda_j^2)^{1/2} \\
&\le C(\rho,R,k)\,(1+n^2+|\lambda_j|^2)^{1/2}.
\end{aligned}$$

We know that $\{\exp(in\varphi)\}_{n\in\mathbb{Z}}$ and its derivative $\{in\exp(in\varphi)\}_{n\in\mathbb{Z}}$ are orthogonal on $(0,2\pi)$. Consequently, we have that the functions $\tilde{v}_{n,\xi_j}$ are orthogonal for the inner product of $H^1(\tilde{M}_{\rho,R})$. The latter estimate and Parseval's identity give us

$$\begin{aligned}\|\hat{u}(j,\cdot)\|^2_{H^1(\tilde{M}_{\rho,R})} &= \Big\|\sum_{n\in\mathbb{Z}}\hat{\psi}(j,n)\tilde{v}_{n,\xi_j}\Big\|^2_{H^1(\tilde{M}_{\rho,R})}\\ &\le \sum_{n\in\mathbb{Z}}|\hat{\psi}(j,n)|^2\|\tilde{v}_{n,\xi_j}\|^2_{H^1(\tilde{M}_{\rho,R})}\\ &\le C(\rho,R,k)\sum_{n\in\mathbb{Z}}(1+n^2+|\lambda_j|^2)^{1/2}|\hat{\psi}(j,n)|^2. \end{aligned} \tag{3.44}$$

We further know that the corresponding L^2-estimate holds as well due to Parseval's identity and the orthogonality results

$$\|\hat{u}(j,\cdot)\|^2_{L^2(\tilde{M}_{\rho,R})} = \Big\|\sum_{n\in\mathbb{Z}}\hat{\psi}(j,n)\tilde{v}_{n,\xi_j}\Big\|^2_{L^2(\tilde{M}_{\rho,R})} \le \sum_{n\in\mathbb{Z}}|\hat{\psi}(j,n)|^2\|\tilde{v}_{n,\xi_j}\|^2_{L^2(\tilde{M}_{\rho,R})}. \tag{3.45}$$

Due to the proof of norm equivalence in Theorem 3.2.6 we know that

$$\|u\|^2_{H^1(M_{\rho,R})} \le C\sum_{j\in\mathbb{N}}\left[\|\hat{u}(j,\cdot)\|^2_{H^1(\tilde{M}_{\rho,R})} + (1+|\lambda_j|^2)\|\hat{u}(j,\cdot)\|^2_{L^2(\tilde{M}_{\rho,R})}\right]. \tag{3.46}$$

Putting (3.44) - (3.46) together, we see

$$\|u\|^2_{H^1(M_{\rho,R})} \le C\sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}}\left[(1+n^2+|\lambda_j|^2)^{1/2} + (1+|\lambda_j|^2)\|\tilde{v}_{n,\xi_j}\|^2_{L^2(\tilde{M}_{\rho,R})}\right]|\hat{\psi}(j,n)|^2.$$

Applying [CH07, Lemma A2] that for $r>0$ implies that

$$\frac{\left|H_n^{(1)}(i\lambda_j r)\right|^2}{\left|H_n^{(1)}(\lambda_j\rho)\right|^2} \le 1, \text{ for all } j\in\mathbb{N},$$

we see that the L^2-norm of $\tilde{v}_{n,\xi_j}$ can be estimated by

$$\|\tilde{v}_{n,\xi_j}\|^2_{L^2(\tilde{M}_{\rho,R})} = 2\pi\int_\rho^R\left|\frac{H_n^{(1)}(\lambda_j r)}{H_n^{(1)}(\lambda_j\rho)}\right|^2\frac{dr}{r} \le \frac{2\pi}{\rho}\int_\rho^R\frac{\left|H_n^{(1)}(\lambda_j r)\right|^2}{\left|H_n^{(1)}(\lambda_j\rho)\right|^2}\,dr \le 2\pi(R-\rho) \quad \text{for all } j\in\mathbb{N}.$$

If $j>J$, (i.e., if $\lambda_j^2<0$, correspond to evanescent modes), then due to [CH07, Lemma A3] which states for $r>\rho$ that

$$\frac{\left|H_n^{(1)}(\lambda_j r)\right|^2}{\left|H_n^{(1)}(\lambda_j\rho)\right|^2} \le \exp(-(r-\rho)|\lambda_j|),$$

we further obtain

$$\|\tilde{v}_{n,\xi_j}\|^2_{L^2(\tilde{M}_{\rho,R})} \le \frac{2\pi}{\rho}\int_\rho^R e^{-(r-\rho)|\lambda_j|}\,dr \le \frac{2\pi}{|\lambda_j|}(1-\exp(-(R-\rho)|\lambda_j|)) \le \frac{4\pi}{|\lambda_j|} \le \frac{C}{(1+|\lambda_j|^2)^{1/2}}.$$

This shows that

$$\|u\|^2_{H^1(M_{\rho,R})} \le C\sum_{j\in\mathbb{N}}\sum_{n\in\mathbb{Z}}(1+n^2+|\lambda_j|^2)^{1/2}|\hat{\psi}(j,n)|^2 = C\|\psi\|^2_{V_3}.$$

(2) The function u satisfies the Helmholtz equation in $\Omega\setminus M_\rho$ in the classical sense, and for this reason also weakly. This follows by construction of the eigenfunctions ψ_j solving (2.12), since for dimension $m=2$ the modes $v_j=\exp(i\lambda_j|x_1|)$ satisfies the one-dimensional Helmholtz

equation $(\partial/\partial x_1 - \lambda_j^2)v_j = 0$ in $\{|x_1| > \rho\}$, since $\tilde{v}_{n,\xi_j}$ solves $(\Delta_{\tilde{x}} - \lambda_j^2)\tilde{v}_{n,\xi_j} = 0$ in $\{|\tilde{x}| > \rho\}$ for $m = 3$ and since the series in (3.41) and (3.42) was shown to converge in $H^1(M_{\rho,R})$. The same argument shows that u satisfies the waveguide boundary conditions and it is obvious that $u|_{\Sigma_\rho} = \psi$. For dimension two it is obvious that the corresponding mode $x \mapsto \exp(i\lambda_j|x_1|)\phi_j(x_2)$ to $\lambda_j^2 > 0$ (i.e. $\lambda_j \in \mathbb{R}_{>0}$) and to $\lambda_j^2 < 0$ (i.e. $\lambda_j \in i\mathbb{R}_{>0}$) satisfies the radiation conditions (3.3). For dimension three, well-known properties of Hankel and Kelvin functions show that $\tilde{v}_{n,\xi_j}$ is a radiating solution to the Helmholtz equation if $1 \leq j \leq J$, i.e., $\lambda_j^2 > 0$, whereas $\tilde{v}_{n,\xi_j}$ is bounded (and even exponentially decaying) if $j \geq J$, i.e., $\lambda_j^2 < 0$. We claimed that u satisfies the radiation conditions (3.3).

To show now uniqueness of u we consider a further radiating solution v to the same exterior boundary value problem and the difference $w = \sum_{j\in\mathbb{N}} w(j)\phi_j$ of u. For arbitrary $j \in \mathbb{N}$, $w(j) \in H^1_{\text{loc}}(\{|\tilde{x}| > \rho\})$ satisfies $(\partial/\partial x_1 - \lambda_j^2)v_j = 0$ in $\{|x_1| > \rho\}$ for $m = 2$ and $(\Delta_{\tilde{x}} - \lambda_j^2)\tilde{v}_{n,\xi_j} = 0$ in $\{|\tilde{x}| > \rho\}$ for $m = 3$. We further know that $w(j)$ vanishes on $\{|\tilde{x}| = \rho\}$. Moreover, if $\lambda_j^2 < 0$ the function $w(j)$ satisfies the Sommerfeld radiation conditions (the first condition in (3.3). Due to [CK13] the function $w(j)$ vanishes, too. Furthermore, if $\lambda_j^2 > 0$ the function $w(j)$ is exponentially decaying in $\tilde{x}$ and belongs in particular to $H^1(\{|\tilde{x}| > \rho\})$. By partial integration we have that

$$\int_{\{|\tilde{x}|>\rho\}} (|\nabla_{\tilde{x}} w_j|^2 + \lambda_j^2 |w_j|^2)\, d\tilde{x} = 0.$$

In consequence, w vanishes as well. This implies uniqueness of the radiating solution. □

The following Lemma gives us now L^2-coercivity of the Dirichlet-to-Neumann operator $\mathbf{\Lambda}$ when applied to $\psi \in V$ for small frequencies.

Lemma 3.3.4. *There exist constants $c > 0$ and $C > 0$ such that the Dirichlet-to-Neumann operator $\mathbf{\Lambda}$ is L^2 coercive at small frequencies. Further for $0 < \omega \leq C$ we have that*

$$-\langle \mathbf{\Lambda}\psi, \psi\rangle_{L^2(\Sigma_\rho)} \geq c\omega \|u\|^2_{L^2(\Sigma_\rho)} \qquad \text{for all } \psi \in V_m.$$

Proof. We assume ω to be such that no propagating modes exist. Indeed this assumption makes sense by the fact that we can find by Remark 2.2.2 bounds that only evanescent modes exist (i.e. $\lambda_j^2 < 0$).

We first treat the three-dimensional case. For $\psi \in V_m$ we use the relation of the Fourier coefficient, (3.38) and the fact that the eigenvectors $\{\phi_j\}_{j\in\mathbb{N}}$ form an orthonormal basis to see that

$$-\langle \mathbf{\Lambda}\psi, \psi\rangle_{L^2(\Sigma_\rho)} = -\int_{\Sigma_\rho} \overline{\psi}\Lambda\psi\, ds = -2\pi\rho \sum_{j=1}^{\infty}\sum_{n=0}^{\infty} \lambda_j \frac{H_n^{(1)'}(\lambda_j\rho)}{H_n^{(1)}(\lambda_j\rho)} |\hat{w}(n,j)|^2.$$

Next, identity (3.39) used in the proof of Theorem 3.3.2 gives us

$$-\langle \mathbf{\Lambda}\psi, \psi\rangle_{L^2(\Sigma_\rho)} = -2\pi\rho \sum_{j=1}^{\infty}\sum_{n=0}^{\infty} \lambda_j \left(\frac{H_{n-1}^{(1)}(\lambda_j\rho)}{H_n^{(1)}(\lambda_j\rho)} - \frac{n}{\lambda_j\rho} \right) |\hat{w}(n,j)|^2.$$

We point out that the argument of the Hankel function is purely imaginary by the choice of the eigenvalues $\lambda_j^2 < 0$ for all $j \in \mathbb{N}$, . By this assumption we have only evanescent modes and we can apply identity

$$\frac{H_{n-1}^{(1)}(z)}{H_n^{(1)}(z)} = i\frac{K_{n-1}^{(1)}(|z|)}{K_n^{(1)}(|z|)} \qquad \text{for } n \in \mathbb{Z}, z \in i\mathbb{R}_{>0},$$

to obtain

$$-\langle \mathbf{\Lambda}\psi, \psi\rangle_{L^2(\Sigma_\rho)} = -2\pi\rho \sum_{j=1}^{\infty}\sum_{n=0}^{\infty} \left(i\lambda_j \frac{K_{n-1}^{(1)}(|\lambda_j|\rho)}{K_n^{(1)}(|\lambda_j|\rho)} - \frac{n}{\rho} \right) |\hat{w}(n,j)|^2.$$

Due to [AGL08, Lemma A.1] we know that it holds

$$\frac{K_{-1}(t)}{K_0(t)} \geq 1, \qquad \frac{K_0(t)}{K_1(t)} \geq 1 - \frac{2}{t}, \qquad \text{and} \qquad 1 \geq \frac{K_n(t)}{K_{n+1}(t)} \geq \frac{t}{t+2n}, \qquad \text{for } n \in \mathbb{Z}, t > 0.$$

Then, together with the fact that for decreasing order $0 > \lambda_1^2 \geq \lambda_2^2 \geq ...$ it holds

$$K_0(|\lambda_j|\rho)/K_1(|\lambda_j|\rho) \geq 1 - 2/(\rho|\lambda_j|) \geq c_1,$$

we have

$$-\frac{\langle \mathbf{\Lambda}\psi, \psi\rangle_{L^2(\Sigma_\rho)}}{2\pi\rho} \geq \sum_{j=1}^{\infty} \left[|\lambda_j||w(0,j)| + c_1|\lambda_j||\hat{w}(1,j)|^2 + \sum_{n=1}^{\infty} |\lambda_j| \left(\frac{|\lambda_j|\rho}{|\lambda_j|\rho + 2n} + \frac{n}{|\lambda_j|\rho} \right) \right] |\hat{w}(n,j)|^2.$$

Using the binomial formula for ρ to be chosen large enough and that $n \mapsto n(|\lambda_j|\rho + 2n)$ and $j \mapsto \lambda_1/(\lambda_1\rho + 2)$ increase in n and j, respectively, we see for $n \in \mathbb{N}$ that

$$|\lambda_j| \left[\frac{|\lambda_j|\rho}{|\lambda_j|\rho + 2n} + \frac{n}{|\lambda_j|\rho} \right] \geq 2|\lambda_j| \left[\frac{n}{|\lambda_j|\rho + 2n} \right]^{1/2} \geq 2|\lambda_j| \left[\frac{1}{|\lambda_j|\rho + 2} \right]^{1/2} \geq 2|\lambda_j|^{1/2} \left[\frac{|\lambda_1|}{|\lambda_1|\rho + 2} \right]^{1/2} > 0.$$

Now, by Theorem 2.2.3 and Remark 2.2.2 we have for constant sound speed c_+ that $\omega/c(x_m) \geq \omega/c_+$ that

$$\lambda_j^2 \geq (\lambda_j^-)^2 = \left(\frac{\omega^2}{c_+^2} - \frac{\pi(2j-1)}{2H} \right)^2 = \frac{\omega^2}{c_+^2} \left[1 - \left(\frac{\pi(2j-1)c_+}{2H\omega} \right)^2 \right] \geq \frac{3\,\omega^2}{c_+^2}, \qquad \text{for } j \in \mathbb{N}. \quad (3.47)$$

Note that $(\lambda_j^-)^2$ denotes the jth-eigenvalues of the orthonormal eigensystem for the constant sound speed c_+. Now, the latter estimate shows that for $\omega > 0$ small enough yield that $\lambda_j^2 \geq c_*\omega^2$ for all $j \in \mathbb{N}$ and some constant $c_* = 3/c_+^2$. Monotonicity of the square root function directly implies for $1 > \omega > 0$ that

$$|\lambda_j| \geq c_*^{1/2}\omega \qquad \text{and} \qquad |\lambda_j|^{1/2} \geq c_*^{1/4}\omega^{1/2} \geq c_*^{1/4}\omega.$$

Moreover, we choose $1 > \omega > 0$ such that

$$\min\left(c_*^{-1/2}, \frac{\pi c_+}{2H} \right) > \omega,$$

then $1 > \lambda_j^2 > 0$ due to (3.47), i.e., $|\lambda_j|^{1/2} > |\lambda_j|$. Consequently, for $\psi \in \mathcal{V}_3$ and some constant $c > 0$ we obtain by Plancherel's identity that

$$-\langle \mathbf{\Lambda}\psi, \psi\rangle_{L^2(\Sigma_\rho)} \geq c \sum_{j=1}^{\infty} \left[|\lambda_j||w(j,0)|^2 + |\lambda_j||\hat{w}(j,1)|^2 + \sqrt{|\lambda_j|} \sum_{n=2}^{\infty} |\hat{w}(j,n)|^2 \right] \geq c\omega \sum_{j=1}^{\infty} \sum_{n\in\mathbb{Z}} |\hat{\psi}(n,j)|^2.$$

Now, we treat the two-dimensional case. We recall that only evanescent modes exist. Then, again for some constants $C > 0$ such that $C \geq \omega > 0$ we have that $0 > \lambda_1^2 \geq \lambda_2^2 \geq ....$ Like in the three-dimensional case by $\omega/c(x_m) \geq \omega/c_+$ we can use estimates for the eigenvalues λ_j^2. The orthogonality of the eigenvectors $\{\phi_j\}_{j\in\mathbb{N}}$ implies that

$$\begin{aligned} -\langle \mathbf{\Lambda}\psi, \psi\rangle_{L^2(\Sigma_\rho)} &= \sum_{j\in\mathbb{N}} |\lambda_j| \left[|\hat{\psi}_+(j)|^2 + |\hat{\psi}_-(j)|^2 \right] \\ &\geq \frac{1}{1+1/c_1} \sum_{j=1}^{\infty} (1 + |\lambda_j|^2)^{1/2} \left[|\hat{\psi}_+(j)|^2 + |\hat{\psi}_-(j)|^2 \right] = c\|\psi\|_{V_2}^2. \end{aligned}$$

This completes the proof. □

Lemma 3.3.5. *We assume $\omega > 0$ such that $J(\omega, c, H)$ propagating eigenmodes exist. Then there exists some constants $C > 0$ such that for all $\psi \in V_m$*

$$-Re\langle \mathbf{\Lambda}\psi, \psi\rangle_{L^2(\Sigma_\rho)} \geq -C \sum_{j=1}^{J} \begin{cases} |\hat{\psi}_+(j)|^2 + |\hat{\psi}_-(j)|^2 & \text{for } m = 2, \\ 2\pi\rho \sum_{n=0}^{\infty} |\hat{\psi}(n,j)|^2, & \text{for } m = 3, \end{cases}$$

and

$$-Re\langle \mathbf{\Lambda}\psi, \psi\rangle_{L^2(\Sigma_\rho)} \geq -C\|u\|^2_{L^2(\Sigma_\rho)}.$$

Proof. Again, we treat first the three-dimensional case. By the identity (3.39) for Hankel functions used in the proof of Theorem 3.3.2, since $\{\phi_j\}_{j\in\mathbb{N}}$ denotes an orthonormal basis and the auxiliary Fourier coefficients $\hat{w}(n,j)$ in (3.38), we obtain that

$$-\mathrm{Re}\langle \mathbf{\Lambda}\psi, \psi\rangle_{L^2(\Sigma_\rho)} = -2\pi\rho \mathrm{Re} \sum_{j=1}^{\infty}\sum_{n=0}^{\infty} \lambda_j \left(\frac{H^{(1)'}_{n-1}(\lambda_j\rho)}{H^{(1)}_n(\lambda_j\rho)} - \frac{n}{\lambda_j\rho}\right) |\hat{w}(n,j)|^2.$$

Furthermore, since for $j > J$ all eigenvalues λ_j are negative (obviously, $\lambda_j > 0$ for $1 \leq j \leq J$), the arguments of the proof of Lemma 3.3.4 show that for $j > J$ all terms in the series are positive and real. Consequently, we obtain

$$\begin{aligned} -\mathrm{Re}\langle \mathbf{\Lambda}\psi, \psi\rangle_{L^2(\Sigma_\rho)} &\geq -2\pi\rho\,\mathrm{Re}\left(\sum_{j=1}^{J}\sum_{n=0}^{\infty} \lambda_j \left(\frac{H^{(1)'}_{n-1}(\lambda_j\rho)}{H^{(1)}_n(\lambda_j\rho)} - \frac{n}{\lambda_j\rho}\right) |\hat{w}(n,j)|^2 \right) \\ &\geq -2\pi\rho \sum_{j=1}^{J}\sum_{n=0}^{\infty} \lambda_j \left| \frac{H^{(1)'}_{n-1}(\lambda_j\rho)}{H^{(1)}_n(\lambda_j\rho)} \right| |\hat{w}(n,j)|^2. \end{aligned}$$

Since the finite set of numbers $\{\lambda_j\rho\}_{j=1}^J \subset \mathbb{R}$ is bounded away from zero, estimate (3.40) implies that

$$\left| \frac{H^{(1)'}_{n-1}(\lambda_j\rho)}{H^{(1)}_n(\lambda_j\rho)} \right| \leq C(\omega,\rho,H) \qquad \text{for } j = 1, ..., J.$$

Therefore, we finally deduce

$$-\mathrm{Re}\langle \mathbf{\Lambda}\psi, \psi\rangle_{L^2(\Sigma_\rho)} \geq -C\sum_{j=1}^{J}\sum_{n=0}^{\infty} |\hat{w}(n,j)|^2 = -C\sum_{j=1}^{J}\sum_{n\in\mathbb{Z}} |\hat{\psi}(n,j)|^2 \geq -C\|u\|^2_{L^2(\Sigma_\rho)}.$$

The proof for dimension two uses the same techniques and follows analogously.

□

To be able to apply analytic Fredholm theory when establishing existence theory for the scattering problem (3.3) and (3.10) we finally show that $\mathbf{\Lambda} = \mathbf{\Lambda}_\omega$ depends analytically (i.e., holomorphically) on the frequency ω.

We point out that since the sound speed depends on the depth of the ocean, we have analytic dependency of the eigenvalue λ_j^2, for $j \in \mathbb{N}$ on the angular frequency ω and not directly on the wave number like in the well-known theory of [AGL08].

Lemma 3.3.6. *For all $\omega_* > 0$ such that $\lambda_j(\omega_*) \neq 0$ for $j \in \mathbb{N}$ and for all $\omega^* > 0$ small enough that the assumptions of Lemma 3.3.4 hold, there is an open connected set $U \subset \mathbb{C}$ containing ω^* and ω_* such that $\omega \mapsto \mathbf{\Lambda}_\omega$ has analytically dependency on the frequency $\omega \in U$.*

Proof. Theorem 8.12 b) from [Muj85,] gives us the equivalence of weak and strong analyticity of a linear bounded operator and the operator $\mathbf{\Lambda}$ maps into the dual space V_m'.

Thus it is hence sufficient to show for dimension two that

$$\langle \mathbf{\Lambda}v, \psi\rangle_{L^2(\Sigma_\rho)} = \sum_{j\in\mathbb{N}} |\lambda_j| \left[\hat{v}_+(j)\overline{\hat{\psi}_+(j)} + \hat{v}_-(j)\overline{\hat{\psi}_-(j)} \right] \qquad \text{for } v, \psi \in V_2',$$

and for dimension three that

$$\langle \mathbf{\Lambda} v, \psi \rangle_{L^2(\Sigma_\rho)} = 2\pi\rho \sum_{j=1}^{\infty} \lambda_j \sum_{n\in\mathbb{Z}}^{\infty} \frac{H_n^{(1)'}(\lambda_j(\omega)\rho)}{H_n^{(1)}(\lambda_j(\omega)\rho)} \hat{v}(n,j)\overline{\hat{\psi}(n,j)} \tag{3.48}$$

$$= 2\pi\rho \sum_{j=1}^{\infty} \lambda_j \sum_{n\in\mathbb{Z}}^{\infty} \left[\frac{H_{n-1}^{(1)'}(\lambda_j(\omega)\rho)}{H_n^{(1)}(\lambda_j(\omega)\rho)} - \frac{n}{\lambda_j(\omega)\rho} \right] \hat{v}(n,j)\overline{\hat{\psi}(n,j)} \qquad \text{for } v, \psi \in V_3',$$

is a holomorphic function of ω in an open connect $U \in \mathbb{C}$ that satisfies the properties claimed in the lemma. From Lemma 2.2.8 we know that for the index functions $l_j, j \in \mathbb{N}$ all eigenvalues $\omega \mapsto \lambda^2_{l_j(\omega)}(\omega)$ can be extended to holomorphic functions in some open neighborhood U_0 of $\mathbb{R}_{>0}$. We set $\delta_1 > 0$ such that $U_1 = \{z \in U_0,\, 0 \le \mathrm{Re}(z) \le \omega^* + 1,\, |\mathrm{Im}(z)| \le \delta_1\} \subset U$ contains ω_* and ω^*, is compact and connected. By Theorem 2.2.8, we know that

$$K_0 = \{\omega \in U_1 : \text{there is } j \in \mathbb{N} \text{ such that } \lambda_j^2(\omega) = 0\} < \infty.$$

and reducing now the parameter δ_1 we can assume without loss of generality that K_0 contains merely real numbers. We recall that the square root function $z \mapsto z^{1/2}$ that was defined for complex numbers via a branch cut at the positive real axis is holomorphic in the slit complex plane $\mathbb{C}\backslash i\mathbb{R}_{\ge 0}$. The roots $\omega \mapsto \lambda_{\ell_j(\omega)}(\omega)$ are hence holomorphic functions in the set $U_2 := \{z \in U_1 : \mathrm{Im}\, z < 0 \text{ if } \omega \in K_0\}$. We moreover restrict this set by defining the open set $U_3 := \{z \in U_2,\, B(z,\delta_2) \subset U_2\}$ for a parameter $\delta_2 > 0$. If δ_2 is small enough the open set U_3 is connected and contains ω^* and ω_*.

We treat now the more challenging three-dimensional case first. Recall that the Hankel function $z \mapsto H_n^{(1)}(z)$ and its derivative $z \mapsto H_n^{(1)'}(z)$ are holomorphic in the domain $\{z \in \mathbb{C},\, z \neq 0,\, -\pi/2 < \arg(z) < \pi\}$. Furthermore, the fraction $z \mapsto H_n^{(1)'}(z)/H_n^{(1)}(z)$ is holomorphic for $z \neq 0$ and $\arg(z) \in [0,\pi)$ since $z \mapsto H_n^{(1)}(z)$ does not possess zeros in this quadrant, too.

Moreover, due to the paragraph on complex zeros of the Hankel function in [AS64, page. 373/374] we know that an infinite number of zeros of $z \mapsto H_n^{(1)}(z)$ in the lower complex half-plane is contained in the domain $\{-\pi < \arg(z) \le -\pi/2\}$, while at most n zeros are contained in $-\pi/2 < \arg(z) \le 0$.

Then [AS64, page 374] or [CS82, Equation 2.8] state that these finitely many zeros lie in the quadrant $-\pi/2 < \arg(z) \le -\epsilon$ for some $\epsilon > 0$, independent of n and we have $z \mapsto H_n^{(1)'}(z)/H_n^{(1)}(z)$ is holomorphic in $\{z \neq 0,\, \arg(z) \in (-\epsilon, \pi+\epsilon)\}$. Since the numbers $i\lambda_j$ are either purely imaginary with positive imaginary part or positive, again reducing the parameter $\delta_1 > 0$ for the construction of $U_{1,2,3}$ claims that the function $\omega \mapsto H_n^{(1)'}(i\lambda_{\ell_j(\omega)}(\omega)\rho)/H_n^{(1)}(i\lambda_{\ell_j(\omega)}(\omega)\rho)$ is holomorphic for $\omega \in U_3$. In consequence, each term in the series in (3.48) is holomorphic in U_3 and can be developed locally into a power series in the frequency ω.

We now state the holomorphy of the entire series by uniform and absolute convergence of the series in the following. We set

$$g_j(\omega) = \sum_{n\in\mathbb{Z}} \frac{H_n^{(1)'}(\lambda_j(\omega)\rho)}{H_n^{(1)}(\lambda_j(\omega)\rho)} \hat{v}(n,j)\hat{\psi}(n,j). \tag{3.49}$$

Again, identity (3.39) of the proof of Theorem 3.3.2 states

$$g_j(\omega) = \lambda_j(\omega) \sum_{n\in\mathbb{Z}} \left[\frac{H_{n-1}^{(1)}(\lambda_j(\omega)\rho)}{H_n^{(1)}(\lambda_j(\omega)\rho)} - \frac{n}{\lambda_j(\omega)\rho} \right] \hat{v}(j,n)\overline{\hat{\psi}(j,n)}$$

$$= \lambda_j(\omega) \sum_{n\in\mathbb{Z}} \frac{H_{n-1}^{(1)}(\lambda_j(\omega)\rho)}{H_n^{(1)}(\lambda_j(\omega)\rho)} \hat{v}(j,n)\overline{\hat{\psi}(j,n)} - R_j(u,v),$$

where

$$R_j(u,v) := \sum_{n\in\mathbb{Z}} \frac{n}{i\rho} \hat{v}(j,n)\overline{\hat{\psi}(j,n)},$$

is a bounded sesquilinear form on V_3 independent of ω. Due to [AGL08, Lemma A.2 & (A10)], since for all $j > J_*(\omega^* + 1, c, H)$ and all $\omega \in U_3 \cap \mathbb{R}$ it holds that $i\lambda_j(\omega) \in i\mathbb{R}_{>0}$, we have

$$\left|\frac{H^{(1)}_{n-1}(\lambda_j(\omega)\rho)}{H^{(1)}_n(\lambda_j(\omega)\rho)}\right| = \left|\frac{K_{n-1}(\lambda_j(\omega)\rho)}{K_n(\lambda_j(\omega)\rho)}\right| \le C \qquad \text{for } \omega \in U_3,\ n \in \mathbb{Z}.$$

Next,the asymptotic expansion of the Hankel functions for large orders in [AS64, Equation 9.3.1] gives us that, for $1 \le j \le J_*(\omega^* + 1, c, H)$ there is a constant $C > 0$ such that the last bound holds uniformly valid for all $j \in \mathbb{N}$. In consequence, we see

$$|g_j(\omega)| \le \sum_{n\in\mathbb{Z}} \left|\frac{H^{(1)'}_{n-1}(\lambda_j\rho)}{H^{(1)}_n(\lambda_j\rho)}\right| |\hat{v}(j,n)\hat{\psi}(j,n)| \le \sum_{n\in\mathbb{Z}} \left(C|\lambda_j(\omega)| + \frac{n}{\rho}\right) |\hat{v}(j,n)\hat{\psi}(j,n)| \le C\|v\|_{V_3}\|\psi\|_{V_3},$$

since $\omega \mapsto \lambda_j(\omega)$ is holomorphic on U_0 and hence bounded on $U_3 \Subset U_0$. In particular, the series (3.49) converges absolutely and uniformly for each $\omega \in U_3$. We further know, since each series term in (3.49) depends analytic on ω each term can locally be represented by its convergent Taylor series with coefficients $d^{(j)}_l(v, \psi)$,

$$g_j(\omega) = \sum_{n\in\mathbb{Z}} \sum_{l=0}^{\infty} d^{(j)}_l(v,\psi)(\omega - \omega^*)^l, \qquad \omega \in U_3,$$

where the $d^j_l(u, v)$ are bounded sesquilinear forms on $V_3 \times V_3$ and the series absolutely convergent in U_3. The limits in n and l can be interchanged, hence the series in $n \in \mathbb{Z}$ converges uniformly. Then, g_j has a convergent Taylor expansion as well and is hence a holomorphic function of $\omega \in U_3$. In consequence, we have that

$$\sum_{l\in\mathbb{N}} \frac{H^{(1)'}_{n-1}(\lambda_j\rho)}{H^{(1)}_n(\lambda_j\rho)} \hat{u}(n,j)\hat{v}(n,j) = \sum_{l=0}^{\infty} d^j_l(u,v)(\omega - \omega^*)^l, \qquad \omega \in U_3.$$

Then analogously, like in the proof of Lemma 3.3.2 it follows that the series in (3.48) are absolutely convergent and we arrive at

$$\langle \mathbf{\Lambda} v, \psi\rangle_{L^2(\Sigma_\rho)} = \sum_{n=0}^{\infty}\sum_{l=0}^{\infty} (\omega - \omega^*)^{n+l} \sum_{j=1}^{\infty} c^j_l d^j_n(v,\psi) + \sum_{j=1}^{\infty} A_j(v,\psi).$$

Since by [Kat95, Chapter 7.4] it follows that the coefficients c^j_l decrease sufficiently fast, indeed the Taylor series expansion of g_j can again be interchanged with the series in $j \in \mathbb{N}$.

The two-dimensional case follows analogously,

$$\langle \mathbf{\Lambda} v, \psi\rangle_{L^2(\Sigma_\rho)} = \sum_{j\in\mathbb{N}} |\lambda_j| \left[\hat{v}_+(j)\overline{\hat{\psi}_+(j)} + \hat{v}_-(j)\overline{\hat{\psi}_-(j)}\right],$$

have an analytically expansion. This finally implies the claim of the lemma.

□

3.4 Existence and Uniqueness of the Solution to Scattering in the Waveguide

In this section we use all prepared tools to provide existence theory for weak solutions of the waveguide scattering problem (3.3), (3.10) and (3.11). Plugging these tools together is rather standard and can be found for constant background sound speed in [AGL08].

We assume that the incident field $u^i \in H^1_{W,\text{loc}}(\Omega)$ solves the variational formulation of the unperturbed Helmholtz equation (3.6) weakly. We moreover assume that $u \in H^1_{W,\text{loc}}(\Omega)$ solves (3.10) for all $v \in H^1_W$ with compact support, such that $u^s = u - u^i$ is radiating solution. We want show now that

$$\Delta u + \frac{\omega^2}{c^2(x_m)} n^2 u = 0 \qquad \text{in } L^2(M_R), \text{ for every } R > 0.$$

Since the boundary of Ω is flat (e.g. $n^2\omega^2/c^2$ is bounded) and $\Delta u \in L^2_{loc}(\Omega)$, we can apply well-known elliptic regularity results from [McL00, Ch. 4] to obtain $u \in H^2_{\text{loc}}(\Omega)$. Furthermore, by assumption that the scattering field $u^s = u - u^i$ satisfies radiation conditions, we know by Theorem 3.3.3 that it holds

$$\left.\frac{\partial u^s}{\partial \nu}\right|_{\Sigma_\rho} = \mathbf{\Lambda}(u^s|_{\Sigma_\rho}) \qquad \text{in } V_m',$$

where ν defines again the exterior unit normal vector to M_ρ. Using the assumption that it holds $u^s = u - u^i$ and by the normal derivative of $u = u^i + u^s$ on Σ_ρ we deduce

$$\frac{\partial u}{\partial \nu} = \frac{\partial u^i}{\partial \nu} + \frac{\partial u^s}{\partial \nu} = \frac{\partial u^i}{\partial \nu} + \mathbf{\Lambda}\left((u - u^i)|_{\Sigma_\rho} \right) \qquad \text{in } V_m'. \tag{3.50}$$

Next, we multiply the Helmholtz equation with $v \in H^1_W(M_\rho)$ and formally integrate by parts to obtain that

$$\begin{aligned}
&\int_{M_\rho} \left[\Delta u + \frac{\omega^2}{c^2(x_m)} n^2 u \right] \overline{v}\, dx \\
&\quad = \int_{\Sigma_\rho} \frac{\partial u}{\partial \nu} \overline{v}\, ds + \int_{\Gamma_{0,\rho}} \frac{\partial u}{\partial \nu} \overline{v}\, ds + \int_{\Gamma_{H,\rho}} \frac{\partial u}{\partial \nu} \overline{v}\, ds - \int_{M_\rho} \nabla u \cdot \nabla \overline{v}\, dx + \int_{M_\rho} \frac{\omega^2}{c^2(x_m)} n^2 u \overline{v}\, dx = 0
\end{aligned}$$

for all $v \in H^1_W(M_\rho)$, where ν defines the unit normal vector corresponding normal to M_ρ. Note that if we assume Dirichlet or Neumann boundary conditions on the obstacle D, indeed the expression holds, since the boundary term of D vanishes, too. By the definition of the boundary terms $\Gamma_{H,\rho}$ and $\Gamma_{0,\rho}$ we finally obtain

$$\int_{M_\rho} \nabla u \cdot \nabla \overline{v}\, dx - \int_{M_\rho} \frac{\omega^2}{c^2(x_m)} n^2 u \overline{v}\, dx = \int_{\Sigma_\rho} \frac{\partial u}{\partial \nu} \overline{v}\, ds \qquad \text{for all } v \in H^1_W(M_\rho).$$

Next, by the definition of the normal derivative of $u = u^i + u^s$ on Σ_ρ in (3.50) we deduce for the right-hand side that

$$\begin{aligned}
\int_{\Sigma_\rho} \frac{\partial u}{\partial \nu} \overline{v}\, ds &= \int_{\Sigma_\rho} \frac{\partial}{\partial \nu} (u - u^i) \overline{v}\, ds + \int_{\Sigma_\rho} \frac{\partial u^i}{\partial \nu} \overline{v}\, ds \\
&= \int_{\Sigma_\rho} \mathbf{\Lambda}(u) \overline{v}\, ds + \int_{\Sigma_\rho} \left[\frac{\partial u^i}{\partial \nu} - \mathbf{\Lambda}(u^i) \right] \overline{v}\, ds.
\end{aligned}$$

Then, the variational formulation of the waveguide problem (3.3), (3.10) and (3.11) is: Find $u \in H^1_W(M_\rho)$ solving

$$B_\omega(u, v) := \int_{M_\rho} \left[\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)} n^2 u\, \overline{v} \right] dx - \int_{\Sigma_\rho} \mathbf{\Lambda}(u)\, \overline{v}\, ds = F(v) \quad \text{for all } v \in H^1_W(M_\rho), \tag{3.51}$$

where

$$F(v) = \int_{\Sigma_\rho} \left[\frac{\partial u^i}{\partial \nu} - \mathbf{\Lambda}(u^i) \right] \overline{v}\, ds \qquad \text{for all } v \in H^1_W(M_\rho). \tag{3.52}$$

We point out that (3.51) can also be considered for arbitrary continuous anti-linear forms $F : H^1_W(M_\rho) \to \mathbb{C}$ that can be used to tackle source problems instead of scattering problems.

To use Fredholm theory, proving existence and uniqueness of solutions, we first show that the sesquilinear form $B_\omega(u,v)$ for $u,v \in H^1_W(M_\rho)$ defined in (3.51) and the anti-linear form F defined in (3.52) are bounded on $H^1_W(M_\rho)$. We further show that $B_\omega(u,v)$ satisfies a Gårding inequality.

Lemma 3.4.1. *a) The sesquilinear form B_ω defined in (3.51) and the anti-linear form F defined in (3.52) are bounded and B_ω satisfies a Gårding inequality.*

This further implies that the Fredholm alternative holds: Whenever the variational problem (3.51) for $u^i = 0$ possesses only the trivial solution (i.e. for $F = 0$),

$$\int_{M_\rho} \left[\nabla u^i \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)} n^2 u\,\overline{v}\right] dx - \int_{\Sigma_\rho} \mathbf{\Lambda}(u^i)\,\overline{v}\,ds = 0, \qquad \text{for all } v \in H^1_W(M_\rho), \tag{3.53}$$

existence and uniqueness of solution in $H^1_W(M_\rho)$ holds for any $F:\, H^1_W(M_\rho) \to \mathbb{C}$ in (3.51).

b) There is $\omega_0 > 0$ such that the variational problem (3.51),

$$\int_{M_\rho} \left[\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)} n^2 u\,\overline{v}\right] dx - \int_{\Sigma_\rho} \mathbf{\Lambda}(u)\,\overline{v}\,ds = F(v), \qquad \text{for all } v \in H^1_W(M_\rho),$$

is uniquely solvable for all incident fields u^i and for all frequencies $\omega \in (0,\omega_0)$.

c) The variational problem (3.51) is uniquely solvable for all $F\,:\, H^1_W(M_\rho) \to \mathbb{C}$ and all frequencies $\omega > 0$ except possibly for a discrete set of exceptional frequencies $\{\omega_\ell\}_{\ell=1}^{L_*} \subset \mathbb{R}_{>\omega_0}$, where $L_* \in \mathbb{N} \cup \{+\infty\}$. If it holds $L_* = \infty$, then $\omega_\ell \to \infty$ for $\ell \to \infty$.

Proof. a) (1) Due to definition of B_ω in (3.51), by the assumption that the background sound speed is bounded (see estimate (2.4)) and due to the Cauchy-Schwarz inequality, one computes that

$$|B_\omega(u,v)| \le \left[1 + \frac{\omega^2}{c_-^2} + \|\mathbf{\Lambda}\|_{V_m \to V_m'}\right] \|u\|_{H^1(M_\rho)} \|v\|_{H^1(M_\rho)}, \quad u,v \in H^1_W(M_\rho).$$

By Lemma 3.3.2 the Dirichlet-to-Neumann operator is bounded and then $B_\omega(u,v)$ is bounded, too. Next, again that the background sound speed is bounded, due to the trace estimate

$$\left\|\frac{\partial u^i}{\partial \nu}\right\|_{H^{-1/2}(\Sigma_\rho)} \le \|\mathrm{div}\nabla u^i\|_{L^2(M_\rho)} + \|\nabla u^i\|_{L^2(M_\rho)} \le C\|u^i\|_{H^1(M_\rho)},$$

and the trace estimate shown in Theorem 3.2.7, we find that the boundedness of F on $H^1_W(M_\rho)$ follows from

$$|F(v)| \le \left[C\|u^i\|_{H^1(M_\rho)} + \|\mathbf{\Lambda}\|_{V_m \to V_m'} \|u^i\|_{H^1(M_\rho)}\right] \|v\|_{H^1(M_\rho)}, \quad v \in H^1_W(M_\rho).$$

(2) The Gårding inequality for small frequencies follows from the lower bound of $\mathbf{\Lambda}$ at arbitrary frequencies. Due to the fact that the background sound speed is bounded $c_- \le c(x_m)$, we see

$$Re(B_\omega(u,u)) \ge \|u\|^2_{H^1_W(M_\rho)} - \left(\frac{\omega^2}{c_-^2}(1 + \|q\|_{L^\infty(M_\rho)}) + 1\right) \|u\|^2_{L^2(M_\rho)} - Re\Big(\int_{\Sigma_\rho} \mathbf{\Lambda}(u)\,\overline{v}\,ds\Big).$$

for $u \in H^1_W(M_\rho)$. Using Lemma 3.3.5 which states for a constant $C > 0$ that

$$-Re\left(\int_{\Sigma_\rho} \mathbf{\Lambda} u\,\overline{v}\,ds\right) \ge -C\|u\|_{L^2(\Sigma_\rho)},$$

to show that

$$Re(B_\omega(u,u)) \ge \|u\|^2_{H^1_W(M_\rho)} - \left(\frac{\omega^2}{c_-^2}(1 + \|q\|_{L^\infty(M_\rho)}) + 1\right) \|u\|^2_{L^2(M_\rho)} - C\|u\|_{L^2(\Sigma_\rho)},$$

for $u \in H^1_W(M_\rho)$ and where $C > 0$. Since the embedding of $H^1_W(M_\rho)$ in $L^2(M_\rho)$ is compact and since further the trace operator from $H^1_W(M_\rho)$ into $L^2(\Sigma_\rho)$ is compact, due to the compact embedding of $H^{1/2}(\Sigma_\rho)$ in $L^2(\Sigma_\rho)$ the latter estimate is indeed a Gårding inequality for the form B_ω. Then, the variational problem (3.51) is Fredholm of index zero. In particular, uniqueness of solution implies existence of solution together with the continuous dependence of the solution on the right-hand side F.

b) We first use the Lemma 3.3.4, which formulates a weak coercivity result for the Dirichlet-to-Neumann operator $\mathbf{\Lambda}$ for small frequencies, and the assumption that the background sound speed is bounded, to obtain

$$\mathrm{Re}(B_\omega(u,u)) \geq \|\nabla u\|^2_{L^2(M_\rho)} - \frac{\omega^2}{c_-^2}(1 + \|q\|_{L^\infty(M_\rho)})\|u\|^2_{L^2(M_\rho)} + c\omega\|u\|^2_{L^2(\Sigma_\rho)}, \quad u \in H^1_W(M_\rho).$$

Due to the Poincaré's inequality we know that

$$\|u\|^2_{L^2(M_\rho)} \leq \frac{H^2}{2}\|\nabla u\|^2_{L^2(M_\rho)^m} \text{ for all } u \in H^1_W(M_\rho).$$

This hence yields

$$\mathrm{Re}(B_\omega(u,u)) \geq \frac{1}{2}\|\nabla u\|^2_{L^2(M_\rho)} + \frac{1}{H^2}\|u\|^2_{L^2(M_\rho)} - \frac{\omega^2}{c_-^2}(1 + \|q\|_{L^\infty(M_\rho)})\|u\|^2_{L^2(M_\rho)}.$$

For small $\omega > 0$ we see immediately that B_ω is coercive on $H^1_W(\Omega)$. Then using Lax-Milgram Lemma to obtain that (3.51) is uniquely solvable for any $F : H^1_W(M_\rho) \to \mathbb{C}$. This finishes this part of the proof.

c) Due to part b) we know that (3.51) is uniquely solvable for $0 < \omega < \omega_0$. For $\omega \geq \omega_0$ we further use that the Dirichlet-to-Neumann operator $\mathbf{\Lambda}$ depends analytically on ω. We fix an arbitrary $\omega^* > 0$ such that $\lambda_j^2(\omega^*) \neq 0$ for all $j \in \mathbb{N}$ and some $\omega_* \in (0, \omega_0)$, Lemma 3.3.6 hence yields that there is an open connected set $U \subset \mathbb{C}$ containing ω_* and ω^* such that $\omega \mapsto \mathbf{\Lambda}_\omega$ has analytically dependency on the frequency $\omega \in U$. In particular, the entire sesquilinear form B_ω depends on the frequency $\omega \in U$, too. Now since part b) holds, for $\omega_* \in U$ the variational problem (3.51) is uniquely solvable. Furthermore, we see that except (possibly) for a countable sequence of exceptional frequencies without accumulation point in U analytic Fredholm theory holds and we know that (3.51) is uniquely solvable for all $\omega \in U$. In particular, there is at most a countable set of real frequencies without finite accumulation point where uniqueness of solution fails. This ends the proof. □

Remark 3.4.2. *Analytic Fredholm theory is not able to prove uniqueness of solution for those frequencies, where some eigenvalue $\lambda_j^2(\omega)$ vanishes; this does however not imply that uniqueness of solution does indeed fail at those frequencies, compare [AGL08].*

Theorem 3.4.3. *Consider that Assumption 3.1.1 holds. If the variational problem (3.51) is uniquely solvable for any incident fields u^i, then any solution $u \in H^1_W(M_\rho)$ can be extended to a weak solution $\tilde{u} \in H^1_{\mathrm{loc}}(\Omega)$ of the waveguide scattering problem (3.3),(3.10) and (3.11) by setting $\tilde{u}|_{M_\rho} = u|_{M_\rho}$ for $m = 2$ by*

$$\tilde{u}(x) = u^i(x) + \begin{cases} \sum_{j=1}^\infty \hat{u}_+(j)\frac{\exp(i\lambda_j x_1)}{\exp(i\lambda_j \rho)}\phi_j(x_2) & x_1 \geq \rho, \\ \sum_{j=1}^\infty \hat{u}_-(j)\frac{\exp(-i\lambda_j x_1)}{\exp(-i\lambda_j \rho)}\phi_j(x_2) & x_1 \leq -\rho, \end{cases} \qquad x = \left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) \text{ in } \Omega \setminus \overline{M_\rho}, \tag{3.54}$$

and for $m = 3$ by

$$\tilde{u}(x) = u^i(x) + \sum_{n \in \mathbb{Z}, j \in \mathbb{N}} \hat{u}(n,j)\frac{H_n^{(1)}(\lambda_j r)}{H_n^{(1)}(\lambda_j \rho)}\exp(in\varphi)\phi_j(x_3) \quad \text{for } x = \left(\begin{smallmatrix} r\cos\varphi \\ r\sin\varphi \\ x_3 \end{smallmatrix}\right) \text{ in } \Omega \setminus \overline{M_\rho}. \tag{3.55}$$

The coefficients $\hat{u}_{\pm}(j)$ *for* $m = 2$ *are defined by*

$$\hat{u}_+(j) = \int_0^H (u - u^i)(\rho, x_2)\phi_j(x_2)\,dx_2 \qquad \textit{for } j, \in \mathbb{N}, \tag{3.56}$$
$$\hat{u}_-(j) = \int_0^H (u - u^i)(-\rho, x_2)\phi_j(x_2)\,dx_2 \qquad \textit{for } j, \in \mathbb{N},$$

and the coefficients $\hat{u}(j,n)$ *for* $m = 3$ *are defined by*

$$\hat{u}(j,n) = \int_0^H \int_0^{2\pi} (u - u^i) \left(\begin{smallmatrix} \rho\cos\varphi \\ \rho\sin\varphi \\ x_3 \end{smallmatrix} \right) e^{-in\varphi} \phi_j \,d\varphi\,dx_3, \qquad j \in \mathbb{N},\, n \in \mathbb{Z}. \tag{3.57}$$

As the restriction to M_ρ *of any solution to the scattering problem (3.10), (3.11) with the radiation and boundedness conditions (3.3) solves the variational problem (3.51), uniqueness of the solution (3.51) implies uniqueness of solution of the scattering problem (3.3), (3.10), (3.11).*

Proof. We first assume that $u \in H^1_W(M_\rho)$ solves (3.51) uniquely. We further choose $v \in H^1_W(M_\rho)$ such that $v|_{\Sigma_\rho} = 0$ and by integration by parts in (3.51) we have that

$$\int_{M_\rho} \left[\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)} n^2 u\,\overline{v} \right] dx = -\int_{M_\rho} \left(\Delta u + \frac{\omega^2}{c^2(x_m)} n^2 u \right) \overline{v}\,dx + \int_{\Gamma_H \cap \{|\tilde{x}| < \rho\}} \frac{\partial u}{\partial x_m} \overline{v}\,ds = 0.$$

Now, using the definition of F in (3.52) and again integrating by parts implies

$$\int_{\Sigma_\rho} \left(\frac{\partial u}{\partial \nu} - \mathbf{\Lambda}(u) \right) \overline{v}\,ds = \int_{\Sigma_\rho} \left(\frac{\partial u^i}{\partial \nu} - \mathbf{\Lambda}(u^i) \right) \overline{v}\,ds \qquad \text{for all } v \in H^1_W(M_\rho). \tag{3.58}$$

We futher denote $u^s \in H^1_W(M_\rho)$ by $u = u^i + u^s$ such that (3.58) and the surjectivity of the trace operator yields

$$\left(\frac{\partial u^s}{\partial \nu} \right)\bigg|_{\Sigma_\rho} = \mathbf{\Lambda}(u^s|_{\Sigma_\rho}) \qquad \text{holds in } V_m'.$$

Next, in $\Omega \setminus \overline{M_\rho}$ we denote u^s by the series in (3.54) for $m = 2$ and in (3.55) for $m = 3$, such that $\tilde{u} = u^i + u^s$ holds in $\Omega \setminus \overline{M_\rho}$. For simplicity, we write $(\cdot)|^{\pm}_{\Sigma_\rho}$ if a trace on Σ_ρ is taken from the inside (-) or from the outside (+) of Ω. We recall that for dimension two we have $\Sigma_\rho := \Sigma_{-\rho} \cup \Sigma_{+\rho}$. Due to the trace estimates from the proof of Theorem 3.2.7 and the radiation conditions in (3.3) we define the coefficients $u_{\pm}(j)$ in (3.56) for dimension two such that it holds

$$u^s|^-_{\Sigma_{\pm\rho}} = (u - u^i)\big|^-_{\Sigma_{\pm\rho}} = \sum_{j \in \mathbb{N}} \hat{u_{\pm}}(j)\phi_j = u^s|^+_{\Sigma_{\pm\rho}} = (u - u^i)\big|^+_{\Sigma_{\pm\rho}}, \qquad \text{in } V_2.$$

Furthermore, the coefficients $u_{\pm}(j,n)$ for dimension three in (3.57) is defined such that

$$u^s|^-_{\Sigma_\rho} = (u - u^i)\big|^-_{\Sigma_\rho} = \sum_{j \in \mathbb{N}} \sum_{n \in \mathbb{Z}} \hat{u}(j,n) e^{in\,\cdot} \phi_j = u^s|^+_{\Sigma_\rho} = (u - u^i)\big|^+_{\Sigma_\rho}, \qquad \text{holds in } V_3.$$

Consequently, we have that $u|^-_{\Sigma_\rho}$ equals the restriction $\tilde{u}|^+_{\Sigma_\rho}$, i.e., the extension $\tilde{u}$ is continuous over Σ_ρ in the trace sense.

Now due to Theorem 3.3.3, due to the construction of $\tilde{u}(x)$ and the Dirichlet-to-Neumann Operator $\mathbf{\Lambda}$ and since it holds $(\partial u^s/\partial\nu)|^-_{\Sigma_\rho} = \mathbf{\Lambda}(u^s|^-_{\Sigma_\rho})$, we know that $\tilde{u}(x) \in H^1_W(\Omega)$ solves the radiation conditions (3.3) to the Helmholtz equation in $\Omega \setminus M_\rho$ with normal normal derivative

$$\frac{\partial \tilde{u}}{\partial \nu}\bigg|^+_{\Sigma_\rho} = \left[\frac{\partial u^i}{\partial \nu} + \frac{\partial u^s}{\partial \nu} \right]\bigg|^+_{\Sigma_\rho} = \left[\frac{\partial u^i}{\partial \nu} \right]\bigg|^+_{\Sigma_\rho} + \mathbf{\Lambda}(u^s|^+_{\Sigma_\rho}) = \left[\frac{\partial u^i}{\partial \nu} \right]\bigg|^-_{\Sigma_\rho} + \mathbf{\Lambda}(u^s|^-_{\Sigma_\rho}) = \left[\frac{\partial \tilde{u}}{\partial \nu} \right]\bigg|^-_{\Sigma_\rho} \qquad \text{in } V_m'.$$

Since the normal derivative of $\tilde{u}$ across Σ_ρ is hence also continuous in the trace sense, the latter function is a weak solution in $H^1_{loc}(\Omega)$ to the waveguide scattering problem (3.3),(3.10) and (3.11). Again using interior elliptic regularity results [McL00, Chapter 4] to show that $\tilde{u} \in H^2_{\text{loc}}(\Omega)$. To this end, uniqueness of this scattering problem finally follows from uniqueness of solution of the variational problem (3.51), since any non-trivial solution to the scattering problem for $u^i = 0$ is a non-trivial solution to the variational problem (3.53).

□

Theorem 3.4.4. *Consider that Assumption 3.1.1 holds. For* $\text{Im}(q) > 0$ *on a non-empty open subset* D *of* M_ρ*, then the variational problem (3.51) and the waveguide scattering problem (3.3),(3.10) and (3.11) are both uniquely solvable for all incident fields* u^i *and all frequencies* $\omega > 0$*, satisfies Assumption 3.1.1.*

Proof. We consider first that $\text{Im}(q) > 0$ on a non-empty open subset $D \subset M_\rho$. We further assume that $u \in H^1_W(M_\rho)$ is solution to (3.51) with vanishing incident field $u^i = 0$ or vanishing right-hand side $F = 0$ equivalently. This u now can be extend by (3.54) for $m = 2$ or (3.55) for $m = 3$ by Theorem 3.4.3 to a solution in $H^1_{W,\text{loc}}(\Omega) \cap H^2_{\text{loc}}(\Omega)$ of the scattering problem with $u^i = 0$. Of course, this extended function satisfies radiation conditions (3.3) to the Helmholtz equation (3.10). Consequently, for simplicity, we call again the extended function u. Taking the imaginary part of (3.51) with $v = u$ and integrating by parts in $M_R \setminus \overline{M_\rho}$ states for $R > \rho$ that

$$\begin{aligned}\text{Im}\, B_\omega(u,u) &= -\int_{M_\rho} \frac{\omega^2}{c^2(x_m)} \text{Im}(q)|u|^2\, dx - \text{Im} \int_{\Sigma_\rho} \mathbf{\Lambda}(u)\, \overline{u}\, ds \\ &= -\int_{M_\rho} \frac{\omega^2}{c^2(x_m)} \text{Im}(q)|u|^2\, dx - \text{Im} \int_{\Sigma_\rho} \frac{\partial u}{\partial \nu}\, \overline{u}\, ds \\ &= \text{Im} \int_{M_R \setminus \overline{M_\rho}} \left[|\nabla u|^2 - \frac{\omega^2}{c^2(x_m)} |u|^2 \right] dx \\ &\qquad - \int_{M_\rho} \frac{\omega^2}{c^2(x_m)} \text{Im}(q)|u|^2\, dx - \text{Im} \int_{\Sigma_R} \frac{\partial u}{\partial \nu}\, \overline{u}\, ds = 0.\end{aligned}$$

For $u \in H^1_{W,\text{loc}}(\Omega) \cap H^2_{\text{loc}}(\Omega)$ the orthonormal expansion

$$u = \sum_{j \in \mathbb{N}} u(j, \tilde{x}) \phi_j(x_m),$$

shows that

$$\int_{\Sigma_R} \frac{\partial u}{\partial \nu}\, \overline{u}\, ds = \sum_{j \in \mathbb{N}} \int_{|\tilde{x}|=R} \frac{\partial u(j,\tilde{x})}{\partial \nu} \overline{u(j,\tilde{x})}\, ds.$$

For propagating modes, as $1 \leq j \leq J$, then $u(j, \tilde{x})$ is a solution to a Helmholtz equation that satisfies the Sommerfeld radiation conditions due to (3.3) and it is well-known (see, e.g.,[CK13]) that this implies that

$$\text{Im} \int_{|\tilde{x}|=R} \frac{\partial u(j,\tilde{x})}{\partial \nu} \overline{u(j,\tilde{x})}\, ds \geq 0,$$

We point out that the latter expression is a multiple of the L^2-norm of the far field pattern of $u(j, \tilde{x})$. For evanescent modes, as $j > J$, $u(j, \tilde{x})$ is a bounded and hence exponentially decreasing solution to a Helmholtz equation (this follows, e.g., from the estimates of the Hankel functions in the proof of Theorem 3.3.2), such that

$$\text{Im} \int_{|\tilde{x}|=R} \frac{\partial u(j,\tilde{x})}{\partial \nu} \overline{u(j,\tilde{x})}\, ds = \text{Im} \int_{|\tilde{x}|>R} \left[|\nabla_{\tilde{x}} u(j,\tilde{x})|^2 + \lambda_j^2 |u(j,\tilde{x})|^2 \right] dx = 0.$$

In consequence, we conclude that

$$0 = \operatorname{Im} B_\omega(u,u) = \int_{M_\rho} \frac{\omega^2 \operatorname{Im}(q)}{c^2(x_3)} |u|^2 \, dx + \operatorname{Im} \int_{\Sigma_R} \frac{\partial u}{\partial \nu} \, \overline{u} \, ds \geq 0$$

and we deduce that u vanishes on the open, nonempty set D. Finally, due to the unique continuation property for solutions to $\Delta u + (\omega^2/c^2(x_m))(1+q)u = 0$ in [JK85, Th. 6.3, Rem. 6.7], we know that u vanishes in all of Ω. This ends the proof.

□

Remark 3.4.5. *(a) If the obstacle D described by the contrast q is replaced by an impenetrable obstacle $D \Subset M_\rho$ with either Neumann, Dirichlet or impedance boundary conditions, the approach from the beginning of this section implies a variational problem for the total field with restriction to $M_\rho \setminus \overline{D}$. For a Neumann or impedance boundary condition this problem is posed in $H^1_W(M_\rho \setminus \overline{D})$. If we have Dirichlet boundary condition, the variational space additionally needs to incorporate homogeneous Dirichlet boundary conditions on ∂D. The existence and uniqueness results of Theorem 3.4.1 and Theorem 3.4.3(a) hold for those scattering problems, too.*

(b) The Gårding inequality from Theorem 3.4.1(a) yields that a conforming Ritz-Galerkin scheme applied to (3.51) converges if the sequence of discrete variational spaces is dense in $H^1_W(M_\rho)$.

Chapter 4

Lippmann-Schwinger Integral Equation

The aim of this chapter is to derive a volumetric integral equation of the second kind, the so called Lippmann-Schwinger equation. We first introduce the Green's function or fundamental solution for the differential operator under investigation for dimensions two and three to have the basic ingredients for the volumetric integral equation. Then, we show that solving the Lippmann-Schwinger equation is equivalent to solve the Helmholtz equation (2.1) with corresponding boundary conditions (2.5) and (2.6) and radiation conditions (3.3). The essential advantage of the integral equation is that boundary conditions and the radiation conditions are already included in the integral representation. We point out that Colton and Kress have shown in [CK13] that if $u \in C^2(\mathbb{R}^m)$ solves the Helmholtz equation

$$\Delta u + \frac{\omega^2}{c^2}(1+q)u = 0,$$

for constant wavenumber ω/c in $\mathbb{R}^m$, and if $u = u^i + u^s$ is sum of an incident field u^i and a radiating scattered field u^s, then u is a solution of

$$u(x) = u^i(x) - \omega^2/c^2 \int_{\mathbb{R}^m} \Phi(x,y)q(y)u(y)\,dy.$$

Here $\Phi(x,y)$ is the fundamental solution of $\mathbb{R}^m$. Further, the converse holds, too. If the background sound speed c depends on the depth of the ocean, this result must be suitably modified.

4.1 Green's function

We begin first by introducing the fundamental solution of the Helmholtz equation in the unbounded region $\mathbb{R}^{m-1}$ that satisfies the radiation conditions (3.3). Then, we continue our studies of the fundamental solution of the Helmholtz equation (2.1) with boundary conditions for dimension $m = 2$ and $m = 3$.

