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Derivation of boundary conditions at a curved contact interface between a free fluid and a porous medium via homogenisation theory

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Abstract

In soil chemistry or marine microbiology (for example when dealing with marine aggregates), one often encounters situations where porous bodies are suspended in a fluid. In this context, the question of boundary conditions for the fluid velocity and pressure at the porous-liquid interface arises. Up to the present, only results for straight interfaces are known.

In this work, the behaviour of a free fluid above a porous medium is investigated, where the interface between the two flow regions is assumed to be curved. By carrying out a coordinate transformation, we obtain the description of the flow in a domain with a straight boundary.

We assume the geometry in this domain to be ε-periodic. Using periodic homogenisation, the effective behaviour of the solution of the transformed partial differential equations in the porous part is obtained, yielding a Darcy law with a non-constant permeability matrix. The boundary layer approach of Jäger and Mikelić is then generalized to construct corrections at the interface.

Finally, this allows us to obtain the fluid behaviour at the porous-liquid interface: Whereas the velocity in normal direction is continuous over the interface, a jump appears in tangential direction. The magnitude of this jump can explicitely be calculated and seems to be related to the slope of the interface. Therefore the results indicate a generalized law of Beavers and Joseph.

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1 Introduction

This work is concerned with boundary conditions for a fluid flow at the interface of a porous medium and a free fluid.

Porous media consist of (at least) two phases, the *porous matrix* – which in our case will be an impermeable solid – and a *void space*, which in this work is assumed to be filled with a viscous incompressible fluid. Usually one has to consider at least two different scales: First, a *macroscale* given by the domain of interest, such as a piece of soil, for instance. In this case the characteristic length of the macroscale is normally in the range of metres to kilometres. Second, a *microscale* at the level of the individual porous particles. In the above example, its characteristic length would be given in the range of micrometres to millimetres. Porous materials play a great role in mathematical modelling and have applications in various fields of interest, e.g. soil chemistry, oil recovery or hardening of concrete (see for example the works of Bear [Bea72] or Logan [Log01] for an introduction). One example for a practical application of this thesis in the field of marine microbiology is the simulation of dissolution of mineral species from sinking marine aggregates:

Marine aggregates are particles found in the pelagic zone of the oceans. They consist of detritus, dead material, living organism and inorganic matter, for exampel clay minerals. Their size ranges from 500 micrometres to some millimetres. (See [AS88] for an introduction to the subject and further information.) Due to the sinking of these aggregates to the seabed, a constant transportation process of chemical and biological material to the sea floor is maintained. According to Fowler and Knauer [FK86], this is the main process driving vertical fluxes in the ocean.

In order to model the transport and aggregation phenomena, one has to know the advective and diffusive exchange of the aggregate with the sourrounding water. Due to [FK86], the aggregates can be considered as a porous medium. Thus the choice of correct fluid exchange conditions at porous-liquid interfaces is of great importance.

The full inclusion of the microscale in simulations or models is often not feasible. However, in many situations one is only interested in an effective model at the level of the macroscale. In order to obtain such an effective model, mostly two methods are used: The *representative elementary volume*-method (REV), and *homogenisation*.

Let $u(x)$ be the quantity of interest at point x (e.g. a fluid velocity or solute concentration); then the REV-ansatz is to substitute u by a local average $\langle u \rangle$, where

$$\langle u \rangle(x) = \int\limits_{B(x)} u(\tau)\ \mathrm{d}\tau.$$

Here $B(x)$ denotes a given open ball with center x. This ball has to be chosen large enough to average over the material properties, but at the same time small enough to capture local changes of u. Details can be found in [Whi99]. However, this method is not mathematical rigorous.

The main idea of homogenisation is to 'replace' the heterogeneous medium by a homogeneous one. Therefore, not a single problem is considered, but a family of problems is constructed in the following way: It is assumed that the microscopic geometry is periodic, given by a repetition of a so-called *reference cell* which is scaled by a factor $\varepsilon > 0$. The homogeneous material is then created by letting $\varepsilon \longrightarrow 0$.
To be more precise, assume that for given $\varepsilon > 0$ the quantity of interest u^ε in the above setting is given as a solution of the equation

$$\mathcal{L}^\varepsilon u^\varepsilon = f^\varepsilon,$$

where $\mathcal{L}^\varepsilon$ is a linear differential operator and f^ε is a given right hand side with $f^\varepsilon \to f$ in a suitable sense.
Then one is interested whether there exists a function u^0 such that $u^\varepsilon \longrightarrow u^0$. Additionally, under certain conditions one can find a differential operator $\mathcal{L}^0$ such that u^0 is a solution of

$$\mathcal{L}^0 u^0 = f.$$

This problem is then thought to govern the macroscopic behaviour and is considered as an effective model.

In lots of situations, u^ε and u^0 are related in the following way: There exists a so-called *asymptotic expansion* of the form

$$u^\varepsilon(x) = u^0(x, \frac{x}{\varepsilon}) + \varepsilon u_1(x, \frac{x}{\varepsilon}) + \varepsilon^2 u_2(x, \frac{x}{\varepsilon}) + \dots$$

with known functions u^0 and u_i, $i = 1, 2, \dots$. They depend on two variables x and $y = \frac{x}{\varepsilon}$, the first in the actual domain and the latter in the reference cell, extended by periodicity. Therefore the characteristics of the porous structure and the geometry on the microscale are not lost as in the REV-method but are captured in the contribution in the second variable.

1.1 Boundary Conditions for Porous Media

A now classical result in the theory of homogenisation states that, starting with the Stokes or Navier-Stokes equation, the effective fluid flow in a porous medium is given by Darcy's

law (see the works of Tartar in [SP80], Allaire in [Hor97] and Mikelić [Mik91]). When dealing with porous bodies inside another fluid, the boundary condition coupling the free fluid flow and the Darcy flow at the porous-liquid interface is of great interest. However, due to the different nature of the governing equations, the derivation of a 'natural' boundary condition is difficult: While the equation for the Darcy velocity consists of a second order equation for the pressure and a first order equation for the velocity, the system of equations governing the free fluid velocity (e.g. the Stokes or Navier-Stokes equation) is of second order for the velocity and of first order for the pressure.
For an incompressible fluid, the flow in the direction normal to the interface has to be continuous due to mass conservation. However, additional conditions at the interface are not clearly available.

From a mechanical point of view, Beavers and Joseph [BJ67] concluded by practical experiments that a jump in the effective velocity appears in tangential direction. This jump is given by

$$\alpha K^{-\frac{1}{2}}(v_F - v_D) \cdot \tau = (\nabla v_F \nu) \cdot \tau, \tag{1.1}$$

where v_F denotes the velocity of the free fluid, v_D denotes the effective Darcy velocity in the porous medium and K is the permeability of the porous medium. The factor α is the so-called *slip coefficient* which has to be determined experimentally. Moreover, ν and τ are the unit normal and tangential vector with respect to the interface separating the porous medium and the free fluid. The Darcy velocity for given fluid viscosity μ is given by

$$v_D = -\frac{1}{\mu} K \nabla p,$$

where p denotes the pressure. Note that the condition mentioned above gives a relation between the velocity of the free fluid at the interface and the *effective* velocity inside the porous medium – it does not impose a condition on the actual fluid velocity inside the porous medium at the interface. See also Figure 1 for a schematic illustration.

Later, Saffman used a statistical model to derive the boundary condition of Beavers and Joseph. In [Saf71], he argued that $v_D \cdot \tau$ is of lesser order than the other terms and arrived at a jump given by

$$v_F \cdot \tau = \frac{1}{\alpha} K^{\frac{1}{2}} (\nabla v_F \nu) \cdot \tau + \mathscr{O}(K).$$

Other boundary conditions were proposed as well: Ochoa-Tapia and Whitaker for example used the REV-method to obtain that the velocity and pressure as well as the normal stress are continuous over the porous-liquid interface, but a jump appears in the tangential stress in the form

$$\big(\nabla \langle v_D \rangle \nu - \nabla \langle v_F \rangle \nu\big) \cdot \tau = \beta K^{-\frac{1}{2}} \langle v_D \rangle \cdot \tau.$$

Here $\langle v_F \rangle$ denotes the averaged free fluid velocity, which is given by a Stokes equation, and $\langle v_D \rangle$ is the averaged velocity in the porous medium, which in this case fulfills a Darcy law with Brinkman correction,

$$\langle v_D \rangle = -\frac{1}{\mu} K \big(\nabla \langle p \rangle - \mu_B \Delta \langle v_D \rangle \big).$$

μ_B is a known constant, and the dimensionless factor β has to be determined experimentally. For details see [OTW95a] and [OTW95b].

However, a rigorous mathematical derivation of the effective fluid behaviour at the boundary was not available until Jäger and Mikelić applied the theory of homogenisation to the problem.
In [JM96] they developed a mathematical boundary layer together with several corrector terms, which allowed them to justify a jump boundary condition. The main tool was the construction of several 'boundary layer functions': These functions have a given value at the interface and decay exponentially outside it. They can be used to correct the influence of spurious terms at the boundary, stemming from the contributions of other functions to the fluid velocity and pressure.

In [JM00], this theory was applied to give a mathematical proof of the Saffman modification of the boundary condition of Beavers and Joseph (see also Section 4 of the Chapter "Homogenization Theory and Applications to Filtration through Porous Media" in [EFM00] for a more comprehensible, simplified version of the proofs), yielding the condition

$$\varepsilon \ (\nabla \, v_F \nu) \cdot \tau = \alpha v_F \cdot \tau + \mathscr{O}(\varepsilon^2)$$

where $\alpha = -\frac{1}{\varepsilon C_D}$ can be calculated explicitely. The constant C_D stems from a boundary layer problem for the Stokes equation, cf. [JM00]. (See also [JMN01] for numerical simulations of the boundary layer functions.)

However, these results suffer from several drawbacks: First, only a planar boundary in the form of a line or a plane is considered (this also applies to the results of Beavers, Joseph and Saffman). Therefore, the effect of a possible curvature of the interface is not known. Second, the result in [JM00] is not a genuine homogenisation result: The ε appearing in the above equation is the scale parameter of the homogenisation setting, but at the same time it is also considered to be the physical parameter of the square root of permeability. Finally, the 'limit equations' still depend on this ε, with estimates given only in the less strong $H^{-\frac{1}{2}}$-norm.

Generalizations of the boundary layers in [JM96] were developed by Neuss-Radu in [NR00]. However, applications only treat reaction-diffusion systems without flow, and explicit results can only be obtained in the case of a layered medium, see [NR01].

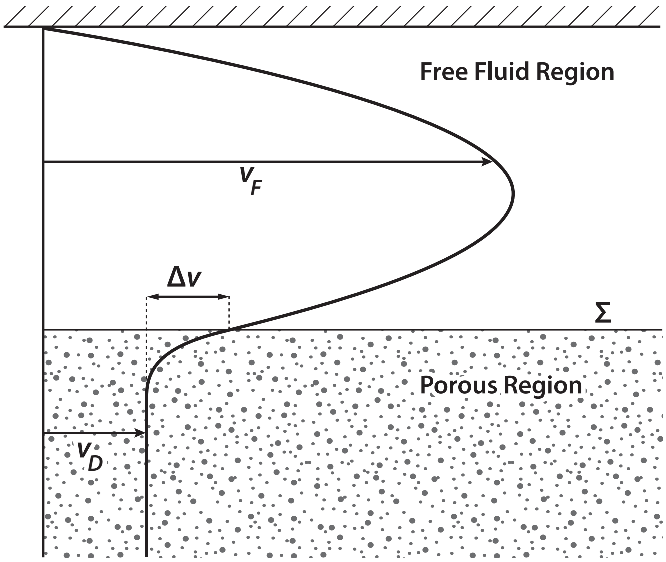

Figure 1: Schematic illustration of the velocity profile for a horizontal flow in a domain consisting of an impermeable upper boundary (with no-slip condition), a free fluid part and a porous region. v_F denotes the velocity in the free fluid domain, whereas v_D is the effective Darcy velocity. The quantity $\Delta v = v_F|_\Sigma - v_D$ corresponds to the jump across the interface as discussed in Equation (1.1).

The main problem which makes the treatment of general settings infeasible is the loss of exponential decay of the boundary layer functions (cf. Section 5): With the generalized definition, Neuss-Radu was able to show in [NR00] that an exponential stabilization is not possible in a general setting. However, all available tools for the treatment of these problems depend on this type of decay[1].

In this work, a new approach towards a generalization of the law of Beavers and Joseph for curved interfaces using constructions similar to [JM96] is proposed.
The main idea is to transform a reference geometry with a straight interface to a domain with a curved interface. It is assumed that the porous part in the reference geometry consists of a periodic array of a scaled reference cells and that the flow in the transformed geometry is governed by the stationary Stokes equation. Therefore one obtains a set of transformed differential equations in the reference configuration. Boundary layer functions for these equations are constructed such that – due to the straight boundary – their exponential decay can be assured.

However, to the author's knowledge the transformed differential equations have never been considered in the context of periodic homogenisation. Therefore, as a first step

[1] Maria Neuss-Radu, private communications.

existence and uniqueness-results as well as effective equations have to be derived – see below for a detailed overview.

Other constructions to obtain the boundary behaviour via homogenisation, especially in the case of reaction-diffusion systems, are also possible; see for example the work of Neuss-Radu and Jäger [NRJ07].

1.2 Overview of the Main Geometries

In this section we describe the geometrical settings which are used throughout this work. For illustrations, the reader is referred to Figure 5 (on page 62), Figure 3 (on page 26) and Figure 6 (page 66).

Let $L > 0$. We consider a fluid flowing in the semi-infinite strip $[0, L] \times \mathbb{R}$ being divided into two parts

$$\Omega_1 := [0, L] \times \mathbb{R}_{>0},$$

corresponding to the free fluid domain, and

$$\Omega_2 := [0, L] \times \mathbb{R}_{<0},$$

corresponding to the porous medium. Both parts are separated by the interface

$$\Sigma := [0, L] \times \{0\}.$$

We assume an ε-periodic geometry in Ω_2: Define a reference cell as

$$Y := [0, 1]^2,$$

containing a connected open set Y_S (corresponding to the solid part of the cell). Its boundary ∂Y_S is assumed to be of class $\mathcal{C}^\infty$ with $\partial Y_S \cap \partial Y = \emptyset$. Let

$$Y^* := Y \backslash \overline{Y_S}$$

be the fluid part of the reference cell.

For given $\varepsilon > 0$ such that $\frac{L}{\varepsilon} \in \mathbb{N}$, let χ be the characteristic function of Y^*, extended by periodicity to the whole $\mathbb{R}^2$. Set $\chi^\varepsilon(x) := \chi(\frac{x}{\varepsilon})$ and define the fluid part of the porous medium as

$$\Omega_2^\varepsilon = \{x \in \Omega_2 \mid \chi^\varepsilon(x) = 1\}.$$

The fluid domain is then given by

$$\Omega^\varepsilon = \Omega_1 \cup \Sigma \cup \Omega_2^\varepsilon.$$

In order to be able to obtain the effective fluid behaviour near Σ, we have to define a number of so-called boundary layer problems, see Section 5. To this end, we introduce the following setting: We consider the domain $[0,1] \times \mathbb{R}$ subdivided as follows:

$$Z^+ = [0,1] \times (0,\infty)$$

corresponds to the free fluid region, whereas the union of translated reference cells

$$Z^- = \bigcup_{k=1}^{\infty} \{Y^* - \binom{0}{k}\} \backslash S$$

is considered to be the void space in the porous part. Here

$$S = [0,1] \times \{0\}$$

denotes the interface between Z^+ and Z^-. Finally, let

$$Z = Z^+ \cup Z^-$$

and

$$Z_{\mathrm{BL}} = Z^+ \cup S \cup Z^-$$

be the fluid domain without and with interface.

1.3 Outline of the Work

We give an overview of what is going to follow:

In Section 2 an outline of stationary coordinate transformations is given together with results how differential operators behave under these transformations. Furthermore, in Lemma 2.7 various identities are derived which will play an important role in the sequel. Finally, the stationary Stokes equation is transformed; yielding the main type of equations we will deal with in this work.

In order to be able to deal with the fluid's behaviour at the interface, the effective fluid velocity stemming from the homogenisation of the transformed Stokes equations needs to be known. This is dealt with in Section 3: We consider the equations in a domain containing a periodic array of holes, with the hole size tending to zero. Note that in this section – as we are only interested in the effective law governing the flow – we deal with a different, bounded geometry, making the proofs easier.
First, a short review of the concept of two-scale-convergence is given, together with some known results from functional analysis. Next, existence and uniqueness results for the transformed Stokes equation are proven. Observe that the arguments in the corresponding

proofs will also be important in subsequent sections, where we will not go into too many details. Finally, the homogenisation itself is carried out, leading to a transformed Darcy's law with variable permeability tensor. Note that no special attention is paid to the thorough introduction of periodic homogenisation; and the reader is assumed to be familiar with the conventions and applications covered in basic textbooks – see for example Donato/Cioranescu [CD99] (with a strong mathematical flavour) or Hornung [Hor97] for applications to porous media.

The cell problem appearing in the above homogenisation turns out to be a partial differential equation depending on a parameter, leading to a family of solutions. Section 4 investigates this dependence. By using the implicit function theorem for Banach spaces, one can show that under some conditions certain differentiability properties in the direction of the parameter carry over to the solution.

Now we have all results at hand to deal with the behaviour at the porous-liquid interface, which is presented in Section 5. Various auxiliary problems for the correction of the flow are constructed and their influence on the right hand side is observed. The derivation is a more formal one: Regularity results are used without thorough treatment, and no special attention is paid to the differentiablity properties of the solutions of the problems. Furthermore, due to the tedious nature of the calculations, only few details are given; and the focus of the exposition lies mostly on the effective velocity. The main results are given in Section 5.6.

Finally, Appendix A collects results needed for the discussion of some auxiliary problems from the foregoing part of the work.

1.4 Notations and Conventions

Some notational conventions: Constants (mostly denoted by $C, C_1, \tilde{C}, \dots$) never depend on the scale parameter ε unless otherwise stated. When deriving inequalities, we frequently use a generic constant which might change in the course of the estimation. $\mathscr{O}$ denotes the well-known Landau symbol, indicating the order of neglected terms.

As for function spaces, we use the usual notation $L^2(\Lambda)$ for functions $f : \Lambda \to \mathbb{R}$ which are square-integrable with respect to the Lebesgue measure on a domain $\Lambda \subset \mathbb{R}^n$, $n \in \mathbb{N}$. The space $L^2_0(\Lambda)$ consists of those $f \in L^2(\Lambda)$ such that $\int_\Lambda f \, \mathrm{d}\lambda = 0$. Sobolev spaces with order of differentiability k and order of integrability 2 are denoted by $H^k(\Lambda)$. Here the subscript 0 indicates a vanishing trace on the boundary (cf. for example [Maz85] for details).

Spaces of continuously differentiable functions are denoted with $\mathcal{C}^k(\Lambda)$ etc., where $k \in \mathbb{N} \cup \{0, \infty\}$ is the order of differentiability. The space $\mathcal{C}^\infty_0(\Lambda)$ consists of infinitely differentiable functions having compact support in Λ.

The subscript # indicates periodicity. For function spaces with respect to the reference cell Y, the periodicity is assumed to hold in both coordinate variables, whereas for the strips Z, Z^+, Z_{BL} and $\Omega, \Omega^\varepsilon$ etc. periodicity is assumed with respect to the first variable only. Finally, the dual of a Banach space X is written as X'.

In the sequel, functions depending on a parameter are considered as well. However, no special attention is paid to notational subtleties, and the notation is switched when necessary. E.g. for $f \in \mathcal{C}^k(\Omega, \mathcal{C}_0^\infty(\Lambda))$ the expressions $f(x,y)$ and $f(x)(y)$ etc. denote the value of $f(x)$ at $y \in \Lambda$, for $x \in \Omega$.

The letter H will be reserved for the *Heaviside-function* $H : \mathbb{R} \longrightarrow \mathbb{R}$,

$$H(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}.$$

For two matrices $A, B \in \mathbb{R}^{n \times n}$, $n \in \mathbb{N}$, their *inner product* is given by

$$A : B = \mathrm{tr}(A^T B) = \sum_{i,j=1}^{n} (A)_{ij} (B)_{ij} \quad \in \mathbb{R},$$

where tr denotes the trace operator. Let $a, b \in \mathbb{R}^n$ be two vectors. The *tensor product* of a and b is defined as

$$a \otimes b = ab^T = (a_i b_j)_{i,j=1,\ldots,n} \quad \in \mathbb{R}^{n \times n}.$$

The unit vectors in $\mathbb{R}^2$ are denoted by e_1 and e_2, and the identity matrix by I.

For a function $u \in H^{\frac{1}{2}+\kappa}(Z)$, $\kappa > 0$, there exist two trace operators on S which we will denote for a moment by γ^{Z^+} and γ^{Z^-}, corresponding to the trace of $u|_{Z^+}$ on S and that of $u|_{Z^-}$ on S, resp. Thus the *jump* of u across S is given by

$$[u]_S = \gamma^{Z^+}(u) - \gamma^{Z^-}(u) \quad \in H^\kappa(S).$$

Finally, for a given Lipschitz boundary $\partial\Lambda$ of a set $\Lambda \subset \mathbb{R}^2$ we denote the outer unit normal vector in $x \in \partial\Lambda$ by $\nu(x)$ and the corresponding unit tangential vector by $\tau(x)$. Hence

$$\tau = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nu.$$

2 Coordinate Transformations

In this section we review the basic concepts of coordinate transformations and apply them to the steady state Stokes equation. However, as we are only concerned with stationary transformations, we do not go into too many details. For an overview of the general notion, the reader is referred to the doctoral dissertation [Mei08] and the references therein.

2.1 Differential Operators under Coordinate Transformations

As a first step, we recall the definition of coordinate transformations and some differential operators and investigate their relations:

Let $\tilde{\Omega} \subset \mathbb{R}^n$ with $n \in \mathbb{N}$ be a Lipschitz domain; let $\tilde{c} : \tilde{\Omega} \longrightarrow \mathbb{R}$ be a scalar function, $\tilde{j} : \tilde{\Omega} \longrightarrow \mathbb{R}^n$ a vector field and $\tilde{M} : \tilde{\Omega} \longrightarrow \mathbb{R}^{n\times n}$ a matrix function. They are assumed to be sufficiently smooth.

2.1 Definition.
The gradient of a vector field is defined as

$$(\nabla \tilde{j})_{ik} = \frac{\partial \tilde{j}_k}{\partial x_i},$$

for $i, k = 1, \ldots, n$ (i.e. $\nabla \tilde{j}$ is the transpose *of the Jacobian matrix of $\tilde{j}$); the divergence of a matrix-valued function is defined column-wise, thus*

$$(\operatorname{div}(\tilde{M}))_k = \sum_{i=1}^{n} \frac{\partial \tilde{M}_{ik}}{\partial x_i},$$

for $k = 1, \ldots, n$; and the Laplacian of a vector field is given by

$$\Delta \tilde{j} = \operatorname{div}(\nabla \tilde{j}).$$

For $n = 2$ we define the two operators

$$\operatorname{Curl}(\tilde{c}) = \begin{pmatrix} -\frac{\partial \tilde{c}}{\partial x_2} \\ \frac{\partial \tilde{c}}{\partial x_1} \end{pmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \nabla \tilde{c}$$

and

$$\operatorname{curl}(\tilde{j}) = \frac{\partial \tilde{j}_2}{\partial x_1} - \frac{\partial \tilde{j}_1}{\partial x_2}.$$

2.2 Remark.
The 'curl'-operators above are two-dimensional variants of the well-known curl operator describing the rotation of three-dimensional vector fields.

We have the relations

$$\operatorname{curl} \nabla \tilde{c} = 0 \qquad \textit{and} \qquad \operatorname{div} \operatorname{Curl} \tilde{c} = 0,$$

and curl *is the formal adjoint of* Curl *(see [Ver07] and [DL90] for details concerning these operators).*

2.3 Definition.
Let $\Omega, \tilde{\Omega} \subset \mathbb{R}^n$ be Lipschitz domains and let $\psi : \Omega \longrightarrow \tilde{\Omega}$. We call ψ a regular orientation-preserving $\mathcal{C}^k$-coordinate transformation *if*

1. *ψ is a $\mathcal{C}^k$-diffeomorphism, and*
2. *There exist $c, C > 0$ such that*

$$c \leq \det F(z) \leq C \quad \forall z \in \Omega,$$

where F denotes the Jacobian matrix of ψ.

If $\det F \equiv 1$, *we call ψ* volume preserving.

We will indicate coordinates in Ω by $z = (z_1, \ldots, z_n)$ and those in $\tilde{\Omega}$ by $x = (x_1, \ldots, x_n)$. Define

$$\begin{aligned} c(z) &:= \tilde{c}(\psi(z)) \\ j(z) &:= \tilde{j}(\psi(z)) \\ M(z) &:= \tilde{M}(\psi(z)). \end{aligned}$$

2.4 Lemma.
Let $\psi : \Omega \longrightarrow \tilde{\Omega}$ be a $\mathcal{C}^1$-coordinate transformation. Denote by F the Jacobian matrix of ψ, and let $J(z) := \det(F(z))$. With the notations and definitions above it holds

1. $F^{-T} \nabla_z c = \nabla_x \tilde{c}$.
2. $\operatorname{div}_z(JF^{-1} j) = (J \circ \psi^{-1}) \operatorname{div}_x(\tilde{j})$.
3. $\operatorname{div}_z(JF^{-1} M) = (J \circ \psi^{-1}) \operatorname{div}_x(\tilde{M})$.

Proof. The first assertion is a simple application of the chain rule, whereas the second one is known as the Piola-transformation (see [Zei88], Chapter 61. Note that Zeidler defines vectors and gradients row-wise, leading to slightly different formulas.) For the matrix divergence the second statement holds column-wise. ♦

Application of this lemma yields:

2.5 Lemma.
Let ψ be a volume-preserving $\mathcal{C}^1$-coordinate transformation. The operators from Definition 2.1 transform according to

1. $\Delta_x(\tilde{c}) = \mathrm{div}_z(F^{-1}F^{-T}\,\nabla_z\,c)$.
2. $\Delta_x(\tilde{j}) = \mathrm{div}_z(F^{-1}F^{-T}\,\nabla_z\,j)$.
3. $\mathrm{div}_x(\tilde{j}) = \mathrm{div}_z(F^{-1}j)$.
4. $\mathrm{div}_x(\tilde{M}) = \mathrm{div}_z(F^{-1}M)$.
5. $\mathrm{Curl}_x(\tilde{c}) = \widetilde{\mathrm{Curl}}_z(c)$,
   with
   $$\widetilde{\mathrm{Curl}}_z(c) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} F^{-T}\,\nabla_z\,c.$$
6. $\mathrm{curl}_x(\tilde{j}) = \widetilde{\mathrm{curl}}_z(j)$,
   with
   $$\widetilde{\mathrm{curl}}_z(j) = \mathrm{curl}_z(F^T j).$$

Proof. For volume-preserving coordinate transformations it holds $J \equiv 1$, thus in that case by the preceding lemma we have $\mathrm{div}_z(F^{-1}j) = \mathrm{div}_x(\tilde{j})$ and $\mathrm{div}_z(F^{-1}M) = \mathrm{div}_x(\tilde{M})$, which gives the third and the fourth statement. The first and the second statement follow by the equalities $\Delta_x(\tilde{c}) = \mathrm{div}_x(\nabla_x\,\tilde{c})$ and $\Delta_x(\tilde{j}) = \mathrm{div}_x(\nabla_x\,\tilde{j})$ and application of the above results to the right hand sides.

The fifth statement follows along the same lines, whereas the sixth can be obtained by a direct calculation of the effect of the transformation on the defining equation. ♦

2.6 Remark.
A simple computation shows that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} F^{-T} = F \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$*; thus it holds*

$$\widetilde{\mathrm{Curl}}_z(c) = F\,\mathrm{Curl}_z(c).$$

2.7 Lemma (Transformed Differential Identities)**.**
Let ψ be a volume-preserving $\mathcal{C}^1$-coordinate transformation as above. Then the following identities hold:

1. $\mathrm{div}_z(F^{-1}c) = F^{-T}\,\nabla_z\,c$.
2. $\mathrm{div}_z(F^{-1}F^{-T}\,\nabla_z(\mathrm{div}_z(F^{-1}j))) = \mathrm{div}_z(F^{-1}\,\mathrm{div}_z(F^{-1}F^{-T}\,\nabla_z\,j))$.

3. $\operatorname{div}_z(F^{-1}\,\widetilde{\operatorname{Curl}}_z(c)) = 0$.

4. $\operatorname{div}_z(F^{-1}(cj)) = c\operatorname{div}_z(F^{-1}j) + F^{-T}\,\nabla c \cdot j$.

5. $\widetilde{\operatorname{curl}}_z(F^{-T}\,\nabla_z c) = 0$.

6. $\widetilde{\operatorname{curl}}_z(\operatorname{div}_z(F^{-1}F^{-T}\,\nabla_z j)) = \operatorname{div}_z(F^{-1}F^{-T}\,\nabla_z \widetilde{\operatorname{curl}}_z(j))$.

7. *If* $\operatorname{div}_z(F^{-1}j) = 0$, *then*

$$F^{-T}\,\nabla_z(\widetilde{\operatorname{curl}}_z(j)) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \operatorname{div}_z(F^{-1}F^{-T}\,\nabla_z j).$$

8. $F^{-T}\,\nabla_z(\operatorname{div}_z(F^{-1}F^{-T}\,\nabla c)) = \operatorname{div}_z(F^{-1}F^{-T}\,\nabla_z(F^{-T}\,\nabla_z c))$.

Proof. To obtain the first statement transform the well-known equation $\operatorname{div}_x(\tilde{c}I) = \nabla_x \tilde{c}$. The second follows from $\Delta_x(\operatorname{div}_x \tilde{j}) = \operatorname{div}_x(\Delta_x \tilde{j})$. Next transform $\operatorname{div}_x(\operatorname{Curl}_x(\tilde{c})) = 0$ and $\operatorname{div}_x(\tilde{c}\tilde{j}) = \tilde{c}\operatorname{div}_x(\tilde{j}) + \nabla_x \tilde{c} \cdot \tilde{j}$. Finally observe that $\operatorname{curl}_x(\nabla_x \tilde{c}) = 0$ as well as $\operatorname{curl}_x(\Delta_x \tilde{j}) = \Delta_x(\operatorname{curl} \tilde{j})$.

If $\operatorname{div}_x(\tilde{j}) = 0$, a simple calculation together with the fact that in this case $\frac{\partial \tilde{j}_1}{\partial x_1} = -\frac{\partial \tilde{j}_2}{\partial x_2}$ shows that $\nabla_x(\operatorname{curl}_x \tilde{j}) = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] \Delta_x(\tilde{j})$, which upon transformation yields the result. For the last statement consider $\nabla_x(\Delta_x \tilde{c}) = \Delta_x(\nabla_x \tilde{c})$. ♦

2.8 Remark.
Let $\nu(x)$ *be the unit normal vector at* $x \in \partial\Omega$. *Then the corresponding transformed unit normal vector is given by*

$$\tilde{\nu}(x) = \|F^{-T}(x)\nu(x)\|^{-1}\,F^{-T}(x)\nu(x).$$

If $n = 2$, *the unit tangential vector* $\tilde{\tau}(x)$ *has the direction* $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] F^{-T}(x)\nu(x) = F(x)\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]\nu(x) = F(x)\tau(x)$, *thus it holds*

$$\tilde{\tau}(x) = \|F(x)\tau(x)\|^{-1}\,F(x)\tau(x).$$

$\|\cdot\|$ *indicates the chosen norm in* $\mathbb{R}^n$.

2.2 Derivation of the Transformed Stokes Problem

We apply the results of the preceding subsection to the following situation in $\mathbb{R}^2$: Let $L > 0$ and define $\tilde{\Omega} = [0, L] \times \mathbb{R}$. Let $g \in \mathcal{C}^\infty(\mathbb{R})$ be a given function such that $g(x + L) = g(x)$

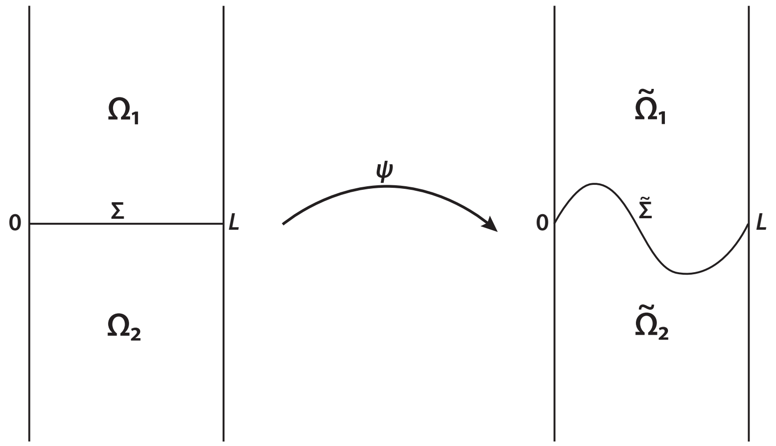

Figure 2: The coordinate transformation ψ transforms the domain Ω with a straight interface Σ to the domain $\tilde{\Omega}$ with curved interface $\tilde{\Sigma}$.

for all $x \in \mathbb{R}$. We consider the graph $\{(x_1, g(x_1)) \mid x_1 \in [0, L]\} \subset \mathbb{R}^2$ to describe an interface $\tilde{\Sigma}$ in $\tilde{\Omega}$, dividing it into two parts $\tilde{\Omega}_1$ and $\tilde{\Omega}_2$:

$$\begin{aligned}
\tilde{\Omega}_1 &:= \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in [0, L], x_2 < g(x_1)\} \\
\tilde{\Omega}_2 &:= \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in [0, L], x_2 > g(x_1)\} \\
\tilde{\Sigma} &:= \{x = (x_1, x_2) \in \mathbb{R}^2 \mid x_1 \in [0, L], x_2 = g(x_1)\}.
\end{aligned}$$

Let $\tilde{S} \subset \tilde{\Omega}_2$ be a closed set (corresponding to a solid part in the domain), such that $\partial\tilde{S} \cap \partial\tilde{\Omega} = \emptyset$.

We want to transform the domain $\Omega = [0, L] \times \mathbb{R}$ with a straight boundary $\Sigma = [0, L] \times \{0\}$, with parts $\Omega_1 = [0, L] \times \mathbb{R}_{>0}$ and $\Omega_2 = [0, L] \times \mathbb{R}_{<0}$ to the above situation. We will indicate coordinates in Ω by $z = (z_1, z_2)$ and those in $\tilde{\Omega}$ by $x = (x_1, x_2)$. Therefore define the transformation (compare Figure 2)

$$\begin{gathered}
\psi : \Omega \longrightarrow \tilde{\Omega} \\
\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \longmapsto \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} z_1 \\ z_2 + g(z_1) \end{pmatrix}.
\end{gathered}$$

Then the Jacobian matrix F of ψ is given by

$$F(z) = \begin{bmatrix} 1 & 0 \\ g'(z_1) & 1 \end{bmatrix}. \tag{2.1}$$

Since $\det F = 1$, ψ is a volume preserving $\mathcal{C}^\infty$-coordinate transformation.

In $\tilde{\Omega}_F := \tilde{\Omega} \backslash \tilde{S}$, we consider a viscous fluid whose flow is governed by a steady state Stokes flow, that is: Find a velocity u and a pressure p with $(\tilde{u}, \tilde{p}) \in H^1(\tilde{\Omega}_F)^2 \times L^2(\tilde{\Omega}_F)/\mathbb{R}$ such that

$$-\Delta_x \tilde{u}(x) + \nabla_x \tilde{p}(x) = \tilde{f}(x) \text{ in } \tilde{\Omega}_F \tag{2.2a}$$
$$\operatorname{div}_x(\tilde{u}(x)) = 0 \text{ in } \tilde{\Omega}_F \tag{2.2b}$$
$$\tilde{u}(x) = 0 \text{ on } \partial \tilde{S} \tag{2.2c}$$
$$\tilde{u}, \tilde{p} \text{ are } L\text{-periodic in } x_1 \tag{2.2d}$$

with a given force $\tilde{f} \in L^2(\tilde{\Omega}_F)$. Here and in the sequel, we will tacitly assume that the fluid viscosity has a constant value of 1 (the case of a different constant viscosity μ can easily be adapted).

By using the transformation formulas from Lemma 2.5, we obtain for $u(z) = \tilde{u}(\psi(z))$, $p(z) = \tilde{p}(\psi(z))$ and $f(z) = \tilde{f}(\psi(z))$:

$$-\operatorname{div}_z(F^{-1}(z)F^{-T}(z)\,\nabla_z\, u(z)) + F^{-T}(z)\,\nabla_z\, p(z) = f(z) \text{ in } \Omega_F \tag{2.3a}$$
$$\operatorname{div}_z(F^{-1}(z)u(z)) = 0 \text{ in } \Omega_F \tag{2.3b}$$
$$u(z) = 0 \text{ on } \partial S \tag{2.3c}$$
$$u, p \text{ are } L\text{-periodic in } z_1 \tag{2.3d}$$

with $S = \psi^{-1}(\tilde{S})$ and $\Omega_F = \Omega \backslash S$.

Variants of this system of equations represent the basic equations with which the rest of this work is concerned.

2.9 Remark.
In continuum mechanics, the quantity $F^{-1}(z)F^{-T}(z)$ corresponds to the inverse of the right Cauchy-Green tensor, see e.g. [MH94].

For the convenience of the reader, important formulas will be highlighted by a grey box, whereas the auxilliary functions in Section 5 are made optically distinguishable by using a frame.

3 Homogenisation of the Transformed Stokes Equation

In this section we carry out a homogenisation procedure for the transformed Stokes equation (2.3). As we do not need results on an unbounded strip, but rather are interested in the effective equation for the velocity and pressure, we consider a simplified situation:

Let Ω be a bounded Lipschitz domain in $\mathbb{R}^2$. We denote by $Y = (0,1)^2$ the reference cell. Let Y_S be a strictly included subset of Y (the solid part) and set $Y^* := Y \backslash \overline{Y_S}$ (the fluid part). Let $\Gamma = \partial Y_S$ be the boundary of the solid part. It is assumed to belong to the class $\mathcal{C}^\infty$. (See also Figure 3 for an illustration.)

Define for $M \subset Y$ and $k \in \mathbb{Z}^2$ the shifted subset

$$M^k = M + \sum_{j=1}^{2} k_j e_j,$$

with e_j denoting the j-th unit vector.

For given $\varepsilon > 0$ we define the following ε-periodic domains:

$$\begin{aligned}
\Omega^\varepsilon &= \bigcup_{k \in \mathbb{Z}^n} (\varepsilon (Y^*)^k \cap \Omega) && \text{the fluid part} \\
\Omega_S^\varepsilon &= \bigcup_{k \in \mathbb{Z}^n} (\varepsilon Y_S^k \cap \Omega) && \text{the solid part} \\
\Gamma^\varepsilon &= \bigcup_{k \in \mathbb{Z}^n} (\varepsilon \Gamma^k \cap \Omega) && \text{the boundary of the solid part}
\end{aligned}$$

In order to avoid technical difficulties, we assume that $\Gamma^\varepsilon \cap \partial\Omega = \emptyset$ for all ε. Changing the name of the variables from z back to x in equation (2.3), we consider the problem

$$- \operatorname{div}(F^{-1}(x) F^{-T}(x) \nabla u^\varepsilon(x)) + F^{-T}(x) \nabla p^\varepsilon(x) = f(x) \text{ in } \Omega^\varepsilon \tag{3.1a}$$

$$\operatorname{div}(F^{-1}(x) u^\varepsilon(x)) = 0 \text{ in } \Omega^\varepsilon \tag{3.1b}$$

$$u^\varepsilon(x) = 0 \text{ on } \Gamma^\varepsilon \tag{3.1c}$$

$$u^\varepsilon(x) = 0 \text{ on } \partial\Omega \tag{3.1d}$$

with a given force $f \in L^2(\Omega)^2$.

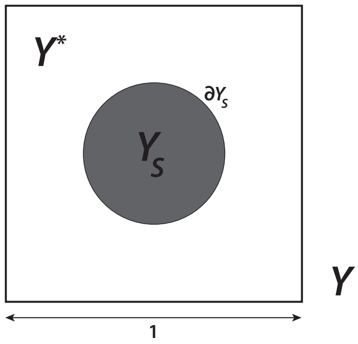

Figure 3: The reference cell, consisting of the solid part Y_S with boundary ∂Y_S, and the fluid part Y^*.

3.1 Remark.
Let $u, \phi \in \mathcal{C}^2(\overline{\Omega^\varepsilon})^2$ and $p \in \mathcal{C}^1(\overline{\Omega^\varepsilon})$. Then the rules of integration by parts for the above type of equation formally read as

$$-\int\limits_{\Omega^\varepsilon} \operatorname{div}(F^{-1}(x)F^{-T}(x)\nabla u(x)) \cdot \phi(x)\, \mathrm{d}x = \int\limits_{\Omega^\varepsilon} F^{-T}(x)\nabla u(x) : F^{-T}(x)\nabla \phi(x)\, \mathrm{d}x - \int\limits_{\partial\Omega^\varepsilon} F^{-1}(x)F^{-T}(x)\nabla u(x)\nu \cdot \phi(x)\, \mathrm{d}\sigma_x$$

and

$$\int\limits_{\Omega^\varepsilon} F^{-T}(x)\nabla p(x)\cdot\phi(x)\, \mathrm{d}x = -\int\limits_{\Omega^\varepsilon} p(x) \operatorname{div}(F^{-1}(x)\phi(x))\, \mathrm{d}x + \int\limits_{\partial\Omega^\varepsilon} F^{-1}(x)\phi(x)\cdot\nu\; p(x)\, \mathrm{d}\sigma_x.$$

Here ':' denotes the inner product, $A : B = \operatorname{tr}(A^T B)$.

3.1 Review of well-known Theorems

For the sake of completeness, in this subsection we review some concepts used to obtain existence and uniqueness results for Problem (3.1), together with theorems which are helpful for the homogenisation procedure.

3.1.1 Two-Scale-Convergence

The notion of two-scale-convergence was introduced by Nguetseng in [Ngu89] and later extended by Allaire, cf. [All92]. It has proven to be extremely useful and is now used as the main tool in the theory of periodic homogenisation. For a comprehensive overview of the concept, the reader is referred to the review article by Dag Lukkassen et al., [LNW02].

Here we will recall the definition and the main results in the Hilbert space $L^2(\Omega)$. Generalizations to L^p-spaces are possible, see for example the above mentioned articles. In addition, special notions of two-scale convergence on surfaces (cf. [NR92], [NR96] and esp. [ADH95]) and on interfaces (in [NRJ07]) have been developed.

3.2 Definition (Two-Scale-Convergence)**.**
Let u^ε be a sequence of fuctions in $L^2(\Omega)$. u^ε converges in two-scale sense *to $u_0 \in L^2(\Omega \times Y)$, if*

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x)\psi(x, \frac{x}{\varepsilon})\, \mathrm{d}x = \int_\Omega \int_Y u_0(x,y)\psi(x,y)\, \mathrm{d}y\, \mathrm{d}x \qquad \forall\, \psi \in \mathcal{C}_0^\infty(\Omega; \mathcal{C}_\#^\infty(Y)). \tag{3.2}$$

We write $u^\varepsilon \xrightharpoonup{2} u$.

3.3 Theorem.
Let u^ε be a bounded sequence of functions in $L^2(\Omega)$. Then there exists a subsequence ε' and a function $u_0 \in L^2(\Omega \times Y)$ such that

$$u^{\varepsilon'} \xrightharpoonup{2} u_0.$$

Proof. See [Ngu89] for the original proof or [All92] for a simplified version. ♦

3.4 Theorem.

1. *Let u^ε be a bounded sequence in $H^1(\Omega)$ such that $u^\varepsilon \rightharpoonup u_0$ in $H^1(\Omega)$ weakly. Then $u^\varepsilon \xrightharpoonup{2} u_0$, and it exists $u_1 \in L^2(\Omega, H^1_\#(Y)/\mathbb{R})$ such that up to a subsequence*

$$\nabla u^\varepsilon \xrightharpoonup{2} \nabla_x u_0 + \nabla_y u_1.$$

2. *Let u^ε and $\varepsilon \nabla u^\varepsilon$ be bounded in $L^2(\Omega)$ and $L^2(\Omega)^2$. Then there exists $u_0 \in L^2(\Omega, H^1_\#(Y))$ such that along a subsequence*

$$u^\varepsilon \xrightharpoonup{2} u_0 \tag{3.3}$$
$$\varepsilon \nabla u^\varepsilon \xrightharpoonup{2} \nabla_y u_0. \tag{3.4}$$

3. *Let u^ε be a divergence-free bounded sequence in $L^2(\Omega)^2$, which two-scale converges to $u_0 \in L^2(\Omega \times Y)^2$. Then u_0 satisfies*

$$\mathrm{div}_y\, u_0 = 0 \tag{3.5}$$
$$\mathrm{div}_x \int_Y u_0\, \mathrm{d}y = 0. \tag{3.6}$$

Proof. See [All92]. ♦

Unfortunately, the maximal class of test functions ψ such that (3.2) holds is not known. In the literature, following classes are often considered (compare [All92], [CD99], [Ngu89], [NR92]):

3.5 Lemma.
Let $u^\varepsilon \xrightarrow{2} u_0$. *Then the convergence* (3.2) *holds true for the following test functions* ψ:

- $\psi \in \mathcal{C}_0^\infty(\Omega, \mathcal{C}_\#^\infty(Y))$.
- $\psi \in L^2(\Omega, \mathcal{C}_\#(Y))$.
- $\psi(x,y) = \psi_1(y)\psi_2(x,y)$, *with* $\psi_1 \in L_\#^\infty(Y)$, $\psi_2 \in L^2(\Omega, \mathcal{C}_\#(Y))$.
- $\psi(x,y) = \psi_1(y)\psi_2(x,y)$, *with* $\psi_1 \in L^2(Y)$, $\psi_2 \in \mathcal{C}_0^\infty(\Omega, \mathcal{C}_\#^\infty(Y))$.

The following lemma shows that a two-scale limit contains more information than a weak limit:

3.6 Lemma.
Let u^ε *be a bounded sequence in* $L^2(\Omega)$ *such that* $u^\varepsilon \xrightarrow{2} u_0$ *with* $u_0 \in L^2(\Omega \times Y)$.

Then $u^\varepsilon \longrightarrow u$ *weakly in* $L^2(\Omega)$, *where* u *is given by*

$$u(x) = \int_Y u_0(x,y)\ \mathrm{d}y.$$

3.1.2 From Functional Analysis

We recall some facts from functional analysis.

The lemma of Lax and Milgram below will be used to obtain a number of existence and uniqueness results:

3.7 Theorem (Lemma of Lax-Milgram)**.**
Let H *be a Hilbert space,* $\mathcal{B} : H \times H \longrightarrow \mathbb{R}$ *be a continuous, coercive bilinear form and let* $b \in H'$ *be a continuous linear functional on* H.

Then there exists a unique $u \in H$ *such that*

$$\mathcal{B}(u,v) = \langle b, v\rangle \quad \forall v \in H.$$

Proof. See for example [Sho97], Theorem I.2.2. ♦

The following two theorems give generalisations of the well-known trace operator of Sobolev spaces. They are used in the construction of some auxiliary functions, cf. the proof of the surjectivity of the transformed divergence operator, Lemma 3.16.

3.8 Theorem (Generalized Trace Theorem)**.**
Let $k \in \mathbb{R}, k \geq 2$, and let Λ be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$ with boundary $\partial\Lambda \in \mathcal{C}^{k+1}$. There exists a continuous linear operator $\mathcal{T} : H^k(\Lambda) \longrightarrow H^{k-\frac{1}{2}}(\partial\Lambda) \times H^{k-1-\frac{1}{2}}(\partial\Lambda)$ with

$$\mathcal{T}(\phi) = (\phi|_{\partial\Lambda}, \frac{\partial\phi}{\partial\nu}|_{\partial\Lambda}) \qquad \textit{for all } \phi \in \mathcal{C}^k(\bar{\Lambda}).$$

Proof. This is a special case of Proposition 8.7 in [Wlo82]. ♦

3.9 Theorem (Generalized Inverse Trace Theorem)**.**
Let Λ be a bounded domain in $\mathbb{R}^n$, $n \in \mathbb{N}$, with boundary $\partial\Lambda \in \mathcal{C}^{k+1}$ with a given $k \in \mathbb{R}, k \geq 2$. Let $\mathcal{T}$ be defined as above.

