ON THE MATHEMATICAL JUSTIFICATION OF THE CONSISTENT-APPROXIMATION APPROACH AND THE DERIVATION OF A SHEAR-CORRECTION-FACTOR FREE REFINED BEAM THEORY

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Kurzzusammenfassung

Die „*konsistente Approximation*“ ist eine Methode, um analytische Theorien für die Statik dünnwandiger Tragwerke aus der unumstrittenen dreidimensionalen Theorie der linearen Elastizität abzuleiten. Die Methode wurde bereits erfolgreich eingesetzt, um verfeinerte Theorien für isotrope und auch anisotrope Platten abzuleiten. Sie beruht darauf, die Euler-Lagrange-Gleichungen einer abgebrochenen Reihenentwicklung der potentiellen Energie aufzustellen.

In dieser Dissertation erweitern wir den Ansatz aus Kienzler (2002) um das gleichzeitige Abbrechen einer Reihenentwicklung der dualen Energie. Aus den Euler-Lagrange-Gleichungen der abgebrochenen Reihenentwicklung der dualen Energie können kompatible Randbedingungen rigoros abgeleitet werden. Die Reihenentwicklungen beider Energien beruhen auf der Taylor-Reihenentwicklung des Verschiebungsfeldes. Wir zeigen, dass das Abklingverhalten der Energie-Summanden anfänglich durch charakteristische Parameter dominiert wird, welche die relative Dünnheit der Struktur beschreiben. Konsequenterweise werden die Reihenentwicklungen nach einer bestimmten Potenz der charakteristischen Parameter abgebrochen.

Für den Fall eines homogenen, eindimensionalen Tragwerks mit Rechteck-Querschnitt erbringen wir die mathematische Legitimierung der Methode durch den Beweis einer a-priori Fehlerabschätzung. Diese impliziert die Konvergenz der Lösung des abgebrochenen eindimensionalen Problems gegen die exakte Lösung der dreidimensionalen Elastizitätstheorie, wenn die Dicke des Tragwerks gegen Null geht. Genauer gesagt klingt der Fehler der Lösung der Nten Approximationsordnung mit der $(N+1)$ten Potenz der charakteristischen Parameter ab, so dass ein wesentlicher Genauigkeitsgewinn für Theorien höherer Approximationsordnung zu erwarten ist, falls das Tragwerk hinreichend dünn ist.

Das nicht abgebrochene eindimensionale Problem ist äquivalent zum dreidimensionalen Problem der linearen Elastizitätstheorie. Wir beweisen, dass dieses Problem für *isotropes* Material aus vier *unabhängigen* Teilproblemen besteht: Ein Stab-, ein Torsions-, und zwei Balken-Probleme mit orthogonalen Belastungsrichtungen. Wir führen eine Zerlegung der Komponentenfunktionen der Lasten nach der Symmetrie bezüglich der Querschnittsachsen ein, wodurch jeder dreidimensionale Lastfall eindeutig zerlegt wird. Jeder Teil dieser Zerlegung kann eindeutig als Triebkraft einer der vier (exakten) eindimensionalen Probleme identifiziert werden. Des Weiteren zeigen wir, wie das Koppelungsverhalten der vier Teilprobleme für jedes beliebige *anisotrope* Material direkt aus der Besetzungsstruktur des Elastizitätstensors abgeleitet werden kann. Da alle Aussagen für das exakte eindimensionale Problem bewiesen werden, gelten sie gleichsam für die abgebrochenen Theorien beliebiger Approximationsordnungen N.

Wir wenden das Verfahren an, um eine Balkentheorie der zweiten Approximationsordnung für isotropes Material abzuleiten, welche sich als neu herausstellt. Die Ableitung der Theorie ist frei von a-priori Annahmen, insbesondere wird keine Schubkorrektur eingeführt. Die Theorie ist im Allgemeinen nicht kompatibel zur Timoshenko-Balkentheorie, da letztere nur eine, die abgeleitete Theorie jedoch drei Lastresultanten enthält, welche im Allgemeinen unabhängig voneinander sind. Des Weiteren berücksichtigt die Timoshenko-Balkentheorie keinerlei Effekte in Breitenrichtung. Dennoch erlaubt die Betrachtung eines einfachen Lastfalls den direkten Vergleich der Differentialgleichungen, wodurch zwei Schubkorrekturfaktoren für die Benutzung in der Timoshenko-Balkentheorie gewonnen werden können.

Abstract

The "*consistent approximation*" technique is a method for the derivation of analytical theories for thin structures from the settled three-dimensional theory of elasticity. The method was successfully applied for the derivation of refined plate theories for isotropic and anisotropic plates. The approach relies on computing the Euler-Lagrange equations of a truncated series expansion of the potential energy.

In this thesis we extend the approach given in Kienzler (2002) towards the simultaneous truncation of a series expansion of the dual energy. The computation of the Euler-Lagrange equations of the truncated series expansion of the dual energy ensures a rigorous derivation of compatible boundary conditions. The series expansions of both energies are gained by Taylor-series expansions of the displacement field. We show that the decaying behavior of the energy summands is initially dominated by characteristic parameters that describe the relative thinness of the structure. Consequently, the energy series are truncated with respect to the power of the characteristic parameters.

For the case of a homogeneous, one-dimensional structural member with rectangular cross-section we proof an a-priori error estimate that provides the mathematical justification for this method. The estimate implies the convergence of the solution of the truncated one-dimensional problem towards the exact solution of three-dimensional elasticity as the thickness goes to zero. Furthermore, the error of the Nth-order one-dimensional theory solution decreases like the $(N+1)$th-power of the characteristic parameter, so that a considerable gain of accuracy could be expected for higher-order theories, if the structure under consideration is sufficiently thin.

The untruncated one-dimensional problem is equivalent to the three-dimensional problem of linear elasticity. We prove that the problem decouples into four *independent* subproblems for *isotropic* material: a rod-, a shaft- and two orthogonal beam-problems. A unique decomposition of any three-dimensional load case with respect to the direction and the symmetries of the load is introduced. It allows us to identify each part of the decomposition as a driving force for one of the four (exact) one-dimensional subproblems. Furthermore, we show how the coupling behavior of the four subproblems can be derived directly from the sparsity scheme of the stiffness tensor for general *anisotropic* materials. Since all propositions are proved for the exact one-dimensional problem, they also hold for any approximative Nth-order theory.

The approach is applied to derive a new second-order beam theory for isotropic material free of a-priori assumptions, which in particular does not require a shear-correction. The theory is in general incompatible with the Timoshenko beam theory, since it contains three in general independent load resultants, whereas Timoshenko's theory only contains one. Furthermore, Timoshenko's theory ignores any effects in width direction. However, the assumption of a simple load case allows for a vis-a-vis comparison of both differential equations and in turn, two shear-correction factors for the use in Timoshenko's theory can be derived.

Contents

1 Introduction

In this introductory section we embed the main results derived in the thesis at hand into the general picture of recent progress achieved in the field of refined theories of thin structures.

1.1 State of the art

The three-dimensional theory of linear elasticity is considered as settled within the applied mechanics community. On the one hand, the modeling equations can be derived in a mathematically rigorous way from first-principles (cf., e.g., Marsden & Hughes, 1983; Zeidler, 1997, chapter 61). On the other hand, a variety of robust numerical solvers is available in the form of Finite Element programs, which allow for the derivation of elastic solutions with the desired accuracy for basically all practical applications that may be treated within the linear theory of elasticity.

Nevertheless, analytical theories for thin elastic structures that are modeled on one-dimensional intervals or two-dimensional areas are of enduring interest. Mainly because of two reasons: The first reason is that one-dimensional theories in particular allow for the derivation of analytical closed-form solutions, which may be directly solvable for target dimensioning parameters. This is the easiest and most effective way to dimension structural members. The second reason is that a sound understanding of the analytical theories is the basis for the derivation of lower-dimensional Finite Elements for the numerical solution of problems involving thin structures. Since the structural member is not triangulated in the thickness direction when using lower-dimensional elements, the computational effort for the derivation of the solution is reduced enormously. Although computer technology evolves fast, this aspect might even gain importance, since the enduring trend towards lightweight construction leads to real-world structures of increasing complexity, consisting of more and more thin structural members.

Engineering mechanics classifies theories for thin structures by their geometry and the applied load case (and in general by the underlying material model, which is linear elastic in the whole contribution). A general two-dimensional thin structure is called shell. Since the three-dimensional problem of a plane thin body with constant thickness decouples for homogeneous monoclinic material (if the plane of symmetry is the mid-plane) into two independent subproblems (cf., e.g., Altenbach et al., 1998), one decomposes the plane shell problem into: The *membrane* problem (or *disk* problem) loaded in-plane, and the transversally loaded *plate* problem. Likewise, a straight thin one-dimensional structure is called: *rod*, if loaded in axial direction, *beam*, if loaded by transversal (shear) forces and bending moments, and *shaft*, if loaded by torsional moments.

The so-called "classical" theories for thin structures treat homogeneous isotropic material. The classical beam theory, called the Euler-Bernoulli beam theory, dates back to the 18th century and carries its name due to major contributions of Leonhard Euler (1707-1783), Jacob Bernoulli (1654-1705) and Daniel Bernoulli (1700-1782) and is nowadays an essential part of every basic course on mechanics of materials (cf., e.g., Schnell et al., 2002; Hibbeler, 2000). The classical plate theory was developed by Kirchhoff (1850). Classical theories for the buckling analysis of plates and shells, geometric linear and nonlinear, are associated with the name von Kármán (1910). All classical theories were modeled by the use of disputable a-priori assumptions and, therefore, questioned for a long period of time, although they showed reasonable results in comparison to experiments when the structure under consideration was sufficiently thin. In particular the modeling approach of the Hungarian aerospace engineer von Kármán, who basically "combined" a membrane and plate theory, led to the phenomena that his equations "play an almost mythical role in applied mathematics", (Ciarlet, 1997, p. 367). Ciarlet himself could do no better than citing the famous Truesdell: “An analyst may regard that theory (von Kármán’s theory of plates) as handed out by some higher power (a Hungarian wizard, say) and study it as a matter of pure analysis. To do so for von Kármán’s theory is particulary tempting because nobody can make sense out of the ‘derivations’ ...”, (Truesdell, 1977, pp. 601-602). Alternative derivation methods were provided (Meenen & Altenbach, 2001) for the von Kármán theory. Nevertheless, nowadays all classical theories are considered as settled mostly due to a series of recent results (Friesecke et al., 2002a,b) that state them to arise as a limit of the three-dimensional elasticity when the thickness goes to zero. Therefore, they provide a rigorous mathematical justification. The proofs use the comparatively young method of Γ-convergence, which was developed by Giorgi (1975).

A long time before the mathematical justification of the classical theories was provided, engineers felt the practical need to develop refined theories for beams and plates that allow for a treatment of moderately thick structures, i.e., they sought for theories that provide higher accuracy. The nowadays most established, lets say, “classical refined” theories are the Timoshenko-beam (Timoshenko, 1921, 1922) and the Reissner-Mindlin plate, which was developed independently by Reissner (1944, 1945) and Mindlin (1951). Both of these theories were developed by the use of a-priori assumptions, partially motivated from experimental observations. Since then the development of refined theories has become a wide field, which is still under heavy development: “Many outstanding mechanicians have contributed to the field, probably over one million research papers have been published as well as over one thousand books [...]. And yet, each week one can find many new papers and reports on the Internet on various plate and shell problems [...]. The main source of such popularity of this field is that plates and shells are basic structural elements of modern technology and everyday life.”, Eremeyev & Pietraszkiewicz (2014).

While many publications still use a-priori assumptions, whole schools of methods have grown that tend to avoid the use of a-priori assumptions. A review at length about recent developments in plate theory can be found, e.g., in Ghugal & Shimpi (2002). A large review article about developments in beam theory is, e.g., Kapania & Raciti (1989a,b). Beside the development of new beam theories, a lot of alternative shear-correction factors for the use in combination with Timoshenko’s theory have been published. The largest collection of shear-correction factors can be found in Kaneko (1975). With regard to newer publications (Hutchinson, 2000; Franco-Villafañe & Méndez-Sánchez, 2014) the collection still seems to contain all shear-correction factors that are established for practical applications.

Today we have in principle two active branches for the structured development of refined theories: A branch based upon Cosserat continua (cf., e.g., Altenbach et al., 2010) and another one that derives lower-dimensional theories from the three-dimensional theory of elasticity by means of series expansions. From the later one we only mention three lines of work of a-priori assumption-free approaches we consider as the most rigorous ones and will refer to them later.

At first the school initiated by Vekua (1955, 1985), which is based on a displacement ansatz with truncated series expansion with respect to a basis of Legendre polynomials. Taking more series coefficients into account leads to more complex theories, so that a hierarchy of increasing complexity is established. The method was frequently applied for the derivation of refined plate and shell theories, e.g., by Poniatovskii (1962), Haimovici (1966), Soler (1969), Khoma (1974) and Zhgenti et al. (1980) to mention only a few and the earliest contributions.

Secondly a so-called restricted-type theory for mixed plate-membrane problems introduced by Steigmann (2008, 2012) and recently extended by Pruchnicki (2014), which combines established modeling approaches of Koiter (1966, 1970a) by arguments taken from contributions based on Γ-convergence.

And finally the so-called “consistent” approach (or uniform-approximation approach), which originates from treatises by Naghdi (1963), Koiter (1970b), Krätzig (1980) and Kienzler (1980). We understand the approach in the way it was applied by Kienzler (2002, 2004), who derived refined theories from the Euler-Lagrange equations of the truncated elastic energy. This approach was extended towards anisotropic material in Schneider et al. (2014) and recently a comparison with a variety of other theories was published (Schneider & Kienzler, 2014b). The approach is similar, but not identical to the "unrestricted approach" introduced in Steigmann (2008) (probably due to common roots, which are the classical treatises of Koiter).

Although all three approaches are free of a-priori assumptions and give rise to hierarchies of truncated theories of increasing complexity, it is unclear whether a specific truncated theory is capable of describing all relevant effects in order to reduce the approximation error significantly. A rigorous mathematical justification is still missing. (Of course this applies more than ever to all non-mentioned theories that do not even follow a rigorous line of reasoning during their derivation.) In general the method of Γ-convergence, which was successfully applied for the mathematical justification of the classical theories, is unlikely to be able to justify refined theories, since, as a limit analysis, it always derives the leading-order approximation, whereas refined theories have to consider effects of different scales.

1.2 The main results of this thesis

The main aim of this thesis is to provide the mathematical justification of the uniform-approximation approach for the derivation of one-dimensional analytic theories, i.e., rod, beam and shaft theories. This is achieved by an a-priori error estimate (cf. (5.27) and theorem 13) that verifies the approximation property. Precisely, the estimate states that the squared approximation error of the solution v of the consistent Nth-order approximation theory compared to the solution u of the exact problem of three-dimensional elasticity decreases with the $2(N+1)$th power of e

$$k \left\| v - u \right\|_X^2 = O\left(e^{2(N+1)}\right),$$

where $e \ll 1$ is basically a geometric constant describing the relative thickness of the cross-section. This estimate implies the convergence of any Nth-order approximation solution v towards the exact solution u for $b, h \longrightarrow 0$ (where h is the thickness and b is the width of the cross-section) and moreover (and more importantly) the estimate states that the accuracy of the solution v of an Nth-order theory increases significantly for every incrementation of $N \in \mathbb{N}_0$.

The thesis treats the case of the derivation of one-dimensional theories, since the proof of the approximation property is more difficult than for the two-dimensional theories. The arguments presented can readily be applied for the derivation of two-dimensional theories, which will be obvious for the reader who is familiar with the article Schneider et al. (2014). Since the Reissner-Mindlin theory (and some other theories, cf., Schneider & Kienzler, 2014b) are equivalent to the second-order consistent plate theory (within the second-order framework, i.e., beside differences of order e^6) this provides also mathematical justification for this established theory, for the first time. In addition, due to Kienzler (2002), the consistent first-order theory equals Kirchhoff's theory, which was already justified by means of Γ-convergence. Also, due to Schneider et al. (2014), there is a consistent second-order plate theory for monoclinic material. For the special case of orthotropic material, the first-order truncation of this theory equals the classical theory of orthotropic plates, which was mainly developed by Huber (1921, 1926, 1929) according to the classical book of Lekhnitskii (1968).

In this thesis, we show that the first-order beam theory is the Euler-Bernoulli theory, which was already proved by means of Γ-convergence. As another main result we derive the second-order consistent beam theory (cf. section 8), which is not known from the literature. In turn Timoshenko's theory turns out to be inconsistent with our approach. However, with some further load-case assumptions we are able to derive a theory comparable to Timoshenko's theory. This allows the identification of two shear-correction factors for use in Timoshenko's theory from the a-priori assumption-free second-order beam theory (which especially does not introduce any shear-correction) introduced in this thesis.

Before the second-order beam theory can be derived, we have to deal with a very fundamental question, we were very surprised not to find an answer for in the existing literature. As already mentioned, one of the most basic concepts of engineering mechanics for thin structures is to classify one- or two-dimensional problems not only according to their geometry, but in addition according to the direction of applied loads. The common definition of beams from a basic mechanics-of-materials text book is "members that are slender and support loadings perpendicular to their longitudinal axis", Hibbeler (2000). It is trivial that one can not take this definition verbally for the load cases of the three-dimensional theory of linear elasticity, since it lacks to mention the corresponding bending moments. (One can apply a bending moment to a beam by one-sided boundary tractions in longitudinal direction, though by the prior definition this would be no beam load case.) So the question arises, what is the (most general) definition of a beam-load case for the three-dimensional theory? As we will show in this thesis even an orthogonal decomposition with respect to the three coordinate directions of the resulting forces and bending moments is insufficient, since the decomposition will not enable us to uniquely decompose every three-dimensional load case into the driving forces of appropriate one-dimensional subproblems. This shows that the classical decomposition has to be extended in order to be compatible with the three-dimensional theory of elasticity.

Another essential point is that a decoupling of the arising one-dimensional subproblems is crucial for a load-case decomposition to be meaningful. For example, by the classical decomposition according to the directions of load, one would decompose a two-dimensional force into a transversal and longitudinal component defining a beam- and a rod-load case, respectively, as illustrated in figure 1. The principle of superposition, i.e., the assumption of small deformations and the

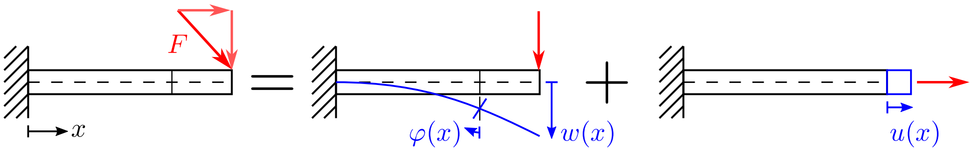

Figure 1: A problem is divided into two subproblems by the orthogonal decomposition of the applied load.

linearity of the material law, is the only justification that is provided for this procedure from generic text books (Hibbeler, 2000, section 8.2). Of course the principle of superposition implies that the overall solution is the sum of the solutions of the subproblems, but it does not state that the deformation quantities of the beam problem w and φ are independent from the rod-load and vice versa. This decoupling is essential, since we would compute wrong displacement quantities from the subproblems on their own, if the decoupling is not given, which would render the whole idea of a decomposition into subproblems senseless. We do not know any result that actually states this form of independence for the classical one-dimensional problems. Moreover, if we allow for anisotropic materials, the problems are indeed not independent. (Cf. section 6.6 for a simple example.)

The situation is clarified by another main result of this thesis (cf. theorem 21). It proves that *all* three-dimensional load cases, on a quasi one-dimensional geometry, can be uniquely decomposed into the driving forces of four one-dimensional subproblems: a rod-, a shaft- and two orthogonal beam-problems, independent of the material properties. To this end, we introduce a detailed definition of the subproblems that considers not only the direction of load, but also the symmetry of the load with respect to the cross-section axes. (Cf. section 8.1 for the rigorous definition of the beam-load case.) We prove that the four subproblems are *decoupled* for *isotropic* material, i.e., that the sets of the unknown displacement quantities of the four problems are disjoint from each other and that the solution of one of the four subproblems is *independent* of the parts of the applied loads that belong to the other subproblems. Furthermore, we provide a fast and elegant way to derive the *coupling* behavior of the four subproblems directly from the sparsity of the stiffness tensor for *any kind of anisotropic* material. E.g., we find the four subproblems to be decoupled for orthotropic material, where the planes of symmetry are given by the coordinate planes, or we have coupled problems of pairwise two classical subproblems for the case of monoclinic materials, where the plane of symmetry is given by two coordinate axes. (Cf. the end of section 6.5.)

1.3 Roadmap, scope and demarcation from prior work

Section 2
In section 2 we recall the basic equations of the three-dimensional theory of linear elasticity, which can be found in any basic text book on "Continuum Mechanics" (cf., e.g., Kienzler & Schröder, 2009), as a point of departure. The section aims at familiarizing the reader with the subsequently used notation.

Section 3
The first main result is the approximation theorem (theorem 13 in section 5.4). The first step towards this goal is the derivation of a general error estimate for the three-dimensional theory of linear elasticity (theorem 11) that is derived in section 3.

The basic line of argumentation in section 3 is mostly adapted from the series of books "Nonlinear Functional Analysis and its Applications" by Zeidler (1990a,b, 1985, 1997) which clearly addresses experienced mathematicians. In contrast, this thesis addresses engineers at the master student's level. Therefore, we skip a lot of regularity questions, which allows us to drop a lot of the mathematical notation overhead. However, none of the essential assumptions or arguments are dropped. Even basic concepts from the calculus of variations are *not* presupposed. We invest a lot of effort, so that the way to theorem 11 may be accepted as sketch of a proof with omitted regularity questions by mathematicians, while being fully understandable for advanced students of engineering sciences.

In order to arrive at theorem 11, we start with the proof of a general theorem (theorem 7 in section 3.2) stating existence and uniqueness of solutions of the linear theory of three-dimensional elasticity, as well as the principle of virtual work (equivalence of the weak problem) and the principle of minimum potential energy. The proof is an extension of theorem 61.D from Zeidler (1997) towards general anisotropy. We basically already presented it in Schneider (2010) and refer to that work (freely online available and permanently hosted by the German national library) for some less important calculations. The first error estimate (theorem 8 in section 3.2), which uses only the potential energy, is derived by a simple calculation once we have all prerequisites available which lead to theorem 7.

All prior publications in the line Kienzler (2002, 2004); Schneider et al. (2014); Schneider &

Kienzler (2014b, 2015) derived the approximative theories by the truncation of the potential energy only. In this thesis we first introduce the simultaneous approximation of the dual energy. The complete set of modeling equations for the three-dimensional problem of linear elasticity written in terms of the displacement, i.e., the Navier-Lamé formulation, is only gained by computing the Euler-Lagrange equations of both problems, the potential energy and the dual energy problem. Likewise, we show in this contribution that the dual energy can be used to derive appropriate displacement boundary conditions for the one-dimensional problems (cf. sections 5.4 and 8.5). Furthermore, the use of dual energy is essential for the proof of theorem 13. The proof of the duality (theorem 9 in section 3.5) is basically taken from (Zeidler, 1997, section 62.16). It is based on a general duality principle, the Friedrichs duality (Friedrichs, 1928). The next theorem (theorem 10 in section 3.5) is merely a stricter form of theorem 9 that actually treats some regularity issues; it may be skipped on a first read. Finally, the proof of theorem 11 is very simple once the duality is proved and already performed in section 3.3 to motivate the introduction of dual energy.

Section 4
In section 4 we introduce the "quasi one-dimensional" geometry (section 4.2), of a beam-like structural member with constant rectangular cross-section, which is the basis for all further investigations. Additionally, we mainly introduce some basic techniques that will be used frequently in the following sections. Section 4.1 introduces the main idea of the consistent approximation, i.e., to sort the summands of the potential energy and dual energy by the amount of energy they contribute to the overall energy (descendent) and to generate approximative theories by calculating the Euler-Lagrange equations of the truncated energy series. As already outlined, the simultaneous truncation of the dual energy is novel. In order to be able to sort the summands by their magnitude we have to introduce the transformation to dimensionless coordinates (section 4.3) and the technique of Taylor-series expansions (section 4.5). The choice of this series expansion and the mathematical consequences are discussed in section 4.4. (Basically the Taylor-series is the only choice that leads to results comparable to classical theories, cf. section 5.5.) Finally we are able to order and truncate the potential energy in section 4.6. This subsection also contains a discussion of the advantages of the truncation approach of the consistent theories and the unrestricted Steigmann approach (which is the same), over the Vekua-type approach. The derived representation is, however, not satisfactory, which originates in the sparsity of the stiffness tensor. To this end we introduce some techniques for the renumbering of finite, nested sums that are introduced in section 4.7. The techniques allow us to reorder the summands by the appearance of the included scaling factors $d^n c^m$, which will be used, e.g., for the investigation of the stress-resultants (section 5.3). All parts of section 4.7 are developed from scratch without the influence of any literature.

Section 5
Section 5 finally provides the first main result, which is theorem 13. The exact one-dimensional representation of the problem of three-dimensional elasticity that is introduced in theorem 13 is intractable in practice, since it consists of an infinite number of equations and unknowns. However, the theorem proves that the consistent truncation of the exact problem leads to tractable problems whose solutions fulfill the desired error estimate (5.27). The problem is formulated in terms of stress resultants, which are introduced in section 5.1 and expressed in terms of displacement coefficients for the exact problem and any Nth-order approximation in section 5.2. The following subsection 5.3 derives a method to decide a-priori which stress-resultants have to be considered in an Nth-order approximation and reveals some dependencies among the

stress-resultants, which will be used in section 8.4. Subsection 5.4 finally provides the proof of theorem 13.

The existence of an exact one-dimensional problem was already published in Schneider (2010), which, furthermore, payed respect to regularity questions. In this thesis the dual energy is truncated simultaneously, for the first time. Also the factorization of the Euler-Lagrange equations of both energies in the right-hand side of the general error estimate is novel and could not be gained by the truncation of the potential energy alone. Eventually, this factorization is the key to the proof of the novel error estimate (5.27) for the consistent Nth-order theories.

In addition subsection 5.6 explains the treatment of mixed boundary conditions on the same face side, which is important for the treatment of some practical applications, like pinned bearings.

Section 6

Section 6 and 7 provide the next main result of the thesis, which is theorem 21. We already published most of the results of both sections in a more general form, allowing general double symmetric cross-sections, in the article Schneider & Kienzler (2015). However, the paper does not use the methods of section 6 for the main proof, but the alternative methods introduced in section 7.3. With the sole exception of the publication Schneider & Kienzler (2015) and a corresponding conference talk (Schneider & Kienzler, 2014a), the mathematical formalism of sections 6 and 7, which is based on the definition of the abstract shift operator $\mathcal{K}$, is entirely developed from scratch for this thesis and does not rely on any other publications.

Section 6.1 introduces the abstract shift operator and illustrates its use by the derivation of a key observation, which allows us already to identify the smallest number of possibly decoupled subproblems. By the use of this operator we are able to prove that the exact equilibrium conditions in terms of stress-resultants always decouple into four independent sets of equations (cf. theorem 15 in section 6.2), which holds independently of any possible material anisotropy. In the next subsection 6.3, we find that any three-dimensional load case can be decomposed into the driving forces of the four subproblems, which we identify as the (exact!) rod, shaft and two beam-problems with orthogonal loading directions, cf. theorem 19. In order to derive the driving forces of a specific subproblem the applied load has to be decomposed with respect to the parities of every component function in the cross-section directions. In section 6.5 we prove that the (exact) one-dimensional equilibrium equations in terms of displacement coefficients decouple into four independent sets of equations, if we have an isotropic material (cf. theorem 20). Also we find a general (easy to use) method to derive the coupling behavior of the four subproblems directly from the sparsity scheme of the stiffness tensor for an arbitrary anisotropic material (cf. theorem 20 and figures 5 and 6). Finally, the main result (theorem 21) is provided in section 6.7. In this subsection we prove that not only the equilibrium conditions are decoupled for an appropriate anisotropic material, but in addition the corresponding (exact) boundary conditions are decoupled, too. Therefore, we have indeed four decoupled mixed boundary value *problems*, if the equilibrium conditions decouple. Since all theorems of section 6 are proved for the exact one-dimensional problem, they also hold for any Nth-order theory that is gained by the consistent truncation of the exact problem.

Finally section 6 provides two easily comprehensible examples that underline the plausibility of the proved theorems. A mixed rod-beam problem that is coupled due to material anisotropy is presented in section 6.6, and a load-case decomposition of a uniform-topside pressure is given in section 6.4, with the surprising outcome that this load-case, which one might mistake for a canonical beam-load case (due to illustrations in every basic course text book on “mechanics of materials”), is actually decomposable into a beam *and* a rod-load case.

Section 7

The main aim of section 7 is to provide a handy derivation technique of the equilibrium equations of an Nth-order consistent theory in terms of displacement coefficients. The basic technique is introduced in section 7.2. It is based on the fact that any equilibrium condition can be written as an infinite sum of differential operators, where each operator is applied to exactly one displacement coefficient. The resulting differential operators for an orthotropic material are given in section 7.4. In order to apply the technique for an Nth-order theory, one first has to derive which displacement coefficients and equations are to consider for the theory, which can be done as outlined in section 7.7. The effort for deriving the equation systems is effectively halved, since the resulting equation system can be made symmetric for orthotropic materials, as we prove in section 7.6 (cf. theorem 22). Also the so-generated equation systems for the two beam problems are equivalent as shown in section 7.9. Furthermore, we show how the technique allows for alternative proofs of the theorems of section 6 in section 7.3, and we prove in section 7.8 that the truncation of the load resultants for the Nth-order theories is consistent with the derivation of the equilibrium conditions.

Section 8

Section 8 provides the last main result of the thesis, the derivation of a consistent second-order beam theory for isotropic material.

One of the greatest merits of the sections 6 and 7 is that they allow for rigorous definitions of the most general load cases for one-dimensional problems. Section 8.1 states the rigorous definition of the beam problem. However, we restrict this most general load-case again, in order to derive a comparable theory.

The equilibrium conditions are given in terms of displacement coefficients in section 8.2 by applying the technique introduced in section 7. The system is reduced to a single ordinary differential equation in one unknown displacement coefficient, the deflection w, by application of the pseudo-reduction technique in section 8.3. The principle technique was already introduced in Kienzler (2002) for the derivation of two-dimensional theories. This simplifies the pseudo reduction significantly, since only one characteristic parameter is involved. We suggested an extension towards one-dimensional theories involving two characteristic parameters in Schneider & Kienzler (2011). However, the technique applied in this thesis is modified again, in order to deal with inconsistencies that may arise from the original approach. We also provide the corresponding stress resultants in terms of the deflection (section 8.4) and the displacement boundary conditions (section 8.5). Finally, the main equation and stress resultants are rewritten in order to achieve better compatibility with the boundary conditions (section 8.6).

The resulting theory is to our best knowledge new and not yet published. In turn, the Timoshenko theory is in general inconsistent with the modeling approach provided here. One main reason is that Timoshenko uses a plain-stress modeling approach, which turns out to be unacceptable for refined theories within the consistent framework. Another reason is that Timoshenko's theory only provides one overall load resultant q, whereas the consistent model requires three, in general independent, load resultants. However, by restricting the load-case even further (especially by neglecting dead weight!), we are also able to derive a theory comparable to the theory of Timoshenko, in order to derive two shear-correction factors for the use in Timoshenko's theory (section 8.7).

2 The three-dimensional theory of linear elasticity

In this section we introduce the notation and give a short summary of the basic equations of the three-dimensional theory of linear elasticity as a point of departure.

2.1 Tensor notation

The physical quantities of linear elasticity are tensor fields. We assume the reader to be familiar with the basic concepts of tensor calculus. For an introduction we refer to (Zeidler, 1997, Chapter 74).

For the sake of simplicity we restrict ourselves to a fixed Cartesian coordinate system in the whole contribution, i.e., the natural basis is a positively oriented orthonormal system, and physical points are denoted by their coordinates $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ with respect to that basis.

In general, if not explicitly stated otherwise, we use the summation convention of tensor calculus in the following form: Lower case indices are tensor indices, which are to be summed up, if they occur twice in a product or in connection with a partial derivative. To this end, the partial derivative of a physical quantity F with respect to the coordinate x_i is denoted by

$$(F)_{|i} := \frac{\partial}{\partial x_i}(F).$$

Latin tensor indices are always from the set $\{1, 2, 3\}$, whereas greek tensor indices are from the set $\{1, 2\}$. If a tensor index is not bound through the summation convention, the equation is valid for any index from the corresponding set. An exception are notations of the form $v = (v_i)$; here v denotes the whole vector, while v_i is the ith component function.

Upper case indices will be used as series indices. We do not use any form of summation convention for these indices.

For an actual curvilinear and/or not normal coordinate system one needs to distinguish between covariant and contravariant tensors (cf., Zeidler, 1997, definition 74.4). We do not distinguish the tensor indices because we have the following special situation for our fixed Cartesian coordinate system (cf., Zeidler, 1997, section 74.5): The twice covariant and the twice contravariant metric tensor fields equal the Kronecker symbol and the Christoffel symbols are identical to zero. Therefore, the covariant derivative (cf., Zeidler, 1997, definition 74.17) becomes the partial derivative.

To obtain an equation that *automatically holds* (cf., Zeidler, 1997, section 74.5) in an arbitrary coordinate system from the equations of this contribution, one has just to proceed the following steps:

- Shift the tensor indices of the equation, so that the bound indices in the equation coincide with the bound indices of the usual co/contravariant summation convention (cf., Zeidler, 1997, definition 74.1) and that the index picture is right (cf., Zeidler, 1997, definition 74.14).
- Replace the partial derivatives by covariant derivatives. To this end, eventually use contraction with a metric tensor field to keep the index picture right.

An example: The stress tensor in the equilibrium condition (2.1) is a two-fold tensor field. If we *choose* to rewrite equation (2.1) for the twice covariant stress tensor in an arbitrary curvilinear coordinate system, we get

$$\sigma_{ji|j} + f_i = 0 \quad \rightsquigarrow \quad \nabla_j \left(g^{jk} \sigma_{ki} \right) + f_i = 0,$$

where g^{jk} denotes the twice contravariant metric tensor field and ∇_j corresponds to the covariant differentiation operator.

2.2 The basic equations of three-dimensional linear elasticity

The problem of three-dimensional linear elasticity is well investigated. There is no discourse in the engineering mechanics community about the modeling equations, since they can be derived in a mathematically rigorous way from first principles (cf., e.g., Marsden & Hughes, 1983; Zeidler, 1997, chapter 61). We assume that the reader is familiar with the basic equations and give only a short summary to introduce the notation. Only in this section we will provide the invariant notations of the basic equations in addition to the index-notation that will be used for the remainder of the contribution. For an introduction to linear elasticity we refer to Kienzler & Schröder (2009).

In the linear theory of elasticity, the equations are formulated on the *undeformed reference configuration* of the elastic body. The specific parametrization of the body in x-coordinates is denoted by Ω_x. (Later on, Ω_ξ will denote the corresponding parametrization in dimensionless ξ-coordinates.) We consider a *stationary* problem, so that the general *equation of motion* (cf., Zeidler, 1997, section 61.3, equation 14) simplifies to the *equilibrium condition* of elastostatics (cf., Kienzler & Schröder, 2009, equation 2.7.4). It connects the divergence of the two-fold *stress tensor* field $\sigma = (\sigma_{ij})$ with the vector field of *volume force* $f = (f_i)$, which is assumed to be given.

$$\begin{aligned} \operatorname{div}(\sigma) + f &= 0 && \text{f.a. } x \in \Omega_x, \\ \sigma_{ji|j} + f_i &= 0 && \text{f.a. } x \in \Omega_x. \end{aligned} \tag{2.1}$$

In general, we will use "f.a." as an abbreviation for "for all". The physical meaning of the (Cauchy) stress tensor is given by Cauchy's formula (cf., Kienzler & Schröder, 2009, equation 2.2.2): For a cut through the body with outer unit normal vector field $n = (n_i)$, the traction vector t_i in every point x is given by $t_i = \sigma_{ji} n_j$. It might be helpful to consider that (2.1) implies

$$\int_{\partial\Theta} t_i \, dA_x = \int_{\partial\Theta} \sigma_{ji} n_j \, dA_x = \int_{\Theta} \sigma_{ji|j} \, dV_x = -\int_{\Theta} f_i \, dV_x,$$

by the use of Cauchy's formula and the divergence theorem (also called "Gauss's theorem"), i.e., for any subregion $\Theta \subset \Omega_x$ the resultant force by volume forces has to be compensated by a stress flux through the boundary of the subregion $\partial\Theta$, which is merely a different, maybe more intuitive, definition of equilibrium.

The stress tensor field is symmetric, i.e., $\sigma_{ij} = \sigma_{ji}$, which can be derived from the equilibrium of moments (also called torques) at an infinitesimal volume element (cf., Kienzler & Schröder, 2009, equation 2.1.25), or by the use of the equilibrium of moments and the equilibrium condition (2.1) for any subregion (cf., Kienzler & Schröder, 2009, equation 2.7.11-14; Zeidler, 1997, section 61.4d).

The *(Green-Lagrangian) strain tensor* is a symmetric two-fold tensor field, that physically describes the local stretching of the body (cf., Kienzler & Schröder, 2009, section 3.1; Zeidler, 1997, section 61.2). It is responsible for the fact that elasticity is in general a nonlinear theory.

Its linearized form, the *strain tensor* field $\varepsilon = (\varepsilon_{ij})$ (cf., Kienzler & Schröder, 2009, equation 3.1.8) is given by

$$\varepsilon := \frac{1}{2}(Du + (Du)^*) \qquad \text{f.a. } x \in \Omega_x,$$
$$\varepsilon_{ij} := \frac{1}{2}(u_{i|j} + u_{j|i}) \qquad \text{f.a. } x \in \Omega_x. \tag{2.2}$$

Here u is the *displacement vector* field, i.e., for every point $x \in \Omega_x$ in the undeformed reference configuration, the coordinate in the deformed body is given by $x + u(x)$ (cf., Kienzler & Schröder, 2009, equation 3.1.1). Obviously the linearized strain tensor is symmetric ($\varepsilon_{ij} = \varepsilon_{ji}$) by definition. The usage of the linearized strain tensor is the main reason for the approximative character of linear elasticity. However, in decades of application it has proven to be the theory to use, if the resulting displacement is sufficiently small.

Finally the constitutive law (or Hooke's law) describes the relation between strain and stress. In linear elasticity it has to be linear by definition. As a linear mapping between two two-fold tensors fields it is a four-fold tensor field $E = (E_{ijrs})$, the so-called *stiffness tensor* field. The three-dimensional *Hooke's law* reads

$$\sigma = E : \varepsilon \qquad \text{f.a. } x \in \Omega_x,$$
$$\sigma_{ij} = E_{ijrs}\varepsilon_{rs} \qquad \text{f.a. } x \in \Omega_x. \tag{2.3}$$

The physical meaning of the stiffness tensor is that it characterizes the elastic material of the body under consideration, therefore it is assumed to be given. The stiffness tensor is in general assumed to fulfill the symmetry relations

$$E_{ijrs} = E_{jirs}, \qquad E_{ijrs} = E_{ijsr}, \qquad E_{ijrs} = E_{rsij}. \tag{2.4}$$

The first two symmetries are suggested by the symmetries of ε and σ. The last one is equivalent to the existence of an elastic potential (cf., Kienzler & Schröder, 2009, equations 4.2.21-23; and E_{pot} in section 3.2).

We will deal with the mixed boundary value problem of linear elasticity. To this end, we assume that the boundary of the body is a disjoint union of two open regions:

$$\partial\Omega_x = \overline{\partial\Omega_{x0}} \cup \overline{\partial\Omega_{xN}}, \qquad \partial\Omega_{x0} \cap \partial\Omega_{xN} = \emptyset, \tag{2.5}$$

where, $\overline{\bullet}$ denotes the set theoretic closure of the set $\bullet$. On $\partial\Omega_{x0}$ we prescribe the displacement vector field $u_0 = (u_{0i})$ and on $\partial\Omega_{xN}$ we prescribe the traction vector field $g = (g_i)$ via

$$u_i = u_{0i} \qquad \text{f.a. } x \in \partial\Omega_{x0}, \tag{2.6}$$
$$\sigma_{ij}n_j = g_i \qquad \text{f.a. } x \in \partial\Omega_{xN}. \tag{2.7}$$

In the standard problem of linear elasticity, E, f, u_0 and g are assumed to be given. Considering the symmetries of ε and σ, the eqs. (2.1) to (2.3) lead to 15 equations for the 15 unknown component functions in (u, σ, ε). If (u, σ, ε) solves the field eqs. (2.1) to (2.3) and fulfills the boundary conditions (2.6) and (2.7), we call it a solution of the three-dimensional problem of linear elasticity.

By insertion of eqs. (2.2) and (2.3) into eq. (2.1) and by use of eq. (2.4), we gain a field equation entirely formulated in terms of the displacement u. By insertion of eqs. (2.2) and (2.3) into eq. (2.7) and by use of eq. (2.4), we gain the corresponding traction boundary condition. This formulation is called the *Navier-Lamé formulation* of linear elasticity.

$$\left(E_{ijrs}u_{r|s}\right)_{|j} = -f_i \qquad \text{f.a. } x \in \Omega_x, \tag{2.8}$$

$$u_i = u_{0i} \qquad \text{f.a. } x \in \partial\Omega_{x0}, \tag{2.9}$$

$$E_{ijrs}u_{r|s}n_j = g_i \qquad \text{f.a. } x \in \partial\Omega_{xN}. \tag{2.10}$$

We call the corresponding problem:

$$\text{Find } u \in \left[C^2(\Omega_x)\right]^3 \cap \left[C(\Omega_x \cup \partial\Omega_{x0})\right]^3 \cap \left[C^1(\Omega_x \cup \partial\Omega_{xN})\right]^3 : \ (2.8), (2.9), (2.10) \tag{Cl}$$

(for given E, f, u_0 and g) the *classical problem* of linear elasticity, and a solution of the problem a *classical solution*, respectively.

2.3 Anisotropic constitutive law

In general, the stiffness tensor field E depends on the coordinate x. By definition the body under consideration is *homogeneous* if, and only if, E is a constant tensor. However, for linguistic simplicity the component functions of tensor fields are frequently denoted as components in the literature, without assuming them to be constant. We will indeed only deal with the case of a homogeneous body in this contribution (cf. (A2) introduced in the next section), mainly for the sake of simplicity.

By definition of the *stress vector* (field)

$$\underline{\sigma} := \left[\sigma_{11}, \sigma_{22}, \sigma_{33}, \sigma_{12}, \sigma_{23}, \sigma_{31}\right]^T, \tag{2.11}$$

the *strain vector* (field)

$$\underline{\varepsilon} := \left[\varepsilon_{11}, \varepsilon_{22}, \varepsilon_{33}, 2\varepsilon_{12}, 2\varepsilon_{23}, 2\varepsilon_{31}\right]^T \tag{2.12}$$

and the *stiffness matrix* (field)

$$\underline{\underline{E}} := \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1112} & E_{1123} & E_{1113} \\ & E_{2222} & E_{2233} & E_{2212} & E_{2223} & E_{2213} \\ & & E_{3333} & E_{3312} & E_{3323} & E_{3313} \\ S & & & E_{1212} & E_{1223} & E_{1213} \\ & Y & & & E_{2323} & E_{2313} \\ & & M. & & & E_{1313} \end{bmatrix}, \tag{2.13}$$

Hooke's law (2.3) can be written in the form

$$\underline{\sigma} = \underline{\underline{E}}\,\underline{\varepsilon} \qquad \text{f.a. } x \in \Omega_x, \tag{2.14}$$

by the use of the first two symmetries of (2.4), which is called the *Voigt notation*. The last symmetry relation of (2.4) is then equivalent to the symmetry of the matrix (2.13). Therefore, it is sufficient to give the upper right triangular matrix, as we already did in (2.13). The most general linear elastic material can be characterized by 21 independent components, which are precisely the elements of the matrix in eq. (2.13). Such a material is called *aeolotropic*, *triclinic* or simply (general) *anisotropic*. The number of independent components is reduced, if the material possesses certain symmetries. In the remainder of the section, we briefly introduce the most common classes of anisotropic materials for engineering applications. The list is, however, incomplete and especially crystallography classifies additional types of anisotropies appearing in nature. For a more detailed introduction to anisotropic materials in engineering applications we refer to Ting (1996).

If the material possesses exactly one plane of reflection symmetry, it is called *monoclinic* or *monotropic*. If, e.g., the plane of symmetry is given by the x_1-x_2-plane, the stiffness matrix is supposed not to change under the corresponding basis transformation given by the matrix

$$\underline{\underline{a}} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

Since a four-fold tensor transforms by (cf., Zeidler, 1997, definition 74.4)

$$E_{\tilde{i}\tilde{j}\tilde{r}\tilde{s}} = a_{\tilde{i}i} a_{\tilde{j}j} a_{\tilde{r}r} a_{\tilde{s}s} E_{ijrs}, \tag{2.15}$$

any component that has an uneven number of tensor indices equal to 3 has to vanish. Therefore, the stiffness matrix is given by

$$\underline{\underline{E}} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1112} & 0 & 0 \\ & E_{2222} & E_{2233} & E_{2212} & 0 & 0 \\ & & E_{3333} & E_{3312} & 0 & 0 \\ S & & & E_{1212} & 0 & 0 \\ & Y & & & E_{2323} & E_{2313} \\ & & M. & & & E_{1313} \end{bmatrix}.$$

Likewise, if the symmetry plane is given by the x_1-x_3-plane, any component that has an uneven number of tensor indices equal to 2 has to vanish. Therefore, the stiffness matrix is given by

$$\underline{\underline{E}} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & E_{1113} \\ & E_{2222} & E_{2233} & 0 & 0 & E_{2213} \\ & & E_{3333} & 0 & 0 & E_{3313} \\ S & & & E_{1212} & E_{1223} & 0 \\ & Y & & & E_{2323} & 0 \\ & & M. & & & E_{1313} \end{bmatrix}.$$

Finally, if the symmetry plane is given by the x_2-x_3-plane, any component that has an uneven number of tensor indices equal to 1 has to vanish. Therefore, the stiffness matrix is given by

$$\underline{\underline{E}} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & E_{1123} & 0 \\ & E_{2222} & E_{2233} & 0 & E_{2223} & 0 \\ & & E_{3333} & 0 & E_{3323} & 0 \\ S & & & E_{1212} & 0 & E_{1213} \\ & Y & & & E_{2323} & 0 \\ & & M. & & & E_{1313} \end{bmatrix}.$$

So every monoclinic material can be characterized by 13 independent material properties. For any other plane of reflection symmetry, including the origin of the coordinate system, the corresponding stiffness matrix can be generated by using eq. (2.15) on one of the matrices above with the orthogonal transformation matrix $\underline{\underline{a}}$ that describes the corresponding rotation (from the plane of symmetry to the actual chosen coordinate system). This does not change the number of independent material properties, however, a stiffness matrix for a monoclinic material will in general not be sparse like the matrices above.

If a material possesses two orthogonal reflection symmetry planes, given by coordinate axes, following the argumentation above, it has to be symmetric with respect to the third plane of symmetry, too. Such a material is called *orthotropic* or *rhombic*. Therefore, the corresponding stiffness matrix becomes

$$\underline{\underline{E}} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ & E_{2222} & E_{2233} & 0 & 0 & 0 \\ & & E_{3333} & 0 & 0 & 0 \\ S & & & E_{1212} & 0 & 0 \\ & Y & & & E_{2323} & 0 \\ & & M. & & & E_{1313} \end{bmatrix},$$

and we have 9 independent material components.

If the material possesses exactly one axis of rotational symmetry, it is called *transversely isotropic*. If, e.g., the axis of symmetry is given by the x_1-axis, the stiffness matrix is supposed not to change under the corresponding basis transformations, which are given by

$$\underline{\underline{a}} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos(\alpha) & \sin(\alpha) \\ 0 & -\sin(\alpha) & \cos(\alpha) \end{bmatrix},$$

where $\alpha \in \mathbb{R}$. Since this implies reflection symmetries with respect to the x_1-x_2-plane and the x_1-x_3-plane, such a material is in particular orthotropic. Furthermore, it can be shown that the independent components reduce further (cf., Kienzler & Schröder, 2009, pp. 125-128), so that the stiffness matrix has the form

$$\underline{\underline{E}} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1122} & 0 & 0 & 0 \\ & E_{2222} & E_{2233} & 0 & 0 & 0 \\ & & E_{2222} & 0 & 0 & 0 \\ S & & & E_{1212} & 0 & 0 \\ & Y & & & \frac{1}{2}\left(E_{2222} - E_{2233}\right) & 0 \\ & & M. & & & E_{1212} \end{bmatrix}.$$

Likewise, we obtain the stiffness matrix

$$\underline{\underline{E}} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ & E_{2222} & E_{1122} & 0 & 0 & 0 \\ & & E_{1111} & 0 & 0 & 0 \\ S & & & E_{1212} & 0 & 0 \\ & Y & & & E_{1212} & 0 \\ & & M. & & & \frac{1}{2}\left(E_{1111} - E_{1133}\right) \end{bmatrix},$$

if the axis of symmetry is given by the x_2-axis and

$$\underline{\underline{E}} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & 0 & 0 & 0 \\ & E_{1111} & E_{1133} & 0 & 0 & 0 \\ & & E_{3333} & 0 & 0 & 0 \\ S & & & \frac{1}{2}\left(E_{1111} - E_{1122}\right) & 0 & 0 \\ & Y & & & E_{2323} & 0 \\ & & M. & & & E_{2323} \end{bmatrix},$$

if the axis of symmetry is given by the x_3-axis. In any case of transversal isotropy, we have 5 independent components remaining.

As in the case of reflection symmetries, the rotational symmetry with respect to a second orthogonal axis already implies the rotational symmetry with respect to any axis (cf., Kienzler & Schröder, 2009, p. 130). Such a material, which is invariant under any rotation, is called *isotropic* material. The stiffness matrix has the form

$$\underline{\underline{E}} = \begin{bmatrix} E_{1111} & E_{1122} & E_{1122} & 0 & 0 & 0 \\ & E_{1111} & E_{1122} & 0 & 0 & 0 \\ & & E_{1111} & 0 & 0 & 0 \\ S & & & \frac{1}{2}\left(E_{1111} - E_{1122}\right) & 0 & 0 \\ & Y & & & \frac{1}{2}\left(E_{1111} - E_{1122}\right) & 0 \\ & & M. & & & \frac{1}{2}\left(E_{1111} - E_{1122}\right) \end{bmatrix},$$

with only 2 independent components remaining (cf., Kienzler & Schröder, 2009, eq. 4.3.40). Usually this constitutive law will be given using the three engineering constants: Young's modulus E (not the stiffness tensor!), Shear modulus G and Poisson's ratio ν, which are not independent, e.g. because of $G = \frac{E}{2(1+\nu)}$. The relation to the tensor components is given by

$$E_{1111} = \frac{(1-\nu)\,E}{(1+\nu)\,(1-2\,\nu)},\ E_{1122} = \frac{\nu\,E}{(1+\nu)\,(1-2\,\nu)},\ \frac{1}{2}\left(E_{1111} - E_{1122}\right) = G = \frac{E}{2\,(1+\nu)}$$

(cf., Kienzler & Schröder, 2009, eq. 4.3.44). Representations using either: shear- and bulk-modulus (G, K), or Lamé's first parameter and Poisson's ratio (λ, ν), or Lamé's first parameter and the shear modulus (λ, G), are also common in the literature. (cf., Kienzler & Schröder, 2009, table 4.1, for conversions among these pairs of parameters.)

In addition, for more general anisotropic materials, like transversely isotropic and orthotropic material (but not monoclinic material), the stiffness matrices are commonly given by the use of direction-dependent engineering constants. The corresponding stiffness matrices can, e.g. be found in (Kienzler & Schröder, 2009, eqs. 4.3.13 and 4.3.38). Furthermore, representations of the constitutive law using the inverse tensor to the stiffness tensor, respectively, the inverse matrix, whose components are called *elastic compliances*, are possible and commonly used. (Every real, symmetric positive definite matrix is invertible, which is a consequence of the finite-dimensional spectral theorem, cf. theorem 1.) At last, direct tensor representations of the stiffness tensor, like

$$\frac{E_{ijrs}}{G} = \delta_{ir}\delta_{js} + \delta_{is}\delta_{jr} + \frac{2\nu}{1-2\nu}\delta_{ij}\delta_{rs}, \tag{2.16}$$

for isotropy, can be found in the literature. A similar representation for transversal isotropy can be found in (Schröder & Neff, 2003, eq. 4.69). However, to find a representation of this form for general kinds of anisotropy is still a partially unsettled problem of representation theory (cf., Zheng, 1994, section 8.5).

3 A general error estimate derived by energy methods

The main goal of this section is to derive an error estimate from the principle of minimum potential energy and the corresponding dual principle.

3.1 Preliminary: The energy norm

Since we mainly address engineers, we skip the technical regularity assumptions for the data (E, f, u_0, g), since they are rarely of interest in engineering applications. This allows us to avoid much of the mathematical notation overhead, so that we can focus on the essential arguments.

Nevertheless, we stress the fact that we will make extensive use of the following two fundamental modeling assumptions, which are generic in linear elasticity: First we assume that we have a part of the boundary where displacement boundary conditions are prescribed

$$\partial\Omega_{x0} \neq \emptyset. \tag{A1}$$

Intuitively one would not doubt that we need this assumption, in order to get a unique solution, since a body that is not supported could perform *any* rigid body motion independently from the load case. Secondly, we assume that the body under consideration is homogeneous, i.e., that the stiffness tensor field E is constant and, furthermore, that the corresponding stiffness matrix is (symmetric) positive definite, i.e., $\underline{x}^T \underline{\underline{E}}\, \underline{x} > 0$, f.a. $\underline{x} \in \mathbb{R}^6$, $\underline{x} \neq \underline{0}$.

$$\underline{\underline{E}} \text{ const. and s.p.d.} \tag{A2}$$

The assumption of the homogeneous body is not needed for this section, but already introduced here for clarity and simplicity of notation. By using the vector notations for σ (2.11) and ε (2.12), we find the inner product of the tensors to be equivalent to the vector scalar product $\sigma_{ij}\varepsilon_{ij} = \underline{\sigma}^T \underline{\varepsilon}$. Using this together with the Voigt notation (2.14), the definition of positive definiteness means that the inner elastic energy density at a point x is positive for every non-zero strain tensor ($\varepsilon = 0 \Leftrightarrow \varepsilon_{ij} = 0$ f.a. $i, j \in \{1, 2, 3\}$)

$$\text{F.a. } \varepsilon \neq 0: \ \frac{1}{2} E_{ijrs}\varepsilon_{ij}\varepsilon_{rs} = \frac{1}{2}\underline{\sigma}^T \underline{\varepsilon} = \frac{1}{2}\underline{\varepsilon}^T \underline{\underline{E}}^T \underline{\varepsilon} = \frac{1}{2}\underline{\varepsilon}^T \underline{\underline{E}}\, \underline{\varepsilon} > 0.$$

Therefore, the inner elastic energy of a body Ω_x (or any subregion) is zero if, and only if, the strain field vanishes identically

$$\int_{\Omega_x} \frac{1}{2} E_{ijrs}\varepsilon_{ij}\varepsilon_{rs}\, dV_x = 0 \Longleftrightarrow \varepsilon \equiv 0 :\Longleftrightarrow \varepsilon_{ij}(x) = 0 \quad \text{f.a. } i, j \in \{1, 2, 3\}, \text{ f.a. } x \in \Omega_x.$$

So the intuitive physical meaning of (A2) is that inner elastic energy is accumulated at every point of the body where the (local) strain does not vanish, independent from the specific "direction" of the strain. In particular, it is not possible to annihilate elastic energy in one part of the body by a specifically chosen strain field in another part of the body. From (A2) one could derive theoretical bounds for the engineering parameters describing the material, like

$$E > 0, \quad -1 < \nu < \frac{1}{2},$$

(cf., Kienzler & Schröder, 2009, eq. 4.3.43) for an isotropic body. These bounds are actually satisfied by real-world materials, which also justifies assumption (A2) a-posteriori.

We start with the following form of a theorem from linear algebra (cf., e.g., Strang, 2003, section 6.4).

Theorem 1 (Finite-dimensional spectral theorem (real case $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$))
Let $n \in \mathbb{N}$ be fix and $\underline{\underline{A}} \in \mathbb{R}^{n\times n}$ a symmetric (i.e., $\underline{\underline{A}} = \underline{\underline{A}}^T$) matrix. Then there exists an orthonormal matrix $\underline{\underline{Q}} \in \mathbb{R}^{n\times n}$ (i.e., $\underline{\underline{Q}}^T\underline{\underline{Q}} = \underline{\underline{Q}}\,\underline{\underline{Q}}^T = id_{\mathbb{R}^{n\times n}}$) and a diagonal matrix $\underline{\underline{\Lambda}} = diag(\lambda_1, ..., \lambda_n) \in \mathbb{R}^{n\times n}$ such that there exists a decomposition

$$\underline{\underline{A}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^T. \tag{3.1}$$

For a more common spelling of the theorem one has to note the following facts: Since $\underline{\underline{Q}}$ is orthonormal, (3.1) is equivalent to $\underline{\underline{A}}\,\underline{\underline{Q}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}$. Let $\underline{q_i}$ for $i \in \{1, ..., n\}$ denote the i-th column vector of $\underline{\underline{Q}}$. Then by reading the prior matrix equation column-wise, (3.1) is equivalent to

$$\underline{\underline{A}}\,\underline{q_i} = \lambda_i\,\underline{q_i} \text{ f.a. } i \in \{1, ..., n\},$$

i.e., the λ_i are real eigenvalues and the $\underline{q_i}$ are the corresponding eigenvectors. So another spelling of the theorem is, that any symmetric $\mathbb{R}^{n\times n}$ matrix has n real eigenvalues and a corresponding orthonormal

$$\underline{\underline{Q}}^T\underline{\underline{Q}} = \underline{\underline{Q}}\,\underline{\underline{Q}}^T = id_{\mathbb{R}^{n\times n}} \iff \begin{cases} \left\|\underline{q_i}\right\|_2^2 = \underline{q_i}^T\underline{q_i} = 1 & \text{f.a. } i \in \{1, ..., n\} \\ (\underline{q_i}; \underline{q_j}) = \underline{q_i}^T\underline{q_j} = 0 & \text{f.a. } i \neq j \end{cases}$$

basis of eigenvectors for $\mathbb{R}^n$.

Since the stiffness matrix $\underline{\underline{E}}$ is not only symmetric, but also assumed to be positive definite (A2), the six eigenvalues of $\underline{\underline{E}}$ are all positive

$$\lambda_i > 0 \text{ f.a. } i \in \{1, ..., 6\},$$

since this is an equivalent definition for positive definiteness in this case (cf., e.g., Strang, 2003, section 6.5).

