Stability, observer design and control of networks using Lyapunov methods

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Abstract

We investigate different aspects of the analysis and control of interconnected systems. Different tools, based on Lyapunov methods, are provided to analyze such systems in view of stability, to design observers and to control systems subject to stabilization. All the different tools presented in this work can be used for many applications and extend the analysis toolbox of networks.

Considering systems with inputs, the stability property input-to-state dynamical stability (ISDS) has some advantages over input-to-state stability (ISS). We introduce the ISDS property for interconnected systems and provide an ISDS small-gain theorem with a construction of an ISDS-Lyapunov function and the rate and the gains of the ISDS estimation for the whole system.

This result is applied to observer design for single and interconnected systems. Observers are used in many applications where the measurement of the state is not possible or disturbed due to physical reasons or the measurement is uneconomical. By the help of error Lyapunov functions we design observers, which have a so-called quasi ISS or quasi-ISDS property to guarantee that the dynamics of the estimation error of the systems state has the ISS or ISDS property, respectively. This is applied to quantized feedback stabilization.

In many applications, there occur time-delays and/or instantaneous "jumps" of the systems state. At first, we provide tools to check whether a network of time-delay systems has the ISS property using ISS-Lyapunov-Razumikhin functions and ISS-Lyapunov-Krasovskii functionals. Then, these approaches are also used for interconnected impulsive systems with time-delays using exponential Lyapunov-Razumikhin functions and exponential Lyapunov-Krasovskii functionals. We derive conditions to assure ISS of an impulsive network with time-delays.

Controlling a system in a desired and optimal way under given constraints is a challenging task. One approach to handle such problems is model predictive control (MPC). In this thesis, we introduce the ISDS property for MPC of single and interconnected systems. We provide conditions to assure the ISDS property of systems using MPC, where the previous result of this thesis, the ISDS small-gain theorem, is applied. Furthermore, we investigate the ISS property for MPC of time-delay systems using the Lyapunov-Krasovskii approach. We prove theorems, which guarantee ISS for single and interconnected systems using MPC.

Contents

Introduction

In this thesis, we provide tools to analyze, to observe and to control networks with regard to stability based on Lyapunov methods.

A network consists of an arbitrary number of interconnected subsystems. We consider such networks, which can be modeled using ordinary differential equations of the form

$$\dot{x}_i(t) = f_i(x_1(t), \dots, x_n(t), u(t)),\ i = 1, \dots, n, \tag{1}$$

which can be seen as one single system of the form

$$\dot{x}(t) \ = \ f(x(t), u(t)), \tag{2}$$

where the time t is continuous, $x(t) = (x_1(t), \dots, x_n(t))^T \in \mathbb{R}^N$, with $x_i(t) \in \mathbb{R}^{N_i},\ N = \sum N_i$, denotes the state of the system and $u \in \mathbb{R}^m$ is a measurable and essentially bounded input function of the system. For example, the dynamics of a logistic network, such as a production network, can be described by a system of the form (1) [48, 15, 12, 13, 103].

In this work, we investigate interconnected systems in view of stability. We consider the notion of input-to-state stability (ISS), introduced in 1989 by Sontag, [114]. ISS means, roughly speaking, that the norm of the solution of a system is bounded for all times by

$$|x(t; x_0, u)| \leq \max \left\{ \beta(|x_0|, t), \gamma^{\text{ISS}}(\|u\|) \right\}, \tag{3}$$

where $x(t; x_0, u)$ denotes the solution of a system with initial value x_0, where $|\cdot|$ denotes the Euclidean norm and $\|\cdot\|$ is the essential supremum norm. The function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ increases in the first argument and tends to zero, if the second argument tends to infinity. The function $\gamma^{\text{ISS}} : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing with $\gamma^{\text{ISS}}(0) = 0$, called a $\mathcal{K}_\infty$-function.

In contrast, instability of a system can lead to infinite states. For example, in case of a logistic system the state can be the work in progress or the number of unsatisfied orders. Instability, by means of an unbounded growth of a state, for example, may cause high inventory costs or loss of customers, if orders will not be satisfied. Hence, for many applications it is necessary to analyze networks in view of stability and to provide tools to check whether a system is stable to avoid such negative outcomes described above.

During the last decades, several stability concepts, such as exponential stability, asymptotic stability, global stability and ISS were established, see [115, 64, 120], for example. Based on ISS, several related stability properties were investigated: input-to-output stability (IOS) [56], integral ISS (iISS) [116] and input-to-state dynamical stability (ISDS) [35]. The ISS property and its variants became important during the recent years for the stability analysis of dynamical systems with disturbances and they were applied in network control, engineering, biological or economical systems, for example. Survey papers about ISS and related stability properties can be found in [118, 11].

