Input-to-state stability of infinite-dimensional control systems

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List of mathematical symbols

$\mathbb{N}$	set of natural numbers
$\mathbb{Z}$	set of integer numbers
$\mathbb{R}$	set of real numbers
$\mathbb{R}_+$	set of nonnegative real numbers
$\mathbb{C}$	set of complex numbers
S^n	$\underbrace{S \times \ldots \times S}_{n \text{ times}}$
x^T	transposition of a vector $x \in \mathbb{R}^n$
$\lvert \cdot \rvert$	the norm in the space $\mathbb{R}^s$, $s \in \mathbb{N}$
∇f	gradient of a function $f : \mathbb{R}^n \to \mathbb{R}$.
$f \circ g$	composition of maps f and g
∂G	boundary of a domain G
$L(X,U)$	space of bounded linear operators from X to U
$L(X)$	$= L(X,X)$
$C(X,U)$	space of continuous functions from X to U with finite norm $\lVert u\rVert_{C(X,U)} := \sup\limits_{x \in X} \lVert u(x)\rVert_U$
$PC(\mathbb{R}_+,U)$	space of piecewise continuous (right-continuous) functions from $\mathbb{R}_+$ to U with finite norm $\lVert u\rVert_{PC(\mathbb{R}_+,U)} = \lVert u\rVert_{C(\mathbb{R}_+,U)}$
$AC(\mathbb{R}_+,U)$	space of absolutely continuous functions from $\mathbb{R}_+$ to U with a finite norm $\lVert u\rVert_{C(X,U)}$
$C(X)$	$= C(X,X)$
$C_0(\mathbb{R})$	$\{f \in C(\mathbb{R}) : \forall \varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \mathbb{R} : \lvert f(s)\rvert < \varepsilon \ \forall s \in \mathbb{R}\backslash K_\varepsilon\}$
μ	Lebesgue measure on $\mathbb{R}$.
$L_\infty(\mathbb{R}_+,\mathbb{R}^m)$	the set of Lebesgue measurable functions with finite norm $\lVert f\rVert_\infty := \operatorname{ess\,sup}_{x \geq 0} \lvert f(x)\rvert = \inf_{D \subset \mathbb{R}_+,\ \mu(D)=0} \sup_{x \in \mathbb{R}_+ \backslash D} \lvert f(x)\rvert$
$C_0^k(0,d)$	space of k times continuously differentiable functions $f : (0,d) \to \mathbb{R}$ with a support, compact in $(0,d)$.
$L_p(0,d)$	space of p-th power integrable functions $f : (0,d) \to \mathbb{R}$ with the norm $\lVert f\rVert_{L_p(0,d)} = \left(\int_0^d \lvert f(x)\rvert^p dx\right)^{\frac{1}{p}}$
$W^{p,k}(0,d)$	Sobolev space of functions $f \in L_p(0,d)$, which have weak derivatives of order $\leq k$, all of which belong to $L_p(0,d)$. Norm in $W^{p,k}(0,d)$ is defined by $\lVert f\rVert_{W^{p,k}(0,d)} = \left(\int_0^d \sum_{1 \leq s \leq k} \left\lvert \frac{\partial^s f}{\partial x^s}(x)\right\rvert^p dx\right)^{\frac{1}{p}}$
$W_0^{p,k}(0,d)$	closure of $C_0^k(0,d)$ in the norm of $W^{p,k}(0,d)$.
$H^k(0,d)$	$= W^{2,k}(0,d)$
$H_0^k(0,d)$	$= W_0^{2,k}(0,d)$

Introduction

No pain - no gain.
Athletes' motto

Input-to-state stability (ISS) has become one of the central concepts for study of the stability of control systems with respect to external inputs. For time-invariant systems of ordinary differential equations (ODE systems) of the form

$$\dot{x} = f(x,u),\ \ x(t) \in \mathbb{R}^n,\ \ u(t) \in \mathbb{R}^m \tag{0.1}$$

the notion of ISS was introduced by E. Sontag in his seminal paper [75]. System (0.1) is called ISS, if for all initial conditions x_0 and all admissible inputs u the state of the system at the moment t is bounded in the following way:

$$|x(t)| \leq \beta(|x(0)|, t) + \gamma(\|u\|_\infty), \quad t \geq 0,$$

where $\|u\|_\infty$ is a norm of an input u, β is an increasing positive definite function w.r.t the first argument and decreasing to zero w.r.t. the second and γ, called gain, is an increasing positive definite function.

Within last two decades it was developed a fairly complete theory of input-to-state stability of time-invariant ODE systems, which central results are depicted in Figure 1.

The fundamental result, that ISS of the system (0.1) is equivalent to the existence of a smooth ISS-Lyapunov function, has been proved in [77] on the basis of results from [59]. This theorem provides the possibility to prove ISS of the system by constructing an ISS-Lyapunov

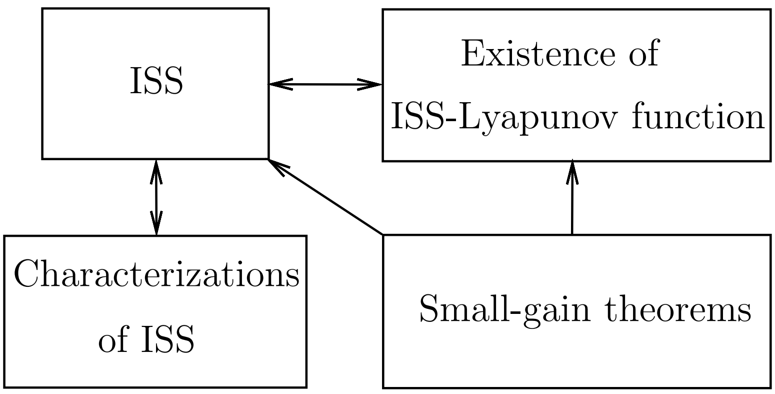

Figure 1: Main results in the ISS theory for ODE systems

function for it. However, a general method for construction of ISS-Lyapunov functions doesn't exist, and search for such a function may be very complicated, especially if the dimension of the state space is large. Small-gain theorems simplify this problem, providing a design of an ISS-Lyapunov function for an interconnection of ISS systems if ISS-Lyapunov functions for the subsystems are known and so-called small-gain condition holds. For interconnections of two nonlinear systems this theorem has been proved in [44] and in [24] it has been generalized to the case of arbitrary interconnections of $n \in \mathbb{N}$ ISS subsystems.

Another type of small-gain theorems (in terms of trajectories) has been proved in [43] and [22] respectively. Last but not least, various characterizations of the ISS property in terms of other stability properties have been derived in [77] and [78].

Such a complete theory exists only for ODE systems which are a subclass of finite-dimensional control systems, i.e. systems with a finite-dimensional state space.

However, many important control systems are infinite-dimensional, in particular, systems based on partial differential equations (PDEs) and time-delay systems.

In contrast to time-delay systems, for which input-to-state stability has been studied extensively for more than decade, the ISS theory for PDEs and systems governed by differential equations in Banach spaces, is a recent field of research. Only few papers have been published at present.

In [62] ISS of certain classes of semilinear parabolic equations have been studied with the help of strict Lyapunov functions. In [68] the construction of ISS-Lyapunov functions for certain time-variant linear systems of hyperbolic equations (balance laws) has been provided. However, the notion of ISS, used in these papers, differs from the usual definition of ISS, see Remark 2.2.5.

Other results have been obtained for general control systems via vector Lyapunov functions. In [51] a general vector Lyapunov small-gain theorem for abstract control systems satisfying weak semigroup property (see also [48], [50]) has been proved. For this class of systems in [49] the trajectory-based small-gain results have been obtained and applied to a chemostat model.

In [42] the results on relations between circle-criterion and ISS for systems, based on equations in Banach spaces, have been proved.

These papers deal with different classes of systems and are obtained on the basis of different mathematical background. In this thesis we are going to develop further the ISS theory for continuous and impulsive infinite-dimensional systems, which may serve as a basis for a further research in this field.

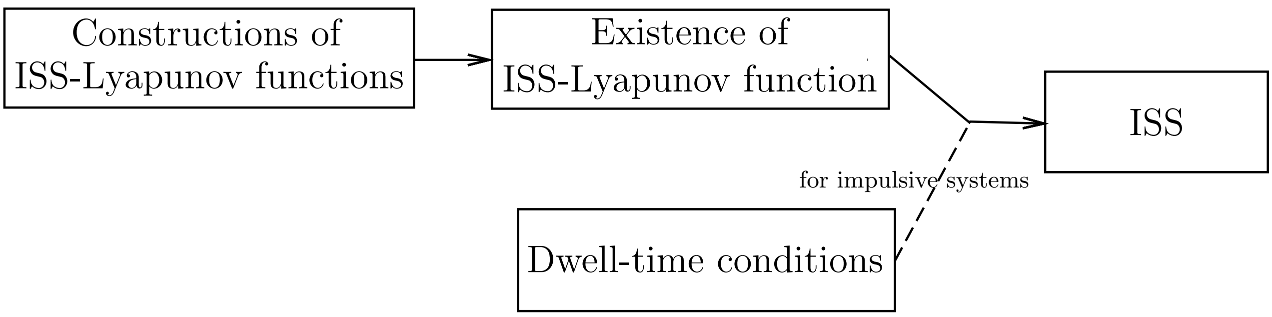

Figure 2: The main aims of the thesis

The main goal of the thesis, as depicted in Figure 2, is a development of the Lyapunov-type sufficient conditions for continuous and impulsive systems as well as methods for construction of ISS-Lyapunov functions for infinite-dimensional systems. We provide two such methods: Lyapunov small-gain theorems for interconnections of infinite-dimensional systems and linearization theorems.

For impulsive systems it is not always possible to prove ISS of the system for all impulse time sequences, and the additional restrictions on the set of impulse time sequences are required to guarantee ISS of the system. These conditions are called dwell-time conditions. We prove, that existence of a Lyapunov function implies ISS of the system provided a dwell-time condition of certain type is satisfied.

Though the results of Chapter 3 are novel already in context of finite-dimensional systems, we prove them for the case of systems, based on differential equations in Banach spaces in order to achieve more generality.

The theoretical results are illustrated on examples of partial and ordinary differential equations. In the next subsections a more detailed overview of results obtained in this work is provided.

ISS of systems with continuous behavior

To study continuous systems, we start with the general axiomatic definition of a control system in Section 1.2, which includes ODE systems, time-delay systems and many classes of partial differential equations as special cases.

For this class of systems in Section 1.3 we introduce stability notions, in particular ISS. We prove in Section 1.4 that these definitions are consistent with the existing definitions for ODE systems and time-delay systems.

In Section 2.2 we define the notion of local ISS-Lyapunov function and prove, that existence of a local ISS-Lyapunov function implies local ISS (LISS) of the system. The consistency of definition of LISS-Lyapunov function with the corresponding definition from the ODE theory is investigated in Section 2.2.

In Chapter 2 we exploit semigroup theory methods and consider infinite-dimensional systems generated by differential equations in Banach spaces:

$$\dot{x} = Ax + f(x,u),$$

where x belongs to a Banach space X, A is the generator of a C_0-semigroup over X and u is an external input.

For such systems we develop two methods for a construction of (L)ISS-Lyapunov functions for the control systems.

To study interconnections of n ISS subsystems

$$\left\{ \begin{array}{l} \dot{x}_i = A_i x_i + f_i(x_1,\ldots,x_n,u),\ x_i(t) \in X_i,\ u(t) \in U, \\ i = 1,\ldots,n, \end{array} \right.$$

we generalize the small-gain theorem for finite-dimensional systems [21], [24] to the infinite-dimensional case. This theorem allows a construction of a Lyapunov function for the whole interconnection if the Lyapunov functions for subsystems are known and the small-gain condition is satisfied. The ISS of the interconnection follows then from the existence of the Lyapunov function for it. The question, whether the small-gain condition, which is only sufficient for existence of a Lyapunov function for interconnection (small-gain condition) can be relaxed, is investigated in Section 1.5.4.

The local ISS of nonlinear control systems can be also investigated in an analogous way (for ODE systems see, e.g., [23]), but also another type of results is possible, namely linearization technique, well-known for the systems without external inputs [36]. We prove, that a system is LISS provided its linearization is ISS in two ways. The first proof holds for systems with a Banach state space, but it doesn't provide a LISS-Lyapunov function. Another proof is based on a converse Lyapunov theorem and provides a LISS-Lyapunov function, but needs that the state space is Hilbert.

The usage of Lyapunov-type sufficient condition as well as of small-gain theorems is illustrated on examples of parabolic partial differential equations.

The most part of the thesis is devoted to the Lyapunov methods for verification of ISS. In order to show that alternative methods can be developed, in Section 2.5 we utilize the notion of monotone control systems introduced in [3] to show that for certain classes of nonlinear reaction-diffusion systems the derivation of ISS property can be significantly simplified, if the system is monotone.

In Section 2.6 we construct a mathematical model of the production network and then analyze its stability via methods of ISS theory. We construct an ISS-Lyapunov function for an interconnection of n subsystems, each of which models a node of the production network. To construct an ISS-Lyapunov function the small-gain theorem is applied.

ISS of abstract impulsive systems

In the modeling of real phenomena often one has to consider systems, which exhibit both continuous and discontinuous behavior.

The general framework for modeling of such phenomena is a hybrid systems theory [33], [30]. Impulsive systems are hybrid systems, in which the jumps occur only at certain moments of time, which do not depend on the state of the system. The first monograph devoted entirely to impulsive systems is [71]. Recent developments in this field can be found, in particular, in [33], [79].

Input-to-state stability of impulsive systems has been investigated in recent papers [38] (finite-dimensional systems) and [11], [60], [89] (time-delay systems).

Chapter 3 is devoted to impulsive systems of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t), u(t)), & t \in [t_0, \infty)\backslash T, \\ x(t) = g(x^-(t), u^-(t)), & t \in T, \end{cases}$$

where T is a sequence of impulse times, at which the state $x \in X$ of a system is changed by a jump. The jump is described by the function g which depends on the values of $x^-(t) = \lim_{s \to t-0} x(s)$ and $u^-(t)$.

The main tool in the ISS theory of impulsive systems are, as in the continuous theory, ISS-Lyapunov functions (properly redefined for impulsive systems). However, if either continuous or discrete dynamics destabilizes the system, the existence of an ISS-Lyapunov function is not enough to prove ISS of the system and one has to impose restrictions on the density of impulse times, which are called dwell-time conditions.

In the current literature only exponential ISS Lyapunov functions (or exponential ISS Lyapunov-Razumikhin functions, exponential ISS Lyapunov-Krasovskii functionals) have been exploited for analysis of ISS of impulsive systems. This restrains the class of systems, which can be investigated by such methods, since an exponential Lyapunov function can be not always constructed.

Another restrictions arise in the study of interconnections of ISS impulsive systems via small-gain theorems. Even if ISS-Lyapunov functions for all subsystems are exponential, an ISS Lyapunov function of the interconnection may be non-exponential, if the gains are nonlinear. Hence for the most cases tools for verification of ISS of an interconnection of impulsive systems do not exist. In Chapter 3 we develop such tools.

We prove, that existence of an ISS Lyapunov function (not necessarily exponential) for an impulsive system implies input-to-state stability of the system over impulsive sequences satisfying nonlinear fixed dwell-time (FDT) condition. Furthermore, for the case, when an impulsive system possesses an exponential Lyapunov function, we generalize the result from [38], by introducing the generalized average dwell-time (gADT) condition and proving, that an impulsive system, which possesses an exponential ISS Lyapunov function is uniform ISS over the class of impulse time sequences, which satisfy the gADT condition. We argue, that gADT condition provides in certain sense tight estimates of the class of impulsive time sequences, for which the system is ISS.

In Section 3.3 we prove a Lyapunov small-gain theorem for interconnections of impulsive systems, analogous to corresponding theorem for infinite-dimensional systems with continuous behavior [19].

Also we prove, that if all subsystems possess exponential ISS Lyapunov functions, and the gains are power functions, then the exponential ISS Lyapunov function for the whole system can be constructed. This result generalizes Theorem 4.2 from [18], where this statement for linear gains has been proved. The relation between small-gain and dwell-time conditions on the stage of selection of gains is discussed in Section 3.3.2.

Additionally, we have shown, how the exponential LISS Lyapunov functions for certain classes of control systems can be constructed via linearization method.

At the end of each chapter we discuss the results and provide possible directions for future research. In Chapter 4 we summarize the results of the whole thesis.

Some of results presented in this work have been already published or submitted for publication: for continuous systems see [19] and [15], for impulsive systems [18] and [20], for applications in logistics see [16].

Chapter 1

System-theoretical framework

In this chapter we introduce the concept of a control system and define stability notions for control systems, in particular, input-to-state stability. In Section 1.4 we prove consistency of our definitions with the notions used in ISS theory for ODE systems and time-delay systems. Then we recall main results from ISS theory of ODE systems, which will serve us as a pattern for development of ISS theory of infinite-dimensional systems in the Chapters 2 and 3. In addition to known theorems, which are stated without proofs, we add some new results. We prove a linearization theorem, which provides a construction of local ISS Lyapunov function for linearizable systems. Then we investigate tightness of a small-gain condition, which plays a crucial role in study of ISS of interconnected systems.

1.1 Notation

The notation for vectors, spaces of numbers and classical function spaces see p. 7.

For arbitrary $x, y \in \mathbb{R}^n$ define the relations "$\geq$" and "$<$" on $\mathbb{R}^n$ by

$$x \geq y \Leftrightarrow \; x_i \geq y_i \; \forall i = 1, \ldots, n,$$

$$x < y \Leftrightarrow \; x_i < y_i \; \forall i = 1, \ldots, n.$$

By "$\not\geq$" we understand the logical negation of "$\geq$", that is $x \not\geq y \Leftrightarrow \exists i \colon \; x_i < y_i$.

For the formulation of stability properties the following classes of comparison functions are useful:

$$\begin{array}{ll}
\mathcal{P} & := \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous}, \; \gamma(0) = 0, \text{ and } \gamma(r) > 0 \text{ for } r > 0\} \\
\mathcal{K} & := \{\gamma \in \mathcal{P} \mid \gamma \text{ is strictly increasing}\} \\
\mathcal{K}_\infty & := \{\gamma \in \mathcal{K} \mid \gamma \text{ is unbounded}\} \\
\mathcal{L} & := \left\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \text{ is continuous and strictly decreasing with } \lim\limits_{t\to\infty} \gamma(t) = 0\right\} \\
\mathcal{KL} & := \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \; \beta(\cdot, t) \in \mathcal{K}, \; \forall t \geq 0, \; \beta(r, \cdot) \in \mathcal{L}, \; \forall r > 0\}
\end{array}$$

Functions of class $\mathcal{P}$ are called positive definite functions.

Note that for all $f \in \mathcal{K}_\infty$ there exists $f^{-1} \in \mathcal{K}_\infty$ and for all $f, g \in \mathcal{K}$ it holds $f \circ g \in \mathcal{K}$, where $\circ$ denotes the composition of the maps f and g. Further properties of comparison functions can be found in [34, p. 95].

1.2 Concept of control system

We start with the axiomatic definition of a continuous control system.

Definition 1.2.1. *The triple $\Sigma = (X, U_c, \phi)$, consisting of*

- *Normed linear spaces $(X, \|\cdot\|_X)$ and $(U, \|\cdot\|_U)$, called state space and space of input values, endowed with the norms $\|\cdot\|_X$ and $\|\cdot\|_U$ respectively.*
- *A normed linear space of admissible input functions $U_c \subset \{f : \mathbb{R}_+ \to U\}$ (with the norm $\|\cdot\|_{U_c}$).*
- *A transition map $\phi : A_\phi \to X$, where $A_\phi \subset \mathbb{R}_+ \times \mathbb{R}_+ \times X \times U_c$.*

is called a control system, if the following properties hold:

1. *Existence: for every $(t_0, \phi_0, u) \in \mathbb{R}_+ \times X \times U_c$ there exists $t > t_0$: $[t_0, t] \times \{(t_0, \phi_0, u)\} \subset A_\phi$.*
2. *Identity property: for every $(t_0, \phi_0, u) \in \mathbb{R}_+ \times X \times U_c$ it holds $\phi(t_0, t_0, \phi_0, u) = \phi_0$.*
3. *Causality: for every $(t, t_0, \phi_0, u) \in A_\phi$, for every $\tilde{u} \in U_c$, such that $u(s) = \tilde{u}(s)$, $s \in [t_0, t]$ it holds $(t, t_0, \phi_0, \tilde{u}) \in A_\phi$ and $\phi(t, t_0, \phi_0, u) = \phi(t, t_0, \phi_0, \tilde{u})$.*
4. *Continuity: for each $(t_0, \phi_0, u) \in \mathbb{R}_+ \times X \times U_c$ the map $t \mapsto \phi(t, t_0, \phi_0, u)$ is continuous.*
5. *Semigroup property: for all $t, s \geq 0$, for all $\phi_0 \in X$, $u \in U_c$ so that $(t, s, \phi_0, u) \in A_\phi$, it follows*
   - *$(r, s, \phi_0, u) \in A_\phi$, $r \in [s, t]$,*
   - *for all $r \in [s, t]$ it holds $\phi(t, r, \phi(r, s, x, u), u) = \phi(t, s, x, u)$.*

Here $\phi(t, s, x, u)$ denotes the state of a system at the moment $t \in \mathbb{R}_+$, if its state at the moment $s \in \mathbb{R}_+$ was $x \in X$ and the input $u \in U_c$ was applied.

The existence property means, that we can start at each moment of time, at each point of a state space and with arbitrary input, and the trajectory will exist, at least locally. In particular, it means that it cannot happen that some input is admissible for one state of a system and is not admissible for another.

We assume throughout the thesis, that for the control systems BIC property (Boundedness-Implies-Continuation property) holds (see [50, p. 4], [51]): for all $(t_0, x_0, u) \in \mathbb{R}_+ \times X \times U_c$ there exists a maximal time of existence of the solution $t_m \in (t_0, \infty]$, such that $[t_0, t_m) \times \{(t_0, x_0, u)\} \subset A_\phi$ and for all $t \geq t_m$ $(t, t_0, x_0, u) \notin A_\phi$. Moreover, if $t_m < \infty$, then for all $M > 0$ there exists $t \in [t_0, t_m)$: $\|\phi(t, t_0, x, u)\|_X > M$.

In other words, the BIC property states that the solution may stop to exist in finite time only because of blow-up phenomena, when the norm of solution goes to infinity in finite time. As examples in this thesis we use mostly systems of parabolic partial differential equations, for which BIC property holds, because of the smoothing action of parabolic systems, see [36].

An important subclass of control systems are time-invariant systems

Definition 1.2.2. *A control system (X, U_c, ϕ) is called time-invariant if for all $\phi_0 \in X$, $u \in U_c$, $t_2 \geq t_1$ and all $s \geq -t_1$ it holds*

$$\phi(t_2, t_1, x, u) = \phi(t_2 + s, t_1 + s, x, u). \tag{1.1}$$

In other words, time-invariance means, that the future evolution of a system depends only on the initial state of the system and on the applied input, but not on the initial time. Since the trajectories of time-invariant systems, corresponding to the same inputs and initial states but for different initial times can be obtained one from another by translation in time, one takes zero as initial time $t_0 := 0$. We denote for short $\phi(t, \phi_0, u) := \phi(t, 0, \phi_0, u)$.

The special cases of abstract control systems are ODE systems, time-delay systems, systems based on parabolic and hyperbolic partial differential equations.

1.3 Stability concepts

We give a list of different stability properties of control systems which we will deal with.

Definition 1.3.1. *An element $\phi_0 \in X$ is called an equilibrium (or fixed point) of a system Σ if $\forall t, t_0 : t \geq t_0$ it holds $\phi(t, t_0, \phi_0, 0) = \phi_0$.*

Definition 1.3.2. *Σ is globally asymptotically stable at zero uniformly with respect to x (0-UGASx), if $\exists \beta \in \mathcal{KL}$, such that $\forall \phi_0 \in X$, $\forall t_0 \geq 0$, $\forall t \geq t_0$ it holds*

$$\|\phi(t, t_0, \phi_0, 0)\|_X \leq \beta(\|\phi_0\|_X, t - t_0). \tag{1.2}$$

The notion 0-UGASx is also called uniform asymptotic stability in the whole (see [34, p. 174]).

Now we introduce one of the main definitions in this work.

Definition 1.3.3. *Σ is called uniformly input-to-state stable (UISS), if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that the inequality*

$$\|\phi(t, t_0, \phi_0, u)\|_X \leq \beta(\|\phi_0\|_X, t - t_0) + \gamma(\|u\|_{U_c}) \tag{1.3}$$

holds $\forall \phi_0 \in X$, $\forall t_0 \geq 0$, $\forall t \geq t_0$ and $\forall u \in U_c$.

In this definition a uniformity means that the functions β and γ do not depend on the initial time t_0. This terminology has been adopted from [58], where ISS of time-variant ODE systems has been studied.

The following stability property is important, in particular, for characterizations of uniform ISS.

Definition 1.3.4. *We call Σ uniformly globally stable (UGS) if there exist functions $\varphi, \gamma \in \mathcal{K}_\infty$, such that for every initial condition $\phi_0 \in X$ and every input $u \in U_c$ and all $t, t_0 : \ t \geq t_0 \geq 0$ it holds*

$$\|\phi(t, t_0, \phi_0, u)\|_X \leq \varphi(\|\phi_0\|_X) + \gamma(\|u\|_{U_c}) \tag{1.4}$$

For time-invariant systems we may assume $t_0 := 0$ and the notion of UISS is reduced to the ISS:

Definition 1.3.5. *A time-invariant system Σ is called input-to-state stable (ISS), if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that the inequality*

$$\|\phi(t, \phi_0, u)\|_X \leq \beta(\|\phi_0\|_X, t) + \gamma(\|u\|_{U_c}) \tag{1.5}$$

holds $\forall \phi_0 \in X$, $\forall t \geq 0$ and $\forall u \in U_c$.

The local version of ISS is defined as follows

Definition 1.3.6. *A time-invariant system Σ is called locally input-to-state stable (LISS), if there exist $\rho_x, \rho_u > 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that the inequality*

$$\|\phi(t, \phi_0, u)\|_X \leq \beta(\|\phi_0\|_X, t) + \gamma(\|u\|_{U_c}) \tag{1.6}$$

holds $\forall \phi_0 : \|\phi_0\|_X \leq \rho_x$, $\forall t \geq 0$ and $\forall u \in U_c$: $\|u\|_{U_c} \leq \rho_u$.

We consider for time-invariant systems in addition to 0-UGASx the following stability property

Definition 1.3.7. *Time-invariant control system Σ is globally asymptotically stable at zero (0-GAS), if it is*

1. *Locally stable: $\forall \varepsilon > 0\ \exists \delta > 0 : \forall x \in X : \|x\|_X < \delta \Rightarrow \|\phi(t, x, 0)\|_X < \varepsilon,\ \ \forall t \geq 0$.*
2. *Globally attractive: $\forall x \in X\ \ \|\phi(t, x, 0)\|_X \to 0,\ t \to \infty$.*

Definition 1.3.8. *If in the Definitions 1.3.2 1.3.3, 1.3.5, 1.3.6 the function β can be chosen as $\beta(r, t) = Me^{-at}r$, $\forall r, t \in \mathbb{R}_+$, for some $a, M > 0$, then Σ is called exponentially 0-UGASx, exponentially UISS (eUISS), eISS and eLISS respectively.*

1.4 Consistency of the introduced stability notions with the existing ones

Since our aim is to develop an ISS theory, which generalizes the current theory for ODE systems and time-delay systems, we have to establish consistency of stability notions introduced in the previous section with standard definitions used for these classes of systems.

One of the most common choices for U_c is the space $U_c := PC(\mathbb{R}_+, U)$. In this case one can use the alternative definition of the UISS property (see, e.g. [51], [38]):

Proposition 1.4.1. *Let $U_c := PC(\mathbb{R}_+, U)$. Then Σ is UISS if and only if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$, such that the inequality*

$$\|\phi(t, t_0, \phi_0, u)\|_X \leq \beta(\|\phi_0\|_X, t - t_0) + \gamma(\sup_{t_0 \leq s \leq t} \|u(s)\|_U) \tag{1.7}$$

holds $\forall \phi_0 \in X$, $\forall t_0 \geq 0$, $\forall t \geq t_0$ and $\forall u \in U_c$.

Proof. Sufficiency is clear, since $\sup\limits_{t_0 \leq s \leq t} \|u(s)\|_U \leq \sup\limits_{t_0 \leq s \leq \infty} \|u(s)\|_U = \|u\|_{U_c}$.

Now let Σ be UISS. Due to causality property of Σ the state $\phi(\tau, t_0, \phi_0, u)$, $\tau \in [t_0, t]$ of the system Σ does not depend on the values of $u(s)$, $s > t$. For arbitrary $t \geq t_0$, $\phi_0 \in X$ and $u \in U_c$ consider another input $\tilde{u} \in U_c$, defined by

$$\tilde{u}(\tau) := \begin{cases} u(\tau), \tau \in [t_0, t], \\ u(t), \tau > t. \end{cases}$$

The inequality (1.3) holds for all admissible inputs, and hence it holds also for $\tilde{u}$. Substituting $\tilde{u}$ into (1.3) and using that $\|\tilde{u}\|_{U_c} = \sup\limits_{t_0 \leq s \leq t} \|u(s)\|_U$, we obtain (1.7). □

The counterparts of this theorem for the cases of ISS and LISS can be easily stated. The similar property (with $\operatorname{ess\,sup}\limits_{t_0 \leq s \leq t} \|u(s)\|_U$ instead of $\sup\limits_{t_0 \leq s \leq t} \|u(s)\|_U$) holds for the class of strongly measurable and essentially bounded inputs $U_c := L_\infty(\mathbb{R}_+, U)$ (which is the standard choice in the case of ODE systems and systems with time-delays), for continuous inputs ($U_c := C(\mathbb{R}_+, U)$) and many other classes of input functions.

Now we are going to prove consistency of our definitions of UISS, ISS and LISS with the definitions, used for time-delay systems (for ODE systems it is clear). Consider a time-invariant time-delay system

$$\dot{x}(t) = f(x^t, u(t)), \quad t > 0. \tag{1.8}$$

Here $x^t \in C([-\theta, 0]; \mathbb{R}^N)$ is the state of the system (1.8) at time t, $x^t(\tau) = x(t + \tau)$, $\tau \in [-\theta, 0]$ and $f : C([-\theta, 0]; \mathbb{R}^N) \times \mathbb{R}^m$ satisfies certain assumptions to guarantee existence and uniqueness of solutions of the system (1.8) (see e.g. [6], [35] and citations therein). System (1.8) defines a time-invariant control system with the state space $X = C([-\theta, 0]; \mathbb{R}^N)$ with the norm $\|\cdot\|_{[-\theta,0]} := \|\cdot\|_{C([-\theta,0];\mathbb{R}^N)}$, input space $U_c = L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ and the transition map $\phi(\cdot, \xi, u)$ defined as a solution of (1.8) subject to initial condition ξ and input u. We will write in this section $x^t = \phi(t, \xi, u)$ for short.

The following proposition shows that the standard definition of LISS for system (1.8) (see e.g. [66]) is equivalent to the Definition 1.3.6.

Proposition 1.4.2. *System (1.8) is LISS if and only if there exist constants $\rho_x, \rho_u > 0$ and functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for every $\xi \in C([-\theta, 0], \mathbb{R}^N)$: $\|\xi\|_{[-\theta,0]} \leq \rho$, every admissible input $\|u\|_\infty \leq \rho_u$ and for all $t \in \mathbb{R}_+$, it holds that*

$$|x(t)| \leq \beta(\|\xi\|_{[-\theta,0]}, t) + \gamma(\|u\|_\infty). \tag{1.9}$$

Proof. If the system (1.8) is LISS according to Definition 1.3.6, then (1.9) holds for the same $\beta, \gamma, \rho_x, \rho_u$ since $|x(t)| \leq \|x^t\|_{[-\theta,0]}$.

In the other direction, let there exist $\rho, \rho_u > 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that (1.9) holds for every initial condition ξ: $\|\xi\|_{[-\theta,0]} \leq \rho$, every external input $u : \|u\|_\infty \leq \rho_u$ and for all $t \geq 0$.

Then for every ξ: $\|\xi\|_{[-\theta,0]} \leq \rho$, every u: $\|u\|_\infty \leq \rho_u$ and all $t > \theta$ it holds

$$\begin{aligned}\|x^t\|_{[-\theta,0]} &= \sup_{\tau \in [-\theta,0]} |x(t+\tau)| \leq \sup_{\tau \in [-\theta,0]} \beta(\|\xi\|_{[-\theta,0]}, t+\tau) + \gamma(\|u\|_\infty) \\ &= \beta(\|\xi\|_{[-\theta,0]}, t-\theta) + \gamma(\|u\|_\infty).\end{aligned}$$

For $t \in [0, \theta]$ it holds

$$\left\|x^t\right\|_{[-\theta,0]} = \max\{ \sup_{t-\theta \leq s \leq 0} |x(s)|, \sup_{0 \leq s \leq t} |x(s)|\} \leq \max\{\|\xi\|_{[-\theta,0]}, \beta(\|\xi\|_{[-\theta,0]}, 0) + \gamma(\|u\|_\infty)\}.$$

Note that for β from (1.9) it holds $r \leq \beta(r, 0)$ for all $r > 0$ (to prove this take in (1.9) $u \equiv 0$ and ξ such that $\|\xi\|_{[-\theta,0]} = |\xi(0)|$).

Therefore we obtain for $t \in [0, \theta]$

$$\left\|x^t\right\|_{[-\theta,0]} \leq \beta(\|\xi\|_{[-\theta,0]}, 0) + \gamma(\|u\|_\infty).$$

Define function $\tilde{\beta}$ for all $r \geq 0$ by

$$\tilde{\beta}(r, t) = \begin{cases} \beta(r, t-\theta), & t > \theta \\ r(\theta - t) + \beta(r, 0), & t \in [0, \theta]. \end{cases}$$

One can simply check that $\tilde{\beta} \in \mathcal{KL}$. Now, for every initial condition $\|\xi\|_{[-\theta,0]} \leq \rho$, every external input $\|u\|_\infty \leq \rho_u$ and for all $t \geq 0$ it holds

$$\left\|x^t\right\|_{[-\theta,0]} \leq \tilde{\beta}(\|\xi\|_{[-\theta,0]}, t) + \gamma(\|u\|_\infty).$$

Therefore the system (1.8) is LISS according to Definition 1.3.6. □

Similar statement can be proved if we take ISS or UGS instead of LISS.

The definition of ISS as in Proposition 1.4.2 was used, in particular, in [66], where it was proved, that the existence of a so-called ISS Lyapunov-Krasovskii functional implies ISS of the system.

Also another definition of ISS is used [80] in the context of time-delay systems, which we call here "weak ISS" (as in [81]).

Definition 1.4.1. *The system (1.8) is called weakly ISS, if there exists $\gamma \in \mathcal{K}$ such that the following two properties hold:*

1. *For all $\varepsilon > 0$ there exists $\delta > 0$ such that $\|\xi\|_{[-\theta,0]} < \delta$ implies $|x(t)| \leq \varepsilon + \gamma(\|u\|_\infty)$, for all $t \geq 0$.*

2. *For each $\varepsilon > 0$, $\eta_x \in \mathbb{R}_+$, $\eta_u \in \mathbb{R}_+$ there exists $T \geq 0$ such that $\|\xi\|_{[-\theta,0]} \leq \eta_x$ and $\|u\|_\infty \leq \eta_u$ imply $|x(t)| \leq \varepsilon + \gamma(\|u\|_\infty)$, $\forall t \geq T$.*

In [80] a theorem was established, which states that the existence of a so-called ISS Lyapunov-Razumikhin function implies weak ISS.

For ODE systems ISS and weak ISS properties are equivalent, as can be proved, in particular, with the help of the characterizations of ISS from [78]. But for time-delay systems ISS implies weak ISS, but the converse implication has not been proved or disproved at the moment, see [82, 81].

Therefore the usage of ISS Lyapunov-Razumikhin functions for verification of (standard) ISS of the system (1.8) has to be justified. We are going to prove a characterization of the ISS property, which will solve this problem.

At first note the following simple fact:

Lemma 1.4.1. *There exist $\gamma \in \mathcal{K}$ such that the second property in Definition 1.4.1 holds if and only if this property holds with $\|x^t\|_{[-\theta,0]}$ instead of $|x(t)|$ for the same γ, ε, η_x, η_u and with $T+\theta$ instead of T.*

We prove the following characterization of the UISS property

Proposition 1.4.3. *The system Σ is UISS if and only if it is*

- *uniformly globally stable,*
- *$\exists \gamma \in \mathcal{K}$ such that for each $\varepsilon > 0$, $\eta_x \in \mathbb{R}_+$, $\eta_u \in \mathbb{R}_+$ there exists $T \geq 0$ such that $\|x\|_X \leq \eta_x$ and $\|u\|_{U_c} \leq \eta_u$ imply $\|\phi(t,t_0,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{U_c})$, $\forall t \geq T + t_0$.*

The proof is similar to the proof of [77, Lemma 2.7].

Proof. We start with necessity. Let Σ be UISS. Then it is UGS with a gain γ and $\varphi(\cdot) := \beta(\cdot, 0)$.

Take arbitrary $\varepsilon > 0$, $\eta_x \in \mathbb{R}_+$. For all $x: \ \|x\|_X \leq \eta_x$ and all $u \in U_c$ it holds

$$\|\phi(t,t_0,x,u)\|_X \leq \beta(\eta_x, t-t_0) + \gamma(\|u\|_{U_c}), \ \forall t \geq t_0.$$

If $\varepsilon > \beta(\eta_x, 0)$, then we choose T as $T := 0$. Otherwise take T as a solution (which for a given η_x is unique) of the equation $\beta(\eta_x, T) = \varepsilon$. The second property is verified.

