Chain Complexes over Principal Ideal Domains

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Phantasie ist wichtiger als Wissen,
denn Wissen ist begrenzt.
(Albert Einstein)

Introduction

Originally, chain complexes arise in algebraic topology to characterise topological spaces by special algebraic invariants called homology groups. Forgetting about underlying spaces, chain complexes are studied abstractly in homological algebra as an algebraic structure.

Given a topological space X, one always obtains a chain complex and so called *singular homology groups* $H_n(X)$ as described in Hatcher (2008). A special case are geometrical simplicial complexes (or more general Δ-complexes) for which a more primitive homology theory exists called *simplicial homology* which yields the same homology groups. Furthermore, a particular chain complex called *cellular* exists for any CW-complex, and its cellular homology groups correspond to the singular homology groups (cf. Hatcher, 2008).

A geometrical simplicial complex consists of simplices and can be built up out of these in a glueing process. If all simplices are glued together in a special way, a simplicial complex is said to be shellable. It is a particular property of shellable simplicial complexes that their complete homological information is well-known (cf. Kozlov, 2008).

The notion of shellability is more generally defined for cell complexes and seems to arise at first in articles by Sanderson (1957) and Rudin (1958) although similar ideas seem to be even older (cf. Bing, 1951, for example). In the beginning, shellability was defined only for pure complexes, but this concept has been generalised to nonpure complexes by Björner and Wachs (1996). Shellability has developed to be an important tool in combinatorics, topology and geometry. For example, the *Upper Bound Conjecture for convex polytopes*[1]

[1]By McMullen and Stanley, the Upper Bound Conjecture was formulated by Motzkin at the November meeting of the American Mathematical Society in Evanston in 1956.

- "Motzkin ... conjectured (implicitly) that if P is any d-dimensional convex polytope with n vertices and f_i i-dimensional faces, then $f_i \leq c_i(n,d)$. ... This is the content of Motzkin's conjecture, known as the *upper bound conjecture* (UBC) *for convex polytopes.*" (cf. Stanley, 1975, p. 136f.),
- "T. S. Motzkin ... formulated what has come to be known as *The Upper Bound Conjecture*." (cf. McMullen, 1970, p. 179),

by Motzkin has been proven[2] by McMullen (1970) using a later published result by Bruggesser and Mani (1971) that the boundary complex of a convex polytope is shellable.

Since simplicial complexes and CW-complexes yield different chain complexes, one may ask whether it is possible to define shellability for abstract chain complexes. Because the homology of shellable simplicial complexes is well-known, one may also ask what special properties a chain complex might have if it comes from a shellable simplicial complex, or, more generally, which properties must be fulfilled by a chain complex to get complete information about its homology.

We are concerned with these questions in the second part of this thesis. It turns out that our notion of shellability for chain complexes does not suffice to determine homology completely, so we specialise shellable chain complexes by further conditions inspired by simplicial complexes.

The first part of this thesis deals with mapping cones of chain complexes and is also motivated by simplicial complexes. Given some simplicial complex Δ, a cone over Δ is obtained by adding a new vertex and extra simplices containing this new vertex. This yields a new simplicial complex which corresponds to a mapping cone over the chain complex C_Δ belonging to Δ. We generalise the notion of a cone to chain complexes abandoning the geometrical idea of an apex and compare it with mapping cones. In particular, we use mapping cones to construct a cone over a given chain complex.

The text comprises five chapters. In Chapter 1 the basic concepts like simplicial complexes, chain complexes and homology are introduced. In particular, homology of shellable simplicial complexes is discussed. At the end of this chapter, we determine the homology module $H_d(C)$ of a finite chain complex C of order d.

The Chapters 2 and 3 contain most of my own results. Chapter 2 deals with acyclic chain complexes, cones and mapping cones. At first, an abstract definition of a cone is given, but mostly, we are concerned with mapping cones which will be used to construct a cone over a given chain complex.

- "In 1957, in an abstract published in the Bulletin of the American Mathematical Society, T. S. Motzkin made the following conjecture. . . . The Upper Bound Conjecture (U.B.C.)." (cf. McMullen and Shephard, 1971, p. 152).

All abstracts of this November meeting are collected by Youngs (1957).

[2] The proof is also published in a book about convex polytopes by McMullen and Shephard (1971). The Upper Bound Conjecture for spheres has been proven by Stanley (1975).

Finally, it turns out that not every mapping cone is a cone matching our definition, but conversely there also exist cones which cannot be obtained as a mapping cone. But it is possible to name certain conditions on which a mapping cone is a cone, see Theorem 2.17.

In Chapter 3 we are concerned with shellable chain complexes. In particular, the existence of a special shelling and shellability of i-skeletons are shown. But in general, there is no information about the homology of shellable chain complexes, therefore we claim additional conditions on them which imitate other properties of simplicial complexes. This leads to the notion of regular and totally regular chain complexes. We obtain complete homological information for totally regular chain complexes which have a specific augmentation map ϵ, this result is noted in Theorem 3.30. In the end, we consider mapping cones over shellable or regular chain complexes. It turns out that the mapping cone over a shellable (or regular) chain complex is shellable (or regular, respectively), too, see Theorem 3.34 and Theorem 3.36. A similar result for totally regular chain complexes follows without further work as a corollary.

A short conclusion of all our results is given in Chapter 4. The following Appendix (Chapter A) contains a short overview about CW-complexes.

Some of the results are joint work with Björn Walker. Primarily, these are the notion of critical basis elements and the proof of Theorem 1.60 as well as the notion of an abstract cone for chain complexes and the proof of Lemma 2.6. Together with most of the content of Chapter 3 these results have been published in Grenzebach and Walker (2014).

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Symbols and Abbreviations

We use the following terms:

$\mathbb{N}$:	natural numbers $0, 1, 2, \ldots$
$\mathbb{Z}$:	integer numbers
$\mathbb{R}$:	real numbers
$\mathbb{F}_q$:	finite field with q elements
$\mathbb{Z}_n$:	the integer numbers mod n
$\emptyset$:	empty set
id:	the identity map from a set to itself
$\cong$:	isomorphic
$\simeq$:	homotopy equivalent
${}^{\#}A$:	number of elements in a finite set A
$A \subseteq B$ or $A \supseteq B$:	set-theoretical containment, not necessarily proper
$A \setminus B$:	relative complement of a set B in a set A
$A \cap B$:	intersection of sets A and B
$A \cup B$:	union of sets A and B
$A \mathbin{\dot{\cup}} B$:	disjoint union of sets A and B
$A \oplus B$:	direct sum of modules A and B
$A \vee B$:	wedge sum of topological spaces A and B
$\langle e_1, \ldots, e_m \rangle$:	module generated by the elements $e_1, \ldots, e_m$
$\langle A \rangle$:	module generated by all elements in a set A

1. Basics: Simplicial and Chain Complexes

In this chapter we will introduce chain complexes. However, we will explain simplicial complexes first since our approach to chain complexes is motivated by them.

1.1. Simplicial Complexes

We introduce some background knowledge about geometric simplicial complexes. For more detailed informations, we refer to the books by Maunder (1970, Section 2.3), Hatcher (2008, Section 2.1), Kozlov (2008, Section 2.2), Spanier (1966, Section 3.1) and Mac Lane (1975, Chapter II.7).

1.1.1. Simplices

Recall that $(n+1)$ points p_0, p_1, $\ldots$, p_n in $\mathbb{R}^m$ for some $m \geq n$ are said to be *affinely independent* if the vectors $p_1 - p_0$, $p_2 - p_0$, $\ldots$, $p_n - p_0$ are linearly independent.

Definition 1.1. A *geometric n-simplex* is the convex hull of $(n+1)$ affinely independent points p_0, p_1, $\ldots$, p_n in $\mathbb{R}^m$ for some $m \geq n$. The points p_0, p_1, $\ldots$, p_n are called *vertices*. The convex hulls of the subsets of $\{p_0, p_1, \ldots, p_n\}$ are called *subsimplices* or *faces* of the n-simplex.

The number n is the *dimension* of the n-simplex. If $n \geq 1$, subsimplices of dimension $(n-1)$ are called *facets*.

Remark 1.2. Because $\emptyset \subseteq \{p_0, p_1, \ldots, p_n\}$, any simplex contains the *empty simplex* as a subsimplex. We denote it by $\emptyset$ and set $\dim(\emptyset) := -1$ for its dimension.

Basic examples for simplices are points, lines, triangles and tetrahedra (simplices of dimension 0, 1, 2 and 3, see Figure 1.1).

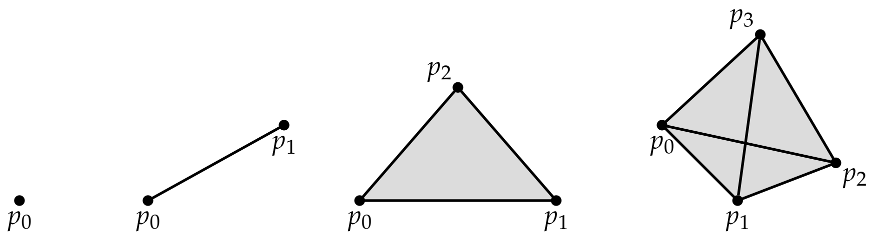


Figure 1.1.: Simplices of dimension 0, 1, 2 and 3 (point, line, triangle and tetrahedron)

If we take the convex hull of the standard unit vectors in $\mathbb{R}^{n+1}$, we get a unique simplex of dimension $n \in \mathbb{N}$, the standard n-simplex:

$$\Delta^n := \left\{ (t_0, t_1, \ldots, t_n) \in \mathbb{R}^{n+1} \;\middle|\; \sum_{k=0}^{n} t_k = 1 \text{ and } t_k \geq 0 \text{ for all } k \right\}.$$

The standard 1-simplex and the standard 2-simplex are shown in Figure 1.2.

Definition 1.3. Let $A := \{p_0, p_1, \ldots, p_n\}$ be the set of vertices of a geometric n-simplex. If A is totally ordered, we call the n-simplex *ordered*.

If $p_0 < p_1 < \cdots < p_n$ with respect to a total ordering $\leq$ of A, we denote the n-simplex by $[p_0, \ldots, p_n]$. The facet of A which does not contain the vertex p_i is denoted by $[p_0, \ldots, \hat{p}_i, \ldots, p_n]$.

Remark 1.4. For example, the indices of the vertices p_i yield a total ordering on a set $A := \{p_0, p_1, \ldots, p_n\}$.

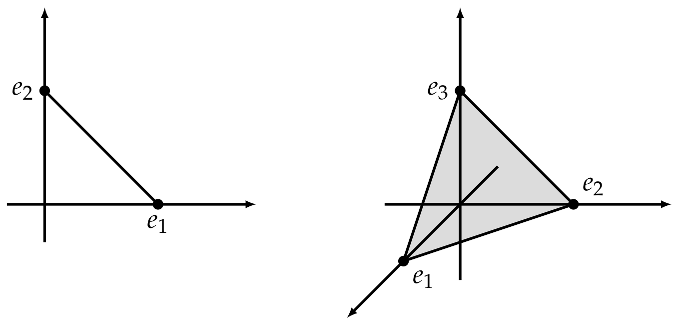


Figure 1.2.: Standard 1-simplex and standard 2-simplex

1.1.2. Geometric Simplicial Complexes

Definition 1.5. A *geometric simplicial complex* Δ is a finite set of simplices in $\mathbb{R}^m$ for some $m \geq 0$ such that

1. every subsimplex of a simplex in Δ is itself a simplex in Δ,
2. the intersection of any two simplices of Δ is a subsimplex of each of them.

For brevity, we will sometimes skip the notion *geometric* and just say *simplicial complex* instead. Examples for simplicial complexes are shown in Figure 1.3.

Definition 1.6. Let Δ be a simplicial complex. A *subcomplex* is a subset Λ of Δ such that all subsimplices of any simplex in Λ are also contained in Λ.

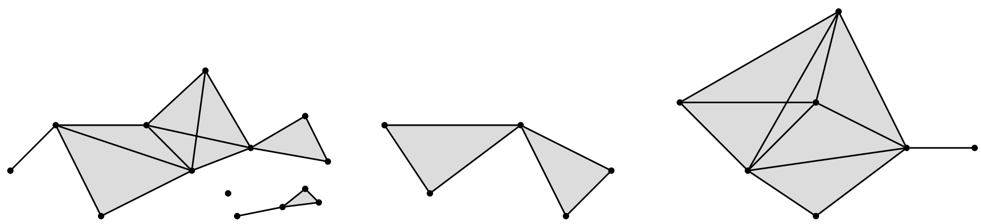
Remark 1.7. By Aleksandrov (1965, p. 152), a simplicial complex is *connected* if it is not the union of two nonempty disjoint subcomplexes. An example for a not connected subcomplex is given in Figure 1.3(a).

Definition 1.8. The *dimension* of a simplicial complex Δ is the maximum of the dimensions of its simplices.

Definition 1.9. A simplex in a simplicial complex Δ is called *maximal* if it is not a subsimplex of any other simplex in Δ.

A simplicial complex is called *pure* if all of its maximal simplices have the same dimension.

An example for a pure simplicial complex is shown in Figure 1.3(b). To determine the dimension of any simplicial complex, it suffices to consider all its maximal simplices.



(a) not connected (b) pure (c) shellable

Figure 1.3.: Examples for simplicial complexes

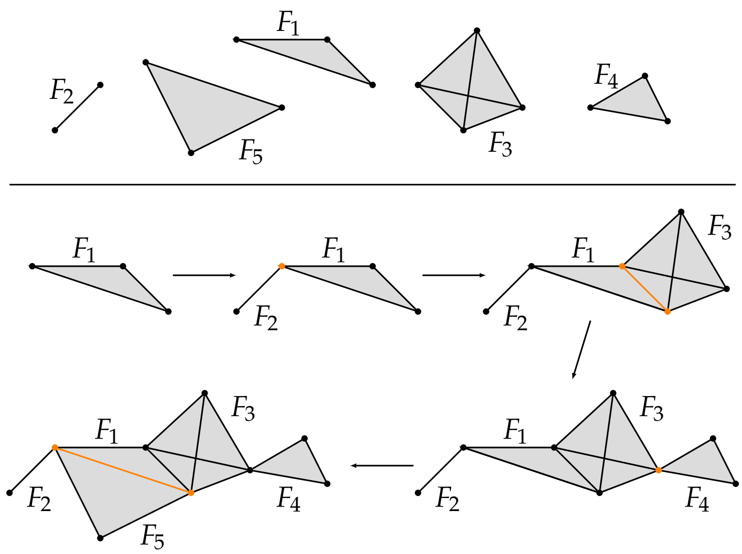


Figure 1.4.: Building up a simplicial complex by glueing simplices. The chosen simplices F_1, F_2, F_3, F_4 and F_5 are shown at the top. Beneath the glueing process is illustrated. For $2 \leq k \leq 5$, any intersection $\left(\cup_{i=1}^{k-1} F_i\right) \cap F_k$ is coloured orange.

One can imagine that a connected geometric simplicial complex can be built up from its maximal simplices by glueing them together. Mathematically, glueing simplices means that equal dimensional proper subsimplices of them are identified.

To build up a geometric simplicial complex in this way, we take t geometric simplices $F_1, F_2, \ldots, F_t$ of arbitrary dimensions which will be the maximal simplices in the constructed complex.

We start with F_1, then we add F_2 by identifying some proper subsimplex of F_2 with a proper subsimplex of F_1 of the same dimension. We proceed in this way, in which any intersection $\left(\bigcup_{i=1}^{k-1} F_i\right) \cap F_k$ must be a simplicial complex. An example for this procedure is shown in Figure 1.4.

If in this process every simplex F_k is glued only along facets, i.e. subsimplices of dimension $(\dim F_k - 1)$, we get a special simplicial complex (cf. Kozlov, 2008, p. 211).

Definition 1.10. Let Δ be a simplicial complex of dimension d. An order of its maximal simplices $F_1, F_2, \ldots, F_t$ is called a *shelling* (or a *shelling order*) if $d = 0$ or if each subcomplex $\left(\bigcup_{i=1}^{k-1} F_i\right) \cap F_k$ is pure and $(\dim F_k - 1)$-dimensional for all $k \in \{2, \ldots, t\}$. If such a shelling exists, the simplicial complex is said to be *shellable*.

Remark 1.11. Every shellable simplicial complex without maximal simplices of dimension 0 is connected. Figure 1.3(c) on page 3 shows an example.

Example 1.12.

1. Every simplex is a shellable simplicial complex with exactly one maximal simplex.

2. Let $[p_0, \ldots, p_n]$ be an ordered n-simplex. The simplicial complex whose maximal simplices are all its facets $[p_0, \ldots, \hat{p}_i, \ldots, p_n] =: \Delta_i$ is shellable because the intersection of any two facets Δ_i and Δ_j is the subsimplex $[p_0, \ldots, \hat{p}_i, \ldots, \hat{p}_j, \ldots, p_n]$ of dimension $(n-2)$. In fact, every ordering of a simplex' facets is a shelling.

For later use we introduce the following notions.

Definition 1.13. A geometric simplicial complex is *ordered* if its set of vertices, i.e. its set of 0-simplices, is totally ordered.

Definition 1.14. Let Δ be a shellable simplicial complex whose maximal simplices F_1, F_2, $\ldots$, F_t are ordered in a in shelling. A maximal simplex F_ℓ is called a *spanning simplex* (with respect to the chosen shelling) if $\dim(F_\ell) = 0$ or if the simplex F_ℓ is glued along all its facets.

Remark 1.15. The ordering of the maximal basis elements in a shelling can always be rearranged in such a way that all spanning simplices come last (cf. Björner and Wachs, 1996, Second Rearrangement Lemma 2.7).

Remark 1.16. We obtain an ordered simplicial complex from any geometric simplicial complex by numbering its vertices.

1.2. Chain Complexes

1.2.1. Preliminaries about Rings and Free Modules

In this section, we introduce some basic terms which are essential for our further work. We refer to the books by Bosch (2004), Lang (2002) and especially by Oeljeklaus and Remmert (1974).

Any ring R is implicitly supposed to be commutative and unital, the unit will be denoted by 1. A principal ideal domain is a commutative and unital ring which is an integral domain and whose ideals are all principal.

Recall that a module M over R is free if there exists a family $(x_i)_{i\in I}$ in M which is linearly independent and generates the module M, i.e. $M = \bigoplus_{i\in I} Rx_i$. Such a family $(x_i)_{i\in I}$ is called a basis of M. Free modules over principal ideal domains have the following property (cf. Lang, 2002, p. 146).

Theorem 1.17. *If M is a free module over a principal ideal domain R, then every submodule of M is free.*

For finitely generated modules, i.e. modules of the form $M = \sum_{i=1}^{n} Re_i$ for some $n \in \mathbb{N}$, we recall the following terms by Oeljeklaus and Remmert (1974, pp. 90 and 110):

- the *generating number* $\operatorname{gen}_R M$ is the minimal number of elements which generate M,
- the *degree of freedom* $\operatorname{dgf}_R M$ is the maximal number of linearly independent elements in M.

The next two theorems can be found in Oeljeklaus and Remmert (1974, pp. 112 and 115) and will be helpful for our calculations in Section 1.6.

Theorem 1.18. *Let R be an integral domain. Let M and N be finitely generated R-modules. For any R-linear map $\varphi\colon M \to N$ holds:*

$$\operatorname{dgf}_R(M) = \operatorname{dgf}_R(\ker\varphi) + \operatorname{dgf}_R(\operatorname{im}\varphi).$$

Theorem 1.19. *Let R be an integral domain. If M is a finitely generated free R-module, then $\operatorname{gen}_R M = \operatorname{dgf}_R M$.*

1.2.2. Main Concepts

We start by defining chain complexes which are our central term. The definition bases on work by Weibel (1994, p. 2), Hilton and Stammbach (1971, p. 117), Kozlov (2008, p. 51), Cartan and Eilenberg (1956, p. 58) and Mac Lane (1975, p. 39).

Definition 1.20. For every $\nu \in \mathbb{Z}$ let C_ν be a module over a ring R. Let

$$\partial_\nu \colon C_\nu \to C_{\nu-1}, \quad \nu \in \mathbb{Z},$$

be R-linear maps such that $\partial_\nu \circ \partial_{\nu+1} = 0$, i.e. $\operatorname{im} \partial_{\nu+1} \subseteq \ker \partial_\nu$. The sequence

$$\dots \xrightarrow{\partial_{i+2}} C_{i+1} \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} C_{-1} \xrightarrow{\partial_{-1}} \dots$$

is a *chain complex* which we denote by C. The R-modules C_ν are called *chain modules*, and the maps ∂_ν are *boundary maps*.

Similar definitions using abelian groups instead of R-modules can be found in the books by Dold (1972, p. 16), Gelfand and Manin (1996, p. 25), Hatcher (2008, p. 106), Massey (1991, p. 254), Maunder (1970, p. 107) and Spanier (1966, p. 157).

Remark 1.21. A chain complex C over R can be seen as a $\mathbb{Z}$-graded R-module with an R-linear map $d \colon C \to C$ such that $d(C_\nu) \subseteq C_{\nu-1}$ and $d \circ d = 0$ (cf. Cartan and Eilenberg, 1956, p. 58).

Remark 1.22. The elements of a chain module C_ν are often called *ν-chains*, see Hilton and Stammbach (1971, p. 118), for example.

Now we characterise some special chain complexes, in parts we follow Spanier (1966, p. 157).

Definition 1.23. Let C be a chain complex over R with chain modules C_ν, $\nu \in \mathbb{Z}$. If every chain module C_ν is free over R with basis Ω_ν, then C is a *free chain complex*. We call $\Omega := \dot{\bigcup}_{\nu\in\mathbb{Z}} \Omega_\nu$ its basis and denote such a chain complex by (C, Ω). If additionally each chain module C_ν is zero or finitely generated, we call (C, Ω) *finitely free*.

Remark 1.24. Sometimes, we will denote a free chain complex (C, Ω) shortly by C, keeping its basis Ω in mind.

Remark 1.25. In the following, we will mostly consider free chain complexes (C, Ω) over a principal ideal domain R with a fixed basis $\Omega := \dot{\bigcup}_{\nu\in\mathbb{Z}} \Omega_\nu$. Then the submodules $\ker(\partial_\nu)$ and $\operatorname{im}(\partial_\nu)$ are free for all $\nu \in \mathbb{Z}$ by Theorem 1.17.

Definition 1.26. Let C be a chain complex with chain modules C_ν, $\nu \in \mathbb{Z}$. If $C_\nu = 0$ for all $\nu < 0$, the chain complex is said to be *nonnegative*[1].

[1] A nonnegative chain complex is called *positive* by Hilton and Stammbach (1971, p. 126), Massey (1991, p. 288) and Mac Lane (1975, p. 41); Cartan and Eilenberg (1956, p. 75) call it *left positive*.

Definition 1.27. Let (C, Ω) be a finitely free and nonnegative chain complex. If $C_d \neq 0$ for some $d \in \mathbb{N}$ and $C_\nu = 0$ for all $\nu > d$, we say that (C, Ω) is *finite of order d*. We write $\operatorname{ord}(C, \Omega) = d$ or shortly $\operatorname{ord}(C) = d$. The zero complex $(Z, \emptyset)$ is said to be finite of order $-1 =: \operatorname{ord}(Z, \emptyset)$.

There exist special maps between chain complexes over R which respect their structure. This term is standard in literature and can, for example, be found in the books by Cartan and Eilenberg (1956, p. 59), Dold (1972, p. 16), Eilenberg and Steenrod (1981, p. 124), Gelfand and Manin (1996, p. 41), Hatcher (2008, p. 111), Hilton and Stammbach (1971, p. 117), Kozlov (2008, p. 52), Massey (1991, p. 255), Maunder (1970, p. 109), Spanier (1966, p. 158) or Weibel (1994, p. 2).

Definition 1.28. Let B, C be chain complexes over R with boundary maps ∂_ν^B and ∂_ν^C, respectively. A *chain map* $f\colon B \to C$ is a collection of R-linear maps $f_\nu\colon B_\nu \to C_\nu$ such that $f_{\nu-1} \circ \partial_\nu^B = \partial_\nu^C \circ f_\nu$ for all $\nu \in \mathbb{Z}$.

A chain map $f\colon B \to C$ is called a *chain isomorphism* if each linear map $f_\nu\colon B_\nu \to C_\nu$ is an isomorphism. Then the chain complexes B and C are said to be *isomorphic*.

For example, let (C, Ω) be a free chain complex. If we permute the basis elements in some basis Ω_ν, we obtain a chain isomorphism $(C, \Omega) \to (C, \widetilde{\Omega})$. In fact, we will never change the basis of any free chain complex (C, Ω) except for such permutations.

Remark 1.29. Taking all chain complexes over some ring R as objects and all chain maps between them as morphisms gives a *category of chain complexes*, cf. Hilton and Stammbach (1971, p. 118) or Spanier (1966, p. 158).

1.2.3. Pure Chain Complexes

Definition 1.30. Let C_ν be a free chain module with basis $\Omega_\nu = \{e_i^\nu \mid i \in I_\nu\}$, $\nu \in \mathbb{Z}$, in a free chain complex (C, Ω) over some ring R. For any element $x = \sum_{i \in I_\nu} a_i e_i^\nu \in C_\nu$, the *support* of x is the set of all basis elements e_i^ν with coefficient $a_i \neq 0$, i.e.

$$\operatorname{supp}(x) := \{e_i^\nu \mid a_i \neq 0\} \subseteq \Omega_\nu.$$

The *boundary* of x is the support of $\partial_\nu(x)$:

$$\operatorname{bd}(x) := \operatorname{supp}\big(\partial_\nu(x)\big) \subseteq \Omega_{\nu-1}.$$

Remark 1.31. Since the module C_ν is free, we have $C_\nu = \bigoplus_{i \in I_\nu} Re_i^\nu$. Therefore, for any element $x = \sum_{i \in I_\nu} a_i e_i^\nu \in C_\nu$, almost every coefficient a_i vanishes. Hence, for any element in a free chain complex, support and boundary are *finite sets*.

Remark 1.32. For any $x \in C_\nu$ the following equivalences hold:

$$\begin{aligned} \operatorname{supp}(x) = \emptyset &\iff x = 0, \\ \operatorname{bd}(x) = \emptyset &\iff x \in \ker(\partial_\nu). \end{aligned}$$

Definition 1.33. Let (C, Ω) be a finite chain complex of order d whose chain modules C_ν have the bases $\Omega_\nu = \{e_1^\nu, \dots, e_{k_\nu}^\nu\}$ for $k_\nu \geq 1$ or $\Omega_\nu = \emptyset$. The chain complex (C, Ω) is called *pure* if each basis element $e_i^\nu \in \Omega$ for $0 \leq \nu \leq (d-1)$ and $1 \leq i \leq k_\nu$ is contained in the boundary of some basis element $e_\ell^{\nu+1}$, $1 \leq \ell \leq k_{\nu+1}$.

Definition 1.34. A basis element $e \in \Omega$ of a free chain complex (C, Ω) is called *maximal* if it is not contained in the boundary of any other basis element.

Remark 1.35. If (C, Ω) is a finite chain complex of order d, all basis elements of Ω_d are maximal. Furthermore, if (C, Ω) is even a pure chain complex, the maximal basis elements are exactly those in Ω_d.

1.2.4. Subcomplexes

For the definition of subcomplexes, we refer to Weibel (1994, p. 6).

Definition 1.36. Let C be a chain complex. A chain complex B is a *subcomplex* of C if each chain module B_ν is a submodule of C_ν and each boundary map $\delta_\nu \colon B_\nu \to B_{\nu-1}$ is the restriction of the boundary map $\partial_\nu \colon C_\nu \to C_{\nu-1}$ to B_ν.

Remark 1.37. Every subcomplex B of a free chain complex (C, Ω) over a principal ideal domain is also free according to Theorem 1.17.

If U, W are subcomplexes of a chain complex C, then their intersection $U \cap W$, whose chain modules are $(U \cap W)_\nu := U_\nu \cap W_\nu$, is also a subcomplex of C. Another subcomplex of C, obtained from U and W as well, is their sum $U + W$, having the chain modules $(U + W)_\nu := U_\nu + W_\nu$.

Every chain complex C contains the following subcomplexes:

- the zero complex Z with all chain modules $Z_\nu = 0$,
- the kernel complex $\ker(\partial)$ with chain modules $\big(\ker(\partial)\big)_\nu = \ker(\partial_\nu)$,
- the image complex $\operatorname{im}(\partial)$ with chain modules $\big(\operatorname{im}(\partial)\big)_\nu = \operatorname{im}(\partial_{\nu+1})$.

In all cases, the restriction of the boundary map to the subcomplex is zero. Furthermore, we get a sequence of subcomplexes

$$Z \subseteq \operatorname{im}(\partial) \subseteq \ker(\partial) \subseteq C.$$

We are interested in some particular subcomplexes of nonnegative chain complexes.

Definition 1.38. Let C be a nonnegative chain complex. For $i \in \mathbb{N}$, the *i-skeleton* $\operatorname{sk}_i(C)$ is a subcomplex of C whose chain modules are $\big(\operatorname{sk}_i(C)\big)_\nu = C_\nu$ for $0 \leq \nu \leq i$ and $\big(\operatorname{sk}_i(C)\big)_\nu = 0$ otherwise. For a free and nonnegative chain complex (C, Ω), the i-skeleton is denoted by $\operatorname{sk}_i(C, \Omega)$.

Remark 1.39. If (C, Ω) is a free and nonnegative chain complex, each i-skeleton $\operatorname{sk}_i(C, \Omega)$ is free with basis $\dot{\bigcup}_{\nu=0}^{i} \Omega_\nu$. If (C, Ω) is even finitely free and nonnegative with $C_i \neq 0$, then $\operatorname{sk}_i(C, \Omega)$ is a finite chain complex of order i. Particularly, if (C, Ω) is a finite chain complex of order d, then the d-skeleton $\operatorname{sk}_d(C, \Omega)$ is equal to (C, Ω). Moreover, for any pure finite chain complex (C, Ω) of order d with $C_i \neq 0$, its i-skeleton $\operatorname{sk}_i(C, \Omega)$ is pure, too.

Now we introduce subcomplexes which are analogues to simplices in a simplicial complex.

Definition 1.40. Let (C, Ω) be a free and nonnegative chain complex with basis Ω. For any $\mu \in \mathbb{N}$ and a basis element $e_i^\mu \in \Omega_\mu$, let $(C_{e_i^\mu}, \Omega_{e_i^\mu})$ denote the free, nonnegative subcomplex of (C, Ω) whose chain modules $\big(C_{e_i^\mu}\big)_\nu$ have the following bases $\big(\Omega_{e_i^\mu}\big)_\nu$:

$$\begin{aligned}
\big(\Omega_{e_i^\mu}\big)_\nu &= \emptyset \qquad \text{for } \nu \geq \mu + 1 \text{ and } \nu < 0,\\
\big(\Omega_{e_i^\mu}\big)_\mu &= \{e_i^\mu\},\\
\big(\Omega_{e_i^\mu}\big)_\nu &= \bigcup_{e \,\in\, (\Omega_{e_i^\mu})_{\nu+1}} \operatorname{bd}(e) \qquad \text{for } 0 \leq \nu \leq \mu - 1.
\end{aligned}$$

Such a subcomplex $(C_{e_i^\mu}, \Omega_{e_i^\mu})$ is called *elementary*.

Remark 1.41. Each elementary subcomplex $(C_{e_i^\mu}, \Omega_{e_i^\mu})$ is a pure and finite chain complex of order μ. In particular, $\left(\Omega_{e_i^\mu}\right)_{\mu-1} = \mathrm{bd}(e_i^\mu)$. We define the *order* of e_i^μ as the order of the finite subcomplex $(C_{e_i^\mu}, \Omega_{e_i^\mu})$, i.e.

$$\mathrm{ord}(e_i^\mu) := \mathrm{ord}(C_{e_i^\mu}, \Omega_{e_i^\mu}) = \mu.$$

If there is some basis element $e_k^\lambda \in \Omega_\lambda$ for which $e_k^\lambda \in \Omega_{e_i^\mu} \cap \Omega_{e_\ell^\kappa}$ holds for some $\lambda < \min\{\mu, \kappa\}$, this implies $\Omega_{e_k^\lambda} \subseteq \Omega_{e_i^\mu} \cap \Omega_{e_\ell^\kappa}$, i.e. the elementary subcomplex $(C_{e_k^\lambda}, \Omega_{e_k^\lambda})$ is contained in the chain complex generated by $\Omega_{e_i^\mu} \cap \Omega_{e_\ell^\kappa}$.

1.2.5. Chain Complexes obtained from Simplicial Complexes

There is a standard way to construct a nonnegative chain complex over some ring from an ordered simplicial complex (cf. Hatcher, 2008, pp. 104–106). Let Δ be a simplicial complex of dimension n whose set of vertices $\{v_0, v_1, \dots, v_k\}$ is ordered such that $v_\ell < v_{\ell+1}$ for all $0 \leq \ell \leq (k-1)$. This simplicial complex generates a finite chain complex (C, Ω) of order n as follows.

For every ν-simplex $[v_{i_0}, v_{i_1}, \dots, v_{i_\nu}]$ with $0 \leq i_0 < i_1 < \cdots < i_\nu \leq k$, there is exactly one basis element $e_{i_0 i_1 \dots i_\nu}$ in the basis Ω_ν of the chain module C_ν. Hence, there are as many basis elements in Ω_ν as ν-simplices in Δ. In particular, every basis element in Ω_n corresponds to a simplex of dimension n in Δ, and the chain module C_0 is generated by $(k+1)$ elements $e_0, \dots, e_k$. Furthermore, we have $C_\nu = 0$ for all $\nu > n$ and all $\nu < 0$.

A boundary map $\partial_\nu \colon C_\nu \to C_{\nu-1}$ for $1 \leq \nu \leq n$ is given by

$$e_{i_0 i_1 \dots i_\nu} \mapsto \sum_{\ell=0}^{\nu} (-1)^\ell e_{i_0 \dots i_{\ell-1} i_{\ell+1} \dots i_\nu}$$

whereas $\partial_0(e_i) = 0$ for all $i \in \{0, 1, \dots, k\}$. For $\nu > n$ and $\nu \leq 0$, each boundary map ∂_ν is the zero map. It is easy to show that $\partial_{\nu-1} \circ \partial_\nu = 0$ for all $\nu \in \mathbb{Z}$ (cf. Hatcher, 2008, p. 105).

1.3. Homology

The notion of homology is central in homological algebra. We follow Gelfand and Manin (1996, p. 25) and Hilton and Stammbach (1971, p. 118).

Definition 1.42. Let C be a chain complex over some ring R with boundary maps $\partial_\nu \colon C_\nu \to C_{\nu-1}$. The *homology modules* of C are

$$H_\nu(C) := {}^{\ker(\partial_\nu)}\!\big/_{\operatorname{im}(\partial_{\nu+1})} \qquad \text{for all } \nu \in \mathbb{Z},$$

and their elements are called *homology classes*.

Remark 1.43. In literature, the homology modules are often called *homology groups*. This name usually occurs when chain complexes are defined with abelian groups instead of R-modules.

Remark 1.44. If (C, Ω) is a free chain complex over a principal ideal domain R, then $\ker(\partial_\nu)$ and $\operatorname{im}(\partial_\nu)$ are free R-modules for all $\nu \in \mathbb{Z}$. However, the homology modules $H_\nu(C)$ are not necessarily free and may contain torsion elements.

By Hilton and Stammbach (1971, p.18), elements of $\ker(\partial_\nu)$ are called *ν-cycles* and elements of $\operatorname{im}(\partial_{\nu+1})$ are called *ν-boundaries*. Using this notions, we can say that each homology class of $H_\nu(C)$ is represented by a ν-cycle. Two ν-cycles in the same homology class are *homologous*, and their difference is a ν-boundary.

Remark 1.45. Let B, C be chain complexes over a ring R and $f \colon B \to C$ a chain isomorphism. Then both chain complexes have the same homology, i.e. $H_\nu(B) \cong H_\nu(C)$ for all $\nu \in \mathbb{Z}$.

In particular, the homology of any free chain complex (C, Ω) will not change if we permute basis elements in some basis Ω_ν.

1.4. Reduced Homology

In this section, we define reduced homology for chain complexes over principal ideal domains. In this case, the reduced homology module of index 0 is without any further conditions related to the usual homology module.