Now let $\lambda_{\min} := \min\{\lambda_1, ..., \lambda_6\} > 0$, $\lambda_{\max} := \max\{\lambda_1, ..., \lambda_6\} > 0$ and $\underline{x} \in \mathbb{R}^6$ be arbitrary and let $\|\bullet\|_2$ denote the euclidian norm on $\mathbb{R}^6$, then

$$\begin{aligned}
\lambda_{\min}\,\|\underline{\varepsilon}\|_2^2 &= \lambda_{\min}\,\underline{\varepsilon}^T\underline{\varepsilon} = \lambda_{\min}\,\underline{\varepsilon}^T\underline{\underline{Q}}\,\underline{\underline{Q}}^T\underline{\varepsilon} = \lambda_{\min}\,\underline{(Q^T\varepsilon)}^T\,\underline{(Q^T\varepsilon)} = \lambda_{\min}\sum_{i=1}^{6}\left(\underline{Q^T\varepsilon}\right)_i^2 \\
&\leq \sum_{i=1}^{6}\lambda_i\left(\underline{Q^T\varepsilon}\right)_i^2 = \left(\underline{Q^T\varepsilon}\right)^T\underline{\underline{\Lambda}}\left(\underline{Q^T\varepsilon}\right) = \underline{\varepsilon}^T\underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^T\underline{\varepsilon} = \underline{\varepsilon}^T\underline{\underline{E}}\,\underline{\varepsilon} \\
&\leq \lambda_{\max}\sum_{i=1}^{6}\left(\underline{Q^T\varepsilon}\right)_i^2 = \lambda_{\max}\,\|\underline{\varepsilon}\|_2^2 \\
\implies \quad \sqrt{\lambda_{\min}}\,\|\underline{\varepsilon}\|_2 &\leq \sqrt{\underline{\varepsilon}^T\underline{\underline{E}}\,\underline{\varepsilon}} \leq \sqrt{\lambda_{\max}}\,\|\underline{\varepsilon}\|_2\,,
\end{aligned} \tag{3.2}$$

i.e., the symmetric billinear form

$$B_{\mathbb{R}^6}(\underline{\varepsilon}, \underline{\gamma}) := \underline{\varepsilon}^T\underline{\underline{E}}\,\underline{\gamma}$$

induces a norm

$$\|\underline{\varepsilon}\|_{E_{\mathbb{R}^6}} = \sqrt{B(\underline{\varepsilon}, \underline{\varepsilon})} = \sqrt{\underline{\varepsilon}^T\underline{\underline{E}}\,\underline{\varepsilon}},$$

which is an equivalent norm on $\mathbb{R}^6$, making $B(\underline{\varepsilon},\underline{\gamma})$ an equivalent scalar product on $\mathbb{R}^6$.

This result can be extended further. If we think of the Navier-Lamé formulation, we would rather be interested in a result stating that the symmetric bilinear form

$$B(u,v) := \int_{\Omega_x} E_{ijrs}\varepsilon_{ij}(u)\varepsilon_{rs}(v)\, dV_x = \int_{\Omega_x} E_{ijrs} u_{i|j} v_{r|s}\, dV_x \tag{3.3}$$

is an equivalent scalar product on a function space for displacement fields, since this would imply a norm estimate for measuring the quality of an approximative solution. This leads us to the concept of weak solutions. We will not go into technical details too much, but we will need some basic definitions.

With the increasing importance of FEM solutions in engineering applications, the associated weak solution theory gained importance in mathematics. As a quick motivation, one might just think of the fact that an FEM solution using, e.g. linear (tetrahedron)-FE-(ansatz)-functions would not be differentiable at the triangulation points, which inspired the theory of "weak" differentiability in mathematics. A weak solution for the three-dimensional problem of elasticity is an element of the function space

$$X := \left[W_2^1(\Omega_x)\right]^3,$$

i.e., every component function is member of the Sobolev space $W_2^1(\Omega_x)$ (of functions that have weak derivatives of order 1, which are members of the Lebesgue space $L_2(\Omega_x)$). We also introduce the corresponding space with vanishing traces on Ω_{x0}

$$X_0 := \{v \in X | Sv_i = 0 \text{ on } \partial\Omega_{x0}\},$$

where S denotes the trace operator. Let for $f, g \in L_2(\Omega_x)$

$$(f,g)_{L_2} := \int_{\Omega_x} f(x)g(x)\, dV_x, \quad \|f\|_{L_2} := \sqrt{\int_{\Omega_x} f(x)^2\, dV_x}$$

denote the standard L_2-scalar product and the associated norm. Then X is a Hilbert space with the scalar-product and associated norm:

$$(u,v)_X := \sum_{i=1}^{3} (u_i, v_i)_{L_2} + \sum_{i,j=1}^{3} \left(u_{i|j}, v_{i|j}\right)_{L_2}, \quad \|v\|_X := \sqrt{\sum_{i=1}^{3} \|v_i\|_{L_2}^2 + \sum_{i,j=1}^{3} \left\|v_{i|j}\right\|_{L_2}^2}.$$

Readers who are not familiar with the concept of Sobolev spaces may just think of once continuous differentiable functions $C^1(\Omega_x)$ instead of $W_2^1(\Omega_x)$. Then X_0 is just the space of functions v that vanish on the Dirichlet boundary, i.e., $v(x) = 0$ f.a. $x \in \partial\Omega_{x0}$. Intuitively one would believe that $\|u - v\|_X$ measures the quality of an approximation $v \in C^1(\Omega_x)$ to the real solution $u \in C^1(\Omega_x)$, since the norm accumulates the differences between u and v and their first derivatives in the whole region Ω_x. For readers who want to familiarize themselves with the concept of Sobolev spaces, we refer to Adams (1975).

To extend the result, we first define the space of displacement fields that are members of X_0 that do not generate any strain

$$Y_0 := \{u \in X_0 | \forall i,j \in \{1,2,3\} :\ \varepsilon_{ij}(u) = 0 \text{ f.a. } x \in \Omega_x\}.$$

Since this is a closed subspace of X_0, we can decompose X_0 as the direct sum of Y_0 and its orthogonal complement ${Y_0}^{\perp} := \{v \in X_0 | (u,v)_X = 0 \text{ f.a. } u \in Y_0\}$

$$X_0 = Y_0 \oplus {Y_0}^{\perp},$$

i.e., every element of X_0 can be written as a uniquely determined sum of an element from Y_0 and one of ${Y_0}^{\perp}$. The subspace Y_0 has a simple physical interpretation:
If $\partial\Omega_{x0} = \emptyset$, we have:

$$u \in Y_0 \Leftrightarrow \exists \alpha, \beta \in \mathbb{R}^3 : \; u(x) = \alpha + \beta \times x,$$

i.e., Y_0 is the space of (infinitesimal) rigid body motions. If $\partial\Omega_{x0} \neq \emptyset$, we have:

$$Y_0 = \{0\},$$

since any rigid body motion violates the homogeneous Dirichlet condition of the space X_0 (cf., Zeidler, 1997, lemma 62.15 and 62.16).

A classical result in linear elasticity is *Korn's inequality* (cf., Zeidler, 1997, theorem 62.F), which states that there is a positive constant $c_{\text{Korn}} > 0$ for which we have

$$\int_{\Omega_x} \varepsilon_{ij}(u)\varepsilon_{ij}(u) \; dV_x \geq c_{\text{Korn}} \|u\|_X^2 \qquad \text{f.a. } u \in {Y_0}^{\perp}.$$

By our assumption (A1) the inequality is valid for all $u \in X_0$. Together with assumption (A2) and equation (3.2) we get a lower bound for the norm induced by B (i.e., the inner elastic energy of a body $\frac{1}{2}B(u,u)$)

$$\begin{aligned} B(u,u) &= \int_{\Omega_x} E_{ijrs}\varepsilon_{ij}(u)\varepsilon_{rs}(u) \; dV_x = \int_{\Omega_x} \underline{\varepsilon^T(u)} \, \underline{\underline{E}} \, \underline{\varepsilon(u)} \; dV_x \\ &\geq \lambda_{\min} \int_{\Omega_x} \underline{\varepsilon^T(u)} \, \underline{\varepsilon(u)} \; dV_x \geq \lambda_{\min} \int_{\Omega_x} \varepsilon_{ij}(u)\varepsilon_{ij}(u) \; dV_x \\ &\geq \lambda_{\min} c_{\text{Korn}} \|u\|_X^2 \qquad \text{f.a. } u \in X_0. \end{aligned} \tag{3.4}$$

A corresponding upper bound

$$B(u,u) \leq c_{\text{cont}} \|u\|_X^2 \qquad \text{f.a. } u \in X,$$

is gained from the continuity of B. The proof results from the use of the Cauchy-Schwarz inequality. For details we refer to (Schneider, 2010, theorem 27). So indeed we obtain

Theorem 2 (Energy norm)
With the assumptions (A1) *and* (A2) *the billinear form*

$$B(u,v) := \int_{\Omega_x} E_{ijrs}\varepsilon_{ij}(u)\varepsilon_{rs}(v) \; dV_x = \int_{\Omega_x} E_{ijrs} u_{i|j} v_{r|s} \; dV_x$$

is an equivalent scalar product on X_0, i.e., for the associated energy norm

$$\|u\|_E = \sqrt{B(u,u)}$$

there are positive constants c_1 and c_2 such that we have

$$c_1 \|u\|_X \leq \|u\|_E \leq c_2 \|u\|_X \qquad \textit{f.a. } u \in X_0.$$

A word of warning: B is *not* an equivalent scalar product on X.

3.2 The principle of minimum potential energy

We need the following basics from the calculus of variations. Let $F : S \to \mathbb{R}$ be some functional (on a real Banach space S), then we say that F has a *local minimum* at a point $u \in S$ if, and only if, there is a neighborhood $U(u) \subset S$ so that

$$F(v) \geq F(u) \quad \text{f.a. } v \in U(u).$$

The local minimum is said to be *strict* if we have "$>$" instead of "$\geq$" in the equation above. A local minimum is said to be a *global* minimum, if $U(u) = S$. Furthermore, we define for $n \in \mathbb{N}$ the *nth variation* of F at the point $u \in S$ in the direction $v \in S$ by

$$\delta^n F(u; v) := \left[\frac{d^n F(u + tv)}{d t^n}\right]_{t=0}. \tag{3.5}$$

For the first variation we write δ instead of δ^1.

The connection between free local minima and the variational derivatives of a functional F is basically similar to the connection between the local minima of a simple, real, scalar function $f : \mathbb{R} \to \mathbb{R}$ and their derivatives, beside the fact that the variational derivatives have to fulfill properties with respect to all directions. The connection is given by the following theorem (cf., e.g., Zeidler, 1985, section 40.2).

Theorem 3 (Calculus of variations)
If F is sufficiently smooth, the following conditions hold for a local minimum.

1. *Necessary condition:*
   If F has a local minimum at u, then

   $$\delta F(u; v) = 0 \qquad \textit{f.a. } v \in S.$$

2. *Sufficient condition:*
   F has a strict local minimum at u, if for an even number $n \geq 2$ we have:

   $$\begin{aligned} &\delta^k F(u; v) = 0 && \textit{f.a. } v \in S,\ k \in \{1, ..., n-1\}, \\ \exists c > 0 :\ &\delta^n F(u; v) \geq c\, \|v\|_S^n && \textit{f.a. } v \in S. \end{aligned}$$

Now, we define the potential energy of an arbitrary function $u \in X$ *that fulfills the prescribed boundary condition* (2.9) on $\partial\Omega_{x0}$ by

$$E_{\text{pot}}(u) := \int_{\Omega_x} \frac{1}{2} E_{ijrs} u_{i|j} u_{r|s}\, dV_x - \int_{\Omega_x} f_i u_i\, dV_x - \int_{\partial\Omega_{xN}} g_i u_i\, dA_x. \tag{3.6}$$

The associated minimization problem of the potential energy, as found in engineering text books (cf., Washizu, 1982, section 2.1), is: *Find a global minimum for the potential energy E_{pot} under all displacement fields $u \in X$ that fulfill the boundary condition* (2.9)*.* To specify the condition that a displacement field has to satisfy the prescribed boundary condition (2.9) in a way compatible to the basic setting of variational calculus, we apply a standard homogenization technique: If we take a fixed extension $u_0 \in X$ of the prescribed boundary condition (2.9) on $\partial\Omega_{x0}$ to the whole region Ω_x, all functions $u \in X$ that fulfill the prescribed boundary condition (2.9) on $\partial\Omega_{x0}$ could be written as a sum $u = v + u_0$, with a uniquely determined $v \in X_0$. If we define

$$\widetilde{E}_{\text{pot}}(u) := E_{\text{pot}}(u + u_0) \qquad \text{f.a. } u \in X_0,$$

the space of admissible displacement fields for $\widetilde{E}_{\text{pot}}$ is simply X_0. The important point is that X_0 is a vector space. In contrast to that: If we take a real scalar $\alpha \in \mathbb{R}$ and two functions $u, v \in X$ that fulfill the prescribed boundary condition (2.9) on $\partial\Omega_{x0}$ for an $u_0 \neq 0$, then neither αu nor $u + v$ fulfill the boundary condition (2.9). The nth variation would not be well-defined this way. However, the minimization problem of the potential energy

$$\text{Find } u \in X_0 : \ \widetilde{E}_{\text{pot}}(u) = \inf_{v \in X_0} \widetilde{E}_{\text{pot}}(v) = E_{\text{pot}}(u + u_0) = \inf_{v \in X_0} E_{\text{pot}}(v + u_0) =: \alpha, \tag{En}$$

is compatible to the basic setting of variational calculus, as defined above. Formulated in another way: The nth variation is a function $\delta^n : (X_0 + u_0) \times X_0 \to \mathbb{R}$, i.e., the first argument (the point) is from the function space fulfilling the boundary conditions (called space of configurations) and the second argument, the direction (also called variation), is from the space fulfilling the corresponding homogeneous boundary conditions (called space of variations).

Defining the linear form

$$F(u) := \int_{\Omega_x} f_i u_i \, dV_x + \int_{\partial\Omega_{xN}} g_i u_i \, dA_x \tag{3.7}$$

and calculating the first two variations of $\widetilde{E}_{\text{pot}}$ or rather E_{pot}

$$\begin{aligned}
E_{\text{pot}}(u + u_0) &= \frac{1}{2} B(u + u_0, u + u_0) - F(u + u_0), \\
\delta E_{\text{pot}}(u + u_0; v) &= \frac{1}{2} \left(B(u + u_0, v) + B(v, u + u_0) \right) - F(v) = B(u + u_0, v) - F(v), \\
\delta^2 E_{\text{pot}}(u + u_0; v) &= B(v, v),
\end{aligned}$$

we find a very special situation, since equation (3.4) gives us

$$\delta^2 \widetilde{E}_{\text{pot}}(u; v) = \delta^2 E_{\text{pot}}(u + u_0; v) = B(v, v) \geq \lambda_{\min} c_{\text{Korn}} \|v\|_X^2 \qquad \text{f.a. } v \in X_0. \tag{3.8}$$

Therefore, by theorem 3 the problem of finding a free local minimum is equivalent to the so-called *stationary* problem of $\widetilde{E}_{\text{pot}}$ or *weak problem* of linear elasticity

$$\text{Find } u \in X_0 : \ \delta E_{\text{pot}}(u + u_0; v) = B(u + u_0, v) - F(v) = 0 \qquad \text{f.a. } v \in X_0. \tag{Wk}$$

“Equivalent” in this context means that every weak solution is a local minimum (because of equation (3.8) and theorem 3, paragraph 2) and every local minimum is a weak solution (because of theorem 3, paragraph 1). In addition by theorem 3, paragraph 2: Every local minimum is automatically a strict one (in this context).

By using the bilinearity of B, the weak problem (Wk) can be rewritten as

$$\text{Find } u \in X_0 : \ B(u, v) = \underbrace{F(v) - B(u_0, v)}_{=: \tilde{F}(v)} \qquad \text{f.a. } v \in X_0. \tag{3.9}$$

B is a continuous billinear form on X (proof in Schneider, 2010, theorem 27), therefore, for fixed $u_0 \in X$: $v \mapsto B(u_0, v)$ is a continuous linear form on X_0. If f and g are sufficiently regular, F is also continuous (proof in Schneider, 2010, theorem 24). So the right-hand side of (3.9), $\tilde{F}$ is a continuous linear form on X_0. Furthermore, by theorem 2: B is a scalar product that makes X_0 a Hilbert space. The space of all continuous linear forms on a Hilbert space H, is called the dual space of H and is denoted by H^*. We can apply the following theorem. (cf., Werner, 2007, thm. V.3.6).

Theorem 4 (Representation theorem of Fréchet-Riesz)
Let H be a Hilbert space. Then $\Phi : H \to H^*, y \mapsto (\bullet, y)$ is a bijective, isometric and conjugate linear mapping. In other words: For every $x^* \in H^*$ there exists a uniquely determined $y \in H$ with $x^*(x) = (x, y)$ and we have $\|x^*\|_{H^*} = \|y\|_H$.

Note that for the real Hilbert space X_0 "conjugate linear" means just "linear". Also the scalar product of a real space is "bilinear, symmetric, positive definite" instead of "sesquilinear, conjugate symmetric, positive definite" for a complex Hilbert space, therefore the order of x and y in the theorem above does not play a role. The application states both the existence *and the uniqueness* of a weak solution $u \in X_0$ at once.

We now can also clarify the connection between the minimization problem (En) and the weak problem (Wk). Equation (3.8) implies that problem (En) has at least one solution $w \in X_0$. This can be denoted by $\{w\} \subset \{\text{sol (En)}\}$, if we denote the set of solutions of the problem (En) by $\{\text{sol (En)}\}$. Also every global minimum is obviously a local minimum, by its mere definition. In addition, we already know that every local minimum is a weak solution. If we denote the set of solutions of the problem (Wk) by $\{\text{sol (Wk)}\}$ this means $\{\text{sol (En)}\} \subset \{\text{sol (Wk)}\}$. At last we know that the weak problem has precisely one solution $u \in X_0$. As a result we have

$$\{w\} \subset \{\text{sol (En)}\} \subset \{\text{sol (Wk)}\} = \{u\},$$

which implies $w = u$ and, therefore, also $\{\text{sol (En)}\} = \{\text{sol (Wk)}\}$, i.e. the problems (En) and (Wk) are equivalent.

So far we showed that problem (En) and (Wk) have the same unique solution $u \in X_0$, if (A1) and (A2) hold true and we have sufficient regularity[1]. If the data u_0, g and f are so smooth that the weak solution reaches the regularity of the classical solution (Cl), e.g., C^∞, which includes all polynomials, we can use integration by parts and apply the variational lemma to reveal the relation to the classical problem (Cl).

Theorem 5 (Integration by parts)
Let $N \in \mathbb{N}$ and let $G \subset \mathbb{R}^N$ be a bounded region with $\partial G \in C^{0,1}$. Let $n = (n_i)$ denote the outer unit normal vector field to the boundary. If n is undefined in some point $x \in \mathbb{R}^N$, e.g., in corners and edges, we set $n(x) := 0$. Then for all $f, g \in W_2^1(G)$ we have

$$\int_G f_{|i} g \, dV = \int_{\partial G} f g \, n_i \, dA - \int_G f g_{|i} \, dV \qquad \text{f.a. } i \in \{1, ..., N\}.$$

Proof cf. Zeidler (1990a), corollary 18.4.

Theorem 6 (Variational lemma)
Let $n \in \mathbb{N}$, $G \subset \mathbb{R}^n$ be nonempty and open, and $f \in L_2(G)$. If

$$\int_G f \phi \, dV = 0 \text{ f.a. } \phi \in C_0^\infty(G)$$

then

$$f = 0 \in L_2(G) \text{ ,i.e., } f(x) = 0 \text{ f.a.a. } x \in G.$$

[1] In Schneider (2010) we proved, that sufficient regularity means precisely, that $\Omega_x \subset \mathbb{R}^3$ is a bounded region with $\partial\Omega_x \in C^{0,1}$, $\partial\Omega_x = \overline{\partial\Omega_{x0}} \cup \overline{\partial\Omega_{xN}}$ and $\partial\Omega_{x0} \cap \partial\Omega_{xN} = \emptyset$, with $\partial\Omega_{x0}$ and $\partial\Omega_{xN}$ relatively open. Furthermore, $u_0 \in \left[W_2^{1/2}(\partial\Omega_{x0})\right]^3$, $g \in \left[W_2^{1/2}(\partial\Omega_{xN})\right]^3$ and $f \in [L_2(\Omega_x)]^3$.

Here "f.a.a." is an abbreviation for "for almost all", which means precisely "for all" with the exception of a set of measure zero. $C_0^\infty(G)$ denotes the space of functions that are elements of $C^n(G)$ f.a. $n \in \mathbb{N}$, with compact support.

Proof cf. Zeidler (1990a), proposition 18.2.

Let $u \in X_0$ be a weak solution and furthermore $u + u_0 \in \left[C^2(\Omega_x)\right]^3 \cap \left[C(\Omega_x \cup \partial\Omega_{x0})\right]^3 \cap \left[C^1(\Omega_x \cup \partial\Omega_{xN})\right]^3$. By application of integration by parts we get

$$\begin{aligned}
0 &= B(u+u_0, v) - F(v) = B(v, u+u_0) - F(v) \\
&= \int_{\Omega_x} E_{ijrs}(u+u_0)_{r|s} v_{i|j}\, dV_x - \int_{\Omega_x} f_i v_i\, dV_x - \int_{\partial\Omega_{xN}} g_i v_i\, dA_x \\
&= -\int_{\Omega_x} E_{ijrs}(u+u_0)_{r|sj} v_i\, dV_x + \int_{\partial\Omega_{x0}} E_{ijrs}(u+u_0)_{r|s} \underbrace{v_i}_{=0} n_j\, dA_x \\
&\quad + \int_{\partial\Omega_{xN}} E_{ijrs}(u+u_0)_{r|s} v_i n_j\, dA_x - \int_{\Omega_x} f_i v_i\, dV_x - \int_{\partial\Omega_{xN}} g_i v_i\, dA_x \\
&= -\int_{\Omega_x} \Big[E_{ijrs}(u+u_0)_{r|sj} + f_i\Big] v_i\, dV_x + \int_{\partial\Omega_{xN}} \Big[E_{ijrs}(u+u_0)_{r|s} n_j - g_i\Big] v_i\, dA_x.
\end{aligned}$$

Now let $e^i \in \mathbb{R}^3$ denote the i-the natural basis vector, i.e., $e^i_i = 1$ and $e^i_j = 0$ for $i \neq j$. Set $v = e^i \phi$ for a fixed $i \in \{1, 2, 3\}$ and arbitrary $\phi \in C_0^\infty(\Omega_x)$. Then $v \in X_0$ is a valid test function for the weak problem (Wk) and we can apply the variational lemma with $G := \Omega_x$ leading to (3.10). Note that "f.a." holds because of the assumed regularity of the integrand. Next, insert equation (3.10) into the equation above and repeat the procedure for $G := \partial\Omega_{xN}$ leading to (3.11). Obviously, $u + u_0 = u_0$ on $\partial\Omega_{x0}$, since $u \in X_0$. Finally we have

$$E_{ijrs}(u+u_0)_{r|sj} = -f_i \qquad \text{f.a. } x \in \Omega_x, \tag{3.10}$$

$$E_{ijrs}(u+u_0)_{r|s} n_j = g_i \qquad \text{f.a. } x \in \partial\Omega_{xN}, \tag{3.11}$$

$$(u+u_0)_i = u_{0i} \qquad \text{f.a. } x \in \partial\Omega_{x0}, \tag{3.12}$$

i.e., $u + u_0$ is a classical solution. Also, if we assume that $u + u_0$ is a classical solution we can multiply the equations (3.10) and (3.11) with an arbitrary test function $v \in X_0$ and perform the corresponding integration and partial integration backwards to gain (Wk). Therefore, given sufficiently smooth data, problem (Wk) and (Cl) are equivalent, too. We sum up our investigation in the following theorem.

Theorem 7 (Equivalent problems)
Given a bounded region $\Omega_x \subset \mathbb{R}^3$ with $\partial\Omega_x \in C^{0,1}$, the boundary decomposition (2.5) *and the data (f and g) with sufficient regularity and under the assumption* (A1) *and* (A2)*, we have for a fixed extension of displacement boundary conditions to the whole region $u_0 \in X$ the following situation:*

The three problems (Cl), (Wk), (En)*:*
Find $u \in X_0$ with:

$$\text{(Cl)} \begin{cases} E_{ijrs}(u+u_0)_{r|sj} &= -f_i \quad \textit{f.a. } x \in \Omega_x, \\ E_{ijrs}(u+u_0)_{r|s} n_j &= g_i \quad \textit{f.a. } x \in \partial\Omega_{xN}, \\ (u+u_0)_i &= u_{0i} \quad \textit{f.a. } x \in \partial\Omega_{x0}, \end{cases}$$

$$\text{(Wk)}\left\{\int_{\Omega_x} E_{ijrs}(u+u_0)_{i|j}v_{r|s}\,dV_x = \int_{\Omega_x} f_i v_i\,dV_x + \int_{\partial\Omega_{xN}} g_i v_i\,dA_x \qquad \textit{f.a. } v\in X_0,\right.$$

$$\text{(En)}\left\{\begin{array}{l} E_{pot}(u+u_0) = \inf_{v\in X_0} E_{pot}(v+u_0) =: \alpha, \\ \textit{where } E_{pot}(v) := \int_{\Omega_x}\frac{1}{2}E_{ijrs}v_{i|j}v_{r|s}\,dV_x - \int_{\Omega_x} f_i v_i\,dV_x - \int_{\partial\Omega_{xN}} g_i v_i\,dA_x,\end{array}\right.$$

have a solution $u \in X_0$, which is the same for all three problems and is, furthermore, uniquely determined (for the given data).

A word of warning: The reader should keep in mind that one actually needs more regular data for a weak solution (Wk) to be a classical solution (Cl), than one requires for the weak solution to exist. Therefore, in a strict mathematical sense, there is a difference between problem (Cl) and problem (Wk). However, problem (Wk) and problem (En) have a common solution under the same regularity assumptions.

Note that the compact theorem above includes a lot of basic results from the classical theory of elasticity. In words:

- **Existence**
  The classical mixed boundary problem of linear elasticity has a solution.

- **Uniqueness**
  The solution is uniquely determined.

- **The principle of virtual work**
  A system is in equilibrium if, and only if, the virtual work of any admissible virtual displacement vanishes. (This is problem (Wk), cf., e.g., Washizu (1982), section 1.4)

- **The principle of minimum potential energy**
  A system is in equilibrium if, and only if, the potential energy E_{pot} is minimal with respect to all admissible displacement fields. (This is problem (En), cf., e.g., Washizu (1982), section 2.1)

Next we want to give a general error estimate for approximative solutions. To this end, we assume that $u \in X_0$ is the solution of theorem 7 and $v \in X_0$ is arbitrary. We will make extensive use of the bilinearity of B and linearity of F. Note that E_{pot} is *not linear*! By the definition of E_{pot} (3.6) we have

$$E_{\text{pot}}(v+u_0) = \frac{1}{2}B(v+u_0, v+u_0) - F(v+u_0).$$

Since $u \in X_0$ is in particular a solution of problem (Wk) and $v \in X_0$ a valid test function of problem (Wk) we additionally have

$$\begin{aligned} E_{\text{pot}}(v+u_0) &= \frac{1}{2}B(v+u_0, v+u_0) - F(v) - F(u_0) \\ &= \frac{1}{2}B(v+u_0, v+u_0) - B(u+u_0, v) - F(u_0). \end{aligned}$$

In particular, the solution $u \in X_0$ is also a valid test function of problem (Wk)

$$E_{\text{pot}}(u+u_0) = \frac{1}{2}B(u+u_0, u+u_0) - B(u+u_0, u) - F(u_0).$$

Now we derive the difference of potential energy

$$\begin{aligned} E_{\text{pot}}(v+u_0) - E_{\text{pot}}(u+u_0) =& \frac{1}{2}B(v+u_0, v+u_0) - B(u+u_0, v) - F(u_0) \\ &- \frac{1}{2}B(u+u_0, u+u_0) + B(u+u_0, u) + F(u_0) \end{aligned}$$

and sort it by terms that contain u_0 and those that do not,

$$\begin{aligned} E_{\text{pot}}(v+u_0) - E_{\text{pot}}(u+u_0) =& \frac{1}{2}B(v,v) - B(u,v) - \frac{1}{2}B(u,u) + B(u,u) \\ &+ \frac{1}{2}B(u_0,v) + \frac{1}{2}B(v,u_0) + \frac{1}{2}B(u_0,u_0) - B(u_0,v) \\ &- \frac{1}{2}B(u_0,u) - \frac{1}{2}B(u,u_0) - \frac{1}{2}B(u_0,u_0) + B(u_0,u) \end{aligned}$$

to see that the terms containing u_0 sum up to 0. Therefore, using the bilinearity of B leads to

$$\begin{aligned} E_{\text{pot}}(v+u_0) - E_{\text{pot}}(u+u_0) =& \frac{1}{2}B(v,v) - B(u,v) + \frac{1}{2}B(u,u) \\ =& \frac{1}{2}B(v,v) - \frac{1}{2}B(u,v) - \frac{1}{2}B(v,u) + \frac{1}{2}B(u,u) \\ =& \frac{1}{2}B(v-u,v) - \frac{1}{2}B(v-u,u) \\ =& \frac{1}{2}B(v-u,v-u). \end{aligned}$$

Since $v - u \in X_0$, we can apply equation (3.4) again to gain

$$\begin{aligned} E_{\text{pot}}(v+u_0) - E_{\text{pot}}(u+u_0) \geq& \frac{1}{2}\,\lambda_{\min}\, c_{\text{Korn}}\, \|v-u\|_X^2 \\ \geq& \frac{1}{2}\,\lambda_{\min}\, c_{\text{Korn}}\, \|(v+u_0)-(u+u_0)\|_X^2\,, \end{aligned}$$

which means that the difference of the potential energies is an upper bound for the error of an approximative solution $v \in X_0$, and it vanishes if, and only if, $v = u$, since u is also a solution of problem (En).

Theorem 8 (Error estimate using potential energy)
Under the assumptions of theorem 7 let $u \in X_0$ be the solution of the three equivalent problems. Then we have

$$\frac{1}{2}\lambda_{min} c_{Korn} \|(v+u_0)-(u+u_0)\|_X^2 \leq E_{pot}(v+u_0) - E_{pot}(u+u_0) \qquad \textit{f.a. } v \in X_0. \quad (3.13)$$

A word of warning: For the proof of (3.13) we used that u is a solution of problem (Wk). Equation (3.13) (again) proves that u is also a strict global minimizer of E_{pot}. Therefore, the assumption that u is a solution is crucial! Equation (3.13) does *not* hold for arbitrary functions $u \in X_0$.

3.3 The general motivation for duality

The drawback of estimate (3.13) is that in general we do not know the potential energy of the actual solution $u + u_0$. Therefore, the estimate can not yet be used e.g. for computing the error of a FEM solution. In order to compensate this, one introduces the dual energy.

We already introduced the minimum problem (En)

$$\text{Find } u \in X_0 : \; E_{\text{pot}}(u + u_0) = \inf_{v \in X_0} E_{\text{pot}}(v + u_0) =: \alpha.$$

Assume we find an associated maximum problem

$$\text{Find } \sigma \in \Sigma : \; E_{\text{dual}}(\sigma) = \sup_{\mu \in \Sigma} E_{\text{dual}}(\mu) =: \beta,$$

formulated in another physical quantity, which will turn out to be the stress tensor σ, such that $\beta = \alpha$. Then we always have

$$E_{\text{dual}}(\mu) \leq \alpha \leq E_{\text{pot}}(v + u_0) \quad \text{f.a. } v \in X_0, \mu \in \Sigma,$$

and we can replace the unknown potential energy of the actual solution $E_{\text{pot}}(u + u_0)$ in estimate (3.13) by an approximation of the dual problem. We get the error estimate

$$\frac{1}{2}\lambda_{\min} c_{\text{Korn}} \, \|(v + u_0) - (u + u_0)\|_X^2 \leq E_{\text{pot}}(v + u_0) - E_{\text{dual}}(\mu) \qquad \text{f.a. } v \in X_0, \mu \in \Sigma.$$

The right-hand side still vanishes if, and only if, v and μ are the solutions of the extremal problems. Since we want to achieve $\alpha = \beta$, we will reformulate the inner elastic energy density in one point of the body in terms of the stress tensor. To this end, we need to invert the constitutive equation.

3.4 Inversion of the constitutive equation

By assumption (A2) and theorem 1, the elasticity matrix $\underline{\underline{E}}$ has a decomposition $\underline{\underline{E}} = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}\,\underline{\underline{Q}}^T$ with $\underline{\underline{\Lambda}} = diag(\lambda_1, ..., \lambda_n)$, $\underline{\underline{Q}}^{-1} = \underline{\underline{Q}}^T$ and $\lambda_i > 0$ f.a. $i \in \{1, ..., 6\}$. Obviously, the inverse of $\underline{\underline{\Lambda}}$ is $\underline{\underline{\Lambda}}^{-1} = diag(\lambda_1^{-1}, ..., \lambda_n^{-1})$ and one easily derives (by computing $\underline{\underline{E}}\,\underline{\underline{D}} = \underline{\underline{D}}\,\underline{\underline{E}} = id$) that

$$\underline{\underline{D}} := \underline{\underline{Q}}\,\underline{\underline{\Lambda}}^{-1}\,\underline{\underline{Q}}^T,$$

is the matrix inverse of $\underline{\underline{E}}$. Therefore, we have proved that $\underline{\underline{E}}$ is invertible and that the orthonormal basis vectors of eigenvectors of $\underline{\underline{E}}$ are also eigenvectors of its inverse. More precisely, if $(\lambda_i, \underline{q_i})$ is an eigenpair of $\underline{\underline{E}}$, then $(\lambda_i^{-1}, \underline{q_i})$ is an eigenpair of $\underline{\underline{D}}$. Furthermore, we have

$$\underline{\underline{D}}^T = \left[\underline{\underline{Q}}\,\underline{\underline{\Lambda}}^{-1}\,\underline{\underline{Q}}^T\right]^T = \underline{\underline{Q}}\,\underline{\underline{\Lambda}}^{-1}\,\underline{\underline{Q}}^T = \underline{\underline{D}},$$

i.e., $\underline{\underline{D}}$ is symmetric and since

$$\lambda_i > 0 \implies \frac{1}{\lambda_i} > 0,$$

we proved that $\underline{\underline{D}}$ is symmetric positive definite

$$\underline{\underline{D}} \text{ s.p.d.}$$

For the tensor inverse D of E, which is defined by

$$D_{ijkl}E_{klrs} = E_{ijkl}D_{klrs} = \delta_{ir}\delta_{js},$$

the symmetry of $\underline{\underline{D}}$ is equivalent to the symmetry relation $D_{ijkl} = D_{klij}$. Together with the symmetries of ε and σ (which are assumed in the setting of the vector notation), this means that D fulfills the same three symmetry relations as E

$$D_{ijkl} = D_{jikl} = D_{ijlk} = D_{klij}.$$

Furthermore by comparison of the vector and tensor notation of the inverse constitutive law

$$\underline{\underline{D}}\,\underline{\sigma} = \underline{\varepsilon} \overset{!}{\iff} D_{ijkl}\sigma_{kl} = \varepsilon_{ij},$$

we find the relation

$$\underline{\underline{D}} = \begin{bmatrix} D_{1111} & D_{1122} & D_{1133} & 2D_{1112} & 2D_{1123} & 2D_{1113} \\ & D_{2222} & D_{2233} & 2D_{2212} & 2D_{2223} & 2D_{2213} \\ & & D_{3333} & 2D_{3312} & 2D_{3323} & 2D_{3313} \\ S & & & 4D_{1212} & 4D_{1223} & 4D_{1213} \\ & Y & & & 4D_{2323} & 4D_{2313} \\ & & M. & & & 4D_{1313} \end{bmatrix}, \tag{3.14}$$

between the matrix and tensor components of D. The inner elastic energy density can be rewritten in terms of stresses by

$$\frac{1}{2}E_{ijrs}\varepsilon_{ij}\varepsilon_{rs} = \frac{1}{2}\underline{\varepsilon}^T\underline{\underline{E}}\,\underline{\varepsilon} = \frac{1}{2}\underline{\sigma}^T\underline{\varepsilon} = \frac{1}{2}\underline{\sigma}^T\underline{\underline{D}}\,\underline{\sigma} = \frac{1}{2}D_{ijrs}\sigma_{ij}\sigma_{rs}.$$

3.5 The principle of dual energy

Now, we are able to formulate the maximum problem of dual energy. For a better understanding of the connection between this problem and the original minimum problem of potential energy, we will also define another problem, called the “free problem”. The free problem will also help us to prove the expected relation between the minimum problem and the maximum problem, i.e., the fact that the extremal values are identical. Therefore, we will finally be able to modify our error estimate by the thoughts of section 3.3. Here we give a classical proof basically taken from (Zeidler, 1997, section 62.16). It is based on a general duality principle, the Friedrichs duality (Friedrichs, 1928).

First, we introduce the maximum problem of dual energy, which reads as

$$\text{Find } \sigma \in \Sigma : \; E_{\text{dual}}(\sigma) = \sup_{\mu\in\Sigma} E_{\text{dual}}(\mu) =: \beta, \tag{Du}$$

as already mention in section 3.3. The dual energy E_{dual} of a stress tensor field σ is defined by

$$E_{\text{dual}}(\sigma) := -\int_{\Omega_x} \frac{1}{2}D_{ijrs}\sigma_{ij}\sigma_{rs}\,dV_x + \int_{\partial\Omega_{x0}} \sigma_{ij}n_j u_{0i}\,dA_x. \tag{3.15}$$

The space of admissible stress fields Σ is given by the symmetric two-fold ($\sigma_{ij} = \sigma_{ji}$) tensor fields on Ω_x which additionally satisfy the side conditions

$$\sigma_{ji|j} = -f_i \qquad \text{f.a. } x \in \Omega_x, \tag{3.16}$$

$$\sigma_{ij} n_j = g_i \qquad \text{f.a. } x \in \partial\Omega_{xN}, \tag{3.17}$$

i.e., all stress fields σ that fulfill the equilibrium condition (2.1) and the stress boundary condition (2.7). (A short reminder: These are inhomogeneous side conditions, therefore, by the thoughts of section 3.2, a variation μ (second argument of the nth variation) would fulfill $\sigma_{ji|j} = 0$ on Ω_x and $\sigma_{ij} n_j = 0$ on $\partial\Omega_{xN}$.)

To understand the connection between this dual problem (Du) and the original problem (En) we also introduce the *free problem* (frequently associated with the names Hu and Washizu (1982) in the engineering literature), which is to find all stationary points of the energy functional

$$\begin{aligned} E_{\text{free}}(v, \gamma, \mu, q) := &\int_{\Omega_x} \frac{1}{2} E_{ijrs} \gamma_{ij} \gamma_{rs} - f_i v_i - \mu_{ij} \left(\gamma_{ij} - \frac{1}{2} \left(v_{i|j} + v_{j|i} \right) \right) \, dV_x \\ &- \int_{\partial\Omega_{x0}} q_i \left(v_i - u_{0i} \right) \, dA_x - \int_{\partial\Omega_{xN}} g_i v_i \, dA_x. \end{aligned}$$

Here v is a one-fold tensor field on Ω_x, whereas, γ and μ are symmetric ($\gamma_{ij} = \gamma_{ji}$, $\mu_{ij} = \mu_{ji}$) two-fold tensor fields on Ω_x and q_i is a one-fold tensor field on $\partial\Omega_{x0}$. If we assume that all fields are sufficiently regular, we can reveal their physical meaning, by computing the first variation

$$\begin{aligned} &\delta E_{\text{free}}((u, \varepsilon, \sigma, t); (v, \gamma, \mu, q)) \\ =& \int_{\Omega_x} \frac{1}{2} E_{ijrs} \left(\varepsilon_{ij} \gamma_{rs} + \gamma_{ij} \varepsilon_{rs} \right) - f_i v_i - \mu_{ij} \left(\varepsilon_{ij} - \frac{1}{2} \left(u_{i|j} + u_{j|i} \right) \right) - \sigma_{ij} \gamma_{ij} \, dV_x \\ &+ \int_{\Omega_x} \sigma_{ij} \frac{1}{2} \left(v_{i|j} + v_{j|i} \right) \, dV_x - \int_{\partial\Omega_{x0}} t_i v_i + q_i \left(u_i - u_{0i} \right) \, dA_x - \int_{\partial\Omega_{xN}} g_i v_i \, dA_x \end{aligned}$$

of the free energy, use the tensor symmetries, apply integration by parts

$$\begin{aligned} =& \int_{\Omega_x} E_{ijrs} \gamma_{ij} \varepsilon_{rs} - f_i v_i - \mu_{ij} \left(\varepsilon_{ij} - \frac{1}{2} \left(u_{i|j} + u_{j|i} \right) \right) - \sigma_{ij} \gamma_{ij} \, dV_x \\ &- \int_{\Omega_x} \sigma_{ij|j} v_i \, dV_x + \int_{\partial\Omega_{x0}} \sigma_{ij} n_j v_i \, dA_x + \int_{\partial\Omega_{xN}} \sigma_{ij} n_j v_i \, dA_x \\ &- \int_{\partial\Omega_{x0}} t_i v_i + q_i \left(u_i - u_{0i} \right) \, dA_x - \int_{\partial\Omega_{xN}} g_i v_i \, dA_x \\ =& \int_{\Omega_x} \left(E_{ijrs} \varepsilon_{rs} - \sigma_{ij} \right) \gamma_{ij} - \left(\sigma_{ij|j} + f_i \right) v_i - \left(\varepsilon_{ij} - \frac{1}{2} \left(u_{i|j} + u_{j|i} \right) \right) \mu_{ij} \, dV_x \\ &+ \int_{\partial\Omega_{x0}} \left(\sigma_{ij} n_j - t_i \right) v_i + \left(u_i - u_{0i} \right) q_i \, dA_x \\ &+ \int_{\partial\Omega_{xN}} \left(\sigma_{ij} n_j - g_i \right) v_i \, dA_x \end{aligned}$$

and finally use the variational lemma in the usual way, in order to compute the Euler-Lagrange equations, which are

$$\sigma_{ij} = E_{ijrs}\varepsilon_{rs} \qquad \text{f.a. } x \in \Omega_x, \tag{3.18}$$
$$\sigma_{ji|j} = -f_i \qquad \text{f.a. } x \in \Omega_x, \tag{3.19}$$
$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i|j} + u_{j|i}\right) \qquad \text{f.a. } x \in \Omega_x, \tag{3.20}$$
$$\sigma_{ij}n_j = t_i \qquad \text{f.a. } x \in \partial\Omega_{x0}, \tag{3.21}$$
$$u_i = u_{0i} \qquad \text{f.a. } x \in \partial\Omega_{x0}, \tag{3.22}$$
$$\sigma_{ij}n_j = g_i \qquad \text{f.a. } x \in \partial\Omega_{xN}. \tag{3.23}$$

We showed that any stationary point of E_{free} fulfills eqs. (3.18) to (3.23). By performing the partial integration backwards, one shows that any tuple $(u, \varepsilon, \sigma, t)$ fulfilling eqs. (3.18) to (3.23) is a stationary point of E_{free}, i.e., the stationary points of E_{free} are precisely the solutions of the basic equations of linear elasticity (compare section 2.2).

Since *every* solution has to fulfill *all* Euler-Lagrange equations, restricting the domain of E_{free} to pairs which a-priori fulfill some of the Euler-Lagrange equations does not change the set of stationary points, i.e., solutions of linear elasticity. However, it changes their “appearance”, since one might eliminate physical quantities by the use of the Euler-Lagrange equations. To be more specific, if we restrict the domain of E_{free} to pairs (v, γ, μ, q) that fulfill eqs. (3.20) and (3.22) a-priori, i.e., $(v, \left(\frac{1}{2}\left(v_{i|j} + v_{j|i}\right)\right), \mu, q)$ where $v \in u_0 + X_0$, we find

$$E_{\text{free}}\left(v, \left(1/2\left(v_{i|j} + v_{j|i}\right)\right), \mu, q\right) = \int_{\Omega_x} \frac{1}{2} E_{ijrs} v_{i|j} v_{r|s} - f_i v_i \, dV_x - \int_{\partial\Omega_{xN}} g_i v_i \, dA_x = E_{\text{pot}}(v),$$

Indeed we assumed (3.20) to hold during the derivation of problem (En) and indeed the set of admissible functions for E_{pot} are the functions fulfilling the boundary condition (3.22) (compare section 3.2). Since E_{pot} is independent of μ and q, the only stationary point of E_{free} is precisely the uniquely determined stationary point of E_{pot}, i.e., the weak solution $u + u_0$ of problem (En).

Now we will show that we arrive at the dual problem (Du), if we assume the complementary set of Euler-Lagrange equations to hold a-priori, i.e., eqs. (3.18), (3.19), (3.21) and (3.23). More precisely, we replace (3.18) by the equivalent (compare section 3.4) inverted constitutive equation

$$\varepsilon_{ij} = D_{ijrs}\sigma_{rs}.$$

By inserting (3.21) and the inverted constitutive equation into the free energy functional, we get

$$\begin{aligned}
&E_{\text{free}}(v, \left(D_{ijrs}\mu_{rs}\right), \mu, \left(\mu_{ij}n_j\right)) \\
&= \int_{\Omega_x} \frac{1}{2} E_{ijrs} D_{ijkl}\mu_{kl} D_{rsmn}\mu_{mn} - f_i v_i - \mu_{ij}\left(D_{ijrs}\mu_{rs} - \frac{1}{2}\left(v_{i|j} + v_{j|i}\right)\right) \, dV_x \\
&\quad - \int_{\partial\Omega_{x0}} \mu_{ij}n_j\left(v_i - u_{0i}\right) \, dA_x - \int_{\partial\Omega_{xN}} g_i v_i \, dA_x \\
&= \int_{\Omega_x} -\frac{1}{2} D_{ijrs}\mu_{ij}\mu_{rs} - f_i v_i + \mu_{ij} v_{i|j} \, dV_x \\
&\quad - \int_{\partial\Omega_{x0}} \mu_{ij}n_j\left(v_i - u_{0i}\right) \, dA_x - \int_{\partial\Omega_{xN}} g_i v_i \, dA_x.
\end{aligned}$$

By formally applying integration by parts

$$
\begin{aligned}
=&\int_{\Omega_x} -\frac{1}{2} D_{ijrs}\mu_{ij}\mu_{rs} - f_i v_i \, dV_x \\
&- \int_{\Omega_x} \mu_{ij|j} v_i \, dV_x + \int_{\partial\Omega_{x0}} \mu_{ij} n_j v_i \, dA_x + \int_{\partial\Omega_{xN}} \mu_{ij} n_j v_i \, dA_x \\
&- \int_{\partial\Omega_{x0}} \mu_{ij} n_j \left(v_i - u_{0i}\right) \, dA_x - \int_{\partial\Omega_{xN}} g_i v_i \, dA_x \\
=&\int_{\Omega_x} -\frac{1}{2} D_{ijrs}\mu_{ij}\mu_{rs} - \left(\mu_{ij|j} + f_i\right) v_i \, dV_x \\
&+ \int_{\partial\Omega_{x0}} \mu_{ij} n_j u_{0i} \, dA_x - \int_{\partial\Omega_{xN}} \left(g_i - \mu_{ij} n_j\right) v_i \, dA_x,
\end{aligned}
$$

and, furthermore, assuming (3.19) and (3.23) to be fulfilled, we get

$$
\begin{aligned}
=&\int_{\Omega_x} -\frac{1}{2} D_{ijrs}\mu_{ij}\mu_{rs} \, dV_x + \int_{\partial\Omega_{x0}} \mu_{ij} n_j u_{0i} \, dA_x \\
=&E_{\text{dual}}(\mu),
\end{aligned}
$$

independent from v. Since the set of stationary points of the dual problem coincides with the set of stationary points of the free problem, which coincides with the set of stationary points of the problem (En), the dual problem also has precisely one stationary point σ, which is connected with the stationary point $\tilde{u} := u + u_0$ of problem (En) by

$$
E_{\text{pot}}(\tilde{u}) = E_{\text{free}}\left(\tilde{u}, \left(1/2\left(\tilde{u}_{i|j} + \tilde{u}_{j|i}\right) = D_{ijrs}\sigma_{rs}\right), \sigma, \left(\sigma_{ij} n_j\right)\right) = E_{\text{dual}}(\sigma),
$$

i.e., the stationary point σ of E_{dual} is defined by $\sigma = (\sigma_{ij}) = (E_{ijrs}\tilde{u}_{r|s})$, where $\tilde{u}$ is the stationary point of E_{pot}.

So far we proved that under the assumptions of theorem 7, we have only one stationary point σ of E_{dual} and the value of the stationary point coincides with the value of the unique solution u of problem (En), i.e., $E_{\text{dual}}(\sigma) = E_{\text{pot}}(u + u_0)$, which was a crucial condition for our thoughts in section 3.3. Furthermore, with the same line of reasoning (as in section 3.2), the stationary problem of E_{dual} is equivalent to the maximum problem (Du): First, the maximum problem of E_{dual} is equivalent to the minimum problem of $-E_{\text{dual}}$. Secondly, the minimum problem is equivalent to the stationary problem by theorem 3, since we have

$$
\begin{aligned}
\delta^2\left(-E_{\text{dual}}\right)(\sigma;\mu) = \int_{\Omega_x} D_{ijrs}\mu_{ij}\mu_{rs} \, dV_x &= \int_{\Omega_x} \underline{\mu}^T \underline{\underline{D}}\, \underline{\mu} \, dV_x \\
&\geq \frac{1}{\lambda_{\max}} \int_{\Omega_x} \underline{\mu}^T \underline{\mu} \, dV_x = \frac{1}{\lambda_{\max}} \|\mu\|_Y^2 .
\end{aligned}
$$

Here we used our thoughts from section 3.4, in particular that $0 < \lambda_{\min} < ... < \lambda_{\max}$ holds for the eigenvalues of $\underline{\underline{E}}$, which implies $0 < \lambda_{\max}^{-1} < ... < \lambda_{\min}^{-1}$, so that the smallest eigenvalue of $\underline{\underline{D}}$ is $\lambda_{\max}^{-1}$. Furthermore, we applied theorem 1 and equation (3.2). Actually

$$
\|\sigma\|_Y := \sqrt{\int_{\Omega_x} \sigma_{ij}\sigma_{ij} \, dV_x}
$$

is a norm on the function space

$$
Y := \left\{\sigma = (\sigma_{ij}) \in \left[L_2(\Omega_x)\right]^{3\times 3} | \sigma_{ij}(x) = \sigma_{ji}(x), \text{ f.a. } x \in \Omega_x, \ i,j \in \{1,2,3\}\right\}.
$$

Therefore, we completed the proof of following theorem.

Theorem 9 (Dual problem)
Under the assumptions of theorem 7, the dual problem (Du) *has a uniquely determined solution* $\sigma \in \Sigma$*, which is connected with the uniquely determined solution* $u \in X_0$ *of theorem 7 by*

$$\sigma_{ij} = E_{ijrs}(u + u_0)_{r|s}.$$

The extremal values of problem (En) *and problem* (Du) *coincide*

$$E_{dual}(\sigma) = E_{pot}(u + u_0).$$

To close the circle we finally compute the Euler-Lagrange equations for the dual problem. To this end, we compute the first variation, use the connection between the solutions given by theorem 9

$$\begin{aligned}\delta E_{\text{dual}}(\sigma;\mu) &= -\int_{\Omega_x} \frac{1}{2} D_{ijrs}\left(\sigma_{ij}\mu_{rs} + \mu_{ij}\sigma_{rs}\right)\, dV_x + \int_{\partial\Omega_{x0}} \mu_{ij} n_j u_{0i}\, dA_x \\ &= -\int_{\Omega_x} D_{ijrs}\sigma_{rs}\mu_{ij}\, dV_x + \int_{\partial\Omega_{x0}} \mu_{ij} n_j u_{0i}\, dA_x \\ &= -\int_{\Omega_x} u_{i|j}\mu_{ij}\, dV_x + \int_{\partial\Omega_{x0}} \mu_{ij} n_j u_{0i}\, dA_x,\end{aligned}$$

formally apply integration by parts

$$\begin{aligned}&= \int_{\Omega_x} u_i \mu_{ij|j}\, dV_x - \int_{\partial\Omega_{x0}} \mu_{ij} n_j u_i\, dA_x - \int_{\partial\Omega_{xN}} \mu_{ij} n_j u_i\, dA_x \\ &\quad + \int_{\partial\Omega_{x0}} \mu_{ij} n_j u_{0i}\, dA_x \\ &= \int_{\partial\Omega_{x0}} \mu_{ij} n_j \left(u_{0i} - u_i\right)\, dA_x\end{aligned}$$

and finally apply the variational lemma in the usual manner, to obtain

$$u_i = u_{0i} \qquad \text{f.a. } x \in \partial\Omega_{x0}. \tag{3.24}$$

Note that the equilibrium equation (3.10) of the classical problem (Cl), as well as the traction boundary condition (3.11), are the Euler-Lagrange equations of the minimum problem of potential energy (En), while the displacement boundary condition (3.24), which is fulfilled automatically by the choice of the function space for the problem (En), is the Euler-Lagrange equation of the dual problem (Du). The complete set of modeling equations for the problem of linear elasticity written in terms of the displacement, i.e., the Navier-Lamé formulation, is only gained by computing the Euler-Lagrange equations of *both* problems and using the relation between their solutions! This fact will be of central importance in the next section.

The proof of theorem 9 is valid since we assumed sufficiently regular data. It paints a remarkable clear and esthetical picture of the connection between the original problem (En) and the dual problem (Du). Unfortunately there is a slight mathematical drawback we want to discuss next.

We already discussed in section 3.2 that one needs more regular data for a weak solution (Wk) to be a classical solution (Cl), than one requires for the mere existence of the weak solution: If we actually only have a weak solution, i.e., an $u \in X_0$ that fulfills (Wk), it can not fulfill (Cl) because $u \in X_0$ does not even have second-order weak derivatives. Likewise, σ defined by

$\sigma_{ij} = E_{ijrs}(u+u_0)_{r|s}$ does not have first-order weak derivatives. We actually have $\sigma \in Y$. So we can not define the space of admissible stress field Σ of the dual problem to be the fields which fulfill (3.19) and (3.23) in a mathematically rigorous setting for weak-solution theory, since σ has no first-order derivatives to fulfill (3.19). To overcome this, we define the side condition for σ as

$$\int_{\Omega_x} \sigma_{ij} \, \frac{1}{2} \left(v_{i|j} + v_{j|i} \right) \, dV_x - \int_{\Omega_x} f_i v_i \, dV_x - \int_{\partial\Omega_{xN}} g_i v_i \, dA_x \qquad \text{f.a. } v \in X_0. \tag{3.25}$$

The space of admissible stress fields is then defined by $\Sigma := \{\sigma \in Y | \sigma \text{ fulfills (3.25)}\}$. If σ has at least first-order weak derivatives, we can apply integration by parts to condition (3.25)

$$\begin{aligned} &\int_{\Omega_x} \sigma_{ij} \, \frac{1}{2} \left(v_{i|j} + v_{j|i} \right) \, dV_x - \int_{\Omega_x} f_i v_i \, dV_x - \int_{\partial\Omega_{xN}} g_i v_i \, dA_x \\ =&\int_{\Omega_x} - \left(\sigma_{ij|j} + f_i \right) \, v_i \, dV_x + \int_{\partial\Omega_{xN}} \left(\sigma_{ij} n_j - g_i \right) v_i \, dA_x \end{aligned}$$

and use the variational lemma to find the condition to be the weak formulation of the conformance of σ with the equilibrium condition (2.1) (or (3.19)) and the stress boundary condition (2.7) (or (3.23)). "Weak form" in this context means that a $\sigma \in Y$ can actually fulfill this condition, without having $W_2^1(\Omega_x)$ component functions. In our setting, where σ is sufficiently smooth, we did not change anything, since we just proved that both definitions are equivalent in this case.

It is getting worse: Even the boundary integral in the definition of E_{dual} in eq. (3.15) is not well-defined, if $\sigma \in Y$. (Since the members of L_2 are actually classes of functions which can differ from each other on sets of measure zero. Therefore, they can not be evaluated pointwise, or be restricted to the boundary, since this is a set of measure zero. One generally introduces the concept of traces to overcome this.) Here we also introduce a similar weak form

$$\int_{\Omega_x} \sigma_{ij} \, \frac{1}{2} \left(u_{0i|j} + u_{0j|i} \right) - f_i u_{0i} \, dV_x - \int_{\partial\Omega_{xN}} g_i u_{0i} \, dA_x,$$

which is well-defined for $\sigma \in \Sigma$ and $u_0 \in X$. Furthermore, if σ has first-order weak derivates we can use integration by parts to obtain

$$\begin{aligned} &\int_{\Omega_x} \sigma_{ij} \, \frac{1}{2} \left(u_{0i|j} + u_{0j|i} \right) - f_i u_{0i} \, dV_x - \int_{\partial\Omega_{xN}} g_i u_{0i} \, dA_x \\ =& - \int_{\Omega_x} \left(\sigma_{ij|j} + f_i \right) u_{0i} \, dV_x + \int_{\partial\Omega_{xN}} \left(\sigma_{ij} n_j - g_i \right) u_{0i} \, dA_x + \int_{\partial\Omega_{x0}} \sigma_{ij} n_j u_{0i} \, dA_x \\ =& \int_{\partial\Omega_{x0}} \sigma_{ij} n_j u_{0i} \, dA_x, \end{aligned}$$

by inserting the strong conformity conditions for the space Σ: (3.19) and (3.23). To sum up, the mathematically rigorous formulation of the dual problem (Du) is

$$\text{Find } \sigma \in \Sigma: \; E_{\text{dual}}(\sigma) = \sup_{\mu \in \Sigma} E_{\text{dual}}(\mu) =: \beta,$$

where E_{dual} is defined by

$$E_{\text{dual}}(\sigma) = - \int_{\Omega_x} \frac{1}{2} D_{ijrs} \sigma_{ij} \sigma_{rs} \, dV_x + \sigma_{ij} \, \frac{1}{2} \left(u_{0i|j} + u_{0j|i} \right) - f_i u_{0i} \, dV_x - \int_{\partial\Omega_{xN}} g_i u_{0i} \, dA_x,$$

for all $\sigma \in \Sigma = \{\sigma \in Y | \sigma \text{ fulfills (3.25)}\}$. Note that this formulation is still equivalent to the first one, if σ has at least first-order weak derivatives. (Also note: Although the dual energy functional is almost identical to the Hellinger-Reissner functional (Hellinger, 1913; Reissner, 1950), which is frequently used for the derivation of Finite Elements, the variational principle of dual energy is entirely different, since the displacement is not considered as a free variable for the dual energy in contrast to the Hellinger-Reissner principle.) With the new definition of the space Σ

$$E_{\text{free}}(v, (D_{ijrs}\mu_{rs}), \mu, (\mu_{ij}n_j)) = E_{\text{dual}}(\mu) \qquad \text{f.a. } \mu \in \Sigma,$$

directly follows by inserting (3.25), without using integration by parts. Therefore, basically under the minimal conditions that are required for the weak solution to exist (cf., Schneider et al., 2014, theorem 4.1), we have the following situation, which is merely a stricter form of theorem 9.