Furthermore, the stability analysis of single systems can be performed in different frameworks such as passivity, dissipativity [108], and its variations [1, 36, 98, 57].

It can be a challenging task to check the stability of a given system or to design a stable system. Lyapunov functions are a helpful tool to investigate the stability of a system, since the existence of a Lyapunov function is sufficient for stability, see [115, 64], for example. Moreover, the necessity of the existence of a Lyapunov function for stability for some stability properties was proved [115, 64]. In [119, 74], it was shown that the ISS property for a system of the form (2) is equivalent to the existence of an ISS-Lyapunov function, which is a locally Lipschitz continuous function $V:\mathbb{R}^N\to\mathbb{R}_+$ that has the properties

$$\begin{aligned}\psi_1(|x|)\leq V(x)&\leq\psi_2(|x|)\,,\ \forall x\in\mathbb{R}^N,\\ V(x)\geq&\chi(|u|)\Rightarrow\nabla V(x)\cdot f(x,u)\leq-\alpha(V(x))\end{aligned}$$

for almost all x and all u, where $\psi_1,\psi_2,\chi\in\mathcal{K}_\infty$, α is a positive definite function and ∇ denotes the gradient of V.

Based on a Lyapunov function, we provide tools to check stability, to design observers and to control networks. To this end, we consider interconnected systems of the form (1). The notion of ISS is a useful property for the investigation of interconnected systems in view of stability, because it can handle internal and external inputs of a subsystem. The ISS estimation of a subsystem is the following:

$$\left|x_i(t;x_i^0,u)\right|\leq\max\left\{\beta_i\left(\left|x_i^0\right|,t\right),\max_{j\neq i}\gamma_{ij}^{\mathrm{ISS}}\left(\|x_j\|_{[0,t]}\right),\gamma_i^{\mathrm{ISS}}(\|u\|)\right\},\tag{4}$$

where $\|\cdot\|_{[0,t]}$ denotes the supremum norm over the interval $[0,t]$, $\gamma_{ij}^{\mathrm{ISS}},\gamma_i^{\mathrm{ISS}}:\mathbb{R}_+\to\mathbb{R}_+$ are $\mathcal{K}_\infty$-functions and are called (nonlinear) gains.

Investigating a whole system in view of stability, it turns out that a network must not possess the ISS property even if all subsystems are ISS. A method to check the stability properties of networks is the so-called small-gain condition. It is based on the gains and the interconnection structure of the system.

For $n=2$ coupled systems an ISS small-gain theorem was proved in [56] and its Lyapunov version in [55], where an explicit construction of the ISS-Lyapunov function for the whole system was shown. For an arbitrary number of interconnected systems, an ISS small-gain theorem was proved in [25, 98] and its Lyapunov version in [28]. For a local variant of ISS, namely LISS, a Lyapunov formulation of the small-gain theorem can be found in [27]. Considering iISS, a small-gain theorem can be found in [49] for two coupled systems and in [50] for n coupled systems. Another approach, using the cycle small-gain condition, which is equivalent to the maximum formulation of the small-gain condition in matrix form, was used in [57, 77] to establish ISS of interconnections. A small-gain theorem considering a mixed formulation of ISS subsystems in summation and maximum formulation was proved in [19, 66]. General nonlinear systems were considered in [58] and [59], where small-gain theorems were proved, using vector Lyapunov functions.

Applying the mentioned tools to check whether a system has the ISS property one can derive the estimation (3) or (4) of the norm of the solution of a system. A stability property equivalent to ISS, which has some advantages over ISS, is the following:

Input-to-state dynamical stability

The definition of ISDS is motivated by the observation that the ISS estimation takes the supremum norm of the input function u into account, despite this the input can change and especially can tend to zero. The ISDS estimation takes essentially only the recent values of the input u into account and past values will be "forgotten" by time. This is known as the so-called "memory fading effect". The ISDS estimation is of the form

$$|x(t;x_0,u)| \leq \max\{\mu(\eta(|x_0|),t), \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu(\gamma^{\text{ISDS}}(|u(\tau)|),t-\tau)\},$$

where the function $\mu:\mathbb{R}_+\times\mathbb{R}_+\to\mathbb{R}_+$ increases in the first argument, tends to zero, if the second argument tends to infinity and has the property $\mu(r,t+s)=\mu(\mu(r,t),s), \forall r,t,s\geq 0$.