There exists a continuous linear extension operator $\mathcal{E} : H^{k-\frac{1}{2}}(\partial\Lambda) \times H^{k-1-\frac{1}{2}}(\partial\Lambda) \longrightarrow H^k(\Lambda)$ such that

$$\mathcal{T} \circ \mathcal{E} = \mathrm{Id}.$$

Proof. See again [Wlo82], Proposition 8.8. ♦

3.10 Proposition (Regularity Results for Elliptic PDE)**.**
Let $r \in \mathbb{N}_0$ be given and let Λ be a bounded domain in $\mathbb{R}^n$ with boundary $\partial\Lambda \in \mathcal{C}^{r+1}$. Assume that A is a given matrix-valued function, $A \in \mathcal{C}^r(\bar{\Lambda}, \mathbb{R}^{n\times n})$ such that there exist constants $0 < k_A < K_A$ with

$$k_A \|\xi\|_2^2 \leq \xi^T A(x)\xi \qquad \forall\ \xi \in \mathbb{R}^n \qquad \forall\ x \in \Lambda$$

and

$$|\xi^T A(x)\eta| \leq K_A \|\xi\|_2 \|\eta\|_2 \qquad \forall\ \xi, \eta \in \mathbb{R}^n \ \ \forall\ x \in \Lambda.$$

Here $\|\cdot\|_2$ denotes the Euclidean norm in $\mathbb{R}^n$.

For $u \in H^1(\Lambda)$ define the differential operator

$$\mathcal{L} : H^1(\Lambda) \longrightarrow H^{-1}(\Lambda)$$
$$\mathcal{L}u = -\operatorname{div}(A\,\nabla\, u).$$

We have:

1. *Let $f \in H^{r-1}(\Lambda)$, $g \in H^{r+\frac{1}{2}}(\partial\Lambda)$ and consider the elliptic Dirichlet problem*

$$\begin{aligned} \mathcal{L}u &= f \quad && \textit{in } \Lambda \\ u &= g \quad && \textit{on } \partial\Lambda. \end{aligned}$$

   Then every weak solution $u \in H^1(\Lambda)$ belongs to the space $H^{r+1}(\Lambda)$.

2. *Let* $f \in H^{r-1}(\Lambda)$ *and* $h \in H^{r-\frac{1}{2}}(\partial\Lambda)$. *Then every weak solution* $u \in H^1(\Lambda)$ *of the elliptic Neumann problem*

$$\begin{aligned} \mathcal{L}u &= f \quad && \textit{in } \Lambda \\ \frac{\partial u}{\partial \nu} &= h \quad && \textit{on } \partial\Lambda \end{aligned}$$

is in $H^{r+1}(\Lambda)$.

Proof. This is a special case of the regularity results found in [Hac96]. ♦

To finish this subsection, we give a result which states that the precise form of the constant appearing in Poincaré's inequality in Ω^ε is of the form $C\varepsilon$, where C only depends on the geometry of the reference cell, but not on ε. This lemma will later also be applied to the set Ω_2^ε of Section 5.

3.11 Lemma.
Let $v \in H_0^1(\Omega^\varepsilon)$. *There exists a constant* C, *independent of* ε, *such that*

$$\|v\|_{L^2(\Omega^\varepsilon)} \le C\varepsilon \, \|\nabla v\|_{L^2(\Omega^\varepsilon)^2} .$$

Proof. The proof will be carried out using a summation-and-scaling argument.

For $w \in H^1(Y^*)$ such that $w = 0$ on ∂Y_S it holds

$$\|w\|^2_{L^2(Y^*)} \le C \, \|\nabla w\|_{L^2(Y^*)^2} ,$$

where the constant C depends on Y^*.

A change of variables $x = \varepsilon y$ implies for $v \in H_0^1(\Omega^\varepsilon)$ and $\tilde{v}(y) := v(\varepsilon y) = v(x)$ the equation

$$\nabla_y \tilde{v}(y) = \nabla_y v(\varepsilon y) = \varepsilon \nabla_x v(x)|_{x=\varepsilon y}.$$

Application of the above inequality now yields

$$\begin{aligned} \|v\|^2_{L^2(\Omega^\varepsilon)} &= \sum_k \int\limits_{\varepsilon(Y^*)^k} |v(x)|^2 \,\mathrm{d}x = \varepsilon \sum_k \int\limits_{(Y^*)^k} |\tilde{v}(y)|^2 \,\mathrm{d}y \\ &\le C\varepsilon \sum_k \int\limits_{(Y^*)^k} |\nabla_y \tilde{v}(y)|^2 \,\mathrm{d}y \\ &= C\varepsilon \sum_k \int\limits_{(Y^*)^k} \varepsilon^2 |\nabla_x v(\varepsilon y)|^2 \,\mathrm{d}y \end{aligned}$$

$$= C\varepsilon^2 \sum_k \int\limits_{\varepsilon(Y^*)^k} |\nabla_x v(x)|^2 \, \mathrm{d}x = C\varepsilon^2 \, \|\nabla v\|^2_{L^2(\Omega^\varepsilon)^2} \, ,$$

where the constant C is the same as above and the sum extends over all $k \in \mathbb{Z}^2$ such that $\varepsilon(Y^*)^k \subset \Omega$. ♦

3.2 Existence and Uniqueness

In this section we prove the existence and uniqueness of the solution of Problem (3.1) for fixed ε. Basically, the approach is the same as in the functional-analytic treatment of the Stokes equation (see e.g. [SP80]).

By multiplying (3.1a) with $\phi \in H^1_{\mathrm{div}}(\Omega^\varepsilon)^2$ where

$$H^1_{\mathrm{div}}(\Omega^\varepsilon)^2 := \{ w \in H^1_0(\Omega^\varepsilon)^2 \mid \mathrm{div}(F^{-1} w) = 0 \},$$

integrating by parts and noting that

$$\int\limits_{\Omega^\varepsilon} F^{-T}(x) \, \nabla p(x) \cdot \phi(x) \, \mathrm{d}x = - \int\limits_{\Omega^\varepsilon} p(x) \, \mathrm{div}(F^{-1}(x)\phi(x)) \, \mathrm{d}x = 0,$$

we obtain the weak formulation of Problem (3.1) in the form

$$\int\limits_{\Omega^\varepsilon} F^{-T}(x) \, \nabla u^\varepsilon(x) : F^{-T}(x) \, \nabla \phi(x) \, \mathrm{d}x = \int\limits_{\Omega^\varepsilon} f(x) \cdot \phi(x) \, \mathrm{d}x \quad \forall \phi \in H^1_{\mathrm{div}}(\Omega^\varepsilon)^2 \tag{3.7}$$

Note that $H^1_{\mathrm{div}}(\Omega^\varepsilon)$ is a Banach space with respect to the norm $\|\nabla \cdot\|_{L^2(\Omega^\varepsilon)^2}$. We need the following lemma for the estimation of the left hand side:

3.12 Lemma.
There exist constants $0 < k_F < K_F$ such that for the eigenvalues $\lambda(x)$ of $F^{-1}(x)F^{-T}(x)$ holds

$$k_F < \lambda(x) < K_F \quad \forall x \in \Omega,$$

i.e. $F^{-1}(x)F^{-T}(x)$ is symmetric and positive definite.

Proof. A calculation of the eigenvalues $\lambda_1(x), \lambda_2(x)$ of $F^{-1}(x)F^{-T}(x)$ yields

$$\lambda_1(x) = 1 + \frac{g'(x_1)^2}{2} + \sqrt{g'(x_1)^2 + \frac{g'(x_1)^4}{4}},$$

$$\lambda_2(x) = 1 + \frac{g'(x_1)^2}{2} - \sqrt{g'(x_1)^2 + \frac{g'(x_1)^4}{4}}.$$

Because of the smoothness of g there exists an $M > 1$ such that $|g(x)| < M$ for all $x \in \Omega$. Obviously $\lambda_i(x) \leq 1 + 2M^2 =: K_F$, $i = 1, 2$.

Choose a k_F small enough such that $M^2 + 2 \leq \frac{1}{k_F}$. Another calculation shows that

$$\lambda_2(x) \geq k_F \quad \Longleftrightarrow \quad g'(x_1)^2 \leq \frac{1}{k_F} + k_F - 2,$$

which gives the desired result since $\lambda_1(x) \geq \lambda_2(x)$. ♦

3.13 Proposition.
Let $\varepsilon > 0$ be fixed and let F be given by (2.1)*. For given $f \in L^2(\Omega)$, the Problem* (3.7) *has a unique solution $u^\varepsilon \in H^1_{\text{div}}(\Omega^\varepsilon)^2$.*

Proof. Define for $u, v \in H^1_{\text{div}}(\Omega^\varepsilon)^2$ the (bi-)linear forms

$$a(u, v) = \int\limits_{\Omega^\varepsilon} F^{-T}(x) \nabla u(x) : F^{-T}(x) \nabla v(x) \, \mathrm{d}x$$

and

$$b(v) = \int\limits_{\Omega^\varepsilon} f(x) \cdot v(x) \, \mathrm{d}x.$$

The continuity of b for $f \in L^2(\Omega)^2$ is standard. In order to apply the lemma of Lax-Milgram, we have to show that a is continuous and coercive.

First note that as a pointwise estimate we have

$$\begin{aligned}
F^{-T}(x) \nabla v(x) : F^{-T}(x) \nabla v(x) &= \mathrm{tr}(\nabla v(x)^T F^{-1}(x) F^{-T}(x) \nabla v(x)) \\
&= \sum_{i=1}^{2} e_i^T \nabla v(x)^T F^{-1}(x) F^{-T}(x) \nabla v(x) e_i \\
&\leq \sum_{i=1}^{2} \|\nabla v(x) e_i\|_2 \left\|F^{-1}(x) F^{-T}(x)\right\|_2 \|\nabla v(x) e_i\|_2 \\
&\leq K_F \sum_{i=1}^{2} e_i^T \nabla v(x)^T \nabla v(x) e_i \\
&= K_F \nabla v(x) : \nabla v(x) = K_F \|\nabla v(x)\|_2^2 ,
\end{aligned}$$

with $\|\cdot\|_2$ the Euclidean vector- and matrixnorm. This gives the continuity of a due to

$$\begin{aligned}
&\int\limits_{\Omega^\varepsilon} |F^{-T}(x) \nabla u(x) : F^{-T}(x) \nabla v(x)| \, \mathrm{d}x \\
&\quad = \int\limits_{\Omega^\varepsilon} \Big| \sum_{i,j=1}^{2} \big(F^{-T}(x) \nabla u(x)\big)_{ij} \big(F^{-T}(x) \nabla v(x)\big)_{ij} \Big| \, \mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&\leq \int_{\Omega^\varepsilon} \Big(\sum_{i,j=1}^{2} \big(F^{-T}(x) \nabla u(x))_{ij}\big)^2 \Big)^{\frac{1}{2}} \Big(\sum_{i,j=1}^{2} \big(F^{-T}(x) \nabla v(x))_{ij}\big)^2 \Big)^{\frac{1}{2}} \,\mathrm{d}x \\
&= \int_{\Omega^\varepsilon} \big(F^{-T}(x) \nabla u(x) : F^{-T}(x) \nabla u(x)\big)^{\frac{1}{2}} \big(F^{-T}(x) \nabla v(x) : F^{-T}(x) \nabla v(x)\big)^{\frac{1}{2}} \,\mathrm{d}x \\
&\leq K_F \int_{\Omega^\varepsilon} \|\nabla u(x)\|_2 \, \|\nabla v(x)\|_2 \,\mathrm{d}x \\
&\leq K_F \, \|\nabla u\|_{L^2(\Omega^\varepsilon)} \, \|\nabla v\|_{L^2(\Omega^\varepsilon)} \,,
\end{aligned}
$$

where the Cauchy-Schwarz inequality in L^2 has been used in the last step.

For the coercivity consider

$$
\begin{aligned}
k_F \, \|\nabla v\|_{L^2(\Omega^\varepsilon)} &\leq \int_{\Omega^\varepsilon} \lambda_2(x) \nabla v(x) : \nabla v(x) \,\mathrm{d}x \\
&\leq \int_{\Omega^\varepsilon} \sum_{i=1}^{2} \lambda_i(x) e_i^T \nabla v(x)^T \nabla v(x) e_i \,\mathrm{d}x \\
&\leq \int_{\Omega^\varepsilon} F^{-T}(x) \nabla v(x) : F^{-T}(x) \nabla v(x) \,\mathrm{d}x.
\end{aligned}
$$

Now the Lax-Milgram lemma implies the proposed result. ♦

Due to (3.7), the solution u^ε fullfills

$$
-\operatorname{div}(F^{-1} F^{-T} \nabla u^\varepsilon) - f \quad \in (H^1_{\operatorname{div}}(\Omega^\varepsilon)^2)^\perp.
$$

We will now characterize the orthogonal complement $(H^1_{\operatorname{div}}(\Omega^\varepsilon)^2)^\perp$ of $H^1_{\operatorname{div}}(\Omega^\varepsilon)^2$ in order to reintroduce the pressure.

3.14 Lemma.
It holds

$$
(H^1_{\operatorname{div}}(\Omega^\varepsilon)^2)^\perp = \{F^{-T} \nabla p \mid p \in L^2(\Omega^\varepsilon)\}.
$$

Proof. Define $G := \{F^{-T} \nabla p \mid p \in L^2(\Omega^\varepsilon)\}$. Let $\phi \in G, u \in H^1_{\operatorname{div}}(\Omega^\varepsilon)^2$ with $\phi = F^{-T} \nabla p$. Then

$$
\langle \phi, u \rangle_{H^{-1}(\Omega^\varepsilon)^2,\, H^1(\Omega^\varepsilon)^2} = \langle F^{-T} \nabla p, u \rangle_{H^{-1}(\Omega^\varepsilon)^2,\, H^1(\Omega^\varepsilon)^2} = -\int_{\Omega^\varepsilon} p \operatorname{div}(F^{-1} u) \,\mathrm{d}x = 0,
$$

such that $\phi \in (H^1_{\operatorname{div}}(\Omega^\varepsilon)^2)^\perp$. Therefore $G \subset (H^1_{\operatorname{div}}(\Omega^\varepsilon)^2)^\perp$.

For the other inclusion we will show that $\operatorname{div}(F^{-1}\cdot) : H^1_0(\Omega^\varepsilon) \longrightarrow L^2_0(\Omega^\varepsilon)$ is surjective and that $-F^{-T} \nabla \cdot$ is its adjoint operator, therefore being injective from $L^2_0(\Omega^\varepsilon)$ to

$\operatorname{im}(-F^{-T}\nabla\cdot)$. Now if $\psi \in (H^1_{\operatorname{div}}(\Omega^\varepsilon)^2)^\perp$, we consider $u \in H^1_0(\Omega^\varepsilon)$ with $\operatorname{div}(F^{-1}u) = 0$. It holds

$$\langle \psi, u \rangle_{H^{-1}(\Omega^\varepsilon)^2,\, H^1_0(\Omega^\varepsilon)^2} = 0.$$

Since u is arbitrary,

$$\psi \perp \ker(\operatorname{div}(F^{-1}\cdot)),$$

and since $\ker(\operatorname{div}(F^{-1}\cdot))^\perp = \operatorname{im}(-F^{-T}\nabla\cdot)$ there exists a $p \in L^2(\Omega^\varepsilon)$ with

$$\psi = F^{-T}\nabla p.$$

The surjectivity of $\operatorname{div}(F^{-1}\cdot)$ is a consequence of Lemma 3.16 below; and the adjointness of the operators can easily be seen from the equation

$$\langle F^{-T}\nabla p, u \rangle_{H^{-1}(\Omega^\varepsilon)^2,\, H^1(\Omega^\varepsilon)^2} = -\int_{\Omega^\varepsilon} p \operatorname{div}(F^{-1}u)\,\mathrm{d}x = (p, \operatorname{div}(F^{-1}u))_{L^2(\Omega^\varepsilon)}. \qquad \blacklozenge$$

Before proving some properties of the divergence operator, we need the following lemma.

3.15 Lemma.
Let $\theta \in H^1(\Omega^\varepsilon)$. Then

$$\begin{aligned} \operatorname{Curl}(\theta)\cdot\nu &= -\nabla\theta\cdot\tau \quad \text{on } \partial\Omega^\varepsilon \\ \operatorname{Curl}(\theta)\cdot\tau &= \nabla\theta\cdot\nu \quad \text{on } \partial\Omega^\varepsilon. \end{aligned}$$

Proof. It holds

$$\operatorname{Curl}(\theta)\cdot\nu = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\nabla\theta\cdot\nu = \nabla\theta\cdot\Big(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^T \nu\Big) = -\nabla\theta\cdot\tau,$$

since the matrix $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]^T$ corresponds to a rotation of $\frac{\pi}{2}$ and thus $\left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]^T \cdot \nu = -\tau$. The second equality follows along the same lines. $\blacklozenge$

Now we are ready to prove the lemma used above:

3.16 Lemma.
Let $G \in L^2(\Omega^\varepsilon)$ with $\int_{\Omega^\varepsilon} G = 0$. There exists a $\phi \in H^1_0(\Omega^\varepsilon)^2$ with

$$\begin{aligned} \operatorname{div}(F^{-1}(x)\phi(x)) &= G(x) \text{ in } \Omega^\varepsilon \\ \phi(x) &= 0 \text{ on } \partial\Omega^\varepsilon \end{aligned}$$

such that

$$\|\phi\|_{H^1(\Omega^\varepsilon)^2} \leq C\,\|G\|_{L^2(\Omega^\varepsilon)}.$$

Thus $\operatorname{div}(F^{-1}\cdot) : H_0^1(\Omega^\varepsilon)^2 \longrightarrow L_0^2(\Omega^\varepsilon)$ *is surjective.*

Proof. We look for ϕ in the form

$$\phi = F\,\nabla\eta + F\,\mathrm{Curl}(\theta)$$

with η satisfying

$$\begin{aligned}\Delta\eta &= G \text{ in } \Omega^\varepsilon\\ \nabla\eta\cdot\nu &= 0 \text{ on } \partial\Omega^\varepsilon.\end{aligned}$$

By considering the weak formulation of this problem

$$-\int\limits_{\Omega^\varepsilon}\nabla\eta : \nabla\psi = \int\limits_{\Omega^\varepsilon} G\cdot\psi \quad \forall\psi\in H_0^1(\Omega^\varepsilon)/\mathbb{R}$$

and using estimates similar to those derived in Propositon 3.13 we see that a unique solution $\eta \in H_0^1(\Omega^\varepsilon)/\mathbb{R}$ exists, satisfying the estimate $\|\nabla\eta\|_{L^2(\Omega^\varepsilon)^2} \le C\,\|G\|_{L^2(\Omega^\varepsilon)}$.
By regularity arguments one can show that

$$\left\|\frac{\partial^2\eta}{\partial x_i\partial x_j}\right\|_{L^2(\Omega^\varepsilon)} \le C\,\|F\|_{L^2(\Omega^\varepsilon)}, \quad i,j\in\{1,2\}.$$

As for θ, it should hold

$$\begin{aligned}\mathrm{Curl}(\theta)\cdot\nu &= -\nabla\theta\cdot\tau = -\nabla\eta\cdot\nu = 0 \text{ on } \partial\Omega^\varepsilon\\ \mathrm{Curl}(\theta)\cdot\tau &= \nabla\theta\cdot\nu = -\nabla\eta\cdot\tau \in H^{\frac{1}{2}}(\Omega^\varepsilon) \text{ on } \partial\Omega^\varepsilon.\end{aligned}$$

By the inverse trace Theorem 3.9 there exists a $\theta\in H^2(\Omega^\varepsilon)$ with $\nabla\theta\cdot\nu|_{\partial\Omega^\varepsilon} = -\nabla\eta\cdot\tau$ and $\theta|_{\partial\Omega^\varepsilon} = 0$ (thus especially $\nabla\theta\cdot\tau = 0$ on $\partial\Omega^\varepsilon$) and

$$\|\theta\|_{H^2(\Omega^\varepsilon)} \le C\,\|\nabla\eta\|_{H^1(\Omega^\varepsilon)}.$$

Now we have $\nabla\eta + \mathrm{Curl}(\theta) = 0$ on $\partial\Omega^\varepsilon$, therefore also $F(\nabla\eta + \mathrm{Curl}(\theta)) = 0$ on the boundary of Ω^ε. ♦

3.17 Remark.
In the sequel, we will use various analogues of this lemma. However, note that the boundedness of the domain is an important prerequisite.

To reintroduce the pressure, notice that by equation (3.7)

$$-\operatorname{div}(F^{-1}F^{-T}\,\nabla u^\varepsilon) - f \quad \in (H^1_{\mathrm{div}}(\Omega^\varepsilon)^2)^\perp.$$

By Lemma 3.14 there exists a pressure $p^\varepsilon \in L^2(\Omega^\varepsilon)$, unique up to a constant, such that

$$- \operatorname{div}(F^{-1}(x)F^{-T}(x) \nabla u^\varepsilon(x)) - f(x) = -F^{-T}(x) \nabla p^\varepsilon(x)$$

holds in Ω^ε. This finishes the considerations about the existence and uniqueness of the transformed Stokes equation.

We have the following regularity result:

3.18 Proposition.
If $f \in H^r(\Omega)^2$, $r \geq 0$, then $u^\varepsilon \in H^{r+2}(\Omega^\varepsilon)^2$ and $p^\varepsilon \in H^{r+1}(\Omega^\varepsilon)$.

We do not give a proof, which can be carried out by adapting the regularity arguments for the usual Stokes equation (see e.g. [Tem77]). For the interior of the domain, one can use the following argument:

Applying $\operatorname{div}(F^{-1}\cdot)$ to Equation (3.1a) gives (by the second formula of Lemma 2.7)

$$\operatorname{div}(F^{-1}F^{-T} \nabla p^\varepsilon) = \operatorname{div}(F^{-1}f) \quad \in H^{r-1}(\Omega^{\varepsilon\prime}).$$

Therefore $p^\varepsilon \in H^{r+1}(\Omega^{\varepsilon\prime})$, where $\Omega^{\varepsilon\prime}$ is a strictly included subdomain of Ω^ε. Because of

$$- \operatorname{div}(F^{-1}F^{-T} \nabla u^\varepsilon) = f - F^{-T} \nabla p^\varepsilon \quad \in H^r(\Omega^{\varepsilon\prime})^2,$$

we conclude that $u^\varepsilon \in H^{r+2}(\Omega^{\varepsilon\prime})^2$.

3.19 Remark.
A careful investigation of the foregoing section shows that a solution of (3.1) exists even for $f \in H^{-1}(\Omega^\varepsilon)^2$.

3.3 A-priori Estimates and Extensions

As usual in the homogenisation of the Stokes equation (see for example [SP80] or [All89]), we scale the velocity by ε^2 and look for $(u^\varepsilon, p^\varepsilon) \in H_0^1(\Omega^\varepsilon)^2 \times L^2(\Omega^\varepsilon)/\mathbb{R}$ such that

$$-\varepsilon^2 \operatorname{div}(F^{-1}(x)F^{-T}(x) \nabla u^\varepsilon(x)) + F^{-T}(x) \nabla p^\varepsilon(x) = f(x) \text{ in } \Omega^\varepsilon \tag{3.8a}$$

$$\operatorname{div}(F^{-1}(x)u^\varepsilon(x)) = 0 \text{ in } \Omega^\varepsilon \tag{3.8b}$$

$$u^\varepsilon(x) = 0 \text{ on } \Gamma^\varepsilon \tag{3.8c}$$

$$u^\varepsilon(x) = 0 \text{ on } \partial\Omega \tag{3.8d}$$

with weak formulation

$$\begin{aligned}\int_{\Omega^\varepsilon} \varepsilon^2 F^{-T}(x)\nabla u^\varepsilon(x) : F^{-T}(x)\nabla \phi(x)\,\mathrm{d}x - \int_{\Omega^\varepsilon} p^\varepsilon(x)\,\mathrm{div}(F^{-1}(x)\phi(x))\,\mathrm{d}x \\ = \int_{\Omega^\varepsilon} f(x)\cdot\phi(x)\,\mathrm{d}x\end{aligned} \tag{3.9}$$

for all $\phi \in H_0^1(\Omega^\varepsilon)^2$.

In order to be able to apply compactness results (like Theorems 3.3 or 3.4) we need estimates of $u^\varepsilon, p^\varepsilon$ on a fixed domain. As Ω^ε varies with ε, we have to define extensions of the velocity and the pressure. For the velocity, we simply extend u^ε by 0 in $\Omega\backslash\Omega^\varepsilon$. Note that $\mathrm{div}(F^{-1}u^\varepsilon) = 0$ still holds for this extension.

The extension of the pressure is more complicated. We adapt the construction of an extension operator, originally proposed by Tartar in [SP80] and later generalized by Allaire, see [All89]. The extension is defined by duality, using the following restriction operator:

3.20 Proposition.
There exists a linear restriction operator

$$\mathcal{R}^\varepsilon : H_0^1(\Omega)^2 \longrightarrow H_0^1(\Omega^\varepsilon)^2$$

such that for $w \in H_0^1(\Omega)^2$:

$$\begin{aligned} w = 0 \ \mathit{on}\ \partial\Omega^\varepsilon\backslash\partial\Omega \quad &\Longrightarrow \quad \mathcal{R}^\varepsilon w = w|_{\Omega^\varepsilon} \\ \mathrm{div}(F^{-1}w) = 0 \ \mathit{in}\ \Omega \quad &\Longrightarrow \quad \mathrm{div}(F^{-1}\mathcal{R}^\varepsilon w) = 0 \ \mathit{in}\ \Omega^\varepsilon \end{aligned}$$

and

$$\|\mathcal{R}^\varepsilon w\|_{L^2(\Omega^\varepsilon)^2} + \varepsilon\,\|\nabla(\mathcal{R}^\varepsilon w)\|_{L^2(\Omega^\varepsilon)^4} \le C(\|w\|_{L^2(\Omega)^2} + \varepsilon\,\|\nabla w\|_{L^2(\Omega)^4}).$$

For the proof we need several lemmas. We begin by repeating the usual definition of the restriction operator for the Stokes equation.

3.21 Lemma.
There exists a linear restriction operator

$$\tilde{\mathcal{R}}^\varepsilon : H_0^1(\Omega)^2 \longrightarrow H_0^1(\Omega^\varepsilon)^2$$

such that

$$\begin{aligned} w = 0 \ \textit{on} \ \partial\Omega^\varepsilon \backslash \partial\Omega \quad &\Longrightarrow \quad \tilde{\mathcal{R}}^\varepsilon w = w|_{\Omega^\varepsilon} \\ \operatorname{div}(w) = 0 \ \textit{in} \ \Omega \quad &\Longrightarrow \quad \operatorname{div}(\tilde{\mathcal{R}}^\varepsilon w) = 0 \ \textit{in} \ \Omega^\varepsilon \end{aligned}$$

and

$$\|\tilde{\mathcal{R}}^\varepsilon w\|^2_{L^2(\Omega^\varepsilon)^2} + \varepsilon^2 \|\nabla(\tilde{\mathcal{R}}^\varepsilon w)\|^2_{L^2(\Omega^\varepsilon)^4} \leq C(\|w\|^2_{L^2(\Omega)^2} + \varepsilon^2 \, \|\nabla \, w\|^2_{L^2(\Omega)^4}).$$

Proof. We construct a similar operator $\tilde{\mathcal{R}}$, defined in the reference cell, and proceed by rescaling and summation:

For given $v \in H^1_0(Y)^2$ we look for $v_{\mathcal{R}} \in H^1(Y^*)^2$, solution of

$$\begin{aligned} -\Delta v_{\mathcal{R}} + \nabla q &= -\Delta v \text{ in } Y^* \\ \operatorname{div}(v_{\mathcal{R}}) = \operatorname{div}(v) + \frac{1}{|Y^*|} \int_{Y_S} &\operatorname{div}(v(y)) \, \mathrm{d}y \text{ in } Y^* \\ v_{\mathcal{R}} &= 0 \text{ on } \partial Y_S \\ v_{\mathcal{R}} &= v \text{ on } \partial Y. \end{aligned}$$

Since

$$\int_{Y_F} \operatorname{div}(v_{\mathcal{R}}) \, \mathrm{d}y = \int_{\partial Y_F} v_{\mathcal{R}} \cdot \nu \, \mathrm{d}\sigma_y,$$

the standard theory for the Stokes equation (see e.g. [Tem77]) yields the existence of a velocity v_R such that

$$\|v_{\mathcal{R}}\|_{H^1(Y^*)^2} \leq C \, \|v\|_{H^1(Y)^2} \, .$$

Now define $\tilde{\mathcal{R}} v := v_{\mathcal{R}}$. Thus $\tilde{\mathcal{R}} : H^1_0(Y)^2 \longrightarrow H^1(Y^*)^2$ is a continuous linear operator.

Rescaling $\tilde{\mathcal{R}}$ to the cell εY^k and applying it to each cell in Ω we obtain the desired operator $\tilde{\mathcal{R}}^\varepsilon : H^1_0(\Omega)^2 \longrightarrow H^1_0(\Omega^\varepsilon)^2$ where – by the usual rescaled estimates for the gradient – we obtain

$$\|\tilde{\mathcal{R}}^\varepsilon w\|^2_{L^2(\Omega^\varepsilon)^2} + \varepsilon^2 \|\nabla(\tilde{\mathcal{R}}^\varepsilon w)\|^2_{L^2(\Omega^\varepsilon)^4} \leq C(\|w\|^2_{L^2(\Omega)^2} + \varepsilon^2 \, \|\nabla \, w\|^2_{L^2(\Omega)^4}). \tag{3.10}$$

♦

We are now able to show the existence of $\mathcal{R}^\varepsilon$:

Proof of Proposition 3.20. Let $w \in H^1(\Omega)^2$ be given and define

$$\mathcal{R}^\varepsilon w = F \tilde{\mathcal{R}}^\varepsilon (F^{-1} w).$$

This gives

$$\begin{aligned}
w = 0 \text{ on } \partial\Omega^\varepsilon \backslash \partial\Omega \quad &\Longrightarrow \quad F^{-1}w = 0 \text{ on } \partial\Omega^\varepsilon \backslash \partial\Omega \\
&\Longrightarrow \quad \tilde{\mathcal{R}}^\varepsilon(F^{-1}w) = F^{-1}w|_{\Omega^\varepsilon} \\
&\Longrightarrow \quad \mathcal{R}^\varepsilon w = w|_{\Omega^\varepsilon} \\
\operatorname{div}(F^{-1}w) = 0 \text{ in } \Omega \quad &\Longrightarrow \quad \operatorname{div}(\tilde{\mathcal{R}}^\varepsilon(F^{-1}w)) = 0 \text{ in } \Omega^\varepsilon \\
&\Longrightarrow \quad \operatorname{div}(F^{-1}R^\varepsilon w) = 0 \text{ in } \Omega^\varepsilon
\end{aligned}$$

and

$$\begin{aligned}
\|\mathcal{R}^\varepsilon w\|^2_{L^2(\Omega^\varepsilon)^2} + \varepsilon^2 \|\nabla(\mathcal{R}^\varepsilon w)\|^2_{L^2(\Omega^\varepsilon)^4} &\leq C(\left\|F^{-1}w\right\|^2_{L^2(\Omega)^2} + \varepsilon^2 \left\|\nabla(F^{-1}w)\right\|^2_{L^2(\Omega)^4}) \\
&\leq C(\|w\|^2_{L^2(\Omega)^2} + \varepsilon^2 \|\nabla w\|^2_{L^2(\Omega)^4}),
\end{aligned}$$

since the second derivatives of the entries of F^{-1} are bounded. ♦

We can now define the extension of the pressure by duality: Let $q^\varepsilon \in H^{-1}(\Omega)^2$ fulfill

$$\langle q^\varepsilon, w\rangle_{H^{-1}(\Omega)^2,\, H^1_0(\Omega)^2} = \langle F^{-T}\nabla p^\varepsilon, \mathcal{R}^\varepsilon w\rangle_{H^{-1}(\Omega^\varepsilon)^2,\, H^1_0(\Omega^\varepsilon)^2} \quad \forall w \in H^1_0(\Omega).$$

Since $\mathcal{R}^\varepsilon$ is continuous and linear from $H^1_0(\Omega)^2$ to $H^1_0(\Omega^\varepsilon)^2$, we conclude that $q^\varepsilon \in H^{-1}(\Omega)^2$ (q^ε can be interpreted as an extension of $F^{-T}\nabla p^\varepsilon$). Next choose an arbitrary $w \in H^1_0(\Omega)^2$ such that $\operatorname{div}(F^{-1}w) = 0$. Then

$$\langle q^\varepsilon, w\rangle_{H^{-1}(\Omega)^2,\, H^1_0(\Omega)^2} = \langle F^{-T}\nabla p^\varepsilon, \mathcal{R}^\varepsilon w\rangle_{H^{-1}(\Omega^\varepsilon)^2,\, H^1_0(\Omega^\varepsilon)^2} = \int_{\Omega^\varepsilon} p^\varepsilon \operatorname{div}(F^{-1}\mathcal{R}^\varepsilon w)\, \mathrm{d}x = 0.$$

Thus $q^\varepsilon \perp w$ and therefore, by Lemma 3.14, $q^\varepsilon = F^{-T}\nabla \tilde{p}^\varepsilon$ with a $\tilde{p}^\varepsilon \in L^2(\Omega)$. Here $\tilde{p}^\varepsilon$ denotes the extended pressure.

In the sequel, we will denote the extensions of u^ε and p^ε by the same symbols.

3.22 Lemma.
For the extended velocity u^ε and pressure p^ε it holds

$$\|u^\varepsilon\|_{L^2(\Omega)^2} + \varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega)^4} \leq C$$

and

$$\|p^\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq C.$$

Proof. Choosing $\phi = u^\varepsilon$ in (3.9) and using the coercivity of the bilinear form gives

$$\varepsilon^2 k_F \|\nabla u^\varepsilon\|^2_{L^2(\Omega)^4} \leq \|f\|_{L^2(\Omega)^2} \|u^\varepsilon\|_{L^2(\Omega)^2} \leq C\varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega)^4},$$

where the Poincaré inequality has been used, giving an additional ε. Thus $\varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega)^4} \leq C$. Use of the Poincaré inequality $\|u^\varepsilon\|_{L^2(\Omega)^2} \leq C\varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega)^4}$ again finishes the estimate of the velocity.

For the estimation of the pressure note that, if $w \in H_0^1(\Omega)^2$,

$$\begin{aligned}\langle F^{-T} \nabla p^\varepsilon, w\rangle_{H^{-1}(\Omega)^2,\, H_0^1(\Omega)^2} &= \langle F^{-T} \nabla p^\varepsilon, \mathcal{R}^\varepsilon w\rangle_{H^{-1}(\Omega^\varepsilon)^2,\, H_0^1(\Omega^\varepsilon)^2} \\ &= -\int\limits_{\Omega^\varepsilon} f \cdot \mathcal{R}^\varepsilon w \,\mathrm{d}x + \varepsilon^2 \int\limits_{\Omega^\varepsilon} F^{-T} \nabla u^\varepsilon : F^{-T} \nabla(\mathcal{R}^\varepsilon w) \,\mathrm{d}x,\end{aligned}$$

thus

$$\begin{aligned}|\langle F^{-T} \nabla p^\varepsilon, w\rangle_{H^{-1}(\Omega)^2,\, H_0^1(\Omega)^2}| &\leq C\Big(\|f\|_{L^2(\Omega)^2} \|\mathcal{R}^\varepsilon w\|_{L^2(\Omega)^2} \\ &\qquad + \varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega)^4}\, \varepsilon \|\nabla(\mathcal{R}^\varepsilon w)\|_{L^2(\Omega)^4}\Big) \\ &\leq C\big(\|f\|_{L^2(\Omega)^2} + \varepsilon \|\nabla u^\varepsilon\|_{L^2(\Omega)^4}\big)\big(\|w\|_{L^2(\Omega)^2} + \varepsilon \|\nabla w\|_{L^2(\Omega)^4}\big) \\ &\leq C(\|w\|_{L^2(\Omega)^2} + \varepsilon \|\nabla w\|_{L^2(\Omega)^4}),\end{aligned}$$

and $F^{-T} \nabla p^\varepsilon$ is bounded in $H^{-1}(\Omega)^2$.

Since $\|p^\varepsilon\|_{L^2(\Omega)/\mathbb{R}} \leq C \|\nabla p^\varepsilon\|_{H^{-1}(\Omega)^2} \leq C \left\|F^{-T} \nabla p^\varepsilon\right\|_{H^{-1}(\Omega)^2}$, we see that p^ε is bounded by a constant independent of ε as well (cf. [Tem77], Proposition 1.2 in Chapter 1). ♦

3.23 Remark.
For further properties about the extended pressure see Hornung et al., [Hor97]: For example, one can show that p^ε equals its extension in Ω^ε, that the weakly convergent subsequences of p^ε are actually strongly convergent; and the derivation of an explicit formula for the extension is possible as well (see also the original work of Lipton and Avellaneda, [LA90]).

3.4 The Limit Equations

Using Theorems 3.3 and 3.4, we see that due to the foregoing lemma there exist a $u_0 \in L^2(\Omega, H^1_\#(Y))^2$ and a $p_0 \in L^2(\Omega \times Y)$ such that along a subsequence (still denoted by ε)

$$u^\varepsilon \xrightarrow{2} u_0 \tag{3.11}$$

$$\varepsilon \nabla u^\varepsilon \xrightarrow{2} \nabla_y u_0 \tag{3.12}$$

$$p^\varepsilon \xrightarrow{2} p_0. \tag{3.13}$$

These limits are characterised in the following theorem:

3.24 Theorem.
The two-scale limits u_0, p_0 are solutions of the homogenised problem

$$\begin{aligned}
&F^{-T}(x)\,\nabla_y\, p_1(x,y) + F^{-T}(x)\,\nabla_x\, p_0(x) && \\
&\qquad - \operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\, u_0(x,y)) = f(x) && \text{in } \Omega\times Y^* \\
&\operatorname{div}_y(F^{-1}(x)u_0(x,y)) = 0 && \text{in } \Omega\times Y^* \\
&\operatorname{div}_x\Big(F^{-1}(x)\int_Y u_0(x,y)\;\mathrm{d}y\Big) = 0 && \text{in } \Omega \\
&u_0(x,y) = 0 && \text{in } \Omega\times Y_S \\
&\Big(\int_Y F^{-1}(x)u_0(x,y)\;\mathrm{d}y\Big)\cdot\nu = 0 && \text{on } \partial\Omega \\
&u_0(x), p_1(x) \text{ are } Y\text{-periodic in } y &&
\end{aligned}$$

We will call this problem the homogenised transformed Stokes equation.

Proof. The proof will be done in several steps:

Step 1) p_0 does not depend on y:

Inserting $\varepsilon\phi(x,\frac{x}{\varepsilon})$ with $\phi \in \mathcal{C}_0^\infty(\Omega, C_\#^\infty(Y))^2$ as a test function in (3.9) yields by the boundedness of f and $\varepsilon\,\nabla\, u^\varepsilon$

$$0 = \lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon} p^\varepsilon(x)\operatorname{div}(F^{-1}(x)\varepsilon\phi(x,\frac{x}{\varepsilon}))\;\mathrm{d}x = \int_\Omega\int_Y p_0(x,y)\operatorname{div}_y(F^{-1}(x)\phi(x,y))\;\mathrm{d}y\;\mathrm{d}x.$$

An integration by parts shows that $F^{-T}(x)\,\nabla_y(p_0(x,y)) = 0$, thus p_0 does not depend on y.

Step 2) Additional conditions:

Test equation (3.8b) with $\phi \in \mathcal{C}_0^\infty(\Omega)$:

$$\begin{aligned}
0 &= \lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon}\operatorname{div}(F^{-1}(x)u^\varepsilon(x))\phi(x)\;\mathrm{d}x = -\lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon} u^\varepsilon(x)\cdot(F^{-T}(x)\,\nabla\,\phi(x))\;\mathrm{d}x \\
&= -\int_\Omega\int_Y u_0(x,y)\cdot F^{-T}(x)\,\nabla_x\,\phi(x)\;\mathrm{d}y\;\mathrm{d}x \\
&= \int_\Omega \operatorname{div}_x\Big(\int_Y F^{-1}(x)u_0(x,y)\Big)\phi(x)\;\mathrm{d}y\;\mathrm{d}x.
\end{aligned}$$

Thus $\operatorname{div}_x(F^{-1}(x)\int_Y u_0(x,y)\;\mathrm{d}y) = 0$ in Ω.

Choosing $\phi \in \mathcal{C}_0^\infty(\Omega, C_\#^\infty(Y))$, $\phi = 0$ in $\Omega \times Y_S$ and testing similarly with $\varepsilon\phi(x, \frac{x}{\varepsilon})$, one obtains

$$
\begin{aligned}
0 &= \lim_{\varepsilon\to 0} \int\limits_{\Omega^\varepsilon} \operatorname{div}(F^{-1}(x)u^\varepsilon(x)) \cdot \varepsilon\phi(x, \frac{x}{\varepsilon}) \, \mathrm{d}x \\
&= -\lim_{\varepsilon\to 0} \int\limits_{\Omega^\varepsilon} u^\varepsilon(x) \cdot F^{-T}(x)\Big(\varepsilon \nabla_x \phi(x, \frac{x}{\varepsilon}) + \nabla_y \phi(x, \frac{x}{\varepsilon})\Big) \, \mathrm{d}x \\
&= -\int\limits_\Omega \int\limits_{Y^*} u_0(x,y) \cdot F^{-T}(x) \nabla_y \phi(x,y) \, \mathrm{d}y \, \mathrm{d}x \\
&= \int\limits_\Omega \int\limits_{Y^*} \operatorname{div}_y(F^{-1}(x)u_0(x,y)) \cdot \phi(x,y) \, \mathrm{d}y \, \mathrm{d}x.
\end{aligned}
$$

We conclude that $\operatorname{div}_y(F^{-1}(x)u_0(x,y)) = 0$ in $\Omega \times Y_F$.

Now testing with $\phi \in \mathcal{C}^\infty(\overline{\Omega})$ gives due to $u^\varepsilon = 0$ on $\partial\Omega^\varepsilon$:

$$
\begin{aligned}
0 &= \lim_{\varepsilon\to 0} \int\limits_{\Omega^\varepsilon} \operatorname{div}(F^{-1}(x)u^\varepsilon(x))\phi(x) \, \mathrm{d}x = -\lim_{\varepsilon\to 0} \int\limits_{\Omega^\varepsilon} u^\varepsilon(x) \cdot F^{-T}(x) \nabla \phi(x) \, \mathrm{d}x \\
&= -\int\limits_\Omega \Big(\int\limits_Y u_0(x,y) \, \mathrm{d}y\Big) \cdot F^{-T}(x) \nabla_x \phi(x) \, \mathrm{d}y \, \mathrm{d}x \\
&= \int\limits_\Omega \operatorname{div}_x\Big(F^{-1}(x) \int\limits_Y u_0(x,y) \, \mathrm{d}y\Big)\phi(x) \, \mathrm{d}y \, \mathrm{d}x \\
&\quad - \int\limits_{\partial\Omega} F^{-1}(x)\Big(\int\limits_Y u_0(x,y) \, \mathrm{d}y\Big) \cdot \nu\phi(x) \, \mathrm{d}\sigma_x.
\end{aligned}
$$

Since $\operatorname{div}_x(F^{-1}(x) \int_Y u_0(x,y) \, \mathrm{d}y) = 0$, we obtain $(\int_Y F^{-1}(x)u_0(x,y) \, \mathrm{d}y) \cdot \nu = 0$ on $\partial\Omega$.

Finally, let $\phi \in \mathcal{C}_0^\infty(\Omega, \mathcal{C}_\#^\infty(Y))$, $\phi = 0$ in $\Omega \times Y^*$. By

$$
0 = \lim_{\varepsilon\to 0} \int\limits_\Omega u^\varepsilon(x)\phi(x, \frac{x}{\varepsilon}) \, \mathrm{d}x = \int\limits_\Omega \int\limits_Y u_0(x,y)\phi(x,y) \, \mathrm{d}y \, \mathrm{d}x = \int\limits_\Omega \int\limits_{Y_S} u_0(x,y)\phi(x,y) \, \mathrm{d}y \, \mathrm{d}x
$$

we get $u_0(x,y) = 0$ in $\Omega \times Y_S$.

Step 3) Obtaining the limit equations:

Choose a test function $\phi \in \mathcal{C}_0^\infty(\Omega, C_\#^\infty(Y))^2$ such that $\phi(x,y) = 0$ in $\Omega \times Y_S$, $\operatorname{div}_x(F^{-1}(x) \int_Y \phi(x,y) \, \mathrm{d}y) = 0$ in Ω and $\operatorname{div}_y(F^{-1}(x)\phi(x,y)) = 0$ in $\Omega \times Y$. Inserting $\phi(x, \frac{x}{\varepsilon})$ in (3.9) yields

$$\int_\Omega F^{-T}(x)\varepsilon\, \nabla u^\varepsilon(x) : F^{-T}(x)\, \nabla_y\, \phi(x, \frac{x}{\varepsilon})\ \mathrm{d}x - \int_\Omega p^\varepsilon(x)\, \mathrm{div}_x(F^{-1}(x)\phi(x, \frac{x}{\varepsilon}))\ \mathrm{d}x$$
$$= \int_\Omega f(x)\cdot \phi(x, \frac{x}{\varepsilon})\ \mathrm{d}x + \mathscr{O}(\varepsilon).$$

It holds

$$\lim_{\varepsilon\longrightarrow 0}\int_\Omega p^\varepsilon(x)\, \mathrm{div}_x(F^{-1}(x)\phi(x, \frac{x}{\varepsilon}))\ \mathrm{d}x = \int_\Omega p_0(x)\, \mathrm{div}_x\Big(F^{-1}(x)\int_Y \phi(x,y)\ \mathrm{d}y\Big)\ \mathrm{d}x = 0,$$

and since $\varepsilon\, \nabla\, u^\varepsilon \xrightarrow{2} \nabla_y\, u_0$, the limit equation for $\varepsilon \longrightarrow 0$ is given by

$$\int_\Omega\int_Y F^{-T}(x)\, \nabla_y\, u_0(x,y) : F^{-T}(x)\, \nabla_y\, \phi(x,y)\ \mathrm{d}y\ \mathrm{d}x = \int_\Omega\int_Y f(x)\cdot\phi(x,y)\ \mathrm{d}y\ \mathrm{d}x. \tag{3.14}$$

By density of the test functions, this equation holds in the space

$$\mathcal{U} = \Big\{\phi \in L^2(\Omega, H^1_\#(Y))^2\ \Big|\ \mathrm{div}_x\Big(F^{-1}(x)\int_Y \phi(x,y)\ \mathrm{d}y\Big) = 0 \text{ in } \Omega, \phi(x,y) = 0 \text{ in } \Omega\times Y_S,$$
$$\mathrm{div}_y(F^{-1}(x)\phi(x,y)) = 0 \text{ in } \Omega\times Y,\ \text{and } \Big(\int_Y F^{-1}(x)\phi(x,y)\ \mathrm{d}y\Big)\cdot\nu = 0 \text{ on } \partial\Omega\Big\}.$$

Finally, we need to reintroduce the pressure: Analogously to [Hor97] one can show that

$$\mathcal{U}^\perp = \big\{F^{-T}(x)\, \nabla_x\, q(x) + F^{-T}(x)\, \nabla_y\, q_1(x,y) \mid q \in H^1(\Omega)/\mathbb{R}, q_1 \in L^2(\Omega, L^2_\#(Y))/\mathbb{R}\big\},$$

giving the existence of $p_* \in H^1(\Omega)/\mathbb{R}, p_1 \in L^2(\Omega, L^2_\#(Y))/\mathbb{R}$ such that

$$\begin{aligned} &-\mathrm{div}_y(F^{-1}(x)F^{-T}(x)\, \nabla_y\, u_0(x,y)) - f(x) \\ &\qquad = -F^{-T}(x)\, \nabla_x\, p_*(x) - F^{-T}(x)\, \nabla_y\, p_1(x,y) \end{aligned} \tag{3.15}$$

in $\Omega\times Y^*$.