First we define

$$\mathbb{C}_{++} = \{z \in \mathbb{C}\backslash\{0\} : \mathrm{Re} z \geq 0, \mathrm{Im} z \geq 0\}.$$

Lemma 4.1.1. *For $\lambda \in \mathbb{C}_{++}$ the fundamental solution of the Helmholtz equation (2.1) in the unbounded region $\mathbb{R}^{m-1}$ is given by*

$$\begin{aligned} x_1 \mapsto & E(\lambda, x_1) := \frac{i}{2\lambda}\exp(i\lambda|x_1|) && \textit{for } m = 2 \textit{ and} \\ \tilde{x} \mapsto & \frac{i}{4}H_0^{(1)}(\lambda|\tilde{x}|) && \textit{for } m = 3, \end{aligned}$$

where $H_0^{(1)}$ denotes the Hankel function of order zero and the fundamental solution satisfies the radiation conditions

$$\lim_{|x_1|\to\infty} [E'(\lambda, x_1) - i\lambda E(\lambda, x_1)] = 0 \qquad \text{for } \lambda \in \mathbb{C}_{++}, m = 2,$$

$$\lim_{|\tilde{x}|\to\infty} \sqrt{|\tilde{x}|} \left[\frac{\partial}{\partial|\tilde{x}|} H_0^{(1)}(\lambda|\tilde{x}|) - i\lambda H_0^{(1)}(\lambda|\tilde{x}|)\right] = 0 \qquad \text{for } \lambda \in \mathbb{C}_{++}, m = 3.$$

Proof. We first treat the case $m = 2$. We exploit results of computing the fundamental solution from [Wal94, Kapitel 5.6]. For fixed $\lambda \in \mathbb{C}\backslash\{0\}$ let c_0 denote an arbitrary constant and we define

$$E(\lambda, x_1) := c_0 \exp(i\lambda|x_1|), \qquad x_1 \in \mathbb{R}.$$

We recall, for fixed $\lambda \in \mathbb{C}\backslash\{0\}$ the function $w \in H^2(\mathbb{R})$ solves the 1D-Helmholtz equation acting on the horizontal axis given by equation (2.10),

$$w_j'' + \lambda^2 w_j = 0 \qquad \text{in } \mathbb{R},$$

and radiation conditions. We set $w = E(\lambda, \cdot)$ in the last equation, multiply it with a test function $\psi \in C_0^\infty(\mathbb{R})$ and do an integration. Then,

$$\int_{-\infty}^{\infty} E(\lambda, \cdot) \left(\frac{\partial^2}{\partial x_1^2} + \lambda^2\right) \psi \, dx_1 = \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) E(\lambda, \cdot) \left(\frac{\partial^2}{\partial x_1^2} + \lambda^2\right) \psi \, dx_1 \quad \text{for all } \psi \in C_0^\infty(\mathbb{R}).$$

We use partial integration to obtain for all $\psi \in C_0^\infty(\mathbb{R})$ that

$$\begin{aligned} &- \int_{-\infty}^{\infty} E(\lambda, \cdot) \left(\frac{\partial^2}{\partial x_1^2} + \lambda^2\right) \psi \, dx_1 \\ &= - \left(\int_{-\infty}^{0} + \int_{0}^{\infty}\right) \left[\frac{\partial}{\partial x_1} E(\lambda, \cdot) \frac{\partial}{\partial x_1} \psi + \lambda^2 E(\lambda, \cdot)\psi\right] dx_1 + \frac{\partial}{\partial x_1} \psi(0) E(\lambda, 0) - \frac{\partial}{\partial x_1} \psi(0) E(\lambda, 0). \end{aligned}$$

Using further the limit that ε tends to zero, we see

$$\begin{aligned} &- \int_{-\infty}^{\infty} E(\lambda, \cdot) \left(\frac{\partial}{\partial x_1^2} + \lambda^2\right) \psi \, dx_1 \\ &\qquad = \lim_{\varepsilon\to 0} \Bigg(- \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) \left[\frac{\partial}{\partial x_1} E(\lambda, \cdot) \frac{\partial}{\partial x_1} \psi + \lambda^2 E(\lambda, \cdot)\psi\right] dx_1 \\ &\qquad\qquad + \frac{\partial}{\partial x_1} E(\lambda, \varepsilon)\psi(\varepsilon) - \frac{\partial}{\partial x_1} E(\lambda, -\varepsilon)\psi(-\varepsilon)\Bigg). \end{aligned}$$

Rigorously computation shows that $E(\lambda, \cdot)$ solves the 1D-Helmholtz equation $\Delta E + \omega^2/c^2 E = 0$. This and and partial integration show

$$\lim_{\varepsilon\to 0} \left(\int_{-\infty}^{-\varepsilon} + \int_{\varepsilon}^{\infty}\right) \frac{\partial}{\partial x_1} E(\lambda, \cdot) \frac{\partial}{\partial x_1} \psi + \lambda^2 E(\lambda, \cdot)\psi \, dx_1 = 0.$$

In consequence, we compute

$$\begin{aligned} - \int_{-\infty}^{\infty} E(\lambda, \cdot) \left[\left(\frac{\partial}{\partial x_1^2} + \lambda^2\right) \psi \, dx_1\right] &= \lim_{\varepsilon\to 0} \left[\frac{\partial}{\partial x_1} E(\lambda, \varepsilon)\psi(\varepsilon) - \frac{\partial}{\partial x_1} E(\lambda, -\varepsilon)\psi(-\varepsilon)\right] \\ &= \lim_{\varepsilon\to 0} \left[i\lambda \exp(i\lambda\varepsilon)\psi(\varepsilon) + i\lambda \exp(-i\lambda\varepsilon)\psi(-\varepsilon)\right]. \end{aligned}$$

We search now the constant c_0 such that

$$\lim_{\varepsilon\to 0} \left[c_0 i\lambda \exp(i\lambda\varepsilon)\psi(\varepsilon) + c_0 i\lambda \exp(-i\lambda\varepsilon)\psi(-\varepsilon)\right] \stackrel{!}{=} \psi(0) \qquad \text{for all } \psi \in C_0^\infty(\mathbb{R}).$$

We see for $c_0 = i/(2\lambda)$ that

$$\lim_{\varepsilon\to 0}\left[c_0 i\lambda \exp(i\lambda\varepsilon)\psi(\varepsilon) + c_0 i\lambda \exp(-i\lambda\varepsilon)\psi(-\varepsilon)\right]$$
$$= \lim_{\varepsilon\to 0}\left[-\frac{1}{2}\exp(i\lambda\varepsilon)\psi(\varepsilon) - \frac{1}{2}\exp(-i\lambda\varepsilon)\psi(-\varepsilon)\right] = \psi(0) \qquad \text{for all } \psi \in C_0^\infty(\mathbb{R}).$$

This finishes the proof for $m = 2$. For the proof that $i/4H_0^{(1)}$ is the fundamental solution of the Helmholtz equation in $\mathbb{R}^2$ we refer the reader e.g. [SV02, Chapter 2]. □

Note that, e.g. Assumption 4.2.11 supposes that all eigenvalues of the eigenvalue problem (2.14) are non-zero, such that the case $\lambda = 0$ does not have to be considered in the last lemma. Our aim is now to determinate the fundamental solution of the Helmholtz equation (2.1) with boundary conditions (2.5), (2.6) and radiation conditions (3.3).

We denote the Dirac distribution on Ω with mass at point x by δ_x,

$$\langle \delta_x, \varphi\rangle = \psi(x) \qquad \text{for } \psi \in C_0^\infty(\Omega).$$

Due to [Wal94, Kapitel 5.6] and with respect to the separation of variables, a fundamental solution G of the Helmholtz equation satisfies

$$\Delta G(x,y) + \frac{\omega^2}{c(x_m)}G(x,y) = -\delta_{\tilde{y}}(\tilde{x})\delta_{y_m}(x_m),$$

that is,

$$\int_\Omega \left[\Delta G(x,y) + \frac{\omega^2}{c(x_m)}G(x,y)\right]\psi(y)\,dy = -\psi(x) \qquad \text{for all } \psi \in C_0^\infty(\Omega).$$

G should also satisfies the radiation conditions. We first treat the case $m = 2$. For a formal computation of G, we expand δ_{y_2} into its Fourier series with respect to the basis $\{\phi_j\}_{j\in\mathbb{N}}$,

$$\delta_{y_2}(x_2) = \sum_{j=1}^\infty c(j)\phi_j(x_2), \qquad \text{with } c(j) \in \mathbb{C}.$$

Next, multiplying the latter equation with $\phi_j(x_2)$, integrating from $x_2 = 0$ to $x_2 = H$ and using the orthogonality of the basis $\{\phi_j\}_{j\in\mathbb{N}}$ to see that

$$c(j) = \phi_j(y_2).$$

Plugging all together and using Lemma 4.1.1, we obtain the fundamental solution of the Helmholtz equation (2.1) with boundary conditions (2.5) and (2.6) in dimension two,

$$G(x,y) = \frac{i}{2}\sum_{j=1}^\infty \frac{1}{\lambda_j}\phi_j(x_2)\phi_j(y_2)\exp(i\lambda_j|x_1 - y_1|), \qquad x_1 \neq y_1. \tag{4.1}$$

This function is from now on called the waveguide Green's function. We point out that for $x_1, y_1 \in \mathbb{R}$, $x_1 = y_1$, such that the Green's function in dimension two is formally defined, however, the derivative in x_1-direction is not continuous anymore, since

$$\frac{\partial}{\partial x_1}\exp(i\lambda_j|x_1 - y_1|) = i\lambda_j\frac{x_1 - y_1}{|x_1 - y_1|}\exp(i\lambda_j|x_1 - y_1|).$$

Note, however, that the series (4.1) is merely absolutely and unconditionally convergent if $x_1 \neq y_1$. For completeness, we further introduce the conjugate Green's function. By the fact that the eigenfunctions are real-valued the complex conjugate Green's function is given by

$$\overline{G(x,y)} = -\frac{i}{2}\sum_{j=1}^\infty \left(\overline{\lambda_j}\right)^{-1}\phi_j(x_2)\phi_j(y_2)\exp(-i\overline{\lambda_j}|x_1 - y_1|), \qquad x_1 \neq y_1. \tag{4.2}$$

Next, we compute the Green's function for the three-dimensional case. Ones more, we can expand δ_{y_3} into its Fourier series with respect to the basis $\{\phi_j\}_{j\in\mathbb{N}}$,

$$\delta_{y_3}(x_3) = \sum_{j=1}^{\infty} c(j)\phi_j(x_3), \qquad \text{with } c(j) \in \mathbb{C}.$$

Lemma 4.1.1 implies that the Green's function in horizontal variable on the unbounded region $\mathbb{R}^2$ is a constant multiple with Hankel function, $\tilde{x} \mapsto i/4\, H_0^{(1)}(\lambda_j|\tilde{x}|)$. Consequently,

$$\frac{i}{4}\sum_{j=1}^{\infty} c(j)\phi_j(x_3)H_0^{(1)}(\lambda_j|\tilde{x}|) = \delta_{\tilde{y}}(\tilde{x})\delta_{y_3}(x_3). \tag{4.3}$$

As for $m = 2$, we multiply the expansion of δ_{y_3} with $\phi_j(x_3)$, integrate from $x_3 = 0$ to $x_3 = H$ and use that the eigenfunctions $\{\phi_j\}_{j\in\mathbb{N}}$ are orthonormal to obtain $c(j) = \phi_j(y_3)$. Thus, (4.3) becomes the Green's function for dimension three,

$$G(x, y) = \frac{i}{4}\sum_{j=1}^{\infty} \phi_j(x_3)\phi_j(y_3)H_0^{(1)}(\lambda_j|\tilde{x} - \tilde{y}|), \qquad \tilde{x} \neq \tilde{y}. \tag{4.4}$$

Again, this series is merely absolutely and unconditionally convergent for $\tilde{x} \neq \tilde{y}$, since λ_j is complex-valued for $j > J$. Since [AS64, Equation 9.1.40],

$$\overline{H_0^{(1)}(z)} = H_0^{(2)}(\overline{z}) \qquad z \in \mathbb{C},$$

we see for the complex conjugate Green's function in dimension three that

$$\overline{G(x, y)} = -\frac{i}{4}\sum_{j=1}^{\infty} \phi_j(x_3)\phi_j(y_3)H_0^{(2)}(\overline{\lambda_j}|\tilde{x} - \tilde{y}|), \qquad \tilde{x} \neq \tilde{y}.$$

We point out that $G(x, y) = G(y, x)$ for $\tilde{x} \neq \tilde{y}$ holds independently of m. The expressions (4.1) and (4.4) are called modal representations. Note that the Green's function solves boundary conditions and radiation conditions by construction. By Section 3.1 we know that there are two kinds of waveguide modes. If $\lambda_j^2 > 0$ the modes are propagating and if $\lambda_j^2 < 0$ they are evanescent. For a homogeneous ocean, where the background sound speed is constant, [AK77] present an alternative expression for the Green's function in dimension three. This representation converges near the singularity $\{\tilde{x} = \tilde{y}\}$ and is called ray representation. The ray representation is derived by the image method, which is only valid for homogeneous media and plane boundaries, see [SW04, Chapter 3]. The idea of this method is to sum up an infinite sequence of families of rays reflected by the sound hard and the sound soft boundaries. In our setting, this method cannot be used to give a ray representation for expression (4.4), since the background sound speed is not constant. Thus, the well-known theory in [CK13, Chapter 8] used to give an equivalent representation of solutions to the Helmholtz equation, with corresponding boundary conditions and radiation conditions via the Lippmann-Schwinger equation does not apply here, at least not straightforwardly.

4.2 The Volumetric Integral Operator

In this section we first introduce the volumetric integral operator $\mathcal{V}$ using the Green's function introduce above. We further show that $\mathcal{V}$ is bounded from $L^2(\Lambda_\rho)$ into $H^2(\Lambda_\rho)$. Then, we can prove that the Helmholtz equation with corresponding boundary and radiation conditions is equivalent the Lippmann-Schwinger equation, using the volumetric integral operator. Existence of classical (i.e., twice differentiable) solutions to the Helmholtz equation for constant and depth-dependent sound speed has been shown via integral equation techniques by Gilbert and Xu in a series of papers in [GX89, Xu92, Xu97, GL97, BGWX04], too. Gilbert and Xu used the fact that the fundamental solution G can be separated into a free space Green's function and a part correcting the boundary conditions, thus using well-known volume integral equation tools. We present here, however, an alternative technique to obtain the required boundedness of the volumetric integral equation we introduce later on.

We first recall the waveguide's Green's function from the last section

$$G(x,y) = \begin{cases} \frac{i}{2}\sum_{j=1}^{\infty} {\lambda_j}^{-1}\phi_j(x_2)\phi_j(y_2)\exp(i\lambda_j|x_1-y_1|) & \text{for } x,y\in\Omega, x_1\neq y_1, m=2 \text{ and} \\ \frac{i}{4}\sum_{j=1}^{\infty}\phi_j(x_3)\phi_j(y_3)H_0^{(1)}(\lambda_j|\tilde{x}-\tilde{y}|) & \text{for } x,y\in\Omega, \tilde{x}\neq\tilde{y}, m=3, \end{cases}$$

and formally define the volume integral operator applied to a function f by

$$\mathcal{V}f = \int_D G(\cdot,y)f(y)dy \qquad \text{for } f\in L^2(D). \tag{4.5}$$

Lemma 4.2.1. *Consider $m = 2$ and the operator $\mathcal{V}_j$, defined by*

$$\begin{aligned} \mathcal{V}_j :&L^2([-\rho,\rho]) \to L^2([-\rho,\rho]), \\ &f \mapsto \int_{-\rho}^{\rho} \frac{i}{2\lambda_j}\exp(i\lambda_j|x_1-y_1|)f(y_1)\,dy_1, \qquad j\in\mathbb{N}. \end{aligned} \tag{4.6}$$

Then $\mathcal{V}_j$ is a bounded operator from $L^2([-\rho,\rho])$ into $L^2([-\rho,\rho])$, and

$$\|\mathcal{V}_j f\|_{L^2([-\rho,\rho])} \leq \frac{C}{j^2}\|f\|_{L^2([-\rho,\rho])}, \tag{4.7}$$

with a constant $C > 0$.

Proof. We first estimate

$$\|\mathcal{V}_j f\|^2_{L^2([-\rho,\rho])} \leq \left(\int_{-\rho}^{\rho}\left|\int_{-\rho}^{\rho}\frac{i}{2\lambda_j}\exp(i\lambda_j|x_1-y_1|)dy_1\right|^2 dx_1\right)\|f\|^2_{L^2([-\rho,\rho])}.$$

Recall that $J(\omega,c,H)\in\mathbb{N}$ denotes the number of positive eigenvalues λ_j^2 and hence $\lambda_j^2 < 0$ for $j > J(\omega,c,H)$ by (2.13). Thus, $i\lambda_j = -|\lambda_j|$ for $j > J$. As $\mathcal{V}_j$ is an integral operator with bounded kernel it is automatically bounded on $L^2([-\rho,\rho])$ so that it is sufficient to consider $j > J$ to prove (4.7). Abbreviating $|\lambda_j| = \mu_j$ and splitting the integral, implies

$$\begin{aligned} &\int_{-\rho}^{\rho}\frac{1}{2|\lambda_j|}\exp\left(-|\lambda_j||x_1-y_1|\right)dy_1 \\ &\qquad = \int_{-\rho}^{x_1}\frac{1}{2\mu_j}\exp(-\mu_j(x_1-y_1))dy_1 + \int_{x_1}^{\rho}\frac{1}{2\mu_j}\exp(-\mu_j(y_1-x_1))dy_1. \end{aligned}$$

By integration we obtain

$$\int_{-\rho}^{x_1}\frac{1}{2\mu_j}\exp(-\mu_j(x_1-y_1))dy_1 = \left[\frac{1}{2\mu_j^2}\exp(-\mu_j(x_1-y_1))\right]_{-\rho}^{x_1} = \frac{1}{2\mu_j^2} - \frac{1}{2\mu_j^2}\exp(-\mu_j(x_1+\rho)), \tag{4.8}$$

and

$$\int_{x_1}^{\rho}\frac{1}{2\mu_j}\exp(-\mu_j(y_1-x_1))dy_1 = \left[\frac{1}{2\mu_j^2}\exp(-\mu_j(y_1-x_1))\right]_{x_1}^{\rho} = \frac{1}{2\mu_j^2} - \frac{1}{2\mu_j^2}\exp(-\mu_j(\rho-x_1)). \tag{4.9}$$

In consequence, we see by taking $\exp(-\mu_j\rho)$ as a common factor that

$$\int_{-\rho}^{\rho}\frac{1}{2\mu_j}\exp(-\mu_j|x_1-y_1|)dy_1 = \frac{1}{\mu_j^2} - \frac{\exp(-\mu_j\rho)}{2\mu_j^2}\left[\exp(\mu_j x_1)+\exp(-\mu_j x_1)\right]. \tag{4.10}$$

In particular, since $x_1 \in [-\rho, \rho]$ we estimate

$$\int_{-\rho}^{\rho} \frac{1}{2\mu_j} \exp(-\mu_j |x_1 - y_1|) dy_1 \leq \frac{1}{\mu_j^2} - \frac{1}{2\mu_j^2} = \frac{1}{2\mu_j^2}. \tag{4.11}$$

This is independent of x_1 and we see now that

$$\int_{-\rho}^{\rho} \left| \int_{-\rho}^{\rho} \frac{i}{2\mu_j} \exp(i\mu_j |x_1 - y_1|) dy_1 \right|^2 dx_1 \leq \frac{\rho}{2\mu_j^4}. \tag{4.12}$$

Now, Lemma 2.2.4 implies that $cj \leq \mu_j \leq Cj$, such that $C > c > 0$ and we deduce

$$\int_{-\rho}^{\rho} \left| \int_{-\rho}^{\rho} \frac{i}{2\mu_j} \exp(i\mu_j |x_1 - y_1|) dy_1 \right|^2 dx_1 \leq \frac{C}{j^4},$$

where $C > 0$ denotes a constant. This completes the proof. □

Lemma 4.2.2. *Consider $m = 2$ and the operator $\mathcal{V}_j$ defined in Lemma 4.2.1. Then $\mathcal{V}_j$ is a bounded operator from $L^2([\rho, \rho])$ into $H^1([-\rho, \rho])$, and*

$$\|\mathcal{V}_j f\|_{H^1([-\rho,\rho])} \leq \frac{C}{j} \|f\|_{L^2([-\rho,\rho])}$$

with a constant $C > 0$.

Proof. Similar to the proof of Lemma 4.2.1, we look at the infinite number of negative eigenvalues λ_j^2 such that $\lambda_j = -\sqrt{|\lambda_j|^2} = -|\lambda_j|^2$. Define

$$\begin{aligned} \mathcal{V}_j' &: L^2([-\rho, \rho]) \to L^2([-\rho, \rho]), \\ f &\mapsto \frac{\partial}{\partial x_1} \int_{-\rho}^{\rho} \frac{i}{2\lambda_j} \exp(i\lambda_j |x_1 - y_1|) f(y_1) dy_1, \qquad j \in \mathbb{N}. \end{aligned}$$

For $f \in \mathbb{C}_0^\infty([-\rho, \rho])$ we can interchange integral and derivation. In particular, for $j > J$ we derive

$$\begin{aligned} \|\mathcal{V}_j' f\|_{L^2([-\rho,\rho])}^2 &= \left\| x_1 \mapsto \int_{-\rho}^{\rho} \frac{\partial}{\partial x_1} \frac{i}{2i|\lambda_j|} \exp\left(-|\lambda_j||x_1 - y_1|\right) f(y_1)\, dy_1 \right\|_{L^2([-\rho,\rho])}^2 \\ &= \left\| x_1 \mapsto \int_{-\rho}^{\rho} -\frac{1}{2} \frac{x_1 - y_1}{|x_1 - y_1|} \exp\left(-|\lambda_j||x_1 - y_1|\right) f(y_1)\, dy_1 \right\|_{L^2([-\rho,\rho])}^2 . \end{aligned}$$

For simplicity, we substitute once more $|\lambda_j| = \mu_j$. As $(x_1 - y_1)/|x_1 - y_1| \leq 1$ and due to the Cauchy-Schwarz inequality we estimate

$$\begin{aligned} &\int_{-\rho}^{\rho} \left| \frac{x_1 - y_1}{|x_1 - y_1|} \frac{1}{2} \exp\left(-\mu_j |x_1 - y_1|\right) f(y_1) \right| dy_1 \\ &\qquad \leq \left(\int_{-\rho}^{\rho} \left|\exp\left(-\mu_j |x_1 - y_1|\right)\right|^2 dy_1 \right)^{1/2} \left(\int_{-\rho}^{\rho} |f(y_1)|^2\, dy_1 \right)^{1/2}. \end{aligned}$$

To use the idea of the proof of Lemma 4.2.1 we first note that

$$\int_{-\rho}^{\rho} \left|\exp\left(-\mu_j |x_1 - y_1|\right)\right|^2 dy_1 \leq 2\mu_j \int_{-\rho}^{\rho} \frac{1}{2\mu_j} \exp\left(-2\mu_j |x_1 - y_1|\right) dy_1.$$

Then, equations (4.8-4.11) and Lemma 2.2.4 yield

$$2\mu_j \int_{-\rho}^{\rho} \frac{1}{2\mu_j} \exp\left(-2\mu_j |x_1 - y_1|\right) dy_1 \leq 2\mu_j \frac{1}{2\mu_j^2} = \frac{1}{\mu_j} \leq \frac{C}{j},$$

where $C > 0$ denotes a constant. Now, the integral with respect to the variable x_1 over $[-\rho, \rho]$ adds only a factor of a constant multiplied with ρ. In consequence,

$$\int_{-\rho}^{\rho} \left| \int_{-\rho}^{\rho} \exp\left(-2\mu_j |x_1 - y_1|\right) dy_1 \right|^2 dx_1 \leq \frac{C}{j^2}.$$

Then,

$$\|\mathcal{V}_j' f\|^2_{L^2([-\rho,\rho])} = \int_{-\rho}^{\rho} \left| \int_{-\rho}^{\rho} \frac{\partial}{\partial x_1} \frac{i}{2i\mu_j} \exp\left(-\mu_j |x_1 - y_1|\right) dy_1 \right|^2 dx_1 \, \|f\|^2_{L^2([-\rho,\rho])} \leq \frac{C}{j^2} \|f\|^2_{L^2([-\rho,\rho])}.$$

To this end, due to Lemma 4.2.1 we obtain

$$\|\mathcal{V}_j f\|^2_{H^1([-\rho,\rho])} = \|\mathcal{V}_j f\|^2_{L^2([-\rho,\rho])} + \|\mathcal{V}_j' f\|^2_{L^2([-\rho,\rho])} \leq \frac{C}{j^2} \|f\|^2_{L^2([-\rho,\rho])}.$$

This completes the proof.

□

Next, we treat the three-dimensional case.

Lemma 4.2.3. *Consider $m = 3$ and the operator $\mathcal{V}_j$, defined by*

$$\mathcal{V}_j : L^2(\tilde{\Lambda}_\rho) \to L^2(\tilde{\Lambda}_\rho),$$
$$f \mapsto \int_{\tilde{\Lambda}_\rho} H_0^{(1)}(\lambda_j |\tilde{x} - \tilde{y}|) f(\tilde{y}) d\tilde{y}, \qquad j \in \mathbb{N}.$$

Then $\mathcal{V}_j$ is a bounded operator from $L^2(\tilde{\Lambda}_\rho)$ into $L^2(\tilde{\Lambda}_\rho)$, and

$$\|\mathcal{V}_j f\|_{L^2(\tilde{\Lambda}_\rho)} \leq \frac{C}{j} \|f\|_{L^2(\tilde{\Lambda}_\rho)}, \tag{4.13}$$

with a constant $C > 0$.

Proof. As V_j is an integral operator with a weakly singular kernel, each $\mathcal{V}_j$ is bounded on $L^2(\tilde{\Lambda}_\rho)$ such that it suffices to prove (4.13) for all $j > J(\omega, c, H)$ such that $\lambda_j^2 < 0$.

Using the relation (A.4) between the Hankel function and the modified Bessel function for imaginary arguments in the Appendix, we obtain for $j > J$ that

$$H_0^{(1)}(i|\lambda_j||\tilde{x} - \tilde{y}|) = \frac{2}{i\pi} K_0(|\lambda_j||\tilde{x} - \tilde{y}|), \qquad \tilde{x} \neq \tilde{y} \in \mathbb{R}^2.$$

We abbreviate $|\lambda_j| = \mu_j$. We further use the definition of the operator $\mathcal{V}_j$, which implies

$$\|\mathcal{V}_j f\|^2_{L^2(\tilde{\Lambda}_\rho)} = \int_{\tilde{\Lambda}_\rho} \left| \int_{\tilde{\Lambda}_\rho} \left| \frac{2}{i\pi} K_0(\mu_j |\tilde{x} - \tilde{y}|) f(\tilde{y}) \right| d\tilde{y} \right|^2 d\tilde{x} \qquad \text{for } j > J, j \in \mathbb{N}.$$

Then,

$$\|\mathcal{V}_j f\|^2_{L^2(\tilde{\Lambda}_\rho)} \leq \frac{2}{\pi} \int_{\tilde{\Lambda}_\rho} \left| \int_{\tilde{\Lambda}_\rho} |K_0(\mu_j |\tilde{x} - \tilde{y}|) f(\tilde{y})| \, d\tilde{y} \right|^2 d\tilde{x} \qquad \text{for } j > J, j \in \mathbb{N}.$$

Next, for ρ sufficient large we split the kernel of $\mathcal{V}_j$ into

$$\kappa_1(\mu_j, \tilde{x}, \tilde{y}) = \begin{cases} K_0(\mu_j |\tilde{x} - \tilde{y}|) & \text{for } 0 < \mu_j |\tilde{x} - \tilde{y}| \leq 1, \\ 0 & \text{for } \mu_j |\tilde{x} - \tilde{y}| > 1, \end{cases}$$

and

$$\kappa_2(\mu_j, \tilde{x}, \tilde{y}) = \begin{cases} 0 & \text{for } 0 < \mu_j |\tilde{x} - \tilde{y}| \leq 1, \\ K_0(\mu_j |\tilde{x} - \tilde{y}|) & \text{for } \mu_j |\tilde{x} - \tilde{y}| > 1. \end{cases}$$

It is well-known that K_0 has a logarithmic singularity at the origin, and that the bound

$$K_0(|z|) \leq |z|^{-1/2} \qquad |z| < 1,$$

holds (the power could of course be chosen as $-\alpha$ for arbitrary $\alpha > 0$ at expense of an additional constant). Thus,

$$|\kappa_1(\mu_j, \tilde{x}, \tilde{y})| \leq \mu_j^{-1/2} |\tilde{x} - \tilde{y}|^{-1/2}.$$

As $Cj \geq \mu_j \geq c_0 j$ due to Lemma 2.2.4,

$$|\kappa_1(\mu_j, \tilde{x}, \tilde{y})| \leq c_0^{-1/2} j^{-1/2} |\tilde{x} - \tilde{y}|^{-1/2} \qquad \text{for } \mu_j |\tilde{x} - \tilde{y}| < 1.$$

Consequently, the Cauchy-Schwarz inequality implies that

$$\begin{aligned} \left| \int_{\tilde{\Lambda}_\rho} |\kappa_1(\mu_j, \tilde{x}, \tilde{y}) f(\tilde{y})| \, d\tilde{y} \right|^2 &\leq \frac{C}{j} \left| \int_{B(\tilde{x}, 1/\mu_j)} \frac{|f(\tilde{y})|}{|\tilde{x} - \tilde{y}|^{-1/2}} \, d\tilde{y} \right|^2 \\ &\leq \frac{C}{j} \left(\int_{\tilde{\Lambda}_\rho} |f(\tilde{y})|^2 \, d\tilde{y} \right) \left(\int_{B(\tilde{x}, Cj)} |\tilde{x} - \tilde{y}|^{-1} \, d\tilde{y} \right), \end{aligned} \tag{4.14}$$

where $B(\tilde{x}, Cj)$ denotes a ball in $\tilde{x}$ with radius Cj. Due to Lemma A.2.2 from the Appendix we see that

$$\int_{B(\tilde{x}, Cj)} \frac{1}{|\tilde{x} - \tilde{y}|} d\tilde{y} \leq \frac{2\pi}{Cj}, \tag{4.15}$$

such that we get that

$$\int_{\tilde{\Lambda}_\rho} \left| \int_{\tilde{\Lambda}_\rho} |\kappa_1(\mu_j, \tilde{x}, \tilde{y}) f(\tilde{y})| \, d\tilde{y} \right|^2 d\tilde{x} \leq \frac{C}{j^2} \|f\|^2_{L^2(\tilde{\Lambda}_\rho)}. \tag{4.16}$$

Now, we estimate the absolute value of κ_2. By the fact that $j \geq J$ and $|\tilde{x} - \tilde{y}| \geq 0$, it holds

$$(j|\tilde{x} - \tilde{y}|)^{1/2} \exp(-(j|\tilde{x} - \tilde{y}|)/2) \leq \max_{s \in (0,\infty)} \{s e^{-s}\} \leq C, \tag{4.17}$$

where $C > 0$. Due to [Bar10, Eq. 3.5] for $x = 1$ and $y > 0$, we have for $\nu \geq 0$, $z \in \mathbb{C}$ and $z > 1$ that

$$K_\nu(1) \exp(1 - z) \sqrt{z} > K_\nu(z) > 0. \tag{4.18}$$

Thus, we obtain an estimate for κ_2,

$$|\kappa_2(\mu_j, \tilde{x}, \tilde{y})| \leq C(\mu_j |\tilde{x} - \tilde{y}|)^{1/2} \exp(-\mu_j |\tilde{x} - \tilde{y}|).$$

Then, once more using that

$$cj|\tilde{x} - \tilde{y}| \leq \mu_j |\tilde{x} - \tilde{y}| \leq Cj|\tilde{x} - \tilde{y}|$$

where $C > c > 0$, we compute that the absolute value of κ_2 is bounded by

$$|\kappa_2(\mu_j, \tilde{x}, \tilde{y})| \leq C(j|\tilde{x} - \tilde{y}|)^{1/2} \exp(-j|\tilde{x} - \tilde{y}|) \leq C \exp(-(j|\tilde{x} - \tilde{y}|)/2). \tag{4.19}$$

Then,

$$\begin{aligned} \left| \int_{\tilde{\Lambda}_\rho} |\kappa_2(\mu_j, \tilde{x}, \tilde{y}) f(\tilde{y})| \, d\tilde{y} \right|^2 &\leq C \left| \int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left| \exp\left(\frac{-j|\tilde{x} - \tilde{y}|}{2} \right) f(\tilde{y}) \right| d\tilde{y} \right|^2 \\ &\leq C \left(\int_{\tilde{\Lambda}_\rho} \left| \exp\left(\frac{-j|\tilde{x} - \tilde{y}|}{2} \right) \right| |f(\tilde{y})|^2 \, d\tilde{y} \right) \left(\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left| \exp\left(\frac{-j|\tilde{x} - \tilde{y}|}{2} \right) \right| d\tilde{y} \right). \end{aligned}$$

Next, due to Lemma A.2.3 in the Appendix we obtain

$$\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left| \exp\left(\frac{-j|\tilde{x}-\tilde{y}|}{2} \right) \right| d\tilde{y} \leq \frac{4\pi}{j\exp(j\rho)} \left[2\rho - \frac{1}{j} \right] + \frac{4\pi}{j^2},$$

if $j > 1/(2\rho)$ is large enough. Thus,

$$\int_{\Lambda_\rho \setminus \tilde{\Lambda}_{c/j}} \left| \exp\left(-j\frac{|\tilde{x}-\tilde{y}|}{2} \right) \right| d\tilde{y} \leq \frac{C}{j},$$

where $C > 0$ and if $j > 1/(2\rho)$ is large enough. In consequence,

$$\left| \int_{\tilde{\Lambda}_\rho} |\kappa_2(\mu_j, \tilde{x}, \tilde{y}) f(\tilde{y})| \, d\tilde{y} \right|^2 \leq \frac{C}{j} \left(\int_{\tilde{\Lambda}_\rho} \left| \exp\left(\frac{-j|\tilde{x}-\tilde{y}|}{2} \right) \right| |f(\tilde{y})|^2 \, d\tilde{y} \right).$$

Thus, Fubini's theorem and Lemma A.2.5 in the Appendix such that for $j > 1/2\rho$ large enough

$$\int_{\tilde{\Lambda}_\rho} \left| \exp\left(-\frac{j|\tilde{x}-\tilde{y}|}{2} \right) \right| d\tilde{x} \leq \frac{4\pi}{j\exp(j\rho)} \left[2\rho - \frac{1}{j} \right] \leq \frac{C}{j},$$

we obtain

$$\int_{\tilde{\Lambda}_\rho} \left| \int_{\tilde{\Lambda}_\rho} |\kappa_2(\mu_j, \tilde{x}, \tilde{y}) f(\tilde{y})| \, d\tilde{y} \right|^2 d\tilde{x} \leq \frac{C}{j} \int_{\tilde{\Lambda}_\rho} \left| \exp\left(\frac{-j|\tilde{x}-\tilde{y}|}{2} \right) \right| \int_{\tilde{\Lambda}_\rho} |f(\tilde{y})|^2 \, d\tilde{x} \, d\tilde{y} \leq \frac{C}{j^2} \|f\|^2_{L^2(\tilde{\Lambda}_\rho)}.$$

In consequence, the latter estimate and estimate (4.16), imply

$$\|\mathcal{V}_j f\|_{L^2(\tilde{\Lambda}_\rho)} \leq \frac{C}{j} \|f\|_{L^2(\tilde{\Lambda}_\rho)},$$

which completes the proof. □

Lemma 4.2.4. *For $m = 3$, the operator $\mathcal{V}_j$ is bounded from $L^2(\tilde{\Lambda}_\rho)$ into $H^1(\tilde{\Lambda}_\rho)$, and*

$$\|\mathcal{V}_j f\|_{H^1(\tilde{\Lambda}_\rho)} \leq \frac{C}{j} \|f\|_{L^2(\tilde{\Lambda}_\rho)}$$

with a constant $C > 0$.

Proof. Like in the proof of Lemma 4.2.3, we first split the Hankel function $H_0^{(1)}$ into a part with imaginary and real argument. In consequence, due to equation (A.4) in the Appendix, we deduce for $j \geq J$ that

$$H_0^{(1)}\left(i|\lambda_j||\tilde{x}-\tilde{y}| \right) = \frac{2}{i\pi} K_0\left(|\lambda_j||\tilde{x}-\tilde{y}| \right).$$

It is well-known that K_0 has a logarithmic singularity at the origin. In consequence, for $f \in \mathbb{C}_0^\infty(\tilde{\Lambda}_\rho)$ we interchange integral and derivative. Again, for simplicity we denote $\mu_j = |\lambda_j|$. All together, we have

$$\left\| \frac{\partial \mathcal{V}_j}{\partial x_k} f \right\|_{L^2(\tilde{\Lambda}_\rho)} = \left\| \tilde{x} \mapsto \int_{\tilde{\Lambda}_\rho} \frac{\partial}{\partial x_k} K_0(\mu_j|\tilde{x}-\tilde{y}|) f(\tilde{y}) d\tilde{y} \right\|_{L^2(\tilde{\Lambda}_\rho)} \qquad \text{for } k = 1, 2.$$

Then, due to equation (A.7) from the Appendix, we obtain

$$\left\| \tilde{x} \mapsto \int_{\tilde{\Lambda}_\rho} \frac{\partial}{\partial x_k} K_0(\mu_j|\tilde{x}-\tilde{y}|) f(\tilde{y}) d\tilde{y} \right\|_{L^2(\tilde{\Lambda}_\rho)} = \left\| \tilde{x} \mapsto \int_{\tilde{\Lambda}_\rho} \mu_j \frac{x_k - y_k}{|\tilde{x}-\tilde{y}|} K_1(\mu_j|\tilde{x}-\tilde{y}|) f(\tilde{y}) \, d\tilde{y} \right\|_{L^2(\tilde{\Lambda}_\rho)},$$

where $k = 1, 2$. Furthermore, by the fact that it holds $(x_k - y_k)/|\tilde{x} - \tilde{y}| \le 1$, we compute

$$\int_{\tilde{\Lambda}_\rho} \mu_j \frac{x_k - y_k}{|\tilde{x} - \tilde{y}|} K_1(\mu_j |\tilde{x} - \tilde{y}|) f(\tilde{y})\, d\tilde{y} \le C \int_{\tilde{\Lambda}_\rho} |\mu_j K_1(\mu_j |\tilde{x} - \tilde{y}|) f(\tilde{y})|\, d\tilde{y}.$$

Now, we abbreviate the derivative of kernel of $\mathcal{V}_j$ and we split it into

$$\kappa_1(\mu_j, \tilde{x}, \tilde{y}) = \begin{cases} \mu_j K_1(\mu_j |\tilde{x} - \tilde{y}|) & \text{for } 0 < \mu_j |\tilde{x} - \tilde{y}| \le 1, \\ 0 & \text{for } \mu_j |\tilde{x} - \tilde{y}| > 1, \end{cases}$$

and

$$\kappa_2(\mu_j, \tilde{x}, \tilde{y}) = \begin{cases} 0 & \text{for } 0 < \mu_j |\tilde{x} - \tilde{y}| \le 1, \\ \mu_j K_1(\mu_j |\tilde{x} - \tilde{y}|) & \text{for } \mu_j |\tilde{x} - \tilde{y}| > 1. \end{cases}$$

It is well-known by [AS64, 9.6.9] that it holds

$$K_1(|z|) \le |z|^{-1}, \qquad \text{for } |z| \le 1.$$

We point out that due to [AS64, 9.6.9], we know that the power could be chosen as $-\alpha$ for arbitrary $\alpha > 0$ such that

$$K_\alpha(z) \le \frac{1}{2}\Gamma(\alpha) \left(\frac{z}{2}\right)^{-\alpha}.$$

In consequence, we obtain that

$$|\kappa_1(\mu_j, \tilde{x}, \tilde{y})| \le C \mu_j |\mu_j \tilde{x} - \tilde{y}|^{-1} \le C |\tilde{x} - \tilde{y}|^{-1},$$

where $C > 0$ is a constant. Now, like in the proof of Lemma 4.2.3 in equation (4.14), Cauchy-Schwarz inequality and Lemma 2.2.4 imply that

$$\left| \int_{\tilde{\Lambda}_\rho} |\kappa_1(\mu_j, \tilde{x}, \tilde{y}) f(\tilde{y})|\, d\tilde{y} \right|^2 \le C \left(\int_{\tilde{\Lambda}_\rho} \frac{|f(\tilde{y})|^2}{|\tilde{x} - \tilde{y}|}\, d\tilde{y} \right) \left(\int_{B(\tilde{x}, Cj)} \frac{1}{|\tilde{x} - \tilde{y}|}\, d\tilde{y} \right),$$

where $B(\tilde{x}, Cj)$ denotes a ball in $\tilde{x}$ with radius Cj. Due to Lemma A.2.2 from the Appendix we see that

$$\int_{B(\tilde{x}, Cj)} \frac{1}{|\tilde{x} - \tilde{y}|} d\tilde{y} \le \frac{2\pi}{Cj},$$

such that we get that

$$\left| \int_{\tilde{\Lambda}_\rho} |\kappa_1(\mu_j, \tilde{x}, \tilde{y}) f(\tilde{y})|\, d\tilde{y} \right|^2 \le \frac{C}{j} \left(\int_{\tilde{\Lambda}_\rho} \frac{|f(\tilde{y})|^2}{|\tilde{x} - \tilde{y}|}\, d\tilde{y} \right).$$

Moreover, Fubini's theorem implies

$$\int_{\tilde{\Lambda}_\rho} \left| \int_{\tilde{\Lambda}_\rho} |\kappa_1(\mu_j, \tilde{x}, \tilde{y}) f(\tilde{y})|\, d\tilde{y} \right|^2 d\tilde{x} \le \frac{C}{j^2} \|f\|^2_{L^2(\tilde{\Lambda}_\rho)}. \tag{4.20}$$

Next, we look at the absolute value of κ_2. Using [Bar10, Eq. 3.5] with $x = 1$ and $y > 0$, we have for $\alpha \ge 0$, $z \in \mathbb{C}$, and $z > 1$ that

$$K_\alpha(1) \exp(1 - z)\sqrt{z} > K_\alpha(z) > 0,$$

and due to estimate (4.17) in the proof of Lemma 4.2.3, we know that

$$(j|\tilde{x} - \tilde{y}|)^{1/2} \exp(-(j|\tilde{x} - \tilde{y}|)/2) \le \max_{s \in (0,\infty)} \{s e^{-s}\} \le C.$$

Thus,

$$\left|\int_{\tilde{\Lambda}_\rho} \kappa_2(\mu_j, \tilde{y}, \tilde{x}) f(\tilde{y}) d\tilde{y}\right|^2 \leq C \left(\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left| j \exp\left(-\frac{j|\tilde{x}-\tilde{y}|}{2}\right) f(\tilde{y})\right| d\tilde{y}\right)^2.$$

Furthermore, once more Cauchy-Schwarz inequality states

$$\left(\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left| j \exp\left(-\frac{j|\tilde{x}-\tilde{y}|}{2}\right) f(\tilde{y})\right| d\tilde{y}\right)^2$$
$$\leq C \left(\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left| j \exp\left(-\frac{j|\tilde{x}-\tilde{y}|}{2}\right)\right| d\tilde{y}\right) \left(\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left| j \exp\left(-\frac{j|\tilde{x}-\tilde{y}|}{2}\right)\right| |f(\tilde{y})|^2 d\tilde{y}\right).$$

Now, using Lemma A.2.4 from the Appendix,

$$\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left| j \exp\left(-j\frac{|\tilde{x}-\tilde{y}|}{2}\right)\right| d\tilde{y} \leq 4\pi \exp(-j\rho) \left[2\rho - \frac{1}{j}\right],$$

to obtain for fixed j that

$$\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left| j \exp\left(-j\frac{|\tilde{x}-\tilde{y}|}{2}\right)\right| d\tilde{y} \leq \frac{C}{j}.$$

Thus, we obtain

$$\int_{\tilde{\Lambda}_\rho} \left(\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left| \exp\left(-j\frac{|\tilde{x}-\tilde{y}|}{2}\right)\right| d\tilde{y}\right) \left(\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left| j \exp\left(-j\frac{|\tilde{x}-\tilde{y}|}{2}\right)\right| |f(\tilde{y})|^2 d\tilde{y}\right) d\tilde{x}$$
$$\leq \frac{C}{j} \int_{\tilde{\Lambda}_\rho} \left(\int_{\tilde{\Lambda}_\rho} j \exp\left(-j\frac{|\tilde{x}-\tilde{y}|}{2}\right) |f(\tilde{y})|^2 d\tilde{y}\right) d\tilde{x}.$$

By Fubini's theorem we find that

$$\int_{\tilde{\Lambda}_\rho} \left|\int_{\tilde{\Lambda}_\rho} \kappa_2(\mu_j, \tilde{x}, \tilde{y}) f(\tilde{y})\, d\tilde{y}\right|^2 d\tilde{x} \leq \frac{C}{j} \int_{\tilde{\Lambda}_\rho} |f(\tilde{y})|^2 \left(\int_{\tilde{\Lambda}_\rho} j \exp\left(-j\frac{|\tilde{x}-\tilde{y}|}{2}\right) d\tilde{x}\right) d\tilde{y}.$$

Then, using for fixed j large enough Lemma A.2.6 from the Appendix,

$$\int_{\tilde{\Lambda}_\rho} \left| j \exp\left(-j\frac{|\tilde{x}-\tilde{y}|}{2}\right)\right| d\tilde{x} \leq 4\pi \exp(-j\rho) \left[2\rho - \frac{1}{j}\right] + \frac{4\pi}{j} \leq \frac{C}{j},$$

to compute that

$$\int_{\tilde{\Lambda}_\rho} \left|\int_{\tilde{\Lambda}_\rho} \kappa_2(\mu_j, \tilde{x}, \tilde{y}) f(\tilde{y}) d\tilde{y}\right|^2 d\tilde{x} \leq \frac{C}{j} \int_{\tilde{\Lambda}_\rho} |f(\tilde{y})|^2\, d\tilde{y} = \frac{C}{j} \|f\|^2_{L^2(\tilde{\Lambda}_\rho)}.$$

Now, the last estimate and estimate (4.20) yields

$$\int_{\tilde{\Lambda}_\rho} \left|\int_{\tilde{\Lambda}_\rho} \kappa(\mu_j, \tilde{x}, \tilde{y}) f(\tilde{y}) d\tilde{y}\right|^2 d\tilde{x} \leq \left(\frac{C}{j} + \frac{C}{j}\right) \|f\|^2_{L^2(\tilde{\Lambda}_\rho)} \leq \frac{C}{j} \|f\|^2_{L^2(\tilde{\Lambda}_\rho)}.$$

Finally, with Lemma 4.2.3 we obtain

$$\|\mathcal{V}_j f\|^2_{H^1(\tilde{\Lambda}_\rho)} = \|\mathcal{V}_j f\|^2_{L^2(\tilde{\Lambda}_\rho)} + \left\|\frac{\partial \mathcal{V}_j}{\partial x_1} f\right\|_{L^2(\tilde{\Lambda}_\rho)} + \left\|\frac{\partial \mathcal{V}_j}{\partial x_2} f\right\|_{L^2(\tilde{\Lambda}_\rho)} \leq \frac{C}{j} \|f\|^2_{L^2(\tilde{\Lambda}_\rho)}.$$

This ends the proof. □

From now, for simplicity, we set $\tilde{\Lambda}_\rho = (-\rho, \rho)$ for $m = 2$, too.

Remark 4.2.5. *(a) The last bounds for the integral operator $\mathcal{V}_j$ can also be shown for $m = 2, 3$ independently via techniques from theory of partial differential equations.*

(b) We could in this section as well work with any open set that contains the support of f instead of the rectangular domains $\tilde{\Lambda}_\rho$. In the subsequent sections, however, rectangular domains become important since our analysis relies on Fourier coefficients.

Lemma 4.2.6. *The operator $\mathcal{V}_j$ previously defined (for $m = 2$ and $m = 3$) satisfies*

$$\|\mathcal{V}_j\|_{L^2(\tilde{\Lambda}_\rho)\to L^2(\tilde{\Lambda}_\rho)} \leq \frac{C}{j^2} \qquad \text{for all } j \in \mathbb{N},$$

where $C > 0$ is a constant independent from j.

Proof. We only treat the case $m = 3$ since the result has been already proved for $m = 2$. As the kernel of $\mathcal{V}_j$ is continuous for $m = 2$ and weakly singular (with logarithmic singularity) for $m = 3$, we have that $\mathcal{V}_j$ is bounded on $L^2(\tilde{\Lambda}_\rho)$. In consequence, $v = \mathcal{V}_j f$ belongs to $L^2(\tilde{\Lambda}_\rho)$ if $f \in L^2(\tilde{\Lambda}_\rho)$. For arbitrary large $\rho > 0$, we extend f by zero to a function in $L^2(\mathbb{R}^{m-1})$.

Since $\mathcal{V}_j$ is the convolution of the radiating or decaying fundamental solution with f, it follows from distribution theory, see [Rud91, Theorem 6.30 & Chapter 8] that $v = \mathcal{V}_j f \in L^2(\tilde{\Lambda}_\rho)$ solves $(\Delta_{\tilde{x}} + \lambda_j^2)v = -f$ in the distributional sense of $\mathbb{R}^{m-1}$ (see also [CK13, SV02] for $m = 3$). Moreover, [Rud91, Theorem 8.12] shows that v belongs to $H^2(\tilde{\Lambda}_\rho)$ for arbitrary $\rho > 0$. For $j > J$ the eigenvalue λ_j^2 is negative by definition of $J \in \mathbb{N}$, such that the kernel of $\mathcal{V}_j$ is exponentially decaying in $\tilde{x}$ The volume potential v hence decays exponentially, too. In consequence, we can integrate by parts to find that

$$\int_{\mathbb{R}^{m-1}} f\overline{v}\, d\tilde{x} = -\int_{\mathbb{R}^{m-1}} (\Delta_{\tilde{x}} v + \lambda_j^2 v)\overline{v}\, d\tilde{x} = \int_{\mathbb{R}^{m-1}} (|\nabla_{\tilde{x}} v|^2 - \lambda_j^2 |v|^2)\, d\tilde{x} \geq |\lambda_j|^2 \int_{\tilde{\Lambda}_\rho} |v|^2\, d\tilde{x}.$$

Since f is supported in $\tilde{\Lambda}_\rho$, the Cauchy-Schwarz inequality hence implies that

$$\|v\|_{L^2(\tilde{\Lambda}_\rho)} \leq \frac{1}{|\lambda_j|^2} \|f\|_{L^2(\tilde{\Lambda}_\rho)}.$$

Due to Lemma 2.2.4 we know $|\lambda_j|^2 \geq cj^2$ for all $j > J$. Then, the operator norms of $\mathcal{V}_j$ for all $j > J$ are bounded, too. Moreover, the operator norms of $\mathcal{V}_j$ for the finite number $j \leq J$ can be simply estimated uniformly by their maximum. This ends the proof. □

Next, we define the integral operator

$$(\mathcal{V}f)(x) = \sum_{j\in\mathbb{N}} \phi_j(x_m) \int_0^H \phi_j(y_m) \mathcal{V}_j f(\cdot, y_m)\, dy_m \qquad \text{for } f \in L^2(\Lambda_\rho). \tag{4.21}$$

Moreover, for $N \in \mathbb{N}$ the corresponding truncated series is denoted by

$$(\mathcal{V}^{(N)}f)(x) = \sum_{j=1}^{N} \phi_j(x_m) \int_0^H \phi_j(y_m) \mathcal{V}_j f(\cdot, y_m)\, dy_m \qquad \text{for } f \in L^2(\Lambda_\rho). \tag{4.22}$$

Lemma 4.2.7. *It holds that $\mathcal{V}$ is bounded on $L^2(\Lambda_\rho)$ and bounded from $L^2(\Lambda_\rho)$ into $H^1(\Lambda_\rho)$.*

Proof. Since the functions $\{\phi_j\}_{j\in\mathbb{N}}$ are orthonormal on $L^2(0, H)$, Parseval's equality implies that the mapping

$$f \mapsto \left(\int_0^H \phi_j(y_m) f(\cdot, y_m)\, dy_m \right)_{j\in\mathbb{N}},$$

is an isometry from $L^2(\Lambda_\rho) = L^2([0,H]; L^2(\tilde{\Lambda}_\rho))$ into $\ell^2(\mathbb{N}; L^2(\tilde{\Lambda}_\rho))$. More precisely, we have

$$\begin{aligned}\|f\|^2_{L^2(\Lambda_\rho)} &= \int_0^H \int_{\tilde{\Lambda}_\rho} |f(\tilde{x}, x_m)|^2 \, d\tilde{x} \, dx_m \\ &= \sum_{j=1}^\infty \int_{\tilde{\Lambda}_\rho} \left| \int_0^H \phi_j(y_m) f(\tilde{x}, y_m) \, dy_m \right|^2 d\tilde{x} \\ &= \sum_{j=1}^\infty \left\| \tilde{x} \mapsto \int_0^H \phi_j(y_m) f(\tilde{x}, y_m) \, dy_m \right\|^2_{L^2(\tilde{\Lambda}_\rho)}.\end{aligned}$$

We next reformulate the L^2-norm of $\mathcal{V}^{(N)} f$ for $f \in L^2(\Omega)$, exploiting again orthogonality of the eigenfunctions ϕ_j,

$$\begin{aligned}\|\mathcal{V}^{(N)} f\|^2_{L^2(\Lambda_\rho)} &= \sum_{j=1}^\infty \int_{\tilde{\Lambda}_\rho} \left| \int_0^H \phi_j(y_m) \big(\mathcal{V}^{(N)} f(\cdot, y_m)\big)(\tilde{x}) \, dy_m \right|^2 d\tilde{x} \\ &= \sum_{j=1}^\infty \int_{\tilde{\Lambda}_\rho} \left| \int_0^H \phi_j(y_m) \sum_{\ell=1}^N \phi_\ell(y_m) \int_0^H \phi_\ell(z_m) \big(\mathcal{V}_\ell f(\cdot, z_m)\big)(\tilde{x}) \, dz_m \, dy_m \right|^2 d\tilde{x} \\ &= \sum_{j=1}^N \int_{\tilde{\Lambda}_\rho} \left| \int_0^H \phi_j(y_m) \big(\mathcal{V}_j f(\cdot, y_m)\big)(\tilde{x}) \, dy_m \right|^2 d\tilde{x}\end{aligned}$$

Now, we interchange the operator $\mathcal{V}_j$ with the inner product of $L^2(0,H)$ due to continuity of the operator $g \mapsto \int_0^H \phi_j g \, dx_m$ from $L^2(\Lambda_\rho)$ into $L^1(\tilde{\Lambda}_\rho)$. Then,

$$\|\mathcal{V}^{(N)} f\|^2_{L^2(\Lambda_\rho)} = \sum_{j=1}^N \left\| \mathcal{V}_j \Big(\int_0^H \phi_j(y_m) f(\cdot, y_m) \, dy_m \Big)(\tilde{x}) \right\|^2_{L^2(\tilde{\Lambda}_\rho)},$$

where In particular, $\mathcal{V} f$ belongs to $L^2(\Lambda_\rho)$ if the limit as $N \to \infty$ of the latter right-hand side exists. We validate that this is indeed the case by estimating

$$\begin{aligned}\|\mathcal{V}^{(N)} f\|^2_{L^2(\Lambda_\rho)} &= \sum_{j=1}^N \left\| \mathcal{V}_j \Big(\int_0^H \phi_j(y_m) f(\cdot, y_m) \, dy_m \Big)(\tilde{x}) \right\|^2_{L^2(\tilde{\Lambda}_\rho)} \\ &\le \sum_{j=1}^N \frac{C^2}{j^4} \left\| \tilde{x} \mapsto \int_0^H \phi_j(y_m) f(\tilde{x}, y_m) \, dy_m \right\|^2_{L^2(\tilde{\Lambda}_\rho)} \\ &\le C \int_{\tilde{\Lambda}_\rho} \sum_{j=1}^N \left| \int_0^H \phi_j(y_m) f(\tilde{x}, y_m) \, dy_m \right|^2 d\tilde{x} \\ &= C \int_{\tilde{\Lambda}_\rho} \|f_N(\tilde{x}, \cdot)\|^2_{L^2(0,H)} \, d\tilde{x} \quad \text{for } f_N(x) = \sum_{j=1}^N \int_0^H \phi_j(y_m) f(\tilde{x}, y_m) \, dy_m \, \phi_j(x_m),\end{aligned}$$

where we exploited the L^2-bounds for the individual operators $\mathcal{V}_j$ from Lemma 4.2.6. Since f_N is a Cauchy sequence in $L^2(\Lambda_\rho)$, the same holds for $\mathcal{V}^{(N)} f$, such that the series defining $\mathcal{V}$ converges in the operator norm of $L^2(\Lambda_\rho)$ to a bounded operator.

The same technique actually shows that the series defining $\mathcal{V}^{(N)} f$ converges in $H^1(\Lambda_\rho)$ to $\mathcal{V} f$, such that $\mathcal{V}$ is even bounded from $L^2(\Lambda_\rho)$ into $H^1(\Lambda_\rho)$. (Recall that we already know from the proof of Lemma 4.2.6 that each series term even belongs to $H^2(\Lambda_\rho)$.) To this end, we note that

$$\nabla_{\tilde{x}} (\mathcal{V}^{(N)} f)(x) = \sum_{j=1}^N \phi_j(x_m) \int_0^H \phi_j(y_m) \nabla_{\tilde{x}} \mathcal{V}_j f(\cdot, y_m) \, dy_m \qquad \text{for } f \in L^2(\Lambda_\rho).$$

Hence,

$$\begin{aligned}\|\nabla_{\tilde{x}}\mathcal{V}^{(N)}f\|^2_{L^2(\Lambda_\rho)} &= \sum_{j=1}^{N}\left\|\nabla_{\tilde{x}}\mathcal{V}_j\Big(\int_0^H \phi_j(y_m)f(\cdot,y_m)\,dy_m\Big)(\tilde{x})\right\|^2_{L^2(\tilde{\Lambda}_\rho)}\\ &\le \sum_{j=1}^{N}\frac{C^2}{j}\left\|\tilde{x}\mapsto\int_0^H \phi_j(y_m)f(\tilde{x},y_m)\,dy_m\right\|^2_{L^2(\tilde{\Lambda}_\rho)}\\ &\le C\int_{\tilde{\Lambda}_\rho}\sum_{j=1}^{N}\left|\int_0^H \phi_j(y_m)f(\tilde{x},y_m)\,dy_m\right|^2 d\tilde{x} = C\int_{\tilde{\Lambda}_\rho}\|f_N(\tilde{x},\cdot)\|^2_{L^2(0,H)}\,d\tilde{x},\end{aligned}$$

such that the same argument used to show L^2 convergence of $\mathcal{V}^{(N)}f$ to $\mathcal{V}f$ shows that $\nabla_{\tilde{x}}\mathcal{V}^{(N)}f$ tends to $\nabla_{\tilde{x}}\mathcal{V}f$ in $L^2(\Lambda_\rho)$. Further,

$$\frac{\partial}{\partial x_m}(\mathcal{V}^{(N)}f)(x) = \sum_{j=1}^{N}\phi_j'(x_m)\int_0^H \phi_j(y_m)\mathcal{V}_jf(\cdot,y_m)\,dy_m \quad \text{for } f\in L^2(\Lambda_\rho), \tag{4.23}$$

and we use the bound $\|\phi_j'\|_{L^2(0,H)}\le Cj$ from Lemma 2.2.4 to estimate

$$\begin{aligned}\left\|\frac{\partial}{\partial x_m}\mathcal{V}^{(N)}f\right\|^2_{L^2(\Lambda_\rho)} &\le \sum_{j=1}^{N}\int_0^H|\phi_j'(x_m)|^2\,dx_m\int_{\tilde{\Lambda}_j}\int_0^H|\phi_j(x_m)|^2\,|\big(\mathcal{V}_jf(\cdot,y_m)\big)(\tilde{x})|^2\,d\tilde{y}\,dy_m\\ &\le C\sum_{j=1}^{N}j^2\int_0^H|\phi_j(x_m)|^2\,dx_m\int_{\tilde{\Lambda}_j}\int_0^H|\big(\mathcal{V}_jf(\cdot,y_m)\big)(\tilde{x})|^2\,d\tilde{y}\,dy_m\\ &\le C\sum_{j=1}^{N}j^2\int_0^H\|\big(\mathcal{V}_jf(\cdot,y_m)\big)(\tilde{x})\|^2_{L^2(\tilde{\Lambda}_\rho)}\,dy_m \qquad (4.24)\\ &\le C\sum_{j=1}^{N}j^2\int_0^H\frac{C^2}{j^4}\|f(\cdot,y_m)\|^2_{L^2(\tilde{\Lambda}_\rho)}\,dy_m \le C\sum_{j=1}^{N}\frac{1}{j^2}\,\|f\|^2_{L^2(\Lambda_\rho)},\end{aligned}$$

where $C>0$ is a constant. Since the series $\sum_{j\in\mathbb{N}}1/j^2$ converges, the series defining the partial derivative of

$$\frac{\partial}{\partial x_m}\mathcal{V}^{(N)}f = \sum_{j=1}^{\infty}\phi_j'(x_m)\int_0^H \phi_j(y_m)\mathcal{V}_jf(\cdot,y_m)\,dy_m,$$

converges in $L^2(\Lambda_\rho)$ as well, since it holds

$$\left\|\frac{\partial}{\partial x_m}(\mathcal{V}f)\right\|^2_{L^2(\Lambda_\rho)} \le C\sum_{j=1}^{N}\frac{1}{j^2}\,\|f\|^2_{L^2(\Lambda_\rho)}.$$

We point out that for $m=3$ the bound from Lemma 4.2.3 would not be sufficient to obtain convergences in (4.24). Finally, the decomposition

$$\|v\|^2_{H^1(\Lambda_\rho)} = \|v\|^2_{L^2(\Lambda_\rho)} + \|\nabla_{\tilde{x}}v\|^2_{L^2(\Lambda_\rho)} + \left\|\frac{\partial v}{\partial x_m}\right\|^2_{L^2(\Lambda_\rho)} \le C\|f\|^2_{L^2(\Lambda_\rho)}$$

implies boundedness of $f\mapsto v=\mathcal{V}f$ from $L^2(\Lambda_\rho)$ into $H^1(\Lambda_\rho)$.

