Regularization of the inverse medium problem

On nonstandard methods for sparse reconstruction

Robin Strehlow

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von Robin Strehlow

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Dr. Kamil S. Kazimierski (Universität Graz)

Für Siu

ABSTRACT

In this thesis, we investigate nonstandard methods for the stable solution of the inverse medium problem. Particularly, we consider the linearization of the model of the scattering process given by the Born approximation and investigate regularization methods that are designed for sparse reconstruction. In numerical experiments we demonstrate that sparsity constraints contribute to meaningful reconstructions from synthetic and even measurement data.

In our investigations, we consider both iterative and variational methods for the solution of the inverse problem. Starting from the Landweber iteration, we discuss existing variants of this approach and develop a novel sparsity-enforcing method which is based on the Bregman projection. Furthermore, we consider a variational regularization scheme. First, we develop a novel parameter choice rule based on the L-curve criterion designed for sparse reconstruction. We then propose to replace the variational problem by some smooth approximation and provide an exhaustive investigation regarding stability of this approach. The theoretical investigations of each of the methods proposed in this work are complemented by a numerical evaluation.

ZUSAMMENFASSUNG

In dieser Dissertation untersuchen wir Methoden zur stabilen Lösung des inversen Streuproblems. Dabei verwenden wir die aus der Born-Approximation hervorgehende Linearisierung des Modells des Streuprozesses und untersuchen Verfahren, welche die Rekonstruktion dünn besetzter Lösungen ermöglichen. In numerischen Experimenten zeigen wir, dass sich auf diese Weise unter Verwendung von synthetischen und sogar echten Daten aussagekräftige Rekonstruktionen finden lassen.

In unseren Untersuchungen berücksichtigen wir sowohl iterative als auch variationelle Verfahren zur Lösung des inversen Problems. Ausgehend vom klassischen Landweber-Verfahren werfen wir einen Blick auf Verallgemeinerungen dieses Ansatzes und entwickeln ein neues Iterationsverfahren mit dünn besetzen Iterierten, welches auf der Bregman-Projektion basiert. Darüber hinaus untersuchen wir ein variationelles Regularisierungsverfahren. Zunächst entwickeln ausgehend vom L-Kurven-Kriterium eine neuartige Parameterstrategie für sparse Rekonstruktionen. Anschließend verfolgen wir den Ansatz, das betrachtete Variationsproblem durch eine glatte Approximation zu ersetzen und untersuchen Stabilitätseigenschaften dieser Methode. Die theoretischen Untersuchungen der in dieser Arbeit vorgestellten Methoden werden jeweils durch eine numerische Evaluation ergänzt.

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CONTENTS

INTRODUCTION

In recent decades, the field of *inverse problems* has been a rapidly growing area of applied mathematics. An inverse problem, figuratively speaking, consists in the determination of a cause based on the observation of its effect. In this fashion, an inverse problem opposes the *direct problem*, which is to determine the effect from its cause. Inverse problems often arise in physics, where only indirect measurements are available. In this setting, instead of the direct measurement of a desired physical quantity one carries out measurements of a derived quantity which is feasibly accessible. Based on a model which describes the physical relationship between both parameters, one can then compute a reconstruction of the desired quantity from the measurement data.

An example is given by the so-called *inverse medium problem*. Here, acoustic waves are emitted into a penetrable inhomogeneous medium. The fact that the frequency of acoustic waves depends on the refractive index of the medium leads to the emergence of a scattered field. The direct problem is thus to determine the scattered field from the known medium. In chapter 2, a brief introduction to the model of this process is introduced. Regarding applications, it is often not possible to access the refractive index directly while the measurement of acoustic waves is feasible. Hence, one may consider the related inverse problem, which consists in the computation of the refractive index based on measurements of the scattered field. Further examples of inverse problems are given by tomographic applications, non-invasive testing and image processing, for instance.

Inverse problems are often modeled by an operator equation of the form

$$A(x) = y\,,$$

where the operator A incorporates the forward model that describes the relationship between the model parameters x and the data y. For given y, the inverse problem consists in the determination of a solution $x^\dagger$ of this equation. In applications, however, usually only an inexact measurement y^δ of the true data y is known. If the operator is ill-posed, a direct inversion of the operator is unsuited for the reconstruction of the true solution $x^\dagger$ since in this case, even a very small error in the data may lead to an arbitrarily large error in the reconstruction. As a remedy, one has to incorporate a regularization that allows for the stable solution of the operator equation. A wide variety of regularization methods has been proposed. They all have in common that incorporating suitable a priori information about the true solution $x^\dagger$ into the regularization scheme contributes to the accuracy of the reconstruction. In this thesis, we focus on the reconstruction of a solution $x^\dagger$ that is sparse in ℓ_2, i.e. a solution which has only finitely many non-zero coefficients. Starting with the influential publication [15], inverse problems with sparsity constraints of this type have received an enormous attention in the recent decade, see [10, 11, 24, 38, 43].

The main objective of this thesis is to provide evidence that methods based on sparsity constraints have the potential to provide appealing reconstructions of a solution of an operator equation of the above form arising in imaging applications. In recent years, this has been suggested in [30, 29], for instance. Exemplarily, we confirm this observation numerically considering the inverse medium problem described above. To this end, we discuss two different approaches for sparse reconstruction and perform numerical experiments that demonstrate their potential. We stress that while throughout this work we keep our focus on the inverse medium problem, the strategies elaborated in this thesis are likely to perform similarly for other imaging applications.

Following the above ambition, in this thesis we put the focus of our investigations on three independent areas. First, in chapter 4 we discuss a novel iterative approach for sparse reconstruction based on the projection onto hyperplanes. Consider the metric projection of $\tilde{x}$ onto some convex C set given by

$$\arg\min_{x \in C} \|x - \tilde{x}\| .$$

To incorporate the assumption of sparsity, we discuss a generalization of the metric projection that is based on the replacement of the norm discrepancy by a more general discrepancy measure. Here, we refer to the *Bregman distance* which for some Banach space X and some mapping $f : X \to \mathbb{R}$ is defined as

$$\Delta_{f,x^*}(x, \tilde{x}) = f(\tilde{x}) - f(x) - \langle x^*, \tilde{x} - x \rangle .$$

At this, we assume that $x \in X$ and $x^* \in \partial f(x) \subset X^*$, where by ∂f we denote the subdifferential of f. It turns out that for $X = \ell_2$ the functional

$$f = \| \cdot \|_{\ell_1} + \frac{\beta}{2} \| \cdot \|_{\ell_2}^2$$

for $\beta > 0$ is of particular importance as the according projection onto hyperplanes favorably lies on the coordinate axes. We discuss basic properties of the Bregman distance with respect to this particular choice of f and provide a geometrical interpretation of the corresponding Bregman projection. Finally, we introduce an iterative regularization method based on the Bregman projection that generates a sparse reconstruction and present a numerical verification of the method based on the inverse medium problem.

As a second focal point of this thesis, in chapter 5 we consider the most prominent variational method for sparse reconstruction and demonstrate its potential when applied to the inverse medium problem. The approach is based on the minimization of the so-called the Tikhonov functional

$$x \mapsto \frac{1}{2}\|A(x) - y^\delta\|^2 + \alpha\|x\|_{\ell_1}$$

for some carefully chosen parameter $\alpha > 0$. The main rationale behind the particular choice of the objective functional is that the corresponding minimizer is known to be sparse. Further, this strategy is known to give rise to a regularization method when complemented with an appropriate parameter choice rule. However, the according choice rules are often merely of theoretical importance. Hence, we present a novel heuristic method for the choice of the regularization parameter α which is based on the L-curve criterion and present a numerical study which shows the relevance of this approach in applications, where we again consider the inverse medium problem.

Concluding the investigations of variational regularization methods, we call attention to a numerical aspect of the Tikhonov functional introduced above. It is clear that the corresponding regularization scheme necessitates the minimization of a non-smooth functional. Gradient-based minimization schemes are hence not applicable. A possible remedy is to approximate the non-smooth regularization term $\|\cdot\|_{\ell_1}$ by some smooth functional. This approach leads to a smooth optimization problem the minimizer of which can be regarded as a replacement of the originally sought for reconstruction. It can be extended to general regularization terms whenever minimization is difficult. In chapter 6, we show that the minimizer of a Tikhonov-type functional can be approximated by the minimizers of a family of functionals that on their part approximate the Tikhonov functional under mild conditions on the structure of the approximation. Further, we show convergence rates with respect to the approximation index. To this end, we develop a general framework for Tikhonov-type functionals that go beyond the particular instance discussed above.

To start with, in chapters 1 and 2 we give present a brief introduction to the general theory of linear inverse problems and the inverse medium problem, respectively. Particularly, a linearized model of the scattering problem based on the Born approximation is introduced. Moreover, in chapter 3 we equip the reader with the basic tools of convex analysis needed in subsequent investigations.

CHAPTER

ONE

A BRIEF INTRODUCTION TO INVERSE PROBLEMS

1.1 Introduction

For the mathematical formulation of an inverse problem, we consider an operator A mapping between topological spaces X and Y. Here, X is the set of model parameters and Y defines the scope of possible measurements. The *direct problem* now denotes the computation of $A(x)$ for given $x \in X$. The *inverse problem*, in contrast, is to determine a solution $x^\dagger \in X$ of the equation

$$A(x) = y \tag{1.1}$$

for a given $y \in Y$. First, the questions of existence and uniqueness of a solution of the operator equation $A(x) = y$ need attention. Beyond, we need to consider the case where the measurement data $y_M \in Y$ is an inexact approximation of y, which may result from noise originating in the measurement process. In this setting, we may seek for a solution x_M of the modified equation $A(x) = y_M$. One natural question is how x_M relates to the solution $x^\dagger$ for the exact problem. If the inverse operator of A is continuous, small perturbations in the measurement data y_M cause a small error in the reconstructed solution x_M. In such a case, we call the problem *well-posed*. However, in applications this property is often not given and thus, small deviations from the true data may lead to an arbitrarily large reconstruction error. This type of problem is referred to as *ill-posed*. We specify this notion in definition 1.1 which goes back to [26].

Definition 1.1. Let $A : X \to Y$ be a mapping between topological spaces X and Y. The problem (A, X, Y) is called *well-posed* if all of the following conditions are fulfilled.

(i) For each $y \in Y$ there is a solution $x \in X$ such that $A(x) = y$.

(ii) For each $y \in Y$ the solution of $A(x) = y$ is unique.

(iii) The inverse mapping $A^{-1} : Y \to X$ is continuous.

Otherwise, the problem is called *ill-posed*.

In practical applications, one usually has to deal with deficiencies inherent in the measurement process such that the acquisition of exact data is unrealistic. Moreover, since for an ill-posed problem even very small measurement errors may lead to unusable results, the tempting strategy to improve the measurement accuracy does not necessarily contribute to a mentionable improvement of the reconstruction of the true solution $x^\dagger$. This substantiates the need for a systematic mathematical approach for the determination of useful reconstructions.

1.2 Linear problems

Historically, the first advances in the theory of inverse problems considered the special case of linear operators. This is due to the exceptional simplicity of this setting, which allows for an comprehensive theory. Further, first important examples arising in applications like *computed tomography* show the relevance of the linearity assumption. Thus, in this section we consider the linear operator equation of the form

$$Ax = y\,, \tag{1.2}$$

where $A : X \to Y$ is a bounded linear operator between Hilbert spaces X and Y. We assume that instead of the exact data $y \in \operatorname{rg}(A)$ in eq. (1.2), we are given a noisy observation

$$y^\delta = y + \eta$$

with bounded noise η, i.e. $\|y - y^\delta\|_Y \le \delta$. The scalar $\delta > 0$ that controls the accuracy of the data is referred to as the *noise level*. It is usually determined by the measurement setup in applications.

The task is now to find an approximation of a solution of eq. (1.2) from the data y^δ. In section 1.1 we have seen that if the problem is ill-posed, the naive approach compute a reconstruction by inverting A does not provide useful results. First, we remark that usually, the noise η does not lie in $\operatorname{rg}(A)$ and hence, the equation $Ax = y^\delta$ does not have a solution, see definition 1.1 item (i). Thus, instead of a solution of eq. (1.2), we seek a minimizing element of the objective functional

$$x \mapsto \|Ax - y^\delta\|_Y\,. \tag{1.3}$$

Further, given that the null space of the operator A is non-trivial the solution of eq. (1.2) is not unique, see definition 1.1 item (ii). A remedy is to select a special solution out of the set of possible choices. Namely, for given $\tilde{x} \in X$, by $x^\dagger$ we denote the unique solution of eq. (1.2) such that the discrepancy $\|x - \tilde{x}\|_X$ is minimal among all solutions x. This solution $x^\dagger$ is called *$\tilde{x}$-minimum-$\|\cdot\|_X$-solution*. Usually we consider the solution that corresponds to the choice $\tilde{x} = 0$, also referred to as the *minimum-$\|\cdot\|_X$-solution*.

One can show that the functional in eq. (1.3) attains its minimizer if and only if $y^\delta \in \mathrm{rg}(A) \otimes \mathrm{rg}(A)^\perp$. Consequently, we can now map any element $y^\delta \in \mathcal{D}(A^\dagger)$ on the according minimum-$\|\cdot\|_X$-solution of eq. (1.2), where $\mathcal{D}(A^\dagger) := \mathrm{rg}(A) \oplus \mathrm{rg}(A)^\perp$. The operator defined by this relation we denote by $A^\dagger$. It is easy to see that $A^\dagger : \mathcal{D}(A^\dagger) \to X$ is linear. Further, if A is invertible, then $A^\dagger$ coincides with A^{-1}. Hence, the operator $A^\dagger$ is called the *generalized inverse* of A.

The strategy behind the generalized inverse suggests that the violation of the assertions in definition 1.1 items (i) and (ii) can be regarded as amendable. Thus, in the following we consider the setting where the inverse operator of A is unbounded, see definition 1.1 item (iii). In this setting, the solution $x^\dagger$ does not depend continuously on the data y^δ and hence we need to stabilize the inversion of A. The main idea is to approximate $A^\dagger$ by a family of continuous operators, which leads to the following notion.

Definition 1.2. Let $x^\dagger$ be the minimum-$\|\cdot\|_X$-solution of the equation $Ax = y$ with exact data $y \in \mathrm{rg}(A)$. For arbitrary $\alpha \in (0, \alpha_0)$ let there be a (possibly non-linear) continuous mapping

$$T_\alpha : Y \to X\,,$$

where $\alpha_0 \in (0, \infty]$. The family $\{T_\alpha\}_{\alpha \in (0,\alpha_0)}$ is called a *regularization* if there is a *parameter choice rule* $\alpha = \alpha(\delta, y^\delta)$ such that it holds

$$\limsup_{\delta \to 0} \{\|T_{\alpha(\delta,y^\delta)} y^\delta - x^\dagger\|_X \,:\, y^\delta \in Y\,,\ \|y - y^\delta\|_Y \le \delta\} \to 0\,. \tag{1.4}$$

A mapping $\alpha : [0, \infty) \times Y \to (0, \alpha_0)$ is a parameter choice rule if and only if there holds

$$\limsup_{\delta \to 0} \{\alpha(\delta, y^\delta) \,:\, y^\delta \in Y\,,\ \|y - y^\delta\|_Y \le \delta\} \to 0\,. \tag{1.5}$$

The pair (T_α, α) is called *regularization method* if eqs. (1.4) and (1.5) hold.

A regularization method can be described by the characterizing property that the reconstruction $x_\alpha^\delta = T_\alpha(y^\delta)$ converges to the minimum-$\|\cdot\|_X$-solution $x^\dagger$ if the noise level δ tends to 0, regardless the specific form of the noise inherent in the data. Note that the desired convergence result depends on the appropriate choice of the parameter $\alpha = \alpha(\delta, y^\delta)$. Next, in section 1.3, we briefly present classical approaches in regularization theory.

1.3 Classical regularization methods

Subsequently, we present classical representatives of the classes of both iterative and variational regularization schemes. Generalizations of these methods will be discussed in chapters 4 and 5, respectively. In this section, we consider a linear operator A mapping between Hilbert spaces X and Y.

1.3.1 Landweber iteration

One prominent method to regularize the problem corresponding to a linear operator equation as in eq. (1.1) is given by the so-called *Landweber iteration* that introduces an iterative scheme for the stable solution of the inverse problem. Here, the discrepancy as in eq. (1.3) is minimized by an iterative minimization scheme. Recall that if the problem is ill-posed and $y^\delta \notin \mathcal{D}(A^\dagger)$, the functional $\|A \cdot - y^\delta\|_Y$ as in eq. (1.3) does not have a minimizer. Thus, it is apparently futile to seek an exact solution to this problem. Instead, we terminate the iteration after a finite number of iterations according to a certain stopping rule.

For iterative methods, the stopping rule plays the role of the parameter choice rule. The task is to find an appropriate n to exit the iteration. On the one hand, one needs to assure that the objective functional is small, i.e. the equation $Ax = y^\delta$ is approximately satisfied. On the other hand, the iteration needs to stop before the reconstruction is over-iterated, which in applications would imply that the solution is overly noisy as a consequence of the noise inherent in the data y^δ.

The Landweber iteration is a simple example for an iterative regularization method. Starting from an initial guess $x_0 \in X$ of the true solution $x^\dagger$ one iteratively follows the direction of the steepest descent which results in the fixed point iteration

$$x_{n+1} = x_n - tA^*(Ax_n - y^\delta)\,, \tag{1.6}$$

where $t > 0$ is chosen appropriately. By T_n, we define a regularization, where by abuse of language, the index n takes the role of α^{-1}. The iteration scheme in eq. (1.6) needs to be complemented with an appropriate stopping rule. One prominent example is given by the discrepancy principle. Accordingly, the stopping index $n = n(\delta, y^\delta)$ is given by the smallest integer $m \in \mathbb{N}$ such that

$$\|Ax_m - y^\delta\|_Y < \tau\delta \tag{1.7}$$

for some fixed parameter $\tau > 1$. The above Landweber iteration scheme combined with the parameter choice according to the discrepancy principle is well-defined and gives rise to a regularization method if $t < 2\|A\|^{-2}$. We do not present a proof for this claim as it is a standard result. It can be found in [45], for instance.

1.3.2 Tikhonov-Phillips regularization

One prominent variational approach to regularize an inverse problem is given by the so-called *Tikhonov-Phillips regularization.* As discussed above, regarding an ill-posed operator A the functional $\|A \cdot - y^\delta\|_Y$ as in eq. (1.3) does not have a minimizer if $y^\delta \notin \mathcal{D}(A^\dagger)$. Particularly, this implies that it necessarily holds $\|x^n\|_X \to \infty$ as $n \to \infty$ for every minimizing sequence $\{x^n\}_{n\in\mathbb{N}}$ of eq. (1.3), see

[45]. To avoid this behavior, the idea is to introduce an additional term to the objective functional that ensures boundedness of every minimizing sequence with respect to the norm. This leads to the functional

$$\Psi_\alpha(x) = \|Ax - y^\delta\|_Y^2 + \alpha \|x\|_X^2 \,, \tag{1.8}$$

which is referred to as the *Tikhonov functional*. Coercivity and weak lower semicontinuity provide the existence of a minimizer of eq. (1.8) which by strict convexity of the functional is uniquely determined. We denote this minimizer by

$$x_\alpha^\delta = \operatorname*{arg\,min}_{x \in X} \Psi_\alpha(x) = \operatorname*{arg\,min}_{x \in X} \|Ax - y^\delta\|_Y^2 + \alpha \|x\|_X^2 \tag{1.9}$$

for arbitrary $\alpha > 0$. By $T_\alpha y^\delta := x_\alpha^\delta$ we can now define a regularization. While in iterative regularization schemes, the choice of the regularization parameter translates to the determination of a proper iteration index to exit the iteration, here the regularization parameter α plays the role of a weight that balances the *fidelity term* $\|Ax - y^\delta\|_Y^2$ against the *regularization term* $\|x\|_X^2$. If the parameter choice rule $\alpha = \alpha(\delta)$ satisfies

$$\alpha \to 0 \,, \ \frac{\delta^2}{\alpha} \to 0 \text{ as } \delta \to 0 \,,$$

then (T_α, α) is a regularization method. An according result can be found in [45], for instance.

CHAPTER
TWO

BORN APPROXIMATION OF THE INVERSE MEDIUM PROBLEM

2.1 Introduction

In the following, we discuss a model which describes the scattering of either electromagnetic waves in transverse magnetic polarization from a penetrable non-magnetic material, or of acoustic waves from a penetrable inhomogeneous medium with constant density. Particularly, we introduce the linearization of the discussed operator given by the Born approximation.

After a brief introduction to the mathematical model of the scattering process and the definition of the operator corresponding to the according inverse problem, in section 2.4 we present reconstructions based on both synthetic and measurement data. Particularly, we show that using the inexact model based on the Born approximation still allows for the determination of reasonable reconstructions.

2.2 The Helmholtz equation

The mathematical model for the propagation of time-harmonic waves in a homogeneous medium is given by the *Helmholtz equation* which reads

$$\triangle u + k^2 u = 0 \text{ in } \mathbb{R}^d \,, \tag{2.1}$$

where d is the dimension of the space under consideration. Here, $k \in \mathbb{R}$ is the *wave number* that by $k = \frac{\omega}{c_0}$ relates to the angular frequency ω of the wave, where c_0 denotes the velocity of propagation. A *time-harmonic wave* is a solution of eq. (2.1). For a deeper insight for the deduction of the Helmholtz equation, see [14].

Any plain wave is easily seen to solve eq. (2.1). To ensure uniqueness of the solution, it has to be complemented by the *Sommerfeld radiation condition* which requires

$$\lim_{|x|\to\infty} |x|^{\frac{d-1}{2}} \left(\frac{\partial}{\partial |x|} - ik \right) u^s(x) = 0 \tag{2.2}$$

uniformly in all directions $\hat{x} = \frac{x}{|x|}$. Figuratively speaking, this assumption ensures that the field is outgoing, i.e. the field has no sinks and no energy is radiated from infinity into the field. The solution of an inhomogeneous Helmholtz equation satisfying the Sommerfeld radiation condition is called *radiating*. Particularly, the radiating fundamental solution Φ of eq. (2.1) can be expressed in a closed form. For the two-dimensional setting $d = 2$ and $x, y \in \mathbb{R}^2$, $x \neq y$ it is given by

$$\Phi(x, y) = \frac{i}{4} H_0^{(1)}(k|y - x|)\,, \tag{2.3}$$

where $H_0^{(1)}$ is the Hankel function.

In eq. (2.1), we only considered the setting of a homogeneous medium. For the inverse medium problem, however, it is essential to incorporate a potentially inhomogeneous medium. Hence, to model a medium such that the refractive index n is dependent on the location, we consider the *modified Helmholtz equation* which is given by

$$\triangle u + k^2 n u = 0\ ,\ \text{in } \mathbb{R}^d\,. \tag{2.4}$$

We assume the medium has a constant density and the refractive index equals one outside some bounded set Ω. Obviously, if $n \equiv 1$ holds even inside of Ω, i.e. the background is air, this equation reduces to eq. (2.1).

2.3 Modeling the inverse medium problem

The inverse medium problem describes the task to recover the refractive index $n = \frac{c_0^2}{c^2}$ of the medium from measurements of the total field, which results from the induction of an incident field. If the medium is absorbing, then the refractive index is complex, where the real part corresponds to the scattering of the wave and the imaginary part corresponds to its absorption, see [14].

We first describe the setup of the inverse medium problem and introduce the modalities of the measurement topology. Let u^i denote a time-harmonic incident field, i.e. a solution of the *Helmholtz equation* as in eq. (2.1). The field is generated by an assemblage of transceivers. It is scattered in the inhomogeneous medium due to changes in the refractive index, which results in a total field u^t that is measured by appropriately located receivers. The goal is now to compute the contrast $q = n - n_b$ with respect to some a priori known background n_b from this measurement data. We assume that the support of q is located in a known bounded set $\Omega \subset \mathbb{R}^d$ we refer to as the *region of interest (ROI)*. This is to say the medium does not differ from the background outside of the ROI. Furthermore, there is a manifold $\Gamma_{TX} \subset \mathbb{R}^d$ outside of Ω on which the transmitters are located as well as a manifold $\Gamma_{RX} \subset \mathbb{R}^d$ on which the receivers are located. A common

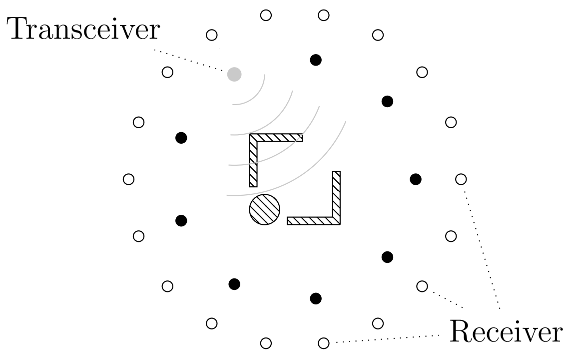


Figure 2.1: Experimental setup consisting of the obstacle, transceivers generating the incident field and receivers to measure the total field.

additional assumption is that the background is air, which gives $n_b \equiv 1$ and, hence, $q \equiv n - 1$. The basic setup is illustrated in Figure 2.1.

The inverse medium problem can now be modeled as follows. First, an incident field u^i is generated, which is scattered in the medium. The receivers then measure the total field u^t, which can be seen as the sum of the incident field and the scattered field u^s, i.e. $u^t = u^i + u^s$. The scattered field u^s can be described by the modified Helmholtz equation as in eq. (2.4). We additionally assume that the scattered field satisfies the Sommerfeld radiation condition as in eq. (2.2). This yields the alternate description

$$u^t - V(qu^t) = u^i \,, \tag{2.5}$$

where by V we denote the radiating volume potential $V : L^{p_2}(\Omega) \to L^{p_3}(\Omega)$ given by

$$(Vu)(x) = -k^2 \int_\Omega \Phi(x, y) u(y)\, dy \,.$$

Here, Φ denotes the fundamental solution of the Helmholtz equation, see eq. (2.3). The identity in eq. (2.5) is referred to as the *Lippmann-Schwinger* form of the scattering problem. For arbitrary $q \in L^r(\Omega)$, we may now define a linear operator $T(q) \in \mathcal{L}(L^{p_3}(\Omega), L^{p_2}(\Omega))$ given by

$$T(q) = (\mathrm{Id} - V(q\,\cdot\,))^{-1} \,, \tag{2.6}$$

which maps the incident field to the total field, i.e. $u^t = T(q)u^i$.

The incident field u_i as a mapping defined on Ω is induced by the transceivers located on the manifold Γ_{TX}. It can be described by the single layer potential

$SL_{TX} : L^{p_1}(\Gamma_{TX}) \to L^{p_2}(\Omega)$ given by

$$(SL_{TX} f_{TX})(x) = \int_{\Gamma_{TX}} \Phi(x,y) f_{TX}(y)\, dS(y)\,, \tag{2.7}$$

which maps a setup of transceivers $f_{TX} \in L^{p_1}(\Gamma_{TX})$ to the time-harmonic wave in the ROI. Usually, f_{TX} can be described as a point source or the sum of several point sources which can be modeled by delta peaks. Further, the total field u^t on the ROI generates a scattered field u^s_{RX} on the measurement manifold Γ_{RX} on which the receivers are located. It can be described using the volume potential operator $V_{RX} : L^{p_3}(\Omega) \to L^{p_4}(\Gamma_{RX})$ defined by

$$(V_{RX} u)(x) = -k^2 \int_{\Omega} \Phi(x,y) u(y)\, dy\,.$$

By this means, we get the relation $u^s_{RX} = V_{RX}(qu^t)$.

Altogether, the forward operator $F : L^r \to \mathcal{L}(L^{p_1}(\Gamma_{TX}), L^{p_4}(\Gamma_{RX}))$ of the scattering problem mapping the contrast q to the operator which describes the relation of the setup of transceivers and the total field measured at the receivers is defined via

$$F(q) = V_{RX}(qT(q)SL_{TX})\,. \tag{2.8}$$

For a sensible choice of the scalars p_1, p_2, p_3, p_4 and r we refer the reader to [36]. Note that the operator in eq. (2.8) is non-linear in q. In section 2.3.1, we discuss a linearization of this model based on the Born approximation.

2.3.1 The Born approximation

In section 2.3 we have developed a model for the inverse medium problem. It is well-known to provide an accurate description of the scattering process and can be used to compute plausible reconstructions for the inverse medium problem. However, the operator in eq. (2.8) is non-linear, which contributes to additional complexity in reconstruction methods based on this model. In order to adapt the problem to the theory developed in section 1.2, we suggest to use a linearized version of eq. (2.8) based on the so-called Born approximation.

The non-linearity of the model in eq. (2.8) is contained in the operator T as in eq. (2.6). It can formally be written as a Neumann series, which leads to the identity

$$T(q) = (Id - V(q\,\cdot\,))^{-1} = Id + V(q\,\cdot\,) + (V(q\,\cdot\,))^2 + \dots\,. \tag{2.9}$$

The idea is to replace the full operator T by the first expansion term in eq. (2.9), i.e. the identity operator Id. By eq. (2.8), this leads to the forward operator

$$A(q) = V_{RX} q SL_{TX}\,. \tag{2.10}$$

The operator A is referred to as the *Born approximation* of the scattering problem and is linear in q. Figuratively speaking, the scattering process inside the ROI is ignored and only the scattering in the exterior is considered. The accuracy of this approximation depends on the parameter k, the wave number. The model is valid if $k^2 q << 1$, see [14]. In chapter 5 we demonstrate that using this linearization, the related inverse problem yields plausible results when applied to real data.

2.4 Numerical realization

In this thesis, we discuss both iterative and variational approaches for the numerical reconstruction of a solution of the inverse problem $Ax = y^\delta$ with A as in eq. (2.10). Particularly, in chapters 4 to 6 we investigate sparsity-enforcing approaches for the solution of the inverse problem. Each of these chapters is concluded by a numerical evaluation of the methods specified therein. At this, we follow two main intentions. First, we want to demonstrate the beneficial behavior of newly developed sparsity-promoting methods when compared to classical approaches considering the stable inversion of the Born approximation of the inverse medium problem as an example for a severely ill-posed and non-injective operator. Furthermore, we want to provide evidence that it is possible to determine reasonable reconstructions from measurement data by means of the Born approximation. To start with, for later reference in this section we present reconstructions according to classical approaches as introduced in section 1.3.

2.4.1 Implementation of the forward operator

Next, we make some remarks regarding the implementation of the Born approximation derived in eq. (2.10). To start with, we propose an alteration of the domain of the operator. Formally, throughout this thesis, we investigate mappings defined on ℓ_2. We argue that this is not a restriction as we may reformulate any operator $A : X \to Y$ mapping between Banach spaces X and Y as a mapping $\ell_2 \to Y$. To this end, for a linearly independent system $\{\psi_k\}_{k\in\mathbb{N}}$ in a Banach space X, let S denote the corresponding synthesis operator that maps an element $x = (x_k)_{k\in\mathbb{N}} \in \ell_2$ onto

$$Su = \sum_{k\in\mathbb{N}} u_k \psi_k \,.$$

Instead of the operator $A : X \to Y$, one may now consider the operator $A \circ S : \ell_2 \to Y$.

Furthermore, the application of numerical methods for the solution of an inverse problem often requires the underlying operator to be defined on real spaces. While the operator presented in eq. (2.10) is a mapping between complex spaces, one can easily translate this into a real setting via

$$\ell_2(\mathbb{R}) \times \ell_2(\mathbb{R}) \to \ell_2(\mathbb{C}) \,, \; (x, y) \mapsto x + iy \,.$$

It is easily seen that this mapping retains main properties such as linearity and continuity and we may thereby assume, without loss of generality, that any operator we consider is defined on a real rather than on a complex space. Consequently, in the subsequent chapters, only the former setting is covered.

We now turn to the numerical aspects of the Born approximation of the scattering problem. Typically, the operator in eq. (2.10) is discretized via collocation. Let Ω be discretized by N^2 points with mesh-size h. Further, let Γ_{TX} be discretized by N_{TX} points and let Γ_{RX} be discretized by N_{RX} points. The operator then reads

$$A : \mathbb{C}^{N\times N} \to \mathbb{C}^{N_{RX}\times N_{TX}} . \tag{2.11}$$

Furthermore, we consider the synthesis operator S corresponding to two different independent systems in $\mathbb{C}^n$. Namely, we consider the discrete wavelet transform corresponding to the Haar wavelet and the biorthogonal Cohen-Daubechies-Feauveau wavelet (CDF) using $9, 7$ filter sizes, see [13]. Together, this leads to the operator

$$K = A \circ S : \mathbb{C}^{N\times N} \to \mathbb{C}^{N_{RX}\times N_{TX}} . \tag{2.12}$$

We implement the wavelet transform as a mapping $\mathbb{C}^{N\times N} \to \mathbb{C}^{N\times N}$ following the lifting scheme, see [55].

2.4.2 Applications to synthetic and measurement data

Subsequently, we present the reconstructions according to both synthetic and real data established by classical approaches, see section 1.3. These results may be seen as a reference point for further numerical experiments incorporating more elaborate approaches as discussed in chapters 4 to 6.

To start with, we consider synthetic data generated by the true solution $x^\dagger$ as depicted in fig. 2.2. We remark that the solution in fig. 2.2 has a sparse representation in both the Haar wavelet basis and the CDF wavelet basis as in section 2.4, i.e. it corresponds to only a few coefficients in the respective bases. For our numerical applications, we refer to a wave number $k = 200$. We choose a comparably large wave number since experiments show that the inversion of the corresponding operator is particularly difficult for high values and hence, by this means one can identify methods that are performing well even in a demanding setting. In fact, the condition number of the operator in eq. (2.11) rises if we increase k. Due to the inaccuracy of the approximation of the operator in eq. (2.11), this data does not correspond to an exact physical description of the scattering process. Nonetheless, based on synthetic data we can evaluate regularization methods regarding their respective properties when applied to a highly ill-posed and non-injective operator.

In fig. 2.3, we present the reconstruction of the true solution $x^\dagger$ as depicted in fig. 2.2 according to classical regularization schemes, see section 1.3. Here, we chose the regularization parameter such that it minimizes the reconstruction error in the least squares sense.

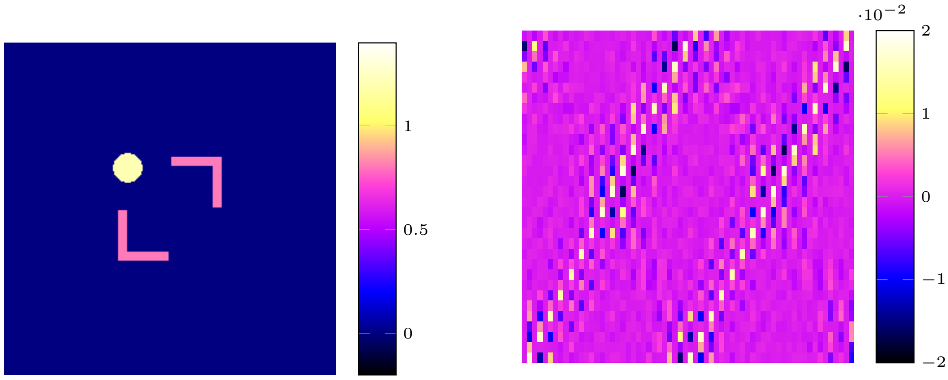


Figure 2.2: The real part of the obstacle $x^\dagger$ to be reconstructed in subsequent numerical applications (left) and the corresponding data $y^\dagger = A(x^\dagger)$ (right).