Step 4) Convergence of the pressure:

What remains to be shown is that the pressure p_* equals the limit p_0. Choose an arbitrary $\phi \in \mathcal{C}^\infty_0(\Omega, C^\infty_\#(Y))^2$ such that $\mathrm{div}_y(F^{-1}(x)\phi(x,y)) = 0$ in $\Omega\times Y$. Inserting $\phi(x, \frac{x}{\varepsilon})$ as test function in (3.9) and passing to the limit yields

$$\int_\Omega \int_Y F^{-T}(x)\,\nabla_y\, u_0(x,y) : F^{-T}(x)\,\nabla_y\,\phi(x,y)\;\mathrm{d}y\;\mathrm{d}x \\ - \int_\Omega \int_Y p_0(x)\,\mathrm{div}_x(F^{-1}(x)\phi(x,y))\;\mathrm{d}y\;\mathrm{d}x = \int_\Omega \int_Y f(x)\cdot\phi(x,y)\;\mathrm{d}y\;\mathrm{d}x.$$

Integrating the first term on the left hand side by parts and inserting equation (3.15), one obtains

$$\int_\Omega \int_Y F^{-T}(x)\,\nabla_x\, p_0(x)\cdot\phi(x,y)\;\mathrm{d}y\;\mathrm{d}x = \int_\Omega \int_Y F^{-T}(x)\,\nabla_x\, p_*(x)\cdot\phi(x,y)\;\mathrm{d}y\;\mathrm{d}x \\ + \int_\Omega \int_Y F^{-T}(x)\,\nabla_y\, p_1(x,y)\cdot\phi(x,y)\;\mathrm{d}y\;\mathrm{d}x.$$

Again, via an integration by parts we see that the last term on the right hand side vanishes since $\mathrm{div}_y(F^{-1}(x)\phi(x,y)) = 0$. Therefore

$$\nabla_x\, p_0 = \nabla_x\, p_*,$$

and we obtain the equality of p_* and p_0 modulo a constant. This finishes the proof of the theorem. ♦

3.25 Remark.
By using estimates similar to those in the proof of Proposition 3.13, one can show the existence and uniqueness (up to constants for the pressures) of a solution of the system in Theorem 3.24. Thus the whole sequence $(u^\varepsilon, p^\varepsilon)$ *converges.*

In order to eliminate the dependence of the homogenised transformed Stokes equation on the variable in Y, we introduce the following cell problem: For $i = 1, 2$ and fixed $x \in \Omega$ let $w_x^i \in H^1_\#(Y)^2$ and $\pi_x^i \in L^2_\#(Y)/\mathbb{R}$ be the solution of

$$\begin{aligned} -\,\mathrm{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\, w_x^i(y)) + F^{-T}(x)\,\nabla_y\,\pi_x^i(y) &= F^{-T}(x)e_i && \text{in } Y^* && (3.16a)\\ \mathrm{div}_y(F^{-1}(x)w_x^i(y)) &= 0 && \text{in } Y^* && (3.16b)\\ w_x^i(y) &= 0 && \text{in } Y_S && (3.16c)\\ w_x^i, \pi_x^i \text{ are } Y\text{-periodic in } y & && && (3.16d) \end{aligned}$$

3.26 Remark.
Again, existence and uniqueness of solutions of the cell problem follow similar to Section 3.2. Note that the Problem (3.16) *corresponds to a transformed Stokes equation, where the*

transformation is given by

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \longmapsto \begin{pmatrix} y_1 \\ y_2 + g'(x_1)y_1 \end{pmatrix}.$$

Thus x plays the role of a parameter. This transformation is also volume-preserving; therefore analogues of Lemmas 2.5 and 2.7 hold.

In the sequel, we write $w^i(x,y) := w^i_x(y)$ and $\pi^i(x,y) := \pi^i_x(y)$. For the differentiability properties of these functions in x-direction see the next section.

Now define

$$u_0(x,y) = \sum_{i=1}^{2} \Big(\big(F^T(x)f(x)\big)_i - \frac{\partial p_0}{\partial x_i} \Big) w^i(x,y) \tag{3.17}$$

$$p_1(x,y) = \sum_{i=1}^{2} \Big(\big(F^T(x)f(x)\big)_i - \frac{\partial p_0}{\partial x_i} \Big) \pi^i(x,y). \tag{3.18}$$

A simple calculation shows that u_0 and p_1 fulfill the homogenised transformed Stokes equation. Define $u(x) := \int_Y u_0(x,y)\,\mathrm{d}y$ and the matrix $A(x)$ by $(A(x))_{ij} = \int_Y w^i_j(x,y)\,\mathrm{d}y$, then $u^\varepsilon \rightharpoonup u$ in $L^2(\Omega)$ weakly (cf. Lemma 3.6) with

$$u(x) = A(x)(F^T(x)f(x) - \nabla p_0(x)) \text{ in } \Omega \tag{3.19a}$$

$$\operatorname{div}(F^{-1}(x)u(x)) = 0 \text{ in } \Omega \tag{3.19b}$$

$$u(x) \cdot F^{-T}(x)\nu(x) = 0 \text{ on } \partial\Omega \tag{3.19c}$$

3.27 Remark.
By application of the inverse coordinate transformation to the above system of equations we see that u satisfies a Darcy law with a non-constant permeability tensor.

We conclude this section by proving some properties of the matrix $F^{-1}A$:

3.28 Lemma.
Fix $x \in \Omega$. Then the matrix $F^{-1}(x)A(x)$ with A defined as above is symmetric and positive definite.

Proof. The weak formulation of the cell problem (3.16) reads

$$\int_{Y^*} F^{-T}(x)\,\nabla_y\, w^i_x : F^{-T}(x)\,\nabla_y\,\phi\;\mathrm{d}y = \int_{Y^*} \phi \cdot F^{-T}(x)e_i\;\mathrm{d}y = \int_{Y^*} (F^{-1}(x)\phi)\cdot e_i\;\mathrm{d}y \tag{3.20}$$

with test functions $\phi \in H^1_\#(Y)^2$ such that $\phi = 0$ on ∂Y_S, ϕ Y-periodic and $\operatorname{div}_y(F^{-1}(x)\phi) = 0$. Choosing w^j_x as test function in the above equation and similarly w^i_x

as test function in the weak formulation for w_x^j leads to

$$\int_{Y^*} (F^{-1}(x) w_x^j) \cdot e_i \, \mathrm{d}y = \int_{Y^*} (F^{-1}(x) w_x^i) \cdot e_j \, \mathrm{d}y,$$

which is equivalent to $(F^{-1}(x)A_j(x)) \cdot e_i = (F^{-1}(x)A_i(x)) \cdot e_j$ (A_i denotes the i-th column of A). Thus $F^{-1}(x)A(x)$ is symmetric.

Next, choose a $\xi \in \mathbb{R}^2$. Then it holds due to equation (3.20)

$$\begin{aligned}
\xi^T(F^{-1}(x)A(x))\xi &= \sum_{i,j} (F^{-1}(x)A(x))_{ij}\xi_i\xi_j = \sum_{i,j} \Big(\int_{Y^*} (F^{-1}(x) w_x^i) \cdot e_j \, \mathrm{d}y \Big) \xi_i \xi_j \\
&= \sum_{i,j} \Big(\int_{Y^*} F^{-T}(x)\xi_i \, \nabla_y \, w_x^i : F^{-T}(x)\xi_j \, \nabla_y \, w_x^j \, \mathrm{d}y \Big) \\
&= \Big\| \sum_i \big(\xi_i F^{-T}(x) \, \nabla_y \, w_x^i \big) \Big\|^2_{L^2(Y^*)^4} \qquad \geq 0.
\end{aligned}$$

Therefore $F^{-1}(x)A(x)$ is positive. As for the positive definiteness, keeping in mind the last equation we have to show the statement $\sum_i (\xi_i F^{-T}(x) \, \nabla_y \, w_x^i) = 0$ a.e. $\Longrightarrow$ $\xi = 0$. Choose a test function ϕ with $\int_{Y^*} \phi \, \mathrm{d}y = F(x)\xi$. (This can be achieved by first constructing ϕ_1 and ϕ_2, Y-periodic solutions of a transformed Stokes flow in Y^* with $\phi_i = 0$ on ∂Y_S, $i = 1, 2$ and given forces f_1 and f_2, such that the vectors $(\int_{Y^*} \phi_1 \, \mathrm{d}y)$ and $(\int_{Y^*} \phi_2 \, \mathrm{d}y)$ are linearly independent. Then define ϕ as a suitable linear combination of ϕ_1 and ϕ_2.) Multiplying (3.20) with ξ_i, one obtains

$$\int_{Y^*} \xi_i F^{-T}(x) \, \nabla_y \, w_x^i : F^{-T}(x) \, \nabla_y \, \phi \, \mathrm{d}y = \int_{Y^*} \xi_i (F^{-1}(x)\phi) \cdot e_i \, \mathrm{d}y = \int_{Y^*} \xi_i \xi_i = |Y^*| \xi_i^2.$$

By summation over i, we see that

$$\sum_i \big(\xi_i F^{-T}(x) \, \nabla_y \, w_x^i \big) = 0 \text{ a.e.} \quad \Longrightarrow \quad \xi_i = 0, \; i = 1, 2,$$

which finishes the proof. ♦

3.29 Remark.
By the definition of the matrix F and the functions w_x^i, the matrix $F^{-1}A$ depends on x only via the x_1-variable, in which it is L-periodic and continuous (see also the next section). By the foregoing lemma, the minimal eigenvalue $\lambda_{\min}(x)$ of $F^{-1}(x)A(x)$ fulfills $\lambda_{\min}(x) > 0$. Since the eigenvalues of a matrix depend continuously on its entries, we see that also

$$\inf_{x \in [0,L]} \lambda_{\min}(x) := C > 0.$$

Analogously, one can find a uniform bound on the greatest eigenvalue $\lambda_{\max}(x)$ *of* $F^{-1}(x)A(x)$, $x \in [0, L]$.

Thus (3.19) *represents an elliptic partial differential equation for* p_0 *with a Neumann boundary condition; and using arguments similar to those in Subsection 3.2, one obtains a solution which is unique up to constants.*

4 Parameter-dependent PDEs

We consider the differentiability of a solution of a partial differential equation with respect to a parameter. The main tool will be the Implicit Function Theorem. Note that in this section we will mostly deal with total derivatives of functions between Banach spaces, i.e. for X, Y Banach spaces and $\mathcal{F} : X \longrightarrow Y$ the total derivative $D\mathcal{F}$ of $\mathcal{F}$ is a bounded linear operator, $D\mathcal{F} \in L(X, Y)$.

In the sequel, let Ω be a domain in $\mathbb{R}^2$ and let Y^* be the fluid part of the reference cell as in the preceding section.

4.1 Theorem (Implicit Function Theorem)**.**
Let X, Y and Z be Banach spaces over $\mathbb{R}$. Let $\mathcal{F} : U(x_0, y_0) \subseteq X \times Y \longrightarrow Z$ be a mapping defined on an open neighbourhood $U(x_0, y_0)$ of $x_0 \in X, y_0 \in Y$ with $\mathcal{F}(x_0, y_0) = 0$. Assume that the total derivative in y-direction $D_y\mathcal{F}$ exists in $U(x_0, y_0)$, and $((D_y\mathcal{F})(x_0, y_0))^{-1}$ exists as a continuous linear operator. Assume also that $\mathcal{F}$ and $D_y\mathcal{F}$ are continuous in (x_0, y_0).

Then the following holds:

1. *There exist $r_0, r > 0$ such that: For all $x \in X$ with $\|x - x_0\|_X \leq r_0$ there exists exactly one $y(x) \in Y$ with $\mathcal{F}(x, y(x)) = 0$ and $\|y(x) - y_0\|_Y \leq r$.*
2. *If $\mathcal{F}$ is m-times continuously differentiable in a neighbourhood of (x_0, y_0), then $y(\cdot)$ is also m-times continuously differentiable in a neighborhood of x_0.*
3. *For the derivative $D_xy(x)$ it holds*

$$D_xy(x) = -D_y\mathcal{F}(x, y(x))^{-1} \circ D_x\mathcal{F}(x, y(x)). \tag{4.1}$$

Proof. See Zeidler [Zei86], Theorem 4.B. ♦

4.1 Application to the Divergence Operator

As a first simple application of the above theorem, we consider a problem similar to one dealing with a divergence-correction, see Section 5.4.1.

Let $h \in \mathcal{C}^m(\Omega, L^2_{0,\#}(Y^*))$ for an $m \in \mathbb{N}$ and let F be given by (2.1). For fixed $x \in \Omega$ consider the problem:

Find $\gamma(x) \in H^1_{0,\#}(Y^*)^2$ with

$$\operatorname{div}_y\big(F^{-1}(x)\gamma(x)\big) = h(x) \text{ in } Y^* \tag{4.2a}$$

$$\gamma(x) = 0 \text{ on } \partial Y_S \tag{4.2b}$$

$$\gamma(x) \text{ is } Y\text{-periodic in } y \tag{4.2c}$$

The operator $\operatorname{div}_y(F^{-1}(x)\cdot)$ maps from $H^1_{0,\#}(Y^*)^2$ to $L^2_{0,\#}(Y^*)$, cf. the proof of Lemma 3.16. Therefore define an operator

$$\begin{gathered}\mathcal{D} : \mathbb{R}^2 \times H^1_{0,\#}(Y^*)^2 \longrightarrow L^2_{0,\#}(Y^*) \\ \mathcal{D}(x,u) = \operatorname{div}_y(F^{-1}(x)u) - h(x)\end{gathered}$$

For a solution $\gamma(x)$ of (4.2) it holds $\mathcal{D}(x,\gamma(x)) = 0$.

We check the requirements of the Implicit Function Theorem in the following lemmas:

4.2 Lemma.
Assume that F as defined in (2.1) is in $\mathcal{C}^m(\mathbb{R})^4$, $m \in \mathbb{N}$. Then the operator $\mathcal{D}$ defined above is continuous.

Proof. Let $x_n \to x$ in $\mathbb{R}^2$ and $u_n \to u$ in $H^1_{0,\#}(Y^*)^2$. Then it holds

$$\begin{aligned}\|\mathcal{D}(x_n,u_n) - \mathcal{D}(x,u)\|_{L^2(Y^*)} &\leq \Big\|\operatorname{div}_y\Big(F^{-1}(x_n)\big(u_n - u\big) - \big(F^{-1}(x_n) - F^{-1}(x)\big)u\Big)\Big\|_{L^2(Y^*)} \\ &\quad + \|h(x_n) - h(x)\|_{L^2(Y^*)} \\ &\leq C \max_{i,j=1,2} |(F^{-1}(x_n))_{ij}| \, \|\nabla_y(u_n - u)\|_{L^2(Y^*)} \\ &\quad + C \max_{i,j=1,2} |(F^{-1}(x_n) - F^{-1}(x))_{ij}| \, \|\nabla u\|_{L^2(Y^*)} \\ &\quad + \|h(x_n) - h(x)\|_{L^2(Y^*)} \\ &\longrightarrow 0.\end{aligned}$$

♦

The operator $\operatorname{div}_y(F^{-1}(x)\cdot)$ is linear and continuous, thus its total derivative is the same operator. Hence we obtain for the derivative of $\mathcal{D}$ in u-direction

$$D_u\mathcal{D}(x,u)[\omega] = \operatorname{div}_y(F^{-1}(x)\omega).$$

4.3 Lemma.
Consider the situation as above. Then $D_u\mathcal{D}$ is continuous and $D_u\mathcal{D}(x,u)$ is continuously invertible.

Proof. Choose sequences $x_n \to x$ in $\mathbb{R}^2$ and $u_n \to u$ in $H^1_{0,\#}(Y^*)^2$. Then it holds

$$\begin{aligned}\big\|\big(D_u\mathcal{D}(x_n,u_n) - D_u\mathcal{D}(x,u)\big)[\omega]\big\|_{L^2_{0,\#}(Y^*)} &= \big\|\operatorname{div}_y\big([F^{-1}(x_n) - F^{-1}(x)]\omega\big)\big\|_{L^2_{0,\#}(Y^*)} \\ &\leq C \max_{i,j=1,2} \big|\big(F^{-1}(x_n) - F^{-1}(x)\big)_{ij}\big| \, \|\nabla \omega\|_{L^2(Y^*)}.\end{aligned}$$

Therefore the operator norm of the difference $D_u\mathcal{D}(x_n,u_n) - D_u\mathcal{D}(x,u)$ is bounded by $C \max_{i,j=1,2} |F^{-1}(x_n) - F^{-1}(x)| \longrightarrow 0$ for $n \to \infty$. Hence $D_u\mathcal{D}$ is continuous.

To see the invertibility, consider the equation $D_u\mathcal{D}(x,u)[\omega] = f$ for given $f \in L^2_{0,\#}(Y^*)$. This corresponds to the problem

$$\begin{aligned} \operatorname{div}_y(F^{-1}(x)\omega) &= f \text{ in } Y^* \\ \omega &= 0 \text{ on } \partial Y_S \\ \omega \text{ is } Y\text{-periodic in } y, \end{aligned}$$

which has a unique solution $\omega \in H^1_{0,\#}(Y^*)$ with

$$\|\nabla\,\omega\|_{L^2(Y^*)} \le C\,\|f\|_{L^2(Y^*)}$$

(see Lemma 3.16). Hence $D_u\mathcal{D}$ is continuously invertible. ♦

Considering the difference quotient and passing to the limit, one sees that

$$D_x\mathcal{D}(x,u)[e_i] = \operatorname{div}_y\Big(\frac{\partial F}{\partial x_i}(x)u\Big) - \frac{\partial h}{\partial x_i}(x)$$

for $i = 1, 2$. The continuity can be shown using the same arguments as above.

Derivatives of $\mathcal{D}$ of higher order can be treated analogously; in u-direction the operator is infinitely differentiable, whereas in x-direction we obtain continuous derivatives up to order m.
Thus the assumptions of the implicit function theorem are fulfilled. This yields the following proposition:

4.4 Proposition.
Let $m \in \mathbb{N}$, $h \in \mathcal{C}^m(\Omega, L^2_{0,\#}(Y^*))$ and assume $F \in \mathcal{C}^m(\Omega)$. For the solution $\gamma(x)$ of Problem (4.2) *it holds*

$$\gamma \in \mathcal{C}^m_{\mathrm{loc}}(\Omega, H^1_{0,\#}(Y^*)^2)$$

Proof. The above lemmas show that we can apply Theorem 4.1, thus it only remains to show that γ is in the above mentioned function space:

The preceding considerations show that for every $x \in \Omega$ there exists a neighborhood U of x such that $\gamma(x)$ is m-times continuously differentiable in U. Thus for every compact subset K of Ω the norms

$$\left\| \frac{\partial^\alpha \gamma(x)}{\partial^{\alpha_1} x_1 \partial^{\alpha_2} x_2} \right\|_{H^1_{0,\#}(Y^*)^2}$$

with $\alpha_1 + \alpha_2 = \alpha$, $\alpha_1, \alpha_2 \ge 0$, $\alpha \le m$ are bounded on K. ♦

Now using Equation (4.1) yields the governing equations for the derivatives $\frac{\partial \gamma}{\partial x_i}$. Due to $\frac{\partial F}{\partial x_2} = 0$ we have

4.5 Corollary.
Define $\omega_1 = \frac{\partial}{\partial x_1}\gamma$ *and* $\omega_2 = \frac{\partial}{\partial x_2}\gamma$*. Then for fixed* $x \in \Omega$*, the functions* $\omega_1(x), \omega_2(x) \in H^1_{0,\#}(Y^*)^2$ *solve the problems*

$$\begin{aligned} \operatorname{div}_y\big(F^{-1}(x)\omega_1(x)\big) &= \frac{\partial h}{\partial x_1}(x) - \operatorname{div}_y\Big(\frac{\partial F}{\partial x_1}(x)\gamma(x)\Big) && \text{in } Y^* \\ \omega_1(x) &= 0 && \text{on } \partial Y_S \\ \omega_1(x) &\text{ is } Y\text{-periodic in } y \end{aligned}$$

and

$$\begin{aligned} \operatorname{div}_y\big(F^{-1}(x)\omega_2(x)\big) &= \frac{\partial h}{\partial x_2}(x) && \text{in } Y^* \\ \omega_2(x) &= 0 && \text{on } \partial Y_S \\ \omega_2(x) &\text{ is } Y\text{-periodic in } y \end{aligned}$$

Proof. By Equation (4.1), ω_i fulfills

$$D_u\mathcal{D}(x,u)[\omega_i] = -D_x\mathcal{D}(x,u)[e_i],$$

which corresponds to the problems above. ♦

4.2 Application to the Cell Problem

Next, we want to apply the Implicit Function Theorem to the following situation: Let $x \in \Omega$ be fixed and let $m \in \mathbb{N}$. We are looking for functions $u(x) \in H^1_{0,\#}(Y^*)^2$ and $p(x) \in L^2_{\#}(Y^*)/\mathbb{R}$ such that for given $f \in \mathcal{C}^m(\Omega, (H^1_{0,\#}(Y^*)^2)')$:

$$\begin{aligned} -\operatorname{div}_y\big(F^{-1}(x)F^{-T}(x)\,\nabla_y\,u(x)\big) + F^{-T}(x)\,\nabla_y\,p(x) &= f(x) && \text{in } Y^* && (4.3a) \\ \operatorname{div}_y(F^{-1}(x)u(x)) &= 0 && \text{in } Y^* && (4.3b) \\ u(x) &= 0 && \text{on } \partial Y_S && (4.3c) \\ u(x), p(x) \text{ are } Y\text{-periodic in } y & && && (4.3d) \end{aligned}$$

(this is the cell problem (3.16) from Section 3). We assume again that the matrix function F as defined in (2.1) is in $\mathcal{C}^m(\mathbb{R})^4$.

We define an operator

$$\begin{aligned} \mathcal{A} : \mathbb{R}^2 \times [H^1_{0,\#}(Y^*)^2 \times L^2_{\#}(Y^*)/\mathbb{R}] &\longrightarrow (H^1_{0,\#}(Y^*)^2)' \times L^2_{0,\#}(Y^*) \\ \mathcal{A}(x,u,p) &:= \begin{pmatrix} -\operatorname{div}_y\big(F^{-1}(x)F^{-T}(x)\,\nabla_y\,u\big) + F^{-T}(x)\,\nabla_y\,p - f(x) \\ \operatorname{div}_y\big(F^{-1}(x)u\big) \end{pmatrix} \end{aligned}$$

For the solution $u(x), p(x)$ of Equation (4.3) it holds $\mathcal{A}(x, u(x), p(x)) = 0$. In order to be able to apply Theorem 4.1, we have to discuss the operator and its total derivative in (u, p)-direction:

4.6 Lemma.
The operator $\mathcal{A}$ is continuous.

Proof. Consider sequences $x_n \to x$ in $\mathbb{R}^2$, $u_n \to u$ in $H^1_{0,\#}(Y^*)^2$ and $p_n \to p$ in $L^2(Y^*)/\mathbb{R}$. We have $F^{-1}F^{-T}(x_n) \longrightarrow F^{-1}F^{-T}(x)$ in $L^\infty(\mathbb{R}^2)$ for $x_n \to x$.
Considering the terms of $\mathcal{A}$ separately yields:

1. As f is continuous in x, $f(x_n) \to f(x)$ in $(H^1_{0,\#}(Y^*)^2)'$.
2. The product of a sequence in L^∞ with one converging in L^2 still gives a sequence converging in L^2; thus an integration by parts shows that for $\phi \in H^1_{0,\#}(Y^*)^2$

$$\int_{Y^*} - \operatorname{div}_y\big(F^{-1}(x_n)F^{-T}(x_n)\,\nabla_y u_n\big)\cdot\phi \,\mathrm{d}y = \int_{Y^*} F^{-T}(x_n)\,\nabla_y u_n : F^{-T}(x_n)\,\nabla_y\phi \,\mathrm{d}y$$
$$\longrightarrow \int_{Y^*} F^{-T}(x)\,\nabla_y u : F^{-T}(x)\,\nabla_y\phi \,\mathrm{d}y = \int_{Y^*} - \operatorname{div}_y\big(F^{-1}(x)F^{-T}(x)\,\nabla_y u\big)\cdot\phi \,\mathrm{d}y.$$

   Therefore

$$- \operatorname{div}_y\big(F^{-1}(x_n)F^{-T}(x_n)\,\nabla_y u_n\big) \quad\longrightarrow\quad - \operatorname{div}_y\big(F^{-1}(x)F^{-T}(x)\,\nabla_y u\big)$$

   in $(H^1_{0,\#}(Y^*)^2)'$.
3. Similarly it holds $F^{-T}(x_n)\,\nabla_y p_n \to F^{-T}(x)\,\nabla_y p$ in $(H^1_{0,\#}(Y^*)^2)'$ due to

$$\int_{Y^*} F^{-T}(x_n)(\nabla_y p_n)\phi \,\mathrm{d}y = -\int_{Y^*} p_n \operatorname{div}_y(F^{-1}(x_n)\phi)\,\mathrm{d}y$$
$$\longrightarrow \quad -\int_{Y^*} p \operatorname{div}_y(F^{-1}(x)\phi)\,\mathrm{d}y = \int_{Y^*} F^{-T}(x)(\nabla_y p)\phi \,\mathrm{d}y.$$

4. Finally, for the divergence it holds

$$\begin{aligned}
&\big\|\operatorname{div}_y\big(F^{-1}(x_n)u_n\big) - \operatorname{div}_y\big(F^{-1}(x)u\big)\big\|_{L^2(Y^*)} \\
&\quad= \Big\|\operatorname{div}_y\Big(F^{-1}(x_n)\big(u_n - u\big) - \big(F^{-1}(x_n) - F^{-1}(x)\big)u\Big)\Big\|_{L^2(Y^*)} \\
&\qquad\le C \max_{i,j=1,2} |(F^{-1}(x_n))_{ij}|\, \|\nabla_y(u_n - u)\|_{L^2(Y^*)} \\
&\qquad\quad + C \max_{i,j=1,2} \big|\big(F^{-1}(x_n) - F^{-1}(x)\big)_{ij}\big|\, \|\nabla u\|_{L^2(Y^*)} \quad\longrightarrow 0. \qquad \blacklozenge
\end{aligned}$$

Note that $-\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\cdot), F^{-T}(x)\,\nabla_y\,\cdot$ and $\operatorname{div}_y(F^{-1}(x)\cdot)$ are continuous linear operators on their respective domains, therefore their derivative is the operator itself. Given $u \in H^1_{0,\#}(Y^*)^2, \pi \in L^2(Y^*)/\mathbb{R}$ we obtain for the total derivative D_{up} of $\mathcal{A}$ in (u,p)-direction:

$$D_{up}\mathcal{A}(x,u,p)[w,\pi] = \begin{pmatrix} -\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\,w) + F^{-T}(x)\,\nabla_y\,\pi \\ \operatorname{div}_y(F^{-1}(x)w) \end{pmatrix}$$

In order to see the existence of $D_{up}\mathcal{A}(x,u,p)^{-1}$ consider for given $G_1 \in (H^1_{0,\#}(Y^*)^2)'$ and $g_2 \in L^2_{0,\#}(Y^*)$ the equation

$$D_{up}\mathcal{A}(x,u,p)[w,\pi] = \begin{pmatrix} G_1 \\ g_2 \end{pmatrix}. \tag{4.4}$$

The following lemma asserts the continuous invertibility of $D_{up}\mathcal{A}(x,u,p)$:

4.7 Lemma.
Equation (4.4) *has a unique solution* $(w,\pi) \in H^1_{0,\#}(Y^*)^2 \times L^2(Y^*)/\mathbb{R}$ *such that*

$$\|w\|_{H^1_{0,\#}(Y^*)^2} + \|\pi\|_{L^2(Y^*)/\mathbb{R}} \leq C(\|G_1\|_{(H^1_{0,\#}(Y^*)^2)'} + \|g_2\|_{L^2(Y^*)}).$$

Therefore $(D_{up}\mathcal{A}(x,u,p))^{-1} : (H^1_{0,\#}(Y^*)^2)' \times L^2_{0,\#}(Y^*) \longrightarrow H^1_{0,\#}(Y^*)^2 \times L^2_{\#}(Y^*)/\mathbb{R}$ *exists as a continuous linear operator.*

Proof. We are looking for a solution of

$$\begin{aligned} -\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\,w) + F^{-T}(x)\,\nabla_y\,\pi &= G_1 \text{ in } Y^* \\ \operatorname{div}_y(F^{-1}(x)w) &= g_2 \text{ in } Y^* \\ w &= 0 \text{ on } \partial Y_S \\ w \text{ is } Y\text{-periodic in } y& \end{aligned}$$

We adapt the method of 'subtracting the divergence', as in the case for the non-homogeneous Stokes equations (cf. [Tem77], [Soh01]): Analogously to Lemma 3.16 there exists a $u_1 \in H^1_{0,\#}(Y^*)^2$ such that

$$\operatorname{div}_y(F^{-1}(x)u_1) = g_2$$

with $\|\nabla_y\,u_1\|_{L^2(Y^*)^4} \leq C\,\|g_2\|_{L^2(Y^*)}$.

Set $v = w - u_1$. The existence theory for the transformed Stokes equation shows that there exists a unique solution of the problem

$$\begin{aligned}
-\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y v) + F^{-T}(x)\,\nabla_y \pi \\
= G_1 - \operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y u_1) \text{ in } Y^* \\
\operatorname{div}_y(F^{-1}(x)v) = 0 \text{ in } Y^* \\
v = 0 \text{ on } \partial Y_S \\
v \text{ is } Y\text{-periodic in } y
\end{aligned}$$

with

$$\|\nabla_y v\|_{L^2(Y^*)^4} + \|\pi\|_{L^2(Y^*)/\mathbb{R}} \le C(\|G_1\|_{(H^1_{0,\#}(Y^*)^2)'} + \|\nabla_y u_1\|_{L^2(Y^*)^4}).$$

Now the lemma follows easily. ♦

We obtain as well

4.8 Lemma.
The total derivative $D_{up}\mathcal{A}$ is continuous.

Proof. Let $x_n \to x$ in $\mathbb{R}^2$ and $u_n \to u$ in $H^1_{0,\#}(Y^*)^2$ as well as $p_n \to p$ in $L^2(Y^*)/\mathbb{R}$. We have to estimate the difference $D_{up}\mathcal{A}(x_n, u_n, p_n) - D_{up}\mathcal{A}(x, u, p)$ in the operator norm. Let $w \in H^1_{0,\#}(Y^*)^2$, $\pi \in L^2(Y^*)/\mathbb{R}$. We start by estimating the terms separately:

1. Consider for $\phi \in H^1_{0,\#}(Y^*)^2$

$$\begin{aligned}
&\Big|\int_{Y^*} \operatorname{div}_y\Big(\big(F^{-1}(x_n)F^{-T}(x_n) - F^{-1}(x)F^{-T}(x)\big)\nabla_y w\Big) \cdot \phi \,\mathrm{d}y\Big| \\
&\quad = \Big|\int_{Y^*} \big(F^{-1}(x_n)F^{-T}(x_n) - F^{-1}(x)F^{-T}(x)\big)\,\nabla_y w : \nabla_y \phi \,\mathrm{d}y\Big| \\
&\quad \le C \max_{i,j=1,2} |\big(F^{-1}(x_n)F^{-T}(x_n) - F^{-1}(x)F^{-T}(x)\big)_{ij}| \\
&\qquad \cdot \|\nabla_y w\|_{L^2(Y^*)^4}\, \|\nabla_y \phi\|_{L^2(Y^*)^4}\,.
\end{aligned}$$

Thus

$$\begin{aligned}
&\Big\|\operatorname{div}_y\Big(\big(F^{-1}(x_n)F^{-T}(x_n) - F^{-1}(x)F^{-T}(x)\big)\,\nabla_y w\Big)\Big\|_{(H^1_{0,\#}(Y^*)^2)'} \\
&\quad \le C \max_{i,j=1,2} |\big(F^{-1}(x_n)F^{-T}(x_n) - F^{-1}(x)F^{-T}(x)\big)_{ij}|\, \|\nabla_y w\|_{L^2(Y^*)^4}\,.
\end{aligned}$$

2. Analogously for $\phi \in H^1_{0,\#}(Y^*)^2$

$$\left|\int_{Y^*} \big(F^{-T}(x_n) - F^{-T}(x)\big)\,\nabla_y\,\pi \cdot \phi \,\mathrm{d}y\right| = \left|\int_{Y^*} \pi\,\mathrm{div}_y\Big(\big(F^{-1}(x_n) - F^{-1}(x)\big)\phi\Big)\,\mathrm{d}y\right|$$
$$\leq C \max_{i,j=1,2} \left|\big(F^{-1}(x_n) - F^{-1}(x)\big)_{ij}\right| \, \|\pi\|_{L^2(Y^*)} \, \|\nabla_y\,\phi\|_{L^2(Y^*)^4}$$

and therefore

$$\left\|\big(F^{-T}(x_n) - F^{-T}(x)\big)\,\nabla_y\,\pi\right\|_{(H^1_{0,\#}(Y^*)^2)'} \leq C \max_{i,j=1,2} \left|\big(F^{-1}(x_n) - F^{-1}(x)\big)_{ij}\right| \, \|\pi\|_{L^2(Y^*)} .$$

3. Finally for the divergence (similar to the above situation) it holds

$$\left\|\mathrm{div}_y\big((F^{-1}(x_n) - F^{-1}(x))w\big)\right\|_{L^2(Y^*)} \leq C \max_{i,j=1,2} \left|\big(F^{-1}(x_n) - F^{-1}(x)\big)_{ij}\right| \, \|\nabla_y\,w\|_{L^2(Y^*)^4} .$$

Now we can estimate

$$\begin{aligned}
&\left\|\big(D_{up}\mathcal{A}(x,u_n,p_n) - D_{up}\mathcal{A}(x,u,p)\big)[w,\pi]\right\|_{(H^1_{0,\#}(Y^*)^2)' \times L^2_{0,\#}(Y^*)} \\
&\leq C\Bigg(\left\|\mathrm{div}_y\Big(\big(F^{-1}(x_n)F^{-T}(x_n) - F^{-1}(x)F^{-T}(x)\big)\,\nabla_y\,w\Big)\right\|_{(H^1_{0,\#}(Y^*)^2)'} \\
&\qquad + \left\|\big(F^{-T}(x_n) - F^{-T}(x)\big)\,\nabla_y\,\pi\right\|_{(H^1_{0,\#}(Y^*)^2)'} + \left\|\mathrm{div}_y\Big(\big(F^{-1}(x_n) - F^{-1}(x)\big)w\Big)\right\|_{L^2(Y^*)}\Bigg) \\
&\leq C\Big(\max_{i,j=1,2} \left|\big(F^{-1}(x_n)F^{-T}(x_n) - F^{-1}(x)F^{-T}(x)\big)_{ij}\right| + \max_{i,j=1,2} \left|\big(F^{-1}(x_n) - F^{-1}(x)\big)_{ij}\right|\Big) \\
&\qquad \cdot \Big(\|\pi\|_{L^2(Y^*)} + \|\nabla_y\,w\|_{L^2(Y^*)^4}\Big).
\end{aligned}$$

Therefore the operator norm of the difference $D_{up}\mathcal{A}(x,u_n,p_n) - D_{up}\mathcal{A}(x,u,p)$ is bounded by

$$C\Big(\max_{i,j=1,2} \left|\big(F^{-1}(x_n)F^{-T}(x_n) - F^{-1}(x)F^{-T}(x)\big)_{ij}\right| + \max_{i,j=1,2} \left|\big(F^{-1}(x_n) - F^{-1}(x)\big)_{ij}\right|\Big) \longrightarrow 0.$$

Thus $D_{up}\mathcal{A}$ is continuous. ♦

In x-direction we obtain by considering the difference quotient and passing to the limit

$$D_x\mathcal{A}(x,u,p)[e_i] = \begin{pmatrix} -\,\mathrm{div}_y\Big(\frac{\partial}{\partial x_i}[F^{-1}(x)F^{-T}(x)]\,\nabla_y\,u\Big) + \big(\frac{\partial}{\partial x_i}F^{-T}(x)\big)\,\nabla_y\,p - \frac{\partial}{\partial x_i}f(x) \\ \mathrm{div}_y\Big(\big(\frac{\partial}{\partial x_i}F^{-1}(x)\big)u\Big) \end{pmatrix}$$

where e_i denotes the i-th unit vector. By considering the partial derivatives and arguing as above, one can show that $\frac{\partial}{\partial x_i}\mathcal{A}(x,u,p)$ is continuous for $i = 1,2$ everywhere in Ω, thus $D_x\mathcal{A}$ has the same property.

Note that the same argument holds for derivatives of higher order in x-direction. Furthermore, as a linear operator, $\mathcal{A}$ is infinitely differentiable in (u, p)-direction. Thus we can apply the Implicit Function Theorem to obtain:

4.9 Proposition.
Assume that $F(x)$ as given by (2.1) is m-times continuously differentiable and that $f \in \mathcal{C}^m(\Omega, L^2_\#(Y^*))$.

Then the solution (u, p) of equation (4.3) is in $\mathcal{C}^m_{\text{loc}}(\Omega, [H^1_{0,\#}(Y^*)^2 \times L^2_\#(Y^*)/\mathbb{R}])$, i.e. $u(x, y)$ and $p(x, y)$ are m-times differentiable in x.

Proof. The arguments above show that the assumptions of the Implicit Function Theorem are fulfilled everywhere in $\Omega \times [H^1_{0,\#}(Y^*)^2 \times L^2_\#(Y^*)/\mathbb{R}]$. Theorem 4.1 therefore gives the existence and differentiability properties of a function $k \in \mathcal{C}^m_{\text{loc}}(\Omega, [H^1_{0,\#}(Y^*)^2 \times L^2_\#(Y^*)/\mathbb{R}])$ such that $\mathcal{A}(x, k(x)) = 0$. By the uniqueness of solutions it must hold

$$k(x)(y) = \begin{pmatrix} u(x,y) \\ p(x,y) \end{pmatrix}. \qquad \blacklozenge$$

4.10 Corollary.
By formula (4.1) we obtain the governing equations for the derivatives: Let

$$\begin{pmatrix} w_1(x,y) \\ \pi_1(x,y) \end{pmatrix} := \frac{\partial}{\partial x_1} \begin{pmatrix} u(x,y) \\ p(x,y) \end{pmatrix} \quad \textit{and} \quad \begin{pmatrix} w_2(x,y) \\ \pi_2(x,y) \end{pmatrix} := \frac{\partial}{\partial x_2} \begin{pmatrix} u(x,y) \\ p(x,y) \end{pmatrix},$$

then in the case of F given by (2.1) it holds for $x \in \Omega$

$$\begin{aligned}
-\operatorname{div}_y\big(F^{-1}(x)F^{-T}(x)\,\nabla_y w_1(x)\big) + F^{-T}(x)\,\nabla_y \pi_1(x) &= \frac{\partial}{\partial x_1} f(x) \\
+\operatorname{div}_y\Big(\frac{\partial}{\partial x_1}\big[F^{-1}(x)F^{-T}(x)\big]\,\nabla_y u(x)\Big) &- \big(\frac{\partial}{\partial x_1} F^{-T}(x)\big)\,\nabla_y p(x) && \textit{in } Y^* \\
\operatorname{div}_y\big(F^{-1}(x) w_1(x)\big) &= -\operatorname{div}_y\Big(\big(\frac{\partial}{\partial x_1} F^{-1}(x)\big) u(x)\Big) && \textit{in } Y^* \\
w_1(x) &= 0 && \textit{on } \partial Y_S \\
w_1(x), \pi_1(x) \ \textit{are } y_1\textit{-periodic in } Y^*
\end{aligned}$$

and

$$\begin{aligned}
-\operatorname{div}_y\big(F^{-1}(x)F^{-T}(x)\,\nabla_y w_2(x)\big) + F^{-T}(x)\,\nabla_y \pi_2(x) &= \frac{\partial}{\partial x_2} f(x) && \textit{in } Y^* \\
\operatorname{div}_y\big(F^{-1}(x) w_1(x)\big) &= 0 && \textit{in } Y^* \\
w_2(x) &= 0 && \textit{on } \partial Y_S \\
w_2(x), \pi_2(x) \ \textit{are } y_1\textit{-periodic in } Y^*
\end{aligned}$$

4.11 Remark.

The above derivation can be extended to problems for a Stokes flow with a jump boundary condition. Schematically, we can make the following considerations: Let S be a given interface in a domain Λ and let $\sigma \in \mathcal{C}^m(\Omega, H^{\frac{1}{2}}(S))$ be a function such that for the parametrized fluid velocity u it holds $[u(x)]_S = \sigma(x)$. One can then define an operator of the form

$$\mathcal{A}' : \mathbb{R}^2 \times [H^1(\Lambda)^2 \times L^2(\Lambda)/\mathbb{R}] \longrightarrow (H^{-1}(\Lambda))^2 \times L^2_0(\Lambda) \times H^{\frac{1}{2}}(S)$$

$$\mathcal{A}'(x,u,p) := \begin{pmatrix} -\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\, u) + F^{-T}(x)\,\nabla_y\, p - f(x) \\ \operatorname{div}_y(F^{-1}(x)u) \\ [u]_S - \sigma(x) \end{pmatrix}$$

and adapt the steps carried out above in order to be able to apply the Implicit Function Theorem. Especially, by considering the governing equations for $\nabla_x\, u$, one arrives at $[\nabla_x\, u]_S = \nabla_x\, \sigma$, thus

$$[\nabla_x\, u]_S = \nabla_x [u]_S.$$

5 Behaviour at the Boundary

In this section, we carry out the mathematical constructions which are necessary to characterise the behaviour of the fluid at the interface Σ. These are mostly generalisations of ideas found in [JM96].
However, as the course of the derivation is lengthy and tedious, the reader should keep in mind the following 'plot':

1. As a first step (Section 5.3.1), the volume force is eliminated by subtracting auxiliary functions defined in the free fluid domain and in the porous part, respectively.

2. This correction introduces a jump of the corrected velocity across Σ. In order to eliminate this jump, a boundary layer function is constructed in Section 5.3.2. This type of function is concentrated around the interface Σ and decays exponentially outside it.

3. When subtracting the boundary layer function in an appropriate way, problems are introduced in the free fluid domain. Therefore another corrector function (which will be called a 'counterflow') is constructed. This counterflow is given in Ω_1 only and corrects the subtraction of the decay constant of the boundary layer function.

4. Finally, the estimates for the pressure are not sufficient yet, leading to another boundary layer- and counterflow problem in Section 5.3.3. This finishes the correction of the volume force.

5. An investigation of the condition on the divergence of the velocity shows that it has to be corrected as well (Section 5.4). In two steps, the influence due to the porous part and due to a boundary layer function are eliminated. For each of these steps, constructions similar to 2. and 3. have to be carried out.

6. To finish with, the influence of all these constructions on the equation is considered and the velocity and pressure are estimated in Section 5.5.

The main goal of this correction process is the elimination of all functions in the weak formulation which do not have a sufficient order in terms of powers of ε. Finally, we arrive at the weak formulation (5.6) which – upon inserting the corrected velocity as test function – allows the derivation of effective estimates.

An overview of the auxiliary functions and their connections is depicted in Figure 4, to which the reader is referred when reading the following subsections. We start by giving an overview of the involved geometries and the main assumption which is necessary to gain information about the behaviour of the velocity at the interface.

5.1 Setting of the Problem

We consider a situation similar to Section 2.2: Let $\Omega_1 = [0, L] \times \mathbb{R}_{>0}$ be the free fluid domain and let $\Omega_2 = [0, L] \times \mathbb{R}_{<0}$ be the porous medium, both separated by an interface $\Sigma = [0, L] \times \{0\}$.
We denote the reference cell by $Y = [0, 1]^2$ and assume that it contains an open set Y_S (the solid part) which is strictly included in Y. Its boundary ∂Y_S is assumed to be of class $\mathcal{C}^\infty$. Let $Y^* = Y \backslash \overline{Y_S}$ be the fluid part of the reference cell.
For given $\varepsilon > 0$ such that $\frac{L}{\varepsilon} \in \mathbb{N}$, we define the ε-periodic geometry as follows: Let χ be the characteristic function of Y^*, extended by periodicity to the whole of $\mathbb{R}^2$, and set $\chi^\varepsilon(x) := \chi(\frac{x}{\varepsilon})$. We define the fluid part of the porous medium as

$$\Omega_2^\varepsilon = \big\{ x \in \Omega_2 \mid \chi^\varepsilon(x) = 1 \big\}.$$

Then the fluid domain Ω^ε is given by (see also Figure 5)

$$\Omega^\varepsilon = \Omega_1 \cup \Sigma \cup \Omega_2^\varepsilon.$$

Let $l \in \mathcal{C}_0^\infty(\Omega)$ be a given volume force, $l \neq 0$ on Σ. The fluid flow is assumed to be governed by the following equations:

$$-\operatorname{div}(F^{-1}(x) F^{-T}(x) \nabla u^\varepsilon(x)) + F^{-T}(x) \nabla p^\varepsilon(x) = L^\varepsilon(x) \qquad \text{in } \Omega^\varepsilon \tag{5.1a}$$

$$\operatorname{div}(F^{-1}(x) u^\varepsilon(x)) = 0 \qquad \text{in } \Omega^\varepsilon \tag{5.1b}$$

$$u^\varepsilon(x) = 0 \qquad \text{on } \partial\Omega^\varepsilon \backslash \partial\Omega \tag{5.1c}$$

$$u^\varepsilon, p^\varepsilon \text{ are } L\text{-periodic in } x_1 \tag{5.1d}$$

with

$$L^\varepsilon = \begin{cases} \varepsilon^2 l & \text{in } \Omega_1 \\ l & \text{in } \Omega_2^\varepsilon \end{cases}.$$

The scaling in L^ε is chosen according to the usual scaling in the homogenisation of fluid flow in porous media, see eg. [SP80].

In the sequel, we will make use of the following 'abuse of notation': For a given function $\phi : \Omega \longrightarrow \mathbb{R}$ we denote by $(1 + x_2)\phi$, $\frac{\phi}{1+x_2}$ etc. the functions $\iota\phi$, $\frac{\phi}{\iota}$ resp., where $\iota(\binom{x_1}{x_2}) = 1 + x_2$. Besides, we will need the following assumption:

5.1 Assumption.
We assume that the following statements are true:

1. *Let $\rho^{i,\mathrm{bl}}(x, y)$ be a boundary layer function which stabilizes exponentially in y towards some constant $C_\rho^\pm(x)$ in $Z^\pm$ (cf. Appendix A.1). We assume that $\rho^{i,\mathrm{bl}}$*

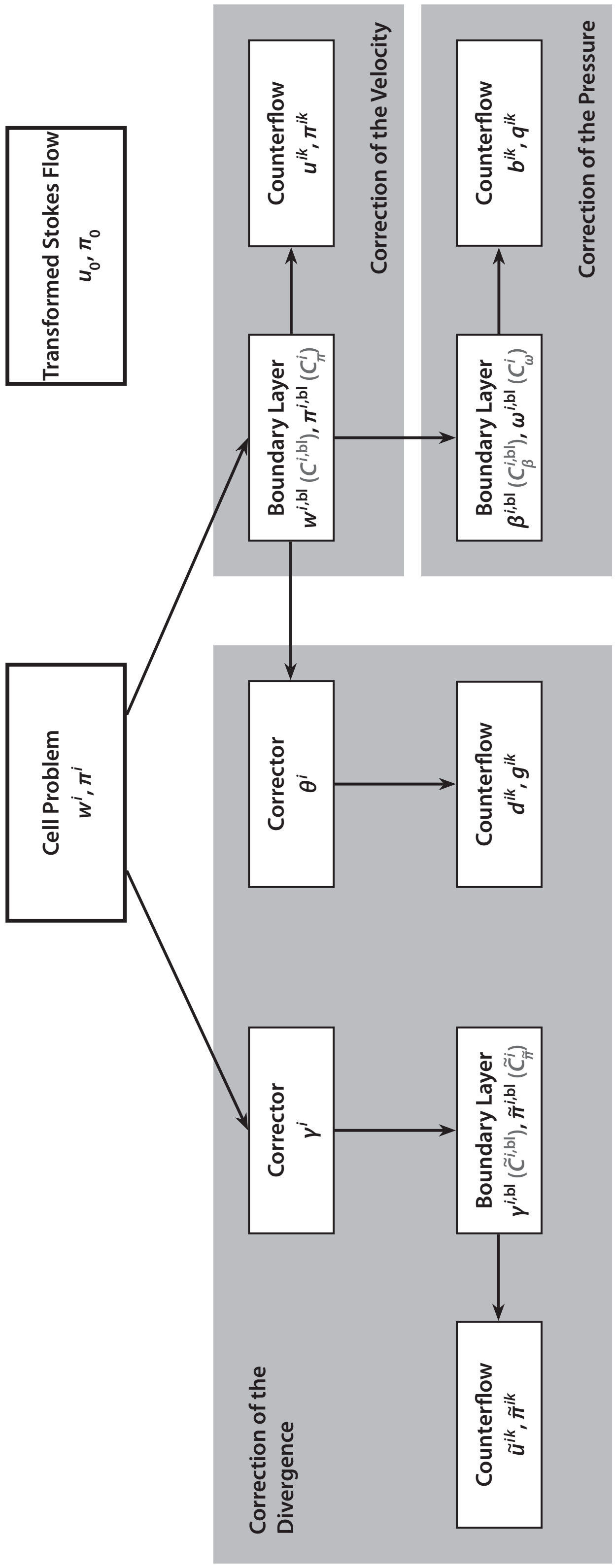

Figure 4: Overview of the auxilliary functions (black) and stabilizing constants (grey) and their relations.

and $C_\rho^\pm$ are differentiable in x, that also $\nabla_x \rho^{i,\mathrm{bl}}$ decays exponentially in y and that the corresponding stabilizing constant is the matching derivative of $C_\rho^\pm$. Thus espescially

- $\nabla_x \rho^{i,\mathrm{bl}}(x,y)$ *stabilizes exponentially in y towards* $\nabla_x C_\rho^\pm(x)$
- $\mathrm{div}_x(\nabla_x \rho^{i,\mathrm{bl}}(x,y))$ *stabilizes exponentially in y towards* $\mathrm{div}_x(\nabla C_\rho^\pm(x))$.