□

Corollary 4.2.8. *For $L^2(\Lambda_\rho)$, which is defined by*

$$L^2(\Lambda_\rho) := \{f:\Lambda_\rho\to\mathbb{C}, f \text{ measurable }, \int_{\Lambda_\rho}|f(x)|^2d\mu<\infty\},$$

we obtain with Fourier coefficients an alternative representation

$$L^2(\Lambda_\rho) = \{f : \Lambda_\rho \to \mathbb{C}, f(x) = \sum_{j\in\mathbb{N}} \hat{f}(j,\tilde{x})\phi_j(x_m), \sum_{j\in\mathbb{N}} \int_{\tilde{\Lambda}_\rho} |\hat{f}(j,\tilde{x})|^2 d\tilde{x} < \infty\},$$

where

$$\sum_{j\in\mathbb{N}} \int_{\tilde{\Lambda}_\rho} |\hat{f}(j,\tilde{x})|^2 d\tilde{x} = \|f\|^2_{L^2(\Lambda_\rho)} < \infty.$$

Then for the volume integral operator $\mathcal{V}$ applied to the function f we have

$$\mathcal{V}f = \sum_{j_m\in\mathbb{N}} \mathcal{V}_{j_m}\left(\tilde{y} \mapsto \hat{f}(j_m,\tilde{y})\right)\phi_{j_m}(x_m), \qquad \text{and}$$

$$\frac{\partial}{\partial x_k}\mathcal{V}f = \sum_{j_m\in\mathbb{N}} \frac{\partial}{\partial x_k}\mathcal{V}_{j_m}\left(\tilde{y} \mapsto \hat{f}(j_m,\tilde{y})\right)\phi_{j_m}(x_m),$$

where $k = 1,2$ (two-dimensional case) or $k = 1,2,3$ (three-dimensional case). It moreover holds

$$\|\mathcal{V}f\|^2_{L^2(\Lambda_\rho)} = \sum_{j\in\mathbb{N}} \|\tilde{x} \mapsto \mathcal{V}_j\hat{f}(j,\tilde{y})\|^2_{L^2(\tilde{\Lambda}_\rho)}.$$

Note that the last corollary gives us a clear separation of $L^2(\Lambda_\rho)$ in horizontal component acting on the variables $\tilde{x}$ and vertical component acting on x_m. In particular, we have the representation

$$L^2(\Lambda_\rho) = L^2([0,H]; L^2(\tilde{\Lambda}_\rho)).$$

We finally show that the volume potential $\mathcal{V}$ is also bounded from $L^2(\Lambda_\rho)$ into $H^2(\Lambda_\rho)$ and additionally defines solutions to the Helmholtz equation.

Theorem 4.2.9. *For $f \in L^2(\Lambda_\rho)$, the potential $v = \mathcal{V}f$ belongs to $H^2(\Lambda_\rho)$ and solves the Helmholtz equation*

$$\Delta v + \frac{\omega^2}{c(x_m)^2} v = -f \qquad \text{in } L^2(\Lambda_\rho).$$

Its extension to Ω by (4.21) belongs to $H^2_{\text{loc}}(\Omega)$ and satisfies boundary conditions (2.5), (2.6) and radiation conditions (3.3).

Proof. We already showed in Lemma 4.2.7 $f \mapsto \mathcal{V}f$ is bounded from $L^2(\Lambda_\rho)$ into $H^1(\Lambda_\rho)$, such that $v = \mathcal{V}f$ belongs to $H^1(\Lambda_\rho)$. From the proof of Lemma 4.2.6 we moreover know that each series term, and hence also truncated series $v_N = \mathcal{V}^{(N)}f$ defined in (4.22), even belongs to $H^2(\Lambda_\rho)$. Due to the construction of the eigenfunction ϕ_j solving $\phi_j'' + \omega^2/c(x_m)^2\,\phi_j = \lambda_j^2\phi_j$, the truncated series v_N moreover solves

$$\begin{aligned}
\Delta v_N(x) + \frac{\omega^2}{c^2(x_m)} v_N(x) = \sum_{j=1}^N \Bigg[\phi_j(x_m)\int_0^H \phi_j(y_m)\Delta_{\tilde{x}}\left(\mathcal{V}_j f(\cdot,y_m)\right)(\tilde{x})\,dy_m \\
+ \left(\phi_j''(x_m) + \frac{\omega^2}{c^2(x_m)}\right)\int_0^H \phi_j(y_m)\left(\mathcal{V}_j f(\cdot,y_m)\right)(\tilde{x})\,dy_m\Bigg] \\
= \sum_{j=1}^N \Bigg[\phi_j(x_m)\int_0^H \phi_j(y_m)\big[-\lambda_j^2\left(\mathcal{V}_j f(\cdot,y_m)\right)(\tilde{x}) - f(\tilde{x},y_m)\big]\,dy_m \\
+ \lambda_j^2\phi(x_m)\int_0^H \phi_j(y_m)\left(\mathcal{V}_j f(\cdot,y_m)\right)(\tilde{x})\,dy_m\Bigg] \qquad (4.25) \\
= -\sum_{j=1}^N (f(\tilde{x},\cdot),\phi_j)\phi_j(x_m) = -f_N(x) \qquad \text{in } L^2(\Lambda_\rho),
\end{aligned}$$

for arbitrary $\rho > 0$. Indeed, interchanging the Laplacian $\Delta_{\tilde{x}}$ with the integral in y_m from 0 to H in the first step is valid, since $\Delta_{\tilde{x}}$ does not act on y_m, since $g \mapsto \int_0^H \phi_j g \, dy_m$ is bounded from $L^2(\Lambda_\rho)$ into $L^2(\tilde{\Lambda}_\rho)$ by the Cauchy-Schwarz inequality and since $\tilde{x} \mapsto (\mathcal{V}_j f(\cdot, y_m))(\tilde{x})$ is twice weakly differentiable. For completeness, by Lemma A.2.7 from the Appendix, we can alternatively compute for $m = 2$ that

$$\int_{\Lambda_{2\rho}} \Delta v_N \psi + \frac{\omega^2}{c^2(x_2)} v_N \psi \, dx = -\int_{\Lambda_{2\rho}} f_N(x)\psi(x) \, dx \qquad \text{for all } \psi \in C_0^\infty(\Lambda_\rho),$$

where

$$f_N(x) = \sum_{j=1}^{N} (f(x_1, \cdot), \phi_j)_{L^2(0,H)} \phi_j(x_2).$$

It is moreover well-known from [SV02] that this identity also holds for $m = 3$.

We conclude that $v_N \in H^1(\Lambda_\rho)$ is a weak solution of the Helmholtz equation with right-hand side $f_N \in L^2(\Lambda_\rho)$ and use interior regularity results, see [McL00, Theorem 4.16], to obtain that

$$\|v_N\|_{H^2(\Lambda_\rho)} \leq C \left[\|v_N\|_{H^1(\Lambda_{2\rho})} + \|f_N\|_{L^2(\Lambda_{2\rho})} \right] \leq C \|f_N\|_{L^2(\Lambda_\rho)}.$$

Hence, v_N is a Cauchy sequence in $H^2(\Lambda_\rho)$ and consequently converges in the norm of $H^2(\Lambda_\rho)$ to $v \in H^2(\Lambda_\rho)$. Taking the limit as $N \to \infty$ of the left- and right-hand side in estimate (4.25) thus shows that v indeed solves the Helmholtz equation $\Delta v + \omega^2/c^2(x_m)\, v = -f$ in $L^2(\Lambda_\rho)$.

Extending $v = \mathcal{V}f$ by (4.21) yields a function in $H^2_{\text{loc}}(\Omega)$, since the series (4.21) converges in $H^2(\Lambda_{\rho^*})$ for arbitrary $\rho^* \geq \rho$. The trace theorem in $H^1(\Lambda_{\rho^*})$ implies that the restriction of v to $\Gamma_0 \cap \partial\Lambda_{\rho^*}$ of Λ_{ρ^*} vanishes, since the eigenfunctions ϕ_j vanish at $x_m = 0$. The same argument shows that $\partial v/\partial x_m \in H^1(\Lambda_{\rho^*})$ vanishes on $\Gamma_H \cap \partial\Lambda_{\rho^*}$, since $\phi_j'(H) = 0$ (recall that $\phi_j' \in C^{0,1/2}$ is Hölder continuous). Consequently, v satisfies the waveguide boundary conditions (2.5) and (2.6).

It remains to show that the extended potential $v = \mathcal{V}f \in H^2_{\text{loc}}(\Omega)$ satisfies the radiation conditions (3.3). From the proof of Lemma 4.2.7 we know that we can write

$$v(x) = \sum_{j \in \mathbb{N}} \phi_j(x_m) \mathcal{V}_j \left(\int_0^H \phi_j(y_m) f(\cdot, y_m) \, dy_m \right)(\tilde{x}) \quad \text{for } f \in L^2(\Lambda_\rho).$$

Our goal now is to verify that the potential $\mathcal{V}_j$ satisfies the radiation conditions

$$\lim_{|\tilde{x}| \to \infty} \sqrt{\tilde{x}} \left(\frac{\partial \mathcal{V}_j}{\partial |\tilde{x}|} - i|\lambda_j| \mathcal{V}_j \right) = 0, \qquad \text{uniformly in } \frac{\tilde{x}}{|\tilde{x}|}.$$

We recall for $m = 2$ that the potential $\mathcal{V}_j$ is defined by

$$\mathcal{V}_j f(x_1) = \frac{i}{2} \frac{1}{\lambda_j} \int_{-\rho}^{\rho} \exp(i\lambda_j |x_1 - y_1|) f(y_1) \, dy_1, \qquad \text{for } f \in L^2(\tilde{\Lambda}_\rho),$$

and its derivative is denoted by

$$\frac{\partial}{\partial x_1} \mathcal{V}_j f(x_1) = \frac{i}{2\lambda_j} \int_{-\rho}^{\rho} \frac{\partial}{\partial x_1} \exp(i\lambda_j |x_1 - y_1|) f(y_1) \, dy_1 \qquad \text{for } f \in L^2(\tilde{\Lambda}_\rho).$$

For $x_1 > \rho$ we see

$$\mathcal{V}_j f(x_1) = \sum_{j \in \mathbb{N}} \exp(i\lambda_j x_1) \frac{i}{2\lambda_j} \int_{-\rho}^{\rho} \exp(-i\lambda_j y_1) f(y_1) \, dy_1 \phi_j(x_2).$$

We plug the latter expression in the definition of the radiation conditions and we see that

$$\lim_{x_1 \to \infty} \left[\frac{\partial}{\partial x_1} \left[\exp(i\lambda_j x_1) \right] - i\lambda_j \exp(i\lambda_j x_1) \right] = \lim_{x_1 \to \infty} \left[i\lambda_j \exp(i\lambda_j x_1) - i\lambda_j \exp(i\lambda_j x_1) \right] = 0.$$

Similar, we obtain the case where $x_1 < -\rho$ and $x_1 \to -\infty$. For $m = 3$ it is well-known that $\mathcal{V}f$ solves the radiation conditions (see e.g. [CK13]). This completes the proof. □

Now, we turn back to the solution of the scattering problem from Chapter 3.1. Recall, that an incident field u^i satisfy the unperturbed Helmholtz equation

$$\Delta u^i + \frac{\omega^2}{c^2(x_m)} u^i = 0 \text{ in } \Omega, \tag{4.26}$$

and boundary conditions

$$u^i = 0 \text{ on } \Gamma_0 := \{x \in \mathbb{R}^m : x_m = 0\} \quad \text{and} \quad \frac{\partial u^i}{\partial x_m} = 0 \text{ on } \Gamma_H := \{x \in \mathbb{R}^m : x_m = H\}, \tag{4.27}$$

and radiation conditions (3.3). We know that the incident field u^i is disturbed by an inhomogeneous medium $D \subset \Omega$ characterized by the refractive index $n^2 : \mathbb{R}^m \to \mathbb{C}$. We further know that a scatterer D creates a scattered field u^s which solves

$$\Delta u^s(x) + \frac{\omega^2}{c^2(x_m)} n^2(x) u^s(x) = -\frac{\omega^2}{c^2(x_m)} q u^i \qquad \text{for } x \in \Omega, \tag{4.28}$$

such that the total field u satisfies

$$u(x) = u^i(x) + u^s(x) \qquad \text{for } x \in \Omega.$$

Therefore, the total field u solves the perturbed Helmholtz equation

$$\Delta u(x) + \frac{\omega^2}{c^2(x_m)} n^2(x) u(x) = 0 \qquad \text{for } x \in \Omega. \tag{4.29}$$

Due to Chapter 3.4 we know that the solution $u \in H^2(\Omega)$ can be found via variational formulation. We show now that one can equivalently determine u via an integral equation of Lippmann-Schwinger type.

If u^i solves the unperturbed Helmholtz equation (4.26), corresponding boundary conditions (4.27) and radiation condition (3.3), we know that u^s solves

$$\Delta u^s + \frac{\omega^2}{c^2(x_m)} u^s = -\frac{\omega^2}{c^2(x_m)} q(u^i + u^s) \qquad \text{in } \Omega,$$

together with radiation condition (3.3) and boundary conditions

$$u^s = 0 \text{ on } \Gamma_0 := \{x \in \mathbb{R}^m : x_m = 0\} \quad \text{and} \quad \frac{\partial u^s}{\partial x_m} = 0 \text{ on } \Gamma_H := \{x \in \mathbb{R}^m : x_m = H\}.$$

Thus, Theorem 4.2.9 motivates to seek u^s as solution to the integral equation

$$u^s = \mathcal{V}\left(\frac{\omega^2}{c^2(x_m)} q(u^i + u^s)\right) \qquad \text{in } \Omega. \tag{4.30}$$

As the contrast q is supported in Λ_ρ, we can restrict this integral equation to Λ_ρ and seeking $v \in L^2(\Lambda_\rho)$ such that

$$v - \mathcal{V}\left(\frac{\omega^2}{c^2(x_m)} q v\right) = -\mathcal{V}\left(\frac{\omega^2}{c^2(x_m)} q u^i\right) \qquad \text{in } L^2(\Lambda_\rho). \tag{4.31}$$

Now, we replace $\omega^2/c^2(x_m)\, q u^i$ on the right-hand side in the last equation, by a general source term $f \in L^2(\Lambda_\rho)$, it yields the more general problem

$$v - \mathcal{V}\left(\frac{\omega^2}{c^2(x_m)} q v\right) = -\mathcal{V}(f) \qquad \text{in } L^2(\Lambda_\rho). \tag{4.32}$$

Equation (4.32) is the so called Lippmann-Schwinger equation. By Lemma 4.2.9, any solution to this integral equation solves the Helmholtz equation. We point out that we can replace the domain of integration in the latter equation by any domain such that the support of q is contained, e.g. $\text{supp}(q) \subset \Lambda_\rho$. The following theorem gives us now that the Helmholtz equation with corresponding boundary condition and radiation conditions can equivalently described by the Lippmann-Schwinger integral equation and that the converse holds, too.

Theorem 4.2.10. *We consider* $q \in L^\infty(\Omega)$ *with* $\mathrm{supp}(q) \subset \Lambda_\rho$ *satisfies* $Im(q) \geq 0$ *and* $f \in L^2(\Lambda_\rho)$.

(a) Let $v \in H^2_{\mathrm{loc}}(\Omega)$ *be a weak solution of our scattering problem then* $v|_{\Lambda_\rho}$ *belongs to* $L^2(\Lambda_\rho)$ *and solves the Lippmann-Schwinger integral equation (4.32), with* $f = \omega^2/c^2(x_m) q u^i$ *in* $L^2(\Lambda_\rho)$.

(b) Let $v \in L^2(\Lambda_\rho)$ *be a solution of the Lippmann-Schwinger integral equation (4.32), then* v *can be extended by*

$$v = \mathcal{V}\left(\frac{\omega^2}{c^2(x_m)} q v - f\right) \tag{4.33}$$

to a solution $v \in H^2_{\mathrm{loc}}(\Omega)$ *of the source problem*

$$\int_\Omega \left[\nabla u \cdot \nabla \overline{v} - \frac{\omega^2}{c^2(x_m)} n^2 u\, \overline{v}\right] dx = \int_{\Lambda_\rho} f \overline{v}\, dx, \tag{4.34}$$

for all $v \in H^1_W(\Omega)$ *with compact support and radiation conditions (3.3).*

(c) If the source problem (4.34) and radiation conditions (3.3) is uniquely solvable for all $f \in L^2(\Lambda_\rho)$, *then the Lippmann-Schwinger equation (4.32) is uniquely solvable in* $L^2(\Lambda_\rho)$ *for all* $f \in L^2(\Lambda_\rho)$.

Proof. a) The proof follows directly from (4.30-4.32) as the volume potential $\mathcal{V}$ solves the Helmholtz equation and satisfies radiation conditions.

b) We first suppose that $v \in L^2(D)$ solves the Lippmann-Schwinger integral equation (4.32). We can now extend v to a radiating solution of the Helmholtz equation in Ω. This holds since $\mathrm{supp}(q) \subset \Lambda_\rho$ and due to Lemma 4.2.9, which implies that (4.33),

$$v = \mathcal{V}\left(\frac{\omega^2}{c^2(x_m)} q v - f\right),$$

belongs to $H^2_{\mathrm{loc}}(\Lambda_\rho)$, satisfies radiation conditions, solves the boundary conditions and the Helmholtz equation. Consequently, we obtain

$$\Delta v + \frac{\omega^2}{c^2(x_m)} v = f - \frac{\omega^2}{c^2(x_m)} q v.$$

In particular,

$$\Delta v + \frac{\omega^2}{c^2(x_m)} (1+q) v = f.$$

This finishes the proof of this part.

(c) We know by Theorem 4.2.7 that the integral operator $f \mapsto v = \mathcal{V} f$ is bounded from $L^2(\Lambda_\rho)$ into $H^1(\Lambda_\rho)$ and we see that $\mathcal{V}$ is a compact operator, such that uniqueness of solution to (4.32) implies existence and bounded invertibility. We point out that multiplication by $q \in L^\infty(\Omega)$ is a bounded operation on $L^2(\Lambda_\rho)$ and the integral operator $f \mapsto v = \mathcal{V}(q\,\omega^2/c^2 f)$ is compact.

If the homogeneous Lippmann-Schwinger equation has only the trivial solution, then the extension of this trivial solution by (4.33) is a radiating weak solution in $H^2_{\mathrm{loc}}(\Omega)$ to

$$\Delta v + \frac{\omega^2}{c^2(x_m)} v = 0,$$

and v solves the boundary conditions (2.5) and (2.6). Uniqueness of solution to the source problem (4.34) and radiation conditions (3.3) hence implies that this extension must vanish, as well as v. This proves the claim of part (c).

□

The subsequent assumption on unique solvability of the underlying source problem (4.34), (3.3) ensures from now on that the integral equation (4.32) is uniquely solvable.

Assumption 4.2.11. *We assume that source problem (4.34) subject to the radiation conditions (3.3) is uniquely solvable for all $f \in L^2(M_\rho)$.*

Roughly speaking, this assumption excludes the discrete set (if it exists) of exceptional frequencies $\omega > 0$ (see Assumption 3.1.1).

Definition 4.2.12. *For $f \in L^2(\Lambda_\rho)$, the Lippman-Schwinger equation on Λ_ρ, is given by*

$$v - \mathcal{V}\left(\frac{\omega^2}{c^2(x_m)}qv\right)\Big|_{\Lambda_\rho} = \mathcal{V}\left(\frac{\omega^2}{c^2(x_m)}qf\right)\Big|_{\Lambda_\rho}. \tag{4.35}$$

We point out that for scattering theory in the following we set $v = u^s$ (the scattered field) and $f = u^i$ (the incident field).

Theorem 4.2.13. *Let $f \in L^2(D)$ extended by zero to Λ_ρ. If $u \in L^2(D)$ solves*

$$u - \mathcal{V}\left(\frac{\omega^2}{c^2(x_m)}qu\right)\Big|_D = f, \tag{4.36}$$

then $v = \mathcal{V}(qu) + f$ solves (4.35). Moreover, we know that $u \in L^2(D)$ solves (4.36) if we restrict a solution $v \in L^2(\Lambda_\rho)$ of (4.35) to D.

The proof follows directly from the proof of Theorem 4.2.10.

4.3 Periodized Green's Function in Dimension Two

We shall discretize the Lippmann-Schwinger equation (4.32) using a collocation method based on eigenfunctions and Fourier expansions. We first periodize the integral operator $\mathcal{V}$ in the horizontal variables and compute its Fourier coefficients with respect to a complete orthonormal system consisting of trigonometric functions in $\tilde{x}$ and the eigenfunctions ϕ_j in x_2. The periodized integral operator is obtained by periodizing the Green's function (4.1),

$$G(x,y) = \frac{i}{2}\sum_{j=1}^{\infty}\frac{1}{\lambda_j}\phi_j(x_2)\phi_j(y_2)\exp(i\lambda_j|x_1-y_1|), \qquad x_1 \neq y_1,$$

on the cubic domain Λ_ρ.

We first define for the eigenvalues λ_j^2, $j \in \mathbb{N}$ the function $E_\rho(x_1, \lambda_j)$ for $x_1 \in \Lambda_\rho$ by

$$E_\rho(x_1,\lambda_j) = \exp(i\lambda_j|x_1|) \qquad \text{for } -\rho < x_1 \leq \rho, \text{ for all } j \in \mathbb{N}. \tag{4.37}$$

Recall that a finite number of eigenvalues λ_j^2 are positive, an infinite number are negative and that any of these eigenvalues vanish on Assumption 3.1.1. For the negative eigenvalues, the value λ_j is imaginary according to (2.13) and then $x_1 \mapsto \exp(i\lambda_j|x_1|)$ decreases exponentially as $x_1 \to \pm\infty$. Recall that the number of the positive eigenvalues $J(\omega, c, H)$ depends on the ocean configuration defined in Theorem 2.2.6. Furthermore, this function $x_1 \mapsto E_\rho(x_1, \lambda_j)$ can be 2ρ-periodically extended from $\tilde{\Lambda}_\rho$ to $\mathbb{R}$ by $E_\rho(x_1 + 2\rho n_1, \lambda_j) = E_\rho(x_1, \lambda_j)$, for $n_1 \in \mathbb{Z}^1$, see Figure 4.1. By abuse of notation the periodic extension is still denoted as E_ρ. We moreover exploit for $j \in \mathbb{N}$ the 2ρ-periodic construction of E_ρ to obtain

$$\int_{\tilde{\Lambda}_\rho}|E_\rho(x_1-y_1,\lambda_j)|^2\,dx_1 = \int_{\tilde{\Lambda}_\rho - y_1}|E_\rho(x_1,\lambda_j)|^2\,dx_1 = \int_{\tilde{\Lambda}_\rho}|E_\rho(x_1,\lambda_j)|^2\,dx_1 \leq \begin{cases} 2\rho & \text{for } j \leq J, \\ \frac{2}{|\lambda_j|} & \text{for } j > J. \end{cases} \tag{4.38}$$

Now, equation (4.38) and the Green's function (4.1) allows us to define the periodized Green's function.

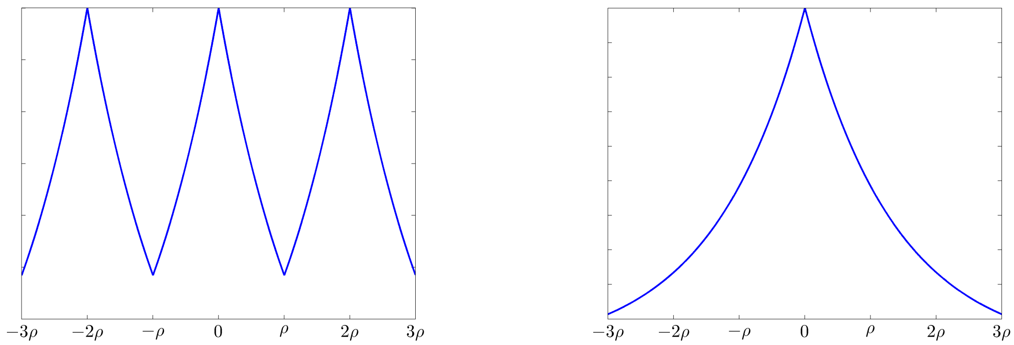


Figure 4.1: Real part of E_ρ (left), real part of exponential function E (right)

Definition 4.3.1. *For $x, y \in \Omega$ the function*

$$G_\rho(x,y) = \frac{i}{2}\sum_{j=1}^{\infty}\frac{1}{\lambda_j}\phi_j(x_2)\phi_j(y_2)E_\rho(x_1-y_1,\lambda_j) \qquad \text{for } x_1 \neq y_1, x, y \in \Omega \tag{4.39}$$

*is called **periodized Green's function** for dimension two.*

We point out that Parseval's equality, the bound for the eigenvalues λ_j^2 from Lemma 2.2.4 and (4.38) yield

$$\|G_\rho\|^2_{L^2(\Lambda_\rho\times\Lambda_\rho)} \leq \sum_{j\in\mathbb{N}}\int_{\tilde{\Lambda}_\rho}\int_{\tilde{\Lambda}_\rho}\frac{1}{2|\lambda_j|^2}|E_\rho(x_1-y_1,\lambda_j)|^2\,dy_1\,dx_1 \leq 2\rho\sum_{j\in\mathbb{N}}\frac{1}{|\lambda_j|^3} < \infty. \tag{4.40}$$

In consequence, we see that every series term

$$G_{\rho,j}(x,y) := \frac{i}{2}\frac{1}{\lambda_j}\phi_j(x_2)\phi_j(y_2)E_\rho(x_1-y_1,\lambda_j),$$

belongs to $L^2(\Lambda_\rho\times\Lambda_\rho)$. We further note that $G_\rho(x,y) = G(x,y)$ for $x_1, y_1 \in (-\rho,\rho]$. We next compute the Fourier coefficients of the periodized Green's function.

We point out that in the following the index j denotes the index of the series term of the Green's function and the index n_2 is related to the Fourier coefficients.

Lemma 4.3.2. *Consider $m = 2$, $j \in \mathbb{N}$ and $n_1 \in \mathbb{Z}$. For fixed $x \in \Lambda_\rho$ the Fourier coefficient of one series term $G_{\rho,j}(x,y)$ satisfies*

$$\hat{G}_{\rho,j}(\boldsymbol{n},x,\cdot) = \delta_{j,n_2}\frac{i\sqrt{\rho}}{\sqrt{2}\lambda_j}\hat{E}_\rho(n_1,\lambda_j)v_{-n_1}(x_1)\,\phi_j(x_2),$$

where the Fourier coefficients of $E_\rho(\lambda_j,\cdot)$ are given by

$$\frac{i\sqrt{\rho}}{\sqrt{2}\lambda_j}\hat{E}_\rho(n_1,\lambda_j) = \begin{cases} (\lambda_j^2-\pi^2\rho^{-2}n_1^2)^{-1}((-1)^{n_1}\exp(i\lambda_j\rho)-1) \\ \qquad\qquad \text{if } \lambda_j \pm n_1\pi\rho^{-1} \neq 0, \\ \frac{i}{2\lambda_j}\left[\rho - \frac{i}{\lambda_j-\pi\rho^{-1}n_1}\left(\exp(i(\lambda_j-\pi\rho^{-1}n_1)\rho)-1\right)\right] \\ \qquad\qquad \text{if } \lambda_j + n_1\pi\rho^{-1} = 0, \\ \frac{i}{2\lambda_j}\left[\frac{i}{\lambda_j+\pi\rho^{-1}n_1}\left(1-\exp(i(\lambda_j+\pi\rho^{-1}n_1)\rho)\right)+\rho\right] \\ \qquad\qquad \text{if } \lambda_j\rho = n_1\pi. \end{cases} \tag{4.41}$$

Proof. We first recall for $m = 2$ the orthonormal basis in $L^2(\Lambda_\rho)$ defined in Equation 3.13,

$$\varphi_{\boldsymbol{n}} = \frac{1}{(2\rho)^{1/2}} \exp\left(i\frac{\pi}{\rho}\, n_1 \cdot \tilde{x}\right) \phi_{n_2}(x_2) \qquad \text{for } \boldsymbol{n} = (n_1, n_2), n_1 \in \mathbb{Z}, n_2 \in \mathbb{N}, x \in \Lambda_\rho,$$

where the orthogonality of the basis $\{\phi_j\}_{j\in\mathbb{N}}$ holds,

$$\int_0^H \phi_j(y_2)\phi_{n_2}(y_2)\, dy = \begin{cases} 1 & \text{if } n_2 = j, \\ 0 & \text{else,} \end{cases}$$

It further holds that the eigenfunction basis $\{\phi_j\}_{j\in\mathbb{N}}$ is real-valued.

We first compute the Fourier coefficient $\hat{G}_{\rho,j}(\boldsymbol{n}, x, \cdot)$ of $y \mapsto \hat{G}_{\rho,j}(x, y)$ for fixed y_1 by

$$\begin{aligned} \hat{G}_{\rho,j}(\boldsymbol{n}, x, \cdot) &= \int_{\Lambda_\rho} \frac{i}{2\lambda_j} \phi_j(x_2)\phi_j(y_2) E_\rho(x_1 - y_1, \lambda_j) \overline{\varphi_{\boldsymbol{n}}(y)}\, dy \\ &= \int_{\Lambda_\rho} \frac{i}{2\lambda_j} \phi_j(x_2)\phi_j(y_2) E_\rho(x_1 - y_1, \lambda_j) \phi_{n_2}(y_2)\, \overline{v_{n_1}(y_1)}\, dy. \end{aligned}$$

Using the fact that $\{\phi_j\}_{j\in\mathbb{N}}$ is an orthonormal basis, we have

$$\hat{G}_{\rho,j}(\boldsymbol{n}, x, \cdot) = \delta_{j,n_2}\, \phi_j(x_2)\, \frac{i}{2\lambda_j} \int_{-\rho}^{\rho} E_\rho(x_1 - y_1, \lambda_j) \overline{v_{n_1}(y_1)}\, dy_1 \qquad \text{for } j \in \mathbb{N},$$

where δ_{j,n_2} is the Kronecker's delta, defined by $\delta_{j,n_2} = 1$ if $j = n_2$ and $\delta_{j,n_2} = 0$ if $j \neq n_2$. We moreover define the Fourier coefficient of E_ρ by

$$\hat{E}_\rho(n_1, \lambda_j) = \int_{-\rho}^{\rho} E_\rho(\cdot, \lambda_j) \overline{v_{n_1}(y_1)}\, dy_1 \qquad \text{for } n_1 \in \mathbb{Z}, j \in \mathbb{N}.$$

We next compute an explicit representation of $\hat{E}_\rho(n_1, \lambda_j)$. Using the definition of the basis function v_{n_1} acting on the horizontal axis, the definition of E_ρ in (4.37) and its periodicity in its second argument, we obtain

$$\begin{aligned} \hat{E}_\rho(n_1, \lambda_j) &= \frac{1}{\sqrt{2\rho}} \int_{-\rho}^{\rho} \exp(i\lambda_j |x_1 - y_1|) \exp\left(-i\pi \frac{n_1}{\rho} y_1\right) dy_1 \\ &= \frac{1}{\sqrt{2\rho}} \int_{-\rho}^{\rho} \exp(i\lambda_j |x_1 - (y_1 + x_1)|) \exp\left(-i\pi \frac{n_1}{\rho} (y_1 + x_1)\right) dy_1. \end{aligned}$$

Further, separation of the integral implies

$$\begin{aligned} \hat{E}_\rho(n_1, \lambda_j) =& \frac{1}{\sqrt{2\rho}} \int_{-\rho}^{\rho} \exp(i\lambda_j |y_1|) \exp\left(-i\pi \frac{n_1}{\rho} y_1\right) dy_1 \exp\left(-i\pi \frac{n_1}{\rho} x_1\right) \\ =& \frac{1}{\sqrt{2\rho}} \Big[\int_{-\rho}^{0} \exp\left(-i\left(\lambda_j + \pi\frac{n_1}{\rho}\right) y_1\right) dy_1 \\ &+ \int_0^{\rho} \exp\left(i\left(\lambda_j - \pi\frac{n_1}{\rho}\right) y_1\right) dy_1 \Big] \exp\left(-i\pi \frac{n_1}{\rho} x_1\right). \end{aligned}$$

We now see that we have to treat three cases. First, integrating the latter integrand, we see for $\lambda_j \pm \pi\rho^{-1} n_1 \neq 0$ that

$$\begin{aligned} &\int_{-\rho}^{0} \exp\left(-i\left(\lambda_j + \pi\frac{n_1}{\rho}\right) y_1\right) dy_1 + \int_0^{\rho} \exp\left(i\left(\lambda_j - \pi\frac{n_1}{\rho}\right) y_1\right) dy_1 \\ =& \frac{i}{\lambda_j + \pi\rho^{-1} n_1} \left[1 - \exp\left(i\left(\lambda_j + \pi\frac{n_1}{\rho}\right)\rho\right)\right] - \frac{i}{\lambda_j - \pi\rho^{-1} n_1} \left[\exp\left(i\left(\lambda_j - \pi\frac{n_1}{\rho}\right)\rho\right) - 1\right] \\ =& \frac{i}{\lambda_j + \pi\rho^{-1}} + \frac{i}{\lambda_j - \pi\rho^{-1} n_1} - \frac{i\exp(i(\lambda_j - \pi\rho^{-1} n_1)\rho)}{\lambda_j - \pi\rho^{-1} n_1} - \frac{i\exp(i(\lambda_j + \pi\rho^{-1} n_1)\rho)}{\lambda_j + \pi\rho^{-1} n_1} \\ =& \frac{2i\lambda_j}{\lambda_j^2 - \pi^2\rho^{-2} n_1^2} \left[1 - (-1)^{n_1} \exp(i\lambda_j\rho)\right], \end{aligned}$$

and we have

$$\frac{i}{2\lambda_j}\hat{E}_\rho(n_1,\lambda_j) = \frac{(-1)^{n_1}\exp(i\lambda_j\rho)-1}{\lambda_j^2-\pi^2\rho^{-2}n_1^2}\, v_{-n_1}(x_1).$$

Second, if $\lambda_j + \pi\rho^{-1}n_1 = 0$ we obtain

$$\begin{aligned}\hat{E}_\rho(n_1,\lambda_j) &= \int_{-\rho}^{0} 1\, dy_1 + \int_0^\rho \exp\left(i\left(\lambda_j - \pi\frac{n_1}{\rho}\right)y_1\right) dy_1\\ &= \rho - \frac{i}{\lambda_j - \pi\rho^{-1}n_1}\left[\exp\left(i\left(\lambda_j-\pi\frac{n_1}{\rho}\right)\rho\right)-1\right],\end{aligned}$$

and we deduce

$$\frac{i}{2\lambda_j}\hat{E}_\rho(n_1,\lambda_j) = \frac{i}{2\lambda_j}\left[\rho - \frac{i}{\lambda_j-\pi\rho^{-1}n_1}\left(\exp(i(\lambda_j-\pi\frac{n_1}{\rho})\rho)-1\right)\right].$$

Last, if $\lambda_j\rho = \pi n_1$ we have

$$\begin{aligned}\hat{E}_\rho(n_1,\lambda_j) &= \int_{-\rho}^{0} \exp\left(-i\left(\lambda_j + \pi\frac{n_1}{\rho}\right)y_1\right) dy_1 + \int_0^\rho 1\, dy_1\\ &= \frac{i}{\lambda_j + \pi\rho^{-1}n_1}\left[1-\exp\left(i\left(\lambda_j+\pi\frac{n_1}{\rho}\right)\rho\right)\right]+\rho,\end{aligned}$$

and we obtain

$$\frac{i}{2\lambda_j}\hat{E}_\rho(n_1,\lambda_j) = \frac{i}{2\lambda_j}\left[\frac{i}{\lambda_j+\pi\rho^{-1}n_1}\left(1-\exp\left(i\left(\lambda_j+\pi\frac{n_1}{\rho}\right)\rho\right)\right)+\rho\right].$$

To this end we can write

$$\hat{G}_{\rho,j}(\boldsymbol{n},x,\cdot) = \delta_{j,n_2}\frac{i\sqrt{\rho}}{\sqrt{2}\lambda_j}\hat{E}_\rho(\lambda_j,n_1)\overline{v_{n_1}(x_1)}\,\phi_j(x_2) \qquad \text{for } j\in\mathbb{N}, n_1\in\mathbb{Z}.$$

This finishes the proof. □

We can now analyze the asymptotic behavior of the coefficients $\hat{G}_{\rho,j}(\boldsymbol{n},x,\cdot)$, defined in Lemma 4.3.2. We first recall the definition the periodic Sobolev space (3.17),

$$H^s(\Lambda_\rho) = \left\{u = \sum_{\boldsymbol{k}\in\mathbb{Z}_+^2}\hat{u}(\boldsymbol{k})\varphi_{\boldsymbol{k}},\ \sum_{\boldsymbol{k}\in\mathbb{Z}_+^2}(1+|k_1|^2+|\lambda_{k_2}|^2)^s|\hat{u}(\boldsymbol{k})|^2<\infty\right\}, \qquad s\in\mathbb{R},$$

with (squared) norm

$$\|u\|^2_{H^s(\Lambda_\rho)} = \sum_{\boldsymbol{k}\in\mathbb{Z}_+^2}(1+|k_1|^2+|\lambda_{k_2}|^2)^s|\hat{u}(\boldsymbol{k})|^2.$$

Lemma 4.3.3. *Consider $m=2$ and $s<1$. Then, the periodized Green's function in dimension two,*

$$G_\rho(x,y) = \sum_{j=1}^\infty G_{\rho,j}(x,y) = \frac{i}{2}\sum_{j=1}^\infty \frac{1}{\lambda_j}\phi_j(x_2)\phi_j(y_2)E_\rho(\tilde{x}-\tilde{y},\lambda_j) \qquad \text{for } \tilde{x}\neq\tilde{y}, x,y\in\Lambda_\rho$$

belongs to $H^s(\Lambda_\rho)\times H^s(\Lambda_\rho)$ and the series converges absolutely in H^s as a function of x or y. Further, it holds

$$\|G_{\rho,j}(x,\cdot)\|^2_{H^s(\Lambda_\rho)} \leq C(s)j^{s-2}, \qquad \text{for } x\in\Lambda_\rho,$$

where $C(s)>0$ is independent of $j\in\mathbb{N}$.

Proof. We first estimate

$$|\hat{G}_{\rho,j}(\boldsymbol{n},x,\cdot)| \leq C \left|\hat{E}_\rho(n_1,\lambda_j)\right| \frac{\|\overline{v_{n_1}}\phi_j\|_\infty}{|\lambda_j|} \qquad j\in\mathbb{N}, n_1\in\mathbb{Z}.$$

Based on the estimate of the vector $v_{\boldsymbol{n}}$ in Corollary 3.2.1 and the estimate of the eigenvector ϕ_j in Corollary 2.2.5, we compute

$$|\hat{G}_{\rho,j}(\boldsymbol{n},x,\cdot)| \leq C \frac{\left|\hat{E}_\rho(n_1,\lambda_j)\right|}{|\lambda_j|}.$$

The second and the third case in (4.41) hold only for a finite number $n_1\in\mathbb{Z}, j\in\mathbb{N}$, since λ_j is real valued only if $j\leq J$. Consequently, we look at $|\hat{E}_\rho(n_1,\lambda_j)|$ where $\lambda_j \pm \rho^{-1}\pi n_1 \neq 0$. By the definition of the coefficient of $\hat{E}_\rho$ in (4.41) we have

$$\frac{|\hat{E}_\rho(n_1,\lambda_j)|}{|\lambda_j|} = \left|\frac{(-1)^{n_1}\exp(i\lambda_j\rho)-1}{\lambda_j^2-\rho^{-2}\pi^2 n_1^2}\right| \leq C\frac{1}{|\lambda_j+\rho^{-1}\pi n_1||\lambda_j-\rho^{-1}\pi n_1|} \quad \text{for } j\in\mathbb{N}, n_2\in\mathbb{Z}.$$

Now, for j sufficiently large, we see that λ_j is situated on the imaginary axis, while n_1 is on the real axis. In consequence,

$$\frac{|\hat{E}_\rho(n_1,\lambda_j)|}{|\lambda_j|} \leq C\left(\frac{1}{\sqrt{|\lambda_j|^2+|\rho^{-1}\pi n_1|^2}}\right)^2 \leq C\left(1+|n_1|^2+|\lambda_j|^2\right)^{-1} \quad \text{for } j\in\mathbb{N}, n_2\in\mathbb{Z}.$$

Plugging all together, we see that for $s<1$

$$\begin{aligned}\sum_{j\in\mathbb{N}}\|G_{\rho,j}(x,\cdot)\|^2_{H^s(\Lambda_\rho)} &= \sum_{n_1\in\mathbb{Z},j=n_2\in\mathbb{N}}(1+|n_1|^2+|\lambda_j|^2)^s|\hat{G}_{\rho,j}(\boldsymbol{n},x,\cdot)|^2\\ &\leq \sum_{n_1\in\mathbb{Z},j\in\mathbb{N}}(1+|n_1|^2+|\lambda_j|^2)^{s-2}\\ &\leq \sum_{n_1\in\mathbb{Z},j\in\mathbb{N}}(1+|n_1|^2+j^2)^{s-2}\end{aligned}$$

Now, due to the fact that $|n_1||j|\leq|n_1|^2+|j|^2$ we know for $a,b>0$ and $1+a\leq b$ that

$$|n_1|^{(1+a)/b}|j|^{(1+a)/b}\leq|n_1|^2+|j|^2. \qquad \text{for } n_1\in\mathbb{Z}, j\in\mathbb{N}.$$

We further see for $a,b>0$ and $1+a\leq b$ that

$$|n_1|^{(1+a)}|j|^{(1+a)}\leq(1+|n_1|^2+|j|^2)^b \qquad \text{for } n_1\in\mathbb{Z}, j\in\mathbb{N}.$$

In particular,for $a,b>0$ and $1+a\leq b$ it holds

$$|n_1|^{-1-a}|j|^{-1-a}\geq(1+|n_1|^2+|j|^2)^{-b} \qquad \text{for } n_1\in\mathbb{Z}, j\in\mathbb{N}.$$

Consequently, for $b=2-s$ and $a=b-1=1-s$ where $s<1$ we have

$$\sum_{n_1\in\mathbb{Z},j\in\mathbb{N}}(1+|n_1|^2+j^2)^{s-2}\leq\sum_{n_1\in\mathbb{Z}}|n_1|^{s-2}\sum_{j\in\mathbb{N}}j^{s-2}\leq 2\sum_{j\in\mathbb{N}}j^{s-2}<\infty.$$

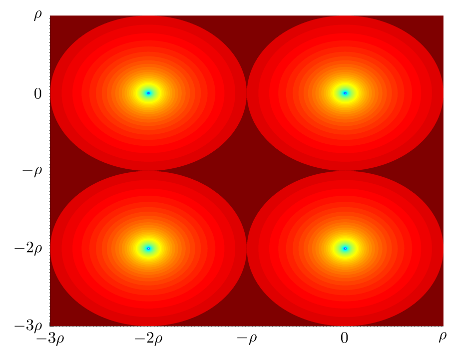
We concluding using the symmetry of the periodized Green's function $G_\rho(x,y)$. ☐

4.4 Periodized Green's Function in Dimension Three

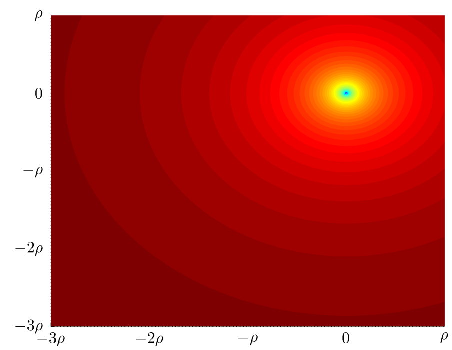
We introduce in this section the periodized Green's function in dimension three. To obtain the periodized Green's function in dimension three we use rather similar techniques as in dimension two. We first define a function $H_\rho(\cdot,\lambda_j):\tilde{\Lambda}_\rho\to\mathbb{C}$ for $j\in\mathbb{N}$ by

$$H_\rho(\tilde{x},\lambda_j)=\begin{cases} H_0^{(1)}(\lambda_j|\tilde{x}|) & \text{if } 0<|\tilde{x}|<\rho,\\ 0 & \text{else,}\end{cases} \qquad j\in\mathbb{N}. \tag{4.42}$$

This function can be 2ρ-biperiodically extended on the horizontal axis from $\tilde{\Lambda}_\rho$ to $\mathbb{R}^2$ by $H_\rho(\tilde{x}+2\rho\tilde{\boldsymbol{n}},\lambda_j)=H_\rho(\tilde{x},\lambda_j)$, for $\tilde{x}\in\tilde{\Lambda}_\rho$, $\tilde{\boldsymbol{n}}\in\mathbb{Z}^2$, see Figure 4.2. We denote this extension again by H_ρ.



(a)



(b)

Figure 4.2: (a) Imaginary part of periodic Hankel function expansion H_ρ. (b) Imaginary part of Hankel function H_0^1.

Furthermore, from [SV02, Equation 2.17] and (4.42), we know that $H_\rho(\tilde{x}-\tilde{y},\lambda_j)$ has a logarithmic singularity at $\tilde{x}=\tilde{y}$. Note that the logarithmic singularity of the Hankel function $H_0^{(1)}$ at the origin implies that $H_\rho(\cdot,\lambda_j)$ belongs to $L^2(\tilde{\Lambda}_\rho)$ for all $j\in\mathbb{N}$. Using H_ρ and the Green's function (4.4), we define the periodized Green's function in dimension three by

$$G_\rho(x,y)=\frac{i}{4}\sum_{j=1}^{\infty}\phi_j(x_3)\phi_j(y_3)H_\rho(\tilde{x}-\tilde{y},\lambda_j), \qquad x,y\in\Omega \text{ and } \tilde{x}\neq\tilde{y}, \tag{4.43}$$

with series coefficients

$$G_{\rho,j}(x,y):=\frac{i}{4}\phi_j(x_3)\phi_j(y_3)H_\rho(\tilde{x}-\tilde{y},\lambda_j) \qquad \text{for } j\in\mathbb{N}, \tag{4.44}$$

such that $G_\rho(x,y)=G(x,y)$ for $x,y\in\Omega$, $0<|\tilde{x}-\tilde{y}|<\rho$. Using the 2ρ-biperiodic construction of H_ρ and the exponential decay of Hankel functions with complex argument, we show that G_ρ belongs point-wise to $L^2(\Lambda_\rho)$ as a function of either $\tilde{x}$ or $\tilde{y}$,

$$\int_{\tilde{\Lambda}_\rho}|H_\rho(\tilde{x}-\tilde{y},\lambda_j)|^2\,d\tilde{x}=\int_{\tilde{\Lambda}_\rho-\tilde{y}}|H_\rho(\tilde{x},\lambda_j)|^2\,d\tilde{x}=\int_{\tilde{\Lambda}_\rho}|H_\rho(\tilde{x},\lambda_j)|^2\,d\tilde{x}<\infty.$$

We next determine Fourier coefficients of the series coefficients of the periodized Green's function in dimension three.

Lemma 4.4.1. *Consider $m=3, j\in\mathbb{N}$ and $\tilde{\boldsymbol{n}}\in\mathbb{Z}^2$. Then, for fixed $x\in\Lambda_\rho$, the $\boldsymbol{n}$-th Fourier coefficient $\hat{G}_{\rho,j}(\boldsymbol{n},x,\cdot)$ of $y\mapsto G_\rho(x,y)$ is denoted by*

$$\hat{G}_{\rho,j}(\boldsymbol{n},x,\cdot)=\frac{i\rho}{2}\,\delta_{j,n_3}\,\hat{H}_\rho(\tilde{\boldsymbol{n}},\lambda_j)\overline{v_{\tilde{\boldsymbol{n}}}}(\tilde{x})\phi_j(x_3). \tag{4.45}$$

where the Fourier coefficient $\hat{H}_\rho(\tilde{\boldsymbol{n}}, \lambda_j)$ *of* $H_\rho(\cdot, \lambda_j)$ *is given by*

$$\hat{H}_\rho(\tilde{\boldsymbol{n}}, \lambda_j) = \begin{cases} i\rho \frac{2+i\pi^2|\tilde{\boldsymbol{n}}|\, H_0^{(1)}(\lambda_j\rho)J_1(\pi|\tilde{\boldsymbol{n}}|)-i\pi\rho\lambda_j H_1^{(1)}(\lambda_j\rho)J_0(\pi|\tilde{\boldsymbol{n}}|)}{\lambda_j^2\rho^2-\pi^2|\tilde{\boldsymbol{n}}|^2} & \text{for } \tilde{\boldsymbol{n}} \neq 0, \lambda_{n_3}\rho \neq \pi|\tilde{\boldsymbol{n}}|, \\ \frac{\pi}{\lambda_j} H_1^{(1)}(\lambda_j\rho) + \frac{2i}{\rho\lambda_j^2} & \text{for } \tilde{\boldsymbol{n}} = 0, \lambda_{n_3}\rho \neq \pi|\tilde{\boldsymbol{n}}|, \\ \frac{\pi^2|\tilde{\boldsymbol{n}}|}{2}\left(J_1(\pi|\tilde{\boldsymbol{n}}|)H_1^{(1)}(\pi|\tilde{\boldsymbol{n}}|) + J_0(\pi|\tilde{\boldsymbol{n}}|)H_0^{(1)}(\pi|\tilde{\boldsymbol{n}}|)\right) & \text{for } \lambda_{n_3}\rho = \pi|\tilde{\boldsymbol{n}}|. \end{cases} \tag{4.46}$$

Proof. We recall the definition of the basis $\varphi_{\boldsymbol{n}}$ in dimension three,

$$\varphi_{\boldsymbol{n}}(x) = \frac{1}{2\rho}\phi_{n_3}(x_3)\exp\left(i\frac{\pi}{\rho}\,\tilde{\boldsymbol{n}}\cdot\tilde{x}\right), \qquad \text{where } \boldsymbol{n} = (\tilde{\boldsymbol{n}}, n_3) \in \mathbb{Z}_+^3, x \in \Lambda_\rho,$$

where for $\{\phi_j\}_{j\in\mathbb{N}}$ the orthogonality holds

$$\int_0^H \phi_j(y_3)\phi_{n_3}(y_3)\,dy_3 = \begin{cases} 1 & \text{if } n_3 = j, \\ 0 & \text{else.} \end{cases}$$

We point out that the basis $\{\phi_j\}_{j\in\mathbb{N}}$ is real valued. We first compute the $\boldsymbol{n}$-th Fourier coefficient of one series term $G_{\rho,j}(x,y)$ defined in (4.44).

Plugging the latter definition of $\varphi_{\boldsymbol{n}}(x)$, the orthogonality result of $\{\phi_j\}_{j\in\mathbb{N}}$ and classical Fourier theory together, yields

$$\begin{aligned} \hat{G}_{\rho,j}(\boldsymbol{n}, x, \cdot) &= \int_{\Lambda_\rho} \frac{i}{4}\phi_j(x_3)\phi_j(y_3)H_\rho(\tilde{x}-\tilde{y}, \lambda_j)\overline{\varphi_{\boldsymbol{n}}(y)}\,dy \\ &= \int_{\Lambda_\rho} \frac{i}{4}\,\phi_j(x_3)\phi_j(y_3)H_\rho(\tilde{x}-\tilde{y}, \lambda_j)\overline{v_{\tilde{\boldsymbol{n}}}(\tilde{y})}\,\phi_{n_3}(y_3)\,dy \\ &= \frac{i}{4}\delta_{j,n_3}\,\phi_j(x_3)\int_{\tilde{\Lambda}_\rho} H_\rho(\tilde{x}-\tilde{y}, \lambda_j)\overline{v_{\tilde{\boldsymbol{n}}}(\tilde{y})}\,d\tilde{y}, \end{aligned}$$

where δ_{j,n_3} denotes the Kronecker's delta, defined by $\delta_{j,n_3} = 1$ if $j = n_3$ and $\delta_{j,n_3} = 0$ if $j \neq n_3$. Due to the definition of the basis $\{v_{\tilde{\boldsymbol{n}}}\}_{\tilde{\boldsymbol{n}}\in\mathbb{Z}^2}$ and the periodicity of H_ρ in its second argument we know

$$\begin{aligned} \int_{\tilde{\Lambda}_\rho} H_\rho(\tilde{x}-\tilde{y}, \lambda_j)\overline{v_{\tilde{\boldsymbol{n}}}}(\tilde{y})\,d\tilde{y} &= \frac{1}{2\rho}\int_{\tilde{\Lambda}_\rho} H_\rho(\tilde{x}-\tilde{y}, \lambda_j)\exp\left(-i\frac{\pi}{\rho}\,\tilde{\boldsymbol{n}}\cdot(\tilde{y}-\tilde{x})\right)\,dy\exp\left(-i\frac{\pi}{\rho}\,\tilde{\boldsymbol{n}}\cdot\tilde{x}\right) \\ &= 2\rho\int_{\tilde{\Lambda}_\rho} H_\rho(\tilde{z}, \lambda_j)\overline{v_{\tilde{\boldsymbol{n}}}(\tilde{z})}\,d\tilde{z}\,\overline{v_{\tilde{\boldsymbol{n}}}(\tilde{x})}. \end{aligned} \tag{4.47}$$

If for $\boldsymbol{n} \in \mathbb{Z}_+^3$ the Fourier coefficient of H_ρ is denoted by

$$\hat{H}_\rho(\tilde{\boldsymbol{n}}, \lambda_j) = \int_{\tilde{\Lambda}_\rho} H_\rho(\tilde{z}, \lambda_j)\overline{v_{\tilde{\boldsymbol{n}}}}(\tilde{z})\,d\tilde{z} = \left(H_\rho(\cdot, \lambda_{n_3}), v_{\tilde{\boldsymbol{n}}}\right)_{L^2(\tilde{\Lambda}_\rho)}, \tag{4.48}$$

then for $x \in \Lambda_\rho$ we have

$$\hat{G}_{\rho,j}(\boldsymbol{n}, x, \cdot) = \frac{i\rho}{2}\,\delta_{j,n_3}\hat{H}_\rho(\tilde{\boldsymbol{n}}, \lambda_j)\overline{v_{\tilde{\boldsymbol{n}}}(\tilde{x})}\phi_{n_3}(x_3). \tag{4.49}$$

We are now interested in the representation of the Fourier coefficient $\hat{H}_\rho(\tilde{\boldsymbol{n}}, \lambda_j)$. The definition of the periodic Hankel function in (4.42) yields

$$\hat{H}_\rho(\tilde{\boldsymbol{n}}, \lambda_j) = \int_{|\tilde{z}|<\rho} H_0^{(1)}(\lambda_j|\tilde{z}|)\overline{v_{\tilde{\boldsymbol{n}}}(\tilde{z})}\,d\tilde{z}, \qquad \tilde{\boldsymbol{n}} \in \mathbb{Z}^2, j \in \mathbb{N}. \tag{4.50}$$

We further define

$$\sigma_{\tilde{\boldsymbol{n}},j} = \frac{\rho^2}{\lambda_j^2\rho^2 - \pi^2|\tilde{\boldsymbol{n}}|^2} \qquad \text{for } \lambda_j\rho \neq \pi|\tilde{\boldsymbol{n}}|, \tag{4.51}$$

and we see that $\sigma_{\tilde{\boldsymbol{n}},j}$ satisfies

$$\left(\Delta_{\tilde{x}} + \lambda_j^2\right) v_{\tilde{\boldsymbol{n}}} = \frac{\lambda_j^2\rho^2 - \pi^2|\tilde{\boldsymbol{n}}|^2}{\rho^2} v_{\tilde{\boldsymbol{n}}}(\tilde{x}).$$

Consequently, the well-known Green's second identity, e.g. [Mon03, Corollary 3.20] implies that

$$\begin{aligned}
&\int_{|\tilde{z}|<\rho} H_0^{(1)}(\lambda_j|\tilde{z}|)\overline{v_{\tilde{\boldsymbol{n}}}}\, d\tilde{z} \\
&= \sigma_{\tilde{\boldsymbol{n}},j} \lim_{\varepsilon\to 0} \int_{\varepsilon<|\tilde{z}|<\rho} H_0^{(1)}(\lambda_j|\tilde{z}|)(\Delta_{\tilde{x}} + \lambda_j^2)\overline{v_{\tilde{\boldsymbol{n}}}}\, d\tilde{z} \\
&= \sigma_{\tilde{\boldsymbol{n}},j} \lim_{\varepsilon\to 0} \left(\int_{|\tilde{z}|=\rho} - \int_{|\tilde{z}|=\varepsilon}\right) \left(H_0^{(1)}(\lambda_j|\tilde{z}|)\frac{\partial\overline{v_{\tilde{\boldsymbol{n}}}}}{\partial\nu} - \overline{v_{\tilde{\boldsymbol{n}}}}\frac{\partial}{\partial\nu}H_0^{(1)}(\lambda_j|\tilde{z}|)\right) ds(\tilde{z}),
\end{aligned} \tag{4.52}$$

where ν denotes the exterior normal vector to the annulus $\{\varepsilon < |\tilde{z}| < \rho\}$ pointing towards infinity. Using [SV02, Chapter 2.2], we first see that

$$\frac{i}{4}\lim_{\varepsilon\to 0}\int_{|\tilde{z}|=\varepsilon} \left(H_0^{(1)}(\lambda_j|\tilde{z}|)\frac{\partial\overline{v_{\tilde{\boldsymbol{n}}}}}{\partial\nu} - \overline{v_{\tilde{\boldsymbol{n}}}}\frac{\partial}{\partial\nu}H_0^{(1)}(\lambda_j|\tilde{z}|)\right) ds(\tilde{z}) = v_{\tilde{\boldsymbol{n}}}(0) = \frac{1}{2\rho}. \tag{4.53}$$

Based on the fact that $H_0^{(1)}(\lambda_j|\tilde{z}|)$ is constant on $|\tilde{z}| = \rho$, we obtain for $\tilde{\boldsymbol{n}} \neq 0$ with [SV02, Section 10.5] that

$$\int_{|\tilde{z}|=\rho} \frac{\partial\overline{v_{\tilde{\boldsymbol{n}}}}}{\partial\nu}\, ds(\tilde{z}) = \frac{-\pi^2|\tilde{\boldsymbol{n}}|}{\rho} J_1(\pi|\tilde{\boldsymbol{n}}|) \quad \text{and} \quad \int_{|\tilde{z}|=\rho} \overline{v_{\tilde{\boldsymbol{n}}}} ds(\tilde{z})\, ds = \pi J_0(\pi|\tilde{\boldsymbol{n}}|),$$

whereas, for grid points $\tilde{\boldsymbol{n}} = 0$, we have

$$\int_{|\tilde{z}|=\rho} \frac{\partial v_0}{\partial\nu}\, ds(\tilde{z}) = 0 \qquad \text{and} \qquad \int_{|\tilde{z}|=\rho} v_0\, ds(\tilde{z}) = \pi.$$

We look now at each component of equation (4.52). For $\tilde{\boldsymbol{n}} \neq 0$ and $\lambda_j\rho \neq \pi|\tilde{\boldsymbol{n}}|$ we obtain

$$\sigma_{\tilde{\boldsymbol{n}},j}\int_{|\tilde{z}|=\rho} H_0^{(1)}(\lambda_j|\tilde{z}|)\frac{\partial\overline{v_{\tilde{\boldsymbol{n}}}}}{\partial\nu} ds(\tilde{z}) = -\sigma_{\tilde{\boldsymbol{n}},j}\frac{\pi^2|\tilde{\boldsymbol{n}}|}{\rho}\, H_0^{(1)}(\lambda_j\rho)J_1(\pi|\tilde{\boldsymbol{n}}|), \tag{4.54}$$

and

$$-\sigma_{\tilde{\boldsymbol{n}},j}\int_{|\tilde{z}|=\rho} \overline{v_{\tilde{\boldsymbol{n}}}}\frac{\partial}{\partial\nu}H_0^{(1)}(\lambda_j|\tilde{z}|)\, ds(\tilde{z}) = \sigma_{\tilde{\boldsymbol{n}},j}\pi\lambda_j\, J_0(\pi|\tilde{\boldsymbol{n}}|)H_1^{(1)}(\lambda_j\rho). \tag{4.55}$$

We now turn to the case that $\tilde{\boldsymbol{n}} = 0$ and $\lambda_j\rho \neq \pi|\tilde{\boldsymbol{n}}|$. Then, we see

$$\frac{i}{4}\sigma_{\tilde{\boldsymbol{n}},j}\int_{|\tilde{z}|=\rho} H_0^{(1)}(\lambda_j|\tilde{z}|)\frac{\partial v_0}{\partial\nu}\, ds(\tilde{z}) = 0, \tag{4.56}$$

and

$$-\sigma_{\tilde{\boldsymbol{n}},j}\int_{|\tilde{z}|=\rho} v_0\,\frac{\partial}{\partial\nu}H_0^{(1)}(\lambda_j|\tilde{z}|)\, ds(\tilde{z}) = \sigma_{\tilde{\boldsymbol{n}},j}\pi\lambda_j H_1^{(1)}(\lambda_j\rho). \tag{4.57}$$