Let us prove sufficiency. Without loss of generality we take $r := \eta_x = \eta_u$ and fix it. From uniform global stability it follows, that there exist $\varphi, \gamma \in \mathcal{K}_\infty$, such that for all $x \in X$: $\|x\|_X \leq r$ and for all $u \in U_c$ it holds

$$\|\phi(t,t_0,x,u)\|_X \leq \varphi(r) + \gamma(\|u\|_{U_c}), \ \forall t \geq t_0. \tag{1.10}$$

Define

$$T(\varepsilon, r) := \inf\{\tau : \|\phi(t,t_0,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{U_c}) \ \forall x, u : \max\{\|x\|_X, \|u\|_{U_c}\} \leq r, \forall t \geq \tau + t_0\}. \tag{1.11}$$

The second assumption of the proposition implies that $T(\varepsilon, r)$ exists and is finite for all $\varepsilon, r > 0$. Note that $T(\varepsilon_1, \cdot) \leq T(\varepsilon_2, \cdot)$, if $\varepsilon_1 \geq \varepsilon_2$ and $T(\cdot, r_1) \geq T(\cdot, r_2)$, if $r_1 \geq r_2$.

Define $\bar{T}_r(s) := \frac{2}{s}\int_{s/2}^{s} T(\varepsilon, r) d\varepsilon$. For every fixed r, $\bar{T}_r$ is a continuous function with $\bar{T}_r(s) \geq T(s,r)$, $\forall s > 0$.

For each $r > 0$, $\bar{T}_r$ is a continuous function. Now for each $r > 0$ and $s > 0$ define

$$T_r(s) := \frac{r}{s} + \sup_{h \geq s} \bar{T}_r(h).$$

For each $r > 0$, T_r is a strictly decreasing function. Thus, it is invertible. For every $r > 0$ define $\psi_r(s) := T_r^{-1}(s)$. We set $\psi_r(0) := \infty$. Note that for all $r > 0$ $\lim_{s \to +0} \psi_r(s) = \infty$.

From (1.11) and from the fact that $T(\varepsilon, r) \leq T_r(\varepsilon)$, we obtain that for all $r > 0$, for all x, u: $\max\{\|x\|_X, \|u\|_{U_c}\} \leq r$, $\forall t \geq T_r(\varepsilon) + t_0$ it follows

$$\|\phi(t,t_0,x,u)\|_X \leq \varepsilon + \gamma(\|u\|_{U_c}).$$

Since $t \geq T_r(\varepsilon) + t_0 \Leftrightarrow \varepsilon \leq T_r^{-1}(t - t_0) = \psi_r(t - t_0)$, then for all $r > 0$ and for all x, u: $\max\{\|x\|_X, \|u\|_{U_c}\} \leq r$ it holds

$$\|\phi(t, t_0, x, u)\|_X \leq \psi_r(t - t_0) + \gamma(\|u\|_{U_c}), \quad \forall t \geq t_0. \tag{1.12}$$

For all $r, t \in \mathbb{R}_+$ define $\hat{\psi}(r, t) := \min\{\inf_{r \geq s} \psi_s(t), \phi(r)\}$, and pick any function $\beta \in \mathcal{KL}$: $\beta(r, t) \geq \hat{\psi}(r, t)$ for all $r, t \geq 0$ (see [77, proof of Lemma 2.7] for the argument, why such function exists). Now $\forall r \geq 0$, $\forall x, u$: $\max\{\|x\|_X, \|u\|_{U_c}\} \leq r$ we have

$$\|\phi(t, t_0, x, u)\|_X \leq \beta(r, t - t_0) + \gamma(\|u\|_{U_c}), \quad \forall t \geq t_0. \tag{1.13}$$

In particular, (1.13) holds for x, u: $\max\{\|x\|_X, \|u\|_{U_c}\} = r$. For such x, u we obtain

$$\begin{aligned}
\|\phi(t, t_0, x, u)\|_X &\leq \beta(\max\{\|x\|_X, \|u\|_{U_c}\}, t - t_0) + \gamma(\|u\|_{U_c}) \\
&= \beta(\|x\|_X, t - t_0) + \beta(\|u\|_{U_c}, t - t_0) + \gamma(\|u\|_{U_c}) \\
&\leq \beta(\|x\|_X, t - t_0) + \gamma_u(\|u\|_{U_c}),
\end{aligned}$$

where $\gamma_u(r) := \beta(r, 0) + \gamma(r)$. This proves UISS of Σ. □

Remark 1.4.2. *In [80, Theorem 1] it was proved that the existence of a Lyapunov-Razumikhin function for* (1.8) *implies the properties mentioned in Proposition 1.4.3, which by Proposition 1.4.3 implies ISS of the system* (1.8).

Remark 1.4.3. *A variation of the Proposition 1.4.3 has been used in [18] for the investigation of ISS of impulsive time-delay systems in terms of exponential ISS-Lyapunov-Razumikhin functions.*

Thus, we have proved, that our definition of (L)ISS is equivalent to the standard definition [66] used in time-delay theory and justified the usage of Lyapunov-Razumikhin framework from [80] for verification of ISS of the time-delay systems.

1.5 ISS theory for time-invariant ODE systems

Before getting into the stability theory of infinite-dimensional control systems we are going to recall some central results from ISS theory of time-invariant ODE systems and to prove some new results extending this theory.

The choice of the results included in this section is oriented to show the similarities and differences between the infinite-dimensional theory, developed in the next chapter and ISS theory for time-invariant ODE systems as well as to provide tools needed in the following exposition.

We do not provide proofs of the known results and give only brief explanations of the theorems. The reader, who is not acquainted with ISS theory, should not be frightened with the amount of notions and formulations in this section. The full understanding will come after development of the infinite-dimensional theory in the next chapter, where the proofs of results, examples and detailed explanations will be provided.

1.5.1 ISS of a single system

We consider a special case of control systems, defined by the time-invariant ODE system

$$\begin{cases} \dot{x} = f(x,u), \ t > 0 \\ x(0) = x_0, \end{cases} \tag{1.14}$$

with $X = \mathbb{R}^n$, $U = \mathbb{R}^m$ and $U_c = L_\infty(\mathbb{R}_+, \mathbb{R}^m)$.

By solution of (1.14) for a given $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ we understand an absolutely continuous function $x : t \mapsto x(t) \in \mathbb{R}^n$, which satisfies $x(0) = x_0$ and the equation (1.14) hold almost everywhere.

We assume that f is Lipschitz continuous w.r.t. the first argument uniformly with respect to the second one. Under this assumption for all $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ and all initial conditions $x_0 \in \mathbb{R}^n$ there exists (at least locally) the unique solution $\phi(\cdot, x_0, u) = x(\cdot)$ of (1.14) (see [1, Paragraph 2.5.]). Note, that absolutely continuous functions are differentiable almost everywhere [54, p.345].

The triple $\Sigma_f = (\mathbb{R}^n, L_\infty(\mathbb{R}_+, \mathbb{R}^m), \phi)$ defines a time-invariant control system.

As a starting point note that for a system (1.14) the notions of 0-GAS and 0-UGASx coincide (see [34, p.109, Theorem 26.3]):

Theorem 1.5.1. *System* (1.14) *is 0-GAS* $\Leftrightarrow \exists \beta \in \mathcal{KL}$ *such that* $\forall x_0 \in \mathbb{R}^n$ *it holds*

$$|\phi(t, x_0, 0)| \leq \beta(|x_0|, t), \quad t \geq 0. \tag{1.15}$$

In the infinite-dimensional theory the situation is completely different, as we will see in Section 2.1.

For linear ODE systems (with $f(x,u) = Ax + Bu$) the following simple fact is well-known

Proposition 1.5.2. *For linear system* (1.14) *the following properties are equivalent: e0-GAS, eISS, 0-GAS, ISS.*

For nonlinear systems an important tool for verification of (L)ISS property are (L)ISS-Lyapunov functions.

Definition 1.5.1. *A smooth function* $V : D \to \mathbb{R}_+$, $D \subset \mathbb{R}^n$, $0 \in int(D) = D \backslash \partial D$ *is called a local ISS-Lyapunov function (LISS-LF) for* (1.14), *if* $\exists \rho_x, \rho_u > 0$, $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ *and* $\alpha \in \mathcal{P}$, *such that:*

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \quad \forall x \in D \tag{1.16}$$

and $\forall x \in \mathbb{R}^n : |x| \leq \rho_x$, $\forall u \in \mathbb{R}^m : |u| \leq \rho_u$ *it holds:*

$$|x| \geq \chi(|u|) \ \Rightarrow \ \nabla V \cdot f(x,u) \leq -\alpha(|x|), \tag{1.17}$$

The function χ *is called Lyapunov gain.*

If in the previous definition $D = \mathbb{R}^n$, $\rho_x = \infty$ *and* $\rho_u = \infty$, *then* V *is called an ISS-Lyapunov function.*

In [77] Sontag and Wang have proved the following fundamental theorem:

Theorem 1.5.3. *System (1.14) is ISS if and only if there exists a smooth ISS-Lyapunov function for (1.14).*

Construction of ISS-Lyapunov functions is in many cases the only way to prove ISS of control systems. For linear systems there exists a general effective method for construction of Lyapunov functions, see [76, p.226]. For nonlinear systems such methods do not exist and often one has to use intuition or to have a good luck (better - both). However, for some subclasses of control systems certain general methods can be developed. One of such ways is a construction of a Lyapunov function for a whole system on the basis of Lyapunov functions for subsystems, which we consider in the next subsection. Another general method, which works only for local ISS, is a linearization method explained in Section 1.5.3.

1.5.2 Interconnections of ISS systems

The main question in the study of stability of interconnected systems is whether the system, which consists of ISS components, is itself ISS. Small-gain theorems play the central role in this study. They provide sufficient conditions for ISS of an interconnection of n ISS subsystems.

Consider the system given by

$$\begin{cases} \dot{x}_i = f_i(x_1,\dots,x_n,u), \\ i = 1,\dots,n. \end{cases} \tag{1.18}$$

Here $u \in L_\infty(\mathbb{R}_+,\mathbb{R}^m)$, $x_i(t) \in \mathbb{R}^{p_i}$, and f_i are Lipschitz continuous w.r.t. x_i uniform with respect to external inputs.

Define $N := p_1 + \ldots + p_n$.

The solution of the whole system (1.18) is an absolute continuous function. Since globally bounded absolutely continuous functions belong to the space $L_\infty(\mathbb{R}_+,\mathbb{R}^N)$, we may consider that the whole input to the i-th subsystem is from the space $L_\infty(\mathbb{R}_+,\mathbb{R}^{N+m-p_i})$. Consequently, i-th subsystem is the control system similar to the whole system.

Small-gain theorem in terms of Lyapunov functions

For the i-th subsystem of (1.18) the definition of an ISS-Lyapunov function can be written as follows.

A smooth function $V_i : \mathbb{R}^{p_i} \to \mathbb{R}_+$ is an ISS-Lyapunov function (ISS-LF) for the i-th subsystem of (1.18), if there exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, $\chi_{ij}, \chi_i \in \mathcal{K}$, $j = 1,\dots,n$, $j \neq i$, $\chi_{ii} := 0$ and a positive definite function α_i, such that:

$$\psi_{i1}(|x_i|) \le V_i(x_i) \le \psi_{i2}(|x_i|), \quad \forall x_i \in \mathbb{R}^{p_i}$$

and $\forall x_i \in \mathbb{R}^{p_i}$, $\forall u \in \mathbb{R}^m$ it holds

$$V_i(x_i) \ge \max\{\max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(|u|)\} \;\Rightarrow\; \nabla V_i(x_i)\cdot f_i(x_1,\dots,x_n,u) \le -\alpha_i(V_i(x_i)). \tag{1.19}$$

The internal Lyapunov gains χ_{ij} characterize the interconnection structure of subsystems. As we will see, the question, whether the interconnection (1.18) is ISS, depends on the properties of the gain operator $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\Gamma(s) := \left(\max_{j=1}^{n} \chi_{1j}(s_j), \dots, \max_{j=1}^{n} \chi_{nj}(s_j)\right), \; s \in \mathbb{R}^n_+. \tag{1.20}$$

To construct an ISS-Lyapunov function for the whole interconnection we will use the notion of Ω-path (see [24, 70]).

Definition 1.5.2. *A function $\sigma = (\sigma_1, \ldots, \sigma_n)^T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, where $\sigma_i \in \mathcal{K}_\infty$, $i = 1, \ldots, n$ is called an Ω-path (with respect to operator Γ), if it possesses the following properties:*

1. *σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;*
2. *for every compact set $P \subset (0, \infty)$ there are finite constants $0 < K_1 < K_2$ such that for all points of differentiability of σ_i^{-1} we have*
$$0 < K_1 \leq (\sigma_i^{-1})'(r) \leq K_2, \quad \forall r \in P;$$
3. $$\Gamma(\sigma(r)) < \sigma(r), \ \forall r > 0. \tag{1.21}$$

The next theorem provides a construction of an ISS-Lyapunov function for an interconnection of ISS subsystems, see [21], [24].

Theorem 1.5.4. *Let for i-th subsystem of (1.18) V_i be the ISS-Lyapunov function with corresponding gains χ_{ij}, $i = 1, \ldots, n$. If there exists an Ω-path $\sigma = (\sigma_1, \ldots, \sigma_n)^T$ corresponding to the operator Γ defined by (1.20), then the ISS-Lyapunov Lyapunov function for the system (1.18) can be constructed as*

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}, \tag{1.22}$$

The Lyapunov gain of the whole system is

$$\chi(r) := \max_i \sigma_i^{-1}(\chi_i(r)). \tag{1.23}$$

In order to apply Theorem 1.5.4 one has to construct the Ω-path or at least prove its existence. To this end we introduce another notion: we say that Γ satisfies *the small-gain condition* if the following inequality holds

$$\Gamma(s) \not\geq s, \ \forall \ s \in \mathbb{R}^n_+ \backslash \{0\}\,. \tag{1.24}$$

Small-gain condition (1.24) can be reformulated in terms of cycles (see [22, p. 16]):

Proposition 1.5.5. *Small-gain condition (1.24) holds if and only if for each cycle in Γ (that is for all $(k_1, ..., k_p) \in \{1, ..., n\}^p$, where $k_1 = k_p$) and for all $s > 0$ it holds*

$$\gamma_{k_1 k_2} \circ \gamma_{k_2 k_3} \circ \ldots \circ \gamma_{k_{p-1} k_p}(s) < s. \tag{1.25}$$

Both formulations of small-gain condition are frequently used in theoretical works. For applications the cyclic formulation seems to be more convenient.

Since often the aim of the analysis is not to construct an ISS-Lyapunov function, but only to prove ISS of the interconnection, one states the small-gain theorem also in the following form

Theorem 1.5.6. *Let for i-th subsystem of (1.18) V_i be the ISS-Lyapunov function with corresponding gains χ_{ij}, $i = 1, \dots, n$. If the corresponding operator Γ defined by (1.20) satisfies the small-gain condition (1.24), then the whole system (1.18) is ISS and possesses ISS-Lyapunov function defined by (1.22).*

This reformulation is possible because of the Theorem 1.5.3.

The small-gain theorem for ODE systems has been proved also in the form of trajectories. One can write the definition of ISS for the i-th subsystem of the system (1.18) as follows

Definition 1.5.3. *The i-th subsystem of (1.18) is called ISS (in maximum formulation), if there exist γ_{ij}, $\gamma_i \in \mathcal{K}$ and $\beta_i \in \mathcal{KL}$, such that for all initial values x_i^0 and all inputs u_i: $\|u_i\|_\infty < \infty$ the inequality*

$$\left|x_i\left(t, x_i^0, x_j : j \neq i, u_i\right)\right| \leq \max\left\{\beta_i\left(\left|x_i^0\right|, t\right), \max_{j\neq i} \gamma_{ij}\left(\|x_j\|_\infty\right), \gamma_i\left(\|u_i\|_\infty\right)\right\} \tag{1.26}$$

is satisfied $\forall t \in \mathbb{R}_+$. γ_{ij} and γ_i are called (nonlinear) gains.

If instead of inequality (1.26) the inequality

$$\left|x_i\left(t, x_i^0, x_j : j \neq i, u_i\right)\right| \leq \beta_i\left(\left|x_i^0\right|, t\right) + \sum_{j\neq i} \gamma_{ij}\left(\|x_j\|_\infty\right) + \gamma_i\left(\|u_i\|_\infty\right) \tag{1.27}$$

holds, then the i-th subsystem of (1.18) is called ISS in summation formulation.

If the system is ISS in summation formulation, then it is ISS also in maximum formulation and vice versa, however, the gains can be different.

In [22] an ISS small gain theorem for networks in terms of trajectories was proved, namely

Theorem 1.5.7. *Let all subsystems of system (1.18) be ISS in maximum formulation. If the corresponding gain operator satisfies the small gain condition (1.24) then the whole system (1.18) is ISS.*

For a summation formulation the same statement holds, but with a stronger small-gain condition:

$$D \circ \Gamma(s) \not\geq s, \ \forall \ s \in \mathbb{R}^n_+ \backslash \{0\}, \tag{1.28}$$

for some $D = diag(id + \alpha_1, \dots, id + \alpha_n)$, $\alpha_i \in \mathcal{K}_\infty$.

This theorem is a generalization of the small-gain theorem for an interconnection of two systems, proved in [43].

1.5.3 Linearization method

The linearization method is an important method for investigation of local asymptotic stability of nonlinear systems in the stability theory of dynamical systems, see e.g. [90, p. 100]. Here we prove the counterpart of this theorem for ODE systems. A more general result will be proved in Section 2.3.

By $P > 0$ we will indicate that matrix P is symmetric and positive definite, and by $P < 0$ that it is symmetric and negative definite.

Theorem 1.5.8. *Let in equation (1.14)*

$$f(x,u) = Bx + Cu + g(x,u), B \in \mathbb{R}^{n\times n},\ C \in \mathbb{R}^{n\times m}$$

where $g(x,u) = o(|x| + |u|)$, for $|x| + |u| \to 0$. If the system

$$\dot{x} = Bx + Cu \tag{1.29}$$

is ISS, then (1.14) is LISS.

Proof. System (1.29) is ISS, and consequently 0-GAS, therefore there exists (see, e.g., [76, Theorem 8, p.231]) a matrix $P > 0$ such that $B^T P + PB = Q < 0$.

We prove, that $V : \mathbb{R}^n \to \mathbb{R}_+$, defined by $V(x) = x^T Px$ is a LISS-Lyapunov function for a system (1.14) for properly chosen gains. Let us compute the Lie derivative of V with respect to the system (1.14), using that $P = P^T$ and that $x^T PBx = x^T B^T P^T x = x^T B^T Px$.

$$\begin{aligned}\dot{V}(x) &= (\nabla V)^T f(x,u) = \left(Px + (x^T P)^T\right)^T (Bx + Cu + g(x,u)) \\ &= x^T (P^T B + PB)x + 2x^T P(Cu + g(x,u)) \\ &\leq x^T (PB + B^T P)x + k|x|\|P\|(\|C\||u| + |g(x,u)|).\end{aligned}$$

Here $k > 0$ is some constant, which depends on the chosen norm of the matrices $\|P\|$, $\|C\|$. Since $g(x,u) = o(|x| + |u|)$ for $|x| + |u| \to 0$, for each $w > 0$ we can find ρ, such that

$$|g(x,u)| \leq w \cdot (|x| + |u|), \quad \forall x : |x| \leq \rho,\ \forall u : |u| \leq \rho.$$

Using this inequality, we continue estimates

$$\dot{V}(x) \leq x^T (PB + B^T P)x + kw\|P\||x|^2 + k\|P\|(\|C\| + w)|x||u|.$$

Take $\chi(r) := \sqrt{r}$. Then for $|u| \leq |x|^2$ we have:

$$\dot{V}(x) \leq x^T (PB + B^T P)x + kw\|P\||x|^2 + k\|P\|(\|C\| + w)|x|^3.$$

Choosing w and ρ small enough, we will have in the right hand side some negative quadratic function of $|x|$ (remember that $PB + B^T P$ is a negative definite matrix). This proves that V is a LISS-Lyapunov function, and consequently, (1.14) is LISS. □

1.5.4 Tightness of small-gain conditions

Theorem 1.5.7 states, that if all the subsystems are ISS in summation formulation then small-gain condition (1.28) is sufficient for input-to-state stability of the whole system. However, the small-gain condition is not necessary for ISS of an interconnection and the question arises, how tight it is. A partial answer is given by the following theorem

Theorem 1.5.9. *Let a gain matrix $\Gamma := (\gamma_{ij})$, $i,j = 1,\dots,n$, $\gamma_{ii} = 0$ be given. If the condition (1.24) is not satisfied, then there exists a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ so that $\forall i = 1,\dots,n$ estimates (1.27) hold $\forall t \geq 0$, but the whole system (1.18) is not 0-GAS.*

Proof. For arbitrary gain matrix Γ, satisfying the assumptions of the theorem, we are going to construct a corresponding system satisfying (1.27), but which is not 0-GAS.

Let Γ does not satisfy (1.24). According to Proposition 1.5.5, there exists some cycle such that the condition (1.25) is violated. Let $\exists s > 0$, such that

$$\gamma_{12} \circ \gamma_{23} \circ \ldots \circ \gamma_{r-1r} \circ \gamma_{r1}(s) \geq s, \tag{1.30}$$

where $2 \leq r \leq n$ (violation of the small-gain condition on another cycles can be treated in the same way).

Due to continuity of γ_{ij}, there exist constants $\varepsilon_i \in [0,1)$, $i = 2, \ldots, r$, such that for functions $\chi_{ij} := (1 - \varepsilon_j)\gamma_{ij}$ and the same s it holds that

$$\chi_{12} \circ \chi_{23} \circ \ldots \circ \chi_{r-1r} \circ \chi_{r1}(s) = s. \tag{1.31}$$

Let us enlarge the domain of definition of functions χ_{ij} to $\mathbb{R}$, defining $\chi_{ij}(-p) = -\chi_{ij}(p)$ $\forall p > 0,\ i,j = 1, \ldots n,\ i \neq j$.

Consider the following system:

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + \chi_{12}\left(x_2(t)\right) \\ \dot{x}_2(t) = -x_2(t) + \chi_{23}\left(x_3(t)\right) \\ \ldots \\ \dot{x}_r(t) = -x_r(t) + \chi_{r1}\left(x_1(t)\right) \\ \dot{x}_{r+1}(t) = -x_{r+1}(t) \\ \ldots \\ \dot{x}_n(t) = -x_n(t) \end{cases} \tag{1.32}$$

For the first equation, using variation of constants formula, we obtain the following estimates:

$$\begin{aligned} |x_1(t)| &\leq |x_1(0)|\, e^{-t} + \left| \int_0^t e^{s-t}\chi_{12}(x_2(s))ds \right| \leq |x_1(0)|\, e^{-t} + e^{-t}\int_0^t e^s \left|\chi_{12}(x_2(s))\right| ds \\ &= |x_1(0)|\, e^{-t} + e^{-t}\int_0^t e^s \chi_{12}(|x_2(s)|)ds \leq |x_1(0)|\, e^{-t} + e^{-t}\int_0^t e^s ds \chi_{12}\left(\|x_2\|_\infty\right) \\ &\leq |x_1(0)|\, e^{-t} + \chi_{12}\left(\|x_2\|_\infty\right) \end{aligned}$$

Similar estimates can be made for all equations. Thus, inequalities (1.27) are satisfied. Now we are going to prove, that the system (1.32) is not 0-GAS.

Fixed points of the system (1.32) are the solutions $(x_1, \ldots, x_n)$ of the following system:

$$\begin{cases} x_1 = \chi_{12}\left(x_2\right) \\ x_2 = \chi_{23}\left(x_3\right) \\ \ldots \\ x_{r-1} = \chi_{r-1r}\left(x_r\right) \\ x_r = \chi_{r1}\left(x_1\right) \\ x_i = 0,\ i = r+1, \ldots, n \end{cases} \tag{1.33}$$

Substituting the i-th equation of (1.33) into the $(i-1)$-th, $i = r, \ldots, 2$, we obtain the equivalent system:

$$\begin{cases} x_1 = \chi_{12} \circ \chi_{23} \circ \ldots \circ \chi_{r1}(x_1) \\ x_2 = \chi_{23} \circ \chi_{34} \circ \ldots \circ \chi_{r1}(x_1) \\ \ldots \\ x_{r-1} = \chi_{r-1r} \circ \chi_{r1}(x_1) \\ x_r = \chi_{r1}(x_1) \\ x_i = 0, \ i = r+1, \ldots, n \end{cases} \tag{1.34}$$

For all solutions $s > 0$ of the equation (1.31), the first equation of the system (1.34) is satisfied with $x_1 = s$, and a point

$$(x_1, \ldots, x_{r-1}, x_r, \ldots, x_n) = (s, \chi_{23} \circ \chi_{34} \circ \ldots \circ \chi_{r-1r} \circ \chi_{r1}(s), \ldots, \chi_{r-1r} \circ \chi_{r1}(s), \chi_{r1}(s), 0, \ldots, 0)$$

is a fixed point for the system (1.32). Hence the system (1.32) has a nonzero fixed point and therefore it is not 0-GAS. □

The counterpart of this result can be proved also for the Lyapunov-type small gain theorem.

Theorem 1.5.10. *Let a matrix of Lyapunov gains $\Gamma := (\gamma_{ij})$, $i, j = 1, \ldots, n$, $\gamma_{ii} = 0$ be given. Let there exist $s > 0$, such that for some cycle in Γ it holds*

$$\gamma_{12} \circ \gamma_{23} \circ \ldots \circ \gamma_{r-1r} \circ \gamma_{r1}(s) > s, \tag{1.35}$$

where $2 \leq r \leq n$ (we can always renumber the nodes to obtain the cycle of the needed form). Then there exist a function $f : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$, and Lyapunov functions V_i for subsystems (in maximum formulation), so that $\forall i = 1, \ldots, n$ it holds (1.19), but the whole system (1.18) is not 0-GAS.

Proof. Take constants $\varepsilon_i \in (0, 1)$, $i = 2, \ldots, r$, such that for the functions $\chi_{ij} := (1 - \varepsilon_i)\gamma_{ij}$ and some $s > 0$ it holds

$$\chi_{12} \circ \chi_{23} \circ \ldots \circ \chi_{r-1r} \circ \chi_{r1}(s) = s.$$

Consider the system (1.32). Take $V_i(x_i) = |x_i|$ as Lyapunov functions for i-th subsystem. For $i = 1, \ldots, r-1$ if

$$V_i(x_i) \geq \gamma_{ii+1}(x_{i+1}) = \frac{1}{1 - \varepsilon_i} \chi_{ii+1}(x_{i+1})$$

holds, then

$$\dot{V}_i(x_i) \leq -V_i(x_i) + (1 - \varepsilon_i)V_i(x_i) = -\varepsilon_i V_i(x_i).$$

Thus, for $i = 1, \ldots, r-1$ V_i is an ISS Lyapunov function for i-th subsystem. In fact, it holds also for all $i = 1, \ldots, n$. Moreover, Γ is a matrix of Lyapunov gains for the system (1.32). According to the proof of the Theorem 1.5.9, (1.32) is not a 0-GAS system. □

We discuss the obtained results in the end of the next section.

1.6 Concluding remarks and open problems

Control-theoretic framework. The axiomatic definition of a control system from Section 1.2 is adopted from [51], but we specialize it to the systems, which satisfy classical semigroup property. Another axiomatic definitions of the control systems are also used in the literature (see [76], [86]).

Overview of existing results. Many results playing an important role in ISS theory have not been mentioned in Section 1.5. In particular, we do not consider characterizations of ISS property [77], [78], the ISS of time-variant ODE systems [58], [52], [27] and the extensions of the theory to the case of input-to-output stability (IOS) [43], integral input-to-state stability (iISS) [4], input-to-state dynamical stability (ISDS) [32] etc. For a survey see [74], [14] and [41].

In ISS theory for time-delay systems two different Lyapunov-type sufficient conditions have been proposed: via ISS Lyapunov-Razumikhin functions [80] and by ISS Lyapunov-Krasovskii functionals [66]. For converse Lyapunov theorems see [47] and [69]. In [51] the general small-gain theorem for abstract systems has been proved and the small-gain results for finite-dimensional and time-delay systems have been provided. However, the theory concerning characterizations of ISS property for time-delay systems is still not complete.

Apart from ODE and time-delay systems another classes of control systems have been considered in view of input-to-state stability, namely discrete systems [45], [2] as well as hybrid, switched and impulsive systems. In papers [8] and [64] different characterizations of ISS property and small-gain theorems for two hybrid finite-dimensional systems with feedback interconnection have been proved. The interconnections of n hybrid systems have been studied, in particular, in [55]. In these works the definition of hybrid system from [31] has been used. The ISS of switched systems was considered in [85]. For a survey of results in stability theory of switched and hybrid systems see [72].

Interconnections of systems. ISS framework is not the only existing tool to study the interconnections of the dynamical systems. In particular, small-gain theorems were originally established within input-output approach to stability of control systems, see [53, Chapter 5].

Another framework is a dissipative systems theory, originated from papers [86], [87] by J. Willems. An important theorem in this framework is that a feedback interconnection of dissipative systems is again dissipative. An important special case of dissipative systems are passive systems [84]. Closely connected to passive systems are port-Hamiltonian systems, widely used in modeling and analysis of finite and infinite-dimensional control systems [26].

The study of interconnections of control systems plays an important role in behavioral approach [88], [67] to dynamical systems theory. The small-gain theorems arise also within this framework, see e.g. [10].

Small-gain theorems. There are several proofs of small-gain theorems in terms of trajectories with maximum formulation for ODE systems. The first proof was given in [22], which uses the small-gain condition in matrix form (1.24). Later another proof was given [46], where the small-gain condition in equivalent cyclic form has been used.

Lyapunov small-gain theorems have been proved not only for max-formulation of ISS property, but also for sum-formulation and some more general cases [24].

Tightness of small-gain condition. In the Theorem 1.5.9 it was proved, that if all the subsystems are ISS in summation formulation, but (1.24) does not hold, then one cannot guarantee 0-GAS of the whole system. On the other hand, in [22, pp. 20-21] it was constructed an example of the system for which the gains are such that (1.24) holds and (1.28) does not hold and which is 0-GAS, but not ISS. Therefore two questions arise:

1. Whether in the case of ISS in summation formulation the condition (1.24) is sufficient for 0-GAS of the interconnection of ISS systems.

2. Whether from violation of (1.28) property for some given gains γ_{ij}, $i, j = 1, \dots, n$ it will follow that there exists a system, which is not ISS and which has gains γ_{ij}, $i, j = 1, \dots, n$.

Another interesting question is to find the classes of systems for which the small-gain condition is necessary for stability. To this end consider two reasons, why the small-gain condition is not necessary for ISS of the interconnected system.

Firstly, the gains may be chosen not tightly, and therefore the system may be ISS, but small gain condition will not hold due to the roughly chosen gains.

Even if the gains are chosen tightly, the small-gain condition is not necessary for ISS of the interconnection. This can be shown by the following example

$$\begin{aligned} \dot{x} &= -x + y + u, \\ \dot{y} &= -x - y + u. \end{aligned}$$

The smallest gains for both subsystems is the identity function, and thus the small-gain condition is not satisfied for all possible choices of gains, but the system is ISS. In this case ISS is reached due to the negative sign of the coefficient of x in the second equation.

To say that the linear dynamical system has positive non-diagonal elements is the same as saying that the system is cooperative (see [73], [3]). Thus, an interesting question is whether for general nonlinear systems the small-gain condition becomes necessary, if the system is cooperative and gains are chosen tightly.

Chapter 2

ISS of infinite-dimensional systems with continuous behavior

For the development of the theory of continuous infinite-dimensional systems we will follow the plan, sketched in Section 1.5, where ISS of time-invariant ODE systems has been considered.

Our first aim is to prove a characterization of input-to-state stability property of linear systems, corresponding to Proposition 1.5.2.

Then we introduce a concept of ISS-Lyapunov function for abstract control system which is the main tool for analyzing of ISS of nonlinear systems. We argue, that our definition of ISS-Lyapunov function is consistent with the standard definition of ISS-Lyapunov function for finite-dimensional systems.

Then we specialize ourselves to the investigation of differential equations over Banach spaces and develop two effective methods for the construction of ISS-Lyapunov functions for certain subclasses of such systems. We will consider throughout this work weak solutions of the equations if not stated otherwise.

In Section 2.3 we prove two linearization theorems for abstract systems. The first of them states that a nonlinear control system is LISS provided its linear approximation is ISS. The second theorem provides us with a form of LISS Lyapunov functions for linearizable nonlinear systems, if their state space is a Hilbert space.

Next, in Section 2.4 we prove a small-gain theorem, which provides us with a construction of an ISS Lyapunov function for an interconnected system if the Lyapunov functions for its subsystems are given, and small-gain condition holds. We show applicability of the small-gain theorem on examples of linear and semilinear reaction-diffusion systems.

To show how non-Lyapunov methods can be applied, we consider in Section 2.5 semilinear monotone reaction-diffusion systems with Neumann boundary conditions. For such systems we apply method of super- and sub-solutions to reduce the proof of ISS of infinite-dimensional systems to the proof of the ISS of its finite-dimensional counterpart without diffusion.

In Section 2.6 we model a production network and analyze its ISS via small-gain theorems in Lyapunov formulation.

In the last section we conclude the results of this chapter and sketch possible directions for future research.

From the reader a basic knowledge of a semigroup theory of bounded operators over Banach spaces and of a theory of linear and nonlinear evolution equations in Banach spaces is required.

Main definitions and results are recalled in Appendix. For a substantial treatment of these questions see the monographs [36], [9], [12], [28]. In the examples we will frequently use Sobolev spaces. For definitions please refer to Section 5.4.

2.1 Linear systems

For linear normed spaces X, Y let $L(X,Y)$ be a space of bounded linear operators from X to Y and $L(X) := L(X,X)$. The norm in these spaces we denote by $\|\cdot\|$. The spectrum of an arbitrary closed linear operator A we denote by $Spec(A)$.

Let X be a Banach space and $\mathcal{T} = \{T(t),\ t \geq 0\}$ be a C_0-semigroup on X with an infinitesimal generator $A = \lim\limits_{t\to+0} \frac{1}{t}(T(t)x - x)$, which domain of definition is a set of $x \in X$ so that the $\lim\limits_{t\to+0} \frac{1}{t}(T(t)x - x)$ exists.

Consider a linear control system with inputs of the form

$$\begin{array}{l} \dot{s} = As + f(u(t)), \\ s(0) = s_0, \end{array} \tag{2.1}$$

where $f : U \to X$ is continuous and so that for some $\gamma \in \mathcal{K}$ it holds

$$\|f(u)\|_X \leq \gamma(\|u\|_U),\ \forall u \in U. \tag{2.2}$$

Remark 2.1.1. *In particular, f can be chosen as a bounded linear operator: $f(u) = Bu$ for some $B \in L(U,X)$. Then $\|f(u)\|_X \leq \|B\|\|u\|_U$.*

We consider weak solutions of the problem (2.1), which are solutions of integral equation, obtained from (2.1) by the variation of constants formula

$$s(t) = T(t)s_0 + \int_0^t T(t-r)f(u(r))dr, \tag{2.3}$$

where $s_0 \in X$.

The space of admissible inputs U_c can be chosen as an arbitrary subspace of the space of strongly measurable functions $f : [0,\infty) \to U$, such that for all $u \in U_c$ the integral in (2.3) exists in the sense of Bochner.

For the examples in this section we will use $U_c := C([0,\infty), U)$. In this case the functions under the sign of integration in (2.3) are strongly measurable according to Proposition 5.2.2 and for all $t \geq 0$

$$\int_0^t \|T(t-r)f(u(r))\|_X dr \ < \ \infty.$$

Thus according to the criterion of Bochner integrability (Theorem 5.2.1), the integral in (2.3) is well-defined in the sense of Bochner.

We are going to generalize Proposition 1.5.2 to the case of infinite-dimensional systems. We need the following lemma:

Lemma 2.1.2. *The following statements are equivalent:*

1. (2.1) is 0-UGASx.

2. *$\mathcal{T}$ is uniformly stable (that is, $\|T(t)\| \to 0, \ t \to \infty$).*

3. *$\mathcal{T}$ is uniformly exponentially stable ($\|T(t)\| \leq Me^{-\omega t}$ for some $M, \omega > 0$ and all $t \geq 0$).*

4. *(2.1) is exponentially 0-UGASx.*

Proof. 1 $\Leftrightarrow$ 2. At first note that for an input-to-state stable system (2.1) $\mathcal{KL}$-function β from the definition of 0-UGASx can be always chosen as $\beta(r,t) = \zeta(t)r$ for some $\zeta \in \mathcal{L}$. Indeed, consider $x \in X : \|x\|_X = 1$, substitute it into (1.2) and choose $\zeta(\cdot) = \beta(1,\cdot) \in \mathcal{L}$. From linearity of $\mathcal{T}$ we have, that $\forall x \in X$, $x \neq 0$ it holds $\|T(t)x\|_X = \|x\|_X \cdot \|T(t)\frac{x}{\|x\|_X}\|_X \leq \zeta(t)\|x\|_X$.

Let (2.1) be 0-UGASx. Then $\exists \zeta \in \mathcal{L}$, such that

$$\|T(t)x\|_X \leq \beta(\|x\|_X, t) = \zeta(t)\|x\|_X \quad \forall x \in X, \ \forall t \geq 0$$

holds. This means, that $\|T(\cdot)\| \leq \zeta(\cdot)$, and, consequently, $\mathcal{T}$ is uniformly stable.