The benefit for logistic systems, for example production networks, is the following: consider the number of unprocessed parts within the system as the state, which have to be stored in a warehouse. By the ISS estimation, which gives an upper bound for the trajectory of the state of the system, we can calculate the size or the capacity of the warehouse to guarantee stability. The costs for warehouses increase by increasing the size or dimension of the warehouse. Consider the case that the influx of parts into the system is large at the beginning of the process, i.e., the number of unprocessed parts in a system is relatively large, and the influx tends to zero or close to zero by time. If the system has the ISS property, the number of unprocessed parts tends also to zero or close to zero by time, which means that the warehouse becomes almost empty by time. Therefore, it is not necessary to provide a huge warehouse to satisfy the upper bound of parts calculated by the ISS estimation. Taking recent values of the input into account by the ISDS estimation, we can calculate tighter estimations in contrast to ISS. The size of the warehouse can be smaller, which avoids high costs caused by the over-dimensioned warehouse.

Another advantage over ISS is that the ISDS property is equivalent to the existence of an ISDS-Lyapunov function, where μ, η and γ^{ISDS} can be directly taken from the definition of an ISDS-Lyapunov function. Considering ISS-Lyapunov functions and the ISS property, the functions of the according definitions are different, in general.

There exist no results for the application of ISDS and its Lyapunov function characterization to networks. This work fills this gap and an ISDS small-gain theorem is proved, which assures that a network consisting of ISDS subsystems is again ISDS under a small-gain condition. An explicit construction of the Lyapunov function and the corresponding gains of the whole system is given. This result was published in [20] and presented at the CDC 2009, [26].

The advantages of the ISDS property will be transfered to observer design:

Observer and quantized output feedback stabilization

In many applications, measurements are used to get knowledge about the systems state. To analyze such systems, we consider systems with an output of the form

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= h(x), \end{aligned} \tag{5}$$

where $y \in \mathbb{R}^P$ is the output.

In view of production networks, it can happen that the measurement of the state of a system is uneconomic or impossible due to physical circumstances or disturbed by perturbations, for example. For these cases, observers are used to estimate the state. An observer for the state of the system (5) is of the form

$$\begin{aligned} \dot{\hat{\xi}} &= F(\bar{y},\hat{\xi},u), \\ \hat{x} &= H(\bar{y},\hat{\xi},u), \end{aligned} \tag{6}$$

where $\hat{\xi} \in \mathbb{R}^L$ is the observer state, $\hat{x} \in \mathbb{R}^N$ is the estimate of the system state x and $\bar{y} \in \mathbb{R}^P$ is the measurement of y that may be disturbed by d: $\bar{y} = y + d$. The state estimation error is given by $\tilde{x} = \hat{x} - x$.

Here, we transfer the idea of ISDS to the design of an observer: the challenge is that the observer of a general system or network should be designed in such a way that the norm of the trajectory of the state estimation error has the ISS property or ISDS property, respectively.

First approaches in the observer design using the (quasi-)ISS property with respect to the state estimation error were performed in [110]. Motivated by the advantages of ISDS over ISS, we introduce the notion of quasi-ISDS observers with respect to the state estimation error of a system. We show that a quasi-ISDS observer can be designed, provided that there exists an error Lyapunov function (see [88, 60]). The design of the observer is the same as for quasi-ISS observers, based on the works [112, 60, 72, 61, 110], for example, but it has the advantage that the estimation of the error dynamics takes only recent disturbances into account (see above for the ISDS property). Namely, if the perturbation of the measurement tends to zero, then the estimation of the norm of the error dynamics tends to zero, which is not the case using the quasi-ISS property.

The approach of the quasi-ISS/ISDS observer design is used here for interconnected systems. We design quasi-ISS/ISDS observers for each subsystem and for the whole system, provided that error Lyapunov functions of the subsystems exist and a small-gain condition is satisfied.

We apply the presented approach to stabilization of single and interconnected systems based on quantized output feedback. The problem of output feedback stabilization was investigated in [62, 63, 60, 71, 72, 110], for example. The question, how to stabilize a system, plays an important role in the analysis of control systems. In this work, we use quantized output feedback stabilization according to the results in [7, 70, 72, 110]. A quantizer is a device, which converts a real-valued signal into a piecewise constant signal, i.e., it maps $\mathbb{R}^P$ into a finite and discrete subset of $\mathbb{R}^P$. It may affect the process output or may also affect the control input.

We show that under sufficient conditions a quantized output feedback law can be designed using quasi-ISS/ISDS observer, which guarantee that a single system, subsystems of a network or the whole system are stable, i.e., the norm of the trajectories of the systems are bounded. Furthermore, we investigate dynamic quantizers, where the quantizers can be adapted by a so-called "zooming" variable. This leads to a feedback law, which provides asymptotic stability of a single system, subsystems of a network or the whole system. The results were partially presented at the CDC 2010, [22].