Putting equations (4.53-4.54) together implies the first case in (4.46), for $\boldsymbol{n} \neq 0$ and $\lambda_j\rho \neq \pi|\tilde{\boldsymbol{n}}|$

$$\hat{H}_\rho(\tilde{\boldsymbol{n}}, \lambda_j) = \frac{i\rho}{\lambda_j^2\rho^2 - \pi^2|\tilde{\boldsymbol{n}}|^2}\left[2 + i\pi^2|\tilde{\boldsymbol{n}}|\, H_0^{(1)}(\lambda_j\rho)J_1(\pi|\tilde{\boldsymbol{n}}|) - i\pi\rho\lambda_j H_1^{(1)}(\lambda_j\rho)J_0(\pi|\tilde{\boldsymbol{n}}|)\right]. \quad (4.58)$$

Next, for $\tilde{\boldsymbol{n}} = 0$ and $\lambda_j\rho \neq \pi|\tilde{\boldsymbol{n}}|$ we find the second case in (4.46),

$$\hat{H}_\rho(\tilde{\boldsymbol{n}}, \lambda_j) = \frac{1}{2\rho}\sigma_{\tilde{\boldsymbol{n}},j}\left[\frac{\pi}{\lambda_j}H_1^{(1)}(\lambda_j\rho) + 4i)\right] = \frac{\pi}{\lambda_j}H_1^{(1)}(\lambda_j\rho) + \frac{2i}{\rho\lambda_j^2}.$$

We assume for a moment that $\rho = \pi|\tilde{\boldsymbol{n}}|/\lambda_j$. Indeed, $\lambda_j \neq 0$ holds by Assumption 4.2.11. We set $\overline{\rho} = \pi|\tilde{\boldsymbol{n}}|/\lambda_j$ and we obtain from (4.58)

$$\lim_{\rho\to\overline{\rho}}\frac{\partial}{\partial\rho}({\lambda_j}^2\rho^2 - \pi^2|\tilde{\boldsymbol{n}}|^2) = \lim_{\rho\to\overline{\rho}} 2\rho\lambda_j^2 = 2\pi|\tilde{\boldsymbol{n}}|. \quad (4.59)$$

Next, we recall the derivative of the Hankel function in equation (A.6) of the Appendix,

$$\frac{\partial}{\partial z}H_1^{(1)}(\varphi(z)) = H_0^{(1)}(\varphi(z))\,\varphi^{'}(z) - \frac{1}{z}H_1^{(1)}(\varphi(z)) \quad \text{and} \quad \frac{\partial}{\partial z}H_0^{(1)}(\varphi(z)) = -\varphi^{'}(z)H_1^{(1)}(\varphi(z)).$$

Thus, we obtain

$$\begin{aligned}
&\lim_{\rho\to\overline{\rho}}\frac{\partial}{\partial\rho}\left[\frac{\rho}{2} - \frac{i\pi^2|\tilde{\boldsymbol{n}}|\rho}{4}\, H_0^{(1)}(\lambda_j\rho)J_1(\pi|\tilde{\boldsymbol{n}}|) + \frac{i\pi\rho^2\lambda_j}{4}H_1^{(1)}(\lambda_j\rho)J_0(\pi|\tilde{\boldsymbol{n}}|)\right] \\
=&\lim_{\rho\to\overline{\rho}}\left[\frac{1}{2} + \frac{i\pi^2|\tilde{\boldsymbol{n}}|}{4}J_1(\pi|\tilde{\boldsymbol{n}}|)\left[\rho\lambda_j H_1^{(1)}(\lambda_j\rho) - H_0^{(1)}(\lambda_j\rho)\right]\right. \\
&\left. + \frac{i\pi}{4}J_0(\pi|\tilde{\boldsymbol{n}}|)\left[2\rho\lambda_j H_1^{(1)}(\lambda_j\rho) + \rho^2\lambda_j\left(\lambda_j H_0^{(1)}(\lambda_j\rho) - \frac{1}{\rho}H_1^{(1)}(\lambda_j\rho)\right)\right]\right] \\
=&\frac{1}{2} + \frac{i\pi^2|\tilde{\boldsymbol{n}}|}{4}J_1(\pi|\tilde{\boldsymbol{n}}|)\left[\pi|\tilde{\boldsymbol{n}}|H_1^{(1)}(\pi|\tilde{\boldsymbol{n}}|) - H_0^{(1)}(\pi|\tilde{\boldsymbol{n}}|)\right] \\
&+ \frac{i\pi}{4}J_0(\pi|\tilde{\boldsymbol{n}}|)\left[2\pi|\tilde{\boldsymbol{n}}|H_1^{(1)}(\pi|\tilde{\boldsymbol{n}}|) + \pi^2|\tilde{\boldsymbol{n}}|^2 H_0^{(1)}(\pi|\tilde{\boldsymbol{n}}|) - \pi|\tilde{\boldsymbol{n}}|H_1^{(1)}(\pi|\tilde{\boldsymbol{n}}|)\right] \\
=&\frac{1}{2} + \frac{i\pi^2|\tilde{\boldsymbol{n}}|}{4}\left(J_1(\pi|\tilde{\boldsymbol{n}}|)\left[\pi|\tilde{\boldsymbol{n}}|H_1^{(1)}(\pi|\tilde{\boldsymbol{n}}|) - H_0^{(1)}(\pi|\tilde{\boldsymbol{n}}|)\right]\right. \\
&\left. + J_0(\pi|\tilde{\boldsymbol{n}}|)\left(H_1^{(1)}(\pi|\tilde{\boldsymbol{n}}|) + \pi|\tilde{\boldsymbol{n}}|H_0^{(1)}(\pi|\tilde{\boldsymbol{n}}|)\right)\right).
\end{aligned}$$

Recall the definition of the Hankel function in equation (A.1) in the Appendix that

$$H_0^{(1)}(z) = J_0(z) + iY_0(z) \quad \text{and} \quad H_1^{(1)}(z) = J_1(z) + iY_1(z),$$

and due to Lemma A.1.1 from the Appendix we claim

$$i\pi^2|\tilde{\boldsymbol{n}}|J_0(\pi|\tilde{\boldsymbol{n}}|)Y_1(\pi|\tilde{\boldsymbol{n}}|) = i\pi^2|\tilde{\boldsymbol{n}}|J_1(\pi|\tilde{\boldsymbol{n}}|)Y_0(\pi|\tilde{\boldsymbol{n}}|) - 2i.$$

To simplify the notation we write in the following $J_\nu := J_\nu(\pi|\tilde{\boldsymbol{n}}|) =$ and $Y_\nu := Y_\nu(\pi|\tilde{\boldsymbol{n}}|)$, for $\nu = 1, 2$. Then, we compute

$$J_0 H_1(\pi|\tilde{\boldsymbol{n}}|) = J_0 J_1 + iJ_1 Y_0 - \frac{2i}{|\tilde{\boldsymbol{n}}|\pi^2} \qquad \text{and} \qquad J_1 H_0(\pi|\tilde{\boldsymbol{n}}|) = J_0 J_1 + iJ_1 Y_0.$$

In consequence, we have

$$
\begin{aligned}
&\frac{1}{2}+\frac{i\pi^2|\tilde{\boldsymbol{n}}|}{4}\Bigg(J_1(\pi|\tilde{\boldsymbol{n}}|)\left[\pi|\tilde{\boldsymbol{n}}|H_1^{(1)}(\pi|\tilde{\boldsymbol{n}}|)-H_0^{(1)}(\pi|\tilde{\boldsymbol{n}}|)\right]\\
&\qquad\qquad+J_0(\pi|\tilde{\boldsymbol{n}}|)\left(H_1^{(1)}(\pi|\tilde{\boldsymbol{n}}|)+\pi|\tilde{\boldsymbol{n}}|H_0^{(1)}(\pi|\tilde{\boldsymbol{n}}|)\right)\Bigg)\\
=&\frac{1}{2}+\frac{i\pi^2|\tilde{\boldsymbol{n}}|}{4}\left(J_1(\pi|\tilde{\boldsymbol{n}}|)\pi|\tilde{\boldsymbol{n}}|H_1^{(1)}(\pi|\tilde{\boldsymbol{n}}|)+J_0(\pi|\tilde{\boldsymbol{n}}|)\pi|\tilde{\boldsymbol{n}}|H_0^{(1)}(\pi|\tilde{\boldsymbol{n}}|)-\frac{2i}{|\tilde{\boldsymbol{n}}|\pi^2}\right)\\
=&\frac{i\pi^3|\tilde{\boldsymbol{n}}|^2}{4}\left(J_1(\pi|\tilde{\boldsymbol{n}}|)H_1^{(1)}(\pi|\tilde{\boldsymbol{n}}|)+J_0(\pi|\tilde{\boldsymbol{n}}|)H_0^{(1)}(\pi|\tilde{\boldsymbol{n}}|)\right).
\end{aligned}
$$

Last, due to equation (4.59) we deduce for $\tilde{\boldsymbol{n}}=0$ and $\lambda_j\rho\neq\pi|\tilde{\boldsymbol{n}}|$ the third case in (4.46),

$$\hat{H}_\rho(\tilde{\boldsymbol{n}},\lambda_j)=\frac{\pi^2|\tilde{\boldsymbol{n}}|}{2}\left(J_1(\pi|\tilde{\boldsymbol{n}}|)H_1^{(1)}(\pi|\tilde{\boldsymbol{n}}|)+J_0(\pi|\tilde{\boldsymbol{n}}|)H_0^{(1)}(\pi|\tilde{\boldsymbol{n}}|)\right).$$

This completes the proof. □

Similar like in dimension two we analyze the asymptotic behavior of the coefficients $\hat{G}_{\rho,j}(\boldsymbol{n},x,\cdot)$, using well-known Bessel and Hankel functions estimates from [AS64].

Lemma 4.4.2. *Consider $m=3$ and $s<1/2$. Then, the periodized Green's function in dimension three (4.43),*

$$G_\rho(x,y)=\sum_{j=1}^{\infty}G_{\rho,j}(x,y)=\frac{i}{4}\sum_{j=1}^{\infty}\phi_j(x_3)\phi_j(y_3)H_\rho(\tilde{x}-\tilde{y},\lambda_j),\qquad x,y\in\Lambda_\rho \text{ and } \tilde{x}\neq\tilde{y},\tag{4.60}$$

belongs to $H^s(\Lambda_\rho)\times H^s(\Lambda_\rho)$ and the series converges absolutely in H^s for $s<1/2$ as a function of x or y. Further, it holds

$$\|G_{\rho,j}(x,\cdot)\|_{H^s}^2\leq C(s)j^{s-3/2},$$

where $C(s)>0$ independent of $j\in\mathbb{N}$ and $x\in\Lambda_\rho$.

Proof. Due to the asymptotic expansion of the Bessel and Hankel function for large arguments in Lemma A.1.2 in the Appendix, we can give in the following an estimate of the absolute value of the periodic Hankel function $\hat{H}_\rho(\tilde{\boldsymbol{n}},\lambda_j)$. Like in dimension two, the second and the third case in (4.46) in Lemma 4.4.1 hold only for a finite number of $\min(\tilde{\boldsymbol{n}},j)$ large enough. Thus, we estimate the first case of (4.46) in Lemma 4.4.1 for $\boldsymbol{n}\neq 0$, $\lambda_j\rho^2\neq\pi^2|\tilde{\boldsymbol{n}}|^2$ and $\tilde{\boldsymbol{n}}\in\mathbb{Z}, j\in\mathbb{N}$ large enough, such that

$$
\begin{aligned}
\left|\hat{H}_\rho(\tilde{\boldsymbol{n}},\lambda_j)\right|&\leq\frac{C}{|\lambda_j^2\rho^2-\pi^2|\tilde{\boldsymbol{n}}|^2|}\left[|\tilde{\boldsymbol{n}}|\,|H_0^{(1)}(\lambda_j\rho)|\,|J_1(\pi|\tilde{\boldsymbol{n}}|)|+|\lambda_j|\,|H_1^{(1)}(\lambda_j\rho)|\,|J_0(\pi|\tilde{\boldsymbol{n}}|)|+1\right]\\
&\leq\frac{|\lambda_j|^{1/2}|\tilde{\boldsymbol{n}}|^{-1/2}}{|\lambda_j^2\rho^2-\pi^2|\tilde{\boldsymbol{n}}|^2|}+\frac{|\lambda_j|^{-1/2}|\tilde{\boldsymbol{n}}|^{1/2}}{|\lambda_j^2\rho^2-\pi^2|\tilde{\boldsymbol{n}}|^2|}+\frac{1}{|\lambda_j^2\rho^2-\pi^2|\tilde{\boldsymbol{n}}|^2|}\\
&\leq C(1+|\tilde{\boldsymbol{n}}|^2+|\lambda_j|^2)^{-3/4},
\end{aligned}\tag{4.61}
$$

where $C>0$ denotes a constant. Now, due to the estimate of the eigenvector ϕ_j in Corollary 2.2.5 and the estimate of the vector $v_{\tilde{\boldsymbol{n}}}$ in Corollary 3.2.1, we obtain

$$|\hat{G}_{\rho,j}(\boldsymbol{n},x,\cdot)|\leq C\left|\hat{H}_\rho(\tilde{\boldsymbol{n}},\lambda_j)\right|\|\overline{v_{\tilde{\boldsymbol{n}}}}\phi_j\|_\infty\leq C\left|\hat{H}_\rho(\tilde{\boldsymbol{n}},\lambda_j)\right|\leq(1+|\tilde{\boldsymbol{n}}|^2+|\lambda_j|^2)^{-3/4}.$$

Thus,

$$\begin{aligned}\sum_{j\in\mathbb{N}} \|G_{\rho,j}(x,\cdot)\|_{H^s(\Lambda_\rho)} &= \sum_{\tilde{n}\in\mathbb{Z}^2 n_3=j\in\mathbb{N}} (1+|\tilde{\boldsymbol{n}}|^2+|\lambda_j|^2)^s|\hat{G}_{\rho,j}(\boldsymbol{n},x,\cdot)|^2 \\ &\leq \sum_{\tilde{n}\in\mathbb{Z}^2, j\in\mathbb{N}} (1+|\tilde{\boldsymbol{n}}|^2+|\lambda_j|^2)^{s-3/2} \\ &\leq \sum_{\tilde{n}\in\mathbb{Z}^2, j\in\mathbb{N}} (1+|\tilde{\boldsymbol{n}}|^2+j^2)^{s-3/2}.\end{aligned}$$

Like in the proof of Lemma 4.3.3 for dimension two, due to the fact that $|\tilde{\boldsymbol{n}}||j| \leq |\tilde{\boldsymbol{n}}|^2+|j|^2$ we know for $a,b>0$ and $1+a\leq b$ that

$$|\tilde{\boldsymbol{n}}|^{(1+a)/b}|j|^{(1+a)/b} \leq |\tilde{\boldsymbol{n}}|^2+|j|^2 \qquad \text{for } \tilde{\boldsymbol{n}}\in\mathbb{Z}^2, j\in\mathbb{N}.$$

We further see for $a,b>0$ and $1+a\leq b$ that

$$|\tilde{\boldsymbol{n}}|^{(1+a)}|j|^{(1+a)} \leq (1+|\tilde{\boldsymbol{n}}|^2+|j|^2)^b \qquad \text{for } \tilde{\boldsymbol{n}}\in\mathbb{Z}^2, j\in\mathbb{N}.$$

In particular, we set $b=3/2-s$ and $a=b-1=1/2-s$ to obtain

$$|\tilde{\boldsymbol{n}}|^{-1-a}|j|^{-1-a} \leq (1+|\tilde{\boldsymbol{n}}|^2+|j|^2)^{-b} \qquad \text{for } \tilde{\boldsymbol{n}}\in\mathbb{Z}^2, j\in\mathbb{N}.$$

Then, we see for $s<1/2$ that

$$\sum_{\tilde{n}\in\mathbb{Z}^2, j\in\mathbb{N}} (1+|\tilde{\boldsymbol{n}}|^2+j^2)^{s-3/2} \leq \sum_{\tilde{n}\in\mathbb{Z}^2}|\tilde{\boldsymbol{n}}|^{s-3/2}\sum_{j\in\mathbb{N}} j^{s-3/2}<\infty.$$

□

The results of the latter lemma shows a decay rate of the Fourier coefficients of the periodized Green's function, which is however not sufficient yet to prove convergence rates for the discretized integral equation we introduce later on. Inspecting the proof of Lemma 4.4.2 one sees that the non-smooth truncation in (4.42) is responsible for the decay rate $s-3/2$. To ensure uniqueness for the discretized integral equation, which we discuss later on, we need a convergence rate of $s-2$. This differs from well-known convergence theory e.g. [LN12] for constant background-speed, too. To improve this decay rate we introduce a cut-off function: For $0<\delta<\rho$, we define by $\chi := \chi_{\rho,\delta}\in C^3(\mathbb{R}^2)$ a function that satisfies

$$\chi(t) = \begin{cases} 0 & \text{for } \rho<|t|, \\ 1 & \text{for } 0\leq|t|\leq\rho-\delta, \end{cases}$$

and $0\leq\chi(t)\leq 1$ for $\rho-\delta\leq|t|\leq\rho$. This cut-off function can be 2ρ-biperiodic extended by

$$\chi_\rho(\tilde{x}+2\rho\tilde{n}) = \chi(|\tilde{x}|) \qquad \text{for } x\in\tilde{\Lambda}_\rho \text{ and } \tilde{n}\in\mathbb{Z}^2.$$

It is obvious that $\chi_\rho\in C^3(\mathbb{R}^2)$. We further note that all partial derivatives of χ_ρ at $\tilde{x}$ vanish if $|\tilde{x}|=\rho$. We moreover represent the introduced cut-off function by its Fourier coefficients

$$\chi_\rho(\tilde{x}) = \sum_{\tilde{\boldsymbol{n}}\in\mathbb{Z}^2}\hat{\chi}_\rho(\tilde{\boldsymbol{n}})v_{\tilde{\boldsymbol{n}}}(\tilde{x}) \qquad \text{with } \hat{\chi}_\rho(\tilde{\boldsymbol{n}}) = \int_{\tilde{\Lambda}_\rho}\chi_\rho(\tilde{x})\overline{v_{\tilde{\boldsymbol{n}}}(\tilde{x})}\,d\tilde{x} \qquad \text{for } \tilde{\boldsymbol{n}}\in\mathbb{Z}^2.$$

We have now the tools to define a second periodized Green's function in dimension three with smoother kernel by

$$G_\rho^{smo}(\tilde{x},\tilde{y}) = \frac{i}{4}\sum_{j=1}^{\infty}\phi_j(x_3)\phi_j(y_3)\chi_\rho(\tilde{x}-\tilde{y})H_\rho(\tilde{x}-\tilde{y},\lambda_j) \qquad \text{for } x,y\in\Omega \text{ and } \tilde{x}\neq\tilde{y}. \tag{4.62}$$

By well-known Fourier theory it follows now that the Fourier coefficient of G_ρ^{smo} is a discrete convolution of the Fourier coefficient of H_ρ and the Fourier coefficient $\hat{\chi}_\rho(\tilde{\boldsymbol{n}})$.

Lemma 4.4.3. *For fixed $x \in \Lambda_\rho$, the Fourier coefficients of $y \mapsto G_\rho^{smo}(x,y)$ are given by*

$$\hat{G}_{\rho,j}^{smo}(\boldsymbol{n}, x, \cdot) = \frac{i}{4}\delta_{j,n_3}\left[\sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}^2} \hat{H}_\rho(\tilde{\boldsymbol{k}}, \lambda_j)\hat{\chi}_\rho(j_1 - k_1, j_2 - k_2)\right]\overline{v_{\tilde{n}}(\tilde{x})}\phi_j(x_3), \qquad \boldsymbol{n} \in \mathbb{Z}_+^3, j \in \mathbb{N}. \tag{4.63}$$

Proof. Due to the convolution structure of the series terms of $G^{smo}\rho$ in the horizontal variables, it is sufficient to replace the Fourier coefficients of $H_\rho(\cdot, \lambda_j)$ in (4.45) by those of $\chi_\rho H_\rho(\cdot, \lambda_j)$. In particular we have for the smooth kernel

$$\begin{aligned}
\left(\widehat{\chi_\rho H_\rho}\right)_j(\tilde{\boldsymbol{n}}) &= \int_{\tilde{\Lambda}_\rho} H_\rho(|\tilde{x} - \tilde{y}|, \lambda_j)\chi_\rho(\tilde{x})\overline{\boldsymbol{v}_{\boldsymbol{n}}(\tilde{x})}\, d\tilde{x} \\
&= 2\rho \sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}^2} \hat{H}_\rho(\tilde{\boldsymbol{k}}, \lambda_j) \int_{\tilde{\Lambda}_\rho} \chi_\rho(\tilde{x})\overline{\boldsymbol{v}_{\boldsymbol{n}}(\tilde{x})}\boldsymbol{v}_{\boldsymbol{k}}(\tilde{x})\, d\tilde{x} \\
&= \frac{1}{2\rho} \sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}^2} \hat{H}_\rho(\tilde{\boldsymbol{k}}, \lambda_j) \int_{\tilde{\Lambda}_\rho} \chi_\rho(\tilde{x}) \exp\left(-i\frac{\pi}{\rho}[(n_1 - k_1)x_1 + (n_2 - k_2)x_2]\right) d\tilde{x} \\
&= \frac{1}{2\rho} \sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}_2} \hat{H}_\rho(\tilde{\boldsymbol{k}}, \lambda_j)\hat{\chi}_\rho(n_1 - k_1, n_2 - k_2) \qquad \text{for all } \tilde{\boldsymbol{n}} \in \mathbb{Z}_+^2, j \in \mathbb{N}.
\end{aligned}$$

This ends the proof. □

If χ_ρ is ℓ times continuously differentiable, then it is well-known that its Fourier coefficients $\hat{\chi}_\rho(\tilde{n})$ decay like $(1 + |\tilde{n}|)^{-\ell}$, see [Kat04, pg. 26, Theorem 4.4]. Consequently, the higher ℓ, the smaller the number of terms in the discrete convolution (4.63) required to obtain an accurate approximation to the entire series. We not that due to [SV02] it is well-known for such χ_ρ and $\ell > m - 1$, the fast Fourier transform provides a mean to compute the Fourier coefficients $\hat{\chi}_\rho(\tilde{n})$ with $|\tilde{\boldsymbol{n}}| \leq N$ in $O(N \log(N))$ steps with a relative ℓ^2-error bounded proportional to $N^{-\ell}$.

To this end, we can now treat decay rate theory using the periodized Green's function with smooth kernel.

Lemma 4.4.4. *Consider $m = 3, l \geq 3$ and fixed x. Then for $s < 1$ it holds*

$$\sum_{j\in\mathbb{N}} \left\| \frac{i}{4}\phi_j(x_3)\phi_j(y_3)\chi_\rho(\tilde{x})H_\rho(\tilde{x} - \tilde{y}, \lambda_j)\right\|_{H^s(\Lambda_\rho)} \leq C(s)\sum_{j\in\mathbb{N}} j^{s-2} \qquad \textit{for } x, y \in \Lambda_\rho, \tilde{x} \neq \tilde{y},$$

where $C(s) > 0$ independent of $j \in \mathbb{N}$.

Proof. The proof of this lemma uses the ideas of the proof of Lemma 4.4.1, however, further relying on the cut-off function χ_ρ in (4.47). Inspecting the proof of Lemma 4.4.1, we see in the following that the Fourier coefficients of $\chi_\rho H_\rho(\cdot, \lambda_j)$ decay faster than those of $H_\rho(\cdot, \lambda_j)$. We first assume that $\sigma_{\tilde{\boldsymbol{n}},j} \neq 0$. Using $\chi_\rho H_\rho(\cdot, \lambda_j)$ instead of $H_\rho(\cdot, \lambda_j)$ in (4.47-4.51), then we can split off the integral into

$$\begin{aligned}
\int_{\tilde{\Lambda}_\rho} \chi_\rho(\tilde{z})H_\rho(\lambda_j, \tilde{z})\overline{\boldsymbol{v}_{\tilde{\boldsymbol{n}}}(\tilde{z})}\, d\tilde{z} &= \lim_{\varepsilon\to 0} \int_{\varepsilon<|\tilde{z}|<\rho} \chi_\rho(\tilde{z})H_\rho(\lambda_j, \tilde{z})\overline{\boldsymbol{v}_{\tilde{\boldsymbol{n}}}(\tilde{z})}\, d\tilde{z} \\
&= \lim_{\varepsilon\to 0}\left(\int_{\varepsilon<|\tilde{z}|<\rho-\delta} + \int_{\rho-\delta<|\tilde{z}|<\rho}\right)\chi_\rho(\tilde{z})H_\rho(\lambda_j, \tilde{z})\overline{\boldsymbol{v}_{\tilde{\boldsymbol{n}}}(\tilde{z})}\, d\tilde{z}.
\end{aligned} \tag{4.64}$$

Furthermore, the well-known Green's second identity yields

$$\begin{aligned}\int_{\tilde{\Lambda}_\rho} &\chi_\rho(\tilde{z})H_\rho(\lambda_j,\tilde{z})\overline{\boldsymbol{v}_{\tilde{\boldsymbol{n}}}(\tilde{z})}\,d\tilde{z}\\ &= \sigma_{\tilde{\boldsymbol{n}},j}\left[\lim_{\varepsilon\to 0}\left(\int_{|\tilde{z}|=\rho-\delta} - \int_{|\tilde{z}|=\varepsilon}\right)\left(H_0^{(1)}(\lambda_j|\tilde{z}|)\frac{\partial\overline{v_{\tilde{\boldsymbol{n}}}}}{\partial\nu} - \overline{v_{\tilde{\boldsymbol{n}}}}\frac{\partial}{\partial\nu}H_0^{(1)}(\lambda_j|\tilde{z}|)\right)ds(\tilde{z})\right.\\ &\qquad\left. + \int_{\rho-\delta<|\tilde{z}|<\rho}(\Delta_{\tilde{x}}+\lambda_j^2)[\chi_\rho(\tilde{z})H_0^{(1)}(\lambda_j|\tilde{z}|)]\overline{v_{\tilde{\boldsymbol{n}}}}d\tilde{z}\right].\end{aligned}$$

From the proof of Lemma 4.4.1 we know that the integral over $\{|\tilde{z}|=\varepsilon\}$ as $\varepsilon\to 0$ tends to $2i/\rho$ and the boundary term on $\{|\tilde{z}|=\rho\}$ vanishes by the definition of the cut-off function. We moreover write for simplicity $H_0^{(1)} := H_0^{(1)}(\lambda_j|\tilde{z}|)$. Then, we have

$$\begin{aligned}\int_{\rho-\delta<|\tilde{z}|<\rho} &(\Delta_{\tilde{x}}+\lambda_j^2)\left[\chi_\rho(\tilde{z})H_0^{(1)}\right]\overline{v_{\tilde{\boldsymbol{n}}}(\tilde{z})}d\tilde{z}\\ &= \int_{\rho-\delta<|\tilde{z}|<\rho}\left[\Delta\chi_\rho(\tilde{z})\cdot H_0^{(1)} + 2\nabla_{\tilde{x}}\chi_\rho\cdot\nabla_{\tilde{x}}H_0^{(1)} + \chi_\rho(\tilde{z})\underbrace{\left(\Delta H_0^{(1)}+\lambda_j^2H_0^{(1)}\right)}_{=0}\right]\overline{v_{\tilde{\boldsymbol{n}}}(\tilde{z})}d\tilde{z},\end{aligned}$$

where it holds $H_0^{(1)}$ solves the two-dimensional homogeneous Helmholtz equation

$$\Delta H_0^{(1)}(\lambda_j|\tilde{z}|) + \lambda_j^2 H_0^{(1)}(\lambda_j|\tilde{z}|) = 0 \qquad \text{for } |\tilde{z}|>\rho-\delta.$$

Next, we write

$$M(\tilde{z},\lambda_j) := \Delta\chi_\rho(\tilde{z})\cdot H_0^{(1)} + 2\nabla_{\tilde{z}}\chi_\rho\cdot\nabla_{\tilde{z}}(\tilde{z})H_0^{(1)} \qquad \text{with } \tilde{z}\in(\rho-\delta,\rho). \tag{4.65}$$

As χ_ρ is at least three times continuously differentiable by assumption, $\Delta\chi_\rho\in C^1(\tilde{\Lambda}_\rho)$ and $\nabla\chi_\rho\in C^2(\tilde{\Lambda}_\rho)$. Additionally, the C^1-norms of $\tilde{z}\mapsto H_0^{(1)}(\lambda_j|\tilde{x}|)$ in $\{|\tilde{z}|\geq\rho-\delta\}$ are uniformly bounded in $j\in\mathbb{N}$, since λ_j is positive and monotonically decreasing for $j>J$, such that $H_0^{(1)}(\lambda_j|\tilde{x}|)$ and each component of $\nabla H_0^{(1)}(\lambda_j|\tilde{x}|)$ are monotonically decreasing sequences in j for $j>J$. In consequence, the C^1-norm of the function M from (4.65) is uniformly bounded in j, and the two-dimensional generalization of [Kat04, pg. 26, Theorem 4.4] implies

$$\left|\hat{M}(\tilde{\boldsymbol{n}},\lambda_j)\right| \leq \frac{C}{1+|\tilde{\boldsymbol{n}}|^{-1}} \qquad \text{for } \tilde{\boldsymbol{n}}\in\mathbb{Z}^2, j\in\mathbb{N},$$

where $C>0$ is independent of $\tilde{\boldsymbol{n}}, j$. In consequence,

$$\int_{\rho-\delta<|\tilde{z}|<\rho} M(\tilde{z})v_{-\boldsymbol{n}}(\tilde{z})d\tilde{z} \leq \frac{C}{1+|\tilde{\boldsymbol{n}}|^{l-2}} \qquad \text{for } \tilde{\boldsymbol{n}}\in\mathbb{Z}^2. \tag{4.66}$$

Plugging (4.64-4.66) together yields uniformly boundedness

$$\left|\int_{\tilde{\Lambda}_\rho}\chi_\rho(\tilde{z})H_\rho(\lambda_j,\tilde{z})\overline{v_{\boldsymbol{n}}(\tilde{z})}\,d\tilde{z}\right| \leq |\sigma_{\tilde{\boldsymbol{n}},j}|\left[\frac{2}{\rho}+\left|\hat{M}(\tilde{\boldsymbol{n}},\lambda_j)\right|\right] \leq |\sigma_{\tilde{\boldsymbol{n}},j}|\left[\frac{2}{\rho}+\frac{C}{1+|\tilde{\boldsymbol{n}}|}\right].$$

For $\min(\tilde{\boldsymbol{n}},j)$ large enough it is well-known from the proof of Lemma 4.4.2 that $|\sigma_{\tilde{\boldsymbol{n}},j}|\leq C(1+|\tilde{\boldsymbol{n}}|^2+|\lambda_j|^2)^{-1}$, the estimate of the eigenvector ϕ_j in Corollary 2.2.5 and the estimate of the vector $v_{\tilde{\boldsymbol{n}}}$ in Corollary 3.2.1 imply

$$\begin{aligned}|\hat{G}_{\rho,j}(\boldsymbol{n},\lambda_j,\cdot)| &\leq C\left|\int_{\tilde{\Lambda}_R}\chi_\rho(\tilde{z})H_\rho(\lambda_j,\tilde{z})\overline{v_{\boldsymbol{n}}(\tilde{z})}\,d\tilde{z}\right| |\|\overline{v_{\tilde{\boldsymbol{n}}}}\phi_j\|_\infty\\ &\leq C|\sigma_{\tilde{\boldsymbol{n}},j}|\left[\frac{2}{\rho}+\frac{C}{1+|\tilde{\boldsymbol{n}}|}\right] \leq (1+|\tilde{\boldsymbol{n}}|^2+|\lambda_j|^2)^{-1}.\end{aligned}$$

Hence, we obtain for $s < 1$ and fixed x that it holds

$$\sum_{j\in\mathbb{N}} \left\| \frac{i}{4}\phi_j(x_3)\phi_j(y_3)\chi_\rho(\tilde{x})H_\rho(\tilde{x}-\tilde{y},\lambda_j) \right\|^2_{H^s(\Lambda_\rho)} = \sum_{\boldsymbol{n}\in\mathbb{Z}^3_+, j=n_3} (1+|\tilde{\boldsymbol{n}}|^2+|\lambda_j|^2)^s |\hat{G}_{\rho,j}(\boldsymbol{n},\lambda_j,\cdot)|^{-2}$$
$$\leq C \sum_{\tilde{\boldsymbol{n}}\in\mathbb{Z}^2_+, j\in\mathbb{N}} (1+|\tilde{\boldsymbol{n}}|^2+|\lambda_j|^2)^{s-2} \leq C\sum_{j\in\mathbb{N}} j^{s-2}.$$

This finishes the proof. □

The cut-off function χ_ρ hence increases here the regularity of the periodized Green's function from $H^s(\Lambda_\rho)$ with $s < 1/2$ (obtained by the proof of Lemma 4.4.3)) to $H^s(\Lambda_\rho)$ with $s < 1$ by increasing the decay rate for of the Fourier coefficients of the Green's function from $-3/2$ to -2.

4.5 Periodized Lippmann-Schwinger-Integral Equation

We have now prepared all tools to introduce the periodized Lippmann-Schwinger equation, that will be shown to be equivalent to its non-periodized variant (4.32). This periodic integral equation is suitable for discretization by a collocation method based on trigonometric polynomials in the horizontal variables $\tilde{x}$ and the eigenfunctions ϕ_j to (2.14) in the vertical variable x_m as we will show later on.

We first define the periodized convolution operator $\mathcal{V}_\rho$ using for $m = 2$ the periodized Green's function $G_\rho(\cdot,y)$ given in (4.39) and for $m = 3$ the periodized Green's function with smooth kernel (using the cut-off function χ_ρ) $G^{smo}_\rho(\cdot,y)$ from equation (4.62). More precisely, the periodic integral operator $\mathcal{V}_\rho : L^2(\Lambda_\rho) \to L^2(\Lambda_\rho)$ is denoted by

$$\mathcal{V}_\rho f = \begin{cases} \int_{\Lambda_\rho} G_\rho(\cdot,y)f(y)\,dy & \text{for } m=2,\\ \int_{\Lambda_\rho} G^{smo}_\rho(\cdot,y)f(y)\,dy & \text{for } m=3. \end{cases}$$

Note hence that in dimension three we rely on the smoothed periodized Green's function G^{smo}_ρ instead of G_ρ, as this choice improves the mapping properties of the periodized volume potential $\mathcal{V}_\rho$. Now, we are interested in the eigenvalues and eigenvectors of the periodized convolution operator $\mathcal{V}_\rho$.

Theorem 4.5.1. *The complete orthonormal system $\{\varphi_n\}_{n\in\mathbb{Z}^m_+}$ is an eigensystem of $\mathcal{V}_\rho$ with corresponding eigenvalues $i/(2\lambda_j)\hat{E}_\rho(n_1,\lambda_j)$ in dimension $m = 2$ and in dimension 3 the corresponding eigenvalues are $i/4\hat{H}^{smo}_\rho$ where*

$$\hat{H}^{smo}_\rho = \sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}^2} \left[\hat{H}_\rho(\tilde{\boldsymbol{k}},\lambda_j)\hat{\chi}_\rho(-(\tilde{\boldsymbol{n}}-\tilde{\boldsymbol{k}}))\right].$$

Proof. We first recall the representation of the periodized Green's function for $m = 2$, given in equation (4.39) and the periodized Green's function with smooth kernel in (4.62),

$$G_\rho(x,y) = \frac{i}{2}\sum_{j=1}^{\infty}\frac{1}{\lambda_j}\phi_j(x_2)\phi_j(y_2)E_\rho(x_1-y_1,\lambda_j), \qquad \text{where } x_1\neq y_1, m=2,$$

$$G^{smo}_\rho(x,y) = \frac{i}{4}\sum_{j=1}^{\infty}\phi_j(x_3)\phi_j(y_3)\chi_\rho(\tilde{x}-\tilde{y})H_\rho(\tilde{x}-\tilde{y},\lambda_j) \qquad \text{where } \tilde{x}\neq\tilde{y}, m=3$$

and due to the proof of Lemma 4.3.2 we certainly obtain the Fourier coefficient of G_ρ in dimension two by

$$\hat{G}_{\rho,j}(\boldsymbol{n},x,\cdot) = \delta_{j,n_2}\,\phi_j(x_2)\,\frac{i\sqrt{\rho}}{\sqrt{2}\lambda_j}\hat{E}_\rho(n_1,\lambda_j)v_{-n_1}(x_1).$$

Then for $m = 2$ the eigendecomposition of $\mathcal{V}_\rho$ it follows that by $\mathcal{V}_\rho \varphi_{\boldsymbol{n}}$, which correspond to a diagonalization process,

$$\mathcal{V}_\rho \varphi_{\boldsymbol{n}}(x) = \int_{\Lambda_\rho} G_\rho(x,y)\varphi_{\boldsymbol{n}}(y)\,dy = \frac{\sqrt{\rho}}{\sqrt{2}\lambda_j}\hat{E}_\rho(n_1,\lambda_j)\varphi_{\boldsymbol{n}}(x), \qquad \boldsymbol{n} \in \mathbb{Z}^2, j \in \mathbb{N},$$

where we exploited that $\{\phi_j\}_{j\in\mathbb{N}}$ and $\{v_{n_1}\}_{n_1\in\mathbb{Z}}$ are orthonormal. For $m = 3$, Lemma 4.4.3 indicates a convolution-type representation of the Fourier coefficients of $y \mapsto G_\rho^{smo}(x,y)$. Due to the proof of Lemma 4.4.3 we know that the term $\sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}^2} \hat{H}_\rho(\tilde{\boldsymbol{k}},\lambda_j)\hat{\chi}_\rho(\tilde{\boldsymbol{n}}-\tilde{\boldsymbol{k}})$ in brackets in (4.63) equals the $\tilde{\boldsymbol{n}}$th Fourier coefficient of $\chi_\rho H_\rho(\lambda_j,\cdot)$. Hence, we see by simple computation that

$$\begin{aligned}
\mathcal{V}_\rho \varphi_{\boldsymbol{n}}(x) &= \int_{\Lambda_\rho} G_\rho^{smo}(x,y)\varphi_{\boldsymbol{n}}(y)\,dy \\
&= \int_{\Lambda_\rho} \sum_{\boldsymbol{k}\in\mathbb{Z}^3_+, j=k_3} \hat{G}^{smo}_{\rho,j}(\boldsymbol{k},x,\cdot)\varphi_{\boldsymbol{k}}(y)\,\varphi_{\boldsymbol{n}}(y)\,dy \\
&= \frac{i}{4}\sum_{\tilde{\boldsymbol{k}}\in\mathbb{Z}^2} \left[\hat{H}_\rho(\tilde{\boldsymbol{k}},\lambda_j)\chi_\rho(-(\tilde{\boldsymbol{n}}-\tilde{\boldsymbol{k}}))\right]\varphi_{\boldsymbol{n}}(x), \qquad \boldsymbol{n}\in\mathbb{Z}^3_+, j\in\mathbb{N}.
\end{aligned}$$

□

Theorem 4.5.2. *The integral operator $\mathcal{V}_\rho$ is bounded from $H^s(\Lambda_\rho)$ into $H^{s+2}(\Lambda_\rho)$.*

Proof. For $m = 2$ we know by Lemma 4.3.3 that the Fourier coefficients of $G_{\rho,j}$ has been estimated by $C(1+|\boldsymbol{n}|^2+|\lambda_j|^2)^{-1}$ for $\tilde{\boldsymbol{n}} \in \mathbb{Z}, j \in \mathbb{N}$. Similarly estimates in dimension three holds for the Fourier coefficients of $G^{smo}_{\rho,j}$ for $\tilde{\boldsymbol{n}} \in \mathbb{Z}^2, j\in\mathbb{N}$ by Lemma 4.4.4. Due to the δ_{j,n_m} in the definition of the Fourier coefficients of $G_{\rho,j}$ and $G^{smo}_{\rho,j}$ in the following we abbreviate $j = n_m$. All together shows that the eigenvalues of $\mathcal{V}_\rho$ decay quadratically independent of dimension m. In particular, it holds

$$|\widehat{\mathcal{V}_\rho f}(\boldsymbol{n})|^2 \le C(1+|\boldsymbol{n}|^2+|\lambda_{n_m}|^2)^{-2} \qquad \text{for } \boldsymbol{n}\in\mathbb{Z}^m.$$

Hence, it follows for $s \in \mathbb{R}$ that

$$\begin{aligned}
\|\mathcal{V}_\rho f\|^2_{H^s(\Lambda_\rho)} &= \sum_{\boldsymbol{n}\in\mathbb{Z}^m_+} (1+|\tilde{\boldsymbol{n}}|^2+|\lambda_{n_m}|^2)^s|\widehat{\mathcal{V}_\rho f(\boldsymbol{n})}|^2 \\
&\le C\sum_{\boldsymbol{n}\in\mathbb{Z}^m_+} (1+|\tilde{\boldsymbol{n}}|^2+|\lambda_{n_m}|^2)^s(1+|\boldsymbol{n}|^2+|\lambda_{n_m}|^2)^{-2}|\hat{f}(\boldsymbol{n})|^2 \\
&\le C\sum_{\boldsymbol{n}\in\mathbb{Z}^m_+} (1+|\tilde{\boldsymbol{n}}|^2+|\lambda_{n_m}|^2)^{s-2}|\hat{f}(\boldsymbol{n})|^2 = C\|\varphi\|^2_{H^{s-2}(\Lambda_\rho)},
\end{aligned}$$

where $C > 0$ denotes a constant. □

Remark 4.5.3. *Analogously, we can show for $s \in \mathbb{R}$ that the integral operator*

$$f \mapsto \int_{\Lambda_\rho} G_\rho(\cdot,y)f(y)\,dy \qquad \text{for } m=3$$

using the Green's function in dimension three without smooth kernel is bounded from $H^s(\Lambda_\rho)$ into $H^{s+3/2}(\Lambda_\rho)$.

Theorem 4.5.4. *Consider $m = 2$. If $f \in L^2(\Lambda_\rho)$ and $\rho > 0$ is sufficient large such that the contrast q is inlcuded in $M_{\rho/2}$, supp$(f) \subset M_{\rho/2}$, then $\mathcal{V}_\rho f$ equals $\mathcal{V}f$ in $M_{\rho/2}$ and converse. Further, let $m = 3$. If $f \in L^2(\Lambda_\rho)$ and $\rho > 0$ is sufficient large that satisfy* supp$(f) \subset M_{(\rho-\delta)/2}$, *then $\mathcal{V}f$ equals $\mathcal{V}_\rho f$ in $M_{(\rho-\delta)/2}$.*

Proof. We first treat the two-dimensional case. For $x \in M_{\rho/2}$ and $f \in C_0^\infty(\Lambda_\rho)$ we obtain by the definition $\mathcal{V}_\rho$ and the definition of the periodized Green's function that

$$\begin{aligned}(\mathcal{V}_\rho f)(x) &= \int_{\Lambda_\rho} G_\rho(x,y) f(y) dy \\ &= \int_{M_{\rho/2}} \frac{i}{2} \sum_{j\in\mathbb{N}} \frac{1}{\lambda_j} \phi_j(x_2)\phi_j(y_2) E_\rho(x_1 - y_1, \lambda_j) f(y)\, dy \\ &= \int_{M_{\rho/2}} \frac{i}{2} \sum_{j\in\mathbb{N}} \frac{1}{2} \phi_j(x_2)\phi_j(y_2) \exp(i\lambda_j |x_1 - y_1|) f(y)\, dy \\ &= \int_{\Lambda_\rho} G(x,y) f(y)\, dy = (\mathcal{V} f)(x).\end{aligned}$$

Owing to the density of $C_0^\infty(\Lambda_\rho)$ in $L^2(\Lambda_\rho)$, $\mathcal{V}_\rho$ is equal $\mathcal{V}$ for any $f \in L^2(\Lambda_\rho)$ and converse.

Let us now turn to dimension three. Let us first note that $|\tilde{x} - \tilde{y}| \leq |\tilde{x}| + |\tilde{y}| < \rho - \delta$ for $x, y \in M_{(\rho-\delta)/2}$, such that $G_\rho^{smo}(x,y)$ equal the waveguide Green's function $G(x,y)$ by definition of the periodic function (4.43). We further need to exploit that $\chi_\rho(t) = 1$ for $|t| < \rho - \delta$. Hence, we show the required identity of $\mathcal{V} f$ and $\mathcal{V}_\rho f$ for $x \in M_{(\rho-\delta)/2}$,

$$\begin{aligned}(\mathcal{V}_\rho f)(x) &= \int_{\Lambda_\rho} G_\rho^{smo}(x,y)\, f(y)\, dy \\ &= \frac{i}{4} \int_{M_{(\rho-\delta)/2}} \sum_{j\in\mathbb{N}} \phi_j(x_3)\phi_j(y_3) \chi_\rho(\tilde{x} - \tilde{y}) H_\rho(\tilde{x} - \tilde{y}, \lambda_j)\, f(y)\, dy \\ &= \frac{i}{4} \int_{M_{(\rho-\delta)/2}} \sum_{j\in\mathbb{N}} \phi_j(x_3)\phi_j(y_3) H_0^{(1)}(\lambda_j |\tilde{x} - \tilde{y}|)\, f(y)\, dy \\ &= \int_{\Lambda_\rho} G(x,y)\, f(y)\, dy = (\mathcal{V} f)(x).\end{aligned}$$

□

In dimension three strengthening the assumption $\text{supp}(q) \subset M_\rho$ to $\text{supp}(q) \subset M_{(\rho-\delta)/2}$ allows next to prove equivalence between the original and the periodized Lippmann-Schwinger equation.

Theorem 4.5.5. *If Assumption 4.2.11 holds and if* $\text{supp}(q) \subset M_{(\rho-\delta)/2}$, *then for any source* $f \in L^2(\Lambda_\rho)$, *the periodized Lippmann-Schwinger equation*

$$u - \mathcal{V}_\rho \left(\frac{\omega^2}{c^2(x_m)} q u \right) = \mathcal{V}_\rho f \qquad \text{in } L^2(\Lambda_\rho) \tag{4.67}$$

is uniquely solvable. If $f \in L^2(\Lambda_\rho)$ *with* $\text{supp}(f) \subset M_{(\rho-\delta)/2}$, *then restricting the solution to (4.67) to* $M_{(\rho-\delta)/2}$ *implies the corresponding restriction of the solution to the Lippmann-Schwinger equation (4.32), or, equivalently, of the solution to the source problem (4.34) and (3.3) with source term* $f|_{M_\rho}$.

Proof. The second part of the claim follows obviously from Theorem 4.5.4 and the equivalence result from Theorem 4.2.10. Hence, we need to show the first part, i.e., existence and uniqueness of solution to (4.67). Due to Riesz theory and the compactness of $\mathcal{V}_\rho$ on $L^2(\Lambda_\rho) = H^0(\Lambda_\rho)$ following from Theorem 4.5.2 and Rellich's compact embedding theorem, this reduces to proving uniqueness of solution to (4.67). This in turn follows by noting that the restriction to $M_{(\rho-\delta)/2}$ of a solution u to (4.67) with $f = 0$ solves, due to Theorem 4.5.4, the original Lippmann-Schwinger equation posed in $L^2(M_{(\rho-\delta)/2})$ with vanishing right-hand side. By Theorem 4.2.10 and Assumption 4.2.11, this is only possible if $u|_{M_{(\rho-\delta)/2}}$ vanishes, which shows that $u \in L^2(\Lambda_\rho)$, solution to (4.67), has to vanish as well since $\text{supp}(q) \subset M_{(\rho-\delta)/2}$, such that $qu \in L^2(\Lambda_\rho)$ vanishes entirely. □

Remark 4.5.6. *In dimension $m = 2$, Theorem 4.5.4 implies that the equivalence result of Theorem 4.5.5 even holds when $(\delta - \rho)/2$ is replaced by ρ. Moreover, we point out that Theorem 4.5.5 holds as well if one defines $\mathcal{V}_\rho$ for $m = 3$ via the non-smoothed periodized kernel G_ρ. However, in this case, $\mathcal{V}_\rho$ is merely bounded from $H^s(\Lambda_\rho)$ into $H^{s+3/2}(\Lambda_\rho)$. For the analysis of the collocation method in three dimension, see Chapter 5.1 later on crucially depends on the fact that $\mathcal{V}_\rho$ is smoothing more than $3/2$ orders, thus we require imperatively the cut-off function χ_ρ to construct G_ρ^{smo}.*

Chapter 5

Numerical Approximation of the Periodized Scattering Problem

5.1 The Collocation Method

In this section, we present a fast numerical solution method, which was first suggested by [SV02] in the context of trigonometric interpolation operators. In this section, we adapt this scheme (we call it Vainikko scheme) to approximate solutions to the introduced Lippmann-Schwinger integral equation, where the sound speed depends on the depth of the ocean. The Vainikko scheme applied to sufficiently smooth contrast uses the representation of Fourier coefficients of the periodized integral operator $\mathcal{V}_\rho$ and fast Fourier transform techniques. This technique avoids to compute the integral defining the integral operator $\mathcal{V}_\rho$ numerically in the domain Λ_ρ.

Let $\boldsymbol{N} := (N_1, N_2) = (\tilde{\boldsymbol{N}}, N_m) \in \mathbb{N}^2$ for $m = 2$ and $\boldsymbol{N} := (N_1, N_2, N_3) = (\tilde{\boldsymbol{N}}, N_m) \in \mathbb{N}^3$ for $m = 3$ denote the discretization parameter and define the grid

$$\mathbb{Z}^2_{\boldsymbol{N}} = \{\boldsymbol{j} \in \mathbb{Z}^2 : -N_1 < j_1 \le N_1,\, 1 \le j_2 \le N_2\} \qquad \text{for } m = 2, \text{ and}$$
$$\mathbb{Z}^3_{\boldsymbol{N}} = \{\boldsymbol{j} \in \mathbb{Z}^3 : -N_k < j_k \le N_k, 1 \le j_3 \le N_3\,, k = 1, 2\} \qquad \text{for } m = 3.$$

The corresponding set of interpolation points are denoted by

$$\left\{x^{(\boldsymbol{N})}_{\boldsymbol{j}} := \left(\rho \frac{j_1}{N_1},\, H\frac{j_2}{N_2}\right) : \boldsymbol{j} \in \mathbb{Z}^2_{\boldsymbol{N}}\right\} \subset [-\rho, \rho] \times (0, H] \qquad \text{for } m = 2 \text{ and}$$
$$\left\{x^{(\boldsymbol{N})}_{\boldsymbol{j}} := \left(\rho \frac{j_1}{N_1},\, \rho\frac{j_2}{N_2},\, H\frac{j_3}{N_3}\right) : \boldsymbol{j} \in \mathbb{Z}^3_{\boldsymbol{N}}\right\} \subset [-\rho, \rho]^2 \times (0, H] \qquad \text{for } m = 3.$$

To simplify the notation we write for dimension three $\boldsymbol{j} = (\tilde{\boldsymbol{j}}, j_3)$ instead of $\boldsymbol{j} = (j_1, j_2, j_3)$. We point out that the grid points $x^{(\boldsymbol{N})}_{\boldsymbol{j}}$ are not contained in the plane $\{x_m = 0\}$ where all eigenfunctions ϕ_{ℓ_m} $(1 \le \ell_m \le N_m)$ vanish, since Dirichlet boundary conditions hold. For simplicity, we further introduce the grid for the horizontal component

$$\mathbb{Z}^{m-1}_{\tilde{\boldsymbol{N}}} = \{\tilde{\boldsymbol{j}} \in \mathbb{Z}^{m-1} : -N_k < j_k \le N_k, 1 \le k \le m-1\} \subset \mathbb{Z}^{m-1} \qquad \text{for } m = 2, 3.$$

We moreover introduce, the interpolation points acting on the horizontal component,

$$\left\{\tilde{x}^{(\boldsymbol{N})}_{j_1} := \left(\rho \frac{j_1}{N_1}\right) : j_1 \in \mathbb{Z}_{\tilde{\boldsymbol{N}}}\right\} \subset [-\rho, \rho] \qquad \text{for } m = 2 \text{ and}$$
$$\left\{\tilde{x}^{(\boldsymbol{N})}_{\tilde{\boldsymbol{j}}} := \left(\rho \frac{j_1}{N_1},\, \rho\frac{j_2}{N_2}\right) : \tilde{\boldsymbol{j}} \in \mathbb{Z}^2_{\tilde{\boldsymbol{N}}}\right\} \subset [-\rho, \rho]^2 \qquad \text{for } m = 3.$$

We point out that these interpolation points are fixed in x_m variable and no interpolation point is contained in the plane $x_m = 0$, where all eigenfunctions ϕ_{ℓ_m} $(1 \le \ell_m \le N_m)$ vanish. Furthermore, we define the discrete eigenvector

$$\boldsymbol{\phi}_{\ell_m}^{(N_m)} = \begin{pmatrix} \phi_{\ell_m}\left(H\frac{1}{N_m}\right) \\ \phi_{\ell_m}\left(H\frac{2}{N_m}\right) \\ \vdots \\ \phi_{\ell_m}(H) \end{pmatrix} \in \mathbb{R}^{N_m}, \qquad \text{where } 1 \le \ell_m \le N_m.$$

Furthermore, we denote the matrix with discretized eigenvectors as columns by

$$\Phi_{N_m} = \left[\boldsymbol{\phi}_1^{(N_m)} \boldsymbol{\phi}_2^{(N_m)} \cdots \boldsymbol{\phi}_{N_m}^{(N_m)}\right]. \tag{5.1}$$

For simplicity, we write in the following $\boldsymbol{\phi}_{\ell_m}$ for $\boldsymbol{\phi}_{\ell_m}^{(N_m)}$, whenever this will not lead to confusion.

Assumption 5.1.1. *The sound speed $c \in C^0([0,H])$ is continuous.*

Lemma 5.1.2. *Suppose that Assumption 5.1.1 holds. Then $\left(\boldsymbol{\phi}_{\ell_m}^{(N_m)}\right)_{\ell_m=1,\dots,N_m}$ forms a basis of $\mathbb{R}^{N_m}$ and Φ_{N_m} is an invertible matrix for all $N_m \in \mathbb{N}^m$.*

Proof. Due to [Kar68], see also [Slo83], it is well-known that for given $w \in C^0([0,H])$ the interpolation problem to find $w_{N_m} = \sum_{j_m=1}^{N_m} \alpha_{j_m}\phi_{j_m} \in C^0([0,H])$ such that $w_{N_m}(H(j_m/N_m)) = w(H(j_m/N_m))$ for $j = 1,\dots,N_m$ is uniquely solvable for the indicated interpolation points and all $N_m \in \mathbb{N}$. (This holds even for arbitrary N_m pairwise disjoint points in $(0,H]$). In particular, for all $N_m \in \mathbb{N}$ and all $j_m = 1,\dots,N_m$ there exists a linear combination $\phi^*_{N_m,j_m} \in \operatorname{span}\{\phi_1,\dots,\phi_{N_m}\}$ such that $\phi^*_{N_m,j_m}(H(\ell_m/N_m)) = \delta_{j_m,\ell_m}$ for $\ell_m = 1,\dots,N_m$. □

Lemma 5.1.2 guarantee that Φ_{N_m} is invertible and $(\boldsymbol{\phi}_{\ell_m})_{\ell_m=1,\dots,N_m}$ are linear independent vectors in $\mathbb{R}^n$. We denote the inverse of Φ_{N_m} by $\Phi_{N_m}^{-1}$, denote the columns of $\Phi_{N_m}^{-1}$ by $\boldsymbol{\phi}_1^{-1},\dots,\boldsymbol{\phi}_{N_m}^{-1}$ and obtain that

$$I = \Phi_{N_m}\Phi_{N_m}^{-1} = \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cdots & \boldsymbol{\phi}_{N_m} \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi}_1^{-1} & \boldsymbol{\phi}_2^{-1} & \cdots & \boldsymbol{\phi}_{N_m}^{-1} \end{bmatrix} = \Phi_{N_m}^{-1}\Phi_{N_m}. \tag{5.2}$$

This implies

$$\sum_{k_m=1}^{N_m} \boldsymbol{\phi}_{k_m}(\ell_m)\boldsymbol{\phi}_{j_m}^{-1}(k_m) = \delta_{\ell_m,j_m} \quad \text{and} \quad \sum_{k_m=1}^{N_m} \boldsymbol{\phi}_{k_m}^{-1}(\ell_m)\boldsymbol{\phi}_{j_m}(k_m) = \delta_{\ell_m,j_m} \qquad (1 \le \ell_m \le N_m). \tag{5.3}$$

Since the eigenvectors have no explicit representation the vectors $\boldsymbol{\phi}_{j_m}^{-1}$ have to be computed numerically by computing the inverse $\Phi_{N_m}^{-1}$. [Sol13] presents a memory optimized and fast iterative method to compute $\Phi_{N_m}^{-1}$. Next, we define

$$\phi^*_{N_m,j_m}(x_m) = \sum_{\ell_m=1}^{N_m} \boldsymbol{\phi}_{j_m}^{-1}(\ell_m)\phi_{\ell_m}(x_m), \qquad \text{where } 1 \le j_m \le N_m, x_m \in [0,H].$$

Thus, $\phi^*_{N_m,j_m}(x_m) \in \operatorname{span}\{\phi_1,\dots,\phi_{N_m}\}$ for $1 \le j_m \le N_m$ and

$$\begin{aligned} \phi^*_{N_m,j_m}\left(H\frac{\ell_m}{N_m}\right) &= \sum_{k_m=1}^{N_m} \boldsymbol{\phi}_{j_m}^{-1}(k_m)\phi_{k_m}\left(H\frac{\ell_m}{N_m}\right) \\ &= \sum_{k_m=1}^{N_m} \boldsymbol{\phi}_{j_m}^{-1}(k_m)\boldsymbol{\phi}_{k_m}(\ell_m) = \delta_{\ell_m,j_m} \qquad (1 \le \ell_m, j_m \le N_m). \end{aligned} \tag{5.4}$$

Next, we consider trigonometric polynomials in the horizontal variables. We denote

$$v^*_{N_1,j_1}(x_1) = \frac{1}{2N_1}\sum_{\ell_1=-N_1}^{N_1-1} \exp\left(i\pi\ell_1\left(\frac{x_1}{\rho}-\frac{j_1}{N_1}\right)\right) \qquad \text{for } m=2 \text{ and}$$

$$v^*_{\tilde{\boldsymbol{N}},\tilde{\boldsymbol{j}}}(\tilde{x}) = \frac{1}{4N_1N_2}\sum_{(\ell_1,\ell_2)\in\mathbb{Z}^2_{\tilde{\boldsymbol{N}}}} \exp\left(i\pi\left[\ell_1\left(\frac{x_1}{\rho}-\frac{j_1}{N_1}\right)+\ell_2\left(\frac{x_2}{\rho}-\frac{j_2}{N_2}\right)\right]\right) \qquad \text{for } m=3.$$

By [SV02, Chapter 8] it is well-known for all $\tilde{\boldsymbol{j}},\tilde{\boldsymbol{k}} \in \mathbb{Z}^{m-1}_{\tilde{\boldsymbol{N}}}$ that

$$v^*_{N_1,j_1}\left(\rho\frac{k_1}{N_1}\right) = \delta_{j_1,k_1} \quad \text{and} \quad \int_{-\rho}^{\rho} v^*_{N_1,j_1}(x_1)\overline{v^*_{k_1}(x_1)}\,dx_1 = \frac{1}{2N_1}\delta_{j_1,k_1} \qquad \text{for } m=2 \text{ and}$$

$$v^*_{\tilde{\boldsymbol{j}}}\left(\rho\frac{k_1}{N_1},\rho\frac{k_2}{N_2}\right) = \delta_{\tilde{\boldsymbol{j}},\tilde{\boldsymbol{k}}} \quad \text{and} \quad \int_{\tilde{\Lambda}_\rho} v^*_{\tilde{\boldsymbol{j}}}(\tilde{x})\overline{v^*_{\tilde{\boldsymbol{k}}}(\tilde{x})}\,d\tilde{x} = \frac{1}{N_1N_2}\delta_{\tilde{\boldsymbol{j}},\tilde{\boldsymbol{k}}} \qquad \text{for } m=3.$$

With these ingredients, we define the linear subspace of $L^2(\Lambda_\rho)$ of trigonometric polynomials by

$$T_{\boldsymbol{N}} := \operatorname{span}\left\{\varphi^*_{\boldsymbol{N},\boldsymbol{j}} = v^*_{N_1,j_1}\phi^*_{N_2,j_2} : \boldsymbol{j}\in\mathbb{Z}^2_{\boldsymbol{N}}\right\} \subset L^2(\Lambda_\rho) = H^0(\Lambda_\rho) \qquad \text{for } m=2 \text{ and}$$

$$T_{\boldsymbol{N}} := \operatorname{span}\left\{\varphi^*_{\boldsymbol{N},\boldsymbol{j}} = v^*_{\tilde{\boldsymbol{N}},\tilde{\boldsymbol{j}}}\phi^*_{N_3,j_3} : \boldsymbol{j}\in\mathbb{Z}^3_{\boldsymbol{N}}\right\} \subset L^2(\Lambda_\rho) = H^0(\Lambda_\rho) \qquad \text{for } m=3. \tag{5.5}$$

Recall the definition of the basis functions acting on the horizontal variables by

$$v_{j_1}(\tilde{x}) := \frac{1}{\sqrt{2\rho}}\exp\left(i\frac{\pi}{\rho}j_1x_1\right) \qquad \text{for } m=2, j_1\in\mathbb{Z}, x_1\in[-\rho,\rho] \text{ and}$$

$$v_{\tilde{\boldsymbol{j}}}(\tilde{x}) := \frac{1}{2\rho}\exp\left(i\frac{\pi}{\rho}\,\tilde{\boldsymbol{j}}\cdot\tilde{x}\right) \qquad \text{for } m=3, \tilde{\boldsymbol{j}}\in\mathbb{Z}^2, \tilde{x}\in\tilde{\Lambda}_\rho.$$

Moreover, [SV02] shows that

$$\operatorname{span}\{v_{j_1}\}_{j_1=-N_1}^{N_1} = \operatorname{span}\{v^*_{j_1}, -N_1\le j_1\le N_1\} \qquad \text{for } m=2 \text{ and}$$

$$\operatorname{span}\{v_{\tilde{\boldsymbol{j}}}\}_{j_{1,2}=-N_{1,2}}^{N_{1,2}} = \operatorname{span}\{v^*_{\tilde{\boldsymbol{j}}}, \tilde{\boldsymbol{j}}\in\mathbb{Z}^2_{\boldsymbol{N}}\} \qquad \text{for } m=3.$$

(The arguments from [SV02] in one dimension are easily extended to two or three dimensions.) Further, by Lemma 5.1.2 holds

$$\operatorname{span}\{\phi_{j_m}\}_{j_m=1}^{N_m} = \operatorname{span}\{\phi^*_{j_m}, j_m = 1,...,N_m\}.$$

By the definition of the grid $\mathbb{Z}^2_{\boldsymbol{N}}$, we see that $\dim T_{\boldsymbol{N}} = 2N_1N_2$ and for the grid $\mathbb{Z}^3_{\boldsymbol{N}}$ we have that $\dim T_{\boldsymbol{N}} = 4N_1N_2N_3$. We point out that the maximum norm is well-defined for functions $u\in T_{\boldsymbol{N}}$,

$$\|u\|_\infty = \max_{x\in\overline{\Lambda_\rho}}|u(x)|.$$

By abuse of notation, we denote the maximum norm of functions on $[-\rho,\rho]^{m-1}$ and on the vertical variable $[0,H]$ by $\|\cdot\|_\infty$.