If $\mathcal{T}$ is uniformly stable, then it follows, that $\exists \zeta \in \mathcal{L}$: $\|T(\cdot)\| \leq \zeta(\cdot)$. Then $\forall x \in X$ $\|T(t)x\|_X \leq \zeta(t)\|x\|_X$.

The equivalence 2 $\Leftrightarrow$ 3 is well-known, see Lemma 5.1.1.

3 $\Leftrightarrow$ 4. Follows from the fact that for some $M, \omega > 0$ it holds that $\|T(t)x\| \leq Me^{-\omega t}\|x\|_X$ $\forall x \in X \Leftrightarrow \|T(t)\| \leq Me^{-\omega t}$ for some $M, \omega > 0$. □

Now we are able to prove the infinite-dimensional counterpart of Proposition 1.5.2:

Proposition 2.1.1. *For systems of the form (2.1) it holds:*

$$(2.1) \text{ is e0-UGAS}x \Leftrightarrow (2.1) \text{ is 0-UGAS}x \Leftrightarrow (2.1) \text{ is eISS} \Leftrightarrow (2.1) \text{ is ISS.}$$

Proof. System (2.1) is e0-UGASx $\Leftrightarrow$ (2.1) 0-UGASx by Lemma 2.1.2.

From eISS of (2.1) it follows ISS of (2.1), and this implies that (2.1) is 0-UGASx by taking $u \equiv 0$. It remains to prove, that 0-UGASx of (2.1) implies eISS of (2.1).

Let system (2.1) be 0-UGASx, then by Lemma 2.1.2, $\mathcal{T}$ is an exponentially stable C_0-semigroup, that is, $\exists M, w > 0$, such that $\|T(t)\| \leq Me^{-wt}$ for all $t \geq 0$. From (2.3) and (2.2) we have

$$\|s(t)\|_X \leq Me^{-wt}\|s_0\|_X + \frac{M}{w}\gamma(\|u\|_{U_c}),$$

and eISS of (2.1) is proved. □

For finite-dimensional linear systems 0-GAS is equivalent to 0-UGASx and ISS to eISS, consequently, Proposition 1.5.2 is a special case of Proposition 2.1.1. However, for infinite-dimensional linear systems 0-GAS and 0-UGASx are not equivalent. Moreover, 0-GAS in general does not imply the bounded-input bounded-state (BIBS) property, defined by

$$\forall x \in X, \ \forall u \in U_c : \ \|u\|_{U_c} \leq M \text{ for some } M > 0 \Rightarrow \|\phi(t,x,u)\|_X \leq R \text{ for some } R > 0.$$

We show this by the following example (another example, which demonstrates this property, can be found in [62]).

Example 2.1.2. *Let $C(\mathbb{R})$ be the space of continuous functions on $\mathbb{R}$, and let $X = C_0(\mathbb{R})$ be the Banach space of continuous functions (with sup-norm), that vanish at infinity:*

$$C_0(\mathbb{R}) = \{f \in C(\mathbb{R}) : \forall \varepsilon > 0 \; \exists \text{ compact set } K_\varepsilon \subset \mathbb{R} : |f(s)| < \varepsilon \; \forall s \in \mathbb{R} \backslash K_\varepsilon\}.$$

For a given $q \in C(\mathbb{R})$ consider the multiplication semigroup T_q (for the properties of these semigroups see [28, pp.24-30]), defined by

$$T_q(t)f = e^{tq}f \quad \forall f \in C_0(\mathbb{R}),$$

and for all $t \geq 0$ we define $e^{tq} : x \in \mathbb{R} \mapsto e^{tq(x)}$.

Let us take $U = X = C_0(\mathbb{R})$ and choose q as $q(s) = -\frac{1}{1+|s|}$, $s \in \mathbb{R}$. Consider the control system, given by

$$\dot{x} = A_q x + u, \tag{2.4}$$

where A_q is the infinitesimal generator of T_q.

Let us show, that the system (2.4) is 0-GAS. Fix arbitrary $f \in C_0(\mathbb{R})$. We obtain

$$\|T_q(t)f\|_{C_0(\mathbb{R})} = \sup_{s\in\mathbb{R}} |(T_q(t)f)(s)| = \sup_{s\in\mathbb{R}} e^{-t\frac{1}{1+|s|}}|f(s)| \leq \sup_{s\in\mathbb{R}} |f(s)| = \|f\|_{C_0(\mathbb{R})}.$$

This shows that the first axiom of 0-GAS property is satisfied.

To show the global attractivity of the system note that $\forall \varepsilon > 0$ there exists a compact set $K_\varepsilon \subset \mathbb{R}$, such that $|f(s)| < \varepsilon \; \forall s \in \mathbb{R}\backslash K_\varepsilon$. For such ε it holds, that $|(T_q(t)f)(s)| < \varepsilon \; \forall s \in \mathbb{R}\backslash K_\varepsilon$, $\forall t \geq 0$. Moreover, there exists $t(\varepsilon)$: $|(T_q(t)f)(s)| < \varepsilon$ for all $s \in K_\varepsilon$ and $t \geq t(\varepsilon)$. Overall, we obtain, that for each $f \in C_0(\mathbb{R})$ and all $\varepsilon > 0$ there exist $t(\varepsilon) > 0$ such that $\|T_q(t)f\|_{C_0(\mathbb{R})} < \varepsilon \; \forall t \geq t(\varepsilon)$. This proves, that system (2.4) is 0-GAS.

Now take constant with respect to time external input $u \in C_0(\mathbb{R})$: $u(s) = a\frac{1}{\sqrt{1+|s|}}$, for some $a > 0$ and all $s \in \mathbb{R}$. The solution of (2.4) is given by:

$$\begin{aligned} x(t)(s) &= e^{-t\frac{1}{1+|s|}}x_0 + \int_0^t e^{-(t-r)\frac{1}{1+|s|}} \frac{a}{\sqrt{1+|s|}} dr \\ &= e^{-t\frac{1}{1+|s|}}x_0 - a\sqrt{1+|s|}(e^{-t\frac{1}{1+|s|}} - 1). \end{aligned}$$

We make a simple estimate, substituting $s = t - 1$ for $t > 1$:

$$\sup_{s\in\mathbb{R}} a \left|\sqrt{1+|s|}(e^{-t\frac{1}{1+|s|}} - 1)\right| \geq a\sqrt{t}(1 - e^{-1}) \to \infty, \quad t \to \infty.$$

For all $x_0 \in C_0(\mathbb{R})$ holds $\|e^{-t\frac{1}{1+|s|}}x_0\|_X \to 0$, $t \to \infty$. Thus, $\|x(t)\|_X \to \infty$, $t \to \infty$, and the system (2.4) possesses unbounded trajectories for arbitrary small inputs. In particular, it is not ISS and according to Proposition 2.1.1 it is not 0-UGASx.

2.1.1 Linear parabolic equations with Neumann boundary conditions

In this subsection we investigate input-to-state stability of a system of parabolic equations with Neumann conditions on the boundary.

Let G be a bounded domain in $\mathbb{R}^p$ with a smooth boundary ∂G, and let Δ be a Laplacian in G. Let also $F \in C(G \times \mathbb{R}^m, \mathbb{R}^n)$, $F(x,0) \equiv 0$.

Consider a parabolic system

$$\begin{cases} \frac{\partial s(x,t)}{\partial t} - \Delta s = Rs + F(x, u(x,t)),\ x \in G, t > 0, \\ s\left(x,0\right) = \phi_0\left(x\right), x \in G, \\ \frac{\partial s}{\partial n}\Big|_{\partial G \times \mathbb{R}_+} = 0. \end{cases} \tag{2.5}$$

Here $\frac{\partial}{\partial n}$ is the normal derivative, $s(x,t) \in \mathbb{R}^n$, $R \in \mathbb{R}^{n \times n}$ and $u \in C(G \times \mathbb{R}_+, \mathbb{R}^m)$ be an external input.

Define an operator $L : C(\overline{G}) \to C(\overline{G})$ by $L := -\Delta$ with the domain of definition

$$D(L) = \{f \in C^2(G) \cap C^1(\overline{G}) : Lf \in C(\overline{G}),\ \frac{\partial f}{\partial n}\Big|_{\partial G} = 0\}.$$

Define the diagonal operator matrix $A = diag(-L, \dots, -L)$ with $-L$ as diagonal elements and $D(A) = (D(L))^n$. The closure $\overline{A}$ of A is an infinitesimal generator of an analytic semigroup on $X = (C(\overline{G}))^n$ (see [73, p. 121]).

Define a space of input values by $U := C(\overline{G}, \mathbb{R}^m)$ and the space of input functions by $U_c := C(\mathbb{R}_+, U)$.

The problem (2.5) may be considered as an abstract differential equation:

$$\dot{s} = (\overline{A} + R)s + f(u(t)),$$

$$s(0) = \phi_0,$$

where $u \in U_c$, $u(t)(x) = u(x,t)$ and $f : U \to X$ is defined by $f(v)(x) := F(x, v(x))$.

One can check, that the map $t \mapsto f(u(t))$ is continuous, and

$$\|f(u)\|_X = \sup_{x \in \overline{G}} |f(u)(x)| = \sup_{x \in \overline{G}} |F(x, u(x))| \leq \sup_{x \in \overline{G}, y : |y| \leq \|u\|_U} |F(x,y)| := \gamma(\|u\|_U).$$

Consequently we have reformulated the problem (2.5) in the form (2.1). Note that $\overline{A} + R$ also generates an analytic semigroup, as a sum of infinitesimal generator of analytic semigroup $\overline{A}$ and bounded operator R.

The following proposition provides the criterion of eISS of the system (2.5).

Proposition 2.1.3. *System (2.5) is eISS $\Leftrightarrow$ R is Hurwitz.*

Proof. We start with sufficiency. Denote by $S(t)$ the analytic semigroup, generated by $\overline{A} + R$.

We are going to find a simpler representation for $S(t)$. Consider (2.5) with $u \equiv 0$. Substituting $s(x,t) = e^{Rt} v(x,t)$ in (2.5) we obtain a simpler problem for v:

$$\begin{cases} \frac{\partial v(x,t)}{\partial t} = Av,\ x \in G, t > 0, \\ v\left(x,0\right) = \phi_0\left(x\right), x \in G, \\ \frac{\partial v}{\partial n}\Big|_{\partial G \times \mathbb{R}_+} = 0. \end{cases} \tag{2.6}$$

In terms of semigroups, this means: $S(t) = e^{Rt} T(t)$, where $T(t)$ is a semigroup generated by $\overline{A}$. It is well-known (see, e.g. [36]), that the growth bound of analytic semigroup $T(t)$ is given by $\sup \Re(Spec(\overline{A})) = \sup_{\lambda \in Spec(\overline{A})} \Re(\lambda)$, where $\Re(z)$ is the real part of a complex number z.

We are going to find an upper bound of spectrum of $\overline{A}$ in $D(A)$. Note that $Spec(A) = Spec(-L)$. Thus, it is enough to estimate the spectrum of $-L$ that consists of all $\lambda \in \mathbb{C}$, such that the following equation has nontrivial solution

$$\left\{ \begin{array}{l} Ls + \lambda s = 0,\ x \in G \\ \frac{\partial s}{\partial n}\Big|_{\partial G} = 0. \end{array} \right. \tag{2.7}$$

Let $\lambda > 0$ be an eigenvalue of $-L$, and $u_\lambda \not\equiv 0$ be the corresponding eigenfunction. If u_λ attains its nonnegative maximum over $\overline{G}$ in some $x \in G$, then according to the strong maximum principle (see [29], p. 333) $u_\lambda \equiv const$ and consequently $u_\lambda \equiv 0$. Thus, u_λ cannot be an eigenfunction. If u_λ attains the nonnegative maximum over $\overline{G}$ at some $x \in \partial G$, then by Hopf's lemma (see [29], p. 330), $\frac{\partial u_\lambda(x)}{\partial n} > 0$. Consequently, $u_\lambda \le 0$ in $\overline{G}$. But $-u_\lambda$ is also an eigenfunction, thus applying the same argument we obtain that $u_\lambda \equiv 0$ in $\overline{G}$, thus $\lambda > 0$ is not an eigenvalue.

Obviously $\lambda = 0$ is an eigenvalue of $-L$, therefore the growth bound of $T(t)$ is 0 and the growth bound of $S(t)$ is $\omega_0 = \sup\{\Re(\lambda) : \exists x \neq 0 : Rx = \lambda x\}$. Thus, if R is Hurwitz, then the system (2.5) is exponentially 0-UGAS$x$and by Proposition 2.1.1 it is eISS.

To prove necessity note that for constant ϕ_0 and $u \equiv 0$ the solutions of (2.5) are for arbitrary $x \in G$ the solutions of $\dot{s} = Rs$, and to guarantee the stability of these solutions R has to be Hurwitz. □

Remark 2.1.3. *In (2.5) the diffusion coefficients are equal to one. In case, when the diffusion coefficients of different subsystems are not equal to each other the statement of Proposition 2.1.3 is in general not true because of Turing instability phenomenon (see [83], [63]).*

2.2 Lyapunov functions for nonlinear systems

To verify both local and global input-to-state stability of nonlinear systems, Lyapunov functions can be exploited. In this section we provide basic notions and results and illustrate them by an example.

Definition 2.2.1. *A continuous function $V : D \to \mathbb{R}_+$, $D \subset X$, $0 \in int(D) = D \backslash \partial D$ is called local ISS-Lyapunov function (LISS-LF) for Σ, if there exist $\rho_x, \rho_u > 0$, functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ and positive definite function α, such that:*

$$\psi_1(\|x\|_X) \le V(x) \le \psi_2(\|x\|_X), \quad \forall x \in D, \tag{2.8}$$

and $\forall x \in X : \|x\|_X \le \rho_x$, $\forall u \in U_c : \|u\|_{U_c} \le \rho_u$ it holds:

$$\|x\|_X \ge \chi(\|u\|_{U_c}) \ \Rightarrow\ \dot{V}_u(x) \le -\alpha(\|x\|_X), \tag{2.9}$$

where the Lie derivative of V corresponding to the input u is given by

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi(t,x,u)) - V(x)). \tag{2.10}$$

The function χ is called ISS-Lyapunov gain for (X, U_c, ϕ).

If in the previous definition $D = X$, $\rho_x = \infty$ and $\rho_u = \infty$, then V is called an ISS-Lyapunov function.

Note, that in general the computation of the Lie derivative $\dot{V}_u(x)$ requires knowledge of the input on some neighborhood of the time instant $t = 0$.

If it is clear from the context, with respect to which input the Lie derivative $\dot{V}_u(x)$ is computed, then we write simply $\dot{V}(x)$.

Theorem 2.2.1. *Let $\Sigma = (X, U_c, \phi)$ be a time-invariant control system, and $x \equiv 0$ be its equilibrium.*

Assume, that for all $u \in U_c$ and for all $s \geq 0$ a function $\tilde{u}$, defined by $\tilde{u}(\tau) = u(\tau + s)$ for all $\tau \geq 0$, belong to U_c and it holds $\|\tilde{u}\|_{U_c} \leq \|u\|_{U_c}$.

If Σ possesses a (L)ISS-Lyapunov function, then it is (L)ISS.

For a counterpart of this theorem for infinite-dimensional dynamical systems (without inputs) see, e.g., [36, p. 84].

Proof. Let the control system $\Sigma = (X, U_c, \phi)$ possess a LISS-Lyapunov function and $\psi_1, \psi_2, \chi, \alpha, \rho_x, \rho_u$ be as in Definition 2.2.1. Take an arbitrary control $u \in U_c$ with $\|u\|_{U_c} \leq \rho_u$ such that

$$I = \{x \in D : \|x\|_X \leq \rho_x,\ V(x) \leq \psi_2 \circ \chi(\|u\|_{U_c}) \leq \rho_x\} \subset int(D).$$

Such u exists, because $0 \in int(D)$.

Firstly we prove, that I is invariant w.r.t. Σ, that is: $\forall x \in I \Rightarrow x(t) = \phi(t, x, u) \in I$, $t \geq 0$.

If $u \equiv 0$, then $I = \{0\}$, and I is invariant, because $x = 0$ is the equilibrium point of Σ. Consider $u \not\equiv 0$.

If I is not invariant w.r.t. Σ, then, due to continuity of ϕ w.r.t. t (continuity axiom of Σ), $\exists t_* > 0$, such that $V(x(t_*)) = \psi_2 \circ \chi(\|u\|_{U_c})$, and therefore $\|x(t_*)\|_X \geq \chi(\|u\|_{U_c})$.

The input to the system Σ after time t^* is $\tilde{u}$, defined by $\tilde{u}(\tau) = u(\tau + t^*)$, $\tau \geq 0$. According to the assumptions of the theorem $\|\tilde{u}\|_{U_c} \leq \|u\|_{U_c}$. Then from (2.9) it follows, that $\dot{V}_{\tilde{u}}(x(t_*)) = -\alpha(\|x(t_*)\|_X) < 0$. Thus, the trajectory cannot escape the set I.

Now take arbitrary x_0: $\|x_0\|_X \leq \rho_x$. If $x_0 \notin I$, then $V(x) > \psi_2 \circ \chi(\|u\|_{U_c})$, which by (2.8) implies that $\|x\|_X > \chi(\|u\|_{U_c})$ and by (2.9) we have the following differential inequality ($x(t)$ is the trajectory, corresponding to the initial condition x_0):

$$\dot{V}(x(t)) \leq -\alpha(\|x(t)\|_X) \leq -\alpha \circ \psi_2^{-1}(V(x(t))).$$

From the comparison principle (see [59], Lemma 4.4 for $y(t) = V(x(t))$) it follows, that $\exists\ \tilde{\beta} \in \mathcal{KL} :\ V(x(t)) \leq \tilde{\beta}(V(x_0), t)$, and consequently:

$$\|x(t)\|_X \leq \beta(\|x_0\|_X, t), \forall t :\ x(t) \notin I, \tag{2.11}$$

where $\beta(r, t) = \psi_1^{-1} \circ \tilde{\beta}(\psi_2^{-1}(r), t)$, $\forall r, t \geq 0$.

From the properties of $\mathcal{KL}$ functions it follows, that $\exists t_1$:

$$t_1 := \inf_{t \geq 0}\{x(t) \in I\}.$$

From the invariance of the set I we conclude, that

$$\|x(t)\|_X \leq \gamma(\|u\|_{U_c}), \ t > t_1, \tag{2.12}$$

where $\gamma = \psi_1^{-1} \circ \psi_2 \circ \chi \in \mathcal{K}$.

Our estimates hold for an arbitrary control u: $\|u\|_{U_c} \leq \rho_u$, thus, combining (2.11) and (2.12), we obtain the claim of the theorem.

To prove, that from existence of ISS-Lyapunov function it follows ISS of Σ, one has to argue as above but with $\rho_x = \rho_u = \infty$. □

Remark 2.2.1. *Assumption on the properties of U_c used in the Theorem 2.2.1 holds for many usual function classes, such as $PC(\mathbb{R}_+, U)$, $L_p(\mathbb{R}_+, U)$, $p \geq 1$, $L_\infty(\mathbb{R}_+, U)$, Sobolev spaces etc.*

Remark 2.2.2. *In case of input spaces $L_p(\mathbb{R}_+, U)$ it may be interesting to consider in the definition of ISS instead of norms in the whole space $L_p(\mathbb{R}_+, U)$ the norms in the space $L_p([0, t], U)$, similarly to the Proposition 1.4.1. In this case one could enlarge*

2.2.1 Density argument

In this subsection we prove a simple lemma, which turns out to be useful in the theory as well as in practice.

Let $\Sigma := (X, U_c, \phi)$ be a control system. Let $\hat{X}$, $\hat{U}_c$ be dense linear normed subspaces of X and U_c respectively, and let $\hat{\Sigma} := (\hat{X}, \hat{U}_c, \phi)$ be the system, generated by the same as in Σ transition map ϕ, but restricted to the state space $\hat{X}$ and space of admissible inputs $\hat{U}_c$.

Assume that ϕ depends continuously on inputs and on initial states, that is $\forall x \in X, \forall u \in U_c, \forall T > 0$ and $\forall \varepsilon > 0$ there exist $\delta > 0$, such that $\forall x' \in X : \|x - x'\|_X < \delta$ and $\forall u' \in U_c : \|u - u'\|_{U_c} < \delta$ it holds

$$\|\phi(t, x, u) - \phi(t, x', u')\|_X < \varepsilon, \quad \forall t \in [0, T].$$

Now we have the following result

Lemma 2.2.3. *Let $\hat{\Sigma}$ be ISS. Then Σ is also ISS with the same β and γ in the estimate (1.5).*

Proof. Since $\hat{\Sigma}$ is ISS, we know that there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty$, such that $\forall \hat{x} \in \hat{X}$, $\forall \hat{u} \in \hat{U}_c$ and $\forall t \geq 0$ it holds

$$\|\phi(t, \hat{x}, \hat{u})\|_X \leq \beta(\|\hat{x}\|_X, t) + \gamma(\|\hat{u}\|_{U_c}). \tag{2.13}$$

Let Σ be not ISS. Then there exist $T > 0$, $x \in X$, $u \in U_c$:

$$\|\phi(T, x, u)\|_X = \beta(\|x\|_X, T) + \gamma(\|u\|_{U_c}) + r, \tag{2.14}$$

where $r = r(T, x, u) > 0$.

From (2.13) and (2.14) we obtain

$$\begin{aligned} \|\phi(T, x, u)\|_X - \|\phi(T, \hat{x}, \hat{u})\|_X \geq \ & (\beta(\|x\|_X, T) - \beta(\|\hat{x}\|_X, T)) \\ & + (\gamma(\|u\|_{U_c}) - \gamma(\|\hat{u}\|_{U_c})) + r. \end{aligned} \tag{2.15}$$

Since $\hat{X}$ and $\hat{U}_c$ are dense in X and U_c respectively, then we can find the sequences $\{\hat{x}_i\} \subset \hat{X}$: $\|x - \hat{x}_i\|_X \to 0$ and $\{\hat{u}_i\} \subset \hat{U}_c$: $\|u - \hat{u}_i\|_{U_c} \to 0$. From (2.15) it follows that $\forall \varepsilon > 0$ there exist $\hat{x}_i$ and $\hat{u}_i$:

$$\|\phi(T, x, u) - \phi(T, \hat{x}_i, \hat{u}_i)\|_X \geq |\|\phi(T, x, u)\|_X - \|\phi(T, \hat{x}_i, \hat{u}_i)\|_X| \geq r - 2\varepsilon. \tag{2.16}$$

This contradicts to the assumption of continuous dependence of ϕ on initial states and inputs. Thus, Σ is ISS. □

2.2.2 ISS-Lyapunov functions for systems with piecewise-continuous inputs

The Definition 2.2.1 differs from the Definition 1.5.1, used in finite-dimensional theory. We are going to prove, that for the ODE systems (with piecewise-continuous inputs) our definition is equivalent to the standard one.

Firstly we reformulate the definition of LISS-LF for the case, when $U_c = PC(\mathbb{R}_+, U)$.

Proposition 2.2.2. *A continuous function $V : D \to \mathbb{R}_+$, $D \subset X$, $0 \in int(D) = D \backslash \partial D$ is a LISS-Lyapunov function for $\Sigma = (X, PC(\mathbb{R}_+, U), \phi)$ if and only if there exist $\rho_x, \rho_u > 0$ and functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\tilde{\chi} \in \mathcal{K}$ and positive definite function α, such that:*

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad \forall x \in D$$

and so that $\forall x \in X : \|x\|_X \leq \rho_x$, $\forall \xi \in U : \|\xi\|_U \leq \rho_u$ it holds, that

$$\|x\|_X \geq \tilde{\chi}(\|\xi\|_U) \ \Rightarrow \ \dot{V}_u(x) \leq -\alpha(\|x\|_X), \tag{2.17}$$

for all $u \in U_c$: $\|u\|_{U_c} \leq \rho_u$ with $u(0) = \xi$.

Proof. Let us begin with sufficiency. Let $u \in U_c = PC(\mathbb{R}_+, U)$, $\|u\|_{U_c} \leq \rho_u$. Take an arbitrary $x \in X$ and assume that $\|x\|_X \geq \chi(\|u\|_{U_c})$. Then $\|x\|_X \geq \chi(\|u(0)\|_U)$ and according to (2.17) for this u it holds $\dot{V}_u(x) \leq -\alpha(\|x\|_X)$. The implication (2.9) is proved and thus V is a LISS-Lyapunov function according to Definition 2.2.1.

Let us prove necessity. Take an arbitrary $u \in U_c$, and for arbitrary $s > 0$ consider the input $u_s \in U_c$ defined by

$$u_s(\tau) := \begin{cases} u(\tau), \tau \in [0, s], \\ u(s), \tau > s. \end{cases}$$

Due to Causality of Σ, $\phi(t, x, u) = \phi(t, x, u_s)$ for all $t \in [0, s]$, and according to the definition of the Lie derivative we obtain $\dot{V}_u(x) = \dot{V}_{u_s}(x)$. Let $u \in U_c$ and $\|u\|_{U_c} \leq \rho_u$. Then also $\|u_s\|_{U_c} \leq \rho_u$ and since V is a LISS-Lyapunov function it follows from (2.9) that

$$\|x\|_X \geq \chi(\|u_s\|_{U_c}) \ \Rightarrow \ \dot{V}_{u_s}(x) \leq -\alpha(\|x\|_X).$$

Then it holds also

$$\|x\|_X \geq \chi(\|u_s\|_{U_c}) \ \Rightarrow \ \dot{V}_u(x) \leq -\alpha(\|x\|_X). \tag{2.18}$$

Since $U_c = PC(\mathbb{R}_+, U)$, it follows that for arbitrary $u \in U_c$ and arbitrary $\varepsilon > 0$ there exists $\tau > 0$ such that $\|u_\tau\|_{U_c} \leq (1 + \varepsilon)\|u(0)\|_U$. Then from (2.18) it follows that

$$\|x\|_X \geq \tilde{\chi}(\|u(0)\|_U) \ \Rightarrow \ \dot{V}_u(x) \leq -\alpha(\|x\|_X),$$

where $\tilde{\chi}(r) = \chi((1+\varepsilon)r)$, for all $r \geq 0$.

Since $u \in U_c$, $\|u\|_{U_c} \leq \rho_u$ has been chosen arbitrarily, the necessity is proved. □

Now consider the ODE system

$$\dot{x}(t) = f(x(t), u(t)),\ x(t) \in \mathbb{R}^n,\ u(t) \in \mathbb{R}^m. \tag{2.19}$$

Let $V : D \to \mathbb{R}_+$, $D \subset \mathbb{R}^n$, $0 \in int(D) = D \backslash \partial D$ be a locally Lipschitz continuous function (and thus it is differentiable almost everywhere by Rademacher's theorem). For such systems $\dot{V}_u(x)$ can be computed for almost all x and the implication (2.17) is resolved to

$$\|x\|_X \geq \chi(\|\xi\|_U) \ \Rightarrow\ \nabla V \cdot f(x,\xi) \leq -\alpha(\|x\|_X).$$

Using this implication instead of (2.17), we obtain the standard definition of LISS-Lyapunov function for finite-dimensional systems. Thus, Definition 2.2.1 is consistent with the existing definitions of LISS-Lyapunov functions for ODE systems.

2.2.3 Example

Consider the following system

$$\begin{cases} \frac{\partial s}{\partial t} = \frac{\partial^2 s}{\partial x^2} - f(s) + u^m(x,t), & x \in (0,\pi),\ t > 0, \\ s(0,t) = s(\pi,t) = 0. \end{cases} \tag{2.20}$$

We assume, that f is locally Lipschitz continuous, monotonically increasing up to infinity, $f(-r) = -f(r)$ for all $r \in \mathbb{R}$ (in particular, $f(0) = 0$), and $m \in (0,1]$.

To reformulate (2.20) as an abstract differential equation we define operator A by $As := \frac{d^2 s}{dx^2}$ with $D(A) = H_0^1(0,\pi) \cap H^2(0,\pi)$.

The norm on $H_0^1(0,\pi)$ we choose as $\|s\|_{H_0^1(0,\pi)} := \left(\int_0^\pi \left(\frac{\partial s}{\partial x}\right)^2 dx\right)^{\frac{1}{2}}$. It is well-known (see, e.g., [36], p.8), that on $H_0^1(0,\pi)$ this norm is equivalent to the original norm (5.6).

Operator A generates an analytic semigroup on $L_2(0,\pi)$. System (2.20) takes the form

$$\frac{\partial s}{\partial t} = As - F(s) + u^m,\ t > 0, \tag{2.21}$$

where F is defined by $F(s(t))(x) := f(s(x,t))$, $x \in (0,\pi)$.

Equation (2.21) defines a control system with the state space $X = H_0^1(0,\pi)$ and input function space $U_c = C(\mathbb{R}_+, L_2(0,\pi))$.

Consider the following ISS-Lyapunov function candidate:

$$V(s) := \int_0^\pi \left(\frac{1}{2}\left(\frac{\partial s}{\partial x}\right)^2 + \int_0^{s(x)} f(y)dy\right) dx. \tag{2.22}$$

We are going to prove, that V is an ISS-Lyapunov function.

Under the above assumptions about function f it holds that $\int_0^r f(y)dy \geq 0$ for every $r \in \mathbb{R}$.

We have to verify the estimates (2.8) for a function V. The estimate from below is easy:

$$V(s) \geq \int_0^\pi \frac{1}{2}\left(\frac{\partial s}{\partial x}\right)^2 dx = \frac{1}{2}\|s\|^2_{H_0^1(0,\pi)}. \tag{2.23}$$

Let us find an estimate from above. We have

$$V(s) = \int_0^\pi \frac{1}{2}\left(\frac{\partial s}{\partial x}\right)^2 dx + \int_0^\pi \int_0^{s(x)} f(y)dy\, dx.$$

According to the embedding theorem for Sobolev spaces (see [29, Theorem 6, p. 270]), every $s \in H_0^1(0,\pi)$ belongs actually to $C^{\frac{1}{2}}(0,\pi)$ (Hölder space with exponent $\frac{1}{2}$). Moreover, there exists a constant C, which does not depend on $s \in H_0^1(0,\pi)$, such that

$$\|s\|_{C^{\frac{1}{2}}(0,\pi)} \leq C\|s\|_{H_0^1(0,\pi)},\ \forall s \in H_0^1(0,\pi). \tag{2.24}$$

Define $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\psi(r) := \frac{1}{2}r^2 + \sup_{s:\ \|s\|_{H_0^1(0,\pi)} \leq r} \int_0^\pi \int_0^{s(x)} f(y)dydx.$$

Inequality (2.24) and the fact that $\|s\|_{C(0,\pi)} \leq \|s\|_{C^{\frac{1}{2}}(0,\pi)}$ for all $s \in C^{\frac{1}{2}}(0,\pi)$ imply

$$\begin{aligned}
\psi(r) &= \frac{1}{2}r^2 + \sup_{s:\ C\|s\|_{H_0^1(0,\pi)} \leq Cr} \int_0^\pi \int_0^{s(x)} f(y)dydx && (2.25)\\
&\leq \frac{1}{2}r^2 + \sup_{s:\ \|s\|_{C(0,\pi)} \leq Cr} \int_0^\pi \int_0^{s(x)} f(y)dydx \leq \frac{1}{2}r^2 + \pi\int_0^{Cr} f(y)dy := \psi_2(r). && (2.26)
\end{aligned}$$

Since f, restricted to positive values of the argument, belongs to $\mathcal{K}_\infty$, ψ_2 is also $\mathcal{K}_\infty$-function.

Finally, for all $s \in H_0^1(0,\pi)$ we have:

$$\frac{1}{2}\|s\|^2_{H_0^1(0,\pi)} \leq V(s) \leq \psi_2(\|s\|_{H_0^1(0,\pi)}), \tag{2.27}$$

and the property (2.8) is verified.

Let us compute the Lie derivative of V

$$\begin{aligned}
\dot{V}(s) &= \int_0^\pi \frac{\partial s}{\partial x}\frac{\partial^2 s}{\partial x \partial t} + f(s(x))\frac{\partial s}{\partial t}dx\\
&= \left[\frac{\partial s}{\partial x}\frac{\partial s}{\partial t}\right]_{x=0}^{x=\pi} + \int_0^\pi \left(-\frac{\partial^2 s}{\partial x^2}\frac{\partial s}{\partial t} + f(s(x))\frac{\partial s}{\partial t}\right)dx.
\end{aligned}$$

From boundary conditions it follows $\frac{\partial s}{\partial t}(0,t) = \frac{\partial s}{\partial t}(\pi,t) = 0$. Thus, substituting expression for $\frac{\partial s}{\partial t}$, we obtain

$$\dot{V}(s) = -\int_0^\pi \left(\frac{\partial^2 s}{\partial x^2} - f(s(x))\right)^2 dx + \int_0^\pi \left(\frac{\partial^2 s}{\partial x^2} - f(s(x))\right)(-u^m)dx.$$

Define

$$I(s) := \int_0^\pi \left(\frac{\partial^2 s}{\partial x^2} - f(s(x))\right)^2 dx.$$

Using the Cauchy-Schwarz inequality for the second term, we have:

$$\dot{V}(s) \le -I(s) + \sqrt{I(s)}\ \|u^m\|_{L_2(0,\pi)}. \tag{2.28}$$

Now let us consider $I(s)$

$$\begin{aligned} I(s) &= \int_0^\pi \left(\frac{\partial^2 s}{\partial x^2}\right)^2 dx - 2\int_0^\pi \frac{\partial^2 s}{\partial x^2} f(s(x))dx + \int_0^\pi f^2(s(x))dx \\ &= \int_0^\pi \left(\frac{\partial^2 s}{\partial x^2}\right)^2 dx + 2\int_0^\pi \left(\frac{\partial s}{\partial x}\right)^2 \frac{\partial f}{\partial s}(s(x))dx + \int_0^\pi f^2(s(x))dx \\ &\ge \int_0^\pi \left(\frac{\partial^2 s}{\partial x^2}\right)^2 dx. \end{aligned}$$

According to Theorem 5.4.2 for $s \in H_0^1(0,\pi) \cap H^2(0,\pi)$ it holds, that

$$\int_0^\pi \left(\frac{\partial^2 s}{\partial x^2}\right)^2 dx \ge \int_0^\pi \left(\frac{\partial s}{\partial x}\right)^2 dx.$$

Overall, we have:

$$I(s) \ge \|s\|^2_{H_0^1(0,\pi)}. \tag{2.29}$$

Let us consider $\|u^m\|_{L_2(0,\pi)}$. Using the Hölder inequality, we obtain:

$$\begin{aligned} \|u^m\|_{L_2(0,\pi)} &= \left(\int_0^\pi u^{2m} \cdot 1\ dx\right)^{\frac{1}{2}} \\ &\le \left(\int_0^\pi u^2\ dx\right)^{\frac{m}{2}} \left(\int_0^\pi 1^{\frac{1}{1-m}} dx\right)^{\frac{1-m}{2}} = \pi^{\frac{1-m}{2}} \|u\|^m_{L_2(0,\pi)}. \end{aligned} \tag{2.30}$$

Now we choose the gain as

$$\chi(r) = a\pi^{\frac{1-m}{2}} r^m,\ a > 1.$$

If $\chi(\|u\|_{L_2(0,\pi)}) \le \|s\|_{H_0^1(0,\pi)}$, we obtain from (2.28), using (2.30) and (2.29):

$$\dot{V}(s) \le -I(s) + \frac{1}{a}\sqrt{I(s)}\|s\|_{H_0^1(0,\pi)} \le (\frac{1}{a} - 1)I(s) \le (\frac{1}{a} - 1)\|s\|^2_{H_0^1(0,\pi)}. \tag{2.31}$$

The above computations are valid for states $s \in \hat{X}$: $\hat{X} := \{s \in C^\infty([0,\pi]) :\ s(0) = s(\pi) = 0\}$ and inputs $u \in \hat{U}_c$, $\hat{U}_c := C(\mathbb{R}_+, C^\infty([0,\pi]))$.

The system $(\hat{X}, \hat{U}_c, \phi)$, where $\phi(\cdot, s, u)$ is a solution of (2.20)for $s \in \hat{X}$ and $u \in \hat{U}_c$, possesses the ISS-Lyapunov function and consequently is ISS according to Proposition 2.2.2.

It is known, that $\hat{X}$ is dense in $H_0^1(0,\pi)$ and $\hat{U}_c$ is dense in $C(\mathbb{R}_+, L_2([0,\pi]))$. According to the Lemma 2.2.3 the system (2.20) is also ISS (with $X = H_0^1(0,\pi)$, $U_c = C(\mathbb{R}_+, L_2(0,\pi))$).

Remark 2.2.4. *In the example we have taken $U = L_2(0,\pi)$ and $X = H_0^1(0,\pi)$. But in case of interconnection with other parabolic systems (when we identify input u with the state of the other system), that have state space $H_0^1(0,\pi)$ (as our system), we have to choose $U = X = H_0^1(0,\pi)$.*

In this case we can continue the estimates (2.30)*, using Friedrichs' inequality* (5.9)*, which now takes the form*

$$\int_0^\pi s^2(x)dx \le \int_0^\pi \left(\frac{\partial s}{\partial x}\right)^2 dx$$

to obtain

$$\|u^m\|_{L_2(0,\pi)} \le \pi^{\frac{1-m}{2}} \|u\|^m_{H^1_0(0,\pi)} \tag{2.32}$$

and choosing the same gains, prove the input-to state stability of (2.21) *w.r.t. spaces* $X = H^1_0(0,\pi)$, $U_c = C(\mathbb{R}_+, H^1_0(0,\pi))$.