Theorem 10 (Dual problem for the weak formulation)
Let $\Omega_x \subset \mathbb{R}^3$ be a bounded region, with boundary decomposition (2.5)*, with $\partial\Omega_{x0}$ and $\partial\Omega_{xN}$ relatively open. Furthermore, let $u_0 \in X$, $g \in \left[W_2^{1/2}(\partial\Omega_{xN})\right]^3$ and $f \in [L_2(\Omega_x)]^3$ be given and assume we have* (A1) *and* (A2)*. Then:*
The problem (En) *is equivalent to its stationary problem* (Wk)*. Both problems have the same uniquely determined solution $u \in X_0$. The dual problem* (Du) *is equivalent to its stationary problem. Both problems have the same uniquely determined solution $\sigma \in \Sigma$. The solutions* (u, σ) *are connected by the condition*

$$\sigma_{ij} = E_{ijrs}(u + u_0)_{r|s}.$$

The extremal values of problem (En) *and problem* (Du) *coincide*

$$E_{dual}(\sigma) = E_{pot}(u + u_0),$$

i.e., we have $\alpha = \beta$.

In a very general setting (monotone potential operators on locally convex spaces) dual problems can be derived by Fenchel's duality principle. With an application in linear elasticity the principle is proved, e.g., in (Zeidler, 1985, chapter 51). The use of this principle results in a proof of a theorem similar to theorem 10 (cf., Zeidler, 1997, section 62.5). Unfortunately this requires the full apparatus of convex analysis and monotone potential operators, generating a lot of notational overhead, which we tried to avoid here.

We already proved in section 3.3 that this implies the following general error estimate.

Theorem 11 (General error estimate)
Let the assumptions of theorem 9 or theorem 10 be fulfilled, then there exists a solution $u \in X_0$ of the weak problem (Wk)*.*
For all functions $v \in X_0$ and $\mu \in \Sigma$ we have:

$$\frac{1}{2}\lambda_{min}c_{Korn}\left\|(v + u_0) - (u + u_0)\right\|_X^2 \leq E_{pot}(v + u_0) - E_{dual}(\mu).$$

4 Consistent approximation for one-dimensional problems

In this section, we introduce the basic concepts of consistent approximation and provide some essential techniques that we will use in the subsequent sections.

4.1 Introduction

At first we want to sketch the basic ideas to generate an approximative, one-dimensional theory. Roughly speaking, theorem 8 tells us that approximating the potential energy E_{pot} means approximating the solution. If we insert a series expansion for the displacement field u in the directions x_2 and x_3 into the potential energy functional, we get a representation of the potential energy as an infinite series $E_{\text{pot}}(u) = \sum_{n=0}^{\infty} E_{\text{pot}}^{n}\left(u_i^{kl}\right)$ in the unknown displacement coefficients u_i^{kl}, which are functions of the coordinate x_1. (This representation will be provided in section 4.6.) If we truncate the series to a finite partial sum, choosing a summation bound $m \in \mathbb{N}$, and derive the corresponding Euler-Lagrange equations by variation with respect to the series coefficients, we get a finite set of ordinary differential equations (ODEs) in finitely many unknown displacement coefficients. By theorem 8 the error of a solution u_m of the ODE-system is given by the remainder of the series

$$\frac{1}{2}\lambda_{\min} c_{\text{Korn}} \left\| u_m - u \right\|_X^2 \leq \sum_{n=0}^{m} E_{\text{pot}}^{n}\left(u_i^{kl}\right) - E_{\text{pot}}(u) = -\sum_{n=m}^{\infty} E_{\text{pot}}^{n}\left(u_i^{kl}\right) \overset{m\to\infty}{\to} 0.$$

We know that the series $\sum_{n=0}^{\infty} E_{\text{pot}}^{n}\left(u_i^{kl}\right) = E_{\text{pot}}(u)$ converges a-priori, since it is a series expansion of the actual solution. The decaying behavior of the remainder is a necessary condition for this convergence. Therefore, the sequence $(u_m)_{m\in\mathbb{N}}$ converges to the actual solution u, i.e., the approximation u_m gets more and more accurate as we take more summands into account.

We already know that we can not actually generate a full set of modeling ODEs *with corresponding boundary conditions* from the Euler-Lagrange equations of the potential energy alone. The missing displacement boundary conditions are derived from the stationary condition of the dual energy E_{dual}, cf. section 3.5. If we denote the stress tensor field of the real solution u by σ, i.e., $\sigma := E_{ijrs} u_{r|s}$, we have $E_{\text{dual}}(\sigma) = E_{\text{pot}}(u)$ by theorem 10 and the error estimate of theorem 11 can be rewritten as

$$\frac{1}{2}\lambda_{\min} c_{\text{Korn}} \left\| v - u \right\|_X^2 \leq E_{\text{pot}}(v) - E_{\text{pot}}(u) - \left(E_{\text{dual}}(\mu) - E_{\text{dual}}(\sigma)\right).$$

So we can derive the missing displacement boundary conditions of a one-dimensional theory from the stationary condition of the truncated series expansion of E_{dual} in the same manner. This will give us a finite set of ODEs with appropriate boundary conditions. If we take more summands into account, the solutions will converge to the actual solution by virtue of theorem 11.

Since the number of summands taken into account will also increase the number of ODEs and unknown displacement coefficients and therefore the complexity of the approximative theory, it is obviously a good idea to sort the summands by the magnitude of energy they contribute to the whole energy functional. If the energy series, which are reordered by magnitude are truncated, this will generate the most accurate theory with a certain complexity, which is the fundamental paradigm of *consistent* approximation. A representation of the reordered sum of potential energy is provided at the end of section 4.7.

To this end, we will transform the energy functionals to a dimensionless formulation. We also need some basics about Taylor series expansions and we will specify our geometry before we can actually proceed with the analysis of magnitude of the summands of E_{pot} in section 4.6.

4.2 The beam geometry

From now on, for the sake of simplicity, we restrict ourselves to a certain geometry: The "quasi one-dimensional" *beam geometry*. In engineering mechanics one understands a "beam problem" as a problem with a certain geometry *and* a certain load case. We intentionally use the denomination "beam geometry" here, because we want to emphasize, that the load case is *not* restricted to the load case of a beam. (Actually, the most general load cases for one-dimensional problems will turn out to be a result of our investigations in section 6.3.)

To be more specific, we restrict ourselves to the geometry of a beam with constant rectangular cross section, illustrated in figure 2: The edges of the cross section shall coincide with the directions of x_2 and x_3, whereas the direction of x_1 shall coincide with the beam axis. The origin of the coordinate system shall be in the middle of the left face side. The measurements shall be:

- l in x_1 direction, i.e., l denotes the beam length,
- b in x_2 direction, i.e., b denotes the beam width and
- h in x_3 direction, i.e., h denotes the beam height.

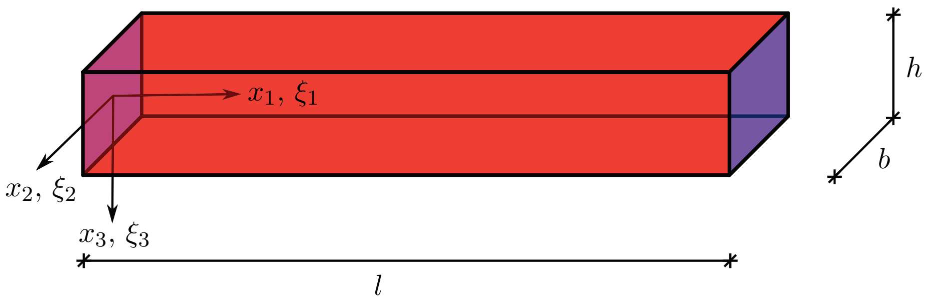

Figure 2: The beam geometry. On the (blue) face sides either displacement boundary conditions or traction boundary conditions are prescribed. On the (red) lateral sides traction boundary conditions are prescribed. The directions of the boundary tractions g, the volume force field f and the prescribed displacements u_0 are not restricted.

We assume that the cross section measurements are much smaller than the beam length, i.e.,

$$b, h \ll l. \tag{A3}$$

As a further restriction to the boundary decomposition: On the lateral sides of the beam geometry boundary tractions g_i shall be prescribed, whereas on the face sides we could either have traction- or displacement-boundary conditions. To fulfill the prior assumption (A1) at least at one face side we shall prescribe displacement boundary conditions. We note this formally by the use of the Cartesian product. By definition, the Cartesian product $A \times B$ of two sets A and B is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Therefore, the definition of the physical body Ω_x becomes

$$\begin{aligned}\Omega_x &:= (0, l) \times \left(-\frac{b}{2}, \frac{b}{2}\right) \times \left(-\frac{h}{2}, \frac{h}{2}\right) \\ &= \left\{ x = (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 \in (0, l)\,,\ x_2 \in \left(-\frac{b}{2}, \frac{b}{2}\right),\ x_3 \in \left(-\frac{h}{2}, \frac{h}{2}\right) \right\}.\end{aligned}$$

Or, if we denote the cross section by A_x, we have:

$$\Omega_x = (0,l) \times A_x, \text{ with } A_x := \left(-\frac{b}{2}, \frac{b}{2}\right) \times \left(-\frac{h}{2}, \frac{h}{2}\right).$$

Furthermore, we define P_{x0}, the x_1-coordinates of face sides where boundary displacements are prescribed and P_{xN}, the x_1-coordinates of face sides where boundary tractions are prescribed, which is the complement of P_{x0}, by

$$P_{x0} := \{0\} \dot\vee \{l\} \dot\vee \{0,l\}, \; P_{xN} := \{0,l\} \setminus P_{x0},$$

here $\dot\vee$ denotes “exclusive or”. The boundary decomposition is then defined by

$$\begin{aligned}
\partial\Omega_{x0} &:= P_{x0} \times \left(-\frac{b}{2}, \frac{b}{2}\right) \times \left(-\frac{h}{2}, \frac{h}{2}\right) = P_{x0} \times A_x, \\
\partial\Omega_{xN} &:= P_{xN} \times \left(-\frac{b}{2}, \frac{b}{2}\right) \times \left(-\frac{h}{2}, \frac{h}{2}\right) \\
&\quad \cup (0,l) \times \left(-\frac{b}{2}, \frac{b}{2}\right) \times \left\{-\frac{h}{2}, \frac{h}{2}\right\} \cup (0,l) \times \left\{-\frac{b}{2}, \frac{b}{2}\right\} \times \left(-\frac{h}{2}, \frac{h}{2}\right).
\end{aligned}$$

For the denotation of boundary conditions, we also introduce the indicator function $\mathbf{1}_S$ of a subset S of M ($S \subset M$), by

$$\begin{aligned}
\mathbf{1}_S : M &\to \{0,1\} \\
x &\mapsto \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}.
\end{aligned}$$

If F is a function on a countable set M, then we extend this definition by

$$\mathbf{1}_S(M)[F] := \sum_{x \in M} \mathbf{1}_S(x) F(x), \tag{4.1}$$

e.g., if $P_{x0} = \{0,l\}$ and $P_{xN} = \emptyset$, then $\mathbf{1}_{P_{x0}}(\{0,l\})[F] = F(0) + F(l)$ and $\mathbf{1}_{P_{xN}}(\{0,l\})[F] = 0$.

4.3 Transformation to dimensionless coordinates

In this contribution we restrict ourselves to a fixed Cartesian coordinate system, i.e., the natural basis is a positively oriented orthonormal system, and physical points of the body are denoted by their coordinates $x = (x_1, x_2, x_3) \in \Omega_x \subset \mathbb{R}^3$. We now introduce the corresponding dimensionless coordinate system, where physical points are denoted by their coordinates $\xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3$. These coordinates are defined by

$$\xi_i := \frac{x_i}{l}, \qquad \text{f.a. } i \in \{1,2,3\}, \tag{4.2}$$

where l is the length of the beam geometry, as introduced in section 4.2.

The definition of ξ (4.2) can be interpreted as a definition of a function, that transforms the parametrization of the given body in x-coordinates Ω_x into a parametrization in ξ-coordinates $\Omega_\xi \subset \mathbb{R}^3$, i.e., $\xi : \Omega_x \to \Omega_\xi, x \mapsto \xi(x)$. Analogously, the inverse mapping is defined by $x : \Omega_\xi \to \Omega_x, \xi \mapsto x(\xi)$. If we think of a scalar physical quantity that can be assigned to each point of the body, e.g. the mass density, or any component function of a tensor field, then it can be interpreted as a function $F : \Omega_x \to \mathbb{R}$, or as a function $\tilde{F} : \Omega_\xi \to \mathbb{R}$. Mathematically F and $\tilde{F}$ are *indeed different* functions, describing *the same* physical quantity, connected by $\tilde{F}(\xi)\big[= \tilde{F}(\xi(x))\big] = F(x)\big[= F(x(\xi))\big]$. However, as usual in physics and engineering applications, we drop this mathematical detail for the sake of simplicity and write $F(x) = F(\xi)$, i.e. F denotes the physical quantity.

The partial derivatives with respect to both coordinates are connected by the chain rule

$$\frac{\partial \tilde{F}}{\partial \xi_j}(\xi) = \frac{\partial}{\partial \xi_j}\left(F(x(\xi))\right) = \sum_{k=1}^{3} \frac{\partial F}{\partial x_k}(x(\xi))\frac{\partial x_k}{\partial \xi_j}(\xi) = \sum_{k=1}^{3} \frac{\partial F}{\partial x_k}(x) l \delta_{kj} = l\frac{\partial F}{\partial x_j}(x),$$

which we note in the usual tensor calculus notation by

$$(F)_{,i} := \frac{\partial}{\partial \xi_i}(F) = l\frac{\partial}{\partial x_i}(F) =: l(F)_{|i}\ , \tag{4.3}$$

where F is some (not necessarily scalar) physical quantity.

If we integrate with respect to, for instance, x_2, where x_1 and x_3 are fixed, integration by substitution yields

$$\int_{g_1}^{g_2} F(x_2)dx_2 = \int_{g_1}^{g_2} \tilde{F}(\xi_2(x_2))dx_2 = l\int_{g_1}^{g_2} \tilde{F}(\xi_2(x_2))\frac{\partial \xi_2}{\partial x_2}(x_2)dx_2 = l\int_{g_1/l}^{g_2/l} \tilde{F}(\xi_2)d\xi_2. \tag{4.4}$$

By application of Fubini's theorem (1907), (cf., e.g., Klenke, 2008, theorem 14.16) we get

$$\int_{\Omega_x} F(x)\, dV_x = l^3 \int_{\Omega_\xi} \tilde{F}(\xi)\, dV_\xi,$$

which we note as

$$\int_{\Omega_x} F\, dV_x = l^3 \int_{\Omega_\xi} F\, dV_\xi,$$

where F denotes some physical quantity.

4.4 A short note on mathematical regularity assumptions

In a mathematically rigorous approach, the corresponding series expansion that is compatible with a weak solution theory, is of course the abstract Fourier series with respect to an orthogonal basis for L_2. As a basis that has the same decaying behavior as monomic polynomials, one can use a basis of scaled Legendre-polynomials, like we did in (Schneider, 2010). In that contribution we showed that the selection of the series expansion has no essential influence on the final theory. In the contribution at hand, we use a basis of monomic polynomials and the corresponding Taylor-series expansion, like in Kienzler (2002, 2004).

To this end, we assume all physical quantities to be real analytic functions, which means by definition that they are representable by their Taylor-series expansion. That the displacement-field solution is real-analytic is implied by assuming real-analytic data (cf., e.g., Fichera, 1973). Since these functions are a subset of the space of smooth-functions C^∞, this assumption is a bit harsh from a mathematical point of view. However, in engineering practice the given data of interest are usually at least piecewise smooth and the points of discontinuities of the derivatives of the data and also the solution are usually known a priori, due to their physical meaning.

Therefore, they could be handled very well by solving the differential equations piecewise and coupling the problems by corresponding boundary conditions, which results in a piecewise smooth solution. In engineering mechanics the treatment of such *multi-domain beams* is a topic of the basic course (cf., Schnell et al., 2002, section 4.5.3). Since the resulting beam theory has a similar ODE and similar stress resultants, a multi-domain beam could be handled in the same manner. The assumption of real analytic quantities is, therefore, reasonable and not more radical than in any other beam theory in engineering mechanics.

The main reason for the choice of the Taylor-series expansion in this thesis, however, is that the resulting equilibrium conditions in stress resultants are classical equations, cf. section 5.5 and 8.4.

4.5 The Taylor series

As commonly known (cf., Bronstein et al., 2001, section 6.1.4.5) a real analytic scalar function depending on one variable x has a representation as a Taylor-series expansion of the form

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0),$$

where $f^{(n)}$ denotes the nth derivative of f and $f^{(0)}$ is understood to be the function f itself, furthermore $n!$ denotes the factorial of the non-negative integer n. The Taylor series above is centered at zero, such a series is also known as a Maclaurin series.
Here we will deal with functions f depending on three independent variables x_1, x_2, x_3, which we expand in two-dimensional Taylor series in x_2- and x_3-direction, centered at the point $(x_2, x_3) = (0, 0)$. Then the coefficients of the two-dimensional Taylor-series expansion still depend on the variable x_1. We have a representation of a form which is usually noted in the literature (cf., Bronstein et al., 2001, section 6.2.2.3) by

$$f(x_1, x_2, x_3) = \sum_{n=0}^{\infty} \frac{1}{n!} \left[\left(x_2 \frac{\partial}{\partial \xi_2} + x_3 \frac{\partial}{\partial \xi_3} \right)^n f(x_1, \xi_2, \xi_3) \right]_{(\xi_2, \xi_3) = (0,0)} .$$

Here $(\bullet)^n$ denotes the n-times composition of the differential operator in round brackets. We rewrite this formulation by inserting the binomial theorem

$$(a+b)^n = \sum_{k=0}^{n} \begin{pmatrix} n \\ k \end{pmatrix} a^k b^{(n-k)}, \qquad \text{with} \begin{pmatrix} n \\ k \end{pmatrix} = \frac{n!}{k!(n-k)!},$$

which gives us

$$f(x_1, x_2, x_3) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} \underbrace{\frac{1}{k!(n-k)!} \left[\frac{\partial^n f}{\partial \hat{x}_2^k \, \hat{x}_3^{n-k}} (x_1, \hat{x}_2, \hat{x}_3) \right]_{(\hat{x}_2, \hat{x}_3) = (0,0)}}_{=: f^{k(n-k)}(x_1)} x_2^k \, x_3^{n-k}. \tag{4.5}$$

For convenience we will call the $f^{k(n-k)}$, introduced in (4.5), the Taylor coefficients, although they are actually functions depending on the variable x_1. The representation (4.5) has the advantage that we can directly identify the Taylor coefficient $f^{k(n-k)}$ belonging to the monomic polynomial $x_2^k \, x_3^{n-k}$.

The partial derivatives of the function f are given by differentiation "under the sum". By renumbering the summands after differentiation, we compute the following differentiation rules, which allow us to differentiate Taylor series only by substitution of the Taylor coefficient:

$$\frac{\partial f}{\partial x_1}(x_1, x_2, x_3) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{\partial f^{k(n-k)}}{\partial x_1}(x_1)\; x_2^k\; x_3^{n-k}, \tag{4.6}$$

$$\begin{aligned}\frac{\partial f}{\partial x_2}(x_1, x_2, x_3) &= \sum_{n=1}^{\infty}\sum_{k=1}^{n} f^{k(n-k)}(x_1)\; k\; x_2^{k-1} x_3^{n-k} \\ &= \sum_{n=0}^{\infty}\sum_{k=1}^{n+1} f^{k(n+1-k)}(x_1)\; k\; x_2^{k-1} x_3^{n+1-k} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} (k+1) f^{k+1(n-k)}(x_1)\; x_2^k\; x_3^{n-k},\end{aligned} \tag{4.7}$$

$$\begin{aligned}\frac{\partial f}{\partial x_3}(x_1, x_2, x_3) &= \sum_{n=1}^{\infty}\sum_{k=0}^{n-1} f^{k(n-k)}(x_1)\; x_2^k\; (n-k) x_3^{n-k-1} \\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n} (n+1-k) f^{k(n+1-k)}(x_1)\; x_2^k\; x_3^{n-k},\end{aligned} \tag{4.8}$$

To have a common notation for the derivatives of a Taylor series we introduce the *shift operator* S, defined on double series indexed scalar functions X^{km} by

$$\begin{aligned} S_1\left[X^{km}\right] &:= \frac{\partial X^{km}}{\partial x_1}, \\ S_2\left[X^{km}\right] &:= (k+1) X^{(k+1)m}, \\ S_3\left[X^{km}\right] &:= (m+1) X^{k(m+1)}, \end{aligned} \tag{4.9}$$

so that we have

$$(f)_{|i}(x_1, x_2, x_3) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} S_i\left[f^{k(n-k)}\right](x_1)\; x_2^k\; x_3^{n-k}. \tag{4.10}$$

We use the word *operator* here, since actually S_i (for *every* i) could be interpreted as a differential operator that is applied to X, because of the Taylor-series mechanism (cf. eq. (4.5)). Nevertheless, it is more helpful to just think of S_i as an object that renumbers series coefficients for the considerations in the following sections.

We will often use the fact that the Taylor coefficients of a function f are uniquely determined. For the motivation of this fact we assume a given series of the form

$$f(x_1, x_2, x_3) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} a^{k(n-k)}(x_1) x_2^k\; x_3^{n-k}. \tag{4.11}$$

The $a^{k(n-k)}$ have to be the $f^{k(n-k)}$, because after the application of the differential operator in the definition of the Taylor coefficients in (4.5), the constant term in the arising series is the corresponding $a^{k(n-k)}$ with the correct factorials as prefactors and the non-constant terms vanish by the insertion of the point $(0,0)$. If we compute it formally, we get indeed

$$\begin{aligned}
&k!(n-k)!\ f^{k(n-k)}(x_1)\\
&= \left[\frac{\partial^n f}{\partial \xi_2^k\ \xi_3^{n-k}}(x_1,\xi_2,\xi_3)\right]_{(\xi_2,\xi_3)=(0,0)}\\
&= \left[\frac{\partial^n}{\partial \xi_2^k\ \xi_3^{n-k}}\left(\sum_{m=0}^{\infty}\sum_{s=0}^{m} a^{s(m-s)}(x_1)\xi_2^s\ \xi_3^{m-s}\right)\right]_{(\xi_2,\xi_3)=(0,0)}\\
&= \left[\sum_{m=0}^{\infty}\sum_{s=0}^{m}\left(\prod_{i=m-s+1}^{m-s+n-k} i\right)\left(\prod_{j=s+1}^{s+k} j\right) a^{(s+k)(m-s+n-k)}(x_1)\xi_2^s\ \xi_3^{m-s}\right]_{(\xi_2,\xi_3)=(0,0)}\\
&= \left(\prod_{i=1}^{n-k} i\right)\left(\prod_{j=1}^{k} j\right) a^{k(n-k)}(x_1)\\
&\quad + \sum_{m=1}^{\infty}\sum_{s=0}^{m}\left(\prod_{i=m-s+1}^{m-s+n-k} i\right)\left(\prod_{j=s+1}^{s+k} j\right) a^{(s+k)(m-s+n-k)}(x_1)\underbrace{\left[\xi_2^s\ \xi_3^{m-s}\right]_{(\xi_2,\xi_3)=(0,0)}}_{=0}\\
&= k!(n-k)!\ a^{k(n-k)}(x_1).
\end{aligned}$$

4.6 Consistent truncation

Now we can proceed with the analysis of the magnitude of the summands of the potential energy E_{pot}. First, we transform to dimensionless ξ-coordinates (cf. section 4.3) and exclude the factor Gl^3, so that the remainder in curly brackets becomes dimensionless. Here G is some characteristic material constant with the physical dimension of stress $\left[N/m^2\right]$ and l is the beam length introduced in section 4.2. For instance, G can be chosen as the shear modulus for isotropy, for a convenient dimensionless representation of Hook's law, as already done in equation (2.16). Also we rewrite the summands in the curly brackets in such way that they are a product of dimensionless quantities (round brackets).

$$\begin{aligned}
E_{\text{pot}}(u) &= \frac{1}{2}B(u,u) - F(u)\\
&= \int_{\Omega_x}\frac{1}{2}E_{ijrs}u_{i|j}u_{r|s}\ dV_x - \int_{\Omega_x} f_i u_i\ dV_x - \int_{\partial\Omega_{xN}} g_i u_i\ dA_x\\
&= Gl^3\Bigg\{\int_{\Omega_\xi}\frac{1}{2}\left(\frac{E_{ijrs}}{G}\right)\left(\frac{u_i}{l}\right)_{,j}\left(\frac{u_r}{l}\right)_{,s} dV_\xi\\
&\qquad - \int_{\Omega_\xi}\left(\frac{lf_i}{G}\right)\left(\frac{u_i}{l}\right) dV_\xi - \int_{\partial\Omega_{\xi N}}\left(\frac{g_i}{G}\right)\left(\frac{u_i}{l}\right) dA_\xi\Bigg\}
\end{aligned}$$

Finally, division by Gl^3 yields the dimensionless equation

$$\begin{aligned}
\left(\frac{E_{\text{pot}}(u)}{Gl^3}\right) &= \int_{\Omega_\xi}\frac{1}{2}\left(\frac{E_{ijrs}}{G}\right)\left(\frac{u_i}{l}\right)_{,j}\left(\frac{u_r}{l}\right)_{,s} dV_\xi\\
&\quad - \int_{\Omega_\xi}\left(\frac{lf_i}{G}\right)\left(\frac{u_i}{l}\right) dV_\xi - \int_{\partial\Omega_{\xi N}}\left(\frac{g_i}{G}\right)\left(\frac{u_i}{l}\right) dA_\xi.
\end{aligned}$$

By insertion of the Taylor-series expansion for the displacement field (see section 4.5)

$$\frac{u_i(\xi)}{l} = \sum_{n=0}^{\infty}\sum_{q=0}^{n} u_i^{q(n-q)}(\xi_1)\xi_2^q\xi_3^{n-q},$$

$$\text{or} \quad \frac{u_r(\xi)}{l} = \sum_{m=0}^{\infty}\sum_{k=0}^{m} u_r^{k(m-k)}(\xi_1)\xi_2^k\xi_3^{m-k}, \tag{4.12}$$

respectively, and making use of the specific beam geometry (see section 4.2), we get

$$\begin{aligned}
&\int_{\Omega_\xi} \frac{1}{2}\Big(\frac{E_{ijrs}}{G}\Big)\Big(\frac{u_i}{l}\Big)_{,j}\Big(\frac{u_r}{l}\Big)_{,s} dV_\xi \\
&= \int_0^1 \int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} \frac{1}{2}\Big(\frac{E_{ijrs}}{G}\Big)\Big(\sum_{n=0}^{\infty}\sum_{q=0}^{n} u_i^{q(n-q)}\xi_2^q\xi_3^{n-q}\Big)_{,j} \\
&\qquad \Big(\sum_{m=0}^{\infty}\sum_{k=0}^{m} u_r^{k(m-k)}\xi_2^k\xi_3^{m-k}\Big)_{,s} d_{\xi_3}d_{\xi_2}d_{\xi_1} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{n}\sum_{k=0}^{m} \frac{1}{2}\Big(\frac{E_{ijrs}}{G}\Big)\int_0^1 S_j\left[u_i^{q(n-q)}\right] S_s\left[u_r^{k(m-k)}\right] d_{\xi_1} \\
&\qquad \int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} \xi_2^{q+k}\xi_3^{n+m-q-k}\, d_{\xi_3}d_{\xi_2}
\end{aligned}$$

for the first dimensionless summand of E_{pot}, where S denotes the shift operator defined in eq. (4.9). Furthermore, we introduce the characteristic parameters

$$d := \frac{b}{\sqrt{12}l} \text{ and } c := \frac{h}{\sqrt{12}l},$$

which basically describe the relative thickness of the beam in ξ_2- and ξ_3-direction. By the basic geometrical assumption for the beam (A3)

$$d, c \ll 1. \tag{4.13}$$

Also we introduce for $k, m \in \mathbb{N}_0$ the shorthand

$$e^{k,m} := \int_{\frac{-b}{2l}}^{\frac{b}{2l}} \int_{\frac{-h}{2l}}^{\frac{h}{2l}} \xi_2^k\xi_3^m \, d\xi_3\, d\xi_2 = \begin{cases} \frac{1}{(k+1)(m+1)}\frac{hb}{l^2}\left(\sqrt{3}d\right)^k\left(\sqrt{3}c\right)^m & \text{if } k \text{ and } m \text{ even} \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

for the scaling factor appearing in the potential energy. The first summand of the potential energy then simply reads as

$$\begin{aligned}
&\int_{\Omega_\xi} \frac{1}{2}\Big(\frac{E_{ijrs}}{G}\Big)\Big(\frac{u_i}{l}\Big)_{,j}\Big(\frac{u_r}{l}\Big)_{,s} dV_\xi \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{n}\sum_{k=0}^{m} \frac{1}{2}\Big(\frac{E_{ijrs}}{G}\Big)e^{q+k,n+m-q-k}\int_0^1 S_j\left[u_i^{q(n-q)}\right] S_s\left[u_r^{k(m-k)}\right] d_{\xi_1}.
\end{aligned}$$

Now we turn our attention to the summands in the potential energy that contain the given loads f and g. First we get

$$
\begin{aligned}
&-\int_{\Omega_\xi}\left(\frac{lf_i}{G}\right)\left(\frac{u_i}{l}\right)dV_\xi-\int_{\partial\Omega_{\xi N}}\left(\frac{g_i}{G}\right)\left(\frac{u_i}{l}\right)dA_\xi\\
=&-\int_0^1\int_{-\frac{b}{2l}}^{\frac{b}{2l}}\int_{-\frac{h}{2l}}^{\frac{h}{2l}}\left(\frac{lf_i}{G}\right)\left(\frac{u_i}{l}\right)d_{\xi_3}d_{\xi_2}d_{\xi_1}\\
&-\int_0^1\int_{-\frac{b}{2l}}^{\frac{b}{2l}}\left[\left(\frac{g_i}{G}\right)\left(\frac{u_i}{l}\right)\right]\left(\xi_1,\xi_2,+\frac{h}{2l}\right)d_{\xi_2}-\int_{-\frac{b}{2l}}^{\frac{b}{2l}}\left[\left(\frac{g_i}{G}\right)\left(\frac{u_i}{l}\right)\right]\left(\xi_1,\xi_2,-\frac{h}{2l}\right)d_{\xi_2}\;d_{\xi_1}\\
&-\int_0^1\int_{-\frac{h}{2l}}^{\frac{h}{2l}}\left[\left(\frac{g_i}{G}\right)\left(\frac{u_i}{l}\right)\right]\left(\xi_1,+\frac{b}{2l},\xi_3\right)d_{\xi_3}-\int_{-\frac{h}{2l}}^{\frac{h}{2l}}\left[\left(\frac{g_i}{G}\right)\left(\frac{u_i}{l}\right)\right]\left(\xi_1,-\frac{b}{2l},\xi_3\right)d_{\xi_3}\;d_{\xi_1}\\
&-\mathbf{1}_{P_{\xi_N}}(\xi_1=\{0,1\})\left[\int_{-\frac{b}{2l}}^{\frac{b}{2l}}\int_{-\frac{h}{2l}}^{\frac{h}{2l}}\left(\frac{g_i}{G}\right)\left(\frac{u_i}{l}\right)d_{\xi_3}d_{\xi_2}\right],
\end{aligned}
$$

by using the specific beam geometry. If we insert the series expansion for the displacement field (4.12) and, furthermore define the *load resultants* $p_i^{k(m-k)}$ for $k,m\in\mathbb{N}_0$ with $k\le m$ by

$$
\begin{aligned}
&p_i^{k(m-k)}(\xi_1)\\
:=&\int_{-\frac{b}{2l}}^{\frac{b}{2l}}\int_{-\frac{h}{2l}}^{\frac{h}{2l}}\left(\frac{lf_i}{G}\right)\xi_2^k\xi_3^{m-k}\,d_{\xi_3}d_{\xi_2}\\
&+\int_{-\frac{b}{2l}}^{\frac{b}{2l}}\left[\frac{g_i}{G}\right]\left(\xi_1,\xi_2,\frac{h}{2l}\right)\xi_2^k\left(\frac{h}{2l}\right)^{m-k}+\left[\frac{g_i}{G}\right]\left(\xi_1,\xi_2,-\frac{h}{2l}\right)\xi_2^k\left(-\frac{h}{2l}\right)^{m-k}d_{\xi_2}\\
&+\int_{-\frac{h}{2l}}^{\frac{h}{2l}}\left[\frac{g_i}{G}\right]\left(\xi_1,\frac{b}{2l},\xi_3\right)\left(\frac{b}{2l}\right)^k\xi_3^{m-k}+\left[\frac{g_i}{G}\right]\left(\xi_1,-\frac{b}{2l},\xi_3\right)\left(-\frac{b}{2l}\right)^k\xi_3^{m-k}\,d_{\xi_3},
\end{aligned}
\tag{4.15}
$$

we get

$$
\begin{aligned}
&-\int_{\Omega_\xi}\left(\frac{lf_i}{G}\right)\left(\frac{u_i}{l}\right)dV_\xi-\int_{\partial\Omega_{\xi N}}\left(\frac{g_i}{G}\right)\left(\frac{u_i}{l}\right)dA_\xi\\
=&-\sum_{m=0}^{\infty}\sum_{k=0}^{m}\Bigg\{\int_0^1 u_i^{k(m-k)}p_i^{k(m-k)}\,d_{\xi_1}\\
&\qquad+\mathbf{1}_{P_{\xi_N}}(\xi_1=\{0,1\})\left[u_i^{k(m-k)}\int_{-\frac{b}{2l}}^{\frac{b}{2l}}\int_{-\frac{h}{2l}}^{\frac{h}{2l}}\left(\frac{g_i}{G}\right)\xi_2^k\xi_3^{m-k}\,d_{\xi_3}d_{\xi_2}\right]\Bigg\}.
\end{aligned}
$$

If we also expand the given loads in corresponding Taylor series

$$\left[\frac{lf_i}{G}\right](\xi_1,\xi_2,\xi_3)\stackrel{!}{=}\sum_{n=0}^{\infty}\sum_{q=0}^{n}f_i^{q(n-q)}(\xi_1)\xi_2^q\xi_3^{n-q},\tag{4.16}$$

$$\left[\frac{g_i}{G}\right](\xi_1,\xi_2,\pm\frac{h}{2l})\stackrel{!}{=}\sum_{n=0}^{\infty}g_i^{n\pm}(\xi_1)\xi_2^n,\tag{4.17}$$

$$\left[\frac{g_i}{G}\right](\xi_1,\pm\frac{b}{2l},\xi_3)\stackrel{!}{=}\sum_{n=0}^{\infty}g_i^{\pm n}(\xi_1)\xi_3^n,\tag{4.18}$$

$$\left[\frac{g_i}{G}\right](0,\xi_2,\xi_3) \stackrel{!}{=} \sum_{n=0}^{\infty}\sum_{q=0}^{n} g_{0i}^{q(n-q)}\xi_2^q\xi_3^{n-q}, \tag{4.19}$$

$$\left[\frac{g_i}{G}\right](1,\xi_2,\xi_3) \stackrel{!}{=} \sum_{n=0}^{\infty}\sum_{q=0}^{n} g_{1i}^{q(n-q)}\xi_2^q\xi_3^{n-q}, \tag{4.20}$$

by insertion into the definition of the load resultants (4.15) we get the representation

$$\begin{aligned}
&p_i^{k(m-k)}(\xi_1)\\
&= \sum_{n=0}^{\infty}\sum_{q=0}^{n} f_i^{q(n-q)} e^{k+q,m-k+n-q}\\
&\quad + \sum_{n=0}^{\infty}\left(g_i^{n+}\left(\frac{h}{2l}\right)^{m-k} + g_i^{n-}\left(-\frac{h}{2l}\right)^{m-k}\right)\int_{-\frac{b}{2l}}^{\frac{b}{2l}} \xi_2^{k+n}\, d_{\xi_2}\\
&\quad + \sum_{n=0}^{\infty}\left(g_i^{+n}\left(\frac{b}{2l}\right)^{k} + g_i^{-n}\left(-\frac{b}{2l}\right)^{k}\right)\int_{-\frac{h}{2l}}^{\frac{h}{2l}} \xi_3^{m-k+n}\, d_{\xi_3}\\
&= \sum_{n=0}^{\infty}\sum_{q=0}^{n} f_i^{q(n-q)}\\
&\qquad \begin{cases} \frac{1}{(k+q+1)(m-k+n-q+1)}\frac{hb}{l^2}\left(\sqrt{3}d\right)^{k+q}\left(\sqrt{3}c\right)^{m-k+n-q} & \text{if } k+q \text{ and } m-k+n-q \text{ even}\\ 0 & \text{otherwise}\end{cases}\\
&\quad + \sum_{n=0}^{\infty}\left(g_i^{n+} + (-1)^{m-k}g_i^{n-}\right)\begin{cases} \frac{1}{k+n+1}\frac{b}{l}\left(\sqrt{3}d\right)^{k+n}\left(\sqrt{3}c\right)^{m-k} & \text{if } k+n \text{ even}\\ 0 & \text{otherwise}\end{cases}\\
&\quad + \sum_{n=0}^{\infty}\left(g_i^{+n} + (-1)^{k}g_i^{-n}\right)\begin{cases} \frac{1}{m-k+n+1}\frac{h}{l}\left(\sqrt{3}d\right)^{k}\left(\sqrt{3}c\right)^{m-k+n} & \text{if } m-k+n \text{ even}\\ 0 & \text{otherwise,}\end{cases}
\end{aligned} \tag{4.21}$$

of the load resultants as power series in c and d. So the summands of the potential energy containing loads have the representation

$$\begin{aligned}
&-\int_{\Omega_\xi}\left(\frac{lf_i}{G}\right)\left(\frac{u_i}{l}\right)dV_\xi - \int_{\partial\Omega_{\xi N}}\left(\frac{g_i}{G}\right)\left(\frac{u_i}{l}\right)dA_\xi\\
&= -\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{n}\sum_{k=0}^{m}\int_0^1 u_i^{k(m-k)} f_i^{q(n-q)}\, d_{\xi_1} e^{k+q,m-k+n-q}\\
&\quad - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\int_0^1 u_i^{k(m-k)}\left(g_i^{n+} + (-1)^{m-k}g_i^{n-}\right)d_{\xi_1}\\
&\qquad \begin{cases} \frac{1}{k+n+1}\frac{b}{l}\left(\sqrt{3}d\right)^{k+n}\left(\sqrt{3}c\right)^{m-k} & \text{if } k+n \text{ even}\\ 0 & \text{otherwise}\end{cases}\\
&\quad - \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\int_0^1 u_i^{k(m-k)}\left(g_i^{+n} + (-1)^{k}g_i^{-n}\right)d_{\xi_1}\\
&\qquad \begin{cases} \frac{1}{m-k+n+1}\frac{h}{l}\left(\sqrt{3}d\right)^{k}\left(\sqrt{3}c\right)^{m-k+n} & \text{if } m-k+n \text{ even}\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

$$-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{n}\sum_{k=0}^{m}\mathbf{1}_{P_{\xi_N}}(\xi_1=\{0,1\})\left[u_i^{k(m-k)}g_{\xi_1 i}^{q(n-q)}\right]e^{k+q,m-k+n-q}.$$

Altogether we found the representation

$$\begin{aligned}
&\left(\frac{E_{\text{pot}}(u)}{Gl^3}\right)\\
&=\int_{\Omega_\xi}\frac{1}{2}\left(\frac{E_{ijrs}}{G}\right)\left(\frac{u_i}{l}\right)_{,j}\left(\frac{u_r}{l}\right)_{,s}dV_\xi-\int_{\Omega_\xi}\left(\frac{lf_i}{G}\right)\left(\frac{u_i}{l}\right)dV_\xi-\int_{\partial\Omega_{\xi N}}\left(\frac{g_i}{G}\right)\left(\frac{u_i}{l}\right)dA_\xi\\
&=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{n}\sum_{k=0}^{m}\frac{1}{2}\left(\frac{E_{ijrs}}{G}\right)\int_0^1 S_j\left[u_i^{q(n-q)}\right]S_s\left[u_r^{k(m-k)}\right]d_{\xi_1}e^{q+k,n+m-q-k}\\
&\quad-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{n}\sum_{k=0}^{m}\int_0^1 u_i^{k(m-k)}f_i^{q(n-q)}\,d_{\xi_1}e^{k+q,m-k+n-q}\\
&\quad-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\int_0^1 u_i^{k(m-k)}\left(g_i^{n+}+(-1)^{m-k}g_i^{n-}\right)d_{\xi_1}\\
&\qquad\qquad\begin{cases}\frac{1}{k+n+1}\frac{b}{l}\left(\sqrt{3}d\right)^{k+n}\left(\sqrt{3}c\right)^{m-k} & \text{if } k+n \text{ even}\\ 0 & \text{otherwise}\end{cases}\\
&\quad-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\int_0^1 u_i^{k(m-k)}\left(g_i^{+n}+(-1)^{k}g_i^{-n}\right)d_{\xi_1}\\
&\qquad\qquad\begin{cases}\frac{1}{m-k+n+1}\frac{h}{l}\left(\sqrt{3}d\right)^{k}\left(\sqrt{3}c\right)^{m-k+n} & \text{if } m-k+n \text{ even}\\ 0 & \text{otherwise.}\end{cases}\\
&\quad-\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{q=0}^{n}\sum_{k=0}^{m}\mathbf{1}_{P_{\xi_N}}(\xi_1=\{0,1\})\left[u_i^{k(m-k)}g_{\xi_1 i}^{q(n-q)}\right]e^{k+q,m-k+n-q}
\end{aligned}\tag{4.22}$$

of the potential energy, which is essentially a power series in c and d.

As already mentioned, truncating the potential energy E_{pot} means approximating the solution. As sketched in section 4.1, we know from theorem 8 that the remainder of the series gives us the approximation error. Therefore, it is obviously a good idea to sort the summands by the magnitude of energy they contribute to the whole energy functional. If we truncate the sorted sum, the most accurate theory with a certain complexity is generated. This is the major target of the consistent approximation approach.

In order to estimate the magnitude of the summands in dependence of the summation indices (n, m, q, k), it is common practice to seek the factor with the fastest decaying behavior. By formula (4.5) the Taylor coefficients appearing in the summands are (partial) derivatives of the corresponding physical quantity F evaluated at the origin with an additional factor. This factor appearing in the Taylor coefficient F^n, respectively $F^{k(n-k)}$, is

$$\frac{1}{n!},\text{ respectively }\frac{1}{k!(n-k)!}=\frac{1}{n!}\begin{pmatrix}n\\k\end{pmatrix}.$$

In the later case the smallest factors are found for $k=n$ or $k=0$ (as one immediately realizes if one thinks of Pascal's triangle), which also leads to a decaying behavior like $1/n!$, as in the first case. These factors have the fastest decaying behavior when it comes to an asymptotic analysis

$(n \to \infty)$. At a first glance, it therefore seems reasonable to generate approximative theories by making the ansatz of truncated series expansions for the displacement field and the load terms. This is a common approach associated with the name Vekua (1985).

However, we propose a different approach based on the basic geometrical assumption for the beam (A3), which implies $d, c \ll 1$. Every summand contains a product of the form $c^n d^m$, which is decaying very fast. More precisely, even for very thick beams we have

$$\frac{b}{l}, \frac{h}{l} < \frac{\sqrt{12}}{10} \approx 0,34 \quad \Longrightarrow \quad d, c < \frac{1}{10},$$

i.e., the product $c^n d^m$ decreases by **at least** one decimal digit in magnitude with every increase of the common power $M := n + m$ by one! However, when it comes to an asymptotic analysis $(M \to \infty)$ this factor does, of course, *not* decay faster than $1/M!$. Nevertheless, we have to consider the fact that for reasonable approximative theories the approximation order of Vekua-type theories is in general very small (definitely < 10), and for small approximation orders the summands' magnitude is still dominated by the factors $c^n d^m$. As an example, if we think of reasonable accuracy in engineering applications, we would aim for three or four accurate decimal digits and whereas $c^n d^m$ for $M = n + m = 4$ is at least smaller than $0,0001$, the smallest Taylor-coefficient prefactors are still $1/4! = 1/24 \approx 0,04$. So it seems to us the most convenient way to truncate the series of potential energy, is to truncate after a certain common power $M = n + m$ of the factors $c^n d^m$, while we keep all summands of the inner sums (with summation indices k and q). Note that every summand of the potential energy contains a factor $c^r d^s$ and that the magnitude of each summand, i.e., the common power $M = r + s$, is precisely $n + m$, in the representation above. Therefore, we derive an approximative theory by

$$\text{replacing the infinite double sum} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \text{ with the finite double sum} \sum_{\{n,m \in N_0 | n+m \leq M\}} .$$

If we truncate in this way, we could expect a significant increase in accuracy (in terms of correct decimal digits) with every increase of M and already sufficiently accurate results for $M = 4$.

We want to stress out that this approach is entirely different from the widely used Vekua approach! Note that, if we make the ansatz of a Taylor series of the displacement field truncated at $n = m = 2$, we would, for example, not consider tuples like $(n, m) = (0, 4)$ or $(4, 0)$ in the potential energy, although the summands for this tuples are of the same magnitude as those for the considered tuple $(2, 2)$ (by the argumentation of the last paragraph). In other words: The consideration of the tuple $(2, 2)$ does not significantly increase the accuracy of the approximation, since other summands which are of the same magnitude are still neglected. In contrast, by the suggested truncation approach one generates the most accurate theory with a certain complexity. Therefore, we call this approach the *consistent* approximation approach.

It is obvious that only even powers of c and d appear in the summands that contain $e^{r,s}$, by the mere definition of $e^{r,s}$ in equation (4.14). In contrast, the summands containing the prescribed mantle tractions $g_i^{\pm n}$ and $g_i^{n\pm}$ could contain odd powers of c and d. This is omitted if we assume, that we only have "even" loads, i.e., if we have

$$g_i^{n+} - g_i^{n-} = 0 \text{ and } g_i^{+n} - g_i^{-n} = 0, \text{ f.a. } n \in \mathbb{N}_0. \tag{A4}$$

The concept of "even" loads will be introduced and discussed in detail in section 6.3. With assumption (A4) it is meaningful to define the *truncation order* N as the half of the greatest considered common power $2N = M$, because the potential energy is a power series in c^2 and

d^2. This means in an Nth-order theory all terms containing factors $c^n d^m$ with $n + m \leq 2N$ will be considered, whereas all terms containing factors with $n + m > 2N$ will be neglected. By assuming (A4) the smallest neglected power is then $2N + 2$. In general, we will denote the negligence of this higher order terms by $+O(e^{2(N+1)})$, according to the fact that the magnitude of $e^{r,s}$ is $M = r + s$. (Note that O is *not* the Landau symbol, since the definition of the Landau symbol explicitly invokes an *asymptotic* analysis. However, we choose a similar symbol, since the ostensive meaning of the symbol is also the negligence of higher order terms.)

The occurrence of odd tractions does not affect our considerations, i.e., assumption (A4) is not at all crucial and only assumed for simplicity in this and in the subsequent subsection. In the presence of a general load case that does not fulfill (A4), we simply lose some accuracy, since the approximation error is then of the order $O(e^{2N+1})$. Alternatively we could simply truncate equation (4.21) with an error of order $O(e^{2(N+1)})$ to gain an overall error of order $O(e^{2(N+1)})$ again, which will be the approach of our choice. Therefore, we could treat any three-dimensional load-case that is admissible by section 4.2. In the next subsection we derive an explicit representation of the truncated potential energy, where the order of the theory N appears as a summation bound.

Let us revisit equation (4.22), again. Definitely the magnitude of $e^{r,s}$ and the case dependent curly brackets are given by the factors of type $c^n d^m$, since the other factors are constant with respect to the summation indices, like $\frac{h}{l}$, or much slower decreasing, like $\frac{1}{k+n+1}$. Also, the dimensionless material constants do not play a role since they do not depend on the summation indices n, m, q and k and are, furthermore, approximately of magnitude 1. All other appearing factors are Taylor coefficients of the unknown displacement field u or the given loads f and g. Again, by formula (4.5) the Taylor coefficients are (partial) derivatives of the corresponding physical quantities evaluated at the origin with an additional scaling factor. We already discussed that the decaying behavior of these factors is not dominant for reasonable approximation orders. However, we have to face the fact that we do not know anything general about the magnitude of the derivatives. One can construct functions with derivatives that increase in magnitude for increasing derivative order. However, typical polynomial given load fields in engineering applications simply vanish at some order and do not differ much in magnitude among decimal digits until then, i.e. they will not play a role compared to $c^n d^m$. Moreover, in general it seems very unlikely that the partial derivatives grow in such a way that they play a significant role. In the absence of a-priori information we simply can not pay respect to their growth behavior for a *general* truncation approach, anyway. But under all circumstances the overall Taylor coefficients can never grow faster than the scaling factors decay, because the decaying behavior of the summands is a necessary condition for the convergence of the sum of potential energy regarded as a power series in c and d, and we know a-priori that the sum has to converge.

4.7 Renumbering the summands by their magnitude

Before we move on with our modeling approach, we first turn our attention to another basic technique we will use frequently. Whether one is able to reveal and understand the structure of a problem crucially depends on its representation. And one of our basic techniques to reveal the underlying structure of our problem will be to renumber our sums, so that the summands are arranged by their magnitude and to “get rid of zeros”, as for example in the first summand of the potential energy which is given in an Nth-order theory by

$$\begin{aligned}
&\int_{\Omega_\xi} \frac{1}{2}\left(\frac{E_{ijrs}}{G}\right)\left(\frac{u_i}{l}\right)_{,j}\left(\frac{u_r}{l}\right)_{,s} dV_\xi \\
&= \sum_{\{n,m\in N_0|n+m\leq 2N\}} \sum_{q=0}^{n}\sum_{k=0}^{m} \frac{1}{2}\left(\frac{E_{ijrs}}{G}\right) e^{q+k,n+m-q-k} \\
&\qquad \int_0^1 S_j\left[u_i^{q(n-q)}\right] S_s\left[u_r^{k(m-k)}\right] d_{\xi_1} + O(e^{2N+2}).
\end{aligned}$$

Roughly speaking, $\frac{3}{4}$ of the summands are zeros because of the distinction of cases in $e^{r,s}$ (4.14). If we define $\gamma\langle n\rangle$ by

$$\gamma\langle n\rangle := \begin{cases} 1 & \text{for } n \text{ even} \\ 0 & \text{for } n \text{ odd} \end{cases}$$

for any $n \in \mathbb{N}_0$, then $e^{q+k,n+m-q-k}$ is zero if, and only if, $\gamma\langle q+k\rangle\gamma\langle n-q+m-k\rangle$ is zero. So, to turn our attention to the fact, whether a summand is zero or not, we find our sum to be of the type

$$\sum_{\{n,m\in N_0|n+m\leq 2N\}} \sum_{q=0}^{n}\sum_{k=0}^{m} \gamma\langle q+k\rangle\gamma\langle n-q+m-k\rangle f(n,m,q,k).$$

We will use this notation frequently, where f always denotes some algebraic term depending on the given summation indices, here n, m, q and k. To be specific, here we have

$$\begin{aligned}
f(n,m,q,k) =& \frac{1}{2(q+k+1)(n-q+m-k+1)} \frac{hb}{l^2}\left(\sqrt{3}d\right)^{q+k}\left(\sqrt{3}c\right)^{n-q+m-k} \\
& \sum_{i,j,r,s=1}^{3}\left(\frac{E_{ijrs}}{G}\right)\int_0^1 S_j\left[u_i^{q(n-q)}\right] S_s\left[u_r^{k(m-k)}\right] d_{\xi_1}.
\end{aligned}$$

The first basic trick is the use of the following lemma.

Lemma 12 *For $a, b \in \mathbb{N}_0$ with $b \geq a$:*

$$\sum_{n=a}^{b} \gamma\langle n\rangle f(n) = \sum_{r=\lceil \frac{a}{2}\rceil}^{\lfloor \frac{b}{2}\rfloor} f(2r).$$

Here, for a real number x the *floor* and *ceiling* functions are defined by

$$\lfloor x\rfloor := \max\{m \in \mathbb{Z}|m \leq x\}, \tag{4.23}$$
$$\lceil x\rceil := \min\{m \in \mathbb{Z}|m \geq x\}. \tag{4.24}$$

Proof The first summand that is actually non-zero is

$$\left\{\begin{array}{cc} a & \text{if } a \text{ even} \\ a+1 & \text{if } a \text{ odd} \end{array}\right\} = 2\left\lceil\frac{a}{2}\right\rceil,$$

whereas, the last summand that is actually non-zero is

$$\left\{\begin{array}{cc} b & \text{if } b \text{ even} \\ b-1 & \text{if } b \text{ odd} \end{array}\right\} = 2\left\lfloor \frac{b}{2} \right\rfloor,$$

which allows us to rewrite the summation indices, such that they are always even.

$$\sum_{n=a}^{b} \gamma\langle n\rangle f(n) = \sum_{n=2\lceil \frac{a}{2}\rceil}^{2\lfloor \frac{b}{2}\rfloor} \gamma\langle n\rangle f(n)$$

Still every second summand is zero. Therefore, we introduce a summation index r with $2r = n$, furnishing by substitution

$$\sum_{n=a}^{b} \gamma\langle n\rangle f(n) = \sum_{r=\lceil \frac{a}{2}\rceil}^{\lfloor \frac{b}{2}\rfloor} \underbrace{\gamma\langle 2r\rangle}_{=1} f(2r).$$

∎

The lemma is not applicable to our sum in a straight forward way, because the argument of γ in our sum is a sum of two, respectively four, summation indices ($\gamma\langle q+k\rangle\gamma\langle n-q+m-k\rangle$). We first make the following observation: The sum of two even numbers is even. Also if a number $k \in \mathbb{Z}$ is even, then $-k$ is even, too. Therefore, we have the two logical implications

$$\begin{aligned} (q+k \text{ even}) \text{ and } (n-q+m-k \text{ even}) &\Rightarrow (n+m \text{ even}), \\ (q+k \text{ even}) \text{ and } (n+m \text{ even}) &\Rightarrow (n-q+m-k \text{ even}), \end{aligned}$$

what proves the equivalence

$$(q+k \text{ even}) \text{ and } (n-q+m-k \text{ even}) \Leftrightarrow (q+k \text{ even}) \text{ and } (n+m \text{ even}),$$

or stated in terms of γ

$$\gamma\langle q+k\rangle\gamma\langle n-q+m-k\rangle = \gamma\langle q+k\rangle\gamma\langle n+m\rangle.$$

This will allow us to apply the lemma successively to the first and the second double sum.

