Universal Logic and the Geography of Thought – Reflections on logical pluralism in the light of culture

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To my Parents.
獻給我的父母。

Acknowledgment

Contents

Abstract

The aim of this dissertation is to provide an analysis for those involved and interested in the interdisciplinary study of logic, particularly Universal Logic. While continuing to remain aware of the importance of the central issues of logic, we hope that the factor of "culture" is also given serious consideration. Universal Logic provides a general theory of logic to study the most general and abstract properties of the various possible logics. As well as elucidating the basic knowledge and necessary definitions, we would especially like to address the problems of motivation concerning logical investigations in different cultures.

First of all, I begin by considering Universal Logic as understood by Jean-Yves Béziau, and examine the basic ideas underlying the Universal Logic project. The basic approach, as originally employed by Universal Logicians, is introduced, after which the relationship between algebras and logics at an abstract level is discussed, i.e., Universal Algebra and Universal Logic. Secondly, I focus on a discussion of the "translation paradox", which will enable readers to become more familiar with the new subject of logical translation, and subsequently comprehensively summarize its development in the literature. Besides helping readers to become more acquainted with the concept of logical translation, the discussion here will also attempt to formulate a new direction in support of logical pluralism as identified by Ruldof Carnap (1934), JC Beall and Greg Restall (2005), respectively. Thirdly, I provide a discussion of logical pluralism. Logical pluralism can be traced back to the principle of tolerance raised by Ruldof Carnap (1934), and readers will gain a comprehensive understanding of this concept from the discussion. Moreover, an attempt will be made to clarify the real and important issues in the contemporary debate between pluralism and monism within the field of logic in general. Fourthly, I study the phenomena of cultural-difference as related to the geography of thought. Two general systems in the geogra-

phy of thought are distinguished, which we here call thought-analytic and thought-holistic. They are proposed to analyze and challenge the universality assumption regarding cognitive processes. People from different cultures and backgrounds have many differences in diverse areas, and these differences, if taken for granted, have proven particularly problematic in understanding logical thinking across cultures. Interestingly, the universality of cognitive processes has been challenged, especially by Richard Nisbett's research in cultural psychology. With respect to these concepts, C-UniLog (appendix A) can also be considered in relation to empirical evidence obtained by Richard Nisbett et al. In the final stage of this dissertation, I will propose an interpretation of the concept of logical translation, i.e., translations between formal logical mode (as cognitive processes in the case of westerners) and dialectical logical mode (as cognitive processes in the case of Asians). From this, I will formulate a new interpretation of the principle of tolerance, as well as of logical pluralism.

Chapter 1

Introduction

Contents

Tarski's account of the concept of logical consequence has gained great importance in modern mathematics, and moreover has founded the role of logic in scientific studies. Universal Logic as a general theory of logics is the view that has followed the most rudimentary stage of Tarski's notion for logical consequence. The conceptual analysis of logic is what the Universal Logic project is about.

There are many questions that one might ask about Tarski-oriented contributions. One – perhaps the most obvious – is why one might suppose the concepts of logical consequence to be mathematical. Another question one might ask is why one might suppose Universal Logic, as a general theory of logics, to be mathematical as well. Yet another question is that of how the logical structures that Universal Logic discusses are mathematical. The short answers might be that a Tarskian contribution is important not for its practical applications, but because a vast quantity of mathematical work assumes that it is true. The aim of this dissertation is to provide not only a theoretical analysis but also application-oriented perspectives for those involved and interested in the interdisciplinary study of logic.

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The very nature of Universal Logic is to provide a general theory of logic to study the most general and abstract properties of the various possible logics. Currently, it is anticipated that many logical researchers will again take an interest in this project. The situation here might be illuminated by analogy with Universal Algebra. Universal Algebra was originally proposed by an interest in a rather precise but intuitive notion to provide a general theory of algebras. Here the notion was that of a conceptually-oriented approach that is closer to our everyday life than the axiomatic approach.

What this dissertation will discuss is the relation between the Universal Logic project and certain core theoretical computer scientific and philosophical notions. The Universal Logic project follows some well-known notions of abstract logic that have their widespread applications.

The first part of this dissertation focuses on such abstract logical investigations. The second part discusses logic translation investigations that have closely followed the techniques adopted in abstract logic. The third part presents logical pluralism, proposed by Rudolf Carnap in 1934 and J. C. Beall and Greg Restall in 2000, respectively. The fourth part concerns cross-cultural logic, a topic rarely considered in formal logic studies, which we discuss in relation to psychology, anthropology, and in particular, modern cognitive science.

1.1 Universal Logic

The meaning of the term **Universal Logic** is easily misunderstood as "a logic" *to unify* all logics literally, however, this is disputed by Jean-Yves Béziau and other pioneer universal logicians, who considered logic on a general and abstract level. On the contrary, according to Universal Logic project, the view that there is only one "universal" logic is not possible, claiming that Universal Logic is a *general theory of different logics*.

In the 1920s, Tarski proposed his theory of consequence operator as a very general theory of logical consequence (see [207], [208], [209], [210]), in the 1930s, Gentzen's sequent calculus considered a family of formal systems sharing a certain style of inference and certain formal properties ([105]), and in the 1970s, Roman Suszko (with Stephen Bloom and Donald Brown) proposed a concept of *abstract logic* $\langle A, \text{Ⓢ} \rangle$ consisting of an algebra A and a closure system Ⓢ ([52], [53]). These three studies considered logics at a general level. They attempted to see logics from a general point of view to

discuss the properties, and relations that different logics should have. None of these logicians said that "abstract logic" is the "universal" logic.

It has been explicitly elucidated that Universal Logic is a general theory to study logical structures. The Universal Logic project we discuss follows the same thinking of the three aforementioned studies by combining a general bivaluation semantics with Gentzen's sequent calculus in order to promote "logic in general" to an even more abstract level. This means, once we recognize which parts of different logics are universal and common, according to the Universal Logic project, we can take them more or less directly to specific logics by the tool kit it provides. Moreover, we could "build" a specific logic of accounting specific situations and problems. (see [29], [31], [34], [38], [39], [46]).

An example raised in [31], states that "the first proofs of completeness for propositional classical logic give the idea that this theorem is depending very much on classical features." People mistakenly think the concept of *maximally consistent*[1] is to depend on classical negation when they study the completeness theorem of classical logic. The fact, however, is "one can present the completeness theorem for classical propositional logic in such a way that the specific part of the proof is trivial." (Ibid, p. 13). This means showing maximal consistency in the proof is trivial. It does not depend on the property of classical negation. It should belong to the universal parts of different logics. In other words, the completeness should be generalized, that is, it should not depend on any specific feature of any given logic.

The Universal Logic project discusses the distinction between what is universal and what is specific for logics. Moreover, it attempts to build a logic for specific problems, just as a doctor prescribes medicine or treatment for an illness. Such a "utopia" provides us a new paradigm to revolutionize the old-fashioned axiomatic approach in **Logic** (compare e.g. [32], [39]).

1.1.1 Philosophical Background of Universal Logic (I): Bourbakism

Universal logicians are neither pure logicians nor pure philosophers and can be regarded as "Philogicians". Universal Logic is not considered as *a logic* but provides a general theory of logics. On one hand, it can be viewed as

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[1]A theory is a maximal by a consistent set of sentences, if it is consistent and none of its proper extensions is consistent.

philosophy-like in terms of a philosophy of logic. On the other hand, it can be thought of as *mathematical-like* in relation to studies concerning mathematical logic. This is a hybrid theory that includes both mathematical and philosophical aspects where *logical structures* are considered as *mathematical structures* within the spirit of **Structuralism**. This position is hardly a new one, but the Universal Logic project has a profound proposal on the subject as follows:

- The idea of Universal Logic says precisely that logic (logical structures) are considered as mathematical structures. Universal Logic is to develop a general theory of logics which is an analogy to the idea of **Universal Algebra**.

Universal Algebra is a field of mathematics which is seen as a special branch of **Model Theory**. Generally speaking, it deals with "structures having operations"; specifically speaking, it studies "algebraic structures" themselves instead of studying various specific examples or models of algebraic structures. For example, it takes "the theory of groups", "the theory of rings", and "the theory of fields" as the objects of study instead of taking "particular groups", "particular rings", and "particular fields" as the objects of study. In other words, Universal Algebra studies the theory of algebraic structures, and is a general theory of algebra. Similarly, Universal Logic claims to study "the theory of logical structures", and is a general theory of logics.[2] Adopting the position of **Bourbakism**, the Universal Logic project proposed that a logical structure is considered as type $\langle \mathcal{S}, \vdash \rangle$ where $\vdash$ is a consequence relation on $\mathcal{P}(\mathcal{S}) \times \mathcal{S}$. Moreover, $\mathcal{S}$ is not specified and any one of the three Bourbakian mother structures could be included in $\mathcal{S}$. This basic conception of logical structure with regard to Bourbakism plays an important role in the Universal Logic Project.

   Before proceeding, we mention the work of Bourbaki, which had the goal of founding all of mathematics from the axiomatic point of view, and to *entirely* reconstruct mathematics through the *concept of structure*. They strove

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[2]Universal Algebra was founded by Garrett Birkhoff who aimed to provide a definition of *abstract algebra*, which is "a set with operations," in a very general manner and then apply this to prove fundamental *universal* results ([48]). Similarly, Universal Logic is concerned with providing a definition of *abstract logic*, which is "a set with consequence operations," in a very general manner and attempting to prove fundamental *universal* results.

for rigour and generality of mathematics and proposed the concept of fundamental mother structures, consisting of the notions of *algebraic structures*, *topological structures*, and *order structures*. Furthermore, they considered "cross-structures" by some crossing processes[3]. This stance is called Bourbakism (compare e.g. [32], [56], [67]).

Béziau and pioneer universal logicians claim to follow Bourbakian ideas on fundamental mother structures, arguing that logical structures should, in a Bourbakian sense, comprise the fourth mother structure ([32], [37], [45]). This means the arbitrary logical structures should be considered as $\langle \mathcal{S}, \vdash \rangle$, where $\mathcal{S}$ can be any of these three mother structures in Bourbakism. For example, for algebraic structures, from the Universal Logic point of view, this implies that algebraic structures have been put on $\mathcal{S}$ with implicit logical structures. To remember the slogan, say, "Universal Logic is a general theory of logic, which considered logics as mathematical structures." It is to say, so far, that the idea of "abstract logics" should be generalized much more to serve as the fourth mother structure in Bourbakism and that it has been considered as a mathematical structure.

1.1.2 Philosophical Background of Universal Logic (II): Logical Structuralism

> "Now what concerns us are not so much historical and sociological considerations about the development of structuralism, but rather the issue of the ultimate view of structuralism as underlying mathematical structuralism and Universal Logic." ([32], p. 138)

The concept of structures is not only restricted to mathematical realms, even if Bourbakism was deeply influenced by **structuralism**, which was fashionable in 1960.[4] It has been claimed that the Universal Logic project, following Bourbakism, is seen as a sort of structuralism. Structuralism is another philosophical background to the Universal Logic project. However,

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[3]See also, [32], p. 137

[4]This history should be clarified much more but we ignore it without affecting our discussion here. One citation sentence in [32], p. 138, is provided for readers: "The notion of structure largely goes beyond the mathematical area, and Bourbaki said himself that he was influenced by such linguists as Benveniste. During the sixties, 'structuralism' was meant as a large movement that mainly occurred in human sciences."

we are not concerned with the merits or demerits of structuralism, nor will we debate whether it was appropriate for the Universal Logic project to take this position.

For the Universal Logic project, it is especially important considering the logical structure, $\langle S; \vdash \rangle$ in the Boubakian sense. We would like to call this position as a sense of **logical structuralism** (structuralism in logic), if we may. It is worth saying again that the logical structures are considered as mathematical structures in the Universal Logic project, and this logical structuralism is seen as mathematical structuralism in Bourbakian sense. Specifying the term "logical structuralism" is technically and philosophically useful in the context of studying the Universal Logic project. Here, logical structuralism is a certain version of Bourbakism, a sort of mathematical structuralism in essence, since logical structures are considered as mathematical structures within the spirit of Universal Logic project. It is useful to notice the roles that logical structures play in mathematics. As we have mentioned, logical structures are claimed to be the fourth mother structure and are seen as a generalization of abstract logics. Thus, universal logicians claim the stance they hold as the **Neo-Bourbakism** (compare e.g. [32], [34]).

Another reason to specify the term "logical structuralism" is to avoid the misunderstanding that Universal Logic is a part of Universal Algebra (compare e.g. [31], [32], [34], [37], [45]). In the past, non-experts in this field have mistakenly viewed Universal Logic to be a part of Universal Algebra. The main reason for this misunderstanding is the development of algebraic logic in the history of mathematical logic. The idea in algebraic logic is to take algebras as models of logics, for example, Lindenbaum-Tarski algebra is the model of classical propositional logic (CPL from now on) and Heyting algebra for intuitionistic propositional logic (IPL from now on) . Using the term "logical structuralism" will draw attention to the fact that Universal Logic, in essence, is different from Universal Algebra.[5]

1.2   Logic Translation

The immediate stimulus for this study comes from several ambiguous concepts found in the analysis about the concept of sub-logic. In some relative discussions about the conceptual analysis of translation paradox, people found the following situation paradoxical: given two logics, one is weaker

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[5]A similar discussion occurs in ([45], pp. 130–133).

than the other in the sense of proving everything the former proves, while at the same time the stronger logic can be translatable to the weaker one ([41], [127], [128], [129] [151] [177])

We present two logics: one is the classical logic $\mathcal{L}_{Classical}$ with the well-known semantic conditions for implication and negation; the other is $\mathcal{L}_{Classical/2}$ which is a logic with classical implication but with only the half part semantic conditions of classical negation: given any truth-assignment such that if φ is 1, then $\neg\varphi$ is 0. We demonstrate the translation paradox in Béziau's case and exploit it to discuss the historical development of the translation of logics between classical logic and intuitionistic logic. Historically speaking, we see intuitionistic logic firstly appears as a *sublogic* of classical logic. By intuition, this means intuitionistic logic, which is a *sub*-logic should be *weaker* than classical logic. However, we will see that CPL can be translated into IPL, indicating that IPL is, in a sense, stronger than CPL. It seems to be that the meaning of *sublogic* and the *strength* of logics are not clear enough.

Many papers about the negative translation from classical to intuitionistic logic have been written, since the proposal of **double negation translation**. Over the last seventy years, there have been various discussions on this logic translation, moreover the general concept of logic translation that has been discussed pertains to the area of *abstract model theory* (see [104], [151]). The other focus is this study is the survey and systematic development of logic translation, begun from 1930, with respect to the study of *general logic* originating from Alfred Tarski. Moreover, we would like to relate it to the discussion about the *deviance* of logics that was proposed by Susan Haack in 1974.

1.2.1 From Sub-logic to Logic Translation

Before proceeding to the discussion of logic translation, a conceptual analysis on the notion of sub-logic should be given. There is a long tradition of attempting to weaken classical logic, e.g. the development of intuitionistic logic. Firstly, the *sublogic* relation is a sort of *contained-relation* of logics at the deductive level. To say the logic $\mathcal{L}$ is a *sub*logic of $\mathcal{L}'$ means that the theorems of the former are a proper subset of the theorems of the latter, i.e., $\mathcal{L}$ is in a sense contained within $\mathcal{L}'$. However, details about the inference rules and axioms behind these logics should be specified individually. For example, "is the principle of excluded middle accepted or not by these two logics?" or "do two logics $\mathcal{L}, \mathcal{L}'$ own the same structure, e.g., (two-sorted)

first-order structure?"

Studies on logic translation can be traced back to Andrey Nikolaevich Kolmogorov (1925), Valery Ivanovich Glivenko (1929), Gerhard Gentzen (1933), and Kurt Gödel (1933). "Translation" has generally played an important role in the history of **structural linguistics**. According to structuralism, the nature of an object is (*relationally*) determined by its own *structure*, that is, its meaning is completely determined by its relationships with other elements within a linguistic system. Thus, for "translations" of two linguistic systems, the following question is addressed:

- How can an object belonging to one structure be identical to an object of another structure?"

Logic translation may accordingly be classified with respect to their concerns on mathematical structures. Béziau [41] (p. 147), for instance, described the following questions:

- How can an object in $\mathcal{N} = \langle \mathbb{N}, +, \times, \leq \rangle$ be identical to an object in $\mathcal{Z} = \langle \mathbb{Z}, +, \times, \leq \rangle$?"

- How can an object in $\mathcal{K} = \langle \mathbb{K}, \neg, \wedge, \vee, \rightarrow, \vdash \rangle$ be identical to an object in $\mathcal{K} = \langle \mathbb{K}, \neg, \wedge, \rightarrow, \vdash \rangle$?"

To discuss Béziau's case the **translation paradox** makes it easier to understand a more general and abstract logic by using the *bivaluation approach*. It also helps us better understand what logic translation is, especially from the abstract logical point of view. Logic translation is a relatively new realm in logical society. Not only have new logical results been generated, but old results and concepts have been re-examined by these translation methods in recent years, for example, "proof methods" used in the "*decidability problem*", originally raised by Michael O. Rabin (1965) ([186]); Dov Gabbay's considerations about "Translation of superintuitionistic logics into normal extensions of $S4$" (2005) ([99]) and "accomplishing belief revision with AGM postulates by translation" (1999) ([101]).

1.2.2   The Senses of Logic Translation

The question of "What is *logic translation*?" should first be addressed before delving into greater depths. We are not concerned here with logic translation in the following sense, for example, CPL; CPL is a class of equivalent

structures ([37], [151]). In this example, let us further clarify the question "What is the *logic translation* of CPL?" In other words, this question seeks the equivalent of the many possibilities of CPL structures. We list these possibilities as follows:

1. CPL is a two-valued truth-table.

2. CPL is a Hilbert-type axiomatic system.

3. CPL is a Gentzen-type proof system.

Thus, we understand that the question should be further clarified: "What is the *logic translation* of CPL-structures?" In other words, to "translate" is to determine the *equivalent relations* between these CPL-structures. We do not want to discuss translation in this sense. Hence, we need to elucidate in which sense the term "logic translation" is employed in our discussions. Here, we consider the following as our study of logic translation:

- The translation between classical logic and intuitionistic logic, discussed against the background of modern philosophy: philosophy of mathematics and philosophy of logic.

- The translation between some deviant systems and extended systems of logic, discussed against the background of modern philosophical logic.

In this study, we will discuss these two aspects as follows:

- We will present an in-depth discussion to elucidate logic translation between classical logic and intuitionistic logic with the help of an abstract example provided by Béziau that has been commonly named the **Béziau's Translation Paradox** ([127], [128]). Further, an in-depth explanation of logic translation will be provided.

- We will describe two investigations on the general systematic research of logic translation in modern mathematical logic. One is an investigation by Kolmogorov-Glivenko-Gödel-Gentzen to Esptein-Wójcicki, which focuses on the consequence relation $\vdash$ (**KGGG-EW**). The other is an investigation by Bloom-Brown-Suszko to *Brazilian logical group*-Esptein-Wójcicki that focuses on the consequence operator Cn (**BBS-BEW**).

1.3 Pluralism in Logic

Only later, when I became acquainted with the entirely different language forms of Principia Mathematica, the modal logic of C. I. Lewis, the intuitionistic logic of Brouwer and Heyting, and the typeless systems of Quine and others, did I recognise the infinite variety of possible language forms. On the one hand, I became aware of the problems connected with the finding of language forms suitable for given purposes; on the other hand, I gained the insight that one cannot speak of "the correct language form", because various forms have different advantages in different respects. The latter insight led me to the principle of tolerance.

(Rudolf Carnap, *Intellectual Autobiography*, p. 68, (1963))

Where is logic heading today? There is a general feeling that the discipline is broadening its scope and agenda beyond classical foundational issues, and maybe even a concern that, like Stephen Leacock's famous horseman, it is 'riding off madly in all directions'. So, what is the resultant vector? There seem to be two broad answers in circulation today. One is logical pluralism, locating the new scope of logic in charting a wide variety of reasoning styles, often marked by non-classical structural rules of inference. This is the new program that I subscribed to in my work on sub-structural logics around 1990, and it is a powerful movement today.

(Johan van Benthem, *Logical dynamics meets logical pluralism?* p. 182 (2008))

Traditionally, there was only one—the logic of Aristotle and the Stoics, as melded together in the Middle Ages–and the question never arose. This century, we have seen a plethora of logics: Frege/Russell (classical) logic, intuitionism, paraconsistent logic, quantum logic. Usually, the advocates of these logics were still logical monists, in the sense that they took it that other logics were wrong.

(Graham Priest, *Doubt truth to be a liar* p. 194 (2006))

Traditionally, human reasoning investigations have been pursued mainly in disciplines related to logics performed by the singularity of one true logic. Classical logic can be thought of as the common agreements among the traditional perspective of philosophy. Just as in the fields of philosophical logic

(non-classical logic), however, the birth of modal logics and non-monotonic logics first raises a huge change of formal logics with different philosophical considerations and varying expressive capabilities. This study is a fundamental investigation into various logic translations, including the Gödel-Gentzen negative translation and the onto-logical translation graph ([150]). However, this is not about defending and endorsing any sense of "logical pluralism" but to propose that logical pluralism could be the foundation or the basic spirit for various logic translation investigations. Here, we outline the general methodology for **Beall and Restall's logical pluralism (BRLP)** and what they have achieved in their version of pluralism. It rests on three main principles: firstly, they formulate a pre-theoretic notion of logical consequence as the General Tarskian Thesis (GTT); secondly, they specify different cases for the pre-theoretical logical consequence; thirdly, they structure a model to unify three cases that they think are "equally good" in a coherent way to justify the plurality of logical consequences.

Of the various definitions of "pluralism", we present the intuition given in the history of philosophy, like Leibniz's philosophy of mind, in which a monism was always proposed as the opposition of pluralism and vice versa in some traditional theoretical topics.

Similarly, the idea of pluralism in logic has been formulated since the principle of tolerance proposed by Rudolf Carnap. In the same spirit, plurality in logic refers to the increasing plurality of logical systems presented in the second half of the 20th century. However, at the end of 20th century, Beall and Restall defended another version of pluralism in logic ([10], [14]). This trend has been widespread in the whole logic society and moreover challenges the old notions on the human intelligence investigations, even directly affecting some interesting research in computer science in applications ([137], [150]).

The often cited definition of the term "pluralism" in academia is not fully captured by BRLP. We will discuss this in detail and relate it to what we believe are the more important aspects of plurality with respect to human reasoning with cognitive differences.

We will argue that BRLP, being precise enough, should have a specific role in modern logic investigations, in particular concerning the coherence exemplification of the general concept of logical consequences. It is a concern that the "tolerance" and "equality of goodness" usually adopted in plural-

ism is not evident in BRLP.[6] Nevertheless, BRLP partially captures certain perspectives of the term "pluralism" as follows:

1. BRLP begins with an analysis of the *disambiguations* of the notion of *logical consequence* and claims that logical consequence possesses a simple structure: "a conclusion is a logical consequence of some premises if and only if that conclusion is true in every circumstance in which the premises are true." ([193], p. 427)

2. BRLP claims that there are three extensions of the term "circumstance": *worlds*, *constructions*, and *situations* that separately generate *classical*, *intuitionistic*, and *relevant* logical consequences, respectively. Further, these three logical consequences are *equally good.*

3. BRLP does not provide different accounts for logical consequence; however, it does provide an irreducible plurality of the notion of logical consequence in applications.

We present a more philosophical and foundational summary in this study and explicate in particular what we mean by the term the pluralism in logic. In the first half of this study, we will sketch BRLP; in the second half, we will discuss the scope of monism in relation to BRLP to present three versions of logical monism with respect to BRLP.

   In order to relate the issues to the aspects of universality in human reasoning and cognitive processes, we will formulate a possible revision for pluralism in logic: **the cognitive processes mergence pluralism** in a boarder sense. We begin this dogma by taking logic as the cognitive process. A fundamental distinction which was understood as the cross-cultural psychological, cultural

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[6]We see political pluralism and value pluralism in the practical philosophy

   "Political pluralism usually starts with the observation that there are different value systems in use in the world, and there are various positions that arise out of observation. Political pluralism is concerned with the question of what sort of restrictions governments can put on people's freedom to act according to their value systems. The strongest version of political pluralism claims that all these value systems are equally true (and thus presumably all ought to be tolerated), a weaker view is that these value systems all ought to be tolerated, and probably the most common version of the view is that some of these systems (the reasonable ones) ought to be tolerated. "

(*Stanford Encyclopedia of Philosophy*–**Value Pluralism**)([185])

psychology, and cognitive scientific perspectives was introduced by Richard Nisbett et al. in [156], [157], [158]. The assumption of treating "logic as cognitive process" should first be kept in mind. After building a background of empirical results in cultural psychology, we will propose a revision of pluralism in logic. Using this assumption, we will present a **cognitive processes mergence pluralism** in an interdisciplinary manner.

1.4   The Geography of Thought

So far we have discussed two projects, both reasonable, for accepting the generalized Tarski's account of logical consequence: the Universal Logic project and BRLP.[7] The first is a conflation of the mathematical perspective of logical structures. Universal Logic is to logic what Universal Algebra is to algebra. It is perfectly obvious when an adequate formulation of the conception of logic is taken, it can yield fresh viewpoints on general logic by revisiting modern logic investigations, such as Béziau's proposal of arbitrary logical structure with bivaluation semantics and the notion of **institutions** that was introduced by Goguen and Burstall in the late 1970 (compare e.g. [111], [112], [113]), moreover it can be widely applied to various computer scientific application domains and foundational domains (e.g. [137], [150]). The second is the philosophical perspective of logical structure that has plural instantiations of the General Tarski Thesis (GTT). Logical pluralism, not restricted to BRLP, has played a role in general logic studies in modern logical society.

This study goes beyond merely adopting different positions. We introduce the cultural psychology perspective to Universal Logic, in an interdisciplinary manner. The same term embodies significant meaning on the principle of tolerance in logical pluralism and stems from the assumption of taking *logic as cognitive process* and the permissible differences of different cultures.

In this introduction, we have presented an integrated account of the contributions of these works. These works on concepts and advocacy of Universal

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[7]We borrow the expression 'Geography of Thought' from a book with the same title: *The Geography of Thought: How Asians and Westerners Think Differently...and Why*([156]). This book is written by the famous social psychologist Richard Nisbett who is working globally on many cutting edge studies about reasoning and cultural-difference. After participating in the second world conference on Universal Logic in Xi-an, China, I came to understand that a completely different notion of Universal Logic had been developed in Mainland China in the late 1990s. Reading this book inspired me to reflect on the underlying reasons for formulating this Chinese universal logic system.

Logic concern the nature and ambitions of the Universal Logic project, and we believe they are sufficient to shed interesting light on the idea of the Universal Logic project and the kind of logical investigations it requires and supports. The Universal Logic project reflects various aspects of logic and extends interpretations in some philosophical logics. In detail, the following aspects have been studied:

- Logic as Mathematical Structure

- Logic as Algebra

- Logic as Cognitive Processing

Our discussions in this dissertation are of fundamental value not only within the Universal Logic movement, but also to the more exact consideration of the concept of logic. Specifically, this thesis:

1. discusses the joint contributions of two influential programs, Universal Logic project and logical pluralism, by discussing logic translation.

2. clarifies and discusses distinct notions of logic translation.

3. specifies and discusses the role that non-western logic could play in logic, by introducing the psychologism viewpoint.

4. provides new insights into the meaning of logic translation by considering the modification of cognitive processes in cultural-psychobiological perspectives by providing new insights into the principle of tolerance and logical pluralism.

1.5 Publications

Parts of this thesis have been published and discussed in the following articles:

- Fu, T.–K., The Usage of "Formal Rules" in the Human Intelligence Investigations, In: Kelemen, J., Romportl, J., and Žáčpvá, E. (eds.): Contemplations, Expectations, Applications, Topics in Intelligent Engineering and Informatics, vol. 4, part 1, pp. 63–74 (2013)

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Chapter 2

On Universal Logic: A General Theory of Logics

Contents

2.1 Introduction

> "*Logica Universalis* (or *Universal Logic, Logique Universelle, Universelle Logik,* in vernacular languages) is not a new logic, but a general theory of logics, considered as mathematical structures." ([31] p. vii)

This chapter addresses a comprehensive discussion on the Universal Logic project. We describe the main features of the Universal Logic project: its connection with Bourbakian structuralism and Universal Algebra. The main point that we will emphasize is that Universal Logic as understood by Béziau is not one absolute system of logic. Universal Logic, we are told, is not itself

a special logic, but a general theory of investigating whatever should be common to all putatively distinct logics.

One of the main results of the Universal Logic project is to combine a general bivaluation semantics with Gentzen's sequent calculus in order to promote the idea of *abstract logic* which has played an important role in the development of modern logic to an even more general level. According to the Universal Logic project,the universal and common parts of various logics are able to be clarified, and moreover be taken more or less directly to specific logics that depend on certain situations or certain problems.

In this chapter, firstly, we discuss the development of Universal Logic, in particular the influences of algebraic logic and Bourbakism. Secondly, we elaborate the main techniques that people usually implement in the Universal Logic project. As a result of the Universal Logic project, while we prove the completeness of a given logic, the *maximal consistency*, which is traditionally considered to depend on specific features, should be attributed to the universal parts of logic instead of taking it as the specific parts of various specific logics. For example, dependence on the features of classical negation in the proof of classical logic completeness is trivial. Moreover, we distinguish that it is a conceptual mistake to claim the logically many-valued semantics by Suszko's Thesis. "Many-valuednesss" should actually be attributed to algebraic values instead of logical values. The logic two-valuedness also has to be classified as the common part of various logics, which does not depend on the specific characteristics of individual logics.

The structure of this chapter is as follows: In section 2.2, we start our discussion on the relationship between algebra and logic, moreover to the relationship between Universal Logic and Universal Algebra moving through the influence of the development of algebraic logic. Section 2.3 then discusses the conceptual approach and Neo-Bourbakism that the Universal Logic project is attempting to take. In section 2.4, we provide a discussion on the strategies and techniques in the Universal Logic project, followed by a discussion on the idea of *logical many-valuedness* being a misleading notion.

2.2   Algebraic Structures and Universal Logic

"Universal Logic is about to expand naturally and will plausibly become soon the mainstream in logic in a short time, supplanting "formal logic", "symbolic logic", or "mathematical logic".

It helps logic and logicians to be again meaningful." ([32], p. 148)

In the introduction, we discussed the motivations for the Universal Logic project: while analogizing to the Universal Algebra, being a s,general theory of algebra *Universal Logic is a general of logics*. However, there is also a more fundamental motivation for the Universal Logic project, namely the nature of logical structures is not captured fully by algebra but instead, it depends on a rapidly changing array of application domains to which modern logicians and computer scientists are devoted, although various techniques in algebraic logic are still predominant in current computer scientific applications (e.g. [24], [66], [80], [134], [152]). To trace the development of mathematical logic to algebraic logic, Universal Logic which adopts various techniques in algebraic logic such as logical matrices (compare e.g. [31], [36] [213], [221]) is always misunderstood as a part of Universal Algebra. It gains importance when considering the suitable status of Universal Logic to avoid being understood as a part of Universal Algebra. To remember the philosophical background of Universal Logic, let us say algebras could be put into the logical structure in the sense of Universal Logic.[1] The concept of the *reduction relation* between algebra structures and logical structures, in this way, would always be a concern.

2.2.1 The Relationship between Logic and Algebra

Algebra and logic are invariably considered together in the literature, whether this is in terms of the **Algebra of Logic** ([54], [55], Boole), **Algebraic Logic** ([51], Blok and Pigozzi), or **Abstract Algebra Logic** (**AAL**)([82], [83], Font and Jansana).[2] A series of clarification has been made for the relation between logic and algebra (see [31], [32], [35], [37]). The reduction relation between algebra and logic and the concept of *algebraizability* are two important concepts we try to clarify. First of all, we exploit the concept of *reduction* between logic and algebra in order to realize the background, and to discuss that Universal Logic is not a part of Universal Algebra. Yet the

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[1]This does not imply that logic should be viewed as algebraic structures, that is, logical structures are algebraic structures. What we see here is to *algebraically instantiate* logic.

[2]The complex history relating to these concepts will, however, not be discussed here. Readers interested in this should consult the references already cited. However, the influence of Abstract Logic and Abstract Algebra Logic on Universal Logic will be discussed later.

Bourbakism that the Universal Logic project has already taken will also provide a reason for us to see the relation between algebra and logic. According to Universal Logic, logical structures are considered mathematical structures, and should be considered in a more general sense than Tarski and Suszko did previously, crossing with three mother structures in Bourbakian sense. In this way, there is no reason to say that "logic as structure" implies "logic as algebra," and there is also no reason to support the claim that "considering logical structures" is the same as "considering algebraic structures"; furthermore, there is no reason to regard an "abstract logical structure" as a kind of "abstract algebraic structure," of which the definition is satisfied with *a set of operations* from a more abstract "Universal Algebra" point of view. In fact, these concepts only appear to be similar.