Remark 2.2.5. *The input-to-state stability for semilinear parabolic PDEs has been studied also in the recent paper [62]. However, the definition of ISS and of ISS-Lyapunov function in that paper are different from used in our paper. In particular, consider the property of* (2.8) *of ISS-Lyapunov function. The corresponding property (2) from [62] is not equivalent to* (2.8) *for* $X := C^2([0,L],\mathbb{R}^n)$ *equipped with the* L_2*-norm (which is chosen as the state space in [62]), since the expression in (2) from [62] cannot be bounded by a function of* L_2*-norm of an element of* X *in general.*

2.3 Linearization

In this section we prove two theorems, stating that a nonlinear system is LISS provided its linearization is ISS. One of them needs less restrictive assumptions, but it doesn't provide us with a LISS-Lyapunov function for the nonlinear system. In the other theorem it is assumed, that the state space is a Hilbert space. This assumption yields a form of LISS-Lyapunov function.

Consider the system

$$\dot{x}(t) = Ax(t) + f(x(t), u(t)),\ \ x(t) \in X, u(t) \in U, \tag{2.33}$$

where X is a Banach space, A is the generator of a C_0-semigroup, $f : X \times U \to X$ is defined on some open set Q, $(0,0)$ is in interior of Q and $f(0,0) = 0$, thus $x \equiv 0$ is an equilibrium point of (2.33).

In this section we assume, that f can be decomposed as

$$f(x,u) = Bx + Cu + g(x,u),$$

where $B \in L(X)$, $C \in L(U,X)$ and for each constant $w > 0$ there exist $\rho > 0$, such that $\forall x : \|x\|_X \le \rho$, $\forall u : \|u\|_U \le \rho$ it holds

$$\|g(x,u)\|_X \le w(\|x\|_X + \|u\|_U). \tag{2.34}$$

Consider also the linear approximation of a system (2.33), given by

$$\dot{x} = Rx + Cu, \tag{2.35}$$

where $R = A + B$ is an infinitesimal generator of a C_0-semigroup (which we denote by T), as the sum of a generator A and bounded operator B.

Our first result of this section:

Theorem 2.3.1. *If (2.35) is ISS, then (2.33) is LISS.*

Proof. If the system (2.35) is ISS, then according to Proposition 2.1.1 and Lemma 2.1.2 the semigroup T is exponentially stable, that is for some $K, h > 0$ it holds $\|T(t)\| \leq Ke^{-ht}$.

For a trajectory $x(\cdot)$ it holds

$$x(t) = T(t)x_0 + \int_0^t T(t-s)\left(Cu(s) + g(x(s), u(s))\right) ds.$$

We have:

$$\|x(t)\|_X \leq Ke^{-ht}\|x_0\|_X + K\int_0^t e^{-h(t-s)}(\|C\|\|u(s)\|_U + \|g(x(s), u(s))\|_X)ds.$$

Take $w > 0$ sufficiently small. Then there exists some $r > 0$, such that (2.34) holds for all x, u: $\|x\|_X \leq r$ and $\|u\|_U \leq r$. Take an initial condition x_0 and an input u such that $\|u\|_{U_c} < r$ and $\|x_0\|_X < r$. Then, due to continuity of the trajectory, there exists $t^* > 0$ such that $\|x(t)\|_X < r$, $t \in [0, t^*]$.

For all $t \in [0, t^*]$ and every $\varepsilon < h$ using "fading-memory" estimates (see, e.g. [50]) and (2.34) we obtain

$$\begin{aligned}\|x(t)\|_X &\leq Ke^{-ht}\|x_0\|_X + K\int_0^t e^{-\varepsilon(t-s)}e^{-(h-\varepsilon)(t-s)}(\|C\|\|u(s)\|_U + w(\|x(s)\|_X + \|u(s)\|_U))ds \\ &\leq Ke^{-ht}\|x_0\|_X + \frac{K}{\varepsilon}\sup_{0\leq s\leq t} e^{-(h-\varepsilon)(t-s)}\left((\|C\| + w)\|u(s)\|_U + w\|x(s)\|_X\right). \qquad (2.36)\end{aligned}$$

Define ψ and v by $\psi(t) := e^{(h-\varepsilon)t}x(t)$ and $v(t) := e^{(h-\varepsilon)t}u(t)$ respectively. Multiplying (2.36) by $e^{(h-\varepsilon)t}$, we obtain:

$$\|\psi(t)\|_X \leq Ke^{-\varepsilon t}\|x_0\|_X + \frac{K}{\varepsilon}(\|C\| + w)\sup_{0\leq s\leq t}\|v(s)\|_U + \frac{K}{\varepsilon}w\sup_{0\leq s\leq t}\|\psi(s)\|_X. \qquad (2.37)$$

Assume that w is so that $1 - \frac{K}{\varepsilon}w > 0$. Taking supremum from the both sides of (2.37), we obtain:

$$\sup_{0\leq s\leq t}\|\psi(s)\|_X \leq \frac{1}{1-\frac{K}{\varepsilon}w}\left(K\|x_0\|_X + \frac{K}{\varepsilon}(\|C\| + w)\sup_{0\leq s\leq t}\|v(s)\|_U\right).$$

In particular,

$$\|\psi(t)\|_X \leq \frac{1}{1-\frac{K}{\varepsilon}w}\left(K\|x_0\|_X + \frac{K}{\varepsilon}(\|C\| + w)\sup_{0\leq s\leq t}\|v(s)\|_U\right).$$

Returning to the variables x, u, we have:

$$\|x(t)\|_X \leq \frac{K}{1-\frac{K}{\varepsilon}w}\left(e^{-(h-\varepsilon)t}\|x_0\|_X + \frac{(\|C\| + w)}{\varepsilon}\sup_{0\leq s\leq t} e^{-(h-\varepsilon)(t-s)}\|u(s)\|_U\right). \qquad (2.38)$$

Taking $\|u\|_{U_c}$ and $\|x_0\|_X$ small enough we guarantee that $\|x(t)\|_X < r$ for all $t \in [0, t^*]$. Because of BIC property it is clear, that t^* can be chosen arbitrarily large. Thus, the last estimate proves LISS of the system (2.33). □

Remark 2.3.1. *Inequality (2.38) is a "fading memory" estimate of a norm of a state. This shows, that the system is not only ISS, but also ISDS, see [32].*

2.3.1 Constructions of LISS-Lyapunov functions

In this subsection we are going to use linearization in order to construct the LISS-Lyapunov function for the nonlinear systems.

In addition to assumptions in the beginning of the Section 2.3 suppose that X is a Hilbert space with a scalar product $\langle \cdot, \cdot \rangle$, and A generates an analytic semigroup on X.

We need the following definitions:

Definition 2.3.1. *A self-adjoint operator $P \in L(X)$ is positive if $\langle Px, x \rangle > 0$ for all $x \in X$, $x \neq 0$.*

Definition 2.3.2. *A self-adjoint operator P on the Hilbert space X is coercive, if $\exists \epsilon > 0$:*

$$\langle Px, x \rangle \geq \epsilon \|x\|_X^2 \quad \forall x \in D(P).$$

The largest of such ϵ is called lower bound of operator P.

Since operator A is an infinitesimal generator of an analytic semigroup and B is bounded, $R = A + B$ also generates an analytic semigroup.

Let system (2.35) be ISS. Then, according to Proposition 2.1.1, (2.35) is exponentially 0-UGASx. By Lemma 2.1.2 this implies that R generates exponentially stable semigroup. By [12, Theorem 5.1.3, p. 217] this is equivalent to the existence of a positive bounded operator $P \in L(X)$, for which it holds that

$$\langle Rx, Px \rangle + \langle Px, Rx \rangle = -\|x\|_X^2, \quad \forall x \in D(R). \tag{2.39}$$

If an operator P is coercive, then a LISS-Lyapunov function for a system (2.33) can be constructed. More precisely, it holds

Theorem 2.3.2. *If the system (2.35) is ISS, and there exist a coercive operator P, satisfying (2.39), then LISS-Lyapunov function of (2.33) can be constructed in the form*

$$V(x) = \langle Px, x \rangle . \tag{2.40}$$

Proof. Since P is bounded and coercive, for some $\epsilon > 0$ it holds

$$\epsilon \|x\|_X^2 \leq \langle Px, x \rangle \leq \|P\| \|x\|_X^2, \quad \forall x \in X,$$

and estimate (2.8) is verified.

Let us compute the Lie derivative of V w.r.t. the system (2.33). Firstly consider the case, when $x \in D(R) = D(A)$. We have

$$\begin{aligned} \dot{V}(x) &= \langle P\dot{x}, x \rangle + \langle Px, \dot{x} \rangle \\ &= \langle P(Rx + Cu + g(x,u)), x \rangle + \langle Px, Rx + Cu + g(x,u) \rangle \\ &= \langle P(Rx), x \rangle + \langle Px, Rx \rangle + \langle P(Cu + g(x,u)), x \rangle + \langle Px, Cu + g(x,u) \rangle . \end{aligned}$$

We continue estimates using the property

$$\langle P(Rx), x \rangle = \langle Rx, Px \rangle ,$$

which holds for positive operators, equality (2.39) and for the last two terms Cauchy-Schwarz inequality in the space X

$$\begin{aligned}\dot{V}(x) &\leq -\|x\|_X^2 + \|P(Cu + g(x,u))\|_X \|x\|_X + \|Px\|_X \|Cu + g(x,u)\|_X \\ &\leq -\|x\|_X^2 + \|P\| \|(Cu + g(x,u))\|_X \|x\|_X + \|P\| \|x\|_X \|Cu + g(x,u)\|_X \\ &\leq -\|x\|_X^2 + 2\|P\| \|x\|_X (\|C\| \|u\|_U + \|g(x,u)\|_X).\end{aligned}$$

For each $w > 0$ $\exists \rho$, such that $\forall x : \|x\|_X \leq \rho,\ \forall u : \|u\|_U \leq \rho$ it holds (2.34). Using (2.34) we continue above estimates

$$\dot{V}(x) \leq -\|x\|_X^2 + 2w\|P\| \|x\|_X^2 + 2\|P\|(\|C\| + w)\|x\|_X \|u\|_U.$$

Take $\chi(r) := \sqrt{r}$. Then for $\|u\|_U \leq \chi^{-1}(\|x\|_X) = \|x\|_X^2$ we have:

$$\dot{V}(x) \leq -\|x\|_X^2 + 2w\|P\| \|x\|_X^2 + 2\|P\|(\|C\| + w)\|x\|_X^3. \tag{2.41}$$

Choosing w and ρ small enough the right hand side can be estimated from above by some negative quadratic function of $\|x\|_X$.

These derivations hold for $x \in D(R) \subset X$. If $x \notin D(R)$, then for all admissible u the solution $x(t) \in D(R)$ and $t \to V(x(t))$ is a continuously differentiable function for all $t > 0$ (these properties follow from the properties of solutions $x(t)$, see Theorem 3.3.3 in [36]).

Therefore, by the mean-value theorem, $\forall t > 0\ \exists t_* \in (0,t)$

$$\frac{1}{t}(V(x(t)) - V(x)) = \dot{V}(x(t_*)).$$

Taking the limit when $t \to +0$ we obtain that (2.41) holds for all $x \in X$.

This proves that V is a LISS-Lyapunov function with $\|x\|_X \leq \rho$, $\|u\|_U \leq \rho$ and consequently (2.33) is LISS. □

Theorem 2.3.2 provides a relatively simple method to prove LISS of the system: by Proposition 2.1.1, eISS of the system (2.35) is equivalent to exponential stability of a semigroup, generated by operator $A+B$, which is (due to analyticity) satisfies spectrum determined growth assumption (see Definition 5.1.6). Thus, if the spectrum of the operator $A+B$ lies in the left half-plane of a complex plane, then the system (2.33) is LISS.

2.4 Interconnections of input-to-state stable systems

In this section we study input-to-state stability of an interconnection of n ISS systems and provide a generalization of Lyapunov small-gain theorem from [21] for the case of infinite-dimensional systems.

Consider the interconnected systems of the following form

$$\left\{ \begin{array}{l} \dot{x}_i = A_i x_i + f_i(x_1, \ldots, x_n, u),\ x_i(t) \in X_i,\ u(t) \in U \\ i = 1, \ldots, n, \end{array} \right. \tag{2.42}$$

where the state space of i-th subsystem X_i is a Banach space and A_i is the generator of C_0 -semigroup on X_i, $i = 1, \ldots, n$. The space U_c we take as $U_c = PC(\mathbb{R}_+, U)$ for some Banach space of input values U.

The state space of the system (2.42) we denote by $X = X_1 \times \ldots \times X_n$, which is Banach with the norm $\|\cdot\|_X := \|\cdot\|_{X_1} + \ldots + \|\cdot\|_{X_n}$.

The input space for the i-th subsystem is $\tilde{X}_i := X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n \times U$. The norm in $\tilde{X}_i$ is given by

$$\|\cdot\|_{\tilde{X}_i} := \|\cdot\|_{X_1} + \ldots + \|\cdot\|_{X_{i-1}} + \|\cdot\|_{X_{i+1}} + \ldots + \|\cdot\|_{X_n} + \|\cdot\|_U.$$

The elements of $\tilde{X}_i$ we denote by $\tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, \xi) \in \tilde{X}_i$.

The transition map of the i-th subsystem we denote by $\phi_i : \mathbb{R}_+ \times X_i \times PC(\mathbb{R}_+, \tilde{X}_i) \to X_i$. The i-th subsystem of a system(3.38) is a control system $\Sigma_i = (X_i, PC(\mathbb{R}_+, \tilde{X}_i), \phi_i)$.

For $x_i \in X_i,\ i = 1, \ldots, n$ define $x = (x_1, \ldots, x_n)$, $f(x, u) = (f_1(x, u), \ldots, f_n(x, u))$. By A we denote the diagonal operator $A := diag(A_1, \ldots, A_n)$, i.e.:

$$A = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_n \end{pmatrix}$$

Domain of definition of A is given by $D(A) = D(A_1) \times \ldots \times D(A_n)$. Clearly A is a generator of C_0-semigroup on X.

We rewrite the system (2.42) in the vector form:

$$\dot{x} = Ax + f(x, u). \tag{2.43}$$

Since the inputs are piecewise continuous functions, then according to Proposition 2.2.2 a function $V_i : X_i \to \mathbb{R}_+$ is an ISS-Lyapunov function for the i-th subsystem, if there exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, $\chi \in \mathcal{K}$ and positive definite function α_i, such that

$$\psi_{i1}(\|x_i\|_{X_i}) \leq V_i(x_i) \leq \psi_{i2}(\|x_i\|_{X_i}), \quad \forall x_i \in X_i \tag{2.44}$$

and $\forall x_i \in X_i$, $\forall \tilde{x}_i \in \tilde{X}_i$, for all $v \in PC(\mathbb{R}_+, \tilde{X}_i)$ with $v(0) = \tilde{x}_i$ it holds the implication

$$\|x_i\|_{X_i} \geq \chi(\|\tilde{x}_i\|_{\tilde{X}_i}) \ \Rightarrow\ \dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)), \tag{2.45}$$

where

$$\dot{V}_i(x_i) = \overline{\lim_{t \to +0}} \frac{1}{t}(V_i(\phi_i(t, x_i, v))) - V_i(x_i)).$$

We are going to rewrite the implication (2.45) in a more suitable form.

Lemma 2.4.1. *A function $V_i : X_i \to \mathbb{R}_+$ is an ISS-Lyapunov function for the i-th subsystem if and only if there exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, $\chi_i, \chi_{ij} \in \mathcal{K}$, $j = 1, \ldots, n$ and a positive definite function α_i, such that (2.44) holds and $\forall x_i \in X_i$, $\forall \tilde{x}_i \in \tilde{X}_i$, for all $v \in PC(\mathbb{R}_+, \tilde{X}_i)$ with $v(0) = \tilde{x}_i$ it holds the implication*

$$V_i(x_i) \geq \max\{\max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(\|\xi\|_U)\} \ \Rightarrow\ \dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)). \tag{2.46}$$

Proof. Using (2.44), we have

$$\begin{aligned}\psi_{i1}^{-1}(V_i(x_i)) &\geq \|x_i\|_{X_i} \geq \chi(\|\tilde{x}_i\|_{\tilde{X}_i}) = \chi\left(\sum_{j=1, j\neq i}^{n} \|x_j\|_{X_j} + \|\xi\|_U\right) \\ &\geq \frac{1}{n+1}\max\{\max_{j=1,j\neq i}^{n}\{\chi(\|x_j\|_{X_j})\}, \chi(\|\xi\|_U)\}\end{aligned}$$

Therefore $\|x_i\|_{X_i} \geq \chi(\|\tilde{x}_i\|_{\tilde{X}_i})$ implies

$$V_i(x_i) \geq \max\{\max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(\|\xi\|_U)\}$$

with

$$\chi_{ij}(r) := \psi_{i1}\left(\frac{1}{n+1}\chi(\psi_{i2}^{-1}(r))\right), \; \chi_i(r) := \psi_{i1}\left(\frac{1}{n+1}\chi(r)\right), \; i \neq j, \; r \geq 0.$$

And from (2.45) it follows that $\forall x_i \in X_i$, $\forall \tilde{x}_i \in \tilde{X}_i$, for all $v \in PC(\mathbb{R}_+, \tilde{X}_i)$ with $v(0) = \tilde{x}_i$ it holds the implication (2.46).

Now let (2.46) holds. Then using (2.44) we obtain

$$\psi_{i2}(\|x_i\|_{X_i}) \geq V_i(x_i) \geq \max\{\max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(\|\xi\|_U)\} \geq \max\{\max_{j=1}^{n} \chi_{ij}(\psi_{j1}(\|x_j\|_{X_j})), \chi_i(\|\xi\|_U)\} \tag{2.47}$$

Define $\chi(r) := \max\{\max_{j=1}^{n} \psi_{i2}^{-1}(\chi_{ij}(\psi_{j1}(r))), \psi_{i2}^{-1}(\chi_i(r))\}$. Thus from (2.47) it follows

$$\|x_i\|_{X_i} \geq \max\{\max_{j=1}^{n} \chi(\|x_j\|_{X_j}), \chi(\|\xi\|_U)\} \geq \chi(\|\tilde{x}_i\|_{\tilde{X}_i}).$$

At last, from (2.46) it follows that $\forall x_i \in X_i$, $\forall \tilde{x}_i \in \tilde{X}_i$, for all $v \in PC(\mathbb{R}_+, \tilde{X}_i)$ with $v(0) = \tilde{x}_i$ it holds the implication (2.45). □

Remark 2.4.2. *Note that we have used in our derivations the certain norm on the space $\tilde{X}_i$. For finite-dimensional $\tilde{X}_i$ such derivations can be made for arbitrary norm in $\tilde{X}_i$ due to equivalence of the norms in a finite-dimensional space. However, for infinite-dimensional systems it is not always true.*

In the following we will use the implication form as in (2.46). Assume, that for all $i = 1, \dots, n$ for Lyapunov function V_i of the i-th system the gains χ_{ij}, $j = 1, \dots, n$ and χ_i are given. In the next theorem we generalize Theorem 1.5.6 to the case of systems in Banach spaces (2.42). The main idea of the proof is similar to that of Theorem 1.5.6, but more abstract notion of Lie derivative for the systems (2.42) needs additional arguments.

Theorem 2.4.1. *Let for each subsystem of (2.42) V_i be the ISS-Lyapunov function with corresponding gains χ_{ij}. If the corresponding operator Γ defined by (1.20) satisfies the small-gain condition (1.24), then the whole system (2.43) is ISS and possesses ISS-Lyapunov function defined by (1.22) where $\sigma = (\sigma_1, \dots, \sigma_n)^T$ is an Ω-path. The corresponding Lyapunov gain is as in (1.23).*

Proof. In order to prove that V is a Lyapunov function it is suitable to divide its domain of definition into subsets on which V takes a simpler form. Thus, for all $i \in \{1, \dots, n\}$ define a set

$$M_i = \left\{x \in X : \sigma_i^{-1}(V_i(x_i)) > \sigma_j^{-1}(V_j(x_j)),\ \forall j = 1, \dots, n,\ j \neq i\right\}.$$

From the continuity of V_i and σ_i^{-1}, $i = 1, \dots, n$ it follows that all M_i are open. Also note that $X = \cup_{i=1}^n \overline{M}_i$ and for all $i \neq j$ holds $M_i \cap M_j = \emptyset$. Define

$$\gamma(r) := \max_{j=1}^n \sigma_j^{-1} \circ \gamma_j(r).$$

Take some $i \in \{1, \dots, n\}$ and pick any $x \in M_i$. Assume that $V(x) \geq \gamma(\|\xi\|_U)$ holds. Then we obtain

$$\sigma_i^{-1}(V_i(x_i)) = V(x) \geq \gamma(\|\xi\|_U) = \max_{j=1}^n \sigma_j^{-1} \circ \gamma_j(\|\xi\|_U) \geq \sigma_i^{-1}(\gamma_i(\|\xi\|_U)).$$

Since $\sigma_i^{-1} \in \mathcal{K}_\infty$ it holds

$$V_i(x_i) \quad \geq \quad \gamma_i(\|\xi\|_U). \tag{2.48}$$

On the other hand, from the condition (1.21) we obtain that

$$\begin{aligned} V_i(x_i) = \sigma_i(V(x)) &\geq \max_{j=1}^n \chi_{ij}\left(\sigma_j\left(V\left(x\right)\right)\right) = \max_{j=1}^n \chi_{ij}\left(\sigma_j\left(\sigma_i^{-1}\left(V_i\left(x_i\right)\right)\right)\right) \\ &> \max_{j=1}^n \chi_{ij}\left(\sigma_j\left(\sigma_j^{-1}\left(V_j\left(x_j\right)\right)\right)\right) = \max_{j=1}^n \chi_{ij}\left(V_j\left(x_j\right)\right). \end{aligned}$$

Combining with (2.48) we obtain

$$V_i(x_i) \geq \max\left\{\max_{j=1}^n \chi_{ij}\left(V_j\left(x_j\right)\right), \gamma_i(\|\xi\|_U)\right\} \tag{2.49}$$

Hence condition (2.46) implies that for all x the following estimate holds

$$\begin{aligned} \frac{d}{dt}V(x) &= \frac{d}{dt}(\sigma_i^{-1}(V_i(x_i))) = \left(\sigma_i^{-1}\right)'(V_i(x_i))\frac{d}{dt}V_i(x_i(t)) \\ &\leq -\left(\sigma_i^{-1}\right)'(V_i(x_i))\alpha_i(V_i(x_i)) = -\left(\sigma_i^{-1}\right)'(\sigma_i(V(x)))\alpha_i(\sigma_i(V(x))). \end{aligned}$$

We set

$$\alpha(r) := \min_{i=1}^n \left\{\left(\sigma_i^{-1}\right)'(\sigma_i(r))\alpha_i(\sigma_i(r))\right\}.$$

Function α is positive definite, because $\sigma_i^{-1} \in \mathcal{K}_\infty$ and all α_i are positive definite functions. Overall, for all $x \in \cup_{i=1}^n M_i$ holds

$$\frac{d}{dt}V(x) \leq -\min_{i=1}^n \left(\sigma_i^{-1}\right)'(\sigma_i(V(x)))\alpha_i(\sigma_i(V(x))) = -\alpha(V(x)).$$

Now let $x \notin \cup_{i=1}^n M_i$. From $X = \cup_{i=1}^n \overline{M}_i$ it follows that $x \in \cap_{i\in I(x)} \partial M_i$ for some index set $I(x) \subset \{1, \dots, n\}$, $|I(x)| \geq 2$.

$$\cap_{i\in I(x)}\partial M_i = \{x \in X : \forall i \in I(x),\ \forall j \notin I(x)\ \sigma_i^{-1}(V_i(x_i)) > \sigma_j^{-1}(V_j(x_j)),$$

$$\forall i,j \in I(x)\ \sigma_i^{-1}(V_i(x_i)) = \sigma_j^{-1}(V_j(x_j))\}.$$

Due to continuity of ϕ we have, that for all $u \in PC(\mathbb{R}_+, U)$, $u(0) = \xi$ there exists $t^* > 0$, such that for all $t \in [0, t^*)$ it follows $\phi(t,x,u) \in \big(\cap_{i\in I(x)} \partial M_i\big) \cup \big(\cup_{i \in I(x)} M_i\big)$.

Then, by definition of the Lie derivative we obtain

$$\begin{aligned} \dot{V}(x) &= \overline{\lim_{t\to+0}} \frac{1}{t}(V(\phi(t,x,u))) - V(x)) \\ &= \overline{\lim_{t\to+0}} \frac{1}{t}\left(\max_{i\in I(x)}\{\sigma_i^{-1}(V_i(\phi_i(t,x,u)))\} - \max_{i\in I(x)}\{\sigma_i^{-1}(V_i(x_i))\}\right) \end{aligned} \tag{2.50}$$

From the definition of $I(x)$ it follows that

$$\sigma_i^{-1}(V_i(x_i)) = \sigma_j^{-1}(V_j(x_j)) \quad \forall i,j \in I(x),$$

and therefore the index i, on which we maximum of $\max_{i\in I(x)}\{\sigma_i^{-1}(V_i(x_i))\}$ is reached, may be always set equal to the index on which the maximum $\max_{i\in I(x)}\{\sigma_i^{-1}(V_i(\phi_i(t,x,u)))\}$ is reached. We continue the estimates (2.50)

$$\dot{V}(x) = \overline{\lim_{t\to+0}} \max_{i\in I(x)}\{\frac{1}{t}\left(\sigma_i^{-1}(V_i(\phi_i(t,x,u))) - \sigma_i^{-1}(V_i(x_i))\right)\}$$

Using Lemma 5.5.1 we obtain

$$\dot{V}(x) = \max_{i\in I(x)}\{\overline{\lim_{t\to+0}} \frac{1}{t}\left(\sigma_i^{-1}(V_i(\phi_i(t,x,u))) - \sigma_i^{-1}(V_i(x_i))\right)\} = \max_{i\in I(x)} \frac{d}{dt}(\sigma_i^{-1}(V_i(x_i))) \leq -\alpha(V(x)).$$

Overall, we have that for all $x \in X$ holds

$$\frac{d}{dt}V(x) = \max_i\{\left(\sigma_i^{-1}\right)'(V_i(x_i))\frac{d}{dt}V_i(x_i(t))\} \leq -\alpha(V(x)),$$

and the ISS-Lyapunov function for the whole interconnection is constructed. ISS of the whole system follows by Proposition 2.2.2 and Theorem 2.2.1. □

Remark 2.4.3. *In the recent paper [51] it was proved a general vector small-gain theorem, that states roughly speaking that if an abstract control system possesses a vector ISS Lyapunov function, then it is ISS. The authors have also shown how from this theorem the small-gain theorems for interconnected systems of ODEs and retarded equations can be derived. It is possible, that the small-gain theorem, similar to the proved in this section, can be derived from the general theorem from [51]. However, it seems, that the constructions in [51] can be provided only for maximum formulation of ISS-Lyapunov functions (as in* (2.46)*). If the subsystems possess the ISS-Lyapunov functions in terms of summations, i.e. instead of* (2.46) *one has*

$$V_i(x_i) \geq \sum_{j=1}^n \chi_{ij}(V_j(x_j)) + \chi_i(\|\xi\|_U) \ \Rightarrow\ \dot{V}_i(x_i) \leq -\alpha_i(V_i(x_i)), \tag{2.51}$$

then it is not clear, how the proofs from [51] can be adapted for this case. In contrast to it, the counterpart of the Theorem 2.4.1 in the summation case can be proved with the method, similar to the used in the proof of the Theorem 2.4.1, see [24]. However, the small-gain condition will have slightly another form.

2.4.1 Interconnections of linear systems

The construction of ISS-Lyapunov function for the interconnections of finite-dimensional input-to-state stable linear systems (see [24]) can be generalized to the case of interconnections of linear systems over Banach spaces.

Consider the following interconnected system

$$\dot{x}_i = A_i x_i(t) + \sum_{j=1}^{n} B_{ij} x_j(t) + C_i u(t),\ i = 1, \dots, n, \tag{2.52}$$

where $x_i(t) \in X_i$, $A_i : X_i \to X_i$ is a generator of an analytic semigroup over X_i defined on $D(A_i) \subset X_i$, $i = 1, \dots, n$. Moreover, $B_{ij} \in L(X_j, X_i)$, $i, j \in \{1, \dots, n\}$ are bounded operators, $u \in U_c = PC(\mathbb{R}_+, U)$ for some Banach space of input values U. We assume, that $B_{ii} = 0$, $i = 1, \dots, n$. Otherwise we can always substitute $\tilde{A}_i = A_i + B_{ii}$.

Let us denote $X = X_1 \times \ldots \times X_n$ and introduce the matrix operators $A = diag(A_1, \dots, A_n) : X \to X$, $B = (B_{ij})_{i,j=1,\dots,n} : X \to X$ and $C = (C_1, \dots, C_n) : U \to X$. Then the system (2.52) can be rewritten in the following form

$$\dot{x}(t) = (A + B)x(t) + Cu(t). \tag{2.53}$$

Now we apply Lyapunov technique developed in this section to the system (2.52). From Theorem 2.1.1 and Lemma 2.1.2 we have, that i-th subsystem of (2.52) is ISS iff the analytic semigroup generated by A_i is exponentially stable. This is equivalent (see Theorem 5.1.2 and Remark 5.1.3) to existence of a positive operator P_i, for which it holds that

$$\langle A_i x_i, P_i x_i \rangle + \langle P_i x_i, A_i x_i \rangle \le -\|x_i\|_{X_i}^2, \quad \forall x_i \in D(A_i). \tag{2.54}$$

Consider a function V_i defined by

$$V_i(x_i) = \langle P_i x_i, x_i \rangle, x_i \in X_i. \tag{2.55}$$

We assume in what follows that P_i is a coercive operator. This implies that

$$a_i^2 \|x_i\|_{X_i}^2 \le V_i(x_i) \le \|P_i\| \|x_i\|_{X_i}^2, \tag{2.56}$$

for some $a_i > 0$.

Differentiating V_i w.r.t. the i-th subsystem of (2.52), we obtain for all $x_i \in D(A_i)$

$$\begin{aligned} \dot{V}_i(x_i) &= \langle P_i \dot{x}_i, x_i \rangle + \langle P_i x_i, \dot{x}_i \rangle \\ &\le (\langle P_i A_i x_i, x_i \rangle + \langle P_i x_i, A_i x_i \rangle) + 2\|x_i\|_{X_i} \|P_i\| \left(\sum_{i \ne j} \|B_{ij}\| \|x_j\|_{X_j} + \|C_i\| \|u\|_U \right). \end{aligned}$$

Operator P_i is self-adjoint, hence it holds that $\langle P_i A_i x_i, x_i \rangle = \langle A_i x_i, P_i x_i \rangle$ and by (2.54) we obtain

$$\dot{V}_i(x_i) \le -\|x_i\|_{X_i}^2 + 2\|x_i\|_{X_i} \|P_i\| \left(\sum_{i \ne j} \|B_{ij}\| \|x_j\|_{X_j} + \|C_i\| \|u\|_U \right).$$

Now take $\varepsilon \in (0,1)$ and let

$$\|x_i\|_{X_i} \geq \frac{2\|P_i\|}{1-\varepsilon}\left(\sum_{i\neq j}\|B_{ij}\|\|x_j\|_{X_j} + \|C_i\|\|u\|_U\right). \tag{2.57}$$

Then we obtain for all $x_i \in D(A_i)$

$$\dot{V}_i(x_i) \leq -\varepsilon\|x_i\|^2_{X_i}.$$

To verify this inequality for all $x_i \in X_i$ we use the same argument, as in the end of the proof of Theorem 2.3.2 (here we use analyticity of a semigroup).

In order to apply Theorem 2.4.1, we replace inequality (2.57) with the following one

$$V_i(x_i) \geq \|P_i\|^3\left(\frac{2}{1-\varepsilon}\right)^2\left(\sum_{i\neq j}\frac{\|B_{ij}\|}{a_j}\sqrt{V_j(x_j)} + \|C_i\|\|u\|_U\right)^2. \tag{2.58}$$

It is easy to see that (2.58) together with (2.56) imply (2.57).

Thus, gains can be defined by:

$$\gamma_{ij}(s) = \left(\frac{2\|P_i\|^{3/2}}{1-\varepsilon}\frac{\|B_{ij}\|}{a_j}\right)\sqrt{s}, \tag{2.59}$$

for all $i \neq j$, $i = 1,\ldots,n$. If the small-gain condition for this choice of gains holds, we can conclude the ISS of the system (2.53).

Now we are going to show, how this technique can be applied for the interconnected linear reaction-diffusion systems.

Example 2.4.2. *Consider the following system*

$$\begin{cases} \frac{\partial s_1}{\partial t} = c_1\frac{\partial^2 s_1}{\partial x^2} + a_{12}s_2, & x\in(0,d),\ t>0,\\ s_1(0,t) = s_1(d,t) = 0; & \\ \frac{\partial s_2}{\partial t} = c_2\frac{\partial^2 s_2}{\partial x^2} + a_{21}s_1, & x\in(0,d),\ t>0,\\ s_2(0,t) = s_2(d,t) = 0. & \end{cases} \tag{2.60}$$

We choose the state space as $X_1 = X_2 = L_2(0,d)$. The operators $A_i = c_i\frac{d^2}{dx^2}$ with $D(A_i) = H^1_0(0,d)\cap H^2(0,d)$, $i = 1,2$ are infinitesimal generators of analytic semigroups for corresponding subsystems.

We are going to prove, that the system(2.60) is 0-UGASx. To this end we construct the ISS-Lyapunov functions V_1, V_2 for both subsystems and apply Theorem 2.4.1.

Note that $Spec(A_i) = \{-c_i\left(\frac{\pi n}{d}\right)^2 \mid n = 1,2,\ldots\}$, $i = 1,2$.

Take $P_i = \frac{1}{2c_i}\left(\frac{d}{\pi}\right)^2 I$, where I is the identity operator on X_i. We have

$$\begin{aligned}\langle A_is, P_is\rangle + \langle P_is, A_is\rangle &= \frac{1}{c_i}\left(\frac{d}{\pi}\right)^2\langle A_is, s\rangle\\ &= \left(\frac{d}{\pi}\right)^2\int_0^d\frac{\partial^2 s}{\partial x^2}s\,dx = -\left(\frac{d}{\pi}\right)^2\int_0^d\left(\frac{\partial s}{\partial x}\right)^2dx\\ &\leq -\|s\|^2_{L_2(0,d)}.\end{aligned}$$

In the last estimate we have used Friedrichs' inequality (5.9).

The Lyapunov functions for subsystems are defined by

$$V_i(s_i) = \langle P_i s_i, s_i \rangle = \frac{1}{2c_i}\left(\frac{d}{\pi}\right)^2 \|s_i\|^2_{L_2(0,d)}, \text{ for } s_i \in X_i.$$

We have the following estimates for derivatives

$$\dot{V}_1(s_1) \leq -\|s_1\|^2_{L_2(0,d)} + \frac{1}{c_1}\left(\frac{d}{\pi}\right)^2 |a_{12}| \|s_1\|_{L_2(0,d)} \|s_2\|_{L_2(0,d)},$$

$$\dot{V}_2(s_2) \leq -\|s_2\|^2_{L_2(0,d)} + \frac{1}{c_2}\left(\frac{d}{\pi}\right)^2 |a_{21}| \|s_1\|_{L_2(0,d)} \|s_2\|_{L_2(0,d)}.$$

We choose the gains in the following way

$$\gamma_{12}(r) = \frac{c_2}{c_1^3}\left(\frac{d}{\pi}\right)^4 \left|\frac{a_{12}}{1-\varepsilon}\right|^2 \cdot r, \quad \gamma_{21}(r) = \frac{c_1}{c_2^3}\left(\frac{d}{\pi}\right)^4 \left|\frac{a_{21}}{1-\varepsilon}\right|^2 \cdot r.$$

We have

$$\begin{aligned} V_1(s_1) \geq \gamma_{12} \circ V_2(s_2) \quad &\Leftrightarrow \quad \sqrt{\frac{c_1}{c_2}\gamma_{12}(1)} \|s_2\|_{L_2(0,d)} \leq \|s_1\|_{L_2(0,d)} \\ &\Leftrightarrow \quad \frac{1}{c_1}\left(\frac{d}{\pi}\right)^2 |a_{12}| \|s_2\|_{L_2(0,d)} \leq (1-\varepsilon)\|s_1\|_{L_2(0,d)}. \end{aligned}$$

Analogously,

$$V_2(s_2) \geq \gamma_{21} \circ V_1(s_1) \quad \Leftrightarrow \quad \frac{1}{c_2}\left(\frac{d}{\pi}\right)^2 |a_{21}| \|s_1\|_{L_2(0,d)} \leq (1-\varepsilon)\|s_2\|_{L_2(0,d)}.$$

Overall, we have the following implications:

$$V_1(s_1) \geq \gamma_{12} \circ V_2(s_2) \quad \Rightarrow \quad \dot{V}_1(s_1) \leq -\varepsilon \|s_1\|^2_{L_2(0,d)},$$

$$V_2(s_2) \geq \gamma_{21} \circ V_1(s_1) \quad \Rightarrow \quad \dot{V}_2(s_2) \leq -\varepsilon \|s_2\|^2_{L_2(0,d)}.$$

The small-gain condition (1.25) for the case of two interconnected systems is reduced to $\gamma_{12} \circ \gamma_{21} < \mathrm{Id}$. *It holds*

$$\gamma_{12} \circ \gamma_{21} < \mathrm{Id} \quad \Leftrightarrow \quad \frac{1}{c_1^2 c_2^2}\left(\frac{d}{\pi}\right)^8 \frac{|a_{12}a_{21}|^2}{(1-\varepsilon)^4} < 1,$$

for arbitrary $\varepsilon > 0$. *Thus, if*

$$|a_{12}a_{21}| < c_1 c_2 \left(\frac{\pi}{d}\right)^4 \tag{2.61}$$

is satisfied, then the whole system (2.60) is 0-UGASx.