Definition 1.46. Let R be a principal ideal domain and (C, Ω) a free and non-negative chain complex over R. If we replace the boundary map $\partial_0 \colon C_0 \to 0$ by an R-linear map $\epsilon \colon C_0 \to R$ with $\epsilon \circ \partial_1 = 0$, we get an *augmented chain* *complex* over R (cf. Hatcher, 2008, p. 110):

$$\dots \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} \dots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} R \xrightarrow{\eta} 0.$$

The homology modules of this chain complex are the *reduced homology modules* $\widetilde{H}_\nu(C)$ for all $\nu \in \mathbb{Z}$. For $\nu \geq 1$, we obtain $\widetilde{H}_\nu(C) = H_\nu(C)$. In particular, we have $\widetilde{H}_0(C) = \ker(\epsilon)/\mathrm{im}(\partial_1)$ and $\widetilde{H}_{-1}(C) = \ker(\eta)/\mathrm{im}(\epsilon)$. Furthermore, $\widetilde{H}_\nu(C) = 0$ for all $\nu \leq -2$. We call ϵ an *augmentation map*.

Remark 1.47. If the augmentation map ϵ is surjective, we obtain $\widetilde{H}_{-1}(C) = 0$ since then $\epsilon(x)$ is a unit in R for some $x \in C_0$. If $\epsilon = 0$, then $\widetilde{H}_{-1}(C) \cong R$. If ϵ is neither the zero map nor surjective, the homology module $\widetilde{H}_{-1}(C)$ is not free and contains only torsion elements.

If $\epsilon = 0$, reduced homology is the same as the usual homology apart from $\widetilde{H}_{-1}(C) \cong R$. But if $\epsilon \neq 0$, which is only possible for $C_0 \neq 0$, the reduced homology module $\widetilde{H}_0(C) = \ker(\epsilon)/\,\mathrm{im}(\partial_1)$ differs from $H_0(C)$. Since $\mathrm{im}(\epsilon)$ is free by Theorem 1.17, we obtain $\mathrm{im}(\epsilon) \cong R$. Hence, we get

$$C_0 = \ker(\partial_0) = \ker(\epsilon) \oplus Re$$

for some $e \in C_0$ with $\epsilon(e)$ generating $\mathrm{im}(\epsilon)$ (cf. Oeljeklaus and Remmert, 1974, p. 108). As $\mathrm{im}(\partial_1) \subseteq \ker(\epsilon)$, we have

$$H_0(C) \cong \widetilde{H}_0(C) \oplus R.$$

Lemma 1.48. *Let (C, Ω) be a free and nonnegative chain complex over a principal ideal domain R. Then its reduced homology module $\widetilde{H}_0(C)$ is independent of the choice of an augmentation map $\epsilon \neq 0$.*

Proof. Let $C_0 \xrightarrow{\epsilon_1} R$ and $C_0 \xrightarrow{\epsilon_2} R$ be nonzero augmentation maps, $\epsilon_1 \neq \epsilon_2$. For the moment, we denote the zeroth reduced homology module by $\widetilde{H}_0(C, \epsilon_i)$ for $i \in \{1, 2\}$. Hence, we get

$$\widetilde{H}_0(C, \epsilon_1) \oplus R \cong H_0(C) \cong \widetilde{H}_0(C, \epsilon_2) \oplus R$$

and therefore $\widetilde{H}_0(C, \epsilon_1) \cong \widetilde{H}_0(C, \epsilon_2)$. □

However, the homology module $\widetilde{H}_{-1}(C)$ depends on the choice of ϵ. For example, we consider the finite chain complex (C, Ω) over $\mathbb{Z}$ whose basis is $\Omega = \Omega_0 = \{e\}$. Let ϵ_1 and ϵ_2 be augmentation maps with $\epsilon_1(e) = 1$ and $\epsilon_2(e) = 2$. Then we get $\widetilde{H}_{-1}(C, \epsilon_1) = 0$ and $\widetilde{H}_{-1}(C, \epsilon_2) \cong \mathbb{Z}_2$.

1.4.1. An Augmentation Map for Simplicial Complexes

Let Δ be an ordered simplicial complex of dimension n. From Δ, we obtain a finite chain complex (C, Ω) of order n over some principal ideal domain R as described in Section 1.2.5. Recall that we have a one-to-one correspondence between the ν-simplices in Δ and the basis elements in $\Omega_\nu \subseteq \Omega$ for every $\nu \in \mathbb{Z}$. In particular, the basis elements in $\Omega_0 = \{e_0, e_1, \dots, e_{k_0}\}$ correspond to the vertices of Δ.

To define an augmentation map, we recall that $\partial_1(e_{i_0 i_1}) = e_{i_1} - e_{i_0}$ for any basis element $e_{i_0 i_1} \in \Omega_1$. Since we need $\epsilon \circ \partial_1 = 0$, we define an R-linear map for any element $x = \sum_{\ell=0}^{k} \alpha_\ell e_\ell \in C_0$ as follows (cf. Hatcher, 2008, p. 110):

$$\epsilon\colon C_0 \to R, \quad x = \sum_{\ell=0}^{k} \alpha_\ell e_\ell \mapsto \sum_{\ell=0}^{k} \alpha_\ell.$$

In particular, $\epsilon(e_\ell) = 1$, so ϵ is surjective.

This is a canonical choice of ϵ. Of course, one can also choose $\epsilon_{-1}(x) = -\sum_{\ell=0}^{k} \alpha_\ell$ or $\epsilon_r(x) = r \sum_{\ell=0}^{k} \alpha_\ell$ for any fixed $r \in R \setminus \{0\}$ instead. Since r is not a zero divisor and thus $\ker(\epsilon_r) = \ker(\epsilon)$, this choice does not change the reduced homology module $\widetilde{H}_0(C)$, but we get torsion in $\widetilde{H}_{-1}(C)$ if r is not a unit in $\mathbb{Z}$.

1.4.2. About Augmentation Maps for Chain Complexes

For any free and nonnegative chain complex over a principal ideal domain R, the augmentation map ϵ does not need to be unique. But in contrast to simplicial complexes there exist chain complexes for which ϵ must be 0. Indeed, it is always possible to take $\epsilon = 0$, but we are interested in $\epsilon \neq 0$ because in that case the reduced homology module $\widetilde{H}_0(C)$ is different from $H_0(C)$. We consider some examples.

1. Let (C, Ω) be a finite chain complex of order 1 over $\mathbb{Z}$ whose chain modules have the bases $\Omega_1 = \{e_1^1\}$ and $\Omega_0 = \{e_1^0\}$. Let $\partial_1(e_1^1) = e_1^0$. We must define $\epsilon(e_1^0) = 0$, hence $\widetilde{H}_0(C) = H_0(C)$ and $\widetilde{H}_{-1}(C) \cong \mathbb{Z}$.

2. Consider the finite chain complex (C, Ω) of order 1 over $\mathbb{Z}$ with chain module bases $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$. Let $\partial_1(e_1^1) = e_1^0 + e_2^0$ and $\partial_1(e_2^1) = e_1^0 + 2e_2^0$. Because $e_2^1 - e_1^1$ maps to e_2^0, we only get $\epsilon = 0$. As above, $\widetilde{H}_0(C) = H_0(C)$ and $\widetilde{H}_{-1}(C) \cong \mathbb{Z}$.

3. Again, let (C, Ω) be a finite chain complex of order 1 over $\mathbb{Z}$ whose chain modules have the bases $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0, e_4^0\}$. Let $\partial_1(e_1^1) = e_1^0 + e_2^0 + 2e_3^0$ and $\partial_1(e_2^1) = e_3^0 + e_4^0$. Then, $H_0(C) \cong \mathbb{Z}^2$. Now, there are at least two different ways to define an augmentation map $C_0 \to \mathbb{Z}$.

   a) ϵ_1 is given by:

$$\epsilon_1(e_1^0) = 1, \qquad \epsilon_1(e_3^0) = -1,$$
$$\epsilon_1(e_2^0) = 1, \qquad \epsilon_1(e_4^0) = 1.$$

   b) ϵ_2 is defined in the following way:

$$\epsilon_2(e_1^0) = 1, \qquad \epsilon_2(e_3^0) = 0,$$
$$\epsilon_2(e_2^0) = -1, \qquad \epsilon_2(e_4^0) = 0.$$

   In both cases $\widetilde{H}_0(C) \cong \mathbb{Z}$ and $\widetilde{H}_{-1}(C) = 0$, as both augmentation maps are surjective.

If there is a basis element $e_i^0 \in \Omega_0$ which is not contained in the boundary of any basis element of Ω_1, then an augmentation map $\epsilon \neq 0$ always exists and can be obtained by defining $\epsilon(e_i^0) = 1$. In particular, ϵ can be defined this way for every finite chain complex of order 0. For chain complexes of order $d \geq 1$ we treat two special cases.

Theorem 1.49. *Let (C, Ω) be a finite chain complex of order $d \geq 1$ over a principal ideal domain R with $\operatorname{char} R = 0$. Let $\Omega_0 = \{e_1^0, \ldots, e_{k_0}^0\}$ be the basis of its chain module C_0 with $k_0 \geq 1$. Then the following statements are equivalent:*

(1) $^{\#}\mathrm{bd}(x) \geq 2$ for every $x \in C_1 \setminus \ker \partial_1$.

(2) An augmentation map $\epsilon\colon C_0 \to R$ exists such that $\epsilon(e_\mu^0) \neq 0$ for all $e_\mu^0 \in \Omega_0$.

Proof. (1) $\Rightarrow$ (2): If $C_1 = 0$, we can set $\epsilon(e_\mu^0) = 1$ for all $e_\mu^0 \in \Omega_0$, and we are done. So let $\Omega_1 = \{e_1^1, \ldots, e_{k_1}^1\}$ with $k_1 \geq 1$. For each e_μ^0, which is not contained in the boundary of any e_λ^1, we define $\epsilon(e_\mu^0) = 1$. Hence, we assume without loss of generality that the 1-skeleton of (C, Ω) is pure. Let

$$\partial_1(e_\lambda^1) = \sum_{\ell=1}^{k_0} a_{\lambda\ell} e_\ell^0 \quad \text{for } 1 \leq \lambda \leq k_1.$$

Because $\epsilon \circ \partial_1 = 0$, we have to solve the following system of linear equations to define ϵ.

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1k_0} \\ a_{21} & a_{22} & \cdots & a_{2k_0} \\ \vdots & \vdots & & \vdots \\ a_{k_1 1} & a_{k_1 2} & \cdots & a_{k_1 k_0} \end{pmatrix} \begin{pmatrix} \epsilon(e_1^0) \\ \epsilon(e_2^0) \\ \vdots \\ \epsilon(e_{k_0}^0) \end{pmatrix} = 0. \tag{1.1}$$

By transformation of the rows we get a block matrix

$$\left(\begin{array}{cccc|ccc} \widetilde{a}_{11} & 0 & \cdots & 0 & \widetilde{a}_{1,j+1} & \cdots & \widetilde{a}_{1,k_0} \\ 0 & \widetilde{a}_{22} & \cdots & 0 & \widetilde{a}_{2,j+1} & \cdots & \widetilde{a}_{2,k_0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \widetilde{a}_{jj} & \widetilde{a}_{j,j+1} & \cdots & \widetilde{a}_{j,k_0} \\ \hline 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{array}\right)$$

for some $1 \leq j \leq k_1$ and $\widetilde{a}_{ii} \neq 0$ for all $1 \leq i \leq j$. Note that $j < k_0$ because otherwise we would get a row

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & \widetilde{a}_{k_0 k_0} \end{pmatrix}$$

having only one entry $\widetilde{a}_{k_0 k_0} \neq 0$. Getting such a row means that there exists an element $x \in C_1$ with $^{\#}\operatorname{bd}(x) = 1$ which is a contradiction!

We define an augmentation map ϵ by choosing values $\epsilon(e_\mu^0)$ for all basis elements $e_\mu^0 \in \Omega_0$ iteratively in the following way.

- At first, we define $\widetilde{a} := \prod_{i=1}^{j} \widetilde{a}_{ii} \neq 0$, choose some $r_{k_0} \in R \setminus \{0\}$ and set $\epsilon(e_{k_0}^0) := \widetilde{a} \cdot r_{k_0}$.
- We proceed by iteration. Assume that for some $(j+1) < k \leq k_0$ elements $r_\ell \in R \setminus \{0\}$ are chosen for all $k \leq \ell \leq k_0$ such that $\epsilon(e_\ell^0) = \widetilde{a} \cdot r_\ell$. We choose some $r_{k-1} \in R \setminus \{0\}$ such that

  $$\widetilde{a}_{i,k-1} r_{k-1} + \sum_{\ell=k}^{k_0} \widetilde{a}_{i,\ell} r_\ell \neq 0$$

  for all indices $1 \leq i \leq j$, for which there is at least one coefficient $\widetilde{a}_{i,t} \neq 0$ for some $(k-1) \leq t \leq k_0$. This is possible because each equation

$\widetilde{a}_{i,k-1}r_{k-1} + \sum_{\ell=k}^{k_0} \widetilde{a}_{i,\ell}r_\ell = 0$ has at most one solution r_{k-1} if there is some $\widetilde{a}_{i,t} \neq 0$. Hence, there are at most j nonzero elements in R which cannot be chosen.

We set $\epsilon(e^0_{k-1}) := \widetilde{a} \cdot r_{k-1}$ then.

- For each $1 \leq \nu \leq j$, we define

$$\epsilon(e^0_\nu) := -\left(\prod_{i \leq j, i \neq \nu} \widetilde{a}_{ii}\right) \sum_{\ell=j+1}^{k_0} \widetilde{a}_{\nu,\ell} r_\ell \neq 0.$$

Then we have $\epsilon(e^0_\mu) \neq 0$ for all $1 \leq \mu \leq k_0$.

(2) $\Rightarrow$ (1): We assume that some $x \in C_1 \setminus \ker \partial_1$ exists such that $^\# \operatorname{bd}(x) = 1$. Then $\partial_1(x) = \rho e^0_i$ for some $e^0_\mu \in \Omega_0$ and $\rho \in R \setminus \{0\}$. Hence, $\epsilon(e^0_\mu) = 0$ holds for every augmentation map ϵ which is a contradiction to (2). □

Remark 1.50. The above theorem is not true for chain complexes defined over some finite field $\mathbb{F}_q$. We show examples for $\mathbb{F}_2$, $\mathbb{F}_3$ and $\mathbb{F}_4$, but in the same manner one can create examples for any finite field.

- Let (C, Ω) be a finite chain complex of order 1 over $\mathbb{F}_2$ whose chain modules have the bases $\Omega_1 := \{e^1_1\}$ and $\Omega_0 := \{e^0_1, e^0_2, e^0_3\}$. Let $\partial_1(e^1_1) = e^0_1 + e^0_2 + e^0_3$, then we must set $\epsilon(e^0_i) = 0$ for some $i \in \{1, 2, 3\}$ and any augmentation map ϵ.

- Let (C, Ω) be a finite chain complex of order 1 over $\mathbb{F}_3$ whose chain modules have the bases $\Omega_1 := \{e^1_1, e^1_2\}$ and $\Omega_0 := \{e^0_1, e^0_2, e^0_3, e^0_4\}$. Let $\partial_1(e^1_1) = e^0_1 + e^0_3 + e^0_4$ and $\partial_1(e^1_2) = e^0_2 + e^0_3 + 2e^0_4$. Then we must set $\epsilon(e^0_1) = 0$ or $\epsilon(e^0_2) = 0$ for any augmentation map with $\epsilon(e^0_3) \neq 0$ and $\epsilon(e^0_4) \neq 0$.

- Let (C, Ω) be a finite chain complex of order 1 over $\mathbb{F}_4$, which may be generated by 1 and σ over $\mathbb{F}_2$. Let its chain modules have the bases $\Omega_1 := \{e^1_1, e^1_2, e^1_3\}$ and $\Omega_0 := \{e^0_1, e^0_2, e^0_3, e^0_4, e^0_5\}$, and let $\partial_1(e^1_1) = e^0_1 + e^0_4 + e^0_5$, $\partial_1(e^1_2) = e^0_2 + e^0_4 + \sigma e^0_5$ and $\partial_1(e^1_3) = e^0_3 + e^0_4 + (\sigma + 1)e^0_5$. Then again it is impossible to get an augmentation map ϵ with $\epsilon(e^0_i) \neq 0$ for all $i \in \{1, 2, 3, 4, 5\}$.

Hence, if R is a principal ideal domain with $\operatorname{char} R \neq 0$, we can only prove a weaker statement.

Theorem 1.51. *Let (C, Ω) be a finite chain complex of order $d \geq 1$ over a principal ideal domain R. Let $\Omega_0 = \{e_1^0, \ldots, e_{k_0}^0\}$ be the basis of its chain module C_0 with $k_0 \geq 1$.*

1. *If ${}^{\#}\operatorname{bd}(x) \geq 2$ for every $x \in C_1 \setminus \ker \partial_1$, then an augmentation map $\epsilon \neq 0$ exists.*

2. *If an augmentation map $\epsilon\colon C_0 \to R$ exists such that $\epsilon(e_\ell^0) \neq 0$ for all $e_\ell^0 \in \Omega_0$, then ${}^{\#}\operatorname{bd}(x) \geq 2$ for every $x \in C_1 \setminus \ker \partial_1$.*

Proof. The proof of the second statement is the same as the proof for $(2) \Rightarrow (1)$ in the preceding Theorem 1.49.

To prove the first statement we have to show that, for any augmentation map ϵ, there is some $e_i^0 \in \Omega_0$ such that $\epsilon(e_i^0) \neq 0$. If the 1-skeleton of (C, Ω) is not a pure finite chain complex of order 1, nothing remains to be done as we can define $\epsilon(e_i^0) = 1$ for any maximal basis element $e_i^0 \in \Omega_0$. Hence, we assume that the 1-skeleton of (C, Ω) is pure and let $\Omega_1 := \{e_1^1, \ldots, e_{k_1}^1\}$ be the basis of C_1. As above let

$$\partial_1(e_i^1) = \sum_{\ell=1}^{k_0} a_{i\ell} e_\ell^0 \quad \text{for } 1 \leq i \leq k_1.$$

To define an augmentation map ϵ we have to solve a system of linear equations as in Equation (1.1) on page 16. The solution of this system cannot be unique because otherwise there must be an element $x \in C_1$ with ${}^{\#}\operatorname{bd}(x) = 1$ which is impossible by assumption. Hence, there is some $e_k^0 \in \Omega_0$ such that $\epsilon(e_k^0) \neq 0$, and therefore $\epsilon \neq 0$. □

1.5. Homology of Shellable Simplicial Complexes

The homology of shellable geometric simplicial complexes is well-known, see Björner and Wachs (1996, Chapter 4) or Björner (1992, Theorem 7.7.2) with a direct proof for pure shellable simplicial complexes, for example. In this section, we consider only homology over $\mathbb{Z}$ because this is the usual case in literature. We will recover the statement of the next theorem in a more general context in Section 3.3.

Theorem 1.52. *Let Δ be a shellable geometric simplicial complex of dimension $d \geq 1$. Let its maximal simplices be ordered in a shelling. For $0 \leq i \leq d$ let there be n_i spanning simplices of dimension i. Then the homology modules of Δ are*

$$\begin{aligned} H_i(\Delta) &\cong \mathbb{Z}^{n_i} \quad \text{for } 1 \leq i \leq d, \\ H_0(\Delta) &\cong \mathbb{Z}^{n_0+1}, \\ H_i(\Delta) &= 0 \quad \text{if } i < 0 \text{ or } i > d. \end{aligned}$$

Remark 1.53. Every simplicial complex Δ of dimension 0 is shellable. If Δ consists of n vertices, we get $H_0(\Delta) \cong \mathbb{Z}^n$ according to Hatcher (2008, Proposition 2.7).

Proof of Theorem 1.52. Let Σ be the set of spanning simplices except the ones of dimension 0. Let $\widehat{\Delta}$ be the subcomplex of Δ which contains all its simplices except the maximal simplices of dimension 0. By Kozlov (2008, Theorem 12.3), the subcomplex $\widehat{\Delta}$ is homotopy equivalent[2] to a wedge of spheres:

$$\widehat{\Delta} \simeq \bigvee_{\sigma \in \Sigma} S^{\dim \sigma}.$$

Because topological spaces which are homotopy equivalent have isomorphic homology modules (cf. Hatcher, 2008, p. 110), we get

$$H_i(\widehat{\Delta}) \cong H_i\Big(\bigvee_{\sigma \in \Sigma} S^{\dim \sigma}\Big)$$

for all $1 \leq i \leq d$. By Hatcher (2008, Corollary 2.25), we obtain

$$H_i(\widehat{\Delta}) \cong \bigoplus_{\sigma \in \Sigma} H_i(S^{\dim \sigma}).$$

The homology of a sphere is well-known, cf. Hatcher (2008, p. 114), Massey (1991, p. 186) or Spanier (1966, p. 190), for example. For $1 \leq i$, we have $H_i(S^i) \cong \mathbb{Z}$ and $H_i(S^n) = 0$ if $i \neq n$. Hence, we get for $1 \leq i \leq d$:

$$H_i(\Delta) = H_i(\widehat{\Delta}) \cong \mathbb{Z}^{n_i}.$$

Because the subcomplex $\widehat{\Delta}$ is path-connected, we obtain $H_0(\widehat{\Delta}) \cong \mathbb{Z}$ and therefore

$$H_0(\Delta) \cong \mathbb{Z}^{n_0+1}. \qquad \square$$

Remark 1.54. An analogous statement for pure shellable simplicial complexes has been proven by Björner (1992, p. 254).

[2]The notion of a homotopy equivalence is explained by Hatcher (2008, p. 3), for example.

1.6. Critical Basis Elements in Free Chain Complexes

In this section let all chain complexes be defined over some principal ideal domain R.

Definition 1.55. Let (C, Ω) be a finitely free chain complex over R whose chain modules C_ν have bases $\Omega_\nu = \emptyset$ or $\Omega_\nu = \{e_1^\nu, \ldots, e_{k_\nu}^\nu\}$ for some $k_\nu \geq 1$. Let Γ be the set of all its maximal basis elements:

$$\Gamma := \{e \in \Omega \mid e \notin \mathrm{bd}(f) \text{ for all } f \in \Omega\}.$$

Let the basis elements in each basis Ω_ν be ordered in such a way that the elements of $\Omega_\nu \setminus \Gamma$ come first. A maximal basis element $e_k^\nu \in (\Omega_\nu \cap \Gamma)$ is called

- *critical* if there exist elements $a_i \in R$ for $1 \leq i \leq (k-1)$ such that

$$\partial_\nu(e_k^\nu) = \sum_{i=1}^{k-1} a_i \partial_\nu(e_i^\nu),$$

- *precritical* if there exist elements $a_i \in R$ for $1 \leq i \leq k$ with $a_k \neq 0$ such that

$$a_k \partial_\nu(e_k^\nu) = \sum_{i=1}^{k-1} a_i \partial_\nu(e_i^\nu),$$

- *noncritical* if e_k^ν is neither critical nor precritical.

Remark 1.56.

1. A critical basis element is always precritical, and for a basis element e_k^ν with $k = 1$ both notions coincide since the empty sum is zero. For $k \geq 2$, a precritical element e_k^ν is critical if the coefficient a_k can be chosen to be a unit in R. Hence, the two notions are equivalent if the principal ideal domain R is a field.

2. $\mathrm{supp}\big(a_k \partial_\nu(e_k^\nu)\big) = \mathrm{supp}\big(\partial_\nu(e_k^\nu)\big)$ if $a_k \neq 0$.

3. It is possible to change the ordering of a basis Ω_ν in such a way that all noncritical basis elements come first.

Critical basis elements are an analogue to spanning simplices in shellable simplicial complexes, as the following lemma shows.

Lemma 1.57. *Let Δ be an ordered simplicial complex of dimension d and (C, Ω) the corresponding finite chain complex of order d over R obtained from Δ as described in Section 1.2.5. Then any basis element $e_k^\nu \in \Omega$ which corresponds to a spanning simplex of Δ is critical.*

Proof. For $0 \leq \nu \leq d$, let each basis Ω_ν be ordered in such a way that the maximal basis elements corresponding to spanning simplices come last, i.e.

$$\Omega_\nu = \{e_1^\nu, \ldots, e_{m_\nu}^\nu, \underbrace{e_{m_\nu+1}^\nu, \ldots, e_{k_\nu}^\nu}_{\text{“spanning simplices”}}\}.$$

Then, for each basis element $e_k^\nu \in \Omega_\nu$ with $(m_\nu + 1) \leq k \leq k_\nu$, we have $\mathrm{bd}(e_k^\nu) \subseteq \bigcup_{i=1}^{m_\nu} \mathrm{bd}(e_i^\nu)$.

We consider the simplicial subcomplex $\widehat{\Delta}$ which consists of all maximal simplices of Δ except its spanning simplices. According to Kozlov (2008, p. 213), this subcomplex is collapsible, so there exists a homotopy equivalence to a single point (cf. Kozlov, 2008, p. 94). Hence, the corresponding subcomplex $(\widehat{C}, \widehat{\Omega})$ of (C, Ω) whose chain modules have the bases $\widehat{\Omega}_\nu := \{e_1^\nu, \ldots, e_{m_\nu}^\nu\}$ has the same reduced homology modules, i.e. $\widetilde{H}_i(\widehat{C}) = 0$ for all $i \in \mathbb{Z}$.

For all basis elements e_k^μ which correspond to a spanning simplex, we consider the subcomplex $\widehat{C}(e_k^\mu)$ whose bases are $\widehat{\Omega}(e_k^\mu)_\nu = \widehat{\Omega}_\nu$ for $\nu \neq \mu$ and $\widehat{\Omega}(e_k^\mu)_\mu = \widehat{\Omega}_\mu \cup \{e_k^\mu\}$. If $\mu = 0$, there is nothing to do. For $\mu \geq 1$, we get

$$\partial_\mu(e_k^\mu) \in \operatorname{im} \partial_\mu|_{\widehat{C}(e_k^\mu)_\mu} \subseteq \ker \partial_{\mu-1}|_{\widehat{C}(e_k^\mu)_{\mu-1}} = \ker \partial_{\mu-1}|_{\widehat{C}_{\mu-1}} = \operatorname{im} \partial_\mu|_{\widehat{C}_\mu},$$

writing ∂_0 instead of ϵ for the augmentation map here if $\mu = 1$. Hence, there is some $x_k^\mu \in \widehat{C}_\mu$ such that $\partial_\mu(x_k^\mu) = \partial_\mu(e_k^\mu)$. Therefore, the basis element e_k^μ is critical. □

In a pure finite chain complex (C, Ω) of order d, all the precritical elements in the chain module basis Ω_d can be seen as the generators of homology in dimension d. If all basis elements of Ω_d are either noncritical or critical, we can name a basis of $H_d(C)$. At first, we treat a special case which does not arise at simplicial complexes of dimension 1 or higher.

Lemma 1.58. *Let (C, Ω) be a pure finite chain complex of order d over R and let all basis elements in $\Omega_d = \{e_1^d, \ldots, e_{k_d}^d\}$ be critical. Then $\Omega_\nu = \emptyset$ for all $\nu \neq d$, and $H_d(C) \cong R^{k_d}$ is generated by $e_1^d, \ldots, e_{k_d}^d$.*

Proof. By induction we get $\partial_d(e_i^d) = 0$ for all $1 \leq i \leq k_d$. As (C, Ω) is pure, there are no basis elements which are not contained in any elementary subcomplex $(C_{e_i^d}, \Omega_{e_i^d})$. □

Remark 1.59. All assumptions of Lemma 1.58 are fulfilled by every finite chain complex of order 0.

To name a basis of $H_d(C)$ for pure finite chain complexes of order $d \geq 1$ which have at least one noncritical basis element in Ω_d, we follow Björner (1992, p. 254), who has performed this task for the special case of shellable simplicial complexes, and generalise his proof to chain complexes. To formulate the theorem we introduce a new notation. Let C_ν be a chain module generated by $\Omega_\nu := \{e_1^\nu, \ldots, e_{k_\nu}^\nu\}$ and consider some $\rho = \sum_{i=1}^{k_\nu} a_i e_i^\nu \in C_\nu$. Then we denote the coefficient of e_i^ν by $\rho(e_i^\nu) := a_i$.

Theorem 1.60. *Let (C, Ω) be a pure finite chain complex of order $d \geq 1$ and Ω_d be a basis of C_d with $k_d \geq 1$ elements. Let there be $n < k_d$ critical elements $g_1, \ldots, g_n$ in Ω_d and all other basis elements be noncritical. Let Ω_d be ordered in such a way that the noncritical elements come first:*

$$\Omega_d = \{e_1, \ldots, e_m, g_1, \ldots, g_n\}, \qquad m + n = k_d.$$

Then the following holds:

1. *$H_d(C) \cong R^n$.*
2. *For $n \geq 1$, there exist unique d-cycles $\rho_1, \ldots, \rho_n$ in $H_d(C) \cong \ker(\partial_d)$ such that $\rho_i(g_j) = \delta_{ij}$.*
3. *For $n \geq 1$, $\{\rho_1, \ldots, \rho_n\}$ is a basis of $H_d(C)$.*

Proof. At first, we consider the case $n = 0$, i.e. the basis Ω_d has only noncritical elements, so $\Omega_d = \{e_1, \ldots, e_m\}$. In particular, we have $\partial_d(e_i) \neq 0$ for all $1 \leq i \leq m$ then. We assume that there exists an element $x \in C_d \setminus \{0\}$ such that $\partial_d(x) = 0$. Then $^\#\mathrm{supp}(x) \geq 2$ holds. Let $x = \sum_{i=1}^{m} a_i e_i$ and define $i_0 := \max\{i \leq m \mid a_i \neq 0\} \geq 2$. So we get $a_{i_0} \partial_d(e_{i_0}) = \sum_{i < i_0} (-a_i) \partial_d(e_i)$. Hence,

e_{i_0} is not noncritical which is a contradiction. Therefore, $\ker(\partial_d) = 0$, i.e. $H_d(C) = 0$.

For $n \geq 1$, the first statement is a consequence of the second and third, so we start proving the second statement using induction. Having $n \geq 1$ critical elements, we get $\Omega_d = \{e_1, \ldots, e_m, g_1, \ldots, g_n\}$.

We consider the subcomplex $(\widehat{C}, \widehat{\Omega})$ of (C, Ω) whose basis is $\widehat{\Omega} := \bigcup_{i=1}^m \Omega_{e_i}$. Its chain modules are $\widehat{C}_d = \langle e_1, \ldots, e_m \rangle$ and $\widehat{C}_\nu = C_\nu$ for $(d-1) \geq \nu \geq 0$. As a chain complex, $(\widehat{C}, \widehat{\Omega})$ is pure and finite of order d having only noncritical elements, because the only critical elements in Ω_d are $g_1, \ldots, g_n$. Therefore, $H_d(\widehat{C}) = 0$.

For every $1 \leq i \leq n$ there is some $\widehat{\rho}_i \in \widehat{C}_d$ such that $\partial_d(\widehat{\rho}_i) = \partial_d(g_i)$. Hence, $\rho_i := g_i - \widehat{\rho}_i \in \ker(\partial_d) \cong H_d(C)$. Since $\widehat{\rho}_i \in \widehat{C}_d = \langle e_1, \ldots, e_m \rangle$, the only critical element contained in $\operatorname{supp}(\rho_i)$ is g_i. In particular, we have

$$\rho_i(g_j) = \delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence, the second statement is proven up to uniqueness.

Let $\sigma_i \in \ker(\partial_d) \cong H_d(C)$ with $\sigma_i(g_j) = \delta_{ij}$. So we get $\sigma_i - \rho_i = \sum_{\ell=1}^m c_\ell^i e_\ell$ with coefficients $c_\ell^i \in R$. We conclude $\sigma_i - \rho_i \in \ker\bigl(\partial_d|_{\widehat{C}_d}\bigr) \cong H_d(\widehat{C}) = 0$. Therefore $\sigma_i = \rho_i$, so we have shown uniqueness. It remains to prove that $\{\rho_1, \ldots, \rho_n\}$ generates $H_d(C)$ and is linearly independent.

Let $\sum_{i=1}^n a_i \rho_i = 0$ with $a_i \in R$. Because $\rho_i = g_i - \widehat{\rho}_i$, we get

$$0 = \sum_{i=1}^n a_i g_i - \underbrace{\sum_{i=1}^n a_i \widehat{\rho}_i}_{\in \widehat{C}_d = \langle e_1, \ldots, e_m \rangle}.$$

As $\{e_1, \ldots, e_m, g_1, \ldots, g_n\}$ is a basis of C_d, we conclude $a_i = 0$ for all i, so the elements $\rho_1, \ldots, \rho_n$ are independent.

Let $\sigma \in H_d(C) \cong \ker(\partial_d) \subseteq C_d$. We set $\tau := \sigma - \sum_{i=1}^n \sigma(g_i)\rho_i \in \ker(\partial_d)$. For $1 \leq j \leq n$, the coefficient $\tau(g_j)$ of g_j in τ is

$$\tau(g_j) = \sigma(g_j) - \sum_{i=1}^n \sigma(g_i) \underbrace{\rho_i(g_j)}_{=\delta_{ij}} = \sigma(g_j) - \sigma(g_j) = 0.$$

Therefore, $\tau \in \langle e_1, \ldots, e_m \rangle = \widehat{C}_d$. So we get $\tau \in \ker\bigl(\partial_d|_{\widehat{C}_d}\bigr) \cong H_d(\widehat{C}) = 0$. This yields $\sigma = \sum_{i=1}^n \sigma(g_i)\rho_i$, i.e. $\{\rho_1, \ldots, \rho_n\}$ generates $H_d(C)$. Because of independence, $\{\rho_1, \ldots, \rho_n\}$ is a basis of $H_d(C)$ and hence, $H_d(C) \cong R^n$. □

In general, there are also precritical elements which are not critical. In this case we only know that $H_d(C) \cong R^n$, but we cannot name a basis. To prove the following theorem we will need the generating number $\operatorname{gen}_R M$ and the degree of freedom $\operatorname{dgf}_R M$ which we introduced in Section 1.2.1 for any finitely generated R-module M.

Theorem 1.61. *Let (C, Ω) be a pure finite chain complex of order $d \geq 1$ and Ω_d be a basis of C_d with $k_d \geq 1$ elements. Let there be $n < k_d$ precritical elements $g_1, \ldots, g_n$ in Ω_d and let Ω_d be ordered in such a way that the noncritical elements come first:*

$$\Omega_d = \{e_1, \ldots, e_m, g_1, \ldots, g_n\}, \qquad m + n = k_d.$$

Then $H_d(C) \cong R^n$.

Proof. The case $n = 0$ is already proven. So let there be $n \geq 1$ precritical elements, i.e. $\Omega_d = \{e_1, \ldots, e_m, g_1, \ldots, g_n\}$. As above we consider the subcomplex $(\widehat{C}, \widehat{\Omega})$ of (C, Ω) with basis $\widehat{\Omega} := \bigcup_{i=1}^m \Omega_{e_i}$. It is pure and finite of order d without precritical elements, hence $H_d(\widehat{C}) \cong \ker(\partial_d|_{\widehat{C}_d}) = 0$.

For every $1 \leq i \leq n$, there is an element $\widehat{\rho}_i \in \widehat{C}_d$ and some $a_i \in R \setminus \{0\}$ such that $\partial_d(\widehat{\rho}_i) = \partial_d(a_i g_i)$ because all g_i are precritical. Therefore we get $\rho_i := a_i g_i - \widehat{\rho}_i \in \ker(\partial_d) \cong H_d(C)$. We show that the elements $\rho_1, \ldots, \rho_n$ are linearly independent.

Let $\sum_{i=1}^n c_i \rho_i = 0$ with $c_i \in R$. Since $\rho_i = a_i g_i - \widehat{\rho}_i$, we get

$$0 = \sum_{i=1}^n c_i a_i g_i - \underbrace{\sum_{i=1}^n c_i \widehat{\rho}_i}_{\in \widehat{C}_d = \langle e_1, \ldots, e_m \rangle}.$$

Because $\{e_1, \ldots, e_m, g_1, \ldots, g_n\}$ is a basis of C_d, we get $c_i a_i = 0$ for all i, so $c_i = 0$. Hence, the elements $\rho_1, \ldots, \rho_n$ are independent, i.e. $\ker(\partial_d)$ contains at least n independent elements. So $\operatorname{dgf}_R(\ker \partial_d) \geq n$.