2. *All the stabilizations are uniform in x, i.e. there exist constants $C, \gamma_0 > 0$ independent of x such that*

$$|\rho^{i,\mathrm{bl}}(x,y) - C_\rho^\pm(x)| \le Ce^{-\gamma_0|y_2|}$$
$$|\nabla_x \rho^{i,\mathrm{bl}}(x,y) - \nabla_x C_\rho^\pm(x)| \le Ce^{-\gamma_0|y_2|}$$

etc., in $Z^\pm$.

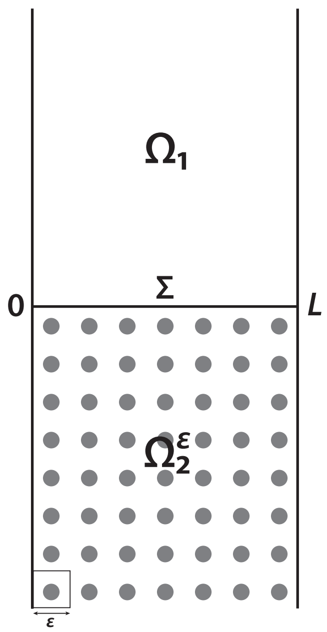

Figure 5: The main geometry, consisting of the free fluid domain Ω_1 and the porous medium Ω_2^ε, separated by the interface Σ.

5.2 Auxiliary Results

5.2 Definition.
We define the following spaces:

$$\begin{aligned}
W_\varepsilon &= \big\{z \in L^2_{\mathrm{loc}}(\Omega^\varepsilon)^2 \mid \nabla z \in L^2(\Omega^\varepsilon)^4, z \in L^2(\Omega_2^\varepsilon), \mathrm{div}(F^{-1}z) = 0 \ \textit{a.e. in } \Omega^\varepsilon, \\
&\qquad z = 0 \ \textit{on } \partial\Omega_2^\varepsilon \backslash \partial\Omega, z \textit{ is } L\textit{-periodic in } x_1\big\} \\
V_i &= \big\{z \in L^2_{\mathrm{loc}}(\Omega_i) \mid \nabla z \in L^2(\Omega_i)^2, z \textit{ is } L\textit{-periodic in } x_1\big\} \quad i = 1,2 \\
\mathcal{W} &= \big\{z \in L^2_{\mathrm{loc}}(\Omega_1)^2 \mid z \in V_1^2, \mathrm{div}(F^{-1}z) = 0 \ \textit{a.e. in } \Omega_1\big\}.
\end{aligned}$$

W_ε is equipped with the norm $\|z\|_{W_\varepsilon} = \|\nabla z\|_{L^2(\Omega^\varepsilon)^4}$.

The unboundedness of the domain poses additional problems. In order to prove existence and uniqueness for a number of auxiliary problems, we need the following Poincaré-type inequalities:

5.3 Lemma.
Let $\phi \in W_\varepsilon$, $z \in V_i$. Then

$$\left\| \frac{\phi}{1+x_2} \right\|_{L^2(\Omega^\varepsilon)^2} \le C(\|\nabla \phi\|_{L^2(\Omega_1)^4} + \sqrt{\varepsilon}\,\|\nabla \phi\|_{L^2(\Omega_2^\varepsilon)^4})$$

$$\left\| \frac{1}{1+x_2}\Big(z - \frac{1}{L}\int_\Sigma z(x_1,0)\ \mathrm{d}x_1\Big) \right\|_{L^2(\Omega_i)} \le C\,\|\nabla z\|_{L^2(\Omega_i)^2}.$$

Proof. See [JM96]. ♦

In Ω_2^ε, we have the usual estimates:

5.4 Lemma.
Let $\phi \in H^1_\#(\Omega_2^\varepsilon)$ with $\phi = 0$ on $\partial\Omega_2^\varepsilon \backslash \partial\Omega_2$. Then there exists constant C independent of ε such that

1. $\|\phi\|_{L^2(\Omega_2^\varepsilon)} \le C\varepsilon\,\|\nabla \phi\|_{L^2(\Omega_2^\varepsilon)^2}$.
2. $\|\phi\|_{L^2(\Sigma)} \le C\varepsilon^{\frac{1}{2}}\,\|\nabla \phi\|_{L^2(\Omega_2^\varepsilon)^2}$.

Proof. The first part follows analogously to Lemma 3.11. The second part can be proved by using the same summation-and-scaling argument based on the usual trace estimate. ♦

We want to derive the variational problem corresponding to (5.1). Let $\phi \in \mathcal{V}$ with

$$\mathcal{V} := \big\{\psi \in \mathcal{C}^\infty_{0,\#}(\Omega^\varepsilon)^2 \mid \mathrm{div}(F^{-1}\psi) = 0\big\}$$

such that $\operatorname{supp}(\phi) \subset \Omega_b$, $\Omega_b := [0, L] \times (-b, b)$. Multiplying the left part of (5.1a) with ϕ and integrating by parts in Ω_b gives due to $\operatorname{div}(F^{-1}\phi) = 0$

$$\int_{\Omega^\varepsilon} -\operatorname{div}(F^{-1}(x)F^{-T}(x)\nabla u^\varepsilon(x)) \cdot \phi(x)\,\mathrm{d}x = \int_{\Omega_b} F^{-T}(x)u^\varepsilon(x) : F^{-T}(x)\nabla\phi(x)\,\mathrm{d}x$$
$$- \int_{\partial\Omega_b} ((F^{-1}(x)F^{-T}(x)\nabla u^\varepsilon(x) - F^{-1}(x)p^\varepsilon)\nu) \cdot \phi(x)\,\mathrm{d}\sigma_x.$$

Note that the boundary integral vanishes because of the periodic boundary conditions and the compact support of h and ϕ, therefore it holds

$$\int_{\Omega^\varepsilon} F^{-T}(x)u^\varepsilon(x) : F^{-T}(x)\nabla\phi(x)\,\mathrm{d}x = \int_{\Omega^\varepsilon} L^\varepsilon(x)\cdot\phi(x)\,\mathrm{d}x \quad \forall\,\phi \in \mathcal{V}.$$

5.5 Lemma.
The closure of $\mathcal{V}$ with respect to $\|\cdot\|_{W_\varepsilon}$ equals W_ε.

Proof. Denote by $\bar{\mathcal{V}}$ the closure of $\mathcal{V}$ with respect to $\|\nabla\cdot\|_{L^2(\Omega^\varepsilon)^4}$. Let $u_m \in \mathcal{V}$, $u \in \bar{\mathcal{V}}$ with $u_m \to u$ in the norm of W_ε. Thus $\nabla u \in L^2(\Omega^\varepsilon)^4$ and $\operatorname{div}(F^{-1}u) = 0$ because of

$$|\operatorname{div}(F^{-1}u)| = |\operatorname{div}(F^{-1}(u - u_m))| \le C\,\|\nabla(u - u_m)\|_{L^2(\Omega^\varepsilon)^4} \longrightarrow 0.$$

Since $\nabla u \in L^2(\Omega^\varepsilon)^4$ we conclude that $u \in L^2_{\mathrm{loc}}(\Omega^\varepsilon)^2$ (see for example [Maz85], Section 1.1.2). Finally, due to the Poincaré inequality in Ω_2^ε

$$\|u\|_{L^2(\Omega_2^\varepsilon)^2} \le C\,\|\nabla u\|_{L^2(\Omega_2^\varepsilon)^4} \le C.$$

Therefore we have $\bar{\mathcal{V}} \subset W_\varepsilon$.

Denote by $\bar{\mathcal{V}}^\perp$ the orthogonal complement of $\bar{\mathcal{V}}$ in W_ε and let $w \in \bar{\mathcal{V}}^\perp$. Since w is orthogonal to functions u with $\operatorname{div}(F^{-1}u) = 0$ we conclude that $w = F^{-T}\nabla p$ with $p \in H^1_{\mathrm{loc}}(\Omega^\varepsilon)$ satisfies

$$\operatorname{div}(F^{-1}F^{-T}\nabla p) = 0 \text{ in } \Omega^\varepsilon$$
$$F^{-T}\nabla p = 0 \text{ on } \partial\Omega^\varepsilon.$$

Thus p is constant and $w = 0$. Therefore $\bar{\mathcal{V}}^\perp = \{0\}$ and the lemma is proved. ♦

Hence the weak formulation of Problem (5.1) is

$$\int_{\Omega^\varepsilon} F^{-T}(x)\nabla u^\varepsilon(x) : F^{-T}(x)\nabla\phi(x)\,\mathrm{d}x = \int_{\Omega^\varepsilon} L^\varepsilon(x)\cdot\phi(x)\,\mathrm{d}x \quad \forall\phi \in W_\varepsilon. \tag{5.2}$$

The properties of the left hand side follow analogously to Section 3.2. Because of Lemma 5.3

$$\left| \int_{\Omega^\varepsilon} L^\varepsilon(x) \cdot \phi(x) \, \mathrm{d}x \right| \leq \left| \int_{\Omega^\varepsilon} (1+x_2) L^\varepsilon(x) \cdot \frac{\phi(x)}{1+x_2} \, \mathrm{d}x \right| \leq C \, \|\nabla \phi\|_{L^2(\Omega^\varepsilon)^4}$$

(since L^ε has a compact support), thus $L^\varepsilon \in W_\varepsilon'$, and we can apply the lemma of Lax-Milgram to obtain a unique solution $u^\varepsilon \in W_\varepsilon$.

Similar to Section 3.2 we obtain the existence of a pressure $p^\varepsilon \in L^2_{\mathrm{loc}}(\Omega^\varepsilon)$ (which is only locally square integrable due to the unbounded domain, see also the proof of Proposition A.3). By using the lifting properties, we get

5.6 Lemma.
Problem (5.2) *has a unique solution* $u^\varepsilon \in W_\varepsilon$, *and there exists a unique* $p^\varepsilon \in L^2_{\mathrm{loc}}(\Omega^\varepsilon)/\mathbb{R}$ *such that* (5.1) *holds. Moreover* $u^\varepsilon \in \mathcal{C}^\infty_{\mathrm{loc}}(\Omega^\varepsilon)^2$, $p^\varepsilon \in \mathcal{C}^\infty_{\mathrm{loc}}(\Omega^\varepsilon)$.

5.3 Correction of the Velocity

Define the space $V_{\mathrm{per}}(\Omega^\varepsilon)$ as

$$V_{\mathrm{per}}(\Omega^\varepsilon) = \left\{ z \in L^2_{\mathrm{loc}}(\Omega^\varepsilon)^2 \mid \nabla z \in L^2(\Omega^\varepsilon)^4, \ z = 0 \text{ on } \partial\Omega^\varepsilon \backslash \partial\Omega, \ z \text{ is } L\text{-periodic in } x_1 \right\}.$$

Similarly, by density of $\mathcal{C}^\infty_{0,\#}(\Omega^\varepsilon)^2$ in $V_{\mathrm{per}}(\Omega^\varepsilon)$ with respect to $\|\cdot\|_{W_\varepsilon}$ (cf. Lemma 5.5), we obtain the following modified weak formulation of (5.1):

$$\begin{aligned} &\int_{\Omega^\varepsilon} F^{-T}(x) \nabla \frac{u^\varepsilon(x)}{\varepsilon^2} : F^{-T}(x) \nabla \phi(x) \, \mathrm{d}x - \int_{\Omega^\varepsilon} \frac{p^\varepsilon}{\varepsilon^2} \operatorname{div}(F^{-1}(x)\phi(x)) \, \mathrm{d}x \\ &\qquad = \int_{\Omega_1} l(x) \cdot \phi(x) \, \mathrm{d}x + \int_{\Omega_2^\varepsilon} \frac{l(x)}{\varepsilon^2} \cdot \phi(x) \, \mathrm{d}x \quad \forall \phi \in V_{\mathrm{per}}(\Omega^\varepsilon). \end{aligned} \tag{5.3}$$

For the definition of the boundary layer functions, we introduce the following geometry (see Figure 6 for an illustration): Set

$$\begin{aligned} Z^- &= \bigcup_{k=1}^{\infty} \{ Y^* - \begin{pmatrix} 0 \\ k \end{pmatrix} \} \backslash S \\ S &= [0,1] \times \{0\} \\ Z^+ &= [0,1] \times (0,\infty) \\ Z &= Z^+ \cup Z^- \\ Z_{\mathrm{BL}} &= Z^+ \cup S \cup Z^-. \end{aligned}$$

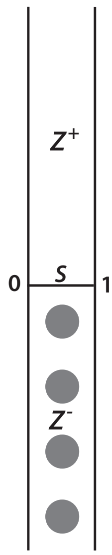

Figure 6: The boundary layer strip Z_{BL}.

Finally, let

$$V = \Big\{ z \in L^2_{\mathrm{loc}}(Z_{\mathrm{BL}})^2 \mid \nabla z \in L^2(Z)^4,\ z \in L^2(Z^-)^2, \\ z = 0 \text{ on } \bigcup_{k=1}^{\infty} \{\partial Y_S - \begin{pmatrix} 0 \\ k \end{pmatrix}\},\ z \text{ is 1-periodic in } x_1 \Big\}.$$

5.3.1 Elimination of the Forces

As a first step, we want to eliminate the force L^ε. We assume that the flow in Ω_1 is dominated by a transformed Stokes flow with no-slip condition on Σ; therefore define u_0 and π_0 by

$$\boxed{\begin{aligned} -\operatorname{div}(F^{-1}(x)F^{-T}(x)\nabla u_0(x)) + F^{-T}(x)\nabla \pi_0(x) &= l && \text{in } \Omega_1 \\ \operatorname{div}(F^{-1}(x)u_0(x)) &= 0 && \text{in } \Omega_1 \\ u_0(x) &= 0 && \text{on } \Sigma \\ u_0, \pi_0 \text{ are } L\text{-periodic in } x_1 & \end{aligned}}$$

For the existence of a solution $(u_0, \pi_0) \in \big(W \times L^2_{\mathrm{loc}}(\Omega_1)\big) \cap \big(\mathcal{C}^\infty_{\mathrm{loc}}(\Omega_1 \cup \Sigma)^2 \times \mathcal{C}^\infty_{\mathrm{loc}}(\Omega_1 \cup \Sigma)\big)$ see Appendix A.5.

The cell problem is defined as in Section 3.4: We are looking for $w^i(x,y), \pi^i(x,y)$ satisfying

$$\boxed{\begin{array}{rl} -\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\nabla_y w^i(x,y)) + F^{-T}(x)\nabla_y \pi^i(x,y) = F^{-T}(x)e_i & \text{in } \Omega\times Y^* \\ \operatorname{div}_y(F^{-1}(x)w^i(x,y)) = 0 & \text{in } \Omega\times Y^* \\ w^i(x,y) = 0 & \text{on } \Omega\times \partial Y_S \\ w^i(x), \pi^i(x) \text{ are } Y\text{-periodic in } y & \end{array}}$$

Here and in the sequel, let all indices i, k, etc. extend over the set $\{1,2\}$.

There exists a unique solution $(w^i,\pi^i) \in \mathcal{C}^\infty_{\mathrm{loc}}(\Omega,[H^1_{0,\#}(Y^*)^2 \times L^2_\#(Y^*)/\mathbb{R}])$ and $(w^i(x),\pi^i(x)) \in \mathcal{C}^\infty(Y^*)^2 \times \mathcal{C}^\infty(Y^*)$. Due to the special form of F, the two functions depend on x only via the x_1-variable. Therefore we obtain $\mathcal{C}^\infty$-regularity in x-direction, and w^i and π^i are bounded from above.

In Ω_2, we expect the flow to be governed by a transformed Darcy's law:

$$\boxed{\begin{array}{rl} \operatorname{div}\Big(F^{-1}(x)A(x)\big(F^T(x)l(x) - \nabla p(x)\big)\Big) = 0 & \text{in } \Omega_2 \\ p(x) = 0 & \text{on } \Sigma \\ p \text{ is } L\text{-periodic in } x_1, & \end{array}}$$

where $A(x)$ is the permeability tensor, $(A(x))_{ij} = \int_{Y^*} w^i_j(x,y)\,dy$. Due to the properties of $F^{-1}A$ (see Remark 3.29), there exists a unique solution $p \in V_2$. By using an analogue of Theorem A.10, we obtain an exponential stabilization of p towards a constant and an exponential stabilization of ∇p towards 0, both for for $x_2 \longrightarrow -\infty$.

Define

$$D^i(x) = \Big((F^T l)_i - \frac{\partial p}{\partial x_i}\Big)(x)$$

and let H denote the Heaviside function. We consider

$$\frac{u^\varepsilon(x)}{\varepsilon^2} - H(x_2)u_0(x) - H(-x_2)\sum_i D^i(x)w^i(x,\frac{x}{\varepsilon})$$

and

$$\frac{p^\varepsilon(x)}{\varepsilon^2} - H(x_2)\pi_0(x) - H(-x_2)\Big[\frac{p}{\varepsilon^2} + \frac{1}{\varepsilon}\sum_i D^i(x)\pi^i(x,\frac{x}{\varepsilon})\Big].$$

Substitution into Equation (5.3) yields

$$\int_{\Omega^\varepsilon} \Big\{ F^{-T}(x)\,\nabla\,\frac{u^\varepsilon(x)}{\varepsilon^2} - H(x_2)F^{-T}(x)\,\nabla\,u_0(x)$$

$$- H(-x_2)\Big[F^{-T}(x)\,\nabla\big(\sum_i D^i(x)w^i(x,\frac{x}{\varepsilon})\big)\Big]\Big\} : F^{-T}(x)\,\nabla\,\phi(x)\;\mathrm{d}x$$

$$-\int_{\Omega^\varepsilon} \Big\{ \frac{p^\varepsilon(x)}{\varepsilon^2} - H(x_2)\pi_0(x) - H(-x_2)\Big[\frac{p}{\varepsilon^2} + \frac{1}{\varepsilon}\sum_i D^i(x)\pi^i(x,\frac{x}{\varepsilon})\Big]\Big\}\,\mathrm{div}(F^{-1}(x)\phi(x))\;\mathrm{d}x$$

$$= \int_{\Sigma} \Big\{ -F^{-1}(x)\sigma_0(x) + F^{-1}(x)\frac{p(x)}{\varepsilon^2} - \mathscr{B}^\varepsilon \Big\} e_2\cdot\phi(x)\;\mathrm{d}\sigma_x + \int_{\Omega_2^\varepsilon} \mathscr{A}_1^\varepsilon\cdot\phi(x)\;\mathrm{d}x,$$

where

$$\sigma_0(x) = \pi_0(x)I - F^{-T}(x)\,\nabla\,u_0(x)$$

$$\mathscr{B}^\varepsilon = F^{-1}(x)F^{-T}(x)\Big(\sum_i(\nabla\,D^i(x))\otimes w^i(x,\frac{x}{\varepsilon})$$

$$+\sum_i\Big(D^i(x)\big[\,\nabla\,w^i(x,\frac{x}{\varepsilon}) - \varepsilon^{-1}\pi^i(x,\frac{x}{\varepsilon})\big]\Big)\Big)$$

and

$$\mathscr{A}_1^\varepsilon = \mathrm{div}\Big(F^{-1}(x)F^{-T}(x)\,\nabla_x\big(\sum_i D^i(x)w^i(x,\frac{x}{\varepsilon})\big)\Big)$$

$$+\frac{1}{\varepsilon}\,\mathrm{div}_x\Big(F^{-1}(x)F^{-T}(x)\sum_i D^i(x)\,\nabla_y\,w^i(x,\frac{x}{\varepsilon})\Big)$$

$$-\frac{1}{\varepsilon}F^{-T}(x)\sum_i(\nabla\,D^i(x))\pi^i(x,\frac{x}{\varepsilon}) - \frac{1}{\varepsilon}F^{-T}(x)\sum_i D^i(x)\,\nabla_x\,\pi^i(x,\frac{x}{\varepsilon}).$$

Then by the Poincaré inequality in Ω_2^ε (cf. Lemma 3.11) and the exponential stabilization of D^i towards 0

$$\Big|\int_{\Omega_2^\varepsilon} \mathscr{A}_1^\varepsilon\cdot\phi(x)\;\mathrm{d}x\Big| \le C\,\|\nabla\,\phi\|_{L^2(\Omega_2^\varepsilon)^4}\,.$$

Similarly

$$\Big|\int_{\Sigma} F^{-1}(x)\sigma_0 e_2\cdot\phi(x)\;\mathrm{d}\sigma_x\Big| \le C\varepsilon^{\frac{1}{2}}\,\|\nabla\,\phi\|_{L^2(\Omega_2^\varepsilon)^4}\,.$$

5.3.2 Continuity of the Traces

The above elimination of the volume force introduced a jump across Σ. In order to be able to use the correction of $\frac{u^\varepsilon}{\varepsilon^2}$ as a test function in (5.3), we have to correct its behaviour on Σ and eliminate that jump.

Define the family of boundary layer functions $(w^{i,\mathrm{bl}}, \pi^{i,\mathrm{bl}})$ by

$$\boxed{\begin{aligned}
-\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\nabla_y w^{i,\mathrm{bl}}(x,y)) + F^{-T}(x)\nabla_y \pi^{i,\mathrm{bl}}(x,y) &= 0 \quad \text{in } \Omega\times Z\\
\operatorname{div}_y(F^{-1}(x) w^{i,\mathrm{bl}}(x,y)) &= 0 \quad \text{in } \Omega\times Z\\
[w^{i,\mathrm{bl}}(x)]_S(y) &= w^i(x,y) \quad \text{on } \Omega\times S\\
[(F^{-1}(x)F^{-T}(x)\nabla_y w^{i,\mathrm{bl}}(x) - F^{-1}(x)\pi^{i,\mathrm{bl}}(x))e_2]_S(y) &\\
= (F^{-1}(x)F^{-T}(x)\nabla_y w^{i}(x) - F^{-1}(x)\pi^{i}(x))e_2(y) &\quad \text{on } \Omega\times S\\
w^{i,\mathrm{bl}}(x,y) = 0 \quad \text{on } \Omega\times \bigcup_{k=1}^\infty\{\partial Y_S - \tbinom{0}{k}\}&\\
w^{i,\mathrm{bl}}(x), \pi^{i,\mathrm{bl}}(x) \text{ are 1-periodic in } y_1&
\end{aligned}}$$

Appendix A.1 gives the existence of $(w^{i,\mathrm{bl}}(x), \pi^{i,\mathrm{bl}}(x)) \in V\cap \mathcal{C}^\infty_{\mathrm{loc}}(Z)^2\times \mathcal{C}^\infty_{\mathrm{loc}}(Z)$. Furthermore, for fixed $x\in\Omega$ there exist constants $\gamma_0>0$, $y^*>0$, $C^{i,\mathrm{bl}}(x)$ and $C^i_\pi(x)$ such that

$$e^{\gamma_0|y_2|}\nabla_y w^{i,\mathrm{bl}}(x) \in L^2(Z)^4,\ e^{\gamma_0|y_2|}w^{i,\mathrm{bl}}(x)\in L^2(Z^-)^2,\ e^{\gamma_0|y_2|}\pi^{i,\mathrm{bl}}(x)\in L^2(Z^-).$$

Moreover we have

$$\begin{aligned}
|w^{i,\mathrm{bl}}(x,y) - C^{i,\mathrm{bl}}(x)| &\le Ce^{-\gamma_0|y_2|}, \quad y_2>y^*\\
|\pi^{i,\mathrm{bl}}(x,y) - C^{i}_\pi(x)| &\le Ce^{-\gamma_0|y_2|}, \quad y_2>y^*\\
|\nabla_x(w^{i,\mathrm{bl}}(x,y) - C^{i,\mathrm{bl}}(x))| &\le Ce^{-\gamma_0|y_2|}, \quad y_2>y^*\\
|\nabla_x(\pi^{i,\mathrm{bl}}(x,y) - C^{i}_\pi(x))| &\le Ce^{-\gamma_0|y_2|}, \quad y_2>y^*.
\end{aligned}$$

Here we used Assumption 5.1: The right hand side is independent of x and the decay carries over to the derivatives in x. Appendix A.1 shows that we have an exponential decay towards 0 in Z^-. Therefore we obtain the following lemma:

5.7 Lemma.
Extend $w^{i,\mathrm{bl}}(x)$ *by* 0 *in* $[0,1] \times \mathbb{R} \backslash Z_{\mathrm{BL}}$. *Then it holds for all* $q \geq 1$

$$\begin{aligned} \left\| w^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon}) - H(x_2) C^{i,\mathrm{bl}}(x) \right\|_{L^q(\Omega)^2} &\leq C \varepsilon^{\frac{1}{q}} \\ \left\| \pi^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon}) - H(x_2) C^i_\pi(x) \right\|_{L^q(\Omega)} &\leq C \varepsilon^{\frac{1}{q}} \\ \left\| \nabla_y w^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon}) \right\|_{L^q(\Omega)^4} &\leq C \varepsilon^{\frac{1}{q}}. \end{aligned}$$

Proof. We only give the proof of the first inequality in Z^+, the others follow analogously. Substituting $y_2 = \frac{x_2}{\varepsilon}$ we obtain for $q \geq 1$

$$\begin{aligned} \int_{Z^+} |w^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon}) - H(x_2) C^{i,\mathrm{bl}}(x)|^q \, \mathrm{d}x &\leq C \int_{Z^+} |e^{-\gamma_0 |\frac{x_2}{\varepsilon}|}|^q \, \mathrm{d}x \\ &\leq C \int_0^\infty e^{-\gamma_0 q |\frac{x_2}{\varepsilon}|} \, \mathrm{d}x_2 = C\varepsilon \int_0^\infty e^{-\gamma_0 q |y_2|} \, \mathrm{d}y_2 \\ &\leq C\varepsilon. \end{aligned}$$

Now taking the q-th root on both sides yields the result. ♦

5.8 Remark.
The preceding lemma justifies the term 'boundary layer function', as $w^{i,\mathrm{bl}}(x)$ *and* $\pi^{i,\mathrm{bl}}(x)$ *are concentrated on* S *and decay exponentially outside this interface.*

Define for given $\delta > 0$

$$D^i_\delta(x) = D^i(x_1, -0) e^{-\delta x_2}$$

In Ω_1 we have to correct the influence of the boundary layer functions by using the counterflow given by

$$\boxed{\begin{aligned} -\operatorname{div}(F^{-1}(x) F^{-T}(x) \nabla u^{ik}(x)) + F^{-T}(x) \nabla \pi^{ik}(x) &= 0 \quad \text{in } \Omega_1 \\ \operatorname{div}(F^{-1}(x) u^{ik}(x)) &= 0 \quad \text{in } \Omega_1 \\ u^{ik}(x_1, +0) = (C^{i,\mathrm{bl}}_k D^i_\delta) e_k(x_1, 0) &\quad \text{on } \Sigma \\ u^{ik}, \pi^{ik} \text{ are } L\text{-periodic in } x_1 & \end{aligned}}$$

Note that the function u^{ik} corresponds to the construction $C^{i,\mathrm{bl}}_k u^{ik}$ in [JM96], and π^{ik} corresponds to $C^{i,\mathrm{bl}}_k \pi^{ik}$. These functions are introduced to correct the term $H(x_2) \sum_i D^i_\delta(x) C^{i,\mathrm{bl}}(x)$ on Σ, see below. There exists a unique solution $(u^{ik}, \pi^{ik}) \in \mathcal{W} \times L^2_{\mathrm{loc}}(\Omega_1)/\mathbb{R}$, cf. Appendix A.5.

We make the following ansatz:

$$U^\varepsilon(x) = \frac{u^\varepsilon(x)}{\varepsilon^2} - H(x_2)u_0(x) - H(-x_2)\sum_i D^i(x)w^i(x,\frac{x}{\varepsilon}) \\ - \sum_i D^i_\delta(x)\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big) - H(x_2)\sum_{i,k} u^{ik}(x)$$

and

$$P^\varepsilon(x) = \frac{p^\varepsilon(x)}{\varepsilon^2} - H(x_2)\pi_0(x) - H(-x_2)\Big[\frac{p}{\varepsilon^2} + \frac{1}{\varepsilon}\sum_i D^i(x)\pi^i(x,\frac{x}{\varepsilon})\Big] \\ - \frac{1}{\varepsilon}\sum_i D^i_\delta(x)\big(\pi^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C^i_\pi(x)\big) - H(x_2)\sum_{i,k}\pi^{ik}(x).$$

Inserting these functions into (5.3) yields

$$\begin{aligned}
&\int_{\Omega^\varepsilon} F^{-T}(x)\,\nabla\, U^\varepsilon(x) : F^{-T}(x)\,\nabla\,\phi(x)\;\mathrm{d}x - \int_{\Omega^\varepsilon} P^\varepsilon(x)\,\mathrm{div}(F^{-1}(x)\phi(x))\;\mathrm{d}x \\
&\quad= \int_\Sigma (-F^{-1}(x)\sigma_0 + \varepsilon^{-2}F^{-1}(x)p)e_2\cdot\phi(x)\;\mathrm{d}\sigma_x + \int_{\Omega_2^\varepsilon} \mathscr{A}_1^\varepsilon\cdot\phi(x)\;\mathrm{d}x \\
&\qquad + \int_\Sigma B_5^\varepsilon\;\mathrm{d}\sigma_x + \int_\Sigma (B_1^\varepsilon + B_2^\varepsilon + B_3^\varepsilon + B_4^\varepsilon)e_2\cdot\phi(x)\;\mathrm{d}\sigma_x \qquad (5.4)\\
&\qquad + \int_{\Omega^\varepsilon} A_1^\varepsilon : F^{-T}\,\nabla\,\phi(x)\;\mathrm{d}x + \int_{\Omega^\varepsilon} A_2^\varepsilon\cdot\phi(x)\;\mathrm{d}x + \int_{\Omega^\varepsilon} A_3^\varepsilon\cdot\phi(x)\;\mathrm{d}x + \int_{\Omega^\varepsilon} A_4^\varepsilon\;\mathrm{d}x
\end{aligned}$$

with

$$\begin{aligned}
A_1^\varepsilon &= -F^{-T}(x)\sum_i \nabla_x(D^i_\delta(x))\otimes\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\\
A_2^\varepsilon &= -D^i_\delta(x)\,\mathrm{div}_x\Big(F^{-1}(x)F^{-T}(x)\,\nabla_x\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\Big)\\
&\quad+D^i_\delta(x)g''(x_1)\,\mathrm{div}_x\Big(\begin{bmatrix} 0 & 0\\ w_1^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_1^{i,\mathrm{bl}}(x) & w_2^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_2^{i,\mathrm{bl}}(x)\end{bmatrix}\Big)\\
A_3^\varepsilon &= \frac{1}{\varepsilon}F^{-T}(x)\sum_i \nabla_x\big(D^i_\delta(x)(\pi^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C^i_\pi(x))\big)\\
A_4^\varepsilon &= -2F^{-T}(x)\,\nabla_x(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)) : F^{-T}(x)\,\nabla(D^i_\delta(x)\phi)\\
&\quad+\begin{bmatrix} 0 & 0\\ w_1^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_1^{i,\mathrm{bl}}(x) & w_2^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_2^{i,\mathrm{bl}}(x)\end{bmatrix} : \nabla(D^i_\delta(x)g''(x_1)\phi(x))\\
&\quad-\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\,\mathrm{div}(F^{-1}(x)F^{-T}(x)\,\nabla\,D^i_\delta(x)\otimes\phi(x)),
\end{aligned}$$

$$
\begin{aligned}
B_1^\varepsilon &= -F^{-1}(x)F^{-T}(x)\,\nabla\big(\sum_i D_\delta^i(x)C^{i,\mathrm{bl}}(x)\big)\\
B_2^\varepsilon &= F^{-1}(x)\Big(F^{-T}(x)\,\nabla[\sum_{i,k} u^{ik}(x)] - \sum_{i,k}\pi^{ik}(x)I\Big)\\
B_3^\varepsilon &= \sum_i F^{-1}(x)F^{-T}(x)\Big[(\nabla\,D_\delta^i(x))\otimes w^i(x,\frac{x}{\varepsilon}) + 2\,\nabla_x\,w^i(x,\frac{x}{\varepsilon})D_\delta^i(x)\\
&\qquad +2(\nabla\,C^{i,\mathrm{bl}}(x))D_\delta^i(x) - \nabla(D_\delta^i(x)C^{i,\mathrm{bl}}(x))\Big]\\
B_4^\varepsilon &= \sum_i \begin{bmatrix} 0 & 0\\ w_1^i(x,\frac{x}{\varepsilon}) & w_2^i(x,\frac{x}{\varepsilon})\end{bmatrix} D_\delta^i(x)g''(x_1) + \begin{bmatrix} 0 & 0\\ C_1^{i,\mathrm{bl}}(x) & C_2^{i,\mathrm{bl}}(x)\end{bmatrix} D_\delta^i(x)g''(x_1)\\
B_5^\varepsilon &= \sum_i F^{-1}(x)F^{-T}(x)\Big[w^i(x,\frac{x}{\varepsilon})\,\nabla\,D_\delta^i(x)e_2\otimes\phi(x) + C^{i,\mathrm{bl}}(x)\,\nabla\,D_\delta^i(x)e_2\otimes\phi(x)\Big].
\end{aligned}
$$

As the derivation of the above terms is not totally standard, we give some hints. We need the following lemma:

5.9 Lemma.
For w sufficiently smooth it holds

$$
\begin{aligned}
&\mathrm{div}_x(F^{-1}(x)F^{-T}(x)\,\nabla_y\,w(x,y))\\
&= \mathrm{div}_y\Big(F^{-1}(x)F^{-T}(x)\,\nabla_x\,w(x,y) - g''(x_1)\begin{bmatrix} 0 & 0\\ w_1(x,y) & w_2(x,y)\end{bmatrix}\Big).
\end{aligned}
$$

Proof. Note that

$$
F^{-1}(x)F^{-T}(x) = \begin{bmatrix} 1 & -g'(x_1)\\ -g'(x_1) & (1+g'(x_1)^2)\end{bmatrix}.
$$

A calculation shows that component-wise

$$
\begin{aligned}
\Big(\mathrm{div}_x(F^{-1}(x)F^{-T}(x)\,\nabla_y\,w(x,y))\Big)_k &= \frac{\partial}{\partial x_1}(\frac{\partial w_k}{\partial y_1} - g'(x_1)\frac{\partial w_k}{\partial y_2})(x,y)\\
&\quad + \frac{\partial}{x_2}(-g'(x_1)\frac{\partial w_k}{\partial y_1} + (1+g'(x_1)^2)\frac{\partial w_k}{\partial y_2})(x,y)\\
&= \frac{\partial}{\partial y_1}(\frac{\partial w_k}{\partial x_1} - g'(x_1)\frac{\partial w_k}{\partial x_2})(x,y)\\
&\quad + \frac{\partial}{y_2}(-g'(x_1)\frac{\partial w_k}{\partial x_1} + (1+g'(x_1)^2)\frac{\partial w_k}{\partial x_2} - g''(x_1)w_k)(x,y)\\
&= \Big(\mathrm{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_x\,w(x,y) - g''(x_1)\begin{bmatrix} 0 & 0\\ w_1(x,y) & w_2(x,y))\end{bmatrix}\Big)_k.
\end{aligned}
$$

♦

When calculating the right hand side of (5.4), the problems are stemming from the volume integral

$$I^\varepsilon = \int\limits_{\Omega^\varepsilon} \operatorname{div}\Big(F^{-1}(x)F^{-T}(x)\,\nabla\big(D^i_\delta(x)(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x))\big)\Big)\phi(x)\;\mathrm{d}x.$$

Using the rules of the transformation lemma 2.7 one obtains

$$\begin{aligned} I^\varepsilon = \int\limits_{\Omega^\varepsilon} &\operatorname{div}\Big(F^{-1}(x)F^{-T}(x)(\nabla D^i_\delta(x)) \otimes \big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\Big)\phi(x) \\ &+ D^i_\delta(x)\operatorname{div}\Big(F^{-1}(x)F^{-T}(x)\,\nabla\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\Big)\phi(x) \\ &F^{-T}(x)\,\nabla\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\cdot F^{-T}(x)\,\nabla D^i_\delta(x)\phi(x)\;\mathrm{d}x. \end{aligned}$$

The first and the last term on the right hand side are integrated by parts, yielding

$$\begin{aligned} &\int\limits_{\Omega^\varepsilon} F^{-T}(x)\,\nabla\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\cdot F^{-T}(x)\,\nabla D^i_\delta(x)\phi(x)\;\mathrm{d}x \\ &\quad= \int\limits_{\Omega^\varepsilon} -\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\operatorname{div}\big(F^{-1}(x)F^{-T}(x)\,\nabla D^i_\delta(x)\otimes\phi(x)\big)\;\mathrm{d}x \\ &\qquad+ \int\limits_{\Sigma} F^{-1}(x)F^{-T}(x)\Big[[w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon})]_S\,\nabla D^i_\delta(x)e_2\otimes\phi(x) \\ &\qquad\qquad+ C^{i,\mathrm{bl}}(x)\,\nabla D^i_\delta(x)e_2\otimes\phi(x)\Big]\;\mathrm{d}\sigma_x. \end{aligned}$$

Note that $[w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon})]_S = w^i(x,\frac{x}{\varepsilon})$.

Next, we treat $\operatorname{div}\Big(F^{-1}(x)F^{-T}(x)\,\nabla\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\Big)$:

$$\begin{aligned} &\operatorname{div}\Big(F^{-1}(x)F^{-T}(x)\,\nabla\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\Big) \\ &\quad= \operatorname{div}\Big(F^{-1}(x)F^{-T}(x)\,\nabla_x\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\Big) \\ &\qquad+ \frac{1}{\varepsilon}\operatorname{div}_x\Big(F^{-1}(x)F^{-T}(x)\,\nabla_y\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\Big) \\ &\qquad+ \frac{1}{\varepsilon^2}\operatorname{div}_y\Big(F^{-1}(x)F^{-T}(x)\,\nabla_y\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\Big). \end{aligned}$$

The last term cancels together with $\frac{1}{\varepsilon^2}F^{-T}(x)\,\nabla_y(D^i_\delta(x)\pi^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}))$, and the first one on the right hand side is integrated by parts. For the remainder the above lemma is used. Remark that due to the presence of the derivative in y-direction the constant $C^{i,\mathrm{bl}}(x)$ can be included in the formula; which gives

$$\begin{aligned}
&\frac{1}{\varepsilon}\operatorname{div}_x\Big(F^{-1}(x)F^{-T}(x)\,\nabla_y\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\Big)\\
&\quad= \frac{1}{\varepsilon}\operatorname{div}_y\Bigg(F^{-1}(x)F^{-T}(x)\,\nabla_x\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\\
&\qquad\quad - g''(x_1)\begin{bmatrix} 0 & 0\\ w_1^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_1^{i,\mathrm{bl}}(x) & w_2^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_2^{i,\mathrm{bl}}(x)\end{bmatrix}\Bigg)\\
&\quad= \operatorname{div}\Bigg(F^{-1}(x)F^{-T}(x)\,\nabla_x\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\\
&\qquad\quad - g''(x_1)\begin{bmatrix} 0 & 0\\ w_1^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_1^{i,\mathrm{bl}}(x) & w_2^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_2^{i,\mathrm{bl}}(x)\end{bmatrix}\Bigg)\\
&\qquad\quad - \operatorname{div}_x\Bigg(F^{-1}(x)F^{-T}(x)\,\nabla_x\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\\
&\qquad\quad - g''(x_1)\begin{bmatrix} 0 & 0\\ w_1^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_1^{i,\mathrm{bl}}(x) & w_2^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_2^{i,\mathrm{bl}}(x)\end{bmatrix}\Bigg).
\end{aligned}$$

Expanding the second terms in the divergences gives, due to the opposed leading sign of the terms

$$\begin{aligned}
&= \operatorname{div}\Big(F^{-1}(x)F^{-T}(x)\,\nabla_x\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\Big)\\
&\quad - g''(x_1)\operatorname{div}\Bigg(\begin{bmatrix} 0 & 0\\ w_1^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_1^{i,\mathrm{bl}}(x) & w_2^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_2^{i,\mathrm{bl}}(x)\end{bmatrix}\Bigg)\\
&\quad - \operatorname{div}_x\Big(F^{-1}(x)F^{-T}(x)\,\nabla_x\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\Big)\\
&\quad - g''(x_1)\operatorname{div}_x\Bigg(\begin{bmatrix} 0 & 0\\ w_1^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_1^{i,\mathrm{bl}}(x) & w_2^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_2^{i,\mathrm{bl}}(x)\end{bmatrix}\Bigg).
\end{aligned}$$

Here, the first two terms can be treated via integration by parts as before. For the last two Assumption 5.1 is used.

The term containing $\mathscr{B}^\varepsilon$ vanishes due to

$$\begin{aligned}
&\Big[\big(F^{-1}(x)F^{-T}(x)\,\nabla\, w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - \varepsilon^{-1}F^{-1}(x)\pi^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon})\big)e_2\Big]_\Sigma\\
&= F^{-1}(x)F^{-T}(x)\,\nabla_x\Big[w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon})\Big]_\Sigma e_2\\
&\qquad + \frac{1}{\varepsilon}\Big[(F^{-1}(x)F^{-T}(x)\,\nabla_y\, w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - F^{-1}(x)\pi^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}))e_2\Big]_\Sigma\\
&= F^{-1}(x)F^{-T}(x)\,\nabla_x\, w^i(x,\frac{x}{\varepsilon})e_2 + \frac{1}{\varepsilon}\Big(F^{-1}(x)F^{-T}(x)\,\nabla_y\, w^i(x,\frac{x}{\varepsilon}) - F^{-1}(x)\pi^i(x,\frac{x}{\varepsilon})\Big)e_2
\end{aligned}$$

$$= F^{-1}(x)[F^{-T}(x)\,\nabla\, w^i(x,\frac{x}{\varepsilon}) - \frac{1}{\varepsilon}\pi^i(x,\frac{x}{\varepsilon})]e_2.$$

Finally, we can estimate the above terms:

$$\left|\int\limits_{\Sigma} B_m^{\varepsilon} e_2 \cdot \phi \, \mathrm{d}\sigma_x\right| \le C\varepsilon^{\frac{1}{2}} \left\|\nabla\,\phi\right\|_{L^2(\Omega_2^{\varepsilon})^4}$$

for $m = 1, \ldots, 4$ and

$$\left|\int\limits_{\Sigma} B_5^{\varepsilon} \, \mathrm{d}\sigma_x\right| \le C\varepsilon^{\frac{1}{2}} \left\|\nabla\,\phi\right\|_{L^2(\Omega_2^{\varepsilon})^4}$$

$$\left|\int\limits_{\Omega^{\varepsilon}} A_1^{\varepsilon} : F^{-T}(x)\,\nabla\,\phi(x)\, \mathrm{d}x\right| \le C\varepsilon^{\frac{1}{2}} \left\|\nabla\,\phi\right\|_{L^2(\Omega^{\varepsilon})^4}$$

$$\left|\int\limits_{\Omega_1} A_2^{\varepsilon} \cdot \phi \, \mathrm{d}x\right| \le C\varepsilon^{\frac{1}{2}} \left\|\phi\right\|_{H^1(\Omega_1)^2}$$

$$\left|\int\limits_{\Omega_2^{\varepsilon}} A_2^{\varepsilon} \cdot \phi \, \mathrm{d}x\right| \le C\varepsilon^{\frac{1}{2}} \left\|\nabla\,\phi\right\|_{L^2(\Omega_2^{\varepsilon})^4}$$

$$\left|\int\limits_{\Omega_1} A_4^{\varepsilon} \cdot \phi \, \mathrm{d}x\right| \le C\varepsilon^{\frac{1}{2}} \left\|\phi\right\|_{H^1(\Omega_1)^2}$$

$$\left|\int\limits_{\Omega_2^{\varepsilon}} A_4^{\varepsilon} \cdot \phi \, \mathrm{d}x\right| \le C\varepsilon^{\frac{1}{2}} \left\|\nabla\,\phi\right\|_{L^2(\Omega_2^{\varepsilon})^4}.$$

5.3.3 Correction of the Pressure

We see that problems arise from A_3^{ε} due to the factor $\frac{1}{\varepsilon}$. Therefore we are going to construct a correction of the pressure in the following way:

Define the following boundary layer problem:

$$\boxed{\begin{aligned}
&-\mathrm{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\beta^{i,\mathrm{bl}}(x,y)) + F^{-T}(x)\,\nabla_y\,\omega^{i,\mathrm{bl}}(x,y)\\
&\qquad = F^{-T}(x)\,\nabla_x\big(D_{\delta}^i(x)(\pi^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon})) - H(x_2)C_{\pi}^i(x)\big) \quad \text{in } \Omega \times Z\\
&\qquad \mathrm{div}_y(F^{-1}(x)\beta^{i,\mathrm{bl}}(x,y)) = 0 \quad \text{in } \Omega \times Z\\
&\qquad [\beta^{i,\mathrm{bl}}(x)]_S(y) = 0 \quad \text{on } \Omega \times S\\
&\big[(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\beta^{i,\mathrm{bl}}(x) - F^{-1}(x)\omega^{i,\mathrm{bl}}(x))e_2\big]_S(y) = 0 \quad \text{on } \Omega \times S\\
&\qquad \beta^{i,\mathrm{bl}}(x,y) = 0 \quad \text{on } \Omega \times \textstyle\bigcup_{k=1}^{\infty}\{\partial Y_S - \binom{0}{k}\}\\
&\qquad \beta^{i,\mathrm{bl}}(x), \omega^{i,\mathrm{bl}}(x) \text{ are 1-periodic in } y_1
\end{aligned}}$$

Appendix A.1 gives the existence of $(\beta^{i,\mathrm{bl}}(x), \omega^{i,\mathrm{bl}}(x)) \in V \cap \mathcal{C}^\infty_{\mathrm{loc}}(Z)^2 \times \mathcal{C}^\infty_{\mathrm{loc}}(Z)$. Furthermore, for fixed $x \in \Omega$ there exist constants $\gamma_0 > 0$, $C^{i,\mathrm{bl}}_\beta(x)$ and $C^i_\omega(x)$ such that

$$e^{\gamma_0|y_2|} \nabla_y \beta^{i,\mathrm{bl}}(x) \in L^2(Z)^4,\ e^{\gamma_0|y_2|}\beta^{i,\mathrm{bl}}(x) \in L^2(Z^-)^2,\ e^{\gamma_0|y_2|}\omega^{i,\mathrm{bl}}(x) \in L^2(Z^-),$$

and we have

$$\begin{aligned} |\beta^{i,\mathrm{bl}}(x,y) - C^{i,\mathrm{bl}}_\beta(x)| &\le Ce^{-\gamma_0|y_2|}, \quad y_2 > y^* \\ |\omega^{i,\mathrm{bl}}(x,y) - C^i_\omega(x)| &\le Ce^{-\gamma_0|y_2|}, \quad y_2 > y^* \\ |\nabla_x(\beta^{i,\mathrm{bl}}(x,y) - C^{i,\mathrm{bl}}_\beta(x))| &\le Ce^{-\gamma_0|y_2|}, \quad y_2 > y^* \\ |\nabla_x(\omega^{i,\mathrm{bl}}(x,y) - C^i_\omega(x))| &\le Ce^{-\gamma_0|y_2|}, \quad y_2 > y^*. \end{aligned}$$

Here we used Assumption 5.1: The right hand side is independent of x and the decay carries over to the derivatives in x.