2.4.2 A nonlinear example

Let us show the applicability of the Theorem 2.4.1 to nonlinear systems. Consider a quasilinear system

$$\begin{cases} \frac{\partial s_1}{\partial t} = c_1 \frac{\partial^2 s_1}{\partial x^2} + s_2^2, \quad x \in (0,d),\ t > 0, \\ s_1(0,t) = s_1(d,t) = 0; \\ \frac{\partial s_2}{\partial t} = c_2 \frac{\partial^2 s_2}{\partial x^2} - b s_2 + \sqrt{|s_1|}, \quad x \in (0,d),\ t > 0, \\ s_2(0,t) = s_2(d,t) = 0. \end{cases} \tag{2.62}$$

We assume, that c_1, c_2, b are positive constants and c_2 is close to zero.

We choose the state space and space of input values for the first subsystem as $X_1 = L_2(0,d)$, $U_1 = L_4(0,d)$ and for the second subsystem as $X_2 = L_4(0,d)$, $U_2 = L_2(0,d)$. The state of the whole system (2.62) is denoted by $X = X_1 \times X_2$.

Define operators $B_i = c_i \frac{d^2}{dx^2}$. These operators (together with Dirichlet boundary conditions) generate an analytic semigroup on $L_2(0,d)$ and $L_4(0,d)$ respectively (see, e.g. [65, Chapter 7]).

For both subsystems we take the set of input functions as $U_{c,i} := C([0,\infty), U_i)$. We consider the mild solutions of the subsystems, that is the solutions s_i, given by the formula (2.3).

Note that $s_2 \in C([0,\infty), L_4(0,d)) \Leftrightarrow s_2^2 \in C([0,\infty), L_2(0,d))$ and $s_1 \in C([0,\infty), L_2(0,d))$ $\Leftrightarrow \sqrt{s_1} \in C([0,\infty), L_4(0,d))$.

Under made assumptions the solution of the first subsystem (when s_2 is treated as input) belongs to $C([0,\infty), H_0^1(0,d) \cap H^2(0,d)) \subset C([0,\infty), L_2(0,d))$ and the solution of the second one belongs to $C([0,\infty), W_0^{4,1}(0,d) \cap W^{4,2}(0,d)) \subset C([0,\infty), L_4(0,d))$.

This implies, that the solution of the whole system is from the space $C([0,T], X)$ for all T such that the solution of the whole system exists on $[0,T]$. The existence and uniqueness of the solution for all times will be proved for the values of parameters which establish ISS of the whole system, since this excludes the possibility of the blow-up phenomena.

To show ISS of both subsystems of (2.62) we choose V_i, $i = 1,2$ as follows

$$V_1(s_1) = \int_0^d s_1^2(x)dx = \|s_1\|^2_{L_2(0,d)},$$

$$V_2(s_2) = \int_0^d s_2^4(x)dx = \|s_2\|^4_{L_4(0,d)}.$$

as ISS-Lyapunov functions for the first and second subsystem respectively.

Consider the Lie derivative of V_1. Using in the second estimation Cauchy-Schwarz inequality, we have:

$$\begin{aligned} \frac{d}{dt} V_1(s_1) &= 2\int_0^d s_1(x,t)\left(c_1 \frac{\partial^2 s_1}{\partial x^2}(x,t) + s_2^2(x,t)\right) dx \\ &\leq -2c_1 \left\| \frac{ds_1}{dx} \right\|^2_{L_2(0,d)} + 2\|s_1\|_{L_2(0,d)} \|s_2\|^2_{L_4(0,d)} \end{aligned}$$

By the Friedrichs' inequality, we obtain the estimation

$$\begin{aligned} \frac{d}{dt} V_1(s_1) &\leq -2c_1 \left(\frac{\pi}{d}\right)^2 \|s_1\|^2_{L_2(0,d)} + 2\|s_1\|_{L_2(0,d)} \|s_2\|^2_{L_4(0,d)} \\ &= -2c_1 \left(\frac{\pi}{d}\right)^2 V_1(s_1) + 2\sqrt{V_1(s_1)}\sqrt{V_2(s_2)} \end{aligned} \tag{2.63}$$

We choose the following gains

$$\chi_{12}(r) := \frac{1}{c_1^2 \left(\frac{\pi}{d}\right)^4 (1-\varepsilon_1)^2} r, \ \forall r > 0,$$

where $\varepsilon_1 \in (0,1)$ - arbitrary constant. Using this gain we obtain from (2.63) that

$$V_1(s_1) \geq \chi_{12}(V_2(s_2)) \quad \Rightarrow \quad \frac{d}{dt} V_1(s_1) \leq -2\varepsilon_1 c_1 \left(\frac{\pi}{d}\right)^2 V_1(s_1).$$

Consider the Lie derivative of V_2:

$$\begin{aligned}\frac{d}{dt} V_2(s_2) &= 4 \int_0^d s_2^3(x,t) \left(c_2 \frac{\partial^2 s_2}{\partial x^2}(x,t) - b s_2(x,t) + \sqrt{|s_1(x,t)|} \right) dx \\ &\leq -12 c_2 \int_0^d s_2^2 \left(\frac{\partial s_2}{\partial x}\right)^2 dx - 4b V_2(s_2) + 4 \int_0^d s_2^3(x,t) \sqrt{|s_1(x,t)|} dx\end{aligned}$$

Applying for the last term the Hölder inequality (5.7) we obtain

$$\frac{d}{dt} V_2(s_2) \leq -4b V_2(s_2) + 4 (V_2(s_2))^{3/4} (V_1(s_1))^{1/4}$$

Let

$$\chi_{21}(r) := \frac{1}{b^4 (1-\varepsilon_2)^4} r, \ \forall r > 0,$$

where $\varepsilon_2 \in (0,1)$ - arbitrary constant. It holds the implication

$$V_2(s_2) \geq \chi_{21}(V_1(s_1)) \quad \Rightarrow \quad \frac{d}{dt} V_2(s_2) \leq -4b\varepsilon_2 V_2(s_2).$$

The small-gain condition (1.25) leads us to the following condition

$$\chi_{12} \circ \chi_{21} < \mathrm{Id} \quad \Leftrightarrow \quad c_1^2 \left(\frac{\pi}{d}\right)^4 (1-\varepsilon_1)^2 b^4 (1-\varepsilon_2)^4 > 1 \quad \Leftrightarrow \quad c_1 \left(\frac{\pi}{d}\right)^2 b^2 > 1.$$

This condition guarantees that the system (2.62) is 0-UGASx.

2.5 Method of super- and subsolutions

Lyapunov functions are the powerful method for verification of stability of the system. However, the construction of ISS-Lyapunov functions for infinite-dimensional systems may be very sophisticated. Hence the development of the alternative tools is also important.

In this section we are going to show, how the monotonicity of certain classes of infinite-dimensional control systems, namely, reaction-diffusion systems with Neumann boundary conditions can be used to simplify verification of the ISS property. We start with the basic definitions.

A subset $K \subset X$ of a Banach space X is called a *positive cone* if $\forall a \in \mathbb{R}_+$, $\forall x, y \in K$ $ax \in K$; $x + y \in K$ and $K \cap (-K) = \{0\}$.

A Banach space X together with a cone $K \subset X$ is called an *ordered Banach space* (see [56]), which we denote (X, K) with an order $\leq$ given by $x \leq y \ \Leftrightarrow \ y - x \in K$. Analogously $x \geq y \ \Leftrightarrow \ x - y \in K$.

Definition 2.5.1. *We call a control system $S = (X, U_c, \phi)$ ordered, if X and U_c are ordered Banach spaces.*

An important for applications subclass of control systems are monotone control systems, introduced in 2003 in the paper [3]. The concept of monotone control system was inspired by the theory of monotone dynamical systems developed by M. Hirsch in 1980-s in a series of papers, beginning with [40] and by H. Smith in his monograph [73].

Definition 2.5.2. *Ordered control system $S = (X, U_c, \phi)$ is called monotone, if $\forall t_0 \in \mathbb{R}_+$, for all $t \geq t_0$, $u_1, u_2 \in U_c : u_1 \leq u_2$, $\forall \phi_1, \phi_2 \in X : \phi_1 \leq \phi_2$ it holds $\phi(t, t_0, \phi_1, u_1) \leq \phi(t, t_0, \phi_2, u_2)$.*

Let G be a bounded domain in $\mathbb{R}^p$ with a smooth boundary and Δ be a Laplacian on G.
Consider the following reaction-diffusion system

$$\begin{cases} \frac{\partial s(x,t)}{\partial t} = c^2 \Delta s + f(s, t, u(x,t)),\ x \in G,\ t > t_0, \\ s\left(x, t_0\right) = \phi_0\left(x\right), x \in G, \\ \frac{\partial s}{\partial n}\Big|_{\partial G \times \mathbb{R}_{\geq t_0}} = 0. \end{cases} \tag{2.64}$$

In the following exposition we identify $\mathbb{R}^n$ with $(\mathbb{R}^n, \mathbb{R}^n_+)$ and choose the state space $X = (C(G, \mathbb{R}^n), C(G, \mathbb{R}^n_+))$. It is clear that $C(G, \mathbb{R}^n_+)$ is a positive cone in $C(G, \mathbb{R}^n)$. As input space we take $U_c = (C(G \times \mathbb{R}_{\geq t_0}, \mathbb{R}^m), C(G \times \mathbb{R}_{\geq t_0}, \mathbb{R}^m_+))$.

We assume that $f : \mathbb{R}^n \times G \times \mathbb{R}_{\geq t_0} \times \mathbb{R}^m \to \mathbb{R}^n$ satisfies certain regularity properties (see, e.g. [57]) such that there exists a unique classical solution s of the problem (2.64) which belongs to $C(\overline{G} \times [t_0, T], \mathbb{R}^n) \cap C^{2,1}(G \times [t_0, T], \mathbb{R}^n)$, and has bounded first and second order derivatives with respect to x in $\overline{G}$.

Here we have defined a space

$$C^{2,1}(M \times [t_0, T], \mathbb{R}^n) := \{s : M \times [t_0, T] \to \mathbb{R}^n | s_x, s_{xx}, s_t \in C(M \times [t_0, T], \mathbb{R}^n)\}.$$

We consider the asymptotic behavior of the solutions s for a problem (2.64), thereby we need their existence and uniqueness for all $T > t_0$.

Under made assumptions (2.64) defines a control system (X, U_c, ϕ).

Consider also the corresponding to (2.64) system without diffusion:

$$\begin{cases} \frac{ds(t)}{dt} = f(s, t, u(t)),\ t > t_0, \\ s\left(t_0\right) = s_0. \end{cases} \tag{2.65}$$

Here $s(t) \in \mathbb{R}^n$ and $u \in U_f := C(\mathbb{R}_{\geq t_0}, \mathbb{R}^m)$. Equations (2.65) define a control system $(\mathbb{R}^n, U_f, \phi_f)$, where $\phi_f(\cdot, t_0, s_0, u)$ is a solution of (2.65) corresponding to the input $u \in U_f$.

We assume that $f(0, t, 0) = 0\ \forall t \geq t_0$, thus, $s \equiv 0$ is an equilibrium point for a system (2.64) as well as for (2.65) (in this case 0 means zero-vector).

Also we assume that the system (2.64) is forward-complete. The following theorem shows, that for monotone system (2.64) the UISS of the local dynamics (i.e. of the system (2.65)) implies the UISS of the system (2.64).

Theorem 2.5.1. *Let system (2.64) be monotone. If* (2.65) *is UISS, then (2.64) is also UISS.*

Proof. For arbitrary $\phi_0 \in X$ we define constant vectors $\phi_+ = (\phi_+^1, \dots, \phi_+^n)^T$, $\phi_- = (\phi_-^1, \dots, \phi_-^n)^T$ by:

$$\phi_+^i = \sup_{x \in G} \phi_0^i(x), \quad \phi_-^i = \inf_{x \in G} \phi_0^i(x),\ i = 1, \dots, n. \tag{2.66}$$

Analogously, for every $u \in U_c$ define the functions $u_+, u_- \in U_f$ by

$$u_+^i(t) = \sup_{x \in G} u^i(x,t), \quad u_-^i(t) = \inf_{x \in G} u^i(x,t),\ i = 1, \dots, m,\ t \geq t_0. \tag{2.67}$$

Consider problem (2.64) with initial conditions and inputs of the following form:

$$\phi_0(x) = \phi_+,\ u(x,t) = u_+(t),\ \forall x \in G,\ \forall t \geq t_0,$$

$$\phi_0(x) = \phi_-,\ u(x,t) = u_-(t),\ \forall x \in G,\ \forall t \geq t_0,$$

Solutions of these problems we denote by $s_+(x,t)$ and $s_-(x,t)$ respectively.

Using that f, u_- and u_+ do not depend on x, we see that $s_+(x,t) \equiv s_+^*(t)\ \forall x \in G$, where s_+^* is a solution of the problem

$$\begin{cases} \frac{ds_+^*(t)}{dt} = f(s_+^*(t), t, u(t)),\ t > t_0, \\ s_+^*(t_0) = \phi_+. \end{cases} \tag{2.68}$$

Thus $\|s_+(\cdot,t)\|_X = |s_+^*(t)|$. Similarly, $\|s_-(\cdot,t)\|_X = |s_-^*(t)|$, where s_-^* is a solution of (2.68) with ϕ_- instead of ϕ_+.

The system (2.65) is UISS, therefore $\exists \beta_* \in \mathcal{KL},\ \gamma_* \in \mathcal{K}$, such that $\forall \phi_-,\ \phi_+ \in \mathbb{R}^n$, $\forall t_0$, $\forall u_-, u_+ \in U_f$, $\forall t \geq t_0$ it holds

$$\|s_-(\cdot,t)\|_X = \left|s_-^*(t)\right| \leq \beta_*(|\phi_-|, t - t_0) + \gamma_*(\|u_-\|_{U_f}). \tag{2.69}$$

$$\|s_+(\cdot,t)\|_X = \left|s_+^*(t)\right| \leq \beta_*(|\phi_+|, t - t_0) + \gamma_*(\|u_+\|_{U_f}). \tag{2.70}$$

From the assumption of monotonicity of (2.64) and from definitions (2.66) and (2.67) we know that for any admissible ϕ_0 for the corresponding solution $s(x,t)$ of the problem (2.64) it holds

$$s_-(\cdot,t) \leq s(\cdot,t) \leq s_+(\cdot,t),\ \forall t \geq t_0.$$

Note that if $a \leq x \leq b$ for some $a, b, x \in \mathbb{R}^n$, then

$$|x| = \sqrt{\sum_{i=1}^n x_i^2} \leq \sqrt{\sum_{i=1}^n (a_i^2 + b_i^2)} \leq \sqrt{\sum_{i=1}^n a_i^2} + \sqrt{\sum_{i=1}^n b_i^2} = |a| + |b|.$$

Thus,

$$\|s(\cdot,t)\|_X \leq \|s_-(\cdot,t)\|_X + \|s_+(\cdot,t)\|_X.$$

The following simple inequalities will be useful for us:

$$|\phi_-| \leq \sqrt{n}\|\phi_0\|_X, \quad |\phi_+| \leq \sqrt{n}\|\phi_0\|_X, \quad \|u_-\|_{U_f} \leq \|u\|_{U_c}, \quad \|u_+\|_{U_f} \leq \|u\|_{U_c}.$$

Using these estimates and inequalities (2.69) and (2.70) we have (in the second inequality we use the monotonicity of γ_*)

$$\begin{aligned}\|s(\cdot,t)\|_X &\leq \beta_*(|\phi_-|,t-t_0)+\beta_*(|\phi_+|,t-t_0)+\gamma_*(\|u_-\|_{U_f})+\gamma_*(\|u_+\|_{U_f})\\ &\leq 2\beta_*(\sqrt{n}\|\phi_0\|_X,t-t_0)+2\gamma_*(\|u\|_{U_c}).\end{aligned}$$

Taking $\beta(r,t) := 2\beta_*(\sqrt{n}r,t)$, $r \in \mathbb{R}_{\geq 0}$, $t \geq 0$ and $\gamma := 2\gamma_*$, we obtain that for solution $s(x,t)$ of the problem (2.64) it holds

$$\|s(\cdot,t)\|_X \leq \beta(\|\phi_0\|_X, t-t_0) + \gamma(\|u\|_{U_c})$$

for all initial functions $\phi_0 \in X$ and external inputs $u \in U_c$. □

Thus, for monotone systems of the form (2.64) we are able to prove ISS simply by proving the ISS of its counterpart without diffusion, which is much easier task, since the finite-dimensional ISS theory is already well-developed.

We have obtained such a result due to the fact that for the system (2.64) the solution, corresponding to space-independent input and space-independent boundary conditions, is also space-independent. For other boundary conditions the proof will not work. It is an interesting task to investigate, how can we use the monotonicity of the systems for investigation of ISS for more general classes of systems.

2.6 Application to stability analysis of production networks

In this section we are going to apply the methods developed in this work to analysis of ISS of production networks.

By definition, production network (or supply chain) is a network of suppliers that produce goods, both, for one another and for generic customers [13].

The dynamics of production network is subject to different perturbations due to changes on market, changes in customers behavior, information and transport congestions, unreliable elements of the network etc.

Typical examples of unstable behavior are unbounded growth of unsatisfied orders or unbounded growth of the queue of the workload to be processed by a machine. This causes a loss of customers and high inventory costs, respectively. To avoid such effects, one needs to investigate stability of a network in advance.

In this chapter we model the production networks with the help of ODE systems and analyze its stability with the help of ISS framework. In the end of the section we discuss possible extensions of results to the cases of time-delay systems (when the time, needed to transport the goods is taken into account) and PDEs.

2.6.1 Description and modeling of a general production network

We consider a production network, consisting of n market entities, which can be raw material suppliers (e.g., extracting or agricultural companies), producers, distributors and consumers etc. Each entity is understood as a subsystem of the whole network. For simplicity we assume that there is only one unified type of material, i.e., all primary products, used in the production network, can be measured as a number of units of this unified material.

The state of the i-th subsystem at time $t \in \mathbb{R}_+$ is the quantity of unprocessed material within the i-th subsystem at time t. It will be denoted by $x_i(t)$. The state of the whole network at time t is denoted by $x(t) = (x_1(t), \dots, x_n(t))^T$. A subsystem can get material from an external source, which is denoted by u_i, and from subsystems of the network (internal inputs).

Let the i-th subsystem processes the raw material from its inventory with the rate $\tilde{f}_{ii}(t, x(t)) \geq 0$ and sends the produced goods (measured in units of unified material) to the j-th subsystem with rate $\tilde{f}_{ji}(t, x(t))$. Thus, the total rate of the distribution from the i-th subsystem to other subsystems is $\sum_{j=1}^n \tilde{f}_{ji}(t, x(t))$ and the rest is sent to some customers not considered in the network.

For general functions $\tilde{f}_{ji}$ it is hard (if possible) to derive stability conditions. Therefore we will investigate the special case $\tilde{f}_{ji}(t, x(t)) = c_{ji}(x(t))\tilde{f}_i(x_i(t)),\ c_{ji}(x) \in \mathbb{R}_+$ and $\tilde{f}_{ii}(t, x(t)) = \tilde{c}_{ii}(x(t))\tilde{f}_i(x_i(t)),\ \tilde{c}_{ii}(x) \in \mathbb{R}_+$, where $\tilde{f}_i(x_i(t)) \in \mathcal{K}_\infty$ is proportional to the processing rate of the system, $c_{ji}(x(t)),\ i \neq j$ are some positive distribution coefficients and $\tilde{c}_{ii}(x(t)) \geq 0$.

Under these assumptions the dynamics of the i-th subsystem is described by ordinary differential equations

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{n} c_{ij}(x(t))\tilde{f}_j(x_j(t)) + u_i(t) - \tilde{c}_{ii}(x(t))\tilde{f}_i(x_i(t)),\ i = 1, \dots, n. \tag{2.71}$$

Denoting $c_{ii} := -\tilde{c}_{ii}$ we can rewrite the above equations in a vector form

$$\dot{x}(t) = C(x(t))\tilde{f}(x(t)) + u(t), \tag{2.72}$$

where $\tilde{f}(x(t)) = (\tilde{f}_1(x_1(t)), ..., \tilde{f}_n(x_n(t)))^T$, $u(t) = (u_1(t), \dots, u_n(t))^T$ and $C(x) \in \mathbb{R}^{n\times n}$. In the next subsection we perform a stability analysis of this model.

2.6.2 Stability analysis

In order to analyze the stability of the system (2.72) we are going to exploit Theorem 1.5.6. We consider network as a composition of n single market entities, construct ISS-Lyapunov functions $V_i(x_i)$ and corresponding gains χ_{ij} for each subsystem (which ensures, that the subsystems are ISS), and seek for conditions, guaranteeing, that the small-gain condition (1.24) holds.

Note that the assumptions that $\tilde{f}_i \in \mathcal{K}_\infty$ and that for all $x > 0$ $c_{ii}(x) < 0$ and $c_{ij}(x) \geq 0,\ i \neq j$ imply that if $x(0) \geq 0$ (that is $x_i(0) \geq 0$ for all $i = 1, \dots, n$) and $u(t) \geq 0$, for all $t > 0$, then $x(t) \geq 0$ for all $t > 0$.

Thus, $\mathbb{R}^n_+ = [0, \infty)^n$ is invariant under the flow of internal and external inputs (if the external inputs are positive). Since we are interested in the stability analysis of production networks it is enough to perform the analysis in $\mathbb{R}^n_+$.

We choose function $V_i(x_i) = |x_i|$ as an ISS-Lyapunov function for i-th entity. Evidently, $V_i(x_i)$ satisfies the condition (1.16).

To prove that the condition (1.17) holds, we choose the functions γ_{ij}, γ_i, (see (1.19)) as

$$\gamma_{ij}(s) := \tilde{f}_i^{-1}\left(\tfrac{a_i}{a_j}\tfrac{1}{1+\delta_j}\tilde{f}_j(s)\right),\ \gamma_i(s) := \tilde{f}_i^{-1}\left(\tfrac{1}{r_i}s\right), \tag{2.73}$$

where δ_j, a_j, $j = 1, \ldots, n$ and r_i are positive reals. It follows from (2.73) that

$$x_i \geq \gamma_{ij}\left(x_j\right) \ \Rightarrow \tilde{f}_j(x_j) \leq \tfrac{a_j}{a_i}(1+\delta_j)\tilde{f}_i(x_i),$$

$$x_i \geq \gamma_i\left(|u_i|\right) \ \Rightarrow |u_i| \leq r_i \tilde{f}_i(x_i).$$

Using the inequalities from the right hand side of the implications above and assuming that the following condition holds

$$\sum_{j=1, j\neq i}^{n} c_{ij}(x)\tfrac{a_j}{a_i}(1+\delta_j) + c_{ii}(x) + r_i \leq -h_i, \ \forall x \in \mathbb{R}^n_+, \ \text{for some } h_i > 0, \tag{2.74}$$

we obtain that for all $x_i \in \mathbb{R}_+ : V_i(x_i) \geq \max\left\{\max_{j\neq i}\gamma_{ij}\left(V_j(x_j)\right), \gamma_i\left(|u_i|\right)\right\}$ it holds that

$$\begin{aligned}\frac{dV_i(x_i(t))}{dt} &= \sum_{j=1}^{n} c_{ij}(x(t))\tilde{f}_j(x_j(t)) + u_i(t) \\ &\leq \left(\sum_{j=1, j\neq i}^{n} c_{ij}(x(t))\tfrac{a_j}{a_i}(1+\delta_j) + c_{ii}(x(t)) + r_i\right)\tilde{f}_i\left(x_i(t)\right) \leq -\mu_i(V_i(x_i(t))),\end{aligned}$$

where $\mu_i(r) := h_i\tilde{f}_i(r)$ and thereby condition (1.17) is satisfied. Thus, under the condition (2.74), $V_i(x_i) = |x_i|$ is an ISS Lyapunov function for the i-th entity with the gains, given by (2.73).

To check whether the interconnected system (2.72) is ISS we need to verify the small-gain condition (we will use cyclic formulation, see Proposition 1.5.5).

Consider a composition $\gamma_{k_1k_2} \circ \gamma_{k_2k_3}$. It holds

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} = \tilde{f}_{k_1}^{-1}\left(\tfrac{a_{k_1}}{a_{k_2}}\tfrac{1}{1+\delta_{k_3}}\tilde{f}_{k_2}\left(\tilde{f}_{k_2}^{-1}\left(\tfrac{a_{k_2}}{a_{k_3}}\tfrac{1}{1+\delta_{k_3}}\tilde{f}_{k_3}\left(s\right)\right)\right)\right) = \tilde{f}_{k_1}^{-1}\left(\tfrac{a_{k_1}}{a_{k_3}}\tfrac{1}{\left(1+\delta_{k_3}\right)\left(1+\delta_{k_2}\right)}\tilde{f}_{k_3}\left(s\right)\right).$$

In the same way we obtain the expression for the cycle condition in (1.25) (here we use that $k_1 = k_p$):

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ \ldots \circ \gamma_{k_{p-1}k_p}(s) = \tilde{f}_{k_1}^{-1}\left(\tfrac{1}{\prod_{i=2}^{p}(1+\delta_{k_i})}\tilde{f}_{k_1}\left(s\right)\right) < s.$$

Thus, the small gain condition (1.25) holds for all $\delta_i > 0$ and by Theorem 1.5.6 the whole system is ISS.

If we assume that the c_{ij} are bounded, i.e., there exists $M > 0$ such that for all $x \in \mathbb{R}^n_+$: $c_{ij}(x) \leq M$ for all $i, j = 1, \ldots, n, \ i \neq j$, then the inequality (2.74) can be simplified. To this end note that

$$\forall w_i > 0 \ \exists \delta_j > 0, j = 1, \ldots, n : \ \sum_{j=1, j\neq i}^{n} c_{ij}(x)\tfrac{a_j}{a_i}\delta_j \leq M\left(\sum_{j=1, j\neq i}^{n}\tfrac{a_j}{a_i}\delta_j\right) < w_i.$$

Using these estimates, we can rewrite (2.74) as

$$\sum_{j=1, j\neq i}^{n} c_{ij}(x)a_j \leq -c_{ii}(x)a_i - \epsilon_i,$$

where $\epsilon_i = a_i(r_i + h_i + w_i)$. In matrix notation, with $a = (a_1, \ldots, a_n)^T$, $\epsilon = (\epsilon_1, \ldots, \epsilon_n)^T$, it takes the form

$$C(x)a < -\epsilon. \tag{2.75}$$

We summarize our investigations in the following proposition.

Proposition 2.6.1. *Consider a network as in (2.71) and assume that the c_{ij} are bounded for all $i, j = 1, \ldots, n$, $i \neq j$. If there exist $a \in \mathbb{R}^n$, $\epsilon \in \mathbb{R}^n$, $a_i > 0$, $\epsilon_i > 0$, $i = 1, \ldots, n$ such that the condition $C(x)a < -\epsilon$ holds for all $t > 0$, then the whole network (2.72) is ISS.*

Remark 2.6.1. *If C is a constant matrix, then the condition $Ca < -\epsilon$ is equivalent to $Ca < 0$ (with a, ϵ as in the proposition above).*

Remark 2.6.2. *Recall, that for the case, when C is a constant matrix with negative elements on the main diagonal and all other elements are nonnegative, C is diagonal dominant (see, e.g., [7]), if it holds $c_{ii} + \sum_{j\neq i} c_{ij} < 0$ for all $i = 1, \ldots, n$. In this case, one can easily prove with help of Gershgorin circle theorem (see [7], Fact 4.10.17), that C is Hurwitz. Similarly, the previous condition can be replaced with another one: there are n numbers $a_i > 0$ such that $c_{ii}a_i + \sum_{j\neq i} c_{ij}a_j < 0$ for all $i = 1, \ldots, n$ (which is equivalent to the existence of a positive vector a such that $Ca < 0$). In this case the matrix is also Hurwitz (see, e.g., [25]). This shows that Proposition 2.6.1 is consistent with the fact, that linear systems are ISS if and only if matrix C is Hurwitz.*

2.6.3 Possible extensions

Models with time-delays. We have considered the basic model of production networks, based on ODEs and where it is assumed that the production rates are $\mathcal{K}_\infty$-functions. Similar argument can be used for more general classes of models.

In particular, consider the model with transportation times. Let the time needed for the transportation of material from the j-th to the i-th entity be $\tau_{ij} \in \mathbb{R}_+$. Then the dynamics of the i-th subsystem can be described by retarded differential equations similar to (2.71):

$$\dot{x}_i(t) = \sum_{j=1, j\neq i}^{n} c_{ij}(x(t))\tilde{f}_j(x_j(t - \tau_{ij})) + u_i(t) - \tilde{c}_{ii}(x(t))\tilde{f}_i(x_i(t)), \; i = 1, \ldots, n. \tag{2.76}$$

Here, the external input and the processing rate do not depend on time-delays, but the internal inputs from other subsystems do. This means that the input of subsystem i at time t from subsystem j is the amount of material sent by the j-th subsystem at the time $t - \tau_{ij}$.

Such systems can be analyzed using Lyapunov-Razumikhin approach, see [16], and similar results have been obtained.

One can consider the equations (2.71) for production rates which are $\mathcal{K}$-functions. In this case we may ask for LISS of the system (ISS cannot be achieved in general), and therefore small-gain theorem for interconnection of LISS subsystems [23], [16] must be used.

Modeling of a transport process. In the model (2.76) a very simple model of transport of goods is used: all the transported goods, which have been sent from i-th node to the j-th node, arrive to the node j after some time τ_{ji}. Of course, during a transport of goods some of the goods can be lost or spoiled, and the transportation rate of the goods between the nodes is bounded. To include these effects into the model, one can model the transportation via transport equations, see [5], [61] and references therein.

Reaction-diffusion models. A model, developed in this section, can be used also for modeling of chemical reaction networks (for such models see [25]). One can easily check, that the system (2.71) is a monotone (cooperative) control system (see [3]). One could extend a model (2.71) by adding diffusion of chemical components:

$$\frac{\partial x_i(t,y)}{\partial t} = c_i \Delta x_i(t,y) + \sum_{j=1, j\neq i}^{n} c_{ij}(x(t,y))\tilde{f}_j(x_j(t,y)) + u_i(t,y) - \tilde{c}_{ii}(x(t,y))\tilde{f}_i(x_i(t,y)), \quad (2.77)$$

where $i = 1, \ldots, n$, $y \in G$, G is some region in $\mathbb{R}^p$. If we use Neumann conditions on the boundary, and if we can prove, that the system (2.77) is monotone control system (for reaction-diffusion system without inputs see [73]), then we can with the help of the theory, developed in Section 2.5 reduce the question of ISS of the system (2.77) to the problem of ISS of a system (2.71), which we have already solved.

2.7 Concluding remarks and open questions

In this chapter we have developed tools for analysis of ISS of abstract control systems.

However, many questions remain to be solved. The first group of questions is a generalization of the results from the finite-dimensional theory to the abstract control systems, in particular:

1. In the papers [77] and [59] the converse ISS Lyapunov theorem has been proved. For abstract systems without inputs of the form (2.33) the converse Lyapunov theorem has been proved in [36]. The fundamental question in ISS theory is to prove the converse ISS Lyapunov theorem for systems of the form (2.33).

2. Important results in the finite-dimensional theory are different characterizations of ISS property [77], [78]. The generalization of all these results (or the counterexamples, if the equivalences do not hold anymore) was not provided even for time-delay systems, see [81].

3. The current proofs of the small-gain theorems in terms of trajectories (for finite-dimensional systems) use explicitly the equivalence between ISS and (Global stability + Asymptotic gain property), which has been proved only for finite-dimensional systems [78]. Therefore the solution of the previous problem will possibly provide the key to the proof of the small-gain theorems in terms of trajectories for systems (2.33).

Another group of problems is "internal" to infinite-dimensional theory:

1. In Section 2.1 we developed linear ISS theory for the case if the function f in (2.1) is bounded for bounded inputs. But for infinite-dimensional systems the more complicated case, when $f(u) = Cu$ for unbounded operator C is also of importance.

2. Most part of the theory, developed in this chapter as well as the results from [51] assume, that inputs are piecewise continuous w.r.t. time. However, often this class is too restrictive

and one studies the weak solutions of the systems of the form (2.33) with the inputs which are from L_p class w.r.t. time, see e.g. [12]. In particular, small-gain theorems for such systems have to be developed.

3. Applications of ISS framework to different classes of systems, in particular to production networks and chemical reaction networks, see Section 2.6.3.

Chapter 3

ISS of infinite-dimensional impulsive systems

In this chapter we extend results of the previous chapter to the case of impulsive systems based on differential equations in Banach spaces.

Impulsive systems combine a continuous and discontinuous dynamics, where the discontinuous dynamics is modeled by an instantaneous jump of the state of the system at some given moments of time, which do not depend on the state of the system.

If both continuous and discontinuous dynamics of the system (taken separately from each other) are ISS, then the resulting dynamics of an impulsive system is also ISS for all impulse time sequences (it is even strongly uniformly ISS, see [38, Theorem 2]).

More interesting is the study of the systems for which continuous or discrete dynamics is not ISS. In this case ISS of the impulsive system cannot be achieved for all sequences of impulse times, and one has to introduce restrictions on the class of impulse time sequences for which ISS can be verified. These conditions are called dwell-time conditions. The study of ISS of finite-dimensional impulsive systems was done in [38], where it was proved that impulsive systems, which possess an exponential ISS-Lyapunov function are uniformly ISS over impulse time sequences, which satisfy so-called average dwell-time (ADT) condition.

In [11] a sufficient condition in terms of Lyapunov-Razumikhin functions is provided, which ensures the uniform ISS of impulsive time-delay systems over impulse time sequences satisfying fixed dwell-time (FDT) condition.

In this chapter we are going to extend the existing results for finite-dimensional impulsive systems in the following directions:

1. We consider not only exponential, but also nonexponential Lyapunov functions, and use the corresponding nonlinear FDT condition.

2. For exponential Lyapunov functions we introduce generalized average dwell-time (gADT) condition.

3. We provide two ways for construction of ISS-Lyapunov functions for impulsive systems (via small-gain theorems and linearization).

4. The results are proved for impulsive systems in Banach spaces.

Our first aim in this chapter is to extend the results of [38] concerning ISS of a single impulsive system in two directions.

We prove, that existence of an ISS Lyapunov function (*not necessarily exponential*) for an impulsive system implies input-to-state stability of the system over impulsive time sequences satisfying *nonlinear FDT condition*. Under slightly weaker FDT condition we prove uniform global stability of the system over corresponding class of impulse time sequences.

Furthermore, for the case, when an impulsive system possesses an exponential ISS Lyapunov function, we generalize the result from [38], by introducing the *generalized average dwell-time (gADT) condition* and proving, that an impulsive system, which possesses an exponential ISS Lyapunov function is ISS for all impulse time sequences, which satisfy generalized ADT condition. We argue that generalized ADT condition provides in certain sense tight estimates of the class of impulsive time sequences, for which the system is ISS.

Then we show, how exponential LISS Lyapunov functions for linearizable control systems can be constructed via linearization method.

Afterwards we investigate ISS of interconnected impulsive systems via small-gain theorems. The first small-gain theorem is analogous to small-gain theorem for continuous systems and states that if subsystems possess ISS-Lyapunov functions (with corresponding gains) and the small-gain condition holds, then an ISS-Lyapunov function for an interconnection can be constructed.

The second small-gain theorem states that if all subsystems possess *exponential* ISS Lyapunov functions, and gains are power functions, then an *exponential* ISS Lyapunov function for the whole system can be constructed (and consequently a stronger result concerning ISS of the interconnection can be obtained). This generalizes Theorem 4.2 from [18], where this theorem for linear gains has been proved.

Remember that a construction of an ISS-Lyapunov function does not guarantee automatically ISS of the interconnected system, because the dwell-time condition of certain type has to be fulfilled.

We investigate relations between different types of dwell-time conditions in the Section 3.2.2 and a relation between small-gain and dwell-time condition on the step of selection of gains is clarified in the Section 3.3.2.

3.1 Preliminaries

Let X and U denote a state space and a space of input values respectively, and let both of them be Banach. Take the space of admissible inputs as $U_c := PC([t_0,\infty),U)$, i.e. the space of piecewise right-continuous functions from $[t_0,\infty)$ to U equipped with the norm

$$\|u\|_{U_c} := \sup_{t\geq t_0} \|u(t)\|_U.$$

Let $T = \{t_1,\ t_2,\ t_3,\ldots\}$ be a strictly increasing sequence of impulse times without finite accumulation points.

Consider a system of the form

$$\begin{cases} \dot{x}(t) = Ax(t) + f(x(t),u(t)), \quad t\in[t_0,\infty)\backslash T, \\ x(t) = g(x^-(t),u^-(t)), \quad t\in T, \end{cases} \tag{3.1}$$

where $x(t) \in X$, $u(t) \in U$, A is an infinitesimal generator of a C_0-semigroup on X and $f, g : X \times U \to X$.

Equations (3.1) together with the sequence of impulse times T define an impulsive system. The first equation of (3.1) describes the continuous dynamics of the system, and the second describes the jumps of the state at impulse times.

We assume that for each initial condition a solution of the problem (3.1) exist and is unique. Note that from the continuity assumptions on the inputs u it follows that $x(t)$ is piecewise-continuous, and $x^-(t) = \lim\limits_{s \to t-0} x(s)$ exists for all $t \geq t_0$.

For a given set of impulse times by $\phi(t, t_0, x, u)$ we denote the state of (3.1) corresponding to the initial value $x \in X$, the initial time t_0 and to the input $u \in U_c$ at time $t \geq t_0$.