Another type of systems is the following:

Time-delay systems

In many applications from areas such as biology, economics, mechanics, physics, social sciences and logistics [5, 65], there occur time-delays. For example, delays appear by considering transportation of material, communication and computational delays in control loops, population dynamics and price fluctuations [94]. A time-delay systems (TDS) is given in the form

$$\begin{aligned} \dot{x}(t) &= f(x^t, u(t)), \\ x^0(\tau) &= \xi(\tau),\ \tau \in [-\theta, 0]\,, \end{aligned}$$

and it is also called a *retarded functional differential equation.* θ is the maximum involved delay and the function $x^t \in C\left([-\theta,0]\,;\mathbb{R}^N\right)$ is given by $x^t(\tau) := x(t+\tau),\ \tau \in [-\theta,0]$, where $C\left([t_1,t_2]\,;\mathbb{R}^N\right)$ denotes the Banach space of continuous functions defined on $[t_1,t_2]$ equipped with the supremum norm. $\xi \in C\left([-\theta,0]\,;\mathbb{R}^N\right)$ is the initial function of the system.

The tool of a Lyapunov function for the stability analysis of systems without time-delays can not be directly applied to TDS. Considering single TDS, a natural generalization of a Lyapunov function is a Lyapunov-Krasovskii functional [44]. It was shown in [87] that the existence of an ISS-Lyapunov-Krasovskii functional is sufficient for the ISS property of a TDS. In contrast to functionals, the usage of a function is more simpler for an analysis. This motivates the introduction of the Lyapunov-Razumikhin methodology for TDS. In [121], the sufficiency of the existence of an ISS-Lyapunov-Razumikhin function for the ISS property of a single TDS was shown. In both methodologies, the necessity is not proved yet.

The ISS property for interconnected systems of TDS has not been investigated so far. In Chapter 4, we provide tools to analyze networks in view of ISS and LISS, based on the Lyapunov-Razumikhin and Lyapunov-Krasovskii approaches, which were presented at the MTNS 2010, [21]. The results are applied to a scenario of a logistic network to demonstrate the relevance of the stability analysis in applications. Further applications of the ISS property for logistic networks can be found in [15, 16, 103, 12, 13], for example.

The Lyapunov-Razumikhin and Lyapunov-Krasovskii approaches will be used for impulsive systems with time-delays:

Impulsive systems

Besides time-delays, also sudden changes or “jumps”, called impulses of the state of a system occur in applications, such as loading processes of vehicles in logistic networks, for example. Such systems are called impulsive systems and they are closely related to hybrid systems, see [43, 100, 34, 66], for example. They combine continuous and discontinuous behaviors of a system:

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)),\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t) &= g(x^-(t), u^-(t)),\ t = t_k,\ k \in \mathbb{N}, \end{aligned}$$

where $t \in \mathbb{R}_+$ and t_k are the impulse times.

The ISS property for hybrid systems was investigated in [8] and for interconnections of hybrid subsystems in [66].

The ISS and iISS properties for impulsive systems were studied in [45] for the delay-free case and in [10] for non-autonomous time-delay systems. Sufficient conditions, which assure ISS and iISS of an impulsive system, were derived using exponential ISS-Lyapunov(-Razumikhin) functions and a so-called “dwell-time condition”. In [45], the average dwell-time condition, introduced in [46] for switched systems, was used, whereas in [10] a fixed dwell-time condition was utilized. The average dwell-time condition takes the average of impulses over an interval into account, whereas the fixed dwell-time condition considers the (minimal or maximal) interval between two impulses.

In impulsive systems, also time-delays can occur. For the stability analysis for such kinds of a system, we provide a Lyapunov-Krasovskii type and a Lyapunov-Razumikhin type ISS theorem for single impulsive time-delay systems using the average dwell-time condition. In contrast to the Razumikhin-type theorem from [10], we consider autonomous time-delay systems and the average dwell-time condition. Our theorem allows to verify the ISS property for larger classes of impulse time sequences, however, we have used an additional technical condition on the Lyapunov gain in our proofs.

Networks of impulsive systems without time-delays and the ISS property were investigated in [66], where a small-gain theorem was proved under the average dwell-time condition for networks. However, time-delays were not considered in the mentioned work.