Next, we define for a function $u\in C^2(\overline{\Lambda_\rho})$ the interpolation projection $Q_{\boldsymbol{N}} : C^2(\overline{\Lambda_\rho}) \to T_{\boldsymbol{N}}$ by

$$(Q_{\boldsymbol{N}}u)\left(x^{(\boldsymbol{N})}_{\boldsymbol{j}}\right) = u\left(x^{(\boldsymbol{N})}_{\boldsymbol{j}}\right) \qquad \text{for all } \boldsymbol{j}\in\mathbb{Z}^m_{\boldsymbol{N}}. \tag{5.6}$$

Consequently, we have

$$u_{\boldsymbol{N}}(x) := (Q_{\boldsymbol{N}}u)(x) = \sum_{\boldsymbol{j}\in\mathbb{Z}^2_{\boldsymbol{N}}} u\left(\rho\frac{j_1}{N_1}, H\frac{j_2}{N_2}\right)\varphi^*_{\boldsymbol{N},\boldsymbol{j}}(x) \qquad \text{for } m=2 \text{ and} \tag{5.7}$$

$$u_{\boldsymbol{N}}(x) := (Q_{\boldsymbol{N}}u)(x) = \sum_{\boldsymbol{j}\in\mathbb{Z}^3_{N}} u\left(\rho\frac{j_1}{N_1},\rho\frac{j_2}{N_2}, H\frac{j_3}{N_3}\right)\varphi^*_{\boldsymbol{N},\boldsymbol{j}}(x) \qquad \text{for } m=3.$$

Lemma 5.1.3. *Suppose that Assumption 5.1.1 holds. Then the interpolation operator $Q_{\boldsymbol{N}}$ is well-defined and bounded on $C^0(\overline{\Lambda_\rho})$ for all $\boldsymbol{N} \in \mathbb{Z}_+^m$.*

Proof. For simplicity we set $m = 3$ since the two-dimensional case follows analogously. Recall that there exists $v^*_{\tilde{\boldsymbol{N}},\tilde{\boldsymbol{j}}} \in \operatorname{span}\{v_{\tilde{\boldsymbol{j}}} : \boldsymbol{j} \in \mathbb{Z}^m_{\tilde{\boldsymbol{N}}}\}$ such that

$$v^*_{\tilde{\boldsymbol{N}},\tilde{\boldsymbol{j}}}\left(\rho\frac{\ell_1}{\boldsymbol{N}_1}, \rho\frac{\ell_2}{\boldsymbol{N}_2}\right) = \delta_{\tilde{\boldsymbol{j}},\tilde{\boldsymbol{\ell}}} \qquad \text{for all } \tilde{\boldsymbol{\ell}} = (\ell_1, \ell_2) \in \mathbb{Z}^2_{\tilde{\boldsymbol{N}}}.$$

The last equation and Lemma 5.1.2 imply that the products $\varphi^*_{\boldsymbol{N},\boldsymbol{j}}(x) = v^*_{\tilde{\boldsymbol{N}},\tilde{\boldsymbol{j}}}(\tilde{x})\phi^*_{N_m,j_m}(x_m)$, $\boldsymbol{j} \in \mathbb{Z}^m_+$, hence belong to $T_{\boldsymbol{N}}$ and satisfy

$$\varphi^*_{\boldsymbol{N},\boldsymbol{j}}(x_{\boldsymbol{i}}^{(\boldsymbol{N})}) = \delta_{\boldsymbol{i},\boldsymbol{j}} \qquad \text{for } \boldsymbol{i}, \boldsymbol{j} \in \mathbb{Z}^m_{\boldsymbol{N}}.$$

For $\boldsymbol{N} \in \mathbb{N}^m$, the interpolation problem (5.6) is hence solved by $u_{\boldsymbol{N}} = \sum_{\boldsymbol{j}\in\mathbb{Z}^m_{\boldsymbol{N}}} u(x_{\boldsymbol{j}}^{(\boldsymbol{N})})\varphi^*_{\boldsymbol{N},\boldsymbol{j}} \in T_{\boldsymbol{N}}$. The interpolation operator $Q_{\boldsymbol{N}}$, $u \mapsto u_{\boldsymbol{N}}$ is linear with finite-dimensional range and hence continuous. □

To prove an error estimate for the difference $Q_{\boldsymbol{N}}u - u$ in the $L^2(\Lambda_\rho)$-norm, we introduce the following interpolation operators related to horizontal and vertical variables. We point out that the last proof gives us that for $\boldsymbol{N} \in \mathbb{Z}^m_+$, first, that the one-dimensional interpolation operator

$$Q^{(\perp)}_{N_m} : w \mapsto w_{N_m} = \sum_{j_m=1}^{N_m} w\left(H\frac{j_m}{N_m}\right)\phi^*_{N_m,j_m} \quad \text{on } C^0([0,H]), \tag{5.8}$$

is well-defined and continuous due to its finite-dimensional range. Second, also

$$Q^{(\sim)}_{\tilde{\boldsymbol{N}}} : v \mapsto v_{\tilde{\boldsymbol{N}}} = \sum_{-\tilde{\boldsymbol{N}}<\tilde{\boldsymbol{j}}\le\tilde{\boldsymbol{N}}} v\left(\tilde{x}^{(\boldsymbol{N})}_{\tilde{\boldsymbol{j}}}\right) v^*_{\tilde{\boldsymbol{N}},\tilde{\boldsymbol{j}}} \quad \text{on } C^0([-\rho,\rho]^{m-1}), \tag{5.9}$$

is well-defined and continuous as well. We point out that the notation of (5.9) holds also for dimension two, if for simplicity we write $\tilde{\boldsymbol{j}}$ instead of j_1 and write $\tilde{\boldsymbol{N}}$ instead of N_1.

Consequently, we can decompose $Q_{\boldsymbol{N}}$ into the product $Q^{(\sim)}_{\tilde{\boldsymbol{N}}}Q^{(\perp)}_{\boldsymbol{N}_m}$. These operators defined (5.8) and (5.9) commute when applied to $u \in C^0(\overline{\Lambda_\rho})$ since the points values of $Q^{(\sim)}_{\tilde{\boldsymbol{N}}}u(\cdot, x_m)$ and of $Q^{(\perp)}_{N_m}u(\tilde{x}, \cdot)$ at the interpolation points $\{x^{(\boldsymbol{N})}_{\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathbb{Z}^m_{\boldsymbol{N}}}$ both equal $u(x^{(\boldsymbol{N})}_{\boldsymbol{j}})$ by construction.

Furthermore, several error estimates for the difference $Q^{(\sim)}_{\tilde{\boldsymbol{N}}}v - v$ are known from trigonometric interpolation theory, see, e.g., [SV02], that we will rely on later on. It is well-known by [SV02, Chapter 10.5.4] that the structure of the basis vector $v^*_{N_1,j_1}(x_1)$ allows us to apply a fast Fourier transform (FFT) computed for dimension one in $\mathcal{O}(N_1 log(N_1))$ operations. For more information of the fast Fourier transform, we refer the reader to the classical article [CT65]. Next, we define the matrix of nodal values $\underline{u}_{\boldsymbol{N}} \in \mathbb{C}^{2N_1\times N_2}$ by

$$\underline{u}_{\boldsymbol{N}}(k_1, k_2) = u\left(\rho\frac{k_1}{N_1}, H\frac{k_2}{N_2}\right), \qquad \boldsymbol{k} \in \mathbb{Z}^2_{\boldsymbol{N}}.$$

Furthermore, we define discrete Fourier coefficients corresponding to the nodal values $\underline{u}_{\boldsymbol{N}}$ by

$$\hat{\underline{u}}_{\boldsymbol{N}}(j_1, j_2) = \frac{\sqrt{2\rho}}{2N_1}\sum_{k_1=-N_1}^{N_1-1}\sum_{k_2=1}^{N_2}\underline{u}_{\boldsymbol{N}}(k_1, k_2)\exp\left(-i\pi k_1\frac{j_1}{N_1}\right)\boldsymbol{\phi}_{j_2}(k_2), \qquad \boldsymbol{j} \in \mathbb{Z}^2_{\boldsymbol{N}}. \tag{5.10}$$

For completeness we finally indicate that

$$\hat{\underline{u}}_{\boldsymbol{N}}\left(k_1 + \lfloor -N_1\rfloor + 1, j_2\right) = \frac{\sqrt{2\rho}}{2N_1}\mathcal{F}_{N_1}\left(\left(\sum_{k_2=1}^{N_2}\underline{u}_{\boldsymbol{N}}(j_1, k_2)\boldsymbol{\phi}_{j_2}(k_2)\right)_{j_1}\right)(k_1),$$

where, $k_1 = 0, ..., 2N_1 - 1$, $j_2 = 1, ...N_2$ and $\mathcal{F}_{N_1}$ is the one-dimensional discrete Fourier transform (1D-FFT) defined for $\underline{v} \in \mathbb{C}^{2N_1}$ by

$$\hat{\underline{v}} := \mathcal{F}_{N_1}(\underline{v})(k_1) = \frac{1}{\sqrt{N_1}} \sum_{j_1=0}^{2N_1-1} \underline{v}(j_1) \exp\left(-2\pi i j_1 \frac{k_1}{N_1}\right), \quad k_1 = 0, 1, ..., 2N_1 - 1.$$

For given Fourier coefficients we can compute the nodal values by

$$\underline{v}\,(j_1) = \frac{\sqrt{N_1}}{\sqrt{2\rho}} \sum_{k_1=-N_1}^{N_1-1} \hat{\underline{v}}(k_1) \exp\left(-2\pi i \left(k_1 + \lfloor -N_1 \rfloor + 1\right) \frac{j_1}{N_1}\right),$$

where $j_1 = 0, 1, ..., 2N_1 - 1$ or shortly

$$\underline{v} = \frac{\sqrt{N_1}}{\sqrt{2\rho}} \mathcal{F}_{N_1}^{-1}(\hat{\underline{v}}).$$

Here, $\mathcal{F}_{N_1}^{-1}$ is the one-dimensional discrete Inverse Fourier transform (1D-IFFT) defined by

$$\mathcal{F}_{N_1}^{-1}(\hat{\underline{v}})(j_1) = \frac{1}{\sqrt{N_1}} \sum_{k_1=0}^{2N_1-1} \hat{\underline{v}}(\left(k_1 + \lfloor -N_1 \rfloor + 1\right), j_2) \exp\left(2\pi i k_1 \frac{j_1}{N_1}\right),$$

where $j_1 = 0, 1, ..., 2N_1 - 1$.

We point out that the Fourier coefficients $\hat{\underline{u}}_{\boldsymbol{N}}(j_1, j_2) \in \mathbb{C}^2$, where $\boldsymbol{j} \in \mathbb{Z}_{\boldsymbol{N}}^2$, are different from the Fourier coefficients introduced in Chapter 3.2, see (3.14), since we used the discrete vector $\boldsymbol{\phi}_{j_2}$ instead of the eigenfunction ϕ_{j_2} in their definition. They are, however, computable once we computed approximations to the discretized eigenvectors $\boldsymbol{\phi}_{j_2}^{(N_2)}$. Moreover, we expect these coefficients to approximate the exact Fourier coefficients from (3.14) up to the discretization error of the eigenvectors $\boldsymbol{\phi}_{j_2}^{(N_2)}$.

By the definition of Φ_{N_2} in (5.4) we have that

$$\underline{u}_{\boldsymbol{N}}(j_1, j_2) = \mathcal{F}_{N_1}^{-1}\left[\left(\hat{\underline{u}}_{\boldsymbol{N}}(\cdot, k_2)\right)_{k_2=1}^{N_2} \cdot \boldsymbol{\phi}_{j_2}^{-1}\right](j_1). = \left(\mathcal{F}_{N_1}^{-1}\left(\hat{\underline{u}}_{\boldsymbol{N}}(\cdot, k_2)\right)(j_1)\right)_{k_2=1}^{N_2} \cdot \boldsymbol{\phi}_{j_2}^{-1}.$$

More precisely,

$$\begin{aligned}
\left(\mathcal{F}_{N_1}^{-1}\left(\hat{\underline{u}}_{\boldsymbol{N}}(\cdot, k_2)\right)(j_1)\right)_{k_2=1}^{N_2} \cdot \boldsymbol{\phi}_{j_2}^{-1} &= \sum_{k_2=1}^{N_2} \mathcal{F}_{N_1}^{-1}\left(\hat{\underline{u}}_{\boldsymbol{N}}(\cdot, k_2)\right)(j_1) \boldsymbol{\phi}_{j_2}^{-1}(k_2) \\
&= \sum_{k_2=1}^{N_2} \boldsymbol{\phi}_{j_2}^{-1}(k_2) \sum_{\ell_2=1}^{N_2} u\left(\rho \frac{j_1}{N_1}, H \frac{\ell_2}{N_2}\right) \boldsymbol{\phi}_{k_2}(\ell_2) \\
&= \sum_{\ell_2=1}^{N_2} u\left(\rho \frac{j_1}{N_1}, H \frac{\ell_2}{N_2}\right) \sum_{k_2=1}^{N_2} \boldsymbol{\phi}_{j_2}^{-1}(k_2) \boldsymbol{\phi}_{k_2}(\ell_2) \\
&= u\left(\rho \frac{j_1}{N_1}, H \frac{j_2}{N_2}\right) = \underline{u}_{\boldsymbol{N}}(j_1, j_2) \qquad \boldsymbol{j} \in \mathbb{Z}_{\boldsymbol{N}}^2.
\end{aligned}$$

Roughly speaking, we found the transformation $\underline{u}_{\boldsymbol{N}} \mapsto \hat{\underline{u}}_{\boldsymbol{N}}$ by doing a 1D FFT and a multiplication with Φ_{N_2} of the nodal values $u\left(\rho \frac{j_1}{N_1}, H \frac{j_2}{N_2}\right)$ for all $\boldsymbol{j} \in \mathbb{Z}_{\boldsymbol{N}}^2$. Conversely, we have $\hat{\underline{u}}_{\boldsymbol{N}} \mapsto \underline{u}_{\boldsymbol{N}}$ by doing a (1D-IFFT) and a multiplication with $\Phi_{N_2}^{-1}$. We point out, that due to the representation the eigenfunctions ϕ_j it is not possible to apply, as for constant speed of sound, a sine transform to compute coefficients with respect to these functions as done in [LN12]. Note that later on, if we compute the eigenvalues and eigenfunction in the vertical direction by a finite element method or a spectral method on a fine mesh, the number of the grid points will be larger than N_2. To get the values on grid points $x_{\boldsymbol{j}}^{(\boldsymbol{N})}$ we use the values on the associated grid points on the fine mesh or we have to do an interpolation.

The presented theory is also applicable if $m = 3$. Next,we define the matrix of nodal values $\underline{u}_{\boldsymbol{N}} \in \mathbb{C}^{2N_1 \times 2N_2 \times N_3}$ by

$$\underline{u}_{\boldsymbol{N}} = u\left(\rho \frac{j_1}{N_1}, \rho \frac{j_2}{N_2}, H \frac{j_3}{N_3}\right), \qquad \boldsymbol{j} \in \mathbb{Z}^3_{\boldsymbol{N}}.$$

Similarly like for $m = 2$, we see

$$\hat{\underline{u}}_{\boldsymbol{N}}(\boldsymbol{j}) = \sum_{k_1=-N_1}^{N_1-1} \sum_{k_2=-N_2}^{N_2-1} \sum_{k_3=1}^{N_3} \underline{u}_{\boldsymbol{N}}\left(\tilde{\boldsymbol{k}}, k_3\right) \boldsymbol{\phi}_{j_3}(k_3) \exp\left(-\pi i \left[k_1 \frac{j_1}{N_1} + k_2 \frac{j_2}{N_2}\right]\right) \qquad (\boldsymbol{j} \in \mathbb{Z}^3_{\boldsymbol{N}}). \tag{5.11}$$

For completeness, for $j_3 = 1, ..., N_3$ hence

$$\begin{aligned}&\hat{\underline{u}}_{\tilde{\boldsymbol{N}}}\left(k_1 + \lfloor -N_1 \rfloor + 1, k_2 + \lfloor -N_2 \rfloor + 1, j_3\right)\\ &\qquad = \frac{4\rho}{4N_1N_2} \mathcal{F}_{N_1,N_2}\left(\left(\left(\sum_{k_3=1}^{N_3} \underline{u}_{\boldsymbol{N}}\left(j_1, j_2, k_3\right) \phi_{j_3}^{(k_3)}\right)_{j_1,j_2}\right)(k_1,k_2)(\underline{u}_{\boldsymbol{N}}(\cdot, j_3)),\end{aligned}$$

where $0 \le k_1 \le 2N_1 - 1$and $0 \le k_2 \le 2N_2 - 1$. Here, $\mathcal{F}_{N_1,N_2}$ denotes the two-dimensional discrete Fourier transform (2D-FFT) defined for $\underline{v} \in \mathbb{C}^{2N_1 \times 2N_2}$ by

$$\mathcal{F}_{N_1,N_2}(\underline{v}(j_1,j_2)) = \frac{1}{\sqrt{N_1N_2}} \sum_{j_1=0}^{2N_1-1} \sum_{j_2=0}^{2N_2-1} \underline{v}(j_1,j_2) \exp\left(-2\pi i \left[j_1 \frac{k_1}{N_1} + j_2 \frac{k_2}{N_2}\right]\right),$$

where $0 \le k_1 \le 2N_1 - 1$ and $0 \le k_2 \le 2N_2 - 1$. Next, for given Fourier coefficients we can now compute the nodal values by

$$\begin{aligned}\underline{u}_{\boldsymbol{N}}\left(j_1, j_2, j_3\right) = \frac{\sqrt{N_1N_2}}{2\rho} &\sum_{k_1=0}^{2N_1-1} \sum_{k_2=0}^{2N_2-1} \left[\sum_{k_3=1}^{N_3} \hat{\underline{u}}_{\boldsymbol{N}}(\tilde{\boldsymbol{j}}, k_3) \boldsymbol{\phi}_{j_3}^{-1}(k_3)\right]\\ &\exp\left(-2\pi i \left[\left(k_1 + \lfloor -N_1 \rfloor + 1\right) \frac{j_1}{N_1} + \left(k_2 + \lfloor -N_2 \rfloor + 1\right) \frac{j_2}{N_2}\right]\right),\end{aligned} \tag{5.12}$$

where $0 \le j_1 \le 2N_1 - 1, 0 \le j_2 \le 2N_2 - 1$ and $j_3 = 1, ..., N_3$ or shortly

$$\underline{u}_{\boldsymbol{N}}(\cdot, j_3) = \frac{\sqrt{N_1N_2}}{2\rho} \mathcal{F}^{-1}_{N_1,N_2}\left(\left(\hat{\underline{u}}_{\boldsymbol{N}}(k_1 + \lfloor -N_1 \rfloor + 1, k_2 + \lfloor -N_2 \rfloor + 1, k_3\right)_{k_3=1}^{N_3} \cdot \boldsymbol{\phi}_{j_3}^{-1}\right),$$

where $0 \le k_1 < 2N_1$, $0 \le k_2 < 2N_2$ and $j_3 = 1, ..., N_3$. Here, $\mathcal{F}^{-1}_{N_1,N_2}$ is the two-dimensional discrete Inverse Fourier transform (2D-IFFT) defined for $\hat{v} \in C^{2N_1 \times 2N_2}$ by

$$\begin{aligned}\mathcal{F}^{-1}_{N_1,N_2}(\hat{v})(\tilde{j}) = \frac{1}{\sqrt{N_1N_2}} &\sum_{k_1=0}^{2N_1-1} \sum_{k_2=0}^{2N_2-1}\\ &\hat{v}\big(\left(k_1 + \lfloor -N_1 \rfloor + 1\right), \left(k_2 + \lfloor -N_2 \rfloor + 1\right)\big) \exp\left(2\pi i \left[k_1 \frac{j_1}{N_1} + k_2 \frac{j_2}{N_2}\right]\right),\end{aligned} \tag{5.13}$$

where $j_1 = 0, 1, ..., 2N_1 - 1$ and $j_2 = 0, 1, ..., 2N_2 - 1$.

We point out that again the Fourier coefficients $\hat{\underline{u}}_{\boldsymbol{N}}(\tilde{\boldsymbol{j}}, j_3) \in \mathbb{C}^3$, where $\boldsymbol{j} \in \mathbb{Z}^3_{\boldsymbol{N}}$ is different from the introduced Fourier coefficients in Chapter 3.2, see (3.14), since we used the discrete vector $\boldsymbol{\phi}_{j_3}$ instead of the eigenvector ϕ_{j_3} in their definition. By the definition of Φ_{N_3} in (5.4) we see that

$$\underline{u}_{\boldsymbol{N}}(\tilde{\boldsymbol{j}}, j_3) = \mathcal{F}^{-1}_{N_1,N_2}\left[\left(\hat{\underline{u}}_{\boldsymbol{N}}(\cdot, k_3)\right)_{k_3=1}^{N_3} \cdot \boldsymbol{\phi}_{j_3}^{-1}\right](\tilde{\boldsymbol{j}}).$$

More precisely,

$$\begin{aligned}
\left(\mathcal{F}^{-1}_{N_1,N_2}\left(\hat{\hat{u}}_{\boldsymbol{N}}(\cdot,k_3)\right)(\tilde{\boldsymbol{j}})\right)_{k_3=1}^{N_3}\cdot\boldsymbol{\phi}^{-1}_{j_3} &= \sum_{k_3=1}^{N_3}\mathcal{F}^{-1}_{N_1,N_2}\left(\hat{\hat{u}}_{\boldsymbol{N}}(\cdot,k_3)\right)(\tilde{\boldsymbol{j}})\boldsymbol{\phi}^{-1}_{j_3}(k_3)\\
&= \sum_{k_3=1}^{N_3}\boldsymbol{\phi}^{-1}_{j_3}(k_3)\sum_{l_3=1}^{N_3}u\left(\rho\frac{j_1}{N_1},\rho\frac{j_2}{N_2},H\frac{l_3}{N_3}\right)\boldsymbol{\phi}_{k_3}(l_3)\\
&= \sum_{l_3=1}^{N_3}u\left(\rho\frac{j_1}{N_1},\rho\frac{j_2}{N_2},H\frac{l_3}{N_3}\right)\sum_{k_3=1}^{N_3}\boldsymbol{\phi}^{-1}_{j_3}(k_3)\boldsymbol{\phi}_{k_3}(l_3)\\
&= u\left(\rho\frac{j_1}{N_1},\rho\frac{j_2}{N_2},H\frac{j_3}{N_3}\right) = \underline{u}_{\boldsymbol{N}}(\boldsymbol{j}) \qquad \boldsymbol{j}\in\mathbb{Z}^3_{\boldsymbol{N}}.
\end{aligned}$$

Roughly speaking, we found analogously as in dimension two the transformation $\underline{u}_{\boldsymbol{N}} \mapsto \hat{\hat{u}}_{\boldsymbol{N}}$ by doing a 2D-FFT and a multiplication with Φ_{N_3} of the nodal values $u(\rho j_1/N_1, \rho j_2/N_2, H j_3/N_3)$ for all $\boldsymbol{j}\in\mathbb{Z}^3_{\boldsymbol{N}}$. Conversely, we have $\hat{\hat{u}}_{\boldsymbol{N}} \mapsto \underline{u}_{\boldsymbol{N}}$ by doing a 2D-IFFT and a multiplication with $\Phi^{-1}_{N_3}$.

5.2 Convergence Estimates for the Interpolation Operator

In this Section, we give error estimates for the interpolation projection, defined in (5.6). For $\boldsymbol{N}\in\mathbb{N}^m$ the orthogonal projection onto $T_{\boldsymbol{N}}$ is denoted by

$$P_{\boldsymbol{N}}: L^2(\Lambda_\rho)\to L^2(\Lambda_\rho) \qquad P_{\boldsymbol{N}}u = \sum_{\boldsymbol{j}\in\mathbb{Z}^m_{\boldsymbol{N}}}\hat{u}(\boldsymbol{j})\varphi_{\boldsymbol{j}}, \qquad \text{where } u = \sum_{\boldsymbol{j}\in\mathbb{Z}^m_+}\hat{u}(\boldsymbol{j})\varphi_{\boldsymbol{j}},$$

defined in (3.14). For error estimates of the orthogonal projection, it is convenient to use the separation in horizontal and vertical components of this orthogonal projection. Let $P^{\sim}_{\boldsymbol{N}}$ denote the orthogonal projection onto trigonometric polynomials of functions in $L^2(\tilde{\Lambda}_\rho)$,

$$P^{\sim}_{\boldsymbol{N}}: L^2(\tilde{\Lambda}_\rho)\to L^2(\tilde{\Lambda}_\rho) \qquad v = \sum_{\tilde{\boldsymbol{j}}\in\mathbb{Z}^{m-1}}\hat{v}_{\tilde{\boldsymbol{j}}}v_{\tilde{\boldsymbol{j}}} \mapsto \sum_{\tilde{\boldsymbol{j}}\in\mathbb{Z}^{m-1}_{\tilde{\boldsymbol{N}}}}\hat{v}_{\tilde{\boldsymbol{j}}}v_{\tilde{\boldsymbol{j}}},$$

and define $P^{\perp}_{\boldsymbol{N}}$ as orthogonal projection onto the first N_m eigenfunction $\phi_1,\ldots,\phi_{N_m}$,

$$P^{\perp}_{\boldsymbol{N}}: L^2(0,H)\to L^2(0,H) \qquad w = \sum_{j_m\in\mathbb{N}}\hat{w}_{j_m}w_{j_m}\mapsto\sum_{j_m=1}^{N_m}\hat{w}_{j_m}w_{j_m}, \qquad N_m\in\mathbb{N}.$$

Recall that for $m=2$ the vectors $\tilde{\boldsymbol{j}}$ and $\tilde{\boldsymbol{N}}$ are scalars.

Lemma 5.2.1. *For $\boldsymbol{N}\in\mathbb{Z}^m_+$ and $r>s\in\mathbb{R}$ it holds that*

$$\begin{aligned}
\|P_{\boldsymbol{N}}u-u\|_{H^s(\Lambda_\rho)} &\le C(r,s)\min(\boldsymbol{N})^{-(r-s)}\|u\|_{H^r(\Lambda_\rho)} && \text{for all } u\in H^r(\Lambda_\rho),\\
\|P^{\sim}_{\tilde{\boldsymbol{N}}}v-v\|_{H^s(\tilde{\Lambda}_\rho)} &\le C(r,s)\min(\tilde{\boldsymbol{N}})^{-(r-s)}\|v\|_{H^r(\tilde{\Lambda}_\rho)} && \text{for all } v\in H^r(\tilde{\Lambda}_\rho),\\
\|P^{\perp}_{N_m}w-w\|_{H^s(0,H)} &\le C(r,s)N_m^{-(r-s)}\|w\|_{H^r(0,H)} && \text{for all } w\in H^r(0,H).
\end{aligned}$$

Proof. We only treat the proof of the first inequality since the other estimates follow in an analogous way. By the definition of the norm defined on $H^s(\Lambda_\rho)$ we see that

$$
\begin{aligned}
\|P_{\boldsymbol N}u-u\|^2_{H^s(\Lambda_\rho)} &= \sum_{\boldsymbol j\in\mathbb Z^m_+\setminus\mathbb Z^m_{\boldsymbol N}} (1+|\tilde{\boldsymbol j}|^2+|\lambda_{j_m}|^2)^s|\hat u(\boldsymbol j)|^2\\
&= \sum_{\boldsymbol j\in\mathbb Z^m_+\setminus\mathbb Z^m_{\boldsymbol N}} (1+|\tilde{\boldsymbol j}|^2+|\lambda_{j_m}|^2)^{s-r}(1+|\tilde{\boldsymbol j}|^2+|\lambda_{j_m}|^2)^r|\hat u(\boldsymbol j)|^2\\
&\le \sup_{\boldsymbol j\in\mathbb Z^m_+\setminus\mathbb Z^m_{\boldsymbol N}}\left[(1+|\tilde{\boldsymbol j}|^2+|\lambda_{j_m}|^2)^{-(r-s)}\right]\sum_{\boldsymbol j\in\mathbb Z^m_+\setminus\mathbb Z^m_{\boldsymbol N}} (1+|\tilde{\boldsymbol j}|^2+|\lambda_{j_m}|^2)^r|\hat u(\boldsymbol j)|^2\\
&= \sup_{\boldsymbol j\in\mathbb Z^m_+\setminus\mathbb Z^m_{\boldsymbol N}}\left[(1+|\tilde{\boldsymbol j}|^2+|\lambda_{j_m}|^2)^{-(r-s)}\right]\|u\|^2_{H^r(\Lambda_\rho)}.
\end{aligned}
$$

The estimate $c_0 j_m^2\le 1+|\lambda_{j_m}|^2$, given by Lemma 2.2.4, yields the convergence rate

$$
\begin{aligned}
\sup_{\boldsymbol j\in\mathbb Z^m_+\setminus\mathbb Z^m_{\boldsymbol N}}\left[(1+|\tilde{\boldsymbol j}|^2+|\lambda_{j_m}|^2)^{-(r-s)}\right] &\le \min(1,c_0^{-(r-s)})\sup_{\boldsymbol j\in\mathbb Z^m_+\setminus\mathbb Z^m_{\boldsymbol N}}\left[|\tilde{\boldsymbol j}|^2+j_m^2\right]^{-(r-s)}\\
&= \min(1,c_0^{-(r-s)})\min(\boldsymbol N)^{-2(r-s)}\\
&= C(r,s)\min(\boldsymbol N)^{-2(r-s)}.
\end{aligned}
$$

The estimates for $P^{\sim}_{\tilde{\boldsymbol N}}$ and $P^{\perp}_{N_m}$ follow analogously. □

The next result shows that functions in $H^s(\Lambda_\rho)$ belong to $C^0(\Lambda_\rho)$ if $s>m/2$, while functions in $H^s(\tilde\Lambda_\rho)$ and $H^s(0,H)$, see (3.18) and (3.19), merely require $s>(m-1)/2$ and $s>1/2$ to be continuous, respectively.

Lemma 5.2.2. *(a) For $s>m/2$, every function $u\in H^s(\Lambda_\rho)$ is continuous and $\|u\|_\infty\le C\|u\|_{H^s(\Lambda_\rho)}$ holds. Moreover,*

$$H^s(\Lambda_\rho)\subset C^0_W(\overline{\Lambda_\rho}) := \{w_0\in C^0(\overline{\Lambda_\rho}),\ w_0|_{\{x_m=0\}}=0\}.$$

(b) For $s>(m-1)/2$, every function in $v\in H^s(\tilde\Lambda_\rho)$ is continuous and $\|v\|_\infty\le C\|v\|_{H^s(\tilde\Lambda_\rho)}$ holds.

(c) For $s>1/2$ every function $w\in H^s(0,H)$ is continuous and $\|w\|_\infty\le C\|w\|_{H^s(0,H)}$ holds. Furthermore,

$$H^s([0,H])\subset C^0_W([0,H]) := \{w_0\in C^0([0,H]),\ w_0(0)=0\}.$$

Proof. (a) Obviously, each finite sum $u_{\boldsymbol N}=\sum_{\boldsymbol j\in\mathbb Z^m_{\boldsymbol N}}\hat u(\boldsymbol j)\varphi_{\boldsymbol j}$ is continuous. Due to Corollary 2.2.5, the eigenvectors $\{\phi_{j_m}\}_{j\in\mathbb N}$ are uniformly bounded, and since $\|v_{\tilde{\boldsymbol j}}\|_\infty = 1/(2\rho)^{m/2}$ there holds that $\|\varphi_{\boldsymbol j}\|_\infty=\|v_{\tilde{\boldsymbol j}}\|_\infty\|\phi_{j_m}\|_\infty\le C$ for all $\boldsymbol j\in\mathbb Z^m_+$ and a constant $C>0$. Next, the maximum norm of $u_{\boldsymbol N}$ can be estimated by

$$
\begin{aligned}
\|u_{\boldsymbol N}\|_\infty &\le \sum_{\boldsymbol j\in\mathbb Z^m_{\boldsymbol N}}|\hat u(\boldsymbol j)|\,\|\varphi_{\boldsymbol j}\|_\infty\\
&\le C\sum_{\boldsymbol j\in\mathbb Z^m_{\boldsymbol N}}|\hat u(\boldsymbol j)|(1+|\tilde{\boldsymbol j}|^2+|\lambda_{j_m}|^2)^{s/2}(1+|\tilde{\boldsymbol j}|^2+|\lambda_{j_m}|^2)^{-s/2}\\
&\le C\Bigg(\sum_{\boldsymbol j\in\mathbb Z^m_{\boldsymbol N}}(1+|\tilde{\boldsymbol j}|^2+|\lambda_{j_m}|^2)^{-s}\Bigg)^{1/2}\Bigg(\sum_{\boldsymbol j\in\mathbb Z^m_{\boldsymbol N}}(1+|\tilde{\boldsymbol j}|^2+|\lambda_{j_m}|^2)^{s}|\hat u(\boldsymbol j)|^2\Bigg)^{1/2}\\
&\le C\Bigg(\sum_{\boldsymbol j\in\mathbb Z^m_{\boldsymbol N}}(1+|\tilde{\boldsymbol j}|^2+|\lambda_{j_m}|^2)^{-s}\Bigg)^{1/2}\|u\|_{H^s(\Lambda_\rho)}.
\end{aligned}
$$

Due to Lemma 2.2.4 it follows once more that $c_0 j_m^2 \le 1 + |\lambda_{j_m}|^2$ holds for the positive constant c_0. Consequently, we see for $s \ge 0$ that

$$(1 + |\tilde{\boldsymbol{j}}|^2 + |\lambda_{j_m}|^2)^{-s} \le \min(1, c_0)^{-s} |\boldsymbol{j}|^{-2s} \text{ for all } \boldsymbol{j} \in \mathbb{Z}_+^m \setminus \{\boldsymbol{0}\}.$$

For $s > m/2$, the series

$$\sum_{\boldsymbol{j} \in \mathbb{Z}_+^m} (1 + |\tilde{\boldsymbol{j}}|^2 + |\lambda_{j_m}|^2)^{-s} \le \min(1, c_0)^{-s} \big[1 + \sum_{0 \neq \boldsymbol{j} \in \mathbb{Z}_{\boldsymbol{N}}^m} |\boldsymbol{j}|^{-2s} \big] =: C < \infty$$

converges. For two vectors $\boldsymbol{N}$, $\boldsymbol{M} \in \mathbb{Z}_+^m$, the difference $u_{\boldsymbol{N}} - u_{\boldsymbol{M}}$ hence satisfies

$$\|u_{\boldsymbol{N}} - u_{\boldsymbol{M}}\|_\infty \le C \left(\sum_{\boldsymbol{j} \in \mathbb{Z}_+^m} (1 + |\tilde{\boldsymbol{j}}|^2 + |\lambda_{j_m}|^2)^{-s} \right)^{1/2} \|u_{\boldsymbol{N}} - u_{\boldsymbol{M}}\|_{H^s(\Lambda_\rho)}.$$

The latter expression tends to zero as $\min(\boldsymbol{N})$, $\min(\boldsymbol{M})$ tend to infinity since $u_{\boldsymbol{N}}$ and $u_{\boldsymbol{M}}$ both tend to u in $H^s(\Lambda_\rho)$.

The same technique shows also parts (b) and (c) of the claim. The zero boundary condition of a function u in $H^s(0,H)$ for $s > 1/2$ is due to the fact that all $\phi_{j_m} \in C^0([0,H])$ vanish at $x_m = 0$. Indeed, the truncated Fourier series $u_{\boldsymbol{N}} = \sum_{\boldsymbol{j} \in \mathbb{Z}_{\boldsymbol{N}}^m} \hat{u}(\boldsymbol{j}) \varphi_{\boldsymbol{j}}$ converges uniformly in $\overline{\Lambda_\rho}$ as $\min(\boldsymbol{N}) \to \infty$, such that $u(\tilde{x}, 0) = \sum_{\boldsymbol{j} \in \mathbb{Z}_+^m} \hat{u}(\boldsymbol{j}) v_{\tilde{\boldsymbol{j}}}(\tilde{x}) \phi_{j_m}(0) = 0$ for $\tilde{x} \in \tilde{\Lambda}_\rho$. □

Next, we investigate the convergence of $Q_{\boldsymbol{N}} u - u$ to zero for functions u in the Sobolev space $H^s(0,H)$. The following theorem differs from well-known error estimates, e.g. [SV02, Theorem 8.3.1] because again the basis functions in the vertical fail to be trigonometric monomials. Furthermore, to be able to apply $Q_{\boldsymbol{N}}$ to u we need $s > m/2$ to ensure continuity.

Theorem 5.2.3. *Suppose that Assumption 5.1.1 holds. Then for all $u \in H^s(0,H)$, $s > m/2$, there is a $C > 0$ such that*

$$\|Q_{\boldsymbol{N}} u - u\|_{L^2(\Lambda_\rho)} \le C \min(\boldsymbol{N})^{-(s-r)} \|u\|_{H^s(\Lambda_\rho)} \qquad \text{with } r > 1/2.$$

We point out that we use in the following $s > r > 1/2$ for the estimate in Theorem 5.2.3.

Proof. (1) Recall first the decomposition of the interpolation operator $Q_{\boldsymbol{N}}$ in horizontal and vertical variables. We have by definition (5.8) and (5.9) the interpolation operators $Q_{\tilde{\boldsymbol{N}}}^{(\sim)} : v \mapsto v_{\tilde{\boldsymbol{N}}}$ on $C^0([-\rho,\rho]^{m-1})$, acting on the horizontal variables $\tilde{x}$, and $Q_{\boldsymbol{N}_m}^{(\perp)} : w \mapsto w_{\boldsymbol{N}_m}$ on $C^0([0,H])$ acting on the vertical variable x_m. Then, we decompose $Q_{\boldsymbol{N}}$ into the product $Q_{\tilde{\boldsymbol{N}}}^{(\sim)} Q_{\boldsymbol{N}_m}^{(\perp)}$ and add a zero term,

$$\begin{aligned} Q_{\boldsymbol{N}} u - u &= Q_{\tilde{\boldsymbol{N}}}^{(\sim)} \left[Q_{\boldsymbol{N}_m}^{(\perp)} u(\tilde{x}, \cdot) \right] - u \\ &= \left[Q_{\tilde{\boldsymbol{N}}}^{(\sim)} \left[Q_{\boldsymbol{N}_m}^{(\perp)} u(\tilde{x}, \cdot) \right] - Q_{\boldsymbol{N}_m}^{(\perp)} u(\tilde{x}, \cdot) \right] + \left[Q_{\boldsymbol{N}_m}^{(\perp)} u(\tilde{x}, \cdot) - u \right]. \end{aligned} \tag{5.14}$$

Next, we estimate the last two terms in brackets separately.

(2) We first investigate the difference $Q_{\boldsymbol{N}_m}^{(\perp)} u(\tilde{x}, \cdot) - u$ from the second bracket in (5.14). Due to Assumption 5.1.1 we have continuous sound speed $c \in C^0([0,H])$ and the given choice of the interpolation points [Slo83, Lemma b) in Section 4, pg. 115] gives for all $w \in C_W^0([0,H]) = \{w_0 \in C^0([0,H]),\ w_0(0) = 0\}$ that

$$\|Q_{\boldsymbol{N}_m}^{(\perp)} w - w\|_{L^2(0,H)} \to 0 \qquad \text{as } N_m \to \infty.$$

Note that the boundary conditions at 0 and H are exchanged in that lemma, which plays no role since we adapted the interpolation points $H(j_m/N_m)$ accordingly. Due to [Slo83, Lemma, in Section 2, pg. 101] this convergence implies the existence of a constant $C>0$ such that

$$\|Q^{(\perp)}_{\boldsymbol{N}_m} w - w\|_{L^2(0,H)} \leq C \min_{w' \in \operatorname{span}\{\phi_{j_m}, 1\leq j_m \leq N_m\}} \|w'-w\|_\infty \qquad \text{for all } w \in C^0_W([0,H]).$$

For $s>r>1/2$ it holds that $w \in H^s(0,H) \subset C^0_W([0,H])$ is continuous due to Lemma 5.2.2, such that

$$\|P^\perp_{N_m} w - w\|_\infty \leq C \|P^\perp_{N_m} w - w\|_{H^r(0,H)} \leq C N_m^{-(s-r)} \|w\|_{H^s(0,H)}.$$

Using this assumption we choose $w' = P^\perp_{N_m} w$ and we compute for $s>r>1/2$ that

$$\|Q^{(\perp)}_{N_m} w - w\|_{L^2(0,H)} \leq C \|P^\perp_{N_m} w - w\|_{H^r(0,H)} \leq C {N_m}^{-(s-r)} \|w\|_{H^s(0,H)}. \tag{5.15}$$

Moreover, if we use the last estimate, then we see for all $u \in H^s(\Lambda_\rho)$ and $s>r>1/2$ that

$$\begin{aligned}\left\|Q^{(\perp)}_{\boldsymbol{N}_m} u - u\right\|^2_{L^2(\Lambda_\rho)} &= \int_{\tilde\Lambda_\rho} \int_0^H \left|Q^{(\perp)}_{\boldsymbol{N}_m} u(\tilde x,\cdot) - u(\tilde x,\cdot)\right|^2 d\tilde x \\ &\leq C {N_m}^{-2(s-r)} \int_{\tilde\Lambda_\rho} \|u(\tilde x,\cdot)\|^2_{H^s(0,H)}\, d\tilde x.\end{aligned}$$

Due to the definition of the (squared) norm $H^s([0,H])$ and the definition of Fourier coefficients acting on vertical variables, we obtain

$$\int_{\tilde\Lambda_\rho} \|u(\tilde x,\cdot)\|^2_{H^s(0,H)}\, d\tilde x = \int_{\tilde\Lambda_\rho} \sum_{j_m\in\mathbb{N}} (1+|\lambda_{j_m}|^2)^s \left|(u(\tilde x,\cdot),\phi_{j_m})_{L^2(0,H)}\right|^2 d\tilde x.$$

Applying Parseval's equality to the horizontal variable gives

$$\begin{aligned}&\int_{\tilde\Lambda_\rho} \sum_{j_m\in\mathbb{N}} (1+|\lambda_{j_m}|^2)^s \left|(u(\tilde x,\cdot),\phi_{j_m})_{L^2(0,H)}\right|^2 d\tilde x \\ &= \sum_{j_m\in\mathbb{N}} (1+|\lambda_{j_m}|^2)^s \sum_{\tilde{\boldsymbol{j}}\in\mathbb{Z}^{m-1}} \left|\left((u(\tilde x,\cdot),\phi_{j_m})_{L^2(0,H)}, v_{\tilde{\boldsymbol{j}}}(\tilde x)\right)_{L^2(\tilde\Lambda_\rho)}\right|^2 \\ &= \sum_{\boldsymbol{j}\in\mathbb{Z}^m_+} (1+|\lambda_{j_m}|^2)^s\, |\hat u(\boldsymbol{j})|^2 \leq \|u\|^2_{H^s(\Lambda_\rho)}.\end{aligned}$$

Then

$$\left\|Q^{(\perp)}_{\boldsymbol{N}_m} u - u\right\|^2_{L^2(\Lambda_\rho)} \leq C {N_m}^{-2(s-r)} \|u\|^2_{H^s(\Lambda_\rho)}. \tag{5.16}$$

(3) We next use well-known trigonometric interpolation operators to treat the difference inside the first bracket in (5.14). For simplicity, we set $v(\tilde x, x_m) = [Q^{(\perp)}_{\boldsymbol{N}_m} u(\tilde x,\cdot)](x_m)$ to obtain

$$\begin{aligned}\left\|Q^{(\sim)}_{\tilde{\boldsymbol{N}}}\left[Q^{(\perp)}_{N_m} u(\tilde x,\cdot)\right] - Q^{(\perp)}_{N_m} u(\tilde x,\cdot)\right\|^2_{L^2(\Lambda_\rho)} &= \left\|Q^{(\sim)}_{\tilde{\boldsymbol{N}}} v - v\right\|^2_{L^2(\Lambda_\rho)} \\ &= \int_0^H \int_{\tilde\Lambda_\rho} \left|\left[Q^{(\sim)}_{\tilde{\boldsymbol{N}}} v(\cdot,x_m)\right](\tilde x) - v(\tilde x,x_m)\right|^2 d\tilde x\, dx_m. \end{aligned} \tag{5.17}$$

Due to [SV02, Th. 8.3.1] we know that for $s>(m-1)$ and $0\leq t\leq s$ there exists a constant $C=C(s,t)$ such that for all $v\in H^s(\tilde\Lambda_\rho)$,

$$\|Q^{(\sim)}_{\tilde{\boldsymbol{N}}} v - v\|^2_{H^t(\tilde\Lambda_\rho)} \leq C \min(\tilde{\boldsymbol{N}})^{-2s} \|v\|^2_{H^s(\tilde\Lambda_\rho)} = C \min(\tilde{\boldsymbol{N}})^{-2s} \sum_{\tilde{\boldsymbol{j}}\in\mathbb{Z}^{m-1}} (1+|\tilde{\boldsymbol{j}}|^2)^s \left|\hat v(\tilde{\boldsymbol{j}})\right|^2.$$

Note that every function $v \in L^2(\tilde{\Lambda}_\rho)$ can be represented as its Fourier series

$$v = \sum_{\tilde{\boldsymbol{j}} \in \mathbb{Z}^{m-1}} \hat{v}(\tilde{\boldsymbol{j}})\, v_{\tilde{\boldsymbol{j}}} \qquad \text{with } \hat{v}(\tilde{\boldsymbol{j}}) = (v, v_{\tilde{\boldsymbol{j}}})_{L^2(\tilde{\Lambda}_\rho)} = \int_{\tilde{\Lambda}_\rho} v\overline{v_{\tilde{\boldsymbol{j}}}} d\tilde{x}. \tag{5.18}$$

For fixed $x_m \in (0, H)$ we apply (5.18) with $t = 0$ to $v(\tilde{x}, x_m) = [Q^{(\perp)}_{\boldsymbol{N}_m} u(\tilde{x}, \cdot)](x_m)$ to see that

$$\|Q^{(\sim)}_{\tilde{\boldsymbol{N}}} v(\cdot, x_m) - v(\cdot, x_m)\|^2_{L^2(\tilde{\Lambda}_\rho)} \leq C \min(\tilde{\boldsymbol{N}})^{-2s} \sum_{\tilde{\boldsymbol{j}} \in \mathbb{Z}^{m-1}} (1 + |\tilde{\boldsymbol{j}}|^2)^s \left| \Big(v(\cdot, x_m), v_{\tilde{\boldsymbol{j}}}\Big)_{L^2(\tilde{\Lambda})} \right|^2, \tag{5.19}$$

and integration over x_m it holds

$$\left\| Q^{(\sim)}_{\tilde{\boldsymbol{N}}} v - v \right\|^2_{L^2(\Lambda_\rho)} \leq C \min(\tilde{\boldsymbol{N}})^{-2s} \sum_{\tilde{\boldsymbol{j}} \in \mathbb{Z}^{m-1}} (1 + |\tilde{\boldsymbol{j}}|^2)^s \int_0^H \left| \Big(v(\cdot, x_m), v_{\tilde{\boldsymbol{j}}}\Big)_{L^2(\tilde{\Lambda})} \right|^2 dx_m.$$

We show now that doing an interpolation projection x_m and then a Fourier series representations in $\tilde{x}$ is equal to doing first a Fourier series representation and then an interpolation projection. By $v(\tilde{x}, x_m) = [Q^{(\perp)}_{\boldsymbol{N}_m} u(\tilde{x}, \cdot)](x_m)$ it follows that

$$\begin{aligned}
\Big(v(\cdot, x_m), v_{\tilde{\boldsymbol{j}}}\Big)_{L^2(\tilde{\Lambda})} &= \Big([Q^{(\perp)}_{N_m} u(\tilde{x}, \cdot)](x_m),\, v_{\tilde{\boldsymbol{j}}}\Big)_{L^2(\tilde{\Lambda}_\rho)} \\
&= \int_{\tilde{\Lambda}_\rho} \sum_{\ell_m=1}^{N_m} u\bigg(\tilde{x}, H\frac{\ell_m}{N_m}\bigg) \phi^*_{N,\ell}(x_m) \overline{v_{\tilde{\boldsymbol{j}}}(\tilde{x})}\, d\tilde{x} \\
&= \sum_{\ell=1}^{N_m} \int_{\tilde{\Lambda}_\rho} u\bigg(\tilde{x}, H\frac{\ell_m}{N_m}\bigg) \overline{v_{\tilde{\boldsymbol{j}}}(\tilde{x})}\, d\tilde{x}\ \phi^*_{N,\ell_m}(x_m) \\
&= Q^{(\perp)}_{\boldsymbol{N}_m} \bigg[x_m \mapsto \Big(u(\cdot, x_m),\, v_{\tilde{\boldsymbol{j}}}\Big)_{L^2(0,H)}\bigg].
\end{aligned} \tag{5.20}$$

Then, we obtain

$$\begin{aligned}
\int_0^H \left| \Big(v(\cdot, x_m), v_{\tilde{\boldsymbol{j}}}\Big)_{L^2(\tilde{\Lambda})} \right|^2 dx_m &= \left\| x_m \mapsto \Big(v(\cdot, x_m), v_{\tilde{\boldsymbol{j}}}\Big)_{L^2(\tilde{\Lambda})} \right\|^2_{L^2(0,H)} \\
&= \left\| x_m \mapsto Q^{(\perp)}_{\boldsymbol{N}_m} \Big(u(\cdot, x_m),\, v_{\tilde{\boldsymbol{j}}}\Big)_{L^2(\tilde{\Lambda}_\rho)} \right\|^2_{L^2(0,H)}.
\end{aligned}$$

We note that (5.15) implies in particular that $Q^{(\perp)}_{N_m}$ is bounded from $H^r(0, H)$ into $L^2(0, H)$ for arbitrary $r > 1/2$, that is, $\|Q^{(\perp)}_{N_m} \hat{u}(\tilde{n}, \cdot)\|_{L^2(0,H)} \leq C\|\hat{u}(\tilde{n}, \cdot)\|_{H^r(0,H)}$. Thus, we find that

$$\begin{aligned}
&\left\| Q^{(\sim)}_{\tilde{\boldsymbol{N}}} \Big[Q^{(\perp)}_{N_m} u(\tilde{x}, \cdot)\Big] - Q^{(\perp)}_{N_m} u(\tilde{x}, \cdot) \right\|^2_{L^2(\Lambda_\rho)} \\
&\leq C \min(\tilde{\boldsymbol{N}})^{-2s} \sum_{\tilde{\boldsymbol{j}} \in \mathbb{Z}^{m-1}} (1 + |\tilde{\boldsymbol{j}}|^2)^s \left\| x_m \mapsto \Big(u(\cdot, x_m),\, v_{\tilde{\boldsymbol{j}}}\Big)_{L^2(\tilde{\Lambda}_\rho)} \right\|^2_{H^r(0,H)} \\
&= C \min(\tilde{\boldsymbol{N}})^{-2s} \sum_{\tilde{\boldsymbol{j}} \in \mathbb{Z}^{m-1}} (1 + |\tilde{\boldsymbol{j}}|^2)^s \sum_{\boldsymbol{j}_m \in \mathbb{N}} (1 + |\lambda_{\boldsymbol{j}_m}|^2)^r \, |\hat{u}(\boldsymbol{j})|^2.
\end{aligned}$$

As $(1 + a)^t \leq C(t)(1 + a^t)$ for $a \geq 0$ and $t > 0$ we see that

$$(1 + |\lambda_{j_m}|^2)^r (1 + |\tilde{\boldsymbol{j}}|^2)^s \leq C(1 + |\lambda_{j_m}|^{2r} + |\tilde{\boldsymbol{j}}|^{2s} + |\lambda_{j_m}|^{2r} |\tilde{\boldsymbol{j}}|^{2s}).$$

Next, by Young's inequality one can show that

$$|\tilde{\boldsymbol{j}}|^{2s}|\lambda_{j_m}|^{2r} \leq \frac{1}{1+r/s}|\tilde{\boldsymbol{j}}|^{2(s+r)} + \frac{1}{1+s/r}|\lambda_{j_m}|^{2(s+r)} \qquad \boldsymbol{j} \in \mathbb{Z}_+^m,$$

Therefore, using the inequality $(1 + a^{2t} + b^{2t}) \leq C(t)(1 + a^t + b^t)^2$ for $s + r = t > 0$ and $a, b \geq 0$ it holds

$$\sum_{\tilde{\boldsymbol{j}} \in \mathbb{Z}^{m-1}} (1+|\tilde{\boldsymbol{j}}|^2)^s \sum_{\boldsymbol{j}_m \in \mathbb{N}} (1+|\lambda_{\boldsymbol{j}_m}|^2)^r \, |\hat{u}(\boldsymbol{j})|^2 \leq C \sum_{\boldsymbol{j} \in \mathbb{Z}_+^m} (1+|\tilde{\boldsymbol{j}}|^2 + |\lambda_{j_m}|^2)^{s+r} \, |\hat{u}(\boldsymbol{j})|^2 = C\|u\|^2_{H^{s+r}(\Lambda_\rho)}.$$

Consequently, we obtain in this part of the proof that

$$\left\| Q_{\tilde{\boldsymbol{N}}}^{(\sim)} \left[[Q_{\boldsymbol{N}_m}^{(\perp)} u(\tilde{x}, \cdot)] - Q_{\boldsymbol{N}_m}^{(\perp)} u(\tilde{x}, \cdot) \right] \right\|^2_{L^2(\Lambda_\rho)} \leq C \min(\tilde{\boldsymbol{N}})^{-2s} \|u\|_{H^{s+r}(\Lambda_\rho)} \qquad \text{for } r > 1/2. \quad (5.21)$$

(4) To this end, plugging together (5.14), (5.16) and (5.21) yields

$$\|Q_{\boldsymbol{N}} u - u\|_{L^2(\Lambda_\rho)} \leq C \left[\min(\tilde{\boldsymbol{N}})^{-(s_1 - r)} \|u\|_{H^{s_1}(\Lambda_\rho)} + \min(\tilde{\boldsymbol{N}})^{-s_2} \|u\|_{H^{s_2+r}(\Lambda_\rho)} \right]$$

for $s_1 > 1/2$, $s_2 > (m-1)/2$ and $r > 1/2$. Choosing $s = s_1 = s_2 + r$ shows the claimed estimate for the projection interpolation. □

Let us finally point out that the latter error estimate concerns the interpolation projection $Q_{\boldsymbol{N}}$ defined via the exact eigenfunctions ϕ_j. Discretizing those as in (5.1) leads to a further discretization error that will, however, not be analyzed or discussed in more detail in this thesis. Note, however, that we derived error estimates for the numerically approximated eigenfunctions in Section 2.3

5.3 The Collocation Method for the Periodized Integral Equation

The error estimate for the interpolation projection $Q_{\boldsymbol{N}}$ shown in the last section is the crucial tool to prove error estimates for a collocation discretization of the periodized Lippmann-Schwinger equation (4.67). This collocation discretization is based on trigonometric polynomials in the horizontal variables $\tilde{x}$ and the eigenfunctions ϕ_j in the vertical variable x_m spanning the finite-dimensional space $T_{\boldsymbol{N}}$ defined in (5.5). We recall that the Assumption 4.2.11 on the unique solvability of the source problem (4.34) and (3.3) is still assumed to hold. In this section we moreover exploit the error estimate for $Q_{\boldsymbol{N}}$ from Theorem 5.2.3, which requires that the sound speed $c \in C^0([0, H])$ to be continuous, see Assumption 5.1.1.

Now, we discretize the introduced periodic integral operator $\mathcal{V}_\rho$ by the interpolation operator $Q_{\boldsymbol{N}} : C^2(\Lambda_\rho) \to T_{\boldsymbol{N}}$. If a discretization parameter $\boldsymbol{N} \in \mathbb{N}^m$ is given, we can search, as a first option, for a collocation discretization of the periodized Lippmann-Schwinger equation (4.67) by requiring that $u_{\boldsymbol{N}} \in T_{\boldsymbol{N}}$ solves the finite-dimensional linear system

$$u_{\boldsymbol{N}} - \mathcal{V}_\rho Q_{\boldsymbol{N}} \left(\frac{\omega^2}{c^2(x_m)} q u_{\boldsymbol{N}} \right) = Q_{\boldsymbol{N}} \mathcal{V}_\rho f \qquad \text{in } T_{\boldsymbol{N}}. \quad (5.22)$$

We recall that $\mathcal{V}_\rho$ maps $T_{\boldsymbol{N}}$ into $T_{\boldsymbol{N}}$, such that (5.22) is well-defined. Note that due to the representation of $\phi^*_{N_m, j_m}$ in (5.4), it follows that $\mathcal{V}_\rho(Q_{\boldsymbol{N}} u) \neq Q_{\boldsymbol{N}} \mathcal{V}_\rho(u)$.

The well-known spectral discretization approach presented by [SV02, Theorem 10.5.4]) is to multiply the Lippmann-Schwinger-Integral equation (4.32) by the contrast. Thus, as a second option is to multiply the periodized Lippmann-Schwinger equation (4.67) by q/c^2 and to search for $w = qu/c^2$, where u is the solution to the periodized Lippmann-Schwinger equation (4.67). We point out that the fraction q/c^2 is well defined by the assumption of the boundedness from below of the sound speed c in (2.3). This implies the modified periodized Lippmann-Schwinger equation to be well-defined

$$w - \frac{\omega^2}{c^2} q\mathcal{V}_\rho w = \frac{q}{c^2}\mathcal{V}_\rho f \qquad \text{in } L^2(\Lambda_\rho). \tag{5.23}$$

We know that if $u \in L^2(\Lambda_\rho)$ solves (4.67), then $(q/c^2)u$ solves (5.23) and we deduce that $u = \omega^2\mathcal{V}_\rho(w) + f$. In consequence, if w is a non-trivial solution to (5.23) for $f = 0$, then u must vanish and the equality $u = \omega^2\mathcal{V}_\rho(w) + f$ shows that $\mathcal{V}_\rho(w) = 0$. Due to the fact that (5.23) yields that $w = 0$, we obtain by Riesz theory existence and uniqueness of solution to (5.23). Thus, the modified periodized Lippmann-Schwinger equation (5.23) possesses a unique solution for all right-hand sides $f \in H^{-2}(\Lambda_\rho)$ if $q/c^2 \in L^\infty(\Lambda_\rho)$. More precisely, solving for u is equivalent to solving for w, because the periodized Lippmann-Schwinger equation (4.67) shows that $u = \omega^2\mathcal{V}_\rho(w) + \mathcal{V}_\rho f$.

Next, applying for $w \in L^2(\Lambda_\rho)$ the interpolation projection $Q_{\boldsymbol{N}}$ to (5.23), we obtain the following discrete problem: find $w_{\boldsymbol{N}} \in T_{\boldsymbol{N}}$ solving

$$w_{\boldsymbol{N}} - \omega^2 Q_{\boldsymbol{N}}\left(\frac{q}{c^2(x_m)}\mathcal{V}_\rho(w)\right) = Q_{\boldsymbol{N}}\left(\frac{q}{c^2(x_m)}\mathcal{V}_\rho f\right) \qquad \text{in } T_{\boldsymbol{N}}. \tag{5.24}$$

Both schemes (5.22) and (5.24) represent a solution to the discrete problem in $T_{\boldsymbol{N}}$ by finitely many eigenfunctions $\varphi_{\boldsymbol{\ell}}$ of $\mathcal{V}_\rho$, such that the application of the periodic integral operator to $v_{\boldsymbol{N}} \in T_{\boldsymbol{N}}$ is easily computed once the representation of $v_{\boldsymbol{N}}$ in the eigenbasis $\{\varphi_{\boldsymbol{\ell}}\}_{\boldsymbol{\ell}\in\mathbb{Z}^m_{\boldsymbol{N}}}$ is known. In particular, we can apply fast operator evaluation to the fully discrete version of (5.22) or (5.24), when solving these linear systems iteratively, as we show later.