Note that the system (3.1) is not time-invariant, that is, $\phi(t_2, t_1, x, u) = \phi(t_2 + s, t_1 + s, x, u)$ doesn't hold for all $\phi_0 \in X$, $u \in U_c$, $t_2 \geq t_1$ and all $s \geq -t_1$.

However, it holds

$$\phi(t_2, t_1, x, u) = \phi_s(t_2 + s, t_1 + s, x, u), \tag{3.2}$$

where ϕ_s is a trajectory corresponding to the system (3.1) with impulse time sequence $T_s := \{t_1 + s,\ t_2 + s,\ t_3 + s, \ldots\}$.

This means that the trajectory of the system (3.1) with initial time t_0 and impulse time sequence T is equal to the trajectory of (3.1) with initial time 0 and impulse time sequence T_{-t_0}. Therefore we will assume in this chapter that t_0 is some fixed moment of time and will investigate the stability properties of the system (3.1) w.r.t. this initial time.

We assume throughout this chapter that $x \equiv 0$ is an equilibrium of the unforced system (3.1), that is $f(0,0) = g(0,0) = 0$.

Let us introduce the stability properties for system (3.1) which we deal with.

Definition 3.1.1. *For a given sequence T of impulse times we call a system (3.1) locally input-to-state stable (LISS) if there exist $\rho > 0$ and $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$, such that $\forall x \in X : \ \|x\|_X \leq \rho$, $\forall u \in U_c : \ \|u\|_{U_c} \leq \rho$, $\forall t \geq t_0$ it holds*

$$\|\phi(t, t_0, x, u)\|_X \leq \beta(\|x\|_X, t - t_0) + \gamma(\|u\|_{U_c}). \tag{3.3}$$

System (3.1) is input-to-state stable (ISS), if (3.3) holds for all $x \in X$, $u \in U_c$.

System (3.1) is called uniformly ISS over a given set $\mathcal{S}$ of admissible sequences of impulse times if it is ISS for every sequence in $\mathcal{S}$, with β and γ independent of the choice of the sequence from the class $\mathcal{S}$.

Definition 3.1.2. *For a given sequence T of impulse times we call system (3.1) globally stable (GS) if there exist $\xi,\ \gamma \in \mathcal{K}_\infty$, such that $\forall x \in X$, $\forall u \in U_c$, $\forall t \geq t_0$ it holds*

$$\|\phi(t, t_0, x, u)\|_X \leq \xi(\|x\|_X) + \gamma(\|u\|_{U_c}). \tag{3.4}$$

The impulsive system (3.1) is uniformly GS over a given set $\mathcal{S}$ of admissible sequences of impulse times if (3.4) holds for every sequence in $\mathcal{S}$, with β and γ independent of the choice of the sequence.

In the next section we are going to find certain sufficient conditions for an impulsive system of the form (3.1) to be ISS.

3.2 Lyapunov ISS theory for an impulsive system

For analysis of (L)ISS of impulsive systems we exploit (L)ISS-Lyapunov functions.

Definition 3.2.1. *A continuous function $V : D \to \mathbb{R}_+$, $D \subset X$, $0 \in int(D)$ is called a LISS-Lyapunov function for (3.1) if $\exists\ \psi_1, \psi_2 \in \mathcal{K}_\infty$, such that*

$$\psi_1(\|x\|_X) \leq V(x) \leq \psi_2(\|x\|_X), \quad x \in D \tag{3.5}$$

holds and $\exists \rho > 0$, $\chi \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$ with $\varphi(x) = 0 \Leftrightarrow x = 0$, such that $\forall x \in X : \ \|x\|_X \leq \rho$, $\forall \xi \in U : \ \|\xi\|_U \leq \rho$ it holds

$$V(x) \geq \chi(\|\xi\|_U) \Rightarrow \begin{cases} \dot{V}_u(x) \leq -\varphi(V(x)) \\ V(g(x,\xi)) \leq \alpha(V(x)), \end{cases} \tag{3.6}$$

for all $u \in U_c$, $\|u\|_{U_c} \leq \rho$ and $u(0) = \xi$. For a given input value $u \in U_c$ the Lie derivative $\dot{V}_u(x)$ is defined by

$$\dot{V}_u(x) = \overline{\lim_{t \to +0}} \frac{1}{t}(V(\phi_c(t,0,x,u)) - V(x)), \tag{3.7}$$

where ϕ_c is a transition map, corresponding to continuous part of the system (3.1), i.e. $\phi_c(t,0,x,u)$ is a state of the system (3.1) at time t, if the state at time $t_0 := 0$ was x, input u was applied and $T = \emptyset$.

If $D = X$ and (3.6) holds for all $x \in X$ and $\xi \in U$, then V is called ISS-Lyapunov function. If in addition

$$\varphi(s) = cs \text{ and } \alpha(s) = e^{-d}s \tag{3.8}$$

for some $c, d \in \mathbb{R}$, then V is called exponential ISS-Lyapunov function with rate coefficients c, d.

If both c and d are positive, then V decreases along the continuous flow and at each jump. In this case an impulsive system is ISS w.r.t. to all impulse time sequences. If both c and d are negative, then we cannot guarantee ISS of (3.1) w.r.t. any impulse time sequence. We are interested in the case of $cd < 0$, where stability properties depend on T. In this case input-to-state stability can be guaranteed under certain restrictions on T. Intuitively, the increase of either c or d leads to less restrictions on T.

Remark 3.2.1. *We would like to emphasize that the solution $\phi(\cdot,0,x,u)$ depends on an impulse time sequence T, but if we take t small enough, then $\phi(s,0,x,u)$, $s \in [0,t]$ does not depend on T because the impulse times do not have finite accumulation points. Therefore the value of $\dot{V}_u(x)$ and the Lyapunov function V itself do not depend on the impulse time sequence.*

Note that our definition of ISS-Lyapunov function is given in the implication form. The next proposition shows another way to introduce an ISS Lyapunov function, which is frequently used in the literature on hybrid systems, see e.g. [64]. We will use it for the formulation of the small-gain theorem in Section 3.3. It is a counterpart of [55, Proposition 2.2.19] where an analogous result for hybrid systems has been proved.

Definition 3.2.2. *Function $g : X \times U \to X$ is called bounded on bounded balls, if for each $\rho > 0$ there exists $K > 0$, so that $\sup_{x \in X: \|x\|_X \leq \rho,\ u \in U: \|u\|_U \leq \rho} \|g(x,u)\|_X \leq K$.*

Proposition 3.2.1. *Let for a continuous function $V : X \to \mathbb{R}_+$ there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, such that (3.5) holds and $\exists \gamma \in \mathcal{K}_\infty$, $\alpha \in \mathcal{P}$ and continuous function $\varphi : \mathbb{R}_+ \to \mathbb{R}$, $\varphi(0) = 0$ such that for all $\xi \in U$ and all $u \in U_c$ with $u(0) = \xi$ it holds*

$$V(x) \geq \gamma(\|\xi\|_U) \quad \Rightarrow \quad \dot{V}_u(x) \leq -\varphi(V(x)) \tag{3.9}$$

and $\forall x \in X, \xi \in U$ it holds

$$V(g(x,\xi)) \leq \max\{\alpha(V(x)), \gamma(\|\xi\|_U)\}. \tag{3.10}$$

Then V is an ISS Lyapunov function. If g is bounded on bounded balls, then also the converse implication holds.

Proof. "$\Rightarrow$" Pick any $\rho \in \mathcal{K}_\infty$ such that $\alpha(r) < \rho(r)$ for all $r > 0$. Then for all $x \in X$ and $\xi \in U$ from (3.10) we have

$$V(g(x,\xi)) \leq \max\{\rho(V(x)), \gamma(\|\xi\|_U)\}.$$

Define $\chi := \max\{\gamma, \rho^{-1} \circ \gamma\} \in \mathcal{K}_\infty$. For all $x \in X$ and $\xi \in U$ such that $V(x) \geq \chi(\|\xi\|_U)$ it follows $\rho(V(x)) \geq \gamma(\|\xi\|_U)$ and hence

$$V(g(x,\xi)) \leq \rho(V(x)).$$

Since $\chi(r) \geq \gamma(r)$ for all $r > 0$, it is clear, that (3.6) holds. Thus, V is an ISS-Lyapunov function.

"$\Leftarrow$" Let g be bounded on bounded balls and let V be an ISS-Lyapunov function for a system (3.1). Then $\exists \chi \in \mathcal{K}$ and $\alpha \in \mathcal{P}$ such that for all $x \in X$ and $\xi \in U$ from $V(x) > \chi(\|\xi\|_U)$ it follows $V(g(x,\xi)) \leq \alpha(V(x))$.

Let $V(x) \leq \chi(\|\xi\|_U)$. Then $\|x\|_X \leq \psi_1^{-1} \circ \chi(\|\xi\|_U)$. Define $S(r) := \{x \in X : \|x\|_X \leq \psi_1^{-1} \circ \chi(r)\}$ and $\omega(r) := \sup\limits_{\|\xi\|_U \leq r,\ x \in S(r)} \psi_2(\|g(x,\xi)\|_X)$. This supremum exists since g is bounded on bounded balls. Clearly, ω is nondecreasing and $\omega(0) = \psi_2(\|g(0,0)\|_X) = 0$. Pick any $\gamma \in \mathcal{K}$: $\gamma \geq \max\{\omega, \chi\}$. Then for all $x \in X$ and $\xi \in U$ inequality (3.10) holds and for all x : $\|x\|_X \geq \gamma(\|\xi\|_U)$ estimate (3.9) holds. □

Similarly one can prove the following proposition (which is not a consequence of a Proposition 3.2.1):

Proposition 3.2.2. *Let for a continuous function $V : X \to \mathbb{R}_+$ there exist $\psi_1, \psi_2 \in \mathcal{K}_\infty$, such that (3.5) holds and $\exists \gamma \in \mathcal{K}_\infty$ and $c, d \in \mathbb{R}$ such that for all $\xi \in U$ and all $u \in U_c$ with $u(0) = \xi$ it holds*

$$V(x) \geq \gamma(\|\xi\|_U) \quad \Rightarrow \quad \dot{V}_u(x) \leq -cV(x)$$

and $\forall x \in X, \xi \in U$ it holds

$$V(g(x,\xi)) \leq \max\{e^{-d}V(x), \gamma(\|\xi\|_U)\}.$$

Then V is an exponential ISS-Lyapunov function. If g is bounded on bounded balls, then also the converse implication holds.

Now we provide a combination of dwell-time and Lyapunov-type conditions that guarantees that system (3.1) is ISS. In contrast to continuous systems the existence of an ISS-Lyapunov function for (3.1) does not automatically imply ISS of the system with respect to all impulse time sequences. In order to find the set of impulse time sequences for which the system is ISS we use the FDT condition (3.11) from [71], where it was used to guarantee global asymptotic stability of finite-dimensional impulsive systems without inputs.

For $\theta > 0$ define the set $S_\theta := \{\{t_i\}_1^\infty \subset [t_0, \infty) \ : \ t_{i+1} - t_i \geq \theta, \ \forall i \in \mathbb{N}\}$, consisting of impulse time sequences with distance between impulse times not less than θ.

Theorem 3.2.3. *Let V be an ISS-Lyapunov function for (3.1) and φ, α are as in the Definition 3.2.1 and $\varphi \in \mathcal{P}$. Let for some $\theta, \delta > 0$ and all $a > 0$ it hold*

$$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} \leq \theta - \delta. \tag{3.11}$$

Then (3.1) is ISS for all impulse time sequences $T \in S_\theta$.

Proof. Fix arbitrary $u \in U_c$, $\phi_0 \in X$ and choose the sequence of impulse times $T = \{t_i\}_{i=1}^\infty$, $T \in S_\theta$. Our aim is to prove ISS of the system (3.1) w.r.t. impulse time sequence T by a direct construction of the functions β and γ from Definition 3.1.1.

For the sake of brevity we denote $x(\cdot) = \phi(\cdot, t_0, \phi_0, u)$ and $y(\cdot) := V(x(\cdot))$.

At first assume that $u \equiv 0$. We are going to bound trajectory from above by a function $\beta \in \mathcal{KL}$.

Since $u \equiv 0$ the following inequalities hold

$$\dot{y}(t) \leq -\varphi(y(t)), \quad t \notin T, \tag{3.12}$$

$$y(t) \leq \alpha(y^-(t)), \quad t \in T. \tag{3.13}$$

Take arbitrary pair $t_i, \ t_{i+1} \in T$. There are two possibilities: either $y(t) > 0$ for all $t \in [t_i, t_{i+1})$ or there exists certain time $\hat{t} \in [t_i, t_{i+1})$: $y(\hat{t}) = 0$ and, since $x = 0$ is an equilibrium point of the system (3.1), $y(t) = 0$ for all $t \geq \hat{t}$.

Let us consider the first case. Integrating, we obtain

$$\int_{t_i}^{t} \frac{dy(\tau)}{\varphi(y(\tau))} \leq -(t - t_i), \quad t \in (t_i, t_{i+1}). \tag{3.14}$$

Fix any $r > 0$ and define

$$F(q) := \int_r^q \frac{ds}{\varphi(s)}, \quad \forall q > 0.$$

Note that $F : (0, \infty) \to \mathbb{R}$ is a continuous strictly increasing function. Thus, it is invertible on $(0, \infty)$ and $F^{-1} : \mathbb{R} \to (0, \infty)$ is also an increasing function.

Changing variables in (3.14) (which is possible since y is bijective on (t_i, t_{i+1})), we can rewrite (3.14) as

$$F(y(t)) - F(y(t_i)) \leq -(t - t_i). \tag{3.15}$$

Consequently, for $t \in [t_i, t_{i+1})$ it holds

$$y(t) \leq F^{-1}\left(F(y(t_i)) - (t - t_i)\right). \tag{3.16}$$

Taking in (3.15) a limit $t \to t_{i+1}$ and recalling that $t_{i+1} - t_i \geq \theta$, we obtain

$$F(y^-(t_{i+1})) - F(y(t_i)) \leq -\theta. \tag{3.17}$$

Using that $y(t_{i+1}) \leq \alpha(y^-(t_{i+1}))$, we obtain the estimate

$$F(y(t_{i+1})) - F(y(t_i)) \leq \big(F(\alpha(y^-(t_{i+1}))) - F(y^-(t_{i+1}))\big) + \big(F(y^-(t_{i+1})) - F(y(t_i))\big).$$

By (3.11) and (3.17) we obtain

$$F(y(t_{i+1})) - F(y(t_i)) \leq (\theta - \delta) - \theta = -\delta.$$

From this inequality we have

$$y(t_{i+1}) \leq F^{-1}(F(y(t_i)) - \delta). \tag{3.18}$$

In particular, $y(t) < y(t_i)$, $t \in (t_i, t_{i+1}]$.

From (3.18) we obtain

$$y(t_{i+1}) \leq F^{-1}(F(F^{-1}(F(y(t_{i-1}))-\delta))-\delta) = F^{-1}(F(y(t_{i-1}))-2\delta) \leq F^{-1}(F(y(t_1))-i\delta). \tag{3.19}$$

The estimate (3.19) is valid for all i: $F(y(t_1)) - i\delta \geq \lim_{q \to +0} F(q)$. Let us denote the maximum of such i by $\hat{i}$ (we set $\hat{i} := \infty$ if such maximum doesn't exist).

Let us construct a function $\tilde{\beta} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$, which provides a bound for a function y. Define

$$\tilde{\beta}(r, t_1 - t_0) = \max\{y(t_1), \alpha(y(t_1))\}, \quad r \geq 0.$$

For all $1 \leq i \leq \hat{i}$ define

$$\tilde{\beta}(r, t_{i+1} - t_0) := F^{-1}(F(\tilde{\beta}(r, t_1 - t_0)) - i\delta), \quad r \geq 0.$$

For any $r > 0$, for all $i \leq \hat{i}$ define $\tilde{\beta}(r, \cdot)$ on $(t_{i-1} - t_i, t_i - t_{i+1})$ as an arbitrary continuous decreasing function, which lies above every solution $y(\cdot)$ of (3.12) with (3.13), corresponding to initial condition $y(t_0) = r$.

If $\hat{i}$ is finite, then define $\tilde{\beta}(r, \cdot)$ on $[t_{\hat{i}} - t_0, \infty)$ as a continuous decreasing to 0 function.

By construction, for all t it holds that

$$y(t) \leq \tilde{\beta}(y_0, t - t_0),$$

where $\tilde{\beta} : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is continuous and decreasing w.r.t. the second argument and $\tilde{\beta}(0, t) \equiv 0$ for all $t \geq 0$. We are going to prove, that for all $r \geq 0$ it holds $\tilde{\beta}(r, t) \to 0$ as soon as $t \to \infty$. If $\hat{i} < \infty$, then it follows from the construction. Thus, let $\hat{i} = \infty$.

To prove this it is enough to prove, that for all $r > 0$ it holds $z_r(t_i) = \tilde{\beta}(r, t_i - t_0) \to 0$, $i \to \infty$.

Let it be false, then due to monotonicity of z_r for some $r > 0$ $\exists \lim\limits_{i \to \infty} z_r(t_i) = b_r > 0$.

Define $c := \min_{b \leq s \leq z_r(0)} \varphi(s)$ and observe by the middle-value theorem that

$$\delta \leq F(z_r(t_i)) - F(z_r(t_{i+1})) = \int_{z_r(t_{i+1})}^{z_r(t_i)} \frac{ds}{\varphi(s)} \leq \frac{1}{c}(z_r(t_i) - z_r(t_{i+1})).$$

Hence for all i it holds

$$z_r(t_i) - z_r(t_{i+1}) \geq c\delta,$$

and the sequence $z_r(t_i)$ does not converge to a positive limit. We obtained a contradiction to $b_r > 0$, thus $z_r(t_i) \to 0$, $i \to \infty$. Thus, $\forall r > 0$ $\tilde{\beta}(r, \cdot) \in \mathcal{L}$.

For all $r, t \geq 0$ define $\beta_1(r,t) := \sup_{0 \leq h \leq r} \tilde{\beta}(h,t)$. Clearly, β_1 is nondecreasing w.r.t. the first argument and $\beta_1(r,t) \geq \tilde{\beta}(r,t)$ for all $r, t \geq 0$.

Define now $\beta_2(r,t) := \frac{1}{r}\int_r^{2r} \beta_1(s,t)ds + re^{-t}$, $\forall r > 0, t \geq 0$. Function $\beta_2 \in \mathcal{KL}$ and $\beta_2(r,t) \geq \beta_1(r,t)$, $\forall r, t \geq 0$. Hence if $u \equiv 0$ then it holds that

$$V(x(t)) \leq \beta_2(V(\phi_0), t - t_0), \quad \forall t \geq 0.$$

Now let u be an arbitrary admissible input. Define

$$I_1 := \{x \in X : V(x) \leq \chi(\|u\|_{U_c})\}.$$

For all $t: \ x(t) \notin I_1$ according to (3.6) the estimates (3.12) and (3.13) hold and consequently

$$V(x(t)) \leq \beta_2(V(\phi_0), t - t_0), \quad \forall t: \ x(t) \notin I_1.$$

Let $t^* := \inf\{t : x(t) \in I_1\}$. From (3.5) we obtain

$$\|x(t)\|_X \leq \beta(\|\phi_0\|_X, t - t_0), \quad t \leq t^*, \tag{3.20}$$

where $\beta(r,t) = \psi_1^{-1}(\beta_2(\psi_2(r), t))$.

At first note that a trajectory can leave I_1 only by a jump. If $\|u\|_{U_c} = 0$, then I_1 is invariant under continuous dynamics, because $x \equiv 0$ is an equilibrium. Let $\|u\|_{U_c} > 0$ and let for some $t > t^*$ we have $x(t) \in \partial I_1$, i.e. $y(t) = \chi(\|u\|_{U_c})$. Then according to the inequality (3.6) it holds $\dot{y}(t) \leq -\varphi(y(t)) < 0$ and thus $y(\cdot)$ cannot leave I_1 at time t.

Define function $\tilde{\alpha} : \mathbb{R}_+ \to \mathbb{R}_+$ by

$$\tilde{\alpha}(x) := \max\{\max_{0 \leq s \leq \chi(x)} \alpha(s), \chi(x)\}, \quad x \in \mathbb{R}_+.$$

Also let us introduce the set

$$I_2 := \{x \in X : V(x) \leq \tilde{\alpha}(\|u\|_{U_c})\} \supseteq I_1.$$

We are going to prove, that $x(t) \in I_2$ for all $t \geq t^*$.

Now let for some $t_k \in T$, $t_k \geq t^*$ it holds $x(t_k) \notin I_1$ and for some $\varepsilon > 0$ $x(t) \in I_1$ for all $t \in (t_k - \varepsilon, t_k)$. Then $x(t_k) \in I_2$ by construction of the set I_2.

But we have proved, that $y(t) < y(t_k)$ as long as $t > t_k$ and $x(t) \notin I_1$. Consequently, $x(t) \in I_2$ for all $t > t^*$.

Thus, for $t > t^*$ it holds

$$V(x(t)) \leq \tilde{\alpha}(\|u\|_{U_c})$$

which implies

$$\|x(t)\|_X \leq \psi^{-1}(\tilde{\alpha}(\|u\|_{U_c})) := \tilde{\gamma}(\|u\|_{U_c}).$$

Function $\tilde{\gamma}$ is positive definite and nondecreasing, thus, it may be always majorized by a $\mathcal{K}$-function γ. Recalling (3.20) we obtain

$$\|x(t)\|_X \leq \beta(\|\phi_0\|_X, t - t_0) + \gamma(\|u\|_{U_c}), \quad \forall t \geq t_0. \tag{3.21}$$

□

Remark 3.2.2. *We haven't proved the uniform ISS of the system* (3.1) *w.r.t.* S_θ*. Although the function* γ *by construction does not depend on the impulse time sequence* $T \in S_\theta$*, the function* β *does depend. However, pick any periodic impulse time sequence* $T = \{t_1, \dots, t_n, \dots\} \in S_\theta$*, that is* $t_{i+1} = t_i + d$ *for some* $d > 0$*. Then from the construction of the function* β *it is clear that* (3.1) *is uniformly ISS over the class* $W = \{T_i, i \geq 1\}$*, where* $T_i = \{t_i, \dots, t_n, \dots\}$*.*

Remark 3.2.3. *If the discrete dynamics does not destabilize the system, i.e.* $\alpha(a) \leq a$ *for all* $a \neq 0$*, then the integral on the right hand side of* (3.11) *is non-positive for all* $a \neq 0$*, and the dwell-time condition* (3.11) *is satisfied for arbitrary small* $\theta > 0$*, that is the system is ISS for all impulse time sequences without finite accumulation points.*

We illustrate the application of our theorem on the following example. Let T be an impulse time sequence. Consider the system Σ, defined by

$$\begin{cases} \dot{x} = & -x^3 + u, \ t \notin T \\ x(t) = & x^-(t) + (x^-(t))^3 + u^-(t), \ t \in T. \end{cases} \tag{3.22}$$

Consider a function $V : \mathbb{R}^+ \to \mathbb{R}^+$, defined by $V(x) = |x|$. We are going to prove, that V is an ISS Lyapunov function of the system (3.22).

The Lyapunov gain χ we choose by $\chi(r) = \left(\frac{r}{a}\right)^{\frac{1}{3}}$, $r \in \mathbb{R}^+$, for some $a \in (0, 1)$.
Condition $|x| \geq \chi(|u|)$ implies

$$\begin{aligned} \dot{V}(x) &\leq -(1-a)(V(x))^3, \\ V(g(x,u)) &\leq V(x) + (1+a)(V(x))^3. \end{aligned}$$

Let us compute the integral on the left hand side of (3.11):

$$I(y,a) = \int_y^{y+(1+a)y^3} \frac{dx}{(1-a)x^3} = \frac{1+a}{2(1-a)} \frac{2+(1+a)y^2}{(1+(1+a)y^2)^2} \leq \frac{1+a}{(1-a)}.$$

For every $\varepsilon > 0$ there exist a_ε such that $I(y, a) \leq 1 + 2\varepsilon$.

Thus, for arbitrary $\varepsilon > 0$ we can choose $\theta := 1 + \varepsilon$. Note, that the smaller θ we take, the larger are the gains. This demonstrates the trade-off between the size of gains and the density of allowable impulse times. This dependence plays an important role in the application of small-gain theorems. See Section 3.3.2 for details.

A counterpart of Theorem 3.2.3 can be proved also for the GS property.

Theorem 3.2.4. *Let all the assumptions of the Theorem 3.2.3 hold with* $\delta := 0$*. Then the system* (3.1) *is globally stable uniformly over* S_θ*.*

Proof. The proof goes along the lines of the proof of the Theorem 3.2.3 up to the inequality (3.18), which holds with $\delta = 0$. Then instead of $\tilde{\beta}$ we introduce $\tilde{\xi} \in \mathcal{K}_\infty$ by $\tilde{\xi}(r) = \max\{r, \alpha(r)\}$, and instead of estimate (3.20) we have

$$\|x(t)\|_X \leq \psi_1^{-1}(\tilde{\xi}(\psi_2(\|\phi_0\|_X))) := \xi(\|\phi_0\|_X). \tag{3.23}$$

Thus, for all $t \geq t_0$ we obtain

$$\|x(t)\|_X \leq \xi(\|\phi_0\|_X) + \gamma(\|u\|_{U_c}), \tag{3.24}$$

Note, that the functions ξ and γ do not depend on t_0 and on the sequence of impulse times T, which implies uniformity. $\square$

Now consider the case, when continuous dynamics destabilizes the system and the discrete one stabilizes it. We only state the results since the proofs are similar to those of Theorems 3.2.3 and 3.2.4.

Define $\tilde{S}_\theta := \{\{t_i\}_1^\infty \subset [t_0,\infty) \ : \ t_{i+1}-t_i \leq \theta, \ \forall i \in \mathbb{N}\}$.

Theorem 3.2.5. *Let V be an ISS-Lyapunov function for (3.1) and φ, α are as in the Definition 3.2.1 with $-\varphi \in \mathcal{P}$. Let for some $\theta, \delta > 0$ and all $a > 0$ it hold*

$$\int_{\alpha(a)}^{a} \frac{ds}{-\varphi(s)} \geq \theta + \delta. \tag{3.25}$$

Then (3.1) is ISS w.r.t. every sequence from $\tilde{S}_\theta$.

Theorem 3.2.6. *Let the assumptions of the Theorem 3.2.5 hold with $\delta := 0$. Then the system (3.1) is GS uniformly over $\tilde{S}_\theta$.*

3.2.1 Sufficient condition in terms of exponential ISS-Lyapunov functions

Theorem 3.2.3 can be used, in particular, for systems possessing exponential ISS-Lyapunov functions, but for this particular class of systems even stronger result can be proved.

For a given sequence of impulse times denote by $N(t,s)$ the number of jumps within the interval $(s,t]$.

Theorem 3.2.7. *Let V be an exponential ISS-Lyapunov function for (3.1) with corresponding coefficients $c \in \mathbb{R}$, $d \neq 0$. For arbitrary function $h : \mathbb{R}_+ \to (0,\infty)$, for which there exists $g \in \mathcal{L}$: $h(x) \leq g(x)$ for all $x \in \mathbb{R}_+$ consider the class $\mathcal{S}[h]$ of impulse time-sequences, satisfying the generalized average dwell-time (gADT) condition:*

$$-dN(t,s) - c(t-s) \leq \ln h(t-s), \quad \forall t \geq s \geq t_0. \tag{3.26}$$

Then the system (3.1) is uniformly ISS over $\mathcal{S}[h]$.

Proof. Pick any h as in the statement of the theorem. Fix arbitrary $u \in U_c$, $\phi_0 \in X$, choose the increasing sequence of impulse times $T = \{t_i\}_{i=1}^\infty \in \mathcal{S}[h]$ and denote $x(t) = \phi(t,t_0,\phi_0,u)$ for short.

Due to the right-continuity of $x(\cdot)$ the interval $[t_0,\infty)$ can be decomposed into subintervals as $[t_0,\infty) = \cup_{i=0}^\infty [t_i^*, t_{i+1}^*)$ (the case, when this decomposition is finite, can be treated in the same way), so that $\forall k \in \mathbb{N} \cup \{0\}$ the following inequalities hold

$$V(x(t)) \geq \chi(\|u\|_{U_c}) \text{ for } t \in [t_{2k}^*, t_{2k+1}^*), \tag{3.27}$$

$$V(x(t)) < \chi(\|u\|_{U_c}) \text{ for } t \in [t_{2k+1}^*, t_{2k+2}^*). \tag{3.28}$$

Let us estimate $V(x(t))$ on the time-interval $I_{2k} = (t_{2k}^*, t_{2k+1}^*]$ for arbitrary $k \in \mathbb{N} \cup \{0\}$.

Within the interval I_{2k} there are $r_k := N(t_{2k}^*, t_{2k+1}^*)$ jumps at times $t_1^k, \dots, t_{r_k}^k$. To simplify the notation, we denote also $t_0^k := t_{2k}^*$.

For $t \in (t_i^k, t_{i+1}^k]$, $i = 0, \dots, r_k$ we have $V(x(t)) \geq \chi(\|u\|_{U_c})$, thus from (3.6) and (3.8) we obtain

$$\dot{V}(x(t)) \leq -cV(x(t)),\ t \in (t_i^k, t_{i+1}^k] \tag{3.29}$$

and thus

$$V(x^-(t_{i+1}^k)) \leq e^{-c(t_{i+1}^k - t_i^k)} V(x(t_i^k)).$$

At the impulse time $t = t_{i+1}^k$ we know from (3.6) and (3.8) that

$$V(x(t_{i+1}^k)) \leq e^{-d} V(x^-(t_{i+1}^k))$$

and consequently

$$V(x(t_{i+1}^k)) \leq e^{-d-c(t_{i+1}^k - t_i^k)} V(x(t_i^k)).$$

For all $t \in I_{2k}$ from (3.29) and previous inequality we obtain the following estimate

$$V(x(t)) \leq e^{-d \cdot N(t, t_{2k}^*) - c(t - t_{2k}^*)} V(x(t_{2k}^*)).$$

Dwell-time condition (3.26) implies

$$V(x(t)) \leq h(t - t_{2k}^*) V(x(t_{2k}^*)), t \in I_{2k}. \tag{3.30}$$

Take $\tau := \inf\{t \geq t_0 : V(x(t)) \leq \chi(\|u\|_{U_c})\}$. We are going to find an upper bound of the trajectory on $[t_0, \tau]$ as a $\mathcal{KL}$-function.

Taking in (3.30) $t_{2k}^* := t_0$ we obtain

$$V(x(t)) \leq h(t - t_0) V(\phi_0). \tag{3.31}$$

According to assumptions of the theorem, $\exists g \in \mathcal{L}$: $h(x) \leq g(x)$ for all $x \in \mathbb{R}_+$. Using (3.5), we obtain that $\forall t \in [t_0, \tau]$ it holds

$$\|x(t)\|_X \leq \psi_1^{-1}(g(t - t_0)\psi_2(\|\phi_0\|_X)) =: \beta(\|\phi_0\|_X, t - t_0).$$

On arbitrary interval of the form $[t_{2k+1}^*, t_{2k+2}^*)$, $k \in \mathbb{N} \cup \{0\}$ we have already the bound on $V(x(t))$ by (3.28). Since t_{2k+2}^* can be an impulse time, we have the estimate

$$V(x(t_{2k+2}^*)) \leq \max\{1, e^{-d}\} \chi(\|u\|_{U_c}).$$

From the properties of h it follows, that $\exists C_\lambda = \sup_{x \geq 0}\{h(x)\} < \infty$. Hence for arbitrary $t > \tau$ we obtain with the help of (3.30) the estimate

$$V(x(t)) \leq C_\lambda \max\{1, e^{-d}\} \chi(\|u\|_{U_c}).$$

Overall, for all $t \geq t_0$ we have

$$\|x(t)\|_X \leq \beta(\|\phi_0\|_X, t - t_0) + \gamma(\|u\|_{U_c}),$$

where $\gamma(r) = \psi_1^{-1}(C_\lambda \max\{1, e^{-d}\}\chi(r))$. This proves, that the system (3.1) is ISS. The uniformity is clear since the functions β and γ do not depend on the impulse time sequence. □

Remark 3.2.4. *Theorem 3.2.7 generalizes Theorem 1 from [38], where this result for the function h with $h(x) = e^{\mu-\lambda x}$ has been proved.*

The condition (3.26) is tight, i.e., if for some sequence T the function $N(\cdot,\cdot)$ does not satisfy the condition (3.26) for every function h from the statement of the Theorem 3.2.7, then one can construct a certain system (3.1) which will not be ISS w.r.t. the impulse time sequence T.

This one can see from the following simple example. Consider

$$\begin{cases} \dot{x} = -cx, & t \notin T, \\ x(t) = e^{-d}x^-(t), & t \in T \end{cases}$$

with initial condition $x(0) = x_0$. Its solution for arbitrary time sequence T is given by

$$x(t) = e^{-dN(t,t_0)-c(t-t_0)}x_0.$$

If T does not satisfy the gADT condition, then $e^{-dN(t,t_0)-c(t-t_0)}$ cannot be estimated from above by $\mathcal{L}$-function, and consequently, the system under consideration is not GAS.

We state also the local version of Theorem 3.2.7:

Theorem 3.2.8. *Let V be an exponential LISS-Lyapunov function for (3.1) with corresponding coefficients $c \in \mathbb{R}$, $d \neq 0$. For arbitrary function $h : \mathbb{R}_+ \to (0,\infty)$, s.t. $\exists g \in \mathcal{L}$: $h(x) \leq g(x)$ for all $x \in \mathbb{R}_+$ there exist a constant $\rho(h)$, such that the system (3.1) is uniformly LISS with this ρ over the class $\mathcal{S}[h]$ of impulse time-sequences, satisfying (3.26).*

Proof. The proof of this result is similar to the proof of Theorem 3.2.7. The only difference is that one has to choose ρ small enough to guarantee that the system evolves on the domain of definition of ISS-Lyapunov function V. □

3.2.2 Relations between different types of dwell-time conditions

For the system (3.1) which possesses an exponential ISS-Lyapunov function we have introduced two different types of dwell-time conditions: generalized ADT condtion (3.26) and fixed dwell-time condition (3.11). In this section we are going to find a relation between these conditions as well as between ADT condition from [38].

Taking in the gADT (3.26) $h(x) = e^{\mu-\lambda x}$ for some $\mu, \lambda > 0$, we obtain the ADT condition from [37], [38]:

$$-dN(t,s) - (c-\lambda)(t-s) \leq \mu, \quad \forall t \geq s \geq t_0. \tag{3.32}$$

The set of impulse time sequences, which satisfies this condition we denote $\mathcal{S}[\mu,\lambda] := \mathcal{S}[e^{\mu-\lambda\cdot}]$.

The gADT condition (3.26) provides for a system (3.1) in addition to jumps, allowed by ADT (3.32) the possibility to jump infinite number of times (on the time-interval of the infinite length), however, these jumps must be "not too close" to each other. Of course, the more extra jumps we allow, the larger are the gain γ and function β, which can be seen from the proof.

For a given sequence of impulse times denote by $N^*(t,s)$ the number of jumps within the time-interval $[s,t]$. The set of impulse time sequences, for which (3.32) holds with $N^*(t,s)$ instead of $N(t,s)$, denote by $\mathcal{S}^*[\mu,\lambda]$. We need the following lemma (see [18, Lemma 3.12.]):

Lemma 3.2.5. *Let $c, d \in \mathbb{R}$, $d \neq 0$ be given. Then $\mathcal{S}[\mu,\lambda] = \mathcal{S}^*[\mu,\lambda]$ for all $\mu,\lambda > 0$.*

Let us show the relation between ADT and FDT conditions.

If the system (3.1) possesses an exponential ISS Lyapunov function with rate coefficients $c, d \in \mathbb{R}$, $d < 0$ then Theorem 3.2.3 guarantees, that for all $\delta > 0$ and $\theta > 0$, such that

$$\int_a^{\alpha(a)} \frac{ds}{\varphi(s)} = \frac{-d}{c} \leq \theta - \delta \tag{3.33}$$

holds the system (3.1) is ISS for the time-sequences from the class S_θ.

Clearly, for all positive numbers θ, δ, satisfying (3.33) there exists $\lambda > 0$, such that the following condition holds with the same θ

$$\frac{1}{\theta} \leq \frac{c - \lambda}{-d}, \tag{3.34}$$

and vice versa.

For a given λ the smallest θ (which corresponds to the largest S_θ) is given by $\theta_* = \frac{-d}{c-\lambda}$.

Next lemma provides an equivalent representation of the set S_{θ_*}.

Lemma 3.2.6. *Let $c > 0$ and $d < 0$ be given. Then it holds $S_{\theta_*} = \mathcal{S}[-d, \lambda]$.*

Proof. Clearly, for arbitrary $T \in S_{\theta_*}$ it holds

$$N^*(t, s) \leq 1 + \frac{c - \lambda}{-d}(t - s), \quad \forall t \geq s \geq t_0,$$

or

$$-dN^*(t, s) - (c - \lambda)(t - s) \leq -d, \quad \forall t \geq s \geq t_0. \tag{3.35}$$

On the contrary, let (3.35) hold. Then for $t - s = k\theta_*$ we obtain $N^*(t, s) \leq k + 1$ and for $t - s \in ((k-1)\theta_*, k\theta_*)$ it follows $N^*(t, s) \leq k$ (since $N^*(t, s)$ is a natural number). This proves that $S_{\theta_*} = \mathcal{S}^*[-d, \lambda]$. From Lemma 3.2.5 the claim of the lemma follows. □

In other words, Theorem 3.11, applied to the exponential ISS Lyapunov functions, states that if the system (3.1) possesses an exponential ISS Lyapunov function V with rate coefficients c, d, then for all $\lambda > 0$ the system (3.1) is ISS for all sequences from the class $\mathcal{S}[-d, \lambda]$.