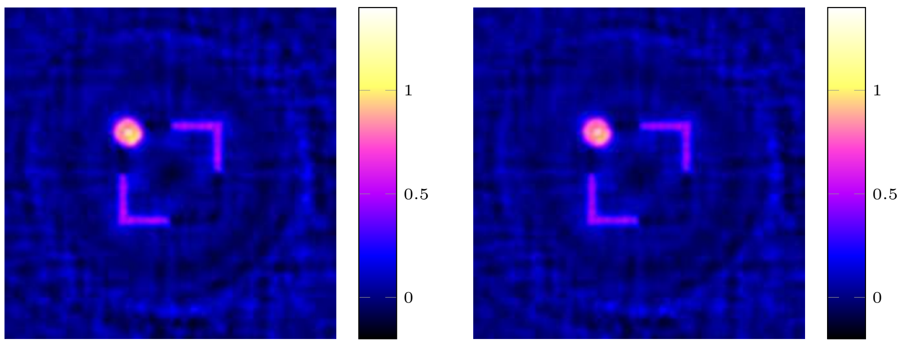


Figure 2.3: The real part of the reconstruction from noisy data with 10% of Gaussian noise according to the Landweber iteration eq. (1.6) (left) and according to Tikhonov-Phillips regularization as in eq. (1.9) (right).

One observes in fig. 2.3 that both reconstructions localize the obstacle depicted in fig. 2.2 quite accurately, yet show an oscillating background and a lack of contrast. In chapters 4 and 5, we evaluate as of how this deficiency can be avoided. To this end, we present generalized methods based on the Landweber iteration and Tikhonov-Phillips regularization, respectively. We show that using these methods, the reconstructions in fig. 2.3 can be notably improved.

Next, we demonstrate the relevance of the above approach for the application to real data. Particularly, we demonstrate that despite it being only an inexact approximation of the full model, the Born approximation of the scattering problem gives rise to appealing reconstructions even for measurement data. In our experiments, we use data provided in the freely accessible Fresnel database, see [5], where the setup of the experiment is guided by the modalities described in section 2.3.

In contrast to the experiments using synthetic data, here we do not refer to a

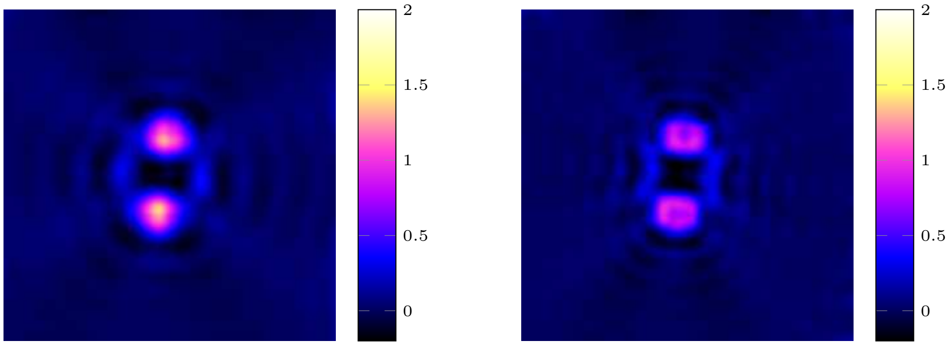


Figure 2.4: The real part of the reconstruction from measurement data according to classical Tikhonov-Phillips regularization as in eq. (1.9) for $k \approx 83$ (left) and $k \approx 125$ (right).

incident field u^i that is generated according to eq. (2.7). Instead, a measurement is taken with an empty ROI to determine the incident field at the receivers. From this measurement data, we then determine the incident field in the ROI. To this end, we follow an approach outlined in [21]. To start with, we remark that the harmonic polynomial of degree N given by

$$u(r, \varphi) = \sum_{n=-N}^{N} c_n e^{in\varphi} H_n^{(1)}(rk) \tag{2.13}$$

is a radiating solution of the Helmholtz equation as in eq. (2.1) in $\mathbb{R}^2 \setminus \{0\}$ corresponding to a point source at the origin, see [14, Thm 2.10]. For $n = -N, \dots, N$ we chose the coefficients $c_n \in \mathbb{C}$ such that the resulting wave u matches the measured values at the coordinates of the receivers in the least squares sense. Throughout this work, we use $N = 7$ for numerical applications. The total field resulting from the scattering at some obstacle in the ROI is measured afterwards. For a more detailed consideration of the experimental setup, we refer the reader to [5].

In the subsequent numerical applications incorporating real data, we refer to data resulting from two circle-shaped obstacles and an induced electromagnetic field with both 4 GHz and 6 GHz, which corresponds to $k \approx 83$ and $k \approx 125$, respectively. An estimation of the contrast q of the obstacle in the ROI is given by $q = 2 \pm 0,3$, see [5]. A reconstruction according to Tikhonov-Phillips regularization is given in fig. 2.4. Here, the regularization parameters were chosen manually in order to obtain a visually appealing reconstruction.

In fig. 2.4, the location of the obstacle is recovered quite nicely. The reconstructions provide evidence that the Born approximation can be used in order to obtain useful solutions of the inverse medium problem. However, there is a notable lack of contrast in both reconstructions. In section 5.5, we present reconstructions obtained by more elaborate methods and discuss as of how the results in fig. 2.4 can be improved.

CHAPTER
THREE

CONVEX ANALYSIS AND BREGMAN PROJECTIONS

3.1 Introduction

In the following, we recall some basic definitions and results from convex analysis which are needed for subsequent investigations. Particularly, we introduce the well-established Bregman distance and the closely related notion of the so-called Bregman projection which generalizes the metric projection. Beyond standard knowledge, in section 3.5 we discuss a particular instance of the Bregman projection and provide a geometrical interpretation for that mapping.

Throughout this chapter let X be a Banach space and let X^* denote the dual space of X. Furthermore, we denote by $\overline{\mathbb{R}} := \mathbb{R} \cup \{\infty\}$ the completed real line.

3.2 Basic notions

The main notion in convex analysis is that of convexity. A mapping $f : X \to \overline{\mathbb{R}}$ is called *convex* if for arbitrary $x, y \in X$ and $t \in (0, 1)$ it holds $f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$. It is called *strictly convex* if the previous estimate is strict. Moreover, by $\mathrm{dom}(f) := f^{-1}(\mathbb{R})$ we denote the *domain* of f. The mapping f is called *proper* if $\mathrm{dom}(f) \neq \emptyset$. Subsequently, we introduce further notions of convex analysis and collect basic results.

Definition 3.1. Let f be convex. For $x \in \mathrm{dom}(f)$ we define the *subdifferential* $\partial f(x) \subset X^*$ of f by

$$x^* \in \partial f(x) \;:\Leftrightarrow\; f(y) - f(x) \geq \langle x^*, y - x\rangle_{X^* \times X} \text{ for all } y \in X\,.$$

Any element $x^* \in \partial f(x)$ is called a *subgradient* of f at x.

The notion of the subgradient generalizes the classical derivative to arbitrary convex functionals. Particularly, if f is differentiable at $x \in X$ the subdifferential is single-valued and we have $\partial f(x) = \{f'(x)\}$. For any convex function f we now specify an associated function f^* which proofs to be a useful tool in convex analysis.

Definition 3.2. Let f be convex. The mapping $f^* : X^* \to \overline{\mathbb{R}}$ given by

$$f^*(x^*) := \sup_{x \in X} \Big(\langle x^*, x \rangle_{X^* \times X} - f(x) \Big)$$

for $x^* \in X^*$ is called the *convex conjugate* of f.

In lemma 3.3 and lemma 3.4 we elaborate as of how the subgradient and the convex conjugate are related. To start with, by means of the convex conjugate we state the important inequality eq. (3.1) which is sometimes referred to as the Young inequality.

Lemma 3.3. *Let f be proper and convex. For arbitrary $x \in X$ and $x^* \in X^*$ it holds*

$$f(x) + f^*(x^*) \geq \langle x^*, x \rangle_{X^* \times X} , \tag{3.1}$$

where we have equality if and only if $x^* \in \partial f(x)$.

Next, the result in lemma 3.4 reveals that for some convex function f the subgradient of the convex conjugate f^* can be regarded as the inverse mapping of ∂f. For a proof of lemma 3.3 and lemma 3.4, we refer to [49, Proposition 4.4.1] and [49, Proposition 4.4.4], respectively.

Lemma 3.4. *Let f be proper and convex. Then, it holds*

$$x \in \operatorname{dom}(f),\ x^* \in \partial f(x) \ \Rightarrow\ x^* \in \operatorname{dom}(f^*),\ x \in \partial f^*(x^*) .$$

If, in addition, X is reflexive and f is lower semicontinuous, the converse holds true as well.

Definition 3.5. Let $x^* \in \partial f(x)$. A convex function f is called *uniformly convex* if for some function $\phi : \mathbb{R} \to \mathbb{R}$ such that $\phi(t) = 0$ if and only if $t = 0$ it holds

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) - t(1-t)\phi(\|x-y\|) . \tag{3.2}$$

If this property holds true for the choice $\phi(t) = \frac{c_0}{2} t^p$ for some scalar $c_0 > 0$ and $1 < p < \infty$, then the mapping f is called *p-convex.*

Figuratively speaking, the notion of uniform convexity yields a description of the vertical gap between the line from $f(x)$ to $f(y)$ and the corresponding values of f in terms of the distance $\|x-y\|$. Any uniform convex mapping is strictly convex.

Lemma 3.6. *Let f, g be functions mapping from X to $\overline{\mathbb{R}}$. If f is convex and g is uniformly convex, then $f+g$ is uniformly convex.*

Proof. Let $x, y \in X$ and $t \in (0,1)$. We estimate

$$\begin{aligned}(f+g)(tx+(1-t)y) &= f(tx+(1-t)y) + g(tx+(1-t)y) \\ &\leq tf(x) + (1-t)f(y) + tg(x) + (1-t)g(y) - t(1-t)\phi(\|x-y\|) \\ &= t(f+g)(x) + (1-t)(f+g)(y) - t(1-t)\phi(\|x-y\|)\,,\end{aligned}$$

which gives the assertion. □

Finally, we cite further results which can be established for the special case of p-convex functionals. The proofs of lemma 3.7 and lemma 3.8 can be found in [33], for instance.

Lemma 3.7. *Let $x, y \in X$ and $x^* \in \partial f(x) \cap \mathrm{dom}(f^*)$, $y^* \in \partial f(y)$. If f is p-convex, then it holds*

$$f^*(x^*) \leq f^*(y^*) + \langle x^* - y^*, x\rangle_{X^*\times X} + c_0^*\|x^* - y^*\|^{p^*}\,, \tag{3.3}$$

where we set

$$c_0^* = \frac{1}{c_0^{p^*-1} p^*}$$

and c_0 is the constant as in definition 3.5.

Lemma 3.8. *If f is p-convex for $p \geq 2$, then f^* is Fréchet-differentiable and it holds*

$$\|\nabla f^*(x^*) - \nabla f^*(y^*)\|^p \leq \left(\frac{\|x^* - y^*\|}{c_0}\right)^{p^*}$$

for all $x^*, y^* \in X^*$, where $\nabla f^* : X^* \to X$ is the gradient of f^*.

3.3 Bregman distance

In this section, we introduce the notion of the Bregman distance, which is a generalized distance measure.

Definition 3.9. For some convex functional f as well as $x \in \mathrm{dom}(f)$, $y \in X$ and $x^* \in \partial f(x)$, we denote by

$$\Delta_{f,x^*}(x,y) := f(y) - f(x) - \langle x^*, y - x\rangle_{X^*\times X} \tag{3.4}$$

the *Bregman distance* with respect to f at x^*.

One immediately observes that for arbitrary $x \in \mathrm{dom}(f)$ and $x^* \in \partial f^*(x)$ the Bregman distance is non-negative and it holds $\Delta_{f,x^*}(x,x) = 0$. However, unlike a proper metric, the Bregman distance is not necessarily symmetric. If f is differentiable, the subdifferential $\partial f(x)$ is single-valued and we can drop the dependency on the special choice of $x^* \in \partial f(x)$. In this setting, we write $\Delta_f(x,y)$ instead of $\Delta_{f,x^*}(x,y)$. Subsequently, we give some examples for prominent choices for f.

Example 3.10. Consider a Hilbert space X and the mapping $f : X \to \overline{\mathbb{R}}$ given by $f = \frac{1}{2}\|\cdot\|_X^2$. The functional f is differentiable with $\partial f(x) = \{x\}$ and for arbitrary $x, y \in \ell_2$ we obtain

$$\Delta_f(x,y) = \frac{1}{2}\|y\|_X^2 - \frac{1}{2}\|x\|_X^2 - \langle x, y-x\rangle_X = \frac{1}{2}\|y-x\|_X^2\,,$$

where in the last step we made use of the polarization identity. This example shows that in this setting, the Bregman distance coincides with the metric induced by the norm in X.

Example 3.11. Consider $X = \ell_2$ and the functional $f : \ell_2 \to \overline{\mathbb{R}}$ given by $f = \|\cdot\|_{\ell_1}$ in ℓ_1 and $f = \infty$ else. Its subdifferential can be given explicitly by $\partial f(x) = \mathrm{Sign}(x) \cap \ell_2$, where Sign is the set-valued *signum operator* given by

$$(\mathrm{Sign}(x))_k = \begin{cases} 1\,, & x_k > 0 \\ [-1,1]\,, & x_k = 0 \\ -1\,, & x_k < 0 \end{cases}. \tag{3.5}$$

It is non-empty if and only if $x \in \ell_0$. For $x^* \in \partial f(x) = \mathrm{Sign}(x) \cap \ell_2$ we now get the relation $\langle x^*, x\rangle = \|x\|_{\ell_1}$ and hence

$$\Delta_{f,x^*}(x,y) = \|y\|_{\ell_1} - \|x\|_{\ell_1} - \langle x^*, y-x\rangle = \|y\|_{\ell_1} - \langle x^*, y\rangle_X \tag{3.6}$$

for arbitrary $y \in \ell_2$. Note that $\Delta_{f,x^*}(x,y)$ vanishes if and only if x and y lie in the same orthant.

The notion of uniform convexity given in definition 3.5 has an important consequence for the Bregman distance which we state in the following lemma. The proof follows easily from eqs. (3.2) and (3.4) and rearranging the components.

Lemma 3.12. *Let f be uniformly convex and let $x \in X$ and $x^* \in \partial f(x)$. Then it holds*

$$\Delta_{f,x^*}(x,y) \geq \phi(\|y-x\|) \tag{3.7}$$

for arbitrary $y \in X$.

If f is p-convex, the estimate in eq. (3.7) reads

$$\Delta_{f,x^*}(x,y) \geq \frac{c_0}{2}\|y-x\|^p \,. \tag{3.8}$$

The inequality in eq. (3.7) particularly implies that the Bregman distance inherits several properties from the norm of the space X. For instance, one immediately sees that the Bregman distance is coercive in both arguments. In lemma 3.13 we collect further properties.

Lemma 3.13. *Let f be a convex functional and let $x \in \operatorname{dom}(f)$ as well as $x^* \in \partial f(x)$. Then, for a convex functional g and $z^* \in \partial g(x)$ it holds*

(i) $x^* + z^* \in \partial(f+g)(x)\,,$

(ii) $\Delta_{f+g,x^*+z^*}(x,y) = \Delta_{f,x^*}(x,y) + \Delta_{g,z^*}(x,y)$

and for $\alpha > 0$ we have

(iii) $\alpha x^* \in \partial(\alpha f)(x)\,,$

(iv) $\Delta_{\alpha f,\alpha x^*}(x,y) = \alpha\Delta_{f,x^*}(x,y)\,,$

where $y \in X$ is arbitrary.

For a proof of lemma 3.13 item (i) we refer to [49, Prop. 4.5.1]. The following assertions can be established simply using the definition in eq. (3.4) and rearranging the terms. The Bregman distance depends on two input arguments. When considering the first argument as fixed, the mapping inherits a couple of properties from the underlying functional f. In lemma 3.15, we collect some corresponding assertions. Beforehand, for later reference we introduce the following notion that is a well-established tool in the context of functional analysis.

Definition 3.14 (Kadec-Klee property)**.** A functional $f : X \to \overline{\mathbb{R}}$ is said to obey the *Kadec-Klee property* if and only if for every sequence $\{x^n\}_{n\in\mathbb{N}}$ in X that is converging weakly to some $\tilde{x} \in X$ it holds

$$f(x^n) \to f(\tilde{x}) \text{ as } n \to \infty \quad \Longrightarrow \quad x^n \to \tilde{x} \text{ as } n \to \infty\,,$$

i.e. convergence of the functional value already implies strong convergence.

We remark that the norm in a Hilbert space X does obey the Kadec-Klee property introduced in definition 3.14 as it is easily seen.

Lemma 3.15. *Let f be weakly lower semicontinuous and let $\{x^n\}_{n\in\mathbb{R}}$ be a sequence converging weakly to $\tilde{x} \in X$. Furthermore, let $x \in \operatorname{dom}(f)$ and $x^* \in \partial f(x)$. Then the following assertions hold true:*

(i) *It holds* $\Delta_{f,x^*}(x,\tilde{x}) \leq \liminf_{n\to\infty} \Delta_{f,x^*}(x,x^n)$ *.*

(ii) *If* $\{\Delta_{f,x^*}(x,x^n)\}_{n\in\mathbb{R}}$ *is bounded, then it holds* $\tilde{x} \in \mathrm{dom}(f)$*.*

(iii) *If* f *satisfies the Kadec-Klee property, see definition 3.14, and it holds* $\Delta_{f,x^*}(x,x^n) \to \Delta_{f,x^*}(x,\tilde{x})$ *, then the sequence* $\{x^n\}_{n\in\mathbb{R}}$ *converges to* $\tilde{x}$ *in* X*.*

Proof. The first assertion item (i) follows immediately from weak lower semicontinuity of f and convergence $\langle x^*, x^n - x\rangle_{X^*\times X} \to \langle x^*, \tilde{x} - x\rangle_{X^*\times X}$. To show item (ii), we consider the identity

$$f(x^n) = \Delta_{f,x^*}(x,x^n) + f(x) + \langle x^*, x^n - x\rangle_{X^*\times X} \,. \tag{3.9}$$

Since $\langle x^*, x^n - x\rangle_{X^*\times X}$ converges and $\Delta_{f,x^*}(x,x^n)$ is bounded by assumption, the right hand side of eq. (3.9) is bounded, which yields boundedness of $\{f(x^n)\}_{n\in\mathbb{R}}$. By weak lower semicontinuity we get the assertion. Finally, to show item (iii) we consider eq. (3.9) and the estimate

$$\begin{aligned}\limsup_{n\to\infty} f(x^n) &\leq \limsup_{n\to\infty} \Delta_{f,x^*}(x,x^n) + f(x) + \limsup_{n\to\infty} \langle x^*, x^n - x\rangle_{X^*\times X} \\ &= f(\tilde{x})\,,\end{aligned}$$

which together with weak lower semicontinuity of f yields convergence $f(x^n) \to f(\tilde{x})$ as $n \to \infty$. By the Kadec-Klee property of f, this yields the assertion. □

3.4 Bregman projection

The metric projection, sometimes referred to as the orthogonal projection, is an essential analytic concept. Based on the Bregman distance introduced in definition 3.9, one can now generalize the notion of the metric projection. This we do in the following definition.

Definition 3.16. Let $C \subset X$ be a convex subset and let f be proper and convex. For $x \in \mathrm{dom}(f)$ and $x^* \in \partial f(x)$ we denote by

$$\Pi_C^{f,x^*}(x) = \arg\min_{y\in C} \Delta_{f,x^*}(x,y) \tag{3.10}$$

the *Bregman projection* with respect to f of x onto C at x^*.

In a Hilbert space and for $f = \frac{1}{2}\|\cdot\|^2$, the Bregman projection reduces to the metric projection, see example 3.10. In this case, we write Π_C instead of Π_C^{f,x^*}. Next, in lemma 3.17 we introduce variational characterizations for the Bregman projection.

Lemma 3.17. *An element $z \in C$ is the Bregman projection with respect to f of x onto a convex set C at $x^* \in \partial f(x)$ if and only if there is $z^* \in \partial f(z)$ such that for arbitrary $y \in C$ the equivalent inequalities*

$$\langle z^* - x^*, y - z \rangle \geq 0\,, \tag{3.11}$$

$$\Delta_{f,\,x^*}(x, y) - \Delta_{f,\,x^*}(x, z) \geq \Delta_{f,\,z^*}(z, x) \tag{3.12}$$

hold true. We call any such z^* an admissible subgradient *for $z = \Pi_C^{f,\,x^*}(x)$.*

For a proof of lemma 3.17, we refer the reader to [39, Lemma 2.2]. To determine the Bregman projection onto a convex set C, one needs to solve a possibly non-smooth constrained optimization problem, which may be difficult for general sets. Hence, further assumptions on the structure of C are required. For $\alpha \in \mathbb{R}$, $\beta \in [0, \infty)$ and $w^* \in X^*$ we define the convex sets

$$H(w^*, \alpha) = \{x \in X \,:\, \langle w^*, x\rangle = \alpha\}\,, \tag{3.13}$$

$$H(w^*, \alpha, \beta) = \{x \in X \,:\, |\langle w^*, x\rangle - \alpha| \leq \beta\}\,. \tag{3.14}$$

The set $H(w^*, \alpha) = H(w^*, \alpha, 0)$ is a *hyperplane* in X. Further, the set $H(w^*, \alpha, \beta)$ we refer to as a *stripe*. It turns out that the Bregman projection onto an intersection of hyperplanes can be computed by solving a smooth unconstrained problem, as stated in lemma 3.18. This result can be found in [39, Lemma 2.4]. For the sake of completeness we present an according proof here as well.

Lemma 3.18. *Let $I \in \mathbb{R}$. For $i \in \{1, \ldots, I\}$ choose $\{w_i^*\}_{i=1}^I \subset X^*$ and $\alpha_i \in \mathbb{R}$. Set $z = \Pi_H^{f,\,x^*}(x)$, where by H we denote the intersection of hyperplanes $H = \bigcap_{i=1}^I H(w_i^*, \alpha_i)$. Then it holds*

$$z = \nabla f^* \left(x^* - \sum_{i=1}^I \tilde{t}_i w_i^* \right)\,,$$

where $\tilde{t} = (\tilde{t}_1, \ldots, \tilde{t}_I)$ is chosen such that it minimizes the functional $h : \mathbb{R}^I \to \mathbb{R}$ given by

$$h(t) = f^* \left(x^* - \sum_{i=1}^I t_i w_i^* \right) + \sum_{i=1}^I t_i \alpha_i\,. \tag{3.15}$$

Further, the element $z^* = x^* - \sum_{i=1}^I \tilde{t}_i w_i^*$ is admissible for z.

Proof. The mapping h is convex, coercive and differentiable and thus it attains its minimum. For $j \in \{1, \ldots, I\}$ we consider the optimality condition $\frac{\partial h}{\partial t_j}(\tilde{t}) = 0$ which can be rewritten as

$$\langle w_j^*, \nabla f \left(x^* - \sum_{i=1}^I \tilde{t}_i w_i^* \right) \rangle_{X^* \times X} = \alpha_j\,.$$

From this, we conclude that $z = \nabla f^*(z^*)$ with $z^* = x^* - \sum_{i=1}^{I} \tilde{t}_i w_i^*$ is contained in $H(w_j^*, \alpha_j)$ and, since j was arbitrary, in H. Further, it is easy to see that $\langle x^* - z^*, y - z\rangle = 0$ an hence, the variational inequality eq. (3.11) is satisfied. Consequently, it holds $z = \Pi_H^{f,x^*}(x)$ and z^* is admissible for z as stated. □

Remark 3.19. The projection onto a stripe $H(w^*, \alpha, \beta)$ can be described by means of projections onto hyperplanes. More precisely it holds

$$\Pi^{f,x^*}_{H(w^*,\alpha,\beta)}(x) = \begin{cases} \Pi^{f,x^*}_{H(w^*,\alpha-\beta)}(x)\,, & \langle w^*, x\rangle - \alpha < \beta \\ x\,, & |\langle w^*, x\rangle - \alpha| \le \beta \\ \Pi^{f,x^*}_{H(w^*,\alpha+\beta)}(x)\,, & \langle w^*, x\rangle - \alpha > \beta \end{cases} .$$

Hence, by lemma 3.18 we can determine the projection onto a stripe by means of the solution of an unconstrained minimization problem.

In [52, 51], the Bregman projection with respect to $\frac{1}{p}\|\cdot\|^p$ is investigated for $p > 1$. The exclusion of $p = 1$ is essential in the according analysis. One reason the case of $p = 1$ is exceptional lies in the fact that the according Bregman distance is not coercive and hence, the existence of a minimizing argument as in eq. (3.10) is not guaranteed, see remark 3.22.

3.5 A geometric interpretation of the Bregman projection

For the subsequent investigations, we refer to the choice of a convex functional f defined on the Hilbert space ℓ_2 which is of particular relevance. We motivate its definition and collect some of its properties. Further, we present a novel geometric interpretation of the corresponding Bregman projection which underpins its importance for sparsity-promoting reconstruction methods.

First, consider the choice $f = \|\cdot\|_{\ell_1}$. Note that while f is convex, it is not p-convex for any p. A general idea to enforce uniform convexity for some convex functional f is to add a term $f + g$ for some uniformly convex mapping g. By lemma 3.6 this amounts to a uniformly convex functional. For $\beta \ge 0$, this idea leads to the definition of the mapping $f_\beta : \ell_2 \to \overline{\mathbb{R}}$ given by

$$f_\beta = \|\cdot\|_{\ell_1} + \frac{\beta}{2}\|\cdot\|_{\ell_2}^2\,. \tag{3.16}$$

The mapping f_β in eq. (3.16) complements the functional $\|\cdot\|_{\ell_1}$ by an additional term. Note that since $\|\cdot\|_{\ell_2}^2$ is 2-convex, by lemma 3.6 we conclude that f_β as in eq. (3.16) is 2-convex if and only if $\beta > 0$. Moreover, the mapping f_β is proper and weakly lower semicontinuous for arbitrary $\beta \ge 0$ and it holds $\mathrm{dom}(f_\beta) = \ell_1$. It is depicted in fig. 3.1(a). Subsequently, we explicitly determine the subgradient as well as the convex conjugate of this particular mapping.

One observes that the mapping f_β is not differentiable in the sense of the Fréchet derivative as it contains the non-smooth expression $\|\cdot\|_{\ell_1}$. However, in example 3.11 we could establish the subdifferential of the latter term explicitly. Recall that for $x \in \ell_2$ it is determined by $\partial\|x\|_{\ell_1} = \operatorname{Sign}(x) \cap \ell_2$. Note that this expression is non-empty if and only if x is sparse, i.e. $\operatorname{dom}(\partial\|\cdot\|_{\ell_1}) = \ell_0$. We may now conclude

$$\partial f_\beta(x) = \operatorname{Sign}(x) \cap \ell_2 + \beta x \tag{3.17}$$

for arbitrary $\beta > 0$. This mapping can easily be inverted. By means of the *shrinkage operator* $S_\alpha = (\operatorname{Id} + \alpha\|\cdot\|_{\ell_1})^{-1}$ which for $\alpha > 0$ can be rewritten as

$$(S_\alpha(x^*))_k = (\operatorname{Id} + \alpha\partial|\cdot|)^{-1}(x_k^*) = \begin{cases} x_k^* - \alpha\,, & x_k^* \geq \alpha \\ 0\,, & x_k^* \in (-\alpha, \alpha) \\ x_k^* + \alpha\,, & x_k^* \leq -\alpha \end{cases}, \tag{3.18}$$

we determine the inverse mapping of ∂f_β by $\partial f_\beta^{-1} = S_{1/\beta}(\beta^{-1}\cdot)$ as a simple calculation shows. Next, by this means in lemma 3.20 we determine the convex conjugate f_β^* of f_β explicitly. It is illustrated in fig. 3.1(b).

Lemma 3.20. *The conjugate mapping $f_\beta^* : \ell_2 \to \overline{\mathbb{R}}$ of f_β is given by*

$$f_\beta^*(x^*) = \frac{1}{2\beta} \sum_{k \in \mathbb{R}} g_{1/\beta}^*\left(\beta^{-1} x_k^*\right) \tag{3.19}$$

for $x^* \in \ell_2$, where

$$g_\alpha^*(t) = \begin{cases} (|t| - \alpha)^2\,, & |t| > \alpha \\ 0\,, & |t| \leq \alpha \end{cases}. \tag{3.20}$$

Proof. Let $x^* \in \ell_2$ and set $x = S_{1/\beta}(\beta^{-1}x^*)$. Note that the latter definition implies $\frac{1}{\beta}x^* \in x + \frac{1}{\beta}\operatorname{Sign}(x)$ and, hence, $x^* - \beta x \in \operatorname{Sign}(x)$. We now define the sequence $\{x^n\}_{n \in \mathbb{R}}$ by

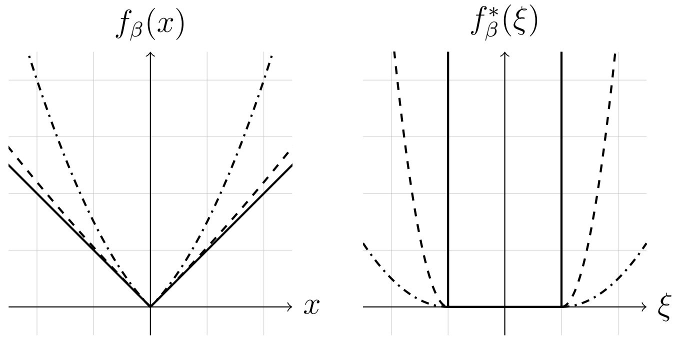
$$x_k^n = \begin{cases} x_k\,, & k \leq n \\ 0\,, & k > n \end{cases}$$

for $n \in \mathbb{R}$. Since it holds $\langle x^* - \beta x, x^n \rangle = \|x^n\|_{\ell_1}$, by the inequality in (3.1) we get

$$f_\beta^*(x^*) \geq \beta\langle x, x^n \rangle - \frac{\beta}{2}\langle x^n, x^n \rangle = \beta\langle x - x^n, x^n \rangle + \frac{\beta}{2}\langle x^n, x^n \rangle = \frac{\beta}{2}\|x^n\|_{\ell_2}^2\,.$$

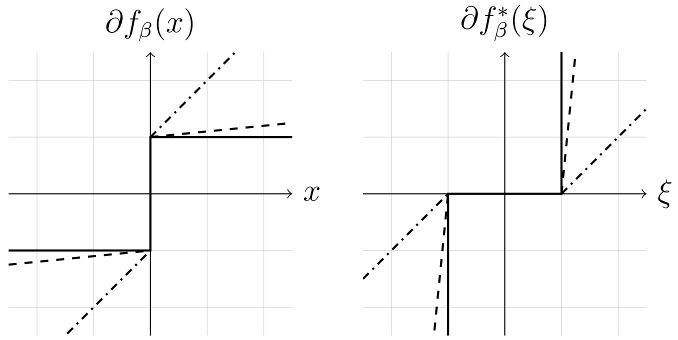
On the other hand, for arbitrary $y \in \ell_1$, from $\langle x^* - \beta x, y \rangle \leq \|y\|_{\ell_1}$ we now conclude

$$\langle x^*, y \rangle - f_\beta(y) \leq \beta\langle x, y \rangle - \frac{\beta}{2}\langle y, y \rangle = \frac{\beta}{2}\langle x, x \rangle - \frac{\beta}{2}\langle x - y, x - y \rangle \leq \frac{\beta}{2}\|x\|_{\ell_2}^2\,.$$



(a) (b)

Figure 3.1: The mapping f_β (left) and its convex conjugate f^*_β (right) for values $\beta = 0$ (solid), $\beta = 0.1$ (dashed) and $\beta = 1$ (dash-dotted).



(a) (b)

Figure 3.2: The mapping ∂f_β (left) and its inverse mapping $\partial f_\beta^{-1} = \nabla f^*_\beta$ (right) for values $\beta = 0$ (solid), $\beta = 0.1$ (dashed) and $\beta = 1$ (dash-dotted).

Note that this estimate is trivial if $y \notin \ell_1$. Taking the supremum over $y \in \ell_2$, we therefore get $f^*_\beta(x^*) \leq \frac{\beta}{2}\|x\|^2_{\ell_2}$. Collectively, this gives equality $f^*_\beta(x^*) = \frac{\beta}{2}\|x\|^2_{\ell_2}$ and hence the assertion. □

While we observed in (3.17) that the subdifferential of f_β is set-valued, we point out to the fact that the conjugate mapping f^*_β is differentiable at any point $x^* \in \ell_2$. This was to be expected due to lemma 3.8. By (3.20), the derivative can easily be computed as

$$\nabla f^*_\beta = S_{1/\beta}(\beta^{-1}\cdot)\,. \tag{3.21}$$

Particularly, the inverse of ∂f_β is single-valued for $\beta > 0$. This is illustrated in fig. 3.2(b).

We now investigate the Bregman projection with respect to the special choice of f_β as in eq. (3.16). To start with, consider the Bregman distance with respect to f_β which for $x \in \mathrm{dom}(f_\beta)$ and $x^* \in \partial f_\beta(x)$ we denote by $\Delta_{\beta,x^*}(x,\cdot)$. Using lemma 3.13 item (ii) and item (iv), this expression can be rewritten as

$$\Delta_{\beta,x^*}(x,y) = \|y\|_{\ell_1} - \langle x^* - \beta x, y\rangle_X + \frac{\beta}{2}\|y-x\|_{\ell_2}^2 , \tag{3.22}$$

where we refer to the results in example 3.11 and example 3.10. Further, for $x \in \ell_0$ and $x^* \in \partial f_\beta(x)$, the Bregman projection of x at x^* with respect to f_β we denote by $\Pi_C^{\beta,x^*}(x)$. We note in remark 3.24 that this projection is sparse if C is an intersection of hyperplanes. Motivated by this important observation, in section 4.3 we discuss a sparsity-promoting reconstruction method that is based on the Bregman projection. To begin with, we show that the Bregman projection is well defined and single-valued if and only if $\beta > 0$.