First, we turn our attention to the first (double) sum with the indices n and m. To sum up all elements with an index sum $n+m \leq 2N$, one might first come up with the idea of line-wise summation, as illustrated in figure 3 (on the left side). However, this is not a good idea, since we want the sum $p := n+m$ to be a summation index, so that the summands are ordered by their magnitude. Furthermore, this will enable us to apply lemma 12. Choosing $p = n+m$ as a summation index corresponds to diagonal-wise summation, as also illustrated in figure 3 (on the right side). In written form we have

$$\sum_{\{n,m\in N_0 | n+m\leq 2N\}} f(n,m) = \sum_{n=0}^{2N} \sum_{m=0}^{2N-n} f(n,m) = \sum_{p=0}^{2N} \sum_{n=0}^{p} f(n,p-n). \tag{4.25}$$

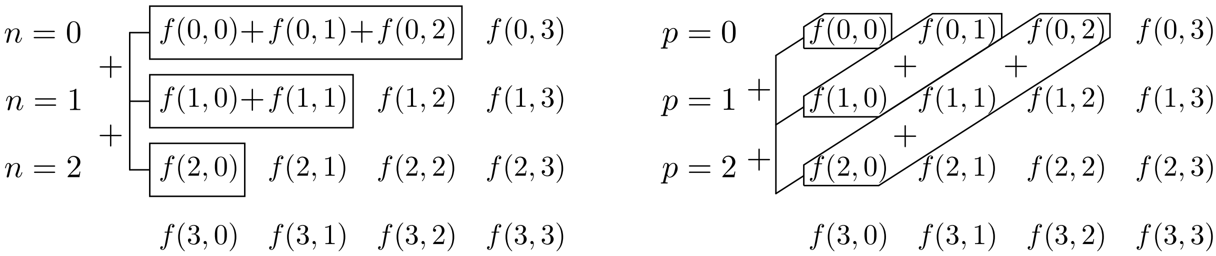

Figure 3: Summation of all elements $f(n,m)$ with $n+m \leq 2N$. Here for $N=1$. On the left: line-wise summation. On the right: diagonal-wise summation. Compare equation (4.25)

If we select the diagonal-wise summation scheme above for the outer double sum in n and m

$$\sum_{\{n,m\in N_0 | n+m\leq 2N\}} \sum_{q=0}^{n} \sum_{k=0}^{m} f(n,m,q,k) = \sum_{p=0}^{2N} \sum_{n=0}^{p} \sum_{q=0}^{n} \sum_{k=0}^{p-n} f(n,p-n,q,k),$$

the inner two sums almost look like the line-wise summation scheme. Precisely, they sum up all elements with index tuples $\{(q,k)\in\mathbb{N}_0^2 | q\leq n,\ k\leq p-n\}$, which is actually a subset of $\{(q,k)\in\mathbb{N}_0^2 | q+k\leq p\}$. If we would deal with the latter set, we could apply the diagonal-wise summation scheme again. By doing this blindly, with the use of the summation indices, let us say $t \coloneqq q+k$ and q, we add some summands, since we actually deal with a subset. But since we have

$$\begin{aligned} k \leq p-n &\Rightarrow t-q \leq p-n \\ &\Rightarrow q \geq t-p+n, \end{aligned}$$

we can reformulate both limiting conditions of the set $\{(q,k)\in\mathbb{N}_0^2 | q\leq n,\ k\leq p-n\}$ as additional conditions for the summation with respect to q, which gives us

$$\begin{aligned} \sum_{\{n,m\in N_0 | n+m\leq 2N\}} \sum_{q=0}^{n} \sum_{k=0}^{m} f(n,m,q,k) &= \sum_{p=0}^{2N} \sum_{n=0}^{p} \sum_{q=0}^{n} \sum_{k=0}^{p-n} f(n,p-n,q,k) \\ &= \sum_{p=0}^{2N} \sum_{n=0}^{p} \sum_{t=0}^{p} \sum_{q=\max\{0,t-p+n\}}^{\min\{t,n\}} f(n,p-n,q,t-q). \end{aligned}$$

Now we can finally apply lemma 12 twice, to obtain

$$\begin{aligned} &\sum_{\{n,m\in N_0 | n+m\leq 2N\}} \sum_{q=0}^{n} \sum_{k=0}^{m} \gamma\langle q+k\rangle \gamma\langle n-q+m-k\rangle f(n,m,q,k) \\ &= \sum_{p=0}^{2N} \sum_{n=0}^{p} \sum_{t=0}^{p} \sum_{q=\max\{0,t-p+n\}}^{\min\{t,n\}} \gamma\langle t\rangle \gamma\langle p\rangle f(n,p-n,q,t-q) \\ &= \sum_{p=0}^{2N} \gamma\langle p\rangle \left(\sum_{t=0}^{p} \gamma\langle t\rangle \left(\sum_{n=0}^{p} \sum_{q=\max\{0,t-p+n\}}^{\min\{t,n\}} f(n,p-n,q,t-q) \right) \right) \end{aligned}$$

$$
\begin{aligned}
&\sum_{\{n,m\in N_0|n+m\le 2N\}} \sum_{q=0}^{n}\sum_{k=0}^{m} \gamma\langle q+k\rangle\gamma\langle n-q+m-k\rangle f(n,m,q,k)\\
&=\sum_{p=0}^{2N}\gamma\langle p\rangle\left(\sum_{t=0}^{p}\gamma\langle t\rangle\left(\sum_{n=0}^{p}\sum_{q=\max\{0,t-p+n\}}^{\min\{t,n\}} f(n,p-n,q,t-q)\right)\right)\\
&=\sum_{\tilde p=0}^{N}\sum_{t=0}^{2\tilde p}\gamma\langle t\rangle\left(\sum_{n=0}^{2\tilde p}\sum_{q=\max\{0,t-2\tilde p+n\}}^{\min\{t,n\}} f(n,2\tilde p-n,q,t-q)\right)\\
&=\sum_{\tilde p=0}^{N}\sum_{\tilde t=0}^{\tilde p}\sum_{n=0}^{2\tilde p}\sum_{q=\max\left\{0,2\tilde t-2\tilde p+n\right\}}^{\min\left\{2\tilde t,n\right\}} f(n,2\tilde p-n,q,2\tilde t-q).
\end{aligned}
$$

Here we indicate the substitution of indices by lemma 12 by $\tilde{\bullet}$. Applying the scheme to the actual sum finally proves

$$
\begin{aligned}
&\int_{\Omega_\xi}\frac{1}{2}\left(\frac{E_{ijrs}}{G}\right)\left(\frac{u_i}{l}\right)_{,j}\left(\frac{u_r}{l}\right)_{,s} dV_\xi\\
&=\sum_{p=0}^{N}\sum_{t=0}^{p}\frac{3^p}{2(2t+1)(2p-2t+1)}\frac{hb}{l^2}d^{2t}c^{2p-2t}\\
&\qquad\sum_{n=0}^{2p}\sum_{q=\max\{0,2t-2p+n\}}^{\min\{2t,n\}}\left(\frac{E_{ijrs}}{G}\right)\int_0^1 S_j\left[u_i^{q(n-q)}\right]S_s\left[u_r^{2t-q(2p-n-2t+q)}\right]d_{\xi_1}\\
&\quad+O(e^{2N+2}).
\end{aligned}
$$

As already outlined, the summands of the potential energy containing the boundary tractions could contain odd powers of the characteristic parameters c and d. If we assume (A4), the first summand depending on the traction on the lateral sides of the beam is of the type

$$
\sum_{\{n,m\in N_0|n+m\le 2N\}}\sum_{k=0}^{m}\gamma\langle k+n\rangle\gamma\langle m-k\rangle f(n,m,k)
$$

after truncation. If we use $\gamma\langle k+n\rangle\gamma\langle m-k\rangle=\gamma\langle k+n\rangle\gamma\langle m+n\rangle$ and diagonal-wise summation for the first double sum, we can apply lemma 12 to obtain

$$
=\sum_{p=0}^{2N}\gamma\langle p\rangle\left(\sum_{n=0}^{p}\sum_{k=0}^{p-n}\gamma\langle k+n\rangle f(n,p-n,k)\right)=\sum_{p=0}^{N}\sum_{n=0}^{2p}\sum_{k=0}^{2p-n}\gamma\langle k+n\rangle f(n,2p-n,k).
$$

By rewriting the inner two sums from line-wise to diagonal-wise summation and application of lemma 12 again we, furthermore, get

$$
=\sum_{p=0}^{N}\sum_{t=0}^{2p}\gamma\langle t\rangle\left(\sum_{n=0}^{t}f(n,2p-n,t-n)\right)=\sum_{p=0}^{N}\sum_{t=0}^{p}\sum_{n=0}^{2t}f(n,2p-n,2t-n).
$$

Using (A4) again, the second summand depending on the traction on the lateral sides of the beam is of the type

$$
\sum_{\{n,m\in N_0|n+m\le 2N\}}\sum_{k=0}^{m}\gamma\langle k\rangle\gamma\langle m-k+n\rangle f(n,m,k)
$$

after truncation. If we use $\gamma\langle k\rangle\gamma\langle m-k+n\rangle = \gamma\langle k\rangle\gamma\langle m+n\rangle$ and diagonal-wise summation for the first double sum, we can apply lemma 12, to obtain

$$=\sum_{p=0}^{2N}\gamma\langle p\rangle\left(\sum_{n=0}^{p}\sum_{k=0}^{p-n}\gamma\langle k\rangle f(n,p-n,k)\right)=\sum_{p=0}^{N}\sum_{n=0}^{2p}\sum_{k=0}^{2p-n}\gamma\langle k\rangle f(n,2p-n,k).$$

By application of lemma 12 again we get

$$=\sum_{p=0}^{N}\sum_{n=0}^{2p}\sum_{t=0}^{\left\lfloor\frac{2p-n}{2}\right\rfloor} f(n,2p-n,2t).$$

Finally, by assuming (A4), we have for the potential energy

$$\begin{aligned}
&\left(\frac{E_{\mathrm{pot}}(u)}{Gl^3}\right)\\
&=\int_{\Omega_\xi}\frac{1}{2}\left(\frac{E_{ijrs}}{G}\right)\left(\frac{u_i}{l}\right)_{,j}\left(\frac{u_r}{l}\right)_{,s}dV_\xi-\int_{\Omega_\xi}\left(\frac{lf_i}{G}\right)\left(\frac{u_i}{l}\right)dV_\xi-\int_{\partial\Omega_{\xi N}}\left(\frac{g_i}{G}\right)\left(\frac{u_i}{l}\right)dA_\xi\\
&=\sum_{p=0}^{N}\sum_{t=0}^{p}\frac{3^p}{(2t+1)(2p-2t+1)}\frac{hb}{l^2}d^{2t}c^{2p-2t}\\
&\qquad\sum_{n=0}^{2p}\sum_{q=\max\{0,2t-2p+n\}}^{\min\{2t,n\}}\left\{\frac{1}{2}\left(\frac{E_{ijrs}}{G}\right)\int_0^1 S_j\left[u_i^{q(n-q)}\right]S_s\left[u_r^{2t-q(2p-n-2t+q)}\right]d_{\xi_1}\right.\\
&\qquad\qquad-\int_0^1 u_r^{k(m-k)}f_i^{q(n-q)}\,d_{\xi_1}\\
&\qquad\qquad\left.-\mathbf{1}_{P_{\xi_N}}(\xi_1=\{0,1\})\left[u_r^{k(m-k)}g_i^{\xi_1 n}\right]\right\}\\
&\quad-\sum_{p=0}^{N}\sum_{t=0}^{p}\frac{3^p}{(2t+1)}\frac{b}{l}d^{2t}c^{2p-2t}\sum_{n=0}^{2t}\int_0^1 u_i^{2t-n(2p-2t)}\left(g_i^{n+}+g_i^{n-}\right)d_{\xi_1}\\
&\quad-\sum_{p=0}^{N}\sum_{n=0}^{2p}\sum_{t=0}^{\left\lfloor\frac{2p-n}{2}\right\rfloor}\frac{3^p}{2p-2t+1}\frac{h}{l}d^{2t}c^{2p-2t}\int_0^1 u_i^{2t(2p-n-2t)}\left(g_i^{+n}+g_i^{-n}\right)d_{\xi_1}\\
&\quad+O(e^{2N+2}).
\end{aligned}\tag{4.26}$$

5 An error estimate for the consistent truncation

In this section we will provide the mathematical justification for the method of consistent approximation, by deriving an a-priori estimate that proves the approximation property of this method, cf. theorem 13. Basically the theorem tells us that a solution of the problem that we get by truncating the (in section 5.1 defined) stress resultants at a certain power of e (given by (5.21),(5.22),(5.25)), approximates the exact solution of three-dimensional elasticity with an error of the same magnitude.

5.1 Notation

In the whole section we will *not* use the summation convention. All tensor indices that were previously bound by the summation convention will now be bound by the explicit use of the summation symbol $\sum$. This allows us to use decompositions like (5.3) below and hopefully avoid confusion.

We introduce $\hat{\xi}^{km}$ for all $k, m \in \mathbb{Z}$ in (5.1) as a shorthand for the monomic products in the cross section directions. This way $\hat{\xi}^{km}$ becomes a double series indexed quantity to which we can apply operators like the shift operator S (already defined in (4.9)), or K and a^{23} defined below. Also we define the conditional differential operator D^1 and χ_j by

$$D_j^1(\bullet) := \begin{cases} \frac{\partial(\bullet)}{\partial \xi_1} & \text{if } j = 1 \\ \bullet & \text{otherwise} \end{cases}, \hat{\xi}^{km} := \begin{cases} \xi_2^k \xi_3^m & \text{if } k, m \in \mathbb{N}_0 \\ 0 & \text{if } k \text{ or } m < 0 \end{cases}, \chi_j := \begin{cases} 1 & \text{if } j = 1 \\ -1 & \text{if } j = 2, 3 \end{cases}. \tag{5.1}$$

In addition, we define the shift operators K and a^{23} defined on double series indexed quantities X^{km} by

$$K_j\left[X^{km}\right] := \begin{cases} \left(X^{km}\right)_{,1} & \text{if } j = 1 \\ -kX^{(k-1)m} & \text{if } j = 2 \\ -mX^{k(m-1)} & \text{if } j = 3 \end{cases} \quad \text{and} \quad a_j^{23}\left[X^{km}\right] := \begin{cases} X^{km} & \text{if } j = 1 \\ kX^{(k-1)m} & \text{if } j = 2 \\ mX^{k(m-1)} & \text{if } j = 3 \end{cases}. \tag{5.2}$$

Note that with the definitions above the shift operator K can be decomposed by

$$K_j\left[X^{km}\right] = \chi_j D_j^1\left(a_j^{23}\left[X^{km}\right]\right). \tag{5.3}$$

Now we define the stress resultants

$$\mathcal{M}_{ij}^{km}(\xi_1) := \int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} \frac{\sigma_{ij}}{G} \xi_2^k \xi_3^m \, d_{\xi_3} d_{\xi_2}, \tag{5.4}$$

where k and m are non-negative integers ($k, m \in \mathbb{N}_0$). For negative integer indices $r, s \in \mathbb{Z}$ we make the convention $\mathcal{M}_{ij}^{rs} := 0$ if $r < 0$ or $s < 0$, i.e. we have

$$\mathcal{M}_{ij}^{km} = \int_{A_\xi} \frac{\sigma_{ij}}{G} \hat{\xi}^{km} \, dA_\xi.$$

We also introduce the corresponding prescribed stress resultants on the face sides that belong to $\partial\Omega_{\xi N}$, for all $k, m \in \mathbb{N}_0$, by

$$\mathcal{M}_{Ni1}^{km} n_1 := \int_{A_\xi} \frac{g_i}{G} \hat{\xi}^{km} \, dA_\xi, \qquad \text{f.a. } \xi_1 \in P_{\xi N}, \tag{5.5}$$

and we will use the definition of the load resultants p_i^{km}, for all $k, m \in \mathbb{N}_0$, already introduced in (4.15), which is by the use of $\hat{\xi}^{km}$ equivalent to

$$p_i^{k(m-k)} = \int_{A_\xi} \left(\frac{lf_i}{G}\right) \hat{\xi}^{km} \, dA_\xi + \int_{\partial A_\xi} \left(\frac{g_i}{G}\right) \hat{\xi}^{km} \, ds_\xi .$$

5.2 The approximation of the stress resultants

By inserting the kinematic relation (2.2) into Hooke's law (2.3) and by using the symmetries of the E-tensor (2.4), we get a representation of the stresses in terms of the deformation field u. Furthermore, by inserting the series expansion for u (4.12) into this representation and by applying the differentiation rules for Taylor series (4.10), we get a representation of the stresses in terms of the deformation coefficients $u_r^{q(n-q)}$

$$\begin{aligned} \frac{\sigma_{ij}}{G} &= \sum_{r,s=1}^{3} \frac{E_{ijrs}}{G} \left(\frac{u_r}{l}\right)_{,s} = \sum_{r,s=1}^{3} \frac{E_{ijrs}}{G} \left(\sum_{n=0}^{\infty} \sum_{q=0}^{n} u_r^{q(n-q)}(\xi_1) \xi_2^q \xi_3^{n-q} \right)_{,s} \\ &= \sum_{n=0}^{\infty} \sum_{q=0}^{n} \underbrace{\left\{ \sum_{r,s=1}^{3} \frac{E_{ijrs}}{G} S_s\big[u_r^{q(n-q)}\big] \right\}}_{=:\omega_{ij}^{q(n-q)}} \xi_2^q \xi_3^{n-q} . \end{aligned} \tag{5.6}$$

By the equation above and the uniqueness of the Taylor coefficients, the $\omega_{ij}^{q(n-q)}$ are the dimensionless Taylor coefficients of the stress components, which consist of linear combinations of formally nine displacement coefficients

$$\begin{aligned} \omega_{ij}^{q(n-q)} &:= \sum_{r,s=1}^{3} \frac{E_{ijrs}}{G} S_s\big[u_r^{q(n-q)}\big] \\ &= \sum_{r=1}^{3} \left\{ \frac{E_{ijr1}}{G} u_{r,1}^{q(n-q)} + \frac{E_{ijr2}}{G} (q+1) u_r^{(q+1)(n-q)} + \frac{E_{ijr3}}{G} (n+1-q) u_r^{q(n+1-q)} \right\}. \end{aligned} \tag{5.7}$$

We use the word "formally" here, because the tensor E is usually sparse.

If we insert the series expansion for u (4.12) into the definition of the stress resultants $\mathcal{M}$ (5.4) with the use of (4.14), we get a series stating $\mathcal{M}$ in terms of the displacement coefficients $u_r^{q(n-q)}$

$$\mathcal{M}_{ij}^{k(m-k)}(\xi_1) = \sum_{n=0}^{\infty} \sum_{q=0}^{n} \omega_{ij}^{q(n-q)} e^{q+k,n-q+m-k} . \tag{5.8}$$

The series in (5.8) has a lot of summands that are actually zero because of the factor $e^{q+k,n-q+m-k}$. We will apply the thoughts of section 4.7 to eliminate these summands. If we look at the series in the abstract way introduced in section 4.7, it is of the form

$$\sum_{n=0}^{\infty} \sum_{q=0}^{n} \gamma\langle q+k \rangle \gamma\langle n-q+m-k \rangle f(n,q).$$

With the use of the already proven identity $\gamma\langle q+k \rangle \gamma\langle n-q+m-k \rangle = \gamma\langle q+k \rangle \gamma\langle n+m \rangle$ we can apply the lemma 12 on both sums in our series

$$\sum_{n=0}^{\infty} \sum_{q=0}^{n} \gamma\langle q+k \rangle \gamma\langle n-q+m-k \rangle f(n,q) = \sum_{n=0}^{\infty} \gamma\langle n+m \rangle \sum_{q=0}^{n} \gamma\langle q+k \rangle f(n,q).$$

With the substitution $\tilde{n} := n + m$ we get

$$= \sum_{\tilde{n}=m}^{\infty} \gamma\langle \tilde{n}\rangle \sum_{q=0}^{\tilde{n}-m} \gamma\langle q+k\rangle f(\tilde{n}-m, q),$$

which allows us to apply the lemma for the outer sum. Since the lemma uses the substitution $2r := \tilde{n} = n + m$, we have $n = 2r - m$ and, therefore,

$$= \sum_{r=\lceil \frac{m}{2}\rceil}^{\infty} \sum_{q=0}^{2r-m} \gamma\langle q+k\rangle f(2r-m, q).$$

Finally, the substitution $\tilde{q} = q + k$

$$= \sum_{r=\lceil \frac{m}{2}\rceil}^{\infty} \sum_{\tilde{q}=k}^{2r-m+k} \gamma\langle \tilde{q}\rangle f(2r-m, \tilde{q}-k)$$

allows us to apply the lemma for the inner sum

$$= \sum_{r=\lceil \frac{m}{2}\rceil}^{\infty} \sum_{s=\lceil \frac{k}{2}\rceil}^{\lfloor r-\frac{m}{2}+\frac{k}{2}\rfloor} f(2r-m, 2s-k).$$

We have therefore proved

$$\sum_{n=0}^{\infty}\sum_{q=0}^{n} \gamma\langle q+k\rangle\gamma\langle n-q+m-k\rangle f(n,q) = \sum_{r=\lceil \frac{m}{2}\rceil}^{\infty} \sum_{s=\lceil \frac{k}{2}\rceil}^{\lfloor r-\frac{m}{2}+\frac{k}{2}\rfloor} f(2r-m, 2s-k).$$

Insertion into (5.8) leads to the exact representation

$$\mathcal{M}_{ij}^{k(m-k)} = \frac{hb}{l^2} \sum_{n=\lceil \frac{m}{2}\rceil}^{\infty} \sum_{q=\lceil \frac{k}{2}\rceil}^{\lfloor n-\frac{m}{2}+\frac{k}{2}\rfloor} \frac{3^n}{(2q+1)(2n-2q+1)} d^{2q} c^{2(n-q)} \omega_{ij}^{(2q-k)(2n-m-2q+k)} \tag{5.9}$$

or

$$\mathcal{M}_{ij}^{k(m-k)} = \frac{hb}{l^2} \sum_{n=\lceil \frac{m}{2}\rceil}^{N} \sum_{q=\lceil \frac{k}{2}\rceil}^{\lfloor n-\frac{m}{2}+\frac{k}{2}\rfloor} \frac{3^n}{(2q+1)(2n-2q+1)} d^{2q} c^{2(n-q)} \omega_{ij}^{(2q-k)(2n-m-2q+k)} + O(e^{2(N+1)}), \tag{5.10}$$

for an Nth-order theory, respectively, if we rename the new summation indices to n and q, again, to avoid interferences with the tensor indices r and s. Formula (5.9) already provides a deep insight into the structure of the stress-resultants.

5.3 Properties of the stress resultants

The representation (5.10) is ideal for a straight forward implementation for calculating the stress-resultants of a specific theory. However, we will renumber the summation indices again, so that both sums start form 0, to get a deeper insight into the structure of the stress-resultants. For an Nth-order theory we get

$$\begin{aligned}\mathcal{M}_{ij}^{k(m-k)} = \frac{hb}{l^2} d^{2\lceil \frac{k}{2}\rceil} c^{2(\lceil \frac{m}{2}\rceil - \lceil \frac{k}{2}\rceil)} \sum_{n=0}^{N-\lceil \frac{m}{2}\rceil} \sum_{q=0}^{\lfloor n+\lceil \frac{m}{2}\rceil - \frac{m}{2} + \frac{k}{2}\rfloor - \lceil \frac{k}{2}\rceil} \\ \Bigg(\frac{3^{n+\lceil \frac{m}{2}\rceil}}{(2q+2\left\lceil \frac{k}{2}\right\rceil+1)(2n+2\lceil \frac{m}{2}\rceil - 2q - 2\left\lceil \frac{k}{2}\right\rceil + 1)} \\ d^{2q} c^{2(n-q)} \omega_{ij}^{(2q+2\lceil \frac{k}{2}\rceil - k)(2n+2\lceil \frac{m}{2}\rceil - m - 2q - 2\lceil \frac{k}{2}\rceil + k)} \Bigg) + O(e^{2(N+1)}). \qquad (5.11)\end{aligned}$$

The sum of the indices of the stress resultants m gives us basically the magnitude $(2\left\lceil \frac{m}{2}\right\rceil)$ of an excludable factor in c^2 and d^2, whereas k indicates how the common power is distributed between d^2 and c^2. For example, $\mathcal{M}_{ij}^{20}$ has an excludable factor of magnitude 2, because the sum of the upper indices is 2. Since the first index is 2, it is d^2. Analogously, $\mathcal{M}_{ij}^{01}$ has an excludable factor c^2 and $\mathcal{M}_{ij}^{12}$ has an excludable factor $c^2 d^2$. Furthermore, m basically defines how many summands the double sum actually has, since the complex-looking upper bound of the second sum simply yields

$$\left\lfloor n + \left\lceil \frac{m}{2} \right\rceil - \frac{m}{2} + \frac{k}{2} \right\rfloor - \left\lceil \frac{k}{2} \right\rceil = \begin{cases} n-1 & \text{if } m \text{ even and } k \text{ odd} \\ n & \text{otherwise} \end{cases}.$$

Basically the double sum "fills up" the stress resultants with summands containing all possible logical combinations of powers of the factors c^2 and d^2, so that the common power of c and d of each summand is less or equal $2N$ after the multiplication with the excludable factor. The greater the common power of the excludable factor $2\left\lceil \frac{m}{2}\right\rceil$ is, the fewer possibilities there are to "fill up" the stress-resultant. $2n$ is the common power of the summand (disregarding the excludable factor) and the sum in q basically arranges the distribution of the common power between d^2 and c^2 through all possible combinations.

We used the word "basically" in the last paragraph, because there is one exception. If m is even and k is odd, or formulated differently, if both upper indices of $\mathcal{M}$ are odd, then the double sum does not contain all possible logical combinations of powers of the factors c^2 and d^2, so that the maximal common power is $2N$. In this case, the upper bound for q results in an extra excludable c^2. For example, $\mathcal{M}_{ij}^{11}$ has an excludable factor $c^2 d^2$.

If we turn our attention to the question whether a stress resultant is to be neglected in an approximative theory, we derive

$$\left\lceil \frac{m}{2} \right\rceil > N \text{ or } (m = 2N \text{ and } k \text{ odd}) \Longrightarrow \mathcal{M}_{ij}^{k(m-k)} = 0 + O(e^{2(N+1)}), \qquad (5.12)$$

directly from the upper bounds of the double sum of equation (5.11). For example we have

$$\mathcal{M}_{ij}^{k(m-k)} = 0 + O(e^6) \qquad \text{if } m \geq 5 \text{ or } (m = 4 \text{ and } k \text{ odd}),$$

for a second-order approximation. In general, equation (5.12) implies that there are only finitely many stress-resultants for any Nth-order theory and their number increases with N.

Another remarkable fact is that there are four classes of the linear combinations $\omega_{ij}^{(g)(h)}$ that can appear in a stress resultant, identified by the divisibility of g and h by 2 (i.e., the parities of g and h). A stress resultant only contains linear combinations of elements of one of these classes. This class is determined by the divisibility of m and k by 2, since the terms in the indices of the linear combinations that depend on m and k yield

$$2\left\lceil\frac{k}{2}\right\rceil - k = \begin{cases} 1 & \text{if } k \text{ odd} \\ 0 & \text{if } k \text{ even} \end{cases}.$$

(Also note that the indices g and h of the $\omega_{ij}^{(g)(h)}$ appearing in (5.11) are always non-negative: g is obviously non-negative. For the second index we have $h = 2n + 2\left\lceil\frac{m}{2}\right\rceil - m - 2q - 2\left\lceil\frac{k}{2}\right\rceil + k$. $2n - 2q$ is non-negative, because of the summation bounds for q, whereas, $2\left\lceil\frac{m}{2}\right\rceil - m - 2\left\lceil\frac{k}{2}\right\rceil + k$ is -1, if and only if, m is even and k is odd and non-negative otherwise. Especially in this case the upper bound for q is $n-1$ and we have $2n - 2q \geq 2$, so that h stays non-negative.) If m and k have different parities for two different stress resultants, the resultants do not have an intersection in the sets of their linear combinations $\omega_{ij}^{(g)(h)}$. If the parities of m and k are the same, there is an intersection, if m differs in the two stress resultants, and the sets are even equal, if m is the same, while the specific value of k has no influence on the sets. However, this generally does not imply proportionality in an Nth-order theory between two stress resultants with the same parities of m and k. By proportionality in an Nth-order theory we understand:

$$a \propto b + O(e^{2(N+1)}) \;:\Longleftrightarrow\; \text{There exists } \alpha = O(e^0) : \alpha a = b + O(e^{2(N+1)}).$$

The reason for the non-proportionality is the denominator of the numerical factor in round brackets, which can not be factorized such that one factor only depends on k and m and the other one only depends on n and q. For example, we have

$$\mathcal{M}_{ij}^{02} = \frac{hb}{l^2}\left(c^2\omega_{ij}^{00} + \frac{9}{5}c^4\omega_{ij}^{02} + d^2c^2\omega_{ij}^{20}\right) + O(e^6),$$
$$\mathcal{M}_{ij}^{20} = \frac{hb}{l^2}\left(d^2\omega_{ij}^{00} + d^2c^2\omega_{ij}^{02} + \frac{9}{5}d^4\omega_{ij}^{20}\right) + O(e^6),$$

so that $\mathcal{M}_{ij}^{02}$ and $\mathcal{M}_{ij}^{20}$ are not proportional, although the m's are equal and the k's are both even. As a trivial fact, we have proportionality, if the double sum has only one summand. For a second-order theory this is the case if we have $m = 4$ or $m = 3$, which gives us

$$m = 4 : \frac{5}{9}\frac{d^2}{c^2}\mathcal{M}_{ij}^{04} \overset{O(e^6)}{=} \mathcal{M}_{ij}^{22} \overset{O(e^6)}{=} \frac{5}{9}\frac{c^2}{d^2}\mathcal{M}_{ij}^{40}, \qquad \mathcal{M}_{ij}^{13} \overset{O(e^6)}{=} \mathcal{M}_{ij}^{31} \overset{O(e^6)}{=} 0 + O(e^6), \tag{5.13}$$
$$m = 3 : \frac{5}{9}\frac{d^2}{c^2}\mathcal{M}_{ij}^{03} \overset{O(e^6)}{=} \mathcal{M}_{ij}^{21}, \qquad \mathcal{M}_{ij}^{12} \overset{O(e^6)}{=} \frac{5}{9}\frac{c^2}{d^2}\mathcal{M}_{ij}^{30}, \tag{5.14}$$

where we used the notation

$$a \overset{O(e^6)}{=} b \;:\Longleftrightarrow\; a = b + O(e^6),$$

for convenience. (As a matter of fact, we also have only one summand for $m = 2$ and $k = 1$, but this generates no proportionalities like the ones above, since there is no other pair (m, k)

with one summand that has the same parities. Therefore, the list of proportionalities one can find without the use of the reduction equations, cf. 8.3, is complete.) On the other hand, for multiplications of stress resultants by c^2 and d^2, we have to retruncate the power series, i.e., decrement the upper bound of n by 1

$$1 - \left\lceil \frac{m}{2} \right\rceil = 2 - \left(\left\lceil \frac{m}{2} \right\rceil + 1 \right) = 2 - \left\lceil \frac{m+2}{2} \right\rceil .$$

Therefore, $e^2 \mathcal{M}_{ij}^{k(m-k)}$ and $\mathcal{M}_{ij}^{\tilde{k}(m+2-\tilde{k})}$ have representations that use the same set of linear combinations ω_{ij}^{gh}, when k and $\tilde{k}$ have the same parity and e is c or d. Again, this leads to relations between these stress-resultants, if the double sum has only one summand. For a second-order theory this gives us

$$m = 2 : c^2 \mathcal{M}_{ij}^{02} \overset{O(e^6)}{=} \frac{c^4}{d^2} \mathcal{M}_{ij}^{20} \overset{O(e^6)}{=} \frac{5}{9} \mathcal{M}_{ij}^{04}, \qquad c^2 \mathcal{M}_{ij}^{11} \overset{O(e^6)}{=} 0 + O(e^6), \tag{5.15}$$

$$m = 1 : c^2 \mathcal{M}_{ij}^{01} \overset{O(e^6)}{=} \frac{5}{9} \mathcal{M}_{ij}^{03}, \qquad c^2 \mathcal{M}_{ij}^{10} \overset{O(e^6)}{=} \mathcal{M}_{ij}^{12}. \tag{5.16}$$

However, these are no proportionality relations. In particular the relations do **not** imply that the series expansions of the two connected stress-resultants are basically equal. Analogously, for the factor e^4, we get another relation, which is

$$m = 0 : c^4 \mathcal{M}_{ij}^{00} \overset{O(e^6)}{=} c^2 \mathcal{M}_{ij}^{02}. \tag{5.17}$$

This relation finally completes the list of relations, if one does not use any information from the equilibrium equations, which will imply further relations, cf. section 8.3 and 8.4 .

Some additional Notes:
In the special case m even and k odd, we have an extra c^2 as excludable factor and the stress resultant actually depends on fewer linear combinations ω_{ij}^{gh}, because of the upper bound for q. For deriving an explicit formula for the number of linear combinations and the actual excludable scaling prefactor in c and d we renumber the sum again. For m even and k odd, we can substitute n with $n+1$, which leads to a case-independent upper bound for q

$$\begin{aligned} \sum_{n=0}^{2-\lceil \frac{m}{2} \rceil} \sum_{q=0}^{n-1} f(n,q) &= \sum_{n=1}^{2-\lceil \frac{m}{2} \rceil} \sum_{q=0}^{n-1} f(n,q) \\ &= \sum_{n=0}^{2-\lceil \frac{m}{2} \rceil -1} \sum_{q=0}^{n} f(n+1,q). \end{aligned}$$

Therefore, we get

$$\sum_{n=0}^{2-\lceil \frac{m}{2} \rceil} \sum_{q=0}^{\lfloor n + \lceil \frac{m}{2} \rceil - \frac{m}{2} + \frac{k}{2} \rfloor - \lceil \frac{k}{2} \rceil} f(n,q) = \sum_{n=0}^{2-\lceil \frac{m}{2} \rceil - \gamma\langle m \rangle \eta\langle k \rangle} \sum_{q=0}^{n} f(n + \gamma\langle m \rangle \eta\langle k \rangle, q),$$

where we introduced the symbol $\eta\langle n \rangle$ for any integer n by

$$\eta\langle n \rangle := \begin{cases} 1 & \text{for } n \text{ odd} \\ 0 & \text{for } n \text{ even} \end{cases} = 1 - \gamma\langle n \rangle. \tag{5.18}$$

Inserting this into formula (5.11), while replacing all occurrences of the floor and ceiling function with the η-symbol, and by the use of the identity

$$\begin{aligned} \eta\langle m\rangle &= (\eta\langle m\rangle)^2 \Rightarrow \\ 2\gamma\langle m\rangle\eta\langle k\rangle + \eta\langle m\rangle - \eta\langle k\rangle &= \eta\langle m\rangle - 2\eta\langle m\rangle\eta\langle k\rangle + \eta\langle k\rangle = (\eta\langle m\rangle - \eta\langle k\rangle)^2 \\ &= \left\{ \begin{array}{ll} 1 & \text{for } (m \text{ even and } k \text{ odd}) \\ 1 & \text{for } (k \text{ even and } m \text{ odd}) \\ 0 & \text{otherwise} \end{array} \right\} = \eta\langle m-k\rangle, \end{aligned}$$

we get

$$\begin{aligned} \mathcal{M}_{ij}^{k(m-k)} &= \frac{hb}{l^2} d^{k+\eta\langle k\rangle} c^{m-k+\eta\langle m-k\rangle} \sum_{n=0}^{N-\frac{m}{2}-\frac{1}{2}\eta\langle k\rangle-\frac{1}{2}\eta\langle m-k\rangle} \sum_{q=0}^{n} \\ &\left(\frac{3^{n+\frac{m}{2}+\frac{1}{2}\eta\langle k\rangle+\frac{1}{2}\eta\langle m-k\rangle}}{(2q+k+\eta\langle k\rangle+1)(2(n-q)+m-k+\eta\langle m-k\rangle+1)} \right. \\ &\left. d^{2q} c^{2(n-q)} \omega_{ij}^{(2q+\eta\langle k\rangle)(2(n-q)+\eta\langle m-k\rangle)} \right) + O(e^{2(N+1)}). \quad (5.19) \end{aligned}$$

The actual excludable factor of a stress resultant $\mathcal{M}_{ij}^{k(m-k)}$ has, therefore, the representation $d^{k+\eta\langle k\rangle} c^{m-k+\eta\langle m-k\rangle}$, independent of the order of the theory. In an Nth-order theory the upper bound for the sum over n is $\iota := N - \frac{m}{2} - \frac{1}{2}\eta\langle k\rangle - \frac{1}{2}\eta\langle m-k\rangle$ and the actual number of linear combinations is computable via

$$\sum_{n=0}^{\iota} n + 1 = \frac{1}{2}\iota^2 + \frac{3}{2}\iota + 1,$$

for non-negative ι.

5.4 A one-dimensional formulation for the approximation error

Already in section 4.6 we computed a representation of the dimensionless potential energy (4.22) by insertion of the series expansion of the displacement field (4.12) and by explicitly using the specific beam geometry of section 4.2. If we use the operator a^{23} introduced in (5.2) instead of the operator S introduced in (4.9), i.e., if we omit to renumber the summation indices as in (4.7) and (4.8), we get the equivalent representation

$$\begin{aligned} &\left(\frac{E_{\text{pot}}(u)}{Gl^3} \right) \\ &= \int_{\Omega_\xi} \sum_{i,j,r,s=1}^{3} \frac{1}{2} \left(\frac{E_{ijrs}}{G} \right) \left(\frac{u_i}{l} \right)_{,j} \left(\frac{u_r}{l} \right)_{,s} dV_\xi \\ &\quad - \int_{\Omega_\xi} \sum_{i=1}^{3} \left(\frac{lf_i}{G} \right) \left(\frac{u_i}{l} \right) dV_\xi - \int_{\partial\Omega_{\xi N}} \sum_{i=1}^{3} \left(\frac{g_i}{G} \right) \left(\frac{u_i}{l} \right) dA_\xi \end{aligned}$$

$$=\frac{1}{2}\sum_{i,j,r,s=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\Big\{\int_0^1\Big(\frac{E_{ijrs}}{G}\Big)D_j^1\Big(u_i^{k(m-k)}\Big)D_s^1\Big(u_r^{q(n-q)}\Big)\,d\xi_1$$
$$\int_{A_\xi} a_j^{23}\Big[\hat{\xi}^{k(m-k)}\Big]a_s^{23}\Big[\hat{\xi}^{q(n-q)}\Big]\,dA_\xi\Big\}$$
$$-\sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\int_0^1 u_i^{k(m-k)}\int_{A_\xi}\Big(\frac{lf_i}{G}\Big)\hat{\xi}^{k(m-k)}\,dA_\xi\,d\xi_1$$
$$-\sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\int_0^1 u_i^{k(m-k)}\int_{\partial A_\xi}\Big(\frac{g_i}{G}\Big)\hat{\xi}^{k(m-k)}\,ds_\xi\,d\xi_1$$
$$-\sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\mathbf{1}_{P_{\xi N}}(\{0,1\})\Big[u_i^{k(m-k)}\int_{A_\xi}\Big(\frac{g_i}{G}\Big)\hat{\xi}^{k(m-k)}\,dA_\xi\Big].$$

Now, the magnitude of the first integral with respect to A_ξ is not necessarily $O(e^{n+m})$, because of the use of a^{23} instead of S, which complicates the truncation with respect to a certain magnitude. However, the magnitude of the first term is still solely given by the integral with respect to A_ξ (cf. the thoughts of section 4.6). Note that we do not change the magnitude of this term, if we use integration by parts solely in ξ_1-direction, since the integration does not affect this integral:

$$=-\frac{1}{2}\sum_{i,j,r,s=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\chi_j\Big\{\int_0^1\Big(\frac{E_{ijrs}}{G}\Big)u_i^{k(m-k)}D_j^1\Big(D_s^1\Big(u_r^{q(n-q)}\Big)\Big)\,d\xi_1$$
$$\int_{A_\xi} a_j^{23}\Big[\hat{\xi}^{k(m-k)}\Big]a_s^{23}\Big[\hat{\xi}^{q(n-q)}\Big]\,dA_\xi\Big\}$$
$$+\frac{1}{2}\sum_{i,r,s=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\sum_{n=0}^{\infty}\sum_{q=0}^{n}\Big\{\Big[\Big(\frac{E_{i1rs}}{G}\Big)u_i^{k(m-k)}D_s^1\Big(u_r^{q(n-q)}\Big)\Big]_{\xi_1=0}^{\xi_1=1}$$
$$\int_{A_\xi}\hat{\xi}^{k(m-k)}a_s^{23}\Big[\hat{\xi}^{q(n-q)}\Big]\,dA_\xi\Big\}$$
$$-\sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\int_0^1 u_i^{k(m-k)}p_i^{k(m-k)}\,d\xi_1$$
$$-\sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\mathbf{1}_{P_{\xi N}}(\{0,1\})\Big[u_i^{k(m-k)}\mathcal{M}_{Ni1}^{k(m-k)}n_1\Big].$$

In contrast, using integration in ξ_2- or ξ_3-direction, would in fact change the magnitude. Note that we already inserted the definitions of the load resultants (4.15) and the prescribed stress resultants (5.5) in the equation above. If we also use the definition of the stress resultants (5.4) and the operator K, defined in (5.2), we furnish

$$\Big(\frac{E_{\mathrm{pot}}(u)}{Gl^3}\Big)=\sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\Big\{\int_0^1 u_i^{k(m-k)}\Big(-\frac{1}{2}\sum_{j=1}^{3}K_j\Big[\mathcal{M}_{ij}^{k(m-k)}(u)\Big]-p_i^{k(m-k)}\Big)\,d\xi_1$$
$$+\mathbf{1}_{P_{\xi N}}(\{0,1\})\Big[u_i^{k(m-k)}\Big(\frac{1}{2}\mathcal{M}_{i1}^{k(m-k)}(u)-\mathcal{M}_{Ni1}^{k(m-k)}\Big)n_1\Big]$$
$$+\mathbf{1}_{P_{\xi 0}}(\{0,1\})\Big[u_i^{k(m-k)}\frac{1}{2}\mathcal{M}_{i1}^{k(m-k)}(u)\,n_1\Big]\Big\}. \tag{5.20}$$

If we recall the definition of the bilinear form B (3.3) and the linear form F (3.7), we can easily derive the first variation of the potential energy by using the representation above, i.e., equation (5.20). Also, we insert the series expansion for the direction of the first variation, i.e. for the virtual displacement $v \in X_0$ (but not for the point $u \in X$). Recall that the virtual displacements have to fulfill homogeneous displacement boundary conditions, therefore the last summand in the curly brackets in (5.20) vanishes and we obtain

$$\begin{aligned}
E_{\text{pot}}(u) =& \frac{1}{2} B(u,u) - F(u) \\
\delta \frac{E_{\text{pot}}}{Gl^3}(u;v) =& \frac{1}{Gl^3}\left(B(u,v) - F(v)\right) \\
=& \sum_{i=1}^{3} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \Bigg\{ \int_0^1 v_i^{k(m-k)} \left(- \sum_{j=1}^{3} K_j \left[\mathcal{M}_{ij}^{k(m-k)}(u) \right] - p_i^{k(m-k)} \right) d\xi_1 \\
& + \mathbf{1}_{P_{\xi N}}(\{0,1\}) \left[v_i^{k(m-k)} \left(\mathcal{M}_{i1}^{k(m-k)}(u) - \mathcal{M}_{Ni1}^{k(m-k)} \right) n_1 \right] \Bigg\}.
\end{aligned}$$

Now let $v(\xi) = \phi(\xi_1)\xi_2^k \xi_3^{m-k}$ for an arbitrary chosen $\phi \in C_0^\infty(0,1)$. The function v is on its own a Taylor series. Furthermore, by insertion of v into the equation above, the last summand vanishes, since ϕ vanishes for $\xi_1 = 0$ or $\xi_1 = 1$. Application of the variational lemma therefore yields the equations (5.21). By inserting (5.21) back into the equation above, only the last summand remains. Since at least one of the points 0 or 1 belongs to the Dirichlet part of the boundary $P_{\xi 0}$, select v now as

$$v = \sum_{m=0}^{\infty} \sum_{k=0}^{m} \left(\mathcal{M}_{i1}^{k(m-k)}(u) - \mathcal{M}_{Ni1}^{k(m-k)} \right) \, n_1 \, \xi_2^k \xi_3^{m-k},$$

where ξ_1 is fixed as the $\xi_1 \in P_{\xi N}$. Then each summand gets non-negative and therefore has to vanish independently from the others, which gives us the traction boundary conditions in terms of the stress resultants (5.22). We already know from theorem 7 that the weak formulation (Wk) is equivalent to the classical problem of three-dimensional elasticity (Cl). Since the weak formulation is satisfied, if the system of the one-dimensional Euler-Lagrange field equations in terms of stress resultants

$$\begin{aligned}
& \text{For all } i \in \{1,2,3\},\, m \in \mathbb{N}_0,\, k \in \{0,...,m\}: \\
& \mathcal{M}_{i1,1}^{k(m-k)} - k\,\mathcal{M}_{i2}^{k-1(m-k)} - (m-k)\,\mathcal{M}_{i3}^{k(m-k-1)} = -p_i^{k(m-k)}, \quad \text{f.a. } \xi_1 \in (0,1) \\
\Leftrightarrow & \sum_{j=1}^{3} K_j \left[\mathcal{M}_{ij}^{k(m-k)}(u) \right] = -p_i^{k(m-k)} \qquad \text{f.a. } \xi_1 \in (0,1), \qquad (5.21)
\end{aligned}$$

as well as the one-dimensional boundary condition in terms of stress resultants

$$\begin{aligned}
& \text{For all } i \in \{1,2,3\},\, m \in \mathbb{N}_0,\, k \in \{0,...,m\}: \\
& \mathcal{M}_{i1}^{k(m-k)}(u)\, n_1 = \mathcal{M}_{Ni1}^{k(m-k)}\, n_1 := \int_{A_\xi} \frac{g_i}{G} \hat{\xi}^{k(m-k)} \, dA_\xi \qquad \text{f.a. } \xi_1 \in P_{\xi N} \qquad (5.22)
\end{aligned}$$

are satisfied, because every test function is uniquely determined by the definition of all Taylor coefficients, we actually derived an exact one-dimensional representation of the three-dimensional minimization problem of the potential energy. (In Schneider (2010) we already gave a similar proof also paying respect to regularity questions.)

Next we take the dual energy, defined in (3.15), and transform it to a dimensionless representation by application of the rules of section 4.3:

$$\begin{aligned}&\left(\frac{E_{\text{dual}}(\sigma)}{Gl^3}\right)\\&= -\int_{\Omega_\xi} \sum_{i,j,r,s=1}^{3} \frac{1}{2} (GD_{ijrs}) \left(\frac{\sigma_{ij}}{G}\right)\left(\frac{\sigma_{rs}}{G}\right) dV_\xi + \int_{\partial\Omega_{\xi 0}} \sum_{i,j=1}^{3} \left(\frac{\sigma_{ij}}{G}\right) n_j \left(\frac{u_{0i}}{l}\right) dA_\xi .\end{aligned}$$

Let u be a displacement field that is associated with σ, i.e. $\sigma_{ij} = \sum_{r,s=1}^{3} E_{ijrs} u_{r|s}$. If we insert the series expansion for u (4.12) and u_0, respectively, and by explicitly using the specific beam geometry of section 4.2, we derive:

$$\begin{aligned}&= -\frac{1}{2}\sum_{i,j}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\int_0^1 D_j^1\left(u_i^{k(m-k)}\right)\int_{A_\xi}\left(\frac{\sigma_{ij}}{G}\right) a_j^{23}\left[\hat{\xi}^{k(m-k)}\right] dA_\xi\, d\xi_1\\&+\sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\mathbf{1}_{P_{\xi 0}}(\{0,1\})\left[u_{0i}^{k(m-k)}\int_{A_\xi}\sum_{j=1}^{3}\left\{\left(\frac{\sigma_{ij}}{G}\right)n_j\right\}\hat{\xi}^{k(m-k)}\, dA_\xi\right].\end{aligned}$$

Again the magnitude of all summands is solely given by the integrals with respect to A_ξ. Therefore, we do not change the magnitude, if we use integration by parts in ξ_1-direction for the first term for $j = 1$:

$$\begin{aligned}&= +\frac{1}{2}\sum_{i}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\int_0^1 u_i^{k(m-k)}\sum_{j}^{3}\left\{\chi_j D_j^1\left(\int_{A_\xi}\left(\frac{\sigma_{ij}}{G}\right)a_j^{23}\left[\hat{\xi}^{k(m-k)}\right] dA_\xi\right)\right\} d\xi_1\\&-\frac{1}{2}\sum_{i}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\left[u_i^{k(m-k)}\int_{A_\xi}\left(\frac{\sigma_{i1}}{G}\right)\hat{\xi}^{k(m-k)}\, dA_\xi\right]_{\xi_1=0}^{\xi_1=1}\\&+\sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\mathbf{1}_{P_{\xi 0}}(\{0,1\})\left[u_{0i}^{k(m-k)}\sum_{j=1}^{3}\left\{\int_{A_\xi}\left(\frac{\sigma_{ij}}{G}\right)\hat{\xi}^{k(m-k)}\, dA_\xi\, n_j\right\}\right].\end{aligned}$$

By insertion of the definition of the stress resultants (5.4), the operator identity (5.3) and the direction of the outer-unit-normal vector n (cf. section 4.2) on the lateral faces, we finally furnish

$$\begin{aligned}&\left(\frac{E_{\text{dual}}(u(\sigma))}{Gl^3}\right)\\&=\sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\Bigg\{\int_0^1 u_i^{k(m-k)}\frac{1}{2}\sum_{j=1}^{3}K_j\left[\mathcal{M}_{ij}^{k(m-k)}(u(\sigma))\right] d\xi_1\\&\qquad -\mathbf{1}_{P_{\xi N}}(\{0,1\})\left[u_i^{k(m-k)}\frac{1}{2}\mathcal{M}_{i1}^{k(m-k)}(u(\sigma))\, n_1\right]\\&\qquad +\mathbf{1}_{P_{\xi 0}}(\{0,1\})\left[\left(u_{0i}^{k(m-k)}-\frac{1}{2}u_i^{k(m-k)}\right)\mathcal{M}_{i1}^{k(m-k)}(u(\sigma))\, n_1\right]\Bigg\}. \qquad (5.23)\end{aligned}$$

Next, we compute the first variation of the dual energy $\delta E_{\text{dual}}(\sigma;\mu)$. To this end, let u be the displacement field associated with σ, i.e., $\sigma_{ij} = \sum_{r,s=1}^{3} E_{ijrs} u_{r|s}$ and let $u_i^{k(m-k)}$ and $u_{0i}^{k(m-k)}$

be the Taylor coefficients of u and u_0, respectively. Note that the variations μ of the dual problem (Du) have to fulfill the homogeneous conditions associated with the three-dimensional equilibrium equation and the three-dimensional stress boundary condition, i.e., $\sum_{j=1}^{3}\mu_{ji|j}=0$ f.a. $\xi\in\Omega_\xi$ and $\sum_{j=1}^{3}\mu_{ij}n_j=0$ f.a. $\xi\in\partial\Omega_{\xi N}$. We just computed that, therefore, the stress resultants associated with the variations fulfill $\sum_{j=1}^{3}K_j\left[\mathcal{M}_{ij}^{k(m-k)}(\mu)\right]=0$ f.a. $\xi_1\in(0,1)$ and $\mathcal{M}_{i1}^{k(m-k)}(\mu)\,n_1=0$ f.a. $\xi_1\in P_{\xi N}$. Therefore, we derive

$$\delta\frac{E_{\text{dual}}}{Gl^3}(\sigma;\mu)=\sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\Big\{\mathbf{1}_{P_{\xi0}}(\{0,1\})\Big[\Big(u_{0i}^{k(m-k)}-u_i^{k(m-k)}\Big)\,\mathcal{M}_{i1}^{k(m-k)}(\mu)\,n_1\Big]\Big\}\tag{5.24}$$

by treating equation (5.23) in an analogous manner as done for equation (5.20). By application of the variational lemma in an analogous manner, we compute the equivalent one-dimensional displacement boundary conditions

For all $i\in\{1,2,3\}$, $m\in\mathbb{N}_0$, $k\in\{0,...,m\}$:

$$u_i^{k(m-k)}=u_{0i}^{k(m-k)}\qquad\text{f.a. }\xi_1\in P_{\xi0}.\tag{5.25}$$

The problem of seeking a displacement field u that fulfills (5.21), (5.22) and (5.25) is well-defined. Its solution minimizes the potential energy and maximizes the dual energy and is the uniquely defined solution of three-dimensional linear elasticity (cf. theorem 7 and 9). We could also verify this statement by using theorem 11, if we insert (5.20) and (5.23)

$$\begin{aligned}
&\frac{1}{2}\frac{\lambda_{\min}\,c_{\text{Korn}}}{Gl^3}\,\|v-u\|_X^2\\
\leq&\Big(\frac{E_{\text{pot}}(v)}{Gl^3}\Big)-\Big(\frac{E_{\text{dual}}(\sigma(v))}{Gl^3}\Big)\\
=&\sum_{i=1}^{3}\sum_{m=0}^{\infty}\sum_{k=0}^{m}\Big\{-\int_0^1 v_i^{k(m-k)}\Big(\sum_{j=1}^{3}K_j\Big[\mathcal{M}_{ij}^{k(m-k)}(v)\Big]+p_i^{k(m-k)}\Big)\,d\xi_1\\
&\qquad+\mathbf{1}_{P_{\xi N}}(\{0,1\})\Big[v_i^{k(m-k)}\Big(\mathcal{M}_{i1}^{k(m-k)}(v)-\mathcal{M}_{Ni1}^{k(m-k)}\Big)n_1\Big]\\
&\qquad+\mathbf{1}_{P_{\xi0}}(\{0,1\})\Big[\Big(v_i^{k(m-k)}-u_{0i}^{k(m-k)}\Big)\mathcal{M}_{i1}^{k(m-k)}(v)\,n_1\Big]\Big\},
\end{aligned}\tag{5.26}$$

but, moreover, the equation tells us how to gain adequate approximative theories. All integrals with respect to the cross-section are present in the stress resultants or the load resultants. If we truncate all stress- and load resultants at a specific approximation order N after inserting the series expansion for the displacement field, i.e., if we neglect all terms that contain a factor d^kc^{k-m} with $m>2N+1$, we end up with a finite set of nontrivial ODEs (5.21) in a finite set of unknown displacement coefficients, since the magnitude of $\mathcal{M}_{ij}^{k(m-k)}$ rises with m, cf. (5.12). For the same reason we get a finite set of nontrivial stress-boundary conditions (5.22). If we, furthermore, consider precisely all equations $u_i^{k(m-k)}=u_{0i}^{k(m-k)}$ (cf. eq. (5.25)), for which the corresponding $\mathcal{M}_{i1}^{k(m-k)}$ is not to be neglected $\mathcal{M}_{i1}^{k(m-k)}\neq0+O(e^{2(N+1)})$, we also get a finite set of displacement-boundary conditions.

A solution of the so-generated approximative problem has potential energy, which is derived by truncating the potential energy of the solution of the exact problem at the same order, since we only used partial integration in ξ_1-direction in the derivation of (5.20). Or formulated differently, the so generated problem equations (5.21) and (5.22) are the Euler-Lagrange equations of the minimization problem of the potential energy truncated at order e^{2N}. Likewise, the truncated dual energy maximization problems Euler-Lagrange equations are the truncated equations (5.25). But, most important of all, equation (5.26) proves that a solution v of the approximation problem only differs from the exact solution u by a difference of order $O(e^{2(N+1)})$, i.e.

$$\frac{1}{2}\frac{\lambda_{\min} c_{\text{Korn}}}{Gl^3} \|v - u\|_X^2 = O(e^{2(N+1)}), \tag{5.27}$$

which proves that the so-generated problem is indeed an approximation of the exact problem. Note that $\lambda_{\min} c_{\text{Korn}}/(2Gl^3) =: k$ is constant and that (5.27) implies $k\|v - u\|_X^2 \leq O\left(\max\{c, d\}^{2(N+1)}\right)$. We sum up our findings in a theorem:

Theorem 13 (A-priori error estimate for the consistent approximation)
Assume all assumptions of theorem 7 and the quasi one-dimensional geometry, defined in section 4.2, especially (A3). *Then the one-dimensional problem of eqs.* (5.21)*,* (5.22) *and* (5.25) *with inserted* (5.10) *and* (5.7) *is equivalent to any of the problems of theorem 7.*
We define the Nth-order problem of consistent approximation by:

- *truncating the field equations* (5.21) *and the stress boundary conditions* (5.22) *by negligence of all stress resultants that are of order $O(e^{2(N+1)})$ by formula* (5.12)*,*
- *truncation of the load resultants* (4.21) *at order $O(e^{2(N+1)})$,*
- *imposing all displacement boundary conditions* (5.25)*, for which the corresponding stress resultants are not to be neglected, i.e.,* $(\mathcal{M}_{i1}^{k(m-k)} \neq 0 + O(e^{2(N+1)}))$,
- *writing the problem in terms of displacement coefficients by insertion of* (5.10) *and* (5.7).

The Nth-order problem of consistent approximation is then given by a finite set of ODEs written in a finite number of unknown displacement coefficients.
Let u be the exact solution of the three-dimensional problem of linear elasticity and v be the displacement field defined by the displacement coefficients of a solution of the Nth-order problem of consistent approximation stated above, then v:

- *minimizes the potential energy truncated at order $O(e^{2(N+1)})$),*
- *maximizes the dual energy truncated at order $O(e^{2(N+1)})$),*
- *fullfills the a-priori error estimate* (5.27).

A handy way of deriving the actual Nth-order problem equations (by insertion of (5.10) and (5.7) into the equilibrium conditions (5.21)) will be provided in section 7.4.

5.5 Some notes on the approach

One could also generate the system (5.21) by multiplying the three-dimensional equilibrium equations (2.1) with monomic polynomials of the form $\xi_2^k \xi_3^{m-k}$ and applying integration by parts. This is a common approach originally developed for the derivation of plate and shell theories and was first applied for the derivation of a beam theory by Cowper (Cowper, 1966). The great advantage of the approach we present here is that one actually knows that the infinite system is an exact representation of *the* three-dimensional theory of linear elasticity, which can be derived completely by first principles. This allows for *general* statements, avoiding the variety of disputable a-priori assumptions that are used for the derivation of classical theories, like the Euler-Bernoulli beam theory. Nevertheless, the first equations of our system correspond directly to classical equations. If we assume the geometry to be loaded only in ξ_3-direction and evaluate (5.21) for the virtual displacements δv_3^{00} and δv_1^{01}, we get the equations

$$\frac{dQ}{dx_1} = -p_3^{00}, \qquad \frac{dM}{dx_1} = Q,$$

(cf. section 8.4) where Q and M denote the classical shear force $Q = \mathcal{M}_{13}^{00} = \int_{A_\xi} \frac{\sigma_{13}}{G}\, dA_\xi$ and bending moment $M = \mathcal{M}_{11}^{01} = \int_{A_\xi} \frac{\sigma_{11}}{G} \xi_3\, dA_\xi$, respectively, and p_3^{00} is the resulting line load (cf. e.g. Schnell et al., 2002, eq. 4.30).

5.6 Generalized boundary conditions

For simplicity we assumed in section 4.2 that at a face side ($\xi_1 \in \{0,1\}$) either the displacement or the traction is prescribed, which led us to the boundary conditions (5.22) and (5.25). This is too restrictive for a large variety of boundary conditions that are relevant in practice. For example one may think of a simple (hinged) support of a beam in the context of the classical Euler-Bernoulli-beam theory: At the simple supported face side $\xi_1 \in \{0,1\}$ one would prescribe the lateral displacement $u_3^{00}(\xi_1) = 0$, but instead of the infinitesimal rotation $u_1^{01}(\xi_1)$ one would prescribe the classical bending moment $\mathcal{M}_{11}^{01}(\xi_1) = 0$. This merely corresponds to a mixture of the boundary conditions (5.22) and (5.25).

Indeed, if we define $\mathbb{P} := \{0,1\} \times \{1,2,3\} \times \mathbb{N}_0 \times \mathbb{N}_0$ and select $P_{\xi 0} \subset \mathbb{P}$ and $P_{\xi N} := \mathbb{P} \setminus P_{\xi 0}$ as the complement of this set, we can define the potential energy of all admissible displacement fields that fulfill (5.25) f.a. $(\xi_1, i, k, m-k) \in P_{\xi 0}$ by (5.20) and the dual energy in terms of admissible stress resultants that fulfill (5.21) and (5.22) f.a. $(\xi_1, i, k, m-k) \in P_{\xi N}$ by (5.23), if we replace every occurrence of $\sum_{i=1}^{3} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \mathbf{1}_{P_{\xi 0}}(\{0,1\})[\bullet]$ by $\mathbf{1}_{P_{\xi 0}}(\mathbb{P})[\bullet]$ and $\sum_{i=1}^{3} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \mathbf{1}_{P_{\xi N}}(\{0,1\})[\bullet]$ by $\mathbf{1}_{P_{\xi N}}(\mathbb{P})[\bullet]$, respectively, cf. (4.1). This way the duality of the problems and the argumentation of section 5.4, i.e., the central error estimate of theorem 13, remains valid and the generalized boundary condition that replaces (5.22) and (5.25) simply becomes

$$\begin{aligned} &\text{F.a. } (\xi_1, i, k, m-k) \in \mathbb{P} : \\ &\quad u_i^{k(m-k)}(\xi_1) = u_{0i}^{k(m-k)}(\xi_1) \text{ or } \mathcal{M}_{i1}^{k(m-k)}(\xi_1)\, n_1 = \mathcal{M}_{Ni1}^{k(m-k)}(\xi_1)\, n_1. \end{aligned} \tag{5.28}$$

However, only the prescription of all Taylor coefficients is equivalent to the prescription of the function itself, therefore, by using (5.28) one loses the ability to write down the boundary conditions in a three-dimensional form without the use of series expansions (like $u = u_0$). Therefore, we decided to use the boundary decomposition of section 4.2 to emphasize the equivalence to the three-dimensional problem. Consequently, we will again refer to the boundary conditions (5.22) and (5.25) instead of (5.28) in the subsequent sections. Nevertheless, all subsequent thoughts are accordingly applicable for the case of mixed boundary conditions (5.28) with the exception that there is no three-dimensional notation for the problem without the use of series expansions.

6 The decoupling of the equilibrium equations

By insertion of (5.9) and (5.7) into the exact (i.e., untruncated) equilibrium conditions (5.21), we get an (infinite) set of series written in terms of (infinitely many) unknown displacement coefficients $u_i^{q(n-q)}$. Deriving all displacement coefficients solves the exact problem of three-dimensional elasticity (Cl). Solving the truncated Nth-order problem gives us an approximative solution, cf. theorem 13.