We first analyze convergence rates for the solution of the collocation method (5.22), using the idea of [SV02, Lemma 10.5.1]) to obtain error estimates. We further show in the following that for uniqueness of the collocation method in dimension three, we require imperatively the cut-off function χ_ρ introduced in Chapter 4.4, leading to sufficiently large convergence rates for the discretized integral equation. We point out, that the error of the approximation of the eigenvalues and the eigenvectors, used for the collocation method is not focus of this chapter. The smoothness of the fraction of the contrast q and the depth-dependent sound speed $c = c(x_m)$ leads to a second crucial quantity for these rates. This motivates the following assumption.

Assumption 5.3.1. *We assume that the contrast q is everywhere defined in Λ_ρ and that $q/c^2 \in H^t(\Lambda_\rho)$.*

Due to Assumption 5.3.1 for $t > m/2$ and $u \in H^t(\Lambda_\rho)$ the multiplication $u \mapsto qu$ is bounded on $H^t(\Lambda_\rho)$,

$$\|qu\|_{H^t(\Lambda_\rho)} \leq C\|u\|_{H^t(\Lambda_\rho)} \qquad \text{for all } u \in H^t(\Lambda_\rho), \tag{5.25}$$

and we further know by the fact that $H^t(\Lambda_\rho)$ is a Banach algebra for $t > m/2$ (e.g [SV02]) that

$$\left\|\frac{q}{c^2}u\right\|_{H^t(\Lambda_\rho)} \leq \left\|\frac{q}{c^2}\right\|_{H^t(\Lambda_\rho)} \|u\|_{H^t(\Lambda_\rho)} \qquad \text{for all } u \in H^t(\Lambda_\rho). \tag{5.26}$$

For dimension one, the proof of estimate (5.25) is shown in [SV02, Lemma 5.13.1] and can analogously be adapted to dimension m.

Corollary 5.3.2. *We assume that Assumption 5.1.1 holds, i.e. the sound speed is continuous, and Assumption 5.3.1 holds, i.e. $q/c^2 \in H^t(\Lambda_\rho)$ and $s \leq 2$. Then for $r > 1/2$, $t > m/2$ and $\boldsymbol{N} \in \mathbb{N}^m$ it holds for all $u \in H^t(\Lambda_\rho)$ that*

$$\left\|\mathcal{V}_\rho\left(\frac{q}{c^2}u\right) - \mathcal{V}_\rho Q_{\boldsymbol{N}}\left(\frac{q}{c^2}u\right)\right\|_{H^s(\Lambda_\rho)} \leq C(r)\min(\boldsymbol{N})^{-(t-r)}\left\|\frac{q}{c^2}\right\|_{H^t(\Lambda_\rho)} \|u\|_{H^t(\Lambda_\rho)},$$

where C is independent of $\boldsymbol{N}$.

Proof. Due to Theorem 4.5.2 we have for $s \leq 2$ that

$$\|\mathcal{V}_\rho u\|_{H^s(\Lambda_\rho)} \leq C\|u\|_{H^{s-2}(\Lambda_\rho)} \qquad \text{for all } u \in H^s(\Lambda_\rho),$$

where $C > 0$ is a constant In consequence, we obtain for $s \leq 2$ that

$$\begin{aligned}\left\|\mathcal{V}_\rho\left(\frac{q}{c^2}u\right) - \mathcal{V}_\rho Q_{\boldsymbol{N}}\left(\frac{q}{c^2}u\right)\right\|_{H^s(\Lambda_\rho)} &\leq C\left\|\frac{q}{c^2}u - Q_{\boldsymbol{N}}\left(\frac{q}{c^2}u\right)\right\|_{H^{s-2}(\Lambda_\rho)} \\ &\leq C\left\|\frac{q}{c^2}u - Q_{\boldsymbol{N}}\left(\frac{q}{c^2}u\right)\right\|_{L^2(\Lambda_\rho)},\end{aligned}$$

where $C > 0$ is a constant. Then, exploiting the convergence of $Q_{\boldsymbol{N}}u - u$ due to Theorem 5.2.3, we see for $r > 1/2$ and $t > m/2$ that

$$\left\|\frac{q}{c^2}u - Q_{\boldsymbol{N}}\left(\frac{q}{c^2}u\right)\right\|_{L^2(\Lambda_\rho)} \leq C\min(\boldsymbol{N})^{-(t-r)}\left\|\frac{q}{c^2}u\right\|_{H^t(\Lambda_\rho)}.$$

To this end, estimate (5.26) yields

$$\left\|\frac{q}{c^2}u - Q_{\boldsymbol{N}}\left(\frac{q}{c^2}u\right)\right\|_{L^2(\Lambda_\rho)} \leq C\min(\boldsymbol{N})^{-(t-r)}\left\|\frac{q}{c^2}\right\|_{H^t(\Lambda_\rho)}\|u\|_{H^t(\Lambda_\rho)},$$

where $C > 0$ is a constant.

□

Lemma 5.3.3. *We assume that Assumptions 5.1.1 and 5.3.1 hold, i.e. the sound speed is continuous, and ($q/c^2 \in H^t(\Lambda_\rho)$), for $t > m/2$ and $\boldsymbol{N} \in \mathbb{N}^m$. Then for $r > 1/2$, there holds that*

$$\left\|\frac{q}{c^2}\mathcal{V}_\rho w - Q_{\boldsymbol{N}}\left[\frac{q}{c^2}\mathcal{V}_\rho w\right]\right\|_{L^2(\Lambda_\rho)} \leq C\min(\boldsymbol{N})^{-(t-r)}\left\|\frac{q}{c^2}\right\|_{H^t(\Lambda_\rho)}\|w\|_{H^{t-2}(\Lambda_\rho)} \text{ for all } w \in H^{t-2}(\Lambda_\rho).$$

Proof. Using the convergence of $Q_{\boldsymbol{N}}u - u$ in Theorem 5.2.3, we know for $r > 1/2$ and $t > m/2$ that

$$\left\|\frac{q}{c^2}\mathcal{V}_\rho w - Q_{\boldsymbol{N}}\left[\frac{q}{c^2}\mathcal{V}_\rho w\right]\right\|_{L^2(\Lambda_\rho)} \leq C\min(\boldsymbol{N})^{-(t-r)}\left\|\frac{q}{c^2}\mathcal{V}_\rho w\right\|_{H^t(\Lambda_\rho)}.$$

Once more, estimate (5.26) gives us for $t > m/2$ that

$$\left\|\frac{q}{c^2}\mathcal{V}_\rho w - Q_{\boldsymbol{N}}\left[\frac{q}{c^2}\mathcal{V}_\rho w\right]\right\|_{L^2(\Lambda_\rho)} \leq C\min(\boldsymbol{N})^{-(t-r)}\left\|\frac{q}{c^2}\right\|_{H^t(\Lambda_\rho)}\|\mathcal{V}_\rho w\|_{H^t(\Lambda_\rho)}.$$

Finally, due to Theorem 4.5.2, for $t > m/2$ it holds $\|\mathcal{V}_\rho w\|_{H^t(\Lambda_\rho)} \leq \|w\|_{H^{t-2}(\Lambda_\rho)}$, such that

$$\begin{aligned}&C\min(\boldsymbol{N})^{-(t-r)}\left\|\frac{q}{c^2}\right\|_{H^t(\Lambda_\rho)}\|\mathcal{V}_\rho w\|_{H^t(\Lambda_\rho)} \\ &\qquad\leq C\min(N)^{-(t-r)}\left\|\frac{q}{c^2}\right\|_{H^t(\Lambda_\rho)}\|w\|_{H^{t-2}(\Lambda_\rho)} \quad \text{for all } w \in H^{t-2}(\Lambda_\rho). \qquad (5.27)\end{aligned}$$

□

The following theorem is based on the idea of [SV02, Lemma 10.5.1] and [LN12] with adaption to the error estimate for the interpolation operator $Q_{\boldsymbol{N}}$ introduced in Theorem 5.2.3 and used in Corollary 5.3.2 and Lemma 5.3.3.

Recall that u denotes the solution to the periodized Lippmann-Schwinger equation (4.67) and that u equals the solution of the source problem (4.34) with the radiation and boundedness conditions (3.3) in $M_{(\rho-\delta)/2}$ if $\operatorname{supp}(f) \subset M_{(\rho-\delta)/2}$, see Theorem 4.5.4. We first investigate convergence of the scheme (5.22).

Theorem 5.3.4. *We assume that Assumptions 4.2.11 and 5.1.1 hold, i.e. unique solvability of source problem (4.34) and the sound speed is continuous and $r > 1/2$.*

(a) If $q/c^2 \in H^s(\Lambda_\rho)$ with $s > m/2$ and $f \in H^{t-2}(\Lambda_\rho)$ with $t > m/2$, then u belongs to $H^{\min(s+2,t)}(\Lambda_\rho)$. If $q/c^2 \in L^\infty(\Lambda_\rho)$ and $f \in H^{t-2}(\Lambda_\rho)$ for some $t \geq 0$, then the periodized Lippmann-Schwinger equation (4.67) possesses a unique solution in $H^{\min(2,t)}(\Lambda_\rho)$.

(b) If Assumption 5.3.1 ($q/c^2 \in H^t(\Lambda_\rho)$) holds and $f \in H^{t-2}(\Lambda_\rho)$ for some $t \in (m/2, 2]$, then there is $N^* \in \mathbb{N}$ such that for $\boldsymbol{N} \in \mathbb{N}^m$ with $\min(\boldsymbol{N}) \geq N^*$ the collocation scheme (5.22) possesses a unique solution $u_{\boldsymbol{N}} \in T_{\boldsymbol{N}}$ that satisfies

$$\|u_{\boldsymbol{N}} - u\|_{H^t} \leq C \left[\|(Q_{\boldsymbol{N}} - I)\mathcal{V}_\rho f\|_{H^t} + \min(\boldsymbol{N})^{-t+r} \left\| \frac{q}{c^2} \right\|_{H^t} \left[\left\| \frac{q}{c^2} \right\|_{H^t} + \|f\|_{H^{t-2}} \right] \right], \qquad (5.28)$$

where for simplicity we abbreviate $H^t := H^t(\Lambda_\rho)$ and $L^2 := L^2(\Lambda_\rho)$ in the latter estimate.

Proof. (a) Since unique solvability of source problem (4.34) holds by Assumption 4.2.11, we obtain by Theorem 4.5.5 existence and uniqueness of the solution to the periodized Lippmann-Schwinger equation in $L^2(\Lambda_\rho)$, independent of dimension m. Using further linearity of the periodized Lippmann-Schwinger equation and exploiting mapping properties from Theorem 4.5.2 that for $t > m/2$ it holds

$$\|\mathcal{V}_\rho u\|_{H^t(\Lambda_\rho)} \leq \|u\|_{H^{t-2}(\Lambda_\rho)} \qquad \text{for all } u \in H^t(\Lambda_\rho),$$

we treat that there exists a constant $C > 0$ such that

$$\begin{aligned} \|u\|_{H^{\min(2,t)}(\Lambda_\rho)} &\leq \omega^2 \left\| \mathcal{V}_\rho \left(\frac{q}{c^2} u \right) \right\|_{H^{\min(2,t)}(\Lambda_\rho)} + \|\mathcal{V}_\rho f\|_{H^{\min(2,t)}(\Lambda_\rho)} \\ &\leq C \left\| \frac{q}{c^2} u \right\|_{H^{t-2}(\Lambda_\rho)} + \|f\|_{H^{t-2}(\Lambda_\rho)}. \end{aligned} \qquad (5.29)$$

The last estimate shows the second part of the claim and for $t \leq 2$ the first part, too. We point out that the operator $\mathcal{V}_\rho$ in dimension three uses the cut-off function χ_ρ to have a smooth kernel to obtain the required mapping property. Now, if $f \in H^{t-2}(\Lambda_\rho)$ for $t > 2$ and $q/c^2 \in H^t(\Lambda_\rho)$, and if $m/2 < s < 2$, we obtain again from Theorem 4.5.2 that

$$\begin{aligned} \|u\|_{H^t(\Lambda_\rho)} &\leq \omega^2 \left\| \mathcal{V}_\rho \left(\frac{q}{c^2} u \right) \right\|_{H^t(\Lambda_\rho)} + \|\mathcal{V}_\rho f\|_{H^t(\Lambda_\rho)} \\ &\leq C \left\| \frac{q}{c^2} u \right\|_{H^{\max(t-2,s)}(\Lambda_\rho)} + C\|f\|_{H^{t-2}(\Lambda_\rho)} \\ &\leq C \left\| \frac{q}{c^2} \right\|_{H^{\max(t-2,s)}(\Lambda_\rho)} \|u\|_{H^{\max(t-2,s)}(\Lambda_\rho)} + C\|f\|_{H^{t-2}(\Lambda_\rho)}. \end{aligned}$$

If $t - 2 \leq 2$, then $\|u\|_{H^{\max(t-2,s)}(\Lambda_\rho)} \leq \|u\|_{H^2(\Lambda_\rho)}$ such that the estimate in (5.29) implies that u belongs to $H^t(\Lambda_\rho)$, because u is H^2-regular as $q/c^2 \in H^t(\Lambda_\rho)$ for $t > 2$. If $t - 2 \geq 2$ and $2 > s > m/2$, then for q sufficient regular we use a bootstrap argument, an iteration technique to bound $\|u\|_{H^{\min(t-2,s)}(\Lambda_\rho)}$, which gives

$$\begin{aligned} \|u\|_{H^t(\Lambda_\rho)} \leq C \left\| \frac{q}{c^2} \right\|_{H^{\max(t-2,s)}(\Lambda_\rho)} & \\ \left[\left\| \frac{q}{c^2} \right\|_{H^{\max(t-4,s)}(\Lambda_\rho)} \|u\|_{H^{\max(t-4,s)}(\Lambda_\rho)} + C\|f\|_{H^{t-2}(\Lambda_\rho)} \right] &+ C\|f\|_{H^{t-2}(\Lambda_\rho)}. \end{aligned} \qquad (5.30)$$

Next, we denote by $p = p(s) \in \mathbb{N}$ the smallest integer such that $t - 2p \leq m/2$ and we see for $2 > s > m/2$ that

$$\|u\|_{H^t(\Lambda_\rho)} \leq C(p) \left\| \frac{q}{c^2} \right\|^p_{H^{\max(t-2,s)}(\Lambda_\rho)} \|u\|_{H^{t-2p}(\Lambda_\rho)} + C(p) \left\| \frac{q}{c^2} \right\|^{p-1}_{H^{\max(t-2,s)}(\Lambda_\rho)} \|f\|_{H^{t-2}(\Lambda_\rho)},$$

such that (5.29) again shows that u belongs to $H^t(\Lambda_\rho)$ if the regularity of f is sufficiently large.

(b) We show by the fact that $\mathcal{V}_\rho$ is a compact operator, the fact that we have $q/c^2 \in H^t(\Lambda_\rho)$, the existence and uniqueness of the solution to the periodized Lippmann-Schwinger equation in $L^2(\Lambda_\rho)$, and the Fredholm alternative, that $I - \omega^2 \mathcal{V}_\rho Q_{\boldsymbol{N}}(q/c^2\,\cdot)$ is an invertible operator on $H^s(\Lambda_\rho)$ for $m/2 < t \leq 2$.

Due to Corollary 5.3.2 it first holds for $t = s$ that

$$\left\| \left[I - \omega^2 \mathcal{V}_\rho Q_{\boldsymbol{N}} \left(\frac{q}{c^2} \cdot \right) \right] - \left[I - \omega^2 \mathcal{V}_\rho \left(\frac{q}{c^2} \cdot \right) \right] \right\|_{H^t(\Lambda_\rho) \to H^t(\Lambda_\rho)} \leq C \omega^2 \min(\boldsymbol{N})^{-(t-r)} \left\| \frac{q}{c^2} \right\|_{H^t(\Lambda_\rho)}.$$

For $r > 1/2$ sufficient small and $\min(\boldsymbol{N}) \geq N^*$ sufficient large, the operator on the left-hand-side of the latter estimate can be made arbitrarily small. By part (a) of this proof, the operator $u \mapsto u - \omega^2 \mathcal{V}_\rho(q/c^2 u)$ is invertible on $H^t(\Lambda_\rho)$. In consequence, for $\min(\boldsymbol{N})$ sufficient large a Neumann series argument implies that

$$u \mapsto A_{\boldsymbol{N}} := u - \omega^2 \mathcal{V}_\rho \left(Q_{\boldsymbol{N}} \left(\frac{q}{c^2} u \right) \right),$$

is invertible on $H^t(\Lambda_\rho)$, too. Moreover, their inverse $A_{\boldsymbol{N}}^{-1}$ are uniformly bounded on $H^t(\Lambda_\rho)$ in the discretization parameter $\boldsymbol{N} \in \mathbb{N}^m$,

$$\left\| \left[I - \omega^2 \mathcal{V}_\rho Q_{\boldsymbol{N}} \left(\frac{q}{c^2} \cdot \right) \right]^{-1} \right\|_{H^t(\Lambda_\rho) \to H^t(\Lambda_\rho)} \leq C \qquad \text{if } \min(\boldsymbol{N}) \geq \boldsymbol{N}^*. \tag{5.31}$$

Next, we apply the operator $u \mapsto u - \omega^2 \mathcal{V}_\rho Q_{\boldsymbol{N}}(q/c^2 u)$ to the difference $u - u_{\boldsymbol{N}}$, where u is solution of the periodized Lippmann Schwinger equation and $u_{\boldsymbol{N}} \in T_{\boldsymbol{N}}$ is solution of the approximated Lippmann Schwinger equation (5.22). Then

$$A(u_{\boldsymbol{N}} - u) = \left[I - \mathcal{V}_\rho Q_{\boldsymbol{N}} \left(\frac{\omega^2}{c^2} q \cdot \right) \right] (u_{\boldsymbol{N}} - u) = (Q_{\boldsymbol{N}} - I) V_\rho f + \mathcal{V}_\rho \left((Q_{\boldsymbol{N}} - I) \frac{\omega^2}{c^2} q u \right).$$

Now, due to the uniform boundedness of $A_{\boldsymbol{N}}^{-1}$ on $H^t(\Lambda_\rho)$, we have for $r > 1/2$ that

$$\|u_{\boldsymbol{N}} - u\|_{H^t(\Lambda_\rho)} \leq C \left\| (Q_{\boldsymbol{N}} - I) \mathcal{V}_\rho f + \mathcal{V}_\rho \left((Q_{\boldsymbol{N}} - I) \frac{\omega^2}{c^2} q u \right) \right\|_{H^t(\Lambda_\rho)}.$$

Next, triangle inequality and the bound of the integral operator $\mathcal{V}_\rho$ given in Theorem 4.5.2 imply

$$\|u_{\boldsymbol{N}} - u\|_{H^t(\Lambda_\rho)} \leq C \left\| (Q_{\boldsymbol{N}} - I) \mathcal{V}_\rho f \right\|_{H^t(\Lambda_\rho)} + \left\| (Q_{\boldsymbol{N}} - I) \frac{\omega^2}{c^2} q u \right\|_{H^{t-2}(\Lambda_\rho)}.$$

Exploiting once more the convergence rate of $\|Q_{\boldsymbol{N}} u - u\|_{L^2(\Lambda_\rho)}$ shown in Theorem 5.2.3 and estimate (5.26), we moreover see

$$\|u_{\boldsymbol{N}} - u\|_{H^t(\Lambda_\rho)} \leq C \Bigg[\|(Q_{\boldsymbol{N}} - I) \mathcal{V}_\rho f\|_{H^t(\Lambda_\rho)} + \min(\boldsymbol{N})^{-t+r} \left\| \frac{q}{c^2} \right\|_{H^t(\Lambda_\rho)} \left[\left\| \frac{q}{c^2} \right\|_{H^t(\Lambda_\rho)} + \|f\|_{H^{t-2}(\Lambda_\rho)} \right] \Bigg].$$

This ends the proof. □

Remark 5.3.5. *Estimate* (5.30) *shows that the second claim of Theorem 5.3.4(a) holds even if we merely suppose that* $q/c^2 \in H^{\max(t-2,s)}(M_\rho)$ *for some* $s \in (m/2, 2)$.

The next result investigates convergence of the alternative scheme (5.24).

Theorem 5.3.6. *We assume that Assumptions 4.2.11, 5.1.1 and 5.3.1 hold, i.e. unique solvability of source problem (4.34)), the sound speed is continuous and ($q/c^2 \in H^t(\Lambda_\rho)$) for $t > m/2$.*

Then there is $N^* \in \mathbb{N}$ such that if $\boldsymbol{N} \in \mathbb{N}^m$ with $\min(\boldsymbol{N}) \geq N^*$ the collocation scheme (5.24) possesses a unique solution $w_{\boldsymbol{N}} \in T_{\boldsymbol{N}}$ for $p = p(t) \in \mathbb{N}$ denotes the smallest integer such that $t - 2 - 2p \leq m/2$, then

$$\left\| w_N - \frac{q}{c^2} u \right\|_{L^2(D)} \leq C \min(N)^{t-r} \left[\|f\|_{H^{t-2}(\Lambda_\rho)} + \left\| \frac{q}{c^2} \right\|_{H^t(\Lambda_\rho)} \left[\left\| \frac{q}{c^2} \right\|^p_{H^{t-2}(\Lambda_\rho)} + \|f\|^p_{H^{t-2}(\Lambda_\rho)} \right] \right]. \tag{5.32}$$

Proof. Due to the uniqueness of solution $w = (q/c^2)u$ we can directly deduce the regularity of w from the regularity of q/c^2 and u: If $q/c^2 \in H^t(\Lambda_\rho)$ and $f \in H^t(\Lambda_\rho)$ with $t > m/2$, then (5.30) implies that $u \in H^{t+2}(\Lambda_\rho)$, such that $w = (q/c^2)u$ belongs to $H^t(\Lambda_\rho)$ as this space is a Banach algebra. Next, Lemma 5.3.3 yields for $q/c^2 \in H^t(\Lambda_\rho)$ with $t > m/2$ that

$$\left\| \left[I - \omega^2 \frac{q}{c^2} \mathcal{V}_\rho \right] - \left[I - \omega^2 Q_{\boldsymbol{N}} \left(\frac{q}{c^2} \mathcal{V}_\rho \right) \right] \right\|_{L^2(\Lambda_\rho) \to L^2(\Lambda_\rho)} \leq C \min(\boldsymbol{N})^{-\min(2,t)+r} \left\| \frac{q}{c^2} \right\|_{H^t(\Lambda_\rho)}$$

for $r > 1/2$.

Similar like in the proof of Theorem 5.3.4 we see that $w \mapsto w - \omega^2 (q/c^2) \mathcal{V}_\rho w$ is invertible on $L^2(\Lambda_\rho)$ and for $\min(\boldsymbol{N}) \geq N^*$ sufficient large we know that $w \mapsto B_{\boldsymbol{N}}(w) := w - \omega^2 Q_{\boldsymbol{N}}((q/c^2)\mathcal{V}_\rho w)$ is invertible on $L^2(\Lambda_\rho)$, too. We moreover see that $B_{\boldsymbol{N}}^{-1}$ is uniformly bounded. Plugging all together shows that the discrete problem in (5.24) possesses a unique solution $w_{\boldsymbol{N}} \in T_{\boldsymbol{N}}$. Analogously like in the proof of Theorem 5.3.4 we further know that

$$B_{\boldsymbol{N}}(w_{\boldsymbol{N}} - w) = \left[I - Q_{\boldsymbol{N}} \left(\frac{\omega^2}{c^2} q \mathcal{V}_\rho(\cdot) \right) \right] (w_{\boldsymbol{N}} - w) = (Q_{\boldsymbol{N}} - I) \mathcal{V}_\rho f + \omega^2 (Q_{\boldsymbol{N}} - I) \left(\frac{q}{c^2} \mathcal{V}_\rho w \right).$$

Consequently, the uniform bound for $B_{\boldsymbol{N}}^{-1}$ in the operator norm of $L^2(\Lambda_\rho)$ implies that

$$\|w_{\boldsymbol{N}} - w\|_{L^2(\Lambda_\rho)} \leq C \min(N)^{-(t-r)} \left[\|\mathcal{V}_\rho f\|_{H^t(\Lambda_\rho)} + \left\| \frac{q}{c^2} \right\|_{H^t(\Lambda_\rho)} \|w\|_{H^t(\Lambda_\rho)} \right],$$

which shows the claim due to (5.30), as $w = (q/c^2)u$ and $\|w_{\boldsymbol{N}} - w\|_{L^2(D)} \leq \|w_{\boldsymbol{N}} - w\|_{L^2(\Lambda_\rho)}$. □

We see by Theorems 5.3.4 and 5.3.6, that the error estimate (5.28) for the first collocation scheme (5.22) provides convergence in $H^t(\Lambda_\rho)$ where $t \in (m/2, 2)$, whereas the error estimate (5.32) corresponding for the alternative scheme (5.24) merely provides convergence in $L^2(\Lambda_\rho)$. On the downside, we have no tool to estimate the interpolation error $\|(Q_{\boldsymbol{N}} - I)f\|_{H^t(\Lambda_\rho)}$ for $t \in (m/2, 2]$ in (5.28) and the convergence rate in (5.28) is limited by two.

In contrast, the error estimate (5.32) for the alternative scheme (5.24) provides a quantified bound depending on q/c^2 and f, and further provides arbitrarily high convergence rates depending on the regularity of these two functions. Due to the imperative need to estimate $(q/c^2)u$ it is impossible to exploit the higher regularity of $u \in H^{t+2}(\Lambda_\rho)$).

Chapter 6

Numerical Computations Using the Collocation Method

In this section we present numerical computations using the introduced collocation method. We recall that Assumption 5.1.1 holds, i.e. the sound speed $c \in C^0([0, H])$ is continuous. We first indicate the fully discrete version of (5.22): For a discretization parameter $\boldsymbol{N} \in \mathbb{N}^m$, find $u_{\boldsymbol{N}} \in T_{\boldsymbol{N}}$ solving the finite-dimensional linear system

$$u_{\boldsymbol{N}} - \mathcal{V}_\rho Q_{\boldsymbol{N}} \left(\frac{\omega^2}{c^2(x_m)} q u_{\boldsymbol{N}} \right) = Q_{\boldsymbol{N}} \mathcal{V}_\rho f \qquad \text{in } T_{\boldsymbol{N}}, \tag{6.1}$$

and (5.24): For $\boldsymbol{N} \in \mathbb{N}^m$ find $w_{\boldsymbol{N}} \in T_{\boldsymbol{N}}$ solving

$$w_{\boldsymbol{N}} - \omega^2 Q_{\boldsymbol{N}} \left(\frac{q}{c^2(x_m)} \mathcal{V}_\rho(w) \right) = Q_{\boldsymbol{N}} \left(\frac{q}{c^2(x_m)} \mathcal{V}_\rho f \right) \qquad \text{in } T_{\boldsymbol{N}}. \tag{6.2}$$

We first define the array of Fourier coefficients by

$$\mathbb{C}^m_{\boldsymbol{N}} := \{ c(\boldsymbol{j}) : \boldsymbol{j} \in \mathbb{Z}^m_{\boldsymbol{N}},\, c(\boldsymbol{j}) \in \mathbb{C} \}.$$

For simplicity, we write the point values of $u_{\boldsymbol{N}}$ and $w_{\boldsymbol{N}} \in T_{\boldsymbol{N}}$ by $\underline{u_{\boldsymbol{N}}} := (u_{\boldsymbol{N}}(x_k^{(\boldsymbol{N})}))_{k \in \mathbb{Z}^m_{\boldsymbol{N}}} \in \mathbb{C}^m_{\boldsymbol{N}}$ and $\underline{w_{\boldsymbol{N}}} \in \mathbb{C}^m_{\boldsymbol{N}}$. Due to Lemma 5.1.2 these point values are uniquely determined by the Fourier coefficients $\hat{u}_{\boldsymbol{N}} := (\hat{u}_{\boldsymbol{N}}(k))_{\boldsymbol{k} \in \mathbb{Z}^m_{\boldsymbol{N}}}$ and $\hat{w}_{\boldsymbol{N}}$ of these functions and vice-versa. To introduce later on the first option for a collocation discretization of the periodized Lippmann-Schwinger equation in (5.22) as a matrix-vector product, for simplicity, we define the element wise multiplication of two elements $a, b \in \mathbb{C}^m_{\boldsymbol{N}}$ by $a \bullet b$, so $(a \bullet b)(\boldsymbol{j}) = a(\boldsymbol{j}) b(\boldsymbol{j})$ where $\boldsymbol{j} \in \mathbb{Z}^m_{\boldsymbol{N}}$. Furthermore, we define the transfer operator by $F_{\boldsymbol{N}}$,

$$F_{\boldsymbol{N}} : \mathbb{C}^m_{\boldsymbol{N}} \to \mathbb{C}^m_{\boldsymbol{N}}, \qquad (u(\boldsymbol{j}))_{\boldsymbol{j} \in \mathbb{Z}^m_{\boldsymbol{N}}} \mapsto (\hat{u}_{\boldsymbol{N}}(\boldsymbol{j}))_{\boldsymbol{j} \in \mathbb{Z}^m_{\boldsymbol{N}}}, \tag{6.3}$$

and its inverse by

$$F^{-1}_{\boldsymbol{N}} : \mathbb{C}^m_{\boldsymbol{N}} \to \mathbb{C}^m_{\boldsymbol{N}}, \qquad (\hat{u}_{\boldsymbol{N}}(\boldsymbol{j}))_{\boldsymbol{j} \in \mathbb{Z}^m_{\boldsymbol{N}}}) \mapsto (u(\boldsymbol{j}))_{\boldsymbol{j} \in \mathbb{Z}^m_{\boldsymbol{N}}}. \tag{6.4}$$

Moreover, we write the eigenvalues of the integral operator $\mathcal{V}_\rho$, given by Theorem 4.5.1, as $\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} = i/(2\lambda_j) \hat{E}_\rho(n_1, \lambda_j)$ in dimension two and as $\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} = i/4 \hat{H}^{smo}_\rho(\tilde{n}, \lambda_j)$ in dimension three. Therefore, we denote the point evaluations of $x \mapsto q(x)/c^2(x_m) =: a(x)$ by $\underline{a}_{\boldsymbol{N}} \in \mathbb{C}^m_{\boldsymbol{N}}$. In consequence, we find for the first option of a collocation discretization of the periodized Lippmann-Schwinger equation in (6.1) that

$$\underline{u}_{\boldsymbol{N}} - \omega^2 F^{-1}_{\boldsymbol{N}} \left[\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} \bullet F_{\boldsymbol{N}} \left[\underline{a}_{\boldsymbol{N}} \bullet \underline{u}_{\boldsymbol{N}} \right] \right] = F^{-1}_{\boldsymbol{N}} \left[\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} \bullet F_{\boldsymbol{N}} \left[\underline{a}_{\boldsymbol{N}} \bullet \underline{f}_{\boldsymbol{N}} \right] \right]. \tag{6.5}$$

Further, we see for the second option of a collocation discretization of the periodized Lippmann-Schwinger equation in (6.2) that

$$\hat{w}_{\boldsymbol{N}} - \omega^2 F_{\boldsymbol{N}} \left[\underline{a}_{\boldsymbol{N}} \bullet F_{\boldsymbol{N}}^{-1} \left[\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} \bullet \hat{w}_{\boldsymbol{N}} \right] \right] = F_{\boldsymbol{N}} \left[\underline{a}_{\boldsymbol{N}} \bullet F_{\boldsymbol{N}}^{-1} \left[\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} \bullet \underline{f}_{\boldsymbol{N}} \right] \right]. \tag{6.6}$$

Using an iterative solver for linear systems of equations like GMRES, we can in principle solve the equation (6.5) or (6.6) for a given source term $f_{\boldsymbol{N}}$ and unknown $u_{\boldsymbol{N}}$ without setting up the large and dense matrix. As the integral operator $\mathcal{V}_\rho$ in (6.5) is smoothing the product $\underline{a}_{\boldsymbol{N}} \bullet \underline{b_{\boldsymbol{N}}}$, we will merely consider that discretization scheme in the following. The disadvantage of (6.6) indeed is its definite requirement for a smooth contrast $a = q/c^2$ in order to approximate a smooth function w. As we are ultimately interested in particular in discontinuous contrasts (without theoretical justification), we hence prefer (6.5) instead of (6.6).

6.1 Numerical Computation of the Transform $F_{\boldsymbol{N}}$

In this section we discuss the numerical computation of the transform $F_{\boldsymbol{N}}$ from (6.3). We first split the transform $F_{\boldsymbol{N}}$ into its "horizontal" and "vertical" parts $F_{\boldsymbol{N}} = F_{\tilde{\boldsymbol{N}}}^{(\sim)} F_{N_m}^{\perp}$. Similar, we split the transform $F_{\boldsymbol{N}}^{-1}$ into the parts $F_{\boldsymbol{N}}^{-1} = (F_{\tilde{\boldsymbol{N}}}^{(\sim)})^{-1} (F_{N_m}^{\perp})^{-1}$. Due to Chapter 5.1 we know that the operator $F_{\tilde{\boldsymbol{N}}}^{(\sim)}$ is a discrete Fourier transform in the horizontal variables (see [SV02, Theorem 10.5.4]), which can be efficiently computed by fast Fourier transform. Analogously, Chapter 5.1 implies that $(F_{\tilde{\boldsymbol{N}}}^{(\sim)})^{-1}$ is a inverse discrete Fourier transform in the horizontal variables and can be computed by inverse fast Fourier transform. Exploiting again Chapter 5.1, we know that the "vertical" transform $F_{N_m}^{\perp}$ requires to compute the matrix of approximated eigenvectors Φ_{N_m} of size $N_m \times N_m$, as well as its inverse, which basically requires to accurately approximate the first N_m eigenfunctions to (2.12). For our numerical implementation, we need to replace the transform $F_{\boldsymbol{N}} : \underline{u}_{\boldsymbol{N}} \mapsto \hat{u}_{\boldsymbol{N}}$ by the fully discrete transform $\underline{u}_{\boldsymbol{N}} \mapsto \hat{\hat{u}}_{\boldsymbol{N}}$ from Section 5.1, see (5.10) and (5.11). Computing the fully discrete coefficients $\hat{\hat{u}}_{\boldsymbol{N}}$ from $\underline{u}_{\boldsymbol{N}}$ requires the matrix Φ_{N_m}, the inverse transform $\hat{\hat{u}}_{\boldsymbol{N}} \mapsto \underline{u}_{\boldsymbol{N}}$ relies on the inverse matrix $\Phi_{N_m}^{-1}$, see (5.2).

The error between the transforms $\underline{u}_{\boldsymbol{N}} \mapsto \hat{u}_{\boldsymbol{N}}$ and $\underline{u}_{\boldsymbol{N}} \mapsto \hat{\hat{u}}_{\boldsymbol{N}}$ depends basically on the discretization parameter $\boldsymbol{N}$ and the accuracy of the approximated eigenvectors. However, we will not attempt to estimate this error in the following, such that a convergence proof for the fully discrete collocation scheme will remain open.

We recall that in Chapter 2.3 we discussed the approximation of eigenfunctions and eigenvalues of the Liouville eigenvalue problem for different sound speed profiles. Independent of the chosen method, it is best to compute the eigenfunctions and eigenvalues in advance in order to economize computation time. Note that the column $\boldsymbol{\phi}_{\ell_m}(k)$ of Φ_{N_m} satisfy

$$\sum_{k_m=1}^{N_m} \boldsymbol{\phi}_{\ell_m}(k) \boldsymbol{\phi}_{j_m}(k) \approx \int_0^H \phi_{\ell_m}(x_m) \phi_{j_m}(x_m)\, dx_m = \delta_{l_m,j_m} \qquad (1 \le \ell_m, j_m \le N_m).$$

The discrepancy between the left- and the right-hand side of the latter approximation basically gives the error between $F_{\boldsymbol{N}}$ and its fully discrete variant $\underline{u}_{\boldsymbol{N}} \mapsto \hat{\hat{u}}_{\boldsymbol{N}}$.

We recall that the inverse of Φ_{N_m} is denoted by $\Phi_{N_m}^{-1}$, which is defined by (5.2) in Chapter 5.1,

$$I = \Phi_{N_m} \Phi_{N_m}^{-1} = \begin{bmatrix} \boldsymbol{\phi}_1 & \boldsymbol{\phi}_2 & \cdots & \boldsymbol{\phi}_{N_m} \end{bmatrix} \begin{bmatrix} \boldsymbol{\phi}_1^{-1} & \boldsymbol{\phi}_2^{-1} & \cdots & \boldsymbol{\phi}_{N_m}^{-1} \end{bmatrix} = \Phi_{N_m}^{-1} \Phi_{N_m}.$$

Therefore, we use for the inverse transform $(F_{N_m}^{\perp})^{-1}$ in the vertical variable the matrix $\Phi_{N_m}^{-1}$. We point out that for constant sound speed [LN12] shows that the multiplication with approximated eigenvectors in the vertical variable can be replaced by a Fourier cosine transform. We further point out that a numerical computation for constant sound speed shows a relative L^2-error of about

Discretization Parameter N_m	2^6	2^7	2^8	2^9
Computation time in sec for scheme (6.7)	0.0039	0.0215	0.171	1.30
Computation time for MATLAB matrix inversion	$< 10^{-3}$	0.0026	0.0071	0.0365
Relative error	$< 10^{-11}$	$< 10^{-15}$	$< 10^{-14}$	$< 10^{-12}$
Iteration steps for scheme in (6.7)	7	8	9	10

Table 6.1: Numerical approximation of $\Phi_{N_m}^{-1}$ using the iterative scheme (6.7)

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Discretization Parameter N_m	2^6	2^7	2^8	2^9
Computation time in sec of scheme (6.8)	0.0062	0.0228	0.1533	1.86
Relative error	$< 10^{-12}$	$< 10^{-15}$	$< 10^{-14}$	$< 10^{-10}$
Iteration steps for scheme in (6.8)	4	5	5	5

Table 6.2: Numerical approximation of $\Phi_{N_m}^{-1}$ using the iterative scheme (6.8)

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10^{-5} between a Fourier cosine transform compared to evaluations at the discretization points of the analytic eigenfunctions as transfer operator. This corresponds to well-known error estimates of a cosine transform.

The matrix Φ_{N_m} is rectangular and has full rank. We point out that Lemma 5.1.2 guarantees that Φ_{N_m} is invertible. Consequently, we can use MATLAB standard routines to compute $\Phi_{N_m}^{-1}$ from Φ_{N_m}. We further note that for a computer environment with memory of 16GB for discretization parameter $N_m > 2^{13}$, the MATLAB routine to invert a matrix produces memory overflow. To avoid memory overflow for large discretization parameters, we use memory optimized and fast iterative methods from [Sol13] instead of the MATLAB routine to compute $\Phi_{N_m}^{-1}$. Using for a nonsingular real or complex matrix Equation 1.3 in [Sol13], we approximate the matrix $\Phi_{N_m}^{-1}$ by a sequence of matrices $A_\ell \in \mathbb{C}^{N_m \times N_m}$ with start value $A_0 = \Phi_{N_m}^T$ by

$$A_{\ell+1} = A_\ell(3I - \Phi_{N_m} A_\ell(3I - \Phi_{N_m} A_\ell)), \tag{6.7}$$

where $\ell = 0, 1, 2, ...$ denotes the iteration index and I denotes the unit matrix of dimension N_m. In Table 6.1 we compare the MATLAB matrix inversion with scheme (6.7).

A second approximation for a nonsingular real or complex matrix is given by equation 2.2 in [Sol13]. More precisely, we approximate $\Phi_{N_m}^{-1}$ by a sequence of matrices $A_\ell \in \mathbb{C}^{N_m \times N_m}$ with start value $A_0 = \Phi_{N_m}^T$ by

$$\begin{aligned} A_{\ell+1} = A_\ell(7I + \Phi_{N_m} A_\ell(-21I + \Phi_{N_m} A_\ell(35I \\ + \Phi_{N_m} A_\ell(-35I + \Phi_{N_m} A_\ell(21I + \Phi_{N_m} A_\ell(-7I + A_\ell)))))), \end{aligned} \tag{6.8}$$

where $\ell = 0, 1, 2, ...$ denotes the iteration index and I denotes the unit matrix of dimension N_m. In Table 6.2 we compare the iterative inversion scheme (6.7) compared to the MATLAB matrix inversion. Furthermore, we stop the iterative inversion scheme (6.7) and (6.8), if the approximation $A_{\ell+1}$ satisfies

$$\|I - A_{\ell+1}\Phi_{N_m}\| < 1.$$

6.2 Optimization in Solving the Collocation Method

In this section we present an optimization technique to reduce memory and computation time in the evaluation of the integral operator when solving the collocation discretization in (6.5) by an iterative scheme. We first recall that the point evaluations of $x \mapsto q(x)/c^2(x_m) =: a(x)$ are denoted by $\underline{a}_{\boldsymbol{N}} \in \mathbb{C}^m_{\boldsymbol{N}}$. We moreover see if $\underline{q}_{\boldsymbol{N}}(\boldsymbol{j}) = 0$ for some $\boldsymbol{j} \in \mathbb{Z}^m_{\boldsymbol{N}}$, then $(\underline{a}_{\boldsymbol{N}} \bullet \underline{u}_{\boldsymbol{N}})(\boldsymbol{j}) = 0$, too. Using an iterative scheme like GMRES to solve the collocation discretization in (6.5), the evaluation of the integral operator

$$u_{\boldsymbol{N}} \mapsto u_{\boldsymbol{N}} - \omega^2 F^{-1}_{\boldsymbol{N}} \left[\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} \bullet F_{\boldsymbol{N}} \left[\underline{a}_{\boldsymbol{N}} \bullet \underline{u}_{\boldsymbol{N}} \right] \right],$$

operates on several zero entries. Furthermore, all computed entries $\underline{u}_{\boldsymbol{N}}$ at indices $\boldsymbol{j}$ where $\underline{a}_{\boldsymbol{N}}(\boldsymbol{j}) = 0$ are in principle superfluous, as they are multiplied by zero in the subsequent operator evaluation.

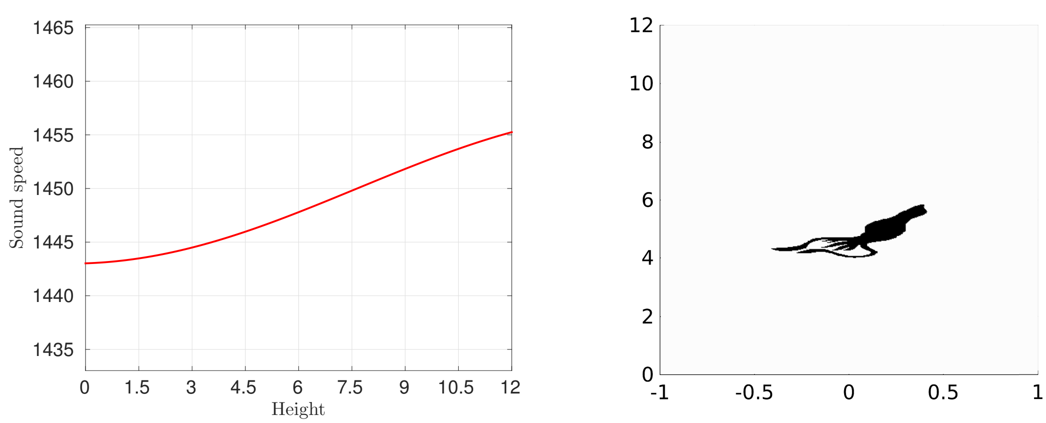
Assuming that $\underline{a}_{\boldsymbol{N}}$ possesses $L \in \mathbb{N}$ non-zero entries, we reduce the memory requirements for the GMRES algorithm by introducing the restriction operator $R_{\boldsymbol{N}} : u_{\boldsymbol{N}} \mapsto v_L$ which maps the non-zero entries of $u_{\boldsymbol{N}}$ to a vector v_L with dimension $L \in \mathbb{N}$. We point out that the dimension of the vector v_L depends on the number of non-zero entries of $a_{\boldsymbol{N}}$. Furthermore, we denote its right inverse by $R^{-1}_{\boldsymbol{N}} : v_L \mapsto u_{\boldsymbol{N}}$, which roughly speaking recovers the full matrix containing non-zero entries, such that $R_{\boldsymbol{N}} \circ R^{-1}_{\boldsymbol{N}} v_L = v_L$. To do numerical computations, we need to save an array of dimension $\boldsymbol{N}$ containing the position of the non-zero entries, to recover the full matrix. Applying this operator to the first option of the collocation discretization in (6.5), then yields the following operator evaluation,

$$v_{\boldsymbol{L}} \mapsto R_{\boldsymbol{N}} \left(F^{-1}_{\boldsymbol{N}} \left[\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} \bullet F_{\boldsymbol{N}} \left[\underline{a}_{\boldsymbol{N}} \bullet R^{-1}_{\boldsymbol{N}}(v_{\boldsymbol{L}}) \right] \right] \right). \tag{6.9}$$

6.3 Numerical Computations

In this section, using the iterative solver GMRES, we now consider in dimension two a numerical example for the collocation discretization (6.5),

$$\underline{u}_{\boldsymbol{N}} - \omega^2 F^{-1}_{\boldsymbol{N}} \left[\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} \bullet F_{\boldsymbol{N}} \left[\underline{a}_{\boldsymbol{N}} \bullet \underline{u}_{\boldsymbol{N}} \right] \right] = F^{-1}_{\boldsymbol{N}} \left[\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} \bullet F_{\boldsymbol{N}} \left[\underline{a}_{\boldsymbol{N}} \bullet \underline{f}_{\boldsymbol{N}} \right] \right].$$



(a) (b)

Figure 6.1: (a) Continuous sound speed profile depending on the depth of the ocean with height $H = 12$. (b) Position of the obstacle D (with form of a squid) with contrast $q = 1$ in the middle of the waveguide.

We first recall that the point evaluations of $x \mapsto a(x) := q(x)/c^2(x_2)$ are denoted by $\underline{a}_{\boldsymbol{N}} \in \mathbb{C}^2_{\boldsymbol{N}}$. We further recall that the kernel coefficient $\hat{\mathcal{V}}_{\rho,\boldsymbol{N}}$ for dimension two is defined in Lemma 4.3.2. We suppose the continuous sound speed from Figure 6.1 a) on $[0, H]$ for an ocean with height $H = 12$ and we assume the angular frequency $\omega = 500$.

In our computations, for $N_1 = 2^9$ the transfer operator in horizontal variable $F^{(\sim)}_{N_1}$ is a 1D FFT and the transfer operator $(F^{(\sim)}_{N_1})^{-1}$ is a 1D IFFT. We further approximate the transfer operator in the vertical variable $F^{\perp}_{N_2}$ by the matrix Φ_{N_2}, with approximated eigenvectors $(\boldsymbol{\phi}_{\ell_2})_{\ell_2=1,\dots,N_2}$ as columns. Using a uniform mesh with discretization parameter $N_2 = 2^9$, the eigenvalues and eigenvectors are approximated by a spectral method discussed in Chapter 2.3. To economize computation time, these eigenvalues and eigenvectors are pre-computed. In this computation the matrix Φ_{N_2} is rectangular and has full rank. For $\rho = 1$, we place the connected obstacle D (with form of a squid) with contrast $q = 1$ on D and 0 outside, i.e. see Figure 6.1 b) in the middle of the ocean, where $D \subset \Lambda_{\rho/2}$ holds.

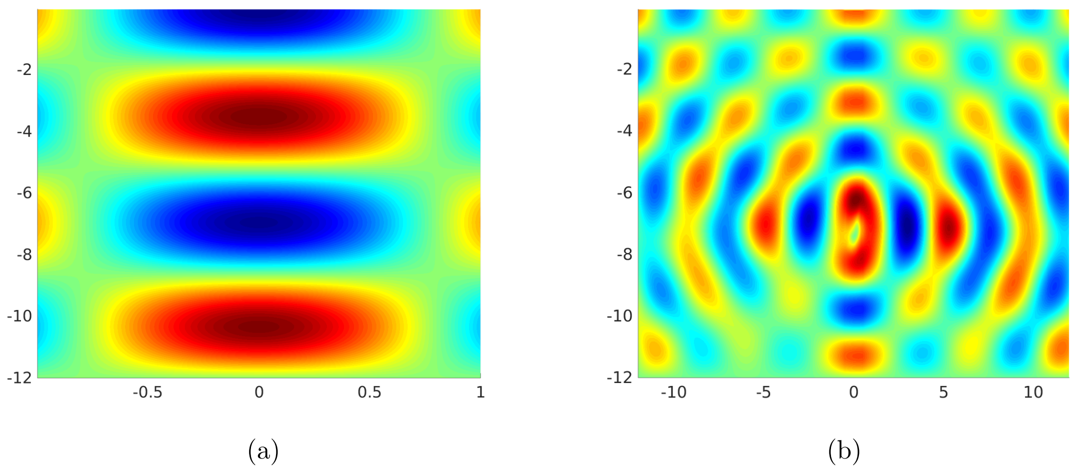


Figure 6.2: (a) Plane wave of mode 4 on $[-1, 1] \times [0, 12]$. (b) Extended scattered field u^s.

Using the approximated eigenvalue λ_4 and the approximated eigenvector $\phi^*_{j_4,2^9}$, we obtain the incident field which is a plane wave in direction $(1, 0)$, i.e. see Figure 6.2 a). We stop the GMRES iteration when the relative residual is less than 10^{-8}. Then, the computation time of the scattered field on a i7 Quad-Core with 2,67 GHz of each core takes 2.53 seconds. Furthermore, we extend the total field u on $\Lambda_{\rho/2}$ by

$$u|_{\Lambda_{\rho/2}} = \mathcal{V}_{\rho,\boldsymbol{N}} \left[\underline{a}_{\boldsymbol{N}} \bullet (\underline{u}^s_{\boldsymbol{N}} + \underline{u}^i_{\boldsymbol{N}}) \right],$$

where $\underline{u}^s_{\boldsymbol{N}}$ is the computed scattered field. Moreover, we use the truncated Green's function to extend the total field outside $\Lambda_{\rho/2}$,

$$u^s_{\boldsymbol{N}} \left(x_1, H\frac{k_2}{N_2} \right) = \sum_{j=1}^{N_2} \hat{\hat{u}}^{s+}_{\boldsymbol{N}}(j)\, \boldsymbol{\phi}_j(k_2) \exp\left(-i\lambda_j \left[x_1 + \frac{\rho}{2} \right] \right), \quad x_1 < -\rho/2, k_2 = 1, \dots, N_2 \text{ and}$$

$$u^s_{\boldsymbol{N}} \left(x_1, H\frac{k_2}{N_2} \right) = \sum_{j=1}^{N_2} \hat{\hat{u}}^{s,-}_{\boldsymbol{N}}(j)\, \boldsymbol{\phi}_j(k_2) \exp\left(i\lambda_j \left[x_1 - \frac{\rho}{2} \right] \right), \quad x_1 > \rho/2, k_2 = 1, \dots, N_2,$$

where $\hat{\hat{u}}^{s,\pm}_{\boldsymbol{N}}$ denotes the Fourier coefficients of the restriction of the computed scattered field on $\Lambda_{\rho/2}$ to $\{x_1 = \pm\rho/2\}$. We point out that the grid in our computation in horizontal variable is chosen such that $x_1 = -\rho/2$ corresponds exactly to the grid points $x^{(\boldsymbol{N})}_j$ with $j_1 = N_1/4$ and $x_1 = \rho/2$ corresponds exactly to the grid points with $j_1 = 3N_1/4$.

Figure 6.2 b) shows the extended scattered field. This extension takes 56.76 seconds on the same computation environment, due to the large number of required evaluations of the waveguide's

Green's function. If we use the scheme with restriction operator R in (6.9) for this example, the computations takes longer since the restriction operation R and the recover operation R^{-1} create more overhead then evaluating the scheme (6.5) acting on the matrix with a couple of zero entries. For example, if we set in the latter computation a uniform mesh with discretization parameter $N_2 = 2^{10}$, the scheme with restriction operator R takes 46.2 seconds instead of 47.8 seconds of evaluating scheme (6.5).

6.4 Optimized Vertical Transform for Small Obstacles

In this section, we present an optimization of the vertical transform process $F^{\perp}_{N_m}$. for the Vainikko scheme (6.5), when the height of the obstacle D is small compared to the height of the ocean. In underwater sound propagation the situation that objects emitting or scattering sound in an ocean are much smaller than the ocean depth appears frequently. This optimized scheme makes high-resolution 3D computations with a small scattering object compared to the depth of a deep ocean possible.

To abbreviate the notation, we assume $m = 2$, however, the optimization scheme applies for $m = 3$, too. We first recall that the point evaluations of $x \mapsto a(x) := q(x)/c^2(x_2)$ are denoted by $\underline{a}_{\boldsymbol{N}} \in \mathbb{C}^2_{\boldsymbol{N}}$. We moreover recall the discretized integral operator used in (6.5),

$$f_{\boldsymbol{N}} \mapsto \mathcal{V}_\rho(\underline{a}_{\boldsymbol{N}} \bullet \underline{f}_{\boldsymbol{N}}) = F^{-1}_{\boldsymbol{N}} \left[\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} \bullet F_{\boldsymbol{N}} \left[\underline{a}_{\boldsymbol{N}} \bullet \underline{f}_{\boldsymbol{N}} \right] \right]. \tag{6.10}$$

By assumption, the scattering object D satisfies $D \subset \Lambda_{\rho/2}$. Consequently, the horizontal support of the contrast is bounded by $\pm\rho/2$, however, the vertical component needs still the whole discretization of the interval $[0, H]$. If the scattering object $D \subset \Lambda_\rho$ is small, the point wise multiplication with $\underline{a}_{\boldsymbol{N}}$ creates a sparse vector for the argument of the discretized integral operator $\mathcal{V}_\rho$ in (6.10). Roughly speaking, for an ocean with $H = 200$ and a scattering object with height of two, situated in the middle of the ocean, $\underline{a}_{\boldsymbol{N}} \bullet \underline{f}_{\boldsymbol{N}}$ needs the discretization of $[0, H]$, however, $\underline{a}_{\boldsymbol{N}} \bullet \underline{f}_{\boldsymbol{N}}$ is sparse. Consequently, for deep oceans the evaluation of the discretized integral operator and the solution of the discretized Lippmann-Schwinger-Integral equation (6.5) is very expensive. Owing to the fact that we are interested only in the domain where scattering occurs, we use that the contrast q is zero outside the scattering object. Recall that the transform $F_{\boldsymbol{N}}$ can be separated into a horizontal and a vertical component, $F^{(\perp)}_{\boldsymbol{N}_m} : \mathbb{C}^{N_m} \to \mathbb{C}^{N_m}$ and $F^{(\sim)}_{\tilde{\boldsymbol{N}}} : \mathbb{C}^{\tilde{N}} \to \mathbb{C}^{\tilde{N}}$. Consequently, we have

$$F_{\boldsymbol{N}}[\underline{a}_{\boldsymbol{N}} \bullet \underline{f}_{\boldsymbol{N}}] = F^{(\sim)}_{N_1} F^{(\perp)}_{N_2} [\underline{a}_{\boldsymbol{N}} \bullet \underline{f}_{\boldsymbol{N}}] = F^{(\sim)}_{N_1} F^{(\perp)}_{N_2} [\underline{v}_{\boldsymbol{N}}], \qquad \text{for } \underline{v}_{\boldsymbol{N}} = \underline{a}_{\boldsymbol{N}} \bullet \underline{f}_{\boldsymbol{N}}.$$

Here the horizontal transform, denoted by $F^{(\sim)}_{N_2}$, can be evaluated by the fast Fourier transform, so it is not focus of this section. We see now that one column of $\underline{v}_{\boldsymbol{N}}$ has the representation

$$v_{N_2} = (0, ..., 0, v_{\alpha_1}, v_{\alpha_1+1}, ..., v_{\alpha_2}, 0, ..., 0)^T \in \mathbb{C}^{N_2}, \tag{6.11}$$

where $v_{\alpha_1} \neq 0$ denotes the first non-zero entry and $v_{\alpha_2} \neq 0$ the last non-zero of v_{N_2}. Note that the indices α_1 and α_2 correspond to the position of the discretized scattering object in Λ_ρ. We recall first the matrix Φ_{N_2} with discretized eigenvectors $\boldsymbol{\phi}^{(N_2)}_j$. Again, we write for simplicity $\boldsymbol{\phi}_j$, where $j = 1, ..., N_2$. Then

$$\Phi_{N_2} = [\boldsymbol{\phi}_1 \boldsymbol{\phi}_2 \cdots \boldsymbol{\phi}_{N_2}],$$

and we denote its transpose by A. By Lemma 5.1.2 the matrix Φ_{N_2} is invertible and if we use all vectors for the vertical transform, than A has full rank and its inverse can computed, such that $I_{N_2 \times N_2} = A^{-1}A$, where $I_{N_2 \times N_2}$ denotes the identity matrix with dimension $N_2 \times N_2$. In the following we write $B = A^{-1}$.

First, we assume that the scattering object D is situated in the middle of the ocean and we use all Fourier coefficients. Based on the separation of vertical and horizontal transform, for one column of $\underline{v}_{\boldsymbol{N}}$ we see that

$$F^{(\perp)}_{\boldsymbol{N_2}}[v_{N_2}] = \begin{bmatrix} \phi_1\left(H\frac{1}{N_2}\right) & \dots & \phi_1(H) \\ \phi_2\left(H\frac{1}{N_2}\right) & \dots & \phi_2(H) \\ \vdots & & \vdots \\ \phi_M\left(H\frac{1}{N_2}\right) & \dots & \phi_M(H) \end{bmatrix} \begin{bmatrix} 0_{\alpha_1} \\ v_{\alpha_1} \\ \vdots \\ v_{\alpha_2} \\ 0_{(N_2-\alpha_2)} \end{bmatrix} = \begin{bmatrix} \phi_1\left(H\frac{\alpha_1}{N_2}\right)v_{\alpha_1} + \ldots + \phi_1\left(H\frac{\alpha_2}{N_2}\right)v_{\alpha_2} \\ \vdots \\ \phi_M\left(H\frac{\alpha_1}{N_2}\right)v_{\alpha_1} + \ldots + \phi_M\left(H\frac{\alpha_2}{N_2}\right)v_{\alpha_2} \end{bmatrix},$$

where 0_{α_j} denotes a zero vector with length $j \in \mathbb{N}$. We see that this transform economizes $M(\alpha_1 + N_2 - \alpha_2)$ multiplications and summations compared to the full transform. Moreover, we define the block matrices,

$$B_{11} \in \mathbb{R}^{(\alpha_2-\alpha_1+1)\times N_2}, A_{11} \in \mathbb{R}^{N_2\times\alpha_1-1}, A_{12} \in \mathbb{R}^{N_2\times(\alpha_2-\alpha_1+1)}, A_{13} \in \mathbb{R}^{N_2\times N_2-\alpha_2}.$$

More precisely, A_{11} contains the first α_1 columns of A, A_{13} contains the last $N_2 - \alpha_2$ columns of A. A_{12} contains the α_1st, $\alpha_1 + 1$st,... and so on up to the α_2nd column of A. The matrix B_{11} is determined by the linear system

$$\begin{bmatrix} 0_{(\alpha_1-1)\times N_2} \\ B_{11} \\ 0_{(N_2-\alpha_2)\times N_2} \end{bmatrix} \begin{bmatrix} A_{11} & A_{12} & A_{13} \end{bmatrix} = \begin{bmatrix} 0_{(\alpha_2-\alpha_1+1)\times(\alpha_1-1)} \\ I_{(\alpha_2-\alpha_1+1)\times(\alpha_2-\alpha_1+1)} \\ 0_{(\alpha_2-\alpha_1+1)\times(N_2-\alpha_2)} \end{bmatrix}^T, \tag{6.12}$$

where $0_{\alpha_1\times N_2}$ denotes an zero matrix with dimension $\alpha_1 \times N_2$.

Roughly speaking, the dimension of A_{11}, A_{13} depends on the zero entries of $\underline{v}_{\boldsymbol{N}}$ and the dimension of A_{12}, B_{11} depends on the non-zero entries of $\underline{v}_{\boldsymbol{N}}$. For full rank of A, since A contains a orthogonal basis, the orthogonality is transfered to the vectors in B, see equation (5.3). If A has not full rank, we use a combination of identity matrix and zero block matrix, given in the right-hand side in the last equation, to deduce orthogonality of the columns of B.

We characterize now the reduced scheme of the discretized Lippmann-Schwinger equation (6.5). For simplicity, we denote for $\boldsymbol{N} \in \mathbb{Z}^2_+$ and $N_2 > \alpha_2 > \alpha_1 > 1$ the vector $\boldsymbol{L} = (N_1, \alpha_2 - \alpha_1 + 1)$. In consequence, there exists nodal values $\underline{u}_{\boldsymbol{L}}$ characterized by

$$\underline{u}_{\boldsymbol{N}} = \left(0_{N_1\times(\alpha_1-1)}, \underline{u}_{\boldsymbol{L}}, 0_{N_1\times(N_2-\alpha_2)}\right).$$

In particular, we obtain the reduced scheme to evaluate the discretized integral operator on $u_{\boldsymbol{L}}$,

$$\underline{u}_{\boldsymbol{L}} \mapsto B_{11}\mathcal{F}^{-1}_{N_1}\left[\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} \bullet A_{12}\mathcal{F}_{N_1}\left[\underline{a}_{\boldsymbol{L}} \bullet \underline{f}_{\boldsymbol{L}}\right]\right]. \tag{6.13}$$

Note that we evaluate this scheme numerically by GMRES.