Considering networks of impulsive systems with time-delays, we prove that under a small-gain condition with linear gains and the dwell-time condition according to [45, 66] the whole system has the ISS property. We use exponential Lyapunov-Razumikhin and exponential Lyapunov-Krasovskii function(al)s. The results regarding impulsive systems with time-delays were partially presented at the NOLCOS 2010, [18], and published in [17].

The analysis of networks with time-delays in view of ISS motivates the investigation of ISS for model predictive control (MPC) of time-delay networks. Furthermore, the advantages of the ISDS property over ISS will be used for MPC of networks:

Model predictive control

Model predictive control (MPC), also known as receding horizon control, is an approach for an optimal control of systems under constraints. For example, MPC can be used to control a system in a optimal way (optimal according to small effort to achieve the goal, for example) such that the solution of the system follows a certain trajectory or that the solution is steered to an equilibrium point, where certain constraints have to be fulfilled.

By the increasing application of automation processes in industry, MPC became more and more popular during the last decades. It has many applications in the chemical, oil or automotive and aerospace industry, for example, see the survey papers [89, 90].

MPC transforms the control problem into an optimization problem: at sampling times $t = k\Delta,\ k \in \mathbb{N},\ \Delta > 0$, the trajectory of a system will be predicted until a prediction horizon. A cost function J will be minimized with respect to a control u and the solution of this optimization problem will be implemented until the next sampling time. Then, the prediction horizon is moved and the procedure starts again.

By the choice of the cost function one has many degrees of freedom for the definition and the achievement of the goals. The MPC procedure results in an optimal control to reach the goals and to satisfy possible constraints. There could be constraints to the state space, the control space or the terminal region of the state of the system. More details about MPC can be found in [78, 9, 38], for example.

However, the stability of MPC is not guaranteed in general, see [91], for example. Therefore, it is desired to derive conditions to assure stability. An overview of existing results regarding (asymptotic) stability and optimality of MPC can be found in [81] and recent results regarding (asymptotic) stability, optimality and algorithms of MPC can be found in [92, 84, 68, 38], for example.

The ISS property for MPC was investigated in [80, 79, 73, 69] for single nonlinear discrete-time systems with disturbances. There, sufficient conditions to guarantee ISS for MPC were established. Interconnections and the ISS property for MPC were analyzed in [93]. The approach in these papers is that the cost function is a Lyapunov function, which implies ISS.

In this thesis, we want to combine the ISDS property and MPC, which is not done yet. Considering single nonlinear continuous-time systems, we show that the cost function of the used MPC scheme is an ISDS-Lyapunov function, which implies ISDS of the system. For interconnections, we apply the ISDS small-gain theorem, which is a result of this thesis, showing that the cost function of the ith subsystem of the interconnection is an ISDS-Lyapunov function for the ith subsystem. We establish the ISDS property for MPC of single and interconnected nonlinear systems.

Considering time-delay systems, the asymptotic stability for MPC was investigated for single nonlinear continuous-time TDS in [29, 96, 95]. Besides asymptotic stability, the determination of the terminal cost, the terminal region and the computation of locally stabilizing controller were performed in these papers, using Lyapunov-Razumikhin and Lyapunov-Krasovskii arguments.

The ISS property for MPC of TDS has not been investigated so far. Here, we want to introduce the ISS property for MPC of nonlinear single and interconnected TDS. We show that the cost function for a single system is an ISS-Lyapunov-Krasovskii functional and apply the ISS-Lyapunov-Krasovskii approach of the chapter regarding TDS. Using the ISS Lyapunov-Krasovskii small-gain theorem for interconnections, which is a result of this thesis, we derive conditions to assure ISS for MPC of networks with TDS.

The tools presented in this work enrich the toolbox for the analysis and control of interconnected systems. They can be used in many applications from different areas, such as logistics and economics, biology, mechanics and physics, or social sciences, for example.

Organization of the thesis

Chapter 1 contains all necessary notions for the analysis and for the main results of this work. The ISDS property is investigated in Chapter 2, where we prove an ISDS small-gain theorem. Chapter 3 is devoted to the quasi-ISS/ISDS observer design for single systems and networks and the application to quantized output feedback stabilization. Time-delay systems are considered in Chapter 4, where ISS small-gain theorems using the Lyapunov-Razumikhin and Lyapunov-Krasovskii approach are proved. They are applied to a scenario of a logistic network in Section 4.3. Proceeding with impulsive systems with time-delays, ISS theorems are given in Chapter 5. The tools within the framework of ISS/ISDS for model predictive control can be found in Chapter 6. Finally, Chapter 7 summarizes the work with an overview of all results combined with open questions and outlooks of possible future research activities.