Proposition 3.21. *Let $x \in \ell_0$ and $x^* \in \partial f_\beta(x)$, where $\beta > 0$. Moreover, let $C \subseteq \ell_2$ be convex such that $C \cap \ell_1$ is non-empty and relatively closed in ℓ_1, i.e. it holds $C \cap \ell_1 = \bar{C} \cap \ell_1$. Here, by $\bar{C}$ we denote the closure of C. Then, the Bregman projection $\Pi_C^{\beta,x^*}(x)$ exists and is uniquely determined.*

Proof. Since $x \in \ell_0$ the subdifferential ∂f_β is non-empty and the choice of x^* is possible. Further, since the Bregman distance $\Delta_{\beta,x^*}(x,\cdot)$ is bounded from below, there is a minimizing sequence $\{x^n\}_{n\in\mathbb{R}} \subseteq C$ such that for the limit it holds $\lim_{n\to\infty} \Delta_{\beta,x^*}(x,x^n) = \inf_{y\in C} \Delta_{\beta,x^*}(x,y)$. Since the sequence $\{\Delta_{\beta,x^*}(x,x^n)\}_{n\in\mathbb{R}}$ is convergent, it is particularly bounded. Thus, by lemma 3.12 we conclude boundedness of $\|x^n\|_{\ell_2}$ which yields the existence of a subsequence also denoted by $\{x^n\}_{n\in\mathbb{R}}$ weakly converging to $\tilde{x} \in \ell_2$. Since the closure $\bar{C}$ is convex and closed by definition, it is particularly weakly closed and thus, we get $\tilde{x} \in \bar{C}$. Moreover, by lemma 3.15 item (ii) we conclude $\tilde{x} \in \ell_1$. Collectively, we obtain $\tilde{x} \in \bar{C} \cap \ell_1 \subseteq C$. Weak lower semicontinuity shown in lemma 3.15 item (i) yields $\Delta_{\beta,x^*}(x,\tilde{x}) \le \inf_{y\in C} \Delta_{\beta,x^*}(x,y)$ which is to say that $\tilde{x}$ is a minimizer of $\Delta_{\beta,x^*}(x,\cdot)$. Finally, strict convexity of $\Delta_{\beta,x^*}(x,\cdot)$ together with convexity of the set C imply uniqueness of the minimizer. □

The result in proposition 3.21 cannot be extended to the setting $\beta = 0$ as the assertions regarding both existence and uniqueness do not remain valid. This we elaborate in remarks 3.22 and 3.23.

Remark 3.22. We have shown existence of the Bregman projection with respect to f_β for arbitrary $\beta > 0$. Considering the situation for $\beta = 0$, however, $\Delta_{0,x^*}(x,\cdot)$ is non-coercive for arbitrary $x \in X$ and $x^* \in \partial f(x)$, see example 3.11, and the existence of a minimizer cannot be guaranteed. In fact, without additional assumptions we cannot assure that the Bregman projection is non-empty. To see this, consider the convex and closed set

$$C = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 > 0 \text{ and } y_2 \leq -y_1^{-1}\} \tag{3.23}$$

as well as $x = (1,1)$ and $x^* = \operatorname{Sign}(x) = (1,1)$. For all $y = (y_1, y_2) \in C$ it holds $y_1 > 0$ and $y_2 < 0$ per definition and hence, we get

$$\Delta_{0,x^*}(x, y) = |y_1| - y_1 + |y_2| - y_2 = 2|y_2| \,.$$

Hence, considering the sequence $\{x^n\}_{n\in\mathbb{R}}$ given by $x^n = (n, -n^{-1})$ we obtain a minimizing sequence such that $\Delta_{0,x^*}(x, x^n) \to 0$. There is, however, no element $y \in C$ such that $\Delta_{0,x^*}(x, y) = 0$. This setting is illustrated in fig. 3.3(b).

Remark 3.23. For $\beta = 0$, we lose strict convexity of the Bregman distance $\Delta_{\beta,x^*}(x,\cdot)$ and hence, the Bregman projection $\Pi_C^{0,x^*}(x)$ is potentially set-valued. In fact, we demonstrate in fig. 3.3(a) that the minimizer of $\Delta_{0,x^*}(x,\cdot)$, if any, is in general is not unique.

The metric projection allows for a particularly intuitive interpretation as it is based on the euclidean distance. Subsequently, we establish a geometric interpretation of the Bregman projection with respect to f_β onto an affine subspace $z+U$ and its relation to the metric projection. To our best knowledge, this is the first time a geometrical interpretation of a non-trivial Bregman projection is given. First, in remark 3.24 we put on record that the according Bregman projection onto the intersection of hyperplanes is sparse.

Remark 3.24. Consider f_β as in eq. (3.16) and the Bregman projection $z = \Pi_C^{\beta,x^*}(x)$ of x with respect to f_β at $x^* \in \partial f_\beta(x)$ onto a convex set $C \subset \ell_2$. If C is an intersection of hyperplanes of the form $H = \bigcap_{i=1}^{I} H(w_i^*, \alpha_i)$, this projection can be expressed by means of the shrinkage operator, see lemma 3.18 and eq. (3.21). As an important consequence, we observe that the projection is sparse.

A projection $p = \Pi_{z+U}^{\beta,x^*}(x)$ is illustrated in fig. 3.3. One observes that for small β, the projection p lies on the coordinate axis, see fig. 3.3(a). Recall that the projection according to f_β is given by the minimizing element of $\Delta_{\beta,x^*}(x,\cdot)$ in $z+U$. The latter expression in eq. (3.22) ensures that p is close to the point x in the sense of the metric distance. The first two terms in eq. (3.22) introduce an additional penalty if the metric projection of x lies in a different orthant than x and shifts it towards the axis. The preimage of some point in $y \in z+U$ illustrates this shift, see fig. 3.4. Particularly, one observes that if y lies on the coordinate axis, the preimage can geometrically be described as a stripe. The width of this stripe is controlled by the parameter β and behaves proportionally to β^{-1}. Particularly, this implies that the projection p preferentially lies on the axis for $\beta > 0$. This interpretation approves the observation that the projection onto intersections of

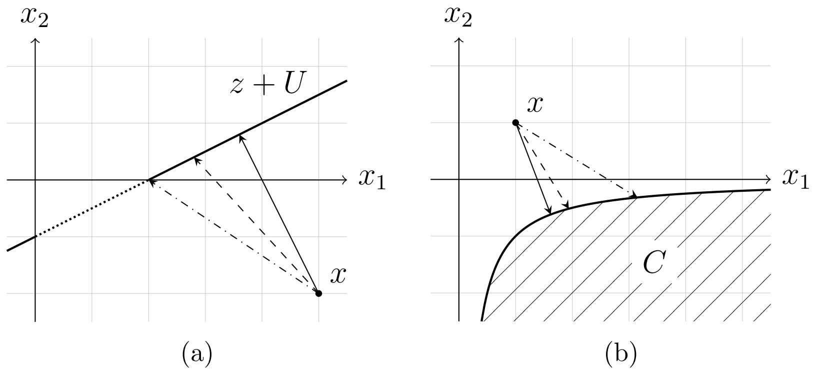


Figure 3.3: The metric projection $\Pi_C(x)$ (solid) onto some convex set C as well as the Bregman projection $\Pi_C^{\beta, x^*}(x)$ for values $\beta = 1$ (dashed) and $\beta = 0.1$ (dash-dotted). Further, in fig. 3.3(a) the set-valued projection $\Pi_C^{0, x^*}(x)$ is depicted (dotted). The corresponding projection does not exist in (b). Note that in (a), the set C is given by an affine subspace we denote by $z + U$.

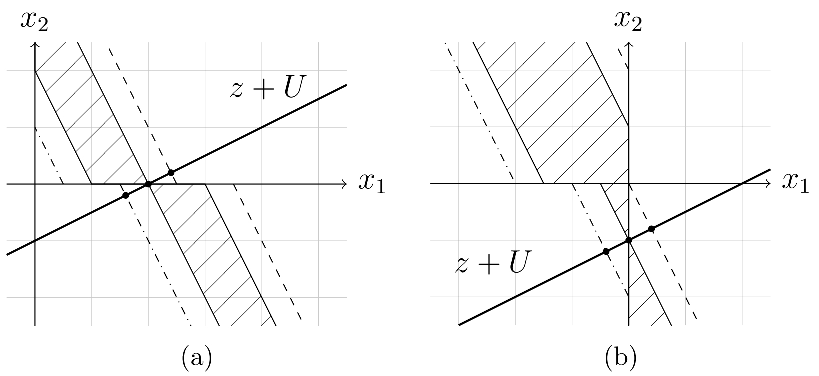


Figure 3.4: The preimage $\{x \in \mathbb{R}^2 : \Pi_{z+U}^{\beta, x^*}(x) = p \text{ and } x^* \in \partial f_\beta(x)\}$ of the Bregman projection onto some affine subspace $z + U$ for $\beta = 1$ and different points $p \in z + U$. One observes that if p lies on a coordinate axis, this set can geometrically be described as a stripe.

hyperplanes is sparse, see remark 3.24.

We further remark that if $\Pi_{z+U}(x) \in \Pi_{z+U}^{0, x^*}(x)$, the Bregman projection and the metric projection coincide. In fact, if it holds $\Pi_C(x) \in \Pi_C^{0, x^*}(x)$, then by (3.22) we can immediately establish the identity $\Pi_C^{\beta, x^*}(x) = \Pi_C(x)$ for arbitrary $\beta > 0$.

We now show a stability result of the Bregman projection with respect to the parameter β. The observed behavior is demonstrated in fig. 3.3 for the two-dimensional situation.

Proposition 3.25. *Let $C \subseteq \ell_2$ be convex such that $C \cap \ell_1$ is non-empty and relatively closed in ℓ_1, i.e. it holds $C \cap \ell_1 = \bar{C} \cap \ell_1$. Here, by $\bar{C}$ we denote the closure of C. Moreover, let $\{\beta_n\}_{n\in\mathbb{R}}$ be a null sequence and let $x \in \ell_0$. For $n \in \mathbb{N}$ consider the Bregman projection $x^n = \Pi_C^{\beta_n, x^{n,*}}(x)$, where $x^{n,*} = x^* + \beta_n x$ for some $x^* \in \partial f_0(x)$. Then, the sequence $\{x^n\}_{n\in\mathbb{R}}$ is a minimizing sequence of $\Delta_{0,x^*}(x, \cdot)$.*

If we additionally assume $\Pi_C^{0,x^*}(x) \neq \emptyset$, then it holds $x^n \to \tilde{z}$ as $n \to \infty$, where $\tilde{z}$ is the unique element in $\Pi_C^{0,x^*}(x)$ such that $\|\cdot - x\|_{\ell_2}$ is minimal.

Proof. The first assertion follows from (3.22) and the estimate

$$\begin{aligned}\limsup_{n\to\infty} \|x^n\|_{\ell_1} - \langle x^*, x^n\rangle &\leq \limsup_{n\to\infty} \|x^n\|_{\ell_1} - \langle x^*, x^n\rangle + \frac{\beta_n}{2}\|x^n - x\|_{\ell_2}^2 \\ &\leq \limsup_{n\to\infty} \|y\|_{\ell_1} - \langle x^*, y\rangle + \frac{\beta_n}{2}\|y - x\|_{\ell_2}^2 \\ &= \|y\|_{\ell_1} - \langle x^*, y\rangle\,,\end{aligned} \tag{3.24}$$

where $y \in C$ is arbitrary. We immediately obtain $\limsup_{n\to\infty} \|x^n\|_{\ell_1} - \langle x^*, x^n\rangle \leq \inf_{y\in C} \|y\|_{\ell_1} - \langle x^*, y\rangle$ which by eq. (3.6) yields the assertion.

We now assume that $\Pi_C^{0,x^*}(x) \neq \emptyset$ and show convergence of the sequence x^n to the distinguished point $\tilde{z}$ as $n \to \infty$. By (3.22) and the minimizing property of x^n we conclude

$$\begin{aligned}\|\tilde{z}\|_{\ell_1} - \langle x^*, \tilde{z}\rangle + \frac{\beta_n}{2}\|x^n - x\|_{\ell_2}^2 &\leq \|x^n\|_{\ell_1} - \langle x^*, x^n\rangle + \frac{\beta_n}{2}\|x^n - x\|_{\ell_2}^2 \\ &\leq \|\tilde{z}\|_{\ell_1} - \langle x^*, \tilde{z}\rangle + \frac{\beta_n}{2}\|\tilde{z} - x\|_{\ell_2}^2\,,\end{aligned}$$

where in the first step we exploit the fact that $\tilde{z} \in \Pi_C^{0,x^*}(x)$ minimizes the expression $\|\cdot\|_{\ell_1} - \langle x^*, \cdot\rangle$ in C. We conclude $\|x^n - x\|_{\ell_2} \leq \|\tilde{z} - x\|_{\ell_2}$ for arbitrary $n \in \mathbb{R}$. Consequently, the sequence $\{x^n\}_{n\in\mathbb{R}}$ is bounded in ℓ_2 and we obtain a subsequence also denoted by $\{x^n\}_{n\in\mathbb{R}}$ that converges weakly to some $\tilde{x} \in \ell_2$. To show strong convergence, repeating the arguments in (3.24) we conclude

$$\limsup_{n\to\infty} \|x^n\|_{\ell_1} - \langle x^*, x^n\rangle \leq \|\tilde{x}\|_{\ell_1} - \langle x^*, \tilde{x}\rangle\,.$$

Together with weak lower semicontinuity of the latter expression this yields convergence of the norm $\|x^n\|_{\ell_1} \to \|\tilde{x}\|_{\ell_1}$ as $n \to \infty$. Since $\|\cdot\|_{\ell_1}$ obeys the Kadec-Klee property, this gives the assertion.

We need to show $\tilde{x} = \tilde{z}$. Firstly, we show that $\tilde{x} \in \Pi_C^{0,x^*}(x)$. Particularly, we need to show $\tilde{x} \in C$. To this end, we remark that $\|x^n\|_{\ell_1} - \langle x^*, x^n\rangle$ and $\langle x^*, x^n\rangle$ are bounded with respect to n, which follows from (3.24) and convergence of $\langle x^*, x^n\rangle$, respectively. Thus, we conclude boundedness of $\|x^n\|_{\ell_1}$. By weak lower semicontinuity of $\|\cdot\|_{\ell_1}$ we get $\tilde{x} \in \ell_1$. Moreover, the closure $\bar{C}$ of C in ℓ_2 is convex and closed per definition and hence, it is weakly closed and thus, we have $\tilde{x} \in \bar{C}$.

From this and relative closedness of C, we conclude $\tilde{x} \in \bar{C} \cap \ell_1 \subseteq C$. Moreover, by (3.24) and weak lower semicontinuity of $\|\cdot\|_{\ell_1}$ we obtain for arbitrary $y \in C$

$$\|\tilde{x}\|_{\ell_1} - \langle x^*, \tilde{x}\rangle \leq \liminf_{n\to\infty} \|x^n\|_{\ell_1} - \langle x^*, x^n\rangle \leq \|y\|_{\ell_1} - \langle x^*, y\rangle \,,$$

from which we conclude that $\tilde{x}$ minimizes $\|\cdot\|_{\ell_1} - \langle x^*, \cdot\rangle$ in C. This finally yields $\tilde{x} \in \Pi_C^{0,x^*}(x)$.

Finally, we need to show that $\tilde{x}$ minimizes $\|\cdot - x\|_{\ell_2}$ in $\Pi_C^{0,x^*}(x)$. To show this, for arbitrary $z \in \Pi_C^{0,x^*}(x)$ we consider the estimate

$$\begin{aligned}
\|\tilde{x} - x\|_{\ell_2}^2 &\leq \liminf_{n\to\infty} \|x^n - x\|_{\ell_2}^2 \\
&= \liminf_{n\to\infty} \frac{2}{\beta_n}\left(\|x^n\|_{\ell_1} - \langle x^*, x^n\rangle + \frac{\beta_n}{2}\|x^n - x\|_{\ell_2}^2\right) \\
&\quad - \frac{2}{\beta_n}\left(\|x^n\|_{\ell_1} - \langle x^*, x^n\rangle\right) \\
&\leq \liminf_{n\to\infty} \frac{2}{\beta_n}\left(\|z\|_{\ell_1} - \langle x^*, z\rangle + \frac{\beta_n}{2}\|z - x\|_{\ell_2}^2\right) \\
&\quad - \frac{2}{\beta_n}\left(\|x^n\|_{\ell_1} - \langle x^*, x^n\rangle\right) \\
&\leq \|z - x\|_{\ell_2}^2 - \frac{2}{\beta_n}\left(\|x^n\|_{\ell_1} - \langle x^*, x^n\rangle - \|z\|_{\ell_1} + \langle x^*, z\rangle\right) \,,
\end{aligned}$$

which by eq. (3.6) and the minimizing property of z yields $\|\tilde{x} - x\|_{\ell_2} \leq \|z - x\|_{\ell_2}$. The set $\Pi_C^{0,x^*}(x)$ is convex as the minimizing set of a convex functional. Moreover, $\|\cdot - x\|_{\ell_2}$ is strictly convex. Thus, the minimizing property uniquely characterizes $\tilde{z}$ and we conclude $\tilde{x} = \tilde{z}$. Since this holds true for any convergent subsequence, the whole sequence converges. □

Note that in proposition 3.25 we required the Bregman projection $\Pi_C^{0,x^*}(x)$ to be non-empty. In fact, this requirement is necessary. In remark 3.22 we present an example such that the Bregman projection $\Pi_C^{\beta,x^*}(x)$ diverges as $\beta \to 0$.

CHAPTER
FOUR

ITERATIVE METHODS

4.1 Introduction

The stable solution of an operator equation of the form $A(x) = y$ as in eq. (1.1) such as the inverse medium problem requires the incorporation of a regularization. In this chapter, we discuss iterative approaches that are designed for sparse reconstruction. Particularly, we propose a novel method for this purpose that is based on the Bregman projection.

The incorporation of the Bregman projection into iteration schemes for the reconstruction of a solution of an operator equation was proposed in [52, 51, 39], for instance. In section 3.5, we discussed a special instance of the Bregman projection for which we provided a geometrical interpretation. Most notably, the according projection onto the intersection of hyperplanes was shown to be sparse, see remark 3.24. Starting from this observation, in the following we develop a novel iterative method for the stable solution of an operator equation as in eq. (1.1). We show its connection to existing methods and show that the method gives rise to a regularization. Furthermore, we demonstrate its potential regarding the inverse medium problem in numerical experiments.

Throughout this chapter, unless otherwise specified by X and Y we denote Banach spaces. Moreover, we consider an operator equation as in eq. (1.1) incorporating a potentially non-linear operator $A : \mathrm{dom}(A) \subset X \to Y$. In assumption 4.1, we collect the needed conditions on A.

Assumption 4.1. For an operator $A : \mathrm{dom}(A) \subset X \to Y$, a proper convex mapping $f : \mathrm{dom}(f) \subset X \to \overline{\mathbb{R}}$ let $x_0 \in \mathrm{dom}(A) \cap \mathrm{dom}(f)$. Assume

(i) There is $R > 0$ such that $B_{2R}(x_0) \subset \mathrm{dom}(A)$ and the operator equation $A(x) = y$ has a solution in $B_R(x_0) \cap \mathrm{dom}(f)$.

(ii) A is continuously Fréchet differentiable on $B_{2R}(x_0)$.

(iii) A is weakly closed, i.e. for any sequence $\{x^n\}_{n\in\mathbb{N}} \subset \mathrm{dom}(A)$ converging to some $\tilde{x} \in X$ such that $A(x^n) \to y$ it holds $\tilde{x} \in \mathrm{dom}(A)$ and $A(\tilde{x}) = y$.

(iv) A satisfies the *tangential cone condition*, i.e. there is $0 \le \eta < 1$ such that

$$\|A(x) - A(\tilde{x}) - A'(\tilde{x})(x - \tilde{x})\| \le \eta \|A(x) - A(\tilde{x})\|$$

for arbitrary $x, \tilde{x} \in X$.

(v) The derivative A' of A is bounded, i.e. there is $C > 0$ such that $\|A'(x)\| \le C$ for arbitrary $x \in B_{2R}(x_0)$.

In the following, we discuss convergence of the proposed methods assuming the operator in eq. (1.1) satisfies assumption 4.1. Obviously, this holds true for any bounded linear operator. Particularly, this special case allows for a strong result which is treated separately in section 4.3.1.

4.2 Generalization of the Landweber iteration

Before we discuss an iterative approach based on the Bregman projection for the stable solution of an operator equation as in eq. (1.1), we introduce a closely related method that generalizes the Landweber iteration as in eq. (1.6). Let X and Y be Banach spaces and let $A : \mathrm{dom}(A) \subset X \to Y$ denote an operator obeying assumption 4.1. For some p-convex mapping $f : \mathrm{dom}(f) \subset X \to \overline{\mathbb{R}}$ consider the iteration defined by

$$x_{n+1}^* = x_n^* - t_n A'(x_n)^* J_r(A(x_n) - y^\delta)\,,\; x_{n+1} = \nabla f^*(x_{n+1}^*) \tag{4.1}$$

for $t_n > 0$ and $n \in \mathbb{N}$ starting with initial choices $x_0 \in \mathrm{dom}(f) \cap \mathrm{dom}(A)$ and $x_0^* \in \partial f(x_0)$. Here, by J_r we denote the *duality mapping* in Y with gauge function $t \mapsto t^r$ which for $r > 1$ is given by

$$J_r(y) = \{y^* \in Y^* \,:\, \langle y^*, y\rangle_{Y^* \times Y} = \|y^*\|_{Y^*}\|y\|_Y\,,\; \|y^*\|_{Y^*} = \|y\|^r\}\,. \tag{4.2}$$

The duality mapping is single-valued if and only if Y is smooth. Further, it is continuous if Y is uniformly smooth. For a deeper insight into this matter, we refer the reader to [53]. We remark that the gradient of f in eq. (4.1) exists due to lemma 3.8. The iteration in eq. (4.1) can be regarded as a generalization of the Landweber iteration in eq. (1.6) as both methods coincide in Hilbert spaces X and Y if t_n is constant in n and f is the norm in X. Recently, in [33] the regularizing properties of the method in eq. (4.1) were proven in a general setting. We cite the according result in theorem 4.2. Similar approaches can be found in [50]. If t_n is constant in n, for the special choice of f as in eq. (3.16) the iteration in eq. (4.1) amounts to the Linearized Bregman iteration which was proposed in

[60] for sparse reconstruction in the setting of a linear operator mapping between finite-dimensional spaces.

Theorem 4.2. *Let X be reflexive and let Y be uniformly smooth. Further, let the conditions in assumption 4.1 hold and let $f : \mathrm{dom}(f) \subset X \to \overline{\mathbb{R}}$ be a proper, lower semicontinuous and p-convex function with $p \geq 2$. For the sequences $\{x_n^*\}_{n\in\mathbb{N}}$ and $\{x_n\}_{n\in\mathbb{N}}$ generated by the iteration in eq. (4.1) with*

$$t_n = t_0 \|A(x_n) - y\|^{p-r} \tag{4.3}$$

for $n \in \mathbb{N}$ and $t_0 > 0$, there exists a solution $\tilde{x}$ of $A(x) = y$ such that $\Delta_{f,x_n^*}(x_n, \tilde{x}) \to 0$ as well as $x_n \to \tilde{x}$ as $n \to \infty$. If, in addition, $\mathcal{N}(F'(x^\dagger)) \subset \mathcal{N}(F'(x))$ for all $x \in B_{2R}(x_0)$, then $\tilde{x} = x^\dagger$, where $x^\dagger$ is the unique solution of $A(x) = y$ such that $\Delta_{f,x_0^*}(x_0, z)$ is minimal among all solutions z of the operator equation.

For a proof of theorem 4.2, we refer the reader to [33, Lemma 3.7]. Further, in the same paper it is demonstrated that if the iteration described in eq. (4.1) is terminated according to the discrepancy principle, see eq. (1.7), this amounts to a regularization scheme under further conditions.

The iteration scheme in eq. (4.1) depends on the proper choice of the step-size t_n as in eq. (4.1). In theorem 4.2, convergence could be established for the adaptive choice rule as in eq. (4.3). Expectedly, numerical experiments suggest that the choice of the descent parameter is crucial for the overall performance of the method, see section 4.4. One idea to accelerate the method proposed in [33, Remark 3.10] is given by

$$t_n = t_0 \frac{\|A(x_n) - y^\delta\|^{p(r-1)}}{\|A'(x_n)^* J_r(A(x_n) - y^\delta)\|^p} \tag{4.4}$$

for some $t_0 > 0$. In [39], the authors propose to accelerate this method using a step-size based on the Bregman projection. Following the latter approach, in section 4.3 we present an iterative method based on Bregman projections and show its relation to the iteration in eq. (4.1).

4.3 Iterative reconstruction based on the Bregman projection

In the following, we present a novel method for sparse reconstruction that is based on the Bregman projection. We follow ideas presented in [52, 39] and generalize the according results to potentially non-linear operators. Particularly, we investigate the regularizing properties of the method.

Throughout this section, let $p > 1$ and let $f : \mathrm{dom}(f) \subset X \to \overline{\mathbb{R}}$ be a proper, lower semicontinuous and p-convex function. In a first step, in algorithm 4.1 we

Algorithm 4.1 General scheme of the Bregman projection iteration

Let $x_0 \in X$ and $x_0^* \in \partial f(x_0)$. Starting with $n = 0$, we choose an integer $I_n \in \mathbb{N}$ and for each $i \in \{1, \ldots, I_n\}$ an element $w_{n,i}^* \in X^*$ such that $w_{n,i}^* = A'(x_n)^* v_{n,i}^*$ for some $v_{n,i}^* \in Y^*$. We assume that it holds

$$|\langle v_{n,i}^*, A(x_n) - y\rangle| > \eta \|v_{n,i}^*\| \|A(x_n) - y\| \tag{4.5}$$

for at least one $i \in \{1, \ldots, I_n\}$. Furthermore, we set

$$\begin{aligned} \alpha_{n,i} &= \langle v_{n,i}^*, A'(x_n)x_n - A(x_n) + y\rangle , \\ \beta_{n,i} &= \eta \|v_{n,i}^*\| \|A(x_n) - y\| , \end{aligned}$$

and define the iterates

$$x_{n+1} = \Pi_{H_n}^{f, x_n^*}(x_n) ,$$

where we project onto the set

$$H_n = \bigcap_{i=1}^{I_n} H(w_{n,i}^*, \alpha_{n,i}, \beta_{n,i}) \tag{4.6}$$

and choose $x_{n+1}^* \in \partial f(x_{n+1})$ such that it is admissible for x_{n+1}. If stopping criterion is satisfied set $x = x_n$, otherwise increment n and start over.

present a general iteration scheme based on the projection onto intersections of stripes.

Remark 4.3. The rationale behind the method in algorithm 4.1 lies in the fact that in every iteration, the true solution $x^\dagger$ lies in the intersection of stripes denoted by H_n. This follows immediately from the observation that for every $i \in \{1, \ldots, I_n\}$ it holds

$$\begin{aligned} |\langle w_{n,i}^*, x^\dagger\rangle - \alpha_{n,i}| &= |\langle v_{n,i}^*, A(x_n) - A(x^\dagger) - A'(x_n)(x_n - x^\dagger)\rangle| \\ &\leq \eta \|v_{n,i}^*\| \|A(x_n) - y\| . \end{aligned}$$

On the other hand, the current iterate x_n is not contained in H_n since it holds

$$|\langle w_{n,i}^*, x_n\rangle - \alpha_{n,i}| = |\langle v_{n,i}^*, A(x_n) - y\rangle| > \eta \|v_{n,i}^*\| \|A(x_n) - y\|$$

for at least one $i \in \{1, \ldots, I_n\}$. Here, we used the assumption in eq. (4.5), which is satisfied for $v_{n,i}^* = J_r(A(x_n) - y)$, for instance. It seems thus intuitive to project on that subset.

Practical purposes give rise to the importance of a feasible approach for the numerical computation of the next iterate in algorithm 4.1, where one needs to determine the Bregman projection onto a convex set. While this matter may be difficult for general convex sets, the setting of hyperplanes allows for an alternate description of the Bregman projection that reduces the task of determining the Bregman projection to an unconstrained smooth optimization problem as it has been shown in lemma 3.18.

We will show convergence of algorithm 4.1 only for certain special situations. However, we can derive a general result that will be useful for the following analysis.

Lemma 4.4. *Let X be reflexive and let Y be smooth. Further, let $\{x_n\}_{n\in\mathbb{N}} \subset X$ and $\{x_n^*\}_{n\in\mathbb{N}} \subset X^*$ be sequences such that x_n^* is an admissible subgradient for x_n for every $n \in \mathbb{N}$ and assume that for some constant $C > 0$ and every solution $\tilde{x}$ of $A(x) = y$ it holds*

$$\Delta_{f,x_{n+1}^*}(x_{n+1}, \tilde{x}) \leq \Delta_{f,x_n^*}(x_n, \tilde{x}) - C\|A(x_n) - y\|^p \tag{4.7}$$

for arbitrary $n \in \mathbb{N}$. Assume that x_0 is chosen such that

$$\Delta_{f,x_0^*}(x_0, x^\dagger) \leq \frac{c_0}{2} R^p \tag{4.8}$$

for some solution $x^\dagger \in B_R(x_0) \cap \mathrm{dom}(f)$ of $A(x) = y$. Then it holds $x_n \in B_{2R}(x_0)$ for every $n \in \mathbb{N}$ and there exists a weakly convergent subsequence of $\{x_n\}_{n\in\mathbb{N}}$ and every such subsequence converges weakly to a solution of the operator equation $A(x) = y$.

Proof. Let $x^\dagger \in B_R(x_0) \cap \mathrm{dom}(f)$ denote some solution of the operator equation $A(x) = y$. By eq. (4.7) we obtain monotonicity of $\{\Delta_{f,x_n^*}(x_n, x^\dagger)\}_{n\in\mathbb{N}}$ with respect to n. We conclude for arbitrary $n \in \mathbb{N}$

$$\frac{c_0}{2}\|x_n - x^\dagger\|^p \leq \Delta_{f,x_n^*}(x_n, x^\dagger) \leq \Delta_{f,x_0^*}(x_0, x^\dagger) \leq \frac{c_0}{2} R^p \,,$$

where in the former step, we used eq. (3.8). We get $\|x_n - x^\dagger\| \leq R$ and, furthermore, it holds $\|x_0 - x^\dagger\| \leq R$ by assumption. Collectively, this implies $\|x_n - x_0\| \leq 2R$, i.e. $x_n \in B_{2R}(x_0)$.

Particularly, the latter relation implies boundedness of the sequence $\{x_n\}_{n\in\mathbb{N}}$. Since the space X is reflexive, this yields the existence of a weakly convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ with limit $\tilde{x} \in X$. By induction, for arbitrary $N \in \mathbb{N}$ from eq. (4.7) we now get the estimate

$$\begin{aligned} 0 &\leq C \sum_{n=0}^{N} \|A(x_n) - y\|^p \\ &\leq \Delta_{f,x_0^*}(x_0, x^\dagger) - \Delta_{f,x_{N+1}^*}(x_{N+1}, x^\dagger) \leq \Delta_{f,x_0^*}(x_0, x^\dagger) \end{aligned}$$

Algorithm 4.2 Bregman projection iteration for linear operators

Let $x_0 \in X$ and $x_0^* \in \partial f(x_0)$. Starting with $n = 0$, we choose an integer $I_n \in \mathbb{N}$ and for each $i \in \{1, \ldots, I_n\}$ an element $w_{n,i}^* \in X^*$ such that $w_{n,i}^* = A^* v_{n,i}^*$ for some $v_{n,i}^* \in Y^*$. Moreover, for $I_n \in N$ we set

$$\alpha_{n,i} = \langle v_{n,i}^*, y \rangle ,$$

and define the iterates

$$x_{n+1}^* = x_n^* - \sum_{i=1}^{I_n} \tilde{t}_{n,i} w_{n,i}^* \ , \ x_{n+1} = \nabla f^*(x_{n+1}^*) \, , \tag{4.9}$$

where $\tilde{t}_n = (t_{n,1}, \ldots, t_{n,I_n})$ is chosen such that it minimizes the mapping h given by

$$h(t) = f^* \left(x_n^* - \sum_{i=1}^{I_n} t_{n,i} w_{n,i}^* \right) + \sum_{i=1}^{I_n} t_{n,i} \alpha_{n,i} \, .$$

If stopping criterion is satisfied set $x = x_n$, otherwise increment n and start over.

---

from which we obtain $\|A(x_n) - y\| \to 0$ as $n \to \infty$. Since by assumption the operator A is weakly closed, we may conclude that $\tilde{x}$ is a solution of the equation $A(x) = y$ which concludes the proof. □

In lemma 4.4 we have established a sufficient condition for the existence of a weakly convergent subsequence of the sequence generated by algorithm 4.1. We stress that the assumptions are met for relevant cases. In the following, we demonstrate the main convergence results. It turns out that both the setting of bounded linear operators and the setting considering the projection onto a stripe are of special interest as they allow for a stronger result than the general case. In the following, we present these cases separately.

4.3.1 Convergence for linear operators

Convergence of the method described in algorithm 4.1 can be shown for the special setting of a linear operator. The method reduces to algorithm 4.2. Convergence is stated in the following theorem. The proof is similar to the one in [51, Proposition 1], where the authors exploit boundedness of ∂f with the special choice $f = \frac{1}{p}\|\cdot\|^p$. We remark that this assumption is in fact not necessary.

Theorem 4.5. *Let A to be a bounded linear operator and let f be weakly lower semicontinuous, p-convex and satisfy the Kadec-Klee property. Assume that for each $n \in \mathbb{N}$, there is $i_n \in \{1, \ldots, I_n\}$ such that $v_{n,i_n}^* = J_r(Ax_n - y)$ and $j_n \in \{1, \ldots, I_n\}$ such that $w_{n,j_n}^* = x_n^* - x_0^*$. Then* *(i) for arbitrary $n \in \mathbb{N}$ it holds*

$$x_{n+1} = \Pi_{H_n}^{f,x_n^*}(x_n) = \Pi_{H_n}^{f,x_0^*}(x_0) \tag{4.10}$$

with $H_n = \bigcap_{i=1}^{I_n} H(w_{n,i}, \alpha_{n,i})$,

(ii) for arbitrary $n \in \mathbb{N}$ and every solution $\tilde{x}$ of $Ax = y$ it holds

$$\Delta_{f,x_{n+1}^*}(x_{n+1}, \tilde{x}) \leq \Delta_{f,x_n^*}(x_n, \tilde{x}) - \frac{c_0}{p}\left(\frac{1}{\|A\|}\|Ax_n - y\|\right)^p , \tag{4.11}$$

where $\{x_n\}_{n\in\mathbb{N}}$ is the sequence generated by algorithm 4.2. Further, $\{x_n\}_{n\in\mathbb{N}}$ converges strongly to $x^\dagger$, where $x^\dagger$ is the unique solution such that $\Delta_{f,x_0^*}(x_0, \tilde{x})$ is minimal among all solutions $\tilde{x}$ of the operator equation.

Proof. The first identity in eq. (4.10) follows immediately from lemma 3.18. Further, to show $x_{n+1} = \Pi_{H_n}^{f,x_0^*}(x_0)$, for $n > 0$ consider $j_n \in \{1, \dots, I_n\}$ such that $w_{n,j_n}^* = x_n^* - x_0^*$ as well as the estimate

$$\langle x_{n+1}^* - x_0^*, z - x_{n+1}\rangle = \langle x_{n+1}^* - x_n^*, z - x_{n+1}\rangle + \langle w_{n,j_n}^*, z - x_{n+1}\rangle \geq 0 \tag{4.12}$$

for arbitrary $z \in H_n$. The first term is non-negative due to $x_{n+1} = \Pi_{H_n}^{f,x_n^*}(x_n)$ and eq. (3.11). The second term vanishes since it holds $z, x_{n+1} \in H_n$ and hence

$$\langle w_{n,j_n}^*, z\rangle = \langle w_{n,j_n}^*, x_{n+1}\rangle = \alpha_{n,j_n} .$$

By eq. (3.11) we see that eq. (4.12) gives the assertion.