Theorem 1.18 is valid for the boundary map $\partial_d \colon C_d \to C_{d-1}$, hence

$$\operatorname{dgf}_R(C_d) = \operatorname{dgf}_R(\ker \partial_d) + \operatorname{dgf}_R(\operatorname{im} \partial_d). \tag{1.2}$$

Since C_d is a free R-module generated by $(m + n)$ elements, we get by Theorem 1.19 that $\operatorname{dgf}_R(C_d) = \operatorname{gen}_R(C_d) = m + n$. This yields

$$\operatorname{dgf}_R(\operatorname{im} \partial_d) = \operatorname{dgf}_R(C_d) - \operatorname{dgf}_R(\ker \partial_d) \leq m. \tag{1.3}$$

We know that $\operatorname{im} \partial_d|_{\widehat{C}_d} \subseteq \operatorname{im} \partial_d$ since $\widehat{C}_d$ is a submodule of C_d. Therefore, we have $\operatorname{dgf}_R(\operatorname{im} \partial_d|_{\widehat{C}_d}) \leq \operatorname{dgf}_R(\operatorname{im} \partial_d)$. Applying Theorem 1.18 to the boundary map $\partial_d\big|_{\widehat{C}_d} \colon \widehat{C}_d \to \widehat{C}_{d-1}$ yields

$$m = \mathrm{dgf}_R(\widehat{C}_d) = \mathrm{dgf}_R \underbrace{\left(\ker \partial_d\big|_{\widehat{C}_d}\right)}_{=0} + \mathrm{dgf}_R\left(\mathrm{im}\, \partial_d\big|_{\widehat{C}_d}\right) \leq \mathrm{dgf}_R(\mathrm{im}\, \partial_d), \tag{1.4}$$

using $\mathrm{dgf}_R(\widehat{C}_d) = \mathrm{gen}_R(\widehat{C}_d) = m$ by Theorem 1.19. By the Equations (1.3) and (1.4) we get $\mathrm{dgf}_R(\mathrm{im}\, \partial_d) = m$. Thus, $\mathrm{dgf}_R(\ker \partial_d) = n$ by Equation (1.2). Since $\ker \partial_d$ is a free R-module, we get $\mathrm{gen}_R(\ker \partial_d) = n$ due to Theorem 1.19. Finally, this implies $H_d(C) \cong R^n$. □

2. Acyclic Chain Complexes, Cones and Mapping Cones

In this chapter, we consider the relationship between cones and mapping cones. Most of this is motivated by geometric simplicial complexes. At first, we introduce a special type of chain complexes named *acyclic* whose homology is very simple: The only nonzero homology module is $H_0(C)$ which is free and generated by a single element. Since a special kind of acyclic simplicial complexes is given by simplicial cones, we generalise the notion of a cone to general chain complexes, forgetting the geometrical description from the simplicial case.

For any simplicial cone there exists a description as a mapping cone of an identity chain map. Most of this chapter deals with mapping cones for chain complexes, and it turns out that not every cone can be regarded as a mapping cone as well as not every mapping cone is a cone itself. So this correspondence in the simplicial case is quite special.

2.1. Acyclic Chain Complexes and Cones

Our definition of acyclic chain complexes refers to Björner (1992, p. 253) or Eilenberg and Steenrod (1981, p. 170).

Definition 2.1. A nonnegative chain complex C over some ring R is *acyclic* if the following holds for its homology groups:

$$H_0(C) \cong R, \qquad H_\nu(C) = 0 \quad \text{for } \nu \geq 1.$$

Similar definitions can be found in Cartan and Eilenberg (1956, p. 75), Massey (1991, p. 288) and Munkres (1984, p. 45). A more general, but unusual definition is given by Hilton and Stammbach (1971, p. 126).

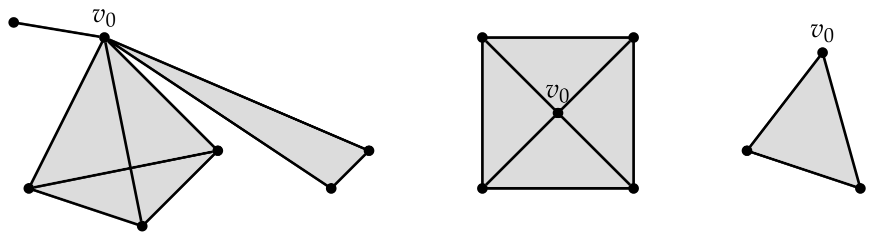


Figure 2.1.: Examples of cones with apex v_0

Remark 2.2. Let (C, Ω) be a free and acyclic chain complex over a principal ideal domain R. If there exists an augmentation map $\epsilon \neq 0$, then $\widetilde{H}_0(C) = 0$ according to Section 1.4. If ϵ is even surjective, then $\widetilde{H}_{-1}(C) = 0$, too.

Some special simplicial complexes being acyclic are *cones*. A simplicial cone has a distinguished vertex v_0 being the apex of the cone, and each maximal simplex S in this simplicial complex has exactly one facet which does not contain the vertex v_0. For example, simplices themselves are cones (cf. Munkres, 1984, p. 44). Further examples are shown in Figure 2.1.

To define the concept of cones for chain complexes, we want to abandon the geometrical idea of an apex. This leads to the following definition.

Definition 2.3. Let (C, Ω) be a finite chain complex of order d over a principal ideal domain R. For all $0 \leq \nu \leq d$, let $\Omega_\nu := \{e_1^\nu, \ldots, e_{k_\nu}^\nu\} \neq \emptyset$ be a basis of the chain module C_ν. The chain complex (C, Ω) is a *cone* if the following conditions are fulfilled:

1. For every $\nu \in \{1, \ldots, d\}$, there is a nonempty subset $S_\nu \subseteq \Omega_\nu$ such that
   a) $\operatorname{supp}(\partial_\nu e_j^\nu) \nsubseteq \bigcup_{e_i^\nu \in S_\nu \setminus \{e_j^\nu\}} \operatorname{supp}(\partial_\nu e_i^\nu)$ for every $e_j^\nu \in S_\nu$,
   b) for every $e_k^\nu \in \Omega_\nu \setminus S_\nu$, there is an element $\tau_k \in C_{\nu+1}$ such that
$$\partial_{\nu+1}\tau_k = c_k e_k^\nu + r_k \qquad \text{with } c_k \text{ unit in } R \text{ and } r_k \in \langle S_\nu \rangle.$$
2. $^\#\operatorname{supp}(\partial_1 x) \geq 2$ for all $x \in C_1 \setminus \ker \partial_1$.
3. There is a subset $\{e\} = S_0 \subseteq \Omega_0$ with $^\# S_0 = 1$ such that the following holds: For every $e_k^0 \in \Omega_0 \setminus S_0$ there is an element $\tau_k \in C_1$ such that
$$\partial_1 \tau_k = c_k e_k^0 + d_k e \qquad \text{with } c_k \text{ unit in } R \text{ and } d_k \neq 0.$$

Remark 2.4.

1. $d_k \neq 0$ in the cone condition 3 follows from the cone condition 2.

2. Let Γ be the set of all maximal basis elements of (C, Ω) as in Definition 1.55. Then $\Gamma \cap \Omega_\nu \subseteq S_\nu$ for all ν. In particular, $\Omega_d = S_d$.

3. $\ker(\partial_\nu) \cap \langle S_\nu \rangle = \{0\}$ for all $1 \leq \nu \leq d$ because of the cone condition 1a.

Remark 2.5. For any cone (C, Ω), we can always define a nonzero augmentation map ϵ in the following way, using the cone condition 3:

- We set $\epsilon(e) := 1$ for the only basis element $e \in S_0 \subseteq \Omega_0$.

- For every basis element $e_k^0 \in \Omega_0 \setminus S_0$, we have an element $\tau_k \in C_1$ such that $\partial_1 \tau_k = c_k e_k^0 + d_k e$ with some unit $c_k \in R$ and $d_k \neq 0$. Therefore, we define $\epsilon(e_k^0) := -c_k^{-1} d_k$.

Since $\epsilon(e) = 1$, this augmentation map is surjective, hence $\widetilde{H}_{-1}(C) = 0$. Furthermore, we have $\epsilon(e_i^0) \neq 0$ for all basis elements $e_i^0 \in \Omega_0$.

Indeed, the above Definition 2.3 implies that cones are acyclic, as the following lemma shows.

Lemma 2.6. *A cone (C, Ω) is acyclic.*

Proof. By definition, a cone is always a finite chain complex of order d for some $d \in \mathbb{N}$.

If $d = 0$, then $\Omega_0 = S_0$. Hence, we get $H_\nu(C) = 0$ for $\nu \geq 1$ and $H_0(C) \cong R$ since ${}^\# S_0 = 1$.

Let $d \geq 1$. At first, we show that $\ker(\partial_\nu) = \operatorname{im}(\partial_{\nu+1})$ for $\nu \geq 1$. For $\nu > d$, there is nothing to do. For $\nu = d$, we get $S_d = \Omega_d$ and conclude $\ker(\partial_d) = \{0\}$ due to Remark 2.4. Therefore, $H_d(C) = 0$.

For $1 \leq \nu \leq (d-1)$, we take an arbitrary element $\sigma \in \ker \partial_\nu$:

$$\sigma = \sum_{i=1}^{k_\nu} a_i e_i^\nu = \sum_{e_i^\nu \in S_\nu} a_i e_i^\nu + \sum_{e_i^\nu \notin S_\nu} a_i e_i^\nu.$$

By definition, there is some $\tau_i \in C_{\nu+1}$ for every $e_i^\nu \notin S_\nu$ such that $\partial_{\nu+1} \tau_i = c_i e_i^\nu + r_i$ with some unit c_i in R and $r_i \in \langle S_\nu \rangle$. Since $\partial_\nu \circ \partial_{\nu+1} = 0$, we have

$\partial_{\nu+1}\tau_i \in \ker(\partial_\nu)$. So we get

$$\underbrace{\sigma - \sum_{e_i^\nu \notin S_\nu} (a_i c_i^{-1})\partial_{\nu+1}\tau_i}_{\in \ker \partial_\nu} = \underbrace{\sum_{e_i^\nu \in S_\nu} a_i e_i^\nu - \sum_{e_i^\nu \notin S_\nu} a_i c_i^{-1} r_i}_{\in \langle S_\nu \rangle}$$

As $\ker(\partial_\nu) \cap \langle S_\nu \rangle = \{0\}$, we get $\sigma - \sum_{e_i^\nu \notin S_\nu} (a_i c_i^{-1})\partial_{\nu+1}\tau_i = 0$. Hence,

$$\sigma = \sum_{e_i^\nu \notin S_\nu} (a_i c_i^{-1})\partial_{\nu+1}\tau_i \in \operatorname{im}(\partial_{\nu+1}),$$

and $\ker(\partial_\nu) \subseteq \operatorname{im}(\partial_{\nu+1})$. Therefore, $\ker(\partial_\nu) = \operatorname{im}(\partial_{\nu+1})$ and $H_\nu(C) = 0$ for all $1 \leq \nu \leq (d-1)$.

For $\nu = 0$, we have to show $H_0(C) \cong R$. By definition, $\ker(\partial_0) = C_0 = \langle \Omega_0 \rangle$ with $\Omega_0 = \{e_1^0, \dots, e_{k_0}^0\}$ and $k_0 \geq 1$ since ${}^\# S_0 = 1$. Furthermore, $\lambda e_i^0 \notin \operatorname{im} \partial_1$ for any $1 \leq i \leq k_0$ and any $\lambda \in R \setminus \{0\}$ because ${}^\# \operatorname{supp}(\partial_1 x) \geq 2$ for every $x \in C_1 \setminus \ker(\partial_1)$. We treat two cases separately:

- ${}^\#\Omega_0 = 1$, so $\Omega_0 = \{e_1^0\}$. Then $\operatorname{im}(\partial_1) = 0$ because of the cone condition 2 of Definition 2.3. So we get

$$H_0(C) = {}^{\ker(\partial_0)}\!\big/\!_{\operatorname{im}(\partial_1)} = {}^{\langle e_1^0 \rangle}\!\big/\!_{0} \cong R.$$

- ${}^\#\Omega_0 \geq 2$, so $\Omega_0 \setminus S_0 \neq \emptyset$. Without loss of generality, let $S_0 = \{e_1^0\}$. For every e_i^0 with $2 \leq i \leq k_0$, some $\tau_i \in C_1$ exists such that

$$\partial_1 \tau_i = c_i e_i^0 + d_i e_1^0 \qquad \text{with } c_i \text{ unit in } R \text{ and } d_i \neq 0.$$

  Without loss of generality we assume that

$$\partial_1 \tau_i = e_i^0 + d_i e_1^0 \qquad \text{with } d_i \neq 0.$$

  All elements $\partial_1 \tau_i$ are linearly independent. Because $\partial_1 \tau_i \in \operatorname{im}(\partial_1)$ for $2 \leq i \leq k_0$ and $\lambda e_1^0 \notin \operatorname{im}(\partial_1)$ for every $\lambda \in R \setminus \{0\}$, we get

$$\begin{aligned} H_0(C) &= {}^{\ker(\partial_0)}\!\big/\!_{\operatorname{im}(\partial_1)} \\ &= {}^{\langle e_1^0, e_2^0, \dots, e_{k_0}^0 \rangle}\!\big/\!_{\operatorname{im}(\partial_1)} \cong \langle e_1^0 \rangle \cong R. \end{aligned}$$

$\square$

For later purpose we have a look at a special finite chain complex (C, Ω) of order 1 whose chain module C_1 is generated by a single element:

$$C_1 = \langle e_1^1 \rangle, \qquad C_0 = \langle e_1^0, \ldots, e_k^0 \rangle \quad \text{with } \mathrm{bd}(e_1^1) = \{e_1^0, \ldots, e_k^0\}.$$

Therefore, $(C, \Omega) = (C_{e_1^1}, \Omega_{e_1^1})$. What is known about the cardinality of the basis Ω_0 if (C, Ω) is acyclic?

C_0 is generated by at least one element (i.e. $k \geq 1$) if (C, Ω) is acyclic. The image $\mathrm{im}(\partial_1)$ is a free submodule of C_0 generated by $\partial_1 e_1^1 = \sum_{i=1}^k a_i e_i^0$ with $a_i \neq 0$ for all i. We distinguish the following cases:

$k = 1$: $C_0 = \langle e_1^0 \rangle$, so $\partial_1 e_1^1 = a_1 e_1^0$ with $a_1 \neq 0$. We get

$$H_0(C) = C_0 / \mathrm{im}(\partial_1) = \langle e_1^0 \rangle / \langle a_1 e_1^0 \rangle \ncong R \quad \text{if } a_1 \neq 0.$$

$k \geq 3$: $C_0 = \langle e_1^0, \ldots, e_k^0 \rangle$, $\mathrm{im}(\partial_1) = \langle \sum_{i=1}^k a_i e_i^0 \rangle$.

We assume $H_0(C) = C_0 / \mathrm{im}(\partial_1) \cong R$. Then any two nonzero homology classes $[e_\ell^0]$, $[e_j^0]$, $\ell \neq j$, of $C_0 / \mathrm{im}(\partial_1)$ are not independent, i.e. there exist elements $x, y \in R \setminus \{0\}$ such that

$$x e_\ell^0 + y e_j^0 = r \sum_{i=1}^k a_i e_i^0 = \sum_{i=1}^k (r a_i) e_i^0 \in \mathrm{im}(\partial_1).$$

Since $\{e_1^0, \ldots, e_k^0\}$ is a basis of C_0 and all $a_i \neq 0$, we conclude $r = 0$. So $x e_\ell^0 + y e_j^0 = 0$ which is a contradiction to the independence of e_ℓ^0 and e_j^0 in C_0. Hence, $H_0(C) \ncong R$.

The case $k = 2$ remains. Indeed, it is possible to get $H_0(C) \cong R$ then. Let $\partial_1 e_1^1 = a_1 e_1^0 + a_2 e_2^0$ with a unit $a_2 \in R$. Then, $a_2^{-1} a_1 e_1^0 + e_2^0 \in \mathrm{im}\, \partial_1$, and we get

$$H_0(C) = \langle e_1^0, e_2^0 \rangle / \langle a_2^{-1} a_1 e_1^0 + e_2^0 \rangle \cong \langle e_1^0 \rangle \cong R.$$

However, it is not necessary that a_1 or a_2 in $\partial_1 e_1^1 = a_1 e_1^0 + a_2 e_2^0$ is a unit. Take $R = \mathbb{Z}$ and $\partial_1 e_1^1 = 2e_1^0 + 3e_2^0$. Then

$$H_0(C) = \langle e_1^0, e_2^0 \rangle / \langle 2e_1^0 + 3e_2^0 \rangle \cong \mathbb{Z}$$

since this factor module is generated by the homology class $[e_1^0 + e_2^0]$.

We summarise:

Lemma 2.7. *Let (C, Ω) be a pure chain complex of order 1 over a principal ideal domain R. Let its chain modules C_0 be finitely generated and C_1 generated by a single element:*

$$C_1 = \langle e_1^1 \rangle, \qquad C_0 = \langle e_1^0, \ldots, e_k^0 \rangle \quad \textit{with } \operatorname{bd}(e_1^1) = \{e_1^0, \ldots, e_k^0\}.$$

If (C, Ω) is acyclic, then C_0 is generated by two elements (so $k = 2$).

The converse is not true. If C_0 is generated by two elements, then (C, Ω) is not necessarily acyclic. We take $R = \mathbb{Z}$ and $\partial_1 e_1^1 = 2e_1^0 + 2e_2^0$ and get

$$H_0(C) = {\langle e_1^0, e_2^0 \rangle}\big/{\langle 2e_1^0 + 2e_2^0 \rangle} \ncong \mathbb{Z}$$

as the factor module is not torsion free: $2[e_1^0 + e_2^0] = [0]$.

To get more familiar with acyclic chain complexes we consider some examples.

Example 2.8.

1. Consider a simplicial complex Δ which is a cone in the common sense. Then the corresponding chain complex (C_Δ, Ω) is also a cone according to our definition: If the distinguished vertex of Δ is v_0, then we choose these basis elements of Ω_ν for S_ν which correspond to ν-dimensional simplices containing the vertex v_0. In particular, $S_0 \cong \{v_0\}$.

   There is no need to set S_ν in this way, as the following example shows.

2. Consider a finite chain complex (C, Ω) of order 2 over $\mathbb{Z}$ with bases $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0\}$ and boundary maps

$$\partial_2(e_1^2) = e_1^1 - e_2^1 + e_3^1, \qquad \begin{aligned} \partial_1(e_1^1) &= e_3^0 - e_2^0, \\ \partial_1(e_2^1) &= e_3^0 - e_1^0, \\ \partial_1(e_3^1) &= e_2^0 - e_1^0. \end{aligned}$$

   We choose $S_2 = \Omega_2 = \{e_1^2\}$, $S_1 = \{e_1^1, e_2^1\}$ and $S_0 = \{e_1^0\}$. Then all cone conditions of Definition 2.3 are fulfilled but $e_1^0 \notin (C_{e_1^1}, \Omega_{e_1^1})$.

3. Let (C, Ω) be a finite chain complex of order 2 over $\mathbb{Z}$ whose chain modules have the bases $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$

with boundary maps

$$\partial_2(e_1^2) = e_1^1 + e_2^1, \qquad \begin{aligned} \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ \partial_1(e_2^1) &= e_1^0 - e_2^0. \end{aligned}$$

The choice of $S_2 = \Omega_2 = \{e_1^2\}$, $S_1 = \{e_1^1\}$ and $S_0 = \{e_1^0\}$ makes (C, Ω) a cone. Notice that this chain complex does not come from a simplicial complex!

4. Again, we consider a finite chain complex (C, Ω) of order 2 over $\mathbb{Z}$. Let $\Omega_2 = \{e_1^2, e_2^2, e_3^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1, e_4^1\}$, $\Omega_0 = \{e_1^0, e_2^0\}$ and

$$\begin{aligned} \partial_2(e_1^2) &= e_1^1 + e_2^1, & \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ \partial_2(e_2^2) &= e_2^1 + e_3^1, & \partial_1(e_2^1) &= e_1^0 - e_2^0, \\ \partial_2(e_3^2) &= e_3^1 + e_4^1, & \partial_1(e_3^1) &= e_2^0 - e_1^0, \\ & & \partial_1(e_4^1) &= e_1^0 - e_2^0. \end{aligned}$$

   We choose $S_2 = \Omega_2 = \{e_1^2, e_2^2, e_3^2\}$, $S_1 = \{e_2^1\}$ and $S_0 = \{e_1^0\}$. Because $\partial_2(e_2^2 - e_3^2) = e_2^1 - e_4^1$, all cone conditions of Definition 2.3 are satisfied.

5. Our last example is also a finite chain complex (C, Ω) of order 2 over $\mathbb{Z}$. Let $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1, e_4^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0, e_4^0\}$ such that

$$\partial_2(e_1^2) = e_1^1 + e_2^1 + e_3^1 + e_4^1, \qquad \begin{aligned} \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ \partial_1(e_2^1) &= e_3^0 - e_2^0, \\ \partial_1(e_3^1) &= e_4^0 - e_3^0, \\ \partial_1(e_4^1) &= e_1^0 - e_4^0. \end{aligned}$$

   This chain complex is acyclic, but not a cone, as we will see now.

   By definition holds: $\mathrm{bd}(e_1^1) \cup \mathrm{bd}(e_3^1) = \Omega_0 = \mathrm{bd}(e_2^1) \cup \mathrm{bd}(e_4^1)$, and using the cone condition 1a of Definition 2.3, we conclude ${}^{\#}S_1 \leq 2$, hence ${}^{\#}(\Omega_1 \setminus S_1) \geq 2$. Since $\mathrm{bd}(e_1^2) = \Omega_1$, we obtain ${}^{\#}\big(\mathrm{bd}(e_1^2) \cap (\Omega_1 \setminus S_1)\big) \geq 2$, so we get a contradiction to the cone condition 1b.

Remark 2.9. Every simplex is also a geometrical cone in the common sense. As described in Example 2.8(1), its corresponding chain complex is also a cone in the sense of Definition 2.3. Hence, any chain complex coming from a simplex is acyclic due to Lemma 2.6.

2.2. Mapping Cones

If there is a chain map f between two chain complexes, a new chain complex called *mapping cone of* f can be constructed. We will introduce this term and show some simple properties of mapping cones of the identity chain map.

Recall that R is always a commutative and unital ring if nothing else is mentioned. Throughout this section, let B and C be chain complexes over R which are equipped, respectively, with the chain modules B_ν and C_ν and the boundary maps ∂_ν^B and ∂_ν^C for all $\nu \in \mathbb{Z}$.

2.2.1. A First Sight

To define mapping cones, we need the notion of a chain map from Definition 1.28. We follow Dold (1972, p. 18), but use the notation of the boundary map by Gelfand and Manin (1996, p. 154) and Weibel (1994, p. 18). Similar definitions of mapping cones are given by Cohen (1973, pp. 8 and 75) and Spanier (1966, p. 166).

Definition 2.10. Let $f\colon B \to C$ be a chain map. The *mapping cone of* f is the chain complex $\mathcal{C}(f)$ whose chain modules are $\mathcal{C}(f)_\nu = C_\nu \oplus B_{\nu-1}$ for $\nu \in \mathbb{Z}$. The boundary map $\delta_\nu\colon \mathcal{C}(f)_\nu \to \mathcal{C}(f)_{\nu-1}$ is given by[1]

$$\delta_\nu \begin{pmatrix} c_\nu \\ b_{\nu-1} \end{pmatrix} = \begin{pmatrix} \partial_\nu^C(c_\nu) + f_{\nu-1}(b_{\nu-1}) \\ -\partial_{\nu-1}^B(b_{\nu-1}) \end{pmatrix} = \begin{pmatrix} \partial_\nu^C & f_{\nu-1} \\ 0 & -\partial_{\nu-1}^B \end{pmatrix} \begin{pmatrix} c_\nu \\ b_{\nu-1} \end{pmatrix} \text{ for } \nu \in \mathbb{Z}.$$

In general, the homology of a mapping cone $\mathcal{C}(f)$ for some chain map $f\colon B \to C$ depends on the homology of B and C. For example, let all $f_\nu = 0$. Then $H_\nu(\mathcal{C}(0)) \cong H_\nu(C) \oplus H_{\nu-1}(B)$.

But in some special case the homology is well-known.

Lemma 2.11. *For the identity chain map* $\mathrm{id}_C\colon C \to C$*, the homology modules of the mapping cone* $\mathcal{C}(\mathrm{id}_C)$ *are* $H_\nu(\mathcal{C}(\mathrm{id}_C)) = 0$ *for all* $\nu \in \mathbb{Z}$.

[1] Indeed, all δ_ν are boundary maps since

$$\begin{pmatrix} \partial_\nu^C & f_{\nu-1} \\ 0 & -\partial_{\nu-1}^B \end{pmatrix} \circ \begin{pmatrix} \partial_{\nu+1}^C & f_\nu \\ 0 & -\partial_\nu^B \end{pmatrix} = \begin{pmatrix} \partial_\nu^C \circ \partial_{\nu+1}^C & \partial_\nu^C \circ f_\nu - f_{\nu-1} \circ \partial_\nu^B \\ 0 & \partial_{\nu-1}^B \circ \partial_\nu^B \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Proof. Let $\binom{x}{y} \in C_\nu \oplus C_{\nu-1}$. Since $\delta_\nu\binom{x}{y} = \binom{\partial_\nu(x)+y}{-\partial_{\nu-1}(y)}$, we get

$$\ker(\delta_\nu) = \left\{ \binom{x}{y} \;\middle|\; y = -\partial_\nu(x) \right\}.$$

But $\binom{x}{-\partial_\nu(x)} = \delta_{\nu+1}\binom{0}{x}$, so $\ker(\delta_\nu) = \operatorname{im}(\delta_{\nu+1})$. Hence, $H_\nu(\mathcal{C}(\mathrm{id}_C)) = 0$ for all $\nu \in \mathbb{Z}$. □

If B and C are nonnegative chain complexes over some principal ideal domain R, one can get a mapping cone for augmented B and some chain map $f\colon B \to C$.

Definition 2.12. Let R be a principal ideal domain. Let B, C be nonnegative chain complexes over R and $f\colon B \to C$ be a chain map. The *mapping cone of* f for augmented B with an augmentation map $\epsilon\colon B_0 \to R$ is the nonnegative chain complex $\widehat{\mathcal{C}}(f)$ whose chain modules are $\widehat{\mathcal{C}}(f)_\nu = C_\nu \oplus B_{\nu-1}$ for $\nu \geq 1$ and $\widehat{\mathcal{C}}(f)_0 = C_0 \oplus R$. Each boundary map $\delta_\nu\colon \widehat{\mathcal{C}}(f)_\nu \to \widehat{\mathcal{C}}(f)_{\nu-1}$ is given by

$$\begin{aligned}
\delta_\nu\begin{pmatrix} c_\nu \\ b_{\nu-1} \end{pmatrix} &= \begin{pmatrix} \partial^C_\nu(c_\nu) + f_{\nu-1}(b_{\nu-1}) \\ -\partial^B_{\nu-1}(b_{\nu-1}) \end{pmatrix} = \begin{pmatrix} \partial^C_\nu & f_{\nu-1} \\ 0 & -\partial^B_{\nu-1} \end{pmatrix}\begin{pmatrix} c_\nu \\ b_{\nu-1} \end{pmatrix} \quad \text{for } \nu \geq 2,\\
\delta_1\begin{pmatrix} c_1 \\ b_0 \end{pmatrix} &= \begin{pmatrix} \partial^C_1(c_1) + f_0(b_0) \\ -\epsilon(b_0) \end{pmatrix} = \begin{pmatrix} \partial^C_1 & f_0 \\ 0 & -\epsilon \end{pmatrix}\begin{pmatrix} c_1 \\ b_0 \end{pmatrix},\\
\delta_0\begin{pmatrix} c_0 \\ r \end{pmatrix} &= 0.
\end{aligned}$$

Lemma 2.13. *For a nonnegative chain complex C over some principal ideal domain R and the identity chain map $\mathrm{id}_C\colon C \to C$, the mapping cone $\widehat{\mathcal{C}}(\mathrm{id}_C)$ is acyclic.*

Proof. As in the proof of Lemma 2.11 we see that $H_\nu(\widehat{\mathcal{C}}(\mathrm{id}_C)) = 0$ for $\nu \geq 2$. Since $\delta_1\binom{x}{y} = \binom{\partial_1(x)+y}{-\epsilon(y)} \in C_0 \oplus R$ for $\binom{x}{y} \in C_1 \oplus C_0$, a similar argumentation yields that $H_1(\widehat{\mathcal{C}}(\mathrm{id}_C)) = 0$.

Since δ_0 is the zero map, $\ker(\delta_0) = C_0 \oplus R$. For any $\binom{y}{r} \in C_0 \oplus R$, we get

$$\binom{y}{r} = \binom{y}{-\epsilon(y)} + \binom{0}{\epsilon(y)+r} = \delta_1\binom{0}{y} + \binom{0}{\epsilon(y)+r},$$

hence $H_0(\mathcal{C}(\mathrm{id}_C)) \cong \langle \binom{0}{1} \rangle \cong R$. So the chain complex $\widehat{\mathcal{C}}(\mathrm{id}_C)$ is acyclic. □

We will see later in Section 2.3.2 that the mapping cone $\widehat{\mathcal{C}}(\mathrm{id}_C)$ is even a cone if the chain complex C fulfils special conditions.

2.2.2. Mapping Cones over Free Chain Complexes

Let $f\colon B \to C$ be a chain map. If both B and C are (finitely) free chain complexes with bases Ω^B and Ω^C, respectively, then the mapping cone $\mathcal{C}(f)$ is also (finitely) free. The basis of its chain module $\mathcal{C}(f)_\nu = C_\nu \oplus B_{\nu-1}$ is

$$\Psi_\nu := \left\{ \begin{pmatrix} e_i^\nu \\ 0 \end{pmatrix} \;\middle|\; e_i^\nu \in \Omega_\nu^C \right\} \cup \left\{ \begin{pmatrix} 0 \\ g_i^{\nu-1} \end{pmatrix} \;\middle|\; g_i^\nu \in \Omega_{\nu-1}^B \right\} \quad \text{for } \nu \in \mathbb{Z}.$$

We set $\Psi := \dot{\bigcup}_{\nu\in\mathbb{Z}} \Psi_\nu$ and denote the (finitely) free mapping cone by $\big(\mathcal{C}(f), \Psi\big)$.

Similarly, for (finitely) free and nonnegative chain complexes (B, Ω^B) and (C, Ω^C) over a principal ideal domain R, the mapping cone $\widehat{\mathcal{C}}(f)$ for augmented B is also (finitely) free. For $\nu \in \mathbb{N}$, its chain modules $\widehat{\mathcal{C}}(f)_\nu$ have the bases

$$\Phi_\nu := \left\{ \begin{pmatrix} e_i^\nu \\ 0 \end{pmatrix} \;\middle|\; e_i^\nu \in \Omega_\nu^C \right\} \cup \left\{ \begin{pmatrix} 0 \\ g_i^{\nu-1} \end{pmatrix} \;\middle|\; g_i^\nu \in \Omega_{\nu-1}^B \right\} \quad \text{for } \nu \geq 1,$$

$$\Phi_0 := \left\{ \begin{pmatrix} e_i^0 \\ 0 \end{pmatrix} \;\middle|\; e_i^0 \in \Omega_0^C \right\} \cup \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

We denote the mapping cone by $\big(\widehat{\mathcal{C}}(f), \Phi\big)$ with basis $\Phi := \dot{\bigcup}_{\nu\in\mathbb{N}} \Phi_\nu$.

To get familiar with free mapping cones, we have a look at some examples.

Example 2.14.

1. Let (B, Ω^B), (C, Ω^C) be two finite chain complexes of order 1 over $\mathbb{Z}$ with chain modules $C_1 = \langle e_1^1, e_2^1 \rangle$, $C_0 = \langle e_1^0 \rangle$, $B_1 = \langle g_1^1 \rangle$ and $B_0 = \langle g_1^0 \rangle$. Let all boundary maps of (B, Ω^B) and (C, Ω^C) be zero.

   We define an embedding $f\colon B \hookrightarrow C$ by $f_1(g_1^1) = e_1^1$ and $f_0(g_1^0) = e_1^0$. The mapping cone $\big(\mathcal{C}(f), \Psi\big)$ of f is a finite chain complex

   $$\{0\} \oplus B_1 \xrightarrow{\delta_2} C_1 \oplus B_0 \xrightarrow{\delta_1} C_0 \oplus \{0\} \xrightarrow{\delta_0} 0$$

   of order 2 with the following bases of its chain modules:

   $$\Psi_2 = \left\{ \begin{pmatrix} 0 \\ g_1^1 \end{pmatrix} \right\}, \qquad \Psi_1 = \left\{ \begin{pmatrix} e_1^1 \\ 0 \end{pmatrix}, \begin{pmatrix} e_2^1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ g_1^0 \end{pmatrix} \right\}, \qquad \Psi_0 = \left\{ \begin{pmatrix} e_1^0 \\ 0 \end{pmatrix} \right\}.$$

   Its boundary maps are

   $$\delta_\nu = \begin{pmatrix} 0 & f_{\nu-1} \\ 0 & 0 \end{pmatrix} \text{ for } \nu \geq 1, \quad \delta_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Therefore, we get the following homology modules:

$$H_2\big(\mathcal{C}(f)\big) = 0, \qquad H_1\big(\mathcal{C}(f)\big) \cong \mathbb{Z}, \qquad H_0\big(\mathcal{C}(f)\big) = 0.$$

So, this mapping cone is not an acyclic chain complex. In particular, it is not a cone.

2. We consider a finite chain complex (C, Ω) of order 1 over $\mathbb{Z}$ with chain modules $C_1 = \langle e_1^1, e_2^1 \rangle$ and $C_0 = \langle e_1^0, e_2^0 \rangle$. Let $\partial_1(e_1^1) = e_1^0 + e_2^0 = -\partial_1(e_2^1)$. We take the elementary subcomplex $(U, \Omega^U) := (C_{e_1^1}, \Omega_{e_1^1})$, which has the chain modules $U_1 = \langle e_1^1 \rangle$ and $U_0 = \langle \mathrm{bd}(e_1^1) \rangle = C_0$, and let $\varepsilon \colon U \hookrightarrow C$ be the canonical embedding.

   Then the mapping cone $\big(\mathcal{C}(\varepsilon), \Psi\big)$ of ε is finite of order 2:

$$\{0\} \oplus U_1 \xrightarrow{\delta_2} C_1 \oplus U_0 \xrightarrow{\delta_1} C_0 \oplus \{0\} \xrightarrow{\delta_0} 0.$$

   The bases of its chain modules are

$$\begin{aligned} \Psi_2 &= \left\{ \begin{pmatrix} 0 \\ e_1^1 \end{pmatrix} \right\}, \\ \Psi_1 &= \left\{ \begin{pmatrix} e_1^1 \\ 0 \end{pmatrix}, \begin{pmatrix} e_2^1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1^0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_2^0 \end{pmatrix} \right\}, \\ \Psi_0 &= \left\{ \begin{pmatrix} e_1^0 \\ 0 \end{pmatrix}, \begin{pmatrix} e_2^0 \\ 0 \end{pmatrix} \right\}, \end{aligned}$$

   and the mapping cone has the boundary maps

$$\delta_2 = \begin{pmatrix} 0 & \varepsilon_1 \\ 0 & -\partial_1 \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} \partial_1 & \varepsilon_0 \\ 0 & 0 \end{pmatrix}, \quad \delta_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

   The homology modules are

   - $H_2\big(\mathcal{C}(\varepsilon)\big) = 0$,
   - $H_1\big(\mathcal{C}(\varepsilon)\big) \cong \mathbb{Z}$ because $\ker(\delta_1) = \left\langle \begin{pmatrix} e_1^1 \\ -e_1^0 - e_2^0 \end{pmatrix}, \begin{pmatrix} e_2^1 \\ e_1^0 + e_2^0 \end{pmatrix} \right\rangle$,
   - $H_0\big(\mathcal{C}(\varepsilon)\big) = 0$.

   Hence, this mapping cone is neither acyclic nor a cone, too.

3. Our third example is a chain complex which is obtained from a simplicial complex. Let (C, Ω) be a finite chain complex of order 2 over $\mathbb{Z}$ whose chain modules have the bases $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0\}$. Let the boundary maps ∂_ν be defined as follows:

$$\begin{aligned} \partial_2(e_1^2) &= e_1^1 - e_2^1 + e_3^1, & \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ & & \partial_1(e_2^1) &= e_3^0 - e_1^0, \\ & & \partial_1(e_3^1) &= e_3^0 - e_2^0. \end{aligned}$$

We consider the elementary subcomplex $(U, \Omega^U) := (C_{e_1^1}, \Omega_{e_1^1})$ whose chain modules are $U_1 = \langle e_1^1 \rangle$ and $U_0 = \langle e_1^0, e_2^0 \rangle$. Let ι be the injective embedding $U \hookrightarrow C$.