We will not spend time discussing the history of the study of the relationship between algebra and logic by George Boole (1815-64), namely Boolean algebra or algebra of logic. Alternatively, we ask readers not to forget their knowledge about Universal Logic they have so far acquired, but to return to a conceptual analysis of the term, "algebraic logic". We believe the discussions below will help readers clarify various possibilities for using this term, and moreover will shed light on the attitude universal logicians should have in the future when they face the term "algebraic logic". Hence, we conclude that logic should not be a part of algebra, moreover we can understand the concept of algebraizability by means of several famous examples.

First of all, Béziau analyzes the term "algebraic logic" in ([34], p. 11) as follows:

> "Algebraic logic is an ambiguous expression which can mean several things. One could think that it is crystal clear and that algebraic logic means the study of logic from an algebraic point of view. But this is itself ambiguous, because this in turn means two things: (1) the study of logic using algebraic tools: (2) logics considered as algebraic structures. (2) implies (1) but not necessarily the converse."

It has become a common position that universal logicians strongly argue against the notion of a reduction of logic to algebra, for the simple and basic reason that algebraic structures are different from logical structures. From the universal logical perspective, algebra is simple one of the options that can be included into the logical structure, with "topology" and "order" being other options. Thus, if logical structures are considered in the sense

of the Universal Logic project, people who hold the idea of reducing logic to algebra should not think of logical structures as only reducible to algebraic structures but also reducible to "topological" or "order" ones.

As we have stated, in the Universal Logic project, logical structures are independent from the three mother structures in a Bourbakian sense. In fact, it is hard to find an explicit statement that proposes the reduction between logic and algebra. The **Lindenbaum-Tarski algebra** (or Lindenbaum algebra) of a logical theory seems to provide a more robust claim for a reduction relationship between algebra and logic. In order not to lose readers with little background in algebraic logic, we first explain the approach used in Lindenbaum-Tarski algebra before continuing our discussion.

Intuitively speaking, the approach the Lindenbaum-Tarski algebra takes firstly is *to factorize* the propositional language to a certain structure, that is, to be a Boolean algebraic structure (or Boolean algebra for short). This structure is an algebraic structure, and is called Lindenbaum-Tarski algebra. In other words, it is a way to factorize to obtain a Lindenbaum-Tarski algebra of *a propositional language*. See the following definition.

Definition 1. Let $PROP$ be a classical propositional language. We define the equivalence relation $\sim$ over formulas F of $PROP$, by x and y are equivalent, when $x \to y$ and $y \to x$ are theorems (or equivalently $x \leftrightarrow y$ is a theorem), denoted as $x \sim y$ if and only if $\vdash x \to y$ and $\vdash y \to x$ (or equivalently $\vdash x \leftrightarrow y$), for any two formulas x, y. Given a theory Γ of $PROP$, the relation $\sim_\Gamma$ is also an equivalence relation of the algebra of the formulas F, where $\sim_\Gamma$ is defined by $x \sim_\Gamma y$ iff $\Gamma \vdash x \leftrightarrow y$. The quotient algebra $F/\sim_\Gamma$ is the Lindenbaum-Tarski algebra , which is determined by Γ.

Let $B = PROP/\sim$ be the set of equivalence classes. We define the operations $\vee_B$, $\wedge_B$, and complementation of $*$ on B as follows:

1. $[x] \vee_B [y] = [x \vee y]$

2. $[x] \wedge_B [y] = [x \wedge y]$

3. $[x]^* = [\neg x]$

We define $0_B = [x \wedge \neg x]$ and $1_B = [x \vee \neg x]$. Then the structure

$$\langle B, \vee_B, \wedge_B, *, 0_B, 1_B \rangle$$

which satisfies axioms (1)-(6), is a Boolean algebra, called the Lindenbaum-Tarski algebra of the propositional language.

(1) $\forall x \forall y (x \wedge_B y = y \wedge_B x)$, $\forall x \forall y (x \vee_B y = y \vee_B x)$ (Commutativity)

(2) $\forall x \forall y \forall z [x \wedge_B (y \wedge_B z) = (x \wedge_B y) \wedge_B z]$, $\forall x \forall y \forall z [x \vee_B (y \vee_B z) = (x \vee_B y) \wedge_B z]$ (Associativity)

(3) $\forall x \forall y [(x \wedge_B y) \vee_B y = y]$, $\forall x \forall y [(x \vee_B y) \wedge_B y = y]$ (Absorption)

(4) $\forall x \forall y \forall z [x \wedge_B (y \vee_B z) = (x \wedge_B y) \vee_B (x \wedge_B z)]$,
     $\forall x \forall y \forall z [x \vee_B (y \wedge_B z) = (x \vee_B y) \wedge_B (x \vee_B z)]$ (Distributivity)

(5) $\forall x (x \wedge_B x^* = 0_B)$, $\forall x (x \vee_B x^* = 1_B)$ (Complements)

Note here that the Lindenbaum-Tarski approach means "the method" used in algebraization of logical systems, e.g. classical propositional logic in this case. It could be further extended to study the algebraization of first-order logic, called **cylindric algebra**. Similarly, by this method, IPL has an algebraization of **Heyting algebras**.

2.2.2 Three Traditions in Algebraic Logic

The previous discussions will directly lead to the following questions:

- Does it mean "logic is reducible to algebra?"

- Or, does it simply mean "logic is algebraizable?"

The Lindenbaum-Tarski approach is a typical example of this type of relation. As in [50], there are many forerunners to the development of algebraic logic which may have had an effect on this reduction relation.

> "Algebraic methods have played an important role in the development of logic. Indeed, it may be argued that modern logic began with the algebraic work of Boole and De Morgan. Their line of investigation was continued by Peirce and Schröder in the latter half of the 19th century, and was taken up again later by Tarski. This work constitutes the first of three distinct traditions that can be traced in the history of algebraic logic." ([50], p. 365)

There are three distinct traditions in the development of taking algebraic methods to develop logic: *the Boolean tradition*, *the logistic tradition*, and *the model-theoretic tradition*. We explain each as follows:

- To the extent that Boolean tradition is concerned with calculus regarding the truth value of propositions (truth and falsity) by way of the operations of logical connectives $\vee$, $\wedge$, $\wedge$, $\neg$, propositions are taken as primitive mathematical elements, and the *calculus of propositions* is akin to the calculus of natural numbers. Boolean tradition, on the other hand, considers *truth* as the primitive logical predicate.

- To the extent that logistic tradition is concerned with the idea of a "non-algebraic" *formalism*, which is known as the *logistic method* of Frege, Peano, Whitehead, and Russell, it consists of a *logistic system* with a formal language and its own deductive apparatus. For example, as shown before, Lindenbaum-Tarski constructed the *algebra of propositional formulas* which comes from logistic systems for CPL that corresponds with *quotient algebra* through defining a relation $\sim$ (*logical equivalence*) on the set of formulas, and $\sim$ forms a *congruence relation* on the formula algebra. Further it is stated that the quotient algebra is *Boolean* ([207], [210], Tarski). We refer to this as an algebraic *representation* of the logistic systems of classical propositional logic, that is, a logical structure can be *represented* or transformed by an algebraic structure. The logistic tradition considers *logical equivalence* as a primitive logical predicate ([83], p. 14).

  As we have explained, in logistic tradition, the *algebraization* of logistic systems is possible, i.e., to have the algebraic representations of logistic systems. Next, we see the role that the model-theoretic tradition plays in the development of algebraic logic.

- A description of the model-theoretic tradition (model theory) is given as follows:

  "The third line of investigation in algebraic logic concerns itself with the algebraic semantics of logical systems on a more general level. It has its roots in the study of abstract consequence relations and their matrix models, begun by Tarski and Łukasiewicz in the 1920's and 30's, and later further developed especially in Poland. Although most of the familiar logistic systems are algebraizable in the sense explained earlier, and therefore can be studied adequately through their associated quasivariety, there are important

> systems to which the process of algebraization cannot be
> applied. Attempts have been made to enlarge the domain
> of logical systems for which a satisfactory semantics can be
> found that still shares many of the attractive feature of a qua-
> sivariety semantics for an algebraizable system." (See, [50],
> p. 369)

By tracing the three traditions of algebraic logic, there is not sufficient
reason to claim that algebraic approaches fully cover all aspects of logic.
In other words, "some logics" which deserve the name *logic* are not known,
from an algebraic point of view, whether taking logics as algebraic structures
(Boolean tradition) or studying logic using algebraic tools (logistic tradition
or model-theoretic tradition) in modern logic. If the reduction we propose
from *logic* to *algebra* precisely refers to "the relation from CPL to *Boolean
algebra*" ([207]), the meaning of reduction then becomes clearer, that is, it
means "Boolean algebra is one of the ways by which classical propositional
logic can be realized," i.e., the former can "reduce" to the latter in a very
weak sense.

So far, the discussion above does not support the radical perspective of
reducing logic to algebra with claims that algebra is all there is to logic,
that is, logic is completely reducible to algebra; a position called *radical
reductionism* with regard to logic. So, if we do not have a radical reductionism
point of view in logic and it is clear that Boolean algebra is not all there is
to CPL, it is possible to employ "CPL *is (Boolean) algebraizable*" to refer
to a weaker meaning of reduction.

Essentially, to claim radical reductionism in logic is counter-intuitive from
the point of view of modern logic. For example, paraconsistent logic $C1$
is equivalent to $C1$–Curry-algebra, i.e., it is *algebraizable* but the former
cannot be reduced to the latter because of the irreducible notion of order
in $C1$-Curry-algebra (see [45], pp. 138–139). $C1$-Curry-algebra is not a *pure*
algebraic structure; it is an *order-algebraic structure*. Hence, as discussed
above, it is better to clarify the relation between logic and algebra by claiming
that *logic is algebraizable* instead of claiming *logic is completely reducible to
algebra*.

Ramon Jansana answered an interesting question in *Stanford Encyclope-
dia of Philosophy* – **Propositional Consequence Relations and Alge-
braic Logic** ([131]):

Q: "What does this mean when a logic is algebraizable?"

A: Generally speaking, when a logic has an *algebraic semantics* we call it *algebraizable.*

This is further elaborated as follows:

> "The term 'algebraic semantics' was (and many times still is) used in the literature in a loose way. To provide a logic with an algebraic semantics was to interpret its language in a class of algebras, define a notion of satisfaction of a formula in an algebra of the class and prove a soundness and completeness theorem, usually for the theorems of the logic only." (Ibid.)

For example, as mentioned, the Lindenbaum-Tarski algebra factorizes classical propositional language to Boolean algebra. Moreover, to define a semantic concept, "satisfaction" in an algebra, Boolean algebra in this case prove some meta-properties of this given logic. What is presented in the citation above is a general statement which discusses a general statement of the way to "algebraize" logical systems, that is, to interpret the formal languages of the logical systems into a class of algebras. The Lindenbaum-Tarski algebra we mentioned previously is only a case of this general statement. It discusses *CPL* only. We do not discuss all the details of this article [131], although some will be discussed in the next section as they are helpful in understanding how universal logicians were inspired by the development from algebraic logic to Universal Algebra. For a precise concept of algebraic semantics for a logic system and to understand algebraic logic comprehensively, the readers are referred to ([51], Blok-Pigozzi, 1989), besides reading this self-contained article.

2.2.3 The Influence of the Development of Abstract Logic on Universal Logic

The approaches algebraic logic have taken did not directly influence the methods universal logicians have taken in the Universal Logic project, however, they directly influenced the development of Universal Algebra and inspired the development of Universal Logic.[3] It is important to see the status of Universal Logic regarding logical structures, which is analogous to that of

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[3]The approach provided by algebraic logic should be fruitful for the development of Universal Logic. A book will soon be published on a recent development of Universal Logic: Hajnal Andréka, István Nemeti, Ildikó Sain, *Universal Algebraic Logic: Dedicated*

Universal Algebra regarding algebraic structures. Universal Algebra is a general theory of algebraic structures and Universal Logic is taken as a general theory of logical structures. In this subsection, we briefly describe the development from algebraic logic to Universal Logic.

First of all, it has been stated that "the general theory of the algebraization of logical systems that has developed is called abstract algebraic logic" ([83], p. 15). In modern mathematical logic, abstract algebraic logic has been seen as a subfield of algebraic logic. It is also a study of the algebraization of logical systems and is an "abstraction" of the Lindenbaum-Tarski approach *generally*.[4] To take the expression we often use, it is to study Lindenbaum-Tarski algebra *in general*. Furthermore, it studies the relationship between the given logical systems and the algebras, *algebraized* from the given logical systems. Abstract algebraic logic did not directly influence either Universal Logic, nevertheless, the relation between abstract algebraic logic and algebraic logic could also be analogous to the relation between Universal Algebra and abstract algebra. It shares similarities with the general theory of logics for the Universal Logic project.[5] Unlike abstract algebraic logic, Universal Algebra inspired the Universal Logic project much more. Universal Algebra was independently developed in two directions, one by Garrett Birkhoff, and the other by the Polish researcher Tarski. Universal Algebra was widely popular in "Polish logic" (see [39], p. 15). Suszko, a representative of the Polish school of thought, proposed a concept of abstract logic $\langle A, \circledS \rangle$, consisting of an algebra A and a closure $\circledS$. This conception of abstract logic was seen as a basic notion in Polish logic. As discussed in previous sections, Universal Logic was easily misunderstood to be a part of Universal Algebra, possibly because abstract logic was considered to be a part of Universal Algebra by

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to the Unity of Science (Studies in Universal Logic), Birkhäuser (December, 2013) and many concepts of algebraic logic were used in studies of logic translation, which are seen as one of the cornerstone of the Universal Logic Project.

[4]There is a similar description in [83], p. 15.

> "One of the goals of AAL is to discover general criteria for a class of algebras (or for a class of mathematical objects closely related to algebras such as, for instance, logical matrices or generalized matrices) to be the algebraic counterpart of a logic, and to develop the methods for obtaining this algebraic counterpart. In this connection, an abstraction of the Lindenbaum-Tarski method plays a major role."

[5]Universal Logic follows Bourbakism. "Logics" here refers to logical structures in abstract, defined as $\langle S, \vdash \rangle$. Logical structures should be seen as mathematical structures.

Roman Suszko himself. Moreover, "it seems that it turned out to be a common idea in Poland" (Ibid, p. 15).[6] However, we do not trace the history of the development of Polish logic, rather a detached description of the development of abstract logic to Universal Logic is given in [39] to which the readers are referred. Readers could consult [222] to understand much more about the logic and philosophy developed in Poland.

One thing the Universal Logic project demonstrates is that logic and algebra are closely related to each other, but Universal Logic should not be misunderstood as a part of Universal Algebra. Clarifying the difference between Universal Logic and Universal Algebra requires the "independent existence" of logical structures. When dealing with logical structures abstracting from the Tarskian abstract logic, logical structures can be seen as the fourth mother structure by following Bourbakism.

2.3   Neo-Bourbakism as a Conceptual Approach

Let us now turn to another inspiration from the "conceptual approach" in Birkhoff's stream of the development of Universal Algebra. In previous sections, we remarked that the Universal Logic project might be taken as an outgrowing of the Polish school of thought. I will begin this section by discussing how, upon adopting conceptual approach, the direct abstraction of the Tarskian account of logical structures gives way to the recognizable arbitrary logical structure with bivaluation. However, this conception of logical structures is strikingly different from that presupposed in the Tarskian account, for reasons that will become obvious. The Universal Logic project adopts the conceptual approach that conforms to our daily life rather than the axiomatic approach in logical society.

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[6]This information on the historical misunderstanding was rephrased from a paragraph in [39], p. 15 as follows. It is provided to help readers understand much more about the history of Universal Algebra and Polish logic.

> "[...]universal algebra was very popular among people working in Polish logic. Suszko himself considered ≪ abstract logic ≫ to be a part of universal algebra and it seems that it turned out to be a common idea in Poland, as suggests the following comment by S. L. Bloom: ≪ Roman taught us the Polish view of logic – as a branch of universal algebra (a novel outlook for us) ≫ (Bloom 1984, p. 313)."

In arriving at the formulation of logical structuralism (structuralism in logics) in the Bourbakian sense that regards logic as a mathematical structure, it is attributed to a restricted version of mathematical structuralism. In other words, the existence of logical structures seems to depend strongly on three mother structures, algebraic structures, topological structures, and order structures. Does this not enhance the reason for those possible radical reductionists to support the reduction of logic to algebra? Universal logicians, however, do not, apparently, wish to see this. They claim that a logical structure consists of the following type:

$$\langle \mathcal{S}, \vdash \rangle$$

where S can be included in any of the three mother structures in the Bourbakian sense. Once we have only this conception of logic, it implies that there are no *pure* logical structures and that logical structures do not possess an *independent* existence. Therefore, the fact that logical structures are considered as mathematical structures means that mathematical structuralism should principally be applicable to logical structures; in other words, logical structuralism can be useful to universal logicians in the sense of acting as a restricted version of mathematical structuralism. However, the fact that logical structures are taken to be mathematical structures does not imply Universal Logic is the same as a mathematical structure. Universal Logic has its own special status: "it is a general theory of logical structures" and universal logicians would "agree with the fact that *logic is algebraizable*" instead of "a *complete* reduction of logic to algebra." We can detect certain meanings that typify logical structuralism in the Universal Logic project, which indicates that it is not just a special case of mathematical structuralism. Moreover, logical structuralism could make the relation between algebra and logical structures clear.

Bourbakism has the goal of founding all mathematics from an axiomatic-formalistic stance. The special status of logical structuralism in relation to the Universal Logic project has caused the Universal Logic project to propose a "Neo-Bourbakian approach", consisting of several criteria:

1. Logical structures are one of the prototypes of the fundamental mother structures based in Bourbakism.

   The first criterion is the main assertion as to why this stance is called "Neo"-Bourbakism. Logical structures are the fourth mother structure,

hence, it leads to the view that Universal Logic is not a part of Universal Algebra.

2. Logic is not reducible to algebra and Universal Logic is not a part of Universal Algebra.

   The second criterion is especially important to Universal Logic to state its own stance.

3. The conceptual approach of Universal Logic aims to capture all logical phenomena.

   The third criterion is important to complete Neo-Bourbakism.

(The conceptual approach)

Having explored the connection between the trend of the Polish school of thought and Neo-Bourbakism, let us now explore the connection between Universal Logic and the conceptual approach – or non-axiomatic approach.

It is natural enough to claim that one ought to follow the conceptual approach. The "ought" here, note, has nothing to do with what morality requires. What we will pursue now is the possibility of a claim about the conceptual approach. The conceptual approach is seen as contrary to the axiomatic approach here. This basic approach, taken by the Universal Logic project, comes from an abstract viewpoint about logical structures. It involves two factors:

To consider "logical structures"

(1) at a general level;

(2) without taking axioms into account.

What these factors illustrate is that attempts to formulate the conceptual approach in terms of the combination of two abstract conceptions, one being *Tarski's consequence operation* that is on "a general level" and the other being *Birkhoff's general theory of algebra* that does not "take axioms into account". It might be thought that Tarskian logical structures become more general in the Universal Logic project by the inspiration of the second factor.

Thus, Universal Logic is regarded as a general theory of logical structures. Here, a logical structure is *arbitrary* and consists of *an arbitrary consequence operator* without any axiom, as in Birkhoff's sense. This conception of *arbitrary logical structures* will be discussed in section 2.4.2.

With this understanding, we can now address the *non-axiomatic* approach, which is often seen as a *conceptual* approach in the Universal Logic project. Béziau states: ([32], p. 136)

> "Such a surprising approach can be called a conceptual one, as opposed to an axiomatic one. Category theory is itself more conceptual than axiomatic. The point is not to produce a large axiomatic system like ZF set-theory from which everything could be deduced; rather, it is to elaborate some concepts that could serve to describe the whole of mathematical phenomena in a unitary fashion."

As Béziau states, the conceptual approach seems to be higher than the axiomatic approach, and, in this regard, beginning with a set of axioms seems to represent an inferior methodology. The term "conceptual" here means to be the opposite of "axiomatic", that is, "non-axiomatic". Specifically speaking, it means not using axiomatic systems to derive everything but using much closer approaches in our daily life, that is free-axiomatic ones. In mathematics, category theory is an example of a non-axiomatic but "more conceptual" approach. A brief historical overview of category theory can be found in [148].

Category theory has occupied an important position in modern mathematics. There are several similarities between the Universal Logic project and category theory. First of all, a general aim of category theory, mentioned in [148], states, "[...] Roughly, it is a general mathematical theory of structures and of systems of structures." Universal Logic employs a similar thought providing "a general theory of structures" as the final target,[7] by restricting the discussion to logical structures aiming to capture all logical phenomena. We do not discuss category theory in detail, however, we need

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[7]Universal Logic aims to develop "a general theory of structures", including mathematical structures. "Universal Logic, like Universal Algebra, is just a part of the general theory of structures, logical abstract nonsense is a subfield of general abstract nonsense. If, as we have suggested, abstraction is the important thing, one could argue that what is really interesting is a general theory of structures, like category theory, and not a theory of specific structures like Universal Logic." ([31], p. 14)

to clarify the conceptual approaches universal logicians use in order to avoid misunderstandings. Though category theory is more conceptual than axiomatic, this does not imply that category theory does not adopt any axiom, on the contrary, it is "the legitimate heir" of the Bourbakian tradition, which has the goal of founding mathematics as a whole from an axiomatic point of view. As stated in [148],

> "[...] It could be argued that category theory represents the culmination of one of the deepest and most powerful tendencies in twentieth century mathematical thought: the search for the most general and abstract ingredients in a given situation. Category theory is, in this sense, the legitimate heir of the Dedekind-Hilbert-Noether-Bourbaki tradition, with its emphasis on the axiomatic method and algebraic structures [...]"

Universal Logic borrows Birkhoff's idea of developing Universal Algebra without taking any axiom into account. In other words, the term "conceptual approach" in the Universal Logic project clearly refers to a "non-axiomatic approach" and it adopts a "non-axiomatic approach" to address its basic stance. Moreover, Béziau himself claims the approach of Universal Logic is a conceptual one ([32], p. 136).

Category theory aims to provide a general mathematical theory of structures; the Universal Logic project aims to provide a general theory of logical structures, where logical structures are seen as mathematical structures. In this way, Universal Logic is not a subfield of category theory, rather they are related to each other. Universal Logic claims to take a "non-axiomatic" approach in the spirit of Neo-Bourbakism. Although category theory might not contribute to the essence of Universal Logic, it has provided powerful mathematical tools for the Universal Logic project. There is nothing in this account which criticizes the conceptual approach. It is, in fact, quite natural as to what we usually do in daily life. Moreover, it may well be that there is no uniform answer to the question as to whether to take an axiomatic approach or not. Some answers may be appropriate in dealing with the assertions of natural sciences in the physical world or in dealing with the statements of pure mathematics in the abstract world, etc. These possibilities also lead in the direction of the conceptual approach, since "not everything is axiomatizable". Adopting something with conceptual understanding is much closer to what we do in everyday life. Even if one knows the entire extension of

the axiomatizable, either in the physical or pure mathematical world, one would not deny that conceptual understanding tends to conform more to our intuitions. These opinions and the understanding of the conceptual approach coincide with the idea of Universal Logic.

In sum, Universal Logic provides a general theory of logical structures, and demonstrates a hierarchical relationship between a *specific theory* and a *general theory*, as well as the *continuous interplay* between "the specific" and "the general" ([34], p. 15), e.g., CPL and Universal Logic, such that this process of alternating between abstract and concrete levels is natural to ordinary humans as it occurs in everyday life. As a universal logician would suggest, *Universal Logic is a new way to view logic* ([32], p. 142), whether to propose a new notion of logical structures, to specify the status of logical structures in mathematics and its independent existence in ontology, or to propose a conceptual approach in studies of logic. In addition, it is also worth mentioning work which *systematically* studies these conception from the philosophy of logic e.g. the well-known concept of **Deviant Logics** which is discussed in later chapters. Maybe this is also a reason why people "choose" to defend the position that the Universal Logic project has taken, meaning many fundamental notions in **Logic**[8] would have to be re-examined and challenged.

2.4   Main Strategies and Techniques of Universal Logic

The literal sense of the term "Universal Logic" seemingly speaks of "a logic" which *unifies* all logics, however, this interpretation has been challenged in our discussions in previous sections. As discussed, Universal Logic is a general theory of studying logical structures themselves instead of studying various specific models or examples of logical structures. In the last section, we showed that a logical structure is considered an abstract logical structure in general, denoted as $\langle S, \vdash \rangle$, that is an *abstraction* of an abstract logical structure, which does not take any axiom into account. Moreover, it claims to take a conceptual approach to the study of logic. In this section, we will discuss the strategies and techniques taken in the Universal Logic project to understand how people can achieve a Neo-Bourbakian paradigm.

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[8] "Logic" is referred to as a discipline.

2.4.1 Strategies

First of all, we examine the strategies used by Universal Logic, which are closely related to our way of thinking in our daily life. As mentioned, it is not necessary to possess an axiomatic mind in order to work with logic. At its core, Universal Logic addresses the following question: "Is there a way of determining the common features that would allow us to unify the study of all particular systems into a science called *logic*?" To answer this question, we first have to ask "What is logic?" Universal Logic suggests the strategy of adopting a more *concrete* approach by answering more concrete questions such as, "What is classical propositional logic?" ([37]) with an *abstract* approach to logic[9], in other words, to see each specific logic from an abstract point of view, as discussed in previous sections. Note here that this style of asking questions, considered basic conceptual analysis in analytic philosophy, is also often used in Universal Logic to address questions such as "What is many-valued logic?" ([44]), "What is paraconsistent logic?" ([40]), "What is the principle of identity?" ([35]), "What is *formal* logic?" ([28]), etc.

The strategy taken implicitly in the Universal Logic project is to "pull back to a more concrete level from an abstract level with an *abstract logic* point of view". This means analyzing each concrete question such as "What is classical propositional logic?" and "What is intuitionistic propositional logic?" with a conception of "abstract logic" in order to answer a basic but general question, "What is logic?" Certainly, Universal Logic proposes a new *conception* of logic, that is arbitrary logical structures of the type $\langle S, \vdash \rangle$, without taking axioms into account. It tries to study several specific logics with respect to this conception. Using this approach, they try to understand the *conceptual analysis* of "logic". We should note here that the "conception" proposed by Universal Logic is not able to realize the "concept" of logic, that is to say we are not able to answer a conceptual question about "what is logic?" comprehensively. It at most provides a method to achieve this conceptual analysis step-by-step.

Whichever is the case, generally speaking, this represents a more practical method toward achieving the goal set by Universal Logic. All Universal Logic researchers presume an abstract logic framework to view different kinds of

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[9]The study of Universal Logic is attributed to an *abstract* level. Logic is an abstract mathematical structure $\langle S, \vdash \rangle$ where S could be put on any three of the Bourbakian structures and $\vdash$ is an abstract consequence operator.

logics, for example, paraconsistent logic, many-valued logic, classical logic, intuitionistic logic, etc. In other words, all logics that we have considered can be addressed from the viewpoint of the Universal Logic project. However, when they try to decipher the meaning of logic, they must first discuss more concrete examples; only then, can they legitimately claim that they could *jump into abstraction* ([32], p. 142) to ascertain the common properties of all varied logics. Moreover, any preconceived ideas can be amended by focusing on the "difference" between logics.[10]

After all, a direct discussion of logic at such an abstract level can lead us astray; for example, it is not possible for a logician to generalize the completeness theorem or prove it without knowing the cases pertaining to the completeness theorem, for example, the completeness theorem for classical logic. Actually, as shown in their work, Universal logicians are trying to prove different meta-properties without using any specific characteristic of a specific logic. As stated by Béziau,

> "In a proof of a completeness theorem for a given logic, one may distinguish the elements of the proof that depend on the specificity of this logic and the elements that do not depend on this peculiarity, that we can call universal,... The first proofs of completeness for propositional classical logic give the idea that this theorem is depending very much on classical features. ... In fact this idea is totally wrong and one can present the completeness theorem for CPL in such a way that the specific part of the proof is trivial, i.e. one can trivialize the completeness theorem. One central aim of a general theory of logics is to get some universal results that can be applied more or less directly to specific logics, this is one reason to call such a theory Universal Logic." ([31], p. 13)

These universal logicians attempt to trivialize the completeness theorem, but they do not stop considering individual logic; instead they also consider it from an abstract point of view, the *pull-back strategy* previously discussed being the preparatory work of the universal logicians. In order to "jump

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[10]As Béziau states in his biographical paper: "... My attention was directed to the common ground between C_1 and classical logic. These two logics are very different and my intuition was that the very essence of logic should not lie in any of their specific differences but on their common features." ([39], p. 6)

into abstract", they have to discuss concrete examples. This back-and-forth movement between the general and the particular plays an important role in the Universal Logic project. "Any abstract theory is not a pure abstraction," stated Béziau himself in [32], p. 148, "it is an abstraction of something else." Similarly, people jumping into abstraction should also begin from a "non-abstract" stage. Universal Logic as an abstract theory, the concepts of which are derived from many observations of specific cases, e.g. CPL, they will also be applicable back to these specific cases. We cite a paragraph by Béziau to conclude this section:

> "What is crucial in Universal Logic is that logics are considered irrespective of the way they are generated, so that one thus makes a jump into abstraction. And this is not surprising at all, it's the most natural thing you could have. CPL can be generated in a hundred different ways, through Hilbert systems, Gentzen systems, tableaux, two-, three- or infinite-valued semantics. What is this object that can be defined in so much different ways? Everybody believes in it, and nobody would venture to claim that classical propositional logic reduces to one particular way of constructing it." (Ibid., p. 142)

Clearly, no one can be confident in claiming which "system" is exactly the final target to reduce CPL. Faced with this situation, the universal logician suggests that people tend to, as in daily life, freely and easily "jump into abstraction." Although this seems very romantic to logicians, "splitting of hairs" is also worthless. Naturally, this approach will bring us closer to the thinking patterns of human beings.

2.4.2 Main Techniques

> "..., J.-Y. Béziau has recently provided a systematic connection between logical bivaluations and structurally standard systems of sequents... With this general result, it is very easy to jump from sequent rules to bivaluation conditions, and thus to provide axiomatization and completeness." ([68], p. 288)

We have discussed the strategies used in the Universal Logic project. In this section, we discuss some main techniques used in it.

(Arbitrary Logical Structures with Bivaluations)

With the help of the ideas from Tarskian logic, it starts at the abstract level by considering an *arbitrary* logical structure as a pair $\mathcal{LS} = \langle S, \vdash_{\mathcal{LS}} \rangle$, where $\vdash_{\mathcal{LS}}$ is an arbitrary consequence relation.

Definition 2. An arbitrary logical structure is a structure of the form $\mathcal{LS} = \langle S, \vdash_{\mathcal{LS}} \rangle$, where S is an arbitrary set and $\vdash_{\mathcal{LS}}$ is an arbitrary relation on $\mathcal{P}(S) \times S$. Equivalently, we could have described it as a pair $\mathcal{LS} = \langle S, Cn_{\mathcal{LS}} \rangle$, where S is an arbitrary set and $Cn_{\mathcal{LS}}$ is an arbitrary mapping from $\mathcal{P}(S) \longrightarrow \mathcal{P}(S)$.

This conception of logical structures was introduced in [46] firstly (also see [31], [34], [38]) in order to capture the most general formalization of logical reasoning that we could call. The study of Universal Logic started by considering the conception of logical structure by Jean-Yves Béziau (1995).[11] Regarding **bivaluations** ([38], Béziau) is crucial, moreover, it can be regarded as the seminal work with respect to modern developments in Universal Logic, which is typified by the following notion. This notion provides an important clue regarding a key step to treat *logic* in Universal Logic. In this regard, it attempts to dissolve the boundary existing between the syntactic and the semantic. This notion will often be referred to in the discussions in the next chapter on logic translation.

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[11]An arbitrary logical structure is said to be Tarskian (normal) when it obeys the following Tarskian conditions: Given a set S of formulas, we say that $\vdash \subseteq \mathcal{P}(S) \times S$ defines a (Tarskian) consequence relation on S if the following conditions hold, for any formulas α and β, and subsets Σ and Δ of S.

1. $\alpha \in \Sigma$ implies $\Sigma \vdash \alpha$. (Reflexivity)

2. $(\Delta \vdash \alpha$ and $\Delta \subseteq \Sigma)$ implies $\Sigma \vdash \alpha$. (Monotonicity)

3. $(\Delta \vdash \alpha$ and $\Sigma, \alpha \vdash \beta)$ implies $\Delta, \Sigma \vdash \beta$. (Cut)

A *compact* Tarskian logic $\mathcal{L}$ is definied as $\langle S, \vdash \rangle$ with *compactness*.

   ∗ If $\Sigma \vdash \alpha$, then $\Gamma \vdash \alpha$, for some finite subset $\Sigma \subseteq \Sigma$. (Compactness)

The Tarskian conditions for consequence relation can be presented in the style of consequence operator $Cn_{\mathcal{LS}}$ as follows: For the theories, T, K, $T \subseteq Cn_{\mathcal{LS}}(T)$; $T \subseteq K$ implies $Cn_{\mathcal{LS}}(T) \subseteq Cn_{\mathcal{LS}}(K)$; $Cn_{\mathcal{LS}}(Cn_{\mathcal{LS}}(T)) = Cn_{\mathcal{LS}}(T)$.