We now show that eq. (4.11) is fulfilled. To this end, for some $i_n \in \{1, \dots, I_n\}$ let $v_{n,i_n} = J_r(Ax_n - y)$. For an arbitrary solution $\tilde{x}$ of $A(x) = y$ the identity $\alpha_{n,i} = \langle v_{n,i}^*, y\rangle = \langle w_{n,i}^*, \tilde{x}\rangle$ leads to

$$\begin{aligned}\Delta_{f,x_{n+1}^*}(x_{n+1}, \tilde{x}) &= f^*\left(x_n^* - \sum_{i=1}^{I_n}\tilde{t}_i w_{n,i}^*\right) + f(\tilde{x}) - \langle x_n^*, \tilde{x}\rangle + \sum_{i=1}^{I_n}\tilde{t}_i\alpha_{n,i} \\ &\leq f^*(x_n^* - tw_{n,i_n}^*) + f(\tilde{x}) - \langle x_n^*, \tilde{x}\rangle + t\alpha_{n,i_n}\end{aligned} \tag{4.13}$$

for arbitrary $t \in \mathbb{R}$, where we used the minimizing property of $\tilde{t} = (\tilde{t}_1, \dots, \tilde{t}_{I_n})$, see lemma 3.18. An application of eq. (3.3) to the first term now yields

$$\begin{aligned}\Delta_{f,x_{n+1}^*}(x_{n+1}, x^\dagger) &\leq f^*(x_n^*) - t\langle w_{n,i_n}^*, x_n\rangle + c_0^*\|tw_{n,i_n}^*\|^{p^*} \\ &\quad + f(x^\dagger) - \langle x_n^*, x^\dagger\rangle + t(\alpha_{n,i_n}) \\ &= \Delta_{f,x_n^*}(x_n, x^\dagger) - t\langle w_{n,i_n}^*, x_n\rangle - \alpha_{n,i_n} \\ &\quad + c_0^* t^{p^*}\|w_{n,i_n}^*\|^{p^*} ,\end{aligned} \tag{4.14}$$

where c_0^* is the constant in eq. (3.3). The latter expression is minimal in t if and only if

$$t_0 = c_0 \left(\frac{\langle w_{n,i_n}^*, x \rangle - \alpha_{n,i_n}}{\|w_{n,i_n}^*\|^{p^*}} \right)^{p/p^*}$$

as elementary calculus shows. By evaluating the right-hand side in eq. (4.14) at t_0 we now obtain the assertion.

We remark that since A is linear, assumption 4.1 is fulfilled. By lemma 4.4 this provides existence of a subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$ converging to some solution $\tilde{x}$ of $Ax = y$. Thus, by weak lower semicontinuity of f and lemma 3.15 item (i) we conclude

$$\begin{aligned}\Delta_{f,x_0^*}(x_0, \tilde{x}) &\le \liminf_{n\to\infty} \Delta_{f,x_0^*}(x_0, x_n) \le \limsup_{n\to\infty} \Delta_{f,x_0^*}(x_0, x_n) \\ &\le \Delta_{f,x_0^*}(x_0, x^\dagger) \le \Delta_{f,x_0^*}(x_0, \tilde{x}),\end{aligned}$$

where we exploited that the solution $x^\dagger$ is contained in H_{n-1}, see remark 4.3, and the identity $x_n = \Pi_{H_{n-1}}^{f,x_0^*}(x_0)$. By uniqueness of the minimizing element $x^\dagger$, this implies $\tilde{x} = x^\dagger$. Consequently, since this holds true for every convergent subsequence, the whole sequence converges. Moreover, the Kadec-Klee property together with convergence $\Delta_{f,x_0^*}(x_0, x_n) \to \Delta_{f,x_0^*}(x_0, x^\dagger)$ by lemma 3.15 item (iii) now yields strong convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ which concludes the proof. □

In theorem 4.5, we have shown convergence of the method in algorithm 4.2 which can be seen as a special case of algorithm 4.1 considering a linear operator A. An analysis of the setting incorporating a potentially nonlinear operator is presented in section 4.3.2, where the iterates are given as the projection onto a stripe.

4.3.2 A general regularization scheme

In the following, we propose an iterative method for the determination of the solution of a potentially non-linear operator equation $A(x) = y$ for both exact and noisy data that is based on Bregman projections onto stripes. Moreover, as the main result of this chapter we show that the presented algorithm complemented by the discrepancy principle, see eq. (1.7), gives rise to a regularization method. In this section, we consider an operator as described in assumption 4.1, where we restrict ourselves to a finite-dimensional setting.

In algorithm 4.3 we adapt the method in algorithm 4.1 to the described setting. We remark that by remarks 3.19 and 4.3 the definition in eq. (4.15) amounts to the iteration $x_{n+1} = \Pi_{H_n}^{f,x_n^*}(x_n)$, where $H_n = H(w_n^*, \alpha_n, \beta_n)$ is a stripe and x_n^* is an admissible subgradient for x_n. It is hence clear that algorithm 4.3 is indeed a special case of algorithm 4.1, where we consider only single-valued sets of descent directions, i.e. $I_n = 1$ for arbitrary $n \in \mathbb{N}$ and the special choice

Algorithm 4.3 Generalized Bregman projection iteration for exact data

Let $x_0 \in X$ and $x_0^* \in \partial f(x_0)$. Starting with $n = 0$, we set

$$\begin{aligned} w_n^* &= A'(x_n)^* J_r(A(x_n) - y)\,, \\ \alpha_n &= \langle J_r(A(x_n) - y), A'(x_n)x_n - A(x_n) + y\rangle\,, \\ \beta_n &= \eta\|A(x_n) - y\|^r\,, \end{aligned}$$

and define the iterates

$$x_{n+1}^* = x_n^* - \tilde{t}w_n^*\,,\ x_{n+1} = \nabla f^*(x_{n+1}^*)\,, \tag{4.15}$$

where $\tilde{t}_n$ is chosen such that it minimizes the mapping h given by

$$h(t) = f^*(x_n^* - tw_n^*) + t(\alpha_n + \beta_n)\,. \tag{4.16}$$

If stopping criterion is satisfied set $x = x_n$, otherwise increment n and start over.

$w_{n,1}^* = A'(x_n)^* J_r(A(x_n) - y)$. Consequently, the result in lemma 4.4 applies. This setting, however, is more convenient due to its theoretical and numerical feasibility. Particularly, the projection in each iteration reduces to a projection onto a hyperplane, see remark 3.19, which establishes direct accessibility in terms of both theory and applications.

Theorem 4.6. *Let X be finite dimensional and let assumption 4.1 be satisfied. Moreover, for some $y \in Y$ let the solution $x^\dagger \in B_R(x_0) \cap \mathrm{dom}(f)$ of the operator equation $A(x) = y$ as in $A(x) = y$ be uniquely determined in $B_{2R}(x_0)$. Then*

(i) for arbitrary $n \in \mathbb{N}$ it holds

$$x_{n+1} = \Pi_{H_n}^{f,x_n^*}(x_n) \tag{4.17}$$

with $H_n = H(w_n^*, \alpha_n, \beta_n)$,

(ii) for arbitrary $n \in \mathbb{N}$ it holds

$$\Delta_{f,x_{n+1}^*}(x_{n+1}, x^\dagger) \leq \Delta_{f,x_n^*}(x_n, x^\dagger) - \frac{c_0}{p}\left(\frac{1-\eta}{\|A'(x_n)\|}\|A(x_n) - y\|\right)^p\,, \tag{4.18}$$

where $\{x_n\}_{n\in\mathbb{N}}$ is the sequence generated by algorithm 4.3 and $\{x_n\}_{n\in\mathbb{N}}$ converges to $x^\dagger$.

Proof. In a first step, to show item (i), we consider the inequality $\langle w_n^*, x_n\rangle >$

$\alpha_n + \beta_n$ which by eq. (4.23) follows from the estimate

$$\langle w_n^*, x_n\rangle - \alpha_n - \beta_n = (1-\eta)\|A(x_n) - y\|^r > 0\,. \tag{4.19}$$

By lemma 3.18 and remark 3.19 we hence get the assertion.

Next, we show that item (ii) holds true. We remark that in each iteration step, $\tilde{t}_n$ is positive. This follows from the observation that for the mapping h as in eq. (4.16) it holds $h'(0) = -\langle w_n^*, x_n\rangle + \alpha_n + \beta_n < 0$, see eq. (4.19). By the minimizing property of $\tilde{t}$, see lemma 3.18, and the relation $\langle w_n^*, x^\dagger\rangle \le \alpha_n + \beta_n$ that follows from the estimate

$$\begin{aligned}\langle w_n^*, x^\dagger\rangle &= \langle J_r(A(x_n) - y), A'(x_n)x^\dagger\rangle \\ &= \langle J_r(A(x_n) - y), A'(x_n)x_n - A(x_n) + A(x^\dagger)\rangle \\ &\quad - \langle J_r(A(x_n) - y), A(x^\dagger) - A(x_n) - A'(x_n)(x^\dagger - x_n)\rangle \\ &\le \langle J_r(A(x_n) - y), A'(x_n)x_n - A(x_n) + A(x^\dagger)\rangle + \eta\|A(x_n) - y\|^r\end{aligned}$$

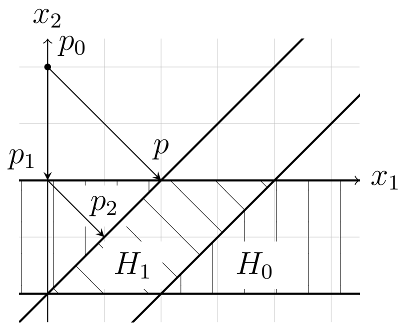
we now obtain for arbitrary $t \in \mathbb{R}$

$$\begin{aligned}\Delta_{f,x_{n+1}^*}(x_{n+1}, x^\dagger) &\le f^*(x_n^* - \tilde{t}w_n^*) + f(x^\dagger) - \langle x_n^*, x^\dagger\rangle + \tilde{t}(\alpha_n + \beta_n) \\ &\le f^*(x_n^* - tw_n^*) + f(x^\dagger) - \langle x_n^*, x^\dagger\rangle + t(\alpha_n + \beta_n)\,.\end{aligned} \tag{4.20}$$

We could thereby re-establish the inequality eq. (4.13) in the proof of theorem 4.5. Starting from this result, the assertion in eq. (4.18) now follows by analogy with the proof of theorem 4.5.

By lemma 4.4 we hence get the existence of a subsequence converging weakly to some solution $\tilde{x}$. It converges strongly since we assumed that X is finite dimensional. Furthermore, it is easily seen that since $x_n \in B_{2R}(x_0)$ for every $n \in \mathbb{N}$, we may conclude $\tilde{x} \in B_{2R}(x_0)$, which by uniqueness of the solution yields $\tilde{x} = x^\dagger$. Hence, since this argument is valid for arbitrary convergent subsequences, the whole sequence converges. This concludes the proof. □

Remark 4.7. In eq. (4.10), we have seen that in the special case of a linear operator and exact data we get the identity $x_{n+1} = \Pi_{H_n}^{f,x_0^*}(x_0)$. This, in turn, could be used to establish strong convergence of the sequence $\{x_n\}_{n\in\mathbb{N}}$ even in a general Banach space setting. While it would be desirable to adopt this technique to establish a corresponding result for general operators as in theorem 4.6, the argument does not hold true in this setting. To see this, it is sufficient to consider the metric projection as a special case of the Bregman projection, see example 3.10, in a two-dimensional setting. Let $x_0 = (0,1)$ and $H_0 = H(w_0, 1, 1)$ as well as $H_1 = H(w_0, 1, 1) \cap H(w_1, 1, 2)$ where $w_0 = (0,1)$ and $w_1 = (1,-1)$. Both sets are illustrated in Figure fig. 4.1(a). A simple calculation yields that $x_1 = \Pi_{H_0}(x_0) = (0,0)$ and, thus, $x_2 = \Pi_{H_1}(x_1) = \frac{1}{\sqrt{2}}(1,-1)$. On the other hand, we have $\Pi_{H_1}(x_0) = (1,0)$. The latter expression does not coincide with x_2, see



(a)

Figure 4.1: The first iterates p_n produced by algorithm 4.3 and the projection $p = \Pi^{p_0}_{H_1}(p_0)$. Note that the latter two projections do not coincide.

Algorithm 4.4 Generalized Bregman projection iteration for noisy data

Let $x_0 \in X$ and $x_0^* \in \partial f(x_0)$. Choose τ such that

$$\tau > \frac{1+\eta}{1-\eta} .$$

Starting with $n = 0$, we set

$$\begin{aligned} w_n^* &= A'(x_n)^* J_r(A(x_n) - y^\delta) , \\ \alpha_n &= \langle J_r(A(x_n) - y^\delta), A'(x_n)x_n - A(x_n) + y^\delta \rangle , \\ \beta_n &= \big(\eta \|A(x) - y^\delta\| + \delta(1+\eta)\big) \|A(x_n) - y^\delta\|^{r-1} , \end{aligned}$$

and define the iterates

$$x_{n+1}^* = x_n^* - \tilde{t} w_n^* , \; x_{n+1} = \nabla f^*(x_{n+1}^*) , \tag{4.21}$$

where $\tilde{t}_n$ is chosen such that it minimizes the mapping h given by

$$h(t) = f^*(x_n^* - t w_n^*) + t(\alpha_n + \beta_n) . \tag{4.22}$$

If it holds

$$\|A(x_n) - y^\delta\| \leq \tau\delta$$

exit the iteration and set $n = n(\delta, y^\delta)$, otherwise increment n and start over.

fig. 4.1, and hence, formula eq. (4.10) can not be reproduced in a general setting. However, in a finite dimensional setting one can still obtain strong convergence of the sequence.

In algorithm 4.4, we now consider the setting of noisy data. Again, as in algorithm 4.1 we remark that the iteration in algorithm 4.4 can be interpreted as the projection $x_{n+1} = \Pi_{H_n}^{f,x_n^*}(x_n)$, where $H_n = H(w_n^*, \alpha_n, \beta_n)$ is a stripe and x_n^* is an admissible subgradient for x_n. Note that the width of the stripe is controlled by the noise level.

Theorem 4.8. *Let X be finite dimensional and let assumption 4.1 be satisfied. Moreover, for some $y \in Y$ let the solution $x^\dagger \in B_R(x_0) \cap \mathrm{dom}(f)$ of the operator equation $A(x) = y$ be uniquely determined in $B_{2R}(x_0)$ and assume $\|y - y^\delta\| \leq \delta$ for data $y^\delta \in Y$. Consider $x_0 \in X$ such that $\|A(x_0) - y^\delta\| > \tau\delta$, where*

$$\tau > \frac{1+\eta}{1-\eta}\,. \tag{4.23}$$

Then

(i) for arbitrary $n \in \mathbb{N}$ it holds

$$x_{n+1} = \Pi_{H_n}^{f,x_n^*}(x_n) \tag{4.24}$$

with $H_n = H(w_n^*, \alpha_n, \beta_n)$,

(ii) for every $x^\dagger$ such that $A(x^\dagger) = y$ with $\|y - y^\delta\| \leq \delta$ it holds

$$\Delta_{f,x_{n+1}^*}(x_{n+1}, x^\dagger) \leq \Delta_{f,x_n^*}(x_n, x^\dagger) - \frac{c_0}{p}\left(\frac{(1-\eta)\|A(x_n) - y^\delta\| - (1+\eta)\delta}{\|A'(x_n)\|}\right)^p , \tag{4.25}$$

where $\{x_n\}_{n\in\mathbb{N}}$ is the sequence generated by algorithm 4.3. Then, the method described in algorithm 4.4 stops for a finite value $n = n(\delta, y^\delta)$ gives rise to a regularization method.

Proof. To show item (i), we consider the inequality $\langle w_n^*, x_n\rangle > \alpha_n + \beta_n$ which by eq. (4.23) follows from the estimate

$$\eta\|A(x_n) - y^\delta\| + \delta(1+\eta) \leq \eta\|A(x_n) - y^\delta\| + \tau\delta(1-\eta) < \|A(x_n) - y^\delta\|\,.$$

By lemma 3.18 and remark 3.19 we hence get the assertion. The second assertion item (ii) follows by analogy with the proof of the relation eq. (4.18) in theorem 4.6.

In a next step, we show that the stopping criterion is fulfilled for a finite iteration index $n = n(\delta, y^\delta)$. To this end, assume that $\|A(x_n) - y^\delta\| > \tau\delta$ for arbitrary $n \in \mathbb{N}$.

We show that this leads to a contradiction. By eq. (4.25) we conclude

$$\begin{aligned}
&\Delta_{f,x^*_{n+1}}(x_{n+1},x^\dagger)\\
&\quad\le \Delta_{f,x^*_n}(x_n,x^\dagger) - \frac{c_0}{p}\left(\frac{(1-\eta)\|A(x_n)-y^\delta\|-(1+\eta)\delta}{\|A'(x_n)\|}\right)^p\\
&\quad\le \Delta_{f,x^*_n}(x_n,x^\dagger) - \frac{c_0}{p}\left(\frac{(1-\eta-\tau^{-1}(1+\eta))\,\|A(x_n)-y^\delta\|}{\|A'(x_n)\|}\right)^p\\
&\quad\le \Delta_{f,x^*_n}(x_n,x^\dagger) - \frac{c_0}{p}\left(\frac{(1-\eta-\tau^{-1}(1+\eta))\,\|A(x_n)-y^\delta\|}{C}\right)^p .
\end{aligned}$$

where $C>0$ such that $\|A'(x)\|\le C$ for arbitrary $x\in X$, see assumption 4.1, and in the latter step, we made use of the fact that the particular choice of τ ensures $1-\eta-\tau^{-1}(1+\eta)>0$. For arbitrary $N\in\mathbb{N}$ we conclude

$$\begin{aligned}
0 &\le D\sum_{n=0}^{N}\|A(x_n)-y^\delta\|^p\\
&\le \Delta_{f,x^*_0}(x_0,x^\dagger) - \Delta_{f,x^*_{N+1}}(x_{N+1},x^\dagger) \le \Delta_{f,x^*_0}(x_0,x^\dagger)\,,
\end{aligned}$$

where $D=\frac{c_0}{p}\cdot\left(C^{-1}\left(1-\eta-\tau^{-1}(1+\eta)\right)\right)^p$. Since $N\in\mathbb{N}$ was arbitrarily large this implies $\|A(x_n)-y^\delta\|\to 0$ as $n\to\infty$ which contradicts the assumption.

Finally, we investigate the regularizing properties of the method. Let $\{\delta_k\}_{k\in\mathbb{N}}$ be a null-sequence. The definition of the index $n(\delta_k,y^{\delta_k})$ provides for every $k\in\mathbb{N}$ $\|A(x^{\delta_k}_{n(\delta_k,y^{\delta_k})})-y^{\delta_k}\|\le\tau\delta_k$ which yields

$$\|A(x^{\delta_k}_{n(\delta_k,y^{\delta_k})})-y\| \le \|A(x^{\delta_k}_{n(\delta_k,y^{\delta_k})})-y^{\delta_k}\| + \|y^{\delta_k}-y\| \to 0 \tag{4.26}$$

as $k\to\infty$. Furthermore, since $x^{\delta_k}_{n(\delta_k,y^{\delta_k})}\in B_{2R}(x_0)$ for every $n\in\mathbb{N}$, see lemma 4.4, we may conclude boundedness of the sequence $\{x^{\delta_k}_{n(\delta_k,y^{\delta_k})}\}_{k\in\mathbb{N}}$. Hence, there is a weakly convergent subsequence which converges strongly since X is finite dimensional. By weak closedness of A and eq. (4.26) we conclude that any such subsequence converges to the unique solution $x^\dagger$ of $A(x)=y$ in $B_{2R}(x_0)$ and consequently, the whole sequence converges. This yields the assertion. □

4.4 Numerical evaluation

Subsequently, we evaluate the method introduced in section 4.3.2 considering an application to the inverse medium problem, where we refer to the linearization given by the Born approximation. Instead of the forward model A as in section 2.3.1, we use the operator $K=A\circ S$ as in eq. (2.12), where S is the synthesis operator corresponding to different choices of the basis. In all subsequent numerical applications, we use the parameters $p=r=2$. Furthermore, we use $\beta=1$ as

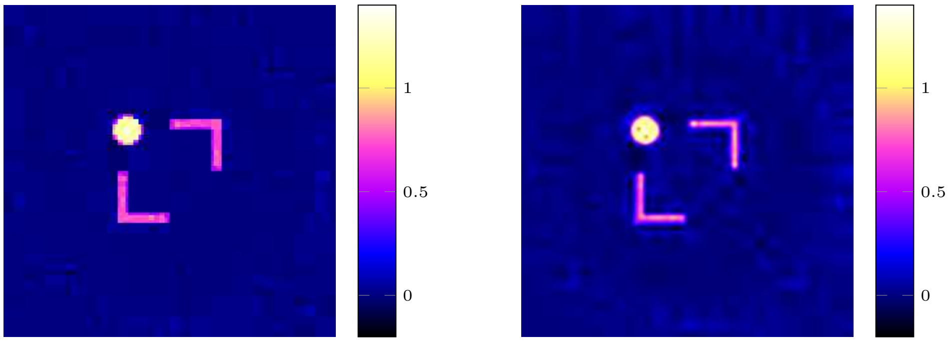


Figure 4.2: The real part of the reconstruction from noisy data with 10% of Gaussian noise according to algorithm 4.4 using the synthesis operator S corresponding to the Haar wavelet basis (left) and the CDF wavelet basis (right).

Method	**Iterations**	**CPU time**
Landweber scheme	1488	1269
Accelerated Landweber scheme	243	213
Bregman projection scheme	87	86

Table 4.1: The number iterations and the CPU time consumed (seconds) until the discrepancy principle is satisfied for the Haar wavelet basis and $f = \|\cdot\|_{\ell_1} + \frac{\beta}{2}\|\cdot\|_{\ell_2}^2$, where $\beta = 0.1$ and $\tau = 1.1$.

this choice could be observed to yield both fast convergence of the algorithm and appealing reconstructions in numerical experiments. In fig. 4.2 we depict the reconstructions according to algorithm 4.4, where the regularization parameter was determined by the discrepancy principle with $\tau = 1.1$. For reference, in fig. 2.3 we provided the reconstruction according to the classical Landweber iteration. It turns out that the Bregman projection-based method yields the more appealing reconstructions by far.

Note that while in theorem 4.8 we required uniqueness of the solution of the operator equation, this property does not hold true in this setting. However, the numeric examples show that despite this deficiency, useful reconstructions can be determined.

In eq. (4.21) it is suggested that algorithm 4.4 can be interpreted as a Landweber-type iteration incorporating an adaptive choice of the descent parameter t_n, see eq. (4.1). In tables 4.1 and 4.2 we demonstrate the numerical advantage of method algorithm 4.4 in terms of iterations over the generalized Landweber method, see theorem 4.2. For comparison, we further incorporate an acceleration of the method in theorem 4.2 based on the adaptive choice of the descent parameter in eq. (4.1) given by eq. (4.4) in our investigations. We refer to this method as the *accelerated Landweber method*. For each method, every iteration requires only one evaluation

Method	**Iterations**	**CPU time**
Landweber scheme	2505	2238
Accelerated Landweber scheme	370	340
Bregman projection scheme	185	193

Table 4.2: The number iterations and the CPU time consumed (seconds) until the discrepancy principle is satisfied for the CDF wavelet basis and $f = \|\cdot\|_{\ell_1} + \frac{\beta}{2}\|\cdot\|_{\ell_2}^2$, where $\beta = 0.1$ and $\tau = 1.1$.

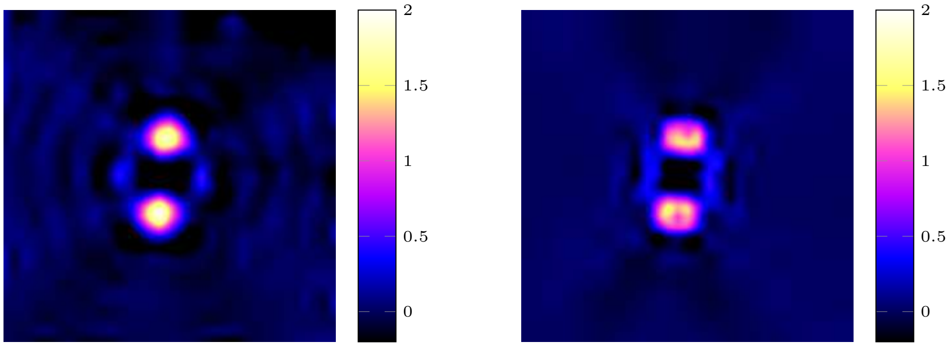


Figure 4.3: The real part of the reconstruction from measurement data according to the regularization method as in algorithm 4.4 for $k \approx 83$ (left) and $k \approx 125$ (right).

of both K and its adjoint operator K^*. It turns out that the Bregman projection iteration in algorithm 4.4 easily outperforms both Landweber-type methods in terms of iterations, while the increased complexity of each iteration step resulting from the optimization problem eq. (4.22) in each step is negligible.

Finally, we present a reconstruction from measurement data according to algorithm 4.4. We use data from the Fresnel database, as we have already described in section 2.4.2. Since the noise level is not given explicitly in the Fresnel database and additionally the model under consideration is only an approximation of the scattering problem, for the application we need to refer to the an estimated value of $\delta = 3$, i.e. about 17% of noise. Again, we use $\beta = 1$. The resulting reconstruction are depicted in fig. 4.3, where the synthesis operator S is chosen according to the CDF wavelet basis and the regularization parameter is determined according to the discrepancy principle with $\tau = 1.1$.

One observes in fig. 4.3 that the reconstructed contrast q establishes the estimated value of 2 ± 0.3 at the obstacle quite accurately. This is a significant improvement over the reconstruction as in fig. 2.4. Further, the obstacle is well localized and there are only minor artifacts in the background.

CHAPTER
FIVE

TIKHONOV-TYPE METHODS

5.1 Introduction

Arguably the most prominent variational approach for the stable solution of an operator equation of the form $A(x) = y$ as in eq. (1.1) is based on the minimization of the so-called Tikhonov functional. The method can be seen as a generalization of the Tikhonov-Phillips regularization, see section 1.3.2. In this chapter, we present the basic concept of Tikhonov-type regularization and introduce a novel heuristic parameter choice rule for sparsity-constrained regularization. Furthermore, we demonstrate the practical relevance of a Tikhonov-type method when applied to the inverse medium problem.

In section 1.3.2, we introduced the classical Tikhonov-Phillips regularization. Following this approach, for Hilbert spaces X and Y and an operator $A : X \to Y$ the reconstruction of a solution $x^\dagger$ of the equation $A(x) = y$ is given by the solution of the variational problem eq. (1.8). Here, to assure stability of the reconstruction method with respect to deviations in the data y, the norm $\|\cdot\|_X$ was introduced as an additional term in the objective functional eq. (1.3). This way, we incorporate additional a priori knowledge about $x^\dagger$ into the model, i.e. we require the according reconstruction given by the minimizer of the functional in eq. (1.8) to be small with respect to the norm. This approach can be generalized by replacing both terms in eq. (1.8) by general functionals, which for $\alpha > 0$ leads to a mapping $\Psi_\alpha : X \to \overline{\mathbb{R}}$ of the form

$$\Psi_\alpha(x) = F(x, y^\delta) + \alpha R(x)\,. \tag{5.1}$$

Further, by

$$x_\alpha^\delta \in \operatorname*{arg\,min}_{x \in X} \Psi_\alpha(x) \tag{5.2}$$

we denote a minimizer of Ψ_α. In eq. (5.1), we consider Banach spaces X and Y and a general *fidelity term* $F : X \times Y \to [0, \infty]$ that measures the discrepancy between x and the measurement data y^δ. Typical choices for F are given by

$\frac{1}{q}\|A(\cdot) - y^\delta\|_{\ell_q}^q$ for $q > 1$ or $\int A(\cdot) - y^\delta \log A(\cdot)$ to name a few, see [10, 17]. The actual choice of F usually is determined by statistical considerations regarding the nature of the noise inherent in the data. In eq. (5.1), furthermore, a *regularization term* $R : X \to \overline{\mathbb{R}}$ is introduced that incorporates a priori knowledge about the true solution of the equation $A(x) = y$. This regularization term R generalizes the Hilbert space norm under consideration in eq. (1.8) and is usually given by a coercive functional to enforce the existence of a minimizer of eq. (5.1). Here, typical instances are given by $\frac{1}{p}\|x\|_{\ell_p}^p$ for $p > 0$ or $\|x\|_{TV}$, see [24, 47]. Again, as in eq. (1.8) the parameter α balances both terms. This general approach allows to incorporate a priori knowledge about the true solution $x^\dagger$ that is more appropriate compared to the classical approach in section 1.3.2.

In the following, we cite a result that provides stability of the method in eq. (5.2). Further, we develop a novel heuristic choice rule for the regularization parameter for Tikhonov-type regularization methods incorporating sparsity constraints that is based on the so-called L-curve criterion. Finally, we demonstrate the potential of a sparsity-promoting method when applied to the inverse medium problem in numerical experiments. Particularly, we demonstrate the relevance of the Born approximation of the scattering problem for the inverse medium problem and show that it is possible to obtain reasonable reconstructions from real data using the linearized model.

To start with, we cite a sufficient criterion for the existence of at least one minimizer of the functional Ψ_α as in eq. (5.2).

Proposition 5.1. *Let* $F(\cdot, y^\delta) : X \to [0, \infty]$ *be proper and lower semi-continuous and let* $R : X \to \overline{\mathbb{R}}$ *be proper, lower semicontinuous and coercive such that* $\mathrm{dom}(F) \cap \mathrm{dom}(R) \neq \emptyset$. *Then the functional in eq.* (5.1) *attains its minimum at at least one point* $x_\alpha^\delta \in X$.

It is standard to conclude the existence of a minimizer for a proper, lower semi-continuous and coercive functional, see [61], for instance. Note that coerciveness of R implies coerciveness of the whole functional.

5.2 Regularization by means of Tikhonov-type functionals

Subsequently, let X and Y be Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. Further let $A : \mathrm{dom}(A) \subset X \to Y$ be a potentially non-linear operator. For $q > 1$, we consider the functional

$$\Psi_\alpha(x) = \frac{1}{q}\|A(x) - y^\delta\|_Y^q + \alpha R(x)\,, \tag{5.3}$$

where $R : X \to [0, \infty]$ is a functional mapping into the extended real line. This functional is obviously of the form eq. (5.1). Note that at this, we do not assume the regularization term R to be convex. Particularly, the prominent non-convex functional $\frac{1}{p}\|x\|_{\ell_p}^p$ is covered even for values $p < 1$. For later use we collect the following assumptions.

Assumption 5.2. Let

(i) X and Y be reflexive Banach spaces.

(ii) $A : \mathrm{dom}(A) \subset X \to Y$ be weakly sequentially closed and bounded on bounded sets.

(iii) $R : X \to \overline{\mathbb{R}}$ be proper.

(iv) $R : X \to \overline{\mathbb{R}}$ be coercive.

(v) $R : X \to \overline{\mathbb{R}}$ is weakly lower semicontinuous.

(vi) $R : X \to \overline{\mathbb{R}}$ obey the Kadec-Klee property, see definition 3.14.

A minimum-R-solution $\tilde{x}$ is a solution of the operator equation $A(x) = y$ such that $R(\tilde{x})$ is minimal among all solutions. We can now state the following result.

Theorem 5.3 (Regularization)**.** *Let assumption 5.2 items (i) to (vi) be fulfilled and assume that $y \in \mathrm{rg}(A)$ such that $\{x \in X \,:\, A(x) = y\} \cap \mathrm{dom}(R) \neq \emptyset$. Then, for a parameter choice $\alpha = \alpha(\delta)$ with*

$$\alpha \to 0\,, \quad \frac{\delta^q}{\alpha} \to 0 \text{ as } \delta \to 0 \tag{5.4}$$

and x_α^δ as in eq. (5.2) there exists a subsequence of x_α^δ which converges to a minimum-R-solution $x^\dagger$ of $A(x) = y$. Moreover it holds that $R(x_\alpha^\delta) \to R(x^\dagger)$.

A proof for theorem 5.3 can be found in [10, Theorem 2.3]. A natural question is whether one may obtain even convergence rates in this setting. The answer to this question, of course, is yes. Usually, to this end we need further assumptions regarding the structure of the regularization term. If R is separable, see section 5.3, there are several results in literature that establish convergence rates, see [10, 23], for instance.

5.3 Sparsity-promoting regularization terms

In theorem 5.3, we considered a rather general setting without further assumptions on the structure of the regularization term. Subsequently, for $X = \ell_2$ we consider a functional that is *separable*, i.e. a mapping $R : \ell_2 \to [0, \infty]$ that for $x = (x_k)_{k\in\mathbb{N}} \in \ell_2$ is determined by a function $\varphi : \mathbb{R} \to [0, \infty]$ according to the series

$$R(x) = \sum_{k\in\mathbb{N}} \varphi(x_k)\,. \tag{5.5}$$

The relevance of functionals of this form is most prominently demonstrated by the regularization

$$\|x\|_{\ell_p}^p = \sum_{k\in\mathbb{N}} |x_k|^p\,. \tag{5.6}$$

Due to its importance in regularization theory, it may be regarded as a prototypical example for a separable regularization term. Particularly, for an operator $A : \ell_2 \to Y$ mapping into a Hilbert space Y in the following we consider the functional

$$\Psi_\alpha(x) = \frac{1}{2}\|A(x) - y^\delta\|_Y^2 + \alpha\|x\|_{\ell_1}\,. \tag{5.7}$$

This functional is widely used in regularization theory and has received enormous attention in recent years, see [11, 24, 38, 43, 10].

Remark 5.4. Note that if $p \leq 1$, then the functional R according to eq. (5.6) is not differentiable even in a finite dimensional setting. This may contribute to potential difficulties in the minimization of the corresponding Tikhonov-type functional as in eq. (5.1). One can, however, replace the functional Ψ_α by some smooth approximation which allows for standard minimization procedures. This approach is presented in chapter 6. Here, we thoroughly investigate stability of the minimizer of the approximated functional with respect to an approximation index and even establish convergence rates.

For further investigations of the functional Ψ_α and the according minimizers we introduce the following notion.

Definition 5.5 (Sparsity)**.** An element $x = (x_k)_{k\in\mathbb{N}} \in \ell_2$ is called *sparse* if and only if it holds $x_k = 0$ for all but finitely many $k \in \mathbb{N}$.

The functional in eq. (5.7) is commonly referred to as *sparsity-promoting* or *sparsity-enforcing*. This can be reasoned with the following rationale. Consider the Hilbert space norm and compare its effect on the solution to the regularization term $\|x\|_{\ell_p}$ as in eq. (5.6) for $p < 2$. One observes that large components of the sequence $x = (x_k)_{k\in\mathbb{N}} \in \ell_2$ contribute to a comparatively small addend, while small components are strongly penalized. This is illustrated in fig. 5.1. Hence, the minimizer of a related Tikhonov-type functional is likely to attain large values for only a few distinct components and small values elsewhere. This structure is sometimes referred to as *almost sparse*. Furthermore, if the operator A is linear,

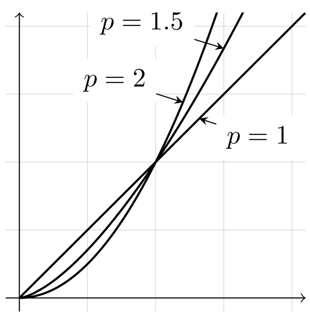


Figure 5.1: The mapping $|x|^p$ for different values of p. For $p < 2$, small values are penalized stronger while large values are penalized less compared to $p = 2$. The structure of the regularization term based on these mappings yields almost sparse and even sparse minimizers.

then the minimizer of eq. (5.7) can be shown to be *sparse* in the strict sense given in definition 5.5. This assertion together with the simple mathematical proof is provided with the next proposition.