Remark 3.2.7. *Note that for $\mu \in (0, -d)$ the set of the impulse time sequences, which are allowed by ADT condition are $\mathcal{S}[\mu, \lambda] = \emptyset$. Indeed, by the ADT condition for small enough $t - s$ we obtain*

$$N(t, s) - \frac{c - \lambda}{-d}(t - s) \leq \frac{\mu}{-d} < 1,$$

i.e. $N(t, s) = 0$. Covering $[0, \infty)$ by small enough intervals, we obtain that $N(t_0, \infty) = 0$, and the impulses are not allowed.

The relations between different types of dwell-time conditions are summarized in Figure 3.1.

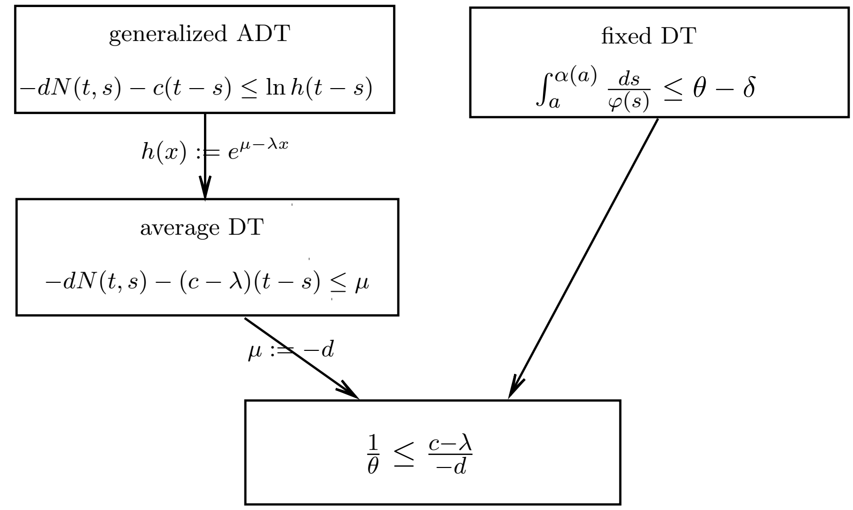

Figure 3.1: Relations between different types of dwell-time conditions

3.2.3 Constructions of exponential LISS Lyapunov functions via linearization

Consider an impulsive system (3.1) on a Hilbert space X with a scalar product $\langle \cdot, \cdot \rangle$, and let A be the infinitesimal generator of an analytic semigroup on X with the domain of definition $D(A)$. Let a function $f : X \times U \to X$ be defined on some open set Q, $(0,0) \in Q$.

In [19, Theorem 3] it was proved that for the system (3.1) with $T = \emptyset$ (when only continuous behavior is allowed) under certain conditions a LISS-Lyapunov function can be constructed.

In this section we prove a counterpart of [19, Theorem 3] for impulsive systems, which allows us to construct an exponential LISS-Lyapunov function for linearizable systems of the form (3.1).

Let us assume, that f and g can be decomposed in the following way

$$f(x,u) = Bx + Cu + f_1(x,u),$$

$$g(x,u) = Dx + Fu + g_1(x,u),$$

where $C, F \in L(U,X)$, $B, D \in L(X)$. Here we denote by $L(U,X)$ a space of linear bounded operators from U to X, $L(X) := L(X,X)$.

Let also for each constant $w > 0$ there exists $\rho > 0$, such that $\forall x : \|x\|_X \leq \rho$, $\forall u : \|u\|_U \leq \rho$ it holds

$$\|f_1(x,u)\|_X \leq w(\|x\|_X + \|u\|_U),$$

$$\|g_1(x,u)\|_X \leq w(\|x\|_X + \|u\|_U).$$

We recall that a self-adjoint operator P on the Hilbert space X is coercive, if $\exists \epsilon > 0$, such that

$$\langle Px, x \rangle \geq \epsilon \|x\|_X^2 \quad \forall x \in D(P).$$

The largest of such ϵ is called the lower bound of an operator P.

Consider a linear approximation of continuous dynamics of a system (3.1):

$$\dot{x} = Rx + Cu, \tag{3.36}$$

where $R = A + B$ is the infinitesimal generator of an analytic semigroup (which we denote by T), as a sum of the generator of an analytic semigroup A and bounded operator B.

We have the following theorem:

Theorem 3.2.9. *If the system (3.36) is ISS and if there exists a bounded coercive operator P, satisfying*

$$\langle Rx, Px\rangle + \langle Px, Rx\rangle = -\|x\|_X^2, \quad \forall x \in D(A),$$

then a LISS-Lyapunov function of (3.1) can be constructed in the form

$$V(x) = \langle Px, x\rangle \,. \tag{3.37}$$

Proof. Since P is bounded and coercive, for some $\epsilon > 0$ it holds

$$\epsilon\|x\|_X^2 \le \langle Px, x\rangle \le \|P\|\|x\|_X^2, \quad \forall x \in X,$$

and the estimate (3.5) is verified.

Define $\chi \in \mathcal{K}_\infty$ by $\chi(r) = \sqrt{r}$, $r \ge 0$. In [19, Theorem 3] it was proved, that for small enough $\rho_1 > 0$, $\forall x : \|x\|_X \le \rho_1$, $\forall u : \|u\|_U \le \rho_1$ it holds

$$\|x\|_X \ge \chi(\|u\|_U) \quad \Rightarrow \quad \dot{V}(x) \le -r\|x\|_X^2 \le -\frac{r}{\|P\|}V(x)$$

for some $r > 0$.

Now we estimate $V(g(x,u))$:

$$\begin{aligned} V(g(x,u)) &= \langle P(Dx + Fu + g_1(x,u)), Dx + Fu + g_1(x,u)\rangle \\ &\le \|P\| \left(\|D\|^2\|x\|_X^2 + \|F\|^2\|u\|_U^2 + 2\|D\|\|F\|\|x\|\|u\|_U \right. \\ &\quad \left. +2(\|D\|\|x\|_X + \|F\|\|u\|_U)w(\|x\|_X + \|u\|_U) + w^2(\|x\|_X + \|u\|_U)^2\right). \end{aligned}$$

One can verify, that $\exists r_2, \rho_2 > 0$, such that $\forall x : \|x\|_X \le \rho_2$, $\forall u : \|u\|_U \le \rho_2$

$$\|x\|_X \ge \chi(\|u\|_U) \quad \Rightarrow \quad V(g(x,u)) \le r_2\|x\|_X^2 \le \frac{r_2}{\varepsilon}V(x).$$

Taking $\rho := \min\{\rho_1, \rho_2\}$, we obtain, that V is an exponential LISS Lyapunov function for a system (3.1). □

3.3 ISS of interconnected impulsive systems

In the previous subsection we have developed a linearization method for construction of LISS-Lyapunov functions for impulsive systems (3.1). Now we are going to provide a method for construction of ISS-Lyapunov functions for interconnected systems which is based on the knowledge of ISS-Lyapunov functions for subsystems.

Let a Banach space X_i be the state space of the i-th subsystem, $i = 1, \dots, n$, and U and $U_c = PC(\mathbb{R}_+, U)$ be the space of input values and of input functions respectively.

Define $X = X_1 \times \ldots \times X_n$, which is a Banach space, which we endow with the norm $\|\cdot\|_X := \|\cdot\|_{X_1} + \ldots + \|\cdot\|_{X_n}$.

The input space for the i-th subsystem is $\tilde{X}_i := X_1 \times \ldots \times X_{i-1} \times X_{i+1} \times \ldots \times X_n \times U$. The norm in $\tilde{X}_i$ is given by

$$\|\cdot\|_{\tilde{X}_i} := \|\cdot\|_{X_1} + \ldots + \|\cdot\|_{X_{i-1}} + \|\cdot\|_{X_{i+1}} + \ldots + \|\cdot\|_{X_n} + \|\cdot\|_U.$$

The elements of $\tilde{X}_i$ we denote by $\tilde{x}_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, \xi) \in \tilde{X}_i$.

Also let $T = \{t_1, \ldots, t_k, \ldots\}$ be a sequence of impulse times for all subsystems (we assume, that all subsystems jump at the same time).

Consider the system consisting of n interconnected impulsive subsystems:

$$\left\{ \begin{array}{l} \dot{x}_i(t) = A_i x_i(t) + f_i(x_1(t), \ldots, x_n(t), u(t)), \quad t \notin T, \\ x_i(t) = g_i(x_1^-(t), \ldots, x_n^-(t), u^-(t)), \quad t \in T, \\ i = \overline{1, n} \end{array} \right. \tag{3.38}$$

Here A_i is the generator of a C_0 - semigroup on X_i, $f_i, g_i : X \times U \to X_i$, and we assume that the solution of all subsystems exists, is unique and forward-complete.

For $x_i \in X_i$, $i = 1, \ldots, n$ define $x = (x_1, \ldots, x_n)^T$, $f(x, u) = (f_1(x, u), \ldots, f_n(x, u))^T$, $g(x, u) = (g_1(x, u), \ldots, g_n(x, u))^T$.

By A we denote the diagonal operator $A := diag(A_1, \ldots, A_n)$, i.e.:

$$A = \begin{pmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & A_n \end{pmatrix}$$

Domain of definition of A is given by $D(A) = D(A_1) \times \ldots \times D(A_n)$. Clearly, A is the generator of C_0-semigroup on X.

We rewrite the system (3.38) in the vector form:

$$\left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + f(x(t), u(t)), \quad t \notin T \\ x(t) = g(x^-(t), u^-(t)), \quad t \in T. \end{array} \right. \tag{3.39}$$

According to the Proposition 3.2.1 for the i-th subsystem of a system (3.38) the definition of an ISS-LF can be written as follows. A continuous function $V_i : X_i \to \mathbb{R}_+$ is an ISS-Lyapunov function for i-th subsystem of (3.38), if three properties hold:

1. There exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$, such that:

$$\psi_{i1}(\|x_i\|_{X_i}) \le V_i(x_i) \le \psi_{i2}(\|x_i\|_{X_i}), \quad \forall x_i \in X_i$$

2. There exist $\chi_{ij}, \chi_i \in \mathcal{K}$, $j = 1, \ldots, n$, $\chi_{ii} := 0$ and $\varphi_i \in \mathcal{P}$, so that for all $x_i \in X_i$, for all $\tilde{x}_i \in \tilde{X}_i$ and for all $v \in PC(\mathbb{R}_+, \tilde{X}_i)$ with $v(0) = \tilde{x}_i$ from

$$V_i(x_i) \ge \max\{\max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(\|\xi\|_U)\}, \tag{3.40}$$

it follows
$$\dot{V}_i(x_i(t)) \leq -\varphi_i\left(V_i(x_i(t))\right), \tag{3.41}$$
where
$$\dot{V}_i(x_i) = \overline{\lim_{t\to+0}} \frac{1}{t}(V_i(\phi_{i,c}(t,0,x_i,v))) - V_i(x_i)),$$
and $\phi_{i,c} : \mathbb{R}_+ \times \mathbb{R}_+ \times X_i \times PC(\mathbb{R}_+, \tilde{X}_i) \to X_i$ is the solution (transition map) of the i-th subsystem of (3.38) for the case if $T = \emptyset$.

3. There exists $\alpha_i \in \mathcal{P}$, such that for gains defined above and for all $x \in X$ and for all $\xi \in U$ it holds
$$V_i(g_i(x,\xi)) \leq \max\{\alpha_i(V_i(x_i)), \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(\|\xi\|_U)\}. \tag{3.42}$$

If $\varphi_i(y) = c_i y$ and $\alpha_i(y) = e^{-d_i} y$ for all $y \in \mathbb{R}_+$, then V_i is called an exponential ISS-Lyapunov function for the i-th subsystem of (3.38) with rate coefficients $c_i, d_i \in \mathbb{R}$.

Now we prove a small-gain theorem for nonlinear impulsive systems. The technique for treatment of the discrete dynamics is adopted from [64] and [17].

Theorem 3.3.1. *Consider the system (3.38). Let V_i be the ISS-Lyapunov function for i-th subsystem of (3.38) with corresponding gains χ_{ij}. If the corresponding operator Γ defined by (1.20) satisfies the small-gain condition (1.24), then an ISS-Lyapunov function V for the whole system can be constructed as in (1.22) where $\sigma = (\sigma_1, \ldots, \sigma_n)^T$ is an Ω-path. The Lyapunov gain of the whole system can be chosen as in (1.23).*

Proof. The part of the proof related to continuous behavior is identical to the proof of [19, Theorem 5]. There it was proved, that $\forall x \in X,\ \xi \in U$ from $V(x) \geq \chi(\|\xi\|_U)$ it follows
$$\frac{d}{dt}V(x) \leq -\varphi(V(x)),$$
for
$$\varphi(r) := \min_{i=1}^{n} \left\{ \left(\sigma_i^{-1}\right)'(\sigma_i(r))\varphi_i(\sigma_i(r)) \right\}. \tag{3.43}$$
Function φ is positive definite, because $\sigma_i^{-1} \in \mathcal{K}_\infty$ and all φ_i are positive definite functions.

Thus, implication (3.9) is verified and it remains to check (3.10) (the estimation of ISS-Lyapunov function on the jumps). With the help of inequality (3.42) we make for all $x \in X$ and $\xi \in U$ the following estimates
$$\begin{aligned}
V(g(x,\xi)) &= \max_i\{\sigma_i^{-1}(V_i(g_i(x,\xi)))\} \\
&\leq \max_i\{\sigma_i^{-1}\left(\max\{\alpha_i(V_i(x_i)), \max_{j=1}^{n} \chi_{ij}(V_j(x_j)), \chi_i(\|\xi\|_U)\}\right)\} \\
&= \max\{\max_i\{\sigma_i^{-1}\circ\alpha_i(V_i(x_i))\}, \max_{i,j\neq i}\{\sigma_i^{-1}\circ\chi_{ij}(V_j(x_j))\}, \max_i\{\sigma_i^{-1}\circ\chi_i(\|\xi\|_U)\}\} \\
&= \max\{\max_i\{\sigma_i^{-1}\circ\alpha_i\circ\sigma_i\circ\sigma_i^{-1}(V_i(x_i))\}, \max_{i,j\neq i}\{\sigma_i^{-1}\circ\chi_{ij}\circ\sigma_j\circ\sigma_j^{-1}(V_j(x_j))\}, \\
&\qquad \max_i\{\sigma_i^{-1}\circ\chi_i(\|\xi\|_U)\}\}.
\end{aligned}$$

Define $\tilde{\alpha} := \max_i\{\sigma_i^{-1} \circ \alpha_i \circ \sigma_i\}$. Since $\alpha_i \in \mathcal{P}$, then $\tilde{\alpha} \in \mathcal{P}$. Pick any $\alpha^* \in \mathcal{K}$: $\alpha^*(r) \geq \tilde{\alpha}(r)$, $r \geq 0$. Then the following estimate holds

$$\max_i\{\sigma_i^{-1} \circ \alpha_i \circ \sigma_i \circ \sigma_i^{-1}(V_i(x_i))\} \leq \alpha^*(\max_i\{\sigma_i^{-1}(V_i(x_i))\}) = \alpha^*(V(x)).$$

Define also $\eta := \max_{i,j\neq i}\{\sigma_i^{-1} \circ \chi_{ij} \circ \sigma_j\}$ and note that according to (1.21)

$$\eta = \max_{i,j\neq i}\{\sigma_i^{-1} \circ \chi_{ij} \circ \sigma_j\} < \max_{i,j\neq i}\{\sigma_i^{-1} \circ \sigma_i\} = id.$$

We continue estimates of $V(g(x,\xi))$:

$$V(g(x,\xi)) \leq \max\{\alpha^*(V(x)), \eta(V(x)), \chi(\|\xi\|_U)\} = \max\{\alpha(V(x)), \chi(\|\xi\|_U)\},$$

where

$$\alpha := \max\{\alpha^*, \eta\}. \tag{3.44}$$

According to Proposition 3.2.1 the function V is an ISS-Lyapunov function of the system (3.1). □

Remark 3.3.1. *Our small-gain theorem has been formulated for Lyapunov functions in the form used in Proposition 3.2.1. According to the Proposition 3.2.1 this formulation can be transformed to the standard formulation, and from the proof it is clear, that the functions α and φ remain the same after the transformation. Next in order to check, whether the system (3.39) is ISS, one should use Theorem 3.2.3.*

3.3.1 Small-gain theorem for exponential ISS-Lyapunov functions

If an exponential ISS-Lyapunov function for a system (3.1) is given, then Theorem 3.2.7 provides us with the tight estimations of the set of impulse time sequences, w.r.t. which the system (3.1) is ISS and hence the exponential ISS-Lyapunov functions are "more valuable", than the general ones.

We may hope, that if ISS-Lyapunov functions for all subsystems of (3.38) are *exponential*, then the expression (1.22) at least for certain type of gains provides the *exponential* ISS-Lyapunov function for the whole system. In this subsection we are going to prove the small-gain theorem of this type.

Firstly note the following fact

Proposition 3.3.2. *Let operator Γ satisfies small-gain condition. Then for arbitrary $a \in int(\mathbb{R}^n_+)$ the function*

$$\sigma(t) = Q(at), \forall t \geq 0 \tag{3.45}$$

satisfies

$$\Gamma(\sigma(r)) \leq \sigma(r), \ \forall r > 0. \tag{3.46}$$

Here $Q : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is defined by

$$Q(x) := MAX\{x, \Gamma(x), \Gamma^2(x), \dots, \Gamma^{n-1}(x)\},$$

with $\Gamma^n(x) = \Gamma \circ \Gamma^{n-1}(x)$, for all $n \geq 2$. The function MAX for all $h_i \in \mathbb{R}^n$, $i = 1, \dots, m$ is defined by

$$z = MAX\{h_1, \dots, h_m\} \in \mathbb{R}^n, \quad z_i := \max\{h_{1i}, \dots, h_{mi}\}.$$

Proof. The result follows from [51, Proposition 2.7 and Remark 2.8] □

Define the following class of functions

$$P := \{f : \mathbb{R}_+ \to \mathbb{R}_+ : \exists a \geq 0,\ b > 0 :\ f(s) = as^b\ \forall s \in \mathbb{R}_+\}.$$

Theorem 3.3.3. *Let V_i be an eISS Lyapunov function for the i-th subsystem of (3.38) with corresponding gains χ_{ij}, $i = 1, \dots, n$. Let also $\chi_{ij} \in P$ and let the small-gain condition (1.24) holds. Then the function $V : X \to \mathbb{R}_+$, defined by (1.22), where the σ is given by (3.45), is an eISS Lyapunov function for the whole system (3.39).*

Proof. Take the Ω-path σ as in (3.45). It satisfies all the conditions of an Ω-path, see Definition 1.5.2, but with $\leq$ instead of $<$ in (1.21). However, the proof of Theorem 3.3.1 is true also for such "quasi"-Ω-path.

According to Theorem 3.3.1 function V, defined by (1.22) is an ISS Lyapunov function. We have only to prove, that it is an exponential one.

For all $f, g \in P$ it follows $f \circ g \in P$, thus for all i it holds that $\sigma_i(t) = \max\{f_1^i(t), \dots, f_{r_i}^i(t)\}$, where all $f_k^i \in P$ and r_i is finite.

Thus, for each i there exists a partition of $\mathbb{R}_+$ into sets S_j^i, $j = 1, \dots, k_i$ (i.e. $\cup_{j=1}^{k_i} S_j^i = \mathbb{R}_+$ and $S_j^i \cap S_s^i = \emptyset$, if $j \neq s$), such that $\sigma_i^{-1}(t) = a_{ij} t^{p_{ij}}$ for some $p_{ij} > 0$ and all $t \in S_j^i$. This partition is always finite, because all $f_j^i \in P$, and two such functions intersect in no more than one point, distinct from zero.

Thus, for all $i \in \{1, \dots, n\}$ define a set

$$M_i = \left\{x \in X : \sigma_i^{-1}(V_i(x_i)) > \sigma_j^{-1}(V_j(x_j)),\ \forall j = 1, \dots, n,\ j \neq i\right\}.$$

Let $x \in M_i$ and $V_i(x_i) \in S_j^i$. Then the condition (1.24) implies (see the proof of [19, Theorem 5])

$$\frac{d}{dt}V(x) \;=\; \frac{d}{dt}(\sigma_i^{-1}(V_i(x_i))) = \frac{d}{ds}(a_{ij}s^{p_{ij}})(V_i(x_i))\frac{d}{dt}(V_i(x_i))$$

Now using (3.41) and (3.8) we have

$$\frac{d}{dt}V(x) \;\leq\; -c_i a_{ij} p_{ij} (V_i(x_i))^{p_{ij}} \leq -cV(x),$$

where $c = \min_{i,j}\{c_i p_{ij}\}$.

We have to prove, that the function α from (3.44) can be estimated from above by linear function. We choose $\alpha^* := \tilde{\alpha} = \max_i\{\sigma_i^{-1} \circ \alpha_i \circ \sigma_i\}$.

For any fixed $t \geq 0$ it holds that $\sigma_i^{-1} \circ \alpha_i \circ \sigma_i(t) = c_i = const$ since α_i are linear and σ_i^{-1} are piecewise power functions. This implies that for some constant k it holds that $\alpha^*(t) \leq kt$ for all $t \geq 0$.

Since function η from the proof of Theorem 3.3.1 satisfies $\eta < id$, it is clear that one can take $\alpha := \max\{k, 1\}\,\mathrm{Id}$, and so and the theorem is proved. □

Remark 3.3.2. *The obtained exponential ISS-Lyapunov function can be transformed to the implication form with the help of Proposition 3.2.2. Then Theorem 3.2.7 can be used in order to verify ISS of the system (3.39).*

Let us demonstrate how one can analyze stability of interconnected impulsive systems on a simple example. Let $T = \{t_k\}$ be a sequence of impulse times. Consider two interconnected nonlinear impulsive systems

$$\begin{aligned} \dot{x}_1(t) = & -x_1(t) + x_2^2(t),\ t \notin T, \\ x_1(t) = & e^{-1}x_1^-(t),\ t \in T \end{aligned}$$

and

$$\begin{aligned} \dot{x}_2(t) = & -x_2(t) + 3\sqrt{|x_1(t)|},\ t \notin T, \\ x_2(t) = & e^{-1}x_2^-(t),\ t \in T. \end{aligned}$$

Both subsystems are uniformly ISS (even strongly uniformly ISS, see [38]) for all impulse time sequences, since continuous and discrete dynamics stabilize the subsystems and one can easily construct exponential ISS Lyapunov functions (with certain Lyapunov gains) with positive rate coefficients for both subsystems. However, arbitrary Lyapunov gains, corresponding to such ISS-Lyapunov functions will not satisfy small-gain condition, since the continuous dynamics of the interconnected system is not stable. Therefore in order to find the classes of impulse time sequences for which the interconnected system is GAS, we have to seek for ISS-Lyapunov functions (and corresponding Lyapunov gains) with one negative rate coefficient.

Take the following exponential ISS-Lyapunov functions and Lyapunov gains for subsystems

$$\begin{aligned} V_1(x_1) = |x_1|, \quad \gamma_{12}(r) = \tfrac{1}{a}r^2, \\ V_2(x_2) = |x_2|, \quad \gamma_{21}(r) = \tfrac{1}{b}\sqrt{r}, \end{aligned}$$

where $a, b > 0$. We have the following implications

$$\begin{aligned} |x_1| \geq \gamma_{12}(|x_2|) \Rightarrow \dot{V}_1(x_1) \leq (a-1)V_1(x_1), \\ |x_2| \geq \gamma_{21}(|x_1|) \Rightarrow \dot{V}_2(x_2) \leq (3b-1)V_2(x_2). \end{aligned}$$

The small-gain condition

$$\gamma_{12} \circ \gamma_{21}(r) = \frac{1}{ab^2}r < r, \quad \forall r > 0 \tag{3.47}$$

is satisfied, if it holds

$$h(a,b) := ab^2 > 1. \tag{3.48}$$

Take an arbitrary constant s such that $\frac{1}{b} < \frac{1}{s} < \sqrt{a}$. Then Ω-path can be chosen as

$$\sigma_1(r) = r, \quad \sigma_2(r) = \frac{1}{s}\sqrt{r},\ \forall r \geq 0.$$

Then

$$\sigma_2^{-1}(r) = s^2r^2,\ \forall r \geq 0.$$

In this case an ISS-Lyapunov function for the interconnection, constructed by small-gain design, is given by

$$V(x) = \max\{|x_1|, s^2|x_2|^2\}, \quad \text{where } \frac{1}{b} < \frac{1}{s} < \sqrt{a} \text{ and } x = (x_1, x_2)^T$$

and we have the estimate

$$V(g(x)) = V(e^{-1}x) \leq e^{-1}V(x). \tag{3.49}$$

Thus, we can take $d = -1$ for the interconnection. The estimates of the continuous dynamics for V are as follows: For $|x_1| \geq s^2x_2^2 > \frac{1}{a}x_2^2 = \gamma_{12}(|x_2|)$ it holds

$$\frac{d}{dt}V(x) = \frac{d}{dt}|x_1| \leq (a-1)|x_1| = (a-1)V(x),$$

and $|x_1| \leq s^2x_2^2 < \gamma_{21}^{-1}(|x_2|)$ implies

$$\frac{d}{dt}V(x) = \frac{d}{dt}\left(s^2x_2^2\right) = \frac{d}{dt}\left(s^2V_2(x_2)^2\right) \leq 2(3b-1)s^2|x_2|^2 = 2(3b-1)V(x).$$

Overall, for all x we have:

$$\frac{d}{dt}V(x) \leq \max\{(a-1), 2(3b-1)\}V(x). \tag{3.50}$$

Function h, defined by (3.48), is increasing w.r.t. both arguments (since $a, b > 0$), hence in order to minimize $c := \max\{(a-1), 2(3b-1)\}$, we have to choose $(a-1) = 2(3b-1)$. Then, from (3.47) we obtain the inequality

$$(1 + 2(3b-1))b^2 > 1.$$

Thus, the best choice for b is $b \approx 0.612$ and V is an exponential ISS-Lyapunov function for an interconnection with rate coefficients with $d = -1$ and $c = 2 \cdot (3 \cdot 0.612 - 1) = 1.672$.

The ISS-Lyapunov function for an interconnection is constructed, and one can apply Theorem 3.2.7 in order to obtain the classes of impulse time sequences for which the interconnection is GAS.

3.3.2 Relation between small-gain and dwell-time conditions

So far we have seen how small-gain and dwell-time conditions can be used to verify stability of interconnected system. The small gain condition (1.24) requires that the gains of subsystems must be small enough so that their cycle compositions are less then the identity, namely

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ \ldots \circ \gamma_{k_{p-1}k_p}(s) < s \tag{3.51}$$

for all $(k_1, ..., k_p) \in \{1, ..., n\}^p$, where $k_1 = k_p$ and for all $s > 0$. The condition in cyclic form (3.51) is equivalent to the condition (1.24), see [22], and is widely used in the literature [46].

In particular a large gain of one subsystem can be compensated by a small gain of another one to satisfy (1.24). A choice of gains depends on the choice of an ISS-Lyapunov function in its turn.

The dwell-time condition is imposed on α and φ from (3.6) or the rate coefficients c and d in case of exponential ISS-Lyapunov functions. It requires that the jumps happen with a certain frequency.

The inequalities (3.6) show how fast the value of $V(x(\cdot))$ changes outside of the region $\{x : V(x) < \gamma(|u|)\}$ with the time t. In the previous example we have seen that the larger is the gain function, the larger the rate coefficients c and d can be chosen and hence the more impulse time sequences satisfy the dwell-time condition (3.26). However in case of interconnected systems large gains are not desired, because of the small-gain condition. Hence there is a trade-off between the size of the gains (which we like to have as small as possible) and the decay rate of $V(x(\cdot))$. This leads to interdependence in the choice of gains and rate coefficients in the stability analysis of interconnected systems. In general case this dependence is rather involved. To shed light on this issue we restrict ourselves in this section to the case of systems possessing exponential ISS-Lyapunov functions with linear gains.

Consider an interconnected impulsive system of the form (3.38), and assume that for each i there is a positive definite and radially unbounded continuous function V_i for the i-th subsystem, such that for almost all $x_i \in X_i$ and all $u \in U$ the following dissipative inequalities hold:

$$\dot{V}_i(x_i) \leq -\tilde{c}V_i(x_i) + \max_{j\neq i}\{\chi_{ij}V_j(x_j), \chi_i(\|u\|_U)\}, \tag{3.52}$$

$$V_i(g_i(x,u)) \leq \max\{e^{-d}V_i(x_i), \chi_i(\|u\|_U)\}, \tag{3.53}$$

where $\chi_{ij} \in \mathbb{R}_+$, $\tilde{c}, d \in \mathbb{R}$, and $\chi_i \in \mathcal{K}$ can be nonlinear functions. We have assumed here for simplicity, that the subsystem affect each other during continuous flow only. At the impulse times the jumps of subsystems are independent on each other.

Let us illustrate the trade-off mentioned above. By the inequalities (3.52) and (3.53) function V_i is an ISS-Lyapunov function in dissipative formulation for the i-th subsystem, see [38]. This form provides us with a freedom to choose the gains during transformation of equation (3.52) from the dissipation form into the implication form which we need in order to apply Theorem 3.3.3.

Let $k \in (0, \infty)$ be the scaling coefficient that allows to adjust the gains to satisfy the small-gain condition. We define

$$\gamma_{ij} := \frac{1}{k}\chi_{ij}, \quad \gamma_i := \frac{1}{k}\chi_i, \quad \Gamma_k := (\gamma_{ij})_{i,j=1,\ldots,n}. \tag{3.54}$$

If

$$\max_{j\neq i}\{\gamma_{ij}V_j(x_j), \gamma_i(\|u\|_U)\} = \frac{1}{k}\max_{j\neq i}\{\chi_{ij}V_j(x_j), \chi_i(\|u\|_U)\} \leq V_i(x_i)$$

holds, then it follows from (3.52) that

$$\dot{V}_i(x_i) \leq (-\tilde{c} + k)\, V_i(x_i) := -c_k V_i(x_i), \tag{3.55}$$

holds for almost all x_i, with $c_k := \tilde{c} - k$.

This shows that V_i is an exponential ISS-Lyapunov function of the i-th subsystem in the sense of Definition 3.2.1 with the rate coefficients c_k and d and gains γ_{ij} for which our small-gain theorem can be applied.

Define the linear operator $\Gamma_k : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by $(\Gamma_k(s))_i = \max_{j\neq i}\{\gamma_{ij}s_j\}$. For this operator the small-gain condition (1.24) is equivalent (see [21]) to

$$\rho(\Gamma_k) < 1 \quad \Leftrightarrow \quad \rho := \rho(\chi_{ij}) < k,$$

where $\rho(\cdot)$ denotes the spectral radius of a matrix.

In this case according to Theorem 3.3.3 an exponential Lyapunov function can be constructed, moreover, an Ω-path can be chosen as a vector of linear functions and the rate coefficients of the ISS-Lyapunov function for a whole system will be c_k and d.

If $k \in (\rho, \tilde{c})$ and $d > 0$, then both rate coefficients of the exponential ISS-Lyapunov functions V_i are positive and hence the system under consideration is ISS for all impulsive time sequences.

Let us consider the case when $d < 0$ and $k \in (\rho, \tilde{c})$, when the rate coefficients are of different signs, and consequently one has to use dwell-time conditions in order to find the classes of impulse time sequences w.r.t. which the system is ISS.

The dwell-time condition (3.32) for $d < 0$ reads in this situation as

$$N(t,s) \leq \frac{1}{-d}(\mu + (c-\lambda)(t-s)) = \mu' + (\frac{c}{-d} - \lambda')(t-s), \ \forall t \geq s \geq t_0, \tag{3.56}$$

where $\mu' = \frac{\mu}{-d}$ and $\lambda' = \frac{\lambda}{-d}$.

For given c, d, λ, μ denote the set of impulse time sequences, which satisfies (3.56) by $\mathcal{S}_{c,d}[\mu, \lambda]$.

Take $c_1, c_2 > 0$ and $d_1, d_2 < 0$ such that $\frac{c_1}{-d_1} > \frac{c_2}{-d_2}$. Then $\forall \lambda_2, \mu_2 > 0 \ \exists \lambda_1, \mu_1 > 0$: $\mathcal{S}_{c_2,d_2}[\mu_2, \lambda_2] \subset \mathcal{S}_{c_1,d_1}[\mu_1, \lambda_1]$. Thus, the set $\mathcal{S}_{c,d}[\mu, \lambda]$ crucially depends on the value of $\frac{c}{-d}$. We will call $\frac{c}{-d}$ the frequency of impulse times.

For the gains as in (3.54) the frequency of impulse times is equal to

$$\omega(k) := \frac{c_k}{-d} = \frac{\tilde{c} - k}{-d}, \tag{3.57}$$

and the possible values of k are contained in $(\rho, \tilde{c})$.

It is clear that ω is decreasing w.r.t. k on the interval $(\rho, \tilde{c})$, as well as the gains Γ_k defined by (3.54). Moreover, $\lim_{k\to\rho} \rho(\Gamma_k) = 1$ and $\lim_{k\to\tilde{c}} \omega(k) = 0$.

In particular, it holds

$$\rho \to \tilde{c} \Leftrightarrow \min_{k\in(\rho,\tilde{c})} \rho(\Gamma_k) = 1 \Leftrightarrow \max_{k\in(\rho,\tilde{c})} \omega(k) = 0.$$

We summarize our investigations in the following proposition:

Proposition 3.3.4. *Let V_i be an ISS-Lyapunov function for the i-th subsystem, $i = 1, \dots, n$ and the inequalities (3.52) and (3.53) hold with $d < 0$ and $\tilde{c} > \rho$ and let the gains are defined as in (3.54). Then the possible values of k are contained in $(\rho, \tilde{c})$, and on this interval the smaller are the gains, the smaller is the frequency of impulses allowed by dwell-time condition. Moreover, if for all admissible gains Γ_k it holds $\rho(\Gamma_k) \to 1$, then the frequency of impulse times $\omega(k) \to 0$ for all $k \in (\rho, \tilde{c})$.*

3.4 Concluding remarks and open questions

Some of the open questions for continuous systems, formulated in Section 2.7, have their direct counterparts in impulsive systems theory. In particular, it is important to develop an ISS theory for less regular input functions (e.g. L_p-functions with respect to time).

Another possible direction for a future work is a transfer of results presented in this chapter to time-delay impulsive systems (in Lyapunov-Krasovskii and Lyapunov-Razumikhin methodology), see [18] for preliminary results in this direction. But there arise also essentially new problems:

Interconnections of systems, each of which possesses its own sequence of impulse times. Consider a system, corresponding to (3.38) (with the same assumptions on the state and input spaces etc.) but so that each subsystem possesses its own sequence of impulse times T_i

$$\left\{ \begin{array}{ll} \dot{x}_i(t) = A_i x_i(t) + f_i(x_1(t), \ldots, x_n(t), u(t)), & t \notin T_i, \\ x_i(t) = g_i(x_1^-(t), \ldots, x_n^-(t), u^-(t)), & t \in T_i, \\ i = \overline{1,n} & \end{array} \right. \tag{3.58}$$

In contrast to the system (3.38), it is not possible to rewrite the system (3.58) in the form (3.1). One can construct the aggregate sequence of impulse times for the whole system as $T := \cup_{i=1}^n T_i$, but the function g for the whole system will still depend on the time-sequences T_i, $i = 1, \ldots, n$. Consequently, the theory, developed in this chapter as well as (at least to a knowledge of the author) in the other literature on ISS of impulsive systems cannot be applied to such systems. Development of such theory is an interesting topic for future research.

Dwell-time conditions for hybrid systems. In the last years the interesting results in ISS of finite-dimensional hybrid systems have been established [8], [64] in the framework, developed in [31]. But the tools developed there (ISS-Lyapunov functions for hybrid systems) make possible only a treatment of systems for which both continuous and discrete dynamics stabilizes the system.

Dwell-time conditions for the case when either continuous or discrete dynamics destabilizes the system have not been developed within ISS theory of hybrid systems.

However, already in the book [71] it was proved (roughly speaking) that a hybrid system without external inputs which possesses a Lyapunov function is GAS for impulse time sequences (which depend on the state of the system) if nonlinear FDT condition holds and certain other conditions are fulfilled.

The question is whether this result can be generalized for the case of ISS of hybrid systems with external inputs. Note, that the framework for study of hybrid systems, used in [71] is different from that from [31], [64].

Chapter 4

Conclusion

In this work we have developed an ISS theory for infinite-dimensional systems. The general framework, which we use, encompasses the ODE systems, systems with time-delays as well as many classes of evolution PDEs and is consistent with the current definitions of ISS for ODEs and time-delay systems, see Section 1.4.

Our guideline was a development of Lyapunov-type sufficient conditions for ISS of the infinite-dimensional systems and elaboration of methods for construction of ISS-Lyapunov functions.

In Section 2.2 we have proved, that existence of an ISS-Lyapunov function implies ISS of general control systems and we have shown, how our definition of ISS-Lyapunov function reduces to the standard one in the case of finite-dimensional systems. For the systems, governed by differential equations in Banach spaces we established in Section 2.4 a small-gain theorem, which provides us with a design of an ISS-Lyapunov function for an interconnection of ISS subsystems, provided the ISS-Lyapunov functions for the subsystems are known and small-gain condition holds. The tightness of the small-gain condition has been investigated as well, see Section 1.5.4. For constructions of local ISS-Lyapunov functions the linearization method has been proposed in Section 2.3, which is a good alternative to Lyapunov methods provided the system is linearizable.