The mapping cone $\big(\mathcal{C}(\iota), \Psi\big)$ of ι is a finite chain complex of order 2:

$$C_2 \oplus U_1 \xrightarrow{\delta_2} C_1 \oplus U_0 \xrightarrow{\delta_1} C_0 \oplus \{0\} \xrightarrow{\delta_0} 0$$

whose chain modules have the bases

$$\begin{aligned} \Psi_2 &= \left\{ \begin{pmatrix} e_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1^1 \end{pmatrix} \right\}, \\ \Psi_1 &= \left\{ \begin{pmatrix} e_1^1 \\ 0 \end{pmatrix}, \begin{pmatrix} e_2^1 \\ 0 \end{pmatrix}, \begin{pmatrix} e_3^1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1^0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_2^0 \end{pmatrix} \right\}, \\ \Psi_0 &= \left\{ \begin{pmatrix} e_1^0 \\ 0 \end{pmatrix}, \begin{pmatrix} e_2^0 \\ 0 \end{pmatrix}, \begin{pmatrix} e_3^0 \\ 0 \end{pmatrix} \right\}. \end{aligned}$$

The boundary maps of $\big(\mathcal{C}(\iota), \Psi\big)$ are

$$\delta_2 = \begin{pmatrix} \partial_2 & \iota_1 \\ 0 & -\partial_1 \end{pmatrix}, \quad \delta_1 = \begin{pmatrix} \partial_1 & \iota_0 \\ 0 & 0 \end{pmatrix}, \quad \delta_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Calculating the homology, we see that $H_\nu\big(\mathcal{C}(\iota)\big) = 0$ for all ν. So, this mapping cone is not acyclic, too.

4. We vary our last example by augmenting the subcomplex (U, Ω^U) by $\mathbb{Z}$ and define an augmentation map $\epsilon \colon U_0 \to \mathbb{Z}$ by $\epsilon(e_1^0) = 1 = \epsilon(e_2^0)$. Thereby, we get the mapping cone $\big(\widehat{\mathcal{C}}(\iota), \Phi\big)$ which is a finite chain complex

$$C_2 \oplus U_1 \xrightarrow{\widehat{\delta}_2} C_1 \oplus U_0 \xrightarrow{\widehat{\delta}_1} C_0 \oplus \mathbb{Z} \xrightarrow{\widehat{\delta}_0} 0$$

of order 2 with the following bases

$$\Phi_2 = \left\{ \begin{pmatrix} e_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1^1 \end{pmatrix} \right\},$$
$$\Phi_1 = \left\{ \begin{pmatrix} e_1^1 \\ 0 \end{pmatrix}, \begin{pmatrix} e_2^1 \\ 0 \end{pmatrix}, \begin{pmatrix} e_3^1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1^0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_2^0 \end{pmatrix} \right\},$$
$$\Phi_0 = \left\{ \begin{pmatrix} e_1^0 \\ 0 \end{pmatrix}, \begin{pmatrix} e_2^0 \\ 0 \end{pmatrix}, \begin{pmatrix} e_3^0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

of its chain modules and the boundary maps

$$\widehat{\delta}_2 = \begin{pmatrix} \partial_2 & \iota_1 \\ 0 & -\partial_1 \end{pmatrix}, \quad \widehat{\delta}_1 = \begin{pmatrix} \partial_1 & \iota_0 \\ 0 & -\epsilon \end{pmatrix}, \quad \widehat{\delta}_0 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Its homology modules are

$$H_2(\widehat{\mathcal{C}}(\iota)) = 0, \qquad H_1(\widehat{\mathcal{C}}(\iota)) = 0, \qquad H_0(\widehat{\mathcal{C}}(\iota)) \cong \mathbb{Z},$$

so this mapping cone is acyclic. Furthermore, $(\widehat{\mathcal{C}}(\iota), \Phi)$ is even a cone by choosing the following subsets of the chain module bases:

$$S_2 = \left\{ \begin{pmatrix} e_1^2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1^1 \end{pmatrix} \right\}, \; S_1 = \left\{ \begin{pmatrix} e_1^1 \\ 0 \end{pmatrix}, \begin{pmatrix} e_2^1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1^0 \end{pmatrix} \right\}, \; S_0 = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

The second example above shows that a mapping cone $(\mathcal{C}(\varepsilon), \Psi)$ of a chain map $\varepsilon\colon U \to C$ needs not to be pure even though the chain complexes (C, Ω) and (U, Ω^U) are pure. However, for a pure chain complex (C, Ω), the mapping cone of the identity chain map id_C is indeed a pure chain complex.

Lemma 2.15. *Let (C, Ω) be a pure and finite chain complex of order d over a ring R. Then the mapping cone $(\mathcal{C}(\mathrm{id}_C), \Psi)$ is pure and finite of order $(d+1)$. If additionally R is a principal ideal domain and a nonzero augmentation map $\epsilon\colon C_0 \to R$ exists, then the mapping cone $(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi)$ for augmented (C, Ω) is pure.*

Proof. It is clear that both mapping cones $(\mathcal{C}(\mathrm{id}_C), \Psi)$ and $(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi)$ are finite of order $(d+1)$. Let $\Omega_\nu := \{e_1^\nu, \dots, e_{k_\nu}^\nu\}$ be the basis of the chain module C_ν for $0 \le \nu \le d$. Then we obtain for the basis elements in Ψ:

$$\delta_{\nu+1} \begin{pmatrix} 0 \\ e_i^\nu \end{pmatrix} = \begin{pmatrix} e_i^\nu \\ -\partial_\nu(e_i^\nu) \end{pmatrix} \quad \text{and} \quad \delta_\nu \begin{pmatrix} e_i^\nu \\ 0 \end{pmatrix} = \begin{pmatrix} \partial_\nu(e_i^\nu) \\ 0 \end{pmatrix} \quad \text{for } 0 \le \nu \le d.$$

Therefore, $\big(\mathcal{C}(\mathrm{id}_C), \Psi\big)$ is pure if (C, Ω) is pure. For $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$, we only have to consider

$$\widehat{\delta}_1 \begin{pmatrix} 0 \\ e_i^0 \end{pmatrix} = \begin{pmatrix} e_i^0 \\ -\epsilon\big(e_i^0\big) \end{pmatrix}$$

because all other boundary maps are the same as in $\big(\mathcal{C}(\mathrm{id}_C), \Psi\big)$. If the augmentation map ϵ is not the zero map, the basis element $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ is not maximal in $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$. Therefore, we get a pure chain complex. $\square$

2.2.3. Trivial Mapping Cones

Definition 2.16. Let B, C be chain complexes over a ring R. A mapping cone $\mathcal{C}(f)$ of a chain map $f \colon B \to C$ is called *trivial* if B or C is the zero chain complex Z with all chain modules $Z_\nu = \{0\}$.

Each chain complex can be regarded as a trivial mapping cone. Let C be an arbitrary chain complex over R and $\iota \colon Z \hookrightarrow C$ be the canonical embedding. Then the mapping cone $\mathcal{C}(\iota)$ is isomorphic to C. Its chain modules are $\mathcal{C}(\iota)_\nu = C_\nu \oplus \{0\}$ with boundary maps $\delta_\nu = \left(\begin{smallmatrix} \partial_\nu & 0 \\ 0 & 0 \end{smallmatrix}\right)$. The homology modules are $H_\nu\big(\mathcal{C}(\iota)\big) = H_\nu(C)$ for all $\nu \in \mathbb{Z}$. Notice that the mapping cone $\mathcal{C}(\iota)$ is always a subcomplex of the mapping cone $\mathcal{C}(\mathrm{id}_C)$ of the identity chain map $\mathrm{id}_C \colon C \to C$.

If we consider the same chain complexes C and Z as above and define a surjective chain map $\sigma \colon C \twoheadrightarrow Z$ via the canonical maps $\sigma_\nu \colon C_\nu \twoheadrightarrow \{0\} = Z_\nu$, we get a mapping cone $\mathcal{C}(\sigma)$ which is slightly different from C. The mapping cone $\mathcal{C}(\sigma)$ has the chain modules $\mathcal{C}(\sigma)_\nu = \{0\} \oplus C_{\nu-1}$ and the boundary maps $\delta_\nu = \left(\begin{smallmatrix} 0 & \sigma_{\nu-1} \\ 0 & -\partial_{\nu-1} \end{smallmatrix}\right)$ for all $\nu \in \mathbb{Z}$. It is isomorphic to the shifted[2] chain complex $C[-1]$ with chain modules $C[-1]_\nu := C_{\nu-1}$ for all $\nu \in \mathbb{Z}$ and boundary maps $\partial[-1]_\nu := -\partial_{\nu-1}$. Therefore, the homology modules of $\mathcal{C}(\sigma)$ and of the shifted chain complex $C[-1]$ are the same, and we obtain $H_\nu\big(\mathcal{C}(\sigma)\big) = H_\nu\big(C[-1]\big) = H_{\nu-1}(C)$ for all $\nu \in \mathbb{Z}$.

In general, neither the trivial mapping cone $\mathcal{C}(\iota)$ nor $\mathcal{C}(\sigma)$ is acyclic or a cone.

[2]In general, shifted chain complexes are defined for any $k \in \mathbb{Z}$: The shifted complex $C[k]$ has chain modules $C[k]_\nu = C_{\nu+k}$ for $\nu \in \mathbb{Z}$. Its boundary maps are $\partial[k]_\nu = (-1)^k \partial_{\nu+k}$. More about this term can be found in the books by Weibel (1994, p. 9) and Gelfand and Manin (1996, p. 154).

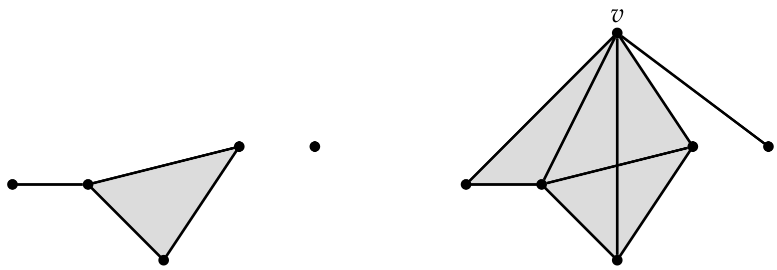


Figure 2.2.: Constructing a cone over a simplicial complex by adding a new vertex v

2.3. Constructing a Cone

By Lemma 2.13 we know that the mapping cone $\widehat{C}(\mathrm{id}_C)$ is acyclic for any nonnegative chain complex C over some principal ideal domain. We will show that $\widehat{C}(\mathrm{id}_C)$ is even a cone for special chain complexes, in imitation of the geometrical construction of a cone over some geometrical simplicial complex (cf. Gelfand and Manin, 1996, p. 26).

2.3.1. The Simplicial Case

Let Δ be a simplicial complex of dimension $d \geq 0$ having k vertices. We construct a geometrical cone over Δ in the following way (cf. Figure 2.2): We add a new vertex v to the complex Δ, and for every i-dimensional simplex S we get a new $(i+1)$-dimensional simplex $\widehat{S}$ whose vertices are v and the vertices of S. This new simplex $\widehat{S}$ contains S as a facet. For example, the geometrical cone over a triangle is a tetrahedron, cf. Figure 2.3.

Therefore, a cone is again a simplicial complex whose vertices are v, which is said to be the *apex* of the cone, and the vertices of Δ. In particular, the

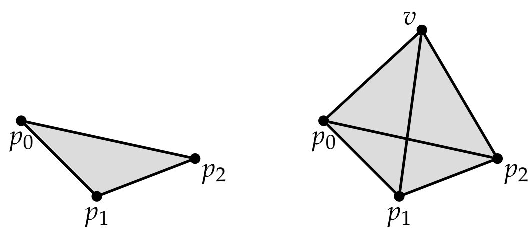


Figure 2.3.: A tetrahedron is a cone over a triangle

original simplicial complex Δ is contained in the cone as a subcomplex (cf. Munkres, 1984, p. 44).

As described in Section 1.2.5, the simplicial complex Δ provides a finite chain complex (C_Δ, Ω) of order d over some principal ideal domain R with free chain modules $(C_\Delta)_\nu$ and boundary maps $\partial_\nu \colon (C_\Delta)_\nu \to (C_\Delta)_{\nu-1}$. The bases Ω_ν of the chain modules have as many elements as there are ν-dimensional simplices in Δ. Therefore, we also describe the cone over Δ as a free chain complex which we denote by $\big(\mathrm{con}(C_\Delta), \Phi\big)$ for now. By construction of a geometrical cone, $\big(\mathrm{con}(C_\Delta), \Phi\big)$ is a finite chain complex of order $(d+1)$. The bases Φ_ν of its chain modules are

$$\begin{aligned} \Phi_0 &\cong \Omega_0 \,\dot\cup\, \{v\}, \\ \Phi_\nu &\cong \Omega_\nu \,\dot\cup\, \Omega_{\nu-1} \text{ for } 1 \le \nu \le d, \\ \Phi_{d+1} &\cong \Omega_d. \end{aligned}$$

Hence, the chain modules of $\big(\mathrm{con}(C_\Delta), \Phi\big)$ are

$$\begin{aligned} \big(\mathrm{con}(C_\Delta)\big)_0 &= (C_\Delta)_0 \oplus R, \\ \big(\mathrm{con}(C_\Delta)\big)_\nu &= (C_\Delta)_\nu \oplus (C_\Delta)_{\nu-1} \text{ for } 1 \le \nu \le d, \\ \big(\mathrm{con}(C_\Delta)\big)_{d+1} &= \{0\} \oplus (C_\Delta)_d. \end{aligned}$$

We define boundary maps $\delta_\nu \colon \big(\mathrm{con}(C_\Delta)\big)_\nu \to \big(\mathrm{con}(C_\Delta)\big)_{\nu-1}$ as

$$\delta_\nu = \begin{pmatrix} \partial_\nu & \mathrm{id}_{\nu-1} \\ 0 & -\partial_{\nu-1} \end{pmatrix} \text{ for } \nu \ge 2, \qquad \delta_1 = \begin{pmatrix} \partial_1 & \mathrm{id}_0 \\ 0 & -\epsilon \end{pmatrix}, \qquad \delta_0 = 0,$$

in which $\epsilon \colon (C_\Delta)_0 \to R$ is the usual augmentation map in the simplicial case. The maps $\mathrm{id}_\nu \colon (C_\Delta)_\nu \to (C_\Delta)_\nu$ denote the identity maps of $(C_\Delta)_\nu$ each, due to the fact that any simplex S of Δ is contained in the boundary of the simplex $\widehat{S}$ which arises by adding the new vertex v to S. This vertex v corresponds to the element $\binom{0}{1} \in \big(\mathrm{con}(C_\Delta)\big)_0$.

Defining the boundary maps δ_ν this way, we describe a geometrical cone over some simplicial complex Δ as a mapping cone of the identity chain map $\mathrm{id}_{C_\Delta} \colon C_\Delta \to C_\Delta$ for augmented (C_Δ, Ω), i.e. $\big(\mathrm{con}(C_\Delta), \Phi\big) = \big(\widehat{\mathcal{C}}(\mathrm{id}_{C_\Delta}), \Phi\big)$. Hence, the chain complex $\big(\mathrm{con}(C_\Delta), \Phi\big)$ is acyclic by Lemma 2.13. We will see that $\big(\mathrm{con}(C_\Delta), \Phi\big)$ is a cone in the sense of Definition 2.3 which will be proven in the next section for a more general case (cf. Remark 2.18).

2.3.2. The General Case for Chain Complexes

A geometrical cone over a simplicial complex Δ corresponds to the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_{C_\Delta}), \Phi\big)$ over the finite chain complex (C_Δ, Ω) which is obtained from Δ and must be defined over a principal ideal domain R. In the same way we can create a mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$ over some finite chain complex (C, Ω) of order d which is defined over some principal ideal domain. Due to Lemma 2.13, the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$ is acyclic. But $\big(\widehat{\mathcal{C}}(\mathrm{id}\, C), \Phi\big)$ is not necessarily a cone. For example, if there is some basis element $e_i^0 \in \Omega_0$ with $\epsilon(e_i^0) = 0$, then we have $\delta_1 \begin{pmatrix} 0 \\ e_i^0 \end{pmatrix} = \begin{pmatrix} e_i^0 \\ 0 \end{pmatrix}$. So ${}^{\#}\mathrm{bd} \begin{pmatrix} 0 \\ e_i^0 \end{pmatrix} = 1$ which is impossible for a cone by definition since $\begin{pmatrix} 0 \\ e_i^0 \end{pmatrix} \in \Phi_1$. Furthermore, if $\Omega_\nu = \emptyset$ for some $0 \le \nu \le d$, then we cannot choose a subset $\emptyset \ne S_\nu \subseteq \Omega_\nu$. However, excluding both of these cases gives us a mapping cone which is a cone.

Theorem 2.17. *Let (C, Ω) be a finite chain complex of order d over a principal ideal domain R. Let its chain modules be $C_\nu \ne 0$ for all $0 \le \nu \le d$, and let there be an augmentation map $\epsilon\colon C_0 \to R$ such that $\epsilon(e_i^0) \ne 0$ for all $e_i^0 \in \Omega_0$. Then the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$ is a cone.*

Remark 2.18. All assumptions in Theorem 2.17 are fulfilled by any chain complex over some principal ideal domain which comes from a simplicial complex. Hence, every chain complex corresponding to a simplicial cone is also a cone in the sense of Definition 2.3.

Proof of Theorem 2.17. By Lemma 2.13 the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$ is acyclic. For $0 \le \nu \le d$, let $\Omega_\nu := \{e_1^\nu, \dots, e_{k_\nu}^\nu\}$ be the basis of the chain module C_ν with $k_\nu \ge 1$ by assumption. Then the bases of the chain modules of the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$ are

$$
\begin{aligned}
\Phi_{d+1} &= \left\{ \begin{pmatrix} 0 \\ e_1^d \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ e_{k_d}^d \end{pmatrix} \right\}, \\
\Phi_\nu &= \left\{ \begin{pmatrix} e_1^\nu \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} e_{k_\nu}^\nu \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e_1^{\nu-1} \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ e_{k_{\nu-1}}^{\nu-1} \end{pmatrix} \right\} \text{ for } 1 \le \nu \le d, \\
\Phi_0 &= \left\{ \begin{pmatrix} e_1^0 \\ 0 \end{pmatrix}, \dots, \begin{pmatrix} e_{k_0}^0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.
\end{aligned}
$$

To get a cone, we choose the following subsets S_i of the bases Φ_i:

$$S_{d+1} = \left\{ \begin{pmatrix} 0 \\ e_1^d \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ e_{k_d}^d \end{pmatrix} \right\} = \Phi_{d+1},$$
$$S_\nu = \left\{ \begin{pmatrix} 0 \\ e_1^{\nu-1} \end{pmatrix}, \dots, \begin{pmatrix} 0 \\ e_{k_{\nu-1}}^{\nu-1} \end{pmatrix} \right\} \text{ for } 1 \leq \nu \leq d,$$
$$S_0 = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

We have to check that these subsets satisfy the three conditions for a cone (cf. Definition 2.3).

1. For any $\nu \in \{1, \dots, d+1\}$ and any $\left(\begin{smallmatrix} 0 \\ e_\ell^{\nu-1} \end{smallmatrix}\right) \in S_\nu$ holds:

   $$\delta_\nu \left(\begin{smallmatrix} 0 \\ e_\ell^{\nu-1} \end{smallmatrix}\right) = \left(\begin{smallmatrix} e_\ell^{\nu-1} \\ 0 \end{smallmatrix}\right) + r_\ell \quad \text{with } r_\ell \in \langle S_{\nu-1} \rangle.$$

   Therefore, the cone conditions 1a and 1b are fulfilled.

2. We have to show that $^\#\mathrm{bd}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \neq 1$ for any $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in C_1 \oplus C_0$.

   At first, we consider the basis elements of Φ_1. By assumption, there exists an augmentation map $\epsilon \colon C_0 \to R$ such that $\epsilon(e_\ell^0) = \lambda_\ell \neq 0$ for all $e_\ell^0 \in \Omega_0$. Therefore, we obtain $\delta_1 \left(\begin{smallmatrix} 0 \\ e_\ell^0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} e_\ell^0 \\ 0 \end{smallmatrix}\right) - \lambda_\ell \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$. Hence, $^\#\mathrm{bd}\left(\begin{smallmatrix} 0 \\ e_\ell^0 \end{smallmatrix}\right) = 2$. Furthermore, we know by Theorem 1.51 that $^\#\mathrm{bd}(e_\ell^1) \geq 2$ for all $e_\ell^1 \in \Omega_1 \setminus \ker(\partial_1)$. So the same holds for all the basis elements $\left(\begin{smallmatrix} e_\ell^1 \\ 0 \end{smallmatrix}\right) \in \Phi_1$ since $\delta_1 \left(\begin{smallmatrix} e_\ell^1 \\ 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} \partial_1(e_\ell^1) \\ 0 \end{smallmatrix}\right)$.

   Now we assume that there is some $\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) \in C_1 \oplus C_0$ such that $^\#\mathrm{bd}\left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = 1$. There are two cases to distinguish:

   a) $\delta_1 \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \mu \left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ for some $\mu \neq 0$.

      We have $\left(\begin{smallmatrix} 0 \\ \mu \end{smallmatrix}\right) = \left(\begin{smallmatrix} \partial_1 & \mathrm{id}_0 \\ 0 & -\epsilon \end{smallmatrix}\right) \left(\begin{smallmatrix} x \\ y \end{smallmatrix}\right) = \left(\begin{smallmatrix} \partial_1(x) + y \\ -\epsilon(y) \end{smallmatrix}\right)$, so we get

      $$\mu = -\epsilon(y) = -\epsilon(-\partial_1(x)) = \epsilon \circ \partial_1(x) = 0,$$

      which is a contradiction.

b) $\delta_1\binom{x}{y} = \mu_\ell\binom{e^0_\ell}{0}$ for some $1 \leq \ell \leq k_0$ and $\mu_\ell \neq 0$. Then

$$\delta_1\left(\binom{x}{y} - \mu_\ell\binom{0}{e^0_\ell}\right) = \mu_\ell\binom{e^0_\ell}{0} - \mu_\ell\binom{e^0_\ell}{-\lambda_\ell} = \underbrace{\mu_\ell\lambda_\ell}_{\neq 0}\binom{0}{1}.$$

So, we are reduced to the impossible first case.

Hence, the cone condition 2 is fulfilled, too.

3. As mentioned above, $\delta_1\binom{0}{e^0_\ell} = \binom{e^0_\ell}{0} - \lambda_\ell\binom{0}{1}$ with $\lambda_\ell \neq 0$ for all basis elements $e^0_\ell \in \Omega_0$. Therefore, the cone condition 3 is also fulfilled.

In total, our choice of the subsets S_i makes $\big(\widehat{C}(\mathrm{id}_C), \Phi\big)$ a cone. $\square$

Remark 2.19. One can generalise this construction in the following way. Let (B, Ω^B), (C, Ω^C) be chain complexes over some principal ideal domain R, both finite of order d, and let $f\colon B \to C$ be a chain isomorphism. Let (B, Ω^B) be augmented by R. Then the mapping cone $\big(\widehat{C}(f), \Phi\big)$ is acyclic, pure if (B, Ω^B) and (C, Ω^C) are pure and $\epsilon\colon B_0 \to R$ is not zero, and a cone if $B_\nu \neq 0$ and $C_\nu \neq 0$ for all $1 \leq \nu \leq d$ and $\epsilon(e) \neq 0$ for all basis elements $e \in (\Omega_B)_0$.

Remark 2.20. For an injective or surjective chain map $f\colon B \to C$, the mapping cone $\widehat{C}(f)$ for augmented B is in general neither acyclic nor a cone. Simple examples over a ring R are obtained by the zero chain complex $(Z, \emptyset)$ and a finite chain complex (P, Π) of order 0 with basis $\Pi = \Pi_0 = \{e\}$, generated by a single element.

- Let $\iota\colon Z \hookrightarrow P$ be the injective embedding. The mapping cone $\big(\widehat{C}(\iota), \Phi\big)$ for augmented Z is finite of order 0. Its chain module $\widehat{C}(\iota)_0 = P_0 \oplus R$ has the basis $\Phi_0 = \big\{\binom{e}{0}, \binom{0}{1}\big\}$. Therefore, $H_0\big(\widehat{C}(\iota)\big) \cong R^2$.

- We consider the surjective chain map $\sigma\colon P \twoheadrightarrow Z$. Let (P, Π) be augmented by R with an augmentation map $\epsilon(e) = 1$. Then the mapping cone $\big(\widehat{C}(\sigma), \Phi\big)$ is finite of order 1, having the bases $\Phi_1 = \big\{\binom{0}{e}\big\}$ of $\widehat{C}(\sigma)_1 = Z_1 \oplus P_0$ and $\Phi_0 = \big\{\binom{0}{1}\big\}$ of $\widehat{C}(\sigma)_0 = Z_0 \oplus R$. Hence, we obtain $H_0\big(\widehat{C}(\sigma)\big) = 0$.

2.3.3. Not Every Cone is a Mapping Cone!

We have seen above that a mapping cone $\widehat{\mathcal{C}}(\mathrm{id}_C)$ for a nonnegative chain complex C over a principal ideal domain R is always acyclic, but only under certain conditions a cone. Can conversely every cone A be regarded as a mapping cone $\widehat{\mathcal{C}}(\mathrm{id}_B)$ of the identity chain map of some subcomplex $B \subseteq A$?

At first, we consider the simplicial case. Let Δ be a simplicial complex of dimension d which is a geometrical cone with apex v_0. The complex Δ contains a subcomplex Λ of dimension $(d-1)$ which consists of all simplices of Δ not containing its apex v_0. Then the corresponding free chain complex $(C_\Delta, \Omega_\Delta)$ is isomorphic to the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_{C_\Lambda}), \Phi\big)$ of the identity chain map $\mathrm{id}_{C_\Lambda} \colon C_\Lambda \to C_\Lambda$. Hence, each chain complex $(C_\Delta, \Omega_\Delta)$ of a simplicial cone Δ can be regarded as a mapping cone of the identity chain map id_{C_Λ} for a special subcomplex $(C_\Lambda, \Omega_\Lambda)$.

In general, this does not work for arbitrary chain complex cones. We consider an example.

Let (C, Ω) be a finite chain complex of order 2 over $\mathbb{Z}$ whose bases are $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$. Let the boundary maps be given by

$$\partial_2(e_1^2) = e_1^1 + e_2^1, \qquad \partial_1(e_1^1) = e_2^0 - e_1^0,$$
$$\partial_1(e_2^1) = e_1^0 - e_2^0.$$

This complex is a cone by choosing $S_2 = \{e_1^2\}$, $S_1 = \{e_1^1\}$ and $S_0 = \{e_1^0\}$. But it cannot be regarded as a mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_B), \Phi\big)$ over a subcomplex $B \subseteq C$ as we will see now.

We assume that (C, Ω) can be described as a mapping cone of an identity chain map. Then we need a chain complex (B, Θ) of order 1 to construct $\big(\widehat{\mathcal{C}}(\mathrm{id}_B), \Phi\big)$:

$$\underbrace{\{0\} \oplus B_1}_{\cong \langle e_1^2 \rangle} \xrightarrow{\delta_2} \underbrace{B_1 \oplus B_0}_{\cong \langle e_1^1, e_2^1 \rangle} \xrightarrow{\delta_1} \underbrace{B_0 \oplus R}_{\cong \langle e_1^0, e_2^0 \rangle} \xrightarrow{\delta_0} 0.$$

Therefore, we need $B_1 = \langle g_1 \rangle$ and $B_0 = \langle g_0 \rangle$. This gives

$$\delta_2 \begin{pmatrix} 0 \\ g_1 \end{pmatrix} = \begin{pmatrix} 0 & \mathrm{id}_{B_1} \\ 0 & -\partial_1 \end{pmatrix} \begin{pmatrix} 0 \\ g_1 \end{pmatrix} = \begin{pmatrix} g_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\alpha g_0 \end{pmatrix}$$

with $\partial_1(g_1) = \alpha g_0$ for some $\alpha \in R$. Hence, we obtain $\epsilon(\alpha g_0) = 0$, i.e. $\alpha = 0$ or $\epsilon(g_0) = 0$. In both cases the chain complexes (C, Ω) and $\big(\widehat{\mathcal{C}}(\mathrm{id}_B), \Phi\big)$ are not isomorphic:

- If $\alpha \neq 0$, then $\delta_1\left(\begin{smallmatrix} 0 \\ g_0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} g_0 \\ 0 \end{smallmatrix}\right)$, so $\left(\widehat{\mathcal{C}}(\mathrm{id}_B), \Phi\right)$ cannot be a cone in the sense of Definition 2.3.
- If $\alpha = 0$, then ${}^{\#}\mathrm{bd}\left(\begin{smallmatrix} 0 \\ g_1 \end{smallmatrix}\right) = 1 \neq 2 = {}^{\#}\mathrm{bd}(e_1^2)$.

Therefore, the chain complex cone (C, Ω) cannot be regarded as a mapping cone of an identity chain map. Any other choice of the subsets S_i does not change the situation.

To describe a chain complex cone (C, Ω) of order d as a mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}), \Phi\right)$, we guess that it might be necessary that the subsets $\Omega_\nu \setminus S_\nu$ generate a subcomplex of (C, Ω). However, this is not sufficient since we need ${}^{\#}\Omega_d = {}^{\#}(\Omega_{d-1} \setminus S_{d-1})$, for example, and maybe further conditions. If any general, sufficient criterion exists at all, it seems to be complicated.

2.4. Mapping Cones over Elementary Chain Complexes

We recall from Definition 1.40 that elementary chain complexes are special subcomplexes of a finitely free and nonnegative chain complex (C, Ω) which are analogues to simplices in a simplicial complex. Such a subcomplex is denoted by $(C_{e_i^\nu}, \Omega_{e_i^\nu})$ for $e_i^\nu \in \Omega$. Because the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ over (C, Ω) is also finitely free and nonnegative, transferring this notation to its elementary subcomplexes yields

$$\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)}, \Phi_{\left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right)}\right) \text{ for } \left(\begin{smallmatrix} a \\ b \end{smallmatrix}\right) \in \Phi.$$

Since this printing looks weird, we will write $\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(a,b)}, \Phi_{(a,b)}\right)$ instead or shortly $\widehat{\mathcal{C}}(\mathrm{id}_C)_{(a,b)}$ if the basis is clear.

We will consider the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_{C_{e_i^\nu}}), \Theta_{(0,e_i^\nu)}\right)$ with basis $\Theta_{(0,e_i^\nu)}$ over an elementary subcomplex $(C_{e_i^\nu}, \Omega_{e_i^\nu})$ which is analogous to a simplex. Since the simplicial case is much more illustrative, we treat it first.

Let Δ be a geometrical simplicial complex of dimension $d \geq 0$. As described in Section 2.3.1, adding a vertex v to Δ yields a geometrical cone over Δ with apex v. If we cut any simplex S from Δ and add the vertex v only to S, we get the simplex (S, v) which is the corresponding simplex in the cone over Δ. An example for this is shown in Figure 2.4.

Since a geometrical cone over a simplicial complex Δ can always be described as a mapping cone $\widehat{\mathcal{C}}(\mathrm{id}_C)$, we ask: Is a similar statement valid for elementary subcomplexes of a finite chain complex of order d? Indeed it is, we even consider a more general case.

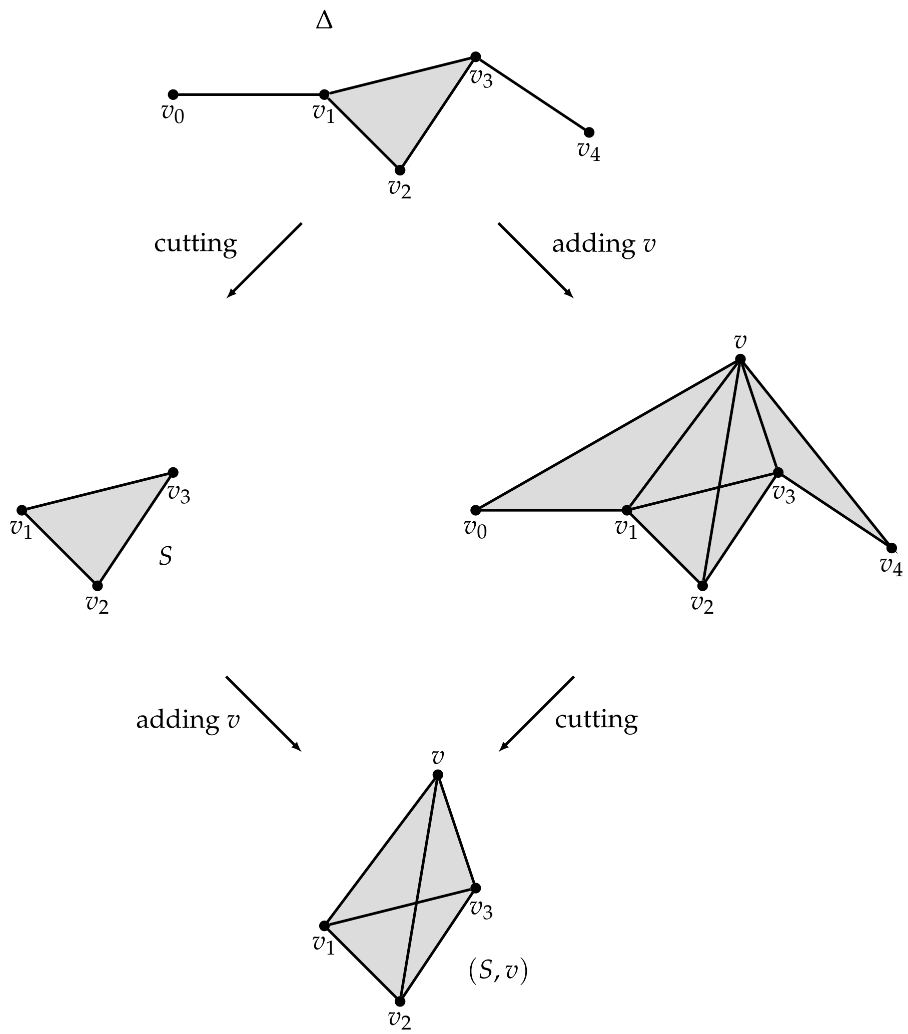


Figure 2.4.: A geometrical cone over a simplex S equals a simplex (S, v) in the cone over Δ

Lemma 2.21. *Let (C, Ω) be a free and nonnegative chain complex over a principal ideal domain R. Let $\partial_\nu(e_i^\nu) \neq 0$ for all basis elements $e_i^\nu \in \Omega \setminus \Omega_0$ and let $\epsilon\colon C_0 \to R$ be an augmentation map such that $\epsilon(e_j^0) \neq 0$ for all $e_j^0 \in \Omega_0$. Then for each basis element $e_i^\nu \in \Omega$ holds:*

$$\big(\widehat{\mathcal{C}}(\mathrm{id}_{C_{e_i^\nu}}), \Theta_{(0,e_i^\nu)}\big) = \big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_i^\nu)}, \Phi_{(0,e_i^\nu)}\big).$$

Remark 2.22. If $\epsilon(e_i^0) = 0$ for some $e_i^0 \in \Omega_0$, the lemma's statement is not valid because then the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_{C_{e_i^0}}), \Theta_{(0,e_i^0)}\big)$ has the chain module bases

$$\Big(\Theta_{(0,e_i^0)}\Big)_1 = \left\{\begin{pmatrix} 0 \\ e_i^0 \end{pmatrix}\right\}, \qquad \Big(\Theta_{(0,e_i^0)}\Big)_0 = \left\{\begin{pmatrix} e_i^0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}$$

whereas the chain module bases of $\big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_i^0)}, \Phi_{(0,e_i^0)}\big)$ are

$$\Big(\Phi_{(0,e_i^0)}\Big)_1 = \left\{\begin{pmatrix} 0 \\ e_i^0 \end{pmatrix}\right\}, \qquad \Big(\Phi_{(0,e_i^0)}\Big)_0 = \left\{\begin{pmatrix} e_i^0 \\ 0 \end{pmatrix}\right\}.$$

If $\partial_\nu(e_i^\nu) = 0$ for some $e_i^\nu \in \Omega_\nu$, $\nu \geq 1$, the lemma is not valid for a similar reason.

Proof of Lemma 2.21. We use induction on the order ν of e_i^ν.