Proposition 5.6. *Let Y be a Hilbert space and consider Ψ_α as in eq.* (5.7)*. Further, let $A : \ell_2 \to Y$ be linear and bounded. Then, every minimizer x_α^δ of eq.* (5.7) *is sparse. Moreover, there is $\alpha_0 > 0$ such that $x_\alpha^\delta = 0$ for arbitrary $\alpha > \alpha_0$.*

Proof. Let x_α^δ be a minimizer of Ψ_α and consider the first order optimality condition which reads

$$0 \in A^*(Ax_\alpha^\delta - y^\delta) + \alpha \operatorname{Sign}(x_\alpha^\delta)\,.$$

This can be rewritten as

$$-A^*(Ax_\alpha^\delta - y^\delta) \in \alpha \operatorname{Sign}(x_\alpha^\delta)\,, \tag{5.8}$$

where Sign as in eq. (3.5). Assuming that x_α^δ is not sparse, from this we obtain $|(A^*(Ax - y^\delta))_k| = \alpha$ for infinitely many $k \in \mathbb{N}$. This contradicts $\operatorname{rg}(A^*) \subset \ell_2$ and hence, we get the first assertion.

To show the second assertion, we remark that x_α^δ is uniformly bounded in ℓ_2 with respect to α for $\alpha > \varepsilon$ with $\varepsilon > 0$ as follows from the estimate

$$\|x_\alpha^\delta\|_{\ell_2} \leq \|x_\alpha^\delta\|_{\ell_1} \leq \alpha^{-1}\|y^\delta\|^2\,, \tag{5.9}$$

where in the latter step we used the minimizing property of x_α^δ which yields $\Psi_\alpha(x_\alpha^\delta) \leq \Psi_\alpha(0)$. The latter expression in eq. (5.9) tends to 0 as $\alpha \to \infty$ and hence, we may conclude that x_α^δ, and by boundedness of A even $A^*(Ax_\alpha^\delta - y^\delta)$, is uniformly bounded in ℓ_2. Particularly, for $\varepsilon > 0$ there is $C > 0$ such that $|(A^*(Ax_\alpha^\delta - y^\delta))_k| < C$ for arbitrary $\alpha > \varepsilon$ and $k \in \mathbb{N}$. From this we get for arbitrary $\alpha > C$ and $k \in \mathbb{N}$ that $|(A^*(Ax_\alpha^\delta - y^\delta))_k| < \alpha$, which by eq. (5.8) yields the identity $(x_\alpha^\delta)_k = 0$. Hence we get $x_\alpha^\delta = 0$, which concludes the proof. □

In proposition 5.6, we have presented a basic proof for sparsity of the reconstruction according to eq. (5.7). More generally, in [23] conditions are established that ensure sparsity of the minimizer of eq. (5.1) for a general regularization term of the form eq. (5.5). Furthermore, in [10] convergence rates for the regularization according to eq. (5.1) incorporating a regularization term of the form eq. (5.5) are presented.

Our main ambition is to evaluate the practical relevance of the sparsity-promoting approach in eq. (5.7) for imaging applications. Thus, in section 5.5 we apply the according regularization scheme to the inverse medium problem using both synthetic and real data. Beforehand, in section 5.4 we study a novel parameter choice rule which is based on the so-called L-curve criterion.

5.4 Parameter choice according to the L-curve criterion

The minimization of a functional of the form eq. (5.1) gives rise to a regularization under quite general assumptions on F and R as we have discussed in section 5.2. We stress that the regularization needs to be complemented with an appropriate parameter choice rule in order to provide useful results, see definition 1.2. In the setting of Tikhonov-type regularization, the regularization parameter α balances the fidelity term against the regularization term. If, on the one hand, the value of α is too small, then the regularization term explodes which may result in noise-cluttered reconstructions. On the other hand, if the value of α too large, then the reconstruction may be only an inaccurate approximation of the sought for solution $x^\dagger$ of the equation $A(x) = y^\delta$. Hence, the regularization parameter needs to be chosen carefully.

One example for an parameter choice rule is given by the so-called L-curve criterion. Consider the Tikhonov-type functional Ψ_α as in eq. (5.1). Further, for each positive value of α by x_α^δ we denote a minimizer of Ψ_α, see eq. (5.2). If we plot the fidelity term $F(x_\alpha^\delta, y^\delta)$ against the regularization term, we obtain a plain curve parametrized by α, i.e.

$$\alpha \mapsto (F(x_\alpha^\delta, y^\delta)\,,\, R(x_\alpha^\delta))\,.$$

Typically, this curve can be separated into two branches homogeneous in direction that connect at some point referred to as the *kink*. Due to the similarity to an L-shape, the curve is usually referred to as the L-curve. In order to emphasize the location of the kink it is standard to scale both axes logarithmically.

The kink plays an important role. Figuratively speaking, changing the regularization parameter α from the value that corresponds to the kink leads to a significant increase of the value of either the fidelity term or the regularization term. Thus, it seems intuitive that the kink corresponds to a value of α for which both terms are small. The idea is hence to choose the regularization parameter α such that it corresponds to the kink of the L-curve. It goes back to [28] and has influenced many approaches since.

Remark 5.7. The L-curve criterion is a representative of the so-called *heuristic* parameter choice rules as the determination of the parameter does not depend on the noise level δ. Consequently, it cannot contribute to a regularization method, see [3]. Indeed, in [56, 27] it was shown that the method presented in [28] fails for appropriately constructed examples. However, starting with the general idea of the L-curve, several methods have been developed that work surprisingly well in applications.

The major task to apply any L-curve based method is the determination of the kink. While it is often visually clear where a kink is located, mathematically this is not as easy. Several characterizations for the kink have been introduced that allow for mathematical treatment. In [28], the kink is defined as the point of maximal curvature, albeit only the setting of classical Tikhonov regularization is covered, see eq. (1.8). A general setting was considered in [42], for instance, where the location of a kink corresponds to a point where the tangent of the curve attains some predefined slope. One major advantage of this approach is that a kink, according to this definition, can be computed applying a fixed point iteration that requires the knowledge of the minimizer x_α^δ for only a few values of α. However, both the fidelity term F and the regularization term R are required to be differentiable for this method to be applicable.

As we have remarked in section 5.3, the regularization term given by eq. (5.6) is a relevant instance of a non-smooth functional. Hence, existing strategies are not applicable, while applications suggest that the L-curve criterion is still appropriate. Indeed, in figs. 5.3(a) and 5.3(d) we see that the overall appearance of the curve remains L-shaped and, moreover, the kink does correspond to a value of the regularization parameter α which approximates the optimal choice accurately. One approach is to replace the non-smooth instance of the regularization term R by some smooth approximation. This allows for the application of known methods cited above. The concept of approximating Tikhonov-type functionals is continued in chapter 6, where we establish stability results for the corresponding minimizers. Subsequently, we follow a different approach. In algorithm 5.1, we propose an iterative method for the determination of the kink for the sparsity-promoting functional in eq. (5.7). In contrast to the approaches in [28, 42], we do not characterize some point on the actual L-curve, but consider an idealized L-shape which allows for a straight-forward definition of the kink. Even though these additional assumptions on the curve are very unlikely to be satisfied, the results provided by this heuristic method are very promising.

5.4.1 The L-curve for sparsity promoting functionals

Consider the fidelity term F and the regularization term R according to the sparsity-promoting functional Ψ_α as introduced in eq. (5.7). In figs. 5.3(a) and 5.3(d) we see that the corresponding L-curve can actually be described as L-shaped. As usual, we use a logarithmic scale to emphasize the kink. However, the kink is only slightly pronounced and it seems comparatively difficult to visually determine a specific point that marks the point of interest. Thus, we propose that instead of a plot of the discrepancy $\frac{1}{2}\|Ax_\alpha^\delta - y^\delta\|^2$ against the regularization term $\|x_\alpha^\delta\|_{\ell_1}$, we suggest to replace the regularization term by $\|x_\alpha^\delta\|_{\ell_0}$. Accordingly, in the following we consider

$$\alpha \mapsto \left(\frac{1}{2}\|Ax_\alpha^\delta - y^\delta\|^2\,,\ \|x_\alpha^\delta\|_{\ell_0}\right) \tag{5.10}$$

instead of

$$\alpha \mapsto \left(\frac{1}{2}\|Ax_\alpha^\delta - y^\delta\|^2\,,\ \|x_\alpha^\delta\|_{\ell_1}\right). \tag{5.11}$$

An according curve is illustrated in figs. 5.3(b) and 5.3(e). We remark that by x_α^δ we still refer to the minimizer eq. (5.2) of Ψ_α and only the component of the curve has changed.

We argue that this replacement is intuitive. In proposition 5.6 we have shown that the minimizer of eq. (5.7) is always sparse and hence, the semi-norm $\|x_\alpha^\delta\|_{\ell_0}$ attains a finite value for arbitrary $\alpha > 0$. Further, it satisfies a favorable asymptotic behavior. In proposition 5.6 we show that $\|x_\alpha^\delta\|_{\ell_0}$ vanishes if $\alpha \to \infty$. If we consider the behavior $\alpha \to 0$, then usually what one observes is that the reconstruction becomes less and less sparse as the error in the data dominates the reconstruction. In other words, it is justified to assume that $\|x_\alpha^\delta\|_{\ell_0}$ becomes arbitrarily large. This behavior can be observed in the corresponding plot of the curve, see figs. 5.3(b) and 5.3(e). Collectively, it is thus sensible to assume that the expression $\|x_\alpha^\delta\|_0$ falls from initially large values to 0 as α is increasing. Furthermore, experiments show that the curve one obtains is again L-shaped. Since the kink is far more distinct using $\|\cdot\|_{\ell_0}$, see figs. 5.3(a) and 5.3(b) or figs. 5.3(d) and 5.3(e), we propose to use the corresponding curve for the determination of the regularization parameter. Next, we propose an iterative method that allows for the efficient determination of the kink in figs. 5.3(b) and 5.3(e).

5.4.2 An iterative algorithm for the determination of the kink

In the following, we propose a simple iterative method to determine the kink of an L-shaped curve. While we recall that such curves occur in parameter choice strategies as discussed above, in the following we do not make assumptions about the way the curve was defined. Instead, for a finite interval $I = [a, b]$ we consider a general curve $c : I \to \mathbb{R}^2$ given by the components $\beta \mapsto (c_1(\beta), c_2(\beta))$. If the curve is actually L-shaped, the task of determining the kink may seem trivial when done

Algorithm 5.1 Determination of the kink for perfect L-curves

Choose initial values β_0^- and β_0^+ such that $\beta_0^- < \beta_0^+$ and set $n = 0$.

(i) Set $\beta_n := \frac{1}{2}(\beta_n^+ + \beta_n^-)$.

(ii) Choose

$$(\beta_{n+1}^-, \beta_{n+1}^+) = \begin{cases} (\beta_n, \beta_n^+), & c_1(\beta_n) = x_0 \\ (\beta_n^-, \beta_n), & c_2(\beta_n) = y_0 \end{cases}. \tag{5.13}$$

(iii) Check if stopping criterion is satisfied, otherwise set $n = n + 1$ and continue with step item (i).

manually, see fig. 5.3. It is, however, not obvious as of how to get a mathematical characterization of this kink as we have elaborated above.

An easy heuristic to determine the kink visually is to divide the curve into two branches based on characteristics such as curvature and direction and let the kink be at the spot where both branches connect. Subsequently, we propose a simple iterative bisection algorithm that incorporates this simple idea. More precisely, we consider curves $c : I \to \mathbb{R}^2$ of the form

$$\beta \mapsto \begin{cases} (x(\beta),\, q(x(\beta)))\,, & \beta \le \beta_0 \\ (p(y(\beta)),\, y(\beta))\,, & \beta > \beta_0 \end{cases} \tag{5.12}$$

for some $\beta_0 \in I$, where x and y are continuous monotonic functions and p, q are polynomials. Such a curve we call *perfect L-curve of degree N*, where N is the degree of the polynomials p and q. Note that we require both branches of the curve to intersect at $\beta = \beta_0$, i.e. we assume continuity of the whole curve. In Figure fig. 5.2, several perfect L-curves of different degree are depicted.

To start with, we consider the case $N = 0$. A perfect L-curve of degree 0 is a curve that consists of two branches, one of which runs horizontally and one of which runs vertically that connect at one point we refer to as the *kink* of the curve, see Figure fig. 5.2. Here, the polynomials in (5.12) are constant, i.e. $p \equiv x_0$ and $q \equiv y_0$ for scalars x_0, y_0. In algorithm 5.1, we propose a simple method to determine this kink iteratively.

We give some explanatory remarks on the mechanics of algorithm 5.1 and present a proof of convergence. Assume $\beta_0^- < \beta_0 < \beta_0^+$. In each iteration step, we consider a tuple of scalars (β_n^-, β_n^+). By induction, we have $c_1(\beta_n^-) = x_0$ and $c_2(\beta_n^+) = y_0$. Moreover, the intermediate point $\beta_n \in (\beta_n^-, \beta_n^+)$ satisfies either $c_1(\beta_n) = x_0$ or $c_2(\beta_n) = y_0$ (or both, which means $\beta_n = \beta_0$ and hence the termination of the iteration). Hence, in each case there are two consecutive points that lie either on a horizontal or a vertical line. In step item (ii) we now choose the tuple corresponding to the points for which this is not the case to proceed with

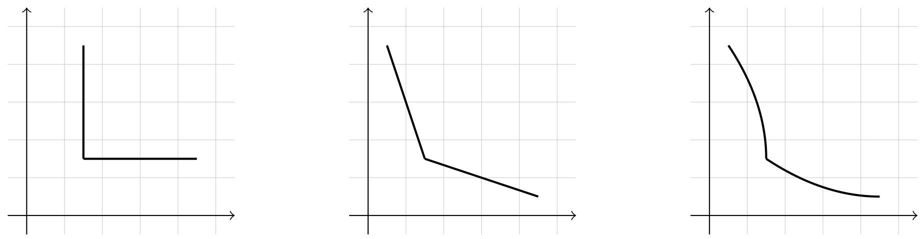


Figure 5.2: Perfect L-curve of order N for $N = 0$ (left), $N = 1$ (center) and $N = 2$ (right).

Algorithm 5.2 Determination of the kink for non-perfect L-curves

Choose initial values β_0^- and β_0^+ such that $\beta_0^- < \beta_0^+$ and set $n = 0$.

(i) Set $\beta_n := \frac{1}{2}(\beta_n^+ + \beta_n^-)$.

(ii) Compute

$$x_n = \tfrac{1}{2}(c_1(\beta_n) + c_1(\beta_n^+)), \quad e_1 = (c_1(\beta_n) - x_n)^2 + (c_1(\beta_n^+) - x_n)^2 \tag{5.14}$$

as well as

$$y_n = \tfrac{1}{2}(c_2(\beta_n) + c_2(\beta_n^-)), \quad e_2 = (c_2(\beta_n) - y_n)^2 + (c_2(\beta_n^-) - y_n)^2 . \tag{5.15}$$

(iii) Choose

$$(\beta_{n+1}^-, \beta_{n+1}^+) = \begin{cases} (\beta_n, \beta_n^+), & e_1 \le e_2 \\ (\beta_n^-, \beta_n), & e_1 > e_2 \end{cases} . \tag{5.16}$$

(iv) Check if stopping criterion is satisfied, otherwise set $n = n + 1$ and continue with step item (i).

in the next iteration step. Note that this procedure ensures that $\beta_n^- \le \beta_0 \le \beta_n^+$ for every $n \in \mathbb{N}$. Moreover, the bisection interval reduces its length by a factor of 2 each step. Collectively we obtain convergence $\beta_n \to \beta_0$.

The method in algorithm 5.1 was designed for a perfect L-curve of degree 0 and is well-defined only under this strong assumption. However, for L-curves arising in applications, this property is most likely not fulfilled. Hence, in algorithm 5.2 we reformulate the method so that it can be applied to arbitrary curves. While we obtain general applicability, we cannot assert convergence to a distinguished point. However, one can observe that the method still provides useful results for curves that are similar to a perfect L-curve.

Regarding algorithm 5.2, since we do not assume that either c_1 or c_2 are constant, we compute the constant fitting term x_n and y_n and compare the least-squares error at the base points $c_1(\beta_n)$, $c_1(\beta_n^+)$ and $c_2(\beta_n)$, $c_2(\beta_n^-)$, respectively. The heuristics follows the idea that the kink is included in the interval corresponding to the larger approximation error. Particularly, for an ideal L-curve we have either $e_1 = 0$ or $e_2 = 0$ and the method reduces to algorithm 5.1.

The approach described in algorithm 5.2 can no longer be shown to converge to a distinguished point. Furthermore, the method provides different values depending on the initial values β_0^- and β_0^+. Even, for an arbitrary curve that is only in some way similar to the perfect L-curve it is not clear as of how to characterize a point we might refer to as a kink. In this setting, however, the method still provides values of β as a limit point that correspond to a point on the curve where one would, visually, locate the kink. While the approach is purely heuristic, it is based on algorithm 5.1 for which we could provide a mathematical proof of convergence.

While easy to understand, the method outlined above provides useful results only for a perfect L-shape of degree 0, i.e. a curve with two branches that are horizontal and vertical, and curves that are visually similar to this. Fortunately, it can be generalized to the setting where the branches are not constant but polynomials of higher order, see (5.12). This is done in algorithm 5.3. We remark that for $N = 0$, this method coincides with algorithm 5.2.

Remark 5.8. The method described in algorithm 5.3 is based on the modeling of both branches of the L-curve using polynomials. While the determination of fitting coefficients is particularly simple in the case, apart from this the usage of polynomials is not mandatory and the approach could be generalized using a different model for the shape of the branches. This way one may incorporate additional knowledge about the typical shape of the branches and in this manner adapt the method to a particular application.

5.5 Numerical evaluation

In the following, we apply standard Tikhonov-type methods for the reconstruction of a solution of the inverse medium problem. Particularly, we evaluate the parameter choice rule as proposed in section 5.4.1. At this, we refer to the operator $K = A \circ S$ as in eq. (2.12), where A is the Born approximation given in eq. (2.11) and S is the synthesis operator corresponding to some chosen wavelet. Moreover, we consider the sparsity-promoting functional $\Psi_\alpha : C^N \to \mathbb{R}$ given by

$$\Psi_\alpha(x) = \frac{1}{2}\|Kx - y^\delta\|_{\mathbb{C}^M}^2 + \alpha\|x\|_1 \,. \tag{5.21}$$

In all numerical applications, the minimization of eq. (5.21) was performed using the well-established method FISTA, see [4].

To start with, we refer to the Haar wavelet basis as well as to synthetic data, see fig. 2.2. The L-curves according to eq. (5.10) and eq. (5.11) are depicted in

Algorithm 5.3 Determination of the kink with polynomials of higher order

Let $N \geq 0$. Choose initial values $\beta_0^1 < \cdots < \beta_0^{N+2}$ such that and set $n = 0$.

(i) Set

$$\gamma_n = (\beta_n^1, \cdots, \beta_n^k, \beta_n, \beta_n^{k+1}, \ldots, \beta_n^{N+2}), \tag{5.17}$$

where $\beta_n = \frac{1}{2}(\beta_n^{k+1} + \beta_n^k)$ and k is chosen such that $\beta_n^{k+1} - \beta_n^k$ is minimal for $k = 1, \ldots, N+1$.

(ii) Compute p_n, the polynomial of degree N fitted for $(c_2(\gamma_n^k),\, c_1(\gamma_n^k))$ for $k = 1, \ldots, N+2$ and q_n, the polynomial of degree N fitted for $(c_1(\gamma_n^k),\, c_2(\gamma_n^k))$ for $k = 2, \ldots, N+3$.

(iii) Compute the approximation error

$$e_1 = \sum_{k=1}^{N+2} (c_1(\gamma_n^k) - p_n(\gamma_n^k))^2 \tag{5.18}$$

and

$$e_2 = \sum_{k=2}^{N+3} (c_2(\gamma_n^k) - q_n(\gamma_n^k))^2 . \tag{5.19}$$

(iv) Choose the branch with worse polynomial fitting

$$(\beta_{n+1}^1, \ldots, \beta_{n+1}^{N+2}) = \begin{cases} (\gamma_n^2, \ldots, \gamma_n^{N+3}), & e_1 \leq e_2 \\ (\gamma_n^1, \ldots, \gamma_n^{N+2}), & e_1 > e_2 \end{cases} . \tag{5.20}$$

(v) Check if stopping criterion is satisfied, otherwise set $n = n+1$ and continue with step item (i).

fig. 5.4. Note that in these plots, to emphasize the kink we used $\frac{1}{2}\|Ax_\alpha^\delta - y^\delta\|^2 + 1$ instead of $\frac{1}{2}\|Ax_\alpha^\delta - y^\delta\|^2$. One observes that the kinks of both L-curves correspond fairly accurately to the regularization parameter α_{opt} of optimal reconstruction error. Furthermore, one can see that the L-curve according to eq. (5.10) visually resembles a perfect L-curve of degree one.

The latter observation motivates the usage of algorithm 5.3 with $N = 1$. In our application, after 10 iterations (Haar wavelet) and 12 iterations (CDF wavelet) the algorithm provided the regularization parameter $\alpha = 2.45 \cdot 10^{-6}$ (Haar wavelet) and $\alpha = 3.16 \cdot 10^{-6}$ (CDF wavelet), where a minimal width of the bisection interval in algorithm 5.3 was used as a stopping criterion. In fig. 5.4, one observes that expectedly, these parameters do correspond to the kink of the L-curve quite accurately.

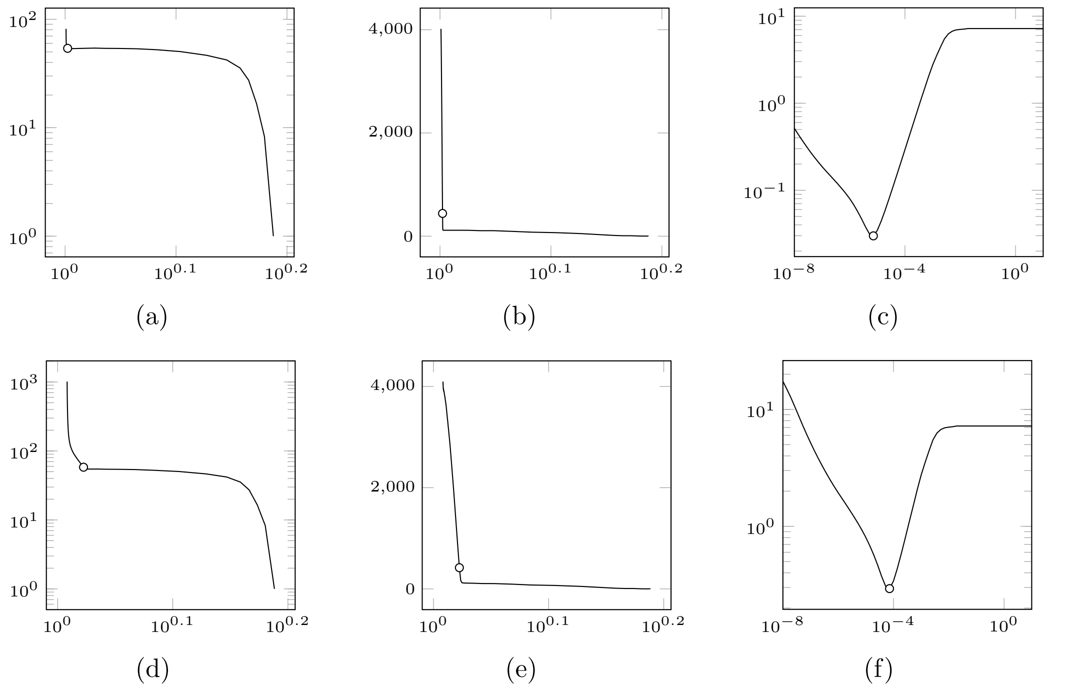


Figure 5.3: The L-curve as in eq. (5.11) (left), the L-curve as in eq. (5.10) (center) and the reconstruction error (right) for noisy data with 1% Gaussian noise (top row) and 10% Gaussian noise (bottom row) according to eq. (5.21). The circle corresponds to the regularization parameter α that minimizes the reconstruction error.

In fig. 5.5 the according reconstructions of the solution $x^\dagger$, see fig. 2.2, are depicted. The results in fig. 5.5 no longer suffer from oscillations in the background as to be observed in fig. 2.3 where the classical Tikhonov-Phillips approach was evaluated.

Next, we apply sparsity-promoting Tikhonov-type approach to real data. We use data from the Fresnel database, the modalities of which we have already described in section 2.4.2. In fig. 5.6, we demonstrated the reconstruction according to the sparsity-promoting functional in eq. (5.21) with the synthesis operator S corresponding to the CDF wavelet basis.

For comparison, in fig. 2.4 the reconstruction according to classical Tikhonov-Phillips regularization was presented. One observes that the reconstructions in fig. 5.6 provides a significantly higher contrast. Recall that the obstacle has an estimated value of $q = 2 \pm 0.3$, i.e. the sparsity-promoting approach gives a better approximation of the sought for solution.

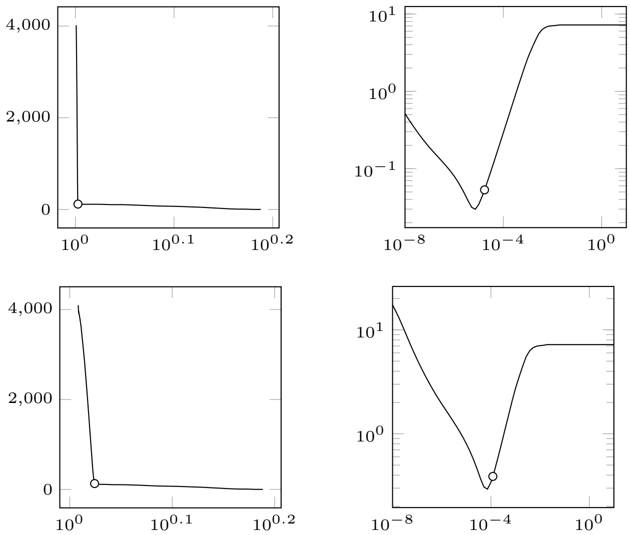


Figure 5.4: The L-curve as in eq. (5.10) (left), and the reconstruction error (right). The circle corresponds to the regularization parameter α determined by algorithm 5.3.

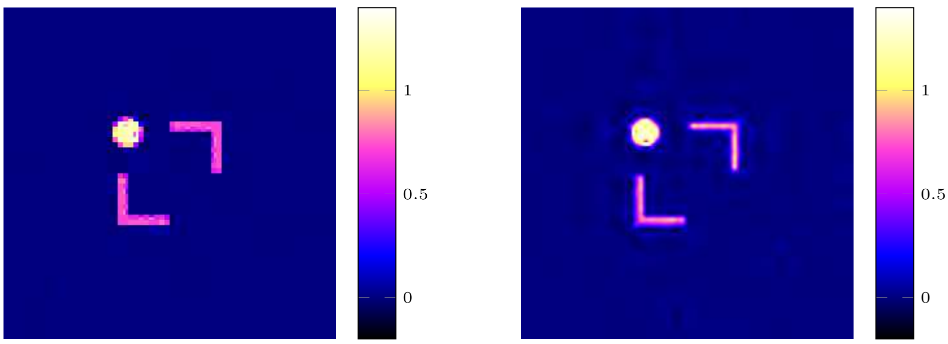


Figure 5.5: The real part of the reconstruction from noisy data with 10% of Gaussian noise according to eq. (5.21) with α chosen according to algorithm 5.3 using the synthesis operator S corresponding to the Haar wavelet basis (left) and the CDF wavelet basis (right).

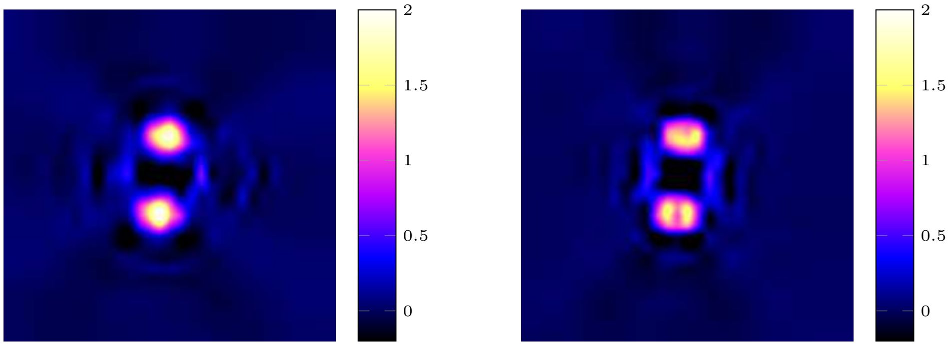


Figure 5.6: The real part of the reconstruction from measurement data according to Tikhonov-type regularization as in eq. (5.21) for $k \approx 83$ (left) and $k \approx 125$ (right).

CHAPTER
SIX

APPROXIMATION OF TIKHONOV-TYPE FUNCTIONALS

6.1 Introduction

The general concept of Tikhonov-type regularization has been introduced in chapter 5 for the reconstruction of a solution of an operator equation as in eq. (1.1). Here, from a numerical point of view, the most challenging task is to minimize a Tikhonov-type functional, see eq. (5.2). Whenever this minimization is difficult, it is tempting to replace the objective functional by a surrogate mapping which is easier to minimize and which yet provides a suitable approximation of the reconstruction. In this section, to elaborate this strategy we introduce the notion of *sensitivity* and establish an according stability result.

Starting with the so-called Tikhonov-Phillips regularization scheme, see section 1.3.2, in chapter 5 we presented a generalized approach. For a fidelity term F, a regularization term R and some scalar $\alpha > 0$ a functional of the form

$$\Psi_\alpha(x) := F(x, y^\delta) + \alpha R(x) \tag{6.1}$$

was proposed. The according reconstruction could then be determined as an minimizer of eq. (6.1) which we denote by

$$x_\alpha^\delta \in \arg\min_{x \in X} \Psi_\alpha(x)\,. \tag{6.2}$$

This approach substantiates the importance of the efficient minimization of functionals of this particular form. If Ψ_α is smooth, gradient-based methods can be applied for numerical minimization (cf. [16, 19, 20, 40]). However, there are examples for which the regularization term R is not smooth. Yet, it is common knowledge that sometimes the term R can be replaced by a similar but much more well-behaved functional in such way that the minimizers of the original and resulting versions of Ψ_α are virtually indistinguishable. In this chapter we elaborate as of how the aforementioned common knowledge is justified.

A prominent example for a non-smooth functional R is given by sparsity enforcing regularization term, see eq. (5.6). Since it is non-smooth for $p < 2$, the according minimization problem cannot be solved by standard means. Note that even in a finite dimensional setting, it is non-smooth for arbitrary $p \leq 1$. Starting with [15], several direct methods to solve this particular problem have been developed, see for instance [4, 6, 35, 22, 25, 59] for $p = 1$ and [8, 12, 44] for the non-convex situation $p < 1$. However, minimization still remains a challenging task, particularly when considering large scale problems or small values of the regularization parameter. Furthermore, one common deficiency is that the according minimization methods are usually highly specialized and fail to incorporate even minor amendments to the objective functional.

A promising alternative way to approach this type of problem is to replace the non-smooth regularization term by a smoothed version. This results in a minimization problem is tractable by means of smooth optimization methods. This was suggested in [1, 18, 57, 62, 41]. This strategy has several potential benefits. Firstly, for a functional the direct minimization of which is numerically challenging, the described approach yields a promising alternative. One example of such a functional is given by the Tikhonov-type functional incorporating the non-convex regularization term R in eq. (5.6) with $0 < p < 1$. Secondly, even if there are methods for the minimization of the functional, the introduction of a smoothness parameter provides an additional possibility to compromise between proximity and minimization speed as it is common belief that smoother problems can be solved faster. Finally, gradient-based methods usually do not make strong assumptions on the structure of the functional and are hence suitable for the setting of Tikhonov-type regularization even with multiple regularization terms or with the replacement of R by $R(T\cdot)$ for some analysis operator T in X. It is hence fair to say that the minimization of smooth functionals is more flexible.

In our investigations, in the following we consider an approximation R_n of R as well as the functional

$$\Psi_{\alpha,n}(x) := F(x, y^\delta) + \alpha R_n(x) \tag{6.3}$$

and the choice of an according minimizer

$$x^\delta_{\alpha,n} \in \operatorname*{arg\,min}_{x \in X} \Psi_{\alpha,n}(x)\,, \tag{6.4}$$

where proximity of the approximation is controlled by the index $n \in \mathbb{N}$. A natural question is as of how the minimizers of the surrogate functional $\Psi_{\alpha,n}$ relate to those of Ψ_α. It seems desirable to establish stability of the minimizer with respect to n, i.e. convergence of the minimizers of $\Psi_{\alpha,n}$ to a minimizer of Ψ_α as $n \to \infty$. The notion of Γ-convergence yields a powerful framework to deal with this type of problem (cf. [7]). However, considering the importance of Tikhonov-type functionals in regularization theory, it seems desirable to adopt the according techniques to this setting. Subsequently, we present a comprehensive theory of this matter where we particularly consider the setting of separable regularization terms as discussed in section 5.3. The corresponding results have first been published in [54]. Furthermore, we present a result providing convergence rates with respect to an approximation parameter, where we continue an approach first presented in [34] and present a generalized result. Based on the inverse medium problem we finally perform numerical experiments that show both feasibility and potential of the approach.

6.2 The notion of sensitivity

Let X be a topological space and let τ be the topology on X. Throughout this section, topological assertions refer to the particular choice τ. In applications, the topology τ is typically given by the weak topology on a reflexive Banach space X. Hence, the latter setting will receive particular attention in section 6.2.2.

Consider functionals $F : X \times Y \to (-\infty, \infty]$ and $R : X \to (-\infty, \infty]$ as well as a sequence of functionals $\{R_n : X \to (-\infty, \infty]\}_{n\in\mathbb{N}}$ defined on X. We obtain the Tikhonov-type functionals Ψ_α and $\Psi_{\alpha,n}$ introduced in eq. (6.1) and eq. (6.3), respectively. Our main focus lies on the regularization term and its approximating functionals. Particularly, we are interested in functionals R and R_n that ensure a certain stability property of the minimizer of the related functionals Ψ_α and $\Psi_{\alpha,n}$ as given in eq. (6.1) and eq. (6.3). For later use we introduce the following notion.

Definition 6.1. Let $\{\Psi_{\alpha,n}\}_{n\in\mathbb{N}}$ be a sequence of functionals on X. We say $\{\Psi_{\alpha,n}\}_{n\in\mathbb{N}}$ is *sensitive* with respect to Ψ_α if and only if for every sequence $x^n \in \arg\min_{x\in X} \Psi_{\alpha,n}(x)$ there is a convergent subsequence of $\{x^n\}_{n\in\mathbb{N}}$, and every convergent subsequence of $\{x^n\}_{n\in\mathbb{N}}$ converges to a minimizer of Ψ_α.

Remark 6.2. Note that in definition 6.1, if the minimizer $\tilde{x}$ of Ψ_α is unique, then we even have convergence $x^n \to \tilde{x}$ of the whole series as $n \to \infty$.

In the following, we want to determine conditions on the regularization term R and R_n that ensure sensitivity of $\Psi_{\alpha,n}$ under general assumptions on the fidelity term F. These conditions are collected in assumption 6.3.