A specific exact equilibrium condition (5.21) will not depend on every displacement coefficient, therefore the system of all equilibrium conditions might decouple into subsystems depending on subsets of displacement coefficients that are disjoint from each other. In fact, the system decouples for instance for isotropic material into four subsystems, which are independent from each other and will be identified as the four classical problems of one-dimensional linear elasticity: the rod-problem, two decoupled beam-problems (one beam loaded in ξ_2-, the other one in ξ_3-direction) and the shaft-problem. For anisotropic material the problems may be coupled.

The coupling behavior will be investigated in several steps in this section. First, we will show that the exact equilibrium conditions written in stress-resultants (5.21) generally decouple into four systems independently of the anisotropy of the material. Next, we will investigate what load-cases are driving the individual problems, leading to an uniquely determined decomposition of every three-dimensional load-case of the "quasi one-dimensional" beam geometry into driving forces for the four subproblems. At last, we will show how the anisotropy couples the four subproblems, which are decoupled for isotropy, for an arbitrary material anisotropy.

Since we will prove every proposition for the exact problem, all proposition are also valid for any Nth-order consistent approximation.

6.1 Notation and a key observation

We start with a key observation that will allow us to deal with the coupling behavior in an abstract and elegant way:

From formula (5.19), which states the stress-resultants in terms of linear combinations of displacement coefficients ω_{ij}^{gh}, we derive directly that a stress resultant $\mathcal{M}_{ij}^{k(m-k)}$ contains the linear combination $\omega_{ij}^{\eta\langle k\rangle\eta\langle m-k\rangle}$ (or it is to be neglected $\mathcal{M}_{ij}^{k(m-k)} = O(e^{2(N+1)})$, in the case of an Nth-order approximation). Let us consider the exact problem, and let "$\bullet$ mod 2" denote the modulo operation, i.e., the operation that finds the remainder of division by 2 and let t_1 denote the parity of k and t_2 the parity of $m-k$, then all stress-resultants in the class

$$\mathcal{M}_{ij}^{t_1 t_2} := \left\{\mathcal{M}_{ij}^{nq}\middle|(n \bmod 2 = k \bmod 2) \quad \text{and} \quad (q \bmod 2 = m-k \bmod 2)\right\}$$

contain the specific linear combination $\omega_{ij}^{(\eta\langle k\rangle)(\eta\langle m-k\rangle)}$, therefore, all stress-resultants in the class $\mathcal{M}_{ij}^{t_1t_2}$ are coupled in terms of displacement coefficients. Hence it is convenient to further investigate how the equilibrium conditions in stress resultants couple these classes of stress-resultants that share the same parity of the upper indices, since we can already derive the smallest sets of possibly decoupled subproblems. Indeed we will find four subproblems that are actually decoupled for, e.g., isotropic material.

In general, we introduce the following notational conventions to be able to handle classes, like $\mathcal{M}_{ij}^{t_1t_2}$, in an elegant way. Let $\mathbb{Z}_2$ denote the quotient set of integers $\mathbb{Z}$ by 2, i.e., the set $\mathbb{Z}_2$ consists of precisely two elements: $e := \{z \in \mathbb{Z} | z \bmod 2 = 0\}$, the set of all even integers, and

$o := \{z \in \mathbb{Z} | z \bmod 2 = 1\}$, the set of all odd integers. If we use e and o as an upper index, the corresponding parity classes are meant, e.g.,

$$\mathcal{M}_{12}^{eo} := \left\{ \mathcal{M}_{12}^{k(m-k)} \middle| \text{ with } k \bmod 2 = 0 \text{ and } (m-k) \bmod 2 = 1 \right\}.$$

In this way, $\mathcal{M}_{ij}^{eo}$ is a class of stress resultants. In contrast $\mathcal{M}_{ij}^{01}$ is one specific stress resultant, which is an element of $\mathcal{M}_{ij}^{eo}$, i.e., we have $\mathcal{M}_{ij}^{01} \in \mathcal{M}_{ij}^{eo}$. As another convention we use the symbols t_k with $k \in \mathbb{N}$ to denote unknown elements of $\mathbb{Z}_2$, like we already did when we introduced $\mathcal{M}_{ij}^{t_1 t_2}$ above, because we did not know the parities of k and $m-k$. In order to be able to denote parity classes by representatives, we introduce $[\bullet]_{\mathbb{Z}_2}$, the mapping of integers to their parity class, i.e., $[\bullet]_{\mathbb{Z}_2} : \mathbb{Z} \to \mathbb{Z}_2$, $z \mapsto z \bmod 2$. In this notation we have, for example, $[8]_{\mathbb{Z}_2} = [2]_{\mathbb{Z}_2} = [0]_{\mathbb{Z}_2} = e$, i.e., it does not matter which even number one chooses as a representative of the class of even numbers e. Three valid choices are $8, 2$ and 0. As an example, we could define the class $\mathcal{M}_{ij}^{t_1 t_2}$ introduced above very fast by writing $\mathcal{M}_{ij}^{[k]_{\mathbb{Z}_2}[m-k]_{\mathbb{Z}_2}}$. In contrast $\mathcal{M}_{ij}^{(\eta\langle k\rangle)(\eta\langle m-k\rangle)}$ denotes one specific member of the class $\mathcal{M}_{ij}^{[k]_{\mathbb{Z}_2}[m-k]_{\mathbb{Z}_2}}$. Our key observation was that all elements of $\mathcal{M}_{ij}^{[k]_{\mathbb{Z}_2}[m-k]_{\mathbb{Z}_2}}$ contain $\omega_{ij}^{(\eta\langle k\rangle)(\eta\langle m-k\rangle)}$.

Next we will define an abstract form of the operator K, which will be of central importance for this section. To this end, note that on $\mathbb{Z}_2$ there is only one bijective mapping to itself that is not the identity. This mapping is to shift the parity, i.e., the mapping that maps e to o and o to e. This mapping is also well-defined by $\mathbb{Z}_2 \to \mathbb{Z}_2$, $[i]_{\mathbb{Z}_2} \mapsto [i+1]_{\mathbb{Z}_2}$, by the use of the mapping $[\bullet]_{\mathbb{Z}_2}$. By the use of the parity shift we define the *abstract shift operator* $\mathcal{K} = (\mathcal{K}_i)$ on tuples of $\mathbb{Z}_2$, i.e., on $\mathbb{Z}_2^2$, in dependence of the lower index $i \in \{1, 2, 3\}$ by

$$\begin{aligned} \mathcal{K}_1 \left([i]_{\mathbb{Z}_2}, [j]_{\mathbb{Z}_2}\right) &:= \left([i]_{\mathbb{Z}_2}, [j]_{\mathbb{Z}_2}\right), \\ \mathcal{K}_2 \left([i]_{\mathbb{Z}_2}, [j]_{\mathbb{Z}_2}\right) &:= \left([i+1]_{\mathbb{Z}_2}, [j]_{\mathbb{Z}_2}\right), \\ \mathcal{K}_3 \left([i]_{\mathbb{Z}_2}, [j]_{\mathbb{Z}_2}\right) &:= \left([i]_{\mathbb{Z}_2}, [j+1]_{\mathbb{Z}_2}\right). \end{aligned} \tag{6.1}$$

We have the following properties of the abstract shift $\mathcal{K}$. Let $\mathcal{K}_i\mathcal{K}_j$ denote the composition of $\mathcal{K}_j$ and $\mathcal{K}_i$ and let $\mathrm{id}_{\mathbb{Z}_2^2}$ be the identity on $\mathbb{Z}_2^2$. (The formal proof is easily performed by just checking the properties for every tuple in $\mathbb{Z}_2^2$. Also, one can immediately comprehend the properties by inspection of the latter, in another context, introduced in figure 6, if one just considers how the abstract shift transfers the tuples $\mathbb{Z}_2^2$ into one another.)

Lemma 14 (Properties of the abstract shift $\mathcal{K}$)

a) $\mathcal{K}_1 = \mathrm{id}_{\mathbb{Z}_2^2}$,

b) $\mathcal{K}_i\mathcal{K}_i = \mathrm{id}_{\mathbb{Z}_2^2}$, *for all* $i \in \{1, 2, 3\}$,

c) $\mathcal{K}_i\mathcal{K}_j = \mathcal{K}_j\mathcal{K}_i$, *for all* $i, j \in \{1, 2, 3\}$.

Let $X^{[k]_{\mathbb{Z}_2}[l]_{\mathbb{Z}_2}}$ be some class of twice upper (and maybe also lower) indexed quantities X, such as $\mathcal{M}_{ij}^{[k]_{\mathbb{Z}_2}[l]_{\mathbb{Z}_2}}$. We define the application of the abstract shift operator to the class $X^{[k]_{\mathbb{Z}_2}[l]_{\mathbb{Z}_2}}$ by $\mathcal{K}_i \left[X^{[k]_{\mathbb{Z}_2}[l]_{\mathbb{Z}_2}}\right] := X^{\mathcal{K}_i\left([k]_{\mathbb{Z}_2}, [l]_{\mathbb{Z}_2}\right)}$. (Note that there are therefore, two definitions for $\mathcal{K}$: the original definition of $\mathcal{K}$ applied to an element of $\mathbb{Z}_2^2$, and the definition above of $\mathcal{K}$ applied to a class definition $X^{[k]_{\mathbb{Z}_2}[l]_{\mathbb{Z}_2}}$. However, both definitions accord with each other naturally and can not be confused because of the type of the argument. This concept is known as *operator overloading* from computer programming.)

6.2 The classification of the stress-resultants

Each equilibrium condition (5.21) for a fixed triple (i, m, k) was gained by the application of the variational lemma with respect to the test function coefficient $v_i^{k(m-k)}$. Rigorously formulated, equation (5.21) for a specific triple (i, k, m) is actually the term we get, when we factor out $v_i^{k(m-k)}$ from the first variation of the potential energy with respect to the virtual displacement $v_i^{k(m-k)} x_2^k x_3^{m-k}$. We will use the formulation: "The variation of the potential energy with respect to the virtual displacement coefficient $v_i^{k(m-k)}$", or, even shorter, "the variation with respect to $\delta v_i^{k(m-k)}$", when we refer to the equilibrium equations (5.21). We will use the optional δ in front of the virtual displacement coefficient in this section to avoid confusion with displacement coefficients, following the classical notation of variational calculus.

By abstraction of (5.21) we derive that the variations with respect to one fixed virtual displacement class $\delta v_{\bar{i}}^{\bar{t_1}\bar{t_2}}$ potentially contain stress resultants from the three classes $\mathcal{K}_j \left[\mathcal{M}_{\bar{i}j}^{\bar{t_1}\bar{t_2}}\right]$, $j \in \{1, 2, 3\}$. (We use the word potentially, since some stress resultants might be zero, or neglected in an Nth-order approximation. In general, the following considerations apply to the exact (untruncated) problem as well as to any Nth-order approximation.) So one tensor index of a stress resultant class, appearing in an equilibrium condition, is the tensor-index of the virtual displacement and the other one indicates the shift. Because of the symmetry relation $\mathcal{M}_{ij}^{kl} = \mathcal{M}_{ji}^{kl}$ a specific stress resultant class $\mathcal{M}_{ij}^{t_1 t_2}$ could therefore appear by variation with respect to the classes $\mathcal{K}_i \left[\delta v_j^{t_1 t_2}\right]$ or $\mathcal{K}_j \left[\delta v_i^{t_1 t_2}\right]$ (if we have $i \neq j$). Furthermore, each equilibrium condition contains exactly one stress resultant class that has two equal tensor indices (the case $i = j$ of the last sentence).

Now, let us consider an arbitrary equilibrium equation that contains a stress resultant from the class $\mathcal{M}_{\bar{i}\bar{i}}^{\bar{t_1}\bar{t_2}}$ and let us consider the distinguished index $\bar{i}$ as well as $\bar{t_1}$ and $\bar{t_2}$ as fixed. Following our general considerations from the last paragraph, the equilibrium condition under consideration was gained by the variation with respect to $\mathcal{K}_{\bar{i}} \left[\delta v_{\bar{i}}^{\bar{t_1}\bar{t_2}}\right]$. Because of (5.21) the other two stress resultant classes appearing in the equilibrium condition are $\mathcal{K}_j \mathcal{K}_{\bar{i}} \left[\mathcal{M}_{\bar{i}j}^{\bar{t_1}\bar{t_2}}\right]$, $\bar{i} \neq j$. Repeating our general considerations, each of these stress resultant classes could appear by variation with respect to two virtual displacement classes:

$$
\begin{aligned}
\mathcal{K}_j \mathcal{K}_j \mathcal{K}_{\bar{i}} \left[\delta v_{\bar{i}}^{\bar{t_1}\bar{t_2}}\right] &= \mathcal{K}_{\bar{i}} \left[\delta v_{\bar{i}}^{\bar{t_1}\bar{t_2}}\right], \quad && \text{the virtual displacement class under consideration} \\
\text{or } \mathcal{K}_{\bar{i}} \mathcal{K}_j \mathcal{K}_{\bar{i}} \left[\delta v_j^{\bar{t_1}\bar{t_2}}\right] &= \mathcal{K}_j \left[\delta v_j^{\bar{t_1}\bar{t_2}}\right], && \text{where } j \neq \bar{i}.
\end{aligned}
$$

The argumentation above can be repeated accordingly, starting with the equilibrium condition that is gained by the variation with respect to $\mathcal{K}_j \left[\delta v_j^{\bar{t_1}\bar{t_2}}\right]$, leading to the same result, where only j and $\bar{i}$ have to be interchanged. Therefore, the equilibrium conditions in stress resultants gained by variations with respect to the classes $\bigcup_{i=1}^{3} \mathcal{K}_i \left[v_i^{\bar{t_1}\bar{t_2}}\right]$ decouple from the other ones. Since the parity tuple $(\bar{t_1}, \bar{t_2}) \in \mathbb{Z}_2^2$ (of the stress resultants with identical lower indices) identifies the problem under consideration, we call it the *problem identifier.* Since the set $\mathbb{Z}_2^2$ has four elements, the equilibrium conditions in stress-resultants *always* decouple into four subproblems (later on identified as: rod, beam loaded in ξ_2 direction, beam loaded in ξ_3 direction and shaft), where each subproblem is identified by one of the four possible pairs $(t_1, t_2) \in \mathbb{Z}_2^2$. We have therefore proved:

Theorem 15 (Classification of virtual displacements and stress resultants)
The equilibrium conditions in terms of stress resultants (5.21) *decouple into four subproblems. We identify each subproblem with one of the four elements of* $\mathbb{Z}_2^2$*:*

$$(e,e),(e,o),(o,e),(o,o).$$

The equilibrium conditions of subproblem $(t_1,t_2)\in\mathbb{Z}_2^2$ *are precisely the first variations of the elastic potential with respect to the virtual displacements from the set* $\bigcup_{i=1}^3 \mathcal{K}_i\left[\delta v_i^{t_1t_2}\right]$*. Furthermore, the equilibrium conditions of subproblem* (t_1,t_2) *are formulated in the stress-resultants from the set* $\bigcup_{i=1}^3\bigcup_{j=1}^3 \mathcal{K}_i\mathcal{K}_j\left[\mathcal{M}_{ij}^{t_1t_2}\right]$.

As an example we consider the subproblem (e,e). We get an equilibrium condition for this subproblem, if we derive the first variation of the elastic potential with respect to a virtual displacement from the class $\mathcal{K}_1\left[\delta v_1^{ee}\right]=\delta v_1^{ee}$. This equation contains potentially three stress resultants from the classes

$$\mathcal{K}_1\left[\delta v_1^{ee}\right]=\delta v_1^{ee}:\mathcal{M}_{11}^{ee},\,\mathcal{M}_{12}^{oe},\,\mathcal{M}_{13}^{eo},$$

because of (5.21). $\mathcal{M}_{11}^{ee}$ is the stress resultant class with equal tensor indices (in this case $1=1$), therefore, its upper indices identify the subproblem as (e,e), since $\mathcal{K}_i\mathcal{K}_i\left[\mathcal{M}_{ii}^{t_1t_2}\right]=\mathcal{M}_{ii}^{t_1t_2}$ (for every i). $\mathcal{M}_{12}^{oe}$ is a stress resultant class with unequal tensor indices ($1\neq 2$). It appears by variation with respect to two virtual displacement classes: $\mathcal{K}_2\left[\delta v_1^{oe}\right]=\delta v_1^{ee}$, giving us the equation already considered and $\mathcal{K}_1\left[\delta v_2^{oe}\right]=\delta v_2^{oe}=\mathcal{K}_2\left[\delta v_2^{ee}\right]$. The corresponding equation contains the stress resultant classes

$$\mathcal{K}_2\left[\delta v_2^{ee}\right]=\delta v_2^{oe}:\mathcal{M}_{12}^{oe},\,\mathcal{M}_{22}^{ee},\,\mathcal{M}_{23}^{oo}.$$

Again, the upper indices of $\mathcal{M}_{22}^{ee}$ identify the subproblem as (e,e). The stress resultant class $\mathcal{M}_{13}^{eo}$ in the equation given by the variation with respect to δv_1^{ee} also appears by variation with respect to $\mathcal{K}_1\left[\delta v_3^{eo}\right]=\delta v_3^{eo}=\mathcal{K}_3\left[\delta v_3^{ee}\right]$. The corresponding equation contains the stress resultant classes

$$\mathcal{K}_3\left[\delta v_3^{ee}\right]=\delta v_3^{eo}:\mathcal{M}_{13}^{eo},\,\mathcal{M}_{23}^{oo},\,\mathcal{M}_{33}^{ee}.$$

Again, the upper indices of $\mathcal{M}_{33}^{ee}$ identify the subproblem identifier as (e,e). We know that the upper indices of $\mathcal{M}_{23}^{oo}$ coincide in the variations with respect to δv_3^{eo} and δv_2^{oe}, since the two possibilities for $\mathcal{M}_{23}^{oo}$ to appear are exactly by variation with respect to $\mathcal{K}_2\left[\delta v_3^{oo}\right]=\delta v_3^{eo}$ and $\mathcal{K}_3\left[\delta v_2^{oo}\right]=\delta v_2^{oe}$. Likewise, this is true for the two other stress resultants with unequal tensor indices. Furthermore, as already stated several times, the stress resultants with equal tensor indices always have their subproblem identifiers as upper indices. So we collected every virtual displacement and every stress resultant class of the problem (e,e) and proved that the corresponding equations decouple from the other three subproblems in terms of stress resultants.

The classifications of all virtual displacement classes of a certain subproblem $\bigcup_{i=1}^3 \mathcal{K}_i\left[\delta v_i^{t_1t_2}\right]$ are given in table 1 below. In the table, we replaced δv by u, since it will turn out later (cf. theorem 20), that the displacement coefficients in which a subproblem is written are also from the set $\bigcup_{i=1}^3 \mathcal{K}_i\left[u_i^{t_1t_2}\right]$ for isotropy. In this way, we do not have to repeat the table. The stress resultants classifications are given in table 2. In both tables, the subproblems are given acronyms (S, $B2$, $B3$, T) corresponding to the classical one-dimensional problems they describe. The correlation is investigated in the following section.

class	(t_1,t_2)	$\mathcal{K}_1[u_1^{t_1t_2}]$	$\mathcal{K}_2[u_2^{t_1t_2}]$	$\mathcal{K}_3[u_3^{t_1t_2}]$
S	(e,e)	u_1^{ee}	u_2^{oe}	u_3^{eo}
B2	(o,e)	u_1^{oe}	u_2^{ee}	u_3^{oo}
B3	(e,o)	u_1^{eo}	u_2^{oo}	u_3^{ee}
T	(o,o)	u_1^{oo}	u_2^{eo}	u_3^{oe}

Table 1: Classification of the displacement coefficients and virtual displacements

class	(t_1,t_2)	$\mathcal{K}_1\mathcal{K}_1[\mathcal{M}_{11}^{t_1t_2}]$	$\mathcal{K}_1\mathcal{K}_2[\mathcal{M}_{12}^{t_1t_2}]$	$\mathcal{K}_1\mathcal{K}_3[\mathcal{M}_{13}^{t_1t_2}]$	$\mathcal{K}_2\mathcal{K}_2[\mathcal{M}_{22}^{t_1t_2}]$	$\mathcal{K}_2\mathcal{K}_3[\mathcal{M}_{23}^{t_1t_2}]$	$\mathcal{K}_3\mathcal{K}_3[\mathcal{M}_{33}^{t_1t_2}]$
S	(e,e)	$\mathcal{M}_{11}^{ee}$	$\mathcal{M}_{12}^{oe}$	$\mathcal{M}_{13}^{eo}$	$\mathcal{M}_{22}^{ee}$	$\mathcal{M}_{23}^{oo}$	$\mathcal{M}_{33}^{ee}$
B2	(o,e)	$\mathcal{M}_{11}^{oe}$	$\mathcal{M}_{12}^{ee}$	$\mathcal{M}_{13}^{oo}$	$\mathcal{M}_{22}^{oe}$	$\mathcal{M}_{23}^{eo}$	$\mathcal{M}_{33}^{oe}$
B3	(e,o)	$\mathcal{M}_{11}^{eo}$	$\mathcal{M}_{12}^{oo}$	$\mathcal{M}_{13}^{ee}$	$\mathcal{M}_{22}^{eo}$	$\mathcal{M}_{23}^{oe}$	$\mathcal{M}_{33}^{eo}$
T	(o,o)	$\mathcal{M}_{11}^{oo}$	$\mathcal{M}_{12}^{eo}$	$\mathcal{M}_{13}^{oe}$	$\mathcal{M}_{22}^{oo}$	$\mathcal{M}_{23}^{ee}$	$\mathcal{M}_{33}^{oo}$

Table 2: Classification of the stress resultants

6.3 The classification of the load resultants

In order to identify the four subproblems by their driving force, we first consider a constant volume load $f = const.$ and free boundary-conditions on the lateral areas of the beam geometry ($g = 0$), i.e. dead weight only. If f acts in ξ_1-direction, i.e., if we only have one non-zero component-function f_1, all non-zero load resultants $p_i^{k(m-k)}$ belong to the rod-problem, by the common definition of a rod. Likewise, if f acts in ξ_i-direction, with $i = \{2, 3\}$, all non-zero load resultants belong to the beam loaded in ξ_i-direction problem. For $f = const.$, f is its own Taylor series. Let $f_i^{00} := \frac{l\tilde{f}_i}{G}$ denote the non-vanishing dimensionless Taylor coefficient, then the definition of the load resultants (4.15) simply reads as $p_i^{k(m-k)} = f_i^{00} e^{k,m-k}$. By the definition of $e^{k,m-k}$ (4.14) the problem is therefore driven by load resultants of the class p_i^{ee} (both series indices are even). These load resultants appear as right-hand sides by variations of the elastic potential with respect to the virtual displacements of the class δv_i^{ee} by equation (5.21). By theorem 15 the class (t_1, t_2) contains all equilibrium equations gathered by variations with respect to the virtual displacements $\mathcal{K}_i\left[\delta v_i^{t_1t_2}\right]$. This identifies the class $(e, e) = \mathcal{K}_1(e, e)$ as the rod problem (acronym: S, German: rod=Stab), the class $(o, e) = \mathcal{K}_2(e, e)$ as the beam-loaded in ξ_2-direction problem (acronym: $B2$, German: beam=Balken) and the class $(e, o) = \mathcal{K}_3(e, e)$ as the beam-loaded in ξ_3-direction problem (acronym: $B3$). The remaining fourth class (o, o) is not driven by dead weight and therefore identifies as the shaft problem. (acronym: T, German: torsion=Torsion, shaft=Welle. Oddly enough, in German the beam-problem is called “Balkenproblem” and the rod-problem is called “Stabproblem”, but the shaft-problem is called “Torsionsproblem”, *not* “Wellenproblem”.)

As a remark: If we had chosen to use series expansions with respect to orthogonal polynomials (abstract Fourier series) instead of using Taylor expansions, the constant volume loads would only lead to non-zero load resultants p_i^{00}, while the other load resultants of the class p_i^{ee} still vanish, by the mere definition of the Fourier-coefficients. That is the main advantage of the Fourier approach aside mathematical regularity questions. However, the classification of the load resultants (or any other quantities) is not affected by the choice of the series expansion.

For the classification of a general load case, we conclude that the parity of the indices of a volume load $f_i^{q(n-q)}$ that actually appears in a given load resultant $p_i^{k(m-k)}$ is defined by the parity of k and $m-k$, because of the distinction of cases in (4.21). Precisely, the parity of q has to coincide with the parity of k and the parity of $m-k$ has to coincide with the parity of $n-q$. Since the problem (t_1, t_2) is driven by the load resultants $\mathcal{K}_i\left[p_i^{t_1 t_2}\right]$, the driving volume-load-Taylor coefficients of the problem (t_1, t_2) are the classes $\mathcal{K}_i\left[f_i^{t_1 t_2}\right]$ (These are three classes since $i \in \{1, 2, 3\}$).

The classes of the driving volume load Taylor coefficients can directly be identified with symmetry relations of the driving-volume load component functions, as we will discuss now. We call a function F that depends on a variable x *even with respect to* x if

$$F(x) = F(-x)$$

is satisfied for all x in the domain of F and *odd with respect to* x if

$$-F(x) = F(-x)$$

is satisfied for all x in the domain of F. Some useful mathematical properties are given in the theorem below, where we use the definitions

$$\begin{aligned} \eta : \mathbb{Z}_2 &\to \{0, 1\}, & \eta : \mathbb{Z}_2^2 &\to \{0, 1\}^2, \\ e &\mapsto 0 & (t_1, t_2) &\mapsto (\eta(t_1), \eta(t_2)) \\ o &\mapsto 1 & & \end{aligned} \tag{6.2}$$

(Note that this is an *operator overloading* for η with respect to eq. (5.18).) Of course, the algebraic operations among elements of $\mathbb{Z}_2$ have to be interpreted by

	$e+\bullet$	$o+\bullet$	$e^\bullet$	$o^\bullet$
e	e	o	o	o
o	o	e	e	o

.

Theorem 16 (Parity decomposition)
A real analytic function is even with respect to x if, and only if, all Taylor coefficients with index tuples that have an odd index with respect to x are zero. Likewise, a real analytic function is odd with respect to x if, and only if, all Taylor coefficients with index tuples that have an even index with respect to x are zero. Furthermore, ***every*** *function F, defined on a domain which is symmetric with respect to x, has an additive decomposition into an even part F^e and an odd part F^o given by*

$$F(x) = \underbrace{\frac{F(x) + F(-x)}{2}}_{=:F^e(x)} + \underbrace{\frac{F(x) - F(-x)}{2}}_{=:F^o(x)},$$

or stated differently by the use of our parity notation

$$F(x) = \sum_{t_1 \in \mathbb{Z}_2} \underbrace{\frac{1}{2} \sum_{t_2 \in \mathbb{Z}_2} (-1)^{\eta\left(t_1^{t_2} + t_2\right)+1} F\Big((-1)^{\eta(t_2)} x\Big)}_{=:F^{t_1}}.$$

Therefore, by successive application of the decomposition with respect to ξ_2 and ξ_3, every component function of the volume load f_i has an additive decomposition into four parts $f_i^{t_1,t_2}$. The parity tuple (t_1, t_2) of each part indicates the parity of the function f^{t_1,t_2} with respect to the variables ξ_2 and ξ_3. t_1 shall indicate the ξ_2-parity and t_2 the ξ_3-parity. With these notational conventions the decomposition reads as

$$
\begin{aligned}
f_i(\xi_1,\xi_2,\xi_3) &= \sum_{t_1,t_2\in\mathbb{Z}_2} \underbrace{\frac{1}{4}\sum_{t_3,t_4\in\mathbb{Z}_2}(-1)^{\eta\left(t_1^{t_3}+t_2^{t_4}+t_3+t_4\right)} f_i\Big(\xi_1,(-1)^{\eta(t_3)}\xi_2,(-1)^{\eta(t_4)}\xi_3\Big)}_{=:f_i^{t_1,t_2}} \\
&= \underbrace{\frac{1}{4}\Big(f_i(\xi_1,\xi_2,\xi_3)+f_i(\xi_1,-\xi_2,\xi_3)+f_i(\xi_1,\xi_2,-\xi_3)+f_i(\xi_1,-\xi_2,-\xi_3)\Big)}_{=:f_i^{e,e}} \\
&\quad+\underbrace{\frac{1}{4}\Big(f_i(\xi_1,\xi_2,\xi_3)+f_i(\xi_1,-\xi_2,\xi_3)-f_i(\xi_1,\xi_2,-\xi_3)-f_i(\xi_1,-\xi_2,-\xi_3)\Big)}_{=:f_i^{e,o}} \\
&\quad+\underbrace{\frac{1}{4}\Big(f_i(\xi_1,\xi_2,\xi_3)-f_i(\xi_1,-\xi_2,\xi_3)+f_i(\xi_1,\xi_2,-\xi_3)-f_i(\xi_1,-\xi_2,-\xi_3)\Big)}_{=:f_i^{o,e}} \\
&\quad+\underbrace{\frac{1}{4}\Big(f_i(\xi_1,\xi_2,\xi_3)-f_i(\xi_1,-\xi_2,\xi_3)-f_i(\xi_1,\xi_2,-\xi_3)+f_i(\xi_1,-\xi_2,-\xi_3)\Big)}_{=:f_i^{o,o}}.
\end{aligned}
$$

Note that the parts $f_i^{t_1,t_2}$ are precisely four functions for every i. They do *not* denote classes of functions. The classes of Taylor coefficients are denoted by $f_i^{t_1t_2}$. The similarity in the notation is intended, since we have $f_i^{t_1,t_2} = 0 \Leftrightarrow f_i^{t_1t_2} = 0 :\Leftrightarrow f_i^{km} = 0$ for all $k, m \in \mathbb{N}_0$ with $k \bmod 2 = t_1$ and $m \bmod 2 = t_2$, because of theorem 16. Since we already derived that the problem (t_1, t_2) is driven by volume-load-Taylor coefficients of the classes $\mathcal{K}_i\left[f_i^{t_1t_2}\right]$ $(i \in \{1, 2, 3\})$, we know that each of the four parts $f_i^{t_3,t_4}$ is a driving force for exactly one of the four problems, precisely the problem $\mathcal{K}_i(t_3, t_4)$.

Therefore, in general it is incorrect to say that, for example, the beam problem is loaded perpendicular to the neutral axes. Instead, every component-function of the volume force is in general a driving force for each of the four problems and, likewise, each of the four problems is driven by volume forces in every direction. In order to have, for example, a pure $B3$-load, (i.e., for a matching material behavior only the $B3$-problem is to be solved; cf. theorem 20) the volume-force in ξ_3-direction has furthermore to fulfill the symmetry relations

$$f_3(\xi_1,\xi_2,\xi_3) = f_3(\xi_1,-\xi_2,\xi_3) \text{ and } f_3(\xi_1,\xi_2,\xi_3) = f_3(\xi_1,\xi_2,-\xi_3),$$

but also non-zero-component functions in ξ_1- and ξ_2-direction are allowed, if they fulfill the symmetry relations

$$
\begin{aligned}
f_1(\xi_1,\xi_2,\xi_3) &= f_1(\xi_1,-\xi_2,\xi_3) \text{ and } f_1(\xi_1,\xi_2,\xi_3) = -f_1(\xi_1,\xi_2,-\xi_3),\\
\text{respectively } f_2(\xi_1,\xi_2,\xi_3) &= -f_2(\xi_1,-\xi_2,\xi_3) \text{ and } f_2(\xi_1,\xi_2,\xi_3) = -f_2(\xi_1,\xi_2,-\xi_3).
\end{aligned}
$$

In this case all additive parts of the component function f_3 vanish, except $f_3^{e,e}$, leading to a Taylor series that only contains coefficients of the class f_3^{ee}. Likewise, the Taylor series of the other component functions only contain coefficients of the classes f_1^{eo} and f_2^{oo}, so we have a pure $B3$-problem to solve.

Of course, non-constant volume loads might not be of great importance for engineering practice and a constant volume load has to be in ξ_3-direction to drive the $B3$-problem, but as we will see, the same thoughts apply to boundary tractions on the lateral surfaces of the beam geometry, leading to more practically relevant implications.

The boundary tractions on the lateral surfaces are expanded into series with respect to only one coordinate, cf. (4.17) and (4.18). In this coordinate direction, again, we derive from (4.21) that the parity of the Taylor-coefficients index of the tractions g appearing in a specific load resultant $p_i^{k(m-k)}$ is given by the parity of the corresponding index of p (i.e., k or $m-k$). All preceding thoughts of the section are accordingly applicable. The other index of p indicates whether the sums or the differences of the Taylor-coefficients of the tractions on opposing sides appear in the power-series representation of p. Precisely we derive from (4.21) that the load resultants of the problem (t_1, t_2), i.e., the $\mathcal{K}_i\left[p_i^{t_1 t_2}\right]$, contain the boundary-traction Taylor-coefficients $\mathcal{K}_i\left[g_i^{t_1+} + (-1)^{\eta(t_2)} g_i^{t_1-}\right]$ and $\mathcal{K}_i\left[g_i^{+t_2} + (-1)^{\eta(t_1)} g_i^{-t_2}\right]$ (for all $i \in \{1,2,3\}$).

In order to identify these sums (respectively differences) of Taylor-coefficients on opposing sides with symmetry relations in the same manner as in the preceding part of the section, we define the following discrete analogon to even, respectively odd functions. One may think of a continuous function that is only evaluated at two discrete points that correspond to opposing boundaries to see the analogy.

For clarity we introduce a similar notation for even and odd parts of a traction g as for f, but with an extension. Again, a $t_1 \in \mathbb{Z}_2$ as an upper index at the left side of the comma indicates the ξ_2-parity, while an $t_3 \in \mathbb{Z}_2$ on the right side of the comma indicates the ξ_3-parity. As an extension a "+" as upper "index" on the left side of the comma means evaluation of g at the positive boundary in ξ_2-direction, i.e., $\xi_2 = \frac{b}{2l}$, while a "$-$" at the same place means evaluation at the negative boundary in ξ_2-direction, i.e., $\xi_2 = -\frac{b}{2l}$. Likewise, a "+" as upper "index" on the right side of the comma means evaluation of g at the positive boundary in ξ_3-direction, i.e., $\xi_3 = \frac{h}{2l}$, while a $-$ at the same place means evaluation at the negative boundary in ξ_3-direction, i.e., $\xi_3 = -\frac{h}{2l}$. For clarity the symbol "$\bullet$" on the left side indicates that the "whole" function, respectively the sum of even and odd part with respect to ξ_2 is meant, while it has the same meaning, but with respect to ξ_3, on the right side of the comma.

We call a tuple $(g_i^{\bullet,\bullet+}, g_i^{\bullet,\bullet-})$ of boundary-tractions component functions on the upper and lower part of the beam geometry (i.e., $g_i^{\bullet,\bullet\pm}(\xi_1,\xi_2) \coloneqq g_i(\xi_1,\xi_2,\pm\frac{h}{2l})$) *even with respect to* ξ_3, if

$$g_i^{\bullet,\bullet+} = g_i^{\bullet,\bullet-}$$

and *odd with respect to* ξ_3, if

$$g_i^{\bullet,\bullet+} = -g_i^{\bullet,\bullet-}.$$

By the mere definition $g_i^{\bullet,\bullet+} - g_i^{\bullet,\bullet-} = 0$ for an even (with respect to ξ_3) traction-load-component function and every even traction-load case in direction ξ_i is fully described by the sum $g_i^{\bullet,\bullet+} + g_i^{\bullet,\bullet-}$. Likewise, $g_i^{\bullet,\bullet+} + g_i^{\bullet,\bullet-} = 0$ for an odd (with respect to ξ_3) traction-load-component function and every odd traction-load case in direction ξ_i is fully described by the difference $g_i^{\bullet,\bullet+} - g_i^{\bullet,\bullet-}$. Furthermore, for **every** boundary-traction-load case the component function tuples $(g_i^{\bullet,\bullet+}, g_i^{\bullet,\bullet-})$ have an additive decomposition into an even $(g_i^{\bullet,e+}, g_i^{\bullet,e-})$ and an odd part $(g_i^{\bullet,o+}, g_i^{\bullet,o-})$ with respect to ξ_3 given by

$$\begin{pmatrix} g_i^{\bullet,\bullet+} \\ g_i^{\bullet,\bullet-} \end{pmatrix} = \begin{pmatrix} g_i^{\bullet,e+} \\ g_i^{\bullet,e-} \end{pmatrix} + \begin{pmatrix} g_i^{\bullet,o+} \\ g_i^{\bullet,o-} \end{pmatrix} := \begin{pmatrix} \frac{g_i^{\bullet,\bullet+}+g_i^{\bullet,\bullet-}}{2} \\ \frac{g_i^{\bullet,\bullet-}+g_i^{\bullet,\bullet+}}{2} \end{pmatrix} + \begin{pmatrix} \frac{g_i^{\bullet,\bullet+}-g_i^{\bullet,\bullet-}}{2} \\ \frac{g_i^{\bullet,\bullet-}-g_i^{\bullet,\bullet+}}{2} \end{pmatrix},$$

respectively,

$$g_i\Big(\xi_1,\xi_2,\pm\frac{h}{2l}\Big) = g_i^{\bullet,\bullet\pm}(\xi_1,\xi_2) = \sum_{t_2\in\mathbb{Z}_2} \underbrace{\frac{1}{2}\sum_{t_4\in\mathbb{Z}_2}(-1)^{\eta\left(t_2^{t_4}+t_4\right)+1} g_i\Big(\xi_1,\xi_2,(\pm1)(-1)^{\eta(t_4)}\frac{h}{2l}\Big)}_{=:g_i^{\bullet,t_2\pm}}.$$

Therefore, every tuple $(g_i^{\bullet,\bullet+}, g_i^{\bullet,\bullet-})$ of boundary-tractions component functions on the upper and lower part of the beam geometry has an additive decomposition into four tuples $(g_i^{t_1,t_2+}, g_i^{t_1,t_2-})$. t_1 indicates the parity of *both* component functions $g_i^{t_1,t_2\pm}$ with respect to the variable ξ_2. t_2 indicates the parity of the tuple $(g_i^{t_1,t_2+}, g_i^{t_1,t_2-})$ with respect to ξ_3. The decomposition is given by

$$g_i\Big(\xi_1,\xi_2,\pm\frac{h}{2l}\Big) = \sum_{t_1,t_2\in\mathbb{Z}_2} \underbrace{\frac{1}{4}\sum_{t_3,t_4\in\mathbb{Z}_2}(-1)^{\eta\left(t_1^{t_3}+t_2^{t_4}+t_3+t_4\right)} g_i\Big(\xi_1,(-1)^{\eta(t_3)}\xi_2,(\pm1)(-1)^{\eta(t_4)}\frac{h}{2l}\Big)}_{=:g_i^{t_1,t_2\pm}},$$

written out,

$$\begin{aligned}
&g_i^{\bullet,\bullet+}(\xi_1,\xi_2) := g_i\Big(\xi_1,\xi_2,+\frac{h}{2l}\Big)\\
&= \underbrace{\frac{1}{4}\Big(g_i^{\bullet,\bullet+}(\xi_1,\xi_2)+g_i^{\bullet,\bullet+}(\xi_1,-\xi_2)+g_i^{\bullet,\bullet-}(\xi_1,\xi_2)+g_i^{\bullet,\bullet-}(\xi_1,-\xi_2)\Big)}_{=:g_i^{e,e+}}\\
&\quad+\underbrace{\frac{1}{4}\Big(g_i^{\bullet,\bullet+}(\xi_1,\xi_2)+g_i^{\bullet,\bullet+}(\xi_1,-\xi_2)-g_i^{\bullet,\bullet-}(\xi_1,\xi_2)-g_i^{\bullet,\bullet-}(\xi_1,-\xi_2)\Big)}_{=:g_i^{e,o+}}\\
&\quad+\underbrace{\frac{1}{4}\Big(g_i^{\bullet,\bullet+}(\xi_1,\xi_2)-g_i^{\bullet,\bullet+}(\xi_1,-\xi_2)+g_i^{\bullet,\bullet-}(\xi_1,\xi_2)-g_i^{\bullet,\bullet-}(\xi_1,-\xi_2)\Big)}_{=:g_i^{o,e+}}\\
&\quad+\underbrace{\frac{1}{4}\Big(g_i^{\bullet,\bullet+}(\xi_1,\xi_2)-g_i^{\bullet,\bullet+}(\xi_1,-\xi_2)-g_i^{\bullet,\bullet-}(\xi_1,\xi_2)+g_i^{\bullet,\bullet-}(\xi_1,-\xi_2)\Big)}_{=:g_i^{o,o+}}
\end{aligned}$$

and

$$\begin{aligned}
&g_i^{\bullet,\bullet-}(\xi_1,\xi_2) := g_i\Big(\xi_1,\xi_2,-\frac{h}{2l}\Big)\\
&= \underbrace{\frac{1}{4}\Big(g_i^{\bullet,\bullet-}(\xi_1,\xi_2)+g_i^{\bullet,\bullet-}(\xi_1,-\xi_2)+g_i^{\bullet,\bullet+}(\xi_1,\xi_2)+g_i^{\bullet,\bullet+}(\xi_1,-\xi_2)\Big)}_{=:g_i^{e,e-}}
\end{aligned}$$

$$+\underbrace{\frac{1}{4}\left(g_i^{\bullet,\bullet-}(\xi_1,\xi_2)+g_i^{\bullet,\bullet-}(\xi_1,-\xi_2)-g_i^{\bullet,\bullet+}(\xi_1,\xi_2)-g_i^{\bullet,\bullet+}(\xi_1,-\xi_2)\right)}_{=:g_i^{e,o-}}$$

$$+\underbrace{\frac{1}{4}\left(g_i^{\bullet,\bullet-}(\xi_1,\xi_2)-g_i^{\bullet,\bullet-}(\xi_1,-\xi_2)+g_i^{\bullet,\bullet+}(\xi_1,\xi_2)-g_i^{\bullet,\bullet+}(\xi_1,-\xi_2)\right)}_{=:g_i^{o,e-}}$$

$$+\underbrace{\frac{1}{4}\left(g_i^{\bullet,\bullet-}(\xi_1,\xi_2)-g_i^{\bullet,\bullet-}(\xi_1,-\xi_2)-g_i^{\bullet,\bullet+}(\xi_1,\xi_2)+g_i^{\bullet,\bullet+}(\xi_1,-\xi_2)\right)}_{=:g_i^{o,o-}}.$$

Again, each of the four tuples $(g_i^{t_1,t_2+}, g_i^{t_1,t_2-})$ generates a driving force for exactly one of the four problems. The tuple $(g_i^{t_1,t_2+}, g_i^{t_1,t_2-})$ fulfills the symmetry relations:

$$\begin{aligned}
&g_i^{t_1,t_2+}(\xi_1,\xi_2)=(-1)^{\eta(t_2)}g_i^{t_1,t_2-}(\xi_1,\xi_2),\\
&g_i^{t_1,t_2+}(\xi_1,\xi_2)=(-1)^{\eta(t_1)}g_i^{t_1,t_2+}(\xi_1,-\xi_2)\text{ and}\\
&g_i^{t_1,t_2-}(\xi_1,\xi_2)=(-1)^{\eta(t_1)}g_i^{t_1,t_2-}(\xi_1,-\xi_2).
\end{aligned}$$

Because of the ξ_3-parity, this tuple generates the scalar driving force $g_i^{t_1,\bullet+}+(-1)^{\eta(t_2)}g_i^{t_1,\bullet-}$. (The factor 1/2 is omitted for brevity.) This scalar driving force vanishes, if and only if, all Taylor coefficients of the classes $g_i^{t_1+}$ and $g_i^{t_1-}$ vanish, which is because of ξ_3-parity equivalent to the vanishing of $g_i^{t_1+}+(-1)^{\eta(t_2)}g_i^{t_1-}$, i.e., $g_i^{n+}+(-1)^{\eta(t_2)}g_i^{n-}=0$ for all $n\in\mathbb{N}_0$ with $n \bmod 2 = t_1$. Therefore, the tuple $(g_i^{t_1,t_2+}, g_i^{t_1,t_2-})$ precisely drives the problem with the class identifier $\mathcal{K}_i(t_1,t_2)$, cf. (5.21) and (4.21).

Of course all thoughts apply accordingly for a tuple $(g_i^{\bullet+,\bullet}, g_i^{\bullet-,\bullet})$ of boundary-traction-component functions on the opposing boundaries of the beam in ξ_2-direction (i.e., $g_i^{\bullet\pm,\bullet}(\xi_1,\xi_3):=g_i(\xi_1,\pm\frac{b}{2l},\xi_3)$). For clarity of notation we quote the decompositions again. The decomposition into an even and odd part in-ξ_2 direction is given by

$$g_i\left(\xi_1,\pm\frac{b}{2l},\xi_3\right)=g_i^{\bullet\pm,\bullet}(\xi_1,\xi_3)=\sum_{t_1\in\mathbb{Z}_2}\underbrace{\frac{1}{2}\sum_{t_3\in\mathbb{Z}_2}(-1)^{\eta\left(t_1^{t_3}+t_3\right)+1}g_i\left(\xi_1,(\pm1)(-1)^{\eta(t_3)}\frac{b}{2l},\xi_3\right)}_{=:g_i^{t_1\pm,\bullet}},$$

leading to the overall additive decomposition

$$g_i\left(\xi_1,\pm\frac{b}{2l},\xi_3\right)=\sum_{t_1,t_2\in\mathbb{Z}_2}\underbrace{\frac{1}{4}\sum_{t_3,t_4\in\mathbb{Z}_2}(-1)^{\eta\left(t_1^{t_3}+t_2^{t_4}+t_3+t_4\right)}g_i\left(\xi_1,(\pm1)(-1)^{\eta(t_3)}\frac{b}{2l},(-1)^{\eta(t_4)}\xi_3\right)}_{=:g_i^{t_1\pm,t_2}},$$

written out,

$$g_i^{\bullet+,\bullet}(\xi_1,\xi_3):=g_i\left(\xi_1,+\frac{b}{2l},\xi_3\right)$$
$$=\underbrace{\frac{1}{4}\left(g_i^{\bullet+,\bullet}(\xi_1,\xi_3)+g_i^{\bullet+,\bullet}(\xi_1,-\xi_3)+g_i^{\bullet-,\bullet}(\xi_1,\xi_3)+g_i^{\bullet-,\bullet}(\xi_1,-\xi_3)\right)}_{=:g_i^{e+,e}}$$

$$+ \underbrace{\frac{1}{4}\Big(g_i^{\bullet+,\bullet}(\xi_1,\xi_3) - g_i^{\bullet+,\bullet}(\xi_1,-\xi_3) + g_i^{\bullet-,\bullet}(\xi_1,\xi_3) - g_i^{\bullet-,\bullet}(\xi_1,-\xi_3)\Big)}_{=:g_i^{e+,o}}$$

$$+ \underbrace{\frac{1}{4}\Big(g_i^{\bullet+,\bullet}(\xi_1,\xi_3) + g_i^{\bullet+,\bullet}(\xi_1,-\xi_3) - g_i^{\bullet-,\bullet}(\xi_1,\xi_3) - g_i^{\bullet-,\bullet}(\xi_1,-\xi_3)\Big)}_{=:g_i^{o+,e}}$$

$$+ \underbrace{\frac{1}{4}\Big(g_i^{\bullet+,\bullet}(\xi_1,\xi_3) - g_i^{\bullet+,\bullet}(\xi_1,-\xi_3) - g_i^{\bullet-,\bullet}(\xi_1,\xi_3) + g_i^{\bullet-,\bullet}(\xi_1,-\xi_3)\Big)}_{=:g_i^{o+,o}}$$

and

$$g_i^{\bullet-,\bullet}(\xi_1,\xi_3) := g_i\Big(\xi_1,-\frac{b}{2l},\xi_3\Big)$$

$$= \underbrace{\frac{1}{4}\Big(g_i^{\bullet-,\bullet}(\xi_1,\xi_3) + g_i^{\bullet-,\bullet}(\xi_1,-\xi_3) + g_i^{\bullet+,\bullet}(\xi_1,\xi_3) + g_i^{\bullet+,\bullet}(\xi_1,-\xi_3)\Big)}_{=:g_i^{e-,e}}$$

$$+ \underbrace{\frac{1}{4}\Big(g_i^{\bullet-,\bullet}(\xi_1,\xi_3) - g_i^{\bullet-,\bullet}(\xi_1,-\xi_3) + g_i^{\bullet+,\bullet}(\xi_1,\xi_3) - g_i^{\bullet+,\bullet}(\xi_1,-\xi_3)\Big)}_{=:g_i^{e-,o}}$$

$$+ \underbrace{\frac{1}{4}\Big(g_i^{\bullet-,\bullet}(\xi_1,\xi_3) + g_i^{\bullet-,\bullet}(\xi_1,-\xi_3) - g_i^{\bullet+,\bullet}(\xi_1,\xi_3) - g_i^{\bullet+,\bullet}(\xi_1,-\xi_3)\Big)}_{=:g_i^{o-,e}}$$

$$+ \underbrace{\frac{1}{4}\Big(g_i^{\bullet-,\bullet}(\xi_1,\xi_3) - g_i^{\bullet-,\bullet}(\xi_1,-\xi_3) - g_i^{\bullet+,\bullet}(\xi_1,\xi_3) + g_i^{\bullet+,\bullet}(\xi_1,-\xi_3)\Big)}_{=:g_i^{o-,o}}.$$

By comparing the parities of the driving forces, i.e., volume force f and traction g, for a specific problem (t_1, t_2), we are finally able to formulate the main result of this subsection in an esthetically short way:

The problem (t_1, t_2) is driven by the parts of the overall load that have in ξ_i-direction the parities given by $\mathcal{K}_i(t_1, t_2)$.

We collect the preceding thoughts of this section by the formulation of three theorems.

Theorem 17 (Volume load-decomposition)
Each volume-load-component function f_i has an unique additive decomposition into four parts given by

$$f_i(\xi_1,\xi_2,\xi_3) = \sum_{t_1,t_2\in\mathbb{Z}_2} \underbrace{\frac{1}{4}\sum_{t_3,t_4\in\mathbb{Z}_2} (-1)^{\eta\left(t_1^{t_3}+t_2^{t_4}+t_3+t_4\right)} f_i\Big(\xi_1,(-1)^{\eta(t_3)}\xi_2,(-1)^{\eta(t_4)}\xi_3\Big)}_{=:f_i^{t_1,t_2}}.$$

The part $f_i^{t_1,t_2}$ has the ξ_2-parity t_1 and the ξ_3-parity t_2, i.e.,

$$\forall \xi \in \Omega_\xi: \qquad f_i^{t_1,t_2}(\xi_1,\xi_2,\xi_3) = (-1)^{\eta(t_1)} f_i^{t_1,t_2}(\xi_1,-\xi_2,\xi_3)$$
$$\textit{and } f_i^{t_1,t_2}(\xi_1,\xi_2,\xi_3) = (-1)^{\eta(t_2)} f_i^{t_1,t_2}(\xi_1,\xi_2,-\xi_3).$$

If, and only if, f_i has the ξ_2-parity t_1 all the complementary parts vanish $\forall t_2 \in \mathbb{Z}_2 :\ f_i^{\mathbb{Z}_2\setminus t_1,t_2} = 0$.
Likewise, if, and only if, f_i has the ξ_3-parity t_2 all the complementary parts vanish $\forall t_1 \in \mathbb{Z}_2 :$ $f_i^{t_1,\mathbb{Z}_2\setminus t_2} = 0$.
The part $f_i^{t_1,t_2}$ vanishes if, and only if, all Taylor coefficients of the class $f_i^{t_1 t_2}$ vanish, i.e.,

$$\forall n, m \in \mathbb{N}_0 \text{ with } (n \text{ mod } 2 = t_1) \text{ and } (m \text{ mod } 2 = t_2) :\ f_i^{nm} = 0.$$

Theorem 18 (Decomposition of the boundary tractions)
Each tuple of boundary-traction-component functions on opposing lateral surfaces of the beam, i.e., the tuples

$$\left(g_i\Big(\xi_1, \xi_2, \frac{h}{2l}\Big), g_i\Big(\xi_1, \xi_2, -\frac{h}{2l}\Big)\right) \text{ and } \left(g_i\Big(\xi_1, \frac{b}{2l}, \xi_3\Big), g_i\Big(\xi_1, -\frac{b}{2l}, \xi_3\Big)\right),$$

have an unique additive decomposition into four parts, given by the tuples $(g_i^{t_1,t_2+}, g_i^{t_1,t_2-})$, or $(g_i^{t_1+,t_2}, g_i^{t_1-,t_2})$ respectively. We have

$$g_i\Big(\xi_1, \xi_2, \pm\frac{h}{2l}\Big) = \sum_{t_1,t_2\in\mathbb{Z}_2} \underbrace{\frac{1}{4} \sum_{t_3,t_4\in\mathbb{Z}_2} (-1)^{\eta\left(t_1^{t_3}+t_2^{t_4}+t_3+t_4\right)} g_i\Big(\xi_1, (-1)^{\eta(t_3)}\xi_2, (\pm 1)(-1)^{\eta(t_4)}\frac{h}{2l}\Big)}_{=:g_i^{t_1,t_2\pm}},$$

or

$$g_i\Big(\xi_1, \pm\frac{b}{2l}, \xi_3\Big) = \sum_{t_1,t_2\in\mathbb{Z}_2} \underbrace{\frac{1}{4} \sum_{t_3,t_4\in\mathbb{Z}_2} (-1)^{\eta\left(t_1^{t_3}+t_2^{t_4}+t_3+t_4\right)} g_i\Big(\xi_1, (\pm 1)(-1)^{\eta(t_3)}\frac{b}{2l}, (-1)^{\eta(t_4)}\xi_3\Big)}_{=:g_i^{t_1\pm,t_2}}$$

respectively. The decomposition tuples have the ξ_2-parity t_1 and the ξ_3-parity t_2, i.e.,

$$\begin{aligned}
g_i^{t_1,t_2+}(\xi_1,\xi_2) &= (-1)^{\eta(t_2)} g_i^{t_1,t_2-}(\xi_1,\xi_2), & g_i^{t_1+,t_2}(\xi_1,\xi_3) &= (-1)^{\eta(t_1)} g_i^{t_1-,t_2}(\xi_1,\xi_3),\\
g_i^{t_1,t_2+}(\xi_1,\xi_2) &= (-1)^{\eta(t_1)} g_i^{t_1,t_2+}(\xi_1,-\xi_2), & g_i^{t_1+,t_2}(\xi_1,\xi_3) &= (-1)^{\eta(t_2)} g_i^{t_1+,t_2}(\xi_1,-\xi_3),\\
g_i^{t_1,t_2-}(\xi_1,\xi_2) &= (-1)^{\eta(t_1)} g_i^{t_1,t_2-}(\xi_1,-\xi_2), & g_i^{t_1-,t_2}(\xi_1,\xi_3) &= (-1)^{\eta(t_2)} g_i^{t_1-,t_2}(\xi_1,-\xi_3).
\end{aligned}$$

If, and only if, the tuple $(g_i(\xi_1, \xi_2, \frac{h}{2l}), g_i(\xi_1, \xi_2, -\frac{h}{2l}))$ has the ξ_2-parity t_1, we have $g_i^{\mathbb{Z}_2\setminus t_1,t_2\pm} = 0$ for all $t_2 \in \mathbb{Z}_2$, i.e., $g_i^{\mathbb{Z}_2\setminus t_1,\bullet+} + (-1)^{\eta(t_2)} g_i^{\mathbb{Z}_2\setminus t_1,\bullet-} = 0$ for all $t_2 \in \mathbb{Z}_2$. If, and only if, the tuple $(g_i(\xi_1, \xi_2, \frac{h}{2l}), g_i(\xi_1, \xi_2, -\frac{h}{2l}))$ has the ξ_3-parity t_2 we have $g_i^{t_1,\bullet+} + (-1)^{\eta(\mathbb{Z}_2\setminus t_2)} g_i^{t_1,\bullet-} = 0$ for all $t_1 \in \mathbb{Z}_2$. If, and only if, the tuple $(g_i(\xi_1, \frac{b}{2l}, \xi_3), g_i(\xi_1, -\frac{b}{2l}, \xi_3))$ has the ξ_2-parity t_1 we have $g_i^{\bullet+,t_2} + (-1)^{\eta(\mathbb{Z}_2\setminus t_1)} g_i^{\bullet-,t_2} = 0$ for all $t_2 \in \mathbb{Z}_2$. If, and only if, the tuple $(g_i(\xi_1, \frac{b}{2l}, \xi_3), g_i(\xi_1, -\frac{b}{2l}, \xi_3))$ has the ξ_3-parity t_2 we have $g_i^{t_1\pm,\mathbb{Z}_2\setminus t_2} = 0$ for all $t_1 \in \mathbb{Z}_2$, i.e., $g_i^{\bullet+,\mathbb{Z}_2\setminus t_2} + (-1)^{\eta(t_1)} g_i^{\bullet-,\mathbb{Z}_2\setminus t_2} = 0$ for all $t_1 \in \mathbb{Z}_2$.
The parts $g_i^{t_1,\bullet\pm}$ vanish if, and only if, all Taylor coefficients of the class $g_i^{t_1\pm}$ vanish, i.e.,

$$\forall n \in \mathbb{N}_0 \text{ with } (n \text{ mod } 2 = t_1) :\ g_i^{n\pm} = 0.$$

The parts $g_i^{\bullet\pm,t_2}$ vanish if, and only if, all Taylor coefficients of the class $g_i^{\pm t_2}$ vanish, i.e.,

$$\forall n \in \mathbb{N}_0 \text{ with } (n \text{ mod } 2 = t_2) :\ g_i^{\pm n} = 0.$$

Theorem 19 (Classification of the driving forces)
The problem (t_1, t_2) is driven by load resultants that are elements of the set $\bigcup_{i=1}^{3} \mathcal{K}_i \left[p_i^{t_1 t_2}\right]$, which contain the volume-load coefficients from the set $\bigcup_{i=1}^{3} \mathcal{K}_i \left[f_i^{t_1 t_2}\right]$ and boundary-traction coefficients from the set $\bigcup_{i=1}^{3} \mathcal{K}_i \left[g_i^{t_1+} + (-1)^{\eta(t_2)} g_i^{t_1-}\right]$ and $\bigcup_{i=1}^{3} \mathcal{K}_i \left[g_i^{+t_2} + (-1)^{\eta(t_1)} g_i^{-t_2}\right]$.

To sum up, the problem (t_1, t_2) is driven by the parts of the overall load (volume force f and traction g) that have in ξ_i-direction the parities given by $\mathcal{K}_i(t_1, t_2)$.