Definition 3. A bivaluation of $\mathcal{LS}$ is any characteristic function $v : X \longrightarrow \{0,1\}$, for $X \subseteq S$. Given a set V of bivaluations of $\mathcal{LS}$, for all $X \subseteq S$ and $\varphi \in S$, the following semantical deduction relation is defined:

$$X \models_V \varphi \text{ iff for all } v \in V, v(X) = 1 \text{ implies } v(\varphi) = 1.$$

This logical two-valuedness originated from R. Suszko and N. C. A. da Costa independently. Suszko ([204]) proposed a method for providing any structural abstract logic with a complete set of bivaluations. *Suszko's Thesis* (Suszko's reduction method) was proposed, which states that any logic with a structural consequence operator conforming to Tarski's standard conditions is logically two-valued, while da Costa's theory of valuations ([69]) was used in the analysis of the paraconsistent calculi comprehensively. Béziau (1995), in this way, analyzed the conditions for an arbitrary logical structure, which is a very abstract concept of logical structures, to obtain a complete set of bivaluations, linking the *theory of valuations* with the *sequent calculus* to show how it is possible to translate conditions defining bivaluations into sequent rules and vice versa ([31], p. 13).

The conceptions of *arbitrary logical structures* and *logical two-valuedness* (bivaluations) have been considered together in the idea of Universal Logic.

Definition 4. A theory Γ such that, if $\Gamma \vdash a$ then $a \in \Gamma$, is said to be closed. A theory is considered as a bivaluation by taking its characteristic function; a bivaluation can be considered as a theory by taking the set of true formulas under this function.

Definition 5. (Béziau 2001) An adequate bivalent semantics for a logic $\mathcal{LS} = \langle S, \vdash_{\mathcal{LS}} \rangle$ is a set of functions **BIV** from S to $\{0,1\}$ such that the semantic deducibility relation $\models$ defined in the usual manner ($T \models a$ iff for every $\beta \in \mathbf{BIV}$, if $\beta(b) = 1$ for every $b \in T$, then $\beta(\alpha) = 1$) by this set is the same as $\vdash$. If $\vdash$ is included in $\models$, we say that the semantic is sound (for $\mathcal{LS}$, and if $\models$ is included in $\vdash$ we say that the semantic is complete (for $\mathcal{LS}$)

Theorem 6. *(Béziau 2001, 1995) The semantics of closed theories of a Tarskian (normal) logic is an adequate semantics for it. If $\mathcal{LS} = \langle S, Cn \rangle$ is a logical structure such that $X \subseteq Cn_{\mathcal{LS}}(S)$, for all $X \subseteq S$ (reflexivity), then $\mathcal{LS}$ has a adequate set of bivaluations.*

Theorem 7. *(Béziau 2001, 1995) A bivalent semantics is sound for a normal logic iff it is included in the semantics of closed theories. If $\mathcal{LS} = \langle S, Cn \rangle$ such that for all $X, Y \in S$, $X \subseteq Y$ implies $X \subseteq Cn_{\mathcal{LS}}(X) \subseteq Cn_{\mathcal{LS}}(Y)$ and $Cn_{\mathcal{LS}}(X \cup Cn_{\mathcal{LS}}(X)) = Cn_{\mathcal{LS}}(X)$, then $\mathcal{LS}$ has a sound set of bivaluations.*

Any set V_s of bivaluations of a logical structure $\mathcal{LS}$ which are adequate and sound for it: $X \models_{Vs} \varphi$ iff $X \vdash_{\mathcal{LS}} \varphi$, is called as a *Suszko set* for a $\mathcal{LS}$.

Note. Theorem 6 and Theorem 7 were mentioned in [38], [42], and the proofs can be found in [213]. Suszko's thesis is connected with the reduction of many-valuedness to two-valuedness (a discussion on this can be found [143] and [68]). Bivaluation provides a more general formulation of Suszko's thesis. Readers should note that what we have presented for Béziau's results should be attributed to Gentzen's sequent calculus, however, this requires in-depth knowledge of Gentzen's sequent calculus. Readers could consult [38], [46] to obtain a comprehensive understanding of this.

The significance of this work is its claim that any semantics can be reduced to a bivalent semantics. In other words, in a general sense, in relation to Universal Logic, it provides a general definition of semantics: a semantics on a given set S is a pair $\langle K, Fun \rangle$, where K is a set and Fun is a function from S to $\mathcal{P}(K)$. The logic induced by the semantics is defined: $\Gamma \Vdash \varphi$ iff $Fun(\Gamma) \subseteq Fun(\varphi)$, where $\Vdash$ represents a general sense of semantic relation, where given any semantics on a set S, we can find a bivalent semantics on S which induces the same logic ([68]). For Universal Logic, the *bivalent semantics* on a set S is a semantics, where K is a set of functions from S to $\{0, 1\}$ (bivaluations) and F is defined as follows: $\beta \in Fun(\varphi)$ iff $\beta(\varphi) = 1$. Here, bivaluations are logical values and not algebraic values, as in Suszko's terminology ([44], [68],[204], [213]). This definition is carried out at the abstract level.

(Lindenbaum-Asser Theorem)

> "Combining action of valuations upon sequent rules, in the spirit of Gentzen's 1932 proof,... with Lindenbaum-Asser theorem, I have given a general version of the completeness theorem, from which it is possible to derive instantaneously many specific completeness theorems..." ([31], p. 13)

Lindenbaum's theorem states that a set S is *maximally consistent* if it is consistent and no proper superset of S is consistent. A consistent theory is a theory which is not trivial, i.e., there exists a formula which is not deducible from it. This means there is no non-trivial strict extensions.

Definition 8. Given a theory Γ, a formula α such that $\Gamma \nvdash_{\mathcal{L}} \alpha$ and further for any strict extension Σ of Γ, $\Sigma \vdash_{\mathcal{L}} \alpha$, we say that Γ is relatively maximal with respect to α in logic $\mathcal{L}$.

From the Universal Logic point of view, when proving a completeness theorem for a given logic, people might be able to make a distinction for elements that depend on the specificity of this logic and the others that do not depend on this specificity in the whole proof. The latter was called the "universal" part. It was argued by universal logicians that a maximally consistent set (in Lindenbaum theorem), which is a useful argument for proving completeness, should be taken as the universal part. In other words, Universal Logic claimed that the concept of a maximally consistent set, which was misunderstood as depending on the classical features of negations, should be trivialized into the universal parts. It follows to generalize the completeness theorem.

Lemma 9. *(Lindenbaum-Asser Theorem) Let $\mathcal{L}$ be a compact Tarskian logic. Given any set of formulas Γ and a formula α such that $\Gamma \nvdash_{\mathcal{L}} \alpha$, there is an extended set Δ, $\Gamma \subseteq \Delta$ such that Δ is relatively maximal in $\mathcal{L}$ (with respect to) α.*

Proof. Consider an enumeration $\{\varphi_n\}_{n \in \mathbb{N}}$ of all formulas and a chain $\Delta_n, n \in \mathbb{N}$ of sets built as two cases,
(a) $\Delta_0 = \Gamma$;
(b)
$$\begin{cases} \Delta_{n+1} = \Delta_n \cup \{\varphi_n\}, & \text{if} \quad \Delta_n, \varphi_n \nvdash_{\mathcal{L}} \alpha; \\ \Delta_{n+1} = \Delta_n, & \text{if} \quad \text{else.} \end{cases}$$
Let $\Delta = \bigcup_{n \in \mathbb{N}}$. We show Δ is relatively maximal w.r.t. α in logic $\mathcal{L}$.

By mathematical induction on the above chain, we get that $\Delta_n \nvdash_{\mathcal{L}} \alpha$. This implies $\Delta \nvdash_{\mathcal{L}} \alpha$. If this were not the case, there would be some finite subset $\Delta^{FIN} \subseteq \Delta$, such that $\Delta^{FIN} \vdash_{\mathcal{L}} \alpha$ by compactness. This implies $\Delta_m \supseteq \Delta^{FIN}$ for some $m \in \mathbb{N}$, such that $\Delta_m \vdash_{\mathcal{L}} \alpha$ by using **Monotonicity** This contradicts $\Delta_m \nvdash_{\mathcal{L}} \alpha$. Now we consider a formula $\beta = \varphi_n$, for some n, where

$\beta \notin \Delta$. From $\Delta_{n+1} \subseteq \Delta$, we get $\beta \notin \Delta_{n+1}$. Then $\Delta_{n+1} = \Delta_n$ and $\Delta_n, \beta \vdash_{\mathcal{L}} \alpha$. Since $\Delta_n \subseteq \Delta$, we must conclude $\Delta, \beta \vdash_{\mathcal{L}} \alpha$ by **Monotonicity**. $\square$

Lemma 10. *Any relatively maximal set of formulas is a closed theory.*

Proof. Given a set of formulas Γ that is relatively maximal with respect to a formula α, then $\Gamma \Vdash_{\mathcal{L}} \beta$ if and only if $\beta \in \Gamma$.

($\Longleftarrow$) By **Reflexivity**.

($\Longrightarrow$) Given some $\beta \notin \Gamma$, we have that $\Gamma \nVdash_{\mathcal{L}} \alpha$ and $\Gamma, \beta \Vdash_{\mathcal{L}} \alpha$, since Γ is relatively maximal with respect to α, as we say. Then we conclude that $\Gamma \nVdash_{\mathcal{L}} \beta$. $\square$

Theorem 11. *(Béziau 2001, [38]) The semantics of relatively maximal theories of a Tarskian compact logic is an adequate semantics for it.*

The Lindenbaum-Asser theorem claimed that, inside any compact Tarskian logic $\mathcal{L}$, every theory can be extended to a relatively maximal theory. Universal Logic, by the Lindenbaum-Asser theorem, takes the concept of "relatively maximal consistent" to replace the concept of "maximally consistent" (or to trivialize the concept of maximally consistent). So far, we have considered a proof of Lindenbaum-Asser theorem. By considering the conception of the arbitrary logical structures, a systematic study of the class of bivaluation semantics which is adequate for a given "abstract logic" is presented. ([38]) From this point of view, it is claimed further to generalize or to trivialize the meta-property, *completeness* pushing it to a more general level.

2.4.3 Logical Two-Valued Semantics and Algebraic Valued Semantics

The idea to generalize logic by means of providing a general notion of logical consequence operators was first raised by Tarski. However, Suszko, da Costa and Béziau achieved a more general theory that can be seen in Polish logic tradition, especially Béziau who used connecting sequent calculus with bivaluation to prove completeness ([38], [46]).

This significant result in Universal Logic not only shows a general completeness without taking any specific characteristic of any specific logic, but it is possible to obtain each individual completeness theorem for each individual logic ([31], p. 13). This is a a powerful tool for logicians, who are devoted to the realm of studying applications of logic. As Béziau states in [32] (p. 146),

"Universal logic considers the world of all possible logics and ways to construct them, so that it gives a way out of many requirements and problems."

This means, once we obtain what should be trivial, universal and common to some sort of reasoning, as shown in Universal Logic, we are able to build "a logic" in this situation. This is an ideal from the perspective of studying applications in the logic discipline, however this utopia (Universal Logic) is not "one universal logic" to explain everything but "a general theory of different logics".

For Suszko, logical two-valuedness (truth and falsity) means a (logical) valuation, which is a function associating one value (true or false) to each formula; algebraic values, on the other hand, are *admissible reference assignments* (compare e.g. [143], [203], [221]). Algebraic valuation of formalized language $\mathcal{L}$, where $\mathcal{L}$ is a free (anarchic) algebra, is a morphism from the free algebra of formulas into any algebraic structure $\mathcal{A}$, which belongs to the whole class of all algebraic structures, $K(\mathcal{L})$. Suszko claimed:

"[...] the domain of them consists of all expressions of definite syntactic category: formulas (sentences), terms (names) and diverse kinds of formators. The size of codomains of algebraic valuations is not a priori limited. In particular, the formulas may have many algebraic values (admissible referents)."
([203], p. 377)

Suszko's logical two-valuedness is based on Tarskian logic. An earlier result by J. Łoś and R. Suszko in [184] (also see [62]) is: if $\mathfrak{M}$ is a matrix, every truth-assignment to the variables that verifies the formulas in the set of all formulas X will define a system of truth-tables of the set of all formulas, denoted as $\mathfrak{M}(X)$, such that the matrix consequence relation, $Cn(X) = \mathfrak{M}(X)$, and every uniform substitution consequence relation is a matrix consequence. This idea is usually used in the Polish tradition of logic, including the birth of the many-valued logics. Clearly speaking, many-valued logics are defined by logical matrices in this way. Having a logical matrix is necessary to generate a many-valued logic, in other words, the generated logic should be with a truth-functional semantics (or the logic is *truth-functional*).

"[...] in case of any logic considered as an inference relation, ⊢,...
one can find sets V of zero-one valued functions defined for all

formulas and, called here logical valuations,... In short, every
logic is (logically) two-valued." ([203], p. 378)

According to Suszko's reduction, given a consequence relation, it is always
possible to find a bivalent semantics with respect to this consequence relation.
However, for Suszko, this bivalent semantics should not be attributed to the
bivalence in the traditional matrix semantics, e.g. two-valued semantics in
CPL. The value 0 and 1, on the contrary, should be the logical values. It
is not the algebraic values that are always seen as the domain of an algebra.
Hence, in the case of $L3$, Suszko has claimed that it has both a two logical-
valued semantics and a three algebraic-valued semantics ([203], [204]).

2.5   Conclusion

In the last century, it has been common for logicians to conflate the notion of
logical consequence deductive calculus since Tarski proposed the consequence
operator. In the Universal Logic project, a new conception of the nature of
logical structures has been studied.

Universal Logic is a general theory of different logics, which allows it to
be a general theory of "many-valued logics". However, the technique that
Universal Logic has taken reduces the many-valuedness to bivalence. In other
words, Universal Logic provides a general theory that includes bivaluation
(bivalent semantics) to treat many-valuedness. It seems to be paradoxical
to claim that a general theory with the property of bivalence is a general
theory of many-valued logics. However, we clearly see that the Universal
Logic project does not "reject" many-valued logic. On the contrary, many-
valued logics should also be treated equally in the Universal Logic project,
i.e. it provides a general theory of different logics, including many-valued
logics, but this general theory studies many-valued logics in a certain sense
of reductionism: the generalization of Suszko's reduction

As discussed in this chapter, the bivalent semantics for Universal Logic
should not be two-valued in the sense of truth-functional matrix seman-
tics. A many-valued truth-functional matrix semantics is logically bivalent
in essence. "Logical matrices do not violate the principle of bivalence" ([36]).
Recalling the theory of valuation (bivaluation) with respect to the arbitrary
logical structures, it is possible to generate a logic whose semantics is with-
out truth-functionality in a very general sense ([43]). In other words, arbi-
trary logical structures and bivaluation, taken in Universal Logic generalize

the truth-functional semantics as in the case of algebraic many-valuededness that we have discussed and in a certain sense is a generalization of Suszko's Thesis. Finally, as logical two-valuedness should be the common part of various logics, it depends not only the specific characteristics of various logics. Suszko's Thesis states: "every structural Tarskian consequence relation is characterized by a bivalent semantics", therefore we obtain a generalization of this reduction with the non-truth-functionality.

Chapter 3

Logic Translation and the Translation Paradox

Contents

3.1 Introduction

Jean-Yves Béziau in his work (*Classical Negation can be expressed by One of its Halves*) ([41]) gave an example of a phenomenon that can be considered the translation paradox. We elaborate on Béziau's example, which concerns classical negation to the half of classical negation, as well as proving relevant background on this discussion. Béziau's work turns out not to deliver new results but is important in the interests of illustrating the development of logic translation that is widely discussed in various modern applications to computer science. In this chapter, we discuss logic translation. We review the translation paradox and discuss the concepts of sublogic and deviant logics. Then, we give a comprehensive survey on the development of logic

translation. The discussion of the translation paradox will enable readers to become more familiar with the development of the new subject of logic translation which is a fundamental aspect of Universal Logic.

The structure of the chapter is as follows: In section 3.2, we discuss the translation paradox using the example provided by Béziau, based on the presentation of two logics: $\mathcal{L}_{Classical}$ and $\mathcal{L}_{Classical/2}$. Section 3.3 then discusses a number of ideas on logic translation in history of modern logic with respect to the abstract logic viewpoint. We also introduce the first systematic investigation of logic translation attributed to Prawitz and Malmnäs in 1968 ([169]), and also to present two other systematic investigations by Wójcicki (1988) ([223]) and Epstein (1990, 2006) ([74], [76]), where we discuss logic translation with respect to the development of the consequence operation that has been widespread in the *Polish school of thought.* In section 3.5, we discuss the deviance of logics and the ideas of sub-logic with the growth of various logics in modern logical society.

3.2   Translation Paradox

3.2.1   The Inclusion Relation

We use the results shown in [41] to understand the translation paradox.

Recalling the definitions shown in Chapter 1 (section 2.4.2): an arbitrary logical structure $\mathcal{L} = \langle S, \vdash_{\mathcal{L}} \rangle$ and the bivaluation, which is a set of functions **BIV** from the set of formulas into the set $\{0, 1\}$.

The logic $\mathcal{L}_{Classical} = \langle S, \rightarrow, \neg, \models_{Classical} \rangle$ with its bivaluation **BIV**$_{Classical}$ means CPL with classical implication $\rightarrow$ and classical negation $\neg$. The two well-known classical semantic conditions for implication and negation are as follows:

$\beta \in$ **BIV**$_{Classical}$ iff given any two formulas φ, $\psi \in S$ the following two conditions hold:

- $\beta(\varphi \rightarrow \psi) = 0$ iff $\beta(\varphi) = 1$ and $\beta(\psi) = 0$

- $\beta(\varphi) = 1$ iff $\beta(\neg\varphi) = 0$.

Other classical connectives could be defined by implication and negation in $\mathcal{L}_{Classical}$ and then define the whole CPL. We consider the logic $\mathcal{L}_{Classical/2} = \langle S', \rhd, \ominus, \models_{Classical/2} \rangle$ with its bivaluation **BIV**$_{Classical/2}$ as follows:

a. S' is a set of formulas constructed by one binary connective $\triangleright$ and one unary connective $\ominus$.

b. $\beta \in \mathbf{BIV}_{Classical/2}$ iff given any two formulas $\varphi, \psi \in S'$ the following two conditions hold:

   (1) $\beta(\varphi \triangleright \psi) = 0$ iff $\beta(\varphi) = 1$ and $\beta(\psi) = 0$

   (2) $\beta(\varphi) = 1$ implies $\beta(\ominus\varphi) = 0$

c. $T \models_{Classical/2} \varphi$ iff for every $\beta \in \mathbf{BIV}_{Classical/2}$, if $\beta(a) = 1$, for every $a \in T$, then $\beta(\varphi) = 1$.

$\mathcal{L}_{Classical/2}$ is a weaker CPL with classical implication $\triangleright$ and "half classical negation" $\ominus$. Note here that "half classical negation" means we take only the following "half" condition: if $\beta(\varphi) = 1$, then $\beta(\ominus\varphi) = 0$ instead of the condition for classical negation: $\beta(\varphi) = 1$ if and only if $\beta(\ominus\varphi) = 0$.

Let us say that two formulas φ and ψ, φ is *logically equivalent* to ψ in $\mathcal{L}_{Classical/2}$ iff $\varphi \models_{Classical/2} \psi$ and $\psi \models_{Classical/2} \varphi$.

Lemma 12. *In $\mathcal{L}_{Classical/2}$, for any formula φ, ψ, ϕ,*

(1) $\varphi \triangleright (\psi \triangleright \phi)$ is logically equivalent to $\psi \triangleright (\varphi \triangleright \phi)$.

(2) $\ominus(\varphi \triangleright (\psi \triangleright \phi))$ is not logically equivalent to $\ominus(\psi \triangleright (\varphi \triangleright \phi))$.

Proof. In $\mathcal{L}_{Classical/2}$, $\varphi \triangleright (\psi \triangleright \phi)$ is *logically equivalent* to $\psi \triangleright (\varphi \triangleright \phi)$ iff for any function $\beta \in \mathbf{BIV}$, if $\beta(\psi \triangleright (\varphi \triangleright \phi)) = 1$, then $\beta(\varphi \triangleright (\psi \triangleright \phi)) = 1$ and if $\beta(\varphi \triangleright (\psi \triangleright \phi)) = 1$, then $\beta(\psi \triangleright (\varphi \triangleright \phi)) = 1$. This can be checked by the *truth-table method*. For (2), consider a specific counterexample that $\ominus(p \triangleright (\ominus p \triangleright q)$ is not equivalent to $\ominus(\ominus p \triangleright (p \triangleright q)$ by the truth-table method again. $\square$

In the following, we define some necessary concepts we will use in the discussion of Béziau's translation paradox.

Definition 13. A function $f : X \to Y$ is **injective (one-to-one)** if for any $x, y \in X$, $f(x) = f(y)$ implies $x = y$. A function $f : X \to Y$ is **surjective (onto)** if for any $y \in Y$, there is a $x \in X$ such that $y = f(x)$. A function is **bijective (one-to-one and onto)** if and only if it is both injective and surjective.

Take f as an *bijection* between the atomic formulas of $\mathcal{A}$ and the atomic formulas of $\mathcal{B}$. Given two algebras *absolutely free algebras* $\langle \mathcal{A}, \rhd, \ominus \rangle$ and $\langle \mathcal{B}, \rightarrow, \neg \rangle$, there is a unique extension g of f which is an isomorphism up to g between these two *absolutely free algebras*, i.e., $g(a \rhd b) = g(a) \rightarrow g(b)$ and $g(\ominus a) = \neg(g(a))$, such that $T \models_{Classical/2} \varphi$ implies $g(T) \models_{Classical} g(\varphi)$. Here g is called a **language-isomorphism**.

Consider an atomic formula a and the bivaluation $\beta \in \mathbf{BIV}$, $\beta(a) = 0$, $\beta(\ominus a) = 0$ and $\beta(\varphi) = 0$, where $\varphi = ((\ominus a \rhd a) \rhd a)$, such that $\not\models_{Classical/2} \varphi$ in $\mathcal{L}_{Classical/2}$. It is known that $((\neg a \rightarrow a) \rightarrow a)$ is a tautology in $\mathcal{L}_{Classical}$, i.e. $\models_{Classical} \varphi$ in $\mathcal{L}_{Classical}$. Hence, $g(T) \models_{Classical} g(\varphi)$ does not imply $T \models_{Classical/2} \varphi$.

Here, we call the logic $\mathcal{L}_{Classical/2}$ is **strictly included**, up to language-isomorphism, in the logic $\mathcal{L}_{Classical}$ in the sense that the relation $\models_{Classical/2}$ is **strictly included** in the relation $\models_{Classical}$. This is explained as follows: ([41], p. 147)

> "[...] it seems that we can say that the logic $\mathcal{L}_{Classical/2}$ is **strictly weaker** than the logic $\mathcal{L}_{Classical}$". One might want to interpret this fact by saying that $\mathcal{L}_{Classical/2}$ is a proper **sublogic** of $\mathcal{L}_{Classical}$ [...]"

Apart from the presentation of these two logics, this is an attempt to interpret the fact that the $\mathcal{L}_{Classical/2}$ is **strictly weaker** than the logic $\mathcal{L}_{Classical}$ as the former is the proper **sub-logic** of the latter to make the paradoxical situation via the understanding of the contained-relation of deductive sense to sub-logic.

3.2.2   The Translation Relation

As a prelude to study Béziau's translation paradox, we set up the general idea of translation between logics, as well as discussing the historical perspective on translation between classical logic and intuitionistic logic.

Definition 14. Given two logics $K_1 = \langle \mathcal{A}, \models_1 \rangle$ and $K_2 = \langle \mathcal{A}', \models_2 \rangle$, K_1, K_2 have the set Σ_1 and Σ_2 (of propositional symbols) as signatures, and a function $\rho : \Sigma_1 \rightarrow \Sigma_2$ between such sets as a signature morphism. A Σ-model M is a mapping from Σ to $\{true, false\}$. α_Σ is a sentence translation

function from the Σ_1-sentence to Σ_2-sentences, and γ is a model translation function from K_2-models to K_1-models, such that $M_2 \models_2 \alpha(\varphi_1)$ if and only if $\gamma(M_2) \models_1 \varphi_1$ holds for any $\varphi_1 \in \mathcal{A}$ and any $M_2 \in K_2$-model.

One can say that the translation from $\mathcal{L}_{Classical}$ to $\mathcal{L}_{Classical/2}$ is as follows:

The logic $\mathcal{L}_{Classical} = \langle \mathcal{S}, \rightarrow, \neg, \models \rangle$ and $\mathcal{L}_{Classical/2} = \langle \mathcal{S}', \rhd, \ominus, \models_{1/2} \rangle$ have sets Σ and Σ' of propositional symbols as signatures, respectively. $\mathcal{L}_{Classical}$-sentences are built from Σ with the propositional connectives $\neg$ and $\rightarrow$, and $\mathcal{L}_{Classical/2}$-sentences are built from Σ with the propositional connectives $\ominus$ and $\rhd$. Take the function Φ from Σ to Σ' as the translation of signature, and the function ρ as the translation of sentences from $\mathcal{L}_{Classical}$-sentences to $\mathcal{L}_{Classical/2}$-sentences as follows:

1. $\rho(p) = p$, for any atomic formula p

2. $\rho(p \rightarrow q) = \rho(p) \rhd \rho(q)$

3. $\rho(\neg p) = \rho(p) \rhd \ominus \rho(p)$,

such that the model translation γ along the Φ make the following hold: for any $\varphi \in \mathcal{S}$ and any $M_{1/2} \in \mathcal{L}_{Classical/2}$-model, $M_{1/2} \models_{Classical_{1/2}} \rho(\varphi)$ if and only if $\gamma(M_{1/2}) \models_{Classical} \varphi$.

We can not get a similar translation from $\mathcal{L}_{Classical/2}$ to $\mathcal{L}_{Classical}$. The sentences of $\mathcal{L}_{Classical/2}$ are the same as in $\mathcal{L}_{Classical}$ but the models are valuations of all sentences that respect the truth-table semantics of the implication $\rhd$ and the negation $\ominus$, which is only with half of the condition in $\mathcal{L}_{Classical/2}$:

- $\varphi \rhd \psi = 1$ if and only if $\varphi = 0$ or $\psi = 1$

- $\ominus \varphi = 0$, if $\varphi = 1$.

(Béziau's Translation Paradox)

As previously discussed, $\models_{Classical/2}$ is strictly included (up to language-isomorphism g) in the relation $\models_{Classical}$. This might imply that $\mathcal{L}_{Classical/2}$ is strictly weaker (up to language-isomorphism g) than $\mathcal{L}_{Classical}$. Here, the immediate connection to the idea of translation is perhaps that $\mathcal{L}_{Classical}$ is thought of as **translatable** to $\mathcal{L}_{Classical/2}$ (but not vice versa). Thus, for

two logics $\mathcal{L}_{Classical}$ and $\mathcal{L}_{Classical/2}$, the situation becomes that $\mathcal{L}_{Classical/2}$ is strictly included into $\mathcal{L}_{Classical}$ which suggests the $\mathcal{L}_{Classical/2}$ is strictly weaker than $\mathcal{L}_{Classical}$, but $\mathcal{L}_{Classical}$ is translatable into $\mathcal{L}_{Classical/2}$. In other words, while $\mathcal{L}_{Classical}$ "specify a copy" of the $\mathcal{L}_{Classical/2}$, $\mathcal{L}_{Classical/2}$ should be at least strong as $\mathcal{L}_{Classical}$. The $\mathcal{L}_{Classical}$ is a "sub-logic" of the $\mathcal{L}_{Classical/2}$ in the sense of being translatable.

The so-called **Béziau's translation paradox** (compare e.g. [127], [128], [129]) actually originated from a quite similar situation that has already been discussed about the development of intuitionistic logic. Graham Priest stated:

> " [...] If an inference is intuitionistically valid, it is therefore classically valid (when $\rightharpoondown$ and $\sqsupset$ are replaced with $\neg$ and $\supset$, respectively). The converse is not true, as we shall see. Hence, intuitionist logic is a sub-logic of classical logic [...] This is not true of intuitionist mathematics in general. Intuitionist mathematics endorses some mathematical principles which are not endorsed in classical mathematics; in fact, they are inconsistent classically. But because intuitionist logic is weaker than classical logic, the principles are intuitionistically consistent. For the record, it is worth noting that there is a certain way of seeing classical logic as a part of intuitionist logic too [...]" ([171], p. 107)

It is this "certain way" that makes the phenomenon philosophically arguable, say, a weaker logic is a "sub-logic" of a stronger logic but the latter is "contained" in the former, which suggests the former is at least as strong as the latter.

3.3 The Development of Logic Translations within the Abstract Logic Tradition

In order to understand logic translation better, we discuss this in further detail. An abstract logical perspective runs between different ideas of translation, ranging from the "rough" to the "rigorous". Béziau's statement is a very specific case to provide an approach to examine the relationship between classical logic and intuitionistic logic from an abstract logical point of view. For example, we do not need to know much about Glivenko's theorem

and what principles intuitionistic logic endorses or not, we only need to have basic knowledge about some version of CPL.[1]

3.3.1 Consequence Relations and Logic Translation

Most ideas about logic translation in literature were founded on the discussion of Tarskian logic. Some background on Tarskian logic has already been mentioned in the previous chapter, but we further offer some definitions to gain a deeper understanding of logic translation. First, we consider a *consequence relation* $\vdash$ to define a logical structure $\mathcal{L} = \langle \mathcal{F}, \vdash \rangle$. Second, we consider a *consequence operator* Cn to define a logical structure $\mathcal{L} = \langle \mathcal{F}, Cn \rangle$. The idea of logic translation, in this way, can be traced back to different sources by these two conceptions of logical structure. Recalling the conception of Tarskian logic, an arbitrary logical structure is said Tarskian when it obeys reflexivity, monotonicity, and cut. Now, we explore various ideas on translation (also see, [65], [76], [84], [223]).

Definition 15. (Rough Translation) A *translation* from one logic $\mathcal{L}_1$ into logic $\mathcal{L}_2$ is defined as mapping: $f : \mathcal{L}_1 \rightarrow \mathcal{L}_2$, that is to map the set of formulas in $\mathcal{L}_1$ to the set of formulas in $\mathcal{L}_2$, *such that for any formula φ, if φ is a theorem of $\mathcal{L}_1$, then $f(\varphi)$ is a theorem of $\mathcal{L}_2$.*

Definition 16. (Revised Rough Translation (i)) Follow **Rough Translation** with a stronger condition:

(a) For any formula φ, φ is a theorem of $\mathcal{L}_1$ *iff* $f(\varphi)$ is a theorem of $\mathcal{L}_2$.

Definition 17. (Revised Rough Translation (ii)) Follow **Rough Translation** with another stronger condition:

(b) For any set of formulas Γ, formula φ, if $\Gamma \vdash_{\mathcal{L}_1} \varphi$ then $f(\Gamma) \vdash_{\mathcal{L}_2} f(\varphi)$.

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[1]Readers might argue that we still need to have knowledge on bivaluation and abstract logic. This is definitely debatable in methodology. But we should realize that understanding "abstract" logic is not harder than learning CPL, moreover, "bivaluation" is as simple as the way we assign truth value for formulas in CPL. In addition, intuitionistic logic, as a deviant logic of classical logic (see, subsection 3.5.2), was born later than classical logic and is seen as a rival of classical logic.

Definition 18. (**Conservative Translation**) Follow **Revised Rough Translation (ii)** with a stronger condition than (b):

(c) For any set of formulas Γ, formula φ, $\Gamma \vdash_{\mathcal{L}_1} \varphi$ *iff* $f(\Gamma) \vdash_{\mathcal{L}_2} f(\varphi)$.

The more noteworthy idea on translation is *schematic mapping* (compare e.g. [65], [76], [84], [223]). Schematic mapping relies on **homomorphism** among formal languages, and involves some *diagrammatic* representation of the structures of language expressions. Here, "some" preserved diagrammatic representations are in a sense *algebraic*, which is in tune with Tarski's paradise.

Definition 19. (Schematic Translation) Let two formal languages L_1 and L_2 with only unary ♠ and binary connectives II be given. If for any atomic formulae $a_0, a_1, \ldots, a_n, \cdots \in L_1$, there are *schemata of formulae A*, $B_{♠}$, $C_{\text{II}} \in L_2$ such that the mapping $r : L_1 \mapsto L_2$ satisfies the following conditions, then r is a *schematic mapping*.

(1) $r(a) = A(a)$, for every atomic formula $a \in L_1$,

(2) $r(♠\varphi) = B_{♠}(r(\varphi))$, for every unary connective ♠ and formula φ of L_1,

(3) $r(\text{II}(\varphi, \psi)) = C_{\text{II}}(r(\varphi), r(\psi))$, for every binary connective II and formula φ, ψ of L_1.

Definition 15-18 with *schematic mapping* are *schematic translation*.