Assumption 6.3. Assume that for the functional $F : X\times Y \to [0, \infty]$ and $y^\delta \in Y$ it holds that

(i) $F(0, y^\delta) < \infty$.

(ii) F is non-negative.

(iii) F is lower semicontinuous with respect to the first component.

Here, we impose only very general assumptions. Considering typical examples, these conditions can be regarded as not restrictive. They are met, for instance, by the choice $\|A(x)-y^\delta\|_Y$ for a weakly continuous and potentially nonlinear operator A mapping between Banach spaces X and Y if we let τ be the weak topology in X. A further example that obeys assumption 6.3 is given by the *Kullback-Leibler divergence*, c.f. [17].

Next, we formulate assumptions on the regularization term and its approximations. For arbitrary $M > 0$ and $n \in \mathbb{N}$ we denote by $E_{M,n} := \{x \in X : R_n(x) \leq M\}$ the levelsets of R_n. By this means, we formulate the following assumptions.

Assumption 6.4. Assume that for $R, R_n : X \to (-\infty, \infty]$ it holds that

(i) $R(0) = R_n(0) = 0$ for all $n \in \mathbb{N}$.

(ii) R_n converges point-wise to R as $n \to \infty$.

(iii) $R(\tilde{x}) \leq \liminf_{n\to\infty} R_n(x^n)$ for $\tilde{x} \in X$ and any convergent sequence $x^n \to \tilde{x}$.

(iv) The set $E_M := \bigcup_{n\in\mathbb{N}} E_{M,n}$ is sequentially precompact for any $M > 0$, i.e. every sequence in E_M has a convergent subsequence.

Note that assumption 6.3 and assumption 6.4 depend on the choice of the topology τ.

Remark 6.5. We remark that a family of mappings R_n satisfying assumption 6.4 item (ii) and item (iii) is easily seen to Γ-converge to R. Particularly, in the setting of Γ-convergence the former condition is generalized to assuming that for each $\tilde{x} \in X$, there exists a convergent sequence $x^n \to \tilde{x}$ such that $\limsup_{n\to\infty} R_n(x^n) \leq R(\tilde{x})$. See [7] for general reference on Γ-convergence. In this paper, however, we require slightly more. This is due to the fact that we want to ensure Γ-convergence of $\{\Psi_{\alpha,n}\}_{n\in\mathbb{N}}$ to Ψ_α without imposing additional assumptions on the fidelity term F. To keep the focus of our investigations on the regularization term, it seems justified to abandon greater generality.

Remark 6.6. From assumptions 6.3 and 6.4 it immediately follows that R_n is coercive and, consequently, that $\Psi_{\alpha,n}$ has at least one minimizer $x_{\alpha,n} \in X$ for all $n \in \mathbb{N}$, see proposition 5.1.

Under the assumptions on the fidelity term F and on the regularization term R with its approximation R_n introduced in assumption 6.3 and assumption 6.4, using definition 6.1 we are now able to formulate a basic sensitivity result. We remark that to establish this result, it is not necessary to assume convexity of the functional Ψ_α or its approximations.

Theorem 6.7 (Sensitivity)**.** *Let F be as in assumption 6.3 and let R and R_n be such that assumption 6.4 is satisfied. Then, the family $\{\Psi_{\alpha,n}\}_{n\in\mathbb{N}}$ is sensitive with respect to Ψ_α.*

Proof. By evaluating $\Psi_{\alpha,n}$ at 0 and the minimizing property of x^n we immediately obtain $\alpha R_n(x^n) \leq F(0, y^\delta)$ which yields $x^n \in E_{\alpha^{-1}F(0,y^\delta)}$. Precompactness now yields the existence of a subsequence also denoted by $\{x^n\}_{n\in\mathbb{N}}$ and some $\tilde{x} \in X$ such that $x^n \to \tilde{x}$ as $n \to \infty$.

We have to show that $\tilde{x}$ is indeed a minimizer of Ψ_α. Let $n \in \mathbb{N}$. For arbitrary $x \in X$, from assumption 6.4 item (iii) together with lower semicontinuity of F we conclude

$$\Psi_\alpha(\tilde{x}) \leq \liminf_{n\to\infty} \Psi_{\alpha,n}(x^n) \leq \liminf_{n\to\infty} \Psi_{\alpha,n}(x) = \Psi_\alpha(x)\,,$$

which gives the assertion. □

Lemma 6.8. *For any sequence of minimizers $x^n \in \arg\min_{x\in X} \Psi_{\alpha,n}(x)$ such that $x^n \to \tilde{x}$ it holds $R_n(x^n) \to R(\tilde{x})$ as $n \to \infty$.*

Proof. Since the fidelity term F is lower semicontinuous with respect to the first component, we get the estimate

$$\begin{aligned}
\limsup_{n\to\infty} \alpha R_n(x^n) &\leq \limsup_{n\to\infty} \Psi_{\alpha,n}(x^n) - \liminf_{n\to\infty} F(x^n, y^\delta) \\
&\leq \limsup_{n\to\infty} \Psi_{\alpha,n}(\tilde{x}) - F(\tilde{x}, y^\delta) \\
&= \limsup_{n\to\infty} \alpha R_n(\tilde{x}) \\
&= \alpha R(\tilde{x})\,,
\end{aligned}$$

which together with the estimate in assumption 6.4 item (iii) yields the assertion. □

Remark 6.9. In the setting where R_n is monotonic with respect to n, we can consider the relaxed conditions

(iii′) R_n is lower semicontinuous for any $n \in \mathbb{N}$

(iv′) $E_{M,n}$ is sequentially precompact for any $n \in \mathbb{N}$ and $M > 0$

and show that under the assumption of monotonicity, the conditions item (iii′) and item (iv′) already imply item (iii) and item (iv), respectively. In the following, we consider the non-decreasing situation. The non-increasing case can be dealt with similarly and is left to the reader.

For $m \in \mathbb{N}$ and an arbitrary convergent sequence $x^n \to \tilde{x}$ by lower semicontinuity we estimate

$$R_m(\tilde{x}) \leq \liminf_{n\to\infty} R_m(x^n) \leq \liminf_{n\to\infty} R_n(x^n)\,,$$

where in the latter step, we used monotonicity of R_n with respect to n. Since this holds true for arbitrary $m \in \mathbb{N}$, we even conclude $R(\tilde{x}) \leq \liminf_{n\to\infty} R_n(x^n)$ and, hence, we get assumption 6.4 item (iii).

One can easily see that precompactness of E_M yields precompactness of the levelsets $E_{M,n}$. In the setting where R_n is monotonic with respect to n, the converse holds true as well. We observe that for arbitrary $M > 0$ and $n \in \mathbb{N}$ it holds $E_{M,n} \subseteq E_{M,1}$ and, hence, $E_M \subseteq E_{M,1}$. Since the latter set is sequentially precompact, so is E_M and thus, assumption 6.4 item (iv) is fulfilled.

Remark 6.10. Let some family of functionals $P_\beta : X \to (-\infty, \infty]$ satisfy assumption 6.4 items (i) to (iii) but not item (iv). A remedy can be the introduction of a second family $Q_\beta : X \to (-\infty, \infty]$ that satisfies assumption 6.4. The weighted sum $R_\beta = P_\beta + \gamma Q_\beta$ can now easily be shown to obey assumption 6.4 for an arbitrary constant $\gamma > 0$. For instance, any constant family $Q_\beta = Q$ with Q coercive matches this requirement.

Example 6.11 (The ℓ_0 penalty)**.** Consider $X = \ell_2$ and the regularization term given by

$$\|x\|_{\ell_0} = \#\{i \in \mathbb{N} : x_i \neq 0\}\,,$$

where by # we measure the cardinality of the index set. This functional is known to lead to a NP-hard minimization problem eq. (6.1) for standard choices of the fidelity term. However, it can be approximated point-wise by $\|\cdot\|_{\ell_{\beta_n}}^{\beta_n}$ for some null sequence $\{\beta_n\}_{n\in\mathbb{N}}$. This family of approximations obeys assumption 6.4 items (i) to (iii) but not item (iv). We leave the simple proof of these assertions to the reader. Consequently, instead of $\|x\|_{\ell_0}$ for $\gamma > 0$ we consider the functional $R := \|\cdot\|_{\ell_0} + \frac{\gamma}{2}\|\cdot\|_{\ell_2}^2$ which has been thoroughly investigated in [58] regarding its regularizing properties. By remark 6.10 it is now easy to see that the family of approximations

$$R_n := \|\cdot\|_{\ell_{\beta_n}}^{\beta_n} + \frac{\gamma}{2}\|\cdot\|_{\ell_2}^2$$

satisfies assumption 6.4.

Example 6.12 (Constrained minimization)**.** If for a proper fidelity term $\Psi : X \mapsto \overline{\mathbb{R}}$ we seek the solution $\arg\min_{x\in U} \Psi(x)$ for some closed subset $U \subset X$, one can rephrase this optimization problem by introducing an additional term P given by

$$P(x) = \begin{cases} 0\,, & x \in U \\ \infty\,, & x \notin U \end{cases}$$

and consider the problem $\arg\min_{x\in X} \Psi(x) + P(x)$ instead. This problem is still difficult to minimize numerically. Hence, we propose to approximate P by a family of functionals P_n given by

$$P_n(x) = \begin{cases} 0\,, & x \in U \\ \frac{1}{\beta_n}\operatorname{dist}(x,U)^2\,, & x \notin U \end{cases}\,,$$

where $\operatorname{dist}(x,U)$ is the distance between x and U and $\{\beta_n\}_{n\in\mathbb{N}}$ denotes some null sequence. Considering a translation of the set U we may say that without loss of generality, the functional in eq. (6.5) satisfies assumption 6.4 item (i). Further, it obeys assumption 6.4 item (ii) as well as item (iii′) but not item (iv′). From remark 6.10 it follows that it is sufficient to complement P and P_n by functionals Q and Q_n, respectively. We thereby conclude that $P_n + Q_n$ fulfills assumption 6.4 given that this holds true for Q_n.

Note that for $X = \ell_2$ and a cuboid $U = \{x \in X \,:\, a_k \le x_k \le b_k\}$ with sequences $\{a_k\}_{k\in\mathbb{N}}$, $\{b_k\}_{k\in\mathbb{N}}$ in $[-\infty,\infty]$ the functional P can be approximated by

$$P_n(x) = \sum_{k\in\mathbb{N}} p_n^{a_k,b_k}(x_k)\,,$$

where the addends are given by

$$p_n^{a_k,b_k}(s) = \begin{cases} \frac{1}{\beta_n}(a_k - s)^2\,, & s < a_k \\ 0\,, & a_k \le s \le b_k \\ \frac{1}{\beta_n}(s-b_k)^2\,, & s > b_k \end{cases}\,. \tag{6.5}$$

Particularly, this explicit form allows for simple gradient-based minimization of the resulting objective functional $\Psi + P_n + Q_n$.

6.2.1 Choice of the regularization parameter

The regularization parameter α which balances the regularization term against the fidelity term has to be chosen carefully in order to get convenient reconstructions of the solution of the regularized operator equation. So far, it has been considered to be constant. Subsequently, we allow sequences $\{\alpha_n\}_{n\in\mathbb{N}}$ and investigate the corresponding sensitivity properties.

Remark 6.13. If the regularization parameter is approximated by a sequence $\{\alpha_n\}_{n\in\mathbb{N}}$, the corresponding functionals $\Psi_{\alpha_n,n}$ can be rewritten as $\hat{\Psi}_{1,n}$ using reweighed versions

$$\hat{R}_n := \alpha_n R_n \tag{6.6}$$

in the corresponding definitions in eq. (6.3). Hence, a regularization parameter that depends on the approximation index can easily be incorporated into the framework developed in section 6.2 and we can apply the result in theorem 6.7 to functionals of the form $\Psi_{\alpha_n,n}$.

One important example is given for the constant case $R_n \equiv R$. The according stability result is known for some specific regularization terms (cf. [31, Theorem 2.1]). Using the results developed in section 6.2 we can elegantly generalize these results. Additionally, we establish sensitivity in the situation when both the regularization parameter and the approximation index vary. To begin with, we show a simple sensitivity result that extends the result from theorem 6.7 to the setting outlined above.

Proposition 6.14. *Let F be as in assumption 6.3 and let R and R_n be such that assumption 6.4 is satisfied. Moreover, let $\{\alpha_n\}_{n\in\mathbb{N}}$ a sequence converging to $\alpha > 0$. Then, the family $\{\Psi_{\alpha_n,n}\}_{n\in\mathbb{N}}$ is sensitive with respect to Ψ_α.*

Proof. Obviously, the functionals R and $\alpha_n R_n$ satisfy assumption 6.4. By theorem 6.7 we get the assertion. □

Example 6.15 (Stability with respect to the regularization parameter)**.** Let R be a semicontinuous functional such that the levelsets are sequentially precompact with $R(0) = 0$. Considering the constant choice $R_n \equiv R$, by proposition 6.14 we immediately establish stability of the minimizer of the functional Ψ_α with respect to the regularization parameter α if $\alpha > 0$. We remark for readers well-accustomed with the notion of Γ-convergence the sensitivity is immediately clear.

In proposition 6.14, we have shown sensitivity of the minimizer for arbitrary convergent sequences α_n. However, the regularization parameter α often is determined adaptively by a parameter choice rule depending on the functional R_n and, hence, on n. A natural question that arises in this context is whether we can establish sensitivity for the resulting approximation.

Subsequently, we consider the case where the regularization parameter is not chosen explicitly, but implicitly according to the discrepancy principle. We say α is chosen according to the discrepancy principle with respect to R if and only if for the minimizer $z \in \arg\min_{x\in X} F(x, y^\delta) + \alpha R(x)$ it holds

$$F(z, y^\delta) = \delta^2\,, \tag{6.7}$$

where $\delta > 0$ is the noise level in the measurement data. Here, we assume existence of a solution of eq. (6.7). For further insight into this matter, we refer the reader to [32]. The following theorem provides a stability result with respect to this particular choice of the parameter.

Theorem 6.16. *Let F be as in assumption 6.3 and let R and R_n be such that assumption 6.4 is satisfied. Assume $R(x) = 0$ if and only if $x = 0$. Moreover, let the noise level $\delta > 0$ be such that $F(z, y^\delta) < \delta^2 < F(0, y^\delta)$ for some $z \in X$ and let α_n be chosen according to the discrepancy principle with respect to R_n for every $n \in \mathbb{N}$. Then, there is a subsequence $\{\alpha_{n_k}\}_{k\in\mathbb{N}}$ of $\{\alpha_n\}_{n\in\mathbb{N}}$ converging to*

$\alpha > 0$, where α is chosen according to the discrepancy principle with respect to R. Moreover, the family $\{\Psi_{\alpha_{n_k},n_k}\}_{k\in\mathbb{N}}$ is sensitive with respect to Ψ_α.

Proof. We show that $c < \alpha_n < C$ for positive constants c and C using an indirect argument. Assume that there is a subsequence such that $\alpha_n \to \infty$. By $\alpha_n R_n(x^n) \leq F(0, y^\delta)$ we conclude $R_n(x^n) \to 0$. Particularly, $\{R_n(x^n)\}_{n\in\mathbb{N}}$ is bounded and hence, by assumption 6.4 we find a convergent subsequence of $\{x^n\}_{n\in\mathbb{N}}$ with limit point $\tilde{x} \in X$. We observe $R(\tilde{x}) \leq \liminf_{n\to\infty} R_n(x^n) = 0$ from which we conclude $\tilde{x} = 0$. We get for arbitrary $m \in \mathbb{N}$

$$\delta^2 = \liminf_{n\to\infty} F(x^n, y^\delta) \geq F(0, y^\delta) \geq F(x^m, y^\delta) + \alpha_m R_m(x^m)\,,$$

which yields $\alpha_m R_m(x^m) \leq 0$ and, consequently, $x^m = 0$. This finally implies $F(0, y^\delta) = \delta^2$, which contradicts the assumption. Hence, α_n is bounded from above.

We now assume that there is a subsequence such that $\alpha_n \to 0$. For arbitrary $x \in X$ we obtain the estimate

$$\begin{aligned} F(z, y^\delta) < \delta^2 &\leq \liminf_{n\to\infty} F(x^n, y^\delta) + \alpha_n R_n(x^n) \\ &\leq \liminf_{n\to\infty} F(x, y^\delta) + \alpha_n R_n(x) = F(x, y^\delta)\,, \end{aligned}$$

which again yields a contradiction. Hence, α_n is also bounded from below.

Collectively, boundedness of $\{\alpha_n\}_{n\in\mathbb{N}}$ yields the existence of a subsequence converging to some $\alpha > 0$. By proposition 6.14 we obtain a subsequence of $\{x^n\}_{n\in\mathbb{N}}$ converging to a minimizer $\tilde{x}$ of Ψ_α. Finally, by analogy with lemma 6.8 we obtain the identity $F(\tilde{x}, y^\delta) = \delta^2$. □

Remark 6.17. In theorem 6.16 we observe that every subsequence of $\{\alpha_n\}_{n\in\mathbb{N}}$ converges to some α, where α is chosen according to the discrepancy principle. Obviously, if the choice of the regularization parameter according to the discrepancy principle is unique, then the whole sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ converges.

6.2.2 Setting in Banach spaces

In a reflexive Banach space X, to show assumption 6.4 it may be helpful to choose τ as the weak topology in X. Recall that particularly, the assumption that the fidelity term F is lower semicontinuous depends on the choice of τ. However, in this setting theorem 6.7 provides sensitivity with respect to the weak topology only. One might still be interested in convergence of the minimizers with respect to the norm in X. To establish this result, we introduce the following notion.

Definition 6.18. Let X be a Banach space. A sequence of functionals $\{R_n\}_{n\in\mathbb{N}}$ on X is said to satisfy the *sequential Kadec-Klee property* if and only if there is

R such that $R_n \to R$ point-wise and for each weakly convergent sequence $x^n \rightharpoonup \tilde{x}$ convergence $R_n(x^n) \to R(\tilde{x})$ implies $x^n \to \tilde{x}$ in X.

We remark that we did not encounter the above definition previously in literature. We introduce the sequential Kadec-Klee property since it will turn out to be very useful in our subsequent analysis. It is closely related to the usual notion of the Kadec-Klee property of a functional R defined on X, see definition 3.14. In definition 6.18, we generalized this property, see example 6.19.

Example 6.19 (Constant sequence)**.** Let R fulfill the Kadec-Klee property. It is obvious that the constant sequence given by $R_n = R$ obeys the sequential Kadec-Klee property.

Example 6.20 (Regularization parameter)**.** Let $\hat{R}$ fulfill the Kadec-Klee property. Then, for any sequence $\{\alpha_n\}_{n\in\mathbb{N}}$ converging to some $\alpha > 0$ the sequence of functionals $\{R_n\}_{n\in\mathbb{N}}$ given by $R_n = \alpha_n \hat{R}$ obeys the sequential Kadec-Klee property.

One may ask whether there are non-trivial families of functionals that obey the sequential Kadec-Klee property. It will turn out that in particular, this property can be established in the setting of separable penalization under mild conditions, see theorem 6.26. In a first step, the subsequent lemma introduces a wide class of approximations of R which satisfy the sequential Kadec-Klee property.

Lemma 6.21. *Let R be a weakly lower semicontinuous functional satisfying the Kadec-Klee property. Moreover, assume that one of the following conditions hold:*

(i) R_n converges uniformly to R on bounded sets as $n \to \infty$.

(ii) R_n converges point-wise to R and is non-decreasing with respect to n.

Then $\{R_n\}_{n\in\mathbb{N}}$ satisfies the sequential Kadec-Klee property.

Proof. Assume $x^n \rightharpoonup \tilde{x}$ such that $R_n(x^n) \to R(\tilde{x})$. In a first step, we assume item (i) and show that it holds $\limsup_{n\to\infty} R(x^n) \le R(\tilde{x})$. Since $\{x^n\}_{n\in\mathbb{N}}$ is weakly convergent it is bounded. Hence, for every $\varepsilon > 0$ exists a $n_0 \in \mathbb{N}$ such that $|R_n(x^k) - R(x^k)| \le \varepsilon$ for all $n \ge n_0$ and $k \in \mathbb{N}$. Particularly, we have $R(x^n) \le R_n(x^n) + \varepsilon$ for $n \ge n_0$ from which we obtain $\limsup_{n\to\infty} R(x^n) \le R(\tilde{x}) + \varepsilon$. Since ε was arbitrarily small, we can even conclude $\limsup_{n\to\infty} R(x^n) \le R(\tilde{x})$. It is easily seen that the latter assertion also holds if we assume item (ii) instead. Together with lower semicontinuity of R we get convergence $R(x^n) \to R(\tilde{x})$ as $n \to \infty$ which by the Kadec-Klee property of R proves the claim. □

Example 6.22 (Elastic net regularization)**.** Consider $X = \ell_2$ and the regularization term $R := \alpha\|\cdot\|_{\ell_1}$ as well as the sequence of approximating functionals R_n

given by $R_n := \alpha\|\cdot\|_{\ell_1} + \frac{\beta_n}{2}\|\cdot\|_{\ell_2}^2$ for some null sequence $\{\beta_n\}_{n\in\mathbb{N}}$. This is known as the elastic net regularization regularization term, which has been analyzed in [31]. The sequence R_n converges uniformly to R on bounded sets as $n \to \infty$ and, moreover, R obeys the Kadec-Klee property. Hence, by lemma 6.21 we conclude that $\{R_n\}_{n\in\mathbb{N}}$ obeys the sequential Kadec-Klee property.

As a direct consequence of theorem 6.7, in the setting where $\{R_n\}_{n\in\mathbb{N}}$ satisfies the sequential Kadec-Klee property we can easily establish sensitivity even in terms of the strong topology.

Corollary 6.23 (Sensitivity in Banach spaces)**.** *Let F be as in assumption 6.3 and let R and R_n be such that assumption 6.4 is satisfied with respect to the weak topology. Moreover, let $\{R_n\}_{n\in\mathbb{N}}$ satisfy the sequential Kadec-Klee property. Then, the family $\{\Psi_{\alpha,n}\}_{n\in\mathbb{N}}$ is sensitive with respect to Ψ_α in the norm topology in X.*

Remark 6.24. In corollary 6.23, we assumed that the set E_M as in assumption 6.4 is weakly sequentially precompact for every $M > 0$. Since in this setting X is a reflexive Banach space, is sufficient to require the weakened assumption that E_M is bounded in X instead.

6.2.3 Sensitivity for separable regularization terms

So far we did not make any assumptions on the structure of R and R_n. In this section, we exploit the special structure of the regularization term in eq. (6.1) arising in sparsity regularization to replace assumption 6.4 by a set of assumptions on an univariate function. In section 6.2.3, we show that the assumptions on the univariate function can be shown resorting to elementary calculus methods.

In the following, we consider the case $X = \ell_2$ and a separable regularization term R, i.e. a functional $\ell_2 \to [0,\infty]$ of the form

$$R(x) = \sum_{k=1}^{\infty} \varphi(x_k) \tag{6.8}$$

for some univariate function φ, where we set $R(x) = \infty$ if and only if for $x \in \ell_2$ the series in eq. (6.8) does not converge. The functional eq. (6.8) gives rise to a regularization scheme under general assumptions on φ, see [10, 23]. An approximation of R can now be found by approximating φ by some family of univariate functions $\{\varphi_\beta\}_{\beta>0}$ and the functional $\ell_2 \to (-\infty,\infty]$ given by

$$R_\beta(x) = \sum_{k=1}^{\infty} \varphi_\beta(x_k)\,, \tag{6.9}$$

where the parameter β controls the tradeoff between smoothness and proximity. Again, we let R_β equal ∞ if and only if the series does not converge. Note that this choice can easily be translated into the integer-indexed setting of eq. (6.3) by setting $R_n = R_{\beta_n}$ for some null sequence $\{\beta_n\}_{n\in\mathbb{N}}$. We now want to apply the result in theorem 6.7 to the setting of a separable regularization term described above. To this end, we choose τ the weak topology on ℓ_2 and impose the following conditions on φ and the approximating functions φ_β.

Assumption 6.25. Assume that it holds

(i) φ and φ_β are non-negative and it holds $\varphi(0) = \varphi_\beta(0) = 0$ for all $\beta > 0$.

(ii) φ is lower semicontinuous.

(iii) φ_β converges uniformly to φ as $\beta \to 0$ on compact sets.

(iv) There is a coercive function ρ_* such that $\rho_* \leq \varphi_\beta$ for every $\beta > 0$ and for every $M > 0$ there is a $C > 0$ such that $\rho_*(s) \geq C|s|^2$ for $s \in (-M, M)$.

(v) There is a function ρ^* such that $\varphi_\beta \leq \rho^*$ for every $\beta > 0$ and for every $M > 0$ there is a $C > 0$ such that $\rho^*(s) \leq C\varphi(s)$ for $s \in (-M, M)$.

While in assumption 6.4 we imposed conditions on functionals defined on a general topological space X, here we state conditions concerning scalar functions mapping between the real line. Note that since the estimate $\rho_*(s) \geq C|s|^2$ holds true only locally, coercivity of ρ_* is in fact an additional assumption.

We now present a sensitivity result for the setting of separable regularization terms. At this, we show that under assumption 6.25, the functionals in eq. (6.8) and eq. (6.9) satisfy assumption 6.4. We remark that by theorem 6.7, this yields sensitivity with respect to the weak topology. In the proof, however, using the sequential Kadec-Klee property we establish even strong convergence of the minimizers.

Theorem 6.26 (Sensitivity of separable approximation)**.** *Let F be as in assumption 6.3 and let φ and φ_β be functions such that assumption 6.25 is satisfied. Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a null sequence. Then, $\{R_{\beta_n}\}_{n\in\mathbb{N}}$ satisfies the sequential Kadec-Klee property and the family $\{\Psi_{\alpha,\beta_n}\}_{n\in\mathbb{N}}$ is sensitive with respect to Ψ_α, where R and R_n as in eq.* (6.8) *and eq.* (6.9)*, respectively.*

Proof. In a first step, we prove that $\{R_{\beta_n}\}_{n\in\mathbb{N}}$ satisfies the sequential Kadec-Klee property for every null sequence $\{\beta_n\}_{n\in\mathbb{N}}$, where and R_β as in eq. (6.9). Subsequently, we show that assumption 6.4 is fulfilled with respect to the weak topology from which we conclude the second assertion.

Let $\{\beta_n\}_{n\in\mathbb{N}}$ be a sequence converging to 0 as $n \to \infty$ and let $x^n \rightharpoonup \tilde{x}$ be a weakly convergent sequence in ℓ_2 such that $R_{\beta_n}(x^n) \to R(\tilde{x})$. Let $\varepsilon > 0$ and choose $k_0 \in \mathbb{N}$ such that $\sum_{k=k_0}^{\infty} \varphi(\tilde{x}_k) \leq \varepsilon$. Moreover, choose $n_0 \in \mathbb{N}$ such that for any $n \geq n_0$ it holds $\sum_{k=k_0}^{\infty} \varphi_{\beta_n}(x_k^n) \leq 2\varepsilon$. This choice is possible since obviously, $R_{\beta_n}(x^n) \to R(\tilde{x})$ particularly implies convergence

$$\sum_{k=k_0}^{\infty} \varphi_{\beta_n}(x_k^n) \to \sum_{k=k_0}^{\infty} \varphi(\tilde{x}_k) .$$

Since $\{x^n\}_{n\in\mathbb{N}}$ is bounded, the coefficients x_k^n and $\tilde{x}_k$ are uniformly bounded by some scalar $M > 0$. We find $C > 0$ such that $|s|^2 \leq C\rho_*(s)$ for $|s| \leq M$. Thus, we conclude

$$\begin{aligned}\sum_{k=k_0}^{\infty} |x_k^n - \tilde{x}_k|^2 &\leq 2 \sum_{k=k_0}^{\infty} |x_k^n|^2 + 2 \sum_{k=k_0}^{\infty} |\tilde{x}_k|^2 \\ &\leq 2C \sum_{k=k_0}^{\infty} \rho_*(x_k^n) + 2C \sum_{k=k_0}^{\infty} \rho_*(\tilde{x}_k) \\ &\leq 2C \sum_{k=k_0}^{\infty} \varphi_{\beta_n}(x_k^n) + 2C \sum_{k=k_0}^{\infty} \varphi(\tilde{x}_k) .\end{aligned}$$

The latter expression can be estimated from above by $6C\varepsilon$. Weak convergence yields convergence $x^n \to \tilde{x}$ in ℓ_2 from which we obtain the first assertion.

Next, to apply corollary 6.23, we show that assumption 6.4 is fulfilled with respect to the weak topology. To this end, let $\{\beta_n\}_{n\in\mathbb{N}}$ be a null sequence and set $R_n = R_n$. It is easy to see that the first condition is satisfied, see assumption 6.25 item (i). Moreover, by the dominated convergence theorem, see [48, Theorem 11.32], we get point-wise convergence $R_n \to R$ for every $x \in \ell_2$ such that $R(x) < \infty$, i.e. assumption 6.4 item (ii) is fulfilled. If $R(x) = \infty$, the assertion follows from Fatou's lemma. Regarding assumption 6.4 item (iii), consider a weakly convergent sequence $x^n \rightharpoonup \tilde{x}$ in ℓ_2. For arbitrary $K \in \mathbb{N}$ by uniform convergence and lower semicontinuity of φ, see assumption 6.25 item (ii) and item (iii), one easily concludes

$$\begin{aligned}\sum_{k=1}^{K} \varphi(\tilde{x}_k) &\leq \liminf_{n\to\infty} \sum_{k=1}^{K} \varphi_{\beta_n}(x_k^n) \\ &\leq \liminf_{n\to\infty} R_{\beta_n}(x^n) ,\end{aligned}$$

where we omit some arguments for readability. Considering the limit $K \to \infty$ immediately gives the assertion.

Following remark 6.24 we now have to show that the set E_M as in assumption 6.4 item (iv) is indeed bounded in ℓ_2. To this end, we observe $E_M \subseteq \{x \in \ell_2 : R_*(x) \leq M\}$ where we set $R_*(x) := \sum_{k\in\mathbb{N}} \rho_*(x_k)$ and show that the latter set is bounded for arbitrary $M > 0$. For $k \in \mathbb{N}$ and $x \in \ell_2$ such that $R_*(x) \leq M$ we obtain $\rho_*(x_k) \leq R_*(x) \leq M$ for arbitrary $k \in \mathbb{N}$ and, hence, $\rho_*(x_k) \leq M$. Note that this estimate is independent of the choice of $x \in E_M$. Coercivity of ρ_* now yields boundedness of the coefficients. We find $C > 0$ such that there holds

$$\sum_{k\in\mathbb{N}} |x_k|^2 \le C \sum_{k\in\mathbb{N}} \rho_*(x_k) \le C R_*(x) \le CM\,,$$

from which we conclude $\|x\|^2 \le CM$. □

Remark 6.27. In remark 6.9, we have pointed out that the assumption of monotonicity of the approximation is of special interest in the general setting. This can be observed in the special case of separable regularization terms as well. We consider the setting where φ_β is monotonic with respect to β.

The first relaxation of assumption 6.25 can be derived in the setting where in addition to monotonicity, we have that both φ and φ_β are continuous. Note that in prominent applications, continuity is provided, see section 6.2.3. We consider the conditions

(ii′) φ and φ_β are continuous for arbitrary $\beta > 0$.

(iii′) φ_β converges point-wise to φ as $\beta \to 0$.

which can be seen as a relaxed version of assumption 6.25 item (ii) and item (iii), respectively. To see this, we remark that from continuity and point-wise convergence by Dini's theorem we immediately obtain uniform convergence on compact sets.

Irrespective of these observations, considering the non-decreasing situation a natural way of finding functions ρ_* and ρ^* as in assumption 6.25 is choosing $\rho_* = \varphi_{\beta^*}$ and $\rho^* = \varphi$ with $\beta^* := \max_{n\in\mathbb{N}} \beta_n$. Thus, it is sufficient to consider the relaxed conditions

(iv′) The function φ is coercive and for every $M > 0$ there is $C > 0$ such that $\varphi(s) \ge C|s|^2$ for $s \in (-M, M)$.

(v′) For every $\beta > 0$ and for every $M > 0$ there is $C > 0$ such that $\varphi_\beta(s) \le C\varphi(s)$ for $s \in (-M, M)$.

as a replacement of condition item (iv) and item (v) in assumption 6.25. The non-decreasing case can be dealt with similarly by switching the role of φ and φ_β. In this setting, we chose $\rho^* = \varphi$ and $\rho^* = \varphi_{\beta^*}$ with β^* as above. The condition now reads

(iv″) For every $\beta > 0$ the function φ_β is coercive and for every $M > 0$ there is $C > 0$ such that $\varphi(s) \ge C|s|^2$ for $s \in (-M, M)$.

and can be used as a replacement of assumption 6.25 item (iv). Note that with this particular choice of ρ^*, the assertion in assumption 6.25 item (v) trivially holds true.

Applications to sparsity promoting regularization terms

In the following, we propose several ways to approximate the function $\varphi = |\cdot|^p$ for values $p > 0$ and, particularly, $p = 1$. The absolute value function is of particular importance in this context since it leads to the sparsity promoting regularization term given in eq. (5.7). The main aim of this section is to show that assumption 6.25 is usually fulfilled for common choices of φ_β and that it is easy to check. We stress that also the non-convex case $0 < p < 1$ fits into the framework developed above. Of course, we can only cover a small selection of possible approximations. However, the following examples illustrate the broad applicability of the framework developed in section 6.2.3. In Figure fig. 6.1, the different choices of φ_β proposed in the subsequent examples are depicted.

Example 6.28. Consider the choice

$$\varphi_\beta(s) := \begin{cases} \frac{|s|^2}{2\beta}, & |s| \geq \beta \\ |s| - \frac{\beta}{2}, & |s| \geq \beta \end{cases} . \tag{6.10}$$

We show that this particular choice satisfies assumption 6.25. It is obvious that it holds $\varphi_\beta(0) = 0$ and that φ_β is non-negative. Moreover, we observe that φ_β is non-increasing with respect to β. Hence, following remark 6.27 it is now easy to show assumption 6.25 item (iii′) and item (iii′). Moreover, item (iv″) holds true by construction.

Example 6.29. For $0 < p < 2$ we consider the mapping

$$\varphi_\beta(s) := \left(s^2 + \beta\right)^{p/2} - \beta^{p/2} . \tag{6.11}$$

We show that this particular choice satisfies assumption 6.25. It is obvious that it holds $\varphi_\beta(0) = 0$ and that φ_β is non-negative. Following remark 6.27, to show monotonicity with respect to β we consider the derivative of $\varphi_\beta(s)$ with respect to β for a fixed value of s. We obtain

$$\frac{\partial}{\partial\beta}\left(\varphi_\beta(s)\right) = \frac{p}{2}\left[\left(s^2 + \beta\right)^{p/2-1} - \beta^{p/2-1}\right] ,$$

which is non-positive for $p < 2$. From this we conclude that φ_β is non-increasing with respect to β for arbitrary s.

We observe that φ and φ_β are continuous and that φ_β converges point-wise to φ, which gives assumption 6.25 item (ii′) and item (iii′). To show that item (iv″) is fulfilled, it is sufficient to show that $s^2\varphi_\beta(s)^{-1}$ is bounded with respect to s on bounded sets. Hence, we are interested in the limit of this expression as $s \to 0$. An application of l'Hôspitals rule yields $\lim_{s\to 0} s^2\varphi_\beta(s)^{-1} = \beta^{1-p/2} < \infty$ for $p < 2$, which gives the above assertion.