Due to the exponential decay to 0 in Z^- (see Appendix A.1 and Lemma 5.7) it holds for all $q \ge 1$

$$\begin{aligned} \left\|\beta^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}_\beta(x)\right\|_{L^q(\Omega)^2} &\le C\varepsilon^{\frac{1}{q}} \\ \left\|\omega^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon}) - H(x_2)C^i_\omega(x)\right\|_{L^q(\Omega)} &\le C\varepsilon^{\frac{1}{q}} \\ \left\|\nabla_y \beta^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon})\right\|_{L^q(\Omega)^4} &\le C\varepsilon^{\frac{1}{q}}. \end{aligned}$$

We define the corresponding counterflow to be governed by

$$\boxed{\begin{aligned} -\operatorname{div}(F^{-1}(x)F^{-T}(x)\nabla b^{ik}(x)) + F^{-T}(x)\nabla q^{ik}(x) = 0 &\quad \text{in } \Omega_1 \\ \operatorname{div}(F^{-1}(x)b^{ik}(x)) = 0 &\quad \text{in } \Omega_1 \\ b^{ik}(x_1, +0) = (C^{i,\mathrm{bl}}_{\beta,k})e_k(x_1, 0) &\quad \text{on } \Sigma \\ b^{ik}, q^{ik} \text{ are } L\text{-periodic in } x_1 & \end{aligned}}$$

The existence of a unique solution $(b^{ik}, q^{ik}) \in \mathcal{W} \times L^2_{\mathrm{loc}}(\Omega_1)/\mathbb{R}$ is given in Appendix A.5.

5.10 Remark.
Another possibility to correct the pressure would be the use of a function Q^i satisfying

$$\begin{aligned} \frac{\partial Q^i(x,y)}{\partial y_1} &= \pi^{i,\mathrm{bl}}(x,y) - C^i_\pi(x) \quad \text{in } \Omega \times ([0,1] \times \mathbb{R}_{>0}) \\ Q^i &\text{ is 1-periodic in } y_1 \end{aligned}$$

and then proceeding as in [JM96]. Note that due to Lemma A.18 a solution is given by

$$Q^i(x,y) = \int_0^{y_1} \pi^{i,\mathrm{bl}}(x)(z,y_2)\ \mathrm{d}z - C_\pi^i(x) y_1.$$

For further details, see the article of Jäger and Mikelić cited above.

We can now define

$$\tilde U^\varepsilon = U^\varepsilon - \varepsilon \sum_i \big(\beta^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2) C_\beta^{i,\mathrm{bl}}(x)\big) - \varepsilon H(x_2) \sum_{i,k} b^{ik}(x)$$

and

$$\tilde P^\varepsilon = P^\varepsilon - \sum_i \big(\omega^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2) C_\omega^{i}(x)\big) - \varepsilon H(x_2) \sum_{i,k} q^{ik}(x).$$

Remark that $[\nabla_x \beta^{i,\mathrm{bl}}]_S = \nabla_x [\beta^{i,\mathrm{bl}}]_S = 0$, thus we obtain

$$\begin{aligned}
&\int_{\Omega^\varepsilon} F^{-T}(x)\nabla \tilde U^\varepsilon(x) : F^{-T}(x)\nabla\phi(x)\ \mathrm{d}x - \int_{\Omega^\varepsilon} \tilde P^\varepsilon(x)\,\mathrm{div}(F^{-1}(x)\phi(x))\ \mathrm{d}x \\
&\qquad = \int_\Sigma \big(-F^{-1}(x)\sigma_0 + \varepsilon^{-2} F^{-1}(x) p(x)\big) e_2 \cdot \phi(x)\ \mathrm{d}\sigma_x + \int_{\Omega_2^\varepsilon} \mathscr{A}_1^\varepsilon \cdot \phi(x)\ \mathrm{d}x \\
&\qquad\quad + \int_\Sigma (B_1^\varepsilon + B_2^\varepsilon + B_3^\varepsilon + B_4^\varepsilon) e_2 \cdot \phi(x)\ \mathrm{d}\sigma_x + \int_\Sigma B_5^\varepsilon\ \mathrm{d}\sigma_x \\
&\qquad\quad + \int_{\Omega^\varepsilon} A_1^\varepsilon : F^{-T}\nabla\phi(x)\ \mathrm{d}x + \int_{\Omega^\varepsilon} A_2^\varepsilon \cdot \phi(x)\ \mathrm{d}x + \int_{\Omega^\varepsilon} A_4^\varepsilon\ \mathrm{d}x \qquad (5.5)\\
&\qquad\quad + \varepsilon \int_{\Omega^\varepsilon} \mathfrak{A}_1^\varepsilon : F^{-T}\nabla\phi(x)\ \mathrm{d}x + \int_{\Omega^\varepsilon} \mathfrak{A}_2^\varepsilon \cdot \phi(x)\ \mathrm{d}x + \int_{\Omega^\varepsilon} \mathfrak{A}_3^\varepsilon \cdot \phi(x)\ \mathrm{d}x \\
&\qquad\quad + \varepsilon \int_\Sigma \mathfrak{B}_1^\varepsilon e_2 \cdot \phi(x)\ \mathrm{d}\sigma_x + \int_\Sigma \mathfrak{B}_2^\varepsilon e_2 \cdot \phi(x)\ \mathrm{d}\sigma_x
\end{aligned}$$

with

$$\begin{aligned}
\mathfrak{A}_1^\varepsilon &= \sum_i \nabla_x(\beta^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2) C_\beta^{i,\mathrm{bl}}(x)) \\
\mathfrak{A}_2^\varepsilon &= \sum_i \mathrm{div}_x\Big(F^{-1}(x)F^{-T}(x)\nabla_y\big(\beta^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2) C_\beta^{i,\mathrm{bl}}(x)\big)\Big) \\
\mathfrak{A}_3^\varepsilon &= \sum_i F^{-T}(x)\nabla_x(\omega^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2) C_\omega^{i}(x)),
\end{aligned}$$

$$\mathfrak{B}_1^\varepsilon = \sum_{i,k} F^{-1}(x)(F^{-T}(x) \nabla b^{ik}(x) - q^{ik}(x) I)$$
$$\mathfrak{B}_2^\varepsilon = F^{-1}(x) C_\omega^i(x).$$

We estimate the terms separately: We have

$$\left| \varepsilon \int\limits_{\Omega^\varepsilon} \mathfrak{A}_1^\varepsilon : F^{-T} \nabla \phi(x) \, \mathrm{d}x \right| \leq C \varepsilon^{\frac{3}{2}} \left\| \nabla \phi \right\|_{L^2(\Omega^\varepsilon)^4}$$
$$\left| \int\limits_{\Omega_1} \mathfrak{A}_3^\varepsilon \cdot \phi(x) \, \mathrm{d}x \right| \leq C \varepsilon^{\frac{1}{2}} \left\| \phi \right\|_{H^1(\Omega_1)^2}$$
$$\left| \int\limits_{\Omega_2^\varepsilon} \mathfrak{A}_3^\varepsilon \cdot \phi(x) \, \mathrm{d}x \right| \leq C \varepsilon^{\frac{3}{2}} \left\| \nabla \phi \right\|_{L^2(\Omega_2^\varepsilon)^4}$$

as well as

$$\left| \varepsilon \int\limits_{\Sigma} \mathfrak{B}_1^\varepsilon e_2 \cdot \phi(x) \, \mathrm{d}\sigma_x \right| \leq C \varepsilon^{\frac{3}{2}} \left\| \nabla \phi \right\|_{L^2(\Omega^\varepsilon)^4}$$
$$\left| \int\limits_{\Sigma} \mathfrak{B}_2^\varepsilon e_2 \cdot \phi(x) \, \mathrm{d}\sigma_x \right| \leq C \varepsilon^{\frac{1}{2}} \left\| \nabla \phi \right\|_{L^2(\Omega_2^\varepsilon)^4} .$$

The term $\int_{\Omega^\varepsilon} \mathfrak{A}_2^\varepsilon \cdot \phi(x) \, \mathrm{d}x$ can be estimated similarly to the preceding paragraph by using Lemma 5.9, hence obtaining

$$\left| \int\limits_{\Omega^\varepsilon} \mathfrak{A}_2^\varepsilon \cdot \phi(x) \, \mathrm{d}x \right| \leq C \varepsilon^{\frac{3}{2}} \left\| \phi \right\|_{H^1(\Omega^\varepsilon)^2} .$$

Finally, we see that the right hand side of (5.5) is bounded by

$$C \left\| \nabla \phi^\varepsilon \right\|_{L^2(\Omega_2^\varepsilon)^4} + C \left\| \phi \right\|_{H^1(\Omega_1)^2} + C \varepsilon^{-2} \left| \int\limits_{\Sigma} p e_2 \phi \, \mathrm{d}\sigma_x \right|,$$

such that setting $p = 0$ on Σ was a sensible choice.

However, $\|\mathrm{div}(F^{-1}\tilde{U}^\varepsilon)\|_{L^2(\Omega^\varepsilon)} \leq C$, introducing problems in the estimation of the term $(\int_{\Omega^\varepsilon} P^\varepsilon \, \mathrm{div}(F^{-1}\phi) \, \mathrm{d}x)$ when inserting $\tilde{U}^\varepsilon$ as a test function. Therefore, as a next step we have to correct the divergence.

5.4 Correction of the Divergence

By calculating the transformed divergence of $\tilde{U}^\varepsilon$, we obtain

$$\begin{aligned}
\operatorname{div}(F^{-1}(x)\tilde{U}^\varepsilon(x)) &= -H(-x_2)\sum_i \operatorname{div}_x(F^{-1}(x)D^i(x)w^i(x,\frac{x}{\varepsilon})) \\
&- \sum_i \Big[D^i_\delta(x)\operatorname{div}_x(F^{-1}(x)w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon})) - H(x_2)D^i_\delta(x)\operatorname{div}(F^{-1}(x)C^{i,\mathrm{bl}}(x)) \\
&\qquad + \big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\cdot F^{-T}(x)\,\nabla\, D^i_\delta(x)\Big] \\
&\qquad\qquad + \varepsilon\sum_i \operatorname{div}_x\Big(F^{-1}(x)\big(\beta^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}_\beta(x)\big)\Big).
\end{aligned}$$

The aim of this subsection is to correct all terms from the above formula which do not have an order of at least $\mathscr{O}(\varepsilon)$.

5.4.1 The Compressibility Effect in the Porous Part

In order to correct the term $\operatorname{div}_x(F^{-1}(x)D^i(x)w^i(x,\frac{x}{\varepsilon}))$ appearing in $\tilde{U}^\varepsilon$, we introduce the following auxiliary problems:

$$\boxed{\begin{aligned}
\operatorname{div}_y(F^{-1}(x)\gamma^i(x,y)) &= \operatorname{div}_x(F^{-1}(x)D^i(x)w^i(x,y)) \\
-\frac{1}{|Y^*|}\operatorname{div}_x&\Big(F^{-1}(x)D^i(x)\Big(\int_{Y^*} w^i(x,y)\ \mathrm{d}y\Big)\Big) \qquad \text{in } \Omega\times Y^* \\
\gamma^i(x,y) &= 0 \qquad \text{on } \Omega\times\partial Y_S \\
\gamma^i(x) &\text{ is } Y\text{-periodic in } y
\end{aligned}}$$

Because the right hand side has zero mean value there exists at least one $\gamma^i(x) \in H^1_{0,\#}(Y^*)^2$ satisfying the equation, cf. the proof of Lemma 3.14. By regularity results and the theory of parameter dependent differential equations, we obtain

$$\gamma^i \in \mathcal{C}^\infty_{\mathrm{loc}}(\Omega, H^1_{0,\#}(Y^*)^2) \quad \text{and} \quad \gamma^i(x) \in \mathcal{C}^\infty(Y^*)^2.$$

Since

$$\operatorname{div}_x(F^{-1}(x)D^i(x)w^i(x,y)) = F^{-T}(x)\,\nabla_x\, D^i(x)\cdot w^i(x,y) + D^i(x)\operatorname{div}_x(F^{-1}(x)w^i(x,y)),$$

we can obtain a solution γ^i in the form

$$\gamma^i(x,y) = \sum_j \big(F^{-T}\,\nabla\, D^i(x)\big)_j \hat{\gamma}^{ij}(x,y) + D^i(x)\tilde{\gamma}^i(x,y),$$

where $\hat{\gamma}^{ij}$ solves

$$\begin{aligned} \operatorname{div}_y(F^{-1}(x)\hat{\gamma}^{ij}(x,y)) &= w^i_j(x,y) - \frac{A_{ij}(x)}{|Y^*|} && \text{in } \Omega\times Y^* \\ \hat{\gamma}^{ij}(x,y) &= 0 && \text{on } \partial Y_S \\ \hat{\gamma}^{ij}(x) &\text{ is } Y\text{-periodic in } y \end{aligned}$$

and $\tilde{\gamma}^i(x)$ fulfills

$$\begin{aligned} \operatorname{div}_y(F^{-1}(x)\tilde{\gamma}^i) = \operatorname{div}_x(F^{-1}(x)w^i(x,y)) & \\ -\frac{1}{|Y^*|}\operatorname{div}_x\Big(F^{-1}(x)\Big(\int_{Y^*} w^i(x,y)\;\mathrm{d}y\Big)\Big) & \quad \text{in } \Omega\times Y^* \\ \tilde{\gamma}^i(x,y) = 0 & \quad \text{on } \Omega\times\partial Y_S \\ \tilde{\gamma}^i(x) \text{ is } Y\text{-periodic in } y & \end{aligned}$$

By the theory of parameter-dependent PDEs in Section 4.1, the solutions of the above problems are in $\mathcal{C}^\infty_{\mathrm{loc}}(\Omega, H^1_{0,\#}(Y^*)^2)$ with $\hat{\gamma}^i(x)$ and $\tilde{\gamma}^i(x)$ in $\mathcal{C}^\infty(Y^*)^2$, due to regularity results. Since the two functions depend on x only via the x_1-variable, we even have $\hat{\gamma}, \tilde{\gamma} \in \mathcal{C}^\infty(\Omega, H^1_{0,\#}(Y^*)^2)$, and $\hat{\gamma}^i$ as well as $\tilde{\gamma}^i$ are bounded from above. Now due to its form, we get an exponential decay of γ^i towards 0 for $x_2 \longrightarrow -\infty$.

Finally, define the corresponding boundary layer

$$\boxed{\begin{aligned} -\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\gamma^{i,\mathrm{bl}}(x,y)) + F^{-T}(x)\,\nabla_y\,\tilde{\pi}^{i,\mathrm{bl}}(x,y) = 0 & \quad \text{in } \Omega\times Z \\ \operatorname{div}_y(F^{-1}(x)\gamma^{i,\mathrm{bl}}(x,y)) = 0 & \quad \text{in } \Omega\times Z \\ [\gamma^{i,\mathrm{bl}}(x)]_S(y) = \gamma^i(x,y) & \quad \text{on } \Omega\times S \\ [(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\gamma^{i,\mathrm{bl}}(x) - F^{-1}(x)\tilde{\pi}^{i,\mathrm{bl}}(x))e_2]_S(y) & \\ = (F^{-1}(x)F^{-T}(x)\,\nabla_y\,\gamma^i(x))e_2(y) & \quad \text{on } \Omega\times S \\ \gamma^{i,\mathrm{bl}}(x,y) = 0 \quad \text{on } \Omega\times \textstyle\bigcup_{k=1}^\infty\{\partial Y_S - \binom{0}{k}\} & \\ \gamma^{i,\mathrm{bl}}(x), \tilde{\pi}^{i,\mathrm{bl}}(x) \text{ are 1-periodic in } y_1 & \end{aligned}}$$

Appendix A.1 gives the existence of $(\gamma^{i,\mathrm{bl}}(x), \tilde{\pi}^{i,\mathrm{bl}}(x)) \in V \cap \mathcal{C}^\infty_{\mathrm{loc}}(Z)^2 \times \mathcal{C}^\infty_{\mathrm{loc}}(Z)$. Furthermore, for fixed $x \in \Omega$ there exist constants $\gamma_0 > 0$, $\tilde{C}^{i,\mathrm{bl}}(x)$ and $\tilde{C}^i_{\tilde{\pi}}(x)$ such that

$$e^{\gamma_0|y_2|}\,\nabla_y\,\gamma^{i,\mathrm{bl}}(x) \in L^2(Z)^4,\;\; e^{\gamma_0|y_2|}\gamma^{i,\mathrm{bl}}(x) \in L^2(Z^-)^2,\;\; e^{\gamma_0|y_2|}\tilde{\pi}^{i,\mathrm{bl}}(x) \in L^2(Z^-),$$

and we have

$$
\begin{aligned}
|\gamma^{i,\mathrm{bl}}(x,y) - \tilde{C}^{i,\mathrm{bl}}(x)| &\le Ce^{-\gamma_0|y_2|}, \quad y_2 > y^* \\
|\tilde{\pi}^{i,\mathrm{bl}}(x,y) - \tilde{C}^{i}_{\tilde{\pi}}(x)| &\le Ce^{-\gamma_0|y_2|}, \quad y_2 > y^* \\
|\nabla_x(\gamma^{i,\mathrm{bl}}(x,y) - \tilde{C}^{i,\mathrm{bl}}(x))| &\le Ce^{-\gamma_0|y_2|}, \quad y_2 > y^* \\
|\nabla_x(\tilde{\pi}^{i,\mathrm{bl}}(x,y) - \tilde{C}^{i}_{\tilde{\pi}}(x))| &\le Ce^{-\gamma_0|y_2|}, \quad y_2 > y^*.
\end{aligned}
$$

Here we used Assumption 5.1: The right hand side is independent of x and the decay carries over to the derivatives in x.

Due to the exponential decay to 0 in Z^- (see Appendix A.1 and Lemma 5.7) it holds for all $q \ge 1$

$$
\begin{aligned}
\left\| \gamma^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon}) - H(x_2)\tilde{C}^{i,\mathrm{bl}}(x) \right\|_{L^q(\Omega)^2} &\le C\varepsilon^{\frac{1}{q}} \\
\left\| \tilde{\pi}^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon}) - H(x_2)\tilde{C}^{i}_{\tilde{\pi}}(x) \right\|_{L^q(\Omega)} &\le C\varepsilon^{\frac{1}{q}} \\
\left\| \nabla_y\, \gamma^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon}) \right\|_{L^q(\Omega)^4} &\le C\varepsilon^{\frac{1}{q}}.
\end{aligned}
$$

Finally, define the counterflow as

$$
\boxed{
\begin{aligned}
-\operatorname{div}(F^{-1}(x)F^{-T}(x)\,\nabla\,\tilde{u}^{ik}(x)) + F^{-T}(x)\,\nabla\,\tilde{\pi}^{ik}(x) = 0 &\quad \text{in } \Omega_1 \\
\operatorname{div}(F^{-1}(x)\tilde{u}^{ik}(x)) = 0 &\quad \text{in } \Omega_1 \\
\tilde{u}^{ik}(x_1, +0) = (\tilde{C}^{i,\mathrm{bl}}_k)e_k(x_1, 0) &\quad \text{on } \Sigma \\
\tilde{u}^{ik}, \tilde{\pi}^{ik} \text{ are } L\text{-periodic in } x_1 &
\end{aligned}
}
$$

Note that the function $\tilde{u}^{ik}$ corresponds to a construction of the form $\tilde{C}^{i,\mathrm{bl}}_k\tilde{u}^{ik}$ in [JM96], and π^{ik} corresponds similarly to $\tilde{C}^{i,\mathrm{bl}}_k\tilde{\pi}^{ik}$. There exists a unique solution $(\tilde{u}^{ik}, \tilde{\pi}^{ik}) \in \mathcal{W} \times L^2_{\mathrm{loc}}(\Omega_1)/\mathbb{R}$, cf. Appendix A.5.

Define

$$
\begin{aligned}
U^\varepsilon_1(x) = \tilde{U}^\varepsilon + \varepsilon H(-x_2)\sum_i \gamma^i(x, \frac{x}{\varepsilon}) + \varepsilon \sum_i \big(\gamma^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon}) - H(x_2)\tilde{C}^{i,\mathrm{bl}}(x)\big) \\
+ \varepsilon H(x_2) \sum_{i,k} \tilde{u}^{ik}(x)
\end{aligned}
$$

and

$$
P^\varepsilon_1 = \tilde{P}^\varepsilon + \sum_i \big(\tilde{\pi}^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon}) - \tilde{C}^{i}_{\tilde{\pi}}(x)\big) + \varepsilon H(x_2) \sum_{i,k} \tilde{\pi}^{ik}(x).
$$

Note that

$$\sum_i \operatorname{div}_x\Big(F^{-1}(x)D^i(x)\Big(\int_{Y^*} w^i(x,y)\,\mathrm{d}y\Big)\Big) = \operatorname{div}_x\Big(F^{-1}(x)\sum_i [A_i(x)D^i(x)]\Big)$$
$$= \operatorname{div}_x\Big(F^{-1}(x)A(x)\big(F^T(x)l - \nabla p(x)\big)\Big) = 0$$

by construction of the transformed Darcy velocity. Here A_i denotes the i-th column of the permeability tensor A. Thus

$$\begin{aligned}\operatorname{div}(F^{-1}(x)U_1^\varepsilon(x)) = &-\sum_i \operatorname{div}_x\Big(F^{-1}(x)D_\delta^i(x)\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big)\Big)\\ &+ \varepsilon H(-x_2)\sum_i \operatorname{div}_x(F^{-1}(x)\gamma^i(x,\frac{x}{\varepsilon}))\\ &+ \varepsilon\sum_i \operatorname{div}_x\Big(F^{-1}(x)\big(\beta^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_\beta^{i,\mathrm{bl}}(x)\big)\Big)\\ &+ \varepsilon\sum_i \operatorname{div}_x\Big(F^{-1}(x)\big(\gamma^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)\tilde{C}^{i,\mathrm{bl}}(x)\big)\Big).\end{aligned}$$

5.4.2 The Compressibility Effect due to the Boundary Layer Functions

We see that the order of $\operatorname{div}(F^{-1}(x)U_1^\varepsilon(x))$ is $\mathscr{O}(\varepsilon^{\frac{1}{2}})$, dominated by the boundary layer term $w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)$. That is why we need to construct an additional correction. Set

$$C_\theta^{i,\mathrm{bl}}(x) = F(x)\Big(\int_{Z_{\mathrm{BL}}} \operatorname{div}_x\Big(F^{-1}(x)D_\delta^i(x)\big[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)\big]\Big)\,\mathrm{d}y\Big)e_2.$$

Find θ^i such that

$$\boxed{\begin{aligned}\operatorname{div}_y(F^{-1}(x)\theta^i(x,y)) = \operatorname{div}_x\Big(F^{-1}(x)D_\delta^i(x)\big[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)\big]\Big) \quad &\text{in } \Omega\times Z\\ \theta^i(x,y) = 0 \quad &\text{on } \Omega\times \textstyle\bigcup_{k=1}^\infty\{\partial Y_S - \binom{0}{k}\}\\ [\theta^i]_S(x,y) = C_\theta^{i,\mathrm{bl}}(x) \quad &\text{on } \Omega\times S\\ \theta^i(x) \text{ is 1-periodic in } y_1 &\end{aligned}}$$

Proposition A.34 ensures the existence of at least one $\theta^i(x) \in H^1(Z)^2 \cap \mathcal{C}^\infty_{\mathrm{loc}}(Z)^2$, having exponential decay towards 0 for $|y_2| \longrightarrow \infty$ as well as for $|x_2| \longrightarrow \infty$. In order to correct the jump of θ^i across Σ in Ω_1, we define the following counterflow:

$$\boxed{\begin{aligned}-\operatorname{div}(F^{-1}(x)F^{-T}(x)\nabla d^{ik}(x)) + F^{-T}(x)\nabla g^{ik}(x) = 0 \quad &\text{in } \Omega_1\\ \operatorname{div}(F^{-1}(x)d^{ik}(x)) = 0 \quad &\text{in } \Omega_1\\ d^{ik}(x_1,+0) = (C_{\theta,k}^{i,\mathrm{bl}})e_k(x_1,0) \quad &\text{on } \Sigma\\ d^{ik}, g^{ik} \text{ are } L\text{-periodic in } x_1 &\end{aligned}}$$

Note again that the function u^{ik} corresponds to the construction $C^{i,\mathrm{bl}}_{\theta,k} d^{ik}$ in [JM96], and π^{ik} corresponds to $C^{i,\mathrm{bl}}_{\theta,k} g^{ik}$. There exists a unique solution $(d^{ik}, g^{ik}) \in \mathcal{W} \times L^2_{\mathrm{loc}}(\Omega_1)/\mathbb{R}$, cf. Appendix A.5.

Now define

$$U_2^\varepsilon(x) = U_1^\varepsilon(x) + \varepsilon \sum_i \theta^i(x, \frac{x}{\varepsilon}) - \varepsilon H(x_2) \sum_{i,k} d^{ik}(x)$$

and

$$P_2^\varepsilon(x) = P_1^\varepsilon(x) - \varepsilon H(x_2) \sum_{i,k} g^{ik}(x).$$

Inserting these functions and calculating the right hand side of the variational equation (5.3) yields

$$\begin{aligned}
&\int_{\Omega^\varepsilon} F^{-T}(x)\, \nabla U_2^\varepsilon(x) : F^{-T}(x)\, \nabla \phi(x)\ \mathrm{d}x - \int_{\Omega^\varepsilon} P_2^\varepsilon(x) \operatorname{div}(F^{-1}(x)\phi(x))\ \mathrm{d}x \\
&\quad = \int_\Sigma (-F^{-1}(x)\sigma_0 + \varepsilon^{-2} F^{-1}(x) p) e_2 \cdot \phi(x)\ \mathrm{d}\sigma_x + \int_{\Omega_2^\varepsilon} \mathscr{A}_1^\varepsilon \cdot \phi(x)\ \mathrm{d}x \\
&\qquad + \int_\Sigma (B_1^\varepsilon + B_2^\varepsilon + B_3^\varepsilon + B_4^\varepsilon) e_2 \cdot \phi(x)\ \mathrm{d}\sigma_x + \int_\Sigma B_5^\varepsilon\ \mathrm{d}\sigma_x \\
&\qquad + \int_{\Omega^\varepsilon} A_1^\varepsilon : F^{-T}\, \nabla \phi(x)\ \mathrm{d}x + \int_{\Omega^\varepsilon} A_2^\varepsilon \cdot \phi(x)\ \mathrm{d}x + \int_{\Omega^\varepsilon} A_3^\varepsilon \cdot \phi(x)\ \mathrm{d}x \\
&\qquad + \int_{\Omega^\varepsilon} A_4^\varepsilon\ \mathrm{d}x + \varepsilon \int_{\Omega^\varepsilon} \mathfrak{A}_1^\varepsilon : F^{-T}\, \nabla \phi(x)\ \mathrm{d}x + \int_{\Omega^\varepsilon} \mathfrak{A}_2^\varepsilon \cdot \phi(x)\ \mathrm{d}x \qquad (5.6) \\
&\qquad + \int_{\Omega^\varepsilon} \mathfrak{A}_3^\varepsilon \cdot \phi(x)\ \mathrm{d}x + \varepsilon \int_\Sigma \mathfrak{B}_1^\varepsilon e_2 \cdot \phi(x)\ \mathrm{d}\sigma_x + \int_\Sigma \mathfrak{B}_2^\varepsilon e_2 \cdot \phi(x)\ \mathrm{d}\sigma_x \\
&\qquad + \varepsilon \int_{\Omega_2^\varepsilon} \mathcal{A}_1^\varepsilon : F^{-T}\, \nabla \phi(x)\ \mathrm{d}x + \varepsilon \int_{\Omega^\varepsilon} \mathcal{A}_2^\varepsilon : F^{-T}\, \nabla \phi(x)\ \mathrm{d}x \\
&\qquad + \varepsilon \int_{\Omega^\varepsilon} \mathcal{A}_3^\varepsilon : F^{-T}(x)\, \nabla \phi(x)\ \mathrm{d}x + \int_{\Omega^\varepsilon} \mathcal{A}_4^\varepsilon \cdot \phi(x)\ \mathrm{d}x \\
&\qquad + \int_{\Omega^\varepsilon} \mathcal{A}_5^\varepsilon \cdot \phi(x)\ \mathrm{d}x + \int_\Sigma \mathcal{B}_1^\varepsilon e_2 \cdot \phi(x)\ \mathrm{d}\sigma_x + \varepsilon \int_\Sigma \mathcal{B}_2^\varepsilon e_2 \cdot \phi(x)\ \mathrm{d}x,
\end{aligned}$$

where

$$\begin{aligned}
\mathcal{A}_1^\varepsilon &= \sum_i F^{-T}(x)\, \nabla \gamma^i(x, \frac{x}{\varepsilon}) \\
\mathcal{A}_2^\varepsilon &= \sum_i F^{-T}(x)\, \nabla \theta^i(x, \frac{x}{\varepsilon}),
\end{aligned}$$

$$\mathcal{A}_3^\varepsilon = \sum_i F^{-T}(x)\,\nabla_x\big(\gamma^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)\tilde{C}^{i,\mathrm{bl}}(x)\big)$$

$$\mathcal{A}_4^\varepsilon = \sum_i F^{-T}\,\nabla_x\big(\tilde{\pi}^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - \tilde{C}^i_{\tilde{\pi}}(x)\big)$$

$$\mathcal{A}_5^\varepsilon = \sum_i \mathrm{div}_x\Big(F^{-1}(x)F^{-T}(x)\,\nabla_y\big(\gamma^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)\tilde{C}^{i,\mathrm{bl}}(x)\big)\Big)$$

and

$$\mathcal{B}_1^\varepsilon = -\sum_i F^{-1}(x)F^{-T}(x)\big[\nabla_y\,\gamma^i(x,\frac{x}{\varepsilon}) + \nabla_y\,\theta^i(x,\frac{x}{\varepsilon})\big]$$

$$\mathcal{B}_2^\varepsilon = -\sum_{i,k} F^{-1}(x)\Big[F^{-T}(x)\,\nabla\,\tilde{u}^{ik}(x) + F^{-T}(x)\,\nabla\,d^{ik}(x) - \tilde{\pi}^{ik}(x)I - g^{ik}(x)I\Big].$$

We obtain the following estimates:

$$\Big|\varepsilon\int_{\Omega_2^\varepsilon} \mathcal{A}_1^\varepsilon : F^{-T}\,\nabla\,\phi(x)\ \mathrm{d}x\Big| \le C\,\|\nabla\,\phi\|_{L^2(\Omega_2^\varepsilon)^4}$$

$$\Big|\varepsilon\int_{\Omega^\varepsilon} \mathcal{A}_2^\varepsilon : F^{-T}\,\nabla\,\phi(x)\ \mathrm{d}x\Big| \le C\,\|\nabla\,\phi\|_{L^2(\Omega^\varepsilon)^4}$$

$$\Big|\varepsilon\int_{\Omega^\varepsilon} \mathcal{A}_3^\varepsilon : F^{-T}(x)\,\nabla\,\phi(x)\ \mathrm{d}x\Big| \le C\varepsilon^{\frac{3}{2}}\,\|\nabla\,\phi\|_{L^2(\Omega^\varepsilon)^4}$$

$$\Big|\varepsilon\int_{\Omega^\varepsilon} \mathcal{A}_4^\varepsilon \cdot \phi(x)\ \mathrm{d}x\Big| \le C\varepsilon^{\frac{3}{2}}\,\|\phi\|_{H^1(\Omega^\varepsilon)^2}$$

and

$$\Big|\varepsilon\int_{\Sigma} \mathcal{B}_j^\varepsilon e_2 \cdot \phi(x)\ \mathrm{d}\sigma_x\Big| \le C\varepsilon^{\frac{3}{2}}\,\|\nabla\,\phi\|_{L^2(\Omega_2^\varepsilon)^4}$$

for $j = 1, 2$.

The estimation of the term $(\frac{1}{\varepsilon}\int_{\Omega^\varepsilon} \mathcal{A}_5^\varepsilon \cdot \phi(x)\ \mathrm{d}x)$ goes along the same lines as that of $\frac{1}{\varepsilon}\,\mathrm{div}_x(F^{-1}(x)F^{-T}(x)\,\nabla_y\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\big))$ above (cf. page 75); therefore we obtain the estimate

$$\Big|\int_{\Omega^\varepsilon} \mathcal{A}_5^\varepsilon \cdot \phi(x)\ \mathrm{d}x\Big| \le C\varepsilon^{\frac{3}{2}}\,\|\phi\|_{H^1(\Omega^\varepsilon)^2}\,.$$

5.5 Estimation of the Corrected Solutions

5.5.1 Estimates for the Velocity

Now we can insert U_2^ε as a test function in (5.6) in order to obtain an estimate in Ω^ε. Using Proposition A.32 and the above inequalities we arrive at

$$\begin{aligned} k_F \left\| \nabla U_2^\varepsilon \right\|_{L^2(\Omega^\varepsilon)^4}^2 &\le \left| \int_{\Omega^\varepsilon} F^{-T} \nabla U_2^\varepsilon : F^{-T} \nabla U_2^\varepsilon \, \mathrm{d}x \right| \\ &\le \frac{C}{\varepsilon} \left[\left\| \nabla U_2^\varepsilon \right\|_{L^2(\Omega^\varepsilon)^4} + C \right] \left\| \operatorname{div}(F^{-1} U_2^\varepsilon) \right\|_{L^2(\Omega^\varepsilon)} + C \left\| \nabla U_2^\varepsilon \right\|_{L^2(\Omega^\varepsilon)^4} , \end{aligned} \tag{5.7}$$

where k_F is the constant from Lemma 3.12.

5.11 Proposition.
There exists a constant C such that

$$\left\| U_2^\varepsilon \right\|_{L^2(\Omega_2^\varepsilon)^2} + \left\| \operatorname{div}(F^{-1} U_2^\varepsilon) \right\|_{L^2(\Omega_2^\varepsilon)} \le C\varepsilon$$

$$\left\| U_2^\varepsilon \right\|_{L^2(\Sigma)^2} \le C\sqrt{\varepsilon}$$

Proof. First notice that $\left\| \operatorname{div}(F^{-1} U_2^\varepsilon) \right\|_{L^2(\Omega^\varepsilon)} \le C\varepsilon$ by construction of the divergence-correction.

Due to the choice of the function spaces and the auxiliary problems it holds $\nabla U_2^\varepsilon \in L^2(\Omega^\varepsilon)^4$. Thus for each fixed $\varepsilon > 0$ there exists a constant $C(\varepsilon)$, depending on ε, such that $\left\| \nabla U_2^\varepsilon \right\|_{L^2(\Omega^\varepsilon)^4} \le C(\varepsilon)$. Consider for a moment only those ε such that $\left\| \nabla U_2^\varepsilon \right\|_{L^2(\Omega^\varepsilon)^4} \ge 1$. For these ε, the inequality (5.7) is equivalent to

$$\left\| \nabla U_2^\varepsilon \right\|_{L^2(\Omega^\varepsilon)^4} \le C + \frac{C}{\left\| \nabla U_2^\varepsilon \right\|_{L^2(\Omega^\varepsilon)^4}} \le 2C.$$

Therefore the whole sequence is bounded by $C' := \max\{2C, 1\}$, independent of ε. Finally, due to Poincaré's inequality,

$$\left\| U_2^\varepsilon \right\|_{L^2(\Omega_2^\varepsilon)^2} \le C\varepsilon \left\| \nabla U_2^\varepsilon \right\|_{L^2(\Omega_2^\varepsilon)^4} \le C\varepsilon,$$

which concludes the proof of the first inequality.

The second one is a consequence of the trace estimate

$$\left\| U_2^\varepsilon \right\|_{L^2(\Sigma)^2} \le C\sqrt{\varepsilon} \left\| \nabla U_2^\varepsilon \right\|_{L^2(\Omega_2^\varepsilon)^4} \le C\sqrt{\varepsilon}. \qquad \blacklozenge$$

In the free fluid domain, we want to use the estimates for the very weak solution of the transformed Stokes equations, see Appendix A.2.

A calculation shows that in Ω_1 we have

$$-\operatorname{div}(F^{-1}F^{-T}\nabla U_2^\varepsilon) + F^{-T}\nabla P_2^\varepsilon = \Phi_1^\varepsilon + \operatorname{div}(F^{-1}\Phi_2^\varepsilon)$$
$$\operatorname{div}(F^{-1}U_2^\varepsilon) = \Theta^\varepsilon,$$

with

$$\begin{aligned}
\Phi_1^\varepsilon(x) = &-\sum_i \operatorname{div}_x\Big(F^{-1}(x)F^{-T}(x)\nabla_y\big(\beta^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_\beta^{i,\mathrm{bl}}(x)\big)\Big)\\
&-\sum_i \operatorname{div}_x\big(F^{-1}(x)F^{-T}(x)\nabla_y\gamma^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon})\big)\\
&+\sum_i F^{-T}(x)\nabla_x\big(\omega^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C_\omega^i(x)\big)\\
&+\sum_i F^{-T}(x)\nabla_x\big(\tilde\pi^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - \tilde C_{\tilde\pi}^i(x)\big)\\
&-\sum_i \operatorname{div}_x\Big(F^{-1}(x)F^{-T}(x)\nabla_x\Big(D_\delta^i(x)\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C^{i,\mathrm{bl}}(x)\big)\Big)\Big)\\
&-\sum_i g''(x_1)\operatorname{div}_x\left(\begin{bmatrix} 0 & 0\\ D_\delta^i(x)(w_1^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C_1^{i,\mathrm{bl}}(x)) & D_\delta^i(x)(w_2^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C_2^{i,\mathrm{bl}}(x))\end{bmatrix}\right)
\end{aligned}$$

as well as

$$\begin{aligned}
\Phi_2^\varepsilon(x) = &\,2\sum_i F^{-T}(x)\nabla_x\Big(D_\delta^i(x)\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C^{i,\mathrm{bl}}(x)\big)\Big)\\
&-\sum_i F(x)g''(x_1)\left(\begin{bmatrix} 0 & 0\\ D_\delta^i(x)(w_1^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C_1^{i,\mathrm{bl}}(x)) & D_\delta^i(x)(w_2^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C_2^{i,\mathrm{bl}}(x))\end{bmatrix}\right)\\
&-\varepsilon\sum_i F^{-T}\nabla_x\big(\beta^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C_\beta^{i,\mathrm{bl}}(x)\big)\\
&-\varepsilon\sum_i F^{-T}(x)\nabla_x\big(\gamma^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - \tilde C^{i,\mathrm{bl}}(x)\big) - \varepsilon\sum_i F^{-T}(x)\nabla\theta^i(x,\frac{x}{\varepsilon})
\end{aligned}$$

and

$$\begin{aligned}
\Theta^\varepsilon(x) = &\,\varepsilon\sum_i \operatorname{div}_x\Big(F^{-1}(x)\big(\gamma^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - \tilde C^{i,\mathrm{bl}}(x)\big)\Big) + \varepsilon\sum_i \operatorname{div}_x(F^{-1}(x)\theta^i(x,\frac{x}{\varepsilon}))\\
&+\varepsilon\sum_i \operatorname{div}_x\Big(F^{-1}(x)\big(\beta^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C_\beta^{i,\mathrm{bl}}(x)\big)\Big),
\end{aligned}$$

where we used Lemma 5.9 on the term (cf. page 75)

$$\frac{1}{\varepsilon}\sum_i \operatorname{div}_x\Big(F^{-1}(x)F^{-T}(x)\nabla_y\Big(D_\delta^i(x)\big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C^{i,\mathrm{bl}}(x)\big)\Big)\Big).$$

Due to the decay of the terms on the right hand sides, we get an estimate of the form $\|\Phi_i^\varepsilon\|_{L^2(\Omega_1)^{2i}} \leq C\varepsilon^{\frac{1}{2}}$, $i = 1, 2$. However, to use the estimates derived from the theory of the very weak solutions, this is not sufficient and we need the following lemma:

5.12 Lemma.
Let w be a boundary layer function of the above type stabilizing to a constant C_w such that

$$|w(\frac{x}{\varepsilon}) - C_w| \leq Ce^{-\gamma_0 \frac{x_2}{\varepsilon}}$$

in Z^+ and let $0 < \delta < \gamma_0$. Then it holds

$$\left\| e^{\delta x_2}(w(\frac{x}{\varepsilon}) - C_w) \right\|_{L^2(\Omega_1)^2} \leq C\varepsilon^{\frac{1}{2}}.$$

Proof. Assume w.l.o.g. $\varepsilon < 1$. By the monotonicity of the exponential function it holds

$$e^{\delta x_2}|w(\frac{x}{\varepsilon}) - C_w| \leq Ce^{\delta x_2 - \gamma_0 \frac{x_2}{\varepsilon}} \leq C^{(\delta - \gamma_0)\frac{x_2}{\varepsilon}}.$$

Since $(\delta - \gamma_0) < 0$, arguing as in the proof of Lemma 5.7 yields the asserted inequality. ♦

Finally, we arrive at the follwing estimates:

$$\|e^{\delta x_2}\Phi_1^\varepsilon\|_{L^2(\Omega_1)^2} \leq C\varepsilon^{\frac{1}{2}}, \quad \|e^{\delta x_2}\Phi_2^\varepsilon\|_{L^2(\Omega_1)^4} \leq C\varepsilon^{\frac{1}{2}}, \quad \|e^{\delta x_2}\Theta^\varepsilon\|_{L^2(\Omega_1)} \leq C\varepsilon^{\frac{1}{2}}.$$

Since $(1 + x_2) \leq Ce^{\delta x_2}$ for a certain constant C, we can use Proposition A.28 to obtain the following result:

5.13 Proposition.
Let Assumption A.22 hold (e.g. the function g describing the boundary is point-symmetric with respect to $\frac{L}{2}$ on $[0, L]$). Then we have the following estimate for U_2^ε:

$$\left\|(1 + x_2)^{-1}U_2^\varepsilon\right\|_{L^2(\Omega_1)^2} \leq C\varepsilon^{\frac{1}{2}}.$$

5.5.2 Estimates for the Pressure

The estimation of the pressure is much easier. As in [JM96], we consider the pressure $P_0^\varepsilon := \varepsilon^2 P_2^\varepsilon$, where p^ε is replaced by the extended pressure $\tilde{p}^\varepsilon$, cf. Appendix A.3. We obtain

5.14 Proposition.
For the pressure P_0^ε it holds

$$\left\|(1 + |x_2|)^{-1}P_0^\varepsilon\right\|_{L^2(\Omega)} \leq C\varepsilon.$$

Proof. The inequality is a consequence of Proposition A.32: We have

$$\left\| \frac{P_2^\varepsilon}{1+|x_2|} \right\|_{L^2(\Omega)} \le \frac{C}{\varepsilon} \Big[\|\nabla U_2^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} + \|(1+|x_2|)\Phi_1^\varepsilon\|_{L^2(\Omega_1)^2} + \varepsilon \, \|\Phi_1^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \|\Phi_2^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} \Big],$$

where Φ_1^ε and Φ_2^ε are defined as above. The norms on the right hand side are bounded independent of ε, thus scaling by ε^2 gives the desired result. ♦

5.6 Main Results

We summarize our results:

5.15 Theorem.
Define the corrected velocity as

$$\begin{aligned} U_0^\varepsilon(x) = {} & \frac{u^\varepsilon(x)}{\varepsilon^2} - H(x_2)u_0(x) - H(-x_2)\sum_i D^i(x) w^i(x, \frac{x}{\varepsilon}) \\ & - \sum_i D_\delta^i(x)\Big(w^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C^{i,\mathrm{bl}}(x)\Big) - H(x_2)\sum_{i,k} u^{ik}(x) \\ & - \varepsilon \sum_i \Big(\beta^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_\beta^{i,\mathrm{bl}}(x)\Big) - \varepsilon H(x_2)\sum_{i,k} b^{ik}(x) \\ & + \varepsilon H(-x_2)\sum_i \gamma^i(x,\frac{x}{\varepsilon}) + \varepsilon \sum_i \Big(\gamma^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)\tilde{C}^{i,\mathrm{bl}}(x)\Big) \\ & + \varepsilon H(x_2)\sum_{i,k}\tilde{u}^{ik}(x) + \varepsilon\sum_i \theta^i(x,\frac{x}{\varepsilon}) - \varepsilon H(x_2)\sum_{i,k} d^{ik}(x) \end{aligned}$$

and the corrected pressure as

$$\begin{aligned} P_0^\varepsilon(x) = {} & \tilde{p}^\varepsilon(x) - H(x_2)\varepsilon^2\pi_0(x) - H(-x_2)\Big[p + \varepsilon\sum_i D^i(x)\pi^i(x,\frac{x}{\varepsilon})\Big] \\ & - \varepsilon\sum_i D_\delta^i(x)\Big(\pi^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - C_\pi^i(x)\Big) - \varepsilon^2 H(x_2)\sum_{i,k}\pi^{ik}(x) \\ & - \varepsilon^2\sum_i\Big(\omega^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - H(x_2)C_\omega^i(x)\Big) - \varepsilon^3 H(x_2)\sum_{i,k} q^{ik}(x) \\ & + \varepsilon^2\sum_i\Big(\tilde{\pi}^{i,\mathrm{bl}}(x,\frac{x}{\varepsilon}) - \tilde{C}_{\tilde{\pi}}^i(x)\Big) + \varepsilon^3 H(x_2)\sum_{i,k}\tilde{\pi}^{ik}(x) - \varepsilon^3 H(x_2)\sum_{i,k} g^{ik}(x). \end{aligned}$$

Let Assumption A.22 hold (e.g. the function g describing the boundary is point-symmetric with respect to $\frac{L}{2}$ on $[0,L]$). Then we have the following estimates, where C is a constant *independent of ε:*

$$\begin{aligned} \left\|(1+x_2)^{-1}U_0^\varepsilon\right\|_{L^2(\Omega_1)^2} &\leq C\varepsilon^{\frac{1}{2}} \\ \left\|\operatorname{div}(F^{-1}U_0^\varepsilon)\right\|_{L^2(\Omega_1)} &\leq C\varepsilon \\ \|U_0^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \left\|\operatorname{div}(F^{-1}U_0^\varepsilon)\right\|_{L^2(\Omega_2^\varepsilon)} &\leq C\varepsilon \\ \|U_0^\varepsilon\|_{L^2(\Sigma)^2} &\leq C\varepsilon^{\frac{1}{2}} \end{aligned}$$

and

$$\left\|(1+|x_2|)^{-1}P_0^\varepsilon\right\|_{L^2(\Omega)} \leq C\varepsilon.$$

Observe that $\sum_i D^i(x)w^i(x,\frac{x}{\varepsilon}) \rightharpoonup A(x)(F^T(x)l(x) - \nabla p(x))$ by the definition of D^i and the cell problems. Furthermore, the boundary layers tend to 0 outside Σ for $\varepsilon \to 0$. Therefore we obtain

5.16 Theorem.
It holds

$$\frac{u^\varepsilon}{\varepsilon^2} \rightharpoonup H(x_2)\big[u_0 + \sum_{i,k} u^{ik}\big] + H(-x_2)A(F^T l - \nabla p) \quad \textit{weakly in } L^2(K)^2$$

and

$$p^\varepsilon \rightharpoonup H(-x_2)\,p \quad \textit{weakly in } L^2(K)^2$$

for all $K \subset \Omega$ such that K is precompact.

Thus the velocity u_F of the free fluid in Ω_1 is given by

$$u_F = u_0 + \sum_{i,k} u^{ik}$$

whereas for the filtration velocity u_D in the porous medium Ω_2 it holds

$$u_D = A(F^T l - \nabla p)$$

(the Stokes velocity u_0, the pressure p and the permeability tensor A are defined on pages 66 and 67, and the counterflow u^{ik} can be found on page 70).