The classification of all driving forces by theorem 19 is given in the tables 3 and 4 below.

class	(t_1, t_2)	$\mathcal{K}_1[f_1^{t_1 t_2}]$	$\mathcal{K}_2[f_2^{t_1 t_2}]$	$\mathcal{K}_3[f_3^{t_1 t_2}]$
S	(e, e)	f_1^{ee}	f_2^{oe}	f_3^{eo}
B2	(o, e)	f_1^{oe}	f_2^{ee}	f_3^{oo}
B3	(e, o)	f_1^{eo}	f_2^{oo}	f_3^{ee}
T	(o, o)	f_1^{oo}	f_2^{eo}	f_3^{oe}

Table 3: Classification of the volume-load-Taylor coefficients.

class	(t_1, t_2)	$\mathcal{K}_1\left[g_1^{t_1+} + (-1)^{\eta(t_2)} g_1^{t_1-}\right]$ $\mathcal{K}_1\left[g_1^{+t_2} + (-1)^{\eta(t_1)} g_1^{-t_2}\right]$	$\mathcal{K}_2\left[g_2^{t_1+} + (-1)^{\eta(t_2)} g_2^{t_1-}\right]$ $\mathcal{K}_2\left[g_2^{+t_2} + (-1)^{\eta(t_1)} g_2^{-t_2}\right]$	$\mathcal{K}_3\left[g_3^{t_1+} + (-1)^{\eta(t_2)} g_3^{t_1-}\right]$ $\mathcal{K}_3\left[g_3^{+t_2} + (-1)^{\eta(t_1)} g_3^{-t_2}\right]$
S	(e, e)	$g_1^{e+} + g_1^{e-}$ $g_1^{+e} + g_1^{-e}$	$g_2^{o+} + g_2^{o-}$ $g_2^{+e} - g_2^{-e}$	$g_3^{e+} - g_3^{e-}$ $g_3^{+o} + g_3^{-o}$
$B2$	(o, e)	$g_1^{o+} + g_1^{o-}$ $g_1^{+e} - g_1^{-e}$	$g_2^{e+} + g_2^{e-}$ $g_2^{+e} + g_2^{-e}$	$g_3^{o+} - g_3^{o-}$ $g_3^{+o} - g_3^{-o}$
$B3$	(e, o)	$g_1^{e+} - g_1^{e-}$ $g_1^{+o} + g_1^{-o}$	$g_2^{o+} - g_2^{o-}$ $g_2^{+o} - g_2^{-o}$	$g_3^{e+} + g_3^{e-}$ $g_3^{+e} + g_3^{-e}$
T	(o, o)	$g_1^{o+} - g_1^{o-}$ $g_1^{+o} - g_1^{-o}$	$g_2^{e+} - g_2^{e-}$ $g_2^{+o} + g_2^{-o}$	$g_3^{o+} + g_3^{o-}$ $g_3^{+e} - g_3^{-e}$

Table 4: Classification of the boundary traction driving forces.

The theorems above allow for a unique additive decomposition of every three-dimensional load case of the "quasi one-dimensional" beam geometry into driving forces of the four subproblems (beside boundary conditions).

6.4 Example: Decomposition of a topside pressure "beam"-load case

We will give an example for the application of the theorems of the last subsection. We assume a constant positive pressure p on the top side of the beam geometry with no other loads, i.e.,

$$f_i = 0,\ g_1 = 0,\ g_2 = 0,\ g_3^{\bullet,\bullet-} = p \text{ and } g_3^{\bullet,\bullet+} = 0.$$

This is a load-case most relevant for engineering practice, which one might mistake as the canonical $B3$-load-case as suggested by almost every illustration in basic course engineering mechanics text books. (See figure 4 for an illustration.) Obviously, the upper-side and lower-side loads are even with respect to ξ_2, since both loads are constant. This gives us $g_3^{o,\bullet\pm} = 0$, which implies $g_3^{o,e\pm} = g_3^{o,o\pm} = 0$. But in ξ_3-direction the load is nor even or odd, which we formally verify by deriving the other four parts of the decomposition

$$\begin{aligned} g_3^{e,e+} &= \frac{1}{4}\Big(0+0+p+p\Big) = \frac{1}{2}p, \\ g_3^{e,e-} &= \frac{1}{4}\Big(p+p+0+0\Big) = \frac{1}{2}p, \\ g_3^{e,o+} &= \frac{1}{4}\Big(0+0-p-p\Big) = -\frac{1}{2}p, \\ g_3^{e,o-} &= \frac{1}{4}\Big(p+p-0-0\Big) = \frac{1}{2}p, \end{aligned}$$

(See figure 4).

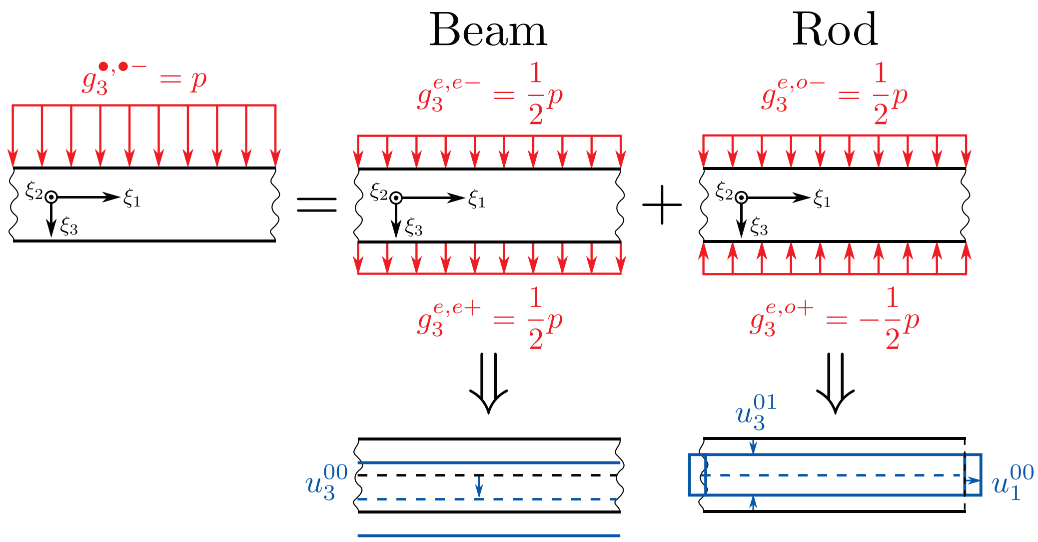

Figure 4: Decomposition of the uniform top side traction p into an even and odd part. The even part is a driving force for the beam-problem and the odd part is a driving force for the rod problem, therefore, the top-side load p is actually a mixed-load case and not a pure beam-load.

Theorem 19 identifies the even part $(g_3^{e,e+}, g_3^{e,e-})$ as the driving force for the $B3$-problem, while the odd part $(g_3^{e,o+}, g_3^{e,o-})$ drives the rod problem. The classification is obviously reasonable, since one would directly assume that the even part of the load would result in an displacement in ξ_3-direction of the neutral axis that does not deform the cross section. This displacement u_3^{00} will indeed turn out to be the canonical $B3$-deformation in accordance with classical beam theories (u_3^{00} equals w in classical beam theories). Also one would directly assume that the odd part $(g_3^{e,o+}, g_3^{e,o-})$ results in a uniform squeezing of the cross-section in ξ_3-direction. This deformation u_3^{01} will turn out to belong to the rod problem. For an isotropic material with non-zero Poisson's ratio $\nu \neq 0$ this will also lead to an uniform displacement in ξ_1-direction that does not further deform the cross-section. This deformation u_1^{00} will turn out to be the canonical rod-displacement also in accordance with the classical theory. (See figure 4 for an illustration of the deformations.)

We will find that the $B3$- and S-problem are decoupled for isotopic material, therefore solving the $B3$-problem alone results in the correct result for the elastic line w. Nevertheless, the original load will also drive a rod problem, which can be solved independently from the beam-problem (superposition). Since the rod problem also results in stresses (although decoupled from the beam-stresses) the stresses one derives from the $B3$-problem alone do not correspond to the actual load, nor does the three-dimensional displacement field, therefore the problem given by the original load is indeed a beam and a rod problem. As another line of argumentation it seems reasonable that the even load is the only one among all loads with $g_3^{e,o+} + g_3^{e,o-} = p$ that does not lead to a squeezing of the cross-section in ξ_3-direction and, therefore it is the only pure $B3$-load, although the solution for the elastic line w only depends on the sum of the upper and lower load $g_3^{e,o+} + g_3^{e,o-}$.

We already mentioned that the displacement coefficients of problem (t_1, t_2) will turn out to be from the set $\bigcup_{i=1}^{3} \mathcal{K}_i \left[u_i^{t_1 t_2}\right]$, cf. theorem 20. This leads to the fact that the $B3$-problem (e, o) only contains displacements in ξ_3-direction that are even in ξ_3-direction (and ξ_2-direction, i.e., u_3^{ee}), in other words, displacements that preserve the cross-section height. This means that the classical assumption of the preserved cross-section height in classical beam theories is only necessary because the load case is defined wrongly! In our setting the preserved cross-section height is an (exactly fulfilled) consequence of the theorems 19 and 20!

6.5 The anisotropic coupling

In this subsection we will investigate how the anisotropy of the material couples the four one-dimensional subproblems written in terms of the displacements-coefficients u_i^{kl}. The field equations are given by the system of all equilibrium equations (5.21), by insertion of (5.9) and the definition of the linear combinations ω_{ij}^{gh} in equation (5.7).

We have to investigate what kind of displacement coefficients appear in a stress-resultant of a certain class $\mathcal{M}_{ij}^{t_1 t_2}$, since we already derived which stress-resultants appear in a specific subproblem, cf. theorem 15. From equation (5.19) we derive that a stress-resultant $\mathcal{M}_{ij}^{t_1 t_2}$ is a series that contains only linear combinations of displacement coefficients that belong to the class $\omega_{ij}^{t_1 t_2}$. The definition of the linear combinations ω_{ij}^{gh} in equation (5.7) uses the shift operator S defined on double series indexed quantities X^{km} defined in equation (4.9). Since we only want to know which classes of displacement coefficients appear in a certain class $\omega_{ij}^{t_1 t_2}$, we could use an abstraction of the shift operator S to an operator defined on classes of double series indexed quantities $X^{t_1 t_2}$, as we have already done in subsection 6.2 with the shift operator K, defined by (5.2). Again, the abstract operator to S turns out to be $\mathcal{K}$ defined by (6.1).

If we have an aelotrop (or triclinic) material, i.e., $E_{ijrs} \neq 0$ for all $i, j, r, s \in \{1, 2, 3\}$, the class $\omega_{ij}^{t_1 t_2}$ contains the displacement coefficient classes $\bigcup_{r=1}^{3} \bigcup_{s=1}^{3} \mathcal{K}_s u_r^{t_1 t_2}$. This follows directly from the abstraction of equation (5.6). Furthermore, we know from theorem 15 that the problem (t_1, t_2) contains the stress-resultants of the classes $\bigcup_{i=1}^{3} \bigcup_{j=1}^{3} \mathcal{K}_i \mathcal{K}_j \left[\mathcal{M}_{ij}^{t_1 t_2}\right]$ and, therefore, the linear combinations of the classes $\bigcup_{i=1}^{3} \bigcup_{j=1}^{3} \mathcal{K}_i \mathcal{K}_j \left[\omega_{ij}^{t_1 t_2}\right]$, so that the problem contains displacement coefficients of the classes

$$\bigcup_{i,j,r,s=1}^{3} \mathcal{K}_i \mathcal{K}_j \mathcal{K}_s u_r^{t_1 t_2} = \bigcup_{r=1}^{3} \bigcup_{(t_3,t_4) \in \mathbb{Z}_2^2} u_r^{t_3 t_4},$$

which are already all classes of displacement coefficients (cf. figure 6 below). This means that all four subproblems are coupled, i.e. we have one combined problem containing all unknown displacement coefficients to solve.

In general, the structure of the tensor E, i.e. the property of tensor-elements to be non-zero, defines how the subproblems are coupled with each other. We repeat our thoughts from the last paragraph for an arbitrary kind of anisotropy. Once again, independently from the kind of anisotropy, we get from theorem 15, that the subproblem (t_1, t_2) contains the stress-resultants of the classes $\bigcup_{i=1}^3 \bigcup_{j=1}^3 \mathcal{K}_i\mathcal{K}_j\,[\mathcal{M}_{ij}^{t_1t_2}]$ and, therefore the linear combinations of the classes $\bigcup_{i=1}^3 \bigcup_{j=1}^3 \mathcal{K}_i\mathcal{K}_j\,[\omega_{ij}^{t_1t_2}]$. By abstraction of (5.7) the subproblem (t_1, t_2) is written in displacement coefficients from the set

$$\bigcup_{\substack{i,j,r,s=1\\E_{ijrs}\neq 0}}^{3} \mathcal{K}_i\mathcal{K}_j\mathcal{K}_s u_r^{t_1t_2} = \bigcup_{\substack{i,j,r,s=1\\E_{ijrs}\neq 0}}^{3} \underbrace{\mathcal{K}_i\mathcal{K}_j\mathcal{K}_r\mathcal{K}_s}_{=:\mathcal{K}_{\text{eff}}(i,j,r,s)} \mathcal{K}_r u_r^{t_1t_2}. \tag{6.3}$$

For the equation above we inserted the identity $\mathcal{K}_r\mathcal{K}_r = \mathrm{id}_{\mathbb{Z}_2^2}$ and made use of the commutativity of the abstract shift operators (cf. lemma 14). Furthermore, to investigate the relation between the sparsity scheme of E and the coupling of the one-dimensional subproblems, we define an object, which we call the *effective shift* operator $\mathcal{K}_{\text{eff}}$, that maps each index quadruple $(i, j, r, s) \in \{1, 2, 3\}^4$ to a mapping $\mathbb{Z}_2^2 \to \mathbb{Z}_2^2$, which is given by the composition of the corresponding abstract shifts $\mathcal{K}_i\mathcal{K}_j\mathcal{K}_r\mathcal{K}_s$, i.e.,

$$\begin{aligned}\mathcal{K}_{\text{eff}} :&\{1,2,3\}^4 \longrightarrow \left(\mathbb{Z}_2^2 \to \mathbb{Z}_2^2\right)\\ &(i,j,r,s) \longmapsto \mathcal{K}_i\mathcal{K}_j\mathcal{K}_r\mathcal{K}_s.\end{aligned}$$

The transformation in (6.3) is motivated by the structure of E for isotropy. In this case all non-zero elements of the tensor E have an effective shift operator that equals the identity

$$E_{ijrs} \neq 0 \Rightarrow \mathcal{K}_{\text{eff}}(i,j,r,s) = \mathrm{id}_{\mathbb{Z}_2^2}\,. \tag{6.4}$$

The calculation of all effective shifts associated to the tensor components of E is given in figure 5. The most general anisotropic material behavior that fulfills (6.4) is orthotropic (or rhombic)

$$\begin{bmatrix} E_{1111} & E_{1122} & E_{1133} & E_{1112} & E_{1123} & E_{1113}\\ & E_{2222} & E_{2233} & E_{2212} & E_{2223} & E_{2213}\\ & & E_{3333} & E_{3312} & E_{3323} & E_{3313}\\ S & & & E_{1212} & E_{1223} & E_{1213}\\ & Y & & & E_{2323} & E_{2313}\\ & & M. & & & E_{1313} \end{bmatrix} \mathrel{\hat{=}} \begin{bmatrix} \mathrm{id}_{\mathbb{Z}_2^2} & \mathrm{id}_{\mathbb{Z}_2^2} & \mathrm{id}_{\mathbb{Z}_2^2} & \mathcal{K}_2 & \mathcal{K}_2\mathcal{K}_3 & \mathcal{K}_3\\ & \mathrm{id}_{\mathbb{Z}_2^2} & \mathrm{id}_{\mathbb{Z}_2^2} & \mathcal{K}_2 & \mathcal{K}_2\mathcal{K}_3 & \mathcal{K}_3\\ & & \mathrm{id}_{\mathbb{Z}_2^2} & \mathcal{K}_2 & \mathcal{K}_2\mathcal{K}_3 & \mathcal{K}_3\\ S & & & \mathrm{id}_{\mathbb{Z}_2^2} & \mathcal{K}_3 & \mathcal{K}_2\mathcal{K}_3\\ & Y & & & \mathrm{id}_{\mathbb{Z}_2^2} & \mathcal{K}_2\\ & & M. & & & \mathrm{id}_{\mathbb{Z}_2^2} \end{bmatrix}$$

Figure 5: On the left: The 21 independent tensor components of E in Voigt's notation, cf. (2.13). On the right: The associated effective shift operators $\mathcal{K}_{\text{eff}}(i, j, r, s)$ for every index quadrupel (i, j, r, s) on the left. For the calculation we reduced the effective shifts as much as possible by using the identities of lemma 14.

material, (cf., e.g. Ting (1996)), if the symmetry axis of the material coincide with the coordinate axis. Therefore, we proved that the displacement coefficients of problem (t_1, t_2) are from the set

$$\bigcup_{r=1}^{3} \mathcal{K}_r u_r^{t_1 t_2},$$

for orthotropic, in particular isotropic, material. We already gave the assignment of the displacement coefficients to the subproblems in table 1. By investigation of the table one immediately realizes that the sets of displacement coefficients of different subproblems are disjoint. We therefore proved that the equilibrium conditions (5.21) decouple into four subproblems for orthotropic, in particular isotropic, material.

For a general anisotropic material we conclude from equation (6.3): A non-zero element E_{ijrs} of the elasticity tensor E results in the coupling of all the subproblems (t_1, t_2) and (t_3, t_4) with each other, for which $\mathcal{K}_{\text{eff}}(i, j, r, s)(t_1, t_2) = (t_3, t_4)$ (or $\mathcal{K}_{\text{eff}}(i, j, r, s)(t_3, t_4) = (t_1, t_2)$, respectively) is true. The problems (t_1, t_2) and (t_3, t_4) are coupled with each other means precisely that the equilibrium conditions in terms of displacement coefficients of problem (t_1, t_2) contain displacement coefficients of the problem (t_3, t_4). (And the equilibrium conditions of problem (t_3, t_4) contain displacement coefficients of the problem (t_1, t_2), respectively. Also the equilibrium conditions might contain further classes of displacement coefficients, if there are further non-zero components of E with unequal effective shift operators.) We summarize our thoughts in the following theorem, before we investigate the actual consequences.

Theorem 20 (Anisotropic coupling and classification of displacement coefficients) *For an* isotropic *material, the field equations of the exact one-dimensional problem written in displacement coefficients (*(5.21)*, by insertion of* (5.19) *and* (5.7)*) decouple into four subproblems. We identify each subproblem by a parity tuple* $(t_1, t_2) \in \mathbb{Z}_2^2$.

The field-equations of problem (t_1, t_2) *are formulated in displacement coefficients from the set* $\bigcup_{i=1}^{3} \mathcal{K}_i \left[u_i^{t_1 t_2}\right]$.

For an anisotropic *material, two problems* (t_1, t_2) *and* (t_3, t_4) *are coupled if, and only if, there exists a non-zero component of the stiffness tensor* $E_{ijrs} \neq 0$ *with assigned effective shift* $\mathcal{K}_{\mathit{eff}}(i, j, r, s)$ *that transfers the problem identifiers into one another, i.e.,*

$$\mathcal{K}_{\mathit{eff}}(i, j, r, s)(t_1, t_2) = (t_3, t_4).$$

For a better understanding of theorem 20, diagram 6 illustrates how the abstract shift operator $\mathcal{K}$ transfers the class identifiers of the four problems into one another. Furthermore, we already derived the effective shift operators for every component of a full elasticity tensor E given in Voigt's notation in figure 5.

As already mentioned, even for an orthotropic (or rhombic) material we have only non-zero elements of the elasticity tensor that have an effective shift $\mathcal{K}_1$. Therefore, we have four decoupled problems in this case and, of course, in the case of a more specific material behavior, like e.g. transversely isotropic (or hexagonal) material. On the other hand, for an aelotrop (or triclinic) material the problems are all coupled with each other, because the tensor E is fully populated. So the interesting cases are materials that are more specific than an aelotrop material but less specific than an orthotropic material; these are the monoclinic materials. They have in general a fully populated tensor E and therefore a fully coupled system of equilibrium conditions, unless the symmetry plane is given by two coordinate axes. By comparison of the structure of E for a specific symmetry plane (compare, e.g., Ting (1996), p.44ff) with figure 5 and consideration of the diagram 6 we derive:

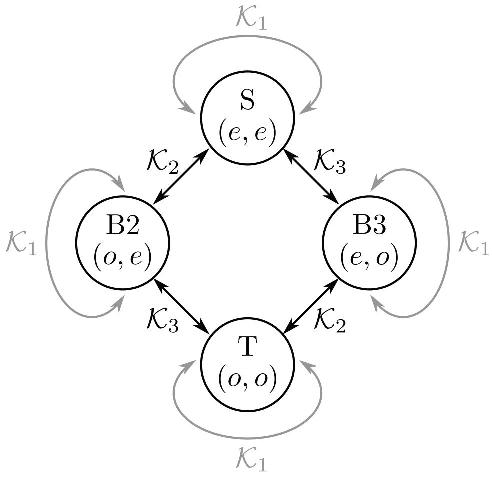

Figure 6: How the abstract shift operator $\mathcal{K}$ transfers the class identifiers in one another.

- For a monoclinic material where the symmetry plane is given by $\xi_1 = 0$, or the axes ξ_2 and ξ_3, respectively, we have non-zero components of E with effective shifts of either $\mathcal{K}_1$ or $\mathcal{K}_2\mathcal{K}_3$. This leads to a coupling of the S and the T-problem and a coupling of the $B2$ and the $B3$-problem, whereas the combined problems $S - T$ and $B2 - B3$ are decoupled.
- For a monoclinic material where the symmetry plane is given by $\xi_2 = 0$, or the axes ξ_1 and ξ_3, respectively, we have non-zero components of E with effective shifts of either $\mathcal{K}_1$ or $\mathcal{K}_3$. This leads to a coupling of the S and the $B3$-problem and a coupling of the $B2$ and the T-problem, whereas the combined problems $S - B3$ and $B2 - T$ are decoupled.
- For a monoclinic material where the symmetry plane is given by $\xi_3 = 0$, or the axes ξ_1 and ξ_2, respectively, we have non-zero components of E with effective shifts of either $\mathcal{K}_1$ or $\mathcal{K}_2$. This leads to a coupling of the S and the $B2$-problem and a coupling of the $B3$ and the T-problem, whereas the combined problems $S - B2$ and $B3 - T$ are decoupled.

6.6 Example: Monoclinic S-$B3$-problem

In order to give a plausibility consideration for at least one of the monoclinic problems, we assume the plane of symmetry to be given by the coordinate axes x_1 and x_3. As a load case we assume a constant volume load in x_1-direction, respectively, a constant traction p in x_1-direction at a positive boundary of a cutting plane, i.e., with an outer normal unit vector in x_1-direction. (See figure 7 for an illustration.) This is a pure rod load case by theorem 19.

The setting of such a monoclinic rod may appear, if we apply a homogenization approach to a laminated beam that consists of two isotropic materials that differ in their stiffness, as illustrated in figure 7a. In this example the x_1-x_3-plane is a plane of symmetry, so that a homogenized replacement material would be monoclinic. If we abstract the rod to a mere wireframe, where the elasticity is presented by linear elastic springs in the edges, the springs on opposing sides of the symmetry plane must have the same stiffness. On the other hand, they would differ in their stiffness on the upper and lower face side of the rod, because of the different local volume fractions of the two materials. Under the assumed uniform normal traction p, we would get a deformation u, as sketched in figure 7a, which is a superposition of the canonical rod deformation, i.e., the constant elongation u_1^{00}, and the $B3$-beam deflection, i.e., u_1^{01}. (Compare figure 1.) Therefore,

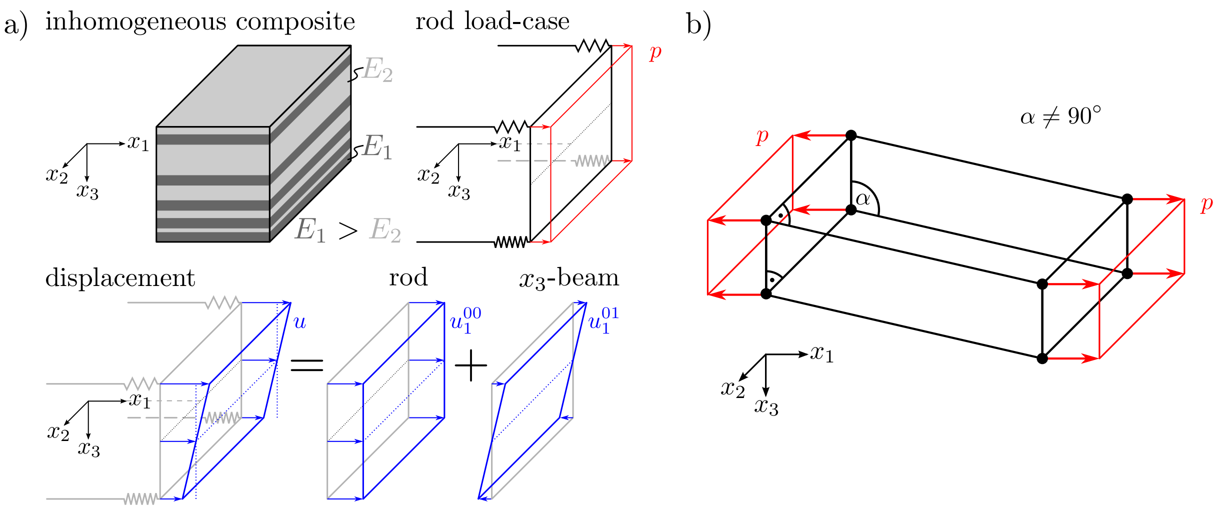

Figure 7: Homogeneous monoclinic rod with symmetry plane x_1-x_3: a) Monoclinic replacement material after the homogenization of a laminated rod. b) Bravais lattice of a monoclinic single crystal.

we have a coupled S-$B3$-problem. Likewise, because of the plane of symmetry one would not assume any $B2$-deformations or a twist of the cross section that belongs to the T-problem under the given load. So we have indeed a S-$B3$-problem that is decoupled from the $B2$-T-problem.

As another line of reasoning, the setting might also appear, if we have a homogeneous monoclinic single crystal with symmetry plane x_1-x_3, which corresponds to a Bravais lattice of the type sketched in figure 7b, where $\alpha \neq 90°$. Applying an uniform normal traction p would lead to a change of the angle α in the deformed configuration and therefore also to a shear deformation of type ε_{13}, in accordance with the first example.

6.7 The classification of the boundary conditions

Theorem 20 deals with the decoupling of the field equations of the exact one-dimensional subproblems. For a decoupling of the *problems* we will also have to deal with the boundary conditions, i.e., equations (5.22) and (5.25). Fortunately this is surprisingly simple.

With theorem 15 and 20 all displacement coefficients and all stress resultants are assigned to one of the four (maybe coupled) subproblems due to the field equations. Since the corresponding traction boundary conditions (5.22) are already formulated in terms of stress resultants and the displacement boundary conditions (5.25) are already formulated in terms of displacement coefficients, all boundary conditions are also assigned automatically.

The prescribed stress resultants of problem (t_1, t_2) are elements from the set $\bigcup_{i=1}^{3} \mathcal{K}_i \left[\mathcal{M}_{i1}^{t_1 t_2}\right]$, which is a subset of the stress resultants that belong to this subproblem, i.e. $\bigcup_{i=1}^{3} \mathcal{K}_i \left[\mathcal{M}_{i1}^{t_1 t_2}\right] \subset \bigcup_{i=1}^{3} \bigcup_{j=1}^{3} \mathcal{K}_i \mathcal{K}_j \left[\mathcal{M}_{ij}^{t_1 t_2}\right]$. If we insert eqs. (4.14), (4.19) and (4.20) into the stress boundary condition (5.22)

$$\mathcal{M}_{i1}^{k(m-k)}(u)\, n_1 = \int_{A_\xi} \frac{g_i}{G}\, \hat{\xi}^{k(m-k)}\, dA_\xi = \sum_{n=0}^{\infty} \sum_{q=0}^{n} g_{\xi_1 i}^{q(n-q)}\, e^{k+q, m-k+n-q} \quad \text{f.a. } \xi_1 \in P_{\xi N},$$

we find in analogy to theorem 17 that the part $\mathcal{K}_i\left[g_i^{t_1,t_2}\right]$ of the parity decomposition

$$g_i(\xi_1,\xi_2,\xi_3) = \sum_{t_3,t_4\in\mathbb{Z}_2} \underbrace{\frac{1}{4}\sum_{t_5,t_6\in\mathbb{Z}_2}(-1)^{\eta\left(t_3^{t_5}+t_4^{t_6}+t_5+t_6\right)} g_i(\xi_1,(-1)^{\eta(t_5)}\xi_2,(-1)^{\eta(t_6)}\xi_3)}_{=:g_i^{t_3,t_4}}$$

of a prescribed traction-component function g_i at a face side $\xi_1 \in P_{\xi N}$ generates the stress boundary conditions for the problem (t_1,t_2). Also we already discovered in theorem 19 that the parts of the overall load (volume force f and traction g) that have in ξ_i-direction the parities given by $\mathcal{K}_i(t_1,t_2)$ are the driving forces for the field equations of problem (t_1,t_2). This means, if f and g fulfill the symmetries $f_i = \mathcal{K}_i\left[f_i^{t_1,t_2}\right]$ and $g_i = \mathcal{K}_i\left[g_i^{t_1,t_2}\right]$ for all $i \in \{1,2,3\}$, all inhomogeneous right-hand sides of the exact equilibrium conditions (5.21) belong to problem (t_1,t_2) (i.e. all non-vanishing load resultants belong to the set $\bigcup_{i=1}^{3}\mathcal{K}_i\left[p_i^{t_1t_2}\right]$) and all inhomogeneous stress boundary conditions belong to problem (t_1,t_2), too.

Likewise, the prescribed displacement coefficients of problem (t_1,t_2) are all elements from the set $\bigcup_{i=1}^{3}\mathcal{K}_i\left[u_i^{t_1t_2}\right]$, which are generated by the parts $\mathcal{K}_i\left[u_{0i}^{t_1t_2}\right]$ for all $i \in \{1,2,3\}$ of u_0 at the points $\xi_1 \in P_{\xi 0}$ due to equation (5.25). If u_{0i} fulfills the symmetries $u_{0i} = \mathcal{K}_i\left[u_{0i}^{t_1t_2}\right]$ for all $i \in \{1,2,3\}$, the only inhomogeneous displacement boundary conditions belong to problem (t_1,t_2).

The (untruncated) problem of the eqs. (5.21), (5.22) and (5.25) is equivalent to the problem of three-dimensional linear elasticity by theorem 13. By theorem 7 the solution u of the three-dimensional theory of linear elasticity exists and is unique. The parity decomposition of this solution is unique, too. By theorem 20 the parts $\mathcal{K}_i\left[u_i^{t_1,t_2}\right]$ for all $i \in \{1,2,3\}$ belong to problem (t_1,t_2), i.e. the parity parts of the solution are uniquely assigned to the subproblems, which therefore have a unique solution, too. If u_0, f and g fulfill the symmetry relations

$$u_{0i} = \mathcal{K}_i\left[u_{0i}^{t_1,t_2}\right], \qquad f_i = \mathcal{K}_i\left[f_i^{t_1,t_2}\right], \qquad g_i = \mathcal{K}_i\left[g_i^{t_1,t_2}\right] \qquad \text{for all } i \in \{1,2,3\}$$

and we have decoupled field equations by theorem 20, e.g. for isotropic material, the unique solution of all subproblems with exception of (t_1,t_2) is given by the vanishing displacement field $u = 0$ (because all right-hand sides and boundary conditions of these problems vanish, which makes $u = 0$ a solution, and we already know that this solution is unique). This means we have indeed a pure (t_1,t_2) problem. Therefore, the parity decomposition of u_0, f and g decomposes the exact one-dimensional problem in four *independent* subproblems, which also holds true for any Nth-order consistent approximation. We sum up our findings in the main result of this section.

Theorem 21 (Decoupling theorem for one-dimensional problems)
Assume all assumptions of theorem 7 and the quasi one-dimensional geometry, defined in section 4.2, especially (A3)*. Then the one-dimensional problem of eqs.* (5.21)*,* (5.22) *and* (5.25) *with inserted* (5.10) *and* (5.7) *is equivalent to any of the problems of theorem 7.*

For isotropic *material and* any *given data u_0, f and g, the one-dimensional mixed boundary value problem* decouples *into four* independent *subproblems, by decomposing the data component functions due to their parities. We identify each subproblem, by its problem identifier $(t_1,t_2) \in \mathbb{Z}_2^2$.*

Problem $(t_1, t_2) \in \mathbb{Z}_2^2$ *is given by all equilibrium conditions* (5.21) *which were gained by the first variations of the elastic potential with respect to virtual displacements from the set*

$$\bigcup_{i=1}^{3} \mathcal{K}_i \left[\delta v_i^{t_1 t_2}\right],$$

prescription of all displacement coefficients at $\xi_1 \in P_{\xi 0}$ *which belong to the set*

$$\bigcup_{i=1}^{3} \mathcal{K}_i \left[u_i^{t_1 t_2}\right]$$

and prescription of all stress resultants at $\xi_1 \in P_{\xi N}$ *which belong to the set*

$$\bigcup_{i=1}^{3} \mathcal{K}_i \left[\mathcal{M}_{i1}^{t_1 t_2}\right].$$

The subproblem's solution u *only depends on the parts of the given: prescribed displacement* u_0*, volume force* f *and the traction* g *(on face sides and lateral faces, i.e. on* $\partial\Omega_N$*) that fulfill the symmetries*

$$u_{0i} = \mathcal{K}_i \left[u_{0i}^{t_1,t_2}\right], \qquad f_i = \mathcal{K}_i \left[f_i^{t_1,t_2}\right], \qquad g_i = \mathcal{K}_i \left[g_i^{t_1,t_2}\right] \qquad \textit{for all } i \in \{1,2,3\}.$$

The displacement field solution u *of problem* (t_1, t_2) *fulfills the symmetries*

$$u_i = \mathcal{K}_i \left[u_i^{t_1,t_2}\right] \qquad \textit{for all } i \in \{1,2,3\},$$

and all stress resultants derived from the subproblems solution are elements of the set

$$\bigcup_{i=1}^{3} \bigcup_{j=1}^{3} \mathcal{K}_i \mathcal{K}_j \left[\mathcal{M}_{ij}^{t_1 t_2}\right].$$

For an anisotropic *material, two subproblems* $(t_1, t_2) \in \mathbb{Z}_2^2$ *and* $(t_3, t_4) \in \mathbb{Z}_2^2$ *are coupled if, and only if, there exists a non-zero component of the stiffness tensor* $E_{ijrs} \neq 0$*, with assigned effective shift* $\mathcal{K}_{\mathit{eff}}(i,j,r,s)$*, that transfers the problem identifiers into one another, i.e.,*

$$\mathcal{K}_{\mathit{eff}}(i,j,r,s)(t_1,t_2) = (t_3,t_4).$$

At last we want to mention that the theorem's statement is not affected if one uses the generalized boundary conditions (5.28) instead of (5.22) and (5.25), with the exception that there is no three-dimensional representation of the problem without the use of series expansions, cf. section 5.6.

7 The one-dimensional equilibrium equations in terms of displacement coefficients

In principle theorem 13 of section 5 already tells us how to compute the Nth-order approximation of the whole one-dimensional problem. In section 6 we enlighted under which circumstances this problem decouples into independent subproblems. Knowing about the decoupling of the subproblems a-priori can be used to reduce the effort of deriving the subproblems' equations enormously.

We introduce a very elegant way of computing the field equations by introduction of the differential operator D_{eq} in equation (7.1). First, one needs to derive which equations and displacement coefficients have to be considered in a certain Nth-order subproblem, cf. section 7.7. Next, one can derive the equations piecewise by evaluation of the formulas of section 7.4, that reveal a great insight into the structure of the equations. The effort of deriving the equations is often halved, because the resulting system can be symmetrized for a lot of anisotropies, cf. section 7.6. (This symmetry also has a certain meaning with respect to *pseudo reductions* (cf. section 8.3) since symmetric matrices are diagonalizable.) Also the two beam problems equations are equivalent, if they are independent subproblems, cf. section 7.9.

In addition, the operator D_{eq} can be used to derive alternative proofs for the decoupling theorems 15 and 20, cf. section 7.3.

7.1 Notation

Like in section 5 we will not use the summation convention in this whole section. Tensor indices that were previously bound by the summation convention will now be bound by the explicit use of the summation symbol $\sum$. Recall that this allows us to use decompositions like (5.3).

We will make frequent use of the notations introduced in section 5.1, especially:

- D_j^1, $\hat{\xi}^{km}$ and χ_j introduced in (5.1),
- K_j and a_j^{23} introduced in (5.2) and decomposition (5.3).

7.2 The differential operator of the one-dimensional equilibrium conditions

The key observation of this section is that we can interpret the one-dimensional equilibrium conditions, where each condition is gained by variation with respect to a specific $v_i^{k(m-k)}$ (cf. eq. (5.21)), as an infinite sum of differential operators, where each operator is applied to exactly one displacement coefficient $u_r^{q(n-q)}$

For all $i \in \{1,2,3\}$, $m \in \mathbb{N}_0$, $k \in \{0,...,m\}$:

$$\sum_{r=1}^{3}\sum_{n=0}^{\infty}\sum_{q=0}^{n} D_{\text{eq}}\left[v_i^{k(m-k)}, u_r^{q(n-q)}\right]\left(u_r^{q(n-q)}\right) = -p_i^{k(m-k)}. \tag{7.1}$$

To this end, we have to define the differential operator $D_{\text{eq}}\left[v_i^{k(m-k)}, u_r^{q(n-q)}\right]$ by

$$D_{\text{eq}}\left[v_i^{k(m-k)}, u_r^{q(n-q)}\right](\bullet) := \sum_{j,s=1}^{3}\frac{E_{ijrs}}{G}\left(\int_{-\frac{b}{2l}}^{\frac{b}{2l}}\int_{-\frac{h}{2l}}^{\frac{h}{2l}} \chi_j a_j^{23}\left[\hat{\xi}^{k(m-k)}\right] a_s^{23}\left[\hat{\xi}^{q(n-q)}\right]\, d_{\xi_3} d_{\xi_2}\right) D_j^1 D_s^1(\bullet), \tag{7.2}$$

which is derived by inserting the definition of the stress resultants (5.4) and the series expansion for the displacement field (4.12) into the equations of equilibrium (5.21).

$$
\begin{aligned}
&\text{For all } i \in \{1,2,3\},\ m \in \mathbb{N}_0,\ k \in \{0,...,m\}: \\
&\sum_{j=1}^{3} K_j \left[\mathcal{M}_{ij}^{k(m-k)}\right] = -p_i^{k(m-k)} \\
&= \sum_{j=1}^{3} K_j \left[\int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} \frac{\sigma_{ij}}{G} \hat{\xi}^{k(m-k)}\, d_{\xi_3} d_{\xi_2}\right] \\
&= \sum_{j=1}^{3} K_j \left[\int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} \sum_{r,s=1}^{3} \frac{E_{ijrs}}{G} \left(\frac{u_r}{l}\right)_{,s} \hat{\xi}^{k(m-k)}\, d_{\xi_3} d_{\xi_2}\right] \\
&= \sum_{j=1}^{3} \int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} \sum_{r,s=1}^{3} \frac{E_{ijrs}}{G} D_j^1 \left(\left(\frac{u_r}{l}\right)_{,s}\right) \chi_j a_j^{23} \left[\hat{\xi}^{k(m-k)}\right] d_{\xi_3} d_{\xi_2} \\
&= \sum_{j=1}^{3} \int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} \sum_{r,s=1}^{3} \frac{E_{ijrs}}{G} D_j^1 \left(\left(\sum_{n=0}^{\infty}\sum_{q=0}^{n} u_r^{q(n-q)} \hat{\xi}^{q(n-q)}\right)_{,s}\right) \chi_j a_j^{23} \left[\hat{\xi}^{k(m-k)}\right] d_{\xi_3} d_{\xi_2} \\
&= \sum_{j=1}^{3} \int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} \sum_{r,s=1}^{3} \frac{E_{ijrs}}{G} D_j^1 \left(\sum_{n=0}^{\infty}\sum_{q=0}^{n} D_s^1 \left(u_r^{q(n-q)}\right) a_s^{23} \left[\hat{\xi}^{q(n-q)}\right]\right) \chi_j a_j^{23} \left[\hat{\xi}^{k(m-k)}\right] d_{\xi_3} d_{\xi_2} \\
&= \sum_{j=1}^{3} \sum_{r,s=1}^{3} \frac{E_{ijrs}}{G} \sum_{n=0}^{\infty}\sum_{q=0}^{n} D_j^1 D_s^1 \left(u_r^{q(n-q)}\right) \int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} a_s^{23} \left[\hat{\xi}^{q(n-q)}\right] \chi_j a_j^{23} \left[\hat{\xi}^{k(m-k)}\right] d_{\xi_3} d_{\xi_2} \\
&= \sum_{j=1}^{3} \sum_{r,s=1}^{3} \sum_{n=0}^{\infty}\sum_{q=0}^{n} \frac{E_{ijrs}}{G} \left(\int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} \chi_j a_j^{23} \left[\hat{\xi}^{k(m-k)}\right] a_s^{23} \left[\hat{\xi}^{q(n-q)}\right] d_{\xi_3} d_{\xi_2}\right) D_j^1 D_s^1 \left(u_r^{q(n-q)}\right) \\
&= \sum_{r=1}^{3} \sum_{n=0}^{\infty}\sum_{q=0}^{n} \underbrace{\sum_{j,s=1}^{3} \chi_j \frac{E_{ijrs}}{G} \left(\int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} a_j^{23} \left[\hat{\xi}^{k(m-k)}\right] a_s^{23} \left[\hat{\xi}^{q(n-q)}\right] d_{\xi_3} d_{\xi_2}\right) D_j^1 D_s^1}_{=D_{\text{eq}}\left[v_i^{k(m-k)},\, u_r^{q(n-q)}\right]} \left(u_r^{q(n-q)}\right)
\end{aligned}
$$

Note that the equations above are well-defined, if we define all displacement coefficients and stress resultants with at least one negative upper index to be constant (especially zero, as we have chosen), since they are always multiplied with zeros arising from the prefactors of a^{23}.

7.3 The anisotropic coupling revisited

Now we will give an alternative proof for the theorems 15 and 20 using equation (7.1). This will also help us to write down the differential operator of equilibrium in an explicit form.

We derive from (7.1) that the equation, which is the variation with respect to $v_i^{k(m-k)}$, contains a term in $u_r^{q(n-q)}$ if, and only if, $D_{\text{eq}}\left[v_i^{k(m-k)}, u_r^{q(n-q)}\right] \neq 0$. Note that the double integral in (7.2) vanishes, if not both sums of the corresponding indices of $\hat{\xi}$ are even after the applications of the shift operators (cf. eq. (4.14)). This means the indices have to have pairwise equal parity after the applications of the shift operators to generate a $D_{\text{eq}}\left[v_i^{k(m-k)}, u_r^{q(n-q)}\right] \neq 0$. Since the abstract form of a^{23} is (again) $\mathcal{K}$, this is equivalent to $\mathcal{K}_j\left([k]_{\mathbb{Z}_2}, [m-k]_{\mathbb{Z}_2}\right) = \mathcal{K}_s\left([q]_{\mathbb{Z}_2}, [n-q]_{\mathbb{Z}_2}\right)$.

In addition, the corresponding material constant has to be non-zero. To sum up, the fact that a variation $v_i^{k(m-k)}$ contains a term in $u_r^{q(n-q)}$, implies

$$\exists(j,s) \in \{1,2,3\}^2 : \mathcal{K}_j\left([k]_{\mathbb{Z}_2}, [m-k]_{\mathbb{Z}_2}\right) = \mathcal{K}_s\left([q]_{\mathbb{Z}_2}, [n-q]_{\mathbb{Z}_2}\right) \text{ and } E_{ijrs} \neq 0.$$

(Here $\exists$ denotes the existential quantification, i.e., $\exists x : P(x)$ means "there is at least one x such that $P(x)$ is true".) In section 6.5 we already showed that we have for an orthotropic material (and in particular for an isotropic material) with non-vanishing Poisson ratios

$$E_{ijrs} \neq 0 \Leftrightarrow \mathcal{K}_i\mathcal{K}_j\mathcal{K}_r\mathcal{K}_s = \text{id} \Leftrightarrow \mathcal{K}_s\mathcal{K}_j = \mathcal{K}_r\mathcal{K}_i.$$

Therefore, the fact that the variation with respect to $v_i^{k(m-k)}$ contains the displacement coefficient $u_r^{q(n-q)}$, implies

$$\begin{aligned}
&\exists(j,s) \in \{1,2,3\}^2 : \mathcal{K}_j\left([k]_{\mathbb{Z}_2}, [m-k]_{\mathbb{Z}_2}\right) = \mathcal{K}_s\left([q]_{\mathbb{Z}_2}, [n-q]_{\mathbb{Z}_2}\right) \text{ and } E_{ijrs} \neq 0\\
\Leftrightarrow &\exists(j,s) \in \{1,2,3\}^2 : \mathcal{K}_s\mathcal{K}_j\left([k]_{\mathbb{Z}_2}, [m-k]_{\mathbb{Z}_2}\right) = \left([q]_{\mathbb{Z}_2}, [n-q]_{\mathbb{Z}_2}\right) \text{ and } \mathcal{K}_s\mathcal{K}_j = \mathcal{K}_r\mathcal{K}_i\\
\Leftrightarrow &\mathcal{K}_r\mathcal{K}_i\left([k]_{\mathbb{Z}_2}, [m-k]_{\mathbb{Z}_2}\right) = \left([q]_{\mathbb{Z}_2}, [n-q]_{\mathbb{Z}_2}\right)\\
\Leftrightarrow &\mathcal{K}_i\left([k]_{\mathbb{Z}_2}, [m-k]_{\mathbb{Z}_2}\right) = \mathcal{K}_r\left([q]_{\mathbb{Z}_2}, [n-q]_{\mathbb{Z}_2}\right) =: (t_1, t_2).
\end{aligned}$$

This means, so to say, variations and displacement coefficients map to the same tuple (t_1, t_2). Since this tuple characterizes the subproblem, we called it the *subproblem identifier* (already in section 6.2). Indeed, if we build the negation of the implication we just derived, we obtain that, if variations and displacement coefficients do not map to the same tuple (t_1, t_2), we get $D_{\text{eq}}\left[v_i^{k(m-k)}, u_r^{q(n-q)}\right] = 0$, i.e., the subproblems are decoupled. Since there are four elements in $\mathbb{Z}_2^2$ and, therefore, four possibilities for the tuple (t_1, t_2), there are four one-dimensional subproblems, whose equilibrium conditions are decoupled. Furthermore, we derive that the equilibrium conditions of problem (t_1, t_2) are gained by variations from the set

$$\bigcup_{i=1}^{3} \mathcal{K}_i v_i^{t_1t_2}, \qquad \bigcup_{r=1}^{3} \mathcal{K}_r u_r^{t_1t_2}$$

and the displacement coefficients of the subproblem are from the "same" set. Since the abstract form of the operator K is $\mathcal{K}$ we furthermore derive from (5.21) that the stress resultants of problem (t_1, t_2) are from the classes $\bigcup_{i=1}^{3}\bigcup_{j=1}^{3} \mathcal{K}_i\mathcal{K}_j\mathcal{M}_{ij}^{t_1t_2}$. This completes an alternative proof of the theorems 15 and 20 for isotropy.

For a monoclinic material, where the plane of symmetry is given by two coordinate-axes, we have

$$E_{ijrs} \neq 0 \Leftrightarrow \mathcal{K}_i\mathcal{K}_j\mathcal{K}_r\mathcal{K}_s = \text{id or } \mathcal{K}_i\mathcal{K}_j\mathcal{K}_r\mathcal{K}_s = \mathcal{K}_{\text{mon}}$$

for a specific $\mathcal{K}_{\text{mon}} \neq \text{id}$, i.e., $\mathcal{K}_{\text{mon}} \in \{\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_2\mathcal{K}_3\}$, where each possibility belongs to one possible plane of symmetry. Therefore the variation with respect to $v_i^{k(m-k)}$ containing the displacement coefficient $u_r^{q(n-q)}$ implies

$$\begin{aligned}
&\exists(j,s) \in \{1,2,3\}^2 : \mathcal{K}_j\left([k]_{\mathbb{Z}_2}, [m-k]_{\mathbb{Z}_2}\right) = \mathcal{K}_s\left([q]_{\mathbb{Z}_2}, [n-q]_{\mathbb{Z}_2}\right) \text{ and } E_{ijrs} \neq 0\\
\Leftrightarrow &\mathcal{K}_i\left([k]_{\mathbb{Z}_2}, [m-k]_{\mathbb{Z}_2}\right) = \mathcal{K}_r\left([q]_{\mathbb{Z}_2}, [n-q]_{\mathbb{Z}_2}\right)\\
&\text{or } \mathcal{K}_{\text{mon}}\mathcal{K}_i\left([k]_{\mathbb{Z}_2}, [m-k]_{\mathbb{Z}_2}\right) = \mathcal{K}_r\left([q]_{\mathbb{Z}_2}, [n-q]_{\mathbb{Z}_2}\right).
\end{aligned}$$

Therefore, since

$$\begin{aligned}&\mathcal{K}_{\text{mon}}\mathcal{K}_i\left([k]_{\mathbb{Z}_2},[m-k]_{\mathbb{Z}_2}\right)=\mathcal{K}_r\left([q]_{\mathbb{Z}_2},[n-q]_{\mathbb{Z}_2}\right)\\ \Leftrightarrow&\mathcal{K}_i\left([k]_{\mathbb{Z}_2},[m-k]_{\mathbb{Z}_2}\right)=\mathcal{K}_{\text{mon}}\mathcal{K}_r\left([q]_{\mathbb{Z}_2},[n-q]_{\mathbb{Z}_2}\right),\end{aligned}$$

in the case of a monoclinic material, where the plane of symmetry is given by two coordinate-axes, two problems are coupled, if their class identifiers are transformed into one another by $\mathcal{K}_{\text{mon}}$, which leads to two problems of pairwise two classes illustrated in figure 6. We already derived this in an even more general context in theorem 20.

For a monclinic material, where the plane of symmetry is not given by two coordinate-axes, or an aelotropic material, all four classes are coupled.

7.4 The equilibrium equations in terms of displacement coefficients

With the knowledge about the decoupling behavior we can write down the differential operator $D_{\text{eq}}\left[v_i^{k(m-k)},u_r^{q(n-q)}\right]$ in dependence of the tuple (i,r). First assume that we have an orthotropic material, then we could only get a non-zero summand for (j,s) in (7.2) if

$$\begin{aligned}&\mathcal{K}_j\mathcal{K}_s=\mathcal{K}_i\mathcal{K}_r\\ \Leftrightarrow&\begin{cases}(j,s)\in G_{\text{id}} & \text{if}\quad (i,r)\in G_{\text{id}}\\ (j,s)\in\{(i,r),(r,i)\} & \text{if}\quad (i,r)\notin G_{\text{id}}\end{cases},\end{aligned}$$

where $G_{\text{id}}:=\{(1,1),(2,2),(3,3)\}$, so that we get the following sums of at most three summands by insertion of (4.14) into (7.2):

$$\begin{aligned}&(i,r)\in G_{\text{id}},\ \text{i.e.,}\ i=r:\ D_{\text{eq}}\left[v_r^{k(m-k)},u_r^{q(n-q)}\right]\\ &=\frac{hb}{l^2}\Bigg[\ \frac{E_{r1r1}}{G}\frac{(\sqrt{3})^{n+m}}{(q+k+1)(n+m-q-k+1)}\,d^{q+k}\,c^{n+m-q-k}\,\frac{\partial^2(\bullet)}{\partial\xi_1^2}\\ &\qquad+\frac{E_{r2r2}}{G}\frac{(-kq)(\sqrt{3})^{n+m-2}}{(q+k-1)(n+m-q-k+1)}\,d^{q+k-2}\,c^{n+m-q-k}\,(\bullet)\\ &\qquad+\frac{E_{r3r3}}{G}\frac{(-(m-k)(n-q))(\sqrt{3})^{n+m-2}}{(q+k+1)(n+m-q-k-1)}\,d^{q+k}\,c^{n+m-q-k-2}\,(\bullet)\Bigg]\end{aligned}\tag{7.3}$$

$$\begin{aligned}&(i,r)=(1,2):\ D_{\text{eq}}\left[v_1^{k(m-k)},u_2^{q(n-q)}\right]\\ &=\frac{hb}{l^2}\Bigg[\ \frac{E_{1122}}{G}\frac{(q)(\sqrt{3})^{n+m-1}}{(q+k)(n+m-q-k+1)}\,d^{q+k-1}\,c^{n+m-q-k}\,\frac{\partial(\bullet)}{\partial\xi_1}\\ &\qquad+\frac{E_{1221}}{G}\frac{(-k)(\sqrt{3})^{n+m-1}}{(q+k)(n+m-q-k+1)}\,d^{q+k-1}\,c^{n+m-q-k}\,\frac{\partial(\bullet)}{\partial\xi_1}\Bigg]\end{aligned}\tag{7.4}$$

$$\begin{aligned}&(i,r)=(2,1):\ D_{\text{eq}}\left[v_2^{k(m-k)},u_1^{q(n-q)}\right]\\ &=\frac{hb}{l^2}\Bigg[\ \frac{E_{2211}}{G}\frac{(-k)(\sqrt{3})^{n+m-1}}{(q+k)(n+m-q-k+1)}\,d^{q+k-1}\,c^{n+m-q-k}\,\frac{\partial(\bullet)}{\partial\xi_1}\\ &\qquad+\frac{E_{2112}}{G}\frac{(q)(\sqrt{3})^{n+m-1}}{(q+k)(n+m-q-k+1)}\,d^{q+k-1}\,c^{n+m-q-k}\,\frac{\partial(\bullet)}{\partial\xi_1}\Bigg]\end{aligned}\tag{7.5}$$

$$(i,r) = (1,3): \; D_{\text{eq}}\left[v_1^{k(m-k)}, u_3^{q(n-q)}\right]$$
$$= \frac{hb}{l^2}\Bigg[\frac{E_{1133}}{G}\frac{(n-q)(\sqrt{3})^{n+m-1}}{(q+k+1)(n+m-q-k)}\, d^{q+k}\, c^{n+m-q-k-1}\, \frac{\partial(\bullet)}{\partial\xi_1}$$
$$+ \frac{E_{1331}}{G}\frac{(-(m-k))(\sqrt{3})^{n+m-1}}{(q+k+1)(n+m-q-k)}\, d^{q+k}\, c^{n+m-q-k-1}\, \frac{\partial(\bullet)}{\partial\xi_1}\Bigg] \tag{7.6}$$

$$(i,r) = (3,1): \; D_{\text{eq}}\left[v_3^{k(m-k)}, u_1^{q(n-q)}\right]$$
$$= \frac{hb}{l^2}\Bigg[\frac{E_{3311}}{G}\frac{(-(m-k))(\sqrt{3})^{n+m-1}}{(q+k+1)(n+m-q-k)}\, d^{q+k}\, c^{n+m-q-k-1}\, \frac{\partial(\bullet)}{\partial\xi_1}$$
$$+ \frac{E_{3113}}{G}\frac{(n-q)(\sqrt{3})^{n+m-1}}{(q+k+1)(n+m-q-k)}\, d^{q+k}\, c^{n+m-q-k-1}\, \frac{\partial(\bullet)}{\partial\xi_1}\Bigg] \tag{7.7}$$

$$(i,r) = (2,3): \; D_{\text{eq}}\left[v_2^{k(m-k)}, u_3^{q(n-q)}\right]$$
$$= \frac{hb}{l^2}\Bigg[\frac{E_{2233}}{G}\frac{(-k(n-q))(\sqrt{3})^{n+m-2}}{(q+k)(n+m-q-k)}\, d^{q+k-1}\, c^{n+m-q-k-1}\, (\bullet)$$
$$+ \frac{E_{2332}}{G}\frac{(-q(m-k))(\sqrt{3})^{n+m-2}}{(q+k)(n+m-q-k)}\, d^{q+k-1}\, c^{n+m-q-k-1}\, (\bullet)\Bigg] \tag{7.8}$$

$$(i,r) = (3,2): \; D_{\text{eq}}\left[v_3^{k(m-k)}, u_2^{q(n-q)}\right]$$
$$= \frac{hb}{l^2}\Bigg[\frac{E_{3322}}{G}\frac{(-q(m-k))(\sqrt{3})^{n+m-2}}{(q+k)(n+m-q-k)}\, d^{q+k-1}\, c^{n+m-q-k-1}\, (\bullet)$$
$$+ \frac{E_{3223}}{G}\frac{(-k(n-q))(\sqrt{3})^{n+m-2}}{(q+k)(n+m-q-k)}\, d^{q+k-1}\, c^{n+m-q-k-1}\, (\bullet)\Bigg] \tag{7.9}$$

Of course these terms have to be truncated for a specific Nth-order theory. Since the magnitude of each term increases with $n+m$, we always have a finite system of equations and unknown displacement coefficients for an Nth-order theory. Also, if the variations and displacement coefficients are sorted by increasing m and n, we will find only zeros at the lower right corner of the equation system (compare the tables of section 8.2), so that a characteristic "triangular-like" form appears. (It is not really a triangular form, since the number of variables for a specific n, or m, increases with n, or m, cf. section 7.7.)

We now take a first glance at the principle form of the formulas (7.2) for more general kinds of anisotropy. We will not give the explicit formulas by means of eqs. (7.3) to (7.9) for more general kinds of anisotropy, since this is beyond the focus of this work.

For the monoclinic materials, where the plane of symmetry is given by two coordinate-axes, we could only get a non-zero summand for (j,s) in (7.2), if

$$\mathcal{K}_j\mathcal{K}_s = \mathcal{K}_i\mathcal{K}_r \text{ or } \mathcal{K}_j\mathcal{K}_s = \mathcal{K}_i\mathcal{K}_r\mathcal{K}_{\text{mon}},$$

for a specific $\mathcal{K}_{\text{mon}} \neq \text{id}$, i.e., $\mathcal{K}_{\text{mon}} \in \{\mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_2\mathcal{K}_3\}$. This is for the specific $\mathcal{K}_{\text{mon}}$ equivalent to

$$\mathcal{K}_{\text{mon}} = \mathcal{K}_2:$$
$$\begin{cases} (j,s) \in G_{\text{id}} \cup \{(1,2),(2,1)\} & \text{if} \quad (i,r) \in G_{\text{id}} \cup \{(1,2),(2,1)\} \\ (j,s) \in \{(1,3),(3,1),(2,3),(3,2)\} & \text{if} \quad (i,r) \in \{(1,3),(3,1),(2,3),(3,2)\} \end{cases} \tag{7.10}$$

$$\mathcal{K}_{\text{mon}} = \mathcal{K}_3 :$$
$$\begin{cases} (j,s) \in G_{\text{id}} \cup \{(1,3),(3,1)\} & \text{if} \quad (i,r) \in G_{\text{id}} \cup \{(1,3),(3,1)\} \\ (j,s) \in \{(1,2),(2,1),(2,3),(3,2)\} & \text{if} \quad (i,r) \in \{(1,2),(2,1),(2,3),(3,2)\} \end{cases} \tag{7.11}$$

$$\mathcal{K}_{\text{mon}} = \mathcal{K}_2\mathcal{K}_3 :$$
$$\begin{cases} (j,s) \in G_{\text{id}} \cup \{(2,3),(3,2)\} & \text{if} \quad (i,r) \in G_{\text{id}} \cup \{(2,3),(3,2)\} \\ (j,s) \in \{(1,2),(2,1),(1,3),(3,1)\} & \text{if} \quad (i,r) \in \{(1,2),(2,1),(1,3),(3,1)\} \end{cases} \tag{7.12}$$

Of course, by the use of eqs. (7.10) to (7.12), we could generate formulas analogously to eqs. (7.3) to (7.9). Basically these formulas would be sums of two corresponding formulas of eqs. (7.3) to (7.9) (Basically!).