3.4 The Systematic Discussion of Logic Translations

As mentioned in the introduction, in the development of logic translation and abstract logic, three systematic discussions arise: firstly, the work of Prawitz and Malmnäs in the article *A survey of some connections between classical, intuitionistic and minimal logic* (1968) ([169]), notably as primary systematic discussion in the literature, which defines the term "translation" at a general level; secondly, the Ryszard Wójcicki's discussions which adopt the abstract logical perspective fashioned in the Polish school of thought ([223]); thirdly, the Richard Epstein's which focus on the translations between propositional

logics ([76]) and the translation within predicate logic ([74]). These systematic discussions and integrations relied on early work on translation, such as ([133], Kolmogorov, 1925), ([107], Glivenko, 1929), ([109], Gödel, 1933), ([106], Gentzen, 1933), and ([52], [53], Bloom, Brown, Suszko, 1973). We will discuss two lines of thought on logic translation which have been the focus of a large amount of literature.

3.4.1 History: $1968 - 1933 - 1929 - 1925$

The first systematic definition of the term "translation" was discussed in ([169], Prawitz and Malmnäs, 1968) as follows:

(a) Consider two logical systems S_1, S_2, an interpretation[2] from S_1 to S_2 is a mapping t from formulas of S_1 to S_2 such that for any formula φ, $\vdash_{S_1} \varphi$ iff $\vdash_{S_2} t(\varphi)$.

(b) For each set $\Gamma \cup \{\varphi\}$ of formulas in S_1, $\Gamma \vdash_{S_1} \varphi$ iff $t(\Gamma) \vdash_{S_2} t(\varphi)$ where $t(\Gamma)$ is the set of replacing all elements ψ of Γ by $t(\psi)$.

The idea in (a) is the same as the **revised rough translation (i)** (Definition 16). With regard to (a), we can say that S_1 is *interpretable* into S_2, and also that S_1 is *interpretable* into S_2 *with respect to derivability*. Note here that (b) does coincide with the **conservative translation** (Definition 18) that was studied later in the literature (see e.g. [60], [84], [151]). As Mossakowski et al. stated: "Prawitz and Malmnäs also use a more permissive notion of conservative translation where the equivalence is only required for $\Gamma = \emptyset$" ([151], p. 98). This amounts to Definition 16. Prawitz and Malmnäs also describe the idea of *schematically interpretable* as mentioned in Definition 19.

These works shown by Prawitz and Malmnäs (1968) is the first survey paper on the early work on translation, including ([133], Kolmogorov, 1925), ([107], Glivenko, 1929), ([109], Gödel, 1933), and ([106], Gentzen, 1933). We provide the gist of these papers without delving into the details, after which, we proceed with our main discussion. These earlier papers focused mainly on the problem of *consistency* and the relation between classical logic and intuitionistic logic. To begin with, let us examine Kolmogorov's idea on translation in [133] reflected in the following quotation:

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[2]Specifically, they used the term "interpretation", whereas we use the term "translation".

"The main purpose of this paper is to prove that classical mathematics is translatable into intuitionistic mathematics. For this purpose, with each formula $\Im$ of mathematics there is associated a translation $\Im^*$ in a perfectly general manner (IV, § 2)." ([133], pp. 414–415)

Note here that we provide Hao Wang's introduction before the English translation of this article; however, in the translation, we replace outdated terms with modern terms.

(Kolmogorov, 1925)

For $\mathcal{H}$ (Hilbert's formal system of CPL) and $\mathcal{B}$ (Brouwer's formal system of IPL), there is a translation $*$ from $\mathcal{H}$ to $\mathcal{B}$ such that for any atomic formula p there is a corresponding formula $(p)^*$ which expresses the double negation of p, denoted as $\neg\neg p$ and formulas $(\neg\varphi)^*$, $(\varphi \to \psi)^*$ are defined as $\neg\neg(\neg\varphi^*)$, $\neg\neg(\varphi^* \to \psi^*)$, respectively.

Theorem 20. *If $\Gamma = \{\Gamma_1, \ldots, \Gamma_n\}$ is a set of axioms in $\mathcal{H}$ and $\Gamma^* = \{\Gamma_1^*, \ldots, \Gamma_n^*\}$, then for any formula φ, $\Gamma \vdash_{\mathcal{H}} \varphi$ implies $\Gamma^* \vdash_{\mathcal{B}} \varphi^*$.*

Theorem 21. *(Glivenko's Theorem, 1929) (Glivenko's translation) An arbitrary propositional formula φ is a theorem of the CPL, i.e. classically provable if and only if $\neg\neg\varphi$ is a theorem of IPL, i.e. intuitionistically provable.*

(Gödel, 1933)

For a system of CPL $\mathcal{A}$, and a system of IPL $\mathcal{H}'$, the translation $\star$ from $\mathcal{A}$ to $\mathcal{H}'$ is defined as follows:

(1) $\varphi^\star =_{df} \varphi$

(2) $(\neg\varphi)^\star =_{df} \neg\varphi^\star$

(3) $(\varphi \wedge \psi)^\star =_{df} \varphi^\star \wedge \psi^\star$

(4) $(\varphi \vee \psi)^\star =_{df} \neg(\neg\varphi^\star \wedge \neg\psi^\star)$

(5) $(\varphi \to \psi)^\star =_{df} \neg(\varphi^\star \wedge \neg\psi^\star)$, for every atomic formula φ, ψ

Theorem 22. $\vdash_{\mathcal{A}} \varphi$ *implies* $\vdash_{\mathcal{H}'} \varphi^\star$.

(Gentzen, 1933)

For a system of CPL $\mathcal{A}$, a system of IPL $\mathcal{I}$, the translation $\blacklozenge$ from $\mathcal{A}$ to $\mathcal{I}$ is defined as follows:

(1') $\varphi^{\blacklozenge} =_{df} \neg\neg\varphi$

(2') $(\neg\varphi)^{\blacklozenge} =_{df} \neg\varphi^{\blacklozenge}$

(3') $(\varphi \wedge \psi)^{\blacklozenge} =_{df} \varphi^{\blacklozenge} \wedge \psi^{\blacklozenge}$

(4') $(\varphi \vee \psi)^{\blacklozenge} =_{df} \neg(\neg\varphi^{\blacklozenge} \wedge \neg\psi^{\blacklozenge})$

(5') $(\varphi \rightarrow \psi)^{\blacklozenge} =_{df} \varphi^{\blacklozenge} \rightarrow \psi^{\blacklozenge}$

Theorem 23. *$\Gamma \vdash_{\mathcal{A}} \varphi$ if and only if $\Gamma^{\blacklozenge} \vdash_{\mathcal{I}} \varphi^{\blacklozenge}$.*

Remark 24. Both Kolmogorov and Gödel's ideas on translation are special cases of Definition 16, and Kolmogorov's idea on translation also satisfies Definition 17. We see that there is a translation $*$ or $\star$ that is considered between two formal systems $\mathcal{H}$ (Hilbert's formalization of propositional calculus) and $\mathcal{B}$ or $\mathcal{H}'$ (Brouwer's or Heyting's formalization of propositional calculus). Gentzen's idea on translation is a *conservative translation* (Definition 18) which indicates the importance of direction, denoted as "$\Leftrightarrow$". Moreover, Gentzen translates *implication*, denoted as "$\rightarrow$" directly rather than in terms of "$\neg$" and "$\wedge$" (Gödel).

3.4.2   History: 1988 − 1973 − 1971 −

The second systematic research on translation was conducted by Ryszard Wójcicki. Wójcicki used on the conception of consequence operators to develop his idea about logic translation, by firstly considering *language translation*, that is, "the map" between languages, and secondly by "preserving" the consequence operators. In other words, it is a *derivability preserving* schematic translation. His idea can be viewed as a systematic study of logic translation from an abstract logical point of view, which can be traced back to Tarski's idea of logical consequence (compare e.g. [65], [84], [210], [223]). Before continuing, we should consider conducted research made by Brown-Bloom-Suszko on *abstract logic* ([52], [53], Bloom, Brown, Suszko, 1973). Brown-Bloom-Suszko's idea is considered to be a pioneering point of view on

abstract logic, which enhances an understanding of abstract logic from Polish school of thought.

Definition 25. (Bloom-Brown-Suszko, 1971, 1973) The objects of the *category of abstract logics* are ordered pairs $\langle \mathcal{S}, Cn \rangle$ consisting of an *abstract algebra* $\mathcal{S}$ and a *closure operator* C_n on $S = |\mathcal{S}|$, the carrier (universe) of $\mathcal{S}$. If S is a non-empty set, Cn is a closure operator on S then $\langle S, Cn \rangle$ is called a *closure space*.

Definition 26. Let $K_1 = \langle S_1, Cn_1 \rangle$, $K_2 = \langle S_2, Cn_2 \rangle$ be two closure spaces. A *mapping* f from S_1 to S_2 is said to be *continuous* if $f^{-1}(Z) \in Cn_1$, for all $Z \in Cn_2$. Here $f^{-1}(Z)$ is the inverse image of Z under f. The set of all continuous maps of K_1 into K_2 is denoted as $Hom(K_1, K_2)$. If both f and its inverse image $\breve{f}$ are continuous, i.e., $f \in Hom(K_1, K_2)$ and $f^{-1}(Z) \in Hom(K_1, K_2)$, then a bijective map $f : S_1 \to S_2$ is called a *homeomorphism* between K_1 and K_2.

Note here that Brown-Bloom-Suzuko's idea on *abstract logic* is a *category-theoretical* viewpoint of logic inspired by topology. Naturally, they consider a *logical morphism* as the translation between two abstract logics from a topological point of view.

Adopting the position of abstract logic taken in the Polish school of thought with respect to consequence operations, Wójcicki's systematic study on the logic translation is as follows:

(Wójcicki, 1988)

Given two propositional languages S_1, S_2 with the same variables, a mapping $t : S_1 \mapsto S_2$ is a translation from S_1 to S_2 iff two conditions are satisfied:

(i) There is a formula $\varphi(p_0) \in S_2$ in one variable p_0 such that for each variable p, $t(p) = \varphi(p)$.

(ii) For each connective ρ_i of S_1 there is a formula $\varphi_i \in S_2$ such that for all terms $\alpha_1, \ldots, \alpha_k \in S_1$, k being the arity of ρ_i, we have that

$$t(\rho_i(\alpha_1, \ldots, \alpha_k)) = \varphi_i(t\alpha_1/p_1, \ldots, t\alpha_k/p_k).$$

For two propositional calculi $\mathcal{C}_1 = (S_1, C_{n_1}), \mathcal{C}_2 = (S_2, C_{n_2})$, if there is a translation t from S_1 into S_2 such that for all $X \subseteq S_1$ and all $\alpha \in S_1$,

$$\alpha \in C_{n_1}(X) \Leftrightarrow t\alpha \in C_{n_2}(t(X))$$

then $\mathcal{C}_1$ has a translation in $\mathcal{C}_2$.

3.4.3 History: $2006 - 1990 -$

The third systematic research on *translation* was conducted by Richard Epstein. Epstein's study on translation is divided into two parts, one to study a general idea on translation between propositional logics ([76], Epstein, 1990) and the other to study the translation *within* classical predicate logic ([74], Epstein, 2006). Epstein's statements regarding *translation* can be considered at two different levels— the level of propositional logic and that of predicate logic level.

Definition 27. (Epstein, 1990) A *validity mapping* of a propositional logic $\mathcal{L}_1$ into a propositional logic $\mathcal{L}_2$ is a map t from language of $\mathcal{L}_1$ to language $\mathcal{L}_2$ such that for every φ,

$$\models_{\mathcal{L}_1} \varphi \Leftrightarrow \models_{\mathcal{L}_2} t(\varphi).$$

Definition 28. (Epstein, 1990) For any theory Γ, formula φ, if the mapping relation from t to $t(\Gamma) = \{t(\varphi) : \varphi \in \Gamma\}$, such that

$$\Gamma \models_{\mathcal{L}_1} \varphi \Leftrightarrow t(\Gamma) \models_{\mathcal{L}_2} t(\varphi)$$

then this mapping relation is a *translation* from logic $\mathcal{L}_1$ to logic $\mathcal{L}_2$, denoted as $\mathcal{L}_1 \hookrightarrow \mathcal{L}_2$.

Definition 29. (Epstein, 2006) Given two theories $\mathbf{T}, \mathbf{R}$ in first-order classical predicate logic and a mapping t from the language of $\mathbf{T}$ to the language of $\mathbf{R}$.

1. t is *validity-preserving* iff for every φ, $\models_{\mathbf{T}} \varphi$ iff $\models_{\mathbf{R}} \varphi$.

2. t is a *translation* of $\mathbf{T}$ into $\mathbf{R}$ iff for every Γ and φ, $\Gamma \models_{\mathbf{T}} \varphi$ iff $t(\Gamma) \models_{\mathbf{R}} t(\varphi)$.

Note here that the term "translation" in ([76], Epstein, 1990) is used to discuss the translation between two logics, $\mathcal{L}_1$ and $\mathcal{L}_2$. In ([74], Epstein, 2006), the term "translation" is used to discuss two different theories, **T** and **R**, both of which are in the same first-order logic. To explain the motivation to consider translation between two theories, Esptein states:

> "[...] Mathematicians typically conceive of reductions of one theory to another as transforming models of one to models of the other, rather than as linguistic mappings." ([74], p. 405)

Recall some basic notions: given a theory Σ and a class of models $\mathcal{S}$. $Th(\mathcal{S}) = \{\varphi \mid$ for every $\mathcal{M}$ in $\mathcal{S}$, $\mathcal{M} \models \varphi\}$. The idea of **model-preserving mapping** as follows:

(Epstein, 2006)

Let $\mathbf{T} = Th(\mathcal{T})$ and $\mathbf{R} = Th(\mathcal{R})$ be two theories with respect to classes of models $\mathcal{T}, \mathcal{R}$ in classical predicate logic, τ is a mapping from language of **T** to language of **R**. If an onto mapping τ from $\mathcal{R}$ to $\mathcal{T}$ such that for every φ in language of **T**, every model $\mathcal{M}$ in $\mathcal{R}$, the satisfaction condition: $\tau(\mathcal{M}) \models \varphi$ iff $\mathcal{M} \models \tau(\varphi)$ holds, then τ is a model-preserving mapping with respect to $\mathcal{T}$ and $\mathcal{R}$.

Theorem 30. *(Epstein, 2006) Every model-preserving mapping is a translation.*

Proof. Assume τ is a model-preserving mapping from **T** to **R** with respect to classes of models $\mathcal{T}, \mathcal{R}$, then we claim τ is also a translation, i.e., we prove $\Gamma \models_\mathbf{T} \varphi$ if and only if $t(\Gamma) \models_\mathbf{R} \tau(\varphi)$ holds.

$\Gamma \models_\mathbf{T} \varphi$
$\iff$ For any model $\mathcal{M} \in \mathcal{T}$, $\mathcal{M} \models \Gamma$ implies $\mathcal{M} \models \varphi$.
$\iff$ For any model $\mathcal{N} \in \mathcal{R}$, $\tau(\mathcal{N}) \models \Gamma$ implies $\tau(\mathcal{N}) \models \varphi$, since τ is onto.
$\iff$ For any model $\mathcal{N} \in \mathcal{R}$, $\mathcal{N} \models \tau(\Gamma)$ implies $\mathcal{N} \models \tau(\varphi)$, since τ is model-preserving.
$\iff$ $\tau(\Gamma) \models_\mathbf{R} \tau(\varphi)$. $\qquad\qquad\square$

So far, we have seen many different ideas about logic translation in the literature. These ideas can roughly be classified into two periods:

(i) Kolmogorov-Glivenko-Gödel-Gentzen to Esptein-Wójcicki period (**KGGG-EW**):This period focuses on logical relation $\vdash$.

(ii) Bloom-Brown-Suszko to *Brazilian*-Esptein-Wójcicki period (**BBS-BEW**): This period focuses on consequence relation Cn.

It is worth mentioning the difference between the two, since it shows that logic has begun to be viewed as a finitary consequence operator in period (ii) and it also highlights the trend of seeing logic *in general* or considering *abstract logic*, as Bloom-Brown-Suszko did. Research conducted in the (**BBS-*B*EW**) period, from the abstract logic viewpoint, is concerned about "consequence operator Cn". Logics is characterized as sets with *consequence operator Cn*, and translation as *continuous functions* between Cn. Although research in the **KGGG-EW** period was also conducted from the abstract logic viewpoint, it differs in that it discusses the concept of logic translation by considering abstract logical structures instead of studying translation of individual logics directly.

3.5 Logic Translation and the Deviance of Logics

Traditionally, investigations into logic translation mainly focused on the relationship between classical logic and intuitionistic logic. The study of logic translation can be carried out to a large extent independently of the discussions of these two logics. Apart from the concerns of these fundamental investigations, some researches into logic translation can be found (e.g. [137], [112], [150]). It is true that, at certain times in the twentieth century, various logicians have endorsed intuitionistic logic. This is particularly true of logicians in the realm of the philosophy of mathematics and logics. Kurt Gödel, for example, studied the translation from IPL to modal system $\mathcal{G}$ ([109]) which is a quite different idea from the discussion in previous sections. However it is fair to say that, at least since the discussions of various non-classical logics in philosophy and theoretical computer sciences, little attention has been paid on the classification of non-classical logics. Hence, we attempt to have a broader discussion than that of logic translation between classical logic and intuitionistic logic. In this section, we will attempt to relate the classification to the study of logic translation by means of a similar discussion on the deviance of logics by Susan Haack (see [116], [118]).

3.5.1 From Expansion/Reduct Cases to Deviant Cases

First of all, we make a distinction between two concepts: expansion and extension.

Given two structures S_1, S_2, S_1 is a structure equipped with a domain $\mathcal{D}_1$ and a certain type of this structure T_1, and S_2 is a structure equipped with a domain $\mathcal{D}_2$ and a certain type of this structure T_2, denoted as $S_1 = \langle \mathcal{D}_1; T_1 \rangle$, $S_2 = \langle \mathcal{D}_2; T_2 \rangle$, respectively.

(i) S_1 is an *expansion* of S_2 or we can say S_2 is *reduct* of S_1 if and only if (1) $D_1 = D_2$ and (2) the type of S_1 is included into S_2, denoted as $T_1 \subseteq T_2$.

(ii) S_1 is an *extension* of S_2 or we say S_2 is a *substructure* of S_1 if and only if (1) $D1 \subseteq D_2$ and (2) $T_1 = T_2$

In the following two examples:

- $\mathcal{Z} = \langle \mathbb{Z}, +, \times, \leq \rangle$ is an extension of $\mathcal{N} = \langle \mathbb{N}, +, \times, \leq \rangle$.

- $\mathcal{K} = \langle \mathbb{K}, \neg, \wedge, \vee, \rightarrow, \vdash \rangle$ is an expansion of $\mathcal{K} = \langle \mathbb{K}, \neg, \wedge, \rightarrow, \vdash \rangle$.

Recalling the definitions of translation, the concept of logic translation might be intuitively defined as the *map preserving consequence relations or consequence operators.* Here, the "source" is selected in translation; then we consider the "target". First of all, given a formal language for the source, we consider its *expansion* by adding a logical constant or its *reduct* by excluding some logical constant for the target. For example, given a formal language FPL for CPL as the *source* language, where $A = \{p_n\}_{n<\omega}$ is a countable collection of propositional atoms and P is the set of propositional formulae created from A by $\neg$, $\wedge$, $\vee$, and $\rightarrow$. Now, if we consider a expansion of FPL by adding $\Box$, $\Diamond$, then P should also expand to be MP that is built from $\{\neg, \wedge, \vee, \rightarrow\} \cup \{\Box, \Diamond\}$; if we consider a reduct of FPL by excluding $\neg$, then P should also reduce to be P^- that is built from $\{\wedge, \vee, \rightarrow\}$. MP is the formal language for modal propositional logic and P^- is the formal language for *positive classical propositional logic (PCPL) (i.e., without negation).*

After determining the structure of the two languages, we consider any potential to conduct translations between the logics described by the source languages and the target languages. Logic translations in this sense are

quite different from the translation between classical logic and intuitionistic logic that we have discussed. A famous example signifies this sort of logic translation as follows:

(Gödel, 1933)

Gödel provides a translation from IPL to his "modal" system $\mathcal{G}$ ([109], p. 301). which is an example of expanding formal language. The modal system $\mathcal{G}$ is an expansion of IPL by adding logical constant B, which is interpreted as standing "provable". Bp means "p is provable". Except axioms of IPL, $\mathcal{G}$ follows other axioms and a new rule of inference.

1. $Bp \rightarrow p$

2. $Bp \rightarrow (B(p \rightarrow q) \rightarrow Bq)$

3. $Bp \rightarrow BBp$

New inference rule: $\frac{A}{BA}$, which means "from A, BA may be inferred." The mapping procedure $\Omega : IPL \rightarrow \mathcal{G}$ is as follows:

(1) $\varphi^{\Omega} =_{df} \varphi$

(2) $(\neg\varphi)^{\Omega} =_{df} \neg B\varphi^{\Omega}$

(3) $(\varphi \wedge \psi)^{\Omega} =_{df} \varphi^{\Omega} \wedge \psi^{\Omega}$

(4) $(\varphi \vee \psi)^{\Omega} =_{df} B\varphi^{\Omega} \vee B\psi^{\Omega}$

(5) $(\varphi \rightarrow \psi)^{\Omega} =_{df} B\varphi^{\Omega} \rightarrow B\psi^{\Omega}$

Theorem 31. $\vdash_{IPL} \varphi$ ***implies*** $\vdash_{\mathcal{G}} \varphi^{\Omega}$.

In Gödel's example, as we saw here, it is assumed that we have a source language (e.g., formal language for IPL), then we expand the languages employed. Now the observation posed by the translation between classical logic and intuitionistic logic, *weakens* the conditions of logical constants in classical logic with the same language. Here, we discuss some *deviant cases*: the translation from CPL to IPL.

3.5.2 Deviance of Logics

As we have seen, discussions of expansions/reducts of languages are mechanical adjustments from a formal language of propositional logic to another formal language; further, only some of the logical terms were excluded from $\{\neg, \vee, \wedge, \rightarrow\}$, for example, positive CPL, or some modal terms were added, namely, to propositional modal logics.

Haack's Deviant Logic (Haack, 1974, 1996)

Although logicians such as Patrick Blackburn and Johan van Benthem considered the **standard translation** of modal language into classical first-order or second-order languages (see [49]), what attracts us is not only the relation of IPL and CPL, but also the relation of other *deviant logics* and CPL, moreover classical first-order logic. Next, we discuss the basic distinction between *extended logics* and *deviant logics* made by Susan Haack (see e.g. [116], [117], [118]). Most related philosophical debates on Haack's discussion are about whether "deviance" is a sufficient condition to *rival* classical logic. Here, we do not discuss these debates. We try to apply this distinction to argue the reliability of the distinction between two concepts: deviant logic translation and extended logic translation. Finally, we discuss the critical opinions proposed by Béziau (in [32]) and what the Universal Logic project expected to do for traditional studies of the philosophy of logic. Recalling Béziau's Translation Paradox, we can see that the method of translation involves adjusting (in this case, weakening) the conditions of logical constants. By Haack's definition, the adjusted logic (weakened logic) in this case is precisely a *deviant logic*. In addition to this, there are other deviant logics, such as paraconsistent logic, relevant logic, free logic, and intuitionistic logic. Consider the following definition by Haack: ([116], p. 4)

(1) "The class of well-formed formula of L_1 and the class of well-formed formula of L_2 coincide, but the class of theorems/valid inferences of L_1 differs from the class of theorems/valid inference of L_2. In this case L_1, L_2 are **deviations** of each other.

(2) The class of well-formed formula of L_1 properly includes the class of well-formed formula of L_2 and the class of theorems/valid inferences of L_1 properly includes the class of theorems/valid inferences of L_2, the

additional theorems/valid inferences of L_1 all containing essentially occurrences of L_1's additional vocabulary. In this case L_1 is an **extension** of L_2."

In other words, for any one of the various non-classical systems of logic, if the set of generated well-formed formulas is equal to that of classical logic but the set of generated theorems is different from generated by classical logic, then we call this non-classical system of logic a "deviant logic". According to Haack, some deviant logics rival the logics that are deviated. We do not debate whether "deviance" is a sufficient or necessary condition for deviant logics "being a rival" to deviated logics. Here, we discuss how all deviant logics (e.g. IPL) disagree in some way with deviated logics (e.g. CPL). For example, paraconsistent logics disagree in some way with classical logic as well as relevance logic, intuitionistic logic, free logic, etc. These represent *case-by-case* differences and there are fewer common disagreements with classical logic. We further promote Haack's definition to the discussion of logic translation, and we try to distinguish different logic translations using Haack's idea of deviant logic. Following the examples of logic translation that we have discussed so far, we can make a distinction in the studies of logic translation using Haack's distinction:

(1) a translation from CPL to IPL is called a *deviant logic translation*, which is between a deviant logic and a deviated logic;

(2) a translation from IPL to modal system $\mathcal{G}$ is called an *extended logic translation*, which is between two logics based on the formal language expansion.

This distinction between deviant logic translation and extended logic translation seems vacuous. However, case (1) discusses two logics in one formal language which generates different well-formed formulae, and case (2) discusses two logics in two formal languages, of which the language expansion generates more well-formed formulae. Many find it difficult to consider that the translation relations in these two cases are of the same kind.

Since we started this section with a discussion of deviant logics, let us finish by relating it to the Universal Logic project. So far, we have seen the second sense of logic translation other than the logic translation between classical logic (deviated logic) and intuitionistic logic (deviant logic). The stimulus to address this distinction is Gödel's example from IPL to modal

system $\mathcal{G}$, with the help of Haack's famous distinction about various non-classical systems of logic. A criticism made by Béziau was that Universal Logic should provide a systematic framework for old ideas in the philosophy of logic. He said that the reasonable distinction made by Haack lacks serious and systematic theory from the Universal Logic point of view as follows:

> "[...] that it doesn't rest upon any serious and systematic theory, but only on some ideas thrown in the air and explained and justified with basic elementary examples." ([32], p. 143).

He suggests that in order to make these distinctions, we need *some new concepts and an entire theory* (Ibid, p. 143) within the framework of Universal Logic. Universal Logic should provide a systematic framework to justify these distinctions with more than only the help of case study examples. Otherwise, as stated by Béziau, the case where CPL is definable within IPL seems to, in some sense, consider IPL an extension of CPL ([32], p. 143), making Haack's distinction unjustifiable. In the same manner, Universal Logic is expected to propose a systematic study on the concepts of logic translation, which will help us better understand the philosophical concepts of Haack's distinction, as well as enhance our understanding of the whole picture of logic translation investigations.

3.6   Conclusion

Traditionally speaking, intuitionistic logic was seen as a sublogic of classical logic, however, the studies on logic translation, especially the phenomenon of the translation paradox, are not clear enough for us to tell which one is stronger or weaker, which one should be seen as an extension of the other, or which one is indeed translatable to the other. As per the distinction made by us in this chapter, the translation from CPL to IPL is concerned with the translation of deviant logics; the translation from IPL to $\mathcal{G}$ is concerned with the translation between extended logics and the original logics. The notion of deviant translation could help us understand the relationship of deviated logic and deviant logic. For example, intuitionistic logic is a deviant of classical logic, and classical logic is a deviant of paraconsistent logic.[3]

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[3]Intuitionistic logic is seen as target logic and classical logic is seen as source logic in the former case; classical logic is seen as target logic and paraconsistent logic is seen as source logic in the latter case.

In other words, there is a deviant relation between intuitionistic logic and classical logic, and there is, as well, another deviant relation between classical logic and paraconsistent logic. However, there is no deviant relation between intuitionistic logic and paraconsistent logic in this way, although there might be a certain deviant relation between intuitionistic logic and paraconsistent intuitionistic logic, where the latter is a deviant of the former. "Deviance" always plays the key role.

The prima facie meaning of sub-super logic is intuitively realized as the subset (superset) in set theory; that is, logic $\mathcal{L}_1$ is a sublogic of logic $\mathcal{L}_2$ which means that "something" of $\mathcal{L}_1$ is weaker/less than (or contained in) $\mathcal{L}_2$. Nevertheless, we realize that sublogic in the aforementioned sense is too vague. In this chapter, we have discussed some clear meanings that support the use of this concept. First, the concept of sublogic should be distinct from that of *the strength of logics*. Secondly, the definition of the concept of sublogic should be clarified, that is, the relation of "weaker than" should be clarified and be distinct from the relation of sublogic. Moreover, the relation of "sublogic" should be treated in a rigorous way. We suggest that the clarification of the meaning of *sublogic* and that of the *strength* of logics should be within the framework of Logic Translation. It is worth noting that the clarification between strength of logics and sublogic should be about deviant logic translation, since the adjustment of logical principles of classical logic causes the birth of deviant logics, moreover the birth of deviant logic translation.

Chapter 4

Pluralism in Logic

Contents

4.1 Introduction

Just as one may ask what insight might be like if logic translation could be addressed as the new subject in the logical community, so one may ask what it must be like if logical pluralism could be treated seriously in the philosophy of logic. This chapter discusses pluralism in logic, a subject made fashionable through the work of J.C. Beall and Greg Restall. Pluralism in logic can be traced back to ideas raised by Ruldof Carnap (1934). In this chapter, we take up not only the stances of pluralism expressed by Beall and Restall's logical pluralism (BRLP), but we also ask what monism must be like if BRLP is regarded as a sort of pluralism. This chapter is in some sense quite independent from the rest of the dissertation, however, it should be treated as a philosophical and fundamental study to the series of logic translation investigations. In this way, we will better understand the role that pluralism could play in the Universal Logic project.

   This chapter addresses philosophical questions related to logic translation, namely it aims to substantiate the fundamental part of logic translation investigations by answering the following three basic questions: (1) What is logical pluralism?; (2) Is it correct only to claim the singularity of classical logic as the only one true logic?; (3) Is there a universality of the mode of cognitive processes? To answer these questions, first of all, we take BRLP as the modern landmark for pluralism usually discussed in logic. Then, we endorse Rudolf Carnap's principle of tolerance as the central dogma to the concept of pluralism in logic, i.e., plurality in various logics within one dogma. Secondly, to structure and discuss BRLP in a critical manner, we base our work on the style of modern analytic philosophy in order to challenge the concept that BRLP does not capture the intuitive concept of plurality and pluralism, like some examples that have been forcefully argued by John Etchemendy on the concept of logical consequence ([77], [78], [79]) and Kurt Gödel ([75], pp. 228–229). In particular, in this chapter, we distinguish the scope of pluralism and monism in logic and some different kinds of formulations will be derived from BRLP.

   Moreover, we employ the cultural psychological perspective to formulate the revision of "logical" pluralism under the assumption of taking "logic as cognitive process". We show how the notion of plurality can be understood to benefit both philosophers and computer scientists ([137], [150]). In the next chapter, we will show how different modes of cognitive processes over logic translation are related to this philosophical proposal to connect the next chapter in order to answer the third basic question.

4.2   Truth-Preservation and GTT

In logic, there is a long tradition of attempting to provide a criterion to the validity of various arguments by the concept of "truth-preservation" and the concept of logical consequence. This was first proposed by Beall and Restall in 1999 (see [12], [13]) in what they call the **GTT(Generalized Tarski Thesis)**.

   (**GTT**): An argument is valid_x if and only if, in every case_x in which the premises are true, so is the conclusion.
BRLP is a pluralism about the concept of logical consequence. That is to say, "there is more than one relation of logical consequence." ([10], p. 25)

In the philosophy of logic, the idea of logical pluralism does not have a long history. It can be traced back to Carnap's principle of tolerance Ruldof Carnap, in '*The logical syntax of language*'([59]), put forward **the principle of tolerance** as follows:

> "It is not our business to set up prohibitions, but to arrive at conventions. ...In logic, there are no morals. Everyone is at liberty to build his own logic, i.e. his own form of language, as he wishes. All that is required of him is that, if he wishes to discuss it, he must state his methods clearly, and give syntactical rules instead of philosophical arguments." ([59],§17)

The idea of logical pluralism has been debated heavily within the intersection of philosophy and logic (compare [10], [21], [81], [173]). Pluralism in logic has become in vogue after BRLP and is seen as a new landmark for the logical pluralist thesis. Carnap argues that people should not be forbidden to choose logics for themselves; in fact, they should be required to provide their reasoning for choosing a particular logic and for the consequences of the choices made. It is apparent that this principle focuses on the "liberty" to choose a language. Moreover, the choice of language needs to be validated with statements about the methods and syntactical rules used when determining whether or not an argument is valid. For instance, we may construct a logic based on the choice of a language, such as CPL, IPL, relevant logic, or paraconsistent logic. This principle allows us to admit different logics in different philosophical situations and different practical applications in computer science.[1]

Although people have attributed the origin of pluralism in logic to Carnap's principle of tolerance, proponents of BRLP criticized Carnap's version of pluralism, stating: "according to Carnap, given an argument, the validity of the argument is determined by the languages we choose." Further, disputes about the validity of this argument become disagreements in our understanding of the argument. Thus, we can interpret this argument as being in Language X or as in Language Y. Therefore, there will be no *real* disagreement on the validity of a given argument because the disagreement is on the choice of language" (See [193], p. 431).