Now the question arises which conditions hold at the interface Σ, coupling u_F and u_D. We have the following result:

5.17 Corollary.
At the interface Σ it holds

$$u_F(x) \cdot F^{-T}(x)e_2 = u_D(x) \cdot F^{-T}(x)e_2 \tag{5.8}$$

and

$$\big(u_F(x) - u_D(x)\big) \cdot e_1 = \sum_i \big(C_1^{i,\mathrm{bl}}(x) - A_{i1}(x)\big)D^i(x) \tag{5.9}$$

as well as

$$\begin{aligned}\big(u_F(x) - u_D(x)\big) \cdot F(x)e_1 &= \sum_i \big(C_1^{i,\mathrm{bl}}(x) - A_{i1}(x)\big)D^i(x) \\ &\quad + g'(x_1)\sum_i \big(C_2^{i,\mathrm{bl}}(x) - A_{i2}(x)\big)D^i(x) \\ &= \Big[\sum_i \big(C^{i,\mathrm{bl}}(x) - A_i(x)\big)D^i(x)\Big] \cdot F(x)e_1\end{aligned} \tag{5.10}$$

where A_i denotes the i-th column of the permeability matrix A. Moreover, the constant $C^{i,\mathrm{bl}}(x) = \begin{pmatrix} C_1^{i,\mathrm{bl}}(x) \\ C_2^{i,\mathrm{bl}}(x)\end{pmatrix}$ is the solution of the following system of equations:

$$\begin{bmatrix} 1 & g'(x_1) \\ -g'(x_1) & 1 \end{bmatrix}\begin{pmatrix} C_1^{i,\mathrm{bl}}(x) \\ C_2^{i,\mathrm{bl}}(x)\end{pmatrix} = \begin{pmatrix} \int_0^1 w^{i,\mathrm{bl}}(x)(y_1,+0) \cdot F(x)e_1 \,\mathrm{d}y \\ \int_S w^i(x,y) \cdot F^{-T}(x)e_2 \,\mathrm{d}\sigma_y \end{pmatrix}.$$

For the pressure we have

$$p(x_1,-0) = 0 \quad \text{on } \Sigma.$$

Proof. Due to Lemma A.21 it holds $C^{i,\mathrm{bl}}(x) \cdot F^{-T}(x)e_2 = A_i(x) \cdot F^{-T}(x)e_2$. Furthermore we have $u_0 = 0$ on Σ. Therefore, by the definition of u^{ik},

$$\begin{aligned}u_F(x_1,+0) \cdot F^{-T}(x_1,0)e_2 &= u_0(x_1,+0) \cdot F^{-T}(x_1,0)e_2 + \sum_{i,k} u^{ik}(x_1,+0) \cdot F^{-T}(x_1,0)e_2 \\ &= \sum_i (C^{i,\mathrm{bl}}D_\delta^i)(x_1,+0) \cdot F^{-T}(x_1,0)e_2 \\ &= \sum_i (D^i A_i)(x_1,-0) \cdot F^{-T}(x)e_2 \\ &= A(x_1,-0)(F^T l(x_1,-0) - \nabla p(x_1,-0)) \cdot F^{-T}(x)e_2 \\ &= u_D(x_1,-0) \cdot F^{-T}(x)e_2.\end{aligned}$$

The second condition is obtained by a simple calculation:

$$\begin{aligned}(u_F - u_D)(x_1,0)\cdot e_1 &= \sum_{i,k} u^{ik}(x_1,+0)\cdot e_1 - (A(F^T f - \nabla p))(x_1,-0)\cdot e_1\\ &= \sum_{i,k}(C_k^{i,\mathrm{bl}} D^i)(x_1,+0)e_k\cdot e_1 - \sum_i (A_{i1}D^i)(x_1,-0)\\ &= \sum_i (C_1^{i,\mathrm{bl}} D^i)(x_1,+0) - \sum_i (A_{i1}D^i)(x_1,-0).\end{aligned}$$

The third statement follows analogously to the second one, by noting that the direction of the transformed tangential vector is given by $F(x)e_1 = \binom{1}{g'(x_1)}$. As for the pressure, the condition holds by definition, see Section 5.3.1.

The conditions for the constants $C^{i,\mathrm{bl}}(x)$ are derived in Lemmas A.17 – A.19 and in Remark A.20. ♦

5.18 Remark.
We make some remarks on the interpretation of the conditions above: The first equation gives the conservation of mass of the incompressible fluid: In the continuity condition (5.8)*, the vector $F^{-T}(x)e_2$ corresponds to the direction of the transformed normal vector of Σ. Thus the velocity is continuous over the interface $\tilde{\Sigma}$ in normal direction.*

Furthermore, condition (5.10) *indicates a jump across Σ in the direction of the transformed tangential vector. The magnitude of the jump is given by the value of the involved constants, which can be calculated explicitely by solving the associated auxiliary problems. As they are defined on reference domains and depend on the function g describing the interface, the local geometry of the porous medium influences the jump condition.*

Assuming the constants to be of the same order of magnitude, there seem to be two contributions to the jump: One is independent of the curved interface plus another one which is proportional to the slope of the interface. For an explanation, the following consideration is suggested: The functions involved are defined to be periodic in x_1-direction – therefore they correspond to a flow which is dominated by a rotation in a two-dimensional torus. Now the bigger the slope of the interface, the more fluid particles get deflected at that interface, leading to an increased jump (in terms of its absolute value).

6 Conclusions

In this work we proposed a generalisation of the boundary layer approach developed by Willi Jäger and Andro Mikelić. This was done in order to be able to deal with *curved* fluid-porous interfaces when deriving boundary conditions for a fluid flow. Previous works have been confined to planar interfaces.
In Section 2 we introduced coordinate transformations and applied them to differential operators. This was done in order to transform a flow situation governed by a steady state Stokes equation in a domain with a curved interface to a domain with a straight interface.
To study the effective fluid behaviour, periodic homogenisation was applied to the transformed Stokes equation in Section 3. After proving existence and uniqueness results for this type of problem, we identified the effective equations to be a transformed Darcy's law with a non-constant permeability matrix.
Since the cell problem consisted of a parameter-dependent family of solutions of partial differential equations, we investigated the dependence of the solution on that parameter in the subsequent Section 4. To sum up, the differentiability properties in the direction of the parameter carry over to the solution under certain conditions.
Concerning the behaviour of the fluid at the interface, in Section 5 numerous interfacial exchange conditions were identified after introducing several auxiliary problems. Their rigorous mathematical treatment is postponed to Appendix A.

Therefore, by using a transformation approach, one can consider more general situations as in [JM96] – without losing the exponential decay of the boundary layer type functions, which is necessary to obtain effective estimates.

The results do not give a generalized law of Beavers and Joseph – however, they do point in that direction: For the coupling of the velocity u_F of a free fluid above a porous medium with effective fluid velocity u_D, the considerations suggest that a jump of the velocities in tangential direction appears across the interface. The value of this jump can explicitely be calculated by considering several auxiliary problems. Assuming that the appearing constants are of the same order of magnitude, this jump is affine linear in the slope of the interface.
As for the normal direction, the calculations show that the velocity is continuous over the interface. Therefore our results do not contradict the conservation of mass for incompressible fluids.

Nevertheless, the approach suffers from some drawbacks, which give room for further research:

- It is assumed that the porous part of the reference configuration consists of a periodic array of cells. Due to the nature of the coordinate transformation, this

periodicity is lost in the actual domain where the Stokes flow is considered. As a result of the degeneration of the periodic structure, a non-constant permeability matrix for the Darcy flow appears.
One possibility to correct this would be the inclusion of another transformation in the process of deriving the boundary behaviour. This transformation is supposed to account for the local deformation of the solid parts in $\tilde{\Omega}_2$. For homogenisation settings including coordinate transformations, see for example the works of Meier and Peter, [Mei08], [Pet06] or [Pet07].

- Second, the transformation is chosen in a way such that *every* point of the domain is translated. However, one might conjecture that the effect of a curved interface is only local, having no influence on points of the porous medium far away from the interface (i.e. with a great x_2-value). To overcome this drawback, one can adapt the approach to a coordinate transformation which applies only locally, e.g. one which transforms $\tilde{\Omega} \cap [0, L] \times [-b, b]$ to the rectangle $[0, L] \times [-b, b]$. Note that this transformation must be volume-preserving in order to keep important properties used in the above considerations.

- Assumption 5.1 was needed to obtain the decay of the boundary layer functions. However, we did not prove the assertions presented therein; nor did we mention sufficient conditions for the assumption to hold.
  One possible starting point to obtain such results might be to include the constant of decay into the partial differential equation describing the problem (e.g. by considering the difference $w^{i,\mathrm{bl}}(x, \frac{x}{\varepsilon}) - C^{i,\mathrm{bl}}(x)$). Then one can use the theory of parameter-dependent PDEs to derive the governing equations for the derivative in x-direction of this difference. Afterwards, it might be possible to use the theory of Landis/Panasenko and Iosif'jan/Oleĭnik to gain information about the decay of that derivative.

- The concept of very weak solutions of the transformed Stokes equation was only treated superficially. Especially, one might try to work out the situation when the compatibility condition used in Lemma A.24 does not hold.

- Finally, in most parts of the work the actual form of the matrix F was not used. Therefore it seems feasible to consider more complicated coordinate transformations such that $-\operatorname{div}(F^{-1}F^{-T}\nabla\,\cdot)$ is a strongly elliptic operator.
  When dealing with a different coordinate transformation, it would be interesting to observe whether the results obtained in this work depend on the actual form of the transformation or whether they are independent of it.

Appendix

A Results for the Corrector Problems

In this section we give the details for various auxiliary problems which are used in Section 5. These results are mostly a generalization of those obtained in [JM96]. We are going to use the same notation for the geometry as above.

When dealing with parameter dependent problems in the reference domains Y^* and Z_{BL}, we will always assume $x \in \Omega$ to be fixed. To simplify the notation, we will drop that parameter in the name of the solution; and we are not going to consider explicitly the dependence on it.

A.1 Boundary Layer Functions

We repeat some definitions:

Let

$$V = \Big\{ z \in L^2_{\mathrm{loc}}(Z_{\mathrm{BL}})^2 \mid \nabla z \in L^2(Z_{\mathrm{BL}})^4,\ z \in L^2(Z^-)^2,\\ z = 0 \text{ on } \bigcup_{k=1}^{\infty} \{\partial Y_S - \binom{0}{k}\},\ z \text{ is 1-periodic in } x_1 \Big\}$$

and

$$V_{\mathrm{div}} = \Big\{ z \in L^2_{\mathrm{loc}}(Z_{\mathrm{BL}})^2 \mid \nabla z \in L^2(Z_{\mathrm{BL}})^4,\ z \in L^2(Z^-)^2,\ z = 0 \text{ on } \bigcup_{k=1}^{\infty} \{\partial Y_S - \binom{0}{k}\},\\ \mathrm{div}_y(F^{-1}(x) z(y)) = 0,\ z \text{ is 1-periodic in } x_1 \Big\}.$$

Define W as the completion of V_{div} with respect to the norm

$$\|z\|_W = \|\nabla z\|_{L^2(Z_{\mathrm{BL}})^4}.$$

The Poincaré inequality in Z^- reads

$$\|z\|_{L^2(Z^-)^2} \le C \, \|\nabla z\|_{L^2(Z^-)^4} \quad \forall z \in V.$$

A.1.1 The Main Auxiliary Problem

For the development of a theory for the boundary layer functions, we start with a more general formulation:

Let $\gamma_1 > 0$, $\sigma \in H^{\frac{1}{2}}(S)^2$, $\rho \in L^2(Z)^2$ and $\rho_1 \in L^2(Z)^4$ be given. Assume that $e^{\gamma_1 |y_2|}\rho \in L^2(Z_{\mathrm{BL}})^2$ and $e^{\gamma_1 |y_2|}\rho_1 \in L^2(Z_{\mathrm{BL}})^4$. Consider the following parameter-dependent problem: Find $\zeta \in W$ such that

$$\begin{aligned} \int_{Z_{\mathrm{BL}}} F^{-T}(x)\,\nabla_y\,\zeta(y) : F^{-T}(x)\,\nabla_y\,\phi(y)\;\mathrm{d}y &= \int_{Z_{\mathrm{BL}}} \rho(y)\cdot\phi(y)\;\mathrm{d}y \\ - \int_{Z_{\mathrm{BL}}} \rho_1(y) : F^{-T}(x)\,\nabla_y\,\phi(y)\;\mathrm{d}y &+ \int_S F^{-1}(x)\sigma(y)\cdot\phi(y)\;\mathrm{d}\sigma_y \quad \forall\,\phi\in W \end{aligned} \tag{A.1}$$

A.1 Proposition.
There exists a unique solution of Problem (A.1).

Proof. The result follows by application of the Lax-Milgram lemma:

Define for $u, \phi \in W$

$$B(u,\phi) = \int_{Z_{\mathrm{BL}}} F^{-T}(x)\,\nabla_y\,\zeta(y) : F^{-T}(x)\,\nabla_y\,\phi(y)\;\mathrm{d}y,$$

$$b(\phi) = \int_{Z_{\mathrm{BL}}} \rho(y)\cdot\phi(y)\;\mathrm{d}y - \int_{Z_{\mathrm{BL}}} \rho_1(y) : F^{-T}(x)\,\nabla_y\,\phi(y)\;\mathrm{d}y + \int_S F^{-1}(x)\sigma(y)\cdot\phi(y)\;\mathrm{d}\sigma_y.$$

The continuity and coercivity of the bilinear form B in W can be proved analogously to the case of the transformed Stokes equation, see Proposition 3.13.

To see that b is bounded, note that

$$\begin{aligned} \left|\int_{Z_{\mathrm{BL}}} \rho(y)\cdot\phi(y)\;\mathrm{d}y\right| &\le \left|\int_{Z^+} \rho(y)\cdot\phi(y)\;\mathrm{d}y\right| + \left|\int_{Z^-} \rho(y)\cdot\phi(y)\;\mathrm{d}y\right| \\ &\le \left\|(1+y_2)\rho\right\|_{L^2(Z^+)^2}\left\|(1+y_2)^{-1}\phi\right\|_{L^2(Z^+)^2} + \left\|\rho\right\|_{L^2(Z^-)^2}\left\|\phi\right\|_{L^2(Z^-)^2} \\ &\le C\left\|\nabla\,\rho\right\|_{L^2(Z_{\mathrm{BL}})^4}, \end{aligned}$$

where we used the standard Poincaré inequality in Z^-, the fact that $|(1+y_2)\rho| \le e^{|y_2|}|\rho| \le \frac{1}{e^{\gamma_1}} e^{\gamma_1|y_2|}|\rho| \le C e^{\gamma_1|y_2|}|\rho|$ and $\left\|(1+y_2)^{-1}\phi\right\|_{L^2(Z^+)^2} \le \left\|\nabla\,\phi\right\|_{L^2(Z^+)}$, see Lemma 5.3. The estimation of the remaining terms is standard. ♦

A.2 Lemma.
Let $\operatorname{div}_y(F^{-1}(x)\rho_1(y)) \in L^2(Z_{\mathrm{BL}})^2$ *and let* ρ, ρ_1, σ *be* 1*-periodic in* y_1*. Then the solution* ζ *of* (A.1) *is in* $H^2_{\mathrm{loc}}(Z)^2$.

A.3 Proposition.
Under the assumptions of Lemma A.2, there exists a pressure field $\kappa \in L^2_{\mathrm{loc}}(Z_{\mathrm{BL}})$ *such that*

$$\begin{aligned}-\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\zeta(y)) + F^{-T}(x)\,\nabla_y\,\kappa(y)&\\ = \rho(y) + \operatorname{div}_y(F^{-1}(x)\rho_1(y)) \quad &\textit{in } W'.\end{aligned}$$

Proof. We are going to use analogues of Lemmas 3.14 and 3.16 for an increasing sequence of sets in order to show that $W^\perp = \{F^{-T}(x)\,\nabla_y\, p \mid p \in L^2_{\mathrm{loc}}(Z_{\mathrm{BL}})\}$.

Define for $l \in \mathbb{N}$ the sets $Z_l^* = [0,1] \times ((0,l) \cup (\bigcup_{k=1}^{l}\{Y^* - \binom{0}{k}\})$ and the space

$$W_l = \Big\{ z \in H^1(Z_l^*)^2 \mid z = 0 \text{ for } y_2 = \pm l \text{ and on } \bigcup_{k=1}^{l}\{\partial Y_S - \begin{pmatrix}0\\k\end{pmatrix}\},\\ z \text{ is 1-periodic in } y_1\Big\}.$$

It is clear that $Z_l^* \subset Z_{l+1}^*$ and that each Z_l^* is a Lipschitz domain.

$\operatorname{div}_l(F^{-1}(x)\cdot) : W_l \longrightarrow L^2_0(Z_l^*)$, $\operatorname{div}_l(F^{-1}(x)\cdot) := \operatorname{div}_y(F^{-1}(x)\cdot)$ is surjective by an analogue of Lemma 3.16, thus $F^{-T}(x)\,\nabla_l(\cdot) := F^{-T}(x)\,\nabla_y(\cdot)$ is injective from $L^2_0(Z_l^*)$ to W_l'.

Now let $f \in V'$ such that $\langle f, \phi\rangle_{H^{-1}(Z_{\mathrm{BL}})^2,\, H^1(Z_{\mathrm{BL}})^2} = 0$ for all $\phi \in W$. Let $u \in \ker(\operatorname{div}_l(F^{-1}(x)\cdot))$ be given and denote by $\tilde{u}$ the extension by 0 outside Z_l^*. Since then $\operatorname{div}_y(F^{-1}(x)\tilde{u}) = 0$ in Z_{BL} we have $\langle f, \tilde{u}\rangle_{H^{-1}(Z_{\mathrm{BL}})^2,\, H^1(Z_{\mathrm{BL}})^2} = 0$. By duality of the extension operation we conclude that $f|_{Z_l^*} \perp \ker(\operatorname{div}_l(F^{-1}(x)\cdot))$. Therefore $f|_{Z_l^*} \in \operatorname{im}(F^{-T}(x)\,\nabla_l\,\cdot)$, and there exists a $p_l \in L^2(Z_l^*)$, unique up to a constant with $f = F^{-T}(x)\,\nabla_y\, p_l$ in Z_l^*.

Since $Z_l^* \subset Z_{l+1}^*$, the difference $p_{l+1} - p_l$ is constant in Z_l^* and we can choose p_{l+1} in such a way that $p_{l+1} = p_l$ in Z_l^*. Thus $f = F^{-T}(x)\,\nabla_y\, p$ with $p \in L^2_{\mathrm{loc}}(Z_{\mathrm{BL}})$.

The pressure κ can now be obtained by observing that – via an integration by parts of (A.1) –, $\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\zeta(y)) + \rho(y) + \operatorname{div}_y(F^{-1}(x)\rho_1(y)) \in W^\perp$. ♦

A.4 Lemma.
Let ζ *and* κ *be defined as above. Under the assumptions of Lemma A.2 we have* $\zeta \in H^2_{\mathrm{loc}}(Z)^2$ *and* $\kappa \in H^1_{\mathrm{loc}}(Z)$.

Finally, we obtain the following strong form of Problem (A.1):

$$
\begin{aligned}
-\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\zeta(y)) + F^{-T}(x)\,\nabla_y\,\kappa(y)& \\
= \rho + \operatorname{div}_y(F^{-1}(x)\rho_1(y)) \text{ a.e. in } Z& \qquad \text{(A.2a)}\\
\operatorname{div}_y(F^{-1}(x)\zeta(y)) = 0 \text{ a.e. in } Z& \qquad \text{(A.2b)}\\
\zeta(y_1, \pm 0) = \zeta_0^{\pm} \text{ on } S& \qquad \text{(A.2c)}\\
\zeta = 0 \text{ on } \textstyle\bigcup_{k=1}^{\infty}\{\partial Y_S - \binom{0}{k}\}& \qquad \text{(A.2d)}\\
\zeta, \kappa \text{ are 1-periodic in } y_1& \qquad \text{(A.2e)}
\end{aligned}
$$

with known functions $\zeta_0^{\pm} \in H^{\frac{3}{2}}(S)^2$.

A.1.2 Exponential Decay

Define for $k \in -\mathbb{N}$ the sets $Z_k = Z^- \cap ([0,1] \times [k, k+1])$ (these domains, as well as other auxiliary sets needed in the course of the derivation, are depicted in Figure 7).

A.5 Proposition.
Let $\bar{\rho} := \rho + \operatorname{div}_y(F^{-1}(x)\rho_1) \in L^2(Z^-)^2$ and let ζ and κ be as above. Define

$$r_k = \frac{1}{|Y^*|} \int\limits_{Z_k} \kappa(y)\ \mathrm{d}y.$$

Then the following estimates hold:

$$
\begin{aligned}
\|\kappa - r_k\|_{L^2(Z_k)} &\le C(\|\nabla\,\zeta\|_{L^2(Z_k)^4} + \|\bar{\rho}\|_{L^2(Z_k)^2})\\
|r_{k+1} - r_k| &\le C(\|\nabla\,\zeta\|_{L^2(Z_k\,\cup\,Z_{k+1})^4} + \|\bar{\rho}\|_{L^2(Z_k\,\cup\,Z_{k+1})^2})
\end{aligned}
$$

Proof. Define the space

$$V_k = \Big\{ z \in H^1(Z_k)^2 \mid z = 0 \text{ on } \partial Z_k \backslash ((\{0\} \cup \{1\}) \times [k, k+1]), z \text{ is 1-periodic in } y_1 \Big\}.$$

Consider Equation (A.2a) on Z_k with $\nabla_y(\kappa - r_k)$ instead of $\nabla_y\,\kappa$. By multiplication with a test function and integration by parts we obtain

$$
\begin{aligned}
\int\limits_{Z_k} F^{-T}(x)\,\nabla_y\zeta(y) : F^{-T}(x)\,\nabla_y\,\phi(y)\ \mathrm{d}y - \int\limits_{Z_k} (\kappa - r_k)\operatorname{div}_y(F^{-1}(x)\phi(y))\ \mathrm{d}y&\\
= \int\limits_{Z_k} (\rho(y) + \operatorname{div}_y(F^{-1}(x)\rho_1(y)) \cdot \phi(y)\ \mathrm{d}y \quad \forall \phi \in V_k.&
\end{aligned}
$$

Analogously to Lemma 3.16 there exists $\phi_k \in V_k$, solution of

$$\operatorname{div}_y(F^{-1}(x)\phi_k(y)) = \kappa - r_k \quad \text{in } Z_k$$

with

$$\|\nabla_y \phi_k\|_{L^2(Z_K)^4} \leq C \|\kappa - r_k\|_{L^2(Z_k)} .$$

C depends only on the geometry of Y^* but not on k.

Inserting ϕ_k in the above equation and remarking that $\left\|F^{-T}(x)\, \nabla_y z\right\|_{L^2} \leq C \|\nabla_y z\|_{L^2}$ yields

$$\begin{aligned} \|\kappa - r_k\|^2_{L^2(Z_k)} &\leq \|\bar{\rho}\|_{L^2(Z_k)^2} \|\nabla_y \phi_k\|_{L^2(Z_k)^4} + C \|\nabla_y \zeta\|_{L^2(Z_k)^4} \|\nabla_y \phi_k\|_{L^2(Z_k)^4} \\ &\leq C\big(\|\bar{\rho}\|_{L^2(Z_k)^2} + \|\nabla_y \zeta\|_{L^2(Z_k)^4}\big) \|\kappa - r_k\|_{L^2(Z_k)} , \end{aligned}$$

thus the first assertion is proved.

Next, set $Z_{k,k+1} = Z_k \cup Z_{k+1}$ and consider $\phi_{k,k+1}$ satisfying

$$\begin{gathered} \operatorname{div}_y(F^{-1}(x)\phi_{k,k+1}(y)) = \begin{cases} 1 & \text{in } Z_k^0 \\ -1 & \text{in } Z_{k+1}^0 \end{cases} \\ \phi_{k,k+1} = 0 \text{ on } (\partial Z_k \cup \partial Z_{k+1}) \backslash ((\{0\} \cup \{1\}) \times [k, k+2]) \\ \phi_{k,k+1} \text{ is 1-periodic in } y_1 \end{gathered}$$

(the existence is assured since the right hand side of the first equation is in $L^2(Z_{k,k+1})$ and has mean value 0).

Testing (A.2a) with $\phi_{k,k+1}$ in $Z_{k,k+1}$ gives

$$\begin{aligned} -\int\limits_{Z_k} \kappa(y)\ \mathrm{d}y + \int\limits_{Z_{k+1}} \kappa(y)\ \mathrm{d}y + \int\limits_{Z_{k,k+1}} F^{-T}(x)\, \nabla_y \zeta(y) : F^{-T}(x)\, \nabla_y \phi_{k,k+1}(y)\ \mathrm{d}y \\ = \int\limits_{Z_{k,k+1}} \bar{\rho}(y) \cdot \phi_{k,k+1}(y)\ \mathrm{d}y. \end{aligned}$$

Note that $\|\phi_{k,k+1}\|_{L^2(Z_{k,k+1})^2} \leq C \|\nabla_y \phi_{k,k+1}\|_{L^2(Z_{k,k+1})^4} \leq C|Z_{k,k+1}|$, thus dividing the equation by $|Y^*|$ gives the estimate

$$|r_{k+1} - r_k| \leq C(\|\nabla \zeta\|_{L^2(Z_{k,k+1})^4} + \|\bar{\rho}\|_{L^2(Z_{k,k+1})^2}),$$

which finishes the proof. ♦

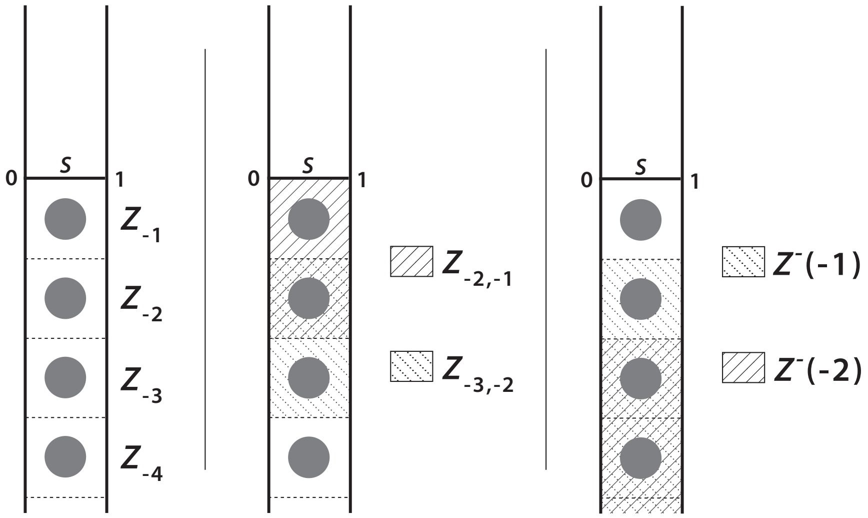

Figure 7: Auxiliary domains based on the boundary layer cell Z_{BL}. *Left*: The translated reference cells Z_k, $k \in -\mathbb{N}$. *Middle*: The two-cell subsets $Z_{k,k+1}$, illustrated with the sets $Z_{-2,-1}$ (shaded with lines) and $Z_{-3,-2}$ (shaded with dots). *Right*: The unbounded strips $Z^-(k)$. In the figure the sets $Z^-(-1)$ (shaded with dots) and $Z^-(-2)$ (shaded with lines) are shown.

A.6 Proposition.
For $k \in -\mathbb{N}$ choose functions $\tilde{\sigma}_k \in \mathcal{C}^\infty(\mathbb{R}_{\leq 0})$, $0 \leq \tilde{\sigma}_k \leq 1$ with $\tilde{\sigma}_k(z) = 0$ for $z \geq k+1$ and $\tilde{\sigma}_k(z) = 1$ for $z \leq k$, $z \in \mathbb{R}_{\geq 0}$, such that $\tilde{\sigma}_k$ and the derivative $\tilde{\sigma}_k'$ are bounded uniformly in k. For $y = \binom{y_1}{y_2} \in [0,1] \times (-\infty, 0]$ define $\sigma_k(y) := \tilde{\sigma}_k(y_2)$.

Let ζ, κ be a solution of Problem (A.2)*. Then it holds*

$$\int_{Z^-} |F^{-T}(x)\,\nabla_y\,\zeta(y)|^2 \sigma_k(y)\;\mathrm{d}y = \int_{Z_k} (\kappa - r_k)\zeta \cdot F^{-T}(x)\,\nabla_y\,\sigma_k(y)\;\mathrm{d}y$$
$$+ \int_{Z^-} \bar{\rho}(y) \cdot \zeta(y)\sigma_k(y)\;\mathrm{d}y - \int_{Z^-} F^{-T}(x)\,\nabla_y\,\zeta(y) : F^{-T}(x)(\zeta(y) \otimes \nabla_y\,\sigma_k(y))\;\mathrm{d}y.$$

Proof. Testing (A.2a) with $\phi \in \mathcal{C}_0^\infty(Z_{\mathrm{BL}})$, $\phi = 0$ on $\bigcup_{k=1}^\infty \{\partial Y_S - \binom{0}{k}\}$ and ϕ 1-periodic in y_1 yields

$$\int_{Z_{\mathrm{BL}}} F^{-T}(x)\,\nabla_y\,\zeta(y) : F^{-T}(x)\,\nabla_y\,\phi(y)\;\mathrm{d}y - \int_{Z_{\mathrm{BL}}} \kappa\,\mathrm{div}_y(F^{-1}(x)\phi(y))\;\mathrm{d}y \int_{Z_{\mathrm{BL}}} \bar{\rho}(y) \cdot \phi(y)\;\mathrm{d}y.$$

Define for $l \leq k-1$ the functions $\sigma_{k,l} = \sigma_k(1-\sigma_l)$. Choosing $\phi = \zeta\sigma_{k,l}$ leads to

$$\begin{aligned}\int\limits_{Z^-} |F^{-T}(x)\,\nabla_y\,\zeta(y)|^2\sigma_{k,l}\;\mathrm{d}y &= \int\limits_{Z^-} \kappa(y)\zeta(y)\cdot F^{-T}(x)\,\nabla_y\,\sigma_{k,l}(y)\;\mathrm{d}y \\ &+ \int\limits_{Z^-} \bar{\rho}(y)\cdot\zeta(y)\sigma_{k,l}(y)\;\mathrm{d}y - \int\limits_{Z^-} F^{-T}(x)\,\nabla_y\,\zeta(y) : F^{-T}(x)(\zeta(y)\otimes\nabla_y\,\sigma_{k,l}(y))\;\mathrm{d}y,\end{aligned}$$

where we used the fact that

$$\begin{aligned}F^{-T}(x)\,\nabla_y(\zeta\sigma_{k,l}) &= F^{-T}(x)(\nabla_y\,\zeta)\sigma_{k,l} + \zeta\otimes F^{-T}(x)\,\nabla_y\,\sigma_{k,l},\\ \operatorname{div}_y(F^{-1}(x)\zeta\sigma_{k,l}) &= \zeta\cdot F^{-T}(x)\,\nabla_y\,\sigma_{k,l} + \sigma_{k,l}\operatorname{div}_y(F^{-1}(x)\zeta).\end{aligned}$$

We want to pass to the limit $l \longrightarrow -\infty$ for fixed k. First observe that $\sigma_{k,l} \longrightarrow \sigma_k$ as well as $\nabla\,\sigma_{k,l} \longrightarrow \nabla\,\sigma_k$ pointwise for $l \longrightarrow -\infty$. As $|\sigma_{k,l}| \leq C$ and $|\nabla\,\sigma_{k,l}| = |(\nabla\,\sigma_k)(1+\sigma_l) - \sigma_k\,\nabla\,\sigma_l| \leq C$ a.e. with a constant C, we obtain that almost everywhere

$$\begin{gathered}\big|\,|F^{-T}(x)\,\nabla_y\,\zeta(y)|^2\sigma_k(y)\big| \leq C|F^{-T}(x)\,\nabla_y\,\zeta(y)|^2\\ |\bar{\rho}(y)\cdot\zeta(y)\sigma_{k,l}(y)| \leq C\,|\bar{\rho}(y)\cdot\zeta(y)|\\ \big|F^{-T}(x)\,\nabla_y\,\zeta(y) : F^{-T}(x)(\zeta(y)\otimes\nabla_y\,\sigma_{k,l}(y))\big| \leq C\,\big|F^{-T}(x)\,\nabla_y\,\zeta(y) : F^{-T}(x)(\zeta(y)\otimes I)\big|\end{gathered}$$

where I denotes the identity matrix. Since the right hand sides are integrable, application of Lebesgue's dominated convergence theorem yields for $l \longrightarrow -\infty$

$$\begin{aligned}\int\limits_{Z^-} |F^{-T}(x)\,\nabla_y\,\zeta(y)|^2\sigma_{k,l}\;\mathrm{d}y &\longrightarrow \int\limits_{Z^-} |F^{-T}(x)\,\nabla_y\,\zeta(y)|^2\sigma_k\;\mathrm{d}y\\ \int\limits_{Z^-} \bar{\rho}(y)\cdot\zeta(y)\sigma_{k,l}(y)\;\mathrm{d}y &\longrightarrow \int\limits_{Z^-} \bar{\rho}(y)\cdot\zeta(y)\sigma_k(y)\;\mathrm{d}y\end{aligned}$$

and

$$\begin{aligned}&\int\limits_{Z^-} F^{-T}(x)\,\nabla_y\,\zeta(y) : F^{-T}(x)(\zeta(y)\otimes\nabla_y\,\sigma_{k,l}(y))\;\mathrm{d}y\\ &\qquad\longrightarrow \int\limits_{Z^-} F^{-T}(x)\,\nabla_y\,\zeta(y) : F^{-T}(x)(\zeta(y)\otimes\nabla_y\,\sigma_k(y))\;\mathrm{d}y.\end{aligned}$$

Finally we have to consider the term $\int_{Z^-} \kappa(y)\zeta(y) \cdot F^{-T}(x) \nabla_y \sigma_{k,l}(y)$. Because of $\nabla \sigma_{k,l}(y) = 0$ a.e. for $y \notin Z_k \cup Z_l$ we have

$$\begin{aligned}
\int_{Z^-} \kappa(y)\zeta(y) \cdot F^{-T}(x) \nabla_y \sigma_{k,l}(y) \, \mathrm{d}y &= \int_{Z_k \cup Z_l} (\kappa(y) - r_k)\zeta(y) \cdot F^{-T}(x) \nabla_y \sigma_{k,l}(y) \, \mathrm{d}y \\
= \int_{Z_k} (\kappa(y) - r_k)\zeta(y) \cdot F^{-T}(x) \nabla_y \sigma_{k,l}(y) \, \mathrm{d}y &+ \int_{Z_l} (\kappa(y) - r_l)\zeta(y) \cdot F^{-T}(x) \nabla_y \sigma_{k,l}(y) \, \mathrm{d}y \\
+ (r_l - r_k) \int_{Z_l} \zeta(y) \cdot F^{-T}(x) \nabla_y \sigma_{k,l}(y) \, \mathrm{d}y.
\end{aligned}$$

For $l \longrightarrow -\infty$ we obtain by using Poincaré's inequality

$$\left| (r_l - r_k) \int_{Z_l} \zeta(y) \cdot F^{-T}(x) \nabla_y \sigma_{k,l}(y) \, \mathrm{d}y \right| \leq C \left\| \nabla \zeta \right\|_{L^2(Z_l)^4} \longrightarrow 0$$

and

$$\begin{aligned}
\left| \int_{Z_l} (\kappa(y) - r_l)\zeta(y) \cdot F^{-T}(x) \nabla_y \sigma_{k,l}(y) \, \mathrm{d}y \right| &\leq C(\left\| \nabla \zeta \right\|_{L^2(Z_l)^4} + \left\| \bar{\rho} \right\|_{L^2(Z_l)^2}) \left\| \nabla \zeta \right\|_{L^2(Z_l)^4} \\
&\longrightarrow 0,
\end{aligned}$$

where we also used the preceding lemma for the last estimate. Thus arguing similarly with Lebesgue's theorem one arrives at

$$\lim_{l \to -\infty} \int_{Z^-} \kappa(y)\zeta(y) \cdot F^{-T}(x) \nabla_y \sigma_{k,l}(y) \, \mathrm{d}y = \int_{Z_k} (\kappa(y) - r_k)\zeta(y) \cdot F^{-T}(x) \nabla_y \sigma_k(y) \, \mathrm{d}y,$$

and the proof is complete. ♦

Define for $k \in -\mathbb{N}$ the sets

$$Z^-(k) := Z^- \cap ([0, 1] \times (-\infty, k]).$$

A.7 Proposition.
Let $\bar{\rho} \in L^2(Z^-)^2$ and let ζ, κ be a solution of problem (A.2). *There exists a constant C_0 independent of k such that*

$$\left\| \nabla \zeta \right\|^2_{L^2(Z^-(k))} \leq C_0^2 \left\| \bar{\rho} \right\|^2_{L^2(Z^-(k))^2}.$$

Proof. We estimate the terms on the right hand side of the previous proposition separately: By the Poincaré inequality

$$\begin{aligned}\left| \int_{Z^-} F^{-T}(x)\,\nabla_y\,\zeta(y) : F^{-T}(x)(\zeta(y)\otimes\nabla_y\,\sigma_{k,l}(y))\;\mathrm{d}y\right| \\ = \left| \int_{Z_k} F^{-T}(x)\,\nabla_y\,\zeta(y) : F^{-T}(x)(\zeta(y)\otimes\nabla_y\,\sigma_{k,l}(y))\;\mathrm{d}y\right| \\ \leq C\,\|\nabla_y\,\zeta\|^2_{L^2(Z_k)^4}\end{aligned}$$

and

$$\left|\int_{Z^-} \bar{\rho}(y)\cdot\zeta(y)\sigma_k(y)\;\mathrm{d}y\right| = \left|\int_{Z^-(k)} \bar{\rho}(y)\cdot\zeta(y)\sigma_k(y)\;\mathrm{d}y\right| \leq C\,\|\nabla_y\,\zeta\|_{L^2(Z^-(k))^4}\,\|\bar{\rho}\|_{L^2(Z^-(k))^2}\,.$$

Using Proposition A.5 gives

$$\left|\int_{Z_k} (\kappa - r_k)\zeta\cdot F^{-T}(x)\,\nabla_y\,\sigma_k(y)\;\mathrm{d}y\right| \leq C\,\|\nabla_y\,\zeta\|^2_{L^2(Z_k)^4} + \|\bar{\rho}\|_{L^2(Z_k)^2}\,\|\nabla\,\zeta\|_{L^2(Z_k)^4}\,.$$

Because of $Z^-(k)\subset Z^-(k+1)$, $Z_k\subset Z^-(k+1)$ and Young's inequality we obtain

$$\begin{aligned}k_F\int_{Z^-} |\,\nabla_y\zeta(y)|^2\sigma_k(y)\;\mathrm{d}y \leq \int_{Z^-} |F^{-T}(x)\,\nabla_y\,\zeta(y)|^2\sigma_k(y)\;\mathrm{d}y \\ \leq C^*\,\|\nabla_y\,\zeta\|^2_{L^2(Z_k)^4} + C\delta\int_{Z^-(k)} |\,\nabla_y\,\zeta(y)|^2\;\mathrm{d}y + \frac{C}{\delta}\,\|\bar{\rho}\|^2_{L^2(Z^-(k\,+\,1))^2}\end{aligned}$$

for $\delta > 0$.

Next observe that

$$\int_{Z^-} |\,\nabla_y\,\zeta(y)|^2\sigma_k(y)\;\mathrm{d}y = \int_{Z^-(k)} |\,\nabla_y\,\zeta(y)|^2\sigma_k(y)\;\mathrm{d}y + \int_{Z_k} |\,\nabla_y\,\zeta(y)|^2\sigma_k(y)\;\mathrm{d}y$$

and

$$\|\nabla_y\,\zeta\|^2_{L^2(Z_k)^4} = \int_{Z^-(k+1)} |\,\nabla_y\,\zeta(y)|^2\;\mathrm{d}y - \int_{Z^-(k)} |\,\nabla_y\,\zeta(y)|^2\;\mathrm{d}y,$$

thus leading to

$$(k_F - C\delta + C^*)\int_{Z^-(k)} |\,\nabla_y\,\zeta(y)|^2\;\mathrm{d}y \leq C^*\,\|\nabla_y\,\zeta\|^2_{L^2(Z^-(k\,+\,1))^4} + C_1\,\|\bar{\rho}\|^2_{L^2(Z^-(k\,+\,1))^2}\,.$$

Choosing δ small enough such that $k_F - C\delta + C^* > 0$ and $k_F > C\delta$ gives the recursion

$$a_k \le \gamma a_{k+1} + F_k, \quad k \in -\mathbb{N}$$

with

$$a_k = \|\nabla \zeta\|^2_{L^2(Z^-(k))}, \qquad \gamma = \frac{C^*}{k_F - C\delta + C^*} < 1,$$
$$F_k = \frac{C_1}{k_F - C\delta + C^*} \|\bar{\rho}\|^2_{L^2(Z^-(k+1))^2}.$$

Since $Z^-(k) \subset Z^-(k+1)$ we also have $F_k \le F_{k+1}$. This implies the claim as in [JM96]. ♦

A.8 Corollary.
Consider the situation as above. Then there exists a constant κ_∞,

$$\kappa_\infty = \lim_{k\to-\infty} \frac{1}{|Y^*|} \int_{Z_k} \kappa(y)\ \mathrm{d}y$$

and a constant C^*, independent of k, such that for $k \in -\mathbb{N}$ holds

$$\|\kappa - \kappa_\infty\|^2_{L^2(Z^-(k))} \le C^* \sum_{l=-\infty}^{k} \|\bar{\rho}\|^2_{L^2(Z^-(l+1))^2}.$$

Proof. Proposition A.5 yields

$$\|\kappa - r_k\|_{L^2(Z_k)} \le C(\|\nabla \zeta\|_{L^2(Z_k)^4} + \|\bar{\rho}\|_{L^2(Z_k)^2})$$
$$|r_{k+1} - r_k| \le C(\|\nabla \zeta\|_{L^2(Z_{k,k+1})^4} + \|\bar{\rho}\|_{L^2(Z_{k,k+1})^2}).$$

We show that r_k is a Cauchy sequence in $\mathbb{R}$, thus providing the existence of κ_∞: By the triangle inequality it holds for $k \in -\mathbb{N}$, $l \le 0$

$$\begin{aligned}
|r_{k+l} - r_k| &\le \sum_{j=l}^{1} |r_{k+j} - r_{k+j-1}| \\
&\le \sum_{j=l}^{1} C(\|\nabla \zeta\|_{L^2(Z_{k+j-1,k+j})^4} + \|\bar{\rho}\|_{L^2(Z_{k+j-1,k+j})^2}) \\
&\le C(\|\nabla \zeta\|_{L^2(Z^-(k-1))^4} + \|\bar{\rho}\|_{L^2(Z^-(k-1))^2}),
\end{aligned}$$

where the last constant is independent of k and l. Since the last term converges to 0 for $k \to -\infty$, we obtain the desired result.

Next observe that

$$\begin{aligned}
\|\kappa-\kappa_\infty\|_{L^2(Z_m)} &\le \|\kappa-r_m\|_{L^2(Z_m)}+\|r_m-\kappa_\infty\|_{L^2(Z_m)}\\
&\le C(\|\nabla\zeta\|_{L^2(Z_m)^4}+\|\bar\rho\|_{L^2(Z_m)^2})+|Y^*|\,|r_m+\kappa_\infty|\\
&\le C(\|\nabla\zeta\|_{L^2(Z^-(m+1))^4}+\|\bar\rho\|_{L^2(Z^-(m+1))^2})\\
&\quad+|Y^*|\lim_{j\to\infty}\left(\sum_{l=0}^{j}|r_{m-l}-r_{m-(l+1)}|+|r_{l-(j+1)}-\kappa_\infty|\right)\\
&\le C\,\|\bar\rho\|_{L^2(Z^-(m+1))^2}+\sum_{l=0}^{\infty}C(\|\nabla\zeta\|_{L^2(Z_{m-(l+1),m-l})^4}+\|\bar\rho\|_{L^2(Z_{m-(l+1),m-l})^2})\\
&\le C\,\|\bar\rho\|_{L^2(Z^-(m+1))^2}+2C(\|\nabla\zeta\|_{L^2(Z^-(m+1))^4}+\|\bar\rho\|_{L^2(Z^-(m+1))^2})\\
&\le C\,\|\bar\rho\|_{L^2(Z^-(m+1))^2}\,,
\end{aligned}$$

where we used the above inequalites and Proposition A.7. Furthermore, note that $\lim_{j\to\infty}|r_{l-(j+1)}-\kappa_\infty|=0$. Thus by

$$\begin{aligned}
\|\kappa-\kappa_\infty\|^2_{L^2(Z^-(k))} &\le \sum_{m=-\infty}^{k}\|\kappa-\kappa_\infty\|_{L^2(Z_m)}\\
&\le C^*\sum_{m=-\infty}^{k}\|\bar\rho\|_{L^2(Z^-(m+1))^2}
\end{aligned}$$

the second assertion holds. ♦

Finally we are able to get a result on the decay of the solutions ζ,κ in the porous part Z^- of Z_{BL}:

A.9 Corollary.
Assume that $e^{\gamma_1|y_2|}\bar\rho\in L^2(Z_{\mathrm{BL}})^2$ *for a* $\gamma_1>0$. *Then there exists a* $\beta>0$ *such that for the solution* ζ,κ *of Problem* (A.2) *holds*

$$\begin{aligned}
\|\nabla\zeta\|_{L^2(Z^-(k))^4} &\le Ce^{-\beta|k|}\\
\|\zeta\|_{L^2(Z^-(k))^2} &\le Ce^{-\beta|k|}\\
\|\kappa-\kappa_\infty\|_{L^2(Z^-(k))} &\le Ce^{-\beta|k|}
\end{aligned}$$

Proof. By the assumption on $\bar\rho$ note that $\|\bar\rho\|_{L^2(Z_k)^2}\le Ce^{-\gamma_1|k|}$. Therefore

$$\begin{aligned}
\|\bar\rho\|_{L^2(Z^-(l))^2} &\le C\sum_{k=-\infty}^{l}e^{-\gamma_1|k|}=Ce^{-\gamma_1|l|}\sum_{k=-\infty}^{0}(e^{\gamma_1})^{-|k|}\\
&=\frac{C}{1-e^{\gamma_1}}e^{-\gamma_1|l|},
\end{aligned}$$

where we used the formula for the geometric series for $e^{\gamma_1} > 1$. Using the same argument once again, one obtains

$$\sum_{l=-\infty}^{k} \|\bar{\rho}\|_{L^2(Z^-(l))} \leq C e^{-\gamma_1 |k|},$$

which gives the first and the last assertion. The second one follows due to Poincaré's inequality. ♦

In order to deal with the behaviour of ζ and κ in Z^+, we are going to use the theory for the exponential decay of solutions of elliptic problems, developed by Landis/Panasenko and Oleĭnik/Iosif'jan, see [LP85] and [OI81].

A.10 Theorem (Exponential Decay)**.**
Let the geometry be given as above. In Z^+ consider the elliptic equation

$$-\operatorname{div}_y(F(y)\,\nabla_y\, u(y)) = f(y)$$

with a given matrix function $F \in L^\infty(Z^+)^4$ satisfying the following ellipticity condition: Let there exist constants $c_1, C_1 > 0$ and $M > 0$ ($M = 1$ in case F is symmetric) such that for all $\eta, \xi \in \mathbb{R}^2$

$$\begin{aligned} c_1|\xi|^2 \leq \xi^T F(y)\xi \leq C_1|\xi|^2 \\ |\eta^T F(y)\xi| \leq M(\eta^T F(y)\eta)^{\frac{1}{2}}(\xi^T F(y)\xi)^{\frac{1}{2}}. \end{aligned}$$

Assume further periodic boundary conditions on $(\{0\} \cup \{1\} \times \mathbb{R}_{\geq 0})$ and Dirichlet and/or Neumann conditions on S such that there exists a solution u with $\nabla u \in L^2(Z^+)$. Let there exist constants $q, Q > 0$ such that $Qe^{qy_2} f \in L^2(Z^+)$.

Then there exist constants $q_1, Q_1 > 0$ and C_u such that

$$\begin{aligned} \|\nabla u\|_{L^2(Z^+ \cap ([0,1] \times [k,\infty)))^2} \leq Q_1 e^{-q_1 k} \\ \|u - C_u\|_{L^2(Z^+ \cap ([0,1] \times [k,\infty)))} \leq Q_1 e^{-q_1 k}. \end{aligned}$$

Furthermore, there exists $y^* > 0$ with

$$|u(y) - C_u| \leq Q_1 e^{-q_1 y_2} \quad \textit{for } y_2 > y^*.$$

Proof. Theorem 10 of [OI81] gives the first two estimates.

Due to the lifting property of elliptic operators, we obtain a solution $u \in H^2(Z^+)$; and because of the embedding $H^2(Z^+) \hookrightarrow \mathcal{C}^0(Z^+)$ there exists a continuous representative. Therefore we can apply Theorem 2 in [LP85] in order to get the pointwise estimate. ♦

A.11 Proposition.
Assume that (ζ, κ) *is a solution of Problem* (A.2) *with* $e^{\gamma_1 y_2}\rho \in H^1(Z^+)^2$, $e^{\gamma_1 y_2}\rho_1 \in H^1(Z^+)^4$ *and* $e^{\gamma_1 y_2}\,\mathrm{div}_y(F^{-1}(x)\rho_1) \in H^1(Z^+)^2$.