For $\nu = 0$, the basis of any subcomplex $(C_{e_i^0}, \Omega_{e_i^0})$ is $\Omega_{e_i^0} = \big(\Omega_{e_i^0}\big)_0 = \{e_i^0\}$. So the nonzero chain modules of the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_{C_{e_i^0}}), \Theta_{(0,e_i^0)}\big)$ have the bases

$$\Big(\Theta_{(0,e_i^0)}\Big)_1 = \left\{\begin{pmatrix} 0 \\ e_i^0 \end{pmatrix}\right\}, \qquad \Big(\Theta_{(0,e_i^0)}\Big)_0 = \left\{\begin{pmatrix} e_i^0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}.$$

Because

$$\delta_1 \begin{pmatrix} 0 \\ e_i^0 \end{pmatrix} = \begin{pmatrix} e_i^0 \\ 0 \end{pmatrix} - \lambda_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ with } \epsilon(e_i^0) = \lambda_i \neq 0,$$

the chain complex $\big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_i^0)}, \Phi_{(0,e_i^0)}\big)$ has exactly the same bases as the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_{C_{e_i^0}}), \Theta_{(0,e_i^0)}\big)$, so both chain complexes coincide.

For $\nu \geq 1$, we assume that the statement is true for all basis elements $e_\ell^\mu \in \Omega$ with $0 \leq \mu < \nu$. Since $\partial_\nu(e_i^\nu) \neq 0$, let $\mathrm{bd}(e_i^\nu) = \{h_1, \ldots, h_k\} \subseteq \Omega_{\nu-1}$. Then the basis $\Omega_{e_i^\nu}$ of the elementary chain complex $(C_{e_i^\nu}, \Omega_{e_i^\nu})$ is

$$\Omega_{e_i^\nu} = \left\{e_i^\nu\right\} \cup \left(\bigcup_{\ell=1}^{k} \Omega_{h_\ell}\right).$$

The mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_{C_{e_i^\nu}}), \Theta_{(0,e_i^\nu)}\right)$ has the basis

$$\Theta_{(0,e_i^\nu)} = \left\{\begin{pmatrix} 0 \\ e_\ell^\mu \end{pmatrix} \;\middle|\; e_\ell^\mu \in \Omega_{e_i^\nu}\right\} \cup \left\{\begin{pmatrix} e_\ell^\mu \\ 0 \end{pmatrix} \;\middle|\; e_\ell^\mu \in \Omega_{e_i^\nu}\right\} \cup \left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}.$$

Because $\mathrm{bd}\left(\begin{smallmatrix} e_i^\nu \\ 0 \end{smallmatrix}\right) = \left\{\left(\begin{smallmatrix} h_\ell \\ 0 \end{smallmatrix}\right) \;\middle|\; 1 \leq \ell \leq k\right\} \subseteq \bigcup_{\ell=1}^{k} \mathrm{bd}\left(\begin{smallmatrix} 0 \\ h_\ell \end{smallmatrix}\right)$, we get

$$\Theta_{(0,e_i^\nu)} = \left\{\begin{pmatrix} 0 \\ e_i^\nu \end{pmatrix}\right\} \cup \left\{\begin{pmatrix} e_i^\nu \\ 0 \end{pmatrix}\right\} \cup \left(\bigcup_{\ell=1}^{k} \Theta_{(0,h_\ell)}\right).$$

Each $\Theta_{(0,h_\ell)}$ is the basis of the chain complex $\left(\widehat{\mathcal{C}}(\mathrm{id}_{C_{h_\ell}}), \Theta_{(0,h_\ell)}\right)$ which is equal to the elementary subcomplex $\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,h_\ell)}, \Phi_{(0,h_\ell)}\right)$ of $\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_i^\nu)}, \Phi_{(0,e_i^\nu)}\right)$ by induction. Since

$$\delta_{\nu+1}\begin{pmatrix} 0 \\ e_i^\nu \end{pmatrix} = \begin{pmatrix} e_i^\nu \\ 0 \end{pmatrix} - \sum_{\ell=1}^{k} a_\ell^{\nu,i} \begin{pmatrix} 0 \\ h_\ell \end{pmatrix} \text{ with all } a_\ell^{\nu,i} \neq 0,$$

the set $\Theta_{(0,e_i^\nu)}$ is also the basis of the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_i^\nu)}, \Phi_{(0,e_i^\nu)}\right)$. □

Lemma 2.23. *If a chain complex (C, Ω) fulfils all assumptions of Lemma 2.21, any elementary subcomplex $\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_i^\nu)}, \Phi_{(0,e_i^\nu)}\right)$ of the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ is acyclic.*

Proof. By Lemma 2.21, we get $\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_i^\nu)}, \Phi_{(0,e_i^\nu)}\right) = \left(\widehat{\mathcal{C}}(\mathrm{id}_{C_{e_i^\nu}}), \Theta_{(0,e_i^\nu)}\right)$, and this mapping cone of the identity chain map $\mathrm{id}_{C_{e_i^\nu}}$ is acyclic by Lemma 2.13. □

2.5. A Final Comment on Cones

There exist different definitions for acyclic chain complexes in literature. As mentioned in Section 2.1, many authors use the same definition as we did, but Dold (1972, p. 17), Weibel (1994, p. 3) and Spanier (1966, p. 163) define them as follows:

Definition. A chain complex C is *acyclic* if $H_\nu(C) = 0$ for all $\nu \in \mathbb{Z}$.

Dold (1972, p. 18) even defines a cone:

Definition. For any chain complex C, the mapping cone $\mathcal{C}(\mathrm{id}_C)$ of the identity chain map $\mathrm{id}_C \colon C \to C$ is called *cone* of C.

According to Lemma 2.11, all homology modules of $\mathcal{C}(\mathrm{id}_C)$ are zero. Hence, this definition of a cone is consistent with the definition of an acyclic chain complex given above as one can still say that a cone is acyclic.

Both definitions of an acyclic chain complex which are given here and in Section 2.1 coincide if we consider *reduced* homology of a free and nonnegative chain complex over a principal ideal domain with a *surjective* augmentation map $\epsilon \neq 0$, cf. Remark 2.2. But both definitions of a cone seem to be totally different.

An advantage of the definitions given here is that they hold for any chain complex whereas our definitions in Section 2.1 are only valid for nonnegative or even finite chain complexes over a principal ideal domain. But our definitions are geometrically motivated as geometrical cones over simplicial complexes are included in our definition of a cone.

3. Shellable and Regular Chain Complexes

In this chapter we generalise the notion of shellability from simplicial complexes to finite chain complexes over some principal ideal domain R. Contrary to simplicial complexes, it turns out that shellability does not suffice to determine the homology of chain complexes. Hence, we try to imitate the properties of chain complexes obtained from shellable simplicial complexes which leads to the notion of regular and totally regular chain complexes. With an additional condition on an augmentation map ϵ, the homology of totally regular chain complexes is known in general. In the end, we will consider mapping cones over shellable and regular chain complexes.

3.1. Shellable Chain Complexes

3.1.1. Definition and First Examples

Definition 3.1. Let (C, Ω) be a finite chain complex of order $d \geq 0$ over a principal ideal domain R. Let $\Gamma \neq \emptyset$ be the set of all maximal basis elements of (C, Ω). An order of the basis elements in $\Gamma := \{g_1, \ldots, g_k\}$ is a *shelling* (or a *shelling order*) if $d = 0$ or if the following conditions hold for $d \geq 1$:

1. For $2 \leq j \leq k$, the set $\Omega_{g_j} \cap \big(\bigcup_{i=1}^{j-1} \Omega_{g_i}\big)$ generates a pure finite chain complex of order $\big(\mathrm{ord}(g_j) - 1\big)$.
2. For $2 \leq j \leq k$ and $\mathrm{ord}(g_j) \geq 1$, the set $(\Omega_{g_j})_{\mathrm{ord}(g_j)-1}$ has a shelling in which the basis elements of the intersection $\Big(\Omega_{g_j} \cap \big(\bigcup_{i=1}^{j-1} \Omega_{g_i}\big)\Big)_{\mathrm{ord}(g_j)-1}$ come first.
3. $(\Omega_{g_1})_{\mathrm{ord}(g_1)-1}$ has a shelling.

The chain complex (C, Ω) is said to be *shellable* then.

Remark 3.2.

1. It must be $\operatorname{ord}(g_1) = d$, otherwise achieving a shelling would be impossible because of the first condition. So we can rewrite the third condition as follows: $(\Omega_{g_1})_{d-1}$ has a shelling.

2. For $0 \leq \nu \leq d$, each chain module C_ν of a shellable chain complex (C, Ω) of order d has a nonempty basis Ω_ν, i.e. $C_\nu \neq 0$. In particular, the zero chain complex Z is not shellable.

3. Due to the shelling conditions 2 and 3, any elementary subcomplex $(C_{e_i^\nu}, \Omega_{e_i^\nu})$ of a shellable chain complex (C, Ω) is shellable. Hence for $\nu \geq 1$, its $(\nu - 1)$-skeleton $\operatorname{sk}_{\nu-1}(C_{e_i^\nu}, \Omega_{e_i^\nu})$ is shellable, too.

4. For a precritical element $g_j \in \Gamma$ holds $\Omega_{g_j} \cap \left(\bigcup_{i=1}^{j-1} \Omega_{g_i}\right) = \Omega_{g_j} \setminus \{g_j\}$. Therefore, it is possible to rearrange the elements in a shelling of Γ such that all precritical elements come last.

5. All facets of a simplex compose a shellable simplicial complex, cf. Example 1.12(2). So any chain complex obtained from a simplicial complex satisfies the shelling conditions 2 and 3 trivially (cf. Björner and Wachs, 1983, Section 4). Hence, shellability of chain complexes contains the simplicial case (cf. Definition 1.10).

6. Shellability for regular cell complexes is defined in Definition A.14 in the Appendix. As explained in Section A.4, a cellular chain complex obtained from a shellable regular cell complex is also shellable.

In contrast to shellable simplicial complexes (cf. Section 1.5), the homology of shellable chain complexes is not known in general. We consider some examples.

Example 3.3.

1. Let (C, Ω) be a finite chain complex of order 1 over $\mathbb{Z}$. Let $\Omega_1 = \{e_1^1\}$ and $\Omega_0 = \{e_1^0, \ldots, e_k^0\}$ for some $k \geq 1$ be the bases of the chain modules C_1 and C_0, respectively, and $\partial_1 e_1^1 = \sum_{i=1}^{k} e_i^0$. This complex is shellable, and its homology modules are

$$H_1(C) = 0, \qquad H_0(C) \cong \mathbb{Z}^{k-1}.$$

If $k = 2$, this chain complex is acyclic and even a cone.

2. Let (C, Ω) be a finite chain complex of order 1 over $\mathbb{Z}$ with chain modules $C_1 = \langle e_1^1, e_2^1 \rangle$ and $C_0 = \langle e_1^0, \ldots, e_k^0 \rangle$ for some $k \geq 1$. We assume that $\partial_1 e_1^1 = \partial_1 e_2^1 = \sum_{i=1}^k e_i^0$. This chain complex is shellable and has the following homology modules:
$$H_1(C) \cong \mathbb{Z}, \qquad H_0(C) \cong \mathbb{Z}^{k-1}.$$
3. Let (C, Ω) be a finite chain complex of order 1 over $\mathbb{Z}$ with chain modules $C_1 = \langle e_1^1, e_2^1 \rangle$ and $C_0 = \langle e_1^0, \ldots, e_k^0 \rangle$ for some $k \geq 2$ such that $\partial_1 e_1^1 = \sum_{i=1}^k e_i^0$ and $\partial_1 e_2^1 = -e_1^0 + \sum_{i=2}^k e_i^0$. Again, this chain complex is shellable. About the homology we know:
$$H_1(C) = 0, \qquad H_0(C) \ncong \mathbb{Z}^\ell \quad \text{for any } \ell \in \mathbb{N}$$
as there are torsion elements in $H_0(C)$, for example $[e_1^0]$.

So we need more conditions on shellable chain complexes to get some information about homology. We will treat this later in Section 3.3.

3.1.2. Monotonically Descending Shellings

In a shellable simplicial complex whose maximal simplices $F_1, \ldots, F_t$ are ordered in a shelling, each simplex F_i can only be glued to proper faces of maximal simplices having the same or higher dimension. So there must exist a shelling $G_1, \ldots, G_t$ in which the dimension of each maximal simplex G_i with $i \geq 2$ is not greater than the dimension of its antecessor G_{i-1}. This has been shown by Björner and Wachs (1996, Rearrangement Lemma 2.6). We will prove the existence of such a special shelling in the more general case of shellable chain complexes over a principal ideal domain. But first such a special shelling gets a name.

Definition 3.4. Let (C, Ω) be a shellable chain complex over a principal ideal domain R and $\Gamma \subseteq \Omega$ be the subset of all its maximal basis elements. A shelling of $\Gamma := \{g_1, \ldots, g_k\}$ is *monotonically descending* if $\operatorname{ord}(g_i) \geq \operatorname{ord}(g_{i+1})$ for all $1 \leq i \leq (k-1)$. A *failure* in a shelling of $\Gamma = \{g_1, \ldots, g_k\}$ is a pair (i, j) with $i < j$ and $\operatorname{ord}(g_i) < \operatorname{ord}(g_j)$.

Clearly, a shelling is monotonically descending if and only if it is a shelling without failures.

If (C, Ω) is a shellable chain complex and *pure*, every shelling of Γ is monotonically descending.

Theorem 3.5. *Let (C, Ω) be a shellable chain complex over a principal ideal domain R, finite of order d, and $\Gamma \subseteq \Omega$ be the subset of all its maximal basis elements. Then a monotonically descending shelling of $\Gamma = \{g_1, \ldots, g_k\}$ exists.*

To prove this theorem, we need the following lemma.

Lemma 3.6. *Let (C, Ω) be a shellable chain complex over a principal ideal domain R with $\operatorname{ord}(C, \Omega) = d$. Let $\Gamma \subseteq \Omega$ be the subset of all its maximal basis elements which is ordered in a shelling with $m \geq 1$ failures. Then it is possible to permute the elements of Γ such that there is a new shelling of Γ with $(m-1)$ failures.*

Proof. Let $\Gamma := \{g_1, \ldots, g_k\}$ ordered in a shelling with $m \geq 1$ failures. Because (C, Ω) is shellable, $\operatorname{ord}(g_1) \geq \operatorname{ord}(g_i)$ for all $2 \leq i \leq k$, but there is a minimal $2 \leq i_0 \leq (k-1)$ such that $\operatorname{ord}(g_{i_0}) < \operatorname{ord}(g_{i_0+1})$. We want to show that we still have a shelling after permuting g_{i_0} and g_{i_0+1}, i.e. the ordered set

$$\{g_1, \ldots, g_{i_0-1}, g_{i_0+1}, g_{i_0}, g_{i_0+2}, \ldots, g_k\}$$

is also a shelling.

At first, we consider the subcomplex generated by

$$\Theta := \left(\bigcup_{i=1}^{i_0} \Omega_{g_i} \right) \cap \Omega_{g_{i_0+1}}.$$

Because of shellability we know:

- Θ generates a pure chain complex of order $\big(\operatorname{ord}(g_{i_0+1}) - 1\big)$. Hence, all maximal basis elements of Θ are contained in $\big(\Omega_{g_{i_0+1}}\big)_{\operatorname{ord}(g_{i_0+1})-1}$.
- $\big(\Omega_{g_{i_0+1}}\big)_{\operatorname{ord}(g_{i_0+1})-1}$ has a shelling in which the basis elements of Θ come first.

We divide the intersection Θ into two parts:

$$\Theta = \left(\left(\bigcup_{i=1}^{i_0-1} \Omega_{g_i} \right) \cap \Omega_{g_{i_0+1}} \right) \cup \Big(\Omega_{g_{i_0}} \cap \Omega_{g_{i_0+1}} \Big).$$

Since $\operatorname{ord}(g_{i_0}) < \operatorname{ord}(g_{i_0+1})$, the set $\Omega_{g_{i_0}} \cap \Omega_{g_{i_0+1}}$ generates a subcomplex of order $t \leq \operatorname{ord}(g_{i_0}) - 1 \leq \operatorname{ord}(g_{i_0+1}) - 2$. Therefore, any maximal basis element e in $\Omega_{g_{i_0}} \cap \Omega_{g_{i_0+1}}$ is contained in the boundary of some other basis element $f \in \Theta$, otherwise Θ would not generate a pure chain complex. Since e is maximal in $\Omega_{g_{i_0}} \cap \Omega_{g_{i_0+1}}$, we conclude that $f \not\in \Omega_{g_{i_0}} \cap \Omega_{g_{i_0+1}}$. Hence, $f \in \left(\bigcup_{i=1}^{i_0-1} \Omega_{g_i}\right) \cap \Omega_{g_{i_0+1}}$, and we get

$$\Omega_{g_{i_0}} \cap \Omega_{g_{i_0+1}} \subseteq \left(\bigcup_{i=1}^{i_0-1} \Omega_{g_i}\right) \cap \Omega_{g_{i_0+1}}.$$

Therefore,

$$\Theta = \left(\bigcup_{i=1}^{i_0-1} \Omega_{g_i}\right) \cap \Omega_{g_{i_0+1}}.$$

Hence, the subcomplex of (C, Ω) generated by $\left(\bigcup_{i=1}^{i_0-1} \Omega_{g_i}\right) \cap \Omega_{g_{i_0+1}}$ is pure of order $\left(\operatorname{ord}(g_{i_0+1}) - 1\right)$ and satisfies all other shelling properties, too.

We consider now the subcomplex whose basis is

$$\Xi := \left(\left(\bigcup_{i=1}^{i_0-1} \Omega_{g_i}\right) \cup \Omega_{g_{i_0+1}}\right) \cap \Omega_{g_{i_0}} = \left(\left(\bigcup_{i=1}^{i_0-1} \Omega_{g_i}\right) \cap \Omega_{g_{i_0}}\right) \cup \left(\Omega_{g_{i_0+1}} \cap \Omega_{g_{i_0}}\right).$$

We have shown above that $\Omega_{g_{i_0}} \cap \Omega_{g_{i_0+1}} \subseteq \left(\bigcup_{i=1}^{i_0-1} \Omega_{g_i}\right) \cap \Omega_{g_{i_0+1}} \subseteq \left(\bigcup_{i=1}^{i_0-1} \Omega_{g_i}\right)$. Since $\Omega_{g_{i_0}} \cap \Omega_{g_{i_0+1}} \subseteq \Omega_{g_{i_0}}$, we conclude

$$\Omega_{g_{i_0}} \cap \Omega_{g_{i_0+1}} \subseteq \left(\bigcup_{i=1}^{i_0-1} \Omega_{g_i}\right) \cap \Omega_{g_{i_0}}.$$

Hence,

$$\Xi = \left(\bigcup_{i=1}^{i_0-1} \Omega_{g_i}\right) \cap \Omega_{g_{i_0}}.$$

Since the basis elements g_i are ordered in a shelling, Ξ generates a pure chain complex of order $\left(\operatorname{ord}(g_{i_0}) - 1\right)$ which also satisfies all other shelling properties.

Therefore, $\{g_1, \ldots, g_{i_0-1}, g_{i_0+1}, g_{i_0}, g_{i_0+2}, \ldots, g_k\}$ is a shelling order of Γ with exactly one failure less. $\square$

Proof of Theorem 3.5. Let the set Γ of maximal basis elements of (C, Ω) be ordered in a shelling. If this shelling is not monotonically descending, we get such a shelling by repeated application of Lemma 3.6. $\square$

3.1.3. *i*-Skeletons of Shellable Chain Complexes

By definition of shellability, every elementary subcomplex $(C_{e_i^\nu}, \Omega_{e_i^\nu})$ of a shellable chain complex (C, Ω) is shellable. For $\nu \geq 1$, its $(\nu-1)$-skeleton $\mathrm{sk}_{\nu-1}(C_{e_i^\nu}, \Omega_{e_i^\nu})$ is also shellable. We will show that this is not only a special property of elementary chain complexes but also holds for any shellable chain complex.

Lemma 3.7. *Let (C, Ω) be a pure shellable chain complex over a principal ideal domain R, finite of order $d \geq 1$. Its $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C, \Omega)$ is shellable, too.*

Proof. For $d = 1$ there is nothing to do because every finite chain complex of order 0 is shellable by definition. Let $d \geq 2$ and $\Omega_d = \{e_1^d, \dots, e_{k_d}^d\}$. We want to show that $\Omega_{d-1} = \bigcup_{i=1}^{k_d} \mathrm{bd}(e_i^d)$ has a shelling, using an inductive argument. By definition we know that $\mathrm{bd}(e_1^d) = \left(\Omega_{e_1^d}\right)_{d-1}$ has a shelling.

We consider $\bigcup_{i=1}^{\ell} \mathrm{bd}(e_i^d) \subseteq \Omega_{d-1}$ for $2 \leq \ell \leq k_d$ and assume that the set $\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)$ has a shelling. We have to show that

$$\left(\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)\right) \cup \left(\Omega_{e_\ell^d}\right)_{d-1} = \left(\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)\right) \cup \mathrm{bd}(e_\ell^d)$$

has a shelling, too.

If $\mathrm{bd}(e_\ell^d) \subseteq \bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)$, we are done. So we assume $\mathrm{bd}(e_\ell^d) \nsubseteq \bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)$. Because the chain complex (C, Ω) is shellable, the set $\mathrm{bd}(e_\ell^d) = \left(\Omega_{e_\ell^d}\right)_{d-1}$ has a shelling in which the elements of

$$\left(\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)\right) \cap \Omega_{e_\ell^d} = \left(\left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\right) \cap \Omega_{e_\ell^d}\right)_{d-1}$$

come first. So let $\left(\Omega_{e_\ell^d}\right)_{d-1} = \{f_1, \dots, f_s, h_1, \dots, h_t\}$ be a shelling such that

$$\left(\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)\right) \cap \Omega_{e_\ell^d} = \{f_1, \dots, f_s\} \text{ with } s \geq 1.$$

As $\mathrm{bd}(e_\ell^d) \nsubseteq \bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d)$, we also have $t \geq 1$.

Let $\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d) = \{e_1, \dots, e_r, f_1, \dots, f_s\}$. This order is not necessarily a shelling but this is not of interest here. We have to show that the set

$$\{e_1, \dots, e_r, f_1, \dots, f_s, h_1, \dots, h_t\}$$

has a shelling. As there exists a shelling for $\{e_1, \ldots, e_r, f_1, \ldots, f_s\}$ by assumption, we must only check the shelling properties for all h_k, $1 \le k \le t$. We do this by another induction and begin with

$$\left(\left(\bigcup_{i=1}^{r} \Omega_{e_i}\right) \cup \left(\bigcup_{i=1}^{s} \Omega_{f_i}\right)\right) \cap \Omega_{h_1} = \left(\left(\bigcup_{i=1}^{r} \Omega_{e_i}\right) \cap \Omega_{h_1}\right) \cup \left(\left(\bigcup_{i=1}^{s} \Omega_{f_i}\right) \cap \Omega_{h_1}\right).$$

Since $\left(\Omega_{e_\ell^d}\right)_{d-1} = \{f_1, \ldots, f_s, h_1, \ldots, h_t\}$ is a shelling, the set $\left(\bigcup_{i=1}^{s} \Omega_{f_i}\right) \cap \Omega_{h_1}$ generates a pure chain complex of order $(d-2)$ and satisfies all three shelling conditions of Definition 3.1.

By assumption, we have $\left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\right) \cap \Omega_{e_\ell^d} = \bigcup_{j=1}^{s} \Omega_{f_j}$. Therefore, we get $\left(\bigcup_{i=1}^{r} \Omega_{e_i}\right) \cap \Omega_{h_1} \subseteq \bigcup_{j=1}^{s} \Omega_{f_j}$ and hence $\left(\bigcup_{i=1}^{r} \Omega_{e_i}\right) \cap \Omega_{h_1} \subseteq \left(\bigcup_{j=1}^{s} \Omega_{f_j}\right) \cap \Omega_{h_1}$. Hence, we conclude

$$\left(\left(\bigcup_{i=1}^{r} \Omega_{e_i}\right) \cup \left(\bigcup_{i=1}^{s} \Omega_{f_i}\right)\right) \cap \Omega_{h_1} = \left(\bigcup_{i=1}^{s} \Omega_{f_i}\right) \cap \Omega_{h_1},$$

so the set $\{e_1, \ldots, e_r, f_1, \ldots, f_s, h_1\}$ has a shelling.

Now, we assume that the set $\{e_1, \ldots, e_r, f_1, \ldots, f_s, h_1, \ldots, h_k\}$ has a shelling for some $1 \le k \le (t-1)$. In particular, the set

$$\left(\bigcup_{i=1}^{r} \Omega_{e_i}\right) \cup \left(\bigcup_{i=1}^{s} \Omega_{f_i}\right) \cup \left(\bigcup_{i=1}^{k} \Omega_{h_i}\right)$$

generates a shellable chain complex of order $(d-1)$. By a similar argument as above we obtain

$$\begin{aligned}
&\left(\left(\bigcup_{i=1}^{r} \Omega_{e_i}\right) \cup \left(\bigcup_{i=1}^{s} \Omega_{f_i}\right) \cup \left(\bigcup_{i=1}^{k} \Omega_{h_i}\right)\right) \cap \Omega_{h_{k+1}} \\
&\quad = \underbrace{\left(\left(\bigcup_{i=1}^{r} \Omega_{e_i}\right) \cap \Omega_{h_{k+1}}\right)}_{\subseteq \left(\bigcup_{i=1}^{s} \Omega_{f_i}\right) \cap \Omega_{h_{k+1}}} \cup \underbrace{\left(\left(\left(\bigcup_{i=1}^{s} \Omega_{f_i}\right) \cup \left(\bigcup_{i=1}^{k} \Omega_{h_i}\right)\right) \cap \Omega_{h_{k+1}}\right)}_{\text{generates a pure complex of order } (d-2)} \\
&\quad = \left(\left(\bigcup_{i=1}^{s} \Omega_{f_i}\right) \cup \left(\bigcup_{i=1}^{k} \Omega_{h_i}\right)\right) \cap \Omega_{h_{k+1}}.
\end{aligned}$$

This set is a basis of a shellable pure chain complex of order $(d-2)$. Therefore, the set $\{e_1, \dots, e_r, f_1, \dots, f_s, h_1, \dots, h_{k+1}\}$ has a shelling.

By induction we conclude that the set

$$\{e_1, \dots, e_r, f_1, \dots, f_s, h_1, \dots, h_t\} = \bigcup_{i=1}^{\ell} \mathrm{bd}(e_i^d)$$

has a shelling. Hence, we get by the first induction that $\Omega_{d-1} = \bigcup_{i=1}^{k_d} \mathrm{bd}(e_i^d)$ has a shelling. $\square$

Theorem 3.8. *Let (C, Ω) be a shellable chain complex over a principal ideal domain, finite of order $d \geq 1$. Its $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C, \Omega)$ is shellable, too.*

Proof. For pure (C, Ω) this statement is already proven, cf. Lemma 3.7. So it suffices to deal with the nonpure case.

Let $\Gamma \subseteq \Omega$ be the subset of all maximal basis elements and let the elements of Γ be ordered in a monotonically descending shelling.

Let $\Omega_d = \{e_1^d, \dots, e_{k_d}^d\} \subseteq \Gamma$ and $\Gamma \cap \Omega_{d-1} = \{g_1^{d-1}, \dots, g_m^{d-1}\}$. In the chosen shelling order of Γ the elements of Ω_d come first, followed by all elements of $\Gamma \cap \Omega_{d-1}$. The basis of the chain module $\big(\mathrm{sk}_{d-1}(C, \Omega)\big)_{d-1} = C_{d-1}$ is

$$\Omega_{d-1} = \left(\bigcup_{i=1}^{k_d} \mathrm{bd}(e_i^d)\right) \cup \{g_1^{d-1}, \dots, g_m^{d-1}\}.$$

Due to Lemma 3.7, the set $\big(\bigcup_{i=1}^{k_d} \mathrm{bd}(e_i^d)\big)$ has a shelling because the subcomplex generated by $\bigcup_{i=1}^{k_d} \Omega_{e_i^d}$ is shellable and pure.

For all $1 \leq i \leq k_d$, let $\widehat{\Omega}_{e_i^d} := \Omega_{e_i^d} \setminus \{e_i^d\}$ be the basis of the $(d-1)$-skeleton of $(C_{e_i^d}, \Omega_{e_i^d})$. Then we get

$$\left(\bigcup_{i=1}^{k_d} \Omega_{e_i^d}\right) \cap \Omega_{g_1^{d-1}} = \left(\bigcup_{i=1}^{k_d} \widehat{\Omega}_{e_i^d}\right) \cap \Omega_{g_1^{d-1}}.$$

This set satisfies all three shelling conditions of Definition 3.1 because the chain complex (C, Ω) is shellable. Hence,

$$\left(\bigcup_{i=1}^{k_d} \mathrm{bd}(e_i^d)\right) \cup \{g_1^{d-1}\} \subseteq \Omega_{d-1}$$

has a shelling. By induction we conclude that Ω_{d-1} has a shelling, too. $\square$

Without further work, we get the next corollary by Theorem 3.8.

Corollary 3.9. *Let (C, Ω) be a shellable chain complex of order d over a principal ideal domain. For $0 \leq i \leq d$, every i-skeleton $\mathrm{sk}_i(C, \Omega)$ of (C, Ω) is shellable.*

Remark 3.10. It is a consequence of the preceding corollary that, if any i-skeleton $\mathrm{sk}_i(C, \Omega)$ of a chain complex (C, Ω) is not shellable, the chain complex itself is not shellable.

3.1.4. Well-ordered Bases

In the proofs of Lemma 3.7 and Theorem 3.8, we used a special ordering of the chain module bases Ω_ν. We emphasize it for later purpose.

Definition 3.11. Let (C, Ω) be a shellable chain complex of order d over a principal ideal domain R and $\Gamma \subseteq \Omega$ be the set of all its maximal basis elements, ordered in a monotonically descending shelling.

For $(d-1) \geq \nu \geq 0$, a basis Ω_ν of a chain module C_ν is called *well-ordered* if the basis $\Omega_{\nu+1} := \{e_1^{\nu+1}, \ldots, e_{k_{\nu+1}}^{\nu+1}\}$ is ordered in a shelling and Ω_ν is ordered as follows:

- The first elements in Ω_ν are the elements of $\mathrm{bd}(e_1^{\nu+1})$, ordered in a shelling.
- For $2 \leq \ell \leq k_{\nu+1}$, the basis elements of $\mathrm{bd}(e_\ell^{\nu+1}) \setminus \bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^{\nu+1})$ follow after all basis elements of $\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^{\nu+1})$. They are ordered in a shelling of $\mathrm{bd}(e_\ell^{\nu+1})$ in which all basis elements of $\left(\bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^{\nu+1})\right) \cap \mathrm{bd}(e_\ell^{\nu+1})$ come first.
- Finally, the elements of $\Omega_\nu \cap \Gamma$ follow, in the same order as they occur in Γ.

If all bases Ω_ν for $0 \leq \nu \leq (d-1)$ are well-ordered, we call Ω *well-ordered*.

Remark 3.12. As shown in the proofs of Lemma 3.7 and Theorem 3.8, a well-ordered basis Ω_ν yields a shelling of Ω_ν. Notice that Ω_ν contains exactly all maximal basis elements of the subcomplex generated by $\bigcup_{i=1}^{k_\nu} \Omega_{e_i^\nu}$.

Remark 3.13. According to Theorem 3.5 and the proofs of Lemma 3.7 and Theorem 3.8, there exists a well-ordered basis for every shellable chain complex over a principal ideal domain.

3.2. Regular Chain Complexes

We have seen above that the notion of shellability is not strict enough to determine the homology of a chain complex. So we need additional conditions trying to imitate the properties of shellable simplicial complexes.

3.2.1. Definition and Examples

Definition 3.14. Let (C, Ω) be a shellable chain complex of order d over a principal ideal domain R. Let $\Gamma \subseteq \Omega$ be the set of all its maximal basis elements, ordered in a monotonically descending shelling, and let Ω be well-ordered. Precisely, let $\Omega_d := \{e_1^d, \ldots, e_{k_d}^d\}$ and $\Omega_\nu := \{e_1^\nu, \ldots, e_{m_\nu}^\nu, e_{m_\nu+1}^\nu, \ldots, e_{k_\nu}^\nu\}$ for $0 \leq \nu \leq (d-1)$ such that the following holds:

- $\Gamma \cap \Omega_\nu = \{e_{m_\nu+1}^\nu, \ldots, e_{k_\nu}^\nu\}$;
- $\Gamma = \{e_1^d, \ldots, e_{k_d}^d\} \,\dot\cup\, \{e_{m_{d-1}+1}^{d-1}, \ldots, e_{k_{d-1}}^{d-1}\} \,\dot\cup \ldots \dot\cup\, \{e_{m_0+1}^0, \ldots, e_{k_0}^0\}$ is a monotonically descending shelling.

Then, Γ has a *regular order* if the following two conditions are fulfilled:

1. For any $e_\ell^\nu \in \Gamma$ (i.e. $(m_\nu + 1) \leq \ell \leq k_\nu$) holds:
   If $\mathrm{bd}(e_\ell^\nu) \subseteq \bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^\nu)$, then e_ℓ^ν is precritical, i.e. there exist elements $a_i \in R$ for $1 \leq i \leq \ell$, $a_\ell \neq 0$, such that
   $$a_\ell \partial_\nu(e_\ell^\nu) = \sum_{i=1}^{\ell-1} a_i \partial_\nu(e_i^\nu).$$
2. For any $e_\ell^\nu \in \Omega_\nu$, $\nu \geq 1$, let $\left(\Omega_{e_\ell^\nu}\right)_{\nu-1} = \mathrm{bd}(e_\ell^\nu) := \{f_1^{\nu-1}, \ldots, f_{n_\ell}^{\nu-1}\}$ be a shelling in which the elements of $\left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^\nu}\right) \cap \Omega_{e_\ell^\nu}$ come first such that for any $1 \leq j \leq n_\ell$ holds:
   If $\mathrm{bd}(f_j^{\nu-1}) \subseteq \bigcup_{i=1}^{j-1} \mathrm{bd}(f_i^{\nu-1})$, then $c_j \partial_{\nu-1}(f_j^{\nu-1})$ is a linear combination of $\partial_{\nu-1}(f_i^{\nu-1})$ for some $c_j \in R \setminus \{0\}$, i.e. there exist elements $c_i \in R$, $1 \leq i \leq j$, $c_j \neq 0$, such that
   $$c_j \partial_{\nu-1}(f_j^{\nu-1}) = \sum_{i=1}^{j-1} c_i \partial_{\nu-1}(f_i^{\nu-1}).$$

The chain complex (C, Ω) is said to be *regular* if the set Γ has a regular order.

Remark 3.15. For the bases $\Omega_\nu = \{e_1^\nu, \ldots, e_{m_\nu}^\nu, e_{m_\nu+1}^\nu, \ldots, e_{k_\nu}^\nu\}$, there are two special cases:

- $m_\nu = k_\nu$. Then $\Gamma \cap \Omega_\nu = \emptyset$.
- $m_\nu = 0$. Then $\Gamma \cap \Omega_\nu = \Omega_\nu$. This is always valid for Ω_d.

Definition 3.16. A regular chain complex (C, Ω) whose elementary subcomplexes $(C_{e_i^\nu}, \Omega_{e_i^\nu})$ are all acyclic is called *totally regular*.

Remark 3.17. It follows directly from the definitions above that any elementary subcomplex $(C_{e_i^\nu}, \Omega_{e_i^\nu})$ of a (totally) regular chain complex (C, Ω) is also (totally) regular.

Remark 3.18. Let (C, Ω) be a finite chain complex of order d over a principal ideal domain obtained from a shellable simplicial complex Δ. Due to Remark 3.2(5), the chain complex (C, Ω) is shellable, too. By Lemma 1.57, every spanning simplex in a shellable simplicial complex yields a critical basis element. Since every simplex is also shellable, the chain complex (C, Ω) fulfils all conditions for regular chain complexes. According to Remark 2.9, every elementary subcomplex $(C_{e_i^\nu}, \Omega_{e_i^\nu})$ in (C, Ω) is acyclic as it corresponds to a simplex in Δ. Hence, any chain complex which is obtained from a shellable simplicial complex is totally regular.

Any finite chain complex of order 0 over a principal ideal domain satisfies trivially all regularity conditions given in Definition 3.14, it is even totally regular. But there also exist more serious examples. We consider some of them to clarify the ideas of *shellable*, *regular* and *totally regular* chain complexes.

Example 3.19.

1. Let (C, Ω) be a finite chain complex of order 1 over $\mathbb{Z}$ whose chain modules C_1 and C_0 have the bases $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$. Let $\partial_1(e_1^1) = 2e_1^0 + e_2^0$ and $\partial_1(e_2^1) = e_1^0 + 2e_2^0$. Then we get a shellable chain complex.

   We have $\mathrm{bd}(e_1^1) = \Omega_0 = \mathrm{bd}(e_2^1)$, but $\partial_1(e_1^1)$ and $\partial_1(e_2^1)$ are linearly independent. So (C, Ω) is not regular.