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[1]For example, by taking Carnap's principle, Kutz et al. ([137]) acknowledge the situation that there are various kinds of ontologies to target at various domains of applications, and moreover, they are formulated in varying logical languages.

This is a direct criticism of the principle of tolerance. In a different way, Beall and Restall developed their own pluralism in logic. Firstly, BRLP considers that logic is a matter of truth-preservation in all cases of GTT. Secondly, to specify a logic is to give an account of the cases in GTT. According to Beall and Restall's assertion, in order to determine the logic, one must specify *the cases*. Thirdly, BRLP claims that there are exactly three different logical consequence relations: classical, relevant, and constructive, and these three are equally good and do not compete with each other.

4.2.1   The Classical case and GTT

> "To construct a logic needs to specify what the *cases$_x$* are and to give an account of *a claim to be truth in a case.*" ([12], p. 476–77)

There are two different approaches of fleshing out GTT so as to develop it as classical logic: (a) the possible-world approach and (b) the Tarski-model approach

(a) The possible-world approach means: (i) to take cases of GTT as *possible worlds*; (ii) to state what account of a given claim is to be true in possible worlds as follows:

For any claim φ, ψ and world w,

- $\neg\varphi$ is true in w iff φ is not true in w.

- $\varphi \wedge \psi$ is true in w iff φ is true in w and ψ is true in w.

- $\varphi \vee \psi$ is true in w iff φ is true in w or ψ is true in w.

- $\forall_x\varphi(x)$ is true iff for every object $a \in w$, $\varphi(a)$ is true in w.

- $\exists_x\varphi(x)$ is true iff for some object $a \in w$, $\varphi(a)$ is true in w.

In this approach, we can confirm that the *truth-preservation* criterion in classical logic is satisfied as follows:

(**GTT**$_{PW}$): An argument is *valid* if and only if, in every possible world, in which the premises are true, so is the conclusion.

Example 32. (Ibid, p. 478)

Considering an argument that states "if an object is red, then is colored," we apply the possible-world approach to it to check its validity. It is easy to see that in any case (possible world), where something is red, it is also colored; it is impossible for something that is red to not be colored.

To elucidate, this specifies a *necessary truth-preservation* account of validity by this possible worlds approach. However, this is not the only way in which GTT can be applied to classical logic in BRLP. The other approach is the Tarskian-Model approach.

(b) **The Tarskian-model approach** means: (i) to take cases of GTT as *Tarskian models*; (ii) to state what account of a given is to be true in every model.

A *Tarskian-model* is a structure $\mathcal{M} = \langle D, I \rangle$ consisting of a nonempty set D as a domain and an interpretation function I, which interprets a name N into the domain D, symbolized as $I(N) \in D$. If N is a n-place predicate, $I(N)$ is a set of ordered n-tuples of D elements. We use model $\mathcal{M}$ to interpret the formal language of first-order logic as follows:

$\mathcal{M}_\beta$ represents a model $\mathcal{M}$ under the assignment β. β is a value assignment to *assign* the value to variables, i.e., $\beta(x)$ is the value of the variable x. If a is a name, then $I_\beta(a) = I(a)$. If β is an assignment to assign the elements in D to variables x, then $I_\beta(x) = \beta(x)$. For the predicate symbol $f \in \mathcal{M}$, any term $t_1, \ldots, t_n, n \in \mathbb{N}$ in this formal language $\mathcal{L}$, $f(t_1, \ldots, t_n)$ is also a term in $\mathcal{L}$, and $f(t_1, \ldots, t_n)$ is true in $\mathcal{M}_\beta$ if and only if $\langle I_\beta(t_1), \ldots, I_\beta(t_n) \rangle \in I(f)$. For any formula φ, ψ,

- $\varphi \wedge \psi$ is true in $\mathcal{M}_\beta$ if and only if φ is true in $\mathcal{M}_\beta$ and ψ is true in $\mathcal{M}_\beta$.

- $\varphi \vee \psi$ is true in $\mathcal{M}_\beta$ if and only if φ is true in $\mathcal{M}_\beta$ or ψ is true in $\mathcal{M}_\beta$.

- $\neg\varphi$ is true in $\mathcal{M}_\beta$ if and only if φ is not true in $\mathcal{M}_\beta$.

- $\forall_x \varphi$ is true in $\mathcal{M}_\beta$ if and only if φ is true in $\mathcal{M}_{\beta'}$ for each x-variant β' of β.

- $\exists_x \varphi$ is true in $\mathcal{M}_\beta$ if and only if φ is true in $\mathcal{M}_{\beta'}$ for some x-variant β' of β.

By this recursive definition, we can determine the truth in the Tarskian model for any sentence of this first-order language. We consider the Tarskian model to be a case in GTT, such that it can be specified as follows:

($\mathbf{GTT}_{TM}$): An argument is valid if and only if, in every model in which the premises are true, so is the conclusion.

$\mathbf{GTT}_{TM}$ and $\mathbf{GTT}_{PW}$ are two accounts of truth-preservation in classical logic. The Tarski-model approach in the study of logical consequence. It states that the validity of an argument is determined by its *logical form*, i.e., validity is a matter of form. The argument in **Example 32** could be formalized as $R_X \vdash C_X$ in first-order language where R_X represents 'X is red', and C_X represents 'X is colored'. It is easy to find a counterexample to make the premise true and the conclusion untrue in the form of this argument, i.e., it is an *invalid* argument. Hence, it does not follow GTT. However, Tarski's model-theoretic approach is an alternative for the necessary truth-preservation criterion.

4.2.2   The Relevant Case and GTT

The relevant case is a restriction of approach (a). In approach (a), GTT generates logical consequence in the classical sense, i.e., $\mathbf{GTT}_{PW}$ refers to classical logical consequence when cases are referred to as possible worlds. In this way, "cases (possible worlds) are complete and consistent with respect to negation". ([10], p. 49) For English users, there is another strong sense of "follow from"; here, " 'from' is taken seriously." (Ibid.) Further, two examples of arguments are always discussed. It is not valid that for any sentence ψ follows *from* any contradiction $\varphi \wedge \neg\varphi$; it is also not valid that any tautology $\varphi \vee \neg\varphi$ follows *from* any sentence ψ. This inevitably leads to a revision of classical logical, resulting in the birth of relevant logic. Beall and Restall propose this account of cases for logical consequence in the sense of relevance to claim that the premises have to be "relevant" to the conclusions. This means that some valid classical logical consequence relations become invalid. The situation approach is claimed to flesh out GTT for relevant logic in BRLP.

 (c) **The situation approach** means: (i) to take cases of GTT as *situations*; (ii) to state what account of a given claim is to be true in situations.

In approach (c), conjunction and disjunction behave in exactly the same

manner as they do in classical logic approach (a). Thus, let us first consider the truth-conditions of these two propositional connectives as follows:

- $\varphi \wedge \psi$ is true in s if and only if φ is true in s and ψ is true in s.

- $\varphi \vee \psi$ is true in s if and only if φ is true in s or ψ is true in s.

With regard to situation approach (c), the main challenge to classical logic is the classical account of the behavior of negation. This challenge will change the truth-condition of the propositional connectives.

The idea of this approach is that "situations may be incomplete". A situation may make some statements true and others false. However, by virtue of being restricted to parts of the world, a situation may fail to make a claim true or make it false ([10], p. 50). Hence, some statements are *undetermined*.

According to BRLP, the possible worlds are taken as "special situations" that are complete and consistent. Given the "undetermined" statements, we can assume that *relevant* negation is different from *classical* negation. Classical negation is "over complete and consistent values of case$_x$." ([10], p. 51) It is "complete and consistent with respect to negation." (Ibid, p. 49) However, situations are not always complete and consistent. For example, in fiction, inconsistent situations do occur.[2] Hence, in this way, incomplete or inconsistent worlds can be considered. In approach (c), the cases refer to "situations" instead of "worlds." Given that possible worlds are considered as special situations, i.e., complete and consistent situations, the truth-condition of classical negation cannot fully capture the *behavior of negation* once we begin to consider negation with respect to situations. In other words, those ignored "situations" have to be reconsidered here. Hence, if a sentence φ is not true, the negation of φ does not have to be true; moreover, if the "true contradiction", $\varphi \wedge \neg\varphi$ holds, ψ does not have to follow from it, for any ψ.

For classical negation, the truth-condition is as follows: $\neg\varphi$ is true in a case$_x$ if and only if φ is not true in case$_x$. In BRLP, it is suggested that *relation of compatibility* is used to analyze the semantics of negation.[3]

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[2]As Beall and Restall suggest, an inconsistent but perfectly intelligible story has been provided by Graham Priest in [180]. As Priest stated in the abstract of this article: "The existence of such a story is used to establish various views about truth in fiction and impossible worlds."

[3]The analysis of the semantic of negation was proposed by Michael Dunn in 1994 and 1996. It is not the only approach of the semantic of negation, but it is preferable in Beall

- $\neg\varphi$ is true in s if and only if for any s', s' is compatible with s, φ is not true in s'.

Further, we formulate the truth-conditions of quantifiers by using the same consideration of compatibility:

- $\forall_x\varphi$ is true in s if and only if for any s', s' is compatible with s, φ is true in s'.

- $\exists_x\varphi$ is true in s if and only if there is some s', s' is compatible with s, φ is true in s'.

The notion of specifying "cases" in GTT to "situations" in BRLP is in a way attempting to follow both approaches (a) and (b), fleshing out GTT, but, moreover, going beyond truth-preservation in "possible worlds." Thus, what is required here is to go beyond "the constraints" of the worlds, and to obtain some kind of "truth-preservation," i.e., "truth-preservation over situations." Recalling the two examples of invalid arguments in classical logic, we can state that the "behavior of negation" constrains possible worlds from being complete and consistent. This means that classical logical consequence, restricted to referring cases in GTT to be possible worlds (approach (a)), makes (constrained) possible worlds complete and consistent with respect to negation. According to BRLP, specifying cases as "situations" enables one to specify "relevant logical consequences," under the condition that "possible worlds" in approach (a) are considered as special "situations." By removing *completeness* and *consistency* in $\mathbf{GTT}_{PW}$, we can state that relevant logical consequence is an *incomplete* and *inconsistent* case of GTT. Hence, the validity of an argument can be expressed as follows:

($\mathbf{GTT}_{RE}$): An argument is valid if and only if in every situation in which the premises are true, so is the conclusion.

Based on the *situation semantics*,[4] "possible worlds" in approach (a) can be extended to situations. Any statement that is true in the possible worlds in

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and Restall's theory (See, [10], p. 52).

[4]As an alternative to the possible world semantics, this work can be traced back to ([16], Barwise, 1989) and ([17], Barwise and Perry, 1983) in the eighties. As we show here, *situations*, unlike *worlds*, are not complete in the sense that every proposition or its negation holds in a world. Further, meaning is related to the relationship between discourse situations.

$\mathbf{GTT}_{PW}$ has been considered in situations. $\mathbf{GTT}_{RE}$ is a more general version of $\mathbf{GTT}_{PW}$ since possible worlds can be seen as special situations: complete and consistent situations. Generally speaking, this approach suggests a wider view to describe the behavior of negation in classical logic.

4.2.3 The Constructive case and GTT

The constructive approach stems from the birth of intuitionistic logic (constructive logic), which is a variant (deviant) of classical logic. In BRLP, it is used to instantiate GTT into constructive $\mathbf{GTT}_{IN}$. It is similar to the system of possible worlds applied in approach (a) for classical logic and those of situations applied in approach (c) for relevant logic.

(d) The constructive approach means: (i) to take cases of GTT as *stages*; (ii) to state what account of a given claim is to be true in stages.

In BRLP, there are three crucial features of the stages. (In [10], p. 62)

- Stages are **incomplete** since it "might neither verify a claim nor its negation; we need not posses total information at each step of our enquiry."

- Stages are **extensible** since a stage, s, "might be followed by (extended by) another stages $s' : s \subseteq s'$."

- Stages are **partially ordered** by this relation since the inclusion relation $\subseteq$ is *reflexive, transitive, anti-symmetric* on the set of stages, $\subseteq$ is a partially ordered relation.

In sum, stages are *incomplete*, *extensible* and *partially ordered* by the relation of inclusion $\subseteq$. Similar to the above three mentioned approaches, we discuss the *truth in a case* for the constructive approach and the instantiate GTT, GTT_{IN}. Before proceeding to check the conditions of the propositional connectives, we first draw attention to another property: *cumulative*. This property represents the relationship between "the truth of propositions φ according to a stage" and "the inclusion relation $\subseteq$". Further, this property demonstrates that the information that is provided in stage s is also inherited to stage s'.

> Given any stage s and formula φ, ψ, if $s \subseteq s'$ and φ is true in s,
> then φ is also true in s'.[5] (*Property of Cumulative*)

Next, we check the truth-conditions of propositional connectives $\vee$, $\wedge$ as follows:

- $\varphi \wedge \psi$ is true in s if and only if φ is true in s and ψ is true in s.

- $\varphi \vee \psi$ is true in s if and only if φ is true in s or ψ is true in s.

The truth-conditions w.r.t. "case" in approaches (a), (b), and (c) are as follows: $\neg\varphi$ is true in "case" if and only if φ is not true in "case". This truth-condition assumes "case" is complete with respect to negation. However, since "case" might be incomplete with respect to negation in intuitionistic logic, it is too strong to assert a similar truth-condition of negation. According to Beall and Restall: "it is too much to say that $\neg\varphi$ is true in s if and only if φ is not true in s'." ([10], p. 63) Hence, the truth-condition of negation in stages (in intuitionistic logic), which might be "incomplete," should be expressed as:

- $\neg\varphi$ is true in s if and only if for any $s \subseteq s'$, φ is not true in s.

However, "consistency" is required in intuitionistic logic. If $\neg\varphi$ is true in s, then φ is not true in s, moreover, if s' is the succeeding stage of s, by the cumulative property, φ is not true in s'.

For implication $\rightarrow$, in classical logic, $\varphi \rightarrow \psi$ is equivalent to $\neg\varphi \vee \psi$, so it is sufficient to let φ *entail* ψ in classical logic. However, this equivalence is too strong to hold in intuitionistic logic. First of all, for the weakest form of entailment, $\varphi \rightarrow \varphi$, there is no further information required to infer φ besides φ itself. $\varphi \rightarrow \varphi$ is the weakest one in all propositions to let φ be sufficient to *entail* φ. We also know that $\varphi \rightarrow \varphi$ is equivalent to $\neg\varphi \vee \varphi$ classically, so $\neg\varphi \vee \varphi$ could also be sufficient to let φ entail φ, or $\neg\varphi$ entail $\neg\varphi$. However, $\neg\varphi \vee \varphi$ is not true in all stages; but it is true in those stages where φ was decided (constructed) in a certain way. $\neg\varphi \vee \varphi$ could not be a

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[5]According to BRLP, we can consider this property in the following two ways: (a) This property provides that if any simple proposition is evaluated, then its evaluated value (truth) at any stage also holds in complex ones by an induction process. (b) This property can be regarded as a global condition which governs all propositions, and we can further check if any truth-conditions and definitions of complex propositions are consistent with the property.

form of entailment. Hence, the truth-condition of implication here is similar to those of cases in approaches (a), (b), (c): $\varphi \to \psi$ is true in s, if and only if, "φ is true in s implies ψ is true in s". With the help of taking cases as stages, $\varphi \to \psi$ is true in s and $s \subseteq s'$, *everything true at s is also true in s'*, we get the truth-condition of implication in approach (d) as follows:

- $\varphi \to \psi$ is true in s if and only if for any $s \subseteq s'$, if φ is true in s', then ψ is true in s'.

When discussing quantifiers, we need to know when an object is available. One needs to check if "an object is available for evaluation at a stage." This condition emphasizes the *sequential constructivity* of stages in GTT_{IN}. Due to this property, the range of a quantifier cannot be affirmatively ensured at stages, since stages are neither "omniscient with respect to the information they warrant" nor "omniscient with respect to the objects they know about." (Ibid, p. 63) For quantifiers $\exists$, we should note that the stages might be incomplete in this approach. This means an object might be "unavailable" to instantiate an existential quantifier. The form $\exists_x \varphi(x)$ needs an *available witness w* such that $\varphi(w)$ is true in s.

- $\exists_x \varphi(x)$ is true in s if and on if $\varphi(w)$ is true in s, for some w which is *available* in s.

An important constraint is that objects available at a stage are also available at all later stages: If $s \subseteq s'$ and an object a is available at s, it is available at s'. For quantifiers $\forall$, it is not enough to say for all *available witnesses w* in s such that the $\varphi(w)$ is true in s. Stages are *sequential*. When we consider the truth-condition of $\forall$, it is possible that counterexamples, which do not occur in the predecessor stages, occur in a later stage. Hence, a constraint is placed on the later stages so as to ensure that the truth-condition is suitable.

- $\forall_x \varphi(x)$ is true in s if and only if for each $s \subseteq s'$ and for each w available at s', $\varphi(w)$ is true in s'.

In light of the above concepts regarding propositional connectives and quantifiers, the validity of an argument can be expressed as follows: ([10] p. 64)

($\textbf{GTT}_{IN}$): (i) Given a system of stages, an argument is valid (with respect to the given system of stages) if and only if, in every stage in which the

premises are true, so is the conclusion; (ii) an argument is *constructively valid* or *intuitionistically valid* if and only if it is valid in **all** systems of stages.

4.2.4   A Unified Model of Negation

BRLP proposes a *model* to unify the three various logical consequences. This idea originates from the fact that many aspects of a thing could be *equally* correct. First of all, we see a metaphor: Hilary claims that "Barack Obama is an African American" and John claims that "Barack Obama is a lawyer." Both of them have described one aspect of Barack Obama, and both claims can be correct at the same time. The same idea is represented in the statements of negation made by BRLP. Classical logic involves providing a description of when negation is true in possible worlds or Tarski models; relevant logic provides a description of when negation is true in situations, and constructive logic provides a description of when negation is true in stages. Each of these refers to one aspect of negation; thus, each addresses incomplete accounts of the same thing.

The same idea can be seen in the statements of negation in BRLP. Classical logic gives a description of when negation is true in possible worlds or Tarski models, relevant logic gives a description of when negation is true in situations, and constructive logic gives a description of when negation is true in stages. Each of these displays one feature of negation; each obtains incomplete accounts of the same thing. BRLP provides "a model" to serve as a candidate for unifying these aspects of negation. This model deals with the relations between worlds, constructions, and situations, and is divided into the following three routes:

  (a) worlds with constructions,

  (b) worlds with situations,

  (c) constructions with situations.

(a) worlds with constructions.

Worlds are regarded as a kind of construction since there is no problem in determining true or false (and not both) for any given proposition in worlds.

In this way, worlds *finalize* the constructions, i.e., worlds will not be extended by any further construction as they are in the final stage. Hence, the model consists of "constructions," some of which are also "worlds." The partially ordered (extension) relation of constructions, $\subseteq$ is also in this model. Worlds w play the end point of ordering. In other words, there is no stage $w \subseteq w', w \neq w'$. Two points need to be explained here. First, in all *constructions*, **worlds** behave as we expect of them, and second, in all *worlds*, **constructions** behave as we expect of them. For the former,

- $\neg\varphi$ is true in stage s if and only if for any world, φ is not true in stage s', where $s \subseteq s'$;

for the latter,

- $\neg\varphi$ is true in world w if and only if for any construction s, $w \subseteq s$, φ is not true in s.

Hence, φ is not true in w since w itself is the only one extended construction of w.

(b) worlds with situations.

As shown previously, "possible worlds" are seen as consistent and complete "situations." It is easy to see how this model can make the possible worlds behave as we expect of them in situations, under the relation of compatibility.

- $\neg\varphi$ is true in situation s if and only if φ is not true in s', where s' is any compatible situation with s.

(c) constructions with situations.

The discussion of (c) finishes this model. According to (a) and (b), worlds are constructions and worlds are situations, respectively. If worlds are both constructions and situations, then some constructions are situations. How can we correlate or connect "constructions" and "situations," and still keep them distinct from one another? This was proposed by Beall and Restall as follows (in [14], p. 10):

1. Take all **constructions** to be **situations**.

2. If a **construction** is compatible with itself, then we have a **situation** which is compatible with a **situation** itself.

3. If a **construction** is compatible with some **situations**, then these **situations** are consistent.

4. "**Constructions** may well be seriously incomplete, but they still carry the information that the situation/world (or possible worlds) is not inconsistent." (Ibid.)

These four items provide us with the means to consider constructions with situations so as to make the model coherent. We verify the truth-condition of constructive negation as well as the truth-condition of relevant negation in the following steps.

5. Given any two consistent and compatible **situations** s and t, if s, t are both compatible with themselves and compatible with each other, there is some **world** w such that s, $t \sqsubseteq w$.[6]

6. If s and t are consistent and compatible, then $\neg\varphi$ is true in s if and only if φ is not true in s; $\neg\varphi$ is true in t if and only if φ is not true in t. So, by *plenitude of worlds assumption*, s, t are part of some **world** w, in which the behavior of negation will also be classical. The truth-condition is: $\neg\varphi$ is true in w if and only if φ is not true in w.

Item 5 is seen as passing the consideration of situations to that of worlds. Further, based on item 6, we elucidate how the behavior of relevant negation can be realized as classical negation.

For any formula φ, **construction** c, **situation** s, we moreover have the following four propositions, each of which builds naturally on the previous ones:

7. If φ fails in any **situation** s, which is compatible with **construction** c, then φ fails in any **extending construction** of c, denoted as c', since any such **construction** is compatible with itself, and hence, is compatible with c.

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[6]This is a plausible assumption made by Beall and Restall, termed the *plenitude of worlds assumption*.

8. If φ fails in c', c' is compatible with c, which is compatible with s, then φ fails in s.

9. If φ fails in c', then if s is a **situation** compatible with φ, it follows that s is consistent, since **constructions** are compatible with consistent **situations** only.

Since c, s are compatible with each other and c, s are both consistent, by the plenitude of worlds assumption, there is some **world** w extending both. But **worlds** are **constructions**, and we know that φ fails in that **world**. So φ must fail in s too. So if φ fails in every **extending construction** of c, it must fail in every **situation** compatible with c too. Moreover, items 7 and 8 demonstrate the relation between situations with **extended construction** c' Following item 8, item 9 provides the "consistency" of **situations**.[7]

4.3 BRLP-Plurality

The discussion so far offers four cases for three logics that were taken in BRLP. However, there is a more fundamental concern for BRLP, namely the plurality of these cases of logical consequence (BRLP-plurality) that strongly depends on different arrays of application in different perspectives in which the philosophical motivations are deployed. Thus far, we have discussed the four cases as "possible worlds," "Tarskian-models," "situations," and "stages". We understand that there are three instantiated logical consequences endorsed by BRLP. Similar to Carnap's principle of tolerance, BRLP-plurality will gain justification by considering a similar criterion. In other words, as a theory about pluralism, BRLP needs to further justify the plurality that is mainly about the concept of logical consequence.

4.3.1 Plurality

When acknowledging that there are four cases of GTT at various concerns, formulated in varying logics, it is obvious that there is a key requirement for BRLP to be considered a type of pluralism. Indeed, according to BRLP, there are four cases where three logical consequences are exemplified. But

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[7]These nine items were rephrased from the statements made in [14], pp. 10–11 so as to finalize this model.

for plurality, the *equally-good* criterion has also been specified by BRLP in [14]:

> "We are pluralists about logical consequence. We hold that there is more than one sense in which arguments may be deductively valid, that these senses are equally good, and equally deserving of the name deductive validity."

Another criterion: *equally-legitimate* has been proposed by Graham Priest in [10], p. 97:

> "[Beall and Restall argue that] we can give the truth conditions for the connectives in different ways. Thus, we may give either intuitionist truth conditions or classical truth conditions. If we do the former, the result is a notion of validity that is constructive, that is, tighter than classical validity, but which it is perfectly legitimate to use for certain ends... We can indeed give different truth conditions. But the results are not equally legitimate. The two give us, in effect, different theories of vernacular connectives: they cannot both be right."

In this regard, there appears to be a different between "equally good" logics and "equally legitimate" (equally right). As we have seen, there are four different theories of negation, classical$_{PW}$, classical$_{TM}$, relevant, and constructive. Priest challenged that all these cannot be right; at the most, only one of them can be right to account for the behavior of negation. Thus, there should be only one true logic. Following this, BRLP primarily needs to defend that the endorsed logics are equally legitimate in order to justify plurality.

The third criterion: *irreducibly plural* has been proposed in relation to BRLP in [193], p. 426:

> "With J. C. Beall, I have argued for and defended a pluralism about logical consequence. We take it that the notion of logical consequence is irreducibly plural in its application. That is, we take it that there are at least two distinct relations of logical consequence."

Although BRLP is being discussed, it is restricted to three logics: classical logic, relevant logic, and intuitionistic logic and thus does not include all logic in the modern logical discipline where there are new, typically inter-disciplinary application areas for logics. Undoubtedly, this unified model of negation might be taken to respond to Priest's challenge that these logics are "equally legitimate (right)" by expressing different aspects of the same thing. However, it is not as suitable when adopting an equally good criterion to justify plurality.

4.3.2 From a Strong Sense of Plurality to Constructing BRLP

The second focus of this chapter is on the justification of weak plurality in BRLP and the related problems and criticisms raised in the literature. Beall and Restall were aware that BRLP is not what is commonly understood as pluralism. It is no wonder that Beall and Restall affirmatively clarify their stance by the question-answer as follows: ([10], p. 25)

Q : What kind of claim is logical pluralism?

A : Logical pluralism is a **pluralism** about logical consequence.

The meaning presented by BRLP is quite different from the common sense of pluralism in other disciplines. First, BRLP suggests a pluralism of **logical consequence** rather than that of **logic**. (In [10], p. 88)

> "We granted the point that we have defended pluralism about logical consequence, but not pluralism about 'logic' understood as the study of consequence relation."

The proposed model in BRLP states that these different negations are different aspects of the behavior of negation. Therefore, they are different aspects of logical consequence which has reflected different specifications of the pre-theoretic (intuitive) notion of logical consequence.These different specifications are not in competition and not rivals.

Second, as we have borrowed the sense of stronger-weaker pluralism adopted in moral pluralism and political pluralism, BRLP states different logical consequences are equally good or equally right, however, it is debatable whether BRLP can be regarded as pluralism in this stronger sense without adopting

the principle of tolerance. Like moral pluralism, if "there are many different logical consequences" which is enough to be considered as a reason to claim a sort of pluralism, then it is no problem for BRLP to be concerned a type of a pluralism in this same way. Thus, BRLP is obviously a type of pluralism to propose that there are many different logical consequences. However, it is hard to capture and justify any plurality. Third, there is no competitor (rival) between the different negations that have characterized the different logical consequences. Beall and Restall refuted the requirement of rivals for pluralism. (Ibid, p. 89)

> "...electron pluralism is the claim that there is a plurality of different electrons – an **un**controversial physical claim. To require **rivalry** in a pluralism is to mischaracterise it."

BRLP, as a strong logical pluralism, states that any two logics (logical consequences) are equally good by the unified model of negation. The manner in which BRLP justifies plurality is to characterize each logic in logical pluralism as each individual logical consequence, and to characterize each logical consequence by a negation. It is not sufficient that the unified model has justified that classical, relevant, and constructive negations are equally good, moreover, classical, relevant, and constructive logical consequences are also equally good; therefore, classical, relevant, and constructive logics are equally good. This justification is not so much as to provide a reason to justify that these four logics are equally good, but to show that there are three different stories for a pre-theoretic notion of logical consequence.

We believe that a lot can be learned in this respect about the justification of plurality discussed here, and the reliable account of plurality where *irreducibly plural* as the criterion can be found in BRLP. In doing so, BRLP will not be an analogy to be realized as those kinds of pluralisms that we have mentioned above, particularly when they have also claimed that their pluralism should not be required to have **rivals** and not endorse the tolerance principle.

The gist that BRLP proposes to the above issues about logical consequence relies strongly on the concept of *plurality*: facing the fact that several logics and formalisms are used for various purposes and applications, we suggest not to take the "equally-good" criterion but instead the criterion of "irreducible plurality", to structure the basic attitude and the translation of logics in various but always formally and well-founded logics. An idea: one-many statement, then, in BRLP, perfectly articulated the basic framework

for this cutting-edge research in logic According to this statement, BRLP acknowledges and approves that its pluralism will require a different treatment than the ordinary pluralism, while the one-many statement maintains that there is precisely one core notion of logical consequence, and there are many true instances of this one core notion.

> "Graham Priest poses the question: 'Logic: One or Many?' Our answer is 'both'. One: There is precisely one core notion of logical consequence, and that notion is captured in schema (V). Many: There are many true instances of (V), each of which specifies a different consequence relation governing our language. This one-many answer is what we call 'pluralism'." ([14], p. 17)[8]

This position takes one-many statement and thus pluralizes the one precise and abstract notion of logical consequence to various concrete instantiated logical consequences. In Beall and Restall's words, these instantiated logical consequences should not be rivals. Again, BRLP is not traditional pluralism. Beall and Restall's answer to the question definitely suggests that their theory is not at all a pluralism in the ordinary sense.

4.4 The Scope of Monism in BRLP

Thus far, we have obtained a clear understanding of BRLP. BRLP has the following three characteristics:

1. BRLP assumes an pre-theoretic notion of logical consequence formulated as GTT;

2. GTT can be instantiated as three "irreducibly plural logical consequences", classical, relevant, intuitionistic logical consequences;

3. these individual logical consequences do not rival each other but are different aspects of the intuitive logical consequences.

In BRLP, logic is a matter of truth-preservation in all "cases." To specify a logic is to give an account about "cases in GTT". BRLP has provided a model to make classical, intuitionistic, and relevant logic compatible. BRLP,

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[8](V) appears in this citation refers to GTT.

thus, sets up a standard for logical pluralists that when people endorse some
logics, they should propose these logics which are compatible with each other,
like a pluralist under BRLP. In other words, if one deviant-BRLP holds only
the spirit of BRLP without endorsing the three are logics that BRLP is
considering, he will also have to provide a model for the logics that they are
considering, similar to the model in BRLP.

So far, we have discussed a logical pluralism: BRLP. However, there is no
opposite statement of pluralism, called "monism in logic" clarified by BRLP
explicitly. Thus, in the next section, we discuss what statements should be
given for monism in the context of BRLP to explore the scopes of monism
and pluralism in logic investigations.

4.4.1   From BRLP to Monism in Logic

In the previous sections, we have analyzed in detail a specific example of
logical pluralism from the literature. We have shown that BRLP has been
taken as the landmark of pluralism in logic and influenced research in other
realms. For example, various applications in computer science strongly in-
volve reviewing various logics, using various logics, and developing various
logics for the various purposes. Certainly, this position can also be found
in the studies of logic translation and the new field of the Universal Logic
project (see [72], [137], [150], [151], [152]).

Nevertheless, we will not become involved in these one by one, since these
also involve completely different application domains and are not necessarily
connected with foundational discussions. What we are concerned about is
how a monist asserts his position properly. As we have compared the Car-
nap's logical pluralism with BRLP (and the deviant-BRLP), the plurality
of different logics, shown in Carnap's logical pluralism, is due to different
languages. Moreover, this kind of plurality has been justified by Carnap's
principle of tolerance. If what we consider here is in the Carnap sense, there
seems to be no real disagreement between these logics since differences are
simply due to different languages.

4.4.2   A Discussion of Priest's BRLP-Challenge

> "Monisms contrast with pluralisms and nihilisms. Where the
> monist for target t counted by unit u holds that t counted by u
> is one, her pluralistic counterpart holds that t counted by u is

many, and her nihilistic counterpart holds that t counted by u is none." (*Stanford Encyclopedia of Philosophy*– **Monism** ([195]))

Based on this citation, the meaning of monism is the opposite to that of pluralism. However, this is not always so obvious in monism and pluralism in logic. Let us first address the following question:

Q: Is it possible, for a logician proposing X-logic, not to assert
    X-logic as the logic?

If the answer is "Yes," then this X-logician might be able to hold some pluralism like Beall and Restall; if the answer is "No," then the X-logician is a monist.

In the following discussion, we deal with the role for the aforementioned question.

"Take some inference $\alpha \vdash \beta$ that is valid in K_1, but not in K_2, and suppose that we know (or assume) α; are we, or are we not entitled to accept β?" ([10], p. 93)

Stephen Read's opinion on this question is also worth considering again. He considers the two specifications of logic K_1, K_2 as classical logic and relevant logic, respectively ([189], p. 194). This statement can thus be expressed as follows:

"For the inference $\alpha \vdash \beta$ which is classical valid but not relevant valid, and α is told true, does it tell us β is true?"