Example 6.30. Consider the choice

$$\varphi_\beta(s) := |s| - \beta \ln\left(1 + \frac{|s|}{\beta}\right) . \tag{6.12}$$

Following remark 6.27, we exploit monotonicity of φ_β with respect to β. To this end, we consider the derivative of $\varphi_\beta(s)$ with respect to β and obtain

$$\frac{\partial}{\partial \beta}\left(\varphi_\beta(s)\right) = \frac{z-1}{z} - \ln(z)\,,$$

where $z := 1 + \frac{1}{\beta}|s|$. This mapping vanishes for $z = 1$ and is decreasing since

$$\frac{\partial}{\partial z}\left(\frac{z-1}{z} - \ln(z)\right) = \frac{1}{z^2} - \frac{1}{z} \leq 0$$

for any $z \geq 1$. Thus, the derivative of $\varphi_\beta(s)$ with respect to β is non-positive for arbitrary s from which we conclude that φ_β is non-increasing.

The only assertions in assumption 6.25 that are not obvious are item (iii′) and item (iv″). To show the first assertion, i.e. point-wise convergence of φ_β to φ, we incorporate l'Hôspitals rule. Another application of l'Hôspitals rule yields $\lim_{s\to 0} s^2 \varphi_\beta(s)^{-1} = 2\beta < \infty$. Repeating the arguments in example 6.29, we conclude item (iv″).

Example 6.31. Consider the choice

$$\varphi_\beta(s) := |s|^{1+\beta}\,. \tag{6.13}$$

This mapping leads to ℓ_p regularization well understood in literature with $p = 1 + \beta$, see eq. (5.6). In [34] it was shown that this approximation leads to a sensitive sequence of functionals. However, these results were established only under more restrictive assumptions.

It is a simple matter to verify assumption 6.25 item (i) – item (iii). In contrast to the approximations introduced above, however, in this setting φ_β is not monotonic with respect to β. Nevertheless, it is easily seen that considering the functions $\rho_* := \min\{|\cdot|, |\cdot|^2\}$ and $\rho^* := \max\{|\cdot|, |\cdot|^2\}$ the according conditions in assumption 6.25 are fulfilled.

6.3 Convergence rates

In section section 6.2, we have developed a general framework to obtain stability of the minimizer of eq. (6.1) with respect to an approximation parameter. Particularly, in section 6.2.3 we investigated the setting where the regularization term

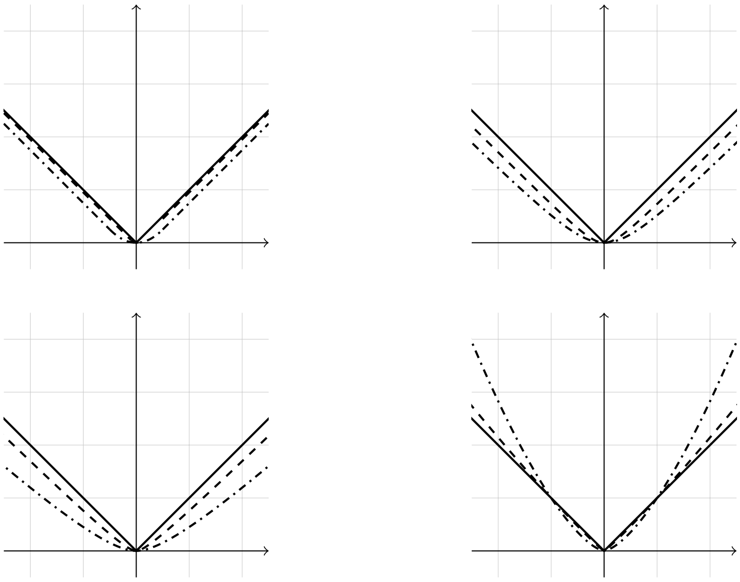


Figure 6.1: The absolute value function (solid) as well as the approximations φ_β given by eq. (6.10) to eq. (6.13) (top left to bottom right) for $\beta = 0.1$ (dashed) and $\beta = 0.5$ (dash-dotted).

is separable and can be approximated depending on a smoothness parameter β. We now address the question whether it is possible to establish convergence rates with respect to the approximation parameter β, i.e. constants $C > 0$ and $r > 0$ such that

$$\|x^\delta_{\alpha,\beta} - x^\delta_\alpha\| \leq C\beta^r\,, \tag{6.14}$$

where $x^\delta_{\alpha,\beta}$ and x^δ_α as in eq. (6.4) and eq. (6.2), respectively. Again, as in section 6.2.3 we focus on separable regularization terms.

As one might expect, to obtain this stronger result, one has to impose more restrictive assumptions compared to the general setting in section 6.2. In contrast to assumption 6.3 where we referred to general possibly nonlinear operators, throughout this section we consider the case of a bounded linear operator A mapping between Hilbert spaces X and Y as well as the functional

$$\Psi_{\alpha,\beta}(x) = \frac{1}{2}\|Ax - y^\delta\|^2_Y + \alpha R_\beta(x) \tag{6.15}$$

that incorporates the norm discrepancy as a fidelity term F. We further assume the existence of a unique minimizer

$$x^\delta_{\alpha,\beta} = \operatorname*{arg\,min}_{x\in X} \Psi_{\alpha,\beta}(x)\,, \tag{6.16}$$

which can be guaranteed by proposition 5.1 and strict convexity, for instance.

Finally, we assume that it holds $\|A\|_{\mathcal{L}(X,Y)} < \sqrt{2}$, which is of technical importance and can be enforced by rescaling the problem.

In our subsequent investigations, the chosen approach is based on a description of the minimizer $x^\delta_{\alpha,\beta}$ of eq. (6.15) as an implicit function of an appropriate equation $F(\beta, x) = 0$. The idea is to obtain differentiability of the mapping $x^\delta_\alpha(\beta) = x^\delta_{\alpha,\beta}$ from an application of the implicit function theorem. Thus, by the general relation

$$D_\beta x^\delta(\beta) = -\left[D_x F(\beta, x^\delta(\beta))\right]^{-1} D_\beta F(\beta, x^\delta(\beta)) \tag{6.17}$$

we obtain the derivative of the minimizer with respect to the smoothness parameter β, where by D_β and D_x we denote the directional derivatives with respect to the corresponding argument. This result particularly provides a convergence rate of linear order, i.e. there holds an equation of type eq. (6.14) with $r = 1$. This approach goes back to [34], where only the special case with R as in eq. (5.6) is considered. In the following, we generalize the result in [34] to a broad class of regularization terms.

6.3.1 The minimizer as an implicit function

To apply the approach described in eq. (6.17), we need to introduce an ansatz function F that satisfies the assumptions of the implicit function theorem (cf. [2, Theorem 8.2]). The starting point is the first order optimality condition of eq. (6.15) given by

$$-A^*(Ax - y^\delta) \in \alpha \partial R_\beta(x)\,. \tag{6.18}$$

The expression ∂R_β here denotes the subgradient of R_β. It might be tempting to choose $F(\beta, x) = A^*(Ax - y^\delta) + \alpha \partial R_\beta(x)$. This definition, however, turns out to be problematic, since the derivative of F incorporates the second derivative of R_β, which does not exist for relevant choices of R_β. Furthermore, the subgradient may be set-valued, for instance if $R = \|\cdot\|_{\ell_1}$. However, one can deduce an equivalent condition by adding x on both sides of eq. (6.18) followed by an application of the inverse mapping of $\mathrm{Id} + \alpha\partial R_\beta$ which leads to the identity $x = (\mathrm{Id} + \alpha \partial R_\beta)^{-1}(x - A^*(Ax - y^\delta))$. Hence we choose

$$F(\beta, x) = x - P_{\alpha,\beta}(x - A^*(Ax - y^\delta))\,. \tag{6.19}$$

Here, we denote by $P_{\alpha,\beta} := (\mathrm{Id} + \alpha\partial R_\beta)^{-1}$ the well known *proximal operator*, which particularly plays a crucial role in iterative thresholding algorithms as analyzed in [15], for instance.

The implicit function theorem requires that ansatz function F as in eq. (6.19) is continuously differentiable. Hence, the structure of F necessitates continuous differentiability of the mapping $(\beta, x) \mapsto P_{\alpha,\beta}(x)$. For notational convenience, in the following we write

$$P_\alpha(\beta, x) = P_{\alpha,\beta}(x) \tag{6.20}$$

for arbitrary $\beta \in I \subset \mathbb{R}$ and $x \in X$. A first rather general step towards convergence rates and, beyond, differentiability of the minimizer $x^\delta_{\alpha,\beta}$ of eq. (6.15) with respect to the approximation parameter β is given in the subsequent lemma. The result will be helpful when investigating the special setting of separable and even sparsity promoting regularization terms in section 6.3.2.

Lemma 6.32. *Let $\beta_0 \in I \subset \mathbb{R}$ and let P_α be continuously differentiable at $(\beta_0, x^\delta_{\alpha,\beta_0} - A^*(Ax^\delta_{\alpha,\beta_0} - y^\delta))$. Moreover, we define an operator $T_{\beta_0} : X \to X$ by the identity*

$$T_\beta := D_x P_\alpha(\beta, x^\delta_{\alpha,\beta_0} - A^*(Ax^\delta_{\alpha,\beta_0} - y^\delta))(\mathrm{Id} - A^*A)\,. \tag{6.21}$$

If $\|T_{\beta_0}\|_{\mathcal{L}(X,X)} < 1$, then the minimizer $x^\delta_\alpha = x^\delta_\alpha(\beta)$ of eq. (6.15) is continuously differentiable with respect to β at β_0 and it holds

$$D_\beta x^\delta(\beta_0) = \sum_{k=0}^{\infty} T^k_{\beta_0} D_\beta P_\alpha(\beta_0, x^\delta_{\alpha,\beta_0} - A^*(Ax^\delta_{\alpha,\beta_0} - y^\delta))\,. \tag{6.22}$$

Proof. The minimizer $x^\delta_{\alpha,\beta}$ of the functional eq. (6.15) is given as an implicit function with respect to the ansatz function F, see eq. (6.19). We need to check the following assumptions of the implicit function theorem.

(i) It holds $F(\beta, x^\delta_{\alpha,\beta}) = 0$ for any $\beta > 0$.

(ii) The mapping F is continuously differentiable.

(iii) The derivative $D_x F(\beta, x^\delta_{\alpha,\beta})$ is regular.

The mapping F was chosen such that (i) holds true by definition. Furthermore, continuous differentiability of F immediately follows from continuous differentiability of proximal operator P_α as in eq. (6.20), which holds true by assumption. This implies (ii) as well as the identities

$$\begin{aligned} D_\beta F(\beta, x) &= -D_\beta P_\alpha(\beta, x - A^*(Ax - y^\delta))\,, \\ D_x F(\beta, x) &= \mathrm{Id} - D_x P_\alpha(\beta, x - A^*(Ax - y^\delta))(\mathrm{Id} - A^*A)\,, \end{aligned} \tag{6.23}$$

where the latter identity follows from an application of the chain rule. To get (iii), we remark that

$$\|D_x P_\alpha(\beta, x - A^*(Ax - y^\delta))(\mathrm{Id} - A^*A)\|_{\mathcal{L}(X,X)} < 1$$

implies that the inverse operator of eq. (6.23) can be determined as the Neumann series

$$[D_x F(\beta, x)]^{-1} = \sum_{k=0}^{\infty} \big(D_x P_\alpha(\beta, x - A^*(Ax - y^\delta))(\mathrm{Id} - A^*A)\big)^k .$$

Hence, the ansatz function F matches the assumptions of the implicit function theorem. The general relation eq. (6.17) now gives the assertion. □

6.3.2 Separable regularization terms

A mapping R that is of particular interest is given by eq. (5.7) known from sparsity regularization. In the following, we want to apply the results in lemma 6.32 to this setting. As we have already seen, the application of lemma 6.32 requires the assumption of continuous differentiability of P_α on an appropriate subset of X. In this section, we show that this assumption can actually be established for relevant choices R and R_β under the assumption that R_β is separable, see section 5.3. Here, as in eq. (5.5) we consider the case $X = \ell_2$ as well as mappings $R : \ell_2 \to (-\infty, \infty]$ and $R_\beta : \ell_2 \to (-\infty, \infty]$ of the form

$$R(x) = \sum_{k\in\mathbb{N}} \varphi(x_k) \tag{6.24}$$

as well as

$$R_\beta(x) = \sum_{k\in\mathbb{N}} \varphi_\beta(x_k) \tag{6.25}$$

for a mapping φ and appropriate approximations φ_β, where $\beta \in I$ for an index set $I \subset \mathbb{R}$ and $\varphi_{\beta_0} = \varphi$ for some $\beta_0 \in I$.

Properties of the proximal operator

For a convex function φ_β and $\beta \in I$ consider the expression

$$P_{\alpha,\beta} := (\mathrm{Id} + \alpha\partial R_\beta)^{-1}$$

as a mapping $\ell_2 \to \ell_2$, where R_β is defined as in eq. (6.25). By means of the canonical basis $\{e^k\}$, we describe $P_{\alpha,\beta}$ by a series expansion

$$P_{\alpha,\beta}(x) = \sum_{k\in\mathbb{N}} p_\alpha\varphi_\beta(x_k)e^k , \tag{6.26}$$

where by

$$p_\alpha\varphi_\beta = (1 + \alpha\partial\varphi_\beta)^{-1} \tag{6.27}$$

we denote the scalar proximal mapping of φ_β. We write $p_\alpha(\beta, s) = p_\alpha\varphi_\beta(s)$. Basic properties of this mapping are discussed in section 6.3.3.

Again, as in eq. (6.20) we use the notation $P_\alpha(\beta, x) := P_{\alpha,\beta}(x)$. We now discuss properties of this mapping. Particularly, we examine the operator P_α regarding differentiability, which has to be assumed for an application of lemma 6.32. More generally, in this section we consider a scalar function f and develop a set of assumptions that ensure continuous differentiability of the series expansion given by

$$F(\lambda, x) = \sum_{k\in\mathbb{N}} f(\lambda, x_k)e^k \,. \tag{6.28}$$

We need to investigate convergence, continuity and, finally, differentiability of the series in eq. (6.28). To start with, we collect several conditions on the function f.

Assumption 6.33. Let $\mu \in \Lambda$ and $x \in \ell_2$. Assume that it holds

(i) The function f is continuously differentiable at (μ, x_k) for every $k \in \mathbb{N}$.

(ii) There is $C_1 > 0$ such that

$$|f(\lambda, u)| \le C_1|u| \tag{6.29}$$

for every (λ, u) in a neighborhood of $(\mu, 0)$.

(iii) There is $C_2 > 0$ such that

$$\left|\frac{\partial f}{\partial \lambda}(\lambda, u)\right| \le C_2|u| \tag{6.30}$$

for every (λ, u) in a neighborhood of $(\mu, 0)$.

(iv) There is $C_3 > 0$ such that

$$\left|\frac{\partial f}{\partial u}(\lambda, u)\right| \le C_3 \tag{6.31}$$

for all $(\lambda, u) \in \Lambda \times \mathbb{R}$

The assertion of local uniform convergence of the series in eq. (6.28) is central to our analysis. The following lemma provides a sufficient criterion on the scalar function f to establish this property.

Lemma 6.34. *Let X be a Hilbert space, Λ a normed vector space, and let $\lambda \in \Lambda$ and $f : \Lambda \times \mathbb{R} \to \mathbb{R}$ such that eq. (6.29). Then for every $x \in \ell_2$ the series $\sum_{k\in\mathbb{N}} f(\lambda, x_k)e^k$ converges uniformly in a neighborhood of (μ, x).*

Proof. Define the partial sums $F_n(\mu, y) := \sum_{k=0}^{n-1} f(\mu, y_k)e^k$ as well as the limit $F(\mu, y) := \lim_{n\to\infty} F_n(\mu, y)$. We remark that F does exist for arbitrary $\mu \in \Lambda$ and

$y \in \ell_2$ which follows from the fact that the partial sums F_n of the series give rise to a Cauchy series. We need to show that F_n converges locally uniformly to F.

Assume $\lambda \in \Lambda$, $x \in \ell_2$ and $\varepsilon > 0$. Since f is locally uniformly Lipschitz continuous at 0, there are $\delta > 0$ and $M > 0$ such that $|u| \leq \delta$ implies $|f(\mu, u)| \leq M|u|$ for any μ in a neighborhood of λ. There is no loss of generality in assuming $\delta \leq M^{-1}\varepsilon$. Furthermore, suppose $x \in \ell_2$ with $\|y - x\| \leq \frac{\delta}{2}$ and choose n_0 so large that for $n \geq n_0$ we have $\sum_{k=n+1}^{\infty} |x_k|^2 \leq \frac{\delta^2}{4}$. Thus, we conclude for $n \geq n_0$

$$\begin{aligned}\sum_{k=n+1}^{\infty} |y_k|^2 &\leq 2 \sum_{k=n+1}^{\infty} |y_k - x_k|^2 + 2 \sum_{k=n+1}^{\infty} |x_k|^2 \\ &\leq 2\,\|y - x\|^2 + \frac{\delta^2}{2} \leq \delta^2\end{aligned}$$

and, particularly, $|x_k| \leq \delta$ for $k \geq n_0$. We get $|f(\mu, y_k)| \leq M|y_k|$ for arbitrary $k \geq n_0$. Assuming $n \geq n_0$, we can now compute

$$\|F_n(\mu, y) - F(\mu, y)\|^2 \leq \sum_{k=n+1}^{\infty} M^2 |y_k|^2 \leq M^2\delta^2 \leq \varepsilon^2 .$$

Since this estimate is independent of μ and x in a neighborhood of λ and x, we get the assertion. □

We now consider the derivative of F. The partial derivatives of the series eq. (6.28) are given by the series of the partial derivatives of the coefficient functions. First we need to state a simple lemma which will help with our investigations.

Lemma 6.35. *Let $I \subset \mathbb{R}$ be an open subset and let $f : I \to \mathbb{R}$ be differentiable. Then, the derivative f' is continuous on $[a, b] \subset I$ for some $a < b$ if and only if for it every $\varepsilon > 0$ there is a $\delta > 0$ such that it holds*

$$|f(t + h) - f(t) - f'(t)h| \leq \varepsilon|h| \tag{6.32}$$

for $|h| \leq \delta$ and arbitrary $t \in I$.

Proof. Let f' be continuous on $[a, b]$ and consider sequences $t_n \to t$, $s_n \to t$ for some $t \in [a, b]$. For every $n \in \mathbb{N}$, the mean value theorem provides existence of $r_n \in [\min(t_n, s_n), \max(t_n, s_n)]$ such that

$$f(t_n) - f(s_n) = f'(r_n)(t_n - s_n) .$$

We observe that $r_n \to t$. By continuity we get $f'(r_n) \to f'(t)$ and we may conclude

$$\frac{f(t_n) - f(s_n)}{t_n - s_n} \to f'(t) .$$

This we use to argue that the mapping

$$g(t,s) := \begin{cases} \frac{f(t)-f(s)-f'(t)(t-s)}{t-s}, & t \neq s \\ 0, & t = s \end{cases}$$

is continuous. Compactness now provides uniform continuity on $[a,b]$ which gives the assertion. □

The latter property in lemma 6.35 could be interpreted as *uniform differentiability* of a mapping f, and we found a simple criterion for this property to hold true.

Lemma 6.36. *Let Λ be a normed vector space, X a Hilbert space, and for $\lambda \in \Lambda$ and $x \in \ell_2$ let f satisfy assumption 6.33. Then, the operator F as in eq.* (6.28) *is continuously differentiable at (λ, x) and it holds*

$$D_\lambda F(\lambda, x) = \sum_{k\in\mathbb{N}} \frac{\partial f}{\partial \lambda}(\lambda, x_k) e^k \tag{6.33}$$

as well as

$$D_x F(\lambda, x) = \sum_{k\in\mathbb{N}} \frac{\partial f}{\partial u}(\lambda, x_k)\langle \cdot\,, e^k\rangle e^k\,. \tag{6.34}$$

Proof. Given $\mu \in \Lambda$ and $y, z \in \ell_2$ we define the partial sums

$$D_n(\mu, y)z := \sum_{k=0}^{n-1} \frac{\partial f}{\partial u}(\mu, y_k) z_k e^k$$

and the point-wise limit $D(\mu, y)z := \lim_{n\to\infty} D_n(\mu, y)z$. Since by assumption $\frac{\partial f}{\partial u}$ is uniformly bounded by some $C > 0$, we have

$$\|D_n(\mu, y)z - D_m(\mu, y)z\|^2 \le C^2 \sum_{k=m+1}^{n} |z_k|^2\,.$$

Thus, $\{D_n(\mu, y)z\}_{n\in\mathbb{N}}$ is a Cauchy series and we obtain an operator $D(\mu, y)$ that is both well-defined on X and bounded. We need to show that $D_x F(\mu, y) = D(\mu, y)$.

To this end, let $\varepsilon > 0$. Since the derivative $\frac{\partial f}{\partial u}$ is continuous with respect to u, by lemma 6.35 we find $\delta > 0$ such that eq. (6.32). We immediately obtain for arbitrary $h \in \ell_2$ such that $\|h\| \le \delta$

$$\begin{aligned}\|F(\mu, y+h) - F(\mu, y) - D(\mu, y)h\|^2 \\ &= \sum_{k=1}^{\infty} \left| f(\mu, y_k + h_k) - f(\mu, y_k) - \frac{\partial f}{\partial u}(\mu, y_k) h_k \right|^2 \\ &\le \varepsilon^2 \sum_{k=1}^{\infty} |h_k|^2 = \varepsilon^2 \|h\|^2 \,,\end{aligned}$$

which yields differentiability of F as well as the relation eq. (6.34).

We now consider $\lambda \in \Lambda$ and $x \in \ell_2$ and show that $D_x F$ is continuous at (λ, x). The coefficients of x are bounded by some $R > 0$. Since the derivative $\frac{\partial f}{\partial u}$ is continuous, it is uniformly continuous on the compact set $U = H \times [-R, R]$, where $H \subset \Lambda$ is some compact neighborhood of λ. Hence, for arbitrary $\varepsilon > 0$ this provides $\delta > 0$ such that

$$\left| \frac{\partial f}{\partial u}(\mu, v) - \frac{\partial f}{\partial u}(\lambda, u) \right| \le \varepsilon \tag{6.35}$$

for arbitrary $(\mu, v) \in U$ such that $|u - v| \le \delta$ and $|\mu - \lambda| \le \delta$. For every $\mu \in \Lambda$ such that $|\mu - \lambda| \le \delta$ and $y \in \ell_2$ such that and $\|y - x\| \le \delta$ from eq. (6.35) we now obtain $\|D_x F(\mu, y) e^k - D_x F(\lambda, x) e^k\| \le \varepsilon$ for arbitrary $k \in \mathbb{N}$. Consequently, we conclude

$$\begin{aligned}\|D_x F(\mu, y) - D_x F(\lambda, x)\|_{\mathcal{L}(\ell_2, \ell_2)} \\ &= \sup_{k \in \mathbb{N}} \|D_x F(\mu, y) e^k - D_x F(\lambda, x) e^k\| \le \varepsilon \,,\end{aligned}$$

which yields continuity at (λ, x). We point out to the fact that we can even conclude uniform continuity on bounded sets with some minor modifications to the above arguments.

We now turn to the derivative with respect to λ. By assumption, the partial derivative with respect to λ is locally uniformly Lipschitz continuous with respect to x. Thus, by lemma 6.34 it follows that the series eq. (6.33) converges locally uniformly with respect to the vector (λ, x). From this and convergence of the mapping F, see lemma 6.34, we get differentiability of the proximal operator with respect to λ as well as the identity eq. (6.33). Moreover, continuity of $\frac{\partial f}{\partial \lambda}$ yields continuity of $D_\lambda F$.

By continuous partial differentiability, we now conclude continuous differentiability of the operator F, which concludes the proof. □

In lemma 6.36, we could develop conditions on the coefficient function f of an operator F of the form eq. (6.28) that ensure continuous differentiability. In section 6.3.3, we show that these conditions can actually be established for the proximal mapping p_α that correspond to a rather general class of choices φ_β.

Particularly, in section 6.3.3, we cover several approximations of the absolute value function. A result similar to lemma 6.36 can be found in [34, Lemma 2.20]. Here, we presented a different proof that allows for the application of lemma 6.36 to a more general class of coefficient functions f.

Results on convergence rates

To apply lemma 6.32, by assumption 6.33 we need to proof differentiability of the proximal operator P_α as in eq. (6.20). Using the results in section 6.3.2, in the setting of separable regularization terms this can be done by means of an analysis of the scalar proximal mapping p_α, see eq. (6.27). In this section, we present a proof of convergence rates under the assumption that p_α does satisfy assumption 6.33. In section 6.3.3, we show that these assumptions can actually be established under reasonable assumptions on the mapping φ_β.

To start with, we remark that an application of lemma 6.32 requires that the operator T_{β_0} in eq. (6.21) satisfies $\|T_{\beta_0}\|_{\mathcal{L}(X,X)} < 1$. To this end, we state the following general lemma, the proof of which is left to the reader.

Lemma 6.37. *Let X and Y be Hilbert spaces. If $\|A\|_{\mathcal{L}(X,Y)} \leq \sqrt{2}$, then it holds $\|(\mathrm{Id} - A^*A)x\| \leq \|x\|$ for any $x \in X$. Moreover, if we instead assume $\|A\|_{\mathcal{L}(X,Y)} < \sqrt{2}$, then equality holds if and only if $x \in \mathcal{N}(A)$.*

Theorem 6.38. *Let p_α as in eq. (6.27) satisfy assumption 6.33. Moreover, assume that the derivative of p_α satisfies*

$$\left| \frac{\partial p_\alpha}{\partial s}(\beta_0, s) \right| < 1 \tag{6.36}$$

for arbitrary $s \in \mathbb{R}$. Then, the minimizer $x_\alpha^\delta = x_\alpha^\delta(\beta)$ of eq. (6.15) is continuously differentiable with respect to β at β_0 and it holds eq. (6.22).

Proof. By lemma 6.36, we have seen that from Assumption assumption 6.33 it follows continuous differentiability of the proximal operator P_α as in eq. (6.20). We need to show that $\|T_{\beta_0}\|_{\mathcal{L}(\ell_2,\ell_2)} < 1$. By assumption, it holds eq. (6.36) for all $s \in \mathbb{R}$. By continuity of the derivative of p_α we get $\|P_\alpha(\beta_0, x^\delta_{\alpha,\beta_0} - A^*(Ax^\delta_{\alpha,\beta_0} - y^\delta))\|_{\mathcal{L}(\ell_2,\ell_2)} < 1$. Further, lemma 6.37 provides $\|\mathrm{Id} - A^*A\|_{\mathcal{L}(\ell_2,\ell_2)} \leq 1$ which yields the desired statement. □

Remark 6.39. The condition in eq. (6.36) can actually be established for a broad class of choices for φ_β. Particularly, it holds true if φ_β is strictly convex at β_0. This matter is addressed in lemma 6.47.

The condition in eq. (6.36), while helpful for several applications, does not apply for the important case $\varphi = |\cdot|$. Hence, to establish differentiability of the minimizer with respect to β in this setting, in the subsequent theorem we have to additionally assume that the operator A satisfies the *FBI property* (finite basis injectivity), i.e. for all finite subsets $J \subset \mathbb{N}$, A is injective on the subspace generated by $\{e^k\}_{k\in J}$. This property is well-established in the context of sparsity regularization and has proven helpful to obtain convergence rates of regularization methods based on the functional eq. (5.7). Moreover, note that under this assumption, the minimizer of the Tikhonov functional is unique, see [9, Theorem 2].

Theorem 6.40. *For $\alpha > 0$, let R_β be chosen as in eq.* (6.25) *and let the approximation φ_β satisfy assumption 6.33. Let $\varphi_{\beta_0} = \varphi = |\cdot|$ and let the operator A satisfy the FBI property (finite basis injectivity) and $\|A\|_{\mathcal{L}(X,Y)} < \sqrt{2}$. Moreover, assume that the minimizer $x^\delta_{\alpha,\beta_0}$ fulfills the condition*

$$|(x^\delta_{\alpha,\beta_0} - A^*(Ax^\delta_{\alpha,\beta_0} - y^\delta))_k| \neq \alpha$$

for all $k \in \mathbb{N}$. Then the minimizer $x^\delta_\alpha = x^\delta_\alpha(\beta)$ is continuously differentiable with respect to β at 0 *and the derivative is given by eq.* (6.22)*.*

Proof. By lemma 6.36, we obtain continuous differentiability of P_α on the set

$$M := \{x \in \ell_2 \,:\, |(x - A^*(Ax - y^\delta))_k| \neq \alpha \text{ for all } k \in \mathbb{N}\}$$

and by assumption, we have $x^\delta_{\alpha,\beta_0} \in M$. We need to establish the strict estimate $\|T_{\beta_0}\|_{\mathcal{L}(\ell_2,\ell_2)} < 1$.

It is well known that the minimizer $x^\delta_{\alpha,\beta_0}$ is sparse, see proposition 5.6. Thus, consider the finite set $J := \{k \in \mathbb{N} \,:\, (x^\delta_{\alpha,\beta_0})_k \neq 0\}$. From the optimality condition eq. (6.18) we get

$$-(A^*(Ax^\delta_{\alpha,\beta_0} - y^\delta))_k \in \alpha \operatorname{sign}((x^\delta_{\alpha,\beta_0})_k)$$

for all $k \in \mathbb{N}$. From this we conclude $|(x^\delta_{\alpha,\beta_0} - A^*(Ax^\delta_{\alpha,\beta_0} - y^\delta))_k| > \alpha$ if and only if $k \in J$. Since the derivative of the shrinkage operator $P_\alpha = (\mathrm{Id} + \alpha\partial\|\cdot\|_{\ell_1})^{-1}$, see eq. (3.18), as a piecewise linear mapping can easily be determined by an indicator function, we get $D_x P_\alpha(\beta_0, x^\delta_{\alpha,\beta_0} - A^*(Ax^\delta_{\alpha,\beta_0} - y^\delta)) = \sum_{k\in J} 1 e^k =: \Pi_J$.

Set $K := \Pi_J(\mathrm{Id} - A^*A)$. Since the range of K is finite-dimensional, K is compact and we conclude existence of the singular value decomposition. Let $\sigma > 0$ be a singular value of K with singular function $u \in \ell_2$. Since by assumption A is injective on the subspace ℓ_J generated by $\{e^k\}_{k\in J}$ and it holds $0 \neq Ku \in \ell_J$, by lemma 6.37 we get the sharp estimate $\sigma^2 = \|K^*Ku\| = \|(\mathrm{Id} - A^*A)Ku\| < \|Ku\| \leq 1$. Since the range of K is finite-dimensional, this implies $\|K\|_{\mathcal{L}(\ell_2,\ell_2)} < 1$. An application of lemma 6.32 now yields the assertion. □

6.3.3 The proximal mapping in one dimension

As we have already seen in eq. (6.26), in the setting of separable regularization terms such as eq. (6.24), it turns out that the proximal operator can be described by means of the proximal mapping as a function defined on the real line. Moreover, in lemma 6.36 we have shown that we can ensure differentiability of an operator of type eq. (6.26) if the coefficient function is appropriately chosen. In the following, we want to establish the according conditions as stated in assumption 6.33 for the scalar proximal mapping. Particularly, we cover the case $\varphi = |\cdot|$, which requires separate treatment.

The next definition introduces an alternate definition of the scalar proximal mapping. In lemma 6.42, we show that this definition coincides with eq. (6.27) in case φ is convex.

Definition 6.41. For any proper function $\varphi : \mathbb{R} \to \bar{\mathbb{R}}$, $\alpha > 0$ and $s \in \mathbb{R}$, the *proximal mapping* is defined as

$$p_\alpha \varphi(s) := \underset{t \in \mathbb{R}}{\arg\min} (s-t)^2 + 2\alpha\varphi(t)\,. \tag{6.37}$$

In the general case, the proximal mapping may be set-valued. See [37, 49] for general reference on set-valued mappings. The following lemma enables us to describe the proximal mapping incorporating the subgradient, which is a generalized notion of the derivative, see [46].

Lemma 6.42. *For any proper function $\varphi : \mathbb{R} \to (-\infty, \infty]$ and $s \in \mathbb{R}$, the proximal mapping satisfies $p_\alpha\varphi(s) \subseteq (1 + \alpha\partial\varphi)^{-1}(s)$. Moreover, equality holds if φ is convex.*

The proof can be found in [37, Lemma 4.1]. Note that in lemma 6.42 we see that for a convex mapping, the definition in eq. (6.37) reduces to the one in eq. (6.27). The next lemma provides yet another reason why the setting of a convex mapping is of particular importance.

Lemma 6.43. *If φ is convex, then the proximal mapping $p_\alpha\varphi$ is single-valued.*

Proof. The definition of the proximal mapping in eq. (6.37) introduces $p_\alpha\varphi(x)$ as the solution of a minimization problem which is strictly convex given that φ is convex. Uniqueness of the minimizer yields the assertion. □

Analysis for differentiable mappings

In this section, for $\alpha > 0$ we consider a convex function $\varphi : \mathbb{R} \to \mathbb{R}$ that is differentiable. In this setting, we may define the mapping

$$q_\alpha\varphi(u) := u + \alpha\varphi'(u)\,. \tag{6.38}$$

By lemma 6.42, this means that q_α is the inverse mapping of p_α. Since we assume differentiability of φ it is well-defined. In the following, we make some general statements regarding the proximal mapping in this setting. Note that the case $\varphi = |\cdot|$ is not covered by the subsequent investigations. In section 6.3.3, this special setting is analyzed separately.

Lemma 6.44. *Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a convex function. If φ is differentiable, then for the proximal mapping it holds*

(i) $p_\alpha\varphi$ is non-decreasing.

(ii) $p_\alpha\varphi$ is Lipschitz continuous and it holds $|p_\alpha\varphi(s) - p_\alpha\varphi(t)| \le |s-t|$.

Proof. Convexity of φ yields monotonicity of the derivative, which yields (i). To show (ii), we set $u := p_\alpha\varphi(s)$ and $v := p_\alpha\varphi(t)$ and consider the mapping $q_\alpha\varphi$ given in eq. (6.38). Again, since the derivative is non-decreasing and we can conclude that $\varphi'(u) - \varphi'(v)$ has the same sign as $u - v$. The assertion now follows from the estimate

$$\begin{aligned}|q_\alpha\varphi(u) - q_\alpha\varphi(v)| &= |u - v + \alpha\varphi'(u) - \alpha\varphi'(v)| \\ &= |u-v| + \alpha|\varphi'(u) - \varphi'(v)| \ge |u-v|\,.\end{aligned}$$

Inserting the definition of u and v now yields the assertion. □

Next, as in eq. (6.27) for an index set $I \subset \mathbb{R}$ and a family of convex functions $\{\varphi_\beta\}_{\beta\in I}$, for $\alpha > 0$ and any $\beta \in I$ we denote the according proximal mapping $p_\alpha\varphi_\beta$ as a function of two arguments by

$$p_\alpha(\beta, s) := p_\alpha\varphi_\beta(s)\,, \tag{6.39}$$

In the following, we investigate this mapping in regard to its dependence on both β and s. First, we present a general continuity result.