Let us look at the effectiveness of the cut-version of the collocation scheme that relies on the operator evaluation in (6.13), i.e. at the behavior of error and runtime and the reduction of zero data elements if $m = 2$, for a a scattering object D of height three situated away from the boundary (see Figure 6.3a). Suppose an inhomogeneous ocean of height $h = 30$, angular frequency $\omega = 500$ and continuous non-constant sound speed given in Figure 6.3b). Using $2^9 = 512$ eigenvalues and eigenvectors and a discretization of the horizontal component of 2^9 points to compute the scattered field u^s in equation (6.5). If we would use matrices A with full rank, then the computation time on a i7 Quad-Core with 2,67 GHz of each core takes 4.45 seconds.

Figure 6.4 a) shows the computation time using the optimized scheme of A and its inverse, if α_1 and α_2 approach the scattering object. Due to the error curve in Figure 6.4 b), we can easily see that the scattering object is situated between $\alpha_1 = 172$ and $\alpha_2 = 340$. Note that the computing time includes the pre-computation of the inverse B_{11} in (6.12). We point out that for a large scattering object the cut-version of the collocation scheme and the pre-computation of the inverse B_{11} take longer than using the full matrices Φ_{N_2} and $\Phi^{-1}_{N_2}$. The diamond in Figure 6.4 a)

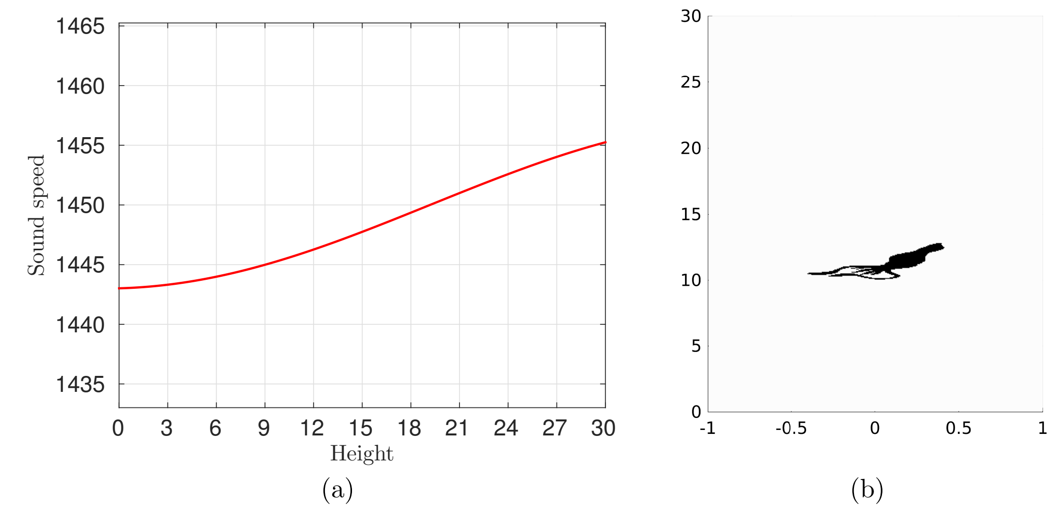


Figure 6.3: (a) Continuous sound speed profile depending on the depth of the ocean. (b) Position of a squid as scattering object D.

denotes the last time value, where the full rank matrices method is faster than the cut scheme. Moreover, we can extend the total field u on $\Lambda_{\rho/2}$ by

$$u|_{\Lambda_{\rho/2}} = \mathcal{V}_{\rho/2}\left[\underline{a}_{\boldsymbol{N}} \bullet (\underline{u}^s_{\boldsymbol{N}} + \underline{u}^i_{\boldsymbol{N}})\right],$$

where $\underline{u}^s_{\boldsymbol{N}}$ is the scattered field computed by the cut scheme. We see in Figure 6.4 b) the relative L^2-error of the extended total field to the total field computed by A with full rank.

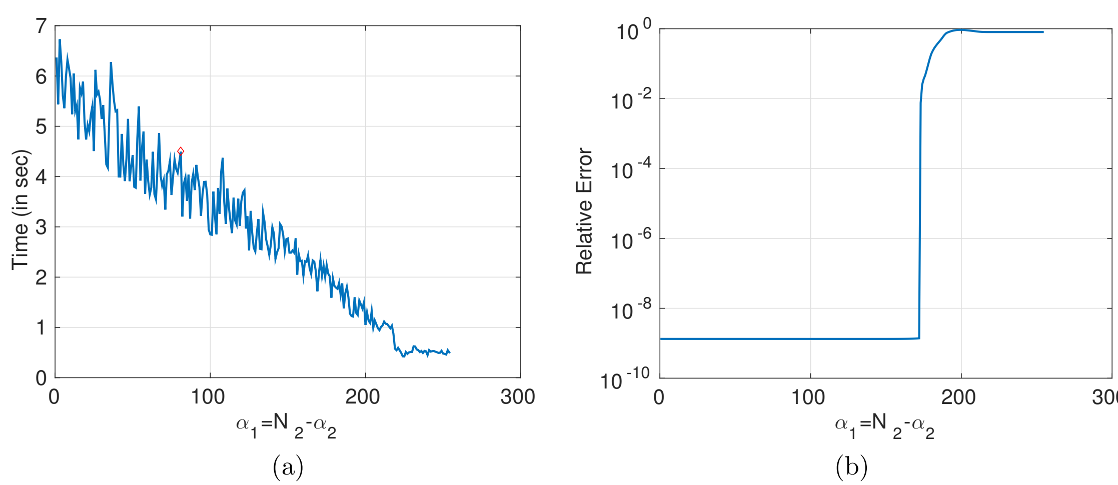


Figure 6.4: (a) Computation time of the total field $u|_{\Lambda_{\rho/2}}$ using cut-version. The red diamond denotes the last value where the full rank matrices method is faster. (b) Relative L^2-error of the total field $u|_{\Lambda_{\rho/2}}$ using cut-version to the non-cut-version.

We point out that, an alternative cut scheme for the vertical transformation, using not all eigenfunctions (corresponding to evanescent modes) has no significant advantage to economize computation costs.

6.5 Convergence of the Discretized Integral Operator

In this section we want to check convergence of the collocation discretization of the periodized integral operator from (5.22) by testing whether this operator satisfies the Helmholtz equation.

We assume that $H = 30$, $\rho = 2$, $\omega/c = 1$. In particular, we have 10 propagating waveguide modes. We choose as contrast the cut-off function acting on the full height and satisfies

$$q(x) = \begin{cases} 0 & \text{for } x_1 \leq -1/2 \\ 4x_1 + 2 & \text{for } -1/2 < x_1 < -1/4 \\ 1 & \text{for } -1/4 \leq x_1 \leq 1/4 \\ -4x_1 + 2 & \text{for } 1/4 < x_1 < 1/2 \\ 0 & \text{for } x_1 \geq 1/2 \end{cases}$$

We denote by $q_{\boldsymbol{N}}$ the point evaluation of the contrast on the grid points $x_j^{(\boldsymbol{N})}$.

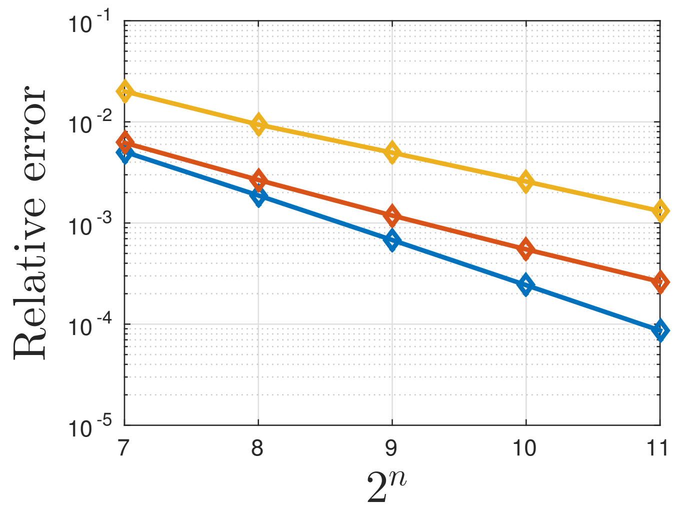


Figure 6.5: Relative L^2-error of the left-hand side and right-hand side of equation (6.14) with discretization parameter $N_2 = 2^n$ where $n = 7, 8, 9, 10, 11$ and the incident field, which is a plane wave in direction $(1, 0)$, correspond to propagating waveguide modes $j = 1$ (blue), $j = 6$ (red), evanescent mode $j = 11$ (yellow).

As the periodized integral operator satisfies the Helmholtz equation in $\Lambda_{\rho/2} = \Lambda_{1/2} = [-1/2, 1/2] \times (0, 30)$, we check numerically whether the discretized and period potential satisfies a discrete Helmholtz equation obtained by a finite difference approximation $\Delta_{\boldsymbol{N}}$ to the Laplace operator.

$$\Delta_{\boldsymbol{N}} F_{\boldsymbol{N}}^{-1} \left[\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} \bullet F_{\boldsymbol{N}} \left[\underline{q}_{\boldsymbol{N}} \bullet \underline{u^i}_{\boldsymbol{N}} \right] \right] + \underline{q}_{\boldsymbol{N}} \bullet F_{\boldsymbol{N}}^{-1} \left[\hat{\mathcal{V}}_{\rho,\boldsymbol{N}} \bullet F_{\boldsymbol{N}} \left[\underline{q}_{\boldsymbol{N}} \bullet \underline{u^i}_{\boldsymbol{N}} \right] \right] \approx \underline{a}_{\boldsymbol{N}} \bullet \underline{u^i}_{\boldsymbol{N}}, \qquad (6.14)$$

where $\underline{u^i}_{\boldsymbol{N}} \in \mathbb{C}^2_{\boldsymbol{N}}$ is the point evaluation of one of the 10 propagating waveguide modes. Using MATLAB's discrete Laplacian, we evaluate the left-hand side of the latter equation for different discretization parameter $N_1 = N_2 = 2^n$, where $n = 7, ..., 11$. For $j = 1, 6, 11$ we use the approximated eigenvalue λ_j^2 and the approximated eigenvector $\phi^*_{2^n,j}$ to obtain the incident field which is a plane wave in direction $(1, 0)$.

We compute the solution $u^s_{\boldsymbol{N}}$ to the collocation discretization (6.5) with $\underline{f}_{\boldsymbol{N}}$ replaced by $\underline{u}^s_{\boldsymbol{N}}$ and evaluate the discrete L^2-norm of

$$\Delta_{\boldsymbol{N}} \underline{u}^s_{\boldsymbol{N}} + \underline{u}^s_{\boldsymbol{N}} + \underline{q}_{\boldsymbol{N}} \bullet \underline{u}^s_{\boldsymbol{N}} + \underline{q}_{\boldsymbol{N}} \bullet \underline{u}^i_{\boldsymbol{N}}.$$

Figure 6.5 b) shows the corresponding relative L^2-error that is

$$\boldsymbol{N} \mapsto \frac{\left(\sum_{\boldsymbol{j}\in\mathbb{Z}^2_{\boldsymbol{N}}} \left|\left[\Delta_{\boldsymbol{N}} \underline{u}^s_{\boldsymbol{N}} + \underline{u}^s_{\boldsymbol{N}} + \underline{q}_{\boldsymbol{N}} \bullet \underline{u}^s_{\boldsymbol{N}} + \underline{q}_{\boldsymbol{N}} \bullet \underline{u}^i_{\boldsymbol{N}}\right]_{\boldsymbol{j}}\right|^2\right)^{1/2}}{\left(\sum_{\boldsymbol{j}\in\mathbb{Z}^2_{\boldsymbol{N}}} \left|\underline{q}_{\boldsymbol{N}} \bullet \underline{u}^i_{\boldsymbol{N}}\right|^2\right)^{1/2}}, \tag{6.15}$$

for the discretization parameter $\boldsymbol{N} = (2^n, 2^n)$ for propagating mode $j = 1$ (blue), $j = 6$ (red) and evanescent mode $j = 11$ (yellow). Moreover, for $j = 1$ we obtain the convergence rate of about 1.49, for $j = 6$ the rate of about 1.06, and for $j = 11$ the rate of about 0.97. To analyze convergence for $m = 3$, MATLAB's discrete Laplacian has to act on the whole three dimensional discretized domain Λ_ρ. For the setting with height $H = 30$, $\rho = 2$, $\omega/c = 1$, discretization parameter $N_1 = N_2 = N_3 = 2^9$ and a cylinder as contrast with diameter one, the MATLAB's discrete Laplacian computation takes about $190GB$ of memory. (For $N_1 = N_2 = N_3 = 2^{10}$ the MATLAB's discrete Laplacian produces memory overflow.) The relative L^2-error of the left-hand side and right-hand side of equation (6.15) with discretization parameter $N_1 = N_2 = N_3 = 2^9$ is about 2 percent, which is insufficient to give reasonable convergence rates of the collocation discretization (6.5) in dimension three.

Chapter 7

Combined Spectral/Multipole Method

7.1 Diagonal Approximation of the Green's function

In this section we present a combined spectral/multipole method applied to ocean acoustics with depth-dependent background sound speed in dimension three. The idea of this method is to construct an union of several domains, where each domain contains a part of the obstacle. In consequence, we avoid one large box with multiple scatterers to compute. Thus, this technique makes computations for multiple scattering objects placed over large distances in the waveguide possible. In the following we use the ideas of [LN12] and [GR87], which discovered a combined spectral/multipole method applied to an ocean with constant background sound speed.

We first introduce well-know tools from [LN12] to handle multiple scatterers in multiple boxes with adaption to the Lippmann-Schwinger on our setting.

Let $L \in \mathbb{N}$ denote the whole number of different disjoint local perturbations $D_\ell \subset \Omega$, for $\ell = 1, ..., L$. Further, let the contrast function, characterizing each D_ℓ for $\ell = 1, ..., L$ be denoted by q_ℓ and

$$q = \sum_{\ell=1}^{L} q_\ell \qquad \text{in } \Omega.$$

It moreover holds $\overline{D_\ell} := \operatorname{supp}(q_\ell)$ in Ω. For simplicity, we introduce the notation $\boldsymbol{q} = (q_1, ..., q_L)^T$ and the elementwise multiplication of two vectors $\boldsymbol{q}$ and $\boldsymbol{v}$ by $\boldsymbol{q} \bullet \boldsymbol{v} = (v_1 q_1, ..., v_L q_L)^T$.

Next, we consider for $\ell = 1, ..., L$, $\rho_\ell > 0$ and $o_\ell \in \mathbb{R}^3$, where $o_\ell = (\tilde{o}_\ell, 0)$, the cylindrical domain

$$M^{(\ell)} := M_{\rho_\ell/2} + o_\ell = \left\{ x = (\tilde{x}, x_3) : |\tilde{x} - \tilde{o}_\ell| < \frac{\rho_\ell}{2}, 0 < x_3 < H \right\}.$$

Furthermore, we suppose for $\ell = 1, ..., L$, $\rho_\ell > 0$ and $o_\ell \in \mathbb{R}^3$ the rectangular domain

$$\Lambda^{(\ell)} := \Lambda_{\rho_\ell} + o_\ell.$$

We point out that ρ_ℓ is chosen to be large enough that $\overline{D_\ell} \subset M^{(\ell)} \subset \Lambda^{(\ell)} \subset \Omega$ holds. Moreover, our standing assumption from now on is that all domains $M^{(\ell)}$ satisfy that

$$\delta_{\min} := \inf_{x_k \in M^{(k)}, x_\ell \in M^{(\ell)}, 1 \leq k \neq \ell \leq L} |\tilde{x}_k - \tilde{x}_\ell| > 0$$

such that the $M^{(\ell)}$ are in particular disjoint. We also introduce

$$\delta_{\max} := \sup_{x_k \in M^{(k)}, x_\ell \in M^{(\ell)}, 1 \leq k \neq \ell \leq L} |\tilde{x}_k - \tilde{x}_\ell| > 0.$$

However, the domains $\Lambda^{(\ell)}$ are not necessarily disjoint. Due to the fact that $\overline{D_\ell} \subset M^{(\ell)}$, there are cut-off functions $\chi_\ell^* \in C_0^\infty(\Lambda^{(\ell)})$ such that

$$\chi_\ell^* = \begin{cases} 1 & \text{in } \overline{D_\ell}, \\ 0 & \text{in } \Lambda^{(\ell)} \backslash M^{(\ell)}, \end{cases}$$

and $0 \leq \chi_\ell^* \leq 1$ in $M^{(\ell)} \backslash \overline{D_\ell}$. (These cut-off functions serve to separate expressions on different domains D_ℓ and have nothing to do with the cut-off function χ_ρ in Chapter 4.4). Furthermore, using this cut-off functions χ_ℓ^*, we introduce the truncation and shift operators T_ℓ^+ and T_ℓ^-,

$$\begin{aligned} T_\ell^+ &: L^2(\Lambda_{\rho_\ell}) \to L^2(\Lambda^{(\ell)}), && (T_\ell^+ u)(x) = (\chi_\ell^* u)(x - o_\ell), && x \in \Lambda^{(\ell)}, \ell = 1, ..., L, \\ T_\ell^- &: L^2(\Lambda^{(\ell)}) \to L^2(\Lambda_{\rho_\ell}), && (T_\ell^- v)(x) = (\chi_\ell^* v)(x - o_\ell), && x \in \Lambda_{\rho_\ell}, \ell = 1, ..., L. \end{aligned}$$

The last operators shift functions from Λ_{ρ_ℓ} to $\Lambda^{(\ell)}$ and reverse. For completeness, the component-wise application of the truncation operators $T_\ell^\pm$ on the vector $\boldsymbol{u} = (u_1, ..., u_L)$ with $u_\ell : \Lambda_\ell \to \mathbb{C}$ is defined as $\boldsymbol{T}^\pm \boldsymbol{u}$. Therefore, we introduce for $t \in \mathbb{R}$ the Sobolev space

$$\boldsymbol{H}^t = \bigoplus_{\ell=1}^{L} H^t(\Lambda_{\rho_\ell}),$$

with norm

$$\|\boldsymbol{u}\|_{\boldsymbol{H}^t} = \sum_{\ell=1}^{L} \|u_\ell\|_{H^t(\Lambda_{\rho_\ell})} \qquad \text{for } \boldsymbol{u} = (u_1, ..., u_L).$$

We point out that for $t = 0$ there holds $\boldsymbol{H}^0 = \bigoplus_{\ell=1}^{L} L^2(\Lambda_{\rho_\ell})$, which we denote by $\boldsymbol{H}$.

Lemma 7.1.1. *For arbitrary $s, t \in \mathbb{R}$ the integral operator $\mathcal{K}_{kl} : L^2(\Lambda_{\rho_k}) \to L^2(\Lambda_{\rho_\ell})$, defined by*

$$\mathcal{K}^{k\ell} u = \chi_k^* \int_{\Lambda^{(k)}} G(\cdot + o_k, y)(\chi_\ell^* u)(y - o_\ell)\, dy \Big|_{\Lambda_{\rho_\ell}} \qquad \text{for } k \neq \ell, \tag{7.1}$$

is continuous and for $s, r \in \mathbb{R}$ there is a constant $C > 0$ such that

$$\|\mathcal{K}^{k\ell} u\|_{H^s(\Lambda_{\rho_k})} \leq C \|u\|_{H^t(\Lambda_{\rho_\ell})}.$$

The proof of this lemma follows for $k \neq \ell$ by smoothness of the integral operator $\mathcal{K}_{kl}$. Furthermore, let $\mathcal{V}_{\rho_\ell}$ denotes for $\ell = 1, ..., L$ the integral operator related to the domain Λ_{ρ_ℓ}. Thus, we define $\boldsymbol{\mathcal{K}}$ by $\mathcal{K}^{\ell\ell} = \mathcal{V}_{\rho_\ell}$ for all $\ell = 1, ..., L$ and the off-diagonal elements of $\boldsymbol{\mathcal{K}}$ by $\mathcal{K}^{k\ell}$ for $\ell, k = 1, ..., L$ and $\ell \neq k$ We know by the mapping properties of Theorem 4.5.2 that $\mathcal{V}_{\rho_\ell}$ is bounded from $H^s(\Lambda_{\rho_\ell})$ into $H^{s+2}(\Lambda_{\rho_\ell})$. Thus $\boldsymbol{\mathcal{K}}$ is bounded for $s \in \mathbb{R}$ from $\boldsymbol{H}^s$ into $\boldsymbol{H}^{s+2}$, too.

Assumption 7.1.2. *We assume for $\ell = 1, ..., L$ that each q_ℓ of the vector of the contrast $\boldsymbol{q} = (q_1, ...q_L)$ is compactly supported in $M^{(\ell)}$ and that $q_\ell / c^2 \in H^s(\Lambda^{(\ell)})$ for $s > 3/2$.*

We want now to reformulate the Lippmann-Schwinger equation (4.35),

$$u - \mathcal{V}\left(\frac{\omega^2}{c^2(x_3)} q u\right)\Big|_{\Lambda_\rho} = \mathcal{V}\left(\frac{\omega^2}{c^2(x_3)} q f\right)\Big|_{\Lambda_\rho},$$

by using the disjoint domains Λ_{ρ_ℓ}, and the vector-valued unknown

$$\boldsymbol{u} = (T_1^-(u|_{\Lambda^{(1)}}), ..., T_L^-(u|_{\Lambda^{(L)}})).$$

We first consider the source term $f \in L^2(D)$ and contrast $q \in L^\infty(D)$ and define $\boldsymbol{f} \in \boldsymbol{H}$ by

$$f_\ell = T_\ell^-(f|_{\Lambda^{(\ell)}}) \qquad \text{for } 1 \leq \ell \leq L.$$

Moreover, we define $\boldsymbol{c}^2 = (c^2|_{\Lambda^{(1)}}, ..., c^2|_{\Lambda^{(L)}})$ and write $1/\boldsymbol{c}^2$ for the vector $\big(1/c^2|_{\Lambda^{(1)}}, ..., 1/c^2|_{\Lambda^{(L)}}\big)$.

Theorem 7.1.3. *Let $\boldsymbol{f} = (T_1^-(f|_{\Lambda^{(1)}}), ..., T_L^-(f|_{\Lambda^{(L)}})) \in \boldsymbol{H}$. Then any solution $\boldsymbol{u} \in \boldsymbol{H}$ solving*

$$\boldsymbol{u} - \boldsymbol{K}\left(\boldsymbol{T}^-\left(\frac{\omega^2}{\boldsymbol{c}^2} \bullet \boldsymbol{q}\right) \bullet \boldsymbol{u}\right) = \boldsymbol{f} \qquad \text{in } \boldsymbol{H}, \tag{7.2}$$

defines a solution $u \in L^2(D)$ to (4.35) with right-hand side f by setting $u|_{D_\ell} = T_\ell^+(u_\ell)|_{D_\ell}$ for $\ell = 1, ..., L$. Any solution to (4.35) yields a solution $\boldsymbol{u}$ to (7.2) by setting $\boldsymbol{v} = (v_{D_1}, ..., v_{D_L})$ and

$$\boldsymbol{u} = \boldsymbol{K}\left(\boldsymbol{T}^-\left(\frac{\omega^2}{\boldsymbol{c}^2} \bullet \boldsymbol{q}\right) \bullet \boldsymbol{T}^-(\boldsymbol{v})\right) + \boldsymbol{f}.$$

The proof of the this theorem follows directly by the construction of the operator $\boldsymbol{K}$ and its application to Theorem 4.2.13. Furthermore, by construction of equation (7.2), uniqueness and regularity results of the Lippmann-Schwinger equation hold for the solution of (7.2), too. For numerical computation, we discretize equation (7.2) using a collocation method.

We moreover recall that $J(\omega, c, H)$ denotes the number of positive eigenvalues to the eigenvalue problem (2.14). Following this definition we know that $J + 1$ is the index of the first negative eigenvalue.

Lemma 7.1.4. *For $1 \leq k \neq \ell \leq L$ and for all $x \in M^{(k)}$, $y \in M^{(\ell)}$ there is a constant $C(\delta_{\min}, J^*) > 0$, independent of k, ℓ, such that it holds*

$$\left| G(x, y) - \frac{i}{4} \sum_{j=1}^{J^*} \phi_j(x_3)\phi_j(y_3) H_0^{(1)}(\lambda_j |\tilde{x} - \tilde{y}|) \right| \leq C \exp(-\delta_{\min} J^*), \qquad \tilde{x} \neq \tilde{y}, J + 1 \leq J^*. \tag{7.3}$$

Proof. Due to [CL05, Lemma 2.2] we know that for $z \in \mathbb{C}$, fixed $\nu \in \mathbb{N}_0$, $0 < \theta \leq |z|$, where $\mathrm{Re}(z) \geq 0$ and $\mathrm{Im}(z) \geq 0$, it holds

$$|H_\nu^{(1)}(z)| \leq \exp\left(-Im(z)\left(1 - \frac{\theta^2}{|z|^2}\right)^{1/2}\right) |H_\nu^{(1)}(\theta)|.$$

Consequently, we deduce for $\nu \in \mathbb{N}_0$ and $j > J$,

$$|H_\nu^{(1)}(\lambda_j |\tilde{x} - \tilde{y}|)| \leq \exp\left(-Im(\lambda_j)|\tilde{x} - \tilde{y}|\left(1 - \left|\frac{\lambda_{J+1}}{\lambda_j}\right|^2\right)^{1/2}\right) |H_\nu^{(1)}(|\lambda_{J+1}||\tilde{x} - \tilde{y}|)|. \tag{7.4}$$

We moreover see, by Lemma 2.2.4 a) that for a constant $c_0 > 0$ and for $j > J + 1$ yields $c_0 j \leq Im(\lambda_j)$. Thus, it holds

$$\begin{aligned} \exp\left(-Im(\lambda_j)|\tilde{x} - \tilde{y}|\left(1 - \left|\frac{\lambda_{J+1}}{\lambda_j}\right|^2\right)^{1/2}\right) &\leq \exp\left(-Im(\lambda_j)\delta_{min}\left(1 - \left|\frac{\lambda_{J+1}}{\lambda_j}\right|\right)\right) \\ &\leq \exp(-Im(\lambda_j)\delta_{min}) \exp(\lambda_{J+1}\delta_{min}) \\ &\leq C \exp(-\delta_{min} j), \end{aligned} \tag{7.5}$$

where $C > 0$ is independent of j. Next, the collection [AS64, Equation 9.6.24] and [Wat66, p.441-444)] give for $z > 0$, where $Re(z) > 0$ and for $\nu \in \mathbb{N}_0$, that

$$|H_\nu^{(1)}(z)|^2 = Y_\nu^2(z) + J_\nu^2(z) = \frac{8}{\pi^2} \int_0^\infty K_0(2z sinh(t)) cosh(2\nu t) dt, \tag{7.6}$$

and we further obtain that $|H_\nu^{(1)}(z)|$ is monotonically decreasing in $z > 0$ for $\nu > 0$. Plugging this and estimate (7.4-7.5) together, we have for $\delta_{min} > 0$ that

$$|H_0^{(1)}(\lambda_j |\tilde{x} - \tilde{y}|)| \leq C \exp(-\delta_{min} j) |H_0^{(1)}(|\lambda_{J+1}|\delta_{min})| \qquad \text{for } j > J,$$

where $C > 0$, see estimate (7.5). Indeed the $\delta_{min} > 0$ on the right-hand side in the latter estimate holds, since for $1 \leq k \neq \ell \leq L$ for all $x \in M^{(k)}$, $y \in M^{(\ell)}$ the difference $\tilde{x} - \tilde{y}$ is bounded. Next we recall that due to Lemma 2.2.4 b) it follows for eigenvector $\phi_j \in H^1([0,H])$ that it holds

$$\max_{x_3 \in [0,H]} \phi_j(x_3) \leq C(H) \qquad \text{for } j \in \mathbb{N}.$$

Thus, we obtain for $J^* > J$ that

$$\begin{aligned}\left| \sum_{j=J^*+1}^{\infty} \phi_j(x_3)\phi_j(y_3) H_0^{(1)}(\lambda_j|\tilde{x} - \tilde{y}|) \right| &\leq \sum_{j=J^*+1}^{\infty} \left| H_0^{(1)}(\lambda_j|\tilde{x} - \tilde{y}|) \right| \\ &\leq C \sum_{j=J^*+1}^{\infty} \exp(-\delta_{min} j) \\ &\leq C \exp(-\delta_{min} J^*) \frac{\exp(-\delta_{min})}{1 - \exp(-\delta_{min})}.\end{aligned}$$

This ends the proof. □

Next, we denote the vector $\tilde{x} \in \mathbb{R}^2$ in cylindrical coordinates as

$$\tilde{x} = \begin{pmatrix} r_x \cos(\varphi_x) \\ r_x \sin(\varphi_x) \end{pmatrix}.$$

Moreover, we define for $x \in M^{(k)}$ and $y \in M^{(\ell)}$ the numbers $\varphi_{k\ell}$ and $r_{k\ell}$ by

$$o_k - o_\ell = \begin{pmatrix} r_{k\ell} \cos(\varphi_{k\ell}) \\ r_{k\ell} \sin(\varphi_{k\ell}) \\ 0 \end{pmatrix}, \qquad 1 \leq k \neq \ell \leq L. \tag{7.7}$$

Furthermore, we introduce for $x \in M^{(k)}$ and $y \in M^{(\ell)}$ functions $r = r(x,y)$ and $\varphi = \varphi(x,y)$ such that

$$(\tilde{y} - \tilde{o}_k) - (\tilde{x} - \tilde{o}_\ell) = \begin{pmatrix} r(x,y) \cos(\varphi(x,y)) \\ r(x,y) \sin(\varphi(x,y)) \end{pmatrix}.$$

(The dependence of these functions on o_k and o_ℓ is suppressed, for simplicity.)

Definition 7.1.5. *For $1 \leq n \leq 2N+1$, $N \in \mathbb{N}$ we define*

$$\begin{aligned} f_n^{\pm}(\tilde{x}, \lambda) &:= \exp\left(\pm i \lambda r_x \cos\left(\frac{2\pi n}{2N+1} - \varphi_x \right) \right), \\ s_n(\tilde{x}, \lambda) &:= \frac{1}{2N+1} \sum_{\nu=-N}^{N} (-i)^{\nu} H_{\nu}^{(1)}(\lambda r_x) \exp\left(i\nu \left(\varphi_x - \frac{2\pi n}{2N+1} \right) \right). \end{aligned}$$

[AP99] and [BH08] discussed a multipole expansion for fundamental solution to the 2D Helmholtz equation for real eigenvalues. [LN12] adapted this idea for complex eigenvalues and an ocean with constant sound speed. Based on these ideas, we introduce now a multipole expansion for an ocean with depth depend background sound speed. The idea of the proof of the following corollary is rather similar to [AP99, Theorem 3.1] and [LN12, Proposition 6.4].

Corollary 7.1.6. *We consider $j \in \mathbb{N}$ and for $1 \leq k \neq \ell \leq L$ that $x \in M^{(k)}$ and $y \in M^{(\ell)}$. We moreover chose $\alpha = \alpha(n) \in [-N, N]$ be such that $\alpha \equiv n(\mod 2N+1)$, where $n \in \mathbb{Z}$ and $N \in \mathbb{N}$. Consequently, for all $j \in \mathbb{N}$ it holds*

$$\begin{aligned} H_0^{(1)}(\lambda_j|\tilde{x} - \tilde{y}|) &= \sum_{n=1}^{2N+1} f_n^-(\tilde{x} - \tilde{o}_k, \lambda_j) s_n(\tilde{o}_k - \tilde{o}_\ell, \lambda_j) f_n^+(\tilde{y} - \tilde{o}_\ell, \lambda_j) \\ &+ \sum_{|n|>N} J_n(\lambda_j r(x,y)) \exp(-in\varphi(x,y)) \left(H_n^{(1)}(\lambda_j r_{kl}) \exp(in\varphi_{kl}) + i^{n-\alpha} H_{\alpha}^{(1)}(\lambda_j r_{kl}) \exp(i\alpha\varphi_{kl}) \right). \end{aligned} \tag{7.8}$$

Proof. For a finite number of indices j it holds $\lambda_j > 0$ and the proof follows directly by the proof of [AP99, Theorem 3.1]. Assume now that $\lambda_j^2 < 0$. Using the modified addition theorem from Lemma A.1.3 from the Appendix,

$$H_0^{(1)}(\lambda_j|\tilde{x}-\tilde{y}|) = \sum_{\nu\in\mathbb{Z}} H_\nu^{(1)}(\lambda_j|\tilde{x}|)J_\nu(\lambda_j|\tilde{y}|)\exp(i\nu(\theta_{\tilde{x}}-\theta_{\tilde{y}})), \qquad \tilde{z} = \tilde{x}+\tilde{y}, |\tilde{x}| > |\tilde{y}|,$$

and using the splitting result

$$\tilde{x}-\tilde{y} = \tilde{o_k} - \tilde{o_\ell} + r(x,y)\begin{pmatrix}\cos(\varphi(x,y))\\ \sin(\varphi(x,y))\end{pmatrix},$$

and

$$\tilde{o_k} - \tilde{o_\ell} = |\tilde{o_k} - \tilde{o_\ell}|\begin{pmatrix}\cos(\varphi_{kl})\\ \sin(\varphi_{kl})\end{pmatrix} = r_{k\ell}\begin{pmatrix}\cos(\varphi_{kl})\\ \sin(\varphi_{kl})\end{pmatrix},$$

we see that

$$H_0^{(1)}(\lambda_j|\tilde{x}-\tilde{y}|) = \sum_{\nu\in\mathbb{Z}} H_\nu^{(1)}(\lambda_j r_{kl})J_\nu(\lambda_j r(x,y))\exp\big(i\nu[\varphi_{kl}-\varphi(x,y)]\big). \tag{7.9}$$

We now replace equation (3.2) in the proof of [AP99, Theorem 3.1] by this identity. Consequently, equation (7.8) can be now extended to complex-valued λ_j on the imaginary axis.

We point out that by a change of variables $t \mapsto t - \varphi(x,y)$ there holds

$$\begin{aligned} J_\nu(\lambda_j r(x,y)) &= \frac{1}{2\pi i^\nu}\int_0^{2\pi} \exp\Big(i[\lambda_j r(x,y)\cos(t) - \nu t]\Big)\,dt \\ &= \frac{1}{2\pi i^\nu}\int_0^{2\pi} \exp\Big(i\lambda_j r(x,y)\cos(t-\varphi(x,y))\Big)\exp\Big(-i\nu[t-\varphi(x,y)]\Big)\,dt. \end{aligned}$$

Next, we exploit

$$\begin{aligned} r(x,y)\cos(t-\varphi(x,y)) &= r(x,y)[\cos(t)\cos(\varphi(x,y)) + \sin(t)\sin(\varphi(x,y))] \\ &= (\tilde{x} - \tilde{o}_k - \tilde{y} + \tilde{o}_\ell)\cdot\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}, \end{aligned}$$

to obtain

$$J_\nu(\lambda_j r(x,y))\exp(-i\nu\varphi(x,y)) = \frac{1}{2\pi i^\nu}\int_0^{2\pi} \exp\left[i\lambda_j[\tilde{x} - \tilde{o}_k - \tilde{y} + \tilde{o}_\ell]\cdot\begin{pmatrix}\cos(t)\\ \sin(t)\end{pmatrix}\right]\exp(-i\nu t)\,dt. \tag{7.10}$$

Next, one uses the trapezoidal rule with $2N+1$ quadrature points to approximate the integral in (7.10), and explicitly computes the resulting error precisely as in [AP99, Theorem 3.1] to obtain (7.8).

□

Assumption 7.1.7. *We assume that there is $\nu \in (0,1)$ such that $r(x,y) < \nu r_{k,\ell}$ for all $x \in M^{(k)}$ and all $y \in M^{(\ell)}$ for $1 \le k \ne \ell \le L$.*

Roughly speaking, the following lemma shows that the second line of equation (7.8) is small if N is large enough. In consequence, we can use the first part of the equation (7.8) to approximate the Hankel function $H_0^1(\lambda_j|\tilde{x}-\tilde{y}|)$.

Lemma 7.1.8. *We assume that Assumption 7.1.7 holds. We chose $\alpha = \alpha(n) \in [-N,N]$ be such that $\alpha \equiv n(\mod 2N+1)$, where $n \in \mathbb{Z}$ and $N \in \mathbb{N}$. For $j \in \mathbb{N}$ there are constants $C = C(\delta_{max},\nu)$ and $N_0 = N_0(\lambda_j,\delta_{max})$ such that $x \in M^{(k)}$ and $y \in M^{(\ell)}$, $1 \le k \ne \ell \le L$, there holds*

$$\Big|\sum_{|n|>N} J_n(\lambda_j r(x,y))\exp(-in\varphi(x,y)) \Big(H_n^{(1)}(\lambda_j r_{kl})\exp(in\varphi_{kl}) + i^{n-\alpha}H_\alpha^{(1)}(\lambda_j r_{kl})\exp(i\alpha\varphi_{kl})\Big)\Big| \le C\nu^N, \quad N \ge N_0. \tag{7.11}$$

Proof. Similar as in the last lemma, for a finite number $j \leq J$ it holds $\lambda_j > 0$, and the estimate follows directly by the proof of [AP99] and [BH08].

Assume now that $j > J$ and thus $\lambda_j^2 < 0$. Then, we follow the proof of [LN12] to obtain the estimate for imaginary λ_j. We know by equation (A.3) from the Appendix that for imaginary eigenvalues λ_j it holds

$$\pi H_n^{(1)}(\lambda_j|\tilde{x}-\tilde{y}|) = -2i \exp\left(-i\frac{n\pi}{2}\right) K_n(|\lambda_j||\tilde{x}-\tilde{y}|), \qquad \text{and}$$
$$J_n(\lambda_j r(x,y)) = \exp\left(i\frac{\pi n}{2}\right) I_n(|\lambda_j| r(x,y)).$$

Due to [AS64, equation 9.6.24], we moreover see that the modified Bessel function K_n is monotonic in n for real arguments. Plugging all this together, we compute for imaginary eigenvalues λ_j that

$$\left| \sum_{|n|>N} J_n(\lambda_j r(x,y)) \exp(-in\varphi(x,y)) \left(H_n^{(1)}(\lambda_j r_{kl}) \exp(in\varphi_{kl}) + i^{n-\alpha} H_\alpha^{(1)}(\lambda_j r_{kl}) \exp(i\alpha\varphi_{kl}) \right) \right|$$
$$\leq 2 \sum_{|n|>N} |J_n(\lambda_j r(x,y))||H_n^{(1)}(\lambda_j r_{kl})| \leq \frac{2}{\pi} \sum_{|n|>N} I_n(|\lambda_j| r(x,y)) K_n(|\lambda_j| r_{kl}).$$

We next use the identities for the Hankel function with only imaginary argument from [LN10, Theorem 1.2],

$$0 < \frac{K_{n+1}(r)}{K_n(r)} \leq \frac{\sqrt{(n+1)^2 + r^2} + (n+1)}{r} \qquad \text{for } r > 0, n \geq 0, \tag{7.12}$$

and for Bessel functions of first kind and order ν with only imaginary argument from [Nâs74] we use

$$0 < \frac{I_{n+1}}{I_n} \leq \frac{r}{n+r} \qquad \text{for } r > 0, n \geq 0. \tag{7.13}$$

We combine (7.12), (7.13) and we use estimate $r(x,y) \leq \nu r_{kl}$ to obtain

$$\begin{aligned}
\frac{I_{n+1}(r)K_{n+1}(r)}{I_n(r)K_n(r)} &\leq \frac{(\lambda_j r(x,y))(\sqrt{(n+1)^2 + |\lambda_j|^2 r_{kl}^2} + n + 1)}{(\lambda_j r(x,y) + n)|\lambda_j| r_k l} \\
&\leq \nu \frac{\sqrt{(n+1)^2 + |\lambda_j|^2 r_{kl}^2} + n + 1}{\lambda_j r(x,y) + n} \\
&\leq \left(\sqrt{\left(\frac{N+2}{N+1}\right)^2 + \frac{|\lambda_j|^2 \delta_{max}}{N+1}} + \frac{N+2}{N+1} \right) := \nu_N.
\end{aligned}$$

Due to the fact that we supposed $0 < \nu < 1/2$, setting $N_0 = N_0(\lambda_j, \delta_{max})$, we see $\nu_{N_0} = \sup_{N \geq N_0} \nu_N < 1$. Consequently, we have for $N \geq N_0$ and for $|n| > N$ the bound of the sum

$$\begin{aligned}
\sum_{|n|>N} I_n(|\lambda_j| r(x,y)) K_n(|\lambda_j| r_{kl}) &\leq I_{N+1}(|\lambda_j| r(x,y)) K_{N+1}(|\lambda_j| r_{kl}) \sum_{n>N} \nu_N^{n-N-1} \\
&\leq \frac{1}{1-\nu_N} I_{N+1}(|\lambda_j| r(x,y)) K_{N+1}(|\lambda_j| r_{kl}).
\end{aligned} \tag{7.14}$$

Due to [AS64, equation 9.3.1] we now can estimate in the latter estimate the part of K_N and I_N,

$$\begin{aligned}
I_{N+1}(|\lambda_j| r(x,y)) K_{N+1}(|\lambda_j| r_{kl}) &\leq |J_{N+1}(|\lambda_j| r(x,y))||H_{N+1}^{(1)}(|\lambda_j| r_{kl})| \\
&\leq \frac{C}{N+1} \left(\frac{\exp(1)|\lambda_j| r(x,y)}{2N+2} \right)^{N+1} \left(\frac{\exp(1)|\lambda_j| r_{k\ell}}{2N+2} \right)^{-N-1} \\
&= \frac{C}{N+1} \left(\frac{r(x,y)}{r_{k\ell}} \right)^{N+1} \leq \frac{C}{\pi} \frac{\nu^{N+1}}{N+1}.
\end{aligned} \tag{7.15}$$

To this end, we plug (7.11), (7.14) and (7.15) together. This ends the proof. □

Due to Lemma 7.1.4 and Lemma 7.1.8 we can give now an error estimate of the truncated series of the Green's function in dimension three,

$$G_{J^*,N}(x,y) = \frac{i}{4}\sum_{j=1}^{J^*}\phi_j(x_3)\phi_j(y_3)\sum_{n=1}^{2N+1} f_n^-(\tilde{x}-o_k,\lambda_j)s_n(o_k-o_\ell,\lambda_j)f_n^+(\tilde{y}-o_\ell,\lambda_j), \tag{7.16}$$

where $x \in M^{(k)}$ and $y \in M^{(\ell)}$.

Corollary 7.1.9. *We assume that Assumption 7.1.7 holds, which defines the constant $\nu \in (0,1)$. Then it exists a constant $C = C(\delta_{min}, \delta_{max}, J^*)$ such that*

$$|G(x,y) - G_{J^*,N}(x,y)| \le C\big(\exp(-\delta_{min}J^*) + J^*\nu^N\big) \qquad J^* > J, N \ge N_0(k_{J^*}, \delta_{max}),$$

for all $x \in M^{(k)}$, $y \in M^{(\ell)}$, $1 \le k \ne \ell \le L$.

Proof. Due to Lemma 7.1.4, we first see the exponential convergence of the truncated series to the Green's function. Then Corollary 7.1.6 yields the representation of the Hankel function and Lemma 7.1.8 shows an estimate for the truncated representation by $J^*\nu^N$. All together, the three lemmas show the proof. □

Before, we give the corresponding corollary for the error estimates of of the partial derivative to $G(x,y) - G_{J^*,N}(x,y)$, we further need estimates for the partial derivative of the eigenfunctions.

Lemma 7.1.10. *Consider $\beta_m \in \mathbb{N}$ such that $\beta_m > 2$. Then for $c \in C^{\beta_m - 2}([0,H])$ there is a constant $c_0 > 0$ such that*

$$\|\phi_j^{(\beta_m)}\|_{L^2([0,H])} \le c_0 j^{\beta} \qquad \text{for all } j \in \mathbb{N}.$$

It further holds

$$|\phi_j^{(\beta_m)}(x_3)| \le c_0 j^{\beta_m+1} \qquad \text{almost everywhere in } (0,H).$$

Proof. Due to the Helmholtz equation (2.14) for $\beta_m = 2$, we see

$$\phi_j'' = \left(\lambda_j^2 - \frac{\omega^2}{c^2(x_3)}\right)\phi_j \qquad \text{almost everywhere in } (0,H).$$

Similar, like in the proof of Lemma 2.2.4 b) we have for all j such that $\lambda_j \ne \omega^2/c_+^2$, that

$$\begin{aligned}
\|\phi_j''\|_{L^2([0,H])} &= \left\|\left(\lambda_j^2 - \frac{\omega^2}{c^2(x_3)}\right)\phi_j\right\|_{L^2([0,H])} \\
&\le \left|\left(\lambda_j^2 - \frac{\omega^2}{c_+^2}\right)\right| \|\phi_j\|_{L^2([0,H])} \\
&\le \lambda_j^2 + \frac{\omega^2}{c_+^2} \le \frac{\pi^2(2j-1)^2}{4H^2} + \omega^2\frac{c_+^2 - c_-^2}{c_+^2 c_-^2} \le c_0 j^2,
\end{aligned}$$

where $c_0 > 0$. Next for $\beta_m = 3$, we see that

$$(\phi_j'')' = \left(\left[\lambda_j^2 - \frac{\omega^2}{c^2(x_3)}\right]\phi_j\right)' = \left[\lambda_j^2 - \frac{\omega^2}{c^2(x_3)}\right]'\phi_j + \left[\lambda_j^2 - \frac{\omega^2}{c^2(x_3)}\right]\phi_j'.$$

In particular, using the triangle inequality, we obtain

$$\left\|\phi_j'''\right\|_{L^2([0,H])} \le \left|\lambda_j^2 + \frac{\omega^2}{c_-^2}\right| \|\phi_j'\|_{L^2([0,H])} + \left|2\omega^2\frac{c'(x_3)}{c^3(x_3)}\right| \|\phi_j\|_{L^2([0,H])}.$$

Due to the proof of Lemma 2.2.4 b) we know that there is a constant $C_0 > 0$ such that

$$\left|\lambda_j^2 + \frac{\omega^2}{c_-^2}\right| \|\phi_j'\|_{L^2([0,H])} \leq Cj^3.$$

If c is sufficient regular for $j \in \mathbb{N}$ we obtain

$$\omega^2 \left\|\frac{c'(x_3)}{c^3(x_3)}\right\|_\infty \|\phi_j\|_{L^2([0,H])} \leq Cj.$$

Thus, it holds for a constant $C > 0$ that

$$\left\|\phi_j'''\right\|_{L^2([0,H])} \leq Cj^3 \qquad \text{for } j \in \mathbb{N}.$$

Furthermore, we discover for $\beta_m > 3$ via a bootstrap argument that there is a constant $C > 0$ such that

$$\left\|\phi_j^{(\beta_m)}\right\|_{L^2([0,H])} \leq Cj^{\beta_m} \qquad \text{for } j \in \mathbb{N}.$$

This proves the first part of the claims. Finally, by the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |\phi_j^{(\beta_m)}(x_3)| &= \left|\int_0^{x_3} \phi_j^{(\beta_m+1)}(s)\, ds\right| \leq \int_0^H |\phi_j^{(\beta_m+1)}(s)|\, ds \\ &\leq \left(\int_0^H |\phi_j^{(\beta_m+1)}(s)|^2\, ds\right)^{1/2} \left(\int_0^H 1\, ds\right)^{1/2} \leq \sqrt{H}\|\phi_j^{(\beta_m+1)}\|_{L^2([0,H])} \leq C\sqrt{H}j^{\beta_m+1}. \end{aligned}$$

This finishes the proof. □

We now can give error estimates for the partial derivative of $G(x,y) - G_{J^*,N}(x,y)$.

Corollary 7.1.11. *We assume that Assumption 7.1.7 holds, which defines the constant $\nu \in (0,1)$. We moreover consider a multi-index $\beta = (\tilde{\beta}, \beta_3) = (\beta_1, \beta_2, \beta_3) \in N_0^3$ with length $|\beta|_1 = \beta_1 + \beta_2 + \beta_3$ where $\beta_1 + \beta_2 \leq 2$ and $\beta_3 \in \mathbb{N}_0$. Then it exists a constant $C = C(\delta_{min}, \delta_{max}, J^*, \nu, |\beta|_1)$ such that*

$$\left|\frac{\partial^{|\beta|_1}}{\partial x^\beta}\left(G(x,y) - G_{J^*,N}(x,y)\right)\right| \leq C\big(\exp(-\delta_{min}J^*) + J^*\nu^N\big) \qquad J^* > J, N \geq N_0(k_{J^*}, \delta_{max}),$$

for all $x \in M^{(k)}$, $y \in M^{(\ell)}$, $1 \leq k \neq \ell \leq L$.

Proof. We first bound the truncation error due to J^* and then the error obtained by neglecting the second line of (7.8) linked to N. For $1 \leq k \neq \ell \leq L$ and for all $x \in M^{(k)}$, $y \in M^{(\ell)}$, there holds that for $|\tilde{x} - \tilde{y}| > \delta_{min}$ such that the series

$$G(x,y) = \frac{i}{4}\sum_{j=1}^\infty \phi_j(x_3)\phi_j(y_3)H_0^{(1)}(\lambda_j|\tilde{x} - \tilde{y}|),$$

converges uniformly and absolutely because λ_j is purely imaginary for $j > J$. We now apply series truncation to

$$\frac{\partial^{|\beta|_1}}{\partial x^\beta}G(x,y) = \frac{i}{4}\sum_{j=1}^\infty \phi_j(y_3)\frac{\partial^{|\beta|_1}}{\partial x^\beta}\big(\phi_j(x_3)H_0^{(1)}(\lambda_j|\tilde{x} - \tilde{y}|)\big), \qquad |\tilde{x} - \tilde{y}| > \delta_{min}, \tag{7.17}$$

and estimate the remainder of the truncated series.

We first look on the vertical derivative acting on the x_3 axis. Due to Lemma 7.1.10 we have estimates for derivatives of the eigenfunctions. We moreover use the estimate (7.4) to estimate derivatives with respect to the variable x_3,

$$\left|\sum_{j=J^*}^{\infty} \frac{\partial^{\beta_3}\phi_j}{\partial x_3^{\beta_3}}(x_3)\phi_j(y_3)H_0^{(1)}(\lambda_j|\tilde{x}-\tilde{y}|)\right| \leq c_0 \left|\sum_{j=J^*}^{\infty} j^{\beta_3+1}\exp(-|Im(\lambda_j)|\delta_{min})\right| \leq C\exp(-\delta_{min}J^*).$$

We point put that we used in the last estimate the monotonicity of $|H_0^{(1)}(z)|$ in z, which follows from estimate (7.6). Next, the horizontal derivatives of $x \mapsto d(x,y) = |\tilde{x}-\tilde{y}|$ satisfy bounds of the form

$$\left|\frac{\partial^{|\tilde{\beta}|_1} d(x,y)}{\partial x^{\tilde{\beta}}}\right| \leq \frac{2}{\delta_{min}} \qquad \text{and} \qquad \left|\frac{\partial d(x,y)}{\partial x_{1,2}}\right| \leq 1 \qquad \text{for } |\tilde{\beta}|_1 \leq 2.$$

The partial derivatives in $\tilde{x}$ direction of $H_0^{(1)}(\lambda_j|\tilde{x}-\tilde{y}|)$ are represented by higher-order Hankel functions and $x \in M^{(k)}$ and $y \in M^{(\ell)}$. Due to estimate (7.4) these terms form an exponentially decaying sequence. The eigenvalues λ_j can be estimated by Lemma 2.2.4 a). All together, we see for the horizontal component that

$$\left|\sum_{j=J^*}^{\infty} \phi_j(y_3)\phi_j(x_3)\frac{\partial^{|\tilde{\beta}|_1}}{\partial \tilde{x}^{\tilde{\beta}}}\left[H_0^{(1)}(\lambda_j|\tilde{x}-\tilde{y}|)\right]\right| \leq C\exp(-\delta_{min}J^*).$$

Next, we estimate for $j = 1, ..., J^*$ the partial derivatives of the remainder terms in the second line of equation (7.8). Due to equation (7.17), for the first $j = 1, ..., J^*$ eigenvalues, we can diagonalize the partial derivative in the second line of equation (7.8),

$$\sum_{|n|>N} \frac{\partial^{|\tilde{\beta}|_1}}{\partial \tilde{x}^{\tilde{\beta}}} J_n(\lambda_j r(x,y))\exp(-in\varphi(x,y))\left(H_n^{(1)}(\lambda_j r_{kl})\exp(in\varphi_{kl}) + i^{n-\alpha}H_\alpha^{(1)}(\lambda_j r_{kl})\exp(i\alpha\varphi_{kl}\right). \tag{7.18}$$

We use the idea of [AP99, Lemma 3.2] and [LN12, Chapter 6] to bound the partial derivatives $J_n(\lambda_j r(x,y))\exp(-in\varphi(x,y))$ in terms of higher-order Bessel functions. We exploit the definition (7.7) to obtain

$$\left|\frac{\partial r(x,y)}{\partial \tilde{x}}\right| \leq 1 \qquad \text{and} \qquad \left|\frac{\partial \varphi(x,y)}{\partial \tilde{x}}\right| \leq \frac{1}{r(x,y)}.$$

This yields for $|\tilde{\beta}| \leq 2$ and $r(x,y) > 0$ that

$$\frac{\partial^{|\tilde{\beta}|_1}}{\partial \tilde{x}^{\tilde{\beta}}} r(x,y) \leq \frac{2}{r(x,y)}, \qquad \frac{\partial^{|\tilde{\beta}|_1}}{\partial \tilde{x}^{\tilde{\beta}}}\varphi(x,y) \leq \frac{3}{r(x,y)^2}.$$

To this end, using well-known tools from [AP99, Lemma 3.2] and [LN12, Chapter 6], for all $j = 1, ..., J^*$ we obtain

$$\left|\frac{\partial^{|\tilde{\beta}|_1}}{\partial \tilde{x}^{\tilde{\beta}}} J_n(\lambda_j r(x,y))\exp(-in\varphi(x,y))\right| \leq \frac{3|\lambda_j|}{2} J_{n-1}(|\lambda_j| r(x,y)) \qquad \text{for } |\tilde{\beta}| \leq 1 \text{ and}$$

$$\left|\frac{\partial^{|\tilde{\beta}|_1}}{\partial \tilde{x}^{\tilde{\beta}}} J_n(\lambda_j r(x,y))\exp(-in\varphi(x,y))\right| \leq 4|\lambda_j|^2 J_{n-2}(|\lambda_j| r(x,y)) \qquad \text{for } |\tilde{\beta}| \leq 2, n \geq 3.$$

All together we deduce the error estimates for the remainder term in (7.18). This finishes the proof. □

7.2 A Combined Spectral/Multipole Method

In this section we use now the introduced tools from the latter section to set up a combined spectral/multipole method for scattering problems involving multiple scatterers. We recall first the truncated Green's function from (7.16): For $x \in M^{(k)}$ and $y \in M^{(\ell)}$ with $1 \leq k \neq \ell \leq L$,

$$G_{J^*,N}(x,y) = \frac{i}{4}\sum_{j=1}^{J^*}\phi_j(x_3)\phi_j(y_3)\sum_{n=1}^{2N+1} f_n^-(\tilde{x}-o_k,\lambda_j)s_n(o_k-o_\ell,\lambda_j)f_n^+(\tilde{y}-o_\ell,\lambda_j).$$

Then, following the definition (7.1) in Lemma 7.1.1, we define the volume integral operator $\mathcal{K}^{k\ell}_{J^*,N}: L^2(\Lambda_{\rho_\ell}) \to L^2(\Lambda_{\rho_k})$ with kernel $G_{J^*,N}$ by

$$\mathcal{K}^{k\ell}_{J^*,N}\varphi = \left[\chi_k^*(\cdot + o_k)\int_{\Lambda_{\rho_\ell}} G_{J^*,N}(\cdot + o_k, \cdot + o_\ell)\chi_\ell^*(y+o_\ell)\varphi(y)\,dy\right]\Bigg|_{\Lambda_{\rho_k}} \qquad \text{for } 1 \leq k \neq \ell \leq L.$$

We further define for $\boldsymbol{\zeta_1}, \boldsymbol{\zeta_2} \in \mathbb{N}^3$ and $1 \leq k \neq \ell \leq L$ the operator $\mathcal{K}^{k\ell}_{J^*,N,\boldsymbol{\zeta_1},\boldsymbol{\zeta_2}}: L^2(\Lambda_{\rho_\ell}) \to L^2(\Lambda_{\rho_k})$, acting on the separable parts of the truncated Green's function $G_{J^*,N}$ and using the interpolation operator $Q_{\boldsymbol{N}}$ from Section 5.1, by

$$\begin{aligned}\mathcal{K}^{k\ell}_{J^*,N,\boldsymbol{\zeta_1},\boldsymbol{\zeta_2}}\varphi =& \frac{i}{4}\sum_{j=1}^{J*}\sum_{n=1}^{2N+1} Q_{\boldsymbol{\zeta_1}}\left[\chi_k^*(x+o_k)\phi_{j_3}(x_3)f_n^-(\tilde{x},\lambda_j)\right]s_n(o_k-o_\ell,\lambda_j) \\ &\times \int_{\Lambda_{\rho_\ell}} Q_{\boldsymbol{\zeta_2}}\left[\chi_\ell^*(y+o_\ell)\phi_{j_3}(y_3)f_n^+(\tilde{y},\lambda_j)\right](y)\varphi(y)\,dy, \qquad \text{in } L^2(\Lambda_{\rho_k}).\end{aligned} \tag{7.19}$$

We first give an error estimate for the discrete schemes for the spectral approximation of the diagonal-terms $\mathcal{K}^{k\ell}_{J^*,N}$ to $\mathcal{K}^{k\ell}$. Then, we present an error estimate for the discrete schemes for the spectral approximation of the diagonal-terms $\mathcal{K}^{k\ell}_{J^*,N,\boldsymbol{\zeta_1},\boldsymbol{\zeta_2}}$ on the separable parts to $\mathcal{K}^{k\ell}$.

Lemma 7.2.1. *We assume that Assumption 7.1.7 holds, which defines the constant $\nu \in (0,1)$. Then it exists a constant C that for all $\varphi \in H^t(\Lambda_{\rho_k})$, $t \geq 0$ such that*

$$\|\left(\mathcal{K}^{k\ell} - \mathcal{K}^{k\ell}_{J^*,N}\right)\varphi\|_{H^{t+2}(\Lambda_{\rho_k})} \leq C\big(\exp(-\delta_{min}J^*) + J^*\nu^N\big)\|\varphi\|_{H^t(\Lambda_{\rho_\ell})}, \tag{7.20}$$

for $J^* \geq J+1, N \geq N_0(\lambda_{J^*}, \delta_{max})$.

Proof. Using well-known interpolation theory for operators from [SV02], it is sufficient to show that for $t \in \mathbb{N}_0$ the estimate (7.20) holds. Due to the Cauchy-Schwarz inequality and Corollary 7.1.11 the case $t=0$ follows immediately.