With regard to this statement, Beall and Restall reply:" We are *classically entitled* to this inference, and we are not relevantly entitled to this inference. Hence, telling α is true provides β is true, since classical logic *is entitled* in this case, so it is valid in the classical sense instead of the relevant sense." Following Beall and Restall, Read stated that what is actually expressed here is not only that the information "β does not follow relevantly from α" but also "we are not entitled to this inference." Read, nevertheless, does not consider this description and realization of Beall and Restall as encompassing the crucial point of what relevant logic expresses. He states: "Relevant logic was not put forward as a mere alternative to classical logic..., So too for intuitionistic reasoning..." ([190], p. 196). Read believes that relevant logic

and intuitionistic logic are based on monism.[9] They were both created as a replacement for classical logic, and thus, they are not just "alternatives." Hence, for Read, the question proposed above has a clear response: it is not possible for a X-logician not to assert X is the logic in cases of relevant logic and intuitionistic logic.

4.4.3   The Complement of Plurality

BRLP challenges monism in logic by recognizing the plurality of logical consequences studied in logic. People as pluralists will address the pluralism in logic that they are endorsing. In the first place, the pre-theoretic notion of logical consequence helps us become clear that some of the deepest ideas are independent of notational systems and grammars. Moreover, some of them believe that one is likely to have a variety of non-rival and compatible analysis for the pre-theoretic notion, and most especially, a variety of pre-theoretic notions of one and the same "logical consequence". BRLP proposes such a theory. Second, Carnap's principle of tolerance, as a quite different logic pluralism, summarizes an approach to endorse this pluralism in logic, especially to hold a quite pragmatical attitude to study a new relative area. The position that takes this pluralism and relates this to **ontological designs** that require different logical languages has been called **onto-logical pluralism** (compare e.g. [136], [137], [138], [150]).

It is easily observed that in the latter case, plurality can be justified for many pragmatic reasons, especially when it is adopted in practical areas. However, as a theoretical, fundamental, and philosophical debate, Beall and Restall affirmatively defends what they would like to propose for the BRLP. In this regard, while various challenges, either from those whom have claimed that they are monists (e.g. Stephen Read) or those who have ever objected to BRLP (see [10], Chapter 8 and Chapter 9), the position of monism also

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[9]Readers can learn that BRLP hold a pluralism for constructive logic, which is taken as the underlying logic of constructive mathematics.

> "The fact that there is a classical proof of an important theorem but no constructive proof does not mean that we should attempt to add new axioms to the constructive theory so that it will be provable (though this might be an interesting endeavour in its own right). We could, as pluralists, just acknowledge that this is one place where constructive mathematics is properly weaker than classical mathematics."

needs to be articulated clearly. For example, Stephen Read stated:

> "There is one true logic, the correct logic of truth-preservation.
> It is given by a correct analysis of the connectives '→' and 'o',
> one which does not conflate them with material implication and
> extensional conjunction, respectively. This analysis reveals the
> true meaning, and extension, of the familiar phrase "impossible
> for the premises to be true and conclusion false" ([191])

Furthermore, Otávio Bueno and Scott Shalkowsk stated:

> "According to logical pluralism, there is more than one gen-
> uine deductive consequence relation, even within a given lan-
> guage. The position is opposed to the more common view, logical
> monism, which privileges a single deductive consequence rela-
> tion." ([57], p. 1)

Obviously, the proponents of BRLP have proposed an opposing theory with
monism, where "pluralism" and "monism" are opposing theories in the com-
mon usage in general. In light of the plurality which has been justified by
BRLP, classical logic, intuitionistic logic, and relevant logic represent three
aspects of the common notion of logical consequence, whereby these aspects
are "equally good". Hence, following this justified plurality, we will discuss
three possible opposite-BRLP version of monism.

4.4.4 The Opposite-BRLP Version of Monism (i)

First of all, monism needs to be addressed in the same line as BRLP, i.e., there
is an opposite-BRLP version of monism. Theoretically speaking, proponents
of BRLP should provide information about what is logical monism, since
Beall and Restall have proposed what they termed "logical pluralism." How-
ever, this is not stated clearly in their writings. Thus, we cannot state that
BRLP can successfully be an alternative to logical monism, unless we clarify
the statements about logical monism as well. Beall and Restall includes some
information about logical monism in their writings: One common viewpoint
of the relation between logical monism and BRPL may be as follows: ([206],
§ I, footnote 2.)

> "Firstly, the contemporary debates between logical monists and logical pluralists are recent phenomena. The rigorous debates between them started with an explicit formulation of logical pluralism and rejection of logical monism by the Australian logicians Beall and Restall (2000)."

Hence, the idea of BRLP is seen as a "rejection" of logical monism. Under the formulation: "GTT+$\mathbf{Case}_x$", what does logical monism truly state? Does it claim that "there is only one true logic" or does it state that "there should be only one true logic?" Here is one such voice ([12], p. 488)

> "Our pluralism holds that some formal logics can fruitfully be seen as different elucidations of (V), the pre-theoretic notion of logical consequence, and that *(V) does not determine one logic, but rather, a number of them.*"

> "To be a pluralist about logical consequence, you need only hold that there is more than 'one true logic'..., ..., (2) A *logic* is given by a specification of the cases appear in GTT." (Ibid, pp. 476–477)

Indeed, based on these two citations, we can formulate a possible statement about the opposite-BRLP version of monism:

[Logical Monism (LM) I]

A pre-theoretic notion of consequence is given in *GTT*. It determines only one logic.

Hence, there is only one case that is spelled out using GTT. This kind of monism in fact, is only based on the statement made in BRLP. To reiterate, the conception of logic in BRLP is as follows: (1) logic is a matter of preserving truth in all cases; and (2) a logic is constructed by using GTT and spelling out cases. This conception is firmly accepted by Beall and Restall in the three tenets ([12], pp. 476–477) or five tenets ([10], p. 35) to propose their logical pluralism. Hence, anyone who proposes logical monism under this conception of logic claims that there is only one case that is[10] spelled

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[10]We do not use "should" or "could" to replace "is." If we do, then this kind of a logical monist will have to further state which case "should be" or "could be" the case clearly.

out. LM I, is derived from the conception of logic in BRLP. If Beall and Restall made any statements about "objecting" to some version of monism in their writings, it is the aforementioned version.

In BRLP, statements related to logical monism are not sufficiently clear. Moreover, BRLP does not state clearly how to object to logical monism At the most, we observe a statement in the abstract of [12], *Logical Pluralism: version of March 28, 2000* ([13]). It states the following:

> "A widespread assumption in contemporary philosophy of logic is that there is *one true logic*, that there is one and only one correct answer as to whether a given argument is deductively valid."

The citation above not only reflects the traditional idea of the development in logic, in which monism in logic holds, but also shows that traditional logicians were monists since they claimed their logic to be the only right one. However, this widespread assumption is different from the monism derived from BRLP, i.e., LM I. Thus, BRLP is attempting to reject this widespread assumption regarding monism. Beall and Restall "intend to" deny this traditional conception of logic.

Thus far, we are clear about the monism that BRLP proposes; however, it is still insufficient to understand what a monist logician should claim. "There is only one correct answer for a given argument being deductively valid" is different from "there is only one correct case to spell out using GTT." There is still a difference between the monism derived from BRLP and the above citation. Let us consider the following question:

> Q: In the citation, is the meaning of what is underlined the same as the meaning of the "clause" that follows?

The answer to the question is "Yes"; however, Beall and Restall's theory cannot address this stance. In other words, LM I can only address whether "there is one logic from GTT" so as to answer whether a given argument is deductively valid. It fails to address whether there is *one true logic*. We broaden the scope of this discussion regarding the aforementioned citation based on the following two cases:

Case 1. Given an argument, there is one and only one underlying criterion to judge its validity. We do not know whether this argument is valid or invalid until we judge it through this one criterion.

Case 2. Given a *valid* argument with its unique underlying criterion of judging
validity, we do not know whether this argument is valid or invalid until
we find a criterion to *explain* it.

With respect to these two cases, first, BRLP will not admit that there is one
and only one underlying criterion. Thus, for case 1, Beall and Restall will
judge the validity of an argument by using classical logic, relevant logic, or
intuitionistic logic, respectively. Further, for case 2, Beall and Restall will
judge whether this validity can also be explained by the other two kinds of
validity. This depends on the arguments themselves. For example, if an
argument was checked as "classically valid," they will check whether it can
also be relevantly valid and intuitionistically valid.

   This broadened discussion is necessary because there are no explicit state-
ments regarding logical monism in the official publications of BRLP. We
could, at the most, find some implicit thoughts about logical monism from
different sources of literature and fragments of writings. Thus far, we have
understood that the meaning of the underlined portion in the citation is
equal to the clause that follows it. However, based on the logical monism
that we derived from GTT+$\mathbf{Case}_x$, i.e., LM I, we can see that LM I cannot
be applied to the logical monism expressed in the citation.

   Under the assumption of the conception of logic in BRLP, LM I suggests
that there is only one spelled out case. As mentioned earlier, a logical monist,
e.g., John, is responsible for stating and justifying the logic that he endorses
(the only one admissible case here). He is able to finish this task immediately,
having no difficulty in claiming "the validity of any given argument" or "any
given valid argument," i.e., LM I is consistent with the widespread logical
monism, as stated in the underlined portion of the aforementioned citation.
However, there is no reason to presume "the conception of logic in BRLP"
when determining the position of monism in logic. Thus, there is a difference
between this position of monism and LM I.

4.4.5   The Opposite-BRLP Version of Monism (ii)

In the previous section, we formulate LM I for BRLP, based on the GTT+$\mathbf{Case}_x$.
We now made another statement about BRLP that implicitly echoes our
opinions in the previous subsection. The citation is as follows:

> "[I]n particular, a pluralist response to these issues goes as
> follows: Many appeals to 'Real Validity' appeals to real validity;

> **they are not, however, appeals to the *only* real validity**.
> Real validity comes from a specification of cases which appear in
> (V)." ([12], p. 481)

Based on this citation, we can clearly observe that Beall and Restall, as pluralists, believe that "real validity" comes from spelling out GTT. They claim that the so-called "Real validity" is appealing to "real validity." Further, as pluralists, they do not appeal to the only "real validity." In other words, the position of monism that pluralists would like to replace, based on this citation, is the only real validity, and they would like to add the following condition to complete their theory about pluralism:

(**RV**): Spelling out GTT obtains different kinds of real validity.

Thus, LM I, which is a position derived from BRLP, does not encompass the idea of logical monism that appears in the above citation. LM I suggests that "there is only specification of GTT"; however, the latter suggests that there is only one real validity. There is also a difference between this position of monism, as shown in the citation, and LM I. Beall and Restall were unable to clarify the position that they wanted to replace.

Thus far, we have observed how two differences appear between "LM I" and the "ideas of monism," as discussed based on the aforementioned two citations related to BRPL. The opposite-BRLP version of monism was unable to entirely encompass the logical monism that people follow in the traditional development of logic.

4.4.6 The Opposite-BRLP Version of Monism (iii)

Two contemporary notions of logical monism as a target of logical pluralism, grew out of BRLP which lies at the GTT. In this chapter, one of the main goals is to find the monism that BRLP should replace. In this regard, there is also another similar investigation regarding the meaning of "logical monism shown in BRLP" given by G. C. Goddu. ([108], section III.) In the first place, Goddu clearly states that there are four possible articulations for logical monism as follows:

- [**LM II**] There is one and only one true logic.

- [**LM III**] There is one and only one correct answer of whether a given argument is deductively valid.

- [**LM IV**] There is one and only one sense in which a given argument is deductively valid.

- [**LM I**] There is one and only one specification of cases appropriate for spelling out (GTT).

Second, Goddu not only supports LM III but also, in the same paper, explicitly proposes that LM III is the core stance of logical monism.

LM II is the primary articulation, following the original question. LM III has been discussed in BRLP, and the relationship between BRLP and LM I has been addressed in section 4.4.4. Interestingly, it appears that LM IV is not controversial since LM I, LM II, and LM III can be interpreted as "the sense" in LM IV. However, if we really interpreted each of these three as "a sense," things would no longer remain interesting.[11]

There are two questions in particular we would like to discuss which are directly related to Goddu's statements:

Assume that an argument may be not deductively valid according to two-valued sentential logic (hereafter **SL**), but deductively valid according to first-order predicate logic (hereafter **PL**). If there is only one correct answer to this argument w.r.t. BRLP,

(1) can there be any explanation that fits all these kinds of situations?

(2) irrespective of whether or not BRLP is reasonable, is it not an example to fail LM III?

Intuitively speaking, it is reasonable to say that any one argument is not valid w.r.t. **SL** but valid w.r.t. **PL**. In addition, we will not regard **SL** or **PL** as not being logics. As Goddu stated, **SL**-validity captures only some aspect of deductive validity, i.e., truth-functional validity. It definitely does not capture deductive validity in its entirety. **PL** provides more information about all the aspects of deductive validity. Moreover, it captures more

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[11]There is a sense ("one," "true," and "logic") in which a given argument is deductively valid (to interpret LM II). Similarly, there is "one and only one" "correct answer"; thus, a given argument is also deductively valid (to interpret LM III). There is also "one and only one" "specification of cases" to spell out (GTT), and thus, a given argument is deductively valid (to interpret LM II).

parts pertaining to the structure of natural language than what **PL** encompasses. In this way, Goddu proposes the notion of partial logic. Partial logic states that **SL** allows us to understand part of the story of deductive validity. It enables us to add more to the story, thereby making the logic richer and bringing us closer to representing the structure of natural language. Goddu's notion of partial logic can be summarized as the following principle:

(**Principle of refining partial logic**): To capture more truly valid arguments is to keep refining partial logics in virtue of adding more structure in it.

Goddu's investigation, based on his aforementioned concept, may temporarily bring an end to the exploration of the content of logical monism. He formulated a plausible principle to explain that we can capture a more truly valid argument by refining "the partial logic." Moreover, he explicitly stated that the stance of monism in logic can be represented by LM III. (Ibid, p. 223) In his opinion, monism or pluralism in logic cannot be determined only by the words presented in BRLP.[12] With his concept of partial logic, Goddu, at the most, presented some "by-product" statement of the pluralism/monism that Beall and Restall's considered. This means that although Goddu proposed a sort of monism (LM III) by studying BRLP, he was unable to answer the question about the content of monism that BRLP would like to replace.

4.4.7 From BRLP to Relativism in Logic?

Adopting a position of logical pluralism either with or without notational tolerance will result in several general debates and objections (see [10], Chapter 8) and several specific ones (see [10], Chapter 9; compare e.g. [57], [81], [194], [226]). Obviously, the idea of BRLP can be debated within the philosophy of logic. In this section, however, we specifically discuss on logical relativity and relativism to which pluralism in logic could be easily attributed. Relativism was usually taken by various applications in ontology language enterprises before the proposal of the **onto-logical translation graph** (See [150]), as stated in the following:

"[...]This corresponds to a relativisation of the physical symbol

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[12] "...Without answers to these questions we cannot yet determine whether Beall and Restall have successfully provided an alternative to logical orthodoxy." (Ibid, p. 218)

> system hypothesis, resulting in a more pragmatic use of ontolo-
> gies in focused application domains and contexts, and a grounding
> of ontological concepts (e.g. using sensor data), thus better cap-
> turing the situated, embodied and embedded nature of human
> concepts[...]" ([137], p. 260)

Relativism might have a pragmatic motivation for applications in computer science, however, the discussion so far offers some non-pragmatic motivation for pluralism in the philosophy of logic. Thus, we will give BRLP-answers to enquiries about being relativism.

In relation to BRLP, Achille Varzi ([216]) made a statement that the three specifications of GTT are "in a relativistic position." In other words, in Varzi's stance, when we inquire whether or not an argument is valid, there will be no unique answer. "There is more than one sense in which an argument can be valid" (Ibid., p. 1). Thus, it has to depend on "what the cases are," i.e., the validity of an argument is "relativistic" with respect to cases.

Proponents of BRLP would not like anybody to claim that BRLP is a sort of relativism;

> "[...] Recall that we are not relativists about logical conse-
> quence, or about logic as such. We do not take logical conse-
> quence to be relative to languages, communities of inquiry, con-
> texts, or anything else. We do not take logic to be relative in this
> way [...] This is a pluralism, not a relativism." ([10], p. 88)

However, based on Varzi's observation, it makes sense to consider relativism as a further development of pluralism in logic. According to Varzi, BRLP states that there are different equally good and nonequivalent ways of "read-ing" GTT. This means that there are different but equally good "senses" of models (cases), possible worlds, Tarskian models, situations, or stages. How-ever, BRLP does not consider the **internal disagreements** between the different instantiated logics. Varzi, in this manner, proposes that his logical relativism is a stronger version of specifying GTT, which also leaves room for internal disagreements. ([216], p. 2)

> "They leave room for disagreement because they are com-
> patible with different ways of characterizing that portion of the
> language that is responsible for the required nexus between the

> premises and the conclusion of a valid argument– –different ways
> of characterizing the 'logical vocabulary'."

Note that, in BRLP, the core conception of logic stems from applying GTT, wherein each logic should be specified by "$case_x$". Thus, different logics are instantiated by applying different cases of GTT. Hence, for any "$logical_x$ consequence relation" with respect to "$case_x$", a "$logical_x$ truth" is either true in all different cases or there are $logical_x$ consequence relations with a zero premise.[13] In this way, a logical truth pluralism should hold. Indeed BRLP will accept being "pluralist" about logical truth ([10], p. 100), however it is debatable to accept further a logical truth relativism. (See [10], p 94)

De facto logical relativism fits well with pluralism with Carnap's tolerance principle, which depends on notational systems and grammars. Moreover, its wide acceptance in various areas of ontology designs is clearly a fact we must acknowledge, as expressed in the statements expressed by Kutz et al. as follows [137]:

> "[...] we believe that from a methodological viewpoint, and especially from its sheer practical usefulness, the advantages of adopting a position of logical pluralism in ontology engineering can hardly be seriously challenged, [...]"

Note that logical pluralism in this citation has adopted Carnap's tolerance principle. Another relative debate, which has been attributed to Priest's challenge, must be discussed here.

> "Take some inference $\alpha \vdash \beta$ that is valid in K_1, but not in K_2, and suppose that we know (or assume) α; are we, or are we not entitled to accept β?"

Beall and Restall respond to this question by claiming that BRLP is not about the plurality of what is true in a case. It is about the plurality of the "logical consequence relation." This means that they would prefer not to take into consideration: "β is true" with respect to K_1; "it is not true" with respect to K_2. This is because if they did, as mentioned previously, they would be accused of adopting relativism with respect to truth.

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[13]In the latter case, "logical truth" is naturally seen as a "degenerate" of logical consequence. (For details, see [192], p. 39)

This famous debate has been discussed widely (compare e.g. [10], [12], [173], [178], [189], [190]). Stephen Read's review ([189]) is particularly worth noting: If the inference: $\alpha \vdash \beta$ is valid in K_1, but not valid in K_2 should we conclude that β is true, given "α is true"? Since Beall and Restall deny that they are relativists with respect to truth, the question, "is β true?" would then have a *determined* response. With regard to this inference, Read argued as follows:

1. it is *apparently* expressing the "information": β is true in K_1;

2. it fails to express any "information": β is true in K_2;

3. it does not, moreover, express any "information": β is *false* in K_2.

The information carried by K_2 is insufficient to determine the truth of β in K_2; on the contrary, the information carried by K_1 definitely tells us about the truth of β, i.e., β is true in K_1. Hence, by BRLP, if K_1 and K_2 are both good on the account of validity, only K_1 can enable us to understand what we want to know about "β is true". Read refers to this as "K_1 trumps K_2". In this manner, he further concludes that K_1 and K_2 are not "equally good" *"in a very real sense"* since K_1 answers the question of whether β is true, but K_2 does not. ([190], p. 195)

However, we believe that Read's argument is not correct to a great extent. This is because although the information carried by K_2 fails to express the truth of whether β is true, it succeeds in expressing that "it is not the case that β *is true*" There is no reason to support that the information carried by K_1 is not "as equal as" K_2, if we consider "good" in a more general sense. Further, K_2 carries even more information than K_1 with respect to quantity. The meaning of the usage of "equally good" here is completely different to that of BRLP. In BRLP, the usage of "equally good" focuses on the "equality" of the different features of the same thing. Each of these features, in the sense that they describe the same thing, is equal to the other. This means that in BRLP, there is no dominant feature regarding negation, no dominant logical consequence, and no dominant logic.

Remark 33. Indeed Beall and Restall are aware that such different logics are based on different philosophical analyses:

> "Rival logical theories, such as intuitionistic logic, paraconsistent logics, relevant logics, connexive logics, and so on, are based

on different philosophical analyses of this basic notion." ([12], p. 488, [192], p. 36)

However, they still regard the three logics as equally good (irreducible). Moreover, as we have discussed in this chapter, given two specifications of logic, classical logic and relevant logic, for the inference $\alpha \vdash \beta$, which is classically valid but not relevantly valid, when α is told true, does it tell us β is true? BRLP includes the statement: we are classically entitled to this inference, and we are not relevantly entitled to this inference. Hence, for telling α is true provides β is true, it is because classical logic is entitled in this case, so it is classically valid instead of relevantly valid.

Taking these positions to deal with various non-classical logics which concern the replacement of classical logic is the speciality of BRLP. Some logical monists, like Stephen Read, think that what relevant logic actually expresses is not only to obtain information, such as "β does not follow relevantly from α", but also "we are not entitled to this inference". Read does not think Beall and Restall's description and realization correctly catches the crucial point that the proponent of relevant logic intends to express in the Priest's challenge, he said: "Relevant logic was not put forward as a mere alternative to classical logic." ([189], p. 196) It is apparent that relevant logic is based on monism. It is to replace classical logic instead of just being an alternative.

4.5 Conclusion

In the introduction, we formulated two questions to address the foundational discussions of logic translation. We moreover sketched the landmark ideas of pluralism in logic that were raised by Beall and Restall to answer these two questions. The core of this chapter examined BRLP which is widely discussed by the logic society and formulated the corresponding position of monism with respect to BRLP. BRLP as a pluralism encompasses some logics that are presumed to rival each other.

Chapter 5

The Geography of Cognitive Modes

Contents

5.1   Introduction

In this chapter, we would like to shed new light on the pluralism in logic, particularly in relation to Universal Logic project and logic translation studies. Before continuing we should consider one different academic research in which the same term "universal logic" appears at first glance to satisfy the aims of classicalism AI – **the good old-fashioned AI (GOFAI)** ([121], [122]).

Compared with the Universal Logic project that was connected with the spirit of Universal Algebra, a universal logic system **C-UniLogic** (see Appendix) which was realized as a "universal" logical system was proposed by

Asian.[1] The development of universal logic systems by some Asian is to pursue the trend of the GOFAI. They propose to build a smart, open, self-adaptive framework to describe the fascinating parts of the human brain with properties that are integrated, flexible, dialectic, and evolutionary. Classicalism AI is seen as an opposite theory connectionism AI. It asserts that the core task in human intelligence investigation is to symbolize and formalize "thoughts" so that they can be understood by computers. In the sense we have considered so far, the development of such a universal logic system is an obvious case with different interests and purposes from the Universal Logic project.

As shown in the previous chapters, logical pluralism has already been described in some application-oriented research in the literature, where it is believed that there is no one true logic, nor is there no one true model theory. In this way, discussion of psychologism, and so of the term *new logic* in relation to cognitive science and computation, in Dov Gabbay's words (1990, 2001), will be carried out in the domains of interdisciplinary and cross-cultural studies.[2] This could shed new light on the development of modern cognitive science and artificial intelligence, in particular, to introduce paraconsistent logics with dialetheism and dialectical logic within these areas. Moreover, it provides us a coherence notion about non-western logic allowing for inconsistency and characterizing the cognitive processing in the Asian to our cross-cultural studies. Yet it presents a universal logic system with dialecticism (C-UniLog) that was raised in the nineties, almost at the same time as Jean-Yves Béziau's formulation of the term Universal Logic.

In this regard, the universality of the modes of cognitive processes will be challenged in this chapter. The argument connects the factors of Cultural–psychobiological and developmental psychological biases with the related factors that we have found in the Universal Logic project. Finally, if logical pluralism may be revised, it is clearly possible for the combination of the three factors, psychologism, cognitive modification, and the tolerance principle to formulate a new meaning of pluralism in logic, namely **cognitive processes**

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[1]The first, the second, and the third editions of the *World Congress and School on Universal Logic* were held in 2005 (Switzerland), 2007 (China), and 2010 (Portugal), respectively. The fourth edition will be held in 2013 (Brazil). Readers should consult the official website of Unilog: `http://www.uni-log.org/` for further details.

[2]The only major critique of psychologism in the development of logic, with the partial meaning of applying psychological techniques to logic, was given by Gottlob Frege and Edmund Husserl.

modification pluralism.

5.2 The Modes of Cognitive Processing

Recall the underlying motivation of the Universal Logic project. There is a substantial point of agreement with Béziau's treatment of this proposal: it is a general theory of logics. A proper definition of logical structure and semantics will ensure that this idea does not depart from the Tarskian logic tradition too much. In the theme of the journal *Logica Universalis*, Special Issue "Is logic universal?", the Universal Logic project has caused linguistics, logicians and philosophers to raise this question again (see [26]). The core of this chapter develops a new direction for the Universal Logic project that shows the appearance of cultural differences that were discovered in logical studies.

Briefly, by employing two cultural streams: the Aristotelian stream and the Confucianism-Taoism-Buddhism stream, there are two ideas of universal logic: the Universal Logic project which follows the Aristotelian stream and a universal logic system (Confucianism-Taoism-Buddhism stream). In the last chapter, we addressed the question: Is there a universality of the modes of cognitive processes? This question requires some empirical study in cultural psychology as presented in this chapter. We can now pinpoint the question.

Here we offer a result for comparing the modes of cognitive processes in different cultures, based on an implementation for cultural psychological studies, which is further applied to the principle of tolerance that will formulate a new sense of pluralism in logic studies. According to the cultural psychology investigations, it will sometimes be more difficult for the Westerner to accept eastern schools of thought, such as Taoism or the concepts of yin and yang due to the fundamental differences in their cognitive processes. There are at least two systems of thoughts that exist in different cultures, one is the holistic system and the other is the analytic system. They also reflect the two entirely different cognitive processes of the Asian and the Westerner, respectively (see [156], [158], [198]).

Several relative contributions in cultural psychology describe how systems of thoughts contribute to the formation of theories in different cultures. The Asian adopts a holistic attitude toward the relationship between a part and the whole; they seldom use the framework of formal logic but instead they use dialectical reasoning. On the contrary, the Westerner adopts an analytic

attitude toward an object and the categories to which it belongs. They use rules to realize the behaviors of an object, e.g., the formal logic (compare e.g. [158], [161]).

Moreover, there is a significant extent to which the Westerner is interested in categorization, which provides rules in addressing various issues during the course of education. Formal logic indeed plays an important role in this process of education and problem solving. On the contrary, the Asian is concerned with the contexts within which an object exists; the world is more complicated for them than for the Westerner. When they face an event, they feel the need to consider many factors and the relationships among these factors. They definitely do not try to understand events through a deterministic framework. For them, formal logic does not play a key role in solving problems. In other words, the fundamental beliefs about the nature of the world that are held by people from different cultures differ (compare e.g. [156], [158], [198]).

5.3   Cognitive Instrumentalism in Logic

We have here two very different conceptions of universal logic, and the situation is considerably more perplexing with cultures. Here no single selection of fixed habits, terms, and ordinary usage of languages, etc. produces a uniformly plausible pattern of cognition modes. With different cultures, the only way to have a reasonable communication and develop understanding is by using cross-cultural comparison. It is clear that with the relatively conservative statement of the general theory of logics compared to the statement of proposing a universal logic system, the definitions produce some intuitively reliable results applied to various areas (e.g. [19], [60], [217]). However, we have no assurance that this is a cross-cultural method to produce the right assessment for every results expressible in different cultures.

5.3.1   The Cognitive Toolkit

In [156], there is a discussion that beliefs about the nature of the social and physical world will affect beliefs about how to obtain new knowledge and the cognitive processes (pp. 35–37). Two cognitive processing modes are observed in cultural psychology: the dialectic logical mode and formal logical mode (see [156], [157], [159], [158], [161]), where the conception that logic

as cognitive process is a common assumption. Thus, adopting the position of taking logic as the cognitive process through which basic beliefs about the nature of the world are formed, there is such an analogy in the Claude Lévi-Strauss's classical *The Savage Mind* (*La Pensée sauvage*): when people endeavor to solve the problems of daily life, it is likened to craftsmen with a cognitive toolkit which provides them with tools to create their piece of art; different cultures reflect different preferences regarding their choice of tools and their mastery in making such a choice, as well as the skills and the appropriateness of the timing associated with their choices. Similarly, in Nisbett and Norenzayan's words:

> "[...] actual possession of particular cognitive processes may differ across cultures in that different cultures may invent composite cognitive structures out of universal primitive ones, thus performing feats of cognitive engineering, as suggested by Dennett's (1995) characterization of culture as a 'crane-making crane' [...] As the mutual interdependence of culture and cognition becomes better understood, "crane-made cranes" such as these will tell us much about the cultural foundations of the cognitive tools of everyday life [...]" ([157]).

Hence, according to [156], [157], [158], there are two representative world-views or reality-views: the holistic world-view and analytic world-view. On one hand, if the world (reality) is largely determined by the relationships between objects and events, then the ability to observe all important elements in the surroundings, the relationships between these elements, and the relationship of the part with the whole is important. For the time being, the development of the process of attention, perception, and reasoning focuses on identifying important events and distinguishing the complicated relationship between events. On the other hand, if outcomes are significantly determined by objects' behaviors which are, in turn, significantly determined by rules and categories, then the ability to identify objects by distinguishing them from their surroundings and their contexts as well as their abilities to infer the rules and categories to which they belong assumes importance. Depending on how to consider the world (reality) functions, the processes of cognition would then involve the development of the corresponding abilities.

5.3.2   The Dialectical logic mode and 'Principles'?

By the analogy that logic is seen as a cognitive tool, then dialectics (dialectic logic mode, thereafter) is a key tool in the toolbox for the Asian. Before we can address the issue of the role of dialectics in various empirical sciences, such as cultural psychology, it is important to discuss the attitude of the Asian to contradiction. With no preference for using formal logic (formal logical mode, thereafter), the Asian does not look for ways to "resolve" contradiction but rather "facilitates the transcending" (or synthesizing) of all contradictory situations. Further, they may accept some opinions that are not consistent or harmonious, but facilitate "edification" and "enlightenment." Another challenge posed in [156], [158] pertains to the Asian's infrequent reliance on formal logic and a greater reliance on experiences in the process of reasoning as compared to that by the Westerner. Moreover, other studies reveal that the Asian has maintained such tendencies nowadays (see e.g. [156], [159]).

The origins of what we consider as the dialectic logic mode in Asian can be traced back to ancient China. Following up the aforementioned statements about cognitive processing, a further investigation was made that Asian are more likely to prefer proverbs that explicitly contain contradictory meanings within them (see [156]). For example, "too humble is half-proud". On the contrary, the Westerner is more likely to prefer proverbs that are free of contradictions. For example, "one against all is certain to fall." Emphasizing these transcending and synthesizing contradictions opens up a promising avenue for assessing these contradictions. We can now consider the abstraction of various principles that govern dialectics, even at the risk of violating its very spirit that no "explicit rules" should be associated with dialectics. Three principles of dialectical reasoning have been formulated. Clearly, there are at least three principles concerning dialectics that we can endorse without hesitation. First is the simple principle of the nature of reality, the principle of *change*: ([156], [158])

 a. **The Principle of Change** (1) The nature of reality is that of constant change, that is, reality is a dynamic and changeable process; (2) a thing will not resemble itself over time because of the fluid nature of reality.

 Second is a principle, so to speak, *about logic*

b. **The Principle of Contradiction** Oppositions, paradoxes, and anomalies are continuously being created. Thus, the old and new, the good and bad, the strong and weak coexist and are dependent on each another for their existence.

This second principle follows from the fact that reality is not precise or cut-and-dried but is full of contradiction ([161], p. 743). The third is a principle follows from the second principle.

c. **The Principle of Relationship** Nothing either in human life or nature is isolated or independent; instead, everything is connected.

Principle (a) puts forth the Asian worldview, principle (b) expresses the consequence of the constantly changing nature of the world, and principle (c) explains that due to the nature of reality, which is characterized by change and opposition, in order to meaningfully consider a part, it is essential to consider its relationship with other parts and with the whole as well. Clearly, if these three principles could accurately characterize dialectics or dialectic logic mode, then it would also be easy to understand the deep-seated reasons for and the effect of culture on the preference of attempting to build a universal logic system to unify various logics that have been described (see Appendix).

Finally, experience is the dimension that the Asian culture stresses should not be ignored. We will not call this the principle of experience. Of course, the accounts above take some formalization to be relativized to a sort of cognitive mode, of "logical mode", and so the underlying principles will be a somewhat formulation of the observations in the Asian cultures, histories, and societies. There is a single way these principles might go, and we will consider it in due course. In relation to experience, it is important to see that the Asian bases their accounts on this rather unlikely principle, in some form or other.

Indeed, the substantial technical and mathematical attraction of the modern account of various logic applications, e.g. many-valued logic, paraconsistent logic, fuzzy logic that are widespread in the Asian is mainly derived directly from principles (a) and (b). Assuming the above-mentioned observations are right, it is experience that allows the direct application of well-known ideas for defining "truth" to the task of defining *logical truth* in the various widespread logical systems mentioned here.