Lemma 6.45. *Let $\beta_0 \in I$ and $s_0 \in \mathbb{R}$ and let the mapping $\beta \mapsto \varphi'_\beta(u_0)$ be continuous at β_0, where $u_0 = p_\alpha(\beta_0, s_0)$. Then, the mapping p_α as in eq.* (6.39) *is continuous at (β_0, s_0).*

Proof. First, we remark that p_α is continuous with respect to β as follows from the estimate

$$\begin{aligned}|p_\alpha(\beta, s_0) - p_\alpha(\beta_0, s_0)| &= |p_\alpha(\beta, s_0) - p_\alpha(\beta, q_\alpha(\beta, p_\alpha(\beta_0, s_0)))| \\ &\le |s_0 - q_\alpha(\beta, p_\alpha(\beta_0, s_0))|\,,\end{aligned}$$

where in the latter step, we made use of lemma 6.44 (ii). By means of the triangle inequality, we can now conclude

$$\begin{aligned}|p_\alpha(\beta, s) - p_\alpha(\beta_0, s_0)| &\le |p_\alpha(\beta, s) - p_\alpha(\beta, s_0)| + |p_\alpha(\beta, s_0) - p_\alpha(\beta_0, s_0)| \\ &\le |s - s_0| + |p_\alpha(\beta, s_0) - p_\alpha(\beta_0, s_0)|\,,\end{aligned}$$

where in the latter step, we once again applied lemma 6.44 (ii). Collectively we get the assertion. □

In lemma 6.45, we have seen that continuity of the proximal mapping can be established under mild conditions. To be more precise, any family of differentiable functions φ_β such that φ'_β converges point-wise to φ' gives rise to a continuous proximal mapping p_α as in eq. (6.39). In assumption 6.33, however, beyond continuity we even assume continuous differentiability of the proximal mapping. Next, we address this matter in a general setting. To this end, for convenience we consider the mapping

$$w(\beta, u) := \varphi'_\beta(u) \tag{6.40}$$

as a mapping defined on $I \times \mathbb{R}$.

Lemma 6.46. *Let $\beta_0 \in I$, $s_0 \in \mathbb{R}$ and let φ_β such that the mapping w as in eq.* (6.40) *is continuously differentiable at (β_0, u_0) with $u_0 = p_\alpha(\beta_0, s_0)$. Then, the proximal mapping p_α as in eq.* (6.39) *is continuously differentiable at (β_0, s_0) and it holds*

$$\frac{\partial p_\alpha}{\partial \beta}(\beta_0, s_0) = -\left(1 + \alpha \frac{\partial w}{\partial u}(\beta_0, u_0)\right)^{-1} \frac{\partial w}{\partial \beta}(\beta_0, u_0)\,, \tag{6.41}$$

$$\frac{\partial p_\alpha}{\partial s}(\beta_0, s_0) = \left(1 + \alpha \frac{\partial w}{\partial u}(\beta_0, u_0)\right)^{-1}. \tag{6.42}$$

Proof. We set $f(\beta, s, u) := u - s + \alpha w(\beta, u)$. Since w is continuously differentiable, so is f. It is easy to see that it holds $f(\beta_0, s_0, u_0) = 0$. Convexity of the mappings φ_β particularly implies $\varphi''_{\beta_0} \geq 0$ and thus, it holds

$$\frac{\partial f}{\partial u}(\beta_0, s_0, u_0) = 1 + \alpha \frac{\partial w}{\partial u}(\beta_0, u_0) \neq 0\,.$$

The implicit function theorem now yields existence and uniqueness of a mapping $u = u(\beta, s)$ as stated in a neighborhood of (β_0, s_0). The partial derivatives of s are obtained from differentiating the defining relation $0 = f(\beta, s, u(\beta, s)) = u(\beta, s) - s + \alpha w(\beta, u(\beta, s))$ which yields

$$0 = \frac{\partial f}{\partial \beta}(\beta, s, u) = \alpha \frac{\partial w}{\partial \beta}(\beta, u) + \left(1 + \alpha \frac{\partial w}{\partial u}(\beta, u)\right) \frac{\partial u}{\partial \beta}(\beta, s)\,,$$

$$0 = \frac{\partial f}{\partial s}(\beta, s, u) = \left(1 + \alpha \frac{\partial w}{\partial u}(\beta, u)\right) \frac{\partial u}{\partial s}(\beta, s) - 1\,,$$

where $u = u(\beta, s)$ for short. The identities in eq. (6.41) and eq. (6.42) follow by rearranging the components. Finally, uniqueness of u implies $u = p_\alpha$ which concludes the proof. □

If we assume the mapping φ_β to be strictly convex, we obtain the following result which provides a simple criterion for the condition in eq. (6.36).

Lemma 6.47. *Let φ_β be convex such that w as in eq.* (6.40) *is differentiable with respect to u. Then it holds*

$$\left|\frac{\partial p_\alpha}{\partial s}(\beta, s)\right| \leq 1\,.$$

The estimate is strict if φ_β is strictly convex.

Proof. The assertion follows from eq. (6.42) and the estimate $\varphi_\beta'' \geq 0$, which follows from convexity. In case φ_β is strictly convex we can even conclude $\varphi_\beta'' > 0$ which implies the second assertion. □

In lemma 6.46, we have provided a criterion for continuous differentiability of the proximal mapping p_α. However, in assumption 6.33 we require even more. Namely, we need to ensure that the derivative with respect to β is locally uniformly Lipschitz continuous. The following lemma provides a simple criterion for the mapping φ_β that is easy to check in applications, see section 6.3.3.

Lemma 6.48. *Let $\beta_0 \in I$ and let φ_β be symmetric such that the mapping w as in eq.* (6.40) *is continuously differentiable at $(\beta_0, 0)$. Further, assume*

$$|s| \leq Cw(\beta, \rho(\beta)|s|) \tag{6.43}$$

for $C > 0$ and some mapping $\rho \geq 0$ for all (β, s) in a neighborhood $U \subset \mathbb{R}^2$ of $(\beta_0, 0)$ as well as

$$\left|\left(\frac{\partial w}{\partial u}(\beta, u)\right)^{-1} \frac{\partial w}{\partial \beta}(\beta, u)\right| \leq \frac{|u|}{\rho(\beta)} \tag{6.44}$$

for arbitrary $u \in \mathbb{R}$. Then, it holds

$$\left|\frac{\partial p_\alpha}{\partial \beta}(\beta, s)\right| \leq \frac{C}{\alpha}|s| \tag{6.45}$$

for all $(\beta, s) \in U$, i.e. the derivative of p_α with respect to β is locally uniformly Lipschitz continuous with respect to s at $(\beta_0, 0)$, where p_α as in eq. (6.39).

Proof. From eq. (6.42) and identity eq. (6.44) we obtain the estimate

$$\left|\frac{\partial p_\alpha}{\partial \beta}(\beta, s)\right| \leq \frac{|p_\alpha(\beta, s)|}{\rho(\beta)}$$

for arbitrary $s \in \mathbb{R}$. Moreover, in an appropriate neighborhood of $(\beta_0, 0)$ by eq. (6.43) we deduce

$$|s| \leq \alpha w(\beta, \tilde{C}\rho(\beta)|s|) \leq \tilde{C}\rho(\beta)|s| + \alpha w(\beta, \tilde{C}\rho(\beta)|s|)\,,$$

where $\tilde{C} = \frac{C}{\alpha}$. Because the proximal mapping is non-decreasing, see lemma 6.44 (i), an application of p_α to both sides yields $|p_\alpha(\beta, s)| \leq \tilde{C}\rho(\beta)|s|$. In the latter step we exploited the fact that $p_{\alpha,\beta}$ is point-symmetric which follows immediately from symmetry of φ_β. □

We now state the main result of this section which provides differentiability of the proximal operator and leads hereby towards applicability of theorem 6.38. We are able to develop conditions on the family of functions φ_β which ensure that the proximal mapping as in eq. (6.39) satisfies assumption 6.33. We remark that this way, by theorem 6.38 the question of differentiability of the minimizer $x^\delta_{\alpha,\beta}$ with respect to β reduces to an analysis of a function mapping between the real line.

Lemma 6.49. *Let $\beta_0 \in I$ and $\tilde{x} \in \ell_2$ and let φ_β be a convex and differentiable mapping such that the mapping w as in eq.* (6.40) *is continuously differentiable at (β_0, u_k) for arbitrary $k \in \mathbb{N}$, where $u_k = p_\alpha(\beta_0, \tilde{x}_k)$. Moreover, assume that there is $C > 0$ and a mapping $\rho > 0$ such that eq.* (6.43) *and eq.* (6.44)*. Then, the proximal operator P as in eq.* (6.26) *is continuously differentiable at $(\beta_0, \tilde{x})$.*

Proof. Consider the mapping eq. (6.26). We need to check that the mapping p_α satisfies the conditions in assumption 6.33. By lemma 6.44, we see that obviously, eq. (6.29) is satisfied. Moreover, lemma 6.47 yields eq. (6.31) and by lemma 6.48 we get eq. (6.30). An application of lemma 6.36, we get the assertion. □

Remark 6.50. In theorem 6.38, we required eq. (6.36). This property can easily be established under the additional assumption that φ_β is even strictly convex, see lemma 6.47.

Applications to sparsity promoting regularization terms

In the following, we want to check the conditions in assumption 6.33 to the important setting where $\varphi = |\cdot|$. We remark that while this choice is of particular interest, it is can not be dealt with by means of lemma 6.49. Particularly, the absolute value function is not differentiable at 0. However, one can easily determine an explicit expression of the proximal mapping $p_\alpha|\cdot|$ that allows for a straight-forward analysis.

For $\alpha > 0$ and $s \in \mathbb{R}$, the *shrinkage function* is given by

$$S_\alpha(s) := p_\alpha|\cdot|(s) = \begin{cases} s - \alpha\,, & s > \alpha \\ 0\,, & -\alpha \leq s \leq \alpha \\ s + \alpha\,, & s < -\alpha \end{cases}\,, \tag{6.46}$$

see eq. (3.18). The explicit formula can easily be verified. While the subgradient of the mapping $|\cdot|$ is set-valued at 0, the shrinkage function as a the proximal mapping of a convex function is in fact single-valued. This can be seen directly in eq. (6.46) or, more generally, by the result lemma 6.43.

In section 6.3.3, we collected some properties of the proximal mapping $p_\alpha\varphi$ in the setting where φ is differentiable. While these results are not applicable for the shrinkage function, we can re-establish some of the results as demonstrated in the following. An analogous result to lemma 6.44 is presented in lemma 6.51.

Lemma 6.51. *For the shrinkage function S_α it holds*

(i) S_α is non-decreasing.

(ii) S_α is Lipschitz continuous and it holds $|S_\alpha(s) - S_\alpha(t)| \le |s-t|$ for arbitrary $s, t \in \mathbb{R}$.

The assertion (i) in lemma 6.51 is obvious as it follows immediately from the explicit formula in eq. (6.46). The second assertion (ii) can be dealt with by means of elementary estimates, see [15], for instance.

Next, we consider the choice $I = (0, \infty)$ and a family of convex functions $\{\varphi_\beta\}_{\beta\in I}$ approximating the absolute value function as $\beta \to 0$. For $\alpha > 0$ and for any $\beta \ge 0$ we denote the according proximal mapping $p_\alpha\varphi_\beta$ as a function of two arguments by

$$p_\alpha(\beta, s) := p_\alpha\varphi_\beta(s)\,, \tag{6.47}$$

where we set $\varphi_0 := |\cdot|$. We show that assumption 6.33 is in fact fulfilled by many popular choices of φ_β, see eqs. (6.10) to (6.13). Since the shrinkage function is not differentiable, in our investigations we cannot rely on lemma 6.46.

To start with, for $\beta > 0$ we consider the mapping in eq. (6.11) given by

$$\varphi_\beta(s) := (s^2 + \beta^2)^{1/2} - \beta$$

and examine the corresponding proximal mapping p_α as in eq. (6.39) in regard to the conditions in assumption 6.33. To this end, for $\beta > 0$ we consider the mapping w as in eq. (6.40), which is given by

$$w(\beta, u) = \frac{u}{(u^2 + \beta^2)^{1/2}}\,. \tag{6.48}$$

Differentiating yields the partial derivatives

$$\frac{\partial w}{\partial \beta}(\beta, u) = -\frac{\beta u}{(u^2 + \beta^2)^{3/2}}\,, \quad \frac{\partial w}{\partial s}(\beta, u) = \frac{\beta^2}{(u^2 + \beta^2)^{3/2}}\,. \tag{6.49}$$

In a first step, we show that p_α is continuous.

Lemma 6.52. *The mapping p_α with φ_β as in eq.* (6.11) *is continuous at* $(0, s_0)$ *for arbitrary $s_0 \in \mathbb{R}$.*

Proof. We show that the proximal mapping p_α is continuous with respect to β. By analogy with the proof of lemma 6.45, from this and lemma 6.51 we can obtain continuity. In this proof, without loss of generality, we assume $s \geq 0$. Moreover, we remark that the case $s = 0$ is obvious and we may thus even assume $s > 0$.

We first consider the case $s \leq \alpha$, which implies $p_\alpha(0, s) = 0$, see eq. (6.46). For $\varepsilon > 0$ and $s \neq 0$ we obtain the limit $\lim_{\beta \to 0} w(\beta, \varepsilon) = 1$, see eq. (6.48). This provides $s \leq \alpha \leq \varepsilon + \alpha w(\beta, \varepsilon)$ for every β in a neighborhood of 0. Since the proximal mapping is non-decreasing with respect to s, see lemma 6.44, from this it follows $p_\alpha(\beta, s) \leq \varepsilon$. We conclude $|p_\alpha(\beta, s) - p_\alpha(0, s)| \leq \varepsilon$ and hence the assertion.

We now consider the case $s > \alpha$ which means $p_\alpha(0, s) = s - \alpha$, see eq. (6.46). Again, we consider $\varepsilon > 0$ and the limit $\lim_{\beta \to 0} w(\beta, s - \alpha + \varepsilon) = 1$, which provides the estimate $s \leq s - \alpha + \varepsilon + \alpha w(\beta, s - \alpha + \varepsilon)$ for β in a neighborhood of 0. By monotonicity this gives $p_\alpha(\beta, s) \leq s - \alpha + \varepsilon$. Similarly, a consideration of the limit $\lim_\beta w(\beta, s - \alpha - \varepsilon)$ yields $p_\alpha(\beta, s) \geq s - \alpha - \varepsilon$ and we collectively obtain $|p_\alpha(\beta, s) - p_\alpha(0, s)| \leq \varepsilon$ and the assertion. □

Next, we show that the assumptions in lemma 6.48 are satisfied.

Lemma 6.53. *The mapping φ_β given by eq.* (6.11) *satisfies the conditions in lemma 6.48 with $\rho(\beta) = \beta$ for arbitrary $C > 1$. Particularly, it holds*

$$\frac{|p_\alpha(\beta, s)|}{\beta} \leq \frac{C}{\alpha}|s| \tag{6.50}$$

for every $\beta > 0$ and every s such that $|s| \leq \sqrt{C^2 - 1}$.

Proof. By eq. (6.48), for arbitrary $\beta > 0$ and $s \in \mathbb{R}$ such that $|s| \leq \sqrt{C^2 - 1}$ we conclude

$$w(\beta, \beta s) = \frac{\beta s}{((\beta s)^2 + \beta^2)^{1/2}} = \frac{s}{(s^2 + 1)^{1/2}} \geq C^{-1} s\,,$$

which yields eq. (6.43) and, particularly, eq. (6.50), see lemma 6.48. Moreover, we observe that it holds

$$\left| \left(\frac{\partial w}{\partial u}(\beta, u) \right)^{-1} \frac{\partial w}{\partial \beta}(\beta, u) \right| = \frac{|u|}{\beta}\,,$$

see eq. (6.49), and, thus, eq. (6.44). □

Lemma 6.54. *The proximal mapping p_α with φ_β as in eq.* (6.11) *is continuously* *differentiable with respect to s at $(0, s_0)$ for any $s_0 \neq \alpha$ and it holds*

$$\lim_{s \to s_0} \frac{p_\alpha(0,s) - p_\alpha(0,s_0)}{s} = \begin{cases} 0\,, & |s_0| < \alpha \\ 1\,, & |s_0| > \alpha \end{cases}\,. \tag{6.51}$$

Proof. The derivative of the shrinkage function as a piecewise linear mapping can easily be established. Note that the derivative does not exist at $s = \pm\alpha$. We need to show that the derivative is continuous for all $s \neq \pm\alpha$. First, we consider the case $|s_0| < \alpha$. From eq. (6.42) and eq. (6.49) we derive the elementary estimate

$$\left|\frac{\partial p_\alpha}{\partial s}(\beta, s)\right| \leq \frac{(u^2 + \beta^2)^{3/2}}{\alpha\beta^2} = \frac{\beta(\frac{u^2}{\beta^2} + 1)^{3/2}}{\alpha}\,,$$

where $u = p_\alpha(\beta, s)$. Furthermore, there is $C > 0$ such that $|u| \leq C\beta|s|$, see eq. (6.50), which gives

$$\left|\frac{\partial p_\alpha}{\partial s}(\beta, s)\right| \leq \frac{\beta(C^2 s^2 + 1)^{3/2}}{\alpha}\,.$$

This provides continuity of the derivative. The case $|s_0| > \alpha$ follows from eq. (6.42) and continuity of p_α, see lemma 6.52. □

Lemma 6.55. *The proximal mapping p_α with φ_β as in eq.* (6.11) *is continuously differentiable with respect to β at $(0, s_0)$ for any $s_0 \neq \alpha$ and it holds*

$$\lim_{\beta \to 0} \frac{p_\alpha(\beta, s_0) - p_\alpha(0, s_0)}{\beta} = \begin{cases} s_0\sqrt{\alpha^2 - s_0^2}^{\,-1}\,, & |s_0| < \alpha \\ \pm\infty\,, & s_0 = \pm\alpha \\ 0\,, & |s_0| > \alpha \end{cases}\,.$$

Proof. First, we remark the case $|s_0| > \alpha$ follows immediately from an application of lemma 6.46. Consider $s_0 < \alpha$. In this setting, since the derivative of the shrinkage function vanishes for $|s| \leq \alpha$, see lemma 6.54, lemma 6.46 is not applicable. Instead, we set $M = s_0\sqrt{\alpha^2 - s_0^2}^{\,-1}$ and show

$$\frac{\partial p_\alpha}{\partial \beta}(0, s_0) = M\,. \tag{6.52}$$

Let $\varepsilon > 0$. In a neighborhood of s_0, for $s < \alpha$ it holds

$$\frac{s}{\sqrt{\alpha - s^2}} \leq M + \varepsilon\,,$$

which implies

$$s \leq \frac{\alpha(M+\varepsilon)}{\sqrt{(M+\varepsilon)^2+1}} \leq (M+\varepsilon)\beta + \frac{\alpha(M+\varepsilon)\beta}{\sqrt{((M+\varepsilon)\beta)^2+\beta^2}}\,. \tag{6.53}$$

By monotonicity of the proximal operator, see lemma 6.44 (i), and identity $p_\alpha(0,s) = 0$, an application of p_α to both sides in eq. (6.53) yields the estimate

$$\frac{p_\alpha(\beta,s) - p_\alpha(0,s)}{\beta} \leq M + \varepsilon\,. \tag{6.54}$$

On the other hand, since $s \mapsto s\sqrt{s^2+1}^{-1}$ is increasing, there is $d > 0$ such that for s in a neighborhood of s_0 one has

$$s \geq d + \frac{M-\varepsilon}{\sqrt{(M-\varepsilon)^2+1}} \geq (M-\varepsilon)\beta + \frac{(M-\varepsilon)\beta}{\sqrt{((M-\varepsilon)\beta)^2+\beta^2}}\,.$$

for all β such that $\beta(M-\varepsilon) < d$. By analogy with eq. (6.54) we conclude

$$\frac{p_\alpha(\beta,s) - p_\alpha(0,s)}{\beta} \geq M - \varepsilon\,. \tag{6.55}$$

Collectively, eq. (6.54) and eq. (6.55) provide the desired statement eq. (6.52).

We now investigate the derivative in eq. (6.52) regarding continuity. Again, let $\varepsilon > 0$. By eq. (6.49), rearranging the components in eq. (6.42) yields

$$\frac{\partial p_\alpha}{\partial \beta}(\beta,s) = \frac{u}{\beta}\left(1 - \frac{(u^2+\beta^2)^{3/2}}{(u^2+\beta^2)^{3/2}+\alpha\beta^2}\right)\,,$$

where $u = p_\alpha(\beta,s)$. We remark that the expression $s \mapsto s(s+\alpha\beta^2)^{-1}$ is increasing. Further, there is $C > 0$ such that $|u| \leq C\beta|s|$, see eq. (6.50). Together, this provides the inequality

$$\frac{(u^2+\beta^2)^{3/2}}{(u^2+\beta^2)^{3/2}+\alpha\beta^2} \leq \frac{(C^2|s|^2+1)^{3/2}\beta}{(C^2|s|^2+1)^{3/2}\beta+\alpha} \leq \varepsilon$$

for (β,s) in a neighborhood of $(0,s_0)$. From this together with eq. (6.54) and eq. (6.55) we obtain the estimate

$$(M-\varepsilon)(1-\varepsilon) \leq \frac{u}{\beta}(1-\varepsilon) \leq \frac{\partial p_\alpha}{\partial \beta}(\beta,s) \leq \frac{u}{\beta} \leq M + \varepsilon$$

for (β,s) in an appropriate neighborhood of $(0,s_0)$. Since ε was arbitrarily chosen, from this we obtain continuity of the derivative.

Finally, we consider the case $s_0 = \pm\alpha$. Since $K\sqrt{K^2+1}{-1} > 1$ for arbitrary

$K > 0$ there is $d > 0$ such that

$$\alpha \geq d + \alpha \frac{K}{\sqrt{K^2+1}} \geq \beta K + \alpha \frac{\beta K}{\sqrt{(\beta K)^2 + \beta^2}}$$

for every $\beta > 0$ such that $\beta K < d$. Again, by analogy with eq. (6.54) from this we conclude

$$\frac{p_\alpha(\beta, s_0) - p_\alpha(0, s_0)}{\beta} \geq K$$

and since K can be chosen arbitrarily large, we obtain the stated identity. □

Subsequently, we want to establish differentiability of the proximal operator P_α as in eq. (6.26) corresponding to the mapping φ_β as in eq. (6.11). This property is a necessary assumption in theorem 6.40. We remark that in the preceding investigations, we have already shown that all assumptions on the proximal mapping p_α as required in assumption 6.33 are actually met. This observation enables us to make the following statement.

Proposition 6.56. *Let φ_β be chosen as in eq.* (6.11) *and let $x \in \ell_2$ such that $x_k \neq \alpha$ for every $k \in \mathbb{N}$. Then, the corresponding proximal operator P_α as in eq.* (6.26) *is continuously differentiable at* $(0, x)$.

Proof. We show that the conditions in assumption 6.33 are satisfied for arbitrary $\beta \geq 0$. The case $\beta > 0$ can be dealt with by analogy with the proof of lemma 6.49. Note that in lemma 6.53, we have shown that the assumptions eq. (6.43) and eq. (6.44) hold true.

The case $\beta = 0$ was investigated in lemma 6.51, where we could re-establish the condition in eq. (6.29). Further, differentiability has been shown in lemma 6.55 and lemma 6.54. Note that the derivative in eq. (6.51) is uniformly bounded by 1. An application of lemma 6.36 now yields the assertion. □

Next, we investigate another prominent approximation of the absolute value function. Namely, for $\beta > 0$ we consider the mapping eq. (6.12) given by

$$\varphi_\beta(s) := |s| - \beta \log(1 + \beta^{-1}|s|)$$

and examine the corresponding proximal mapping p_α as in eq. (6.39) in regard to the conditions in assumption 6.33. Here, we follow the investigations of the mapping eq. (6.11). Accordingly, for $\beta > 0$ we consider the mapping w as in eq. (6.40), which is given by

$$w(\beta, u) = \frac{u}{\beta + |u|}\,. \tag{6.56}$$

Differentiating yields the partial derivatives

$$\frac{\partial w}{\partial \beta}(\beta, u) = -\frac{u}{(\beta + |u|)^2}\,, \quad \frac{\partial w}{\partial s}(\beta, u) = \frac{\beta}{(\beta + |u|)^2}\,. \tag{6.57}$$

Again, we first show that p_α is continuous.

Lemma 6.57. *The mapping p_α with φ_β as in eq.* (6.12) *is continuous at* $(0, s_0)$ *for arbitrary $s \in \mathbb{R}$.*

The proof of lemma 6.57 can be done by analogy with the proof of lemma 6.52. Next, we show that the assumptions in lemma 6.48 are satisfied.

Lemma 6.58. *The mapping φ_β given by eq.* (6.12) *satisfies the conditions in lemma 6.48 with $\rho(\beta) = \beta$ for arbitrary $C > 1$. Particularly, it holds*

$$\frac{|p_\alpha(\beta, s)|}{\beta} \le \frac{C}{\alpha}|s| \tag{6.58}$$

for every $\beta > 0$ and every s such that $|s| \le C - 1$.

Proof. By eq. (6.56), for arbitrary $\beta > 0$ and $s \in \mathbb{R}$ such that $|s| \le C - 1$ we conclude

$$w(\beta, \beta s) = \frac{\beta s}{\beta + |\beta s|} = \frac{s}{1 + s} \ge C^{-1} s\,,$$

which yields eq. (6.43) and, particularly, eq. (6.58), see lemma 6.48. Moreover, we observe that it holds

$$\left| \left(\frac{\partial w}{\partial u}(\beta, u) \right)^{-1} \frac{\partial w}{\partial \beta}(\beta, u) \right| = \frac{|u|}{\beta}\,,$$

see eq. (6.57), and, thus, eq. (6.44). □

Lemma 6.59. *The proximal mapping p_α with φ_β as in eq.* (6.12) *is continuously differentiable with respect to s at $(0, s_0)$ for any $s_0 \ne \alpha$ and it holds*

$$\lim_{s \to s_0} \frac{p_\alpha(0, s) - p_\alpha(0, s_0)}{s} = \begin{cases} 0\,, & |s_0| < \alpha \\ 1\,, & |s_0| > \alpha \end{cases}. \tag{6.59}$$

Lemma 6.60. *The proximal mapping p_α with φ_β as in eq.* (6.12) *is continuously* *differentiable with respect to* β *at* $(0, s_0)$ *for any* $s_0 \neq \alpha$ *and it holds*

$$\lim_{\beta \to 0} \frac{p_\alpha(\beta, s_0) - p_\alpha(0, s_0)}{\beta} = \begin{cases} s_0(\alpha - |s_0|)^{-1}, & |s_0| < \alpha \\ \pm\infty, & s_0 = \pm\alpha \\ -\operatorname{sign}(s_0)\alpha(\alpha - |s_0|)^{-1}, & |s_0| > \alpha \end{cases}.$$

The proof of lemma 6.59 and lemma 6.60 can be done by analogy with the proof of lemma 6.55.

Again, we could show that all conditions in assumption 6.33 on the proximal mapping p_α corresponding to φ_β as in eq. (6.12) are fulfilled. This we state the following result.

Proposition 6.61. *Let* φ_β *be chosen as in eq.* (6.12) *and let* $\tilde{x} \in \ell_2$ *such that* $\tilde{x}_k \neq \alpha$ *for every* $k \in \mathbb{N}$. *Then, the corresponding proximal operator* P_α *as in eq.* (6.26) *is continuously differentiable at* $(0, \tilde{x})$.

Proof. We show that the conditions in assumption 6.33 are satisfied for arbitrary $\beta \geq 0$. The case $\beta > 0$ can be dealt with by analogy with the proof of lemma 6.49. Note that in lemma 6.58, we have shown that the assumptions eq. (6.43) and eq. (6.44) hold true.

The case $\beta = 0$ was investigated in lemma 6.51, where we could re-establish the condition in eq. (6.29). Further, differentiability has been shown in lemma 6.55 and lemma 6.59. Note that the derivative in eq. (6.59) is uniformly bounded by 1. An application of lemma 6.36 now yields the assertion. □

Remark 6.62. The mapping eq. (6.10) is an example for an approximation of the absolute value function such that the conditions as in assumption 6.33 are not fulfilled. Particularly, the derivative of the mapping w is not continuous.

Remark 6.63. The mapping p_α with φ_β as in eq. (6.13) was investigated in [34]. This particular choice of φ_β is an example for which the conditions eq. (6.43) and eq. (6.44) in lemma 6.48 are not satisfied. Nevertheless the authors can establish the conditions as in assumption 6.33. Particularly, this demonstrates that lemma 6.48 provides a sufficient, but not a necessary condition.

6.4 Numerical evaluation

In the following, we investigate the numerical aspects of the strategy to approximate Tikhonov-type functionals discussed throughout this chapter. As above, we consider the operator $K = A \circ S$ as in eq. (2.12), where A is the Born approximation given in eq. (2.11) and S is the synthesis operator corresponding to the CDF

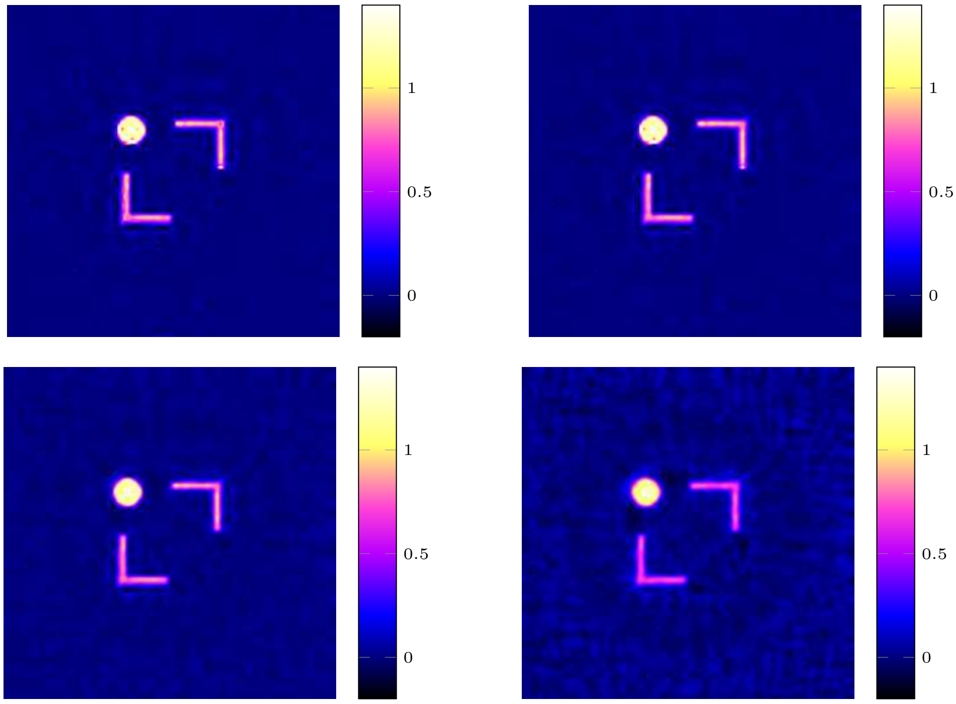


Figure 6.2: The real part of the reconstruction from noisy data with 10% of Gaussian noise according to eq. (6.60) with φ_β as in eq. (6.12) and $\beta = 0$, $\beta = 10^{-2}$, $\beta = 10^{-1}$ and $\beta = 1$ (top left to bottom right). One can observe that the reconstructions are virtually indistinguishable for small values of β.

wavelet coefficients. We present reconstructions according to the Tikhonov-type functional $\Psi_{\alpha,\beta} : \mathbb{C}^N \to \overline{\mathbb{R}}$ given by

$$\Psi_{\alpha,\beta} = \frac{1}{2}\|Kx - y^\delta\|_Y^2 + \alpha \sum_{k=1}^{N} \varphi_\beta(x_k) \tag{6.60}$$

with φ_β the approximation of the absolute value function as in eq. (6.12).

In fig. 6.2, the real part of the reconstructions according to eq. (6.60) with $\alpha = 10^{-6}$ are depicted. The minimization of eq. (6.60) for $\beta > 0$ was performed using the method SESOP-TN, see [62]. One can observe that the reconstruction corresponding to the choice $\beta = 0$, i.e. $\varphi_0 = \varphi = |\cdot|$, is accurately approximated for small values of β. This observation is expected due to the results in theorems 6.26 and 6.40. In fact, the reconstructions are visually indistinguishable if β is small enough, which in this case means $\beta < 10^{-2}$. While we provide this numerical result only for the particular choice of φ_β as in eq. (6.12), we remark that this behavior is typical and can be reproduced for any approximation listed in section 6.2.3.

One major conclusion from the above observation is that it is not essential to use the true regularization term or a most accurate approximation. While the reconstructions according to the approximated functionals are no longer sparse, in topographic applications and the like we are not interested in a sparse set of coefficients but in visually appealing reconstructions. Thus, it is feasible to resort to some coarse approximation without loss. This is particularly promising if one may expect numerical benefits from this replacement.

CONCLUSION

In this thesis, we have evaluated the application of sparsity-promoting reconstruction methods to the inverse problem using the linearized model given by the Born approximation of the scattering process. We have provided evidence that the Born approximation can be used to obtain meaningful reconstructions from both synthetic and even measurement data using different ranges of frequencies. Particularly, we have demonstrated that sparsity constraints provide enhanced reconstructions when compared to classical approaches, particularly regarding high-frequency incident fields.

In our investigations, we considered both an iterative and a variational approach for sparse reconstruction. Both methods performed similar in terms of quality of the reconstruction. We considered the Born approximation, which can be regarded as a prototypical example for an ill-posed and non-injective operator. One may assume that the methods evaluated in our numerical experiments perform similarly when applied to other imaging applications. Further research is necessary whether this assumption is valid.

In chapters 3 and 4, we investigated the Bregman projection in general and, particularly, a special instance of particular interest. We demonstrated that the according projection onto intersections of hyperplanes is sparse and, moreover, provided a simple geometrical interpretation that supports this result. Based on this observation, we developed a novel iteration scheme and provided a proof for the regularizing properties of the method. The method could be shown to outperform existing approaches when applied to the linear problem based on the Born approximation of the scattering process. An interesting question is how the method performs when applied to other, possibly nonlinear operators.

Regarding a variational approach, in chapters 5 and 6, we considered a Tikhonov-type regularization scheme incorporating sparsity constraints. We introduced a novel parameter choice rule based on the L-curve criterion that performed well in numerical applications. While the method was introduced for sparse reconstruction, the presented iteration for the determination of the kink of the L-curve is independent of this application and could be applied to other situations where

L-shaped curves are involved.

Finally, we discussed the approach of replacing the Tikhonov-type functional by some smooth approximation and provided a stability result regarding the minimizers of the according problems. We have developed a general framework to deal with this type of question which particularly includes the setting of separable regularization terms. Further, we even provided convergence rates with respect to the approximation index. The approach was evaluated numerically. In imaging applications where we do not seek for a sparse set of coefficients but for visually appealing reconstructions, the approach has proven useful as it was shown to give rise to visually indistinguishable reconstructions. At the same time, minimization of the cost functional is more flexible. Further research could follow the ambition to evaluate the numerical benefits of this approach in more detail.

This work underpins the belief that sparsity-promoting methods contribute to high resolution reconstructions. Thus, as a final concluding remark, we encourage the usage of sparsity-promoting methods when considering imaging or tomographic applications such as the inverse medium problem.

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