Chapter 1

Preliminaries

In this chapter, all notations and definitions are given, which are necessary for the following chapters. More precisely, the definition of ISS and its Lyapunov function characterization for single systems are included. Considering interconnected systems, we recall the main theorems regarding ISS of networks.

By x^T we denote the transposition of a vector $x \in \mathbb{R}^n,\ n \in \mathbb{N}$, furthermore $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_+^n$ denotes the positive orthant $\{x \in \mathbb{R}^n : x \geq 0\}$, where we use the standard partial order for $x, y \in \mathbb{R}^n$ given by

$$\begin{aligned} x \geq y \ &\Leftrightarrow x_i \geq y_i,\ i = 1, \ldots, n, \\ x \ngeq y \ &\Leftrightarrow \exists i : x_i < y_i \ \text{ and} \\ x > y \ &\Leftrightarrow x_i > y_i,\ i = 1, \ldots, n. \end{aligned}$$

For a nonempty index set $I \subset \{1, \ldots, n\}$, $n \in \mathbb{N}$, we denote by $\#I$ the number of elements of I and $y_I := (y_i)_{i \in I}$ for $y \in \mathbb{R}_+^n$. A projection P_I from $\mathbb{R}_+^n$ into $\mathbb{R}_+^{\#I}$ maps y to y_I. By $B(x, r)$ we denote the open ball with respect to the Euclidean norm around x of radius r.

$|\cdot|$ denotes the Euclidean norm in $\mathbb{R}^n$. The essential supremum norm of a (Lebesgue-) measurable function $f : \mathbb{R} \to \mathbb{R}^n$ is the smallest number K such that the set $\{x : f(x) > K\}$ has (Lebesgue-) measure zero and it is denoted by $\|f\|$.

$|x|_\infty$ denotes the maximum norm of $x \in \mathbb{R}^n$ and ∇V is the gradient of a function $V : \mathbb{R}^n \to \mathbb{R}_+$. We denote the set of essentially bounded (Lebesgue-) measurable functions u from $\mathbb{R}$ to $\mathbb{R}^m$ by

$$\mathrm{L}_\infty(\mathbb{R}, \mathbb{R}^m) := \{u : \mathbb{R} \to \mathbb{R}^m \text{ measurable} \mid \ \exists\ K > 0 : \ |u(t)| \leq K, \text{ for almost all (f.a.a.) } t\},$$

where f.a.a. means for all t except the set $\{t : |u(t)| > K\}$, which has measure zero.

For $t_1, t_2 \in \mathbb{R}$, $t_1 < t_2$, let $C\left([t_1, t_2]\,; \mathbb{R}^N\right)$ denote the Banach space of continuous functions defined on $[t_1, t_2]$ with values in $\mathbb{R}^N$ and equipped with the norm $\|\phi\|_{[t_1,t_2]} := \sup_{t_1 \leq s \leq t_2} |\phi(s)|$ and takes values in $\mathbb{R}^N$. Let $\theta \in \mathbb{R}_+$. The function $x^t \in C\left([-\theta, 0]\,; \mathbb{R}^N\right)$ is given by $x^t(\tau) := x(t+\tau),\ \tau \in [-\theta, 0]$. $PC\left([t_1, t_2]\,; \mathbb{R}^N\right)$ denotes the Banach space of piecewise right-continuous functions defined on $[t_1, t_2]$ equipped with the norm $\|\cdot\|_{[t_1,t_2]}$ and takes values in $\mathbb{R}^N$.

For a function $v : \mathbb{R}_+ \to \mathbb{R}^m$ we define its restriction to the interval $[s_1, s_2]$ by

$$v_{[s_1,s_2]}(t) := \begin{cases} v(t) & \text{if } t \in [s_1, s_2], \\ 0 & \text{otherwise,} \end{cases} \qquad t,\ s_1,\ s_2 \in \mathbb{R}_+.$$

We define the following classes of functions:

Definition 1.0.1.

$$\begin{aligned}
\mathcal{P} &:= \{f : \mathbb{R}^n \to \mathbb{R}_+ \mid f(0) = 0,\ f(x) > 0,\ x \neq 0\}, \\
\mathcal{K} &:= \{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \mid \gamma \textit{ is continuous, } \gamma(0) = 0 \textit{ and strictly increasing}\}, \\
\mathcal{K}_\infty &:= \{\gamma \in \mathcal{K} \mid \gamma \textit{ is unbounded}\}, \\
\mathcal{L} &:= \left\{\gamma : \mathbb{R}_+ \to \mathbb{R}_+ \,\middle|\, \gamma \textit{ is continuous and decreasing with } \lim_{t\to\infty} \gamma(t) = 0\right\}, \\
\mathcal{KL} &:= \{\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+ \mid \beta \textit{ is continuous, } \beta(\cdot, t) \in \mathcal{K},\ \beta(r, \cdot) \in \mathcal{L},\ \forall t, r \geq 0\}.
\end{aligned}$$

We will call functions of class $\mathcal{P}$ positive definite.