This is an important selling point for the account of dialectics. Our ordinary concept of truth, even logical truth involves various notions that are notoriously difficult to pin down, notions such as *necessary*, *possible*, and so forth. But if the modern *philosophical logic* (non-classical logic) account is correct, what it achieves is a truly remarkable description of obscure notions to mathematically tractable ones. If this is right, the development of philosophical logic shows that we can in fact sidestep all of these difficult concepts, and that we can given a mathematically precise definition for them if we can define the notion of truth, even logical truth for the formal languages – or what comes to the same thing, if we can figure out these relative notions to some reliable explanations.

This is a tremendous advantage, one we should not undervalue. Furthermore, it is an advantage not well-shared by Westerners. As Nisbett states:

> "It is precisely because the Chinese mind is so rational that it refuses to become rationalistic and... to separate form from content." ([140] and [156], p. 165)

If experiences should not be ignored, then such kinds of thoughts should be considered seriously. The Chinese do not encourage people to think about things and events that are too abstract; rather, they encourage people to be more practical. In accordance with principle (a), the fact that everything changes is core to their vision of the world and is especially influenced by three traditional philosophies – Taoism, Confucianism, and Buddhism ([156], [157], [158]). Due to "changes", in the context of the knowledge system in the Asian, people could believe that a proposition and its negative proposition could to be at the same time. Consistency is not a necessary condition in this kind of belief system, even when it is claimed that the existence of a proposition implies the existence of its negative proposition, or it is claimed that the negation of proposition *appears soon* after the proposition.

These notions are unacceptable in the traditional Western system of thought. When the Asian engages in daily reasoning, they appeal to dialectical notions from the very outset ([158], [161]), specifically, in assessing the counterfactual statements or engaging in the counterfactual reasoning that plays a key role in Western society, they appeal to the non-Westerner pattern. Ben Goertzel makes a similar statement as follows:

> "[...] After all, every Chinese mathematician uses reductio ad absurdum, a theorem-proving strategy which is explicitly counterfactual in nature. Obviously Chinese mathematicians develop

> a mental 'schema' for applying counterfactual reasoning to mathematical statements. [...] My informal survey indicated that Chinese people, even those who speak reasonable English, are simply not comfortable thinking counterfactually about commonplace situations. Counterfactual reasoning in mathematical proofs would seem to be, psychologically, a different "routine" from counterfactual reasoning regarding politics and everyday life. This is an intriguing example of mental 'modularization'. Just as a person who reasons logically about chess need not reason logically about her boyfriend's activities, a person who reasons counterfactually in mathematics need not reason counterfactually about commonplace real-world events." ([110], pp. 92–93)

Interestingly, the birth of paraconsistent logic in modern formal logic displays similar ideas found in the thoughts of Asian ([172], [173], [174], [175], [176], [183]). Further, the path taken by paraconsistent logic is similar to that taken by dialectics, namely, both attempt to overcome or transcend two contradictions in a situation instead of attempting to resolve them.[3] Hence, within the cultural-anthropological context, while logic is treated as a cognitive process, the details of the story of logic are completely different from what we learn in our courses on logic.

In contrast, the development of formal logic tries to capture the ordinary truth of statements and intuitive validity of arguments by means of treating logical forms suitably. A philosophical criticism about Tarski's model theory which plays an important role in modern formal logic has been presented by John Etchemendy, namely that Tarski's model-theoretic definition with representational semantics which equates the logical truth of a sentence with the ordinary truth of another sentence is misleading (compare [77], [78], [79]). The problem with such a reflection is that the mere truth expressed in such (Western) development of logical studies can, in general, guarantee nothing more than the truth of its logical forms. It cannot guarantee that the arguments have any intuitive validity and ordinary truth of statements. In particular, it cannot guarantee that their formal logical distinctive features, whether proof-theoretic or model-theoretic, which usually were thought to set logical truth, capture the common, daily, run-of-the-mill truths.

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[3]According to Nisbett's observation, the Chinese cultivated a different dialecticism from Hegelian, which claims that the "thesis is followed by antithesis, and resolved by synthesis." ([156], p. 27)

The non-Western pattern of thought simply shows that the truth of the ordinary statements did not completely depend on the specific meanings of the selections of formal methods. People do not always think of truth in the formal logical way, and the ordinary concept, as relativized to a completely formal methodological selection cannot be fully captured only in this way.

5.4   From Cross-Cultural and Interdisciplinary Notions of Logic to a New Sense of Pluralism

According to the considerations that we have put forth, as pointed out by Nisbett, because of differences in cognitive processes, we believe that we cannot arbitrarily comment on people with cultures that differ from ours. We are not concerned here with the merits or demerits of the Western pattern of thought, and in particular we will also not spend time discussing those non-Western pattern of thought. On a daily basis, logicians, as human beings, face problems on daily reasoning and thinking. Although logicians know much about and have worked extensively with inferences and logic, some daily issues are inevitably encountered in the daily course of life. The cross-cultural logics investigation also gains importance when considering the problem of cognitive processing in the empirical studies mentioned in this chapter. Moreover, our discussion so far will offer not only a pragmatic motivation but also empirical evidence to support our formulation of a new sense of logical pluralism.

To relate the issues between the Universal Logic project and the idea of a universal logic system to the cross-cultural aspect that is the focus of this chapter, it is apparent that this issue has gone beyond the confines of logic that goes back to the Gottlob Frege and Edmund Husserl's *anti-psychologism*[4](compare e.g. [100], [130], [135]). The contemporary notions of 'Logic' were claimed by Dov Gabbay to be a result of the interaction with

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[4]A more general usage of the term 'psychologism', is given in [135]: "many authors use the term 'psychologism' for what they perceive as the mistake of identifying non-psychological with psychological entities." To relate to the issue raised in philosophy of logic,[...] 'Psychologism' then refers (approvingly) to positions that apply psychological techniques to traditional philosophical problems." Briefly, in logic, people who think that logical laws are not identical to psychological laws or do not apply psychological techniques to logic, would be viewed as adopting anti-psychologism.

computing ([103]), a decade later, the term *new logic* was specified as being related to cognitive science clearly defending a *psychologism* position ([100], p. 170).

> "Still, it is fair to ask us to say with clarity where we see the difference, such as may be, between logic and psychology. Where, in particular, is the divide between the new logic and theoretical cognitive psychology? Our answer to this is that much of what the new logic comprehends is theoretical cognitive science, including relevant parts of computer science and any other discipline that bears in a principled way on how cognitive processes operate."

The general idea of logic which emerged from the tradition within mathematics in the first decades of the 20th century, with the birth of relative interactions between algebra and logic in mathematical logic (see chapter 1; compare e.g. [45], [50], [51], [55], [54]), could have a whole different look. It has been suggested that Universal Logic as a mathematical theory, which is connected the Universal Algebra, be developed by employing psychological techniques to logic, especially since the idea of a universal logic system has taken the same term "universal logic" but with an opposite meaning. This suggestion moreover has directly led to the following "general and ambiguous questions": "Is logic universal?" Five questions in particular are raised in the Special Issue of *Logica Universalis* concerning aspects of universality in human reasoning: ([27], p. 161–162)

1. Do all human beings have the same capacity of reasoning? Do men, women, children, Papuans, yuppies, reason in the same way?

2. Does reasoning evolve? Did human beings reason in the same way two centuries ago? In the future, will human beings reason in the same way? Are computers changing our way of reasoning? Is a mathematical proof independent of time and culture?

3. Do we reason in different ways depending on the situation? Do we use the same logic for everyday life, in physics, and in questions to do with the economy?

4. Do the different systems of logic reflect the diversity of reasoning?

5. Is there any absolute true way of reasoning?

The two ideas of "universal logic": the Universal Logic project and a universal logic system are born from entirely different academic research and can be distinguished from each other. Nevertheless, the succession of the Universal Logic project that expresses a general theory of logics comes from a series of studies on paraconsistent logics ([39]) in which the underlying thought is to reject the *principle of explosion*, also known as *ex falso quodlibet* or *ex contradictione sequitur quodlibet* meaning "from a contradiction, anything follows" which is the opposite to the Western tradition.[5] While the Westerner attempts to emphasize formal reasoning in critical thinking, argumentation, and debate in their "system of education," Asians try not to treat every proposition as "bivalent", namely, either true or false. The difference in the behaviors and thoughts of the Asian and the Westerner has been reflected in many aspects of daily life. This can be explained by the distinction between holistic and analytic attitudes ([156], [157], [158]). The role of Chinese dialecticism in the modern world should be taken seriously. Moreover, various positions with a not so confined notion of 'logic' should be adopted, while interdisciplinary studies have become widespread in academia.

5.4.1   Revision of Pluralism in Logic

While interdisciplinary information science is predominant, there are many technologies in use, with different cultural-related considerations, user communities, and tool support (e.g. [47], [58], [115], [144], [146], [201], [215], [228]). As stated by Kutz et al.,

> "Logical pluralism de facto permeates almost all areas of knowledge engineering; its wide acceptance is clearly a fact we have to acknowledge. Indeed, we believe that from a methodological viewpoint, and especially from its sheer practical usefulness, the advantages of adopting a position of logical pluralism in ontology engineering can hardly be seriously challenged [...]" ([137], p. 262)

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[5]The principle of explosion: Call $\models$ is explosive if it validates $\{\varphi, \neg\varphi\} \models \psi$ for every φ and ψ. Paraconsistent logics can be defined in the most general sense as any logic with a non-explosive consequence relation. A branch of paraconsistent logic called *dialethesim* argued that it is the view that some contradiction are true, i,e, there are sentences, statements, propositions, or anyone taking to be truth-bearers φ, such that both φ and $\neg\varphi$ are true.

In this manner, the mutual exclusivity of the distinction between logic and psychology, especially to logic and psychology about cognition, including relevant parts of computer science and any other discipline will be eliminated. The trend of scientific and technological developments in the modern era acknowledges that there was no good reason to encumber the very ideas of widespread logical languages "in applications" and various interdisciplinary studies about various logic. Hence, the relationship between logic and psychology can be considered seriously, the stance of relating non-psychological with psychological entities was not a mistake in everyday practice. For example, the notion of *reflective equilibrium* might have been taken as a challenge to anti-psychologism. The notion of reflective equilibrium proposed by Nelson Goodman in *Fact, Fiction, and Forecast* (1955) is a method to justify rules of inductive logic. The conception of reflective equilibrium can be seen as a mechanism which puts the rules of inference (inductive or deductive) and those inferences that we have judged to be acceptable in a broad range of particular cases, in the same basket, named reflective equilibrium, while we justify these rules. It emphasizes (a) no inference rule is acceptable as a logical principle unless it is compatible with those instances of inference reasoning which we admit to be acceptable, or we will abandon this inference rule; (b) at the same time, if we find these individual inferences are incompatible with inference rules that we generally accept and unwillingly abandon, then we have to revise our views about those individual inferences we might have taken as acceptable initially.[6]

Note that the value of mentioning reflective equilibrium not only lies in being a practical example of endorsing psychologism but also in the cultural differences discussed in this chapter, and moreover supports our formulation of the pluralism in logic. Stephen Stich in *the Fragmentation of Reason* (1990) observed that the cultural difference will shatter reflective equilibrium (see [202]; compare e.g. [156], [158]). In this way, if we cannot agree with the principles (inference rules in reflective equilibrium) that are used to judge the correctness of an individual inference, then we cannot help but wonder whether such a disagreement is merely due to personal preferences regarding the underlying principles.

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[6]See also, *Stanford Encyclopedia of Philosophy* – "Reflective Equilibrium". ([70])

5.4.2   Some Cultural–psychobiological Perspectives on Modification of Cognition

So far, our proposal that logical studies involve the cognitive processing dimension and cross-cultural empirical evidence would not seem odd. For example, Graham Priest stated: ([173], p. 150)

> "I believe that the development of modern formal paraconsistent logics is one of the most significant intellectual developments of the twentieth century. In challenging entrenched Western attitudes to inconsistency that are over 2,000 years old, it has the potential to ricochet through all of our intellectual life – and empirical science wears no bullet-proof vest. As we have seen, empirical scientists have always tolerated, and operated within, inconsistency in certain way."

Three ideas, then, in the combination, summarize our formulation of pluralism in logic: **psychologism**, **modification of cognition**, and the **principle of tolerance**. First, with regard to the BRLP, it involves the identification of consequence relations, i.e., to identify them with the concept of logical consequence. In other words, it appears that the proposal of logical pluralism could also be seen as one possible method for dealing with the problem pertaining to the identification of the contemporary notions of 'Logic'.

Second, by adopting the position of psychologism, we further make an assumption that logic is a cognitive process in the cultural psychological perspective ([156], [157], [158]). There are at least two underlying modes of cognitive processes in cultural psychological perspective. Third, with the new trend and investigation of logic translation applications, e.g., ontology design and onto-logical translation graph ([137], [150]), we suggest to address the study on the translation between the two underlying modes of cognitive processes by referring to the cultural psychology research in a broader sense of applications, that is, to formulate the translation between a formal logical mode and dialectical logical mode. Another suggestion, the *descent-with-modification*[7] of cognitive modules from a psychobiological viewpoint, can

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[7]The phrase *descent-with-modification* comes from the Charles Darwin's classic *On the Origin of Species* (1859) where Darwin developed two main ideas, one being descent with modification and the other natural selection. The former explains life's unity and diversity , the latter is the explanation for adaptive evolution.

be found in [147]. In accordance with the descent-with-modification modularity, current cognitive modules are understood to be shaped by evolutionary changes from ancestral cognitive modules. It moreover argues that descent-with-modification modularity is compatible with a range of data in neuroimaging studies, and this conception of modularity may have important implications for the practice and conception of cognitive scientific fields such as linguistics and psychological (developmental) disorders. The main notions expressed in the descent-with-modification modularity approach to cognitive architecture and processing may be summarized as follows:

- Evolution has, through descent-with-modification, created the entire array of living organisms on the planet earth, including human beings; descent-with-modification has not constrained biology to any single narrow niche.

- Descent-with-modification provides the constraint, it does not strictly limit the results, but limits the process: the question is always where one could go from here (or get here in the first place), not about where one could get by starting from scratch. As such, the more we know about ancestral systems, the more we are able to put the notion of descent-with-modification to use.

- The adult cognitive (and neural) structure should still be presumed to be a reflection of both experience and genetic factors– the position that cognitive modules are understood to be shaped by evolutionary changes from ancestral cognitive modules does not in any way lessen the role of experience but instead to suggest a different way of thinking about the nature of the genetic contribution.

- Descent-with-modification made us stand a better chance of understanding cognitive and neural function if we studied particular cognitive domains not as entities unto themselves, but through a comparison with other current systems that might plausibly have descended with modification from common ancestral systems.

"In the On the Origin of Species, Darwin (1859) argued that 'existing forms of life are the descendants by true generation of pre-existing forms': evolution never starts from scratch, but instead by modifying the structures of organisms already in place." ([147], pp. 446–447)

In an interdisciplinary manner, we have intersected the realms of culture psychology, with the realm of cognitive science, and an investigation have been addressed whether the cognitive differences will disappear in the emerging world of ethnic differences ([156], *Epilogue*). Formulating a logic translation between the formal logical mode and dialectical logical mode will shed new light on the meaning of the principle of tolerance that is used to characterize pluralism in logic. We believe that the *institution theory* which provides the idea for logic translations between different logics in use with a certain sense of logical pluralism[8] will also serve as a framework for the translation between the two modes mentioned here by translating the syntax and semantics of different formalism. This formulation of pluralism in logic brings the term closer to the stronger meaning of pluralism. The cultural psychological assumption that logic is taken as the cognitive process enlarged our perspectives of pluralism in logic. It is similar to take "logic as algebra" in algebraic logic pertaining to mathematical enterprises, and "logic as structure" in the structuralism pertaining to philosophical enterprises, "logic as cognitive process" that should be pertain to cognitive scientific or psychological enterprises will also enlarge the perspectives in the Universal Logic project.

In regard to logical pluralism, we have addressed some relatively basic problems in former chapter as follows:

1. Is it correct to claim the singularity of logic?

2. What is the debate between monism and pluralism in logic?

3. Is there a universality in the mode of cognitive processes?

The third question will be discussed. To adopt the aforementioned cultural psychological viewpoint, that is "logic as cognitive process", our answer should clearly be "no". Nonetheless, this answer does not in any way reject

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[8]Goguen and Roşu stated:

> "Mathematicians, and even logicians, have not shown much interest in the theory of institutions, perhaps because their tendency toward Platonism inclines them to believe that there is just one true logic and model theory [...] On the other hand, computer scientists, having been forcibly impressed with the need to work with a number of different logics, often for very practical reasons [...]" ([111], p. 294)

universality in the other sense, where universality pertains to various objective and systematic scientific findings for common features for the cognition of *Homo sapiens*. However, it does not presuppose that there is a unique way of reasoning or one mode of cognitive processes.

Let us now return to the principle of tolerance which will subsequently formulate a different version of pluralism in logic. The tolerance principle we consider must still be within the framework of regarding logic as a cognitive process. In relation to Carnapian principle of tolerance, it stated that people should not be forbidden to choose a logic for themselves, albeit people should provide sound reasons for supporting their choice and the consequences of their choice. "The liberty to choose" characterizes the core essence of the principle of tolerance. Similarly, in BRLP, with the other plurality (irreducibly plural) the common concept of logical consequence has been articulated to claim that classical logic, intuitionistic logic, and relevant logic exemplify the notion of logical consequence equally. As mentioned in this chapter, in cultural psychological and psychobiological perspectives, the modification of cognition for different typical and representative cognitive processes has been discussed: firstly, *cognitive processing* had being changed due to the change in the reality of societies; secondly, descent-with-modification of *cognitive structures* with genetic factors which states cognitive modules are shaped by evolutionary changes from the ancestral cognitive modules. To mark the principle of tolerance as encompassing these variants, including cultural differences, it is possible to structure a logical pluralism as the **cognitive processes modification pluralism** in a broader sense.

5.5 A Universal Logic System with Dialecticism

This formulation of pluralism in logic may be carried out by reviewing the differences between a large extent of the Universal Logic project and a universal logic system. The notion of a universal logic system with dialecticism was introduced by Hucan He et al. in the 1990s. As mentioned in the beginning of this chapter, apart from supporting the classicalism of AI, He et al. aimed to develop a "continuous" and "controllable" logical system such that it is possible for it to be developed as a logic for dealing with different possible cases. We called such a universal logic as the C-UniLog (see Appendix).

First of all, it will be helpful to discuss the concept of logic they hold:

- Logic is a sort of criterion or device that can be applied for judging and regulating doctrines and theories. Logic can be found in all doctrines and theories by abstracting and extracting judgments and deductive rules. It cannot stand alone outside the development of doctrines and theories ([125], p. 1).

- Theories should conform to the logics that they own.

- An existing logic needs to be refined or expanded; otherwise, it must be able to explain new scientific theories and discoveries.

In the literature, the development of C-UniLog is still at a nascent stage ([125], p. iv, pp. 21–22). Based on two main characteristics: *the continuity of the domain* and *controllable propositional connectives operations*, attempts to develop a logic to deal with all circumstances. C-UniLog is with mathematical formal logic (rigid logic) as the kernel, whereby various flexible logics can be changed arbitrarily according to the application requirements.[9] Supposedly, He et al. attempted to deal with and to limit themselves to a rarer pragmatic situation by applying their conception of logic.

What role does, or should, the C-UniLog play in the concept of Universal Logic? This is the question that is to be addressed in this section. The question is hardly a new one, but the development of the Universal Logic project makes a profound statement on the subject as follows:

> "Let us immediately reject some misunderstanding; universal logic, as I understand it, is not one universal logic. In fact, from the viewpoint of universal logic the existence of one universal logic is not even possible, and this is a result that can easily be shown. One might thus say somehow ironically the following: according to universal logic there is no universal logic." ([32])

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[9]He et al. clearly refer the rigid logics based on the modern mathematical formal logic that can only be used in a close, hologram and binary world such as the world of mathematics. It is not necessary in the contrary sense that flexible logics refer to the logics that are capable of handling contradictions and uncertainties (compare e.g. [123], [125]). A similar usage of these terms can be found in [218], p. 39 where rigidness means "justifiability" in inference steps and flexibility means "context-dependency" of the inference processes for the activities of a reasoning system.

Proponents of C-UniLog realize that their subject has important implications for empirical sciences, but discussions on applications of C-UniLog have focused largely on non-Western society. It therefore seems inappropriate to address the question directly.

I will first address the issue of the speciality of C-UniLog: dialecticism. Next, we will look at the role that dialecticism has played in modern science, and the relation of this to C-UniLog. This will raise the question of how the theoretical status of C-UniLog *ought* to be treated in science and if such acceptance should be more than provisional? These topics will be addressed in this section. An outcome of this discussion will be that, in the light of developments in Universal Logic and logical pluralism, we may well have to change our attitude to variants of certain kinds in logic; such a change would open new possibilities in science itself.

5.5.1 From Fuzzy Logic to a Universal Logic System

To understand C-UniLog, we need to distinguish clearly between it and the Universal Logic project. To do this, we need a clear idea of what the former approach is all about. A good place to begin is with the basic understanding of fuzzy logics begun in the 1965 proposal of fuzzy set theory by Lotfi Zadeh. In this well-known article: *Fuzzy sets*, Zadeh defends his own approach to the calculus of logic, drawing on an idea to use a membership function with a range covering the interval $[0, 1]$ operating on the domain of all possible values that characterize or define reasoning that is approximate rather than fixed and exact (see [119], [227]).

I am not concerned here with the merits and history of probability theory and fuzzy logics, and in particular I will not spend time considering any formal system of fuzzy logic in mathematical logic and other applications. But it is worthwhile taking seriously the limitations of fuzzy logics that have been discussed by He et al. as follows:

- Probability theory is a logical system with a continuous truth value, which introduces the continuous changeability of variables, AND-operation and OR-operation. It successfully demonstrates the changeability of these two operations in three special situations, *principles of minimal correlated*, *maximal correlated*, and *independent correlated*, with whose behaviors of operation-models.[10]

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[10]This paragraph is taken from the words and ideas expressed in [126], pp. 76–77.

- As probability theory fails to be a universal logic system, it fails to develop the changeability of the OR-operation, and the AND-operation in a general case, hence probability theory, at present, is just a special instance of C-UniLog. (For related studies, see [219].)

- Although fuzzy logic develops the continuous changeability of the OR-operation and the AND-operation, unfortunately, fuzzy logic only employs the principle of maximal correlation in its early development, thereby missing the opportunity to be developed as a universal logic. In addition, it claims that fuzzy logic is only a special instance of universal logic in the case of $h = 1$. (See Appendix.)

- Triangular norm (T-norm) theory studies the changeability of the OR-operation and the AND-operation, and determines the different operation models of the OR-operation and the AND-operation based on the three correlation principles. He et al. claimed that nobody has ever connected these with a controllable continuous logical system; hence missing the opportunity to be developed as a possible universal logic.

C-UniLog can be viewed as a theory of generalizing *"fuzzy logic in the broad sense (older, better known, heavily applied but not asking deep logical questions) serving mainly as apparatus for fuzzy control, analysis of vagueness in natural language and several other application domains"* ([119]) with a pragmatic ability to be applied to many fields, including control theory and artificial intelligence. [11]

5.5.2   C-UniLog as a Meta-theory

C-UniLog, of course, is particularly concerned with studying the so-call general principles of logic, but the following point might be made about the status of the methodology. To be a theory of analysis over the logic structure, the domain, the connectives, the quantifiers, and the value, a universal logic structure and a universal logic generator are formulated, and C-UniLog should be considered as a **meta-theory** that closely related to the family of fuzzy logic. By giving the specific parameters of the structure, the related

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[11]He et al. claimed that C-UniLog pertains to the stream of second revolution of logic, which tried to break through **three laws and one character** (TLOC) (see Appendix and [123], pp. 86–88).

various logics can be reasonably produced (see [125], chapter 12). In this regard, similar to the Universal Logic project, C-UniLog can become another general theory of logics.

For C-UniLog, there is many much work in introducing quantifiers, which is completely different from the materials in our first course of logic (see Definition 38 in Appendix). They not only introduce new quantifiers to describe every possible circumstance but also attempt to reduce the old, previously acknowledged "quantifiers" and "logical terms," $\forall$, $\exists$, $\Box$, and $\Diamond$, to a different theoretical level. In this sense, C-UniLog can naturally be understood as a meta-theory, especially to the family of fuzzy logics. Purely from the meta-logical point of view, the position of C-UniLog is similar to what people have taken as mathematical logic, i.e., to analogize as mathematical theories of studying the mathematics of logics. If we consider this viewpoint, C-UniLog can be a semi-mathematical theory of logics, which may be regarded as an artificial-intelligential theory of logic.[12]

5.5.3 Dialecticism

A number of modern logicians and philosophers, in the past few decades, have attempted to provide mathematical foundations for dialectical logic (e.g. [23], [187], [188]). *Dialecticism* (Dialectics) in a general sense is the study of those propositions that have intrinsic contradictions and extrinsic uncertainties, and to look for the formal relationship among them. *Dialetheism* which is the view that some propositions of the form φ and $\neg\varphi$ are both true, where $\neg$ is realized as the negation, should be treated as a sub-filed of dialecticism ([183]). It is usually considered a branch of paraconsistent logics. Many logics with dialecticism have appeared in special areas of artificial intelligence and law such as building theories of defeasible reasoning (compare e.g. [98], [132], [166], [167], [168]).

Though the construction of inconsistent mathematical theories or formal paraconsistent logics is relatively newer than dialecticism, there are already inconsistent arithmetic ([179], [181]) and inconsistent mathematics ([149]). In C-UniLog, an appropriate mathematical theory is found and should pertain to the family of fuzzy logics. Hence, a likely way for dialecticism to

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[12]It cannot be regarded as a mathematical theory of logic since mathematics does not allow for the concept of contradiction. C-UniLog aims to introduce the dialecticism mechanism of flexibilization to include inconsistency. Unless we revise the content of mathematics, we cannot possibly claim that this type of theory is a mathematical theory.

be introduced into mathematical logic is via what is called as the process of flexibilization. He et al. believe that the development of mathematical dialectic logic is possible in a step-by-step manner. They also believe that it is different from mathematical formal logic that has only an equivalent form, as the transforming laws of mathematical dialectic logic are infinite and are changing. He et al. state:

> "[...] It is impossible to find the general law in limited time as it has infinite inequivalent forms. However, relatively speaking, it is possible to implement the idea of universal logics according to the research compendium of universal logics: gradually advancing from the bottom level, finding the general laws of universal logics in some levels and sides, including these laws with the flexibilized method [...]" ([125], p. viii)

The question of how to introduce dialectics to mathematical logic is technically sophisticated (see Appendix). We do not intend to provide a detailed answer to this question by using their approach, which is to establish flexible logics in which there are various new principles that dialectic logic could have discovered. The main steps of this approach may be summarized as follows:

- First, abstract the *truth value* range for flexible logics, operation model clusters of flexible propositional connectives and flexible quantifiers from the real world.

- Second, prove the logical properties with respect to these operation model clusters to build the *flexible propositional (quantifier) logics* .

- Third, abstract the mathematical theories pertaining to these flexible logics.

Note that mathematical logic is treated as a sort of rigid logic. Mathematical logic consists of four core fields: set theory, model theory, recursion theory, and proof theory, which was termed the **Four Theories** by He et al. Introducing the mechanism of dialectics to re-formulate every rigid logic will generate individual flexible logic. Thus, there is a conjecture that it is necessary to adjust the Four Theories to obtain the new Four Theories (compare [123], p. 96 and [125], chapter 1).

5.6 Conclusion

C-UniLog and the Universal Logic project are developing similar but different revision methods for the same thing, namely, modern logic to put forward their ideas of universal logic. In this regard, we have discussed related works by considering some empirical studies in cultural psychology pertaining to cognitive processing, logic, and reasoning. Based on the spirit of C-UniLog, the dialectical mechanism to be introduced into mathematical logic flexibilizes mathematical logic, i.e. to mathematicalize dialectical logic.

A general challenge that has particularly been addressed is the coming of the second revolution of mathematical logic ([123]). As He et al. observed, the limitation of logic is similar to other disciplines and theories as follows:

> "Any theory will experience a process from simple to complex. In the beginning, because of the lack of theoretical foundation and research experiences, simplification is necessary for establishing the corresponding elementary theory" (Ibid, p. 89).

In this regard, the "neglect of complexity" was taken as a limitation or even a defect in the development of theories. However, as we have discussed in this chapter, this is common in western culture or in the cognitive mode of the Westerner. Richard Nisbett in [156] used a metaphor to describe this. If we consider the hypothesis that "the universe is pretzel-shaped," then it is better that the universe is "pretzel-shaped," else there would be no opportunity for us to discover the shape of the universe. Further, by holding the "pretzel-shaped" hypothesis: "we're better off starting with a straight line and modifying it as it becomes clear that the linear hypothesis is too simple" (Ibid, p. 209).

This description is a typical example. The Asian is more inclined to believe that the world is "complicated", which of course it is. Hence, the same notion can be found in their theories. Westerners, on the other hand, prefer to begin with a simple hypothesis to view the world. However, this is definitely not a defect, rather it is merely a "limitation" Of course, this word should not be understood in a negative sense.

Chapter 6

Summary and Further Research

Contents

6.1 Summary

With the various logics which have been proposed in the development of modern logic, Universal Logic could be further seen as a theory of trying to make an "identification" to different logics in order to answer a basic question: "what is logic?" This process of identification identifies the commonality and differences between various logics, moreover it could check the connections and relationships between different logics. In the past, different logics were generated by different considerations, some of these were raised to challenge the appropriateness of classical logic, e.g., intuitionistic logic.

After conducting a comprehensive investigation on the development of Universal Logic project which aims to identify abstract logical consequence, we found BRLP which is seen as a landmark in the modern pluralism in logic is qualified as an identification of abstract logical consequence. The abstract logical consequence relation was developed not only for the aforementioned practical reasons, but also for the theoretical and philosophical considerations. Moreover, various related interdisciplinary investigations pertaining either to the Universal Logic project or BRLP can be found in various areas.

We would like to emphasize that the study of logic translation plays a key role in the Universal Logic project. The notions of translation in particular have been employed by many applications of academic research. We incorporate our distinction between deviant logic translation and extended logic translation with Haack's distinction between deviant logics to support discussions in the philosophy of logic. As mentioned, according the Universal Logic project, the study of deviant logic lacks a foundation of systematic and serious theories. We admit that this criticism is also applicable to our distinction of logic translation. However, it is true that with respect to classical logic, modal logic is not considered to be similar to intuitionistic logic; further, it is true that the translation from IPL to modal system $\mathcal{G}$ is not considered similar to that from CPL to IPL. Thus, we have to admit that there should be some distinction for the concept of logic translation.

Readers may be aware of the insufficiency of the connection between the work on logic translation and the Universal Logic project. As discussed in previous chapters, one of the main goals of the Universal Logic project is to identify the "trivial" parts, which should be common to various systems of logic. This will allow us not only to ascertain the common aspects of logic but also determine their differences. One way to do this is to compare them; the work on logic translation provides the possible methods for such comparison. For example, Wójcicki has shown the reconstructability of classical propositional calculi in intuitionistic logic (([224]), [225]). As described in the contest at **the 2nd World Congress and School on Universal Logic**,

> "However Wójcicki has shown that classical logic cannot be reconstructed within intuitionistic logic - his concept of reconstructibilty being a stronger concept of translation. Gödel's translation of classical logic into intuitionistic logic shows that intuitionism is not in a sense safer than classical logic, but maybe this has to be relativised due to Wójcicki's result." ([30], p. 56)

In addition to clarifying the meaning of translation, the main objective of the citation is to express that differences between logics can be indicated through translation. Moreover, some related issues in philosophy of logic can be reexamined by providing a systematic methodology. For example, the translation paradox discussed in this dissertation was solved by Mossakowski et al. in the paper *"What is a logic translation?"* ([151]) by introducing the institution theory to study logic translations from an abstract perspective. Also, it was claimed in the paper *"13 Questions about Universal Logics"*

([32]) that the widely accepted notion about deviant logics in philosophy of logic should be treated in a more systematic way.

To study the foundational issues of logic translation, logical pluralism is naturally considered. As we have shown, BRLP represents a style of logical pluralism in the philosophy of logic. In order to justify plurality, there is a model which allows a systematic analysis of the three endorsed logics in BRLP.

Essentially, when we are out of the philosophical domain but with a integrative viewpoint, we believe that all logics used in the logic society today, e.g. paraconsistent logic, many-valued logic, quantum logic, and free logic, etc. might be able to study some deviant-BRLP models in the same way. Furthermore, this approach allows in particular a classification for different families of logics in an abstract way. The conclusion of logician Stephen Read's book review of *Logical Pluralism* ([189]) states the following:

> "Logical Pluralism presents a challenge to the proponents of alternative logic, and even to classical logicians who find alternative logics interestingly mistaken. In a short book, the authors not only raise deep issues, they also provide neat thumbnail sketches of a range of logics. It is clear from my response that I think their position is mistaken, but theirs is a challenge that must be met, and meeting it adequately is not easy. Every logician should read this book."