   Furthermore, $\partial_1(2e_1^1 - e_2^1) = 3e_1^0$. But $e_1^0 \notin \operatorname{im} \partial_1$, hence the homology module $H_0(C) = C_0 / \operatorname{im} \partial_1$ is not torsion free. Therefore, $H_0(C) \not\cong \mathbb{Z}$, i.e. the chain complex (C, Ω) is not acyclic.

2. Let (C, Ω) be a finite chain complex of order 1 over $\mathbb{Z}$ with the following bases of its chain modules: $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0\}$. Let $\partial_1(e_1^1) = 2e_1^0 + e_2^0$ and $\partial_1(e_2^1) = e_1^0 + e_2^0$.

   This complex is shellable, and every elementary subcomplex $(C_{e_i^\nu}, \Omega_{e_i^\nu})$ is acyclic. Furthermore, we have $\mathrm{bd}(e_1^1) = \{e_1^0, e_2^0\} = \mathrm{bd}(e_2^1)$, but $\partial_1(e_1^1)$ and $\partial_1(e_2^1)$ are linearly independent. Hence, (C, Ω) is not regular.

   We compute its homology modules:

   - $\ker \partial_1 = 0$, so $H_1(C) = 0$.
   - $\ker \partial_0 = \langle e_1^0, e_2^0, e_3^0 \rangle$ and $\mathrm{im}\, \partial_1 = \langle e_1^0 + e_2^0, 2e_1^0 + e_2^0 \rangle$, thus $H_0(C) \cong \mathbb{Z}$.

   Therefore, this chain complex is acyclic, but it is not a cone due to $\partial_1(e_1^1 - e_2^1) = e_1^0$.

3. We consider a finite chain complex (C, Ω) of order 2 over $\mathbb{Z}$ with bases $\Omega_2 = \{e_1^2, e_2^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1, e_4^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0\}$ of its chain modules. Let

$$\begin{aligned} \partial_2(e_1^2) &= e_1^1 - e_2^1 - e_3^1, & \partial_1(e_1^1) &= 2e_1^0 + 3e_2^0 + e_3^0,\\ \partial_2(e_2^2) &= e_1^1 - 2e_2^1 - e_4^1, & \partial_1(e_2^1) &= e_1^0 + e_2^0 + e_3^0,\\ & & \partial_1(e_3^1) &= e_1^0 + 2e_2^0,\\ & & \partial_1(e_4^1) &= e_2^0 - e_3^0. \end{aligned}$$

   This chain complex is shellable but not regular because $\mathrm{bd}(e_1^1) = \mathrm{bd}(e_2^1)$ whereas $\partial_1(e_1^1)$ and $\partial_1(e_2^1)$ are linearly independent. Furthermore, it is acyclic and even a cone by choosing $S_2 = \{e_1^2, e_2^2\}$, $S_1 = \{e_3^1, e_4^1\}$ and $S_0 = \{e_2^0\}$.

4. We consider a finite chain complex (C, Ω) of order 2 over $\mathbb{Z}$ whose chain modules have the bases $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$. Let

$$\begin{aligned} \partial_2(e_1^2) &= e_1^1 + e_2^1 + e_3^1, & \partial_1(e_1^1) &= e_1^0 - e_2^0,\\ & & \partial_1(e_2^1) &= e_1^0 - e_2^0,\\ & & \partial_1(e_3^1) &= 2(e_2^0 - e_1^0). \end{aligned}$$

   This chain complex is shellable and regular. But it is not acyclic since $\ker \partial_1 = \langle (e_1^1 - e_2^1), (e_1^1 + e_2^1 + e_3^1) \rangle$ and $\mathrm{im}\, \partial_2 = \langle e_1^1 + e_2^1 + e_3^1 \rangle$ and therefore $H_1(C) \cong \mathbb{Z}$. In particular, (C, Ω) is not totally regular because then $(C_{e_1^2}, \Omega_{e_1^2}) = (C, \Omega)$ must be acyclic.

5. Let (C, Ω) be a finite chain complex of order 1 over $\mathbb{Z}$. Let $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0\}$ be the bases of its chain modules with $\partial_1(e_1^1) = e_1^0$ and $\partial_1(e_2^1) = e_1^0 + e_2^0 + e_3^0$. We observe that this chain complex is acyclic, shellable and regular, but not a cone. As both subcomplexes $(C_{e_1^1}, \Omega_{e_1^1})$ and $(C_{e_2^1}, \Omega_{e_2^1})$ are not acyclic, this chain complex is not totally regular.

   If we change the order of the basis elements e_1^1 and e_2^1, we get an ordering of its maximal basis elements which is not regular.

6. We consider a finite chain complex (C, Ω) of order 2 over $\mathbb{Z}$ with the chain module bases $\Omega_2 = \{e_1^2, e_2^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$. Let the boundary maps ∂_2 and ∂_1 defined by:

$$\begin{aligned} \partial_2(e_1^2) &= 2e_1^1 + e_2^1 + e_3^1, & \partial_1(e_1^1) &= e_1^0 - e_2^0, \\ \partial_2(e_2^2) &= e_1^1 + e_3^1, & \partial_1(e_2^1) &= e_2^0 - e_1^0, \\ & & \partial_1(e_3^1) &= e_2^0 - e_1^0. \end{aligned}$$

   This chain complex is shellable. But the natural order of its maximal basis elements e_1^2 and e_2^2 is not regular because $\operatorname{bd}(e_2^2) \subseteq \operatorname{bd}(e_1^2)$. However, by swapping e_1^2 and e_2^2, we get a regular order! Therefore, this chain complex is regular, but not totally regular as $(C_{e_1^2}, \Omega_{e_1^2})$ is not acyclic. Its homology modules are the following:

   - $H_2(C) = 0$ since $\ker \partial_2 = 0$,
   - $H_1(C) = 0$ according to $\ker \partial_1 = \langle (e_1^1 + e_2^1), (e_1^1 + e_3^1) \rangle = \operatorname{im} \partial_2$,
   - $H_0(C) \cong \mathbb{Z}$ because $\ker \partial_0 = \langle e_1^0, e_2^0 \rangle$ and $\operatorname{im} \partial_1 = \langle e_1^0 - e_2^0 \rangle$.

   Hence, the chain complex (C, Ω) is acyclic. But it is not a cone because $\operatorname{bd}(e_2^2) \subseteq \operatorname{bd}(e_1^2)$.

7. Let (C, Ω) be a finite chain complex of order 2 over $\mathbb{Z}$ having the bases $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0\}$. Let

$$\begin{aligned} \partial_2(e_1^2) &= e_1^1 + e_2^1 - e_3^1, & \partial_1(e_1^1) &= e_1^0 + e_2^0, \\ & & \partial_1(e_2^1) &= e_2^0 + e_3^0, \\ & & \partial_1(e_3^1) &= e_1^0 + 2e_2^0 + e_3^0. \end{aligned}$$

   It is shellable and regular but not totally regular because the elementary subcomplex $(C_{e_3^1}, \Omega_{e_3^1})$ is not acyclic. Though, the chain complex (C, Ω)

is acyclic and even a cone if we choose $S_2 = \{e_1^2\}$, $S_1 = \{e_1^1, e_2^1\}$ and $S_0 = \{e_2^0\}$.

8. We recall Example 2.8(5) for an acyclic chain complex. Let (C, Ω) be a finite chain complex of order 2 over $\mathbb{Z}$ having the bases $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1, e_4^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0, e_4^0\}$ such that

$$\begin{aligned} \partial_2(e_1^2) &= e_1^1 + e_2^1 + e_3^1 + e_4^1, & \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ & & \partial_1(e_2^1) &= e_3^0 - e_2^0, \\ & & \partial_1(e_3^1) &= e_4^0 - e_3^0, \\ & & \partial_1(e_4^1) &= e_1^0 - e_4^0. \end{aligned}$$

   We know that this chain complex is not a cone. But it is shellable, regular and even totally regular.

9. We reuse Example 2.8(4) for a chain complex cone. Let (C, Ω) be a finite chain complex of order 2 over $\mathbb{Z}$ whose chain modules have the bases $\Omega_2 = \{e_1^2, e_2^2, e_3^2\}$, $\Omega_1 = \{e_1^1, e_2^1, e_3^1, e_4^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$. Let

$$\begin{aligned} \partial_2(e_1^2) &= e_1^1 + e_2^1, & \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ \partial_2(e_2^2) &= e_2^1 + e_3^1, & \partial_1(e_2^1) &= e_1^0 - e_2^0, \\ \partial_2(e_3^2) &= e_3^1 + e_4^1, & \partial_1(e_3^1) &= e_2^0 - e_1^0, \\ & & \partial_1(e_4^1) &= e_1^0 - e_2^0. \end{aligned}$$

   This chain complex is shellable and regular. Since every subcomplex $(C_{e_j^\nu}, \Omega_{e_j^\nu})$ is acyclic, we conclude that this chain complex is even totally regular.

10. We recall Example 2.8(3) for a cone. Let (C, Ω) be a finite chain complex of order 2 over $\mathbb{Z}$ with the chain module bases $\Omega_2 = \{e_1^2\}$, $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$ such that

$$\begin{aligned} \partial_2(e_1^2) &= e_1^1 + e_2^1, & \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ & & \partial_1(e_2^1) &= e_1^0 - e_2^0. \end{aligned}$$

    Hence, this chain complex is shellable and totally regular.

11. We modify our last example in some detail. Let (C,Ω) be a finite chain complex of order 2 over $\mathbb{Z}$ with bases $\Omega_2 = \{e_1^2, e_2^2\}$, $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0\}$ such that

$$\begin{aligned} \partial_2(e_1^2) &= e_1^1 + e_2^1, & \partial_1(e_1^1) &= e_2^0 - e_1^0, \\ \partial_2(e_2^2) &= e_1^1 + e_2^1, & \partial_1(e_2^1) &= e_1^0 - e_2^0. \end{aligned}$$

This chain complex is still totally regular, but it is not acyclic, since there is a critical basis element e_2^2.

Remark 3.20. As well as for shellable chain complexes, the homology of regular chain complexes is not clear, in general. So the chain complexes given in Example 3.3(1) and 3.3(2), are all regular and finite of order 1 but differ in their homology.

3.2.2. *i*-Skeletons of Regular Chain Complexes

Similar to shellable chain complexes we will show that i-skeletons of regular chain complexes are also regular.

Lemma 3.21. *Let (C,Ω) be a pure regular chain complex over a principal ideal domain R, finite of order $d \geq 1$. The $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C,\Omega)$ is regular, too. If (C,Ω) is even totally regular, then $\mathrm{sk}_{d-1}(C,\Omega)$ is also totally regular.*

Proof. For $d = 1$, the statement is clear because every finite chain complex of order 0 over a principal ideal domain is totally regular as mentioned above. So let $d \geq 2$. Since (C,Ω) is a pure chain complex, the set of its maximal basis elements is $\Gamma = \Omega_d = \{e_1^d, \ldots, e_{k_d}^d\}$, ordered in a regular order.

The $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C,\Omega)$ is a pure finite subcomplex of order $(d-1)$ with basis $\Omega \setminus \Omega_d$. Let $\widehat{\Gamma} := \Omega_{d-1} = \bigcup_{i=1}^{k_d} \mathrm{bd}(e_i^d)$ be the set of its maximal basis elements. We have to show that a regular order exists for $\widehat{\Gamma}$.

By Lemma 3.7, the subcomplex $\mathrm{sk}_{d-1}(C,\Omega)$ is shellable. Furthermore, all bases Ω_ν of $\mathrm{sk}_{d-1}(C,\Omega)$ are ordered in the same way as they occur in (C,Ω). Hence, $\mathrm{sk}_{d-1}(C,\Omega)$ satisfies the second regularity condition of Definition 3.14, since it inherits this property from (C,Ω). So we only have to check whether the maximal basis elements of $\mathrm{sk}_{d-1}(C,\Omega)$ fulfil the first regularity condition.

At first, we consider the subcomplex of $\mathrm{sk}_{d-1}(C,\Omega)$ which is generated by $\widehat{\Omega}_{e_1^d} := \Omega_{e_1^d} \setminus \{e_1^d\}$. Hence, $\big(\widehat{\Omega}_{e_1^d}\big)_{d-1} = \mathrm{bd}(e_1^d)$, and because (C,Ω) is regular, there exists a shelling of $\mathrm{bd}(e_1^d)$ such that the second regularity condition of Definition 3.14 is fulfilled. So the elements in $\big(\widehat{\Omega}_{e_1^d}\big)_{d-1}$ satisfy the first regularity condition, i.e. the chosen shelling of $\mathrm{bd}(e_1^d)$ is a regular order. If $k_d = 1$, we are done by now.

If $k_d > 1$, let $1 \leq \ell < k_d$. We assume that the basis elements in $\bigcup_{i=1}^{\ell} \big(\Omega_{e_i^d}\big)_{d-1}$, which may be ordered in a shelling, satisfy the first regularity condition, i.e. the subcomplex generated by $\bigcup_{i=1}^{\ell} \big(\Omega_{e_i^d} \setminus \{e_i^d\}\big)$ is regular. So we have to show that this is also valid for the basis elements of $\big(\bigcup_{i=1}^{\ell} (\Omega_{e_i^d})_{d-1}\big) \cup \big(\Omega_{e_{\ell+1}^d}\big)_{d-1}$.

If $e_{\ell+1}^d$ is precritical, then $\big(\bigcup_{i=1}^{\ell} (\Omega_{e_i^d})_{d-1}\big) \cup \big(\Omega_{e_{\ell+1}^d}\big)_{d-1} = \bigcup_{i=1}^{\ell} \big(\Omega_{e_i^d}\big)_{d-1}$, and there is nothing left to do. Therefore, we assume that $e_{\ell+1}^d$ is not precritical. Let

$$\big(\Omega_{e_{\ell+1}^d}\big)_{d-1} = \{f_1, \ldots, f_s, h_1, \ldots, h_t\}$$

such that $\big(\bigcup_{i=1}^{\ell} (\Omega_{e_i^d})_{d-1}\big) \cap \big(\Omega_{e_{\ell+1}^d}\big)_{d-1} = \{f_1, \ldots, f_s\}$. Then $t \geq 1$ and, because of shellability, $s \geq 1$. By assumption, the first regularity condition is fulfilled by the elements of $\bigcup_{i=1}^{\ell} \big(\Omega_{e_i^d}\big)_{d-1}$, so we have only to consider $h_1, \ldots, h_t$. So let

$$\bigcup_{i=1}^{\ell} \big(\Omega_{e_i^d}\big)_{d-1} = \{e_1, \ldots, e_r, f_1, \ldots, f_s\}.$$

This ordering is not necessarily a shelling but this is not of interest here. Let there be some $1 \leq k \leq t$ such that

$$\mathrm{bd}(h_k) \subseteq \left(\bigcup_{i=1}^{r} \mathrm{bd}(e_i)\right) \cup \left(\bigcup_{i=1}^{s} \mathrm{bd}(f_i)\right) \cup \underbrace{\left(\bigcup_{i=1}^{k-1} \mathrm{bd}(h_i)\right)}_{=\emptyset \text{ if } k=1}.$$

As the chain complex (C,Ω) is shellable, we know that

$$\left(\bigcup_{i=1}^{\ell} \Omega_{e_i^d}\right) \cap \Omega_{e_{\ell+1}^d} = \bigcup_{i=1}^{s} \Omega_{f_i}.$$

Therefore, $\mathrm{bd}(h_k) \cap \big(\bigcup_{i=1}^{r} \mathrm{bd}(e_i)\big) \subseteq \big(\bigcup_{i=1}^{s} \mathrm{bd}(f_i)\big)$, hence

$$\mathrm{bd}(h_k) \subseteq \left(\bigcup_{i=1}^{s} \mathrm{bd}(f_i)\right) \cup \left(\bigcup_{i=1}^{k-1} \mathrm{bd}(h_i)\right).$$

Because the second regularity condition holds for the elements in $\left(\Omega_{e^d_{\ell+1}}\right)_{d-1}$, there exist elements $a_1, \ldots, a_s, b_1, \ldots, b_k$ in R with $b_k \neq 0$ such that

$$\begin{aligned} b_k \partial_{d-1}(h_k) &= \sum_{i=1}^{s} a_i \partial_{d-1}(f_i) + \sum_{i=1}^{k-1} b_i \partial_{d-1}(h_i) \\ &= \sum_{i=1}^{r} 0 \cdot \partial_{d-1}(e_i) + \sum_{i=1}^{s} a_i \partial_{d-1}(f_i) + \sum_{i=1}^{k-1} b_i \partial_{d-1}(h_i). \end{aligned}$$

Hence, the first regularity condition is fulfilled, so $\mathrm{sk}_{d-1}(C, \Omega)$ is a regular chain complex.

Additionally, every elementary subcomplex $(C_{e^v_\ell}, \Omega_{e^v_\ell})$ of $\mathrm{sk}_{d-1}(C, \Omega)$ is acyclic if this holds for (C, Ω). So, if (C, Ω) is totally regular, then $\mathrm{sk}_{d-1}(C, \Omega)$ is totally regular, too. $\square$

Theorem 3.22. *Let (C, Ω) be a regular chain complex, finite of order $d \geq 1$. Then its $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C, \Omega)$ is also regular. If (C, Ω) is even totally regular, $\mathrm{sk}_{d-1}(C, \Omega)$ is also totally regular.*

Proof. Lemma 3.21 deals with this statement for pure regular chain complexes so there is only the nonpure case to do.

It is clear that the $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C, \Omega)$ of (C, Ω) satisfies the second regularity condition of Definition 3.14 so we only have to prove the first one.

Let $\Gamma \subseteq \Omega$ be the subset of all maximal basis elements such that the elements of Γ are ordered in a regular order. Let $\Omega_d = \{e^d_1, \ldots, e^d_{k_d}\} \subseteq \Gamma$ and $\Gamma \cap \Omega_{d-1} = \{g^{d-1}_1, \ldots, g^{d-1}_m\}$. As a regular order is always monotonically descending, in the chosen regular order of Γ the elements of Ω_d come first, followed by all elements of $\Gamma \cap \Omega_{d-1}$. The basis of the chain module $\left(\mathrm{sk}_{d-1}(C, \Omega)\right)_{d-1}$ is

$$\Omega_{d-1} = \left(\bigcup_{i=1}^{k_d} \mathrm{bd}(e^d_i) \right) \cup \{g^{d-1}_1, \ldots, g^{d-1}_m\}.$$

The subcomplex $(\widehat{C}, \widehat{\Omega})$ of (C, Ω) with basis $\bigcup_{i=1}^{k_d} \Omega_{e^d_i}$ is pure and regular. The basis of its chain module $\widehat{C}_{d-1}$ is $\widehat{\Omega}_{d-1} = \bigcup_{i=1}^{k_d} \mathrm{bd}(e^d_i)$.

By Lemma 3.21, the elements in $\widehat{\Omega}_{d-1}$ satisfy the first regularity condition of Definition 3.14. The same holds for $\{g^{d-1}_1, \ldots, g^{d-1}_m\}$ because (C, Ω) is regular. Hence, the $(d-1)$-skeleton $\mathrm{sk}_{d-1}(C, \Omega)$ is regular.

If (C, Ω) is totally regular, then any elementary subcomplex $(C_{e^v_i}, \Omega_{e^v_i})$ of $\mathrm{sk}_{d-1}(C, \Omega) \subseteq (C, \Omega)$ is acyclic, so $\mathrm{sk}_{d-1}(C, \Omega)$ is totally regular then. $\square$

The next two corollaries follow directly from Theorem 3.22.

Corollary 3.23. *Let (C, Ω) be a regular chain complex, finite of order d. For every $0 \leq i \leq d$, its i-skeleton $\mathrm{sk}_i(C, \Omega)$ is regular.*

Corollary 3.24. *Let (C, Ω) be a totally regular chain complex, finite of order d. For every $0 \leq i \leq d$, its i-skeleton $\mathrm{sk}_i(C, \Omega)$ is totally regular.*

3.2.3. About Reduced Homology

For totally regular chain complexes defined over some principal ideal domain, we will see that reduced and usual homology are different because the augmention map ϵ is not forced to be zero. This follows from the next lemma.

Lemma 3.25. *Let (C, Ω) be a totally regular chain complex of order d over a principal ideal domain R. Then $^{\#}\mathrm{bd}(x) \geq 2$ for all $x \in C_1 \setminus \ker(\partial_1)$.*

Proof. Because (C, Ω) is totally regular, each basis Ω_ν of a chain module C_ν is well-ordered, and each elementary subcomplex $(C_{e_i^\nu}, \Omega_{e_i^\nu})$ is acyclic. So let $\Omega_1 := \{e_1^1, \ldots, e_{k_1}^1\}$ and $\Omega_0 := \{e_1^0, \ldots, e_{k_0}^0\}$ be well-ordered. In particular, these orderings are shellings, and every elementary subcomplex $(C_{e_i^1}, \Omega_{e_i^1})$ is acyclic for $1 \leq i \leq k_1$. Hence, by Lemma 2.7, $^{\#}\mathrm{bd}(e_i^1) = 2$ for all $e_i^1 \in \Omega_1$.

We assume that there exists an element $x = \sum_{i=1}^{k_1} a_i e_i^1 \in C_1$ with $^{\#}\mathrm{bd}(x) = 1$. We define $i_0 := \max\{1 \leq i \leq k_1 \mid a_i \neq 0\} \geq 2$, so we get:

$$x = \sum_{i=1}^{i_0-1} a_i e_i^1 + a_{i_0} e_{i_0}^1.$$

Because the 1-skeleton of (C, Ω) is shellable, $\mathrm{bd}(e_{i_0}^1) \cap \left(\bigcup_{i=1}^{i_0-1} \mathrm{bd}(e_i^1)\right) \neq \emptyset$. So we distinguish two cases since $^{\#}\mathrm{bd}(e_{i_0}^1) = 2$.

- $\mathrm{bd}(e_{i_0}^1) \subseteq \left(\bigcup_{i=1}^{i_0-1} \mathrm{bd}(e_i^1)\right)$.

  Because the 1-skeleton of (C, Ω) is also totally regular due to Corollary 3.24, there are elements $\lambda_i \in R$ for $1 \leq i \leq i_0$, $\lambda_{i_0} \neq 0$, such that

$$\lambda_{i_0} \partial_1(e_{i_0}^1) = \sum_{i=1}^{i_0-1} \lambda_i \partial_1(e_i^1).$$

Consider now

$$y := \sum_{i=1}^{i_0-1} (a_i\lambda_{i_0} + a_{i_0}\lambda_i)e_i^1 \in C_1.$$

We get

$$\begin{aligned}\partial_1(y) &= \sum_{i=1}^{i_0-1} (a_i\lambda_{i_0} + a_{i_0}\lambda_i)\partial_1(e_i^1) \\ &= \sum_{i=1}^{i_0-1} a_i\lambda_{i_0}\partial_1(e_i^1) + a_{i_0}\lambda_{i_0}\partial_1(e_{i_0}^1) = \partial_1(\lambda_{i_0}x)\end{aligned}$$

and therefore ${}^{\#}\mathrm{bd}(y) = 1$ since $\partial_1(\lambda_{i_0}x) = \lambda_{i_0}\partial_1(x)$.

- ${}^{\#}\big(\mathrm{bd}(e_{i_0}^1) \cap \big(\bigcup_{i=1}^{i_0-1} \mathrm{bd}(e_i^1)\big)\big) = 1$.

  Then $\mathrm{bd}(x) = \mathrm{bd}(e_{i_0}^1) \setminus \mathrm{bd}\big(\sum_{i=1}^{i_0-1} a_i e_i^1\big)$. So we get ${}^{\#}\mathrm{bd}\big(\sum_{i=1}^{i_0-1} a_i e_i^1\big) = 1$.

In both cases, we get an element of C_1 which is a linear combination of $e_1^1, \dots, e_{i_0-1}^1$ having only one element in its boundary. Iterating this way leads to a contradiction because ${}^{\#}\mathrm{bd}(e_i^1) = 2$ for all $e_i^1 \in \Omega_1$. □

Hence, according to Theorem 1.51, there exists an augmentation map $\epsilon \neq 0$ for any totally regular chain complex (C, Ω). Because its reduced homology module $\widetilde{H}_0(C)$ is independent of the choice of an augmentation map $\epsilon \neq 0$ due to Lemma 1.48, we get $H_0(C) \cong \widetilde{H}_0(C) \oplus R$. But we cannot say anything about its reduced homology module $\widetilde{H}_{-1}(C)$ as this depends strongly on the choice of ϵ.

3.3. Homology of Totally Regular Chain Complexes

By defining totally regular chain complexes, we tried to imitate the intrinsic properties of shellable simplicial chain complexes whose homology is well-known (cf. Section 1.5). But the properties of totally regular chain complexes do not suffice to describe their homology in general. In fact, one more condition concerning the augmentation map ϵ is needed. We will start with pure chain complexes and compute their homology for a special case at first.

Theorem 3.26. *Let (C, Ω) be a pure totally regular chain complex of order $d \geq 1$ over a principal ideal domain R. Let there be only noncritical basis elements in Ω_d, and let there be an augmentation map ϵ such that $\epsilon(e_i^0)$ is a unit in R for every $e_i^0 \in \Omega_0$. Then the chain complex (C, Ω) is acyclic.*

Remark 3.27. The chain complex (C, Ω) above is pure and totally regular without precritical basis elements. Let $\Omega_d = \{e_1^d, \ldots, e_{k_d}^d\}$ be ordered in a regular order. Due to the first regularity condition (cf. Definition 3.14), there exists an element $e_{j_\ell}^{d-1} \in \mathrm{bd}(e_\ell^d)$ for any $e_\ell^d \in \Omega_d$ with $2 \leq \ell \leq k_d$ such that

$$e_{j_\ell}^{d-1} \notin \bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d).$$

In the theorem, we postulated the existence of an augmentation map ϵ which maps every basis element $e_i^0 \in \Omega_0$ to a unit in R. This is necessary to prove a general statement although there exist totally regular chain complexes which are acyclic but do not have such an augmentation map. Totally regular chain complexes without such a map ϵ need not to be acyclic, though. We consider three examples.

Example 3.28. We regard a pure finite chain complex (C, Ω) of order 1 over $\mathbb{Z}$ with bases $\Omega_1 = \{e_1^1, e_2^1\}$ and $\Omega_0 = \{e_1^0, e_2^0, e_3^0\}$ of the chain modules C_1 and C_0, respectively. A boundary map ∂_1 can be defined in different ways:

1. Let $\partial_1(e_1^1) = 2e_1^0 + e_2^0$ and $\partial_1(e_2^1) = e_2^0 + 3e_3^0$. Then (C, Ω) is a totally regular chain complex and acyclic since $H_0(C)$ is generated by the homology class $[e_1^0 - e_3^0]$. But there does not exist an augmentation map ϵ with $\epsilon(e_i^0) = \pm 1$ for all $i \in \{1, 2, 3\}$.

2. Let $\partial_1(e_1^1) = 2e_1^0 + e_2^0$ and $\partial_1(e_2^1) = e_2^0 + 2e_3^0$. This chain complex is also totally regular but not acyclic since $H_0(C) \cong \mathbb{Z} \oplus \mathbb{Z}_2$. As above, there is no augmentation map ϵ with $\epsilon(e_i^0) = \pm 1$ for all $i \in \{1, 2, 3\}$.

3. Let $\partial_1(e_1^1) = e_1^0 + e_2^0$ and $\partial_1(e_2^1) = e_2^0 + e_3^0$. Then we get a totally regular and acyclic chain complex and can define an augmentation map ϵ via $\epsilon(e_1^0) = 1$, $\epsilon(e_2^0) = -1$ and $\epsilon(e_3^0) = 1$.

Proof of Theorem 3.26. We use induction on the order d.

We begin with $d = 1$. Let $\Omega_1 := \{e_1^1, \dots, e_{k_1}^1\}$ be a basis of C_1 in a regular order with $k_1 \geq 1$ and $\Omega_0 := \{e_1^0, \dots, e_{k_0}^0\}$ be a well-ordered basis of C_0.

For $1 \leq k \leq k_1$, let (Q_k, Φ_k) be the subcomplex of (C, Ω) whose basis is $\Phi_k := \bigcup_{i=1}^{k} \Omega_{e_i^1}$. Because of the regular ordering of Ω_1, each subcomplex (Q_k, Φ_k) is totally regular itself. In particular, $(Q_{k_1}, \Phi_{k_1}) = (C, \Omega)$ and $(Q_1, \Phi_1) = (C_{e_1^1}, \Omega_{e_1^1})$, which is acyclic by assumption. We will show by induction that all subcomplexes (Q_k, Φ_k) are acyclic. For this, we need further subcomplexes of (C, Ω).

For $2 \leq \ell \leq k_1$, let (P_ℓ, Ψ_ℓ) be the subcomplex of (C, Ω) whose basis is $\Psi_\ell := \left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^1}\right) \cap \Omega_{e_\ell^1}$. Since (C, Ω) is shellable, ${}^{\#}\left(\left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^1}\right) \cap \Omega_{e_\ell^1}\right)_0 \geq 1$ for all $2 \leq \ell \leq k_1$. So (P_ℓ, Ψ_ℓ) is finite of order 0 and therefore totally regular.

Furthermore, all basis elements $e_i^1 \in \Omega_1$ are noncritical by assumption. Hence,

$$
{}^{\#}\left(\left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^1}\right) \cap \Omega_{e_\ell^1}\right)_0 < {}^{\#}\left(\Omega_{e_\ell^1}\right)_0 \quad \text{for all } 2 \leq \ell \leq k_1.
$$

Since every subcomplex $(C_{e_i^1}, \Omega_{e_i^1})$ is acyclic by assumption, each chain module $(C_{e_i^1})_0$ is generated by exactly two elements according to Lemma 2.7, i.e. ${}^{\#}(\Omega_{e_i^1})_0 = 2$. Therefore, ${}^{\#}\left(\left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^1}\right) \cap \Omega_{e_\ell^1}\right)_0 = 1$ for all $2 \leq \ell \leq k_1$, i.e. each subcomplex (P_ℓ, Ψ_ℓ) is acyclic.

For the induction step, we assume that, for some $1 < \ell \leq k_1$, the subcomplex $(Q_{\ell-1}, \Phi_{\ell-1})$ is acyclic. By our definitions above, we know that

$$
\begin{aligned}
(Q_\ell)_\nu &= \left\langle \left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^1}\right)_\nu \cup \left(\Omega_{e_\ell^1}\right)_\nu \right\rangle \\
&= \left\langle \left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^1}\right)_\nu \right\rangle + \left\langle \left(\Omega_{e_\ell^1}\right)_\nu \right\rangle = (Q_{\ell-1})_\nu + \left(C_{e_\ell^1}\right)_\nu
\end{aligned}
$$

and $(P_\ell)_\nu = (Q_{\ell-1})_\nu \cap (C_{e_\ell^1})_\nu$ for all $\nu \in \mathbb{Z}$. Hence, we get the following short exact sequence

$$
0 \longrightarrow (P_\ell)_\nu \xrightarrow{\varphi_\nu} (Q_{\ell-1})_\nu \oplus (C_{e_\ell^1})_\nu \xrightarrow{\psi_\nu} (Q_\ell)_\nu \longrightarrow 0 \tag{3.1}
$$

with $\varphi_\nu(x) = (x, -x)$ and $\psi_\nu(x, y) = x + y$ for all $\nu \in \mathbb{Z}$ (cf. Hatcher, 2008, p. 149).

By assumption, there exists an augmentation map $\epsilon\colon C_0 \to R$ such that $\epsilon(e_i^0)$ is a unit in R for all $e_i^0 \in \Omega_0$, i.e. each restriction $\epsilon\big|_{\langle e_i^0\rangle}$ is surjective. Each subcomplex (P_ℓ, Ψ_ℓ), $(Q_{\ell-1}, \Phi_{\ell-1})$, $(C_{e_\ell^1}, \Omega_{e_\ell^1})$ and (Q_ℓ, Φ_ℓ) can be augmented by the restriction of ϵ (cf. Hatcher, 2008, p. 150).

Since all φ_ν and ψ_ν are chain maps, we obtain a long exact sequence of reduced homology modules (cf. Hatcher, 2008, p. 116):

$$\begin{aligned} \ldots \longrightarrow \widetilde{H}_m(P_\ell) \xrightarrow{\varphi_{*n}} \widetilde{H}_m(Q_{\ell-1}) \oplus \widetilde{H}_m(C_{e_\ell^d}) \xrightarrow{\psi_{*n}} \widetilde{H}_m(Q_\ell) \xrightarrow{\delta_n} \widetilde{H}_{m-1}(P_\ell) \xrightarrow{\varphi_{*n-1}} \\ \xrightarrow{\varphi_{*n-1}} \ldots \xrightarrow{\delta_1} \widetilde{H}_0(P_\ell) \xrightarrow{\varphi_{*0}} \widetilde{H}_0(Q_{\ell-1}) \oplus \widetilde{H}_0(C_{e_\ell^d}) \xrightarrow{\psi_{*0}} \widetilde{H}_0(Q_\ell) \xrightarrow{\delta_0} \\ \xrightarrow{\delta_0} \widetilde{H}_{-1}(P_\ell) \xrightarrow{\varphi_{*-1}} \widetilde{H}_{-1}(Q_{\ell-1}) \oplus \widetilde{H}_{-1}(C_{e_\ell^d}) \xrightarrow{\psi_{*-1}} \widetilde{H}_{-1}(Q_\ell) \xrightarrow{\delta_{-1}} 0. \end{aligned} \tag{3.2}$$

For any subcomplex $(D, \Omega^D) \subseteq (C, \Omega)$, the homology module $\widetilde{H}_{-1}(D) = 0$ because each restriction $\epsilon\big|_D$ is surjective. So the reduced homology modules of (P_ℓ, Ψ_ℓ), $(Q_{\ell-1}, \Phi_{\ell-1})$ and $(C_{e_\ell^d}, \Omega_{e_\ell^d})$ are all zero since each of these subcomplexes is acyclic. We conclude that (Q_ℓ, Φ_ℓ) is acyclic, too. Hence, by induction, $(C, \Omega) = (Q_{k_1}, \Phi_{k_1})$ is an acyclic chain complex, i.e. $H_\nu(C) = 0$ for $\nu \geq 1$ and $H_0(C) \cong R$.

Now let $d \geq 2$. Let $\Omega_d := \{e_1^d, \ldots, e_{k_d}^d\}$ be ordered in a regular order with $k_d \geq 1$. As above, let (Q_k, Φ_k) be the subcomplex of (C, Ω) for $1 \leq k \leq k_d$ whose basis is $\Phi_k := \bigcup_{i=1}^k \Omega_{e_i^d}$. Since Ω_d is regularly ordered, each subcomplex (Q_k, Φ_k) is totally regular, too. In particular, $(Q_1, \Phi_1) = (C_{e_1^d}, \Omega_{e_1^d})$ and $(Q_{k_d}, \Phi_{k_d}) = (C, \Omega)$. By induction we will see that all (Q_k, Φ_k) are acyclic, using that (Q_1, Φ_1) is acyclic by assumption. Again, we need further subcomplexes for the induction step.

For $2 \leq \ell \leq k_d$, let (P_ℓ, Ψ_ℓ) be the subcomplex of (C, Ω) whose basis is $\Psi_\ell := \left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\right) \cap \Omega_{e_\ell^d}$. About this chain complex we know:

1. (P_ℓ, Ψ_ℓ) is a pure finite chain complex of order $(d-1)$ and shellable since (C, Ω) is shellable.

2. Every elementary subcomplex $(C_{e_i^\nu}, \Omega_{e_i^\nu})$ of (P_ℓ, Ψ_ℓ) is shellable and acyclic.

3. There holds $\Omega_{e_i^\nu} \subseteq \left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\right) \cap \Omega_{e_\ell^d}$ for every $e_i^\nu \in \left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\right) \cap \Omega_{e_\ell^d}$. Hence, the second regularity condition of Definition 3.14 holds for (P_ℓ, Ψ_ℓ).

4. (P_ℓ, Ψ_ℓ) is a subcomplex of $(C_{e_\ell^d}, \Omega_{e_\ell^d})$. Since e_ℓ^d fulfils the second regularity condition and $\left(\left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\right) \cap \Omega_{e_\ell^d}\right)_{d-1} \subseteq (\Omega_{e_\ell^d})_{d-1}$ holds, the subcomplex (P_ℓ, Ψ_ℓ) satisfies the first regularity condition.