There exist $\beta > 0$, $y^* > 0$, *a vector* $C_\zeta \in \mathbb{R}^2$ *and a constant* C_κ *such that*

$$\begin{aligned} \|\nabla\zeta\|_{L^2(Z^+\cap([0,1]\times[k,\infty)))^4} &\le Ce^{-\beta k} \\ \|\zeta - C_\zeta\|_{L^2(Z^+\cap([0,1]\times[k,\infty)))^2} &\le Ce^{-\beta k} \\ \|\kappa - C_\kappa\|_{L^2(Z^+\cap([0,1]\times[k,\infty)\;))} &\le Ce^{-\beta k} \end{aligned}$$

and

$$\begin{aligned} |\zeta(y) - C_\zeta| &\le Ce^{-\beta y_2} \qquad \textit{for } y_2 > y^* \\ |\kappa(y) - C_\kappa| &\le Ce^{-\beta y_2} \qquad \textit{for } y_2 > y^* \end{aligned}$$

Proof. Set $\xi = \widetilde{\mathrm{curl}}(\zeta)$. By taking the $\widetilde{\mathrm{curl}}$ of Equation (A.2) we obtain due to Lemma 2.7:

$$\mathrm{div}_y(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\xi(y)) = -\,\widetilde{\mathrm{curl}}\Big(\rho(y) + \mathrm{div}_y\big(F^{-1}(x)\rho_1(y)\big)\Big) \qquad \text{in } Z^+.$$

The right hand side decays exponentially, thus by the preceding theorem we obtain

$$\begin{aligned} \Big\|\nabla(\widetilde{\mathrm{curl}}\,\zeta)\Big\|_{L^2(Z^+\cap([0,1]\times[k,\infty)))^2} &\le Ce^{-\beta k} \\ \Big\|\widetilde{\mathrm{curl}}\,\zeta - C_C\Big\|_{L^2(Z^+\cap([0,1]\times[k,\infty)))} &\le Ce^{-\beta k} \end{aligned}$$

with some constants $\beta > 0$ and C_C.

Using the 7th assertion of the transformation lemma 2.7 we see that

$$F^{-T}(x)\,\nabla_y(\xi(y)) = F^{-T}(x)\,\nabla_y\big(\widetilde{\mathrm{curl}}\,\zeta(y)\big) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \mathrm{div}_y\big(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\zeta(y)\big).$$

Therefore

$$\mathrm{div}_y\big(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\zeta(y)\big) = h(y) \qquad \text{in } Z^+,$$

h being a known function with $e^{\beta y_2}h \in L^2(Z^+)$. Theorem A.10 now shows that the first two asserted inequalities about the decay of ζ and $\nabla\zeta$ hold.

By taking the transformed divergence of Equation (A.2a) one obtains

$$\mathrm{div}_y\big(F^{-1}(x)F^{-T}(x)\,\nabla_y\,\kappa(y)\big) = -\,\mathrm{div}_y\Big(F^{-1}(x)\big[\rho(y) + \mathrm{div}_y(F^{-1}(x)\rho_1(y))\big]\Big) \qquad \text{in } Z^+.$$

Again, the right hand side decays exponentially, and the estimate for κ is proved. The remaining two inequalities follow easily. ♦

At the end of this paragraph, we want to obtain some information about the constant C_ζ:

A.12 Lemma.
For the solution ζ of Problem (A.2) *it holds for all $z<0$*

$$\int_0^1 \zeta(y_1,z)\cdot F^{-T}(x)e_2 \,\mathrm{d}y_1 = \int_0^1 \zeta(y_1,0)\cdot F^{-T}(x)e_2 \,\mathrm{d}y_1 = 0,$$

and $\int_S \zeta \cdot F^{-T}(x)e_2 \,\mathrm{d}\sigma_y = 0$.

Proof. By density it is enough to show the claim for $\zeta \in W \cap \mathcal{C}_0^\infty(Z_{\mathrm{BL}})^2$.

Integration of the equation $\mathrm{div}_y(F^{-1}(x)\zeta(y))$ over $[0,1]\times(z,0)$ and application of Stokes theorem yields due to the periodic boundary conditions

$$\int_0^1 \zeta(y_1,0)\cdot F^{-T}(x)e_2 \,\mathrm{d}y_1 = \int_0^1 \zeta(y_1,z)\cdot F^{-T}(x)e_2 \,\mathrm{d}y_1 =: K_\zeta \quad \forall z<0.$$

Since $\zeta \in W$ it holds

$$\int_{-\infty}^0 \Big(\int_0^1 \zeta(y_1,t)\cdot F^{-T}(x)e_2 \,\mathrm{d}y_1\Big)^2 \,\mathrm{d}t \le C \int_{-\infty}^0 \int_0^1 |\zeta(y_1,t)|^2 \,\mathrm{d}y_1 \,\mathrm{d}t < \infty,$$

and thus $K_\zeta = 0$. ♦

A.13 Lemma.
For the constant C_ζ it holds $C_\zeta \cdot F^{-T}(x)e_2 = 0$.

Proof. Arguing as in the proof of the above lemma, we obtain

$$\int_0^1 \zeta(y_1,k)\cdot F^{-T}(x)e_2 \,\mathrm{d}y_1 = \int_0^1 \zeta(y_1,-k)\cdot F^{-T}(x)e_2 \,\mathrm{d}y_1$$

for all $k>0$. Now the left hand side converges exponentially to $\int_0^1 C_\zeta \cdot F^{-T}(x)e_2 \,\mathrm{d}y_1 = C_\zeta \cdot F^{-T}(x)e_2$, whereas the right hand side converges to 0. ♦

A.1.3 Application to the Stokes Boundary Layer Problems

We apply the results of the foregoing section to the problem: Find $(w^{i,\mathrm{bl}}, \pi^{i,\mathrm{bl}}) \in V \times L^2_{\mathrm{loc}}(Z_{\mathrm{BL}})$ such that for fixed $x \in \Omega$ it holds

$$
\begin{aligned}
-\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\nabla_y w^{i,\mathrm{bl}}(y)) + F^{-T}(x)\nabla_y \pi^{i,\mathrm{bl}}(y) &= 0 \quad \text{in } Z && \text{(A.3a)}\\
\operatorname{div}_y(F^{-1}(x)w^{i,\mathrm{bl}}(y)) &= 0 \quad \text{in } Z && \text{(A.3b)}\\
[w^{i,\mathrm{bl}}]_S(y) &= w^i(y) \quad \text{on } S && \text{(A.3c)}\\
[(F^{-1}(x)F^{-T}(x)\nabla_y w^{i,\mathrm{bl}} - F^{-1}(x)\pi^{i,\mathrm{bl}})e_2]_S(y) & \\
= (F^{-1}(x)F^{-T}(x)\nabla_y w^i - F^{-1}(x)\pi^i)e_2(y) & \quad \text{on } S && \text{(A.3d)}\\
w^{i,\mathrm{bl}}(y) = 0 \quad \text{on } \textstyle\bigcup_{k=1}^{\infty}\{\partial Y_S - \binom{0}{k}\}& && \text{(A.3e)}\\
w^{i,\mathrm{bl}}, \pi^{i,\mathrm{bl}} \text{ are 1-periodic in } y_1 & && \text{(A.3f)}
\end{aligned}
$$

where the functions w^i, π^i denote the solution of the cell problem for the transformed Stokes equation given by

$$
\begin{aligned}
-\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\nabla_y w^i(y)) + F^{-T}(x)\nabla_y \pi^i(y) &= F^{-T}(x)e_i \quad \text{in } Y^*\\
\operatorname{div}_y(F^{-1}(x)w^i(y)) &= 0 \quad \text{in } Y^*\\
w^i(y) &= 0 \quad \text{on } \partial Y_S\\
w^i, \pi^i \text{ are } Y\text{-periodic in } y &
\end{aligned}
$$

Regularity results imply the existence of a solution $w^i \in \mathcal{C}^\infty(Y^*)$.

We eliminate the jump of $w^{i,\mathrm{bl}}$ on S by setting

$$\gamma^{i,\mathrm{bl}}(y) = w^{i,\mathrm{bl}}(y) - H(y_2)R(y) - e_2 H(y_2)\hat{A}_i,$$

where $\hat{A}_i = \int_0^1 w^i(y_1, 0)\cdot F^{-T}(x)e_2 \,\mathrm{d}y_1$. R is defined in the following lemma:

A.14 Lemma.
The problem

$$
\begin{aligned}
\operatorname{div}_y(F^{-1}(x)R(y)) &= 0 \quad \textit{in } Z^+\\
R(y) &= w^i(y_1, -0) - e_2\hat{A}_i \quad \textit{on } S\\
R \textit{ is 1-periodic in } y_1 &
\end{aligned}
$$

has a solution $R \in H^3(Z^+)^2$ such that there exists a $\gamma_0 > 0$ with $e^{\gamma_0 y_2}R \in H^3(Z^+)^2$. Furthermore $R|_S \in \mathcal{C}^\infty(\bar{S})^2$.

Proof. Similar to Lemma 3.16 we split R in the sum

$$R(y) = F(x)\nabla_y \eta(y) + F(x)\operatorname{Curl}_y(\theta(y)).$$

Thus we get the condition

$$\begin{aligned} \Delta_y \eta(y)) = 0 &\quad \text{in } Z^+ \\ \frac{\partial \eta}{\partial \nu} = F^{-1}(x)(w^i(y_1,-0) - e_2 \hat{A}_i)\cdot e_2 &\quad \text{on } S \\ \eta \text{ is 1-periodic in } y_1 & \end{aligned}$$

The existence of a solution follows from standard theory, since the compatibility condition $\int_S F^{-1}(x)(w^i(y_1,-0) - e_2\hat{A}_{i2}) \cdot e_2 \, \mathrm{d}\sigma_y = 0$ holds. The above problem corresponds to the situation of Theorem A.10 with a Neumann boundary condition; therefore there exist $\gamma > 0$ and a constant C_η such that $\nabla \eta$ and $\eta - C_\eta$ decay exponentially with rate $e^{\gamma y_2}$. Choose γ_0 with $0 < \gamma_0 < \gamma$; then a simple computation shows that $e^{\gamma_0 y_2} \nabla \eta \in L^2(Z^+)$ and $e^{\gamma_0 y_2}(\eta - C_\eta) \in L^2(Z^+)$.

Similarly, by differentiating the governing equation for η we obtain

$$e^{\gamma_0 y_2} \frac{\partial^k \eta}{\partial^{k_1} y_1 \partial^{k_2} y_2} \in L^2(Z^+)^2$$

for $k_1 + k_2 = k, k_1, k_2 \in \mathbb{N}$.

For θ we require

$$\begin{aligned} &\mathrm{Curl}_y(\theta) \cdot e_2 = 0 \\ &\mathrm{Curl}_y(\theta) \cdot e_1 = -F^{-1}(x)(w^{i,\mathrm{bl}}(y_1,-0) - e_2\hat{A}_i)\cdot e_1 \quad \in H^{\frac{3}{2}}(S). \end{aligned}$$

Analogously to Lemma 3.16 we obtain the existence of $\theta \in H^4(Z^+)$, by application of the generalized inverse trace theorem. Note that we can apply Theorem 3.9 in a neighbourhood of S, yielding a θ with compact support (thus having esp. an exponential decay). ♦

Inserting $\gamma^{i,\mathrm{bl}}$ into the equations for $w^{i,\mathrm{bl}}$ we see that the former entity satisfies

$$\begin{aligned} -\operatorname{div}_y(F^{-1}(x)F^{-T}(x)\nabla_y \gamma^{i,\mathrm{bl}}(y)) + F^{-T}(x)\nabla_y \pi^{i,\mathrm{bl}}(y) & \\ = H(y_2)\operatorname{div}_y(F^{-1}(x)F^{-T(x)}\nabla_y R(y)) &\quad \text{in } Z \\ \operatorname{div}_y(F^{-1}(x)\gamma^{i,\mathrm{bl}}(y)) = 0 &\quad \text{in } Z \\ [\gamma^{i,\mathrm{bl}}]_S(y) = 0 &\quad \text{on } S \\ [(F^{-1}(x)F^{-T}(x)\nabla_y \gamma^{i,\mathrm{bl}} - F^{-1}(x)\pi^{i,\mathrm{bl}})e_2]_S(y) & \\ = (F^{-1}(x)F^{-T}(x)\nabla_y w^i - F^{-1}(x)\pi^i)e_2(y) + F^{-1}(x)F^{-T}(x)\nabla_y R(y) &\quad \text{on } S \\ \gamma^{i,\mathrm{bl}}(y) = 0 &\quad \text{on } \textstyle\bigcup_{k=1}^\infty \{\partial Y_S - \binom{0}{k}\} \\ \gamma^{i,\mathrm{bl}}, \pi^{i,\mathrm{bl}} \text{ are 1-periodic in } y_1 & \end{aligned}$$

with variational formulation

$$\begin{aligned}\int\limits_{Z_{\mathrm{BL}}} F^{-T}(x)\,\nabla_y\,\gamma^{i,\mathrm{bl}}(y) : F^{-T}(x)\,\nabla_y\,\phi(y)\;\mathrm{d}y &= \int\limits_{Z^+} \mathrm{div}_y\big(F^{-1}(x)F^{-T}(x)\,\nabla_y\,R(y)\big)\cdot\phi(y)\;\mathrm{d}y \\ &\quad - \int\limits_S F^{-1}(x)\big[F^{-T}(x)\,\nabla_y\,w^i - \pi^i I\big](y)e_2\cdot\phi(y)\;\mathrm{d}\sigma_y \qquad \text{(A.4)} \\ &\quad - \int\limits_S F^{-1}(x)F^{-T}(x)\,\nabla_y\,R(y)e_2\cdot\phi(y)\;\mathrm{d}\sigma_y \qquad \forall\,\phi\in W.\end{aligned}$$

This corresponds to Equation (A.1). By the above results about the exponential decay in Z we therefore obtain the following proposition:

A.15 Proposition.
Problem (A.4) *has a unique solution with* $\gamma^{i,\mathrm{bl}}\in W$, $\gamma^{i,\mathrm{bl}}\in\mathcal{C}^\infty_{\mathrm{loc}}(Z)^2$. *There exists a constant* $\gamma_0>0$ *and a vector* $C^i_\gamma\in\mathbb{R}^2$ *such that*

$$\begin{aligned} e^{\gamma_0|y_2|}\,\nabla_y\,\gamma^{i,\mathrm{bl}} &\in L^2(Z)^4 \\ e^{\gamma_0|y_2|}\big(\gamma^{i,\mathrm{bl}} - H(y_2)C^i_\gamma\big) &\in L^2(Z)^2. \end{aligned}$$

Furthermore there exists $\pi^{i,\mathrm{bl}}\in\mathcal{C}^\infty_{\mathrm{loc}}(Z)$ *together with constants* $\gamma_1>0$, C^i_∞ *and* C^i_π *with*

$$e^{\gamma_1|y_2|}\big(\pi^{i,\mathrm{bl}} - H(y_2)C^i_\pi - H(-y_2)C^i_\infty\big)\in L^2(Z).$$

A.16 Corollary.
Set $w^{i,\mathrm{bl}} := \gamma^{i,\mathrm{bl}} + H(y_2)R + e_2H(y_2)\hat{A}_i$. *Choose the free constant in the pressure* $\pi^{i,\mathrm{bl}}$ *in such a way that* $C^i_\infty = 0$.

Then $(w^{i,\mathrm{bl}},\pi^{i,\mathrm{bl}})$ *is a solution of Problem* (A.3)*; and there exist constants* $\gamma>0, C^i_\pi$ *and a constant vector* $C^{i,\mathrm{bl}}$ *such that*

$$\begin{aligned} e^{\gamma|y_2|}\,\nabla_y\,w^{i,\mathrm{bl}} &\in L^2(Z)^4 \\ e^{\gamma|y_2|}(w^{i,\mathrm{bl}} - H(y_2)C^{i,\mathrm{bl}}) &\in L^2(Z)^2 \\ e^{\gamma|y_2|}(\pi^{i,\mathrm{bl}} - H(y_2)C^i_\pi) &\in L^2(Z). \end{aligned}$$

With the help of Lemma A.13 we obtain the value of $C^{i,\mathrm{bl}}\cdot F^{-T}(x)e_2$:

A.17 Lemma.
It holds

$$C^{i,\mathrm{bl}}\cdot F^{-T}(x)e_2 = \hat{A}_i, \qquad \text{(A.5)}$$

where $\hat{A}_i = \int_0^1 w^i(y_1,0)\cdot F^{-T}(x)e_2\;\mathrm{d}y_1$.

Proof. By application of Lemma A.13 it holds $C_\gamma^i \cdot F^{-T}(x)e_2 = 0$. Now consider the equation $\gamma^{i,\mathrm{bl}}(y) = w^{i,\mathrm{bl}}(y) - H(y_2)R(y) - e_2 H(y_2)\hat{A}_i$. Since R stabilizes exponentially to 0 in Z^+ and the last term of the right hand side is constant, we get the following condition for the stabilization:

$$C^{i,\mathrm{bl}} \cdot F^{-T}(x)e_2 = C_\gamma^i \cdot F^{-T}(x)e_2 + e_2\hat{A}_i \cdot F^{-T}(x)e_2 = \hat{A}_i$$

because of $(F^{-1}(x))_{2,2} = 1$. ♦

Finally, we can obtain the complete information about the constants:

A.18 Lemma.
For all $0 < a < b$ it holds

$$\int_0^1 \pi^{i,\mathrm{bl}}(y_1, a)\ \mathrm{d}y_1 = \int_0^1 \pi^{i,\mathrm{bl}}(y_1, b)\ \mathrm{d}y_1.$$

Thus the constant C_π^i arising in the stabilization of the pressure is given by

$$C_\pi^i = \int_0^1 \pi^{i,\mathrm{bl}}(y_1, +0)\ \mathrm{d}y_1.$$

Proof. Due to (A.3b) and the actual entries of $F^{-1}(x)$ it holds

$$\frac{\partial}{\partial y_1} w_1^{i,\mathrm{bl}} + \frac{\partial}{\partial y_2} w_2^{i,\mathrm{bl}} - g'(x_1)\frac{\partial}{\partial y_2} w_1^{i,\mathrm{bl}} = 0. \tag{A.6}$$

Note that $F^{-T}(x)\,\nabla_y\,\pi^{i,\mathrm{bl}}(y) = \mathrm{div}_y(F^{-1}(x)\pi^{i,\mathrm{bl}}(y))$ (cf. Lemma 2.7), thus Equation (A.3a) reads column-wise

$$\mathrm{div}_y\Big(F^{-1}(x)\big(F^{-T}(x)\,\nabla_y\,w_1^{i,\mathrm{bl}}(y) - \pi^{i,\mathrm{bl}}(y)e_1\big)\Big) = 0$$

and

$$\mathrm{div}_y\Big(F^{-1}(x)\big(F^{-T}(x)\,\nabla_y\,w_2^{i,\mathrm{bl}}(y) - \pi^{i,\mathrm{bl}}(y)e_2\big)\Big) = 0.$$

Let $0 < a < b$. Now integration of the equation

$$\begin{aligned} 0 = {} & \mathrm{div}_y\Big(F^{-1}(x)\big(F^{-T}(x)\,\nabla_y\,w_2^{i,\mathrm{bl}}(y) - \pi^{i,\mathrm{bl}}(y)e_2\big)\Big) \\ & - g'(x_1)\,\mathrm{div}_y\Big(F^{-1}(x)\big(F^{-T}(x)\,\nabla_y\,w_1^{i,\mathrm{bl}}(y) - \pi^{i,\mathrm{bl}}(y)e_1\big)\Big) \end{aligned}$$

over the rectangle $[0,1]\times[a,b]$ yields due to Stokes' theorem and the periodicity of $w^{i,\mathrm{bl}}$ and $\pi^{i,\mathrm{bl}}$ in y_1-direction

$$
\begin{aligned}
0 = &\int_0^1 \big(F^{-T}(x)\,\nabla_y\, w_2^{i,\mathrm{bl}} - \pi^{i,\mathrm{bl}} e_2\big)(y_1,b)\cdot F^{-T}(x) e_2 \,\mathrm{d}y_1 \\
&- \int_0^1 \big(F^{-T}(x)\,\nabla_y\, w_2^{i,\mathrm{bl}} - \pi^{i,\mathrm{bl}} e_2\big)(y_1,a)\cdot F^{-T}(x) e_2 \,\mathrm{d}y_1 \\
&- g'(x_1)\int_0^1 \big(F^{-T}(x)\,\nabla_y\, w_1^{i,\mathrm{bl}} - \pi^{i,\mathrm{bl}} e_1\big)(y_1,b)\cdot F^{-T}(x) e_2 \,\mathrm{d}y_1 \\
&+ g'(x_1)\int_0^1 \big(F^{-T}(x)\,\nabla_y\, w_1^{i,\mathrm{bl}} - \pi^{i,\mathrm{bl}} e_1\big)(y_1,a)\cdot F^{-T}(x) e_2 \,\mathrm{d}y_1.
\end{aligned}
$$

Now we have (by using the actual form of $F^{-T}(x)$)

$$
\pi^{i,\mathrm{bl}}(y) e_2 \cdot F^{-T}(x) e_2 = \pi^{i,\mathrm{bl}}(y) \quad \text{and} \quad \pi^{i,\mathrm{bl}}(y) e_1 \cdot F^{-T}(x) e_2 = -g'(x_1)\pi^{i,\mathrm{bl}}(y)
$$

as well as

$$
\begin{aligned}
F^{-T}(x)\,\nabla_y\, w_1^{i,\mathrm{bl}}(y)\cdot F^{-T}(x) e_2 &= (1+g'(x_1)^2)\frac{\partial}{\partial y_2} w_1^{i,\mathrm{bl}}(y) - g'(x_1)\frac{\partial}{\partial y_1} w_1^{i,\mathrm{bl}}(y) \\
F^{-T}(x)\,\nabla_y\, w_2^{i,\mathrm{bl}}(y)\cdot F^{-T}(x) e_2 &= (1+g'(x_1)^2)\frac{\partial}{\partial y_2} w_2^{i,\mathrm{bl}}(y) - g'(x_1)\frac{\partial}{\partial y_1} w_2^{i,\mathrm{bl}}(y) \\
&= (1+g'(x_1)^2)\Big(g'(x_1)\frac{\partial}{\partial y_2} w_1^{i,\mathrm{bl}}(y) - \frac{\partial}{\partial y_1} w_1^{i,\mathrm{bl}}\Big) \\
&\quad - g'(x_1)\frac{\partial}{\partial y_1} w_2^{i,\mathrm{bl}},
\end{aligned}
$$

where the identity (A.6) was used in the last equation. Substituting these results in the above equation, one obtains

$$
\begin{aligned}
0 = &\int_0^1 \Big(-g'(x_1)\frac{\partial}{\partial y_1} w_2^{i,\mathrm{bl}} - (1+g'(x_1)^2)\frac{\partial}{\partial y_1} w_1^{i,\mathrm{bl}} + g'(x_1)\frac{\partial}{\partial y_1} w_1^{i,\mathrm{bl}}\Big)(y_1,b)\,\mathrm{d}y_1 \\
&- \int_0^1 \Big(-g'(x_1)\frac{\partial}{\partial y_1} w_2^{i,\mathrm{bl}} - (1+g'(x_1)^2)\frac{\partial}{\partial y_1} w_1^{i,\mathrm{bl}} + g'(x_1)\frac{\partial}{\partial y_1} w_1^{i,\mathrm{bl}}\Big)(y_1,a)\,\mathrm{d}y_1 \\
&- (1+g'(x_1)^2)\int_0^1 \pi^{i,\mathrm{bl}}(y_1,b)\,\mathrm{d}y_1 + (1+g'(x_1)^2)\int_0^1 \pi^{i,\mathrm{bl}}(y_1,a)\,\mathrm{d}y_1.
\end{aligned}
$$

The first two integrals vanish due to the fundamental theorem of calculus and the periodic boundary conditions. We divide by $(1 + g'(x_1)^2)$ to obtain

$$\int_0^1 \pi^{i,\mathrm{bl}}(y_1, a) \,\mathrm{d}y_1 = \int_0^1 \pi^{i,\mathrm{bl}}(y_1, b) \,\mathrm{d}y_1 \qquad \forall\, 0 < a < b.$$

This proves the first statement.

To obtain the second one, notice that for $k > 0$ due to Jensen's inequality

$$\begin{aligned}\int_k^\infty \Big| \int_0^1 \pi^{i,\mathrm{bl}}(y_1, y_2) \,\mathrm{d}y_1 - C_\pi^i \Big|^2 \,\mathrm{d}y_2 \le \int_{[0,1]\times(k,\infty)} &|\pi^{i,\mathrm{bl}}(y_1, y_2) - C_\pi^i|^2 \,\mathrm{d}y \\ &\longrightarrow 0 \quad \text{for } k \to \infty\end{aligned}$$

because of the exponential stabilization of $\pi^{i,\mathrm{bl}}$; therefore $\int_0^1 \pi^{i,\mathrm{bl}}(y_1, b) \,\mathrm{d}y_1$ converges to C_π^i for $b \to \infty$. Now letting $a \to +0$ yields the result. ♦

A.19 Lemma.
For the constant $C^{i,\mathrm{bl}}$ appearing in the exponential stabilization of the velocity $w^{i,\mathrm{bl}}$ it holds

$$C^{i,\mathrm{bl}} \cdot F(x)e_1 = \int_0^1 w^{i,\mathrm{bl}}(y_1, +0) \cdot F(x)e_1 \,\mathrm{d}y_1. \tag{A.7}$$

Proof. Let $b > 0$. Similarly to the above lemma, we multiply the equation

$$\begin{aligned}0 = &\operatorname{div}_y\Big(F^{-1}(x)\big(F^{-T}(x)\,\nabla_y\, w_1^{i,\mathrm{bl}}(y) - \pi^{i,\mathrm{bl}} e_1\big)\Big) \\ &+ g'(x_1)\operatorname{div}_y\Big(F^{-1}(x)\big(F^{-T}(x)\,\nabla_y\, w_2^{i,\mathrm{bl}}(y) - \pi^{i,\mathrm{bl}} e_2\big)\Big)\end{aligned}$$

by y_2 and integrate over $[0,1]\times[0,b]$. Integration by parts then yields

$$\begin{aligned}0 = &- \int_{[0,1]\times[0,b]} F^{-1}(x)\big(F^{-T}(x)\,\nabla_y\, w_1^{i,\mathrm{bl}}(y) - \pi^{i,\mathrm{bl}}(y) e_1\big)\cdot e_2 \,\mathrm{d}y \\ &- g'(x_1) \int_{[0,1]\times[0,b]} F^{-1}(x)\big(F^{-T}(x)\,\nabla_y\, w_2^{i,\mathrm{bl}}(y) - \pi^{i,\mathrm{bl}}(y) e_2\big)\cdot e_2 \,\mathrm{d}y \\ &+ \int_0^1 b\big(F^{-T}(x)\,\nabla_y\, w_1^{i,\mathrm{bl}} - \pi^{i,\mathrm{bl}} e_1\big)(y_1, b)\cdot F^{-T}(x) e_2 \,\mathrm{d}y_1 \\ &+ g'(x_1)\int_0^1 b\big(F^{-T}(x)\,\nabla_y\, w_2^{i,\mathrm{bl}} - \pi^{i,\mathrm{bl}} e_2\big)(y_1, b)\cdot F^{-T}(x) e_2 \,\mathrm{d}y_1.\end{aligned}$$

As in the proof of the preceeding lemma we have $F^{-1}(x)\pi^{i,\mathrm{bl}}(y)e_2 \cdot e_2 = \pi^{i,\mathrm{bl}}(y)e_2 \cdot F^{-T}(x)e_2 = \pi^{i,\mathrm{bl}}(y)$ and $\pi^{i,\mathrm{bl}}(y)e_1 \cdot F^{-T}(x)e_2 = -g'(x_1)\pi^{i,\mathrm{bl}}(y)$, thus the terms containing the pressure $\pi^{i,\mathrm{bl}}$ cancel out and we have

$$\begin{aligned}
0 = &-\int\limits_{[0,1]\times[0,b]} F^{-1}(x)F^{-T}(x)\,\nabla_y\, w_1^{i,\mathrm{bl}}(y)\cdot e_2 \,\mathrm{d}y \\
&- g'(x_1)\int\limits_{[0,1]\times[0,b]} F^{-1}(x)F^{-T}(x)\,\nabla_y\, w_2^{i,\mathrm{bl}}(y)\cdot e_2 \,\mathrm{d}y \\
&+ \int\limits_0^1 bF^{-T}(x)\,\nabla_y\, w_1^{i,\mathrm{bl}}(y_1,b)\cdot F^{-T}(x)e_2 \,\mathrm{d}y_1 \\
&+ g'(x_1)\int\limits_0^1 bF^{-T}(x)\,\nabla_y\, w_2^{i,\mathrm{bl}}(y_1,b)\cdot F^{-T}(x)e_2 \,\mathrm{d}y_1.
\end{aligned}$$

Another integration by parts of the volume terms now yields

$$\begin{aligned}
0 = &-\int\limits_0^1 w_1^{i,\mathrm{bl}}(y_1,b)F^{-1}(x)F^{-T}(x)e_2\cdot e_2 \,\mathrm{d}y_1 \\
&+ \int\limits_0^1 w_1^{i,\mathrm{bl}}(y_1,+0)F^{-1}(x)F^{-T}(x)e_2\cdot e_2 \,\mathrm{d}y_1 \\
&- g'(x_1)\int\limits_0^1 w_2^{i,\mathrm{bl}}(y_1,b)F^{-1}(x)F^{-T}(x)e_2\cdot e_2 \,\mathrm{d}y_1 \\
&+ g'(x_1)\int\limits_0^1 w_2^{i,\mathrm{bl}}(y_1,+0)F^{-1}(x)F^{-T}(x)e_2\cdot e_2 \,\mathrm{d}y_1 \\
&+ \int\limits_0^1 bF^{-T}(x)\,\nabla_y\, w_1^{i,\mathrm{bl}}(y_1,b)\cdot F^{-T}(x)e_2 \,\mathrm{d}y_1 \\
&- g'(x_1)\int\limits_0^1 bF^{-T}(x)\,\nabla_y\, w_2^{i,\mathrm{bl}}(y_1,b)\cdot F^{-T}(x)e_2 \,\mathrm{d}y_1.
\end{aligned}$$

When passing to the limit $b \to \infty$, the last two integrals vanish since $\nabla_y\, w^{i,\mathrm{bl}}$ decays exponentially to 0. The terms $w_1^{i,\mathrm{bl}}$ and $w_2^{i,\mathrm{bl}}$ converge to $C_1^{i,\mathrm{bl}}$ and $C_2^{i,\mathrm{bl}}$, repectively.

Thus

$$\int_0^1 \Big(C_1^{i,\mathrm{bl}} + g'(x_1)C_2^{i,\mathrm{bl}}\Big) F^{-1}(x)F^{-T}(x)e_2 \cdot e_2 \,\mathrm{d}y_1$$
$$= \int_0^1 \Big(w_1^{i,\mathrm{bl}} + g'(x_1)w_2^{i,\mathrm{bl}}\Big)(y_1,+0) F^{-1}(x)F^{-T}(x)e_2 \cdot e_2 \,\mathrm{d}y_1.$$

Since $F^{-1}(x)F^{-T}(x)e_2 \cdot e_2 = 1 + g'(x_1)^2$ we can divide the above equation by $1 + g'(x_1)^2$, leading to

$$C_1^{i,\mathrm{bl}} + g'(x_1)C_2^{i,\mathrm{bl}} = \int_0^1 (w_1^{i,\mathrm{bl}} + g'(x_1)w_2^{i,\mathrm{bl}})(y_1,+0)\,\mathrm{d}y_1.$$

This is equation (A.7). ♦

A.20 Remark.
With the help of (A.5) *and* (A.7) *it is possible to obtain the value of* $C^{i,\mathrm{bl}}$*: Using the exact form of* $F(x)$ *and* $F^{-T}(x)$*, the above conditions read*

$$\begin{array}{rcccl} C_1^{i,\mathrm{bl}} & + & g'(x_1)C_2^{i,\mathrm{bl}} & = & \int_0^1 w^{i,\mathrm{bl}}(y_1,+0)\cdot F(x)e_1 \,\mathrm{d}y \\ -g'(x_1)C_1^{i,\mathrm{bl}} & + & C_2^{i,\mathrm{bl}} & = & \hat{A}_i(x) \end{array}.$$

Since the determinant of the coefficient matrix of the left hand side fulfills

$$\det\begin{bmatrix} 1 & g'(x_1) \\ -g'(x_1) & 1 \end{bmatrix} = (1 + g'(x_1)^2) \neq 0,$$

the above linear system of equations has a unique solution $C^{i,\mathrm{bl}} = \begin{pmatrix} C_1^{i,\mathrm{bl}} \\ C_2^{i,\mathrm{bl}} \end{pmatrix}$.

We finish this subsection by showing the relation between the permeability tensor A and the term $\hat{A}_i$:

A.21 Lemma.
Let $A_i = (\int_{Y^*} w^i \,\mathrm{d}y)$ *be the i-th column of the permeability tensor and set* $\hat{A}_i = (\int_S w^i \cdot F^{-T}(x)e_2 \,\mathrm{d}\sigma_y)$ *as above. Then it holds*

$$\hat{A}_i = A_i \cdot F^{-T}(x)e_2 = C^{i,\mathrm{bl}} \cdot F^{-T}(x)e_2.$$

Proof. We argue as above: Integration of the condition $\operatorname{div}_y(F^{-1}(x)w^i) = 0$ over the set $(0,1) \times (0,b)$ for $b \in (0,1)$ and application of Stokes' theorem yields due to the periodic boundary conditions

$$\int_0^1 w^i(y_1, 0) \cdot F^{-T}(x) e_2 \, \mathrm{d}y_1 = \int_0^1 w^i(y_1, b) \cdot F^{-T}(x) e_2 \, \mathrm{d}y_1.$$

Now integrate this equation over the interval $(0,1)$ with respect to b to obtain

$$\begin{aligned} \hat{A}_i &= \int_0^1 \int_0^1 w^i(y_1, 0) \cdot F^{-T}(x) e_2 \, \mathrm{d}y_1 \, \mathrm{d}b = \int_0^1 \int_0^1 w^i(y_1, b) \cdot F^{-T}(x) e_2 \, \mathrm{d}y_1 \, \mathrm{d}b \\ &= \Big(\int_{Y^*} w^i \, \mathrm{d}y \Big) \cdot F^{-T}(x) e_2 = A_i \cdot F^{-T}(x) e_2 \end{aligned}$$

The second equality follows due to Lemma A.17. ♦

A.2 Very Weak Solutions of the Transformed Stokes Equation

In order to get sufficient estimates in Ω_1, we develop a theory of very weak solutions for the transformed Stokes equation.

Consider the problem: Find (B, β) such that

$$-\operatorname{div}(F^{-1}(x)F^{-T}(x)B(x)) + F^{-T}(x) \nabla \beta(x) = G_1 + \operatorname{div}(F^{-1}(x)G_2) \quad \text{in } \Omega_1 \tag{A.8a}$$

$$\operatorname{div}(F^{-1}(x)B(x)) = \Theta(x) \quad \text{in } \Omega_1 \tag{A.8b}$$

$$B(x) = \xi(x) \quad \text{on } \Sigma \tag{A.8c}$$

$$B, \beta \text{ are } L\text{-periodic in } x_1 \tag{A.8d}$$

with $\xi \in L^2(\Sigma)^2$, $e^{\gamma_0 x_2}\Theta \in L^2(\Omega_1)$, Θ L-periodic in x_1 and $G_1 \in L^2(\Omega_1)^2$, $G_2 \in L^2(\Omega_1)^4$ such that $e^{\gamma_0 x_2}(|G_1| + |G_2|) \in L^2(\Omega_1)$ for some $\gamma_0 > 0$.

For the development of the theory, we assume that there exists a solution (B, β) satisfying

$$\begin{gathered} e^{\gamma_0 x_2} \nabla B \in L^2(\Omega_1)^4, \;\; e^{\gamma_0 x_2}(B - B_\infty) \in L^2(\Omega_1)^2, \;\; e^{\gamma_0 x_2}(\beta - \beta_\infty) \in L^2(\Omega_1) \\ |\nabla B(x)| \le e^{-\gamma_0 x_2}, \;\; |B(x) - B\infty| \le e^{-\gamma_0 x_2}, \;\; |\beta(x) - \beta_\infty| \le e^{-\gamma_0 x_2} \end{gathered}$$

for $x_2 > x^*$ with given constant (vectors) x^*, γ_0, B_∞ and β_∞.

Due to the effect of the curved boundary, we need the following assumption:

A.22 Assumption.
We assume that for the function Ψ_2 as defined in Lemma A.24 it holds

$$\int_{\Sigma} F^{-1}F^{-T} \nabla \Psi_2 \cdot e_2 \, d\sigma_x = 0.$$

A sufficient condition for this assumption to hold is that the function g defining the boundary (cf. Section 2.2) is point-symmetric with respect to the point $\frac{L}{2}$ on $[0, L]$, see below.

A.23 Proposition.
Let f, h be given functions with $(1+x_2)f \in L^2(\Omega_1)^2$, $(1+x_2)h \in L^2(\Omega_1)$, $(1+x_2)|\nabla h| \in L^2(\Omega_1)$ and $\int_{\Omega_1} h \, dx = 0$.
There exists a unique solution for the problem

$$-\operatorname{div}(F^{-1}(x)F^{-T}(x)\nabla \Phi(x)) + F^{-T}(x)\nabla \pi(x) = f(x) \text{ in } \Omega_1 \tag{A.9a}$$
$$\operatorname{div}(F^{-1}(x)\Phi(x)) = h \text{ in } \Omega_1 \tag{A.9b}$$
$$\Phi(x) = 0 \text{ on } \Sigma \tag{A.9c}$$
$$\Phi, \pi \text{ is } L\text{-periodic in } x_1 \tag{A.9d}$$

with $\nabla \Phi \in L^2(\Omega_1)^4$, $\frac{\phi}{1+x_2} \in L^2(\Omega_1)^2$, $\nabla \pi \in L^2(\Omega_1)^2$ and $\frac{\pi}{1+x_2} \in L^2(\Omega_1)$.

Proof. We again use a decomposition approach and look for Ψ_1 satisfying

$$-\operatorname{div}(F^{-1}(x)F^{-T}(x)\nabla \Psi_1(x)) = h(x) \text{ in } \Omega_1 \tag{A.10a}$$
$$F^{-1}(x)F^{-T}(x)\nabla \Psi_1(x) \cdot e_2 = 0 \text{ on } \Sigma \tag{A.10b}$$
$$\Psi_1 \text{ is } L\text{-periodic in } x_1 \tag{A.10c}$$

Similar to Lemma A.33 we get the existence of a solution $\Psi_1 \in L^2_{\text{loc}}(\Omega_1)$ which is unique up to a constant, with

$$\nabla \Psi_1 \in L^2(\Omega_1),$$
$$\frac{\partial^2 \Psi_1}{\partial x_i \partial x_j} \in L^2(\Omega_1),$$
$$\frac{\partial^3 \Psi_1}{\partial x_i \partial x_j \partial x_k} \in L^2(\Omega_1),$$

and by Lemma 5.3

$$\frac{1}{1+x_2}\Big(\Psi_1 - \frac{1}{L}\int_{\Sigma} \Psi_1 \, d\sigma_x\Big) \in L^2(\Omega_1).$$

Now let $\theta_0(x) := -x_2(F^{-1}F^{-T}\nabla\Psi_1 \cdot e_1)(x_1,+0)e^{-x_2}$ and define

$$\Psi_2 = \widetilde{\mathrm{Curl}}(\theta_0).$$

Then $\Psi_2 \in H^2(\Omega_1)^2$, $\mathrm{div}(F^{-1}\Psi_2) = 0$ in Ω_1 and Ψ_2 is L-periodic in x_1. Observe that

$$-\frac{\partial}{\partial x_2}\theta_0(x_1,x_2)|_{x_2=0} = (F^{-1}F^{-T}\nabla\Psi_1 \cdot e_1)(x_1,+0)$$

and

$$\frac{\partial}{\partial x_1}\theta_0(x_1,x_2)|_{x_2=0} = 0 = (F^{-1}F^{-T}\nabla\Psi_1 \cdot e_2)(x_1,+0).$$

Therefore $\mathrm{Curl}(\theta) = F^{-1}F^{-T}\nabla\Psi_1$ on Σ, and we obtain

$$\widetilde{\mathrm{Curl}}(\theta_0) = F\,\mathrm{Curl}(\theta_0) = F^{-T}\nabla\Psi_1 \quad \text{on } \Sigma.$$

Next we are looking for Ψ_3 and π with

$$\begin{aligned} -\mathrm{div}(F^{-1}(x)F^{-T}(x)\nabla\Psi_3(x)) + F^{-T}(x)\nabla\pi(x) = f(x)& \\ +\mathrm{div}(F^{-1}(x)h(x)) + \mathrm{div}(F^{-1}(x)F^{-T}(x)\nabla\Psi_2(x)) \text{ in } \Omega_1& \\ \mathrm{div}(F^{-1}(x)\Psi_3(x)) = 0 \text{ in } \Omega_1& \\ \Psi_3(x) = 0 \text{ on } \Sigma& \\ \Psi_3 \text{ is } L\text{-periodic in } x_1& \end{aligned}$$

The weak formulation of the above problem in

$$\begin{aligned} \tilde{W} = \big\{\phi \in L^2_{\mathrm{loc}}(\Omega_1)^2 \mid \nabla\phi \in L^2(\Omega_1)^4, \phi = 0 \text{ on } \Sigma,& \\ \mathrm{div}(F^{-1}\phi) = 0 \text{ in } \Omega_1, \phi \text{ is } L\text{-periodic in } x_1\big\}& \end{aligned}$$

reads

$$\int_{\Omega_1} F^{-T}\nabla\Psi_3 : F^{-T}\nabla\phi\,\mathrm{d}x = \int_{\Omega_1} f\cdot\phi\,\mathrm{d}x - \int_{\Omega_1} F^{-T}\nabla\Psi_2 : F^{-T}\nabla\phi\,\mathrm{d}x \quad \forall\phi \in \tilde{W}$$

(here the term containing h vanishes due to $\int_{\Omega_1}\mathrm{div}(F^{-1}h)\phi\,\mathrm{d}x = \int_{\Omega_1}F^{-T}\nabla h\cdot\phi\,\mathrm{d}x = \int_{\Omega_1}h\,\mathrm{div}(F^{-1}\phi)\,\mathrm{d}x = 0$). Using Lemma 5.3, we obtain the existence of a solution Ψ_3 with $\frac{\Psi_3}{1+x_2} \in L^2(\Omega_1)^2$ and $\nabla\Psi_3 \in L^2(\Omega_1)^4$.

A simple calculation using the transformed differential equalities shows that

$$\Phi = F^{-T}\nabla\Psi_1 + \Psi_2 + \Psi_3$$

solves the Problem (A.9) in weak sense.

We see that

$$\begin{aligned} \nabla \Phi &\in L^2(\Omega_1)^4, \\ \frac{\Phi}{1+x_2} &\in L^2(\Omega_1)^2, \\ \frac{\partial^2 \Phi}{\partial x_i \partial x_j} &\in L^2(\Omega_1)^2. \end{aligned}$$

The pressure can be reintroduced in the usual way; by regularity results we obtain $\pi \in H^1_{\mathrm{loc}}(\Omega_1)$. Taking the transformed divergence of Equation (A.9a), one obtains

$$\begin{aligned} \operatorname{div}(F^{-1}F^{-T}\nabla \pi) &= \operatorname{div}(F^{-1}F^{-T}\nabla h) + \operatorname{div}(F^{-1}f) \\ \pi &\in H^{\frac{1}{2}}(\Sigma) \\ \pi &\text{ is } L\text{-periodic in } x_1 \end{aligned}$$

Thus by elliptic regularity theory

$$\nabla \pi \in L^2(\Omega_1)^2 \quad \text{and} \quad \frac{\pi}{1+x_2} \in L^2(\Omega_1). \qquad \blacklozenge$$

A.24 Lemma.
Assume that $(1+x_2)\Theta \in L^2(\Omega_1)$. The problem

$$\begin{aligned} \operatorname{div}(F^{-1}F^{-T}\nabla w) &= \Theta \ \text{ in } \Omega_1 \\ F^{-1}F^{-T}\nabla w \cdot e_2 &= 0 \ \text{ on } \Sigma \\ w &\text{ is } L\text{-periodic in } x_1 \end{aligned}$$

has a solution $w \in L^2_{\mathrm{loc}}(\Omega_1)/\mathbb{R}$ such that $\frac{\partial^2 w}{\partial x_i \partial x_j} \in L^2(\Omega_1)$.

Proof. Define Ψ_1 by

$$\begin{aligned} \operatorname{div}(F^{-1}F^{-T}\nabla \Psi_1) &= \Theta - \frac{1}{L(1+x_2)^2}\Big(\int_{\Omega_1} \Theta \, \mathrm{d}x\Big) \text{ in } \Omega_1 \\ F^{-1}F^{-T}\nabla \Psi_1 \cdot e_2 &= 0 \text{ on } \Sigma \\ \Psi_1 &\text{ is } L\text{-periodic in } x_1 \end{aligned}$$

This corresponds to (A.10), thus we get a solution $\Psi_1 \in L^2(\Omega_1)/\mathbb{R}$ with $\nabla \Psi_1 \in L^2(\Omega_1)^2$.

Next we are looking for Ψ_2 with

$$\begin{aligned} \operatorname{div}(F^{-1}F^{-T}\nabla\Psi_2) &= \frac{1}{L(1+x_2)^2}\Big(\int_{\Omega_1}\Theta\,\mathrm{d}x\Big) \text{ in } \Omega_1 \\ \Psi_2 &= 0 \text{ on } \Sigma \\ \Psi_2 &\text{ is } L\text{-periodic in } x_1 \end{aligned}$$

Since due to Lemma 5.3

$$\left|\int_{\Omega_1}\frac{1}{L(1+x_2)^2}\Big(\int_{\Omega_1}\Theta\,\mathrm{d}x\Big)\phi\,\mathrm{d}x\right| \le C\left\|\frac{1}{L(1+x_2)}\Big(\int_{\Omega_1}\Theta\,\mathrm{d}x\Big)\right\|_{L^2(\Omega_1)}\|\nabla\phi\|_{L^2(\Omega_1)^2},$$

we obtain the existence of a unique solution Ψ_2 with $\nabla\Psi_2 \in L^2(\Omega_1)^2$.