Yet, as we have discussed, BRLP has not comprehensively explained the exact stances that it will replace. The modern logical society should reflect on the validity of any given argument, to study the essence of validity, the concept of logical consequence, and further, to realize the applications of all these concepts.

Logical monists set up a unique criterion to claim the validity of an argument; logical pluralists set up more than one criterion. It is difficult for logicians to merely look at and not address the various logics that are widely discussed and developed in various scientific domains. BRLP excites not only those working in philosophical studies but also those working in various scientific domains related to logic. It reminds people to reflect on foundational positions when conducting work on "reality" and "applications".

The pluralistic or even the relativistic approach will not only serve as the foundational complement of the earlier chapters about the Universal Logic approach, where logic translation plays a key role, but also serves as the

foundation, we believe, to some areas of computer science, e.g. ontology design and some other interdisciplinary areas.

Theoretical computer scientists, mathematical logicians, and even mathematicians, have shown much interest in and shared the spirit of the Universal Logic project, perhaps because their tendency toward pragmatism inclines them to believe that there is not just one true logic. We have already mentioned some of the model-theoretic studies for Universal Logic that were explored, for example, the categorical abstract model theory based on Goguen and Burstall's notion of institution ([111], [112], [113]), institution-independent model theory ([72]), the axiomatization of the abstract model theory ([18]), and an abstract model theory framework for Universal Logic ([104]).

In the literature, an abstract framework for the study of sub-logic that we have introduced allows a systematic analysis of conceptual and technical problems, both of which are from the **Abstract Model Theory** viewpoint (compare e.g. [104], [151] [154]). Moreover, the abstract model theory point of view has been suggested as an appropriate direction for the Universal Logic project.

> "Although abstract logic and abstract model theory are expressions which look similar, they refer to two different traditions, ..., The combination of abstract model theory with abstract logic is surely an important step towards the development of Universal Logic." ([33], viii,)

Pluralism or relativism which was born for many application-oriented reasons has been widely accepted in various logic studies. In the early part of the 20th century, it was not common for logicians and computer scientists to conflate the general theory of logics with the term Universal Logic. According to Béziau's view, a general theory of logics and its notion of arbitrary logical structures was essentially dependent on Tarskian logic and Birkhoff's notion of Universal Algebra by taking the conceptual approach. Moreover, one of the main results of the Universal Logic project is to combine general bivaluation semantics with Gentzen's sequent calculus in order to promote the idea of *abstract logic* which plays an important role in the development of modern logic to an even more general level.

As mentioned in the previous chapters, the main work of the Universal Logic project is to identify universal and common parts of various logics, such that people can moreover apply them more or less directly to specific

logics, depending on certain situations of certain problems. It is worth mentioning again, that as a result of Universal Logic, the property of maximal consistency, which is traditionally considered to depend on specific features, when we prove the completeness of a given logic, should be attributed to the universal parts of a logic instead of taking it as specific parts of various specific logics.

We wish to emphasize certain points in our work which, though not entirely new, have not been comprehensively discussed in the literature about the Universal Logic project:

1. The notion of arbitrary logical structure accompanied with bivaluation for the Universal Logic project was already noted in the early papers on Universal Logic ([38], [46]), and it is further emphasized here through the discussion of the translation paradox. This fact makes it possible to claim logical structures are on a par with other mathematical mother structures in traditional mathematical structuralism, Bourbakism, and the relationship between logic and algebra are clarified, of which [35] and [45] are worth mentioning. This can be regarded as the primary aspect in the Universal Logic Project, leading to the so-called Neo-Bourbakism, which identifies the status of a logical structure as being at the same level as the other three mother structures in the Bourbakian sense.

2. There have been various logics in the development of the modern logic society. A conception of an arbitrary logical structure has been studied in the Universal Logic project. Universal Logic, which has claimed to be a general theory of logics, has to face the explosion of various logic systems generated by different considerations and applications in order to answer a basic question: "what is logic?" Logic translation investigations could serve the purpose of identifying the commonality and differences between various logics, as well as presenting the relationships between different logics, e.g. the onto-logical translation graph ([150]).

3. In many cases, various logics were generated by different considerations, some of which were raised to challenge the appropriateness of classical logic. The development of intuitionistic logic is the succinct of classical logic without the law of excluded middle. Thus, it was traditionally seen as a sublogic of classical logic. In every single example

we have discussed, the logic translation studies have a notion of inclusion between classical logic and intuitionistic logic. However, in the progress of logic translation between **KGGG-EW** and **BBS-BEW**, the translation paradox shows that it is not clear enough to indicate the meaning of sublogic, which one is stronger or weaker, which one should be seen as an extension of the other (Chapter 2). As a result of the Universal Logic project, there will be systematic and fruitful studies on logic translation (compare e.g. [137], [150], [151], [152], [154]). Logical pluralism serves as the primary fundamental investigation for foundation studies on logic translation.

4. Clearly, the understanding of fundamental studies on logic translation has changed considerably over the decades. This is not to say that such techniques have been, or ever should be, abandoned, rather we now see them as serving a different purpose, especially in computer science. Two fundamental questions have been addressed: (1) Which translation is the best to show that a logic is a sublogic of another? (2) In which sense can a logic be said to be weaker, stronger or safer than another, through a translation? The **institution theory**, which is a kind of **abstract model theory**, was used to answer these questions, with the following three outcomes in [151]:

First, it generated some results about the interaction of different types of translations with different kinds of logical connectives and quantifiers. Second, it showed that some well-known translations between (variants of) classical and intuitionistic logic can be turned into semantic translations. Third, it provided a simpler and more conceptual explanation of the faithfulness of logic translation. In the mathematical regard, the institution theory definitely shed new light on logic translation investigations in general logic. Moreover, in practice, the faithfulness of translations between logics could be a good base in an interdisciplinary manner for applications. From the proof-theoretic perspective, Gödel et al. in the **KGGG-EW** presented double-negation translation (or negative translation) between classical first-order logic and intuitionistic first-order logic. It gains importance, e.g., within the philosophy of mathematics.

The last part of our work has been concerned with the nature of logic in an interdisciplinary manner, its status as a corrigible theory with a certain sub-

ject matter, and its claims to uniqueness. For example, "logic as algebra" in algebraic logic pertaining to the mathematical enterprise, and "logic as structure" in structuralism pertaining to the philosophical enterprise. With this, we revised the nature of several anthropological and psychological notions–especially the introduction of "logic as cognitive process" on the differences between different cultures.

Traditionally, translation in a pragmatic sense encounters linguistic and cultural problems. From this perspective, the outcome of the investigation is an interdisciplinary and interesting one. Consistency and non-contradiction have often been taken to be so central to the notions in question that it has been felt that Westerner traditions could not operate without them. As we have seen, they can. But perhaps more importantly, the discussions show that these core notions are not as crucial for those of non-western cultures. Many of the traditional theories of truth in western culture may be allowed to be neglected by non-western cultures, negation is not a standard contradictory-forming function, and the view of rationality is not similar to contemporary western analytical philosophy, e.g. epistemology and philosophy of science, and the nature of logic has collapsed into some faddish pluralism or relativism in philosophical logics. Perhaps the most surprising thing, then, is how easily considerations of inconsistency and tolerance to contradiction can be found in various applications areas, e.g. Paraconsistent Artificial Neural Networks ([1], [2], [3], [4], [5], [6], [7], [145], [199]). This might make the traditional view of the centrality of consistency and that of non-contradiction to the Westerner even more surprising. These two turn out to be views that can not be sustained.

A formal logical mode for the westerner and a dialectic logic mode for the Asian have subsequently formulated a different pluralism in logic in the following three ways: the first is the consideration of logic translation studies in a broader sense; the second is the combination of cross-cultural empirical results from the Cultural–psychobiological viewpoint on the modification of cognition; and the third is a new consideration of the principle of tolerance from the cultural psychological perspective. We reconsider the "principle of tolerance" within the framework of regarding logic as a cognitive process. Two typical and representative cognitive processes have changed and been modified due to changes in the reality of societies in cultural-psychology form, and moreover some descend-with-modification of cognitive cultures can support these changes and modifications from some psycho-biological viewpoints. To mark the principle of tolerance as encompassing these variants

means the cognitive processes do not fall in either the formal logical mode or dialectical logical mode. Thus, a kind of **cognitive processes modification pluralism** in a broader sense of logic has been presented.

In the end, we doubt that for the comparison of the relationship between C-UniLog and the Universal Logic project connected with the spirit of Universal Algebra is a serious issue. We revisited the general and ambiguous question: "Is logic universal?" and the five specific questions that were addressed in the Special Issue of the journal, *Logical Universalis*. We framed our discussions of these questions in an interdisciplinary manner under the assumption of logic as cognitive process. Core to this dissertation is the development of an interdisciplinary formulation of logical pluralism based on the combination of psychologism, a modification of cognition, and the tolerance principle discusses these questions. Briefly, we reorganize and answer some of these questions which can be summarized as follows:

Do all human beings have the same capacity of reasoning? Do men, women, children reason in the same way?

No. There is no one common reasoning capacity that will fit all human beings. Particularly, when 'reasoning capacity' or 'thinking' (in general) is examined in relation to children, it was found that children's thinking does not develop in the same way as adults. This has been taken to mean that before a certain age children are not capable of understanding things in certain ways and this is not a reflection on their intelligence. This shows the difference of capacity of reasoning between adults and children (compare, e.g. [15], [73], [162], [165], [160], [197], [200]).

Does reasoning evolve? Did human beings reason in the same way two centuries ago? In the future, will human beings reason in the same way? Do we reason in different ways depending on the situation?

Our answer is "Yes" for all four questions. From a psychobiological perspective, cognitive modules is descent-with-modification, that is to say current cognitive structures are shaped by evolutionary changes from ancestral cognitive structures. Moreover, from the cultural-psychological perspective, people perceive and think, with the change of social practices and temporary states of social orientation, In this two perspective, reasoning today has

evolved two centuries ago and will continue to evolve in the future. Moreover, Nisbett states that some hints or clues of **a specific situation** are sufficient to switch to another totally different inference strategy. ([155], p. xi)

Do the different systems of logic reflect the diversity of reasoning? Is there any absolute true way of reasoning?

For the first question, the answer is "yes". While 'different systems' attempt to refer to the various specific reasoning cases between the spectrum where the dialectical mode is on one side and the formal logical mode is on the other, it is not difficult to see the diversity between these two representations. The answer is still "yes", even when we do not clarify logic in this bi-polarized way but to focus on logic application studies, the need for diversity of reasoning has been acknowledged in various application areas nowadays. For the second question, the answer is "no". There is no absolute true way of reasoning by following the answers to these questions here. *Reflective equilibrium* might have been seen as a common and reasonable thinking method for various situations, especially to achieve a state of balance among a set of beliefs by mutually adjusting the general principles and particular judgments in daily life. However, cultural diversity shatters the reliability of this method, where cultural differences make people generate different preference to achieve such a state of balance. Taking these general answers to these fundamental questions concerning the universality to the modes of cognitive processes is our starting point. We have presented a formulation of logical pluralism for the Universal Logic project that rests mainly on two assumptions, namely an endorsement of the modification of cognition with the tolerance principle and an adoption of a certain psychologism. The combination of these ideas led to the introduction and study of **cognitive processes modification pluralism**.

This abstract dogma for the study of the Universal Logic project that we introduced as a general theory of logic and the C-UniLog that we introduced as a flexible logic derived from dialecticalizing the mathematical logic (rigid logic) allows a systematic analysis of conceptual and application problems in **logical engineering** that has been considered both by these two ideas of Universal Logic. In the first place, Béziau observed that the trend of universal logic gained prominence in the 1980s, when practical questions in the new development of different fields, such as AI, linguistics, and computer science, were raised. Further, he stated that issues about mathematical foundations

had already been eclipsed in logic at the time, and hence, many revised versions of logic were proposed ([32], p. 140). In addition, Béziau stated the following: "universal logic plays a crucial role with respect to AI, expert systems and automated reasoning, since it helps to develop systems adapted to the most various data: that is called 'logical engineering'." ([32], p. 147). There should be no specific logic that can be adapted to every situation and every problem; in other words, "there is no miraculous universal logic" (ibid, p. 147).

Essentially, the logics used in logical engineering today are in the spirit of Universal Logic, such as the proposal about the **onto-logical engineering** that applies structuring and modularity principles based on institution theory thus leading to the theory of **hyperontologies** ([137]). It allows ontology designers to build their ontologies. This is considerably similar to the opinions about C-UniLog with respect to the development of AI that aims to use flexible and changeable logics by including mathematical logic as the kernel (the rigid aspect) to solve all uncertain and contradictory situations. In other words, they hope to construct a family of flexible logics for each different situation. Indeed, logical engineering developments support the view that Universal Logic allows various approaches that have been used in the logical society. This relativistic attitude to logical studies and interdisciplinary studies suggests a methodology that will be useful for various application areas.

6.2 Future Research

This dissertation dealt with Universal Logic, logic translation, logical pluralism and logic across cultures. The notion of logic translation is at the heart of the Universal Logic movement; the SFB/TR 8 Spatial Cognition and the DFKI Bremen are deeply involved in this movement. While continuing to remain aware of the importance of the central issues of logic, we hope that psycho-social factors and cultures are also given serious consideration, particularly with the rise of new disciplines. Universal Logic provides a general theory of logic to study the most general and abstract properties of the various possible logics. Following this work, we have tried to bring some additional perspectives to the logical menagerie, starting with but not limited to the logic translation between intuitionistic logic and classical logic, and moreover suggested some potential studies in an interdisciplinary manner. It

seems that our work has raised some more questions, as is often the case, than it has answered. We summarize the future directions and extensions as follows:

Logical Pluralism and Logical Dynamics

A general challenge that will need to be addressed in the future is the integration of various interesting logical pluralism proposals (e.g. [8], [21], [57], [81], [170], [196]). A sense of logical pluralism which proposed the increasing plurality of logical systems was earlier presented by Johan van Benthem in [22], p. 373), in this sense, such plurality is much closer to the case of modal logics.

> "[...] This pluralist view is much closer to what has always been standard practice in modal logic, where no single system has ever commanded general allegiance. It is also closer to the spirit of the earlier-mentioned linguistic applications. Linguistic competence is usually measured through a variety of grammatical mechanisms, which may be different for different syntactic tasks. And the same might be the appropriate view in logic, viewed as a description of human cognitive competence."

Johan van Benthem further stated:

> "[...] There seem to be two broad answers in circulation today. One is logical pluralism, locating the new scope of logic in charting a wide variety of reasoning styles, often marked by non-classical structural rules of inference. This is the new program that I subscribed to in my work on sub-structural logics around 1990, and it is a powerful movement today." ([21], p. 182.)

With regard to the work above, the meaning of plurality in logic relies on sub-structural logics and various modal logical systems that are generated by choosing modal terms as logical constants. Here plurality exactly coincides with Carnap's principle of tolerance. However, it is different from the pluralism that we have elucidated in BRLP.[1]

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[1]As Priest said: "What Johan calls logical pluralism is rather different from what many self-ascribed logical pluralists call by that name. For him, logical pluralism is the study of sub-structural logics–logics obtained, generally speaking, by taking a sequent calculus for classical logic, and then modifying or eliminating some of its structural rules, such as Weakening and Contraction."

For the logical dynamics program, it is observed that the main issue for logic has become the study of "the variety of informational tasks performed by intelligent interacting agents, of which inference is only one among many, involving observation, memory, questions and answers, dialogue, or general communication" which replaces a study of the variety of reasoning styles and notions of consequence. Investigations of logical systems "should deal with a wide variety of these, making information-carrying events first-class citizens in their set-up." Related discussions took place at various logical conferences on pluralism[2], and subsequent discussions made by Johan van Benthem ([21]) and Graham Priest ([170]) appeared in *The Australasian Journal of Logic*, which primarily discusses logical pluralism and logical dynamics.

The General Theory of Structures and Cognitive Structures

Concrete future work concerns interfaces to cognitive science. We propose a study of the cognitive structures in an interdisciplinary way such that the Universal Logic project will also benefit. The primary objective is to study the possible opportunities for developing communication between logic (information), psychology (rationality), and argumentation theory. Thus, we plan to conduct a study to consider the cognitive structures of humans.

From a cross-cultural perspective, with the assumption of logic as cognitive process, we propose to study the formal logical mode for westerners and the dialectic logical mode for the Asian as the first work. For the formal logical mode, there seems to be some candidates to serve this purpose, including classical logic, intuitionistic logic, relevant logic, and linear logic and so on.[3] For the dialectic logic mode, prescribing a clear idea of dialectical logic is the primary goal.

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[2](1) Logical Pluralism Conference, Tartu, Estonia, 27-31 Aug 2008; (2) The first Arché Conference on Foundations of Logical Consequence, 11-15 June 2010; (3)The Arché 'Logic or Logics?' Mini-Course, St Andrews, 27-29 September 2010 (4)The Second Arché Conference on the Foundations of Logical Consequence, St Andrews, 8-10 June 2012

[3]"Some people regard the wider landscape of deduction as it is unfolding right now as one more step in 'foundational' research, looking for 'the correct base logic' governing human reasoning. Thus, people might convert from being a classical logician to a disciple of intuitionism, and then as the century comes to a close, to being a believer in linear logic. ...In many contexts, what they are doing is still much more adequately described by classical or intuitionistic systems, But, for other purposes, linear logics or relevance logics may be much closer to the mark." ([22], pp. 373–374)

Ideally, we hope the study of cognitive structures fits the spirit of Universal Logic because the Universal Logic project is attempting to contribute to the development of a general theory of structure.[4] With regard to the Universal Logic project, in developing "a general theory of structures", it has already focused on several realms of mathematics, e.g., abstract algebra. Apart from this, we can consider specifying the targeted structures that we are going to study instead of only discussing "structure" in a vague sense. For example, the revision of mathematical logic attempts to characterize the cognitive structures of children ([15], [73], [162], [165], [160], [197], [200]) and study the general psychological problems of logical-mathematical thought, introduced by psychologist Jean Piaget and logician E.W. Beth ([25]).

Finally, to consider separate structures suggests that we are not directly addressing the general theory of structures. The development of **Category Theory** is worth mentioning. Category Theory deals with mathematical structures in an abstract manner and investigates the relationships between structures with many more mature results than the Universal Logic project. The study of the structures of human cognitive processes will be carried out in an interdisciplinary manner. It should be not only contribute to the fields of mathematics and logics but also to psychology and cognitive science.

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[4] "Universal Logic can contribute itself to the development of a general theory of structures in stating and solving such crucial issues as for example identity between logical structures. When and how two mathematical structures are identical is a problem of crucial import in the theory of structures."([32], p. 139.)

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Appendix A

C-UniLog: A Chinese Universal Logic System

Contents

This appendix briefly addresses the methodology in C-UniLog, which mainly relates to the flexiblization of mathematical formal logic. We present the corresponding methodology, following [123] and chapter 11 in [125]. Essentially, C-UniLog can be taken as a generalization of the family of fuzzy logics, aimed at substantiating positive answers to various problems for uncertain situations in reality. To support the answers in a principled way, an idea of dialectical mathematical logic by the endorsement of cross-cultural logic investigations is presented. Moreover, if the motivations for the C-UniLog proposal are reasonable, then its development is expected to lead to the new **Four Theories**: model theory, proof theory, set theory, and recursion theory becomes necessary as He et al. have suggested.

A.1   Methodology

He et al. stated that three laws and one character (TLOC) have restricted the scope of applying mathematical logic to a "close hologram two-valued reasoning process in a determined world." ([123], p. 85)

- The law of bivalence: given a proposition p, p is either true or false.

- The law of contradiction: given a proposition p, p and its negation $\neg p$ cannot both be true.

- The law of excluded middle: given a proposition p, either p is true or its negation $\neg p$ is required to be true.

- The character of closeness evidences (**CE**): all evidence requires in reasoning is known and static.

Further, they considered whether classical mathematical logic itself or the three laws and CE are a kind of "approximate description" of the real world. Thus, they termed all logics following these three laws and CE, or any logic following classical mathematical logic as **rigid logics**. On the contrary, they claimed that there should be some specific logics (rules) for capturing uncertain or contradictory situations. We are living in an *intellectualized information* era. They, thus stated that logic should consider many pragmatic problems. Consequently, they proposed the flexibilization of mathematical formal logic by flexibilizing the basic elements, which refer to the Four Theories in mathematical logic.

A.2   Flexible Truth Value

The truth value of a flexible proposition in flexible logic ranges as a continuous value in $[0, 1]$.

Definition 34. A general form of flexible truth value is an *arbitrary multiple dimensional ultra ordered space*, denoted as follows:

$$W = \{\perp\} \cup [0, 1]^n \langle \alpha \rangle, n > 0$$

* $[0, 1]$ is *the base space* of the domain of W, it could be degenerated to any discrete valued space even if to be $\{0, 1\}$.

* n is the space dimension of W where $n = 1, 2, 3, \dots$ generally but does not exclude $n > 0$.

* $\perp$ denotes "something has no definition" or "something is beyond the scope of discussion".

* α is finite symbol string which might be an empty string ε. It denotes the additional characteristics of proposition or predicate.

A.3 Flexible Proposition Connectives Operation

Flexible proposition connectives include (universal) *conjunction, disjunction, negation, implication, equivalence*, which are also found in classical propositional logic. Further, flexible proposition connectives include *universal average* and *universal combination*, which are used to describe the uncertainties of the relation between flexible propositions by introducing the notions of *generalized correlativity, measure error*, and *favoritism*.

Flexible proposition connectives include (universal) *conjunction, disjunction, negation, implication, equivalence*. In addition, it also includes *universal average* and *universal combination*, which are used to describe the uncertainties of the relation between flexible propositions with the help of introducing the notions of *generalized correlativity, measure error* and *favoritism*.

(a) **Generalized Correlative Coefficient**, denoted as h, $h \in [0, 1]$ represents **generalized correlativity** between two propositions. When $h = 1$, it represents that two propositions are *attracted*; when $h = 0.75$, it represents that two propositions are *independently relative*; when $h = 0.5$, it represents that two propositions are *repulsive*; when $h = 0$, it represents that two propositions are *antagonistic*.

(b) **Error Coefficient**, denoted as k, $k \in [0, 1]$ represents **measure error** of the truth value of a proposition. When $k = 1$, it represents that there is a *maximal positive error*; when $k = 0.5$, it represents that there is *no error*; when $k = 0$, it represents that there is a *maximal negative error*.

(c) **Favoritism Efficient** (Property of Inequality), denoted as p, $p \in [0, 1]$ represents **favoritism**. When $p = 1$, it represents that there is *the maximal left favoritism*; when $p = 0.5$, it represents that there is *no favoritism*; when $p = 0$, it represents there is *the maximal right favoritism*.

C-UniLog requires the existence of a "logic creator" to create concrete logics to satisfy concrete requirements. First, it constructs standard **propositional universal logical systems** to achieve this. In this system, there are four generation mechanisms for generating *universal propositional connectives*: **(1) Generation Bases**, **(2) Generators**, **(3) Extended-Ordered**, and **(4) Batse Space Transformation**.

In (1), each propositional universal connective has a *generation base*, i.e., the operation model of each propositional connective. It means there is no error on the truth value of propositions and whose generalized correlativity between propositions is the maximal repulsive. The components of (1) are as follows: (i) seven generation *base models* for propositional universal connectives, (universal)-NEGATION, CONJUNCTION, DISJUNCTION, IMPLICATION, EQUIVALENCE, AVERAGE, COMBINATION; (ii) of the many different expressions with the same model, there are two commonly used ones, *NOT-AND expression of base models* and *NOT-OR expression of base models* (See, [125], pp. 257–258).

In (2), it defines the different levels of notions on the *generator integrity clusters* that are used to deal with the practical problems in the real world, i.e., the world wherein error is never equal to 0 and generalized correlativity is equal to non-neutral. (See [125], pp. 259–260.)

The third generation mechanism, (3), requires that propositional connectives operation models be studied based on a standard base space $[0, 1]$. This is done so as to prescribe how to construct these propositional universal connectives operation models on partially-ordered space $[0, 1]^n$, $n = 2, 3, ...$ and hyper-ordered space $\{\perp\} \cup [0, 1]^n < \alpha >, n = 1, 2, 3...$ (See, [125], pp. 260–261).

In (4), He discusses the transformation methods for various propositional connectives operation models. The purpose is to pursue a *practical appli-*

cation form, derived from the base space, which is from the standard base space $[0, 1]$ to the *optional* base space $[a, b], a \leq b, a, b \in \mathbb{R}$ (See, [125], pp. 260–261).

Definition 35. The behavior of propositional universal connectives *binary* operation models $U_\neg$, $U_\wedge$, $U_\vee$, $U_\rightarrow$, $U_{\circledS}$, $U_{\copyright}$ are as follow:[1]

$(U_\neg)$ $U_\neg(x, k) = N(x, k) = (1 - x^n)^{1/n}$

$(U_\wedge)$ $U_\wedge(x, y, h, k) = T(x, y, h, k) = \Gamma^1[(x^{nm} + y^{nm} - 1)^{1/nm}]$

$(U_\vee)$ $U_\vee(x, y, h, k) = S(x, y, h, k) = (1 - \Gamma^1[((1 - x^n)^m + (1 - y^n)^m - 1)^{1/m})^{1/n}]$

$(U_\rightarrow)$ $U_\rightarrow = I(x, y, h, k) = ite\{1|\ x \leq y; 0|\ m \leq 0$ and $y = 0$ and $x \neq 0;$ $\Gamma^1[(1 - x^{nm} + y^{nm})^{1/nm}]\}$

$(U_\Leftrightarrow)$ $U_\Leftrightarrow = Q(x, y, h, k) = (1 \pm |\ x^{nm} - y^{nm}\ |)$, where if $h \geqslant 0.75$, then $+$; else $-$.

$(U_{\circledS})$ $U_{\circledS} = M(x, y, h, k) = (1 - (((1 - x^n)^m + (1 - y^n)^m)/2)^{1/m})^{1/n}$

$(U_{\copyright})$ $U_{\copyright} = C^e(x, y, h, k) = ite\{\Gamma^e[x^{nm} + y^{nm} - e^{nm}]\}\ |\ x + y \leq 2e; (1 - (\Gamma^{e'}[((1 - x^n)^m + (1 - y^n)^m) - (1 - e^n)^m])^{1/m})^{1/n}\ |\ x + y > 2e; e\}$, where $e' = N(e, k)$.

Definition 36. The behavior of propositional universal connectives *multivariate* operation models $U_\wedge$, $U_\vee$, $U_{\circledS}$ are as follows:

$(U_\wedge)$ $U_\wedge(x, y, h, k) = T(x_1, x_2, \ldots, x_l, h, k) = \Gamma^1[(x_1^{nm} + x_2^{nm} + \cdots + x_l^{nm} - (l - 1))^{1/nm})]$

$(U_\vee)$ $U_\vee(x, y, h, k) = S(x_1, x_2, \ldots, x_l, h, k) = (1 - \Gamma^1[((1 - x_1^n)^m + (1 - x_2^n)^m + \cdots + (1 - x_l^n)^m - (l - 1))^{1/m}])^n)^{1/n}$

$(U_{\circledS})$ $U_{\circledS} = M(x_1, x_2, \ldots, x_l, h, k) = (1 - (((1 - x_1^n)^m + (1 - x_2^n)^m + (1 - x_l^n)^m)/l)^{1/m})^{1/n}$

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[1]where $S = ite\{\beta|\ \alpha; \gamma\}$ is a conditional expression which represents "if α is the case, then $S = \beta$; if α is not the case, then $S = \gamma$". $\Gamma^1[x]$ represents the restriction of x to $[0, 1]$, i.e., $\Gamma^1[x] = ite\{1|\ x > 1; 0|\ x \leq 0$ or x is a imaginary number; $x\}$.

Note. We should note that only three connectives, $U_\wedge, U_\vee, U_\circledS$, have multivariate models. This is still an incomplete definition for the system of propositional universal connectives.

Note. For the expression of value of each k, n, m in Definitions 35 and 36, refer to [125], §.11.3.

Remark 37. Definitions 35 and 36 involve considerable mathematical notions, particularly the **triangular norm**, used in fuzzy logic. Thus, it can be stated that people who wish to understand C-UniLog in detail need to have an advanced understanding of fuzzy logic. Readers without the related background can consult Chapters 2 to 4 in [125], [214] and [120]. Apparently, the context of thinking in C-UniLg originates from a similar context in fuzzy logic. The presentation of Definitions 34, 35, and 36 is to provide an approximate idea of the basic concept and its design in C-UniLog. Further details have not been provided here.

According to C-UniLog, Definitions 34, 35, and 36 will form a propositional universal logic system which is an exchangeable flexible propositional logic under the base of h and k. Moreover, an exchangeable flexible propositional logic becomes an unexchangeable one after favoritism efficient p is introduced to it. "Exchangeable flexible propositional logic is the cornerstone of flexible logic as a whole." ([65], p. 93) This is because the idea of "propositional (sentential) logic" forms the basis of any given traditional logic. Moreover, as we can observe, the two basic elements of traditional logic, truth value and propositional connectives, can be flexibilized, as represented by these two definitions, in order to include correlated contradiction and uncertainty in flexible logic. So, in such a condition, it will further have a propositional universal logic.

A.4   Flexible Quantifiers

The third step in achieving C-UniLog is to establish flexible quantifiers, which is to describe the uncertainties of the *restriction degree*. In addition to $\forall, \exists$ C-UniLog introduces four new quantifiers, $\uparrow_\circ^k$, $\$^k$, $\oint^\alpha$, $\circ_\dagger^k$, $\int^\alpha$, which are defined in the W.

Definition 38. The flexible quantifiers in **C-UniLog** are defined.

a. $\uparrow_\circ^k$ (Threshold Meta Quantifier):

It is a flexible quantifier, where $k \in [0, 1]$ is a threshold of propositional truth value. $\uparrow_\circ^k \varphi$ denotes the truth value of proposition φ which has an error with level k. [2]

b. $\$^k$ (Hypothesis Quantifier):

This quantifier represents the hypothesis propositions. This denotes that the judgment that "happened later" resulted from the hypothesis propositions, where $k \in [0, 1]$ represents the trust degree of the hypothesis.

c. $\oint^{x:\ c}$ (Scope Quantifier): This quantifier represents the restriction of the scope of individual variables. This denotes restricting individual variables of the later predicate to a definite scope. $c \in [0, 1] \cup \{+, !\}$ means the scope of the quantifier can range over some part of the domain, e.g., $c \geq 0.5$ means that it ranges over a major part of the domain, and $c < 0.5$ suggests that it ranges over a minor part of the domain, (even if it could range over the entire domain (when $c = 1$)). When $c > 0$, it denotes *existence* in the domain, symbolized as $\oint^{x:+}$ This can be understood as the existential quantifier of classical propositional logic.

Similarly, when $c = 1$, it corresponds to the universal quantifier $\forall_x$; when $c \geq 0.5$, it corresponds to the certainty quantifier $\Box x$, and when $c < 0.5$, it corresponds to the possibility quantifier $\Diamond x$.

d. $\circ_\dagger^{x*d}$ (Position Quantifier): This quantifier represents the *relative position* of the individual variables and special points. The quantifier divides the domains into three segments, according to a specified point $d \in D$ as $x < d, x = d, x > d$, where $* \in \{<, =, >\}$.

e. $\int^{x:\ c}$ (Transition Quantifier): This quantifier represents the change in the *transition characters* for the truth-value distribution of predicates. This means changing the *distribution transition characters* on the x-axis for the following truth-value of the predicates, where $c \in \mathbb{R}^+$. When $c > 1$, the edge of the fuzzy set is *actued*; when $c < 1$, the edge

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[2]This is the only quantifier concerned with the standard propositional calculus of universal logic. ([125], p. 133) This concept is quite different from all the known traditional logics. Logics with any quantifier will not belong to the propositional level (0-level) but will belong to the predicate level, which is used to range over the domain discussed.

of the fuzzy set is *passivated*; and when $c = 1$, the edge of fuzzy set is *invariable.*

A.5   The Flexible Reasoning Models Establishment and the Generation of Logics

The next stage of C-UniLog focuses on the establishment of *flexible reasoning models*, which are used for the descriptions of the uncertainties in the processes of reasoning There is a need to develop this next stage for the following reasons: ([123], p. 95)

1. reasoning models in classical mathematical logic are *deductive* and *monotonic.*

2. there are deductive reasoning models for the three elements, flexible truth value, flexible propositional connectives, and flexible quantifiers.

3. there are various reasoning models, including inductive reasoning, analogical reasoning, hypothesis reasoning, discovery reasoning, evolutionary reasoning, etc., defined on the basis of the above three elements in flexible logic.

As a result, these reasoning models are not absolutely separated but transformed into each other under some conditions. This **model flexibility** will play a key role in describing uncertainty and contradiction. The final stage of g of universal logic is through coordination transformation to generate logics in other truth value ranges. (Ibid, § 5.5.)