Hence, (P_ℓ, Ψ_ℓ) is a pure totally regular chain complex of order $(d-1)$, and $\epsilon\big|_{P_\ell}(e_i^0)$ is a unit in R for every basis element $e_i^0 \in \big((\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}) \cap \Omega_{e_\ell^d}\big)_0$. If (P_ℓ, Ψ_ℓ) has no precritical elements in $\big((\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}) \cap \Omega_{e_\ell^d}\big)_{d-1}$, we can apply our induction hypothesis and conclude that (P_ℓ, Ψ_ℓ) is acyclic. So we have to show that (P_ℓ, Ψ_ℓ) has no precritical elements in $\big((\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}) \cap \Omega_{e_\ell^d}\big)_{d-1}$.

For $2 \leq \ell \leq k_d$, let $\big((\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}) \cap \Omega_{e_\ell^d}\big)_{d-1} = \{g_1, \ldots, g_{m_\ell}\}$, ordered in a regular order. If $m_\ell = 1$, there is nothing to do, so let $m_\ell \geq 2$. We assume that there is some $2 \leq j \leq m_\ell$ such that g_j is precritical, i.e. there exist elements $a_i \in R$ for $1 \leq i \leq j$ with $a_j \neq 0$ such that

$$a_j \partial_{d-1}(g_j) = \sum_{i=1}^{j-1} a_i \partial_{d-1}(g_i).$$

Hence, $a_j g_j - \sum\limits_{i=1}^{j-1} a_i g_i \in \ker(\partial_{d-1})$.

The chain complex (C, Ω) has no precritical elements in Ω_d by assumption, therefore

$$\left(\left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\right) \cap \Omega_{e_\ell^d}\right)_{d-1} \subsetneqq (\Omega_{e_\ell^d})_{d-1}.$$

Otherwise, we would get

$$\left(\Omega_{e_\ell^d}\right)_{d-1} = \mathrm{bd}(e_\ell^d) \subseteq \left(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\right)_{d-1} = \bigcup_{i=1}^{\ell-1} \left(\Omega_{e_i^d}\right)_{d-1} = \bigcup_{i=1}^{\ell-1} \mathrm{bd}(e_i^d),$$

and then, by the first regularity condition of Definition 3.14, e_ℓ^d would be precritical which is a contradiction to our assumption.

Therefore, we obtain $\partial_d(e_\ell^d) = \sum\limits_{i=1}^{m_\ell} b_i g_i + r$ with $\sum\limits_{i=1}^{m_\ell} b_i g_i \in (P_\ell)_{d-1}$ and some $r \in \langle (\widehat{\Omega}_{e_\ell^d})_{d-1} \rangle$ with $\big(\widehat{\Omega}_{e_\ell^d}\big)_{d-1} := \big(\Omega_{e_\ell^d}\big)_{d-1} \setminus \big(\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}\big)$. In particular, we have $r \neq 0$ since $\big(\Omega_{e_\ell^d}\big)_{d-1} = \mathrm{bd}(e_\ell^d) \nsubseteq \bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}$.

Furthermore, we know $\partial_{d-1} \circ \partial_d(e_\ell^d) = 0$, i.e. $\partial_d(e_\ell^d) \in \ker(\partial_{d-1})$. Because $r \neq 0$, there are two linearly independent elements in $\ker(\partial_{d-1})$, namely $\partial_d(e_\ell^d)$ and $a_j g_j - \sum_{i=1}^{j-1} a_i g_i$. Those two elements are even contained in $\big(C_{e_\ell^d}\big)_{d-1}$, hence $\ker \partial_{d-1}\big|_{C_{e_\ell^d}}$ is generated by at least two elements.

We know that $\left(C_{e_\ell^d}\right)_d = \langle e_\ell^d \rangle$, and thus $\operatorname{im} \partial_d\big|_{C_{e_\ell^d}}$ is generated by a single element.

Hence, we conclude that $H_{d-1}(C_{e_\ell^d}) \neq 0$, which is a contradiction to our assumption that $(C_{e_\ell^d}, \Omega_{e_\ell^d})$ is acyclic. Therefore, $\left((\bigcup_{i=1}^{\ell-1} \Omega_{e_i^d}) \cap \Omega_{e_\ell^d}\right)_{d-1}$ contains no precritical basis elements.

By an analogous argument as above for $d = 1$, using short exact sequences of chain complexes as in Equation (3.1) on page 73 and a long exact sequence in reduced homology as in Equation (3.2), we obtain via induction that the chain complex (C, Ω) is acyclic. $\square$

Theorem 3.29. *Let (C, Ω) be a pure totally regular chain complex of order $d \geq 1$ over a principal ideal domain R. Let the basis $\Omega_d = \{e_1^d, \ldots, e_{k_d}^d\}$ of C_d have $n < k_d$ precritical elements. Let there be an augmentation map $\epsilon \colon C_0 \to R$ such that $\epsilon(e_i^0)$ is a unit in R for every $e_i^0 \in \Omega_0$. Then the homology modules of (C, Ω) are*

$$\begin{aligned} H_d(C) &\cong R^n, \\ H_i(C) &= 0 \quad \text{for } i \neq 0, d, \\ H_0(C) &\cong R. \end{aligned}$$

Proof. We can assume that all noncritical elements in Ω_d come first in the regular order. Otherwise we can change the order such that the precritical (and critical) elements of Ω_d come last, this has no influence on the shellability and regularity of (C, Ω). Let $m := k_d - n$, then we have

$$\Omega_d = \{\underbrace{e_1^d, \ldots, e_m^d}_{\text{noncritical}}, \underbrace{e_{m+1}^d, \ldots, e_{k_d}^d}_{\text{precritical}}\}.$$

With respect to Theorem 1.61 we get $H_d(C) \cong R^n$.

Consider now the subcomplex $(\widehat{C}, \widehat{\Omega}) \subseteq (C, \Omega)$ with basis $\widehat{\Omega} := \bigcup_{i=1}^m \Omega_{e_i^d}$ which is pure and finite of order d. Its chain modules are $\widehat{C}_d = \langle e_1^d, \ldots, e_m^d \rangle$ and $\widehat{C}_\nu = C_\nu$ for $0 \leq \nu \leq (d-1)$. Since (C, Ω) is a totally regular chain complex, so is $(\widehat{C}, \widehat{\Omega})$. Contrary to (C, Ω), the chain complex $(\widehat{C}, \widehat{\Omega})$ has no precritical elements, so it is acyclic due to Theorem 3.26, i.e. $H_0(\widehat{C}) \cong R$ and $H_i(\widehat{C}) = 0$ for $i \neq 0$.

Because $\widehat{C}_{d-1} = C_{d-1}$ and $H_{d-1}(\widehat{C}) = 0$, we get

$$\operatorname{im} \partial_d\big|_{\widehat{C}_d} \subseteq \operatorname{im} \partial_d \subseteq \ker \partial_{d-1} = \ker \partial_{d-1}\big|_{\widehat{C}_{d-1}} = \operatorname{im} \partial_d\big|_{\widehat{C}_d}.$$

Therefore, $\operatorname{im}\partial_d\big|_{\widehat{C}_d} = \operatorname{im}\partial_d$. Since $\widehat{C}_\nu = C_\nu$ for $0 \leq \nu \leq (d-1)$, we conclude:

$$\begin{aligned} H_i(C) &= H_i(\widehat{C}) = 0 \quad \text{for } 1 \leq i \leq d-1, \\ H_0(C) &= H_0(\widehat{C}) \cong R. \end{aligned}$$

□

Theorem 3.30. *Let (C, Ω) be a totally regular chain complex of order $d \geq 1$ over a principal ideal domain R. Let Γ be the subset of Ω which contains all maximal basis elements. For any $0 \leq \nu \leq d$, let $\Omega_\nu := \{e_1^\nu, \ldots, e_{k_\nu}^\nu\}$ be a well-ordered basis of the chain module C_ν. For $0 \leq \nu \leq d$, let there be $n_\nu < k_\nu$ precritical elements in $\Gamma \cap \Omega_\nu$. Let $\epsilon\colon C_0 \to R$ be an augmentation map such that $\epsilon(e_i^0)$ is a unit in R for every $e_i^0 \in \Omega_0$. Then the homology modules of (C, Ω) are*

$$\begin{aligned} H_i(C) &\cong R^{n_i} \quad \textit{for } 1 \leq i \leq d, \\ H_0(C) &\cong R^{n_0+1}, \\ H_i(C) &= 0 \quad \textit{if } i < 0 \textit{ or } i > d. \end{aligned}$$

Remark 3.31. Any chain complex (C, Ω) over a principal ideal domain which is obtained from a simplicial complex is totally regular by Remark 3.18. From Section 1.4.1 we know that each simplicial complex has an augmentation map ϵ such that $\epsilon(e_i^0)$ is a unit in R for every $e_i^0 \in \Omega_0$. Hence, this theorem recovers the statement of Theorem 1.52.

Proof of Theorem 3.30. Let Γ be ordered in a regular order. Because each basis Ω_ν is well-ordered, the elements of Γ come last. Let

$$\Omega_\nu := \{\underbrace{e_1^\nu, \ldots, e_{m_\nu}^\nu}_{\notin \Gamma}, \underbrace{e_{m_\nu+1}^\nu, \ldots, e_{k_\nu}^\nu}_{\in \Gamma}\}.$$

So we even have $n_\nu \leq (k_\nu - m_\nu)$. Furthermore, let all precritical elements in $\Gamma \cap \Omega_\nu$ come last in the ordering of each Ω_ν.

We consider the subcomplex $(C^d, \Omega^d) \subseteq (C, \Omega)$ with basis $\Omega^d := \bigcup_{i=1}^{k_d} \Omega_{e_i^d}$, which is pure, finite of order d and totally regular. According to Theorem 3.29, the homology modules of (C^d, Ω^d) are

$$\begin{aligned} H_d(C^d) &\cong R^{n_d}, \\ H_i(C^d) &= 0 \quad \text{for } i \neq 0, d, \\ H_0(C^d) &\cong R. \end{aligned}$$

Since $(C^d)_d = C_d$, we get $H_d(C) \cong R^{n_d}$.

For $1 \leq \mu \leq (d-1)$, we consider the subcomplex $(C^\mu, \Omega^\mu) \subseteq \mathrm{sk}_\mu(C, \Omega)$ with basis $\Omega^\mu := \bigcup_{i=1}^{k_\mu} \Omega_{e_i^\mu}$, which is pure and finite of order μ. We know that each μ-skeleton of (C, Ω) is shellable and totally regular according to the Lemmata 3.7 and 3.21. Let $\Omega_{\mathrm{sk}_\mu} \subseteq \Omega$ be the basis of $\mathrm{sk}_\mu(C, \Omega)$ and $\Gamma_{\mathrm{sk}_\mu} := \{e \in \Omega_{\mathrm{sk}_\mu} \mid e \notin \mathrm{bd}(f) \text{ for all } f \in \Omega_{\mathrm{sk}_\mu}\}$ be the set of its maximal basis elements, ordered in a regular order. Since a regular order is always monotonically descending, the basis elements $e_1^\mu, \ldots, e_{k_\mu}^\mu$ come first in the regular order of Γ_{sk_μ}. Therefore, each subcomplex (C^μ, Ω^μ) is also shellable and totally regular.

Let there be ℓ_μ precritical basis elements in $\{e_1^\mu, \ldots, e_{m_\mu}^\mu\} \subseteq \Omega_{\mathrm{sk}_\mu}$. By Theorem 3.29 we get

$$\begin{aligned} H_\mu(C^\mu) &\cong R^{n_\mu + \ell_\mu}, \\ H_i(C^\mu) &= 0 \quad \text{for } i \neq 0, \mu, \\ H_0(C^\mu) &\cong R. \end{aligned}$$

Hence, we also obtain $H_\mu(C^{\mu+1}) = 0$. Since $(C^{\mu+1})_{\mu+1} = C_{\mu+1}$, we get

$$\operatorname{im} \partial_{\mu+1} = \operatorname{im} \partial_{\mu+1}\big|_{(C^{\mu+1})_{\mu+1}} = \ker \partial_\mu\big|_{\langle e_1^\mu, \ldots, e_{m_\mu}^\mu \rangle}.$$

Since the subcomplex $(\widehat{C}^\mu, \widehat{\Omega}^\mu) \subseteq (C^\mu, \Omega^\mu)$ with basis $\widehat{\Omega}^\mu := \bigcup_{i=1}^{m_\mu} \Omega_{e_i^\mu}$ is pure, finite of order μ and totally regular, we obtain by Theorem 3.29:

$$\ker \partial_\mu\big|_{\langle e_1^\mu, \ldots, e_{m_\mu}^\mu \rangle} \cong H_\mu(\widehat{C}^\mu) \cong R^{\ell_\mu}.$$

By assumption holds $\Gamma \cap \Omega_\mu = \{e_{m_\mu+1}^\mu, \ldots, e_{k_\mu}^\mu\}$, therefore $e_i^\mu \notin \operatorname{im} \partial_{\mu+1}$ for $(m_\mu + 1) \leq i \leq k_\mu$. We conclude:

$$H_\mu(C) = {}^{\ker \partial_\mu}\!\big/_{\operatorname{im} \partial_{\mu+1}} \cong R^{n_\mu} \quad \text{for } 1 \leq \mu \leq (d-1).$$

For $\mu = 0$, we consider the subcomplex $(C^1, \Omega^1) \subseteq \mathrm{sk}_1(C, \Omega)$ with basis $\Omega^1 := \bigcup_{i=1}^{k_1} \Omega_{e_i^1}$ which is pure, finite of order 1 and totally regular. Hence, $H_0(C^1) = \langle e_1^0, \ldots, e_{m_0}^0 \rangle / \operatorname{im} \partial_1 \cong R$ due to Theorem 3.29. Since $e_i^0 \notin \operatorname{im} \partial_1$ for $(m_0 + 1) \leq i \leq k_0$, we get

$$H_0(C) = {}^{C_0}\!\big/_{\operatorname{im} \partial_1} \cong R^{n_0 + 1}.$$

For $i > d$ or $i < 0$, it is clear that $H_i(C) = 0$ because $C_i = 0$. □

Corollary 3.32. *Let (C, Ω) be a totally regular chain complex of order $d \geq 1$ over a field K with $\operatorname{char} K = 0$. Let Γ be the subset of Ω which contains all maximal basis elements. For any $0 \leq \nu \leq d$, let $\Omega_\nu := \{e_1^\nu, \ldots, e_{k_\nu}^\nu\}$ be a well-ordered basis of the chain module C_ν. For $0 \leq \nu \leq d$, let there be $n_\nu < k_\nu$ precritical elements in $\Gamma \cap \Omega_\nu$. Then the homology modules of (C, Ω) are*

$$
\begin{aligned}
H_i(C) &\cong K^{n_i} \quad \text{for } 1 \leq i \leq d,\\
H_0(C) &\cong K^{n_0+1},\\
H_i(C) &= 0 \quad \text{if } i < 0 \text{ or } i > d.
\end{aligned}
$$

Proof. According to Lemma 3.25, $^{\#}\operatorname{bd}(x) \geq 2$ for all $x \in C_1 \setminus \ker(\partial_1)$. Hence, there exists a augmentation map $\epsilon\colon C_0 \to K$ such that $\epsilon(e_i^0) \neq 0$ for all $e_i^0 \in \Omega_0$ due to Theorem 1.49. Since $\epsilon(e_i^0) \in K$ is a unit, the corollary is a consequence of Theorem 3.30. □

3.4. Mapping Cones over Shellable Chain Complexes

If we construct a geometrical cone over a shellable simplicial complex, the new simplicial complex is also shellable because the dimension of every maximal simplex increases by one. What happens if we construct a mapping cone of the identity chain map over a shellable chain complex?

Theorem 3.33. *Let (C, Ω) be a finite chain complex of order d over a principal ideal domain R which is pure and shellable. Let the set $\Omega_d = \{e_1^d, \ldots, e_{k_d}^d\}$ of all its maximal basis elements be ordered in a shelling. Let there be an augmentation map $\epsilon\colon C_0 \to R$ such that $\epsilon(e_i^0) \neq 0$ for all $e_i^0 \in \Omega_0 = \{e_1^0, \ldots, e_{k_0}^0\}$. Let $\big(\widehat{C}(\mathrm{id}_C), \Phi\big)$:*

$$
\{0\} \oplus C_d \xrightarrow{\delta_{d+1}} C_d \oplus C_{d-1} \xrightarrow{\delta_d} \ldots \xrightarrow{\delta_2} C_1 \oplus C_0 \xrightarrow{\delta_1} C_0 \oplus R \xrightarrow{\delta_0} 0
$$

be the mapping cone of the identity chain map $\mathrm{id}_C\colon (C, \Omega) \to (C, \Omega)$ whose boundary maps are

$$
\delta_\nu = \begin{pmatrix} \partial_\nu & \mathrm{id}_{\nu-1} \\ 0 & -\partial_{\nu-1} \end{pmatrix} \text{ for } \nu \geq 2, \qquad \delta_1 = \begin{pmatrix} \partial_1 & \mathrm{id}_0 \\ 0 & -\epsilon \end{pmatrix}, \qquad \delta_0 = 0. \tag{3.3}
$$

Then the mapping cone $\big(\widehat{C}(\mathrm{id}_C), \Phi\big)$ is shellable and a cone.

Proof. Since the augmentation map ϵ is nonzero on all basis elements of Ω_0, the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ is a cone by Theorem 2.17.

For $d = 0$, the only nonzero chain module in the chain complex (C, Ω) is C_0 with basis $\Omega_0 := \{e^0_1, \ldots, e^0_{k_0}\} \neq \emptyset$. The mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ of id_C over (C, Ω) is

$$\{0\} \oplus C_0 \xrightarrow{\delta_1} C_0 \oplus R \xrightarrow{\delta_0} 0$$

with the boundary maps as in Equation (3.3). Its chain modules $\widehat{\mathcal{C}}(\mathrm{id}_C)_1$ and $\widehat{\mathcal{C}}(\mathrm{id}_C)_0$ have the bases

$$\Phi_1 = \left\{ \begin{pmatrix} 0 \\ e^0_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ e^0_{k_0} \end{pmatrix} \right\}, \qquad \Phi_0 = \left\{ \begin{pmatrix} e^0_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} e^0_{k_0} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

Since $\delta_1 \begin{pmatrix} 0 \\ e^0_i \end{pmatrix} = \begin{pmatrix} e^0_i \\ 0 \end{pmatrix} - r_i \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ with some $r_i \in R \setminus \{0\}$ for all $1 \leq i \leq k_0$, the chain complex $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ is pure.

It is also shellable since for all $2 \leq i \leq k_0$ holds:

$$\Phi_{(0,e^0_i)} \cap \left(\bigcup_{k=1}^{i-1} \Phi_{(0,e^0_k)} \right) = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

For $d \geq 1$, we use induction on the order d, beginning with $d = 1$. We consider the chain complex (C, Ω): $C_1 \to C_0 \to 0$ whose chain modules have the bases $\Omega_0 = \{e^0_1, \ldots, e^0_{k_0}\}$ and $\Omega_1 = \{e^1_1, \ldots, e^1_{k_1}\}$, which is ordered in a shelling. The mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ over (C, Ω) is the chain complex

$$\{0\} \oplus C_1 \xrightarrow{\delta_2} C_1 \oplus C_0 \xrightarrow{\delta_1} C_0 \oplus R \xrightarrow{\delta_0} 0$$

of order 2 with the usual boundary maps δ_ℓ as in Equation (3.3). The bases of its chain modules $\widehat{\mathcal{C}}(\mathrm{id}_C)_\nu$ are

$$\begin{aligned}
\Phi_2 &= \left\{ \begin{pmatrix} 0 \\ e^1_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ e^1_{k_1} \end{pmatrix} \right\}, \\
\Phi_1 &= \left\{ \begin{pmatrix} e^1_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} e^1_{k_1} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ e^0_1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ e^0_{k_0} \end{pmatrix} \right\}, \\
\Phi_0 &= \left\{ \begin{pmatrix} e^0_1 \\ 0 \end{pmatrix}, \ldots, \begin{pmatrix} e^0_{k_0} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.
\end{aligned}$$

The basis elements of Φ are mapped by the boundary maps δ_ν as follows:

$$\begin{aligned}
\delta_2 \begin{pmatrix} 0 \\ e_i^1 \end{pmatrix} &= \begin{pmatrix} e_i^1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \partial_1 e_i^1 \end{pmatrix}, \\
\delta_1 \begin{pmatrix} e_i^1 \\ 0 \end{pmatrix} &= \begin{pmatrix} \partial_1 e_i^1 \\ 0 \end{pmatrix}, \\
\delta_1 \begin{pmatrix} 0 \\ e_i^0 \end{pmatrix} &= \begin{pmatrix} e_i^0 \\ 0 \end{pmatrix} - \lambda_i \begin{pmatrix} 0 \\ 1 \end{pmatrix} \text{ with } \lambda_i := \epsilon\left(e_i^0\right) \neq 0.
\end{aligned}$$

Since (C, Ω) is a pure chain complex and $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ is contained in the boundary of any basis element $\left(\begin{smallmatrix} 0 \\ e_i^0 \end{smallmatrix}\right)$, the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ is also a pure chain complex. Hence, we have to consider the bases $\Phi_{(0,e_i^1)}$ of the elementary subcomplexes $\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_i^1)}, \Phi_{(0,e_i^1)}\right)$ of order 2 for all $1 \leq i \leq k_1$ to check shellability of the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$. The bases of the chain modules of any elementary subcomplex $\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_i^1)}, \Phi_{(0,e_i^1)}\right)$ are

$$\begin{aligned}
\left(\Phi_{(0,e_i^1)}\right)_2 &= \left\{ \begin{pmatrix} 0 \\ e_i^1 \end{pmatrix} \right\}, \\
\left(\Phi_{(0,e_i^1)}\right)_1 &= \left\{ \begin{pmatrix} e_i^1 \\ 0 \end{pmatrix} \right\} \cup \left\{ \begin{pmatrix} 0 \\ e_k^0 \end{pmatrix} \;\middle|\; e_k^0 \in \mathrm{bd}(e_i^1) \right\}, \\
\left(\Phi_{(0,e_i^1)}\right)_0 &= \left\{ \begin{pmatrix} e_k^0 \\ 0 \end{pmatrix} \;\middle|\; e_k^0 \in \mathrm{bd}(e_i^1) \right\} \cup \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.
\end{aligned}$$

For $2 \leq i \leq k_1$, we get

$$\begin{aligned}
\Phi_{(0,e_i^1)} \cap \left(\bigcup_{k=1}^{i-1} \Phi_{(0,e_k^1)} \right) &= \left\{ \begin{pmatrix} 0 \\ e_\ell^0 \end{pmatrix} \;\middle|\; e_\ell^0 \in \mathrm{bd}(e_i^1) \cap \left(\bigcup_{k=1}^{i-1} \mathrm{bd}(e_k^1) \right) \right\} \\
&\quad \cup \left\{ \begin{pmatrix} e_\ell^0 \\ 0 \end{pmatrix} \;\middle|\; e_\ell^0 \in \mathrm{bd}(e_i^1) \cap \left(\bigcup_{k=1}^{i-1} \mathrm{bd}(e_k^1) \right) \right\} \cup \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.
\end{aligned}$$

This set generates a pure chain complex of order 1 since

$$\mathrm{bd} \begin{pmatrix} 0 \\ e_\ell^0 \end{pmatrix} = \left\{ \begin{pmatrix} e_\ell^0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

for every $1 \leq \ell \leq k_0$. Therefore, the shelling condition 1 of Definition 3.1 is fulfilled. Moreover, we see that $\left(\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right)$ is the only element which the boundaries of any two different basis elements $\binom{0}{e^0_\ell}$ and $\binom{0}{e^0_k}$ have in common. Hence, for $1 \leq i \leq k_1$, every ordering of the set

$$\left\{ \begin{pmatrix} 0 \\ e^0_k \end{pmatrix} \middle| \, e^0_k \in \mathrm{bd}(e^1_i) \right\} \subseteq \left(\Phi_{(0,e^1_i)} \right)_1$$

is a shelling. So taking $\binom{e^1_i}{0}$ as the last element in an ordering of $\left(\Phi_{(0,e^1_i)}\right)_1$ yields a shelling because the boundary of $\binom{e^1_i}{0}$ contains all basis elements of $\left(\Phi_{(0,e^1_i)}\right)_0$ except $\binom{0}{1}$. Therefore, every subcomplex $(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e^1_i)}, \Phi_{(0,e^1_i)})$ is shellable, and for $i = 1$ the shelling condition 3 of Definition 3.1 is fulfilled. For $2 \leq i \leq k_1$, we can choose a shelling of $\left\{ \binom{0}{e^0_k} \middle| \, e^0_k \in \mathrm{bd}(e^1_i) \right\}$ in which the elements of

$$\left(\Phi_{(0,e^1_i)} \cap \left(\bigcup_{j=1}^{i-1} \Phi_{(0,e^1_j)} \right) \right)_1 \subseteq \left\{ \begin{pmatrix} 0 \\ e^0_k \end{pmatrix} \middle| \, e^0_k \in \mathrm{bd}(e^1_i) \right\}$$

come first. Then the shelling condition 2 of Definition 3.1 is fulfilled, too, and therefore the mapping cone $(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi)$ is shellable.

We proceed by induction and assume that the theorem's statement is true for all pure chain complexes of order a with $1 \leq a \leq d$ for some $d \geq 1$. Let (C, Ω) be a pure shellable chain complex

$$C_{d+1} \xrightarrow{\partial_{d+1}} C_d \xrightarrow{\partial_d} \dots \xrightarrow{\partial_3} C_2 \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\epsilon} R \longrightarrow 0$$

of order $(d+1)$ such that $\epsilon(e^0_\ell) \neq 0$ for all $e^0_\ell \in \Omega_0$. According to Remark 3.13, we assume that all bases Ω_ν of the chain modules C_ν are well-ordered.

We consider the mapping cone $(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi)$

$$\{0\} \oplus C_{d+1} \xrightarrow{\delta_{d+2}} C_{d+1} \oplus C_d \xrightarrow{\delta_{d+1}} \dots \xrightarrow{\delta_2} C_1 \oplus C_0 \xrightarrow{\delta_1} C_0 \oplus R \xrightarrow{\delta_0} 0$$

with the usual boundary maps δ_ℓ. According to Lemma 2.15, the mapping cone $(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi)$ is a pure chain complex of order $(d+2)$. We will need some subcomplexes of this mapping cone to prove its shellability.

We know that the d-skeleton $\mathrm{sk}_d(C, \Omega)$ is pure, finite of order d and shellable. Hence, the mapping cone $\widehat{\mathcal{C}}(\mathrm{id}_{\mathrm{sk}_d(C,\Omega)})$

$$\{0\} \oplus C_d \xrightarrow{\delta_{d+1}} C_d \oplus C_{d-1} \xrightarrow{\delta_d} \dots \xrightarrow{\delta_2} C_1 \oplus C_0 \xrightarrow{\delta_1} C_0 \oplus R \xrightarrow{\delta_0} 0$$

over the d-skeleton $\mathrm{sk}_d(C,\Omega)$ is also pure, finite of order $(d+1)$ and shellable by induction. Furthermore, the subcomplex

$$C_{d+1}\oplus\{0\}\xrightarrow{\delta_{d+1}}C_d\oplus\{0\}\xrightarrow{\delta_d}\dots\xrightarrow{\delta_2}C_1\oplus\{0\}\xrightarrow{\delta_1}C_0\oplus\{0\}\xrightarrow{\delta_0}0 \tag{3.4}$$

of $\left(\widehat{\mathcal{C}}(\mathrm{id}_C),\Phi\right)$ is isomorphic to (C,Ω) and therefore shellable.

First we prove that the $(d+1)$-skeleton $\mathrm{sk}_{d+1}\left(\widehat{\mathcal{C}}(\mathrm{id}_C),\Phi\right)$

$$C_{d+1}\oplus C_d\xrightarrow{\delta_{d+1}}C_d\oplus C_{d-1}\xrightarrow{\delta_d}\dots\xrightarrow{\delta_2}C_1\oplus C_0\xrightarrow{\delta_1}C_0\oplus R\xrightarrow{\delta_0}0$$

of $\left(\widehat{\mathcal{C}}(\mathrm{id}_C),\Phi\right)$ is shellable. Because the $(d+1)$-skeleton $\mathrm{sk}_{d+1}\left(\widehat{\mathcal{C}}(\mathrm{id}_C),\Phi\right)$ is a pure finite chain complex due to Remark 1.39, we have to consider the basis Φ_{d+1} of its chain module $\left(\widehat{\mathcal{C}}(\mathrm{id}_C)\right)_{d+1}=C_{d+1}\oplus C_d$ which is

$$\Phi_{d+1}=\left\{\begin{pmatrix}0\\e_1^d\end{pmatrix},\dots,\begin{pmatrix}0\\e_{k_d}^d\end{pmatrix}\right\}\cup\left\{\begin{pmatrix}e_1^{d+1}\\0\end{pmatrix},\dots,\begin{pmatrix}e_{k_{d+1}}^{d+1}\\0\end{pmatrix}\right\}.$$

Since the mapping cone $\widehat{\mathcal{C}}(\mathrm{id}_{\mathrm{sk}_d(C)})$ is shellable, we already know that the basis

$$\left\{\begin{pmatrix}0\\e_1^d\end{pmatrix},\dots,\begin{pmatrix}0\\e_{k_d}^d\end{pmatrix}\right\}$$

of its chain submodule $\{0\}\oplus C_d$ has a shelling. By definition of the boundary maps, the elements of Φ_{d+1} are mapped by δ_{d+1} as follows:

$$\begin{aligned}\delta_{d+1}\begin{pmatrix}0\\e_j^d\end{pmatrix}&=\begin{pmatrix}e_j^d\\0\end{pmatrix}-\begin{pmatrix}0\\\partial_d e_j^d\end{pmatrix},\\ \delta_{d+1}\begin{pmatrix}e_\ell^{d+1}\\0\end{pmatrix}&=\begin{pmatrix}\partial_{d+1}e_\ell^{d+1}\\0\end{pmatrix}=\sum_{i=1}^{k_d}a_i^{d+1,\ell}\begin{pmatrix}e_i^d\\0\end{pmatrix}\quad\text{with some } a_i^{d+1,\ell}\in R.\end{aligned}$$

Hence, for all $1\le\ell\le k_{d+1}$ holds:

$$\mathrm{bd}\begin{pmatrix}e_\ell^{d+1}\\0\end{pmatrix}\subseteq\bigcup_{j=1}^{k_d}\mathrm{bd}\begin{pmatrix}0\\e_j^d\end{pmatrix}.$$

So for $\ell=1$ we get

$$\left(\bigcup_{j=1}^{k_d}\Phi_{(0,e_j^d)}\right)\cap\Phi_{(e_1^{d+1},0)}=\Phi_{(e_1^{d+1},0)}\setminus\left\{\begin{pmatrix}e_1^{d+1}\\0\end{pmatrix}\right\}.$$

This set is a basis of the d-skeleton $\mathrm{sk}_d\big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(e_1^{d+1},0)}, \Phi_{(e_1^{d+1},0)}\big)$ of the elementary subcomplex $\big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(e_1^{d+1},0)}, \Phi_{(e_1^{d+1},0)}\big)$ and therefore generates a pure subcomplex which is finite of order d. Because $\big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(e_1^{d+1},0)}, \Phi_{(e_1^{d+1},0)}\big)$ is also an elementary subcomplex of the shellable chain complex in Equation (3.4) on page 83, we conclude that the set

$$\big(\Phi_{(e_1^{d+1},0)}\big)_d = \left(\left(\bigcup_{j=1}^{k_d} \Phi_{(0,e_j^d)}\right) \cap \Phi_{(e_1^{d+1},0)}\right)_d$$

has a shelling.

We proceed with $2 \le \ell \le k_{d+1}$ and get

$$\begin{aligned}
&\left(\left(\bigcup_{j=1}^{k_d} \Phi_{(0,e_j^d)}\right) \cup \left(\bigcup_{i=1}^{\ell-1} \Phi_{(e_i^{d+1},0)}\right)\right) \cap \Phi_{(e_\ell^{d+1},0)} \\
&\qquad = \underbrace{\left(\left(\bigcup_{j=1}^{k_d} \Phi_{(0,e_j^d)}\right) \cap \Phi_{(e_\ell^{d+1},0)}\right)}_{=\Phi_{(e_\ell^{d+1},0)} \setminus \left\{\binom{e_\ell^{d+1}}{0}\right\}} \cup \underbrace{\left(\left(\bigcup_{i=1}^{\ell-1} \Phi_{(e_i^{d+1},0)}\right) \cap \Phi_{(e_\ell^{d+1},0)}\right)}_{\subseteq \Phi_{(e_\ell^{d+1},0)} \setminus \left\{\binom{e_\ell^{d+1}}{0}\right\}} \\
&\qquad = \Phi_{(e_\ell^{d+1},0)} \setminus \left\{\binom{e_\ell^{d+1}}{0}\right\}.
\end{aligned}$$

As above, this set is a basis for the d-skeleton $\mathrm{sk}_d\big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(e_\ell^{d+1},0)}, \Phi_{(e_\ell^{d+1},0)}\big)$ of the elementary subcomplex $\big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(e_\ell^{d+1},0)}, \Phi_{(e_\ell^{d+1},0)}\big)$, hence it generates a pure finite subcomplex of order d. Since $\big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(e_\ell^{d+1},0)}, \Phi_{(e_\ell^{d+1},0)}\big)$, which is isomorphic to $(C_{e_\ell^{d+1}}, \Omega_{e_\ell^{d+1}})$, is contained in the shellable chain complex of Equation (3.4) on page 83, we conclude that the set

$$\big(\Phi_{(e_\ell^{d+1},0)}\big)_d = \left(\left(\left(\bigcup_{j=1}^{k_d} \Phi_{(0,e_j^d)}\right) \cup \left(\bigcup_{i=1}^{\ell-1} \Phi_{(e_i^{d+1},0)}\right)\right) \cap \Phi_{(e_\ell^{d+1},0)}\right)_d$$

has a shelling in which the elements of $\big(\bigcup_{i=1}^{\ell-1} \Phi_{(e_i^{d+1},0)}\big) \cap \Phi_{(e_\ell^{d+1},0)}$ come first. Hence, we obtain that the $(d+1)$-skeleton $\mathrm{sk}_d\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$ of the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$ is shellable.

Now we have to prove that the basis

$$\Phi_{d+2} = \left\{ \begin{pmatrix} 0 \\ e_1^{d+1} \end{pmatrix}, \ldots, \begin{pmatrix} 0 \\ e_{k_{d+1}}^{d+1} \end{pmatrix} \right\}$$

has a shelling. The ordering of these elements is the same as in the shelling of $\Omega_{d+1} = \{e_1^{d+1}, \ldots, e_{k_{d+1}}^{d+1}\}$. If $k_{d+1} = 1$, we are done because we have already proven that the $(d+1)$-skeleton of $(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi)$ is shellable.

So let $k_{d+1} \geq 2$. We observe that, for $1 \leq k \leq k_{d+1}$, the basis of each elementary subcomplex $\big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_k^{d+1})}, \Phi_{(0,e_k^{d+1})}\big)$ is

$$\Phi_{(0,e_k^{d+1})} = \left\{ \begin{pmatrix} 0 \\ e_\ell^\nu \end{pmatrix} \,\middle|\, e_\ell^\nu \in \Omega_{e_k^{d+1}} \right\} \cup \left\{ \begin{pmatrix} e_\ell^\nu \\ 0 \end{pmatrix} \,\middle|\, e_\ell^\nu \in \Omega_{e_k^{d+1}} \right\} \cup \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.$$

So, for $2 \leq j \leq k_{d+1}$, we get

$$\begin{aligned} \left(\bigcup_{i=1}^{j-1} \Phi_{(0,e_i^{d+1})} \right) \cap \Phi_{(0,e_j^{d+1})} = & \left\{ \begin{pmatrix} 0 \\ e_\ell^\nu \end{pmatrix} \,\middle|\, e_\ell^\nu \in \left(\bigcup_{i=1}^{j-1} \Omega_{e_i^{d+1}} \right) \cap \Omega_{e_j^{d+1}} \right\} \\ & \cup \left\{ \begin{pmatrix} e_\ell^\nu \\ 0 \end{pmatrix} \,\middle|\, e_\ell^\nu \in \left(\bigcup_{i=1}^{j-1} \Omega_{e_i^{d+1}} \right) \cap \Omega_{e_j^{d+1}} \right\} \cup \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

By definition of the boundary maps we know that $\binom{0}{1} \in \mathrm{bd}\binom{0}{e_j^0}$ for every $1 \leq j \leq k_0$ and $\binom{e_\ell^\nu}{0} \in \mathrm{bd}\binom{0}{e_\ell^\nu}$ for each $e_\ell^\nu \in \Omega$. Therefore, this set generates a pure finite chain complex of order $(d+1)$ since the set $\big(\bigcup_{i=1}^{j-1} \Omega_{e_i^{d+1}}\big) \cap \Omega_{e_j^{d+1}}$ generates a pure finite chain complex of order d. Hence, the shelling condition 1 of Definition 3.1 is fulfilled.