Note that, if $\gamma \in \mathcal{K}_\infty$, then there exists the inverse function $\gamma^{-1} : \mathbb{R}_+ \to \mathbb{R}_+$ with $\gamma^{-1} \in \mathcal{K}_\infty$, see [98], Lemma 1.1.1.

To introduce interconnected systems, we consider nonlinear systems described by ordinary differential equations of the form

$$\dot{x}(t) \;=\; f(x(t), u(t)), \tag{1.1}$$

where $t \in \mathbb{R}_+$ is the (continuous) time, $\dot{x}$ denotes the derivative of $x \in \mathbb{R}^N$, the input $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ and $f : \mathbb{R}^{N+m} \to \mathbb{R}^N$, $N, m \in \mathbb{N}$. We assume that the initial value $x(t_0) = x_0$ is given and without loss of generality we consider $t_0 = 0$. Systems of the form (1.1) are examples of dynamical systems according to [115, 64, 48].

For the existence and uniqueness of a solution of a system of the form (1.1), we need the notion of a locally Lipschitz continuous function.

Definition 1.0.2. *Let $f : D \subset \mathbb{R}^N \to \mathbb{R}^N$ be a function.*

(i) f satisfies a Lipschitz condition in D, if there exists a $L \geq 0$ such that it holds

$$\forall\ x_1, x_2 \in D :\ |f(x_1) - f(x_2)| \leq L|x_1 - x_2|.$$

L is called Lipschitz constant and f is called Lipschitz continuous.

(ii) f is called locally Lipschitz continuous in D, if for each $x \in D$ there exists a neighborhood $U(x)$ such that the restriction $f|_{D\cap U}$ satisfies a Lipschitz condition in $D \cap U$.

Since we are dealing with locally Lipschitz continuous functions, we recall the following theorem.

Theorem 1.0.3 (Theorem of Rademacher)**.** *Let $f : \mathbb{R}^N \to \mathbb{R}^N$ be a function, which satisfies a Lipschitz condition in $\mathbb{R}^N$. Then, f is differentiable in $\mathbb{R}^N$ almost everywhere (a.e.) (which means everywhere except for the set with (Lebesgue-)measure zero).*

A proof can be found in [30], page 216, for example.

To have existence and uniqueness of a solution of (1.1) we use the following theorem:

Theorem 1.0.4 (Carathéodory conditions)**.** *Consider a system of the form* (1.1)*. Let the function f be continuous and for each $R > 0$ there exists a constant $L_R > 0$ such that it holds*

$$|f(x_1, u) - f(x_2, u)| \leq L_R |x_1 - x_2|$$

for all $x_1, x_2 \in \mathbb{R}^N$ and $u \in L_\infty(\mathbb{R}, \mathbb{R}^m)$ with $|x_1|, |x_2|, |u| \leq R$. Then, for each $x_0 \in \mathbb{R}^N$ and $u \in L_\infty(\mathbb{R}, \mathbb{R}^m)$ there exists a maximal (open) interval I with $0 \in I$ and a unique absolute continuous function $\xi(t)$, which satisfies

$$\xi(t) = x_0 + \int_0^t f(x(\tau), u(\tau))\, \mathrm{d}\tau, \ \forall t \in I.$$

The proof can be found in [117], Appendix C.

We denote the unique function ξ from Theorem 1.0.4 by $x(t; x_0, u)$ or $x(t)$ in short and call it *solution* of the system (1.1) with initial value $x_0 \in \mathbb{R}^N$ and $u \in \mathrm{L}_\infty(\mathbb{R}_+, \mathbb{R}^m)$. For the existence and uniqueness of solutions of systems of the form (1.1), we assume in the rest of the thesis that the function $f : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^N$ satisfies the conditions in Theorem 1.0.4, i.e., f is continuous and locally Lipschitz in x uniformly in u.