For a monoclinic material, where the plane of symmetry is not given by two-coordinate axes, or an aelotropic material, there are no restrictions for the tuple (j, s), so that we really get sums of nine summands, which are basically the sums of all four types of formulas (Basically!).

7.5 Prolog: The symmetry of the equilibrium conditions of three-dimensional linear elasticity

Let us recall the situation for the three-dimensional theory of linear elasticity. The weak problem (Wk) is equivalent to the problem of finding a strict minimizer (En) of the potential energy given by

$$E_{\text{pot}}(u + u_0) := \frac{1}{2}B(u + u_0, u + u_0) - F(u + u_0).$$

In the context of the calculus of variations, the weak problem is the condition that the first variation of the elastic potential

$$\delta E_{\text{pot}}(u + u_0; v) = B(u + u_0, v) - F(v)$$

vanishes for all v in a proper function space X_0, i.e.,

$$B(u + u_0, v) = F(v) \quad \text{f.a. } v \in X_0.$$

The functions v are called *test functions* in the context of the weak-solution theory, *directions* or *variations* in the context of variational calculus and *virtual displacements* in the context of engineering mechanics. The classical equilibrium conditions (3.10) are gained by applying integration by parts to the weak formulation, inserting the stress boundary conditions and applying the variational lemma. In the context of variational calculus they are the Euler-Lagrange equations, which have to be satisfied for every local minimizer of the elastic potential.

The symmetry we want to consider in the next subsection is induced by the symmetry of the bilinear form B

$$B(u, v) = B(v, u),$$

which is obvious by the definition of B and the symmetry of E. If we consider the equilibrium condition for a fixed i (which was gained by variation with respect to v_i) as a sum of differential operators $D_{\text{eq}}[v_i, u_r]$ applied to the component functions of the displacement u_r

$$\sigma_{ij|j} = E_{ijrs}u_{r|sj} = \sum_{r=1}^{3} \underbrace{\sum_{j,s=1}^{3} E_{ijrs}D_sD_j(u_r)}_{=:D_{\text{eq}}[v_i,u_r]} = -f_i,$$

where $D_i(\bullet) := \bullet_{|i}$, we find the symmetry relation

$$D_{\text{eq}}[v_r, u_i] = \sum_{j,s=1}^{3} E_{rjis} D_s D_j = \sum_{s,j=1}^{3} E_{rsij} D_j D_s = \sum_{s,j=1}^{3} E_{ijrs} D_s D_j = D_{\text{eq}}[v_i, u_r],$$

by the use of the symmetries of E. Note that this symmetry does **not** imply the symmetry of the first variation $\delta E_{\text{pot}}(u; v)$ because of F, or the right-hand side f_i of the equilibrium conditions, respectively.

The symmetry of the three-dimensional problem of linear elasticity is propagated from B, because the application of integration by parts to the weak formulation, which is used to gain the equilibrium conditions, is merely a changeover of the differential operator D_j. Therefore, we could expect a similar symmetry for the equilibrium conditions of the one-dimensional problem stated in terms of the displacement coefficients $u_r^{q(n-q)}$, if we would not have used integration by parts only for $j = 1$ during the derivation of the equilibrium conditions (5.21) in section 5.4. Of course this treatment was crucial for the modeling of a one-dimensional theory, since we only got boundary conditions at the cross sections for $\xi_1 = 0$ and $\xi_1 = 1$, while the other three-dimensional stress boundary conditions are transformed into driving forces of the field equations, cf. (4.15). However, it is surprising that the symmetry relation among the differential operators is (sort of) preserved for at least some kinds of anisotropy, which will be investigated in the next subsection.

7.6 The symmetry of the one-dimensional equilibrium conditions

If we denote the summands of the definition of D_{eq} in (7.2) by ω_{js}

$$\begin{aligned} &D_{\text{eq}}\left[v_i^{k(m-k)}, u_r^{q(n-q)}\right](\bullet) \\ &= \sum_{j,s=1}^{3} \underbrace{\chi_j \frac{E_{ijrs}}{G} \left(\int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} a_j^{23}\left[\xi^{k(m-k)}\right] a_s^{23}\left[\xi^{q(n-q)}\right] d_{\xi_3} d_{\xi_2} \right) D_j^1 D_s^1(\bullet)}_{=:\omega_{js}}, \end{aligned}$$

we find by renaming of the summation indices, reordering of the summands, using the symmetries of E, using the permutabilities of the product in the integrand and the differential operators, and finally by the use of the identity (5.3)

$$\begin{aligned} &D_{\text{eq}}\left[v_r^{q(n-q)}, u_i^{k(m-k)}\right](\bullet) \\ &= \sum_{j,s=1}^{3} \frac{E_{rjis}}{G} \left(\int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} \chi_j a_j^{23}\left[\xi^{q(n-q)}\right] a_s^{23}\left[\xi^{k(m-k)}\right] d_{\xi_3} d_{\xi_2} \right) D_j^1 D_s^1(\bullet) \\ &= \sum_{s,j=1}^{3} \frac{E_{rsij}}{G} \left(\int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} \chi_s a_s^{23}\left[\xi^{q(n-q)}\right] a_j^{23}\left[\xi^{k(m-k)}\right] d_{\xi_3} d_{\xi_2} \right) D_s^1 D_j^1(\bullet) \\ &= \sum_{j,s=1}^{3} \frac{E_{ijrs}}{G} \left(\int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} a_j^{23}\left[\xi^{k(m-k)}\right] \chi_s a_s^{23}\left[\xi^{q(n-q)}\right] d_{\xi_3} d_{\xi_2} \right) D_j^1 D_s^1(\bullet) \\ &= \sum_{j,s=1}^{3} \chi_j \chi_s \underbrace{\chi_j \frac{E_{ijrs}}{G} \left(\int_{-\frac{b}{2l}}^{\frac{b}{2l}} \int_{-\frac{h}{2l}}^{\frac{h}{2l}} a_j^{23}\left[\xi^{k(m-k)}\right] a_s^{23}\left[\xi^{q(n-q)}\right] d_{\xi_3} d_{\xi_2} \right) D_j^1 D_s^1(\bullet)}_{=\omega_{js}}, \end{aligned} \tag{7.13}$$

where

$$\chi_j \chi_s = \begin{cases} 1 & \text{if } (j,s) \in G_{\text{id}} \cup \{(2,3),(3,2)\} \\ -1 & \text{if } (j,s) \in \{(1,2),(2,1),(1,3),(3,1)\}. \end{cases}$$

For orthotropic material we have furthermore

$$\omega_{js} = 0 \Leftarrow \begin{cases} (j,s) \notin G_{\text{id}} & \text{if} \quad (i,r) \in G_{\text{id}} \\ (j,s) \notin \{(i,r),(r,i)\} & \text{if} \quad (i,r) \notin G_{\text{id}}, \end{cases}$$

and therefore

$$\begin{cases} D_{\text{eq}}\left[v_i^{k(m-k)}, u_r^{q(n-q)}\right] = D_{\text{eq}}\left[v_r^{q(n-q)}, u_i^{k(m-k)}\right] & \text{if } (i,r) \in G_{\text{id}} \cup \{(2,3),(3,2)\} \\ D_{\text{eq}}\left[v_i^{k(m-k)}, u_r^{q(n-q)}\right] = -D_{\text{eq}}\left[v_r^{q(n-q)}, u_i^{k(m-k)}\right] & \text{if } (i,r) \in \{(1,2),(2,1),(1,3),(3,1)\}. \end{cases} \tag{7.14}$$

Note that the tuples (i,r) with the change of the sign are those, where precisely one index equals 1. Therefore, we could generate a symmetric ODE-system, if we multiply each equilibrium condition, which is gained by variation with respect to a virtual displacement v_1^{km} (with tensor index 1) by -1. In this case the change of the sign for the tuples $(1,2),(2,1),(1,3)$ and $(3,1)$ is compensated, since only one equation is multiplied by -1, while on the other hand, for the tuple $(1,1)$ both affected equations are multiplied by -1 in accordance with the symmetry relation for this tuple. The equations for all other tuples (i,r) are not affected, which is also in accordance with the already fulfilled symmetry-relation.

For the monoclinic materials, where the plane of symmetry is given by two coordinate-axes, we derive from eqs. (7.10) to (7.12) that only in the case of $\mathcal{K}_{\text{mon}} = \mathcal{K}_2$ it is possible to generate a symmetric system, since we also have (7.14), like in the case for orthotropic material. Therefore, the same approach as for orthotropic material generates a symmetric system, if we have a monotropic material, where the plane of symmetry is given by the cross section.

For the other kinds of monotropic material and also for aelotropy, we always have a mixed case of summands ω_{ij} that change their signs, and other summands that do not change their signs. Therefore, there is no relation analogous to (7.14) that holds for the whole sum, and it is not possible to generate symmetric systems for more general kinds of anisotropy. We sum up our findings in a theorem.

Theorem 22 (Symmetry of D_{eq})
If all non-zero components of E have associated effective shifts $\text{id}_{\mathbb{Z}_2^2}$ or $\mathcal{K}_2$

$$E_{ijrs} \neq 0 \Longrightarrow \left(\mathcal{K}_{\mathit{eff}}(i,j,r,s) = \text{id}_{\mathbb{Z}_2^2} \ \text{ or } \ \mathcal{K}_{\mathit{eff}}(i,j,r,s) = \mathcal{K}_2\right),$$

i.e., especially for

- *orthotropic material, where the planes of symmetry are given by coordinate axes,*
- *monoclinic material, where the plane of symmetry is given by the cross section,*

a symmetric ODE-system is generated, if we multiply each equilibrium condition, which is gained by variation with respect to a virtual displacement v_1^{km} (with tensor index 1) by -1.

If there is an $E_{ijrs} \neq 0$ with an effective shift that is not $\text{id}_{\mathbb{Z}_2^2}$ or $\mathcal{K}_2$, then it is impossible to generate a symmetric ODE-system.

7.7 The truncation of the series expansion of the displacement field for a Nth-order theory

In this subsection we want to derive which displacement coefficients $u_r^{q(n-q)}$ and virtual displacements $\delta v_i^{k(m-k)}$ are to be considered in an Nth-order theory.

An equilibrium condition, which is the variation with respect to $\delta v_i^{k(m-k)}$, has to be considered, if there is any $u_r^{q(n-q)}$, such that $D_{\text{eq}}\big[v_i^{k(m-k)}, u_r^{q(n-q)}\big]$ is not zero or to be neglected due to the consistent truncation. Likewise, a displacement coefficient $u_r^{q(n-q)}$ has to be considered, if there is any equilibrium condition, which is the variation with respect to $\delta v_i^{k(m-k)}$, such that $D_{\text{eq}}\big[v_i^{k(m-k)}, u_r^{q(n-q)}\big]$ is not zero or to be neglected due to the consistent truncation. We already now that the displacement coefficients and virtual displacements that have to be considered for a specific problem are generally "from the same classes", i.e., for problem (t_1, t_2) we have to consider the set of virtual displacements $\bigcup_{i=1}^3 \mathcal{K}_i\left[\delta v_i^{t_1 t_2}\right]$ and the set $\bigcup_{i=1}^3 \mathcal{K}_i\left[u_i^{t_1 t_2}\right]$ of displacement coefficients (cf. theorems 15 and 20). Furthermore, we know that the magnitude of $D_{\text{eq}}\big[v_i^{k(m-k)}, u_r^{q(n-q)}\big]$ depends on the sum of all four series (i.e., upper) indices, which is $n+m$, cf. (7.2) and (4.14). Therefore, it follows from (7.13) that the virtual displacements that have to be considered are the same as the displacement coefficients, i.e., *the equilibrium equation, which is the variation with respect to $\delta v_r^{q(n-q)}$, has to be considered if, and only if, $u_r^{q(n-q)}$ has to be considered.* So we always get a "quadric" system (i.e., the number of equations equals the number of unknowns) of ODEs for any Nth-order theory.

It is sufficient to investigate the case of an orthotropic material, since we know from theorem 20, how the subproblems are coupled with each other for a more general kind of anisotropy. In this case, the sets of displacement coefficients, or virtual displacements, that have to be considered are simply given by the union of the quantities that have to be considered in the case of uncoupled subproblems.

We start with the investigation of the 0-th order theories. From eqs. (7.3) to (7.9) we derive that $u_i^{k(m-k)}$ has to be considered if, and only if, there is a $u_r^{q(n-q)}$ with

$$\begin{aligned}
(i,r) \in G_{\text{id}}:\ & [n+m \le 2 \text{ and } (k \neq 0 \wedge q \neq 0) \vee ((m-k) \neq 0 \wedge (n-q) \neq 0)] \\
& \text{or } [n+m=0] \\
(i,r) \in \{(1,2),(2,1)\}:\ & n+m \le 1 \text{ and } (k \neq 0 \vee q \neq 0) \\
(i,r) \in \{(1,3),(3,1)\}:\ & n+m \le 1 \text{ and } ((n-q) \neq 0 \vee (m-k) \neq 0) \\
(i,r) \in \{(2,3),(3,2)\}:\ & n+m \le 2 \\
& \text{and } (k \neq 0 \wedge (n-q) \neq 0) \vee ((q \neq 0 \wedge (m-q) \neq 0).
\end{aligned} \tag{7.15}$$

Obviously we have in general to fulfill $n+m \le 2$. Beside pairs among the canonical class representatives, i.e. the set

$$\left\{\left(u_i^{\eta(t_1,t_2)}, u_r^{\eta(t_3,t_4)}\right)\middle| t_1, t_2, t_3, t_4 \in \mathbb{Z}_2,\ i, r \in \{1,2,3\}\right\},$$

there are only the pairs (u_i^{02}, u_r^{00}) and (u_i^{20}, u_r^{00}) with $n+m \le 2$, but they both violate the additional conditions of (7.15). By testing all canonical class representatives with each other, we find that the representatives of type u_i^{11}, $i \in \{1,2,3\}$ *are precisely those that we have not to consider in the* 0*th-order theories*, i.e., we have to consider

$$\left\{u_i^{k(m-k)}\middle|(k, m-k) \in \{(0,0),(0,1),(1,0)\},\ i \in \{1,2,3\}\right\}.$$

For the first-order theories only the right-hand sides of the inequalities in (7.15) have to be increased by 2, while the additional conditions remain unchanged. In general, the right-hand sides have to be increased by $2N$ for an Nth-order theory. This implies an iteration rule: If $u_r^{q(n-q)}$ has to be considered in an Nth-order theory, then $u_r^{q(n-q)}$ itself, as well as $u_r^{q+2(n-q)}$ and $u_r^{q(n-q+2)}$, have to be considered in a $(N+1)$th-order theory (compare eqs. (7.3) to (7.9)). Therefore, the representatives of the form u_i^{11} have to be considered in the first-order theories, since both indices of u_i^{11} are non-zero and from the other displacement classes, which are already present in the 0th-order theories, the coefficients with the additional summand 2 could be used to fulfill any of the additional conditions of (7.15). We denote for every problem class (t_1, t_2) the orders $O_{\min}(i)$ of the theory, where the canonical representatives of the class $\mathcal{K}_i[u_i^{t_1 t_2}]$ (cf. theorem 20), with the tensor index i, has to be considered for the first time in table 5 (cf. table 1). Once the

class	(t_1, t_2)	$O_{\min}(1)$	$O_{\min}(2)$	$O_{\min}(3)$
S	(e, e)	0	0	0
B2	(o, e)	0	0	1
B3	(e, o)	0	1	0
T	(o, o)	1	0	0

Table 5: $O_{\min}(i)$ denotes the order of the theory where the canonical representatives of the class $\mathcal{K}_i[u_i^{t_1 t_2}]$ have to be considered for the first time.

canonical class representative for a specific i has to be considered in an Nth-order theory, we find all displacement coefficients with tensor index i, that have to be considered in an $(N+1)$th-order theory, by application of the iteration rule. Therefore, the series expansion for the displacement field of the consistent Nth-order theory of problem (t_1, t_2) could be selected as the finite sum

$$\frac{u_i}{l} = \sum_{m=0}^{N-O_{\min}(i)} \sum_{k=0}^{m} u_i^{\eta(\mathcal{K}_i(t_1,t_2))+(2k,2(m-k))} \xi^{\eta(\mathcal{K}_i(t_1,t_2))+(2k,2(m-k))}, \tag{7.16}$$

since we only make an error of order $O(e^{2(N+1)})$ by doing so, cf. theorem 13.

As an example we get for the $B3$-theory with $N = 2$:

$$\begin{aligned} i = 1 &: u_1^{\eta(\mathcal{K}_1(e,o))} = u_1^{\eta(e,o)} = u_1^{01}, \\ i = 2 &: u_2^{\eta(\mathcal{K}_2(e,o))} = u_2^{\eta(o,o)} = u_2^{11}, \\ i = 3 &: u_3^{\eta(\mathcal{K}_3(e,o))} = u_3^{\eta(e,e)} = u_3^{00}, \end{aligned}$$

and therefore

$$\begin{array}{lllll} i=1: & u_1/l = & u_1^{01}\xi_3^1 & +u_1^{03}\xi_3^3 + u_1^{21}\xi_2^2\xi_3^1 & +u_1^{05}\xi_3^5 + u_1^{23}\xi_2^2\xi_3^3 + u_1^{41}\xi_2^4\xi_3^1, \\ i=2: & u_2/l = & 0 & +u_2^{11}\xi_2^1\xi_3^1 & +u_2^{13}\xi_2^1\xi_3^3 + u_2^{31}\xi_2^3\xi_3^1, \\ i=3: & u_3/l = & \underbrace{u_3^{00}}_{\text{0th-order}} & \underbrace{+u_3^{02}\xi_3^2 + u_3^{20}\xi_2^2}_{\text{to add for 1st-order}} & \underbrace{+u_3^{04}\xi_3^4 + u_3^{22}\xi_2^2\xi_3^2 + u_3^{40}\xi_2^4.}_{\text{to add for 2nd-order}} \end{array}$$

Some of the displacement coefficients are illustrated in fig. 8. For the number of unknown displacement coefficients, which coincides with the number of equations, we get

$$\sum_{i=1}^{3} \sum_{m=0}^{N-O_{\min}(i)} (m+1) = \sum_{i=1}^{3} \frac{(N+1-O_{\min}(i))(N+2-O_{\min}(i))}{2}.$$

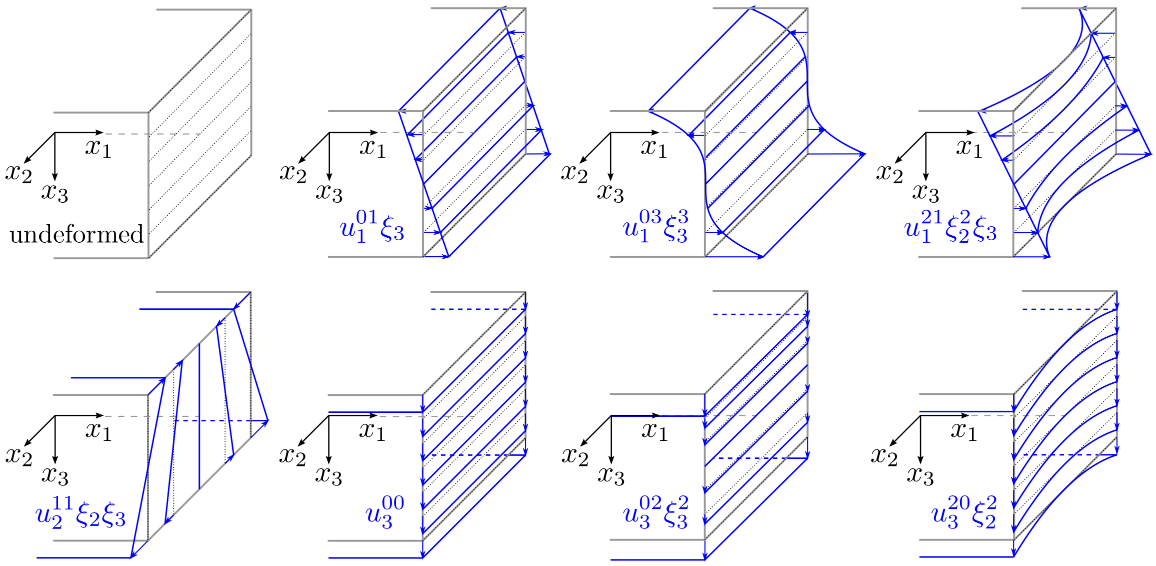

Figure 8: Illustration of some displacement coefficients that belong to the beam problem.

So the number of equations and unknowns of the Nth-order theories grows like $O(N^2)$. For $N = 1, 2, 3$ the specific numbers are displayed in figure 6. It is noteworthy that the problem S has in general more unknown displacement coefficients than all other problems, since it does not contain a class of type u_i^{oo}.

	$N=0$	$N=1$	$N=2$	$N=3$
$B2$, $B3$, T:	2	7	15	26
S:	3	9	18	30

Table 6: Number of equations and unknown displacement coefficients for theories of order $N \leq 3$.

7.8 The consistency of the load-resultant truncation

We already derived in section 7.7 that among the canonical class representatives, the representatives of type u_i^{11}, $i \in \{1, 2, 3\}$, are precisely those which are not to be considered in the 0-th order theories. This means

$$O_{\min}(i) = \begin{cases} 1 & \text{if} \quad \eta\left(\mathcal{K}_i(t_1, t_2)\right) = (1, 1) \\ 0 & \text{otherwise} \end{cases} .$$

Therefore, by equation (7.16), all displacement coefficients u_i^{kl} and virtual displacements δv_i^{kl}, which have to be considered in an Nth-order theory, fulfill

$$k + l \leq 2N + 1.$$

By equation (5.21) the load resultants p_i^{kl} driving the field equations that have to be considered also fulfill the inequality above.

On the other hand, in order to gain the desired accuracy of $O(e^{2(N+1)})$, the load resultants (4.21) have to be truncated at order $O(e^{2(N+1)})$, by theorem 13. This is achieved by truncating the infinite sums over n in (4.21) for the load resultant $p_i^{k(m-k)}$ at $n = 2N + 1 - m$. This implies

$$m > 2N + 1 \Longrightarrow p_i^{k(m-k)} = 0 + O(e^{2(N+1)}).$$

Therefore, only the equations to be considered have right-hand sides that are not to be neglected in an Nth-order approximation, i.e. the truncation of the load resultants is consistent with the truncation criteria of the field equations.

7.9 The equivalence of the problems $B2$ and $B3$

Since the problems $B2$ and $B3$ describe a beam problem, one would expect the two problems to be "equivalent", if the problems are decoupled from the other ones, e.g. if we have an orthotropic material. Equivalent in this setting means that any equation of the $B2$-problem could be transformed into a corresponding equation of the $B3$-problem and vice versa. In fact this is true and will be shown in this subsection.

In principle one only has to interchange the coordinate axes ξ_2 and ξ_3 to interchange the problems $B2$ and $B3$, i.e., every occurrence of a tensor index that equals 2 has to be exchanged with 3 and vice versa. Because of the notational agreements we made, this would also interchange the meaning of h and b and, therefore, also c and d (compare section 4.2). Also the first index of a displacement coefficient would refer to the power of ξ_3 and not ξ_2 (compare (4.12)). Therefore, if we want to derive the ODEs of one of the problems given the ODEs of the other one (where the components of the tensor E are not already inserted), while using the same notation, we have to interchange:

- the tensor indices 2 and 3,
- c and d
- and the order of the indices of the displacement coefficients.

As an example: Given the tables of section 8.2 (of the $B3$-problem), one would have to interchange the displacement coefficients in the first line to those of the $B2$-problem, by interchanging the order of the series indices and interchanging every 2 as a tensor index with a 3 and vice versa. The same has to be performed with the load resultants. For the entries of the tables one would only have to interchange c and d, since the table is only for isotropic material. This would generate the table for the $B2$-problem. Likewise, by applying all changes once more, we would arrive at the original table of the $B3$-problem again.

In order to verify that these procedure in fact leads to the right problem, we could perform these changes at the eqs. (7.3) to (7.9). In detail: Select two differential operators that should interchange into one another. For example: The operators for $(i, r) = (2, 1)$ and $(i, r) = (3, 1)$, or twice $(i, r) \in G_{\text{id}}$, or $(i, r) = (2, 3)$ and $(i, r) = (3, 2)$. Take one of the operators and interchange: k with $m - k$, q with $n - q$, c with d and 2 with 3. The resulting operator now equals the second one, or is invariant under this change, in the case that $(i, r) \in G_{\text{id}}$ was selected, respectively. Therefore, the described substitution method indeed generates the correct problem.

8 The second-order $B3$-theory

In this section we want to treat the second-order $B3$-problem (with problem identifier (e, o)) for isotropic material (2.16).

8.1 Definition of the $B3$-problem

One of the greatest merits of the sections 6 and 7 is that they allow for rigorous definitions of the most general load cases for one-dimensional problems. The definition of the $B3$-problem, we derived is: a "quasi" one-dimensional (cf. section 4.2) thin (A3) structure, that is subject to

- displacement boundary conditions u_0,
- volume forces f,
- boundary tractions g,

that have in ξ_i-direction the parities given by $\mathcal{K}_i(e, o)$, i.e. whose component functions:

- in ξ_1-direction are *even* in ξ_2-direction and *odd* in ξ_3-direction,
- in ξ_2-direction are *odd* in ξ_2-direction and *odd* in ξ_3-direction,
- in ξ_3-direction are *even* in ξ_2-direction and *even* in ξ_3-direction

(cf. theorem 21 and tables 1, 3 and 4). The displacement field u that results from this problem then in turn also fulfills the symmetry relations stated above, due to theorem 21.

Since the definition is valid for the exact (untruncated) $B3$-problem, it is also valid for any Nth-order consistent approximation.

Note that this is indeed a generalization of the classical orthogonal decomposition into resulting forces and moments. Not only that it turned out that each problem is driven by loads in any direction (with appropriate symmetry), in addition the classical decompositions lack in general a whole class of load resultants. An exemplary illustration for that fact is given by fig. 9. The loads of the class p_3^{ee} are the ones that have resulting forces, but no moments. The loads of the class p_1^{eo} are the bending moments, which do not have a resulting force. But loads from the trapezoidal-stress-resultant class p_2^{oo} do neither have a resulting force or moment. Nevertheless,

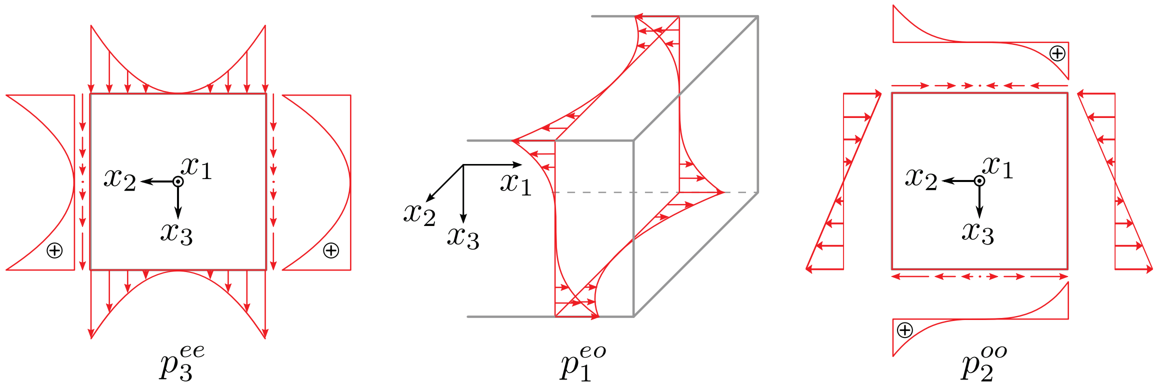

Figure 9: Exemplary illustration of one specific boundary traction-field, for a fixed longitudinal coordinate x_1, for each of the three load resultant classes that belong to the $B3$-problem.

they are proper driving forces for the beam problem and have an effect. The resulting trapezoidal displacements u_2^{oo} (cf. fig. 8 for an illustration) indeed belong to the beam problem and are also generated by a pure bending load case. We like to emphasize that this transversal deformation is not at all a theoretical construct, it is actually measurable and considered for the placement of strain sensors in practice (cf., e.g., Gevatter, 2000, pages 39–40). Furthermore, the twice-odd load resultant classes p_i^{oo} are essential so that *any* three-dimensional load case can be *uniquely* decomposed into the driving forces of the four subproblems, i.e. they are essential for a load-case decomposition that is actually compatible to the three-dimensional theory of linear elasticity.

8.2 The second-order field equations

The field equations that have to be generated for a second-order approximation, i.e. the corresponding virtual displacements $\delta v_i^{k(m-k)}$, as well as the displacement coefficients that have to be considered can be computed a-priori as described in section 7.7. Then the equations may be derived most conveniently by evaluation of equation (7.1), i.e. eqs. (7.3) to (7.9), and truncation of the equations at order $O(e^6)$. The tables on pages 107 and 108 show us the system of the 15 linear second-order (differential order) ODEs, which are the field equations of the second-order (approximation order) $B3$-problem. They are written in terms of the 15 unknown displacement coefficients u_i^{kl} of the first lines of the two tables. Every line below corresponds to one ODE that is the sum of all terms (across both tables) and is generated by insertion of the displacement coefficient of the first line into the tables entry below. The symbol $()'$ denotes differentiation with respect to ξ_1, whereas all summands of a table entry that lack a differentiation symbol are to be multiplied with the displacement coefficient above. The column RHS contains the right hand sides. The second column of the first table contains the virtual displacements $\delta v_i^{k(m-k)}$ that correspond to the equation of the same line. The sign before the virtual displacement indicates whether or not the equation was multiplied by -1 in order to get a symmetric ODE system, cf. theorem 22. For convenience the equations are labeled with (a)-(o). The labels are repeated in the first column of the first table and the last column of the second table. For brevity the equations are multiplied with l^2/bh, since this factor occurs in every stress resultant, cf. (4.14).

		u_3^{00}	u_1^{01}	u_2^{11}	u_3^{02}	u_3^{20}	u_1^{03}	u_1^{21}
(a)	$+\delta v_3^{00}$	$()''$	$()'$	0	$c^2()''$	$d^2()''$	$3c^2()'$	$d^2()'$
(b)	$-\delta v_1^{01}$	$()'$	$1-2\frac{1-\nu}{1-2\nu}c^2()''$	$-\frac{2\nu}{1-2\nu}c^2()'$	$c^2()'-2\frac{2\nu}{1-2\nu}c^2()'$	$d^2()'$	$3c^2-2\frac{1-\nu}{1-2\nu}\frac{9}{5}c^4()''$	$d^2-2\frac{1-\nu}{1-2\nu}c^2d^2()''$
(c)	$+\delta v_2^{11}$	0	$-\frac{2\nu}{1-2\nu}c^2()'$	$-2\frac{1-\nu}{1-2\nu}c^2-d^2+c^2d^2()''$	$-2\frac{2\nu}{1-2\nu}c^2$	$-2d^2$	$-\frac{2\nu}{1-2\nu}\frac{9}{5}c^4()'$	$-\frac{2\nu}{1-2\nu}c^2d^2()'+2c^2d^2()'$
(d)	$+\delta v_3^{02}$	$c^2()''$	$c^2()'-2\frac{2\nu}{1-2\nu}c^2()'$	$-2\frac{2\nu}{1-2\nu}c^2$	$\frac{9}{5}c^4()''-8\frac{1-\nu}{1-2\nu}c^2$	$c^2d^2()''$	$3\frac{9}{5}c^4()'-2\frac{2\nu}{1-2\nu}\frac{9}{5}c^4()'$	$c^2d^2()'-2\frac{2\nu}{1-2\nu}c^2d^2()'$
(e)	$+\delta v_3^{20}$	$d^2()''$	$d^2()'$	$-2d^2$	$c^2d^2()''$	$\frac{9}{5}d^4()''-4d^2$	$3c^2d^2()'$	$\frac{9}{5}d^4()'$
(f)	$-\delta v_1^{03}$	$3c^2()'$	$3c^2-2\frac{1-\nu}{1-2\nu}\frac{9}{5}c^4()''$	$-\frac{2\nu}{1-2\nu}\frac{9}{5}c^4()'$	$3\frac{9}{5}c^4()'-2\frac{2\nu}{1-2\nu}\frac{9}{5}c^4()'$	$3c^2d^2()'$	$9\frac{9}{5}c^4$	$3c^2d^2$
(g)	$-\delta v_1^{21}$	$d^2()'$	$d^2-2\frac{1-\nu}{1-2\nu}c^2d^2()''$	$2c^2d^2()'-\frac{2\nu}{1-2\nu}c^2d^2()'$	$c^2d^2()'-2\frac{2\nu}{1-2\nu}c^2d^2()'$	$\frac{9}{5}d^4()'$	$3c^2d^2$	$4c^2d^2+\frac{9}{5}d^4$
(h)	$+\delta v_2^{13}$	0	$-\frac{2\nu}{1-2\nu}\frac{9}{5}c^4()'$	$-3c^2d^2-2\frac{1-\nu}{1-2\nu}\frac{9}{5}c^4$	$-2\frac{2\nu}{1-2\nu}\frac{9}{5}c^4$	$-6c^2d^2$	0	0
(i)	$+\delta v_2^{31}$	0	$-3\frac{2\nu}{1-2\nu}c^2d^2()'$	$-6\frac{1-\nu}{1-2\nu}c^2d^2-\frac{9}{5}d^4$	$-6\frac{2\nu}{1-2\nu}c^2d^2$	$-2\frac{9}{5}d^4$	0	0
(j)	$+\delta v_3^{04}$	$\frac{9}{5}c^4()''$	$\frac{9}{5}c^4()'-4\frac{2\nu}{1-2\nu}\frac{9}{5}c^4()'$	$-4\frac{2\nu}{1-2\nu}\frac{9}{5}c^4$	$-16\frac{1-\nu}{1-2\nu}\frac{9}{5}c^4$	0	0	0
(k)	$+\delta v_3^{40}$	$\frac{9}{5}d^4()''$	$\frac{9}{5}d^4()'$	$-4\frac{9}{5}d^4$	0	$-8\frac{9}{5}d^4$	0	0
(l)	$+\delta v_3^{22}$	$c^2d^2()''$	$c^2d^2()'-2\frac{2\nu}{1-2\nu}c^2d^2()'$	$-2c^2d^2-2\frac{2\nu}{1-2\nu}c^2d^2$	$-8\frac{1-\nu}{1-2\nu}c^2d^2$	$-4c^2d^2$	0	0
(m)	$-\delta v_1^{05}$	$5\frac{9}{5}c^4()'$	$5\frac{9}{5}c^4$	0	0	0	0	0
(n)	$-\delta v_1^{23}$	$3c^2d^2()'$	$3c^2d^2$	0	0	0	0	0
(o)	$-\delta v_1^{41}$	$\frac{9}{5}d^4()'$	$\frac{9}{5}d^4$	0	0	0	0	0

u_2^{13}	u_2^{31}	u_3^{04}	u_3^{40}	u_3^{22}	u_1^{05}	u_1^{23}	u_1^{41}	RHS	
0	0	$\frac{9}{5}c^4()''$	$\frac{9}{5}d^4()''$	$c^2d^2()''$	$5\frac{9}{5}c^4()'$	$3c^2d^2()'$	$\frac{9}{5}d^4()'$	$-\frac{l^2}{bh}p_3^{00}$	(a)
$-\frac{2\nu}{1-2\nu}\frac{9}{5}c^4()'$	$-3\frac{2\nu}{1-2\nu}c^2d^2()'$	$\frac{9}{5}c^4()' - 4\frac{2\nu}{1-2\nu}\frac{9}{5}c^4()'$	$\frac{9}{5}d^4()'$	$c^2d^2()' - 2\frac{2\nu}{1-2\nu}c^2d^2()'$	$5\frac{9}{5}c^4$	$3c^2d^2$	$\frac{9}{5}d^4$	$\frac{l^2}{bh}p_1^{01}$	(b)
$-3c^2d^2 - 2\frac{1-\nu}{1-2\nu}\frac{9}{5}c^4$	$-\frac{9}{5}d^4 - 6\frac{1-\nu}{1-2\nu}c^2d^2$	$-4\frac{2\nu}{1-2\nu}\frac{9}{5}c^4$	$-4\frac{9}{5}d^4$	$-2c^2d^2 - 2\frac{2\nu}{1-2\nu}c^2d^2$	0	0	0	$-\frac{l^2}{bh}p_2^{11}$	(c)
$-2\frac{2\nu}{1-2\nu}\frac{9}{5}c^4$	$-6\frac{2\nu}{1-2\nu}c^2d^2$	$-16\frac{1-\nu}{1-2\nu}\frac{9}{5}c^4$	0	$-8\frac{1-\nu}{1-2\nu}c^2d^2$	0	0	0	$-\frac{l^2}{bh}p_3^{02}$	(d)
$-6c^2d^2$	$-2\frac{9}{5}d^4$	0	$-8\frac{9}{5}d^4$	$-4c^2d^2$	0	0	0	$-\frac{l^2}{bh}p_3^{20}$	(e)
0	0	0	0	0	0	0	0	$\frac{l^2}{bh}p_1^{03}$	(f)
0	0	0	0	0	0	0	0	$\frac{l^2}{bh}p_1^{21}$	(g)
0	0	0	0	0	0	0	0	$-\frac{l^2}{bh}p_2^{13}$	(h)
0	0	0	0	0	0	0	0	$-\frac{l^2}{bh}p_2^{31}$	(i)
0	0	0	0	0	0	0	0	$-\frac{l^2}{bh}p_3^{04}$	(j)
0	0	0	0	0	0	0	0	$-\frac{l^2}{bh}p_3^{40}$	(k)
0	0	0	0	0	0	0	0	$-\frac{l^2}{bh}p_3^{22}$	(l)
0	0	0	0	0	0	0	0	$\frac{l^2}{bh}p_1^{05}$	(m)
0	0	0	0	0	0	0	0	$\frac{l^2}{bh}p_1^{23}$	(n)
0	0	0	0	0	0	0	0	$\frac{l^2}{bh}p_1^{41}$	(o)

8.3 The pseudo reduction of the second-order ODE system

As outlined in section 8.1, a result of the sections 6 and 7 is that it is *wrong* that a beam is only loaded in ξ_3-direction. However, since normal beam theories only pay regard to loads in ξ_3-direction, we also consider only load resultants that act in ξ_3-direction $p_3^{k(m-k)}$ in order to derive comparable results. For the generalized theory one has to pay respect to all right-hand sides given in the tables of section 8.2.

These tables give us a complete and treatable description of the second-order $B3$-beam bending problem field equations. Because of the linearity of these ODEs, we get an equivalent ODE system, if we replace the original system by linear combinations of the original equations in such a way that the transformation matrix for the system is invertible. Therefore, it seems natural to seek for an easier representation of the system in which not every equation depends on every displacement coefficient. The "triangular" form (cf. section 7.4) of the system suggests, furthermore, that it might be possible to reduce the system to a single main ODE in only one main variable (which is to be solved) and a set of reduction equations (which express all other displacement coefficients in terms of the main variable), by successive elimination from the bottom to the top of the system. This *pseudo reduction* is indeed possible and will be performed in this section. The reduced modeling ODE will have a form very similar to classical beam theories. They are formulated only in the transversal displacement of the cross-section u_3^{00}, which will also be our main variable. Therefore, we introduce $w := u_3^{00}$ as an abbreviation.

We have to consider the fact that the equations of the tables of section 8.2 are actually truncated power series ($+O(e^6)$). Therefore, we can (in general) not perform multiplications or divisions by the parameters c and d without changing the accuracy of the equations. The resulting main ODE and all intermediate equations shall all be uniform approximations, i.e. accurate except for terms of order $O(e^6)$. As a consequence, we have to treat products of different magnitudes of characteristic parameters with the same displacement coefficient as formally independent variables during the elimination. (A neat example will be given in the next paragraph.) In order to preserve the original magnitude of the scaling factor, we introduce variables of the form enu_i^{kl} where u_i^{kl} indicates the displacement coefficient and n is the common power of the scaling factor e^n. Whenever an equation of this form is multiplied by a scaling factor $d^k c^l$ in a way so that the common power $k + l$ matches n, the equation is correct except for terms of order $O(e^6)$.

For example, the ODEs (m), (n) and (o) give us

$$e4u_1^{01} = -(e4w)'. \tag{8.1}$$

Multiplication with the factor $3c^2d^2$ of correct magnitude leads to the correct second-order equation (n):

$$3c^2d^2(u_3^{00})' + 3c^2d^2u_1^{01} = 0 + O(e^6).$$

Equation (8.1) can, e.g. be used for the insertion in equation (l) in order to eliminate $c^2d^2(u_1^{01})'$. The same is true for the equations (h) - (k), or the second summands in u_1^{01} in the equations (f) and (g). It can *not* be used for the insertion in equation (e) or the first summands in u_1^{01} in the equations (f) and (g), since the resulting equations would not be uniformly approximated, i.e. not accurate except for terms of order $O(e^6)$. Nevertheless, one can multiply equations by factors of characteristic parameters and neglect again all terms of the magnitude $O(e^6)$ to generate valid second-order equations, which are in general linear independent from the original equation with respect to the enu_i^{kl}-variables. In this sense equation e^2(f) is linear independent from equation

(f) and e^2(g) is linear independent from equation (g). In fact e^2(f) and e^2(g) are equivalent to (8.1), i.e., the system (m), (n), (o), e^2(f), e^2(g) is of rank 1.

One could also generate an infinite number of new second-order equations by divisions of factors of characteristic parameters, if the orignial equation is computed with accordingly higher approximation order a-prori. We found, however, that the resulting system becomes inconsistent by doing so, whereas we will see that all (only finitely many of them are not trivial) equations that can be generated by multiplications are in accordance with the original equations.

By building the difference of equation (a) multiplied with e^4 and the differentiated (with respect to ξ_1) equation (b) multiplied with e^4, i.e., by equation e^4(a)-e^4(b)$'$, we derive $e^4 p_3^{00} = O(e^6)$. This indicates that the load resultant p_3^{00} is already of magnitude $O(e^2)$. With regard to (4.21) we consequently assume

$$p_i^{lk} = O(e^{l+k+2}), \tag{8.2}$$

which in turn leads to the negligence of p_3^{40}, p_3^{22} and p_3^{04}. Only the right-hand sides of (a), (d) and (e), which are p_3^{00}, p_3^{20} and p_3^{02}, are considered.

The system (h), (i), (j), (k), (l), e^2(c), e^2(d), e^2(e) is by insertion of (8.1) a rank 3 system for the variables $e4u_2^{11}$, $e4u_3^{02}$ and $e4u_3^{20}$. Three linear independent equations are, e.g., (h), (i) and (l). Solving the systems yields

$$e4u_2^{11} = \nu(e4w)'', \tag{8.3}$$

$$e4u_3^{02} = \frac{1}{2}\nu(e4w)'', \tag{8.4}$$

$$e4u_3^{20} = -\frac{1}{2}\nu(e4w)'', \tag{8.5}$$

and the equations (j), (k), e^2(c), e^2(d) and e^2(e) are solved identically by insertion of eqs. (8.1) and (8.3) to (8.5).

By insertion of eqs. (8.1) and (8.3) to (8.5), the system (f), (g) and e^2(b) is a rank 3 system for the unknown variables $e2u_1^{01}$, $e4u_1^{03}$ and $e4u_1^{21}$. Solving the system gives us

$$e2u_1^{01} = -(e2w)' - 3(1+\nu)c^2(e4w)''' + 5\nu\frac{d^2}{d^2+5c^2}c^2(e4w)''', \tag{8.6}$$

$$e4u_1^{03} = \frac{\nu+2}{6}(e4w)''', \tag{8.7}$$

$$e4u_1^{21} = -\frac{1}{2}\nu\frac{5c^2-d^2}{d^2+5c^2}(e4w)'''. \tag{8.8}$$

Note that eq. (8.6) has to be multiplied by a factor of magnitude $O(e^2)$ to give a valid second-order equation.

As a next step the insertion of eqs. (8.4) to (8.8) into e^2(a) leads to

$$-2(1+\nu)c^2(e4w)'''' = -\frac{l^2}{hb}e2p_3^{00}. \tag{8.9}$$

Let I be the geometrical moment of inertia for a rectangular cross-section in x-coordinates

$$\frac{bh^3}{12} = I := \int_{A_x} x_3^2 \, dA_x = l^4 \int_{A_\xi} \xi_3^2 \, dA_\xi,$$

then we have

$$2(1+\nu)c^2 = \frac{E}{G}c^2 = \frac{E}{G}\frac{h^2}{12l^2} = \frac{E}{G}\frac{bh^3}{12l^4}\frac{l^2}{hb} = \frac{E}{G}\frac{I}{l^4}\frac{l^2}{hb} \tag{8.10}$$

and eq. (8.9) corresponds to

$$\frac{E}{G}\frac{I}{l^4}(e4w)'''' = e2p_3^{00}, \tag{8.11}$$

a dimensionless form of the Euler-Bernoulli equation for the displacement of the neutral axis multiplied by a factor of magnitude e^2. To convince ourselves of the correctness, let us assume that we only have tractions that are constant in ξ_2-direction at the top and bottom-side of the beam (no tractions on the other lateral sides and no volume force f). Then by equation (4.17) we have

$$\left[\frac{g_i}{G}\right]\left(\xi_1, \xi_2, \pm\frac{h}{2l}\right) \stackrel{!}{=} \sum_{n=0}^{\infty} g_i^{n\pm}(\xi_1)\xi_2^n = g_i^{0\pm}(\xi_1)$$

and the canonical beam load q has to be defined by

$$\begin{aligned} q(x_1) &:= \int_{-b/2}^{b/2} g_3(x_1, x_2, h/2)\, dx_2 + \int_{-b/2}^{b/2} g_3(x_1, x_2, -h/2)\, dx_2 \\ &= G\left(g_3^{0+}(x_1) + g_3^{0-}(x_1)\right) b, \end{aligned}$$

therefore, by equation (4.21)

$$p_3^{00}(\xi_1) = \left(g_3^{0+}(\xi_1) + g_3^{0-}(\xi_1)\right)\frac{b}{l} = \frac{q(\xi_1)}{Gl}$$

and by insertion into (8.11), insertion of $w(x_1) = u_3(x_1, 0, 0)/l$ (cf. (4.5) and (4.12)) and transformation to x-coordinates (cf. section 4.3), we get indeed

$$EI\,\frac{d^4\, u_3(x_1, 0, 0)}{dx_1^4} = q(x_1) + O(e^4), \tag{8.12}$$

i.e., the Euler-Bernoulli beam equation in classical notation.

Furthermore, eq. (8.9) indicates that the load resultants are of magnitude $O(e^2)$, i.e., $p_i^{lk} = O(e^2)$, again and it allows us to replace all derivatives of an order greater or equal than four of $e4w$ by the given load resultants.

After the insertion of eqs. (8.3) to (8.8), the equations (c), (d) and (e) form an underdetermined rank 3 system for the 8 variables $e2u_2^{11}$, $e2u_3^{02}$, $e2u_3^{20}$, $e4u_2^{13}$, $e4u_2^{31}$, $e4u_3^{04}$, $e4u_3^{40}$ and $e4u_3^{22}$, so that we could not derive a set of reduction equations presenting every variable independently from the others in terms of w and the given right-hand sides. However, this is not necessary for the pseudo reduction. We only have to be able to eliminate every occurrence of these variables in the remaining equations of the original system, i.e., equations (a) and (b), and also in every stress resultant $\mathcal{M}_{ij}^{kl}$, in order to get a theory entirely formulated in w. This is indeed possible. For a pleasant representation, we seek for a system of three linear combinations of the 8 variables that are free of material parameters, i.e., independent of ν, and could therefore be regarded as alternative variables ϕ_i. We transform the system $((c),(d),(e))^T$ by multiplication with the matrix

$$M := \begin{bmatrix} -\frac{1}{2}(1-\nu) & \frac{1}{4}\nu & \frac{1}{4}(1-\nu) \\ \frac{1}{4}\nu & -\frac{1}{8}(1-\nu) & -\frac{1}{8}\nu \\ 0 & 0 & \frac{1}{2} \end{bmatrix},$$

that is – because of $\det(M)=0 \Leftrightarrow \nu = 1/2$ – invertible for isotropic material, to obtain

$$\begin{aligned}
\phi_1 &:= d^2e2u_2^{11} + 3c^2d^2e4u_2^{13} + \frac{9}{5}d^4e4u_2^{31} + 2d^2e2u_3^{20} + \frac{36}{5}d^4e4u_3^{40} + 2c^2d^2e4u_3^{22} \\
&= -\left(1+\frac{\nu\,(3\,d^2+5\,c^2)}{d^2+5\,c^2}\right)c^2d^2(e4w)'''' \\
&\quad + \frac{1}{2}\frac{l^2}{hb}p_3^{20}, \qquad (8.13) \\
\phi_2 &:= c^2e2u_2^{11} + \frac{9}{5}c^4e4u_2^{13} + 3c^2d^2e4u_2^{31} \\
&= \nu c^2(e2w)'' + \frac{1}{10}\frac{c^2}{d^2+5c^2} \\
&\quad \left(\left(15\nu\,(10\nu+9)\,c^4 + \left(-20\nu^2+27\nu+25\right)c^2d^2 + \left(20\nu-30\nu^2+5\right)d^4\right)(e4w)''''\right) \\
&\quad - \frac{1}{4}\frac{l^2}{hb}\left(\nu p_3^{02} + (1-\nu)p_3^{20}\right), \qquad (8.14) \\
\phi_3 &:= c^2e2u_3^{02} + \frac{18}{5}c^4e4u_3^{04} + c^2d^2e4u_3^{22} \\
&= \frac{1}{2}\nu c^2(e2w)'' + \frac{1}{20}\frac{c^2}{d^2+5c^2} \\
&\quad \left(\left(\left(150\,\nu^2+120\,\nu-15\right)c^4 + \left(-20\,\nu^2-\nu-3\right)c^2d^2 - 5\,\nu\,(1+6\,\nu)\,d^4\right)(e4w)''''\right) \\
&\quad + \frac{1}{8}\frac{l^2}{hb}\left((1-\nu)p_3^{02} + \nu p_3^{20}\right). \qquad (8.15)
\end{aligned}$$

Note that by insertion of eq. (8.9) into eq. (8.13) ϕ_1 is completely determined by the given load resultants and therefore a known quantity, whereas ϕ_2 and ϕ_3 depend on $e2w$.

With the additional abbreviation

$$\begin{aligned}
\Phi &:= (e0w)' + e0u_1^{01} + c^2(e2u_3^{02})' + d^2(e2u_3^{20})' + 3c^2e2u_1^{03} + d^2e2u_1^{21} \\
&\quad + \frac{9}{5}c^4(e4u_3^{04})' + \frac{9}{5}d^4(e4u_3^{40})' + c^2d^2(e4u_3^{22})' \\
&\quad + 9c^4e4u_1^{05} + 3c^2d^2e4u_1^{23} + \frac{9}{5}d^4e4u_1^{41} \qquad (8.16)
\end{aligned}$$

and the definitions of ϕ_2 and ϕ_3 above equation (b) reads as

$$\begin{aligned}
&\Phi - \frac{2\nu}{1-2\nu}(\phi_2)' - \frac{4\nu}{1-2\nu}(\phi_3)' \\
&- \frac{1-\nu}{1-2\nu}\left(2c^2(e2u_1^{01})'' + \frac{18}{5}c^4(e4u_1^{03})'' + 2c^2d^2(e4u_1^{21})''\right) = 0. \qquad (8.17)
\end{aligned}$$

By insertion of eqs. (8.6) to (8.8), (8.14) and (8.15) into eq. (8.17) we obtain an eq. for Φ:

$$\begin{aligned}\Phi =& -2(1+\nu)c^2(e2w)''' \\ & -2(1+\nu)\frac{c^2}{d^2+5c^2}\left(3\,(5\,\nu+4)\,c^4-\frac{2}{5}\,(5\,\nu-6)\,c^2d^2-\nu\,\frac{3\,\nu+1}{1+\nu}d^4\right)(e4w)''''' \\ & +\frac{\nu}{2}\frac{l^2}{hb}\left(p_3^{02\prime}-p_3^{20\prime}\right).\end{aligned} \tag{8.18}$$

Since equation (a) simply reads as

$$(\Phi)' = -\frac{l^2}{hb}p_3^{00}$$

we can transform (a) into an equation entirely formulated in w by insertion of eq. (8.18). Furthermore, the sixth-order derivatives of $e4w$ can be written as second-order derivatives of the given load resultant p_3^{00} by the use of eq. (8.9), which leads us finally to the main ODE of the pseudo reduction

$$\begin{aligned}2(1+\nu)c^2(e2w)'''' =& \frac{l^2}{hb}\Bigg[p_3^{00}+\frac{\nu}{2}\left((p_3^{02})''-(p_3^{20})''\right)-\frac{1}{d^2+5c^2} \\ & \left(3\,(5\,\nu+4)\,c^4-\frac{2}{5}\,(5\,\nu-6)\,c^2d^2-\nu\,\frac{3\,\nu+1}{1+\nu}d^4\right)(e2p_3^{00})''\Bigg].\end{aligned} \tag{8.19}$$

By the use of eq. (8.10), we get

$$\begin{aligned}\frac{E}{G}\frac{I}{l^4}(w)'''' =& p_3^{00}+\frac{\nu}{2}\left((p_3^{02})''-(p_3^{20})''\right) \\ & -\frac{1}{d^2+5c^2}\left(3\,(5\,\nu+4)\,c^4-\frac{2}{5}\,(5\,\nu-6)\,c^2d^2-\nu\,\frac{3\,\nu+1}{1+\nu}d^4\right)(p_3^{00})'' \\ & +O(e^6).\end{aligned} \tag{8.20}$$

After solving the equation (8.20) all quantities of this section are derivable from the solution by differentiation. As already proved earlier in this subsection, by neglecting all terms of order $O(e^4)$, eq. (8.20) equals the classical Euler-Bernoulli equation.

8.4 The stress resultants of the second-order approximation

We already found a way to derive which stress resultants have to be considered in an Nth-order theory in section 5.3. By formula (5.12), it is independent of the tensor indices i and j, whether a stress resultant $\mathcal{M}_{ij}^{k(m-k)}$ has to be considered or not, and the upper index pairs $(k, m-k)$ that have to be considered for a second-order approximation are

m	$(k, m-k)$:			
0	(0,0)			
1	(1,0)	(0,1)		
2	(2,0)	(1,1)	(0,2)	
3	(3,0)	(2,1)	(1,2)	(0,3)
4	(4,0)	(2,2)	(0,4)	

By matching these pairs with the parity scheme $\mathcal{K}_i\mathcal{K}_j[\mathcal{M}_{ij}^{eo}]$ of the $B3$-problem, cf. table 2, we find the stress resultants of the second-order $B3$-problem to be

$$\begin{aligned}
\mathcal{M}_{11}^{eo}: &\quad \mathcal{M}_{11}^{01},\, \mathcal{M}_{11}^{21},\, \mathcal{M}_{11}^{03}\\
\mathcal{M}_{12}^{oo}: &\quad \mathcal{M}_{12}^{11}\\
\mathcal{M}_{13}^{ee}: &\quad \mathcal{M}_{13}^{00},\, \mathcal{M}_{13}^{20},\, \mathcal{M}_{13}^{02},\, \mathcal{M}_{13}^{40},\, \mathcal{M}_{13}^{22},\, \mathcal{M}_{13}^{04}\\
\mathcal{M}_{22}^{eo}: &\quad \mathcal{M}_{22}^{01},\, \mathcal{M}_{22}^{21},\, \mathcal{M}_{22}^{03}\\
\mathcal{M}_{23}^{oe}: &\quad \mathcal{M}_{23}^{10},\, \mathcal{M}_{23}^{30},\, \mathcal{M}_{23}^{12}\\
\mathcal{M}_{33}^{eo}: &\quad \mathcal{M}_{33}^{01},\, \mathcal{M}_{33}^{21},\, \mathcal{M}_{33}^{03}.
\end{aligned}$$

Indeed all of these stress resultants appear in the second-order equilibrium conditions (5.21):

$$\begin{aligned}
\mathcal{M}_{13,1}^{00} &= -p_3^{00}, && (a)\\
\mathcal{M}_{11,1}^{01} - \mathcal{M}_{13}^{00} &= -p_1^{01}, && (b)\\
\mathcal{M}_{12,1}^{11} - \mathcal{M}_{22}^{01} - \mathcal{M}_{23}^{10} &= -p_2^{11}, && (c)\\
\mathcal{M}_{13,1}^{02} - 2\mathcal{M}_{33}^{01} &= -p_3^{02}, && (d)\\
\mathcal{M}_{13,1}^{20} - 2\mathcal{M}_{23}^{10} &= -p_3^{20}, && (e)\\
\mathcal{M}_{11,1}^{03} - 3\mathcal{M}_{13}^{02} &= -p_1^{03}, && (f)\\
\mathcal{M}_{11,1}^{21} - 2\mathcal{M}_{12}^{11} - \mathcal{M}_{13}^{20} &= -p_1^{21}, && (g)\\
\cancelto{O(e^6)}{\mathcal{M}_{12,1}^{13}} - \mathcal{M}_{22}^{03} - 3\mathcal{M}_{23}^{12} &= -p_2^{13}, && (h)\\
\cancelto{O(e^6)}{\mathcal{M}_{12,1}^{31}} - 3\mathcal{M}_{22}^{21} - \mathcal{M}_{23}^{30} &= -p_2^{31}, && (i)\\
\mathcal{M}_{13,1}^{04} - 4\mathcal{M}_{33}^{03} &= -p_3^{04}, && (j)\\
\mathcal{M}_{13,1}^{40} - 4\mathcal{M}_{23}^{30} &= -p_3^{40}, && (k)\\
\mathcal{M}_{13,1}^{22} - 2\mathcal{M}_{23}^{12} - 2\mathcal{M}_{33}^{21} &= -p_3^{22}, && (l)\\
\cancelto{O(e^6)}{\mathcal{M}_{11,1}^{05}} - 5\mathcal{M}_{13}^{04} &= -p_1^{05}, && (m)\\
\cancelto{O(e^6)}{\mathcal{M}_{11,1}^{23}} - 2\cancelto{O(e^6)}{\mathcal{M}_{12}^{13}} - 3\mathcal{M}_{13}^{22} &= -p_1^{23}, && (n)\\
\cancelto{O(e^6)}{\mathcal{M}_{11,1}^{41}} - 4\cancelto{O(e^6)}{\mathcal{M}_{12}^{31}} - \mathcal{M}_{13}^{40} &= -p_1^{41}, && (o)
\end{aligned}$$

together with the five stress resultants

$$\mathcal{M}_{11}^{05} \overset{O(e^6)}{=} \mathcal{M}_{11}^{23} \overset{O(e^6)}{=} \mathcal{M}_{11}^{41} \overset{O(e^6)}{=} \mathcal{M}_{12}^{13} \overset{O(e^6)}{=} \mathcal{M}_{12}^{31} \overset{O(e^6)}{=} 0$$

that have to be neglected due to formula (5.12). By (8.2) we immediately derive

$$\mathcal{M}_{13}^{40} \overset{O(e^6)}{=} \mathcal{M}_{13}^{22} \overset{O(e^6)}{=} \mathcal{M}_{13}^{04} \overset{O(e^6)}{=} 0 \tag{8.21}$$

from (m), (n) and (o). Insertion of (8.21) into (j) and (k) gives us in addition

$$\mathcal{M}_{33}^{03} \overset{O(e^6)}{=} \mathcal{M}_{23}^{30} \overset{O(e^6)}{=} 0. \tag{8.22}$$

Finally insertion of eqs. (8.2) and (8.22) into (i) gives us

$$\mathcal{M}_{22}^{21} \overset{O(e^6)}{=} 0. \tag{8.23}$$

The proportionalities of (5.14) and the eqs. (8.22) and (8.23) furthermore imply

$$\mathcal{M}_{22}^{03} \overset{O(e^6)}{=} \mathcal{M}_{23}^{12} \overset{O(e^6)}{=} \mathcal{M}_{33}^{21} \overset{O(e^6)}{=} 0. \tag{8.24}$$

In turn the equilibrium conditions in terms of the stress resultants (h)-(o) are identically fulfilled, since all quantities in these equations have to be neglected. However, note that these equations (as well as the condition that a load resultant vanishes) provided nontrivial equations for the pseudo reduction in displacement coefficients, cf. section 8.3.