We now treat the case where $t > 0$ and note that for $t \in \mathbb{N}$ there are constants $c_0 > 0$ and $c_1 > 0$ such that

$$c_0\|\varphi\|^2_{H^t(\Lambda_{\rho_\ell})} \leq \sum_{|\beta|_1 \leq t}\left\|\frac{\partial^{|\beta|_1}}{\partial x^\beta}\right\|^2_{L^2(\Lambda_{\rho_\ell})} \leq c_1\|\varphi\|^2_{H^t(\Lambda_{\rho_\ell})}.$$

Since G and $G_{J^*,N}$ depends on $\tilde{x}-\tilde{y}$ one computes for $\tilde{\beta} \in \mathbb{N}_0^2$ that

$$\frac{\partial^{|\tilde{\beta}|_1}}{\partial\tilde{x}^{\tilde{\beta}}}G(x,y) = (-1)^{\tilde{\beta}}\frac{\partial^{|\tilde{\beta}|_1}}{\partial\tilde{y}^{\tilde{\beta}}}G(x,y) \quad \text{and} \quad \frac{\partial^{|\tilde{\beta}|_1}}{\partial\tilde{x}^{\tilde{\beta}}}G_{J^*,N}(x,y) = (-1)^{\tilde{\beta}}\frac{\partial^{|\tilde{\beta}|_1}}{\partial\tilde{y}^{\tilde{\beta}}}G_{J^*,N}(x,y). \tag{7.21}$$

Therefore, let $\beta \in N_0^3$ such that $|\beta|_1 \leq t+2$ and for simplicity we write for the difference of the

Green's function and its truncation one $\mathcal{G}(x,y) = G(x,y) - G_{J^*,N}(x,y)$. Due to (7.21), we obtain

$$\frac{\partial^{|\beta|_1}}{\partial x^\beta}\left(\mathcal{K}^{k\ell} - \mathcal{K}^{k\ell}_{J^*,N}\right)\varphi = \sum_{\gamma\le\beta}\binom{\beta}{\gamma}\frac{\partial^{|\beta-\gamma|_1}}{\partial x^{\beta-\gamma}}\chi_k^*(\cdot+o_k)\int_{\Lambda_{\rho_\ell}}\frac{\partial^{|\gamma|_1}}{\partial x^\gamma}\mathcal{G}(\cdot+o_k, y+o_\ell)\chi_\ell^*(y-o_\ell)\varphi(y)\,dy$$
$$= \sum_{\gamma\le\beta}(-1)^{|\tilde\gamma|}\binom{\beta}{\gamma}\frac{\partial^{|\beta-\gamma|_1}}{\partial x^{\beta-\gamma}}\chi_k^*(\cdot+o_k)$$
$$\int_{\Lambda_{\rho_\ell}}\frac{\partial^{\gamma_3}}{\partial x_3^{\gamma_3}}\frac{\partial^{|\tilde\gamma|_1}}{\partial \tilde y^{\tilde\gamma}}\mathcal{G}(\cdot+o_k, y+o_l)\chi_\ell^*(\cdot+o_\ell)\varphi(y)\,dy.$$

Using integration by parts t times, we integrate partial derivatives acting on $\mathcal{G}$ upon φ. More precisely, we split $\tilde\gamma = \gamma_1+\gamma_2$ such that $|\gamma_1|_1 \le 2$ and $|\gamma_2|_1 \le t$. As the cut-off function χ_ℓ^* eliminate the boundary terms on $\partial\Lambda_{\rho_\ell}$ and we obtain

$$\int_{\Lambda_{\rho_\ell}}\frac{\partial^{\gamma_3}}{\partial x_3^{\gamma_3}}\frac{\partial^{|\tilde\gamma|_1}}{\partial \tilde y^{\tilde\gamma}}\mathcal{G}(\cdot+o_k, y+o_\ell)\chi_\ell^*(\cdot+o_\ell)\varphi(y)\,dy$$
$$= (-1)^{|\tilde\gamma|_1}\int_{\Lambda_{\rho_\ell}}\frac{\partial^{\gamma_3}}{\partial x_3^{\gamma_3}}\frac{\partial^{|\gamma_1|_1}}{\partial \tilde x^{\gamma_1}}\mathcal{G}(\cdot+o_k, y+o_\ell)\frac{\partial^{|\gamma_2|_1}}{\partial \tilde y^{\gamma_2}}\chi_\ell^*(\cdot+o_\ell)\varphi(y)\,dy.$$

Now, exploiting again Corollary 7.1.11 and Cauchy-Schwarz inequality, we can estimate that

$$\left\|\int_{\Lambda_{\rho_\ell}}\frac{\partial^{\gamma_3}}{\partial x_3^{\gamma_3}}\frac{\partial^{|\gamma_1|_1}}{\partial \tilde x^{\gamma_1}}\mathcal{G}(\cdot+o_k, y+o_\ell)\frac{\partial^{|\gamma_2|_1}}{\partial \tilde y^{\gamma_2}}\chi_k^*(\cdot+o_\ell)\varphi(y)\,dy\right\|^2_{L^2(\Lambda^{(k)})}$$
$$\le C\Big(\exp(-\delta_{min}J^*) + J^*\nu^N\Big)^2\|\varphi\|^2_{H^t(\Lambda_{\rho_\ell})},$$

where $J^* > J, N \ge N_0(\lambda_{J^*}, \delta_{max})$. □

Theorem 7.2.2. *We assume that Assumption 7.1.7 holds, which defines the constant $\nu \in (0,1)$. Then it exists a constant $C = C(J^*, N)$ such that for all $\varphi \in H^s(\Lambda_{\rho_\ell})$, $s \ge 3/2, r > 1/2$, there holds*

$$\|(\mathcal{K}^{k\ell} - \mathcal{K}^{k\ell}_{J^*,N,\boldsymbol{\zeta_1},\boldsymbol{\zeta_2}})\varphi\|_{L^2(\Lambda_{\rho_k})}$$
$$\le C\left(\exp(-\delta_{min}J^*) + J^*\nu^N + (J^*)^s\left[\min(\boldsymbol{\zeta_1})^{-(s-r)} + \min(\boldsymbol{\zeta_2})^{-(s-r)}\right]\right)\|\varphi\|_{L^2(\Lambda_{\rho_\ell})}, \quad (7.22)$$

for $J^* > J, N \ge N_0(\lambda_{J^*}, \delta_{max}), \boldsymbol{\zeta_1}, \boldsymbol{\zeta_2} \in \mathbb{N}^3$, $\nu \in (0,1)$ and $\varphi \in L^2(\Lambda_{\rho_\ell})$.

Proof. The error estimate of $\mathcal{K}^{k\ell}_{J^*,N,\boldsymbol{\zeta_1},\boldsymbol{\zeta_2}}$ relies on the smoothness of the kernel $G_{J^*,N}$, which relies on the smoothness of

$$f_n^\pm(\tilde x, \lambda) = \exp\left(\pm i\lambda r_x\cos\left(\frac{2\pi n}{2N+1} - \varphi_x\right)\right)$$
$$= \exp\left(\pm i\lambda r_x\left[\sin\left(\frac{2\pi n}{2N+1}\right)\cos(\varphi_x) - \cos\left(\frac{2\pi n}{2N+1}\right)\sin(\varphi_x)\right]\right)$$
$$= \exp\left(\pm i\lambda\left[\sin\left(\frac{2\pi n}{2N+1}\right)x_1 - \cos\left(\frac{2\pi n}{2N+1}\right)x_2\right]\right).$$

Therefore, using Lemma 2.2.4 a), we obtain for the derivatives of $f_n^\pm$ with respect to $\tilde x$ that

$$\left|\frac{\partial^{|\tilde\beta|_1}}{\partial \tilde x^{\tilde\beta}}f_n^\pm(\tilde x, \lambda_j)\right| \le |\lambda_j|^{|\beta|_1}\exp(Im(\lambda_j)\rho_j) \le j^{|\beta|_1}\exp(Im(\lambda_j)\rho_j), \qquad x \in \Lambda_{\rho_\ell}, \tilde\beta \in \mathbb{N}_0^2.$$

Furthermore, we know that for $n \in \mathbb{Z}$, $j \in \mathbb{N}$, $s \in \mathbb{R}$, $s \geq 0$ there exists a constant C independent of $\ell = 1, ..., L$ such that

$$\|\chi_\ell^*(\cdot + o_\ell)\phi_j(x_3) f_n^\pm(\tilde{x}, \lambda_j)\|_{H^s(\Lambda_{\rho_\ell})} \leq C(s) j^s \exp(Im(\lambda_j)\rho_\ell). \tag{7.23}$$

The same estimate holds for Λ_{ρ_k}, too. For $x \in \Lambda_{\rho_k}$ we have

$$\begin{aligned}\left([\mathcal{K}^{k\ell}_{J^*,N} - \mathcal{K}^{k\ell}_{J^*,N,\boldsymbol{\zeta_1},\boldsymbol{\zeta_2}}]\varphi\right)(x) &= \chi_k^*(\cdot + o_k) \int_{\Lambda_{\rho_\ell}} (I - Q_{\boldsymbol{\zeta_2},y})[G_{J^*,N}(x + o_k, y + o_\ell)\chi_\ell^*(y + o_\ell)]\varphi(y)\, dy \\ &+ (I - Q_{\boldsymbol{\zeta_1},x}) \left(\chi_k^*(\cdot + o_k) \int_{\Lambda_{\rho_\ell}} Q_{\boldsymbol{\zeta_2},y}[G_{J^*,N}(x + o_k, y + o_\ell)\chi_\ell^*(y + o_\ell)]\varphi(y)\, dy\right),\end{aligned} \tag{7.24}$$

where $Q_{\boldsymbol{\zeta_1},x}$ denotes the interpolation operator applied to the variable x, whereas $Q_{\boldsymbol{\zeta_2},y}$ is applied to the variable y. Using (7.24), and the triangle inequality, we see that

$$\begin{aligned}&\|(\mathcal{K}^{k\ell}_{J^*,N} - \mathcal{K}^{k\ell}_{J^*,N,\boldsymbol{\zeta_1},\boldsymbol{\zeta_2}})\varphi\|_{L^2(\Lambda_{\rho_k})} \\ &\quad\leq \left\|\chi_k^*(\cdot + o_k) \int_{\Lambda_{\rho_\ell}} (I - Q_{\boldsymbol{\zeta_2},y})[G_{J^*,N}(x + o_k, y + o_\ell)\chi_\ell^*(y + o_\ell)]\varphi(y)\, dy\right\|_{L^2(\Lambda_{\rho_k})} \\ &\quad\quad + \left\|(I - Q_{\boldsymbol{\zeta_1},x}) \left[\chi_k^*(\cdot + o_k) \int_{\Lambda_{\rho_\ell}} Q_{\boldsymbol{\zeta_2},y}[G_{J^*,N}(x + o_k, y + o_\ell)\chi_\ell^*(y + o_\ell)]\varphi(y)\, dy\right]\right\|_{L^2(\Lambda_{\rho_k})} \\ &\quad\leq C \left\|\chi_k^*(\cdot + o_k) \int_{\Lambda_{\rho_\ell}} (I - Q_{\boldsymbol{\zeta_2},y})[G_{J^*,N}(x + o_k, y + o_\ell)\chi_\ell^*(y + o_\ell)]\varphi(y)\, dy\right\|_{L^2(\Lambda_{\rho_k})} \\ &\quad\quad + \left\|\int_{\Lambda_{\rho_\ell}} (I - Q_{\boldsymbol{\zeta_1},x})[\chi_k^*(\cdot + o_k) Q_{\boldsymbol{\zeta_2},y} G_{J^*,N}(x + o_k, y + o_\ell)\chi_\ell^*(y + o_\ell)]\varphi(y)\, dy\right\|_{L^2(\Lambda_{\rho_k})}.\end{aligned}$$

For simplicity, we denote

$$g_k^\pm(x, \lambda_j) = \chi_k^*(x + o_k)\phi_j(x_3) f_n^\pm(\tilde{x}, \lambda_j),$$

and we use the definition of the truncated Green's function from (7.16) to see that

$$\begin{aligned}&\left\|\int_{\Lambda_{\rho_\ell}} (Q_{\boldsymbol{\zeta_1},x} - I)[\chi_k^*(\cdot + o_k) Q_{\boldsymbol{\zeta_2},y} G_{J^*,N}(x + o_k, y + o_\ell)\chi_\ell^*(y + o_\ell)]\varphi(y)\, dy\right\|_{L^2(\Lambda_{\rho_k})} \\ &= \left\|\frac{i}{4} \int_{\Lambda_{\rho_\ell}} \sum_{j=1}^{J^*} \sum_{n=1}^{2N+1} (Q_{\boldsymbol{\zeta_1},x} - I)\right. \\ &\quad\quad \left.\left[\chi_k^*(\cdot + o_k)\phi_j(x_3) f_n^-(\tilde{x}, \lambda_j) s_n(o_k - o_\ell, \lambda_j)\right] Q_{\boldsymbol{\zeta_2},y} \left[f_n^+(\tilde{y}, \lambda_j)\phi_j(y_3)\chi_\ell^*(y + o_\ell)\right] \varphi(y)\, dy\right\|_{L^2(\Lambda_{\rho_k})} \\ &= \frac{1}{4}\left\|\sum_{j=1}^{J^*} \sum_{n=1}^{2N+1} (Q_{\boldsymbol{\zeta_1},x} - I)\left[g_k^-(x, \lambda_j)\right] s_n(o_k - o_\ell, \lambda_j) \int_{\Lambda_{\rho_\ell}} Q_{\boldsymbol{\zeta_2},y}\left[g_\ell^+(x, \lambda_j)\right] \varphi(y)\, dy\right\|_{L^2(\Lambda_{\rho_k})}.\end{aligned}$$

Next, the Cauchy-Schwarz inequality implies that

$$\begin{aligned}
&\left\| \int_{\Lambda_{\rho_\ell}} (Q_{\boldsymbol{\zeta_1},x} - I)[\chi_k^*(\cdot + o_k) Q_{\boldsymbol{\zeta_2},y} G_{J^*,N}(x+o_k, y+o_\ell)\chi_\ell^*(y+o_l)]\varphi(y)\,dy \right\|_{L^2(\Lambda_{\rho_k})} \\
&\qquad \leq C \sum_{j=1}^{J^*} \sum_{n=1}^{2N+1} |s_n(o_k - o_\ell, \lambda_j)| \\
&\qquad \left\| (Q_{\boldsymbol{\zeta_1},x} - I)\left[g_k^-(x,\lambda_j) \int_{\Lambda_{\rho_\ell}} Q_{\boldsymbol{\zeta_2},y}[g_\ell^+(y,\lambda_j)]\varphi(y)\,dy \right] \right\|_{L^2(\Lambda_{\rho_k})} \|\varphi\|_{L^2(\Lambda_{\rho_\ell})} \\
&\qquad \leq C \sum_{j=1}^{J^*} \sum_{n=1}^{2N+1} |s_n(o_k - o_\ell, \lambda_j)| \\
&\qquad \left\| (Q_{\boldsymbol{\zeta_1}} - I)\left[g_k^-(\cdot,\lambda_j) \right] \right\|_{L^2(\Lambda_{\rho_k})} \left\| Q_{\boldsymbol{\zeta_2}}[g_\ell^+(\cdot,\lambda_j)] \right\|_{L^2(\Lambda_{\rho_\ell})} \|\varphi\|_{L^2(\Lambda_{\rho_\ell})}.
\end{aligned}$$

Recall that for all $s \geq 0$ it holds

$$\begin{aligned}
\left\| g_k^-(\cdot,\lambda_j) \right\|_{H^s(\Lambda_{\rho_k})} &\leq C(s) j^s \exp(Im(\lambda_j)\rho_k), \quad \text{and} \\
\left\| g_\ell^-(\cdot,\lambda_j) \right\|_{H^s(\Lambda_{\rho_\ell})} &\leq C(s) j^s \exp(Im(\lambda_j)\rho_\ell).
\end{aligned}$$

Using the convergence of $(Q_{\boldsymbol{\zeta_2}} - I)$ in Theorem 5.2.3, we compute for $r > 1/2$ and $s > 3/2$ that

$$\begin{aligned}
\|Q_{\boldsymbol{\zeta_2}}[g_\ell^+(\cdot,\lambda_j)]\|_{L^2(\Lambda_{\rho_\ell})} &\leq \left\| [Q_{\boldsymbol{\zeta_2}} - I] g_\ell^+(\cdot,\lambda_j) \right\|_{L^2(\Lambda_{\rho_\ell})} + \left\| g_\ell^+(\cdot,\lambda_j) \right\|_{L^2(\Lambda_{\rho_\ell})} \\
&\leq C \left(\min(\boldsymbol{\zeta_2})^{-(s-r)} \left\| [g_\ell^+(\cdot,\lambda_j)] \right\|_{H^s(\Lambda_{\rho_\ell})} + \exp(Im(\lambda_j)\rho_\ell) \right) \\
&\leq C \left(\min(\boldsymbol{\zeta_2})^{-(s-r)} j^s \exp(Im(\lambda_j)\rho_\ell) + \exp(Im(\lambda_j)\rho_\ell) \right) \leq C_1.
\end{aligned}$$

Analogously, we estimate for $r > 1/2$ and $s > 3/2$ that

$$\left\| (Q_{\boldsymbol{\zeta_1}} - I)\left[g_k^-(\cdot,\lambda_j) \right] \right\|_{L^2(\Lambda_{\rho_k})} \leq C \min(\boldsymbol{\zeta_1})^{-(s-r)} j^s \exp(Im(\lambda_j)\rho_k).$$

Plugging all this together, we have

$$\begin{aligned}
&\left\| \int_{\Lambda_{\rho_\ell}} (Q_{\boldsymbol{\zeta_1},x} - I)[\chi_k^*(\cdot + o_k) Q_{\boldsymbol{\zeta_2},y} G_{J^*,N}(x+o_k, y+o_\ell)\chi_\ell^*(y+o_\ell)]\,dy \right\|_{H^s(\Lambda_{\rho_k})} \\
&\qquad \leq C_2 \sum_{j=1}^{J^*} \sum_{n=1}^{2N+1} |s_n(o_k - o_\ell, \lambda_j)| \left(\min(\boldsymbol{\zeta_1})^{-(s-r)} j^s \exp(Im(\lambda_j)\rho_k) \right) C_1 \|\varphi\|_{L^2(\Lambda_{\rho_\ell})}.
\end{aligned}$$

We define now $\rho_{max} = \max\{\rho_\ell, \ell = 1, ..., L\}$ and, for simplicity, we introduce the constant

$$C_3 = C_3(J^*, N) = C \sum_{j=1}^{J^*} \sum_{n=1}^{2N+1} |s_n(o_k - o_\ell, \lambda_j)| \exp(Im(\lambda_j)\rho_k). \tag{7.25}$$

Then,

$$\begin{aligned}
&\left\| \int_{\Lambda_{\rho_\ell}} (Q_{\boldsymbol{\zeta_1},x} - I)[\chi_k^*(x + o_k) Q_{\boldsymbol{\zeta_2},y} G_{J^*,N}(x+o_k, y+o_\ell)\chi_\ell^*(y+o_\ell)]\,dy \right\|_{H^s(\Lambda_{\rho_k})} \\
&\qquad \leq C_3(J^*, N) \min(\boldsymbol{\zeta_1})^{-(s-r)} (J^*)^s \|\varphi\|_{L^2(\Lambda_{\rho_k})}.
\end{aligned}$$

Now, using the truncated Green's function from (7.16), the definition of ρ_{max} and the Cauchy-Schwarz inequality, for $r > 1/2$ and $s > 3/2$ we have

$$\begin{aligned}
&\left\| \int_{\Lambda_{\rho_\ell}} (I - Q_{\boldsymbol{\zeta_2},y})[G_{J^*,N}(x+o_k, y+o_\ell)\chi^*_\ell(y+o_\ell)]\varphi(y)\,dy \right\|_{L^2(\Lambda_{\rho_k})} \\
&\le C \sum_{j=1}^{J^*} \sum_{n=1}^{2N+1} |s_n(o_k - o_\ell, \lambda_j)| \left\| g_k^-(x,\lambda_j) \int_{\Lambda_{\rho_\ell}} (Q_{\boldsymbol{\zeta_2},y} - I)\left[g_\ell^+(y,\lambda_j)\right]\varphi(y)\,dy \right\|_{L^2(\Lambda_{\rho_k})} \\
&\le C \sum_{j=1}^{J^*} \sum_{n=1}^{2N+1} |s_n(o_k - o_\ell, \lambda_j)| \left\| g_k^-(x,\lambda_j)\right\|_{L^2(\Lambda_{\rho_k})} \left\|(Q_{\boldsymbol{\zeta_2},y} - I)\left[g_\ell^+(y,\lambda_j)\right]\right\|_{L^2(\Lambda_{\rho_\ell})} \|\varphi\|_{L^2(\Lambda_{\rho_\ell})} \\
&\le C \sum_{j=1}^{J^*} \sum_{n=1}^{2N+1} |s_n(o_k - o_\ell, \lambda_j)| \exp(Im(\lambda_j)\rho_\ell)(\min(\boldsymbol{\zeta_2})^{-(s-r)} j^s \exp(Im(\lambda_j)\rho_k)) \|\varphi\|_{L^2(\Lambda_{\rho_\ell})} \\
&\le C_3(J^*,N)(J^*)^s \min(\boldsymbol{\zeta_2})^{-(s-r)} \|\varphi\|_{L^2(\Lambda_{\rho_\ell})}.
\end{aligned}$$

Plugging all together, we see

$$\begin{aligned}
&\|(\mathcal{K}^{k\ell}_{J^*,N} - \mathcal{K}^{k\ell}_{J^*,N,\boldsymbol{\zeta_1},\boldsymbol{\zeta_2}})\varphi\|_{L^2(\Lambda_{\rho_k})} \\
&\qquad\qquad \le\ C_3(J^*,N)(J^*)^s \left[\min(\boldsymbol{\zeta_1})^{-(s-r)} + \min(\boldsymbol{\zeta_2})^{-(s-r)}\right] \|\varphi\|_{L^2(\Lambda_{\rho_\ell})}
\end{aligned}$$

where the constant C_3 is defined in (7.25). Next, Lemma 7.2.1 yields

$$\begin{aligned}
&\|\left(\mathcal{K}^{k\ell} - \mathcal{K}^{k\ell}_{J^*,N}\right)\varphi\|_{L^2(\Lambda_{\rho_k})} \\
&\qquad \le C\left(\exp(-\delta_{min}J^*) + J^*\nu^N + (J^*)^s\left[\min(\boldsymbol{\zeta_1})^{-(s-r)} + \min(\boldsymbol{\zeta_2})^{-(s-r)}\right]\right)\|\varphi\|_{L^2(\Lambda_{\rho_\ell})},
\end{aligned}$$

for $J^* \ge J+1, N \ge N_0(\lambda_{J^*}, \delta_{max})$ and $\nu \in (0,1)$. Combining the last estimates finishes the proof. □

Now we introduce the discretization parameter $\boldsymbol{\zeta} = (\boldsymbol{\zeta_1}, ..., \boldsymbol{\zeta_L})$. Following the definition of the linear subspace of $L^2(\Lambda_{\rho_k})$ for basis functions in (5.5), we introduce a finite dimensional product space

$$\boldsymbol{T_\zeta} = \oplus_{\ell=1}^L T_{\boldsymbol{\zeta}_\ell}(\Lambda_{\rho_\ell}).$$

Thus, we can now define the interpolation operator

$$\boldsymbol{Q_\zeta} : \boldsymbol{H}^0 \to \boldsymbol{T_\zeta}, \qquad \boldsymbol{Q_\zeta u} = (Q_{\boldsymbol{\zeta_1}}u_1, ..., Q_{\boldsymbol{\zeta_L}}u_L)^T.$$

We moreover introduce for $\boldsymbol{\zeta}^* = (\boldsymbol{\zeta_1^*}, ..., \boldsymbol{\zeta_L^*})$ a $L \times L$ matrix

$$\boldsymbol{\mathcal{K}}_{J^*,N,\boldsymbol{\zeta}^*} = \begin{pmatrix} \mathcal{V}_{\rho_1} & \cdots & \mathcal{K}^{1L}_{J^*,N,\boldsymbol{\zeta_1^*},\boldsymbol{\zeta_L^*}} \\ \vdots & \ddots & \vdots \\ \mathcal{K}^{L1}_{J^*,N,\boldsymbol{\zeta_L^*},\boldsymbol{\zeta_1^*}} & \cdots & \mathcal{V}_{\rho_L} \end{pmatrix}.$$

Finally, we obtain the collocation method of the spectral/multipole discretization (7.2): Find $\boldsymbol{u_\zeta} \in \boldsymbol{T_\zeta}$ solving

$$\boldsymbol{u_\zeta} - \boldsymbol{\mathcal{K}}_{J^*,N,\boldsymbol{\zeta}^*}\boldsymbol{Q_\zeta}\left[\boldsymbol{T}^-\left(\frac{\omega^2}{\boldsymbol{c}^2} \bullet \boldsymbol{q}\right) \bullet \boldsymbol{u_\zeta}\right] = \boldsymbol{Q_\zeta f}. \tag{7.26}$$

Next, we analyze the discrete scheme (7.26). Due to Section 5.3, we already know this scheme in the case $L = 1$, i.e., we basically know what happens for the diagonal terms (see Lemma 5.3.3). In consequence, we look now on the off-diagonal terms.

For given Fourier coefficients $\hat{\varphi}$ of $\varphi \in T_{\zeta_\ell}(\Lambda_{\rho_\ell})$, we compute the Fourier coefficients of $\mathcal{K}^{k\ell}_{J^*,N,\boldsymbol{\zeta}_k,\boldsymbol{\zeta}_\ell}\varphi$. Due to (7.19), for simplicity, for $k,\ell = 1,...,L$, $j = 1,...,J^*$ and $n = 1,...,2N+1$, we denote

$$b^-_{\zeta,n,k} := Q_{\zeta_k}\left[\chi^*_k(x+o_k)\phi_{j_3}(x_3)f^-_n(\tilde{x},\lambda_j)\right] \in T_{\zeta_k}(\Lambda_{\rho_k}), \text{ and}$$
$$b^+_{\zeta,n,\ell} := Q_{\zeta_\ell}\left[\chi^*_\ell(x+o_\ell)\phi_{j_3}(x_3)f^+_n(\tilde{x},\lambda_j)\right] \in T_{\zeta_\ell}(\Lambda_{\rho_\ell}),$$

and by $\hat{b}^\pm_{j,n,k}$ we denote the corresponding Fourier coefficients. Note that $\hat{b}^\pm$ is a column vector with length corresponding to the dimension of $T_{\zeta_k}(\Lambda_{\rho_k})$. For simplicity, we denote this dimension by $\mathcal{J}_k$. Related to the Fourier coefficients we consider the matrix $B^\pm_k \in \mathbb{C}^{J^*(2N+1)\times\mathcal{J}_k}$ containing the vectors $\hat{b}^\pm$ as rows by,

$$B^\pm_k = \begin{bmatrix} \hat{b}^\pm_{1,1,k} & \hat{b}^\pm_{1,2,k} & \cdots & \hat{b}^\pm_{1,2N+1,k} & \hat{b}^\pm_{2,1,k} & \cdots & \hat{b}^\pm_{2,2N+1,k} & \cdots & \hat{b}^\pm_{J^*,2N+1,k} \end{bmatrix}.$$

We point out that $B^\pm_k$ needs $\mathcal{J}_k J^*(2N+1)$ complex numbers to store, where $2N+1$ denotes the truncation index of the diagonal approximation of the Hankel function defined in Corollary 7.1.6 and J^* denotes the number of positive eigenvalues.

Furthermore, we denote for $k,\ell = 1,...,L$, $j = 1,...,J^*$ the vector $S_{k\ell} \in \mathbb{C}^{J^*(2N+1)}$,

$$\begin{aligned} S_{k\ell} = [&s_1(o_k - o_\ell,\lambda_1), s_2(o_k - o_\ell,\lambda_1), ..., s_{2N+1}(o_k - o_\ell,\lambda_1),\\ &s_1(o_k - o_\ell,\lambda_2), s_2(o_k - o_\ell,\lambda_2), ..., s_{2N+1}(o_k - o_\ell,\lambda_2), ..., s_1(o_k - o_\ell,\lambda^*_{\mathcal{J}}),\\ &s_2(o_k - o_\ell,\lambda^*_{\mathcal{J}}), ..., s_{2N+1}(o_k - o_\ell,\lambda^*_{\mathcal{J}})]^T. \end{aligned}$$

Note that $S_{k\ell}$ needs $J^*(2N+1)$ complex numbers to store. Plugging all together, we obtain for $k,\ell = 1,...,L$, $j = 1,...,J^*$ and $n = 1,...,2N+1$ that

$$(\widehat{\mathcal{K}^{k\ell}_{J^*,N,\boldsymbol{\zeta}_k,\boldsymbol{\zeta}_\ell}\varphi}) = (B^-_k)^T[S_{k\ell} \bullet (B^+_\ell\,(\hat{\varphi}(\boldsymbol{j}))_{\boldsymbol{j}\in\mathbb{Z}^3_{\boldsymbol{\zeta}_k}})]. \tag{7.27}$$

We point out that the computation of the Fourier coefficients of $\mathcal{K}^{k\ell}_{J^*,N,\boldsymbol{\zeta}_k,\boldsymbol{\zeta}_\ell}\varphi$ takes $O(J^*N(\mathcal{J}^*_k+\mathcal{J}^*_\ell$ matrix-vector operations. We further note that we have to store $(\mathcal{J}_k + \mathcal{J}_\ell + 1)J^*(2N+1)$ complex entries, due to the dimensions of $S_{k\ell}$, and $B^\pm_k$.

To abbreviate the notation we now denote

$$\min(\boldsymbol{\zeta}) := \min_{1\le k\le 3, 1\le \ell\le L} \zeta_{k,\ell} \qquad \text{for } \boldsymbol{\zeta} \in \mathbb{R}^{3\times L}.$$

Theorem 7.2.3. *We assume that Assumption 7.1.2 holds, i.e.* $\boldsymbol{T}^-(\boldsymbol{q}/\boldsymbol{c}^2) \in \boldsymbol{H}^s$ *for* $s > 3/2$, *that Assumption 4.2.11 holds, i.e. the source problem (4.34) is uniquely solvable such that the Lippmann-Schwinger integral equation is also uniquely solvable, and that Assumption 7.1.7 holds, which defines the constant* $\nu \in (0,1)$. *We consider for* $s > 3/2$ *that* $\boldsymbol{f} \in \boldsymbol{H}^s$ *and choose* $r \in (1/2, s)$.

For $\boldsymbol{\zeta} \in \mathbb{N}^{3\times L}$, $\boldsymbol{\zeta} \ge \boldsymbol{\zeta^*}$ *and if* $J^*, N \ge N_0(\lambda_{J^*}, \delta_{max}), \min(\boldsymbol{\zeta}), \min(\boldsymbol{\zeta^*})$ *are large enough, then there is a unique solution* $\boldsymbol{u}_{\boldsymbol{\zeta}} \in \boldsymbol{T}_{\boldsymbol{\zeta}}$ *of the discrete problem (7.26). Further, it holds for arbitrary* $r > 1/2$, $r \ge s \ge 0$ *that*

$$\begin{aligned} \|\boldsymbol{u}_{\boldsymbol{\zeta}} - \boldsymbol{u}\|_{\boldsymbol{H}^s} \le{}& \|(Q_{\boldsymbol{\zeta}} - I)f\|_{\boldsymbol{H}^s}\\ &+ \Big(\exp(-\delta_{min}J^*) + J^*\nu^N + (J^*)^s\min(\boldsymbol{\zeta^*})^{-(s-r)} + \min(\boldsymbol{\zeta})^{-(s-r)}\Big)\\ &\qquad \|\boldsymbol{T}^-\|_{\boldsymbol{H}^s}\left[\left\|\boldsymbol{T}^-\left(\frac{\boldsymbol{q}}{\boldsymbol{c}^2}\right)\right\|_{\boldsymbol{H}^s} + \|\boldsymbol{f}\|_{\boldsymbol{H}^s}\right]. \end{aligned}$$

Proof. Due to the Assumptions 4.2.11 and Theorem 4.5.5 the periodized Lippmann-Schwinger equation (4.67) is uniquely solvable in $L^2(\Lambda_\rho)$ for any right-hand side, too. Thus, the unique solvability of the periodized Lippmann-Schwinger equation implies existence and uniqueness of the system of integral equation (7.2).

If $\boldsymbol{f} \in \boldsymbol{H}^s$ and if Assumption 7.1.2 holds, then the solution $\boldsymbol{u} \in \boldsymbol{H}$ to (7.2) belongs to $\boldsymbol{H}^{s+2}$ by Theorem 5.3.4 a). Using the estimate from Corollary 5.3.2 we obtain the following estimate for the difference of the diagonal terms of $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{K}}_{J^*,N,\boldsymbol{\zeta}^*}Q_{\boldsymbol{\zeta}}$: For $\boldsymbol{v} \in \boldsymbol{H}^s$ and $3/2 < s \leq 2$,

$$\begin{aligned}
&\sum_{l=1}^{L} \left\| \mathcal{K}^{\ell\ell}\left(T_\ell^-\left(\frac{\omega^2}{c^2}q_\ell\right)v_\ell\right) - (\mathcal{K}_{J^*,N,\boldsymbol{\zeta}^*})_{\ell\ell}\left(Q_{\boldsymbol{\zeta}_\ell}\left[T_\ell^-\left(\frac{\omega^2}{c^2}q_\ell\right)v_\ell\right]\right)\right\|_{H^s(\Lambda_{\rho_\ell})} \\
=&\sum_{l=1}^{L} \left\| \mathcal{V}_{\rho_\ell}\left(T_\ell^-\left(\frac{\omega^2}{c^2}q_\ell\right)v_\ell\right) - \mathcal{V}_{\rho_\ell}Q_{\boldsymbol{\zeta}_\ell}\left[T_\ell^-\left(\frac{\omega^2}{c^2}q_\ell\right)v_\ell\right]\right\|_{H^s(\Lambda_{\rho_\ell})} \\
\leq&\sum_{l=1}^{L} \left\| (I - Q_{\boldsymbol{\zeta}_\ell})\left[T_\ell^-\left(\frac{\omega^2}{c^2}q_\ell\right)v_\ell\right]\right\|_{H^s(\Lambda_{\rho_\ell})} \\
\leq& C\omega^2 \min(\boldsymbol{\zeta}_\ell)^{-(s-r)} \left\|\frac{q_\ell}{c^2}\right\|_{H^s(\Lambda_{\rho_\ell})} \|v_\ell\|_{H^s(\Lambda_{\rho_\ell})}.
\end{aligned}$$

Moreover, we deduce that

$$\begin{aligned}
\sum_{l=1}^{L} \left\| \left[I - \mathcal{K}^{\ell\ell}\left(T_\ell^-\left(\frac{\omega^2}{c^2}q_\ell\right)v_\ell\right)\right]v_\ell - \left[I - \mathcal{V}_{\rho_\ell}Q_{\boldsymbol{\zeta}_\ell}\left[\left(T_\ell^-\left(\frac{\omega^2}{c^2}q_\ell\right)v_\ell\right)\right]\right]\right\|_{(H^s(\Lambda_{\rho_\ell})\to H^s(\Lambda_{\rho_\ell}))} \\
\leq C\omega^2 \min(\boldsymbol{\zeta})^{-(s-r)} \left\|\boldsymbol{T}^-\left(\frac{\boldsymbol{q}}{\boldsymbol{c}^2}\right)\right\|_{\boldsymbol{H}^s}. \qquad (7.28)
\end{aligned}$$

Now, we consider the off-diagonal terms in the operator matrices $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\mathcal{K}}_{J^*,N,\boldsymbol{\zeta}^*}Q_{\boldsymbol{\zeta}_\ell}$ and note that Theorem 7.2.2 implies that for $\boldsymbol{v} = (v_1, ..., v_L) \in \boldsymbol{H}^s$ and $\boldsymbol{\zeta} \geq \boldsymbol{\zeta}^*$ there holds

$$\begin{aligned}
&\sum_{1\leq k\neq\ell\leq L} \left\| \mathcal{K}^{k\ell}\left(T_\ell^-\left(\frac{\omega^2}{c^2}q_\ell\right)v_\ell\right) - (\mathcal{K}_{J^*,N,\boldsymbol{\zeta}^*})_{k\ell}Q_{\boldsymbol{\zeta}_\ell}\left[T_\ell^-\left(\frac{\omega^2}{c^2}q_\ell\right)v_\ell\right]\right\|_{H^s(\Lambda_{\rho_k})} \\
\leq&\sum_{1\leq k\neq\ell\leq L} \left\| \mathcal{K}^{k\ell}\left(T_\ell^-\left(\frac{\omega^2}{c^2}q_\ell\right)v_\ell\right) - \mathcal{K}^{k\ell}_{J^*,N,\zeta_k^*,\zeta_\ell^*}\left[T_\ell^-\left(\frac{\omega^2}{c^2}q_\ell\right)v_\ell\right]\right\|_{H^s(\Lambda_{\rho_k})} \\
&\qquad + \left\| \mathcal{K}^{k\ell}_{J^*,N,\zeta_k^*,\zeta_\ell^*}(I - Q_{\boldsymbol{\zeta}_\ell})\left[T_\ell^-\left(\frac{\omega^2}{c^2}q_\ell\right)v_\ell\right]\right\|_{H^s(\Lambda_{\rho_k})} \\
\leq& C\omega^2 \sum_{1\leq k\neq\ell\leq L} \left[\exp(-\delta_{min}J^*) + J^*\nu^N + (J^*)^s \min(\boldsymbol{\zeta^*})^{-(s-r)}\right] \left\|\boldsymbol{T}^-\left(\frac{\boldsymbol{q}}{\boldsymbol{c}^2}\right)\right\|_{\boldsymbol{H}^s} \|\boldsymbol{v}\|_{\boldsymbol{H}^s} \\
&\qquad + \min(\boldsymbol{\zeta})^{-(s-r)} \left\|\boldsymbol{T}^-\left(\frac{\boldsymbol{q}}{\boldsymbol{c}^2}\right)\right\|_{\boldsymbol{H}^s} \|\boldsymbol{v}\|_{\boldsymbol{H}^s} \\
\leq& \left[\exp(-\delta_{min}J^*) + J^*\nu^N + (J^*)^s \min(\boldsymbol{\zeta^*})^{-(s-r)} + \min(\boldsymbol{\zeta})^{-(s-r)}\right] \left\|\boldsymbol{T}^-\left(\frac{\boldsymbol{q}}{\boldsymbol{c}^2}\right)\right\|_{\boldsymbol{H}^s} \|\boldsymbol{v}\|_{\boldsymbol{H}^s}
\end{aligned}$$

Reformulating the latter estimate as an estimate for the off-diagonal terms of $I - \boldsymbol{\mathcal{K}}$ and $I - \boldsymbol{\mathcal{K}}_{J^*,N,\boldsymbol{\zeta}^*}Q_{\boldsymbol{\zeta}}$ equation (7.28) yields that for $3/2 \leq s \leq 2$, there holds

$$\begin{aligned}
&\left\| (I - \mathcal{K})\left(\boldsymbol{T}^-\left(\frac{\omega^2}{\boldsymbol{c}^2}\bullet\boldsymbol{q}\right)\bullet\cdot\right) - (I - \mathcal{K}_{J^*,N,\boldsymbol{\zeta}^*})Q_{\boldsymbol{\zeta}}\left[\boldsymbol{T}^-\left(\frac{\omega^2}{\boldsymbol{c}^2}\bullet\boldsymbol{q}\right)\bullet\cdot\right]\right\|_{\boldsymbol{H}^s\to\boldsymbol{H}^s} \\
&\qquad \leq C\left[\exp(-\delta_{min}J^*) + J^*\nu^N + (J^*)^s \min(\boldsymbol{\zeta^*})^{-(s-r)} + \min(\boldsymbol{\zeta})^{-(s-r)}\right] \left\|\boldsymbol{T}^-\left(\frac{\boldsymbol{q}}{\boldsymbol{c}^2}\right)\right\|_{\boldsymbol{H}^s}
\end{aligned}$$

Now, the same arguments as in the proof of Theorems 5.3.4 and 5.3.6 show that the discrete problem (7.26) is uniquely solvable for $\min(\boldsymbol{\zeta}^*)$ and $\min(\boldsymbol{\zeta})$ large enough, and if J^* and N are large enough. The indicated error estimate follows as well as in Theorem 5.3.4 b),

$$\begin{aligned}
&\|\boldsymbol{u}_{\boldsymbol{\zeta}} - \boldsymbol{u}\|_{\boldsymbol{H}^s} \leq \|(Q_{\boldsymbol{\zeta}} - I)f\|_{\boldsymbol{H}^s} \\
&+ \left[\exp(-\delta_{min}J^*) + J^*\nu^N + (J^*)^s \min(\boldsymbol{\zeta^*})^{-(s-r)} + \min(\boldsymbol{\zeta})^{-(s-r)}\right] \left\|\boldsymbol{T}^-\left(\frac{\boldsymbol{q}}{\boldsymbol{c}^2}\right)\right\|_{\boldsymbol{H}^s} \|\boldsymbol{u}\|_{\boldsymbol{H}^s},
\end{aligned}$$

for $3/2 < s \le 2$, since $\|\boldsymbol{u}\|_{\boldsymbol{H}^s}$ is bounded by

$$\left\|\boldsymbol{T}^{-}\left(\frac{\boldsymbol{q}}{\boldsymbol{c}^2}\right)\right\|_{\boldsymbol{H}^s} + \|\boldsymbol{f}\|_{\boldsymbol{H}^s}.$$

□

Appendix A

Auxiliary Results

This section provides useful identities and estimates for Bessel and Hankel functions. We further introduce estimates for exponential functions. Moreover, we present a couple of technical proofs for results on integral operators.

A.1 Identities and Estimates for Special Functions

We first introduce the Bessel function of the first kind and order $\nu \in \mathbb{N}$ given by

$$J_\nu(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{k!(\nu+k)!} \left(\frac{x}{2}\right)^{\nu+2k} \qquad x \in \mathbb{R}.$$

We know that for $\nu \in \mathbb{N}$ it holds $J_\nu \in C^\infty(\mathbb{R})$ and $J_\nu(0) = J_\nu'(0) = 0$. Furthermore, we denote the Bessel functions of the second kind and order $\nu \in \mathbb{N}$ also called Neumann functions by

$$\begin{aligned} Y_\nu(x) =& \frac{2}{\pi}\left(\ln\frac{x}{2} + C\right) J_\nu(x) \\ &- \frac{1}{\pi}\left(\sum_{k=0}^{\nu-1} \frac{(\nu-1-k)!}{k!}\left(\frac{x}{2}\right)^{\nu-2k} + \sum_{k=0}^{\infty} \frac{(-1)^k}{k!(\nu+k)!}\left(\frac{x}{2}\right)^{\nu+2k}\right)(h_{k+\nu} + h_k) \end{aligned}$$

for all $x \in (0,\infty)$. Here, C denotes the Euler constant

$$C := \lim_{k\to\infty} \sum_{\ell=1}^{k} \ell^{-1} - \ln k \qquad \text{and} \qquad h_k = \sum_{\ell=1}^{k} \ell^{-1} \qquad (k = 1, 2, ...),$$

where $h_0 = 0$. We point out that for $x > 0$ the Bessel functions $Y_\nu(x)$ and $J_\nu(x)$ are real-valued. Moreover, the Bessel function of third kind and order ν are defined by

$$H_\nu^{(1)}(z) = J_\nu(z) + iY_n(z) \qquad \text{and} \qquad H_\nu^{(2)}(z) = J_\nu(z) - iY_n(z). \tag{A.1}$$

For a sufficiently smooth function φ the Hankel function of kind one is given by

$$H_\nu^{(1)}(\varphi(z)) = J_\nu(\varphi(z)) + iY_n(\varphi(z)).$$

Lemma A.1.1. *For the Bessel function and the Neumann function it holds that*

$$J_0(\varphi(z))Y_1(\varphi(z)) = J_1(\varphi(z))Y_0(\varphi(z)) - \frac{2}{\pi\varphi(z)},$$

and

$$J_1(\varphi(z))Y_0(\varphi(z)) = J_0(\varphi(z))Y_1(\varphi(z)) + \frac{2}{\pi\varphi(z)}.$$

For completeness we introduce the Bessel function of first kind and order ν with imaginary argument by I_ν. In consequence, for $\nu \in \mathbb{N}$, $Re(z) = 0$ and $Im(z) > 0$ we see that

$$J_\nu(z) = I_\nu(z).$$

Furthermore, the Hankel function of first kind and order ν with imaginary argument is defined by K_ν^0. In particular, for $\nu \in \mathbb{N}$, $Re(z) = 0$ and $Im(z) > 0$ we see that

$$H_\nu^k(z) = K_\nu^k(z), \qquad k = 1, 2.$$

Due to [AS64, Eq. 9.6.3, Eq. 9.6.4] and [JZ07, 8.407] we know that for $-\pi < argz \leq \pi/2$ it holds for $\nu \in \mathbb{N}$, $Re(z) = 0$ and $Im(z) > 0$ that

$$I_\nu(z) = \exp\left(-\frac{i}{2}\nu\pi\right) J_\nu(iz), \tag{A.2}$$

and

$$K_\nu(z) = \frac{i}{2}\pi \exp\left(\frac{i}{2}\pi\nu\right) H_\nu^{(1)}(iz). \tag{A.3}$$

In consequence, we have for $\nu = 0, 1$ the relation

$$H_0^{(1)}(iz) = \frac{2}{i\pi} K_0(z) \qquad \text{and} \qquad H_1^{(1)}(iz) = -\frac{2}{\pi} K_1(z). \tag{A.4}$$

Now, we discuss the derivative of the Hankel function. Using [Wat95, Chapter 3.60.9], we obtain for $z \in C$ and z away from zero the identity

$$\frac{1}{z}\frac{d}{dz} z^\ell H_l^{(1)}(z) = z^\ell H_l^{(1)}(z). \tag{A.5}$$

Moreover, due to [AS64, Equation 9.1.27] we compute for a sufficiently smooth function φ that it holds

$$\frac{\partial}{\partial z} H_0^{(1)}(\varphi(z)) = -H_1^{(1)}(\varphi(z))\, \varphi^{'}(z), \tag{A.6}$$

and

$$\frac{\partial}{\partial z} H_1^{(1)}(\varphi(z)) = H_0^{(1)}(\varphi(z))\, \varphi^{'} - \frac{1}{z} H_1^{(1)}(\varphi(z)).$$

Then, due to [AS64, Equation 9.6.27], we obtain for the derivative of the modified Bessel function with imaginary argument that

$$\frac{\partial}{\partial z} K_0^{(1)}(\varphi(z)) = -K_1^{(1)}(\varphi(z))\, \varphi^{'}(z), \tag{A.7}$$

where φ is a sufficiently smooth function. We now introduce the following characterizations, which can be found in [AS64].

Lemma A.1.2. *For fixed ν and $|z| \to \infty$ the Bessel function converges to*

$$J_\nu(z) = \sqrt{2/(\pi z)}\{\cos(z - \nu\pi/2 - \pi/4)\} \qquad \textit{for } (|\arg z| < \pi),$$

the Neumann function converges to

$$Y_\nu = \sqrt{2/(\pi z)}\{\sin(z - \nu\pi/2 - \pi/4)\} \qquad \textit{for } (|\arg z| < \pi),$$

and the Hankel function converges to

$$H_\nu^{(1)}(z) = \sqrt{2/(\pi z)} \exp(i(z - \nu\pi/2 - \pi/4)) \qquad \textit{for } (-\pi < \arg z < 2\pi).$$

Lemma A.1.3. *Let $\tilde{x}, \tilde{y} \in \mathbb{R}^2$ and θ denotes the angle between $\tilde{x}$ and $\tilde{y}$. Then holds for $\lambda \in \mathbb{C}\backslash\{0\}$ that*

$$H_0^{(1)}(\lambda|\tilde{x}-\tilde{y}|) = \sum_{\nu\in\mathbb{Z}} H_j^{(1)}(\lambda|\tilde{x}|)J_\nu(\lambda|\tilde{y}|)\exp(i\nu\theta), \qquad \tilde{z} = \tilde{x}+\tilde{y}, |\tilde{x}| > |\tilde{y}|. \tag{A.8}$$

Proof. We use the addition theorem from [CK13, Equation 3.88] given by

$$H_0^{(1)}(\lambda|\tilde{x}-\tilde{y}|) = H_0(\lambda|\tilde{x}|)J_0(\lambda|\tilde{y}|) + 2\sum_{\nu=1}^{\infty} H_\nu^{(1)}(\lambda|\tilde{x}|)J_\nu(\lambda|\tilde{y}|)\cos(\nu\theta),$$

where $|x| > |y|$ and θ denotes the angle between $\tilde{x}$ and $\tilde{y}$. Then using the technique from [LS12, Chapter 5.12] to obtain

$$H_0(\lambda|\tilde{x}|)J_0(\lambda|\tilde{y}|) + 2\sum_{\nu=1}^{\infty} H_\nu^{(1)}(\lambda|\tilde{x}|)J_\nu(\lambda|\tilde{y}|)\cos(\nu\theta) = \sum_{\nu\in\mathbb{Z}} H_\nu^{(1)}(\lambda|\tilde{x}|)J_\nu(\lambda|\tilde{y}|)\exp(i\nu(\theta_{\tilde{x}}-\theta_{\tilde{y}}),$$

This finishes the proof. ☐

For more details of Bessel and Hankel functions in scattering we refer the reader to [CK13, Chapter 3.4] and [SV02, Chapter 2], however, the collections [AS64], [JZ07] and [Wat95] give more concise treatments of Bessel and Hankel functions properties.

A.2 Identities and Estimates for Integral Operators

The next few lemmas give us estimates for integrals of exponential functions.

Lemma A.2.1. *For $x_1 \in [-\rho,\rho]$ and $\mu \geq 0$ where $\rho > 0$ there holds*

$$\left\| \int_{-\rho}^{\rho} \mu \exp(-\mu|x_1-y_1|)\, dy_1 \right\|_{L^2([-\rho,\rho])} \leq C,$$

where $C > 0$ is independent of $\mu \geq 0$.

Proof. We use for simplicity $\mu = |\lambda_j|$ and $x_1' = \mu x_1$ and $y_1' = \mu y_1$. Consequently,

$$\begin{aligned}\int_{-\rho}^{\rho} \mu \exp(-\mu|x_1-y_1|)\, dy_1 &= \int_{-\rho\mu}^{\rho\mu} \mu \exp(-|x_1'-y_1'|/2)\,\frac{dy_1}{\mu}\\ &= \int_{-\rho\mu}^{x_1} \exp(-(x_1'-y_1')/2)\, dy_1 + \int_{x_1}^{\rho\mu} \exp(-(y_1'-x_1')/2)\, dy_1.\end{aligned}$$

By integration we deduce

$$\begin{aligned}\int_{-\rho}^{\rho} \mu \exp(-\mu|x_1-y_1|)\, dy_1 &= 2\left[\exp(-(x_1'-y_1')/2)\right]_{-\rho\mu}^{x_1} - 2\left[\exp(-(y_1'-x_1')/2)\right]_{x_1}^{\rho\mu}\\ &= 4 - 2\exp(-x_1'/2-\rho\mu/2) - 2\exp(-\rho\mu/2+x_1'/2).\end{aligned}$$

Now, we see

$$\begin{aligned}&\int_{-\rho\mu}^{\rho\mu} \left[4 - 2\exp(-x_1'/2-\rho\mu/2) - 2\exp(-\rho\mu/2+x_1'/2)\right]\frac{dx_1}{\mu}\\ &= \frac{1}{\mu}\left[4x_1 + 2\exp(-x_1'/2-\rho\mu/2) - 2\exp(-\rho\mu/2+x_1'/2)\right]_{-\rho\mu}^{\rho\mu}\\ &= 8\rho\mu + 4\exp(-\rho\mu) - 4 \leq C,\end{aligned}$$

for some $C > 0$ independent of $\mu \geq 0$. Then,

$$\left\| \int_{-\rho}^{\rho} \mu \exp(-\mu|x_1 - y_1|)\, dy_1 \right\|_{L^2([-\rho,\rho])} \leq C,$$

where $C > 0$ independent of μ. This estimate finals the proof. □

Moreover, due to [JZ07, 8.432.8] we get the integral representation

$$K_\nu(xz) = \sqrt{\frac{\pi}{2z}} \frac{x^\nu \exp(-xz)}{\Gamma(\nu + 1/2)} \int_0^\infty \exp(-xt) t^{\nu - 1/2} \Big(1 + \frac{t}{2z}\Big)^{\nu - 1/2} dt \tag{A.9}$$

for $|\arg z| < \pi, \mathrm{Re}\nu > -\frac{1}{2}, x > 0$.

Lemma A.2.2. *Recall that $\tilde{\Lambda}_\rho = (-\rho, \rho)^{m-1}$. For $\tilde{y} \in \tilde{\Lambda}_\rho$ and $0 < \alpha < 2$ it holds that*

$$\int_{B(\tilde{y},cj)} \frac{1}{|\tilde{x} - \tilde{y}|^\alpha}\, d\tilde{x} \leq \frac{2\pi}{2-\alpha}(Cj)^{2-\alpha}, \qquad j \in \mathbb{N},$$

where $c > 0$ is independent of j and ρ and C independent of j.

Proof. We have

$$\int_{B(\tilde{y},cj)} \frac{1}{|\tilde{x} - \tilde{y}|^\alpha}\, d\tilde{x} \leq \int_{B(0,Cj)} \frac{1}{|\tilde{x}|^\alpha} d\tilde{x},$$

for some $C > 0$ independent of j. By converting of Cartesian coordinates into polar coordinates, we obtain that

$$\int_{B(0,Cj)} \frac{1}{|\tilde{x}|^\alpha}\, d\tilde{x} = 2\pi \int_0^{Cj} r^{-\alpha} r\, dr.$$

We point out that, the integral on the left-hand-side is only defined for $\alpha > 0$ and the integral on the right hand-side holds only for $\alpha < 2$. Next by integration we see

$$2\pi \int_0^{Cj} r^{-\alpha} r\, dr = 2\pi \left[\frac{1}{2-\alpha} r^{2-\alpha}\right]_0^{Cj} = \frac{2\pi}{2-\alpha}(Cj)^{2-\alpha}, \qquad (j \in \mathbb{N}).$$

□

Lemma A.2.3. *For $\tilde{y} \in \tilde{\Lambda}_\rho$ and if $j > 2\rho$ it holds that*

$$\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \Big| \exp\Big(-\frac{j}{2}|\tilde{x} - \tilde{y}|\Big)\Big| d\tilde{x} \leq \frac{4\pi}{j \exp(j\rho)} \left[2\rho - \frac{1}{j}\right] \qquad (j \in \mathbb{N}),$$

where $c > 0$.

Proof. We first convert Cartesian coordinates into polar coordinates. Then, for $j > 2\rho$ we have

$$\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left|\exp\left(-\frac{j|\tilde{x} - \tilde{y}|}{2}\right)\right| d\tilde{x} \leq 2\pi \int_{c/j}^{2\rho} \left| r \exp\left(-\frac{jr}{2}\right)\right| dr.$$

Next, due to partial integration we have

$$\begin{aligned} 2\pi \int_{c/j}^{2\rho} \left| r \exp\left(-\frac{jr}{2}\right)\right| dr &= \frac{4\pi}{j} \left[r \exp\left(-\frac{jr}{2}\right)\right]_{c/j}^{2\rho} - \frac{2\pi}{j} \int_{c/j}^{2\rho} \left|\exp\left(-\frac{jr}{2}\right)\right| dr \\ &= \frac{8\pi\rho}{j} \exp(-j\rho) - \frac{4\pi}{j^2} \exp\left(\frac{c}{2}\right) - \frac{4\pi}{j^2} \left[\exp\left(-\frac{jr}{2}\right)\right]_{c/j}^{2\rho}. \end{aligned}$$

Now, one computes that

$$2\pi \int_{c/j}^{2\rho} \left| r \exp\left(-\frac{jr}{2}\right)\right| \, dr = \frac{8\pi\rho}{j}\exp(-j\rho) - \frac{4\pi}{j^2}\exp\left(\frac{c}{2}\right) - \frac{4\pi}{j^2}\exp(-\rho j) + \frac{4\pi}{j^2}\exp\left(\frac{c}{2}\right)$$
$$= \frac{4\pi}{j\exp(j\rho)}\left[2\rho - \frac{1}{j}\right].$$

□

Lemma A.2.4. *For $\tilde{y} \in \tilde{\Lambda}_\rho$ and if $j > 2\rho$ it holds that*

$$\int_{\tilde{\Lambda}_\rho \setminus \tilde{\Lambda}_{c/j}} \left| j \exp\Big(-\frac{j}{2}|\tilde{x} - \tilde{y}|\Big)\right| d\tilde{x} \leq 4\pi \exp(-j\rho)\left[2\rho - \frac{1}{j}\right] \qquad (j \in \mathbb{N}),$$

where $c > 0$.

The proof follows directly from Lemma A.2.3.

Lemma A.2.5. *For $\tilde{y} \in \tilde{\Lambda}_\rho$ and if $j > 2\rho$ it holds that*

$$\int_{\tilde{\Lambda}_\rho} \left| \exp\left(-\frac{j|\tilde{x} - \tilde{y}|}{2}\right)\right| d\tilde{x} \leq \frac{4\pi}{j\exp(j\rho)}\left[2\rho - \frac{1}{j}\right] + \frac{4\pi}{j^2} \qquad (j \in \mathbb{N}).$$

Proof. We first convert Cartesian coordinates into polar coordinates. Then, for $j > 2\rho$ we have

$$\int_{\tilde{\Lambda}_\rho} \left| \exp\left(-\frac{j|\tilde{x} - \tilde{y}|}{2}\right)\right| d\tilde{x} \leq 2\pi \int_0^{2\rho} \left| r \exp\left(-\frac{jr}{2}\right)\right| \, dr.$$

Therefore, using partial integration, we see

$$2\pi \int_0^{2\rho} \left| r\exp\left(-\frac{jr}{2}\right)\right| \, dr = \frac{4\pi}{j}\left[r\exp\left(-\frac{jr}{2}\right)\right]_0^{2\rho} - \frac{2\pi}{j}\int_0^{2\rho} |\exp\left(-\frac{jr}{2}\right)| \, dr$$
$$= \frac{8\pi\rho}{j}\exp(-j\rho) - \frac{4\pi}{j^2}\left[\exp\left(-\frac{jr}{2}\right)\right]_0^{2\rho}.$$

Moreover, one computes that

$$2\pi \int_0^{2\rho} \left| r\exp\left(-\frac{jr}{2}\right)\right| \, dr = \frac{8\pi\rho}{j}\exp(-j\rho) - \frac{4\pi}{j^2}\exp(-j\rho) + \frac{4\pi}{j^2}$$
$$= \frac{4\pi}{j\exp(j\rho)}\left[2\rho - \frac{1}{j}\right] + \frac{4\pi}{j^2}.$$

□

Lemma A.2.6. *For $\tilde{y} \in \tilde{\Lambda}_\rho$ and if $j > 2\rho$ it holds that*

$$\int_{\tilde{\Lambda}_\rho} \left| j \exp\Big(-\frac{j}{2}|\tilde{x} - \tilde{y}|\Big)\right| d\tilde{x} \leq 4\pi \exp(-j\rho)\left[2\rho - \frac{1}{j}\right] + \frac{4\pi}{j} \qquad (j \in \mathbb{N}),$$

where $C > 0$.

The proof follows directly from Lemma A.2.5.

Finally, the following lemma introduces an integral identity.

Lemma A.2.7. *Consider for $N \in \mathbb{N}$ and $f \in L^2(\Lambda_R)$ the truncated series*

$$v_N = \sum_{j=1}^{N} \phi_j(x_2) \int_0^H \phi_j(y_2) \mathcal{V}_j f(\cdot, y_2)\, dy_2,$$

where $\mathcal{V}_j : L^2([-R,R]) \to L^2([-R,R])$ denotes the integral operator from (4.6),

$$f \mapsto \int_{-R}^{R} \frac{i}{2\lambda_j} \exp(i\lambda_j |x_1 - y_1|) f(y_1)\, dy_1, \qquad j \in \mathbb{N}.$$

Then for $R > 0$ it holds that

$$\int_{\Lambda_R} \Delta v_N \psi + \frac{\omega^2}{c^2(x_2)} v_N \psi \, dx = -\int_{\Lambda_R} \sum_{j=1}^{N} (f(x_1, \cdot), \phi_j)_{L^2(0,H)} \phi_j(x_2) \psi \, dx, \qquad \psi \in C_0^\infty(\Lambda_R).$$

Proof. Using the definition of $\mathcal{V}_j$, we first see

$$\begin{aligned}
&\int_{\Lambda_R} \Delta v_N \psi + \frac{\omega^2}{c^2(x_2)} v_N \psi \, dx \\
=&\frac{i}{2} \int_{\Lambda_R} \left(\frac{\partial^2}{\partial x_1^2} \left[\int_0^H \sum_{j=1}^{N} \int_{-R}^{R} \frac{\exp(i\lambda_j |x_1 - y_1|)}{\lambda_j} f(y_1, y_2) dy_1\, \phi_j(y_2) dy_2 \right] \psi_1(x_1) \right) \phi_j(x_2) \psi_2(x_2)\, dx \\
&+ \frac{i}{2} \int_{\Lambda_R} \int_{\Lambda_R} \sum_{j=1}^{N} \lambda_j^2 \phi_j(x_2) \phi_j(y_2) \frac{\exp(i\lambda_j |x_1 - y_1|)}{\lambda_j} f(y)\, dy\, \psi(x)\, dx. \qquad \text{(A.10)}
\end{aligned}$$

We point out that since the sum is finite we can interchange the sum integral. Next, for simplicity we substitute $z_1 = x_1 - y_1$ and we define

$$\Phi(z_1) := \frac{\exp(i\lambda_j |z_1|)}{\lambda_j}.$$

Then, we have the convolution

$$(\Phi * f)(x_1, y_2) = \int_{-R}^{R} \frac{\exp(i\lambda_j |x_1 - y_1|)}{\lambda_j} f(y_1, y_2)\, dy_1,$$

where $*$ denotes the convolution operator. We derive the function Φ in x_1-direction and we deduce

$$\frac{\partial \Phi}{\partial z_1} = i \operatorname{sgn}(z_1) \exp(i\lambda_j |z_1|) \qquad \text{and} \qquad \frac{\partial^2 \Phi}{\partial z_1^2} = 2i\delta_0 - \lambda_j^2 \Phi,$$

in the sense of distributions, see [Wal94]. Consequently, by derivation in x_1-direction of the convolution operator it follows that

$$\frac{\partial^2}{\partial x_1^2} (\Phi * f) = \frac{\partial^2 \Phi}{\partial x_1^2} * f = -\lambda_j^2 (\Phi * f) + 2if.$$

The last equation implies for the part of the derivation in x_1 direction in (A.10) that

$$\begin{aligned}
&\frac{\partial^2}{\partial x_1^2} \int_0^H \left[\sum_{j=1}^{N} \int_{-R}^{R} \left[\frac{1}{\lambda_j} \exp(i\lambda_j |x_1 - y_1|) f(y_1, y_2) \right] dy_1\, \phi_j(y_2) \right] dy_2 \\
&\qquad = -\frac{i}{2} \sum_{j=1}^{N} \int_{\Lambda_R} \lambda_j \exp(i\lambda_j |x_1 - y_1|) f(y) \phi_j(y_2)\, dy - \sum_{j=1}^{N} \int_0^H f(x_1, y_2) \phi_j(y_2)\, dy_2.
\end{aligned}$$

We insert this in (A.10) and obtain that

$$\int_{\Lambda_R} \Delta v_N \psi + \frac{\omega^2}{c^2(x_2)} v_N \psi \, dx = - \int_{\Lambda_R} \sum_{j=1}^{N} (f(x_1, \cdot), \phi_j)_{L^2(0,H)} \phi_j(x_2) \psi \, dx.$$

This ends the proof. □

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