Finally, we correct the behaviour at Σ by considering Ψ_3 with

$$\begin{aligned} \operatorname{div}(F^{-1}F^{-T}\nabla\Psi_3) &= 0 \text{ in } \Omega_1 \\ F^{-1}F^{-T}\nabla\Psi_3\cdot e_2 &= -F^{-1}F^{-T}\nabla\Psi_2\cdot e_2 \text{ on } \Sigma \\ \Psi_3 &\text{ is } L\text{-periodic in } x_1 \end{aligned}$$

By Assumption A.22, the compatibility condition for this nonhomogeneous Neumann problem,

$$\int_\Sigma F^{-1}F^{-T}\nabla\Psi_3\cdot e_2\,\mathrm{d}\sigma_x = 0,$$

is fullfilled and we get $\nabla\Psi_3 \in L^2(\Omega_1)$, and Ψ_3 is unique up to a constant. By setting $w = \Psi_1+\Psi_2+\Psi_3$ we obtain the desired function. ♦

In the sequel we need the following spaces:

$$\begin{aligned} W_2 &= \big\{z \in L^2(\Omega_1)^2 \mid (1+x_2)z \in L^2(\Omega_1)^2, z \text{ is } L\text{-periodic in } x_1\big\} \\ W_3 &= \big\{z \in L^2(\Omega_1) \mid (1+x_2)z \in L^2(\Omega_1), (1+x_2)|\nabla z| \in L^2(\Omega_1), \\ &\qquad z \text{ is } L\text{-periodic in } x_1\big\} \end{aligned}$$

A.25 Proposition.
Let f, h be as above and let w and Φ, π be defined as in the preceeding lemmas. Assume that Problem (A.8) *has a solution (B,β) with the mentioned properties.*

Then it holds

$$\int_{\Omega_1} (B - F^{-T}\nabla w)\cdot f \,\mathrm{d}x - \langle \beta - \Theta, h\rangle_{W_3', W_3} = \int_{\Omega_1} G_1 \cdot \Phi \,\mathrm{d}x - \int_{\Omega_1} G_2 : F^{-T}\nabla\Phi \,\mathrm{d}x$$
$$+ \int_{\Sigma} F^{-1}(F^{-T}\nabla\Phi - \pi I)e_2 \cdot (\xi - F^{-T}\nabla w)\,\mathrm{d}\sigma_x.$$

Proof. Consider $\lim_{b\to\infty}\int_{\Omega_b}[(B - F^{-T}\nabla w)\cdot f - (\beta-\Theta)h]\,\mathrm{d}x$ where $\Omega_b = [0,L]\times[0,b]$. Integration by parts, the 8^{th} equation of Lemma 2.7 and passing to the limit yield

$$\int_{\Omega_1} (B - F^{-T}\nabla w)\cdot f \,\mathrm{d}x - \langle \beta - \Theta, h\rangle_{W_3', W_3} = \int_{\Omega_1} G_1 \cdot \Phi \,\mathrm{d}x - \int_{\Omega_1} G_2 : F^{-T}\nabla\Phi \,\mathrm{d}x$$
$$+ \int_{\Sigma} F^{-1}(F^{-T}\nabla\Phi - \pi I)e_2 \cdot (\xi - F^{-T}\nabla w)\,\mathrm{d}\sigma_x$$
$$+ \lim_{b\to\infty}\int_0^L F^{-1}(F^{-T}\nabla B - F^{-T}\nabla(F^{-T}\nabla w) - \beta I + \Theta I)e_2\cdot\Phi(x_1,b)\,\mathrm{d}x_1$$
$$- \lim_{b\to\infty}\int_0^L F^{-1}(F^{-T}\nabla\Phi - \pi I)e_2\cdot(B - F^{-T}\nabla w)(x_1,b)\,\mathrm{d}x_1.$$

B and ∇w stabilize exponentially to some constant vector (by assumption/by application of Theorem A.10). Hence

$$(B - F^{-T}\nabla w)(x_1,b) \longrightarrow B_\infty \in \mathbb{R} \quad \text{for } b\to\infty.$$

Thus

$$\lim_{b\to\infty}\int_0^L F^{-1}(F^{-T}\nabla\Phi - \pi I)e_2\cdot(B - F^{-T}\nabla w)(x_1,b)\,\mathrm{d}x_1$$
$$= \int_0^L F^{-1}(F^{-T}\nabla\Phi - \pi I)e_2\cdot B_\infty\,\mathrm{d}x_1.$$

We choose a free constant in π in such a way that this term vanishes. Next, we integrate the equation $\operatorname{div}(F^{-1}\Phi) = h$ over Ω_b to obtain

$$\lim_{b\to\infty}\int_0^L F^{-1}\Phi(x_1,b)e_2\,\mathrm{d}x_1 = \lim_{b\to\infty}\int_{\Omega_b}\operatorname{div}(F^{-1}\Phi) = \int_{\Omega_1} h = 0.$$

Since $(F^{-T}\nabla B - F^{-T}\nabla(F^{-T}\nabla w) - \beta I + \Theta I)(x_1, b)$ stabilizes towards some constant for $b \to \infty$, the first limit term is equal to zero as well. ♦

A.26 Definition.
We call $(V, Q) \in W_2' \times W_3'$ *a* very weak solution *of Problem* (A.8) *if for all* $f \in W_2$ *and for all* $h \in W_3$ *the identity*

$$\int_{\Omega_1} (V - F^{-T}\nabla w) \cdot f \,\mathrm{d}x - \langle Q - \Theta, h\rangle_{W_3', W_3} = \int_{\Omega_1} G_1 \cdot \Phi \,\mathrm{d}x - \int_{\Omega_1} G_2 : F^{-T}\nabla\Phi \,\mathrm{d}x$$
$$+ \int_{\Sigma} F^{-1}(F^{-T}\nabla\Phi - \pi I)e_2 \cdot (\xi - F^{-T}\nabla w) \,\mathrm{d}\sigma_x \tag{A.11}$$

holds, where Φ, π *and* w *are defined as above.*

A.27 Lemma.
With the above assumptions, there exist a unique very weak solution.

Proof. This lemma follows by application of Riesz' representation theorem. The above equation reads

$$\int_{\Omega_1} V \cdot f \,\mathrm{d}x - \langle Q, h\rangle_{W_3', W_3} = \int_{\Sigma} F^{-1}(F^{-T}\nabla\Phi - \pi I)e_2 \cdot (\xi - F^{-T}\nabla w) \,\mathrm{d}\sigma_x \tag{A.12}$$
$$+ \int_{\Omega_1} G_1 \cdot \Phi \,\mathrm{d}x - \int_{\Omega_1} G_2 : F^{-T}\nabla\Phi \,\mathrm{d}x + \int_{\Omega_1} F^{-T}\nabla w \cdot f \,\mathrm{d}x + \langle\Theta, h\rangle_{W_3', W_3},$$

where the right hand side is linear in f and h. Furthermore, we obtain the estimate (cf. the proof of the following proposition)

$$\left|\int_{\Sigma} F^{-1}(F^{-T}\nabla\Phi - \pi I)e_2 \cdot (\xi - F^{-T}\nabla w) \,\mathrm{d}\sigma_x\right| \le C\big(\|\xi\|_{L^2(\Sigma)^2} + \|\Theta\|_{L^2(\Omega_1)}\big)\,\|f\|_{L^2(\Omega_1)^2}$$

as well as

$$\left|\int_{\Omega_1} G_1 \cdot \Phi \,\mathrm{d}x\right| + \left|\int_{\Omega_1} G_2 : F^{-T}\nabla\Phi \,\mathrm{d}x\right| \le C\big(\|(1+x_2)G_1\|_{L^2(\Omega_1)^2} + \|G_2\|_{L^2(\Omega_1)^4}\big)\,\|f\|_{L^2(\Omega_1)^2}$$

and

$$\left|\int_{\Omega_1} F^{-T}\nabla w \cdot f \,\mathrm{d}x\right| + |\langle\Theta, h\rangle_{W_3', W_3}| \le C\,\|\Theta\|_{L^2(\Omega_1)}\big(\|f\|_{L^2(\Omega_1)^2} + \|h\|_{L^2(\Omega_1)}\big).$$

Therefore the right hand side of (A.12) is a continuous linear functional in $W_2 \times W_3$. ♦

A.28 Proposition.
Let (B,β) be a solution of

$$
\begin{aligned}
-\operatorname{div}(F^{-1}(x)F^{-T}(x)B(x)) - F^{-T}(x)\nabla\beta(x) &= G_1 + \operatorname{div}(F^{-1}(x)G_2) \ \text{in } \Omega_1\\
\operatorname{div}(F^{-1}(x)B(x)) &= \Theta(x) \ \text{in } \Omega_1\\
B(x) &= \xi(x) \ \text{on } \Sigma\\
B,\beta \text{ are } L\text{-periodic in } x_1 &
\end{aligned}
$$

where $\xi \in L^2(\Sigma)^2, e^{\gamma_0 x_2}|\Theta| \in L^2(\Omega_1)$, Θ L-periodic in x_1 and $e^{\gamma_0 x_2}(|G_1|+|G_2|) \in L^2(\Omega_1)$ for some $\gamma_0 > 0$ as well as

$$
\begin{aligned}
&e^{\gamma_0 x_2}\nabla B \in L^2(\Omega_1)^4,\ e^{\gamma_0 x_2}(B - B_\infty)^2 \in L^2(\Omega_1),\ e^{\gamma_0 x_2}(\beta-\beta_\infty) \in L^2(\Omega_1)\\
&\quad |\nabla B(x)| \le e^{-\gamma_0 x_2},\ |B(x) - B\infty| \le e^{-\gamma_0 x_2},\ |\beta(x)-\beta_\infty| \le e^{-\gamma_0 x_2}.
\end{aligned}
$$

Then we have the following estimate:

$$
\begin{aligned}
\left\|(1+x_2)^{-1}B\right\|_{L^2(\Omega_1)^2} \le C\Big(&\|(1+x_2)\Theta\|_{L^2(\Omega_1)} + \|(1+x_2)G_1\|_{L^2(\Omega_1)^2}\\
&+ \|G_2\|_{L^2(\Omega_1)^4} + \|\xi\|_{L^2(\Sigma)^2}\Big)
\end{aligned}
$$

Proof. By the foregoing derivation we see that (B,β) is a very weak solution. Choose $f = (1+x_2)^{-2}B$ and $h = 0$ as test functions in (A.11) to obtain

$$
\begin{aligned}
\int_{\Omega_1}(1+x_2)^{-2}B\cdot(B - F^{-T}\nabla w)\,\mathrm{d}x &= \int_\Sigma F^{-1}(F^{-T}\nabla\Phi - \pi I)e_2\cdot(\xi - F^{-T}\nabla w)\,\mathrm{d}\sigma_x\\
&+ \int_{\Omega_1} G_1\cdot\Phi\,\mathrm{d}x - \int_{\Omega_1} G_2 : F^{-T}\nabla\Phi\,\mathrm{d}x.
\end{aligned}
$$

We estimate the terms seperately: We have

$$
\begin{aligned}
\left|\int_{\Omega_1} G_1\cdot\Phi\,\mathrm{d}x\right| &\le \left|\int_{\Omega_1}(1+x_2)G_1\cdot(1+x_2)^{-1}\Phi\,\mathrm{d}x\right| \le C\,\|(1+x_2)G_1\|_{L^2(\Omega_1)^2}\,\|\nabla\Phi\|_{L^2(\Omega_1)^4}\\
&\le C\,\|(1+x_2)G_1\|_{L^2(\Omega_1)^2}\left\|(1+x_2)^{-2}B\right\|_{L^2(\Omega_1)^2}\\
&\le C\,\|(1+x_2)G_1\|_{L^2(\Omega_1)^2}\left\|(1+x_2)^{-1}B\right\|_{L^2(\Omega_1)^2}
\end{aligned}
$$

and

$$\left|\int_{\Omega_1} G_2 : F^{-T}\,\nabla\,\Phi\;\mathrm{d}x\right| \le C\,\|G_2\|_{L^2(\Omega_1)^4}\,\|\nabla\,\Phi\|_{L^2(\Omega_1)^4}$$
$$\le C\,\|G_2\|_{L^2(\Omega_1)^4}\,\left\|(1+x_2)^{-1}B\right\|_{L^2(\Omega_1)^2}$$

together with

$$\left|\int_{\Sigma} F^{-1}(F^{-T}\,\nabla\,\Phi - \pi I)e_2\cdot\xi\;\mathrm{d}\sigma_x\right| \le C\,\left\|F^{-T}\,\nabla\,\Phi - \pi I\right\|_{H^1(\Omega_\Sigma)^4}\,\|\xi\|_{L^2(\Sigma)^2}$$
$$\le C\,\left\|(1+x_2)^{-1}B\right\|_{L^2(\Omega_1)^2}\,\|\xi\|_{L^2(\Sigma)^2}\,,$$

where Ω_Σ is a bounded domain containing Σ, due to elliptic regularity theory. Finally

$$\left|\int_{\Sigma} F^{-1}(F^{-T}\,\nabla\,\Phi - \pi I)e_2\cdot F^{-T}\,\nabla\,w\;\mathrm{d}\sigma_x\right| \le C\,\left\|(1+x_2)^{-1}B\right\|_{L^2(\Omega_1)^2}\,\left\|F^{-T}\,\nabla\,w\right\|_{L^2(\Omega_2)}$$
$$\le C\,\left\|(1+x_2)^{-1}B\right\|_{L^2(\Omega_1)^2}\,\|\Theta\|_{L^2(\Omega_1)}$$

as well as

$$\left|\int_{\Omega_1}(1+x_2)^{-2}B\cdot F^{-T}\,\nabla\,w\;\mathrm{d}x\right| \le C\,\left\|(1+x_2)^{-1}B\right\|_{L^2(\Omega_1)^2}\,\|\Theta\|_{L^2(\Omega_1)}\,. \qquad \blacklozenge$$

We conclude this subsection by showing that for a point-symmetric function g describing the interface Σ the Assumption A.22 holds: Let g be defined as in Section 2.2 and assume further that g is point-symmetric with respect to $\frac{L}{2}$ on $[0, L]$, i.e.

$$g(x) = -g(L-x) \quad \forall x \in [0, \frac{L}{2}].$$

A.29 Lemma.
Under the above assumptions, it holds

$$(F^{-1}F^{-T})(x_1, x_2) = (F^{-1}F^{-T})(L - x_1, x_2)$$

Proof. It holds for $x_1 \in [0, \frac{L}{2})$

$$
\begin{aligned}
(F^{-1}F^{-T})(L - x_1, x_2) &= \begin{bmatrix} 1 & -g'(L-x_1) \\ -g'(L-x_1) & 1 + g'(L-x_1)^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & (g(L-x_1))' \\ (g(L-x_1))' & 1 + ((g(L-x_1))')^2 \end{bmatrix} \\
&= \begin{bmatrix} 1 & -g'(x_1) \\ -g'(x_1) & 1 + g'(x_1)^2 \end{bmatrix} = (F^{-1}F^{-T})(x_1, x_2).
\end{aligned}
$$

♦

Next, we show that the function Ψ_2 as appearing in the proof of Lemma A.24 is axially symmetric with respect to the axis $\{x_2 = \frac{L}{2}\}$:

A.30 Lemma.
Define Ψ_2 *by*

$$\operatorname{div}(F^{-1}F^{-T}\,\nabla\,\Psi_2) = \frac{1}{L(1+x_2)^2}\int_{\Omega_1} \Theta \,\mathrm{d}x \;\; in\ \Omega_1 \tag{A.13a}$$

$$\Psi_2 = 0 \;\; on\ \Sigma \tag{A.13b}$$

$$\Psi_2 \text{ is } L\text{-periodic in } x_1 \tag{A.13c}$$

Then it holds almost surely

$$\Psi_2(x_1, x_2) = \Psi_2(L - x_1, x_2) \quad in\ [0, \frac{L}{2}] \times \mathbb{R}_{\geq 0}.$$

Proof. First notice that the right hand side is independent of x_1 and thus trivially symmetric with respect to $\{x_2 = \frac{L}{2}\}$.

As shown above, there exists a unique solution of the problem (A.13) which will be denoted by Ψ_2. Now define

$$\tilde{\Psi}_2(x_1, x_2) = \begin{cases} \Psi_2(x_1, x_2) & \text{for } x \in [0, \frac{L}{2}] \times \mathbb{R}_{\geq 0} \\ \Psi_2(L - x_1, x_2) & \text{for } x \in (\frac{L}{2}, L] \times \mathbb{R}_{\geq 0} \end{cases}.$$

By definition, $\tilde{\Psi}_2$ solves (A.13) in $[0, \frac{L}{2}] \times \mathbb{R}_{\geq 0}$ and $\tilde{\Psi}_2 = 0$ on Σ. Now for $x \in (\frac{L}{2}, L] \times \mathbb{R}_{>0}$ it holds

$$
\begin{aligned}
\operatorname{div}_x(F^{-1}(x)F^{-T}(x)\,\nabla_x\,\tilde{\Psi}_2(x)) &= (-1)^2 \operatorname{div}_z(F^{-1}(z)F^{-T}(z)\,\nabla_z\,\tilde{\Psi}_2(z))|_{z=(L-x_1,x_2)} \\
&= \frac{1}{L(1+x_2)^2}\int_{\Omega_1} \Theta \,\mathrm{d}x.
\end{aligned}
$$

Thus $\tilde{\Psi}_2$ is a solution of Problem (A.13) and by the uniqueness we get $\Psi_2 = \tilde{\Psi}_2$, which yields the assertion. ♦

Finally, we are ready to prove the compatibility condition for Ψ_2:

A.31 Lemma.
For Ψ_2 as defined above, it holds

$$\int_\Sigma F^{-1}F^{-T}\,\nabla\,\Psi_2 \cdot e_2 \; \mathrm{d}\sigma_x = 0$$

Proof. Observe that $-\nabla\,\Psi_2(z_1, z_2) = \nabla\,\Psi_2(L - z_1, z_2)$ and thus by the above lemmas we obtain

$$\begin{aligned}
&\int_{\frac{L}{2}}^{L} (F^{-1}F^{-T})(x_1,0)\,\nabla\,\Psi_2(x_1,0)\cdot e_2 \; \mathrm{d}x_1 \\
&\qquad = -\int_{\frac{L}{2}}^{0} (F^{-1}F^{-T})(L-x_1,0)\,\nabla\,\Psi_2(L-x_1,0)\cdot e_2 \; \mathrm{d}x_1 \\
&\qquad = -\int_{0}^{\frac{L}{2}} (F^{-1}F^{-T})(x_1,0)\,\nabla\,\Psi_2(x_1,0)\cdot e_2 \; \mathrm{d}x_1.
\end{aligned}$$

Now the assertion follows easily. ♦

A.3 Extension of the Pressure, Pressure Estimates

In this section we consider the following transformed Stokes problem: Find (α, ζ) such that

$$\begin{aligned}
&-\operatorname{div}(F^{-1}(x)F^{-T}(x)\,\nabla\,\alpha^\varepsilon(x)) + F^{-T}(x)\,\nabla\,\zeta^\varepsilon(x) \\
&\qquad\qquad = \Phi_1^\varepsilon(x) + \operatorname{div}(F^{-1}(x)\Phi_2^\varepsilon(x)) \quad \text{in } \Omega^\varepsilon && \text{(A.14a)} \\
&\qquad\qquad \operatorname{div}(F^{-1}(x)\alpha^\varepsilon(x)) = \Phi_3^\varepsilon(x) \quad \text{in } \Omega^\varepsilon && \text{(A.14b)} \\
&\qquad\qquad \alpha^\varepsilon(x) = 0 \quad \text{on } \partial\Omega^\varepsilon \backslash \partial\Omega && \text{(A.14c)} \\
&\qquad\qquad \alpha^\varepsilon, \zeta^\varepsilon \text{ are } L\text{-periodic in } x_1 && \text{(A.14d)}
\end{aligned}$$

with $(1 + |x_2|)\Phi_1^\varepsilon \in L^2(\Omega^\varepsilon)^2$, $\Phi_2^\varepsilon \in L^2(\Omega^\varepsilon)^4$ and $\Phi_3^\varepsilon \in L^2(\Omega^\varepsilon)$. The main aim is to obtain estimates on ζ^ε depending on α^ε and the functions on the right hand side.

Define the function space

$$H^\varepsilon = \big\{\phi \in L^2_{\text{loc}}(\Omega^\varepsilon) \mid \nabla \phi \in L^2(\Omega^\varepsilon),\ \phi = 0 \text{ on } \partial\Omega^\varepsilon \backslash \partial\Omega,\ \phi \text{ is } L\text{-periodic in } x_1\big\}.$$

The weak formulation of the above problem reads

$$\int_{\Omega^\varepsilon} F^{-T}(x)\,\nabla\,\alpha^\varepsilon(x) : F^{-T}(x)\,\nabla\,\phi(x)\;\mathrm{d}x - \int_{\Omega^\varepsilon} \zeta^\varepsilon(x)\,\mathrm{div}(F^{-1}(x)\phi(x))\;\mathrm{d}x$$
$$= \int_{\Omega^\varepsilon} \Phi_1^\varepsilon(x)\cdot\phi(x)\;\mathrm{d}x - \int_{\Omega^\varepsilon} \Phi_2^\varepsilon(x) : F^{-T}(x)\,\nabla\,\phi(x)\;\mathrm{d}x \quad \forall\phi \in H^\varepsilon.$$

Thus by using the Poincaré inequality in Ω_2^ε and Lemma 5.3 in Ω_1 we obtain the estimate

$$\begin{aligned}\left|\int_{\Omega^\varepsilon} \zeta^\varepsilon\,\mathrm{div}(F^{-1}\phi)\;\mathrm{d}x\right| \leq & C\Big[\|\nabla\,\alpha^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} + \|(1+x_2)\Phi_1^\varepsilon\|_{L^2(\Omega_1)^2} \\ & + \varepsilon\,\|\Phi_1^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \|\Phi_2^\varepsilon\|_{L^2(\Omega^\varepsilon)^4}\Big]\,\|\nabla\,\phi\|_{L^2(\Omega^\varepsilon)^4}\,. \end{aligned} \tag{A.15}$$

But in order to get useful estimates we have to find relations on a fixed domain. Therefore we have to extend α^ε and ζ^ε. For the extension of α^ε we simply set $\alpha^\varepsilon = 0$ on $\Omega\backslash\Omega^\varepsilon$. The pressure can be extended similar to Section 3.3 by using the restriction operator $\mathcal{R}^\varepsilon$ in Ω_2^ε, which yields

$$\int_{\Omega} \tilde{\zeta}^\varepsilon\,\mathrm{div}(F^{-1}\phi)\;\mathrm{d}x = \int_{\Omega^\varepsilon} \zeta^\varepsilon\,\mathrm{div}(F^{-1}\mathcal{R}^\varepsilon\phi)\;\mathrm{d}x.$$

Here $\tilde{\zeta}^\varepsilon$ denotes the extended pressure.

In the sequel, we need the spaces

$$W_1 = \big\{z \in L^2(\Omega) \mid (1+|x_2|)z \in L^2(\Omega)\big\}$$
$$V_3 = \left\{z \in L^2(\Omega) \mid \nabla\,z \in L^2(\Omega)^2,\ \frac{1}{1+|x_2|}z \in L^2(\Omega), z \text{ is } L\text{-periodic in } x_1\right\}.$$

A.32 Proposition.
Let $(\alpha^\varepsilon, \zeta^\varepsilon)$ be given by (A.14)*. Let $\tilde{\zeta}^\varepsilon$ be the extension of ζ^ε and choose a free constant in $\tilde{\zeta}^\varepsilon$ such that $\int_\Omega (1+x_2)^{-2}\tilde{\zeta}^\varepsilon = 0$. Then it holds*

$$\begin{aligned}\left\|\frac{\tilde{\zeta}^\varepsilon}{1+|x_2|}\right\|_{L^2(\Omega)} \leq \frac{C}{\varepsilon}\Big[&\|\nabla\,\alpha^\varepsilon\|_{L^2(\Omega^\varepsilon)} + \|(1+|x_2|)\Phi_1^\varepsilon\|_{L^2(\Omega_1)^2} \\ & + \varepsilon\,\|\Phi_1^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \|\Phi_2^\varepsilon\|_{L^2(\Omega^\varepsilon)^4}\Big]\end{aligned}$$

Proof. Let $f \in W_1$ and set

$$h = f - \frac{1}{2L}(1+|x_2|)^{-2}\Big(\int_\Omega f \, \mathrm{d}x\Big).$$

We have

$$\int_\Omega h \, \mathrm{d}x = \Big(\int_\Omega f \, \mathrm{d}x\Big)\Big(1 - \frac{1}{2}\cdot 2\int_0^\infty (1+|x_2|)^{-2}\Big) = 0,$$

since $\int_0^\infty (1+|x_2|)^{-2} \, \mathrm{d}x = \big[-(1+|x_2|)^{-1}\big]_0^\infty = 1$ as well as $(1+|x_2|)h \in L^2(\Omega)$. The following lemma shows that there exists a unique $w \in V_3$, solution of

$$\begin{aligned} \operatorname{div}(F^{-1}F^{-T}\nabla w) &= h \text{ in } \Omega \\ \nabla w &\in L^2(\Omega)^2 \\ w \text{ is } L\text{-periodic in } x_1 \end{aligned}$$

with $w \in H^2(\Omega)$. Thus $\phi := F^{-T}\nabla w$ solves

$$\begin{aligned} \operatorname{div}(F^{-1}\phi) &= h \text{ in } \Omega \\ \phi \in L^2(\Omega), \nabla \phi &\in L^2(\Omega) \\ \phi \text{ is } L\text{-periodic in } x_1 \end{aligned}$$

and we conclude that $\operatorname{div}(F^{-1}\cdot)$ is surjective from $H^1_\#(\Omega)$ to $W_1/\mathbb{R}$. Now

$$\begin{aligned} \int_\Omega \tilde{\zeta}^\varepsilon f \, \mathrm{d}x &= \int_\Omega \tilde{\zeta}^\varepsilon h \, \mathrm{d}x + \frac{1}{2L}\Big(\int_\Omega f \, \mathrm{d}x\Big)\int_\Omega \tilde{\zeta}^\varepsilon (1+|x_2|)^{-2} \, \mathrm{d}x \\ &= \int_\Omega \tilde{\zeta}^\varepsilon h \, \mathrm{d}x = \int_{\Omega^\varepsilon} \zeta^\varepsilon \operatorname{div}(F^{-1}\mathcal{R}^\varepsilon \phi) \, \mathrm{d}x, \end{aligned}$$

which yields by equation (A.15)

$$\begin{aligned} \Big|\int_\Omega \tilde{\zeta}^\varepsilon h \, \mathrm{d}x\Big| = \Big|\int_\Omega \tilde{\zeta}^\varepsilon f \, \mathrm{d}x\Big| \le C\Big[&\|\nabla \alpha^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} + \|(1+x_2)\Phi_1^\varepsilon\|_{L^2(\Omega_1)^2} \\ &+ \varepsilon\,\|\Phi_1^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \|\Phi_2^\varepsilon\|_{L^2(\Omega^\varepsilon)^4}\Big]\,\|\nabla(\mathcal{R}^\varepsilon\phi)\|_{L^2(\Omega^\varepsilon)^4}. \end{aligned}$$

Due to

$$\|\nabla(\mathcal{R}^\varepsilon\phi)\|_{L^2(\Omega^\varepsilon)^4} \le \frac{C}{\varepsilon}\Big(\|\phi\|_{L^2(\Omega)^2} + \varepsilon\,\|\nabla\phi\|_{L^2(\Omega)^4}\Big) \le \frac{C}{\varepsilon}\,\|(1+|x_2|)h\|_{L^2(\Omega)^2}$$

(see the next lemma, where $F^{-T}\nabla w = \phi$), one obtains

$$\left|\int_\Omega \tilde{\zeta}^\varepsilon f \,\mathrm{d}x\right| \le \frac{C}{\varepsilon}\Big[\|\nabla \alpha^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} + \|(1+x_2)\Phi_1^\varepsilon\|_{L^2(\Omega_1)^2} + \varepsilon\,\|\Phi_1^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \|\Phi_2^\varepsilon\|_{L^2(\Omega^\varepsilon)^4}\Big]\,\|(1+|x_2|)h\|_{L^2(\Omega)^2}\,.$$

Therefore it holds for all $(1+|x_2|)h \in L^2(\Omega)^2/\mathbb{R}$

$$\left|\int_\Omega \Big(\frac{\tilde{\zeta}^\varepsilon}{1+|x_2|}\Big)(1+|x_2|)h \,\mathrm{d}x\right| \le \frac{C}{\varepsilon}\Big[\|\nabla \alpha^\varepsilon\|_{L^2(\Omega^\varepsilon)^4} + \|(1+x_2)\Phi_1^\varepsilon\|_{L^2(\Omega_1)^2} + \varepsilon\,\|\Phi_1^\varepsilon\|_{L^2(\Omega_2^\varepsilon)^2} + \|\Phi_2^\varepsilon\|_{L^2(\Omega^\varepsilon)^4}\Big]\,\|(1+|x_2|)h\|_{L^2(\Omega)^2}\,,$$

which implies the result by duality. ♦

A.33 Lemma.
Let a function h be given with $(1+|x_2|)h \in L^2(\Omega)$ and $\int_\Omega h \,\mathrm{d}x = 0$. The problem

$$\begin{gathered}\operatorname{div}(F^{-1}F^{-T}\nabla w) = h \text{ in } \Omega\\ F^{-T}\nabla w \in L^2(\Omega)\\ w \text{ is } L\text{-periodic in } x_1\end{gathered}$$

has a unique solution $w \in V_3$ such that

$$\|\nabla w\|_{L^2(\Omega)^2} + \sum_{i,j}\left\|\frac{\partial^2 w}{\partial x_i \partial x_j}\right\|_{L^2(\Omega)} \le C\,\|(1+|x_2|)h\|_{L^2(\Omega)}$$

and

$$\int_0^L w(x_1,0)\,\mathrm{d}x_1 = 0.$$

Proof. The weak formulation of the above problem reads

$$\int_\Omega F^{-T}\nabla w \cdot F^{-T}\nabla \phi \,\mathrm{d}x = \int_\Omega h\phi \,\mathrm{d}x \quad \forall \phi \in \mathcal{C}^\infty_\#(\Omega)\cap V_3.$$

We want to use the lemma of Lax-Milgram, thus we have to show that $h \in V_3'$. Consider

$$\begin{aligned}\left|\int_\Omega h\phi \,\mathrm{d}x\right| &= \left|\int_\Omega h(\phi - \frac{1}{L}\int_\Sigma \phi \,\mathrm{d}\sigma_x)\,\mathrm{d}x\right|\\ &\le \|(1+|x_2|)h\|_{L^2(\Omega)}\left\|\frac{1}{1+|x_2|}\Big(\phi - \frac{1}{L}\int_\Sigma \phi \,\mathrm{d}\sigma_x\Big)\right\|_{L^2(\Omega)}\\ &\le C\,\|\nabla \phi\|_{L^2(\Omega)^2} \le C\,\|\phi\|_{V_3}\end{aligned}$$

due to the assumption on h and Lemma 5.3. Therefore $h \in V_3'$ and there exists a $w \in V_3$ (unique up to a constant) which satisfies the differential equation. We fix the constant in w by stipulating the mean value over Σ to be 0. The estimates now follow from standard regularity theory for elliptic problems. ♦

A.4 Auxiliary Problems for the Divergence-Correction

In this section we consider the auxiliary problems associated with the correction of the transformed divergence of U^ε. Define

$$K^{i,\mathrm{bl}}(x) = F(x)\Big(\int\limits_{Z_{\mathrm{BL}}} \operatorname{div}_x\Big(F^{-1}(x)D_\delta^i(x)\big[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)\big]\Big)\,\mathrm{d}y\Big)e_2.$$

A.34 Proposition.
The problem: Find θ^i such that

$$\begin{aligned}
\operatorname{div}_y(F^{-1}(x)\theta^i(y)) &= \operatorname{div}_x\Big(F^{-1}(x)D_\delta^i(x)\big[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)\big]\Big) && \text{in } Z\\
\theta^i(y) &= 0 && \text{on } \textstyle\bigcup_{k=1}^\infty\{\partial Y_S - \binom{0}{k}\}\\
[\theta^i]_S(y) &= K^{i,\mathrm{bl}}(x) && \text{on } S\\
\theta^i &\text{ is 1-periodic in } y_1
\end{aligned}$$

has at least one solution $\theta^i \in H^1(Z)^2 \cap \mathcal{C}^\infty_{\mathrm{loc}}(Z)$.

Proof. We argue similarly to Lemma 3.16 and carry out the following ansatz:

$$\theta^i(y) = F(x)\,\nabla_y\,\eta(y) + F(x)\operatorname{Curl}_y \xi(y),$$

where for η it holds

$$\begin{aligned}
\Delta_y\eta(y) &= \operatorname{div}_x\Big(F^{-1}(x)D_\delta^i(x)\big[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)\big]\Big) && \text{in } Z\\
\nabla_y\,\eta(y)\cdot\nu &= 0 && \text{on } \textstyle\bigcup_{k=1}^\infty\{\partial Y_S - \binom{0}{k}\}\\
[\nabla_y\,\eta(y)\cdot e_2]_S(y) &= \int\limits_{Z_{\mathrm{BL}}} \operatorname{div}_x(F^{-1}(x)D_\delta^i(x)[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)])\,\mathrm{d}y && \text{on } S\\
[\eta]_S(y) &= 0 && \text{on } S\\
\eta &\text{ is 1-periodic in } y_1
\end{aligned}$$

We investigate solvability in the space $W_D/\mathbb{R}$, with

$$W_D = \big\{z \in L^2_{\mathrm{loc}}(Z_{\mathrm{BL}}) \mid \nabla z \in L^2(Z_{\mathrm{BL}}),\ z \text{ is 1-periodic in } y_1\big\}.$$

Define the linear functional

$$\mathcal{L}(\phi) = \int\limits_{Z_{\mathrm{BL}}} \Big[\mathrm{div}_x\big(F^{-1}(x)D^i_\delta(x)[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)]\big)\Big]\phi(y)\ \mathrm{d}y$$
$$- \int\limits_0^1 \Big(\int\limits_{Z_{\mathrm{BL}}} \mathrm{div}_x\big(F^{-1}(x)D^i_\delta(x)[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)]\big)\ \mathrm{d}y\Big)\phi(y_1,0)\ \mathrm{d}y_1.$$

Since $\mathcal{L}(1) = 0$ the linear functional is well defined on $W_D/\mathbb{R}$, and by the properties of D^i and $w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)$ it is continuous. An integration by parts shows that the weak formulation of the above equation reads

$$\int\limits_{Z_{\mathrm{BL}}} \nabla_y \eta \cdot \nabla_y \phi\ \mathrm{d}y = \mathcal{L}(\phi).$$

Thus we get a solution η, unique up to a constant.

Next, we search for ξ satisfying

$$\begin{aligned} \mathrm{Curl}(\xi)\cdot\nu &= -\frac{\partial\xi}{\partial y_1} = 0 &&\text{on } \textstyle\bigcup_{k=1}^\infty\{\partial Y_S - \binom{0}{k}\} \\ \mathrm{Curl}(\xi)\cdot\tau &= \frac{\partial\xi}{\partial y_2} = -\nabla\eta\cdot\tau &&\text{on } \textstyle\bigcup_{k=1}^\infty\{\partial Y_S - \binom{0}{k}\}. \end{aligned}$$

Application of the inverse trace Theorem 3.9 to each cell $Y - \binom{0}{k}$ in Z^- and setting $\xi = 0$ in Z^+ yields the existence of ξ similar to the proof of Lemma 3.16. ♦

A.35 Remark.
We give some remarks:

1. *Since the right hand side of the equation for η decays exponentially, we can apply Theorem A.10 and obtain an exponential stabilization of η towards some constant and a stabilization of $\nabla\eta$ towards 0. As the construction of ξ is local, the decay carries over to this auxiliary function as well, and we obtain an exponential stabilization of θ^i to 0 in y for $|y_2| \longrightarrow \infty$.*

2. *Arguing as in the case of the functions γ^i (cf. Section 5.4.1), we can obtain an exponential decay to 0 in x: Since*

$$\begin{aligned} &\mathrm{div}_x\Big(F^{-1}(x)D^i_\delta(x)\big[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)\big]\Big) \\ &\quad = F^{-T}(x)\,\nabla_x D^i_\delta(x)\cdot[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)] \\ &\qquad + D^i_\delta(x)\,\mathrm{div}_x\big(F^{-1}(x)[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)]\big) \end{aligned}$$

we can construct θ^i *by setting*

$$\theta^i = \sum_j \big(F^{-T}(x)\,\nabla_x\, D^i_\delta(x)\big)_j \hat{\theta}^{ij} + D^i_\delta(x)\tilde{\theta}^i,$$

where $\hat{\theta}^{ij}$ *satisfies*

$$\begin{aligned}
\operatorname{div}_y(F^{-1}(x)\hat{\theta}^{ij}(y)) &= [w^{i,\mathrm{bl}}_j(x,y) - H(y_2)C^{i,\mathrm{bl}}_j(x)] && \text{in } Z\\
\hat{\theta}^{ij}(y) &= 0 && \text{on } \textstyle\bigcup_{k=1}^\infty \{\partial Y_S - \binom{0}{k}\}\\
[\hat{\theta}^{ij}]_S(y) &= F(x)\Big(\int_{Z_{\mathrm{BL}}} w^{i,\mathrm{bl}}_j(x,y) - H(y_2)C^{i,\mathrm{bl}}_j(x)\ \mathrm{d}y\Big)e_2 && \text{on } S\\
\hat{\theta}^{ij} &\text{ is 1-periodic in } y_1
\end{aligned}$$

and for $\tilde{\theta}^i$ *it holds*

$$\begin{aligned}
\operatorname{div}_y(F^{-1}(x)\tilde{\theta}^i(y)) &= \operatorname{div}_x\big(F^{-1}(x)[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)]\big) && \text{in } Z\\
\tilde{\theta}^i(y) &= 0 && \text{on } \textstyle\bigcup_{k=1}^\infty \{\partial Y_S - \binom{0}{k}\}\\
[\tilde{\theta}^i]_S(y) &= F(x)\Big(\int_{Z_{\mathrm{BL}}} \operatorname{div}_x\big(F^{-1}(x)[w^{i,\mathrm{bl}}(x,y) - H(y_2)C^{i,\mathrm{bl}}(x)]\big)\ \mathrm{d}y\Big)e_2 &&\\
& && \text{on } S\\
\tilde{\theta}^i &\text{ is 1-periodic in } y_1
\end{aligned}$$

Due to the construction of D^i_δ, θ^i *decays exponentially to* 0 *for* $|x_2| \longrightarrow \infty$.

Therefore we obtain

A.36 Proposition.
The above problem has at least one solution $\theta^i \in V$ *such that there exists a* $\gamma_0 > 0$ *with*

$$e^{\gamma_0|y_2|}\theta^i \in H^1(Z).$$

A.5 Results for Counterflow-like Problems in the Free Fluid Domain

In this section we give the existence and uniqueness results for the counterflow problems as well as a problem arising in the elimination of the forces in Section 5.3.1.

First consider the problem: Find $(u_0, \pi_0) \in \mathcal{W}_S \times L^2_{\text{loc}}(\Omega_1)$ such that

$$\begin{aligned} -\operatorname{div}(F^{-1}(x)F^{-T}(x)\nabla u_0(x)) + F^{-T}(x)\nabla \pi_0(x) &= l && \text{in } \Omega_1 \\ \operatorname{div}(F^{-1}(x)u_0(x)) &= 0 && \text{in } \Omega_1 \\ u_0(x) &= 0 && \text{on } \Sigma \\ u_0, \pi_0 \text{ are } L\text{-periodic in } x_1 & && \end{aligned}$$

for a given volume force $l \in \mathcal{C}_0^\infty(\Omega_1 \cup \Sigma)$. Here the space $\mathcal{W}_S$ is defined by

$$\begin{aligned} \mathcal{W}_S = \big\{ & z \in L^2_{\text{loc}}(\Omega_1)^2 \mid \nabla z \in L^2(\Omega_1)^4, \operatorname{div}(F^{-1}z) = 0 \text{ a.e. in } \Omega_1, z = 0 \text{ on } \Sigma, \\ & z \text{ is } L\text{-periodic in } x_1 \big\}. \end{aligned}$$

We obtain the following weak formulation:

$$\int_{\Omega_1} F^{-T}(x)\nabla u_0(x) : F^{-T}(x)\nabla \phi(x)\, \mathrm{d}x = \int_{\Omega_1} l(x)\cdot\phi(x)\, \mathrm{d}x \quad \forall \phi \in \mathcal{W}_S \tag{A.16}$$

A.37 Lemma.
There exists a unique solution $(u_0, \pi_0) \in \mathcal{W}_S \times L^2_{\text{loc}}(\Omega_1)/\mathbb{R}$ *of Problem* (A.16).

Proof. The continuity and coercivity of the bilinear form defined by the left hand side of (A.16) follows as above. Define for $\phi \in \mathcal{W}_S$

$$b(\phi) = \int_{\Omega_1} l \cdot \phi \, \mathrm{d}x.$$

We will show that $b \in \mathcal{W}_S'$, thus the lemma of Lax-Milgram yields the existence of a unique u_0. The reintroduction of the pressure then follows by employing the usual method.

Note that for $\phi \in W_S$ it holds $\int_\Sigma \phi \, \mathrm{d}\sigma = 0$. Therefore by the second inequality of Lemma 5.3

$$\begin{aligned} \left| \int_{\Omega_1} l\cdot\phi\, \mathrm{d}x \right| &= \left| \int_{\Omega_1} \Big(\frac{1}{1+x_2}\phi - \frac{1}{L}\int_0^L \phi(x_1,0)\, \mathrm{d}x_1 \Big) \cdot (1+x_2) l \, \mathrm{d}x \right| \\ &\le \|\nabla \phi\|_{L^2(\Omega_1)^4} \|(1+x_2)l\|_{L^2(\Omega_1)^2} \\ &\le C \|\nabla \phi\|_{L^2(\Omega_1)^4}, \end{aligned}$$

where we also used the fact that l has compact support. Thus $b \in \mathcal{W}_S'$, which finishes the proof. ♦

A.38 Remark.
By regularity results we obtain $u_0 \in \mathcal{C}^\infty_{\mathrm{loc}}(\Omega_1 \cup \Sigma)^2$ *and* $\pi_0 \in \mathcal{C}^\infty_{\mathrm{loc}}(\Omega_1 \cup \Sigma)$.

Finally, we consider the type of equation arising in the definition of the counterflow problems: We are looking for $(u,p) \in \mathcal{W} \times L^2_{\mathrm{loc}}(\Omega_1)/\mathbb{R}$ such that

$$-\operatorname{div}(F^{-1}(x)F^{-T}(x)\nabla u(x)) + F^{-T}(x)\nabla p(x) = 0 \quad \text{in } \Omega_1 \tag{A.17a}$$
$$\operatorname{div}(F^{-1}(x)u(x)) = 0 \quad \text{in } \Omega_1 \tag{A.17b}$$
$$u(x_1,+0) = \xi(x_1) \quad \text{on } \Sigma \tag{A.17c}$$
$$u,p \text{ are } L\text{-periodic in } x_1 \tag{A.17d}$$

where $\xi \in H^{\frac{1}{2}}(\Sigma)^2$.

The space $\mathcal{W}$ has been defined in Section 5,

$$\begin{aligned}\mathcal{W} = \big\{ z \in L^2_{\mathrm{loc}}(\Omega_1)^2 \mid \nabla z \in L^2(\Omega_1)^4, \operatorname{div}(F^{-1}z) = 0 \text{ a.e. in } \Omega_1,\\ z \text{ is } L\text{-periodic in } x_1 \big\}.\end{aligned}$$

By the usual inverse trace theorem for Sobolev spaces, there exists a $\eta \in H^1_\#(\Omega_1)^2$ with compact support such that $\eta = \xi$ on Σ. Next, in analogy to the proof of Lemma 3.16 there exists a $w \in H^1_\#(\Omega_1)^2$ with compact support and $w = 0$ on Σ such that

$$\operatorname{div}(F^{-1}(x)w(x)) = \operatorname{div}(F^{-1}(x)\eta(x)).$$

Now set $v := u - \eta + w$; then w solves

$$\begin{aligned} -\operatorname{div}(F^{-1}(x)F^{-T}(x)\nabla v(x)) + F^{-T}(x)\nabla p(x) \qquad & \\ = -\operatorname{div}\big(F^{-1}(x)F^{-T}(x)\nabla[\eta(x)+w(x)]\big) & \quad \text{in } \Omega_1 \\ \operatorname{div}(F^{-1}(x)u(x)) = 0 & \quad \text{in } \Omega_1 \\ u(x_1,+0) = 0 & \quad \text{on } \Sigma \end{aligned}$$

and the right hand side of the first equation is in $(H^1_{0,\#}(\Omega_1)^2)'$ with compact support. The arguments used in the proof of Lemma A.37 show that a solution $(v,p) \in \mathcal{W}_S \times L^2_{\mathrm{loc}}(\Omega_1)$ of the above problem exists. Thus we obtain:

A.39 Lemma.
There exists a unique solution $(u,p) \in \mathcal{W} \times L^2_{\mathrm{loc}}(\Omega_1)/\mathbb{R}$ *of Problem* (A.17).

B Function Spaces

This appendix contains a list of the non-standard function spaces from Section 5 and Appendix A (in order of appearance).

Function Spaces defined in Section 5

$$
\begin{aligned}
W_\varepsilon &= \big\{z \in L^2_{\mathrm{loc}}(\Omega^\varepsilon)^2 \mid \nabla z \in L^2(\Omega^\varepsilon)^4,\ z \in L^2(\Omega_2^\varepsilon),\ \operatorname{div}(F^{-1}z) = 0 \text{ a.e. in } \Omega^\varepsilon, \\
&\qquad z = 0 \text{ on } \partial\Omega_2^\varepsilon \backslash \partial\Omega,\ z \text{ is } L\text{-periodic in } x_1\big\} \\
V_i &= \big\{z \in L^2_{\mathrm{loc}}(\Omega_i) \mid \nabla z \in L^2(\Omega_i)^2,\ z \text{ is } L\text{-periodic in } x_1\big\} \\
\mathcal{W} &= \big\{z \in L^2_{\mathrm{loc}}(\Omega_1)^2 \mid z \in V_1^2,\ \operatorname{div}(F^{-1}z) = 0 \text{ a.e. in } \Omega_1\big\} \\
\mathcal{V} &= \big\{\psi \in \mathcal{C}^\infty_{0,\#}(\Omega^\varepsilon) \mid \operatorname{div}(F^{-1}\psi) = 0\big\} \\
V_{\mathrm{per}}(\Omega^\varepsilon) &= \big\{z \in L^2_{\mathrm{loc}}(\Omega^\varepsilon)^2 \mid \nabla z \in L^2(\Omega^\varepsilon)^4,\ z = 0 \text{ on } \partial\Omega^\varepsilon \backslash \partial\Omega,\ z \text{ is } L\text{-periodic in } x_1\big\}
\end{aligned}
$$

Function Spaces defined in Appendix A

$$
\begin{aligned}
V &= \Big\{z \in L^2_{\mathrm{loc}}(Z_{\mathrm{BL}})^2 \mid \nabla z \in L^2(Z)^4,\ z \in L^2(Z^-)^2, \\
&\qquad z = 0 \text{ on } \bigcup_{k=1}^{\infty}\{\partial Y_S - \binom{0}{k}\},\ z \text{ is 1-periodic in } x_1\Big\} \\
V_{\mathrm{div}} &= \Big\{z \in L^2_{\mathrm{loc}}(Z_{\mathrm{BL}})^2 \mid \nabla z \in L^2(Z_{\mathrm{BL}})^4,\ z \in L^2(Z^-)^2, z = 0 \text{ on } \bigcup_{k=1}^{\infty}\{\partial Y_S - \binom{0}{k}\}, \\
&\qquad \operatorname{div}_y(F^{-1}(x)z(y)) = 0,\ z \text{ is 1-periodic in } x_1\Big\}
\end{aligned}
$$

W is the completion of V_{div} with respect to the norm $\|z\|_W = \|\nabla z\|_{L^2(Z_{\mathrm{BL}})}$

$$
\begin{aligned}
W_l &= \Big\{z \in H^1(Z_l^*)^2 \mid z = 0 \text{ for } y_2 = \pm l \text{ and on } \bigcup_{k=1}^{l}\{\partial Y_S - \binom{0}{k}\}, \\
&\qquad z \text{ is 1-periodic in } y_1\Big\} \\
V_k &= \Big\{z \in H^1(Z_k)^2 \mid z = 0 \text{ on } \partial Z_k \backslash ((\{0\} \cup \{1\}) \times [k, k+1]),\ z \text{ is 1-periodic in } y_1\Big\} \\
\tilde{W} &= \big\{\phi \in L^2_{\mathrm{loc}}(\Omega_1)^2 \mid \nabla \phi \in L^2(\Omega_1)^4, \phi = 0 \text{ on } \Sigma, \\
&\qquad \operatorname{div}(F^{-1}\phi) = 0 \text{ in } \Omega_1,\ \phi \text{ is } L\text{-periodic in } x_1\big\} \\
W_2 &= \big\{z \in L^2(\Omega_1)^2 \mid (1 + x_2)z \in L^2(\Omega_1)^2,\ z \text{ is } L\text{-periodic in } x_1\big\} \\
W_3 &= \big\{z \in L^2(\Omega_1) \mid (1 + x_2)z \in L^2(\Omega_1), (1 + x_2)|\nabla z| \in L^2(\Omega_1), z \text{ is } L\text{-periodic in } x_1\big\}
\end{aligned}
$$

$$
\begin{aligned}
H^{\varepsilon} &= \big\{\phi \in L^2_{\mathrm{loc}}(\Omega^{\varepsilon}) \mid \nabla \phi \in L^2(\Omega^{\varepsilon}),\ \phi = 0 \text{ on } \partial\Omega^{\varepsilon}\backslash\partial\Omega,\ \phi \text{ is } L\text{-periodic in } x_1\big\} \\
W_1 &= \big\{z \in L^2(\Omega) \mid (1+|x_2|)z \in L^2(\Omega)\big\} \\
V_3 &= \Big\{z \in L^2(\Omega) \mid \nabla z \in L^2(\Omega)^2,\ \frac{1}{1+|x_2|} z \in L^2(\Omega), z \text{ is } L\text{-periodic in } x_1\Big\} \\
W_D &= \big\{z \in L^2_{\mathrm{loc}}(Z_{\mathrm{BL}}) \mid \nabla z \in L^2(Z_{\mathrm{BL}}),\ z \text{ is 1-periodic in } y_1\big\} \\
\mathcal{W}_S &= \big\{z \in L^2_{\mathrm{loc}}(\Omega_1)^2 \mid \nabla z \in L^2(\Omega_1)^4,\ \mathrm{div}(F^{-1}z) = 0 \text{ a.e. in } \Omega_1,\ z = 0 \text{ on } \Sigma, \\
&\qquad z \text{ is } L\text{-periodic in } x_1\big\}
\end{aligned}
$$

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