We express the remaining relevant stress resultants in terms of w and the known load resultants. The general procedure for the remainder of this subsection is to use the eqs. (5.6) and (5.10) and the equations of the pseudo reduction of the preceding subsection.

At first we derive the classical bending moment $\mathcal{M}_{11}^{01}$ by insertion of the reduction eqs. (8.6) to (8.9), (8.14) and (8.15)

$$\begin{aligned}
\frac{l^2}{hb}\mathcal{M}_{11}^{01} &\overset{O(e^6)}{=} \frac{2}{1-2\nu}\Big[\; c^2\big((1-\nu)u_1^{01\prime} + \nu u_2^{11} + 2\nu u_3^{02}\big) \\
&\qquad + c^2 d^2\big((1-\nu)u_1^{21\prime} + 3\nu u_2^{31} + 2\nu u_3^{22}\big) \\
&\qquad + \frac{9}{5}c^4\big((1-\nu)u_1^{03\prime} + \nu u_2^{13} + 4\nu u_3^{04}\big) \quad \Big] \\
&\overset{O(e^6)}{=} \frac{2\nu}{1-2\nu}\phi_2 + \frac{4\nu}{1-2\nu}\phi_3 + 2\frac{1-\nu}{1-2\nu}\Big(c^2 u_1^{01\prime} + c^2 d^2 u_1^{21\prime} + \frac{9}{5}c^4 u_1^{03\prime}\Big) \\
&\overset{O(e^6)}{=} -2(1+\nu)c^2 w'' \\
&\quad - 2(1+\nu)\frac{c^2}{d^2+5c^2}\left(3\,(5\,\nu+4)\,c^4 - \frac{2}{5}\,(5\,\nu-6)\,c^2 d^2 - \nu\,\frac{3\,\nu+1}{1+\nu}d^4\right) w'''' \\
&\quad - \frac{l^2}{hb}\frac{1}{2}\nu(p_3^{20} - p_3^{02}) \\
\frac{l^2}{hb}\mathcal{M}_{11}^{01} &\overset{O(e^6)}{=} -2(1+\nu)c^2 w'' \\
&\quad - \frac{1}{d^2+5c^2}\left(3\,(5\,\nu+4)\,c^4 - \frac{2}{5}\,(5\,\nu-6)\,c^2 d^2 - \nu\,\frac{3\,\nu+1}{1+\nu}d^4\right)\frac{l^2}{hb}p_3^{00} \\
&\quad - \frac{l^2}{hb}\frac{1}{2}\nu(p_3^{20} - p_3^{02}).
\end{aligned} \tag{8.25}$$

By the proportionalities of the higher moments to the classical bending moment, which are given by the eqs. (5.14) and (5.16) and multiplication of eq. (8.25) with c^2 and truncation, or by insertion of the reduction eqs. (8.1), (8.3) and (8.4), respectively, we derive

$$\begin{aligned}
\frac{l^2}{hb}c^2\mathcal{M}_{11}^{01} &\overset{O(e^6)}{=} \frac{l^2}{hb}\frac{5}{9}\mathcal{M}_{11}^{03} \overset{O(e^6)}{=} \frac{l^2}{hb}\frac{c^2}{d^2}\mathcal{M}_{11}^{21} \\
&\overset{O(e^6)}{=} \frac{2}{1-2\nu}c^4\big((1-\nu)u_1^{01\prime} + \nu u_2^{11} + 2\nu u_3^{02}\big) \\
&\overset{O(e^6)}{=} -2(1+\nu)c^4 w''.
\end{aligned} \tag{8.26}$$

By the use of equilibrium equation (b) we can derive the classical shear force by differentiation of the classical moment $\mathcal{M}_{13}^{00} = \mathcal{M}_{11,1}^{01}$. Alternatively insertion of the reduction eqs. (8.9) and (8.18) also yields

$$\begin{aligned}
\frac{l^2}{hb}\mathcal{M}_{13}^{00} &\overset{O(e^6)}{=} u_1^{01} + u_3^{00\prime} + c^2(3u_1^{03} + u_3^{02\prime}) + d^2(u_1^{21} + u_3^{20\prime}) \\
&\quad + \frac{9}{5}c^4(5u_1^{05} + u_3^{04\prime}) + c^2d^2(3u_1^{23} + u_3^{22\prime}) + \frac{9}{5}d^4(u_1^{41} + u_3^{40\prime}) \\
&\overset{O(e^6)}{=} \Phi \\
&\overset{O(e^6)}{=} -2(1+\nu)c^2w''' \\
&\quad - 2(1+\nu)\frac{c^2}{d^2+5c^2}\left(3\,(5\,\nu+4)\,c^4 - \frac{2}{5}\,(5\,\nu-6)\,c^2d^2 - \nu\,\frac{3\,\nu+1}{1+\nu}d^4\right)w''''' \\
&\quad + \frac{\nu}{2}\frac{l^2}{hb}\left(p_3^{02\prime} - p_3^{20\prime}\right) \\
\frac{l^2}{hb}\mathcal{M}_{13}^{00} &\overset{O(e^6)}{=} -2(1+\nu)c^2w''' \\
&\quad - \frac{1}{d^2+5c^2}\left(3\,(5\,\nu+4)\,c^4 - \frac{2}{5}\,(5\,\nu-6)\,c^2d^2 - \nu\,\frac{3\,\nu+1}{1+\nu}d^4\right)\frac{l^2}{hb}p_3^{00\prime} \\
&\quad + \frac{\nu}{2}\frac{l^2}{hb}\left(p_3^{02\prime} - p_3^{20\prime}\right).
\end{aligned} \tag{8.27}$$

The higher shear forces $\mathcal{M}_{13}^{20}$ and $\mathcal{M}_{13}^{02}$ already contain more than one linear combination ω_{ij}^{rs}, therefore we can not derive linear dependencies a-priori, like we did for the higher moments above. However, we derive by insertion of eqs. (8.4) to (8.8)

$$\begin{aligned}
\frac{l^2}{hb}\mathcal{M}_{13}^{20} &\overset{O(e^6)}{=} d^2(u_1^{01} + u_3^{00\prime}) + d^2c^2(3u_1^{03} + u_3^{02\prime}) + \frac{9}{5}d^4(u_1^{21} + u_3^{20\prime}) \\
\frac{l^2}{hb}\mathcal{M}_{13}^{20} &\overset{O(e^6)}{=} -2\frac{d^2c^2}{d^2+5c^2}(d^2(1+3\nu) + 5c^2(1+\nu))w'''
\end{aligned} \tag{8.28}$$

and

$$\begin{aligned}
\frac{l^2}{hb}\mathcal{M}_{13}^{02} &\overset{O(e^6)}{=} c^2(u_1^{01} + u_3^{00\prime}) + \frac{9}{5}c^4(3u_1^{03} + u_3^{02\prime}) + d^2c^2(u_1^{21} + u_3^{20\prime}) \\
\frac{l^2}{hb}\mathcal{M}_{13}^{02} &\overset{O(e^6)}{=} -\frac{6}{5}(1+\nu)c^4w'''.
\end{aligned} \tag{8.29}$$

Also the trapezoidal stress $\mathcal{M}_{12}^{11}$ has to be proportional to the higher shear forces by equilibrium equation (g). Insertion of eqs. (8.3) and (8.8) yields

$$\frac{l^2}{hb}\mathcal{M}_{12}^{11} \overset{O(e^6)}{=} d^2c^2(2u_1^{21} + u_2^{11\prime}) \overset{O(e^6)}{=} \frac{2\nu c^2d^4}{d^2+5c^2}w'''. \tag{8.30}$$

A posteriori we therefore reveal the relation

$$\begin{aligned}
\frac{l^2}{hb}c^2\mathcal{M}_{13}^{00} &\overset{O(e^6)}{=} \frac{l^2}{hb}\frac{5}{3}\mathcal{M}_{13}^{02} \overset{O(e^6)}{=} \frac{l^2}{hb}\frac{d^2+5c^2}{d^2}c^2\frac{1+\nu}{d^2(1+3\nu)+5c^2(1+\nu)}\mathcal{M}_{13}^{20} \\
&\overset{O(e^6)}{=} -\frac{l^2}{hb}\frac{d^2+5c^2}{d^2}\frac{c^2}{d^2}\frac{1+\nu}{\nu}\mathcal{M}_{12}^{11} \\
&\overset{O(e^6)}{=} c^2(u_1^{01} + u_3^{00\prime}) + c^4(3u_1^{03} + u_3^{02\prime}) + c^2d^2(u_1^{21} + u_3^{20\prime}) \\
&\overset{O(e^6)}{=} -2(1+\nu)c^4w'''.
\end{aligned} \tag{8.31}$$

The remaining three stress resultants, which do not correspond to cutting planes in ξ_1-direction, are directly determined by the given load resultants. We derive: $\mathcal{M}_{22}^{01}$ by the use of the eqs. (8.6) to (8.9), (8.14) and (8.15)

$$\begin{aligned}
\frac{l^2}{hb}\mathcal{M}_{22}^{01} &\overset{O(e^6)}{=} \frac{2}{1-2\nu}\Big[\; c^2(\nu u_1^{01\prime} + (1-\nu)u_2^{11} + 2\nu u_3^{02}) \\
&\qquad + c^2d^2(\nu u_1^{21\prime} + 3(1-\nu)u_2^{31} + 2\nu u_3^{22}) \\
&\qquad + \frac{9}{5}c^4(\nu u_1^{03\prime} + (1-\nu)u_2^{13} + 4\nu u_3^{04}) \;\Big] \\
&\overset{O(e^6)}{=} \frac{2(1-\nu)}{1-2\nu}\phi_2 + \frac{4\nu}{1-2\nu}\phi_3 + 2\frac{\nu}{1-2\nu}\left(c^2u_1^{01\prime} + c^2d^2u_1^{21\prime} + \frac{9}{5}c^4u_1^{03\prime}\right) \\
&\overset{O(e^6)}{=} 2(1+\nu)c^2\frac{d^2}{d^2+5c^2}\left(c^2\frac{5}{2} + d^2\frac{1}{2}\frac{1+5\nu}{1+\nu}\right)w'''' - \frac{l^2}{hb}\frac{1}{2}p_3^{20} \\
\frac{l^2}{hb}\mathcal{M}_{22}^{01} &\overset{O(e^6)}{=} \frac{d^2}{d^2+5c^2}\left(c^2\frac{5}{2} + d^2\frac{1}{2}\frac{1+5\nu}{1+\nu}\right)\frac{l^2}{hb}p_3^{00} - \frac{l^2}{hb}\frac{1}{2}p_3^{20},
\end{aligned} \tag{8.32}$$

$\mathcal{M}_{23}^{10}$ by the use of the eqs. (8.9) and (8.13)

$$\begin{aligned}
\frac{l^2}{hb}\mathcal{M}_{23}^{10} &\overset{O(e^6)}{=} d^2(u_2^{11} + 2u_3^{20}) + c^2d^2(3u_2^{13} + 2u_3^{22}) + \frac{9}{5}d^4(u_2^{31} + 4u_3^{40}) \\
&\overset{O(e^6)}{=} \phi_1 \\
&\overset{O(e^6)}{=} -\left(1 + \frac{\nu(3d^2+5c^2)}{d^2+5c^2}\right)c^2d^2w'''' + \frac{1}{2}\frac{l^2}{hb}p_3^{20} \\
\frac{l^2}{hb}\mathcal{M}_{23}^{10} &\overset{O(e^6)}{=} -\frac{1}{2(1+\nu)}\left(1 + \frac{\nu(3d^2+5c^2)}{d^2+5c^2}\right)d^2\frac{l^2}{hb}p_3^{00} + \frac{1}{2}\frac{l^2}{hb}p_3^{20}
\end{aligned} \tag{8.33}$$

and $\mathcal{M}_{33}^{01}$ by the use of the eqs. (8.6) to (8.9), (8.14) and (8.15)

$$\begin{aligned}
\frac{l^2}{hb}\mathcal{M}_{33}^{01} &\overset{O(e^6)}{=} \frac{2}{1-2\nu}\Big[\; c^2(\nu u_1^{01\prime} + \nu u_2^{11} + 2(1-\nu)u_3^{02}) \\
&\qquad + c^2d^2(\nu u_1^{21\prime} + 3\nu u_2^{31} + 2(1-\nu)u_3^{22}) \\
&\qquad + \frac{9}{5}c^4(\nu u_1^{03\prime} + \nu u_2^{13} + 4(1-\nu)u_3^{04}) \;\Big] \\
&\overset{O(e^6)}{=} \frac{2\nu}{1-2\nu}\phi_2 + \frac{4(1-\nu)}{1-2\nu}\phi_3 + 2\frac{\nu}{1-2\nu}\left(c^2u_1^{01\prime} + c^2d^2u_1^{21\prime} + \frac{9}{5}c^4u_1^{03\prime}\right) \\
&\overset{O(e^6)}{=} -2(1+\nu)c^2\frac{3}{10}c^2w'''' + \frac{1}{2}\frac{l^2}{hb}p_3^{02} \\
\frac{l^2}{hb}\mathcal{M}_{33}^{01} &\overset{O(e^6)}{=} -\frac{3}{10}c^2\frac{l^2}{hb}p_3^{00} + \frac{1}{2}\frac{l^2}{hb}p_3^{02}.
\end{aligned} \tag{8.34}$$

To sum up, we illustrate the dependencies of the stress resultants on the derivatives of the elastic line solution $w(\xi_1)$ and the given loads in table 7.

	stress resultants:	dependencies:
classical bending moment:	$\mathcal{M}_{11}^{01}$	$e^2 w''$, given loads
higher moments:	$\mathcal{M}_{11}^{21}$, $\mathcal{M}_{11}^{03}$	$e^4 w''$
classical shear force:	$\mathcal{M}_{13}^{00}$	$e^2 w'''$, given loads
higher shear forces in ξ_3-direction:	$\mathcal{M}_{13}^{20}$, $\mathcal{M}_{13}^{02}$	$e^4 w'''$
trapezoidal shear stress (in ξ_2-direction):	$\mathcal{M}_{12}^{11}$	$e^4 w'''$
transversal normal- and shear-moments:	$\mathcal{M}_{22}^{01}$, $\mathcal{M}_{23}^{10}$, $\mathcal{M}_{33}^{01}$	given loads
all other stress resultants:	$\mathcal{M}_{ij}^{k(m-k)}$	$= 0 + O(e^6)$

Table 7: Dependencies of the second-order stress resultants

8.5 Boundary conditions of the second-order approximation

The stress boundary conditions are given by equation (5.22). By comparison of the stress resultants that belong to the second-order $B3$-problem to equation (5.22), we find that we formally have to prescribe: $\mathcal{M}_{11}^{01}, \mathcal{M}_{11}^{21}, \mathcal{M}_{11}^{03}, \mathcal{M}_{13}^{00}, \mathcal{M}_{13}^{20}, \mathcal{M}_{13}^{02}, \mathcal{M}_{12}^{11}$. The remaining stress resultants of the problem are already determined by the given loads. Because of the eqs. (8.26) and (8.31) only two of the mentioned stress resultants could be prescribed independently. Table 7 outlines best that all stress resultants are determined once we prescribe the classical shear force and bending moment

$$\mathcal{M}_{13}^{00}(u)\, n_1 = \mathcal{M}_{N13}^{00}(u)\, n_1 := \int_{A_\xi} \frac{g_3}{G}\, dA_\xi, \qquad \text{f.a. } \xi_1 \in P_{\xi N},$$

$$\mathcal{M}_{11}^{01}(u)\, n_1 = \mathcal{M}_{N11}^{01}(u)\, n_1 := \int_{A_\xi} \frac{g_1}{G}\, \xi_3\, dA_\xi, \qquad \text{f.a. } \xi_1 \in P_{\xi N}.$$

The amount of two stress-boundary conditions per boundary is consistent with the forth-order (differential order) ODE (8.20). If one is actually interested to prescribe non-vanishing higher stress-resultants, one has to derive generalized boundary conditions, as we do in the remainder of this section for the displacement boundary conditions.

We derived the general displacement boundary condition (5.25) for the exact one-dimensional problem from the condition that the first variation of the dual energy (5.24) vanishes. But for a correct second-order approximation ($N = 2$) the variation only has to vanish except for terms of order $O(e^6)$, cf. theorem 13, so that the desired error estimate (5.27) remains valid. Since the stress resultants of the virtual stresses μ have to fulfill the homogeneous equilibrium conditions (5.21), cf. section 5.4, as well as the linear dependencies that result from the second-order approximation, cf. eqs. (5.13) to (5.16), they also fulfill the dependencies given by the eqs. (8.26) and (8.31). Therefore, we can derive the correct second-order displacement boundary conditions by insertion of the eqs. (8.26) and (8.31) for the stress resultants of the virtual stresses into the first variation of the dual energy (5.24) and by the use of the variational lemma with respect to the independent virtual stress resultants $\mathcal{M}_{13}^{00}(\mu)$ and $\mathcal{M}_{11}^{01}(\mu)$.

$$\begin{aligned}
\delta \frac{E_{\text{dual}}}{Gl^3} &= \sum_{i=1}^{3} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \Big\{ \mathbf{1}_{P_{\xi 0}}(\{0,1\}) \Big[\Big(u_{0\,i}^{k(m-k)} - u_i^{k(m-k)} \Big) \mathcal{M}_{i1}^{k(m-k)}(\mu)\, n_1 \Big] \Big\} \\
&\overset{O(e^6)}{=} \mathbf{1}_{P_{\xi 0}}(\{0,1\}) n_1 \Big[(u_{0\,1}^{01} - u_1^{01}) \mathcal{M}_{11}^{01} + (u_{0\,1}^{21} - u_1^{21}) \mathcal{M}_{11}^{21} + (u_{0\,1}^{03} - u_1^{03}) \mathcal{M}_{11}^{03} \\
&\qquad + (u_{0\,2}^{11} - u_2^{11}) \mathcal{M}_{12}^{11} \\
&\qquad + (u_{0\,3}^{00} - u_3^{00}) \mathcal{M}_{13}^{00} + (u_{0\,3}^{20} - u_3^{20}) \mathcal{M}_{13}^{20} + (u_{0\,3}^{02} - u_3^{02}) \mathcal{M}_{13}^{02} \Big]
\end{aligned}$$

$$
\begin{aligned}
&\overset{O(e^6)}{=} \mathbf{1}_{P_{\xi 0}}(\{0,1\}) n_1 \Big[(u_{0\,1}^{01} - u_1^{01}) \mathcal{M}_{11}^{01} + (u_{0\,1}^{21} - u_1^{21}) d^2 \mathcal{M}_{11}^{01} + (u_{0\,1}^{03} - u_1^{03}) \frac{9}{5} c^2 \mathcal{M}_{11}^{01} \\
&\qquad + (u_{0\,2}^{11} - u_2^{11}) \left(- \frac{d^4}{d^2 + 5c^2} \frac{\nu}{1+\nu} \right) \mathcal{M}_{13}^{00} \\
&\qquad + (u_{0\,3}^{00} - u_3^{00}) \mathcal{M}_{13}^{00} \\
&\qquad + (u_{0\,3}^{20} - u_3^{20}) \frac{d^2}{d^2 + 5c^2} \frac{d^2(1+3\nu) + 5c^2(1+\nu)}{1+\nu} \mathcal{M}_{13}^{00} \\
&\qquad + (u_{0\,3}^{02} - u_3^{02}) \frac{3}{5} c^2 \mathcal{M}_{13}^{00} \Big] \\
&\overset{O(e^6)}{=} \mathbf{1}_{P_{\xi 0}}(\{0,1\}) n_1 \Big[\Big(\Big(u_{0\,1}^{01} + d^2 u_{0\,1}^{21} + \frac{9}{5} c^2 u_{0\,1}^{03} \Big) - \Big(u_1^{01} + d^2 u_1^{21} + \frac{9}{5} c^2 u_1^{03} \Big) \Big) \mathcal{M}_{11}^{01} \\
&\qquad + \Big(\Big(u_{0\,3}^{00} + \frac{d^2}{d^2 + 5c^2} \frac{d^2(1+3\nu) + 5c^2(1+\nu)}{1+\nu} u_{0\,3}^{20} \\
&\qquad\qquad + \frac{3}{5} c^2 u_{0\,3}^{02} - \frac{d^4}{d^2 + 5c^2} \frac{\nu}{1+\nu} u_{0\,2}^{11} \Big) \\
&\qquad\qquad - \Big(u_3^{00} + \frac{d^2}{d^2 + 5c^2} \frac{d^2(1+3\nu) + 5c^2(1+\nu)}{1+\nu} u_3^{20} \\
&\qquad\qquad\quad + \frac{3}{5} c^2 u_3^{02} - \frac{d^4}{d^2 + 5c^2} \frac{\nu}{1+\nu} u_2^{11} \Big) \Big) \mathcal{M}_{13}^{00} \Big]
\end{aligned}
$$

We introduce shorthands for the prescribed displacement quantities and write them in terms of w using the reduction equations. To this end, note that by the eqs. (8.26) and (8.31) we have

$$
\frac{l^2}{hb} c^2 \mathcal{M}_{11}^{01} \overset{O(e^6)}{=} -2(1+\nu) c^4 w'', \qquad \frac{l^2}{hb} c^2 \mathcal{M}_{13}^{00} \overset{O(e^6)}{=} -2(1+\nu) c^4 w''',
$$

so that we can exclude from both stress resultants a factor c^2 (also cf. section 5.3) to derive

$$
c^2 \widetilde{w} := c^2 u_3^{00} + \frac{c^2 d^2}{d^2 + 5c^2} \frac{d^2(1+3\nu) + 5c^2(1+\nu)}{1+\nu} u_3^{20} + \frac{3}{5} c^4 u_3^{02} - \frac{c^2 d^4}{d^2 + 5c^2} \frac{\nu}{1+\nu} u_2^{11}, \quad (8.35)
$$

$$
c^2 \widetilde{w} \overset{O(e^6)}{=} c^2 w + \frac{1}{10} \frac{c^2}{d^2 + 5c^2} \frac{\nu}{1+\nu} (15(1+\nu)c^4 - 22(1+\nu)c^2 d^2 - 5(1+5\nu) d^4) w'', \quad (8.36)
$$

$$
c^2 \widetilde{\psi} := c^2 u_1^{01} + c^2 d^2 u_1^{21} + \frac{9}{5} c^4 u_1^{03},
$$

$$
c^2 \widetilde{\psi} \overset{O(e^6)}{=} - c^2 w' + \Big(\frac{1}{2} \nu c^2 d^2 - \frac{3}{10} (8+9\nu) c^4 \Big) w''', \quad (8.37)
$$

by the use of the eqs. (8.3) to (8.8). With the variables $\widetilde{w}$ and $\widetilde{\psi}$, the first variation of the dual energy simply reads

$$
\delta \frac{E_{\text{dual}}}{Gl^3} \overset{O(e^6)}{=} \mathbf{1}_{P_{\xi 0}}(\{0,1\}) \Big[(\widetilde{\psi}_0 - \widetilde{\psi}) \mathcal{M}_{11}^{01} \, n_1 + (\widetilde{w}_0 - \widetilde{w}) \mathcal{M}_{13}^{00} \, n_1 \Big],
$$

so that we gain the generalized boundary condition (cf. section 5.6 and eq. (5.28))

$$
\begin{aligned}
&\text{F.a. } \xi_1 \in \{0,1\} : \\
&\qquad \widetilde{w}(\xi_1) = \widetilde{w}_0(\xi_1) \quad \text{or} \quad \mathcal{M}_{13}^{00}(\xi_1) \, n_1 = \mathcal{M}_{N13}^{00}(\xi_1) \, n_1 \\
&\text{and } \widetilde{\psi}(\xi_1) = \widetilde{\psi}_0(\xi_1) \quad \text{or} \quad \mathcal{M}_{11}^{01}(\xi_1) \, n_1 = \mathcal{M}_{N11}^{01}(\xi_1) \, n_1.
\end{aligned} \quad (8.38)
$$

Note that $c^2\widetilde{\psi}$ is actually the second-order approximation of the averaged exact infinitesimal rotation of the cross-section. Alternatively, we can define $c^2\widetilde{\psi}$ by

$$c^2\widetilde{\psi} := \frac{1}{hbl^2}\int_{\frac{-b}{2}}^{\frac{b}{2}}\int_{\frac{-h}{2}}^{\frac{h}{2}} u_1 x_3 \, dx_3 \, dx_2 = \frac{l^2}{hb}\int_{\frac{-b}{2l}}^{\frac{b}{2l}}\int_{\frac{-h}{2l}}^{\frac{h}{2l}} \left(\frac{u_1}{l}\right)\xi_3 \, d\xi_3 \, d\xi_2$$
$$= \frac{l^2}{hb}\sum_{n=0}^{\infty}\sum_{q=0}^{n} u_1^{q(n-q)}(\xi_1)\, e^{q,n-q+1} \overset{O(e^6)}{=} c^2 u_1^{01} + c^2 d^2 u_1^{21} + \frac{9}{5}c^4 u_1^{03}.$$

However, there is no easy physical interpretation for $\widetilde{w}$, mostly due to the fact that it considers displacements in two coordinate directions that, furthermore, have to be weighted in order to obtain a consistent theory. Nevertheless, any three-dimensional displacement boundary condition for u_3 and u_2 can be transfered into a boundary condition for $\widetilde{w}$ by computation of the series expansion at the face-sides and insertion of the coefficients into (8.35). Of course the trivial case $u_3 = 0$ and $u_2 = 0$ f.a. $(\xi_2, \xi_3) \in A_\xi$ and $\xi_1 \in \{0, 1\}$ fixed corresponds to $\widetilde{w} = 0$ at $\xi_1 \in \{0, 1\}$. Therefore, we (nevertheless) have the situation that $\widetilde{w} = 0$ and $\widetilde{\psi} = 0$ corresponds to the standard fixed support, and $\widetilde{w} = 0$ and $\mathcal{M}_{11}^{01} = 0$ corresponds to a hinged support, where the hinge is located at the points $(\xi_2, \xi_3) \in (-\frac{b}{2l}, \frac{b}{2l}) \times \{0\}$ and a rigid plate at the face side enforces $\widetilde{w} = 0$.

8.6 The final theory in terms of $\widetilde{w}$

For convenience, we will rewrite the ODE to solve (8.20) as well as all stress resultants in terms of $\widetilde{w}$ rather than w.

To this end, by differentiation of (8.36) and insertion of (8.9) we derive

$$\begin{aligned} -2(1+\nu)c^2(e2w)'' &\overset{O(e^6)}{=} -2(1+\nu)c^2(e2\widetilde{w})'' + 2(1+\nu)\frac{1}{10}\frac{c^2}{d^2+5c^2}\frac{\nu}{1+\nu} \\ &\qquad (15(1+\nu)c^4 - 22(1+\nu)c^2d^2 - 5(1+5\nu)d^4)e4w'''' \\ &\overset{O(e^6)}{=} -2(1+\nu)c^2(e2\widetilde{w})'' + \frac{1}{10}\frac{1}{d^2+5c^2}\frac{\nu}{1+\nu} \\ &\qquad (15(1+\nu)c^4 - 22(1+\nu)c^2d^2 - 5(1+5\nu)d^4)\frac{l^2}{hb}e2p_3^{00} \end{aligned} \tag{8.39}$$

and multiplication of (8.36) with a factor of order e^2 and truncation yields

$$e4w \overset{O(e^6)}{=} e4\widetilde{w}. \tag{8.40}$$

By insertion of (8.39) into the main ODE (8.19), we derive the main ODE written in $\widetilde{w}$ as

$$2(1+\nu)c^2\widetilde{w}'''' \overset{O(e^6)}{=} \frac{l^2}{hb}\left[p_3^{00} + P''\right], \tag{8.41}$$

if we introduce the shorthand P for the non-classical (i.e. not present in the Euler-Bernoulli beam theory) load term

$$P(\xi_1) := \frac{\nu}{2}\left(p_3^{02}(\xi_1) - p_3^{20}(\xi_1)\right) - \frac{1}{10}\left((24+27\nu)c^2 - 5\nu d^2\right)p_3^{00}(\xi_1). \tag{8.42}$$

With the use of equation (8.10) we can rewrite (8.41) as

$$\frac{E}{G}\frac{I}{l^4}\widetilde{w}'''' \overset{O(e^6)}{=} p_3^{00} + P''. \tag{8.43}$$

The boundary conditions are given by (8.38), which requires us to rewrite $c^2\widetilde{\psi}$ in terms of $\widetilde{w}$. This can be achieved by differentiation of (8.36) and insertion of the resulting equation and equation (8.40) into equation (8.37)

$$c^2\widetilde{\psi} \overset{O(e^6)}{=} -c^2\widetilde{w}' + \Big(-2\frac{6}{5}(1+\nu)c^4 - \frac{2\nu^2}{1+\nu}\frac{c^2d^4}{5c^2+d^2}\Big)\widetilde{w}'''. \tag{8.44}$$

Insertion of the eqs. (8.10) and (8.39) into (8.25) and (8.27) yields the major stress resultants in terms of $\widetilde{w}$, which are required for the formulation of the boundary conditions (8.38)

$$\mathcal{M}_{11}^{01} \overset{O(e^6)}{=} -\frac{E}{G}\frac{I}{l^4}\widetilde{w}'' + P, \tag{8.45}$$

$$\mathcal{M}_{13}^{00} \overset{O(e^6)}{=} -\frac{E}{G}\frac{I}{l^4}\widetilde{w}''' + P' \overset{O(e^6)}{=} \mathcal{M}_{11}^{01}{}'. \tag{8.46}$$

The higher, dependent moments

$$c^2\mathcal{M}_{11}^{01} \overset{O(e^6)}{=} \frac{5}{9}\mathcal{M}_{11}^{03} \overset{O(e^6)}{=} \frac{c^2}{d^2}\mathcal{M}_{11}^{21} \overset{O(e^6)}{=} -\frac{E}{G}\frac{I}{l^4}c^2\widetilde{w}'', \tag{8.47}$$

as well as the higher dependent shear forces and the trapezoidal stress resultant $\mathcal{M}_{12}^{11}$

$$\begin{aligned} c^2\mathcal{M}_{13}^{00} &\overset{O(e^6)}{=} \frac{5}{3}\mathcal{M}_{13}^{02} \overset{O(e^6)}{=} \frac{d^2+5c^2}{d^2}c^2\frac{1+\nu}{d^2(1+3\nu)+5c^2(1+\nu)}\mathcal{M}_{13}^{20} \\ &\overset{O(e^6)}{=} -\frac{d^2+5c^2}{d^2}\frac{c^2}{d^2}\frac{1+\nu}{\nu}\mathcal{M}_{12}^{11} \overset{O(e^6)}{=} -\frac{E}{G}\frac{I}{l^4}c^2\widetilde{w}''', \end{aligned} \tag{8.48}$$

are derived by insertion of (8.40) and (8.10) into (8.26) and (8.31). Finally note that the remaining stress resultants $\mathcal{M}_{22}^{01}$, $\mathcal{M}_{23}^{10}$ and $\mathcal{M}_{33}^{01}$, which are not to be neglected in the second-order consistent theory, are directly determined in terms of the given loads by the eqs. (8.32) to (8.34).

8.7 Comparison to Timoshenko's beam theory

Maybe the most established refined beam theory was developed by Timoshenko (1921, 1922). The ODE of the static problem is usually noted by

$$EI\frac{d^4w_T(x_1)}{dx_1^4} = q(x_1) - \frac{EIK}{AG}\frac{d^2q(x_1)}{dx_1^2},$$

in the literature, or the second-order (differential order) ODE for the additional deflection is given (cf. Timoshenko & Young, 1962, eq. (8.8)). Here $q(x_1)$ is the overall given load in ξ_3-direction, A is the area of the cross section, K is the shear-correction factor and EI is the bending stiffness. Frequently the ODE is also formulated using the reciprocal of the shear-correction factor $\kappa := 1/K$ introduced here.

We already restricted ourselves to a simplified load case that allows only loads in ξ_3-direction in section 8.3, although the correct definition of the beam problem (developed in the sections 6 and 7 and given in short form in section 8.1) allows for much more general load cases. Nevertheless, we ended up with a theory involving three load resultants (p_3^{00}, p_3^{20} and p_3^{02}), whereas the Timoshenko theory only involves the overall load q. Even if we restrict ourselves further, to loads that are constant in the cross section directions ξ_2 and ξ_3, i.e.,

$$\left[\frac{lf_3}{G}\right](\xi_1,\xi_2,\xi_3) = \sum_{n=0}^{\infty}\sum_{q=0}^{n} f_3^{q(n-q)}(\xi_1)\xi_2^q\xi_3^{n-q} \stackrel{!}{=} f_3^{00}(\xi_1),$$

$$\left[\frac{g_3}{G}\right]\left(\xi_1,\xi_2,\pm\frac{h}{2l}\right) = \sum_{n=0}^{\infty} g_3^{n\pm}(\xi_1)\xi_2^n \stackrel{!}{=} g_3^{0\pm}(\xi_1),$$

$$\left[\frac{g_3}{G}\right]\left(\xi_1,\pm\frac{b}{2l},\xi_3\right) = \sum_{n=0}^{\infty} g_3^{\pm n}(\xi_1)\xi_3^n \stackrel{!}{=} g_3^{\pm 0}(\xi_1),$$

cf. the eqs. (4.16) to (4.18), the evaluation of (4.21) yields

$$p_3^{00} \stackrel{O(e^6)}{=} f_3^{00}\frac{hb}{l^2} + (g_3^{0+}+g_3^{0-})\frac{b}{l} + (g_3^{+0}+g_3^{-0})\frac{h}{l},$$

$$p_3^{20} \stackrel{O(e^6)}{=} f_3^{00}\frac{hb}{l^2}d^2 + (g_3^{0+}+g_3^{0-})\frac{b}{l}d^2 + (g_3^{+0}+g_3^{-0})\frac{h}{l}3d^2,$$

$$p_3^{02} \stackrel{O(e^6)}{=} f_3^{00}\frac{hb}{l^2}c^2 + (g_3^{0+}+g_3^{0-})\frac{b}{l}3c^2 + (g_3^{+0}+g_3^{-0})\frac{h}{l}c^2,$$

hence no linear dependencies that justify a theory in only one load resultant. Therefore, the Timoshenko theory is in general inconsistent with the modeling approach provided here.

Another, not less important reason, for the Timoshenko theory to be inconsistent is that it models the theory as a plane problem, i.e. no deformations (nor loads and stresses) in ξ_2-direction are considered. The deformation u_2^{11}, however, is already to be considered in the first-order consistent theory (cf. the table on page 107) and therefore already present within the framework of the Euler-Bernoulli theory. We like to emphasize that this transversal deformation is not at all a theoretical construct, it is actually measurable and considered for the placement of strain sensors in practice (cf., e.g., Gevatter, 2000, pages 39–40). To avoid any misunderstanding, nevertheless, it may be acceptable to model a plate theory as a plane problem. The plain stress assumption is indeed first-order consistent (cf. Kienzler, 1980, eq. 5.3.5), i.e., acceptable in the context of the Euler-Bernoulli theory. However, it is evident, e.g. from table 7 that it is inconsistent in the context of a second-order theory, i.e. for refined theories.

We have to restrict ourselves even further, to a setting also neglecting dead weight and tractions on the lateral sides

$$\left[\frac{g_3}{G}\right]\left(\xi_1,\xi_2,\pm\frac{h}{2l}\right) = \sum_{n=0}^{\infty} g_3^{n\pm}(\xi_1)\xi_2^n \stackrel{!}{=} g_3^{0\pm}(\xi_1), \qquad f \stackrel{!}{=} 0,\ g\left(\xi_1,\pm\frac{b}{2l},\xi_3\right) \stackrel{!}{=} 0,$$

to derive the linear dependencies

$$p_3^{20} = d^2p_3^{00},\ p_3^{02} = 3c^2p_3^{00} \text{ and } p_3^{00}(\xi_1) = \left(g_3^{0+}(\xi_1)+g_3^{0-}(\xi_1)\right)\frac{b}{l} = \frac{q(\xi_1)}{Gl}, \tag{8.49}$$

where we defined the overall load resultants by

$$q(x_1) := \int_{-b/2}^{b/2} g_3(x_1,x_2,h/2)\,dx_2 + \int_{-b/2}^{b/2} g_3(x_1,x_2,-h/2)\,dx_1$$
$$= G\left(g_3^{0+}(x_1)+g_3^{0-}(x_1)\right)b,$$

in order to get a theory in only one load resultant. Note that such a theory is inappropriate for the treatment of dead weight, since the linear dependencies of the dead weight terms f_3^{00} are different, therefore, *the dead load can not be included into the overall resultant q.*

We already derived in section 8.3 (cf. eq. (8.12)) that the first-order approximation of our approach yields

$$EI\,\frac{d^4\,u_3(x_1,0,0)}{dx_1^4} = q(x_1) + O(e^4),$$

i.e., the Euler-Bernoulli equation. If we insert (8.49) into the second-order approximation ODE (8.43) in terms of $\widetilde{w}$, we obtain

$$\frac{EI}{l^3}\widetilde{w}'''' = q - \frac{6}{5}(2+\nu)c^2q'' + O(e^6), \tag{8.50}$$

by multiplication with Ghb/l. This equation is comparable to the Timoshenko ODE and hence can be used to derive a shear-correction factor for this theory. (However, note that the consistent theory does not involve any a-priori assumption, we did in particular not introduce any shear-correction terms!) By the use of

$$I = \frac{bh^3}{12},\ A = bh,\ G = \frac{E}{2(1+\nu)} \implies \frac{EI}{AG} = 2(1+\nu)c^2l^2,$$

and transformation to x-coordinates (cf. section 4.3) the Timoshenko equation reads as

$$\frac{EI}{l^3}\left(\frac{w_T}{l}\right)'''' = q - 2K(1+\nu)c^2q'', \tag{8.51}$$

which gives us the shear-correction factor

$$K_{\widetilde{w}} = \frac{3}{5}\frac{2+\nu}{1+\nu} \tag{8.52}$$

by comparison of (8.51) with (8.50). However, the left-hand sides of (8.51) with (8.50) are not exactly comparable, especially since $\widetilde{w}$ also contains a displacement coefficient in ξ_2-direction, i.e., u_2^{11}, cf. (8.35). So it might be more convenient to multiply the original second-order equation in $w(x_1) = u_3(x_1,0,0)/l$ with the factor Ghb/l to obtain

$$\begin{aligned} \frac{EI}{l^3}(w)'''' &\overset{O(e^6)}{=} EI\,\frac{d^4\,u_3(x_1,0,0)}{dx_1^4} \\ &\overset{O(e^6)}{=} q - 2K_w(1+\nu)c^2q'' \overset{O(e^6)}{=} q(x_1) - 2K_w(1+\nu)c^2l^2\frac{d^2q(x_1)}{dx_1^2}, \end{aligned}$$

by insertion of (8.49), where the shear-correction factor

$$\begin{aligned} K_w &:= \frac{15c^4(5\nu+8)(1+\nu) - 2c^2d^2(1+\nu)(5\nu-12) - 5\nu d^4(5\nu+1)}{20(1+\nu)^2(5c^2+d^2)c^2} \\ &= \frac{3}{20}\frac{5\nu+8}{1+\nu} + 5\frac{\nu^2d^2}{(1+\nu)^2(5c^2+d^2)} - \frac{\nu d^2(5\nu+1)}{4c^2(1+\nu)^2} \\ &= \underbrace{\frac{3}{20}\frac{5\nu+8}{1+\nu}}_{\text{Olsson (1935)}} - \beta^2\frac{\nu}{1+\nu}\left[\frac{1}{4} + \frac{\beta^2}{\beta^2+5}\frac{\nu}{1+\nu}\right] \end{aligned} \tag{8.53}$$

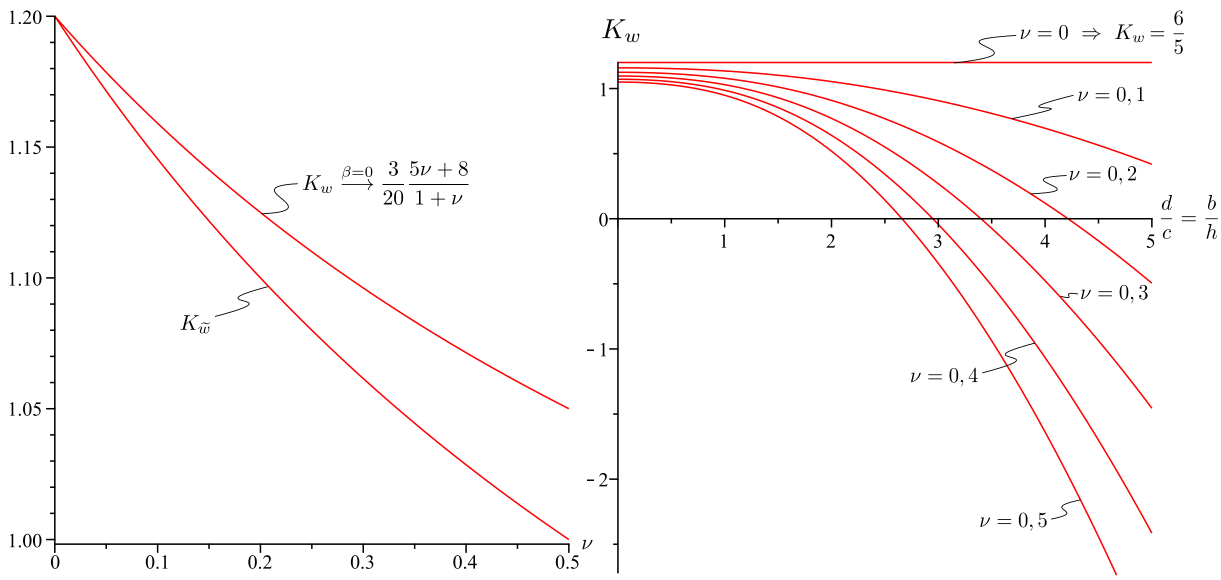

Figure 10: The shear-correction factors' dependencies of their parameters. On the left: $K_{\widetilde{w}}$ and K_w for $\beta = 0$ as functions of Poission's ratio ν. On the right: K_w as function of the cross-section-aspect ratio β for several values of ν.

depends on the cross-section-aspect ratio $\beta := \frac{d}{c} = \frac{b}{h}$. Indeed the limit value $\beta = 0$ is a known shear-correction factor. Kaneko (1975) dedicates the factor to Olsson (1935). However, in this article Olsson states that he "concluded" the factor from the work of v. Kármán & Seewald (1927).

We illustrate the dependencies of $K_{\widetilde{w}}$ and K_w of the parameters ν and β in figure 10. Both of them equal Timoshenko's classical shear-correction factor $\frac{6}{5}$ for $\nu = 0$. Compared to literature values (cf., Kaneko, 1975) both of them are comparatively small for usual values of ν, which may be due to the fact that the trapezoidal displacement v_2^{11} is usually neglected for the derivation of shear-correction factors.

$\mathcal{K}_w$ turns negative for large values of β. To convince ourselves of the plausibility, let us get some numbers. For a fixed support beam loaded by a singular force at the free end a German standard basic course text book (cf. Schnell et al., 2002, section 4.6.2) computes the additional shear deflection to be about 3% of the Euler-Bernoulli beam deflection using Timoshenkos approach ($K = 6/5$, $\nu = 0,3$, $l/h = 5$). By comparing the plate modulus of the Kirchhoff (1850) theory $\frac{Eh^3}{12(1-\nu^2)}$ with the bending stiffness EI of the Euler-Bernoulli beam, we find the plate displacement to be $w_{\text{plate}} = (1-\nu^2)\, w_{\text{beam}}$, if the force per width is identical (cf., e.g., Eschenauer & Schnell, 1993, section 8.2.4.a), i.e. for $\nu = 0,3$ the deflection of the plate is reduced by about 10%. The reason for the plate stiffening is the prevention of the trapezoidal transverse displacement v_2^{11} (cf., e.g., Altenbach et al., 1998, section 3.2.3). Therefore it seems reasonable for the shear-correction factor K_w, which involves this displacement coefficient, to turn negative for large values of β.

9 Discussion and outlook

9.1 Discussion

A vast amount of refined theories for thin structures is available in the literature. It may be due to the success of the "classical refined" theories (cf. section 1.1), which were motivated by disputable a-priori assumptions, or due to the general truth that all theories have to be validated against experiments in the end, that many authors did not seem to care much about the "legitimation" of their modeling approaches. However, especially since the three-dimensional theory of elasticity is settled (without any doubt), we think that any modeling approach for thin structures should always follow a rigorous line of reasoning, starting from the three-dimensional problem. The "legitimation" of the method of consistent approximation provided in (Kienzler, 2002) could be roughly summarized by the following reasoning: Since the approach derives the theories for thin structures by the truncation of the exact elastic potential and does not use any a-priori assumptions (i.e. assumptions that are not already present in the three-dimensional theory of elasticity) and, furthermore, the first-order theories turn out to be the established (and lately mathematically justified) classical theories, there is no reason not to believe in the correctness of the higher-order theories derived by the same approach. This is already a legitimation that may be regarded as sufficient by the majority of engineers, however, the recent success of the method of Γ-convergence and some comments from the mathematics community inspired us to seek for an even more rigorous legitimation.

First of all, already theorem 8 (basically taken from Zeidler (1997)) basically tells us that the approximation of the potential energy leads to the approximation of the displacement field solution. The shortcoming is, however, that we can not derive displacement boundary conditions from the truncation of the potential energy. So we have to truncate the dual energy too, in order to derive the full set of modeling Euler-Lagrange equations. That is the extension of the principle method of consistent approximation we provide in this thesis.

There are lots of ways to perform a series expansion. We chose the Taylor-series expansion in this contribution because of a simple reason: Only the Taylor-series expansion leads to equilibrium equations in terms of stress resultants (5.21) that contain the ones known from classic theories, cf. section 5.5. Only from a mathematician's point of view this might be considered as a shortcoming, since one has to assume real analytic data. However, regularity questions are hardly of interest for engineering applications *for a reason* which is that, speaking of a multi-field problem, piecewise real analytic functions are sufficient for the treatment of practical applications, cf. section 4.4. The mathematically appropriate (L_2) orthogonal basis was chosen in Schneider et al. (2014). However, it turned out that the resulting reduced plate theory is equivalent to the one choosing the Taylor-series (Kienzler, 2004). Therefore, we think it is better to choose the expansion that leads to simpler equilibrium conditions, which, furthermore, contain the ones known from classic theories.

In turn we also break down the mathematical preliminaries of section 3 to the essential assumptions and arguments, avoiding much of mathematical notation overhead. Since the (basis) three-dimensional error estimate (theorem 11) is basically from Zeidler (1997), we hope that the way towards the first main result (theorem 13 in section 5.4), may be accepted as a sketch of proof with omitted regularity questions by mathematicians, while being fully understandable for the average master student of engineering sciences.

One has some choices of how to truncate the series expansion. At the end of section 4.6 we discuss that a truncation with respect to the asymptotically dominant term, which leads to a truncated displacement ansatz (Vekua-type theory), is not the best choice, since reasonable accuracy in engineering applications is already achieved with (comparatively) low-order theories, and for low approximation orders the (geometry-dependent) characteristic parameters c and d dominate the decaying behavior. This leads in turn to the uniform approximation approach.

The application of this approach results in the first main result, the approximation theorem 13. The included estimate (5.27) implies the convergence of any Nth-order approximation solution v towards the exact solution u for $b, h \longrightarrow 0$ (where h is the thickness and b is the width of the cross-section) which is the sort of mathematical justification that is provided by limit analysis approaches, like Γ-convergence. Moreover (and more importantly), the estimate states that the accuracy of the solution v increases significantly for every incrementation of N by 1, since the error decreases with $\max\{c, d\}$ and $c, d \ll 1$ for thin structures. (So basically (!) the evaluation of $\max\{c, d\}^{N+1}$ allows an estimate of the accuracy in terms of decimal digits.)

For us it seems evident, that a comparable estimate holds for the derivation of two-dimensional theories,

$$k \, \|v - u\|_X^2 = O\left(c^{2(N+1)}\right),$$

since the case of the derivation of one-dimensional theories is more difficult. The arguments presented can readily be applied. (The estimate for the derivation of two-dimensional theories will be published in an upcoming paper.) Since the Reissner-Mindlin theory (and some other theories, cf., Schneider & Kienzler, 2014b) are equivalent to the second-order consistent plate theory (within the second-order framework, i.e. beside differences of order c^6) (Kienzler, 2004) this provides mathematical justification for this established theory, for the first time. In addition, due to (Kienzler, 2002) the consistent first-order theory equals Kirchhoff's theory, which was already justified by means of Γ-convergence. Also, due to Schneider et al. (2014) there is a consistent second-order plate theory for monoclinic material. For the special case of orthotropic material the first-order truncation of this theory equals the classical theory of orthotropic plates, which was according to the classical book of Lekhnitskii (1968) mainly developed by Huber (1921, 1926, 1929).

In this thesis, we showed that the first-order beam theory is the Euler-Bernoulli theory, which is in accordance with proofs in literature that already justified that theory. As another main result we derived the second-order consistent beam theory (cf. section 8), which is not known from the literature. In turn Timoshenko's theory turns out to be inconsistent with our approach. One main reason is that Timoshenko uses a plain-stress modeling approach, which turns out to be unacceptable for refined theories within the consistent framework. Another reason is that Timoshenko's theory only contains one overall load resultant, whereas the consistent approach has three in general independent load resultants. However, with some further load-case assumptions (In ξ_2-direction constant top- and bottom-side traction only; no dead weight; cf. section 8.7), we were able to derive a theory comparable to Timoshenko's theory that allowed the identification of two shear-correction factors for the use in this theory. However, note that the consistent second-order beam theory is free of a-priori assumptions, in particular it does not introduce any shear-correction. The identification just compares the resulting differential equations.

The last main result of this thesis is theorem 21, which is already published in a more general form in Schneider & Kienzler (2015). The theorem states how *all* three-dimensional load cases, for a quasi one-dimensional geometry, can be uniquely decomposed into the driving forces of the four (exact) one-dimensional subproblems: a rod-, a shaft- and two orthogonal beam-problems (independent of the material properties). To this end, we introduced detailed definitions of the subproblems that consider not only the direction of load, but also the symmetry of the load with respect to the cross-section axes. (Cf. section 8.1 for the rigorous definition of the beam-load case.) The theorem states that the four subproblems are *decoupled* for *isotropic* material, i.e. that the sets of the unknown displacement coefficients of the solutions of the four problems are disjoint from each other (or as formulated in the final theorem: the solution's component functions have different parities with respect to the cross-section directions) and that the solution of one of the four subproblems is *independent* of the parts of the applied loads that belong to the other subproblems.

Indeed the one-dimensional representation introduced in theorem 21 is equivalent to the three-dimensional theory of elasticity. Therefore, the subproblems' load-case definitions are *the only* load-case definitions compatible with the three-dimensional theory of elasticity.

As outlined in the introduction, the independence of the subproblems is crucial for the general procedure of engineering mechanics to define problems by the load case. Otherwise, it would be senseless to solve only one subproblem! Therefore, we are honestly surprised not to find any literature stating this fundamental independence.

Furthermore, the theorem provides a fast and elegant way to derive the *coupling* behavior of the four subproblems directly from the sparsity of the stiffness tensor for *any kind of anisotropic* material. For instance, we found the four subproblems to be decoupled for orthotropic material, where the planes of symmetry are given by the coordinate planes, or we have coupled problems of pairwise two classical subproblems for the case of monoclinic materials, where the plane of symmetry is given by two coordinate axes (cf. the end of section 6.5), whereas aelotropy leads to one coupled problem containing all four subproblems.

The theorem has lots of interesting consequences, like:

- There is simply nothing like a monoclinic beam-theory, since the problem always couples with at least one other problem. So completely novel type of theories have to be modeled for the coupled problems.
- Every subproblem (rod, beam, shaft) is driven by loads in every coordinate direction.
- There is a whole class of loads p_i^{oo} that does not have resulting forces or moments, but still an effect.
- And maybe most surprisingly at all: The classical assumption of the preservation of the cross section height in classical beam theories is only necessary because the load case is defined wrongly! The "canonical" beam load is actually a mixed rod-beam load case, cf. 6.4. The preservation of the cross-section height is an (exactly fulfilled) consequence of the theorem 21! The cross-section squeezing results from the rod part of the "canonical" beam load, which can be computed independently.

9.2 Outlook

Of course a thesis is never really finished. Some further lines of work are provided in this section.

- First of all, some example calculations using the new beam theory should be provided. And maybe the best way of checking the proved approximation property would be a test of the analytic solutions against high accuracy three-dimensional Finite Element solutions.
- We made the simplifying assumption that the beam is only loaded in ξ_3-direction for the derivation of the final second-order beam theory, although theorem 21 proved that the beam-load case is far more general, cf. section 8.1. Beside the simplification of the pseudo

reduction, we introduced the assumption to derive comparable theories. Taking all load resultants into account should already lead to a generalized Euler-Bernoulli beam and of course a generalized second-order theory. It would be tempting to investigate, which amount of insight these theories would provide for engineering mechanics. For instance, since the last step of pseudo reduction is to compute (a)−(b)′, the first derivative of the distributed moment $p_1^{01\prime}$ has the same effect as the overall distributed force p_3^{00}.

- The method provided in this thesis also allows for the derivation of (refined) rod- and shaft-theories. A comparison of higher-order theories to established approaches, e.g. for warping torsion would be no less tempting.
- This applies equally to theories for anisotropic materials, like a transversal-isotropic (refined) beam theory
- and for coupled problems, e.g. for monoclinic materials, as well.
- If one is actually interested to prescribe non-vanishing higher stress resultants, one could derive generalized boundary conditions for the stress resultants, as we did in section 8.5 for the displacement-boundary conditions.
- If it would be possible to express all linear combinations ω_{ij}^{kl} that occur in a certain theory in terms of the main variable(s) (i.e., w for the first and second-order beam) by the use of the corresponding reduction equations (i.e., the equations of section 8.3 for the second-order beam), this would provide analytical solutions for the full three-dimensional stress tensor field σ_{ij}, due to equation (5.6).
- The generalized definitions of the exact subproblems provided by theorem 21 may be used as the definitions of the most general Almansi-Michell problems (Almansi, 1901; Michell, 1901), which could be studied in the sense of de Saint-Venant solutions (de Saint-Venant, 1856).
- The results of theorem 21 have been extended towards general two-fold symmetric cross-sections in Schneider & Kienzler (2015). The refined beam theory could be refined towards this more general class of cross-sections as well. The principle procedure is the same: Although $e^{k,m}$ would have a more complex representation than (4.14), the principle decaying behavior is the same.
- The maybe most tempting question is: What if we have a general non-symmetric cross-section? If we move the origin of the coordinate system to the center of mass of the cross section and rotate towards main-axes, we still have $e^{k,m} = 0$ for k and m even and $k + m \leq 2$. This would lead to a decoupling of the first-order theories, but the second-order theories could be coupled due to the cross-section geometry. This should also lead to a whole new type of second-order theories.

Würdigung studentischer Arbeiten

Erklärung gemäß des Merkblattes zum Promotionsverfahrens der Geschäftsstelle des Promotionsausschusses des Fachbereichs 4 der Universität Bremen, Punkt 2 e):

In der vorliegenden Arbeit sind Ergebnisse enthalten, die im Rahmen der Betreuung der folgenden studentischen Arbeit entstanden sind:

- Jens Laube, Herleitung einer konsistenten Theorie für den Balken mit Rechteckquerschnitt. Studienarbeit im Studiengang Produktionstechnik, Universität Bremen, 2011.

In der obenstehenden Arbeit wurde das ursprügliche Pseudo-Reduktionsverfahren aus Schneider & Kienzler (2011) verwendet. Mit den Reduktionsgleichungen aus der Arbeit lässt sich ein Widerspruch konstruieren (vgl. den Kommentar zu Divisionen in Abschnitt 8.3). Konkret besteht dieser zwischen Gleichung (5.57) und Gleichung (5.56), wenn man die Gleichungen (5.40) und (5.46) in diese einsetzt. Wir möchten ausdrücklich betonen, dass dieser Widerspruch aus dem Verfahren und nicht etwa aus einem Fehler von Herrn Laube herrührt. Der Widerspruch führte zu der Entwicklung des in Abschnitt 8.3 vorgestellten Verfahrens, welches nur Multiplikationen mit charakteristischen Parametern benutzt und die enu_i^{kl}-Variablen einführt, durch welche Widersprüche dieser Art prinzipiell ausgeschlossen sind.

English version of the declaration according to the guidelines of the PhD committee of the department 4 of the University of Bremen, section 2 e):

The thesis contains results that originated from the supervision of the student thesis:

- Jens Laube, Derivation of a consistent theory for beams with rectangular cross-section (in German), “Studienarbeit” for the degree program bachelor of production engineering, University of Bremen, 2011.

The work mentioned above used the original pseudo-reduction procedure introduced in the article Schneider & Kienzler (2011). The reduction equations of this thesis became contradictory (cf. the comment concerning divisions in section 8.3). To be specific, the contradiction results from the equations (5.57) and (5.56), if one inserts the equations (5.40) and (5.46) into the later one. We like to emphasize that the contradiction results from the procedure itself and not from any mistake of Mr. Laube. The contradiction led to the development of the procedure introduced in section 8.3, that solely uses multiplications with characteristic parameters and introduces the enu_i^{kl}-variables, which eliminate the possibility of this type of contradictions in general.

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