We have to prove now that, for each $1 \leq k \leq k_{d+1}$, the basis $\big(\Phi_{(0,e_k^{d+1})}\big)_{d+1}$ has a shelling in which the elements of $\big(\big(\bigcup_{i=1}^{k-1} \Phi_{(0,e_i^{d+1})}\big) \cap \Phi_{(0,e_k^{d+1})}\big)_{d+1}$ come first if $k \geq 2$.

By definition, each set $\Phi_{(0,e_k^{d+1})}$ is a basis of the elementary subcomplex $\big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_k^{d+1})}, \Phi_{(0,e_k^{d+1})}\big)$. According to Lemma 2.21, we get

$$\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_k^{d+1})}, \Phi_{(0,e_k^{d+1})} \right) = \left(\widehat{\mathcal{C}}(\mathrm{id}_{C_{e_k^{d+1}}}), \Theta_{(0,e_k^{d+1})} \right).$$

By assumption, each elementary subcomplex $(C_{e_k^{d+1}}, \Omega_{e_k^{d+1}})$ of (C, Ω) is shellable, pure and finite of order $(d+1)$. We have already proven that the $(d+1)$-skeleton $\mathrm{sk}_{d+1}\big(\widehat{\mathcal{C}}(\mathrm{id}_{C_{e_k^{d+1}}}), \Theta_{(0,e_k^{d+1})}\big)$ of the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_{C_{e_k^{d+1}}}), \Theta_{(0,e_k^{d+1})}\big)$

is shellable. Since $\left(\Phi_{(0,e_k^{d+1})}\right)_{d+2} = \left\{\binom{0}{e_k^{d+1}}\right\}$, the whole elementary subcomplex $\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_k^{d+1})}, \Phi_{(0,e_k^{d+1})}\right)$ is shellable, and its chain module basis

$$\left(\Phi_{(0,e_k^{d+1})}\right)_{d+1} = \left\{\binom{0}{e_\ell^d} \;\middle|\; e_\ell^d \in \mathrm{bd}(e_k^{d+1})\right\} \cup \left\{\binom{e_k^{d+1}}{0}\right\}$$

has a shelling in which the element $\binom{e_k^{d+1}}{0}$ comes last and the elements $\binom{0}{e_\ell^d}$ are ordered in the same way as the elements e_ℓ^d in $\left(\Omega_{e_k^{d+1}}\right)_d = \mathrm{bd}(e_k^{d+1})$.

By assumption, each basis Ω_ν is well-ordered, so in the shelling of $\mathrm{bd}(e_k^{d+1})$ the elements of $\left(\bigcup_{i=1}^{k-1} \mathrm{bd}(e_i^{d+1})\right) \cap \mathrm{bd}(e_k^{d+1})$ come first. The same holds for the set $\left(\Phi_{(0,e_k^{d+1})}\right)_{d+1}$, because for $2 \leq k \leq k_{d+1}$ we get

$$\left(\left(\bigcup_{i=1}^{k-1} \Phi_{(0,e_i^{d+1})}\right) \cap \Phi_{(0,e_k^{d+1})}\right)_{d+1} = \left\{\binom{0}{e_\ell^d} \;\middle|\; e_\ell^d \in \left(\bigcup_{i=1}^{k-1} \mathrm{bd}(e_i^{d+1})\right) \cap \mathrm{bd}(e_k^{d+1})\right\}.$$

Hence, all shelling conditions are fulfilled. Therefore, each mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ over a pure shellable chain complex (C, Ω) is shellable. □

Theorem 3.34. *Let (C, Ω) be a shellable finite chain complex of order d over a principal ideal domain R and $\Gamma \subseteq \Omega$ be the set of its maximal basis elements, ordered in a monotonically descending shelling. Let $\epsilon\colon C_0 \to R$ be an augmentation map such that $\epsilon(e_i^0) \neq 0$ for all $e_i^0 \in \Omega_0 = \{e_1^0, ..., e_{k_0}^0\}$. Let $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ be the mapping cone of the identity chain map $\mathrm{id}_C\colon (C, \Omega) \to (C, \Omega)$:*

$$\{0\} \oplus C_d \xrightarrow{\delta_{d+1}} C_d \oplus C_{d-1} \xrightarrow{\delta_d} \ldots \xrightarrow{\delta_2} C_1 \oplus C_0 \xrightarrow{\delta_1} C_0 \oplus R \xrightarrow{\delta_0} 0.$$

Then the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ is shellable and a cone.

Remark 3.35. The set of the maximal basis elements of the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ is

$$\Gamma(\mathrm{id}_C) = \left\{\binom{0}{e_i^\nu} \;\middle|\; e_i^\nu \in \Gamma\right\}$$

since any basis element $\binom{e_i^\nu}{0}$ for $0 \leq \nu \leq d$ and $1 \leq i \leq k_\nu$ is in the boundary of $\binom{0}{e_i^\nu}$, and the basis element $\binom{0}{1}$ is contained in the boundary of every element $\binom{0}{e_i^0} \in \Phi_1$. In particular, $\Gamma(\mathrm{id}_C) \cap \Phi_0 = \emptyset$.

Proof of Theorem 3.34. If the chain complex (C, Ω) is pure, we are done due to Theorem 3.33. For finite chain complexes of order $d = 0$ there is nothing left to do because every finite chain complex of order 0 is pure. Therefore, let (C, Ω) be nonpure and finite of order $d \geq 1$.

The mapping cone $(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi)$ is a cone by Theorem 2.17 since $\epsilon(e_i^0) \neq 0$ for all $e_i^0 \in \Omega_0$.

Since the shelling of Γ is monotonically descending, the basis elements of $\Omega_d = \{e_1^d, \ldots, e_{k_d}^d\} \subseteq \Gamma$ come first in this shelling. Hence, the subcomplex $C_{\Gamma,d}$ with basis $\bigcup_{i=1}^{k_d} \Omega_{e_i^d}$ is shellable and pure of order d. So the mapping cone $\widehat{\mathcal{C}}(\mathrm{id}_{C_{\Gamma,d}})$ is finite of order $(d+1)$ and shellable due to Theorem 3.33. By Lemma 3.7, its d-skeleton $\mathrm{sk}_d\big(\widehat{\mathcal{C}}(\mathrm{id}_{C_{\Gamma,d}})\big)$ is shellable, too.

Let $\Gamma \cap \Omega_{d-1} = \{g_1^{d-1}, \ldots, g_{m_{d-1}}^{d-1}\}$, which may be ordered as in the monotonically descending shelling of Γ. First we show that, for any $1 \leq \ell \leq m_{d-1}$, the set

$$Y_\ell := \left(\left(\bigcup_{i=1}^{k_d} \Phi_{(0, e_i^d)} \right) \cup \underbrace{\left(\bigcup_{j=1}^{\ell-1} \Phi_{(0, g_j^{d-1})} \right)}_{=\emptyset \text{ if } \ell=1} \right) \cap \Phi_{(0, g_\ell^{d-1})}$$

generates a pure chain complex of order $(d-1)$. Since

$$\Phi_{(0, e_i^\nu)} = \left\{ \begin{pmatrix} 0 \\ e_r^\mu \end{pmatrix} \,\middle|\, e_r^\mu \in \Omega_{e_i^\nu} \right\} \cup \left\{ \begin{pmatrix} e_r^\mu \\ 0 \end{pmatrix} \,\middle|\, e_r^\mu \in \Omega_{e_i^\nu} \right\} \cup \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}$$

for every $e_i^\nu \in \Omega$, we get

$$\begin{aligned} Y_\ell = & \left\{ \begin{pmatrix} 0 \\ e_r^\mu \end{pmatrix} \,\middle|\, e_r^\mu \in \left(\left(\bigcup_{i=1}^{k_d} \Omega_{e_i^d} \right) \cup \left(\bigcup_{j=1}^{\ell-1} \Omega_{g_j^{d-1}} \right) \right) \cap \Omega_{g_\ell^{d-1}} \right\} \\ & \cup \left\{ \begin{pmatrix} e_r^\mu \\ 0 \end{pmatrix} \,\middle|\, e_r^\mu \in \left(\left(\bigcup_{i=1}^{k_d} \Omega_{e_i^d} \right) \cup \left(\bigcup_{j=1}^{\ell-1} \Omega_{g_j^{d-1}} \right) \right) \cap \Omega_{g_\ell^{d-1}} \right\} \cup \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}. \end{aligned}$$

This set indeed generates a pure chain complex of order $(d-1)$ because each $\begin{pmatrix} e_r^\mu \\ 0 \end{pmatrix}$ is in the boundary of $\begin{pmatrix} 0 \\ e_r^\mu \end{pmatrix}$ and the set

$$\left(\left(\bigcup_{i=1}^{k_d} \Omega_{e_i^d} \right) \cup \left(\bigcup_{j=1}^{\ell-1} \Omega_{g_j^{d-1}} \right) \right) \cap \Omega_{g_\ell^{d-1}}$$

generates a pure chain complex of order $(d-2)$ by assumption.

It remains to show that each basis set $\big(\Phi_{(0,g_\ell^{d-1})}\big)_{d-1}$ has a shelling in which the elements of $(Y_\ell)_{d-1}$ come first. We recall

$$\big(\Phi_{(0,g_\ell^{d-1})}\big)_{d-1} = \left\{ \begin{pmatrix} 0 \\ e_i^{d-2} \end{pmatrix} \,\middle|\, e_i^{d-2} \in \mathrm{bd}(g_\ell^{d-1}) \right\} \cup \left\{ \begin{pmatrix} g_\ell^{d-1} \\ 0 \end{pmatrix} \right\},$$

$$(Y_\ell)_{d-1} = \left\{ \begin{pmatrix} 0 \\ e_i^{d-2} \end{pmatrix} \,\middle|\, e_i^{d-2} \in \left(\left(\left(\bigcup_{i=1}^{k_d} \Omega_{e_i^d} \right) \cup \left(\bigcup_{j=1}^{\ell-1} \Omega_{g_j^{d-1}} \right) \right) \cap \Omega_{g_\ell^{d-1}} \right)_{d-2} \right\}.$$

From the proof of Theorem 3.33 we know that a shelling of $\big(\Phi_{(0,g_\ell^{d-1})}\big)_{d-1}$ exists in which the element $\begin{pmatrix} g_\ell^{d-1} \\ 0 \end{pmatrix}$ comes last and the other elements are ordered in the same way as the elements in $\big(\Omega_{g_\ell^{d-1}}\big)_{d-2} = \mathrm{bd}(g_\ell^{d-1})$.

Because $\big(\Omega_{g_\ell^{d-1}}\big)_{d-2}$ has a shelling in which the elements of

$$\left(\left(\left(\bigcup_{i=1}^{k_d} \Omega_{e_i^d} \right) \cup \left(\bigcup_{j=1}^{\ell-1} \Omega_{g_j^{d-1}} \right) \right) \cap \Omega_{g_\ell^{d-1}} \right)_{d-2}$$

come first, the elements of $(Y_\ell)_{d-1}$ come first in the analogous shelling of $\big(\Phi_{(0,g_\ell^{d-1})}\big)_{d-1}$. So we get shellability of the mapping cone over the subcomplex $C_{\Gamma,d-1}$ with basis $\big(\bigcup_{i=1}^{k_d} \Omega_{e_i^d}\big) \cup \big(\bigcup_{i=1}^{m_{d-1}} \Omega_{g_i^{d-1}}\big)$.

Proceeding in this way we can add the basis elements of $\Gamma(\mathrm{id}_C) \cap \Phi_{d-1}$, $\Gamma(\mathrm{id}_C) \cap \Phi_{d-2}, \ldots, \Gamma(\mathrm{id}_C) \cap \Phi_1$. In the end we get a monotonically descending shelling of the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$. $\square$

Theorem 3.36. *Let (C, Ω) be a regular finite chain complex of order d over a principal ideal domain R and $\Gamma \subseteq \Omega$ be the set of its maximal basis elements, ordered in a monotonically descending shelling. Let there be an augmentation map $\epsilon\colon C_0 \to R$ such that $\epsilon(e_i^0) \neq 0$ for all $e_i^0 \in \Omega_0 = \{e_1^0, ..., e_{k_0}^0\}$. Let $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$ be the mapping cone of the identity chain map id_C:*

$$\{0\} \oplus C_d \xrightarrow{\delta_{d+1}} C_d \oplus C_{d-1} \xrightarrow{\delta_d} \ldots \xrightarrow{\delta_2} C_1 \oplus C_0 \xrightarrow{\delta_1} C_0 \oplus R \xrightarrow{\delta_0} 0.$$

Then the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$ is regular and a cone.

Proof. Due to Theorem 2.17, the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ is a cone since the augmentation map ϵ is nonzero on Ω_0.

Moreover, by Theorem 3.34, the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ is shellable. In the proof of that theorem we constructed a monotonically descending shelling of $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$, and due to Remark 3.13 there exists a well-ordered basis for the mapping cone. So we only have to check both conditions for regularity (cf. Definition 3.14).

According to Remark 3.35, the set of the maximal basis elements of the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ is

$$\Gamma(\mathrm{id}_C) = \left\{ \begin{pmatrix} 0 \\ e_i^\nu \end{pmatrix} \;\middle|\; e_i^\nu \in \Gamma \right\}.$$

By definition of the mapping cone's boundary maps, we get for any maximal basis element $\left(\begin{smallmatrix} 0 \\ e_i^\nu \end{smallmatrix}\right) \in \Gamma(\mathrm{id}_C)$ that

$$\delta_{\nu+1} \begin{pmatrix} 0 \\ e_i^\nu \end{pmatrix} = \begin{pmatrix} e_i^\nu \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ \partial_\nu e_i^\nu \end{pmatrix}.$$

Hence, for any maximal basis element we have

$$\mathrm{bd} \begin{pmatrix} 0 \\ e_i^\nu \end{pmatrix} \not\subseteq \bigcup_{k=1}^{i-1} \mathrm{bd} \begin{pmatrix} 0 \\ e_k^\nu \end{pmatrix},$$

so the first condition for regularity is fulfilled.

Therefore, the second condition for regularity remains to be checked. We begin with some basis element $\left(\begin{smallmatrix} 0 \\ e_k^{\nu-1} \end{smallmatrix}\right) \in \Phi_\nu$ for $1 \leq \nu \leq (d+1)$. Its boundary is

$$\left(\Phi_{(0,e_k^{\nu-1})}\right)_{\nu-1} = \left\{ \begin{pmatrix} 0 \\ e_i^{\nu-2} \end{pmatrix} \;\middle|\; e_i^{\nu-2} \in \mathrm{bd}(e_k^{\nu-1}) \right\} \cup \left\{ \begin{pmatrix} e_k^{\nu-1} \\ 0 \end{pmatrix} \right\}.$$

We know that

$$\mathrm{bd} \begin{pmatrix} 0 \\ e_i^{\nu-2} \end{pmatrix} \not\subseteq \bigcup_{j \neq i} \mathrm{bd} \begin{pmatrix} 0 \\ e_j^{\nu-2} \end{pmatrix}$$

for any basis element $e_i^{\nu-2} \in \mathrm{bd}(e_k^{\nu-1})$ since $\left(\begin{smallmatrix} e_i^{\nu-2} \\ 0 \end{smallmatrix}\right) \in \mathrm{bd}\left(\begin{smallmatrix} 0 \\ e_j^{\nu-2} \end{smallmatrix}\right)$ if and only if $i = j$. But

$$\mathrm{bd} \begin{pmatrix} e_k^{\nu-1} \\ 0 \end{pmatrix} \subseteq \bigcup_{e_i^{\nu-2} \in \mathrm{bd}(e_k^{\nu-1})} \mathrm{bd} \begin{pmatrix} 0 \\ e_i^{\nu-2} \end{pmatrix}.$$

By definition of the boundary map δ_ν, we get

$$\delta_\nu \begin{pmatrix} 0 \\ e_k^{\nu-1} \end{pmatrix} = \begin{pmatrix} e_k^{\nu-1} \\ 0 \end{pmatrix} + \sum_{e_i^{\nu-2} \in \mathrm{bd}(e_k^{\nu-1})} a_i^{\nu-1,k} \begin{pmatrix} 0 \\ e_i^{\nu-2} \end{pmatrix}$$

with nonzero coefficients $a_i^{\nu-1,k} \in R$. Since $\delta_{\nu-1} \circ \delta_\nu = 0$, we obtain

$$\delta_{\nu-1} \begin{pmatrix} e_k^{\nu-1} \\ 0 \end{pmatrix} = \sum_{e_i^{\nu-2} \in \mathrm{bd}(e_k^{\nu-1})} -a_i^{\nu-1,k} \delta_{\nu-1} \begin{pmatrix} 0 \\ e_i^{\nu-2} \end{pmatrix}.$$

Therefore, the second condition for regularity is fulfilled for any basis element $\begin{pmatrix} 0 \\ e_k^{\nu-1} \end{pmatrix} \in \Phi$.

We proceed with some basis element $\begin{pmatrix} e_\ell^\nu \\ 0 \end{pmatrix} \in \Phi_\nu$ for $1 \leq \nu \leq d$. Its boundary is

$$\left(\Phi_{(e_\ell^\nu,0)}\right)_{\nu-1} = \left\{ \begin{pmatrix} e_i^{\nu-1} \\ 0 \end{pmatrix} \,\middle|\, e_i^{\nu-1} \in \mathrm{bd}(e_\ell^\nu) \right\}.$$

We know that the elementary subcomplex $\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(e_\ell^\nu,0)}, \Phi_{(e_\ell^\nu,0)}\right)$ of the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ is isomorphic to the elementary subcomplex $(C_{e_\ell^\nu}, \Omega_{e_\ell^\nu})$ of (C, Ω). Since (C, Ω) is a regular chain complex by assumption, the second condition for regularity is fulfilled by the basis elements of $(C_{e_\ell^\nu})_{\nu-1}$ and hence by the basis elements in $\left(\Phi_{(e_\ell^\nu,0)}\right)_{\nu-1}$.

Therefore, the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ fulfils the second regularity condition, so it is itself regular. □

Corollary 3.37. *Let the same conditions be fulfilled as in Theorem 3.36. If the chain complex (C, Ω) is even totally regular, the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ is totally regular and a cone.*

Proof. According to Theorem 3.36, the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$ is regular and a cone.

By assumption, every elementary subcomplex $(C_{e_i^\nu}, \Omega_{e_i^\nu})$ of (C, Ω) is acyclic. So for the elementary subcomplexes of the mapping cone $\left(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\right)$, we get the following:

- Any subcomplex $\left(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(e_\ell^\nu,0)}, \Phi_{(e_\ell^\nu,0)}\right)$ is isomorphic to $(C_{e_i^\nu}, \Omega_{e_i^\nu})$, so it is acyclic.

- By Lemma 2.23, any subcomplex $\big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,e_\ell^\nu)}, \Phi_{(0,e_\ell^\nu)}\big)$ is acyclic.
- The elementary subcomplex $\big(\widehat{\mathcal{C}}(\mathrm{id}_C)_{(0,1)}, \Phi_{(0,1)}\big)$ is acyclic since its basis is $\Phi_{(0,1)} = \big(\Phi_{(0,1)}\big)_0 = \big\{\left(\begin{smallmatrix}0\\1\end{smallmatrix}\right)\big\}$.

Hence, every elementary subcomplex of the mapping cone $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$ is acyclic. Therefore, $\big(\widehat{\mathcal{C}}(\mathrm{id}_C), \Phi\big)$ is totally regular. $\square$

4. Conclusion

We summarise our results which generalise the notion of a cone and of shellability to finite chain complexes. Our starting point have been geometric simplicial complexes for which shellability and cones are well-known. Shellable simplicial complexes have a certain homology (cf. Section 1.5), and a cone over a shellable simplicial complex is also shellable.

We have generalised the notion of a cone as well as shellability for finite chain complexes over a principal ideal domain. In both generalisations the simplicial case is included in the following sense: A chain complex which is obtained from a simplicial cone (or a shellable simplicial complex) is also a cone (or a shellable chain complex) with respect to our definitions.

In particular, our definition of a cone does not need a distinguished vertex as an apex. Nevertheless, a cone by our definition is an acyclic chain complex. We have furthermore investigated the connection between cones and mapping cones $\widehat{\mathcal{C}}(\mathrm{id}_C)$ of the identity chain map $\mathrm{id}_C \colon C \to C$ since any chain complex obtained from a simplicial cone can be described as such a mapping cone and vice versa, i.e. for each chain complex C, which is obtained from a simplicial complex, the mapping cone $\widehat{\mathcal{C}}(\mathrm{id}_C)$ is a cone.

In general, this concordance does not hold. We presented am example for a chain complex cone which cannot be regarded as a mapping cone of the identity chain map (cf. Section 2.3.3). Furthermore, a mapping cone $\widehat{\mathcal{C}}(\mathrm{id}_C)$ of the identity chain map is only a cone if the chain complex C fulfils certain conditions, see Theorem 2.17.

For the generalisation of the notion of shellability, we adapted the familiar definition made for regular cell complexes (cf. Definition A.14). A chain complex which is obtained from a shellable simplicial complex or a shellable regular cell complex is itself shellable, so our definitions for chain complexes generalises not only the simplicial case but also shellability of regular cell complexes. But in contrast to shellable simplicial complexes, the homology of shellable chain complexes is not known in general. This leads us to introduce regular and totally regular chain complexes by formulating additional conditions for shellable chain complexes. With an extra condition on an augmentation map, the homology of totally regular chain complexes is determined, our result is noted in Theorem 3.30. In particular, we implicitly name certain properties of shellable simplicial complexes since any chain complex obtained from a shellable simplicial complex is totally regular (cf. Remark 3.18) and its standard augmentation map fulfils the needed extra condition.

In the end, shellability and mapping cones are connected, and we consider the mapping cone $\widehat{\mathcal{C}}(\mathrm{id}_C)$ over a shellable, regular or totally regular chain complex C. It turns out that the mapping cone $\widehat{\mathcal{C}}(\mathrm{id}_C)$ is not only a cone but also shellable, regular or totally regular, respectively! These results can be found in Theorem 3.34, Theorem 3.36 and Corollary 3.37. So our definitions of a chain complex cone and of shellable, regular and totally regular chain complexes fit together.

For all our main results, the augmentation map ϵ seems to have an important role in the background since we always need the image $\epsilon(e_i^0)$ of each basis element $e_i^0 \in \Omega_0$ to be nonzero or even a unit! Then, due to Theorem 1.51, the boundary $\mathrm{bd}(x)$ of each element x in the chain module C_1 is either empty or consists of at least two elements.

Appendix A. CW-Complexes

Our definition of shellability of chain complexes given in Chapter 3 is motivated by shellability of regular cell complexes. Therefore, we give a short overview of CW-complexes which are more general than cell complexes and have been introduced by Whitehead (1949).

A.1. Terms and Notions

For the basics, we follow Hatcher (2008), Kozlov (2008) and Björner *et al.* (1999). First, we recall the notion of a *unit disk* $D^m := \{x \in \mathbb{R}^m \mid |x| \leq 1\}$ for $m \in \mathbb{N}$. Its *interior* is denoted by $\operatorname{int} D^m := \{x \in \mathbb{R}^m \mid |x| < 1\}$, a unit disk's *boundary* is $\partial D^m := \{x \in \mathbb{R}^m \mid |x| = 1\}$. In particular, for $m = 0$, we get $\partial D^0 = \emptyset$ and $D^0 = \{0\} = \operatorname{int} D^0$.

Definition A.1. Let $m \in \mathbb{N}$. An *(open) m-cell* σ^m is a topological space which is homeomorphic to $\operatorname{int} D^m$. For an open m-cell σ^m, the number m is called its *dimension*.

Definition A.2. Let X be a Hausdorff space which is a disjoint union of open m-cells σ_α^m with $m \in \mathbb{N}$. The space X is called a *CW-complex* if the following conditions are satisfied:

1. For any α, there exists a continuous map $\Phi_\alpha \colon D_\alpha^m \to X$ such that
   a) the restriction $\Phi_\alpha\big|_{\operatorname{int} D_\alpha^m} \colon \operatorname{int} D_\alpha^m \to X$ is a homeomorphism onto an open cell σ_α^m,
   b) for $m \geq 1$, $\Phi_\alpha(\partial D_\alpha^m)$ is contained in the union of a finite number of open cells of dimension less than m.
2. A subset $A \subseteq X$ is closed in X if and only if $A \cap \overline{\sigma_\alpha^m}$ is closed in $\overline{\sigma_\alpha^m}$ for each α, where $\overline{\sigma_\alpha^m}$ denotes the closure of σ_α^m in X.

The maps Φ_α are called *characteristic maps*.

Remark A.3. For any m-cell σ_α^m in a CW-complex X, its closure $\overline{\sigma_\alpha^m}$ is in general *not* homeomorphic to the disk D_α^m.

Definition A.4. A *subcomplex* of a CW-complex X is a subspace $Y \subseteq X$ which is a union of open cells of X such that the closure of each cell in Y is contained in Y.

Definition A.5. For $n \in \mathbb{N}$, the *n-skeleton* X^n of a CW-complex X is the subcomplex consisting of all cells σ_α^m of X with $m \leq n$. For all $k \in \mathbb{Z}$, $k < 0$, we set $X^k := \emptyset$.

Any CW-complex can be constructed by building up its n-skeletons successively for all $n \in \mathbb{N}$. Precisely, a CW-complex is a topological space X which is constructed in the following way (cf. Hatcher, 2008, p. 5 and p. 519):

1. Start with a discrete set X^0 whose points are the 0-cells of X.

2. Build up the n-skeleton X^n from X^{n-1} inductively by attaching n-cells σ_α^n via continuous maps $\varphi_\alpha\colon \partial D_\alpha^n \to X^{n-1}$. Then the n-skeleton X^n is the quotient space of the disjoint union $X^{n-1} \dot\cup \left(\dot\bigcup_\alpha D_\alpha^n\right)$ of X^{n-1} with a collection of n-disks D_α^n under the identifications $x \sim \varphi_\alpha(x)$ for $x \in \partial D_\alpha^n$. Let each n-skeleton be equipped with the quotient topology. Each n-cell σ_α^n is the homeomorphic image of $\operatorname{int} D_\alpha^n$ under the quotient map.

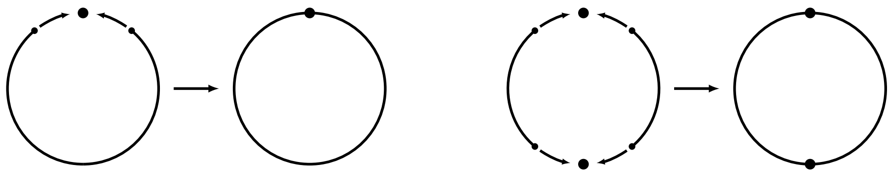
3. The space $X = \bigcup_n X^n$ is given the weak topology: A set $A \subset X$ is open (closed) in X if and only if $A \cap X^n$ is open (closed) in X^n for all $n \in \mathbb{N}$.

Remark A.6. Let $\iota_n\colon X^n \hookrightarrow X$ denote the embedding of the n-skeleton into X. Then we get $\Phi_\alpha\big|_{\partial D_\alpha^m} = \iota_{m-1} \circ \varphi_\alpha$ for the restriction of the characteristic map Φ_α of the m-cell σ_α^m to the disk's boundary ∂D_α^m.

Definition A.7. If the restriction $\Phi_\alpha\big|_{\partial D_\alpha^m}\colon \partial D_\alpha^m \to X$ of each characteristic map to the disk's boundary is a homeomorphism onto $\Phi_\alpha(\partial D_\alpha^m)$, a CW-complex is called *regular*.

Remark A.8. For any regular CW-complex, each characteristic map Φ_α is a homeomorphism.

Remark A.9. For regular CW-complexes, condition 1b) of Definition A.2 can be written as "For $m \geq 1$, $\Phi_\alpha(\partial D_\alpha^m)$ is the union of a finite number of cells of dimension less than m.", according to Björner *et al.* (1999, p. 202).



(a) not regular

(b) regular

Figure A.1.: The circle S^1 as cell complex

Definition A.10. Let X be a regular CW-complex with open cells σ_α^m. The *boundary* $\operatorname{bd}\sigma_\alpha^m$ of a cell σ_α^m is the set of all open cells σ_β^n which are contained in $\Phi_\alpha(\partial D_\alpha^m)$, these cells are called *faces* of σ_α^m. An open cell σ_α^m is called *maximal* if it is not contained in the boundary of any other cell.

Definition A.11. A *cell complex* (or *finite CW-complex*) is a CW-complex which consists of a finite number of open cells σ_α^m.

Definition A.12. The *dimension* of a cell complex is the maximum of the dimensions of its cells.

Definition A.13. A regular cell complex is called *pure* if all its maximal cells have the same dimension.

In Figure A.1, two cell structures on the circle S^1 are shown. The cell complex in Figure A.1(a) consists of a single vertex, where both ends of a single edge are glued on. Since both boundary points of the edge are identified, this cell complex is not regular. In contrast, the cell complex in Figure A.1(b), which contains two vertices and two edges, is regular since both boundary points of each edge are not identified.

A.2. Shellability of Regular Cell Complexes

We follow Björner and Wachs (1997, Definition 13.1). For each cell σ in a regular CW-complex let $\delta\sigma$ denote the subcomplex consisting of all proper faces of σ.

Definition A.14. Let X be a regular cell complex of dimension d. A linear ordering $\sigma_1, \sigma_2, \ldots, \sigma_t$ of its maximal cells is called a *shelling* (or a *shelling order*) if either $d = 0$, or if $d \geq 1$ and the following conditions are satisfied:

1. $\delta\sigma_k \cap \left(\bigcup_{i=1}^{k-1} \delta\sigma_i\right)$ is a pure subcomplex of dimension $(\dim \sigma_k - 1)$, for all $2 \leq k \leq t$,

2. $\delta\sigma_k$ has a shelling in which the $(\dim \sigma_k - 1)$-cells of $\delta\sigma_k \cap \left(\bigcup_{i=1}^{k-1} \delta\sigma_i\right)$ come first, for all $2 \leq k \leq t$,

3. $\delta\sigma_1$ has a shelling.

A regular cell complex having a shelling is said to be *shellable*.

Similar to shellable simplicial complexes, which are homotopy equivalent to a wedge of spheres (cf. Section 1.5), the following holds for shellable regular cell complexes (cf. Björner and Wachs, 1997, Corollary 13.3):

Theorem A.15. *A shellable regular cell complex has the homotopy type of a wedge of spheres.*

A.3. Oriented CW-Complexes

For any CW-complex X, we consider the relative singular homology modules $H_k(X^n, X^{n-1})$ of its n-skeletons. According to Hatcher (2008, Lemma 2.34), $H_k(X^n, X^{n-1}) = 0$ for $k \neq n$, and $H_n(X^n, X^{n-1})$ is a free $\mathbb{Z}$-module whose basis elements e_α^n are in one-to-one correspondence with the n-cells σ_α^n of X.

In particular, we get for $n \geq 1$ (cf. Hatcher, 2008, Proposition 2.22 and Corollary 2.25):

$$H_n(X^n, X^{n-1}) \cong \widetilde{H}_n\left(X^n / X^{n-1}\right) \cong \bigoplus_\alpha \widetilde{H}_n\Big(\underbrace{\overline{\sigma_\alpha^n} / \left(\overline{\sigma_\alpha^n} \setminus \sigma_\alpha^n\right)}_{n\text{-sphere}}\Big).$$

Since $\widetilde{H}_n\left(\overline{\sigma_\alpha^n} / \left(\overline{\sigma_\alpha^n} \setminus \sigma_\alpha^n\right)\right) \cong \mathbb{Z}$, a generator can be chosen in two ways which are negatives of each other. A generator of this homology module is called an *orientation* of the cell σ_α^n for $n \geq 1$ (cf. Massey, 1991, p. 239).

For $n = 0$, we get by Hatcher (2008, Proposition 2.6):

$$H_0(X^0, X^{-1}) \cong H_0(X^0) \cong \bigoplus_\alpha H_0(\sigma_\alpha^0).$$

Since each 0-cell σ_α^0 is a single point, the question of choosing an orientation does not arise. We consider the augmentation map $\epsilon\colon \bigoplus_\alpha H_0(\sigma_\alpha^0) \to \mathbb{Z}$ and choose a generator e^0_α for each $H_0(\sigma^0_\alpha) \cong \mathbb{Z}$ such that $\epsilon(e^0_\alpha) = 1$ (cf. Massey, 1991, p. 240).

Following Whitehead (1978, p. 82), we call a CW-complex *oriented* if each of its n-cells σ^n_α for $n > 0$ has an orientation.

A.4. Cellular Chain Complexes

Any oriented CW-complex X gives rise to a *cellular chain complex* over $\mathbb{Z}$ (cf. Hatcher, 2008, Chapter 2)

$$\dots \xrightarrow{d_{n+2}} H_{n+1}(X^{n+1}, X^n) \xrightarrow{d_{n+1}} H_n(X^n, X^{n-1}) \xrightarrow{d_n} H_{n-1}(X^{n-1}, X^{n-2}) \xrightarrow{d_{n-1}} \dots$$

whose chain modules are the relative homology modules $H_n(X^n, X^{n-1})$ described above. Each of them is a free $\mathbb{Z}$-module having as many basis elements e^n_α as there are n-cells σ^n_α in X. In particular, $H_n(X^n, X^{n-1}) = 0$ for all $n < 0$. Hence, the boundary map

$$d_0 \colon H_0(X^0, X^{-1}) \to H_{-1}(X^{-1}, X^{-2}) = 0$$

is the zero map. For $d = 1$, the boundary map

$$d_1 \colon H_1(X^1, X^0) \to H_0(X^0, X^{-1}) \cong H_0(X^0)$$

is given by $d_1(e^1_\alpha) = \pm(e^0_{\beta_1} - e^0_{\beta_0})$ if the corresponding 0-cells $\sigma^0_{\beta_1}$ and $\sigma^0_{\beta_0}$ are faces of σ^1_α (cf. Hatcher, 2008, p. 140). The sign depends on the orientation of the CW-complex. In particular, if both boundary cells $\sigma^0_{\beta_1}$ and $\sigma^0_{\beta_0}$ are the same, we get $d_1(e^1_\alpha) = 0$.

For $n > 1$, the boundary map

$$d_n \colon H_n(X^n, X^{n-1}) \to H_{n-1}(X^{n-1}, X^{n-2})$$

is given by $d_n(e^n_\alpha) = \sum_{e^{n-1}_\beta} \deg(\chi^n_{\alpha\beta}) e^{n-1}_\beta$, where $\deg(\chi^n_{\alpha\beta}) \in \mathbb{Z}$ is the degree (cf. Hatcher, 2008, p. 134) of the map

$$\chi^n_{\alpha\beta} \colon S^{n-1} \cong \partial D^n_\alpha \xrightarrow{\varphi_\alpha} X^{n-1} \longrightarrow X^{n-1} \big/ \left(X^{n-1} \setminus \sigma^{n-1}_\beta\right) \cong S^{n-1}$$

in which φ_α denotes the attaching map of the n-cell σ^n_α described in Section A.1. In literature, $\deg(\chi^n_{\alpha\beta})$ is often called *incidence number* of the cells

σ_α^n and σ_β^{n-1}, cf. Geoghegan (2008, p. 54) or Lundell and Weingram (1969, p. 162), for example.

For any regular CW-complex X holds due to Geoghegan (2008, p. 138) or Massey (1991, p. 244):

$$\deg(\chi_{\alpha\beta}^n) = \begin{cases} \pm 1 & \text{if } \sigma_\beta^{n-1} \in \operatorname{bd} \sigma_\alpha^n, \\ 0 & \text{otherwise.} \end{cases}$$

Hence, we obtain the following equivalence for the cells in an oriented regular chain complex and the basis elements in its cellular chain complex for all $n \geq 1$:

$$\sigma_\beta^{n-1} \in \operatorname{bd} \sigma_\alpha^n \iff e_\beta^{n-1} \in \operatorname{bd} e_\alpha^n.$$

Therefore, if an oriented regular cell complex is shellable, the corresponding cellular chain complex is also shellable as specified in Definition 3.1.

Remark A.16. Any regular cell complex can be oriented by specifying incidence numbers for it (cf. Massey, 1991, Theorem 7.2). Hence, our definition of shellability for chain complexes generalises this notion defined for regular cell complexes.

By Geoghegan (2008, p. 64), the homology of a regular cell complex is independent of the choice of its orientation.

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