For impulsive systems we developed Lyapunov-type stability conditions for impulsive systems for the case when the ISS-Lyapunov function is of general type (nonexponential) as well as when the ISS-Lyapunov function is exponential. To provide the classes of impulse time sequences, for which the system is ISS, we have used nonlinear fixed dwell-time condition from [71] and generalized average dwell-time (gADT) condition, which contains ADT condition from [38] as a special case. The small-gain theorems as well as linearization method have been generalized to the case of impulsive systems in Sections 3.2.3 and 3.3.

Altogether these results provide us with a firm basis for investigation of input-to-state stability of general control systems. However, it is only a first step in construction of a whole ISS theory of infinite-dimensional systems. If we look on the Figure 1 in Introduction we see, that two big problems remain open.

A broad field, full of nontrivial problems, are the characterizations of ISS for infinite-dimensional systems. Sontag and Wang solved this problem in papers [77] and [78] for finite-dimensional systems (with $X = \mathbb{R}^n$ and $U_c = L_\infty(\mathbb{R}_+, \mathbb{R}^m)$). For the infinite-dimensional case the complexity of the problem increases significantly not only because the state space X becomes an arbitrary Banach space, but also because the regularity of the inputs may play an important role, and the type of the system itself may become important. It is possible, that some characterizations can be proved for the general control systems as in Definition 1.2.1, some - only for the systems governed by differential equations in Banach spaces, or more special classes of systems. The converse Lyapunov theorem is another desired fundamental theoretical result, which is beyond the scope of this thesis.

These two questions do not deplete the vast field of problems, opened for a spirit of research. I recall here only some possible directions for a future investigation.

Most part of results in this thesis as well as in another papers on ISS theory of infinite-dimensional systems [62], [68], [51] have been proved for either piecewise-continuous or continuous inputs. On the one side it is quite restrictive for many applications, in particular, for PDEs, on the other it doesn't give us a full right to say that the current infinite-dimensional theory generalizes the corresponding theory for ODEs and time-delay systems, since in these theories usually the class of essentially bounded Lebesgue measurable inputs is used.

Another important issue is how general can be the systems for which the small-gain theorems, which provide the construction of a single Lyapunov function for a system can be proved. We have proved them for the systems of differential equations in Banach spaces. However, in [51] the general vector Lyapunov small-gain theorem has been proved for substantially more general class of the systems, however, without construction of a single ISS-Lyapunov function for the system. Is it possible to prove "constructive" small-gain theorems for such general control systems? Can one generalize in the same way our results on impulsive systems (when either continuous or discontinuous behavior is destabilizing)?

The theory of interconnected impulsive systems has been developed in the Chapter 3 under assumption that the impulse time sequences for all subsystems are the same. Under this supposition we could generalize the small-gain theorems for the impulsive systems. Dropping this assumption out, we obtain a more general class of systems, than the impulsive systems, considered in the current literature on ISS of impulsive systems. How will look the theory for this new class of impulsive systems?

Many other problems have been mentioned in the last sections of preceding chapters. I hope the other researchers will find this field fruitful and promising, and that this work has contributed to our understanding of the ISS theory.

Chapter 5

Appendix

5.1 Semigroups of bounded operators

In this section we introduce basic definitions and state known results from semigroup theory, needed in our exposition.

Main definitions

Let X be a Banach space, and $L(X)$ be the space of bounded linear operators, defined on X.

Definition 5.1.1 (Strongly continuous semigroup)**.** *A family of operators $\{T(t),\ t \geq 0\} \subset L(X)$, is called a strongly continuous semigroup (for short C_0-semigroup), if it holds that*

1. *$T(0) = I$.*

2. *$T(t+s) = T(t)T(s)$, $\forall t, s \geq 0$.*

3. *For all $x \in X$ function $t \mapsto T(t)x$ belongs to $C([0,\infty), X)$.*

Take $U_c := \{0\}$, i.e. the input space consists of only one element, and define $\phi(t,x,0) := T(t)x$, $t \geq 0$. It is easy to see, that $\Sigma := (X, \{0\}, \phi)$ is a control system according to Definition 1.2.1.

We will deal with special classes of C_0-semigroups:

Definition 5.1.2. *A C_0-semigroup T is called an analytic semigroup if instead of 3. it holds that*

- *$T(t)x \to x$ as $t \to +0$, for all $x \in X$.*

- *$t \mapsto T(t)x$ is real analytic for all $t \in (0,\infty)$ for all $x \in X$.*

Definition 5.1.3. *The linear operator L (possibly unbounded), defined by $Lx = \lim_{t\to+0} \frac{1}{t}(T(t)x - x)$ with domain of definition $D(L) = \{x \in X : \lim_{t\to+0} \frac{1}{t}(T(t)x - x)$ exists$\}$ is called the infinitesimal generator of a C_0-semigroup.*

Stability of semigroups

Here we provide the stability notions for C_0-semigroups $T := \{T(t),\ t \geq 0\}$. They will be needed in Section 2.1.

Definition 5.1.4. *A strongly continuous semigroup T is called*

1. *Exponentially stable, if $\exists \omega > 0$, such that $\lim_{t\to\infty} e^{\omega t}\|T(t)\| = 0$.*
2. *Uniformly stable, if $\lim_{t\to\infty} \|T(t)\| = 0$.*
3. *Strongly stable, if $\lim_{t\to\infty} \|T(t)x\|_X = 0 \quad \forall x \in X$.*

Note that strong stability of a semigroup T is what we call attractivity of a corresponding dynamical system $\Sigma := (X, \{0\}, \phi)$. It holds

Lemma 5.1.1 (Proposition 1.2, p. 296 in [28])**.** *A C_0-semigroup is uniformly stable iff it is exponentially stable.*

Uniform stability implies strong stability, but the converse implication doesn't hold in general.

We need the methods for checking of the exponential stability of the C_0-semigroups.

Lemma 5.1.2. *Let $\omega_0 := \inf_{t>0}(\frac{1}{t}\log\|T(t)\|)$ be well-defined. Then $\forall \omega > \omega_0$ there exists M_ω: $\|T(t)\| \leq M_\omega e^{\omega t}$.*

Definition 5.1.5. *The constant ω_0 from the previous lemma is called growth bound of a C_0 semigroup.*

Denote by $\Re(\lambda)$ the real part of a complex number λ.

Definition 5.1.6. *Let T be C_0-semigroup and A be its generator. If $\omega_0 = \sup_{\lambda\in Spec(A)} \Re(\lambda)$, then we say, that $T(t)$ satisfies the spectrum determined growth assumption.*

In contrast to the finite-dimensional case, not all C_0-semigroups satisfy the spectrum determined growth assumption (see [12], p.222 and Exercise 5.6 in the same book). However, it holds

Proposition 5.1.1 (see Theorem 5.1.1 from [36])**.** *Analytic semigroups satisfy the spectrum determined growth assumption.*

Another method for the proof of exponential stability of semigroups is the Lyapunov method.

In what follows let X be a Hilbert space with inner product $\langle\cdot,\cdot\rangle$.

Definition 5.1.7. *A self-adjoint operator P on the Hilbert space X is positive, if*

$$\langle Px, x\rangle > 0 \ \forall x \in D(P)\backslash\{0\}.$$

The following criterion is of great importance in particular for the proof of linearization theorem for nonlinear systems with inputs.

Theorem 5.1.2 (see [12], p. 217). *Suppose that A is the infinitesimal generator of a C_0 semigroup $T(t)$ on a Hilbert space X. Then T is exponentially stable iff there exists a positive operator $P \in L(X)$, such that*

$$\langle Ax, Px\rangle + \langle Px, Ax\rangle = -\|x\|_X^2, \ \forall x \in D(A). \tag{5.1}$$

In this case a Lyapunov function V can be chosen as $V(x) = \langle Px, x\rangle$.

Remark 5.1.3. *An equivalent formulation of the Theorem 5.1.2 can be obtained if one takes "$\leq$" instead of "$=$" in (5.1). The proof is similar to the proof of Theorem 5.1.2.*

5.2 Bochner integration theory

For the work with infinite-dimensional systems we use Bochner integration theory.

Within this section X is a Banach space, B is a Borel σ-field over $\mathbb{R}$ and μ is a Lebesgue measure on $(\mathbb{R}, B)$. Also f and f_i, $i = \overline{1,n}$ denote functions from $\mathbb{R}$ to X if not mentioned otherwise.

Definition 5.2.1. *We call f a countably valued function if the image $Im(f)$ of f is a countable set and for all $x_i \in Im(f)$ it follows $A_i := \{s \in \mathbb{R} : f(s) = x_i\} \in B$.*

Definition 5.2.2. *We say that a sequence of functions f_i converges to f almost everywhere if $\exists S \in B$: $\mu(S) = 0$ and $\forall x \in \mathbb{R}\backslash S$ $\lim\limits_{n\to\infty} \|f_n(x) - f(x)\|_X = 0$.*

Definition 5.2.3. *We call f strongly measurable iff there exists a sequence of countably valued functions which converges to f a.e.*

Now we define a Bochner integral

Definition 5.2.4. *For a countably-valued function f its Bochner integral over the set $E \subset \mathbb{R}$ is defined by*

$$(B)\int_E f d\mu = \sum_i x_i \mu(E \cap A_i).$$

Definition 5.2.5. *A function f we call Bochner-integrable over E if there exist a sequence of countably-valued functions f_i which converges to f a.e. such that*

$$\lim_{i\to\infty} (L) \int_E \|f(x) - f_i(x)\|_X dx = 0,$$

where $(L)\int$ denotes a usual Lebesgue integral. In this case Bochner-integrable of f is defined by

$$(B)\int_E f d\mu = \lim_{i\to\infty} (B) \int_E f_i d\mu.$$

By definition, a Bochner-integrable function has to be strongly measurable. Even more:

Theorem 5.2.1 (Theorem 3.7.4. in [39]). *A function $f : \mathbb{R} \to X$ is Bochner-integrable iff it is strongly measurable and $(L)\int_{\mathbb{R}} \|f(x)\|dx < \infty$.*

Proposition 5.2.2 (See p. 84 in [39]). *If the function $f : [a, b] \to X$ is continuous then it is strongly measurable.*

5.3 Differential equations in Banach spaces

Let A be a generator of a semigroup $T(t)$.

Consider the following problem:

$$\begin{cases} \dot{x} = Ax + f(t), \\ x(0) = x_0. \end{cases} \tag{5.2}$$

Definition 5.3.1. *A function $x \in C^1([0,T],X)$ is called a strong solution of a problem (5.2) on time-interval $[0,T]$, if $x(t) \in D(A)$ for $t \in [0,T]$ and x satisfies (5.2) for $t \in [0,T]$.*

Theorem 5.3.1 (Corollary 1.2 in [90])**.** *Let $x_0 \in D(A)$ and $f \in C^1([0,T],X)$. Then the strong solution of (5.2) on time-interval $[0,T]$ is given by:*

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(s)ds. \tag{5.3}$$

Definition 5.3.2. *For arbitrary Bochner integrable function f we call the function $x \in C([0,T],X)$, given by a formula (5.3) a weak solution of a problem (5.2).*

Now we turn to consider semilinear problems in abstract spaces, where A is a generator of C_0-semigroup:

$$\begin{cases} \dot{x} = Ax + f(x), \\ x(0) = x_0. \end{cases} \tag{5.4}$$

Definition 5.3.3. *A function $f : X \to X$ is called Lipschitz continuous on bounded subsets of X, if $\forall M > 0$ exists $L(M) > 0$, such that:*

$$\|f(y) - f(x)\| \leq L(M)\|y - x\|, \forall x, y \in B_M,$$

where B_M is a ball in X with center at 0 and radius M.

We consider a weak form of a problem (5.4):

$$x(t) = T(t)x_0 + \int_0^t T(t-s)f(x(s))ds \tag{5.5}$$

It holds (see [9], p. 56):

Theorem 5.3.2. *Let f be Lipschitz on bounded domains, and let $M > 0$ and x_0 be such that $\|x\| \leq M$. Then $\exists T > 0$, such that there exists a unique solution $u \in C([0,T],X)$ of (5.5) on $[0,T]$.*

5.4 Function spaces and inequalities

In the applications of the general theory to the problems arising in partial differential equations, we exploit the following function spaces

- $C_0^k(0,d)$ is the space of k times continuously differentiable functions $f:(0,d)\to\mathbb{R}$ with compact in $(0,d)$ support.

- $L_p(0,d)$ is the space of p-th power integrable functions $f:(0,d)\to\mathbb{R}$ with the norm

$$\|f\|_{L_p(0,d)} = \left(\int_0^d |f(x)|^p dx\right)^{\frac{1}{p}}.$$

- $W^{p,k}(0,d)$ is the Sobolev space of functions from the space $L_p(0,d)$, which have weak derivatives of order $\le k$, all of which belong to $L_p(0,d)$. Norm in $W^{p,k}(0,d)$ is defined by

$$\|f\|_{W^{p,k}(0,d)} = \left(\int_0^d \sum_{1\le s\le k} \left|\frac{\partial^s f}{\partial x^s}(x)\right|^p dx\right)^{\frac{1}{p}}. \tag{5.6}$$

- $W_0^{p,k}(0,d)$ is the closure of $C_0^k(0,d)$ in the norm of $W^{p,k}(0,d)$.

- $H^k(0,d) = W^{2,k}(0,d)$, $H_0^k(0,d) = W_0^{2,k}(0,d)$.

Inequalities

Let U be an open bounded region in $\mathbb{R}^n$. We state basic inequalities in L_p and Sobolev spaces used throughout Chapter 2.

Hölder's inequality. Assume $a \le p,q \le \infty$ and $\frac{1}{p}+\frac{1}{q}=1$. Then if $u\in L_p(U)$, $v\in L_q(U)$, then

$$\int_U |u(x)v(x)|dx \le \|u\|_{L_p(U)}\|v\|_{L_q(U)}. \tag{5.7}$$

Cauchy-Schwarz' inequality. The special case of Hölder's inequality for $p=q=2$ is called Cauchy-Schwarz' inequality

$$\int_U |u(x)v(x)|dx \le \|u\|_{L_2(U)}\|v\|_{L_2(U)}. \tag{5.8}$$

We need the following version of Friedrich's inequality

Theorem 5.4.1 (Friedrich's inequality)**.** *For every $u\in H_0^1(U)$ holds that*

$$\int_U u^2 dx \le \frac{1}{\mu_1}\int_U |\nabla u|^2 dx, \tag{5.9}$$

where μ_1 is the smallest (positive) eigenvalue of the following eigenvalue problem

$$\begin{cases} \Delta u + \mu u = 0, \\ u(x) = 0, \ x\in\partial U. \end{cases} \tag{5.10}$$

Proof. Firstly let $u \in C_c^\infty(U)$ be a smooth function with compact support in U. Consider an eigenvalue problem (5.10). Let the eigenvalues of this problem and corresponding orthonormal eigenvectors are given as μ_i and ϕ_i, $i = 1, \dots, \infty$ respectively, and μ_i is increasing. Then $u = \sum_{i=1}^\infty a_i\phi_i$. Since the function u is smooth, we can apply Green's formula [29, Theorem 3, p. 628]

$$\int_U |\nabla u|^2 dx = -\int_U u\Delta u dx = \sum_{i=1}^\infty a_i^2\mu_i \geq \mu_1 \sum_{i=1}^\infty a_i^2 = \mu_1 \int_U u^2 dx.$$

To verify (5.9) for all $u \in H_0^1(U)$, one can use the approximation technique, see [29, Chapter 5, par. 5.3]. □

Another Friedrichs'-like inequality will be useful:

Theorem 5.4.2. *For every $u \in H^2(U) \cap H_0^1(U)$ holds*

$$\int_U |\nabla u|^2 dx \leq \frac{1}{\mu_1}\int_U |\Delta u|^2 dx, \tag{5.11}$$

where μ_1 is the smallest (positive) eigenvalue of the problem (5.10)

Proof. For every $u \in C_c^\infty(U)$ integrating by parts and using Cauchy-Schwarz inequality and inequality (5.9), we obtain:

$$\int_U |\nabla u|^2 dx = -\int_U u\Delta u dx \leq \left(\int_U u^2 dx\right)^{\frac{1}{2}} \left(\int_U (\Delta u)^2 dx\right)^{\frac{1}{2}} \leq \tag{5.12}$$

$$\left(\frac{1}{\mu_1}\int_U |\nabla u|^2 dx\right)^{\frac{1}{2}} \left(\int_U (\Delta u)^2 dx\right)^{\frac{1}{2}}. \tag{5.13}$$

For all functions, which are not constant a.e. we obtain (5.11) dividing both parts of the above inequality by $\left(\int_U |\nabla u|^2 dx\right)^{\frac{1}{2}}$. For functions, which are constants a.e. (5.11) is trivial. To prove the needed inequality for all $u \in H^2(U) \cap H_0^1(U)$, one can use approximation technique. □

In particular, if $U = [0, l]$, then the eigenvalues of (5.10) are $\mu_n = \left(\frac{\pi n}{l}\right)^2$, $n = 1, \dots, \infty$, and we have:

Corollary 5.4.3 (Wirtinger's inequality)**.** *For every $u \in H^2([0, l]) \cap H_0^1([0, l])$ holds that*

$$\int_0^l |\nabla u|^2 dx \leq \frac{l^2}{\pi^2}\int_0^l |\Delta u|^2 dx, \tag{5.14}$$

5.5 Some lemmas from analysis

Lemma 5.5.1. *Let $\{x_k^1\}_{k=1}^\infty$, …, $\{x_k^m\}_{k=1}^\infty$ be sequences of real numbers. Let the limit $\lim_{k\to\infty} \max_{1\leq i\leq m}\{x_k^i\}$ exist. Then it holds that*

$$\lim_{k\to\infty} \max_{1\leq i\leq m}\{x_k^i\} = \max_{1\leq i\leq m}\{\overline{\lim_{k\to\infty}} x_k^i\}, \tag{5.15}$$

where $\overline{\lim}_{k\to\infty} x_k^i$ is the upper limit of the sequence x_k^i.

Proof. For all $k \in \mathbb{N}$ define $i(k) = \arg\max_{1\le i\le m}\{x_k^i\}$ - the index of the maximal element of $\{x_k^i\}$, $i = 1, \dots, n$ (if there are more than one maximal element, than take arbitrary index). Then $\max_{1\le i\le m} x_k^i = x_k^{i(k)}$ for all $k \in \mathbb{N}$. Extract from the sequence $\{x_k^{i(k)}\}$ the maximal subsequences of the form $\{x^j_{n^j_k}\}$, $j = 1, \dots, m$, where n_k^j is the monotone increasing sequence of indexes. At least some of $\{x^j_{n^j_k}\}$, $j = 1, \dots, m$ are infinite (without loss of generality let it be $\{x^1_{n^1_k}\}$).

The sequence $\{x_k^{i(k)}\}$ is convergent, hence all its subsequences are convergent and have the same limit value. Thus we obtain

$$\lim_{k\to\infty} \max_{1\le i\le m}\{x_k^i\} = \lim_{k\to\infty} x_k^{i(k)} = \lim_{k\to\infty} x^1_{n^1_k} \le \overline{\lim_{k\to\infty}}\, x_k^1 \le \max_{1\le i\le m} \overline{\lim_{k\to\infty}}\, x_k^i. \tag{5.16}$$

To obtain the reverse inequality, take any sequence $\{x^i_{n_k}\}$, such that

$$\lim_{k\to\infty} x^i_{n_k} = \max_{1\le i\le m} \overline{\lim_{k\to\infty}}\, x_k^i.$$

We have that $\max_{1\le i\le m}\{x^i_{n_k}\} \ge x^i_{n_k}$, and so

$$\lim_{k\to\infty} \max_{1\le i\le m}\{x_k^i\} \ge \max_{1\le i\le m}\{\overline{\lim_{k\to\infty}}\, x_k^i\}. \tag{5.17}$$

From (5.16) and (5.17) we obtain (5.15). □

Corollary 5.5.1. *Let $f_i : \mathbb{R} \to \mathbb{R}$ are defined and bounded in some neighborhood D of $t = 0$. Then it holds*

$$\overline{\lim_{t\to 0}} \max_{1\le i\le m}\{f_i(t)\} = \max_{1\le i\le m}\{\overline{\lim_{t\to 0}} f_i(t)\} \tag{5.18}$$

Proof. Under made assumptions the upper limits in both parts of the equation (5.18) exist. From $\max_{1\le i\le m}\{f_i(t)\} \ge f_i(t)\ \forall i = 1, \dots, m$, for all $t \in D$. Thus,

$$\overline{\lim_{t\to 0}} \max_{1\le i\le m}\{f_i(t)\} \ge \max_{1\le i\le m}\{\overline{\lim_{t\to 0}} f_i(t)\}$$

To prove the converse inequality, we use Lemma 5.5.1.

$$\overline{\lim_{t\to 0}} \max_{1\le i\le m}\{f_i(t)\} = \sup_{t_{n_k}\to 0} \lim_{k\to\infty} \max_{1\le i\le m}\{f_i(t_{n_k})\}$$

$$= \sup_{t_{n_k}\to 0} \max_{1\le i\le m}\{\overline{\lim_{k\to\infty}} f_i(t_{n_k})\} \le \max_{1\le i\le m}\{\overline{\lim_{t\to 0}} f_i(t)\},$$

where the sup is taken over all convergent to 0 sequences t_{n_k}. □

Bibliography

[1] V. M. Alekseev, V. M. Tikhomirov, and S. V. Fomin. *Optimal control*. Contemporary Soviet Mathematics. Consultants Bureau, New York, 1987.

[2] David Angeli. Intrinsic robustness of global asymptotic stability. *Systems Control Lett.*, 38(4-5):297–307, 1999.

[3] David Angeli and Eduardo D. Sontag. Monotone control systems. *IEEE Trans. Automat. Control*, 48(10):1684–1698, 2003. New directions on nonlinear control.

[4] David Angeli, Eduardo D. Sontag, and Yuan Wang. A characterization of integral input-to-state stability. *IEEE Trans. Automat. Control*, 45(6):1082–1097, 2000.

[5] D. Armbruster, P. Degond, and C. Ringhofer. A model for the dynamics of large queuing networks and supply chains. *SIAM Journal on Applied Mathematics*, 66(3):896–920, 2006.

[6] George Ballinger and Xinzhi Liu. Existence, uniqueness and boundedness results for impulsive delay differential equations. *Appl. Anal.*, 74(1-2):71–93, 2000.

[7] Dennis S. Bernstein. *Matrix mathematics*. Princeton University Press, Princeton, NJ, second edition, 2009. Theory, facts, and formulas.

[8] C. Cai and A.R. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, 58(1):47–53, 2009.

[9] Thierry Cazenave and Alain Haraux. *An introduction to semilinear evolution equations*, volume 13 of *Oxford Lecture Series in Mathematics and its Applications*. The Clarendon Press Oxford University Press, New York, 1998.

[10] Anna L. Chen, Gui-Qiang Chen, and Randy A. Freeman. Stability of nonlinear feedback systems: a new small-gain theorem. *SIAM J. Control Optim.*, 46(6):1995–2012, 2007.

[11] Wu-Hua Chen and Wei Xing Zheng. Brief paper: Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays. *Automatica (Journal of IFAC)*, 45(6):1481–1488, 2009.

[12] Ruth F. Curtain and Hans Zwart. *An introduction to infinite-dimensional linear systems theory*, volume 21 of *Texts in Applied Mathematics*. Springer-Verlag, New York, 1995.

[13] C.F. Daganzo. *A Theory of Supply Chains*, volume 526 of *Lecture Notes in Economics and Mathematical Systems*. Springer-Verlag, Berlin, 2003.

[14] S. Dashkovskiy, D. Efimov, and E. Sontag. Input to state stability and allied system properties. *Automation and Remote Control*, 72:1579–1614, 2011.

[15] S. Dashkovskiy and A. Mironchenko. On the uniform input-to-state stability of reaction-diffusion systems. In *Proceedings of the 49th IEEE Conference on Decision and Control, Atlanta, Georgia, USA, Dec. 15-17, 2010*, pages 6547–6552, 2010.

[16] Sergey Dashkovskiy, Michael Görges, Michael Kosmykov, Andrii Mironchenko, and Lars Naujok. Modeling and stability analysis of autonomously controlled production networks. *Logistics Research*, 3:145–157, 2011.

[17] Sergey Dashkovskiy and Michael Kosmykov. Stability of networks of hybrid ISS systems. In *Proceedings of the 48th IEEE Conference on Decision and Control, Shanghai, China, Dec. 16-18, 2009*, pages 3870–3875, 2009.

[18] Sergey Dashkovskiy, Michael Kosmykov, Andrii Mironchenko, and Lars Naujok. Stability of interconnected impulsive systems with and without time-delays using Lyapunov methods. *Nonlinear Analysis: Hybrid Systems.*, 6(3):899–915, 2012.

[19] Sergey Dashkovskiy and Andrii Mironchenko. Input-to-state stability of infinite-dimensional control systems. Submitted to MCSS.

[20] Sergey Dashkovskiy and Andrii Mironchenko. Dwell-time conditions for robust stability of impulsive systems. In *Accepted to 20th International Symposium on Mathematical Theory of Systems and Networks, Melbourne, Australia, Jul. 09-13, 2012*, 2012.

[21] Sergey Dashkovskiy, Björn S. Rüffer, and Fabian R. Wirth. An ISS Lyapunov function for networks of ISS systems. In *Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems (MTNS), Kyoto, Japan, July 24-28, 2006*, pages 77–82, 2006.

[22] Sergey Dashkovskiy, Björn S. Rüffer, and Fabian R. Wirth. An ISS small gain theorem for general networks. *Math. Control Signals Systems*, 19(2):93–122, 2007.

[23] Sergey N. Dashkovskiy and Björn S. Rüffer. Local ISS of large-scale interconnections and estimates for stability regions. *Systems & Control Letters*, 59(3-4):241 – 247, 2010.

[24] Sergey N. Dashkovskiy, Björn S. Rüffer, and Fabian R. Wirth. Small Gain Theorems for Large Scale Systems and Construction of ISS Lyapunov Functions. *SIAM Journal on Control and Optimization*, 48(6):4089–4118, 2010.

[25] Patrick De Leenheer, David Angeli, and Eduardo D. Sontag. Monotone chemical reaction networks. *J. Math. Chem.*, 41(3):295–314, 2007.

[26] Vincent Duindam, Alessandro Macchelli, Stefano Stramigioli, and Herman Bruyninckx, editors. *Modeling and control of complex physical systems. The port-Hamiltonian approach.* Springer-Verlag, Berlin, 2009.

[27] H. A. Edwards, Y. Lin, and Y. Wang. On input-to-state stability for time varying nonlinear systems. In *Proceedings of the 39th IEEE Conference on Decision and Control, 2000*, volume 4, pages 3501–3506. IEEE, 2000.

[28] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*, volume 194 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 2000. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara, C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt.

[29] L. C. Evans. *Partial Differential Equations*, volume 19 of *Graduate Studies in Mathematics*. American Mathematical Society, 1998.

[30] R. Goebel, R.G. Sanfelice, and A.R. Teel. Hybrid dynamical systems: robust stability and control for systems that combine continuous-time and discrete-time dynamics. *IEEE Control Systems Magazine*, 29(2):28–93, 2009.

[31] R. Goebel and A. R. Teel. Solutions to hybrid inclusions via set and graphical convergence with stability theory applications. *Automatica J. IFAC*, 42(4):573–587, 2006.

[32] Lars Grüne. Input-to-state dynamical stability and its Lyapunov function characterization. *IEEE Trans. Automat. Control*, 47(9):1499–1504, 2002.

[33] Wassim M. Haddad, Vijay Sekhar Chellaboina, and Sergey G. Nersesov. *Impulsive and hybrid dynamical systems*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2006.

[34] Wolfgang Hahn. *Stability of motion*. Translated from the German manuscript by Arne P. Baartz. Die Grundlehren der mathematischen Wissenschaften, Band 138. Springer-Verlag New York, Inc., New York, 1967.

[35] Jack K. Hale and Sjoerd M. Verduyn Lunel. *Introduction to functional-differential equations*, volume 99 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1993.

[36] Daniel Henry. *Geometric theory of semilinear parabolic equations*, volume 840 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1981.

[37] J. P. Hespanha and A. S. Morse. Stability of switched systems with average dwell-time. *Proceedings of the 38th IEEE Conference on Decision and Control, 1999.*, 3:2655–2660 vol.3, 1999.

[38] João P. Hespanha, Daniel Liberzon, and Andrew R. Teel. Lyapunov conditions for input-to-state stability of impulsive systems. *Automatica J. IFAC*, 44(11):2735–2744, 2008.

[39] Einar Hille and Ralph S. Phillips. *Functional analysis and semi-groups*. American Mathematical Society, Providence, R. I., 1974. Third printing of the revised edition of 1957, American Mathematical Society Colloquium Publications, Vol. XXXI.

[40] Morris W. Hirsch. Systems of differential equations which are competitive or cooperative. I. Limit sets. *SIAM J. Math. Anal.*, 13(2):167–179, 1982.

[41] B. Jayawardhana, H. Logemann, and E. P. Ryan. The circle criterion and input-to-state stability. *Control Systems, IEEE*, 31(4):32–67, August 2011.

[42] Bayu Jayawardhana, Hartmut Logemann, and Eugene P. Ryan. Infinite-dimensional feedback systems: the circle criterion and input-to-state stability. *Commun. Inf. Syst.*, 8(4):413–414, 2008.

[43] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Math. Control Signals Systems*, 7(2):95–120, 1994.

[44] Zhong-Ping Jiang, Iven M. Y. Mareels, and Yuan Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica J. IFAC*, 32(8):1211–1215, 1996.

[45] Zhong-Ping Jiang and Yuan Wang. Input-to-state stability for discrete-time nonlinear systems. *Automatica J. IFAC*, 37(6):857–869, 2001.

[46] Zhong-Ping Jiang and Yuan Wang. A generalization of the nonlinear small-gain theorem for large-scale complex systems. In *Proceedings of the 7th World Congress on Intelligent Control and Automation (WCICA).*, pages 1188–1193, June 2008.

[47] Iasson Karafyllis. Lyapunov theorems for systems described by retarded functional differential equations. *Nonlinear Analysis: Theory, Methods & Applications*, 64(3):590 – 617, 2006.

[48] Iasson Karafyllis. A system-theoretic framework for a wide class of systems. II. Input-to-output stability. *J. Math. Anal. Appl.*, 328(1):466–486, 2007.

[49] Iasson Karafyllis and Zhong-Ping Jiang. New Results in Trajectory-Based Small-Gain with Application to the Stabilization of a Chemostat, February 2010.

[50] Iasson Karafyllis and Zhong-Ping Jiang. *Stability and stabilization of nonlinear systems.* Communications and Control Engineering Series. Springer-Verlag London Ltd., London, 2011.

[51] Iasson Karafyllis and Zhong-Ping Jiang. A vector small-gain theorem for general non-linear control systems. *IMA Journal of Mathematical Control and Information*, 28(3):309–344, June 2011.

[52] Iasson Karafyllis and J. Tsinias. Nonuniform in time input-to-state stability and the small-gain theorem. *IEEE Trans. Automat. Control*, 49(2):196–216, 2004.

[53] Hassan K. Khalil. *Nonlinear systems.* Prentice Hall, New York, 2002.

[54] A.N. Kolmogorov and S. V. Fomin. *Elementy teorii funktsiy i funktsionalnogo analiza (in Russian).* Nauka, 1976.

[55] M. Kosmykov. *Hybrid dynamics in large-scale logistics networks.* PhD thesis, Department of Mathematics and Computer Science, University of Bremen, 2011.

[56] M. A. Krasnoselskiy. *Positive solutions of operator equations.* Translated from the Russian by Richard E. Flaherty; edited by Leo F. Boron. P. Noordhoff Ltd. Groningen, 1964.

[57] O. A. Ladyženskaja, V. A. Solonnikov, and N. N. Uralceva. *Linear and quasilinear equations of parabolic type.* Translated from the Russian by S. Smith. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, Rhode Island, 1967.

[58] Y. Lin, Y. Wang, and D. Cheng. On nonuniform and semi-uniform input-to-state stability for time-varying systems. In *IFAC World Congress, Prague*, 2005.

[59] Yuandan Lin, Eduardo D. Sontag, and Yuan Wang. A smooth converse Lyapunov theorem for robust stability. *SIAM J. Control Optim.*, 34(1):124–160, 1996.

[60] Jun Liu, Xinzhi Liu, and Wei-Chau Xie. Input-to-state stability of impulsive and switching hybrid systems with time-delay. *Automatica*, 47(5):899–908, May 2011.

[61] M. La Marca. *Control Of Continuum Models Of Production Systems.* PhD thesis, Arizona State University, 2008.

[62] F. Mazenc and C. Prieur. Strict Lyapunov functions for semilinear parabolic partial differential equations. *Mathematical Control and Related Fields*, 1:231–250, June 2011.

[63] J. D. Murray. *Mathematical biology. II Spatial models and biomedical applications.*, volume 18 of *Interdisciplinary Applied Mathematics.* Springer-Verlag, New York, third edition, 2003.

[64] D. Nesic and A.R. Teel. A Lyapunov-based small-gain theorem for hybrid ISS systems. In *Proceedings of the 47th IEEE Conference on Decision and Control, Cancun, Mexico, Dec. 9-11, 2008*, pages 3380–3385, 2008.

[65] A. Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44 of *Applied Mathematical Sciences.* Springer-Verlag, New York, 1983.

[66] P. Pepe and Z.-P. Jiang. A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. *Systems Control Lett.*, 55(12):1006–1014, 2006.

[67] Jan Willem Polderman and Jan C. Willems. *Introduction to mathematical systems theory*, volume 26 of *Texts in Applied Mathematics.* Springer-Verlag, New York, 1998. A behavioral approach.

[68] Christophe Prieur and Frederic Mazenc. ISS-lyapunov functions for time-varying hyperbolic systems of balance laws. *Mathematics of Control, Signals, and Systems (MCSS)*, pages 1–24, 2012.

[69] Zhu Qiao and Hu Guang-Da. Converse Lyapunov Theorem of Input-to-State Stability for Time-delay Systems. *Acta Automatica Sinica*, 36(8):1131–1136, 2010.

[70] B. S. Rüffer. Monotone inequalities, dynamical systems, and paths in the positive orthant of Euclidean n-space. *Positivity.*, 14(2):257–283, 2010.

[71] A. M. Samoilenko and N. A. Perestyuk. *Impulsive differential equations*, volume 14 of *World Scientific Series on Nonlinear Science. Series A: Monographs and Treatises*. World Scientific Publishing Co. Inc., River Edge, NJ, 1995.

[72] Robert Shorten, Fabian Wirth, Oliver Mason, Kai Wulff, and Christopher King. Stability criteria for switched and hybrid systems. *SIAM Rev.*, 49(4):545–592, 2007.

[73] Hal L. Smith. *Monotone dynamical systems*, volume 41 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1995. An introduction to the theory of competitive and cooperative systems.

[74] E. D. Sontag. Input to state stability: basic concepts and results. In *Nonlinear and optimal control theory*, volume 1932 of *Lecture Notes in Math.*, pages 163–220, Berlin, 2008. Springer.

[75] Eduardo D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34(4):435–443, 1989.

[76] Eduardo D. Sontag. *Mathematical control theory*, volume 6 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 1998. Deterministic finite-dimensional systems.

[77] Eduardo D. Sontag and Yuan Wang. On characterizations of the input-to-state stability property. *Systems Control Lett.*, 24(5):351–359, 1995.

[78] Eduardo D. Sontag and Yuan Wang. New characterizations of input-to-state stability. *IEEE Trans. Automat. Control*, 41(9):1283–1294, 1996.

[79] Ivanka Stamova. *Stability analysis of impulsive functional differential equations*, volume 52 of *de Gruyter Expositions in Mathematics*. Walter de Gruyter GmbH & Co. KG, Berlin, 2009.

[80] Andrew R. Teel. Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Trans. Automat. Control*, 43(7):960–964, 1998.

[81] Shanaz Tiwari and Yuan Wang. Razumikhin-type Small-Gain Theorems for Large-Scale Systems with Delay. In *Proceedings of the 49th IEEE CDC, Atlanta, GA, USA, Dec. 15-17, 2010*, pages 7407–7412, 2010.

[82] Shanaz Tiwari, Yuan Wang, and Zhong-Ping Jiang. A Nonlinear Small-Gain Theorem for Large-Scale Time Delay Systems. In *Proceedings of the 48th IEEE CDC, Shanghai, China, Dec. 16-18, 2009*, pages 7204–7209, 2009.

[83] A. M. Turing. The Chemical Basis of Morphogenesis. *Philosophical Transactions of the Royal Society of London. Series B, Biological Sciences*, 237(641):37–72, August 1952.

[84] Arjan van der Schaft. *L_2-gain and passivity techniques in nonlinear control*. Communications and Control Engineering Series. Springer-Verlag London Ltd., London, second edition, 2000.

[85] L. Vu, D. Chatterjee, and D. Liberzon. Input-to-state stability of switched systems and switching adaptive control. *Automatica J. IFAC*, 43(4):639–646, 2007.

[86] Jan C. Willems. Dissipative dynamical systems. I. General theory. *Arch. Rational Mech. Anal.*, 45:321–351, 1972.

[87] Jan C. Willems. Dissipative dynamical systems. II. Linear systems with quadratic supply rates. *Arch. Rational Mech. Anal.*, 45:352–393, 1972.

[88] Jan C. Willems. Paradigms and puzzles in the theory of dynamical systems. *IEEE Trans. Automat. Control*, 36(3):259–294, 1991.

[89] Shujie Yang, Bao Shi, and Shuyan Hao. Input-to-state stability for discrete-time nonlinear impulsive systems with delays. *Int. J. Robust. Nonlinear Control*, page n/a, 2011.

[90] Jerzy Zabczyk. *Mathematical control theory*. Modern Birkhäuser Classics. Birkhäuser Boston Inc., Boston, MA, 2008.

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