1.1 Input-to-state stability

It is desirable to have knowledge about the systems behavior. For example, in applications it is needed to know that the trajectory of a system with bounded external input remains in a ball around the origin for all times whatever the input is. This leads to the notion of (L)ISS, introduced by Sontag [114]:

Definition 1.1.1 (Input-to-state stability)**.** *The System* (1.1) *is called locally input-to-state stable (LISS), if there exist $\rho > 0, \rho_u > 0$, $\beta \in \mathcal{KL}$ and $\gamma^{ISS} \in \mathcal{K}_\infty$ such that for all $|x_0| \leq \rho$, $\|u\| \leq \rho_u$ and all $t \in \mathbb{R}_+$ it holds*

$$|x(t; x_0, u)| \leq \max\left\{\beta(|x_0|, t), \gamma^{ISS}(\|u\|)\right\}. \tag{1.2}$$

γ^{ISS} is called gain. If $\rho = \rho_u = \infty$, then system (1.1) *is called input-to-state stable (ISS).*

(L)ISS establishes an estimation of the norm of the trajectory of a system. On the one hand, this estimation takes the initial value into account by the term $\beta(|x_0|, t)$, which tends to zero if t tends to infinity. On the other hand, it takes the supremum norm of the input into account by the term $\gamma^{\mathrm{ISS}}(\|u\|)$.

Note that we get an equivalent definition of LISS or ISS, respectively, if we replace (1.2) by

$$|x(t; x_0, u)| \leq \beta(|x_0|, t) + \gamma^{\mathrm{ISS}}(\|u\|), \tag{1.3}$$

where β and γ^{ISS} in (1.2) and (1.3) are different in general. It is known for ISS systems that if $\limsup_{t\to\infty} u(t) = 0$ then also $\lim_{t\to\infty} x(t) = 0$ holds, see [114, 118], for example. However, with $t \to \infty$, (1.2) provides only a constant positive bound for $u \not\equiv 0$.

The relationship between ISS and other stability concepts was shown in [120]. One of these concepts is the 0-global asymptotic stability (0-GAS) property, which we use in the following chapters and is defined as follows (see [120]):

Definition 1.1.2. *The system* (1.1) *with $u \equiv 0$ is called 0-global asymptotically stable (0-GAS), if there exists $\beta \in \mathcal{KL}$ such that for all x_0 and for all $t \in \mathbb{R}_+$ it holds*

$$|x(t; x_0, \underline{0})| \leq \beta(|x_0|, t),$$

where $\underline{0}$ denotes the input function identically equal to zero on $\mathbb{R}_+$.

It is not always an easy task to find the functions β and γ^{ISS} to verify the ISS property of a system. As for systems without inputs, Lyapunov functions are a helpful tool to check whether a system of the form (1.1) possesses the ISS property.

Definition 1.1.3. *A locally Lipschitz continuous function $V : D \to \mathbb{R}_+$, with $D \subset \mathbb{R}^N$ open, is called a local ISS-Lyapunov function of the system* (1.1)*, if there exist $\rho > 0, \rho_u > 0$, $\psi_1, \psi_2 \in \mathcal{K}_\infty, \tilde{\gamma}^{ISS} \in \mathcal{K}$ and $\alpha \in \mathcal{P}$ such that $B(0, \rho) \subset D$ and*

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|), \ \forall x \in D, \tag{1.4}$$

$$V(x) \geq \tilde{\gamma}^{ISS}(|u|) \Rightarrow \nabla V(x) \cdot f(x, u) \leq -\alpha(V(x)) \tag{1.5}$$

for almost all $x \in B(0, \rho) \setminus \{0\}$ and all $|u| \leq \rho_u$. If $\rho = \rho_u = \infty$, then the function V is called an ISS-Lyapunov function of the system (1.1)*. $\tilde{\gamma}^{ISS}$ is called (L)ISS-Lyapunov gain.*

The function V can be interpreted as the "energy" of the system. The condition (1.4) states that V is positive definite and radially bounded by two $\mathcal{K}_\infty$-functions. The meaning of the condition (1.5) is that outside of the region $\{x : \ V(x) < \tilde{\gamma}^{\text{ISS}}(|u|)\}$ the "energy" of the system is decreasing. In particular, for every given external input with finite norm, the energy of the system is bounded, which implies, by (1.4) that the trajectory of system also remains bounded for all times $t > 0$. However, there is no general method to find a Lyapunov function for arbitrary nonlinear systems.

The equivalence of ISS and the existence of an ISS-Lyapunov function was shown in [119, 74]:

Theorem 1.1.4. *The system* (1.1) *possesses the ISS property if and only if there exists an ISS-Lyapunov function for the system* (1.1)*.*

With the help of this theorem one can check, whether a system has the ISS property: the existence of an ISS-Lyapunov function for the system is sufficient and necessary for ISS.

Note that the ISS-Lyapunov gain $\tilde{\gamma}^{\text{ISS}}$ and the gain γ^{ISS} in the definition of ISS are different in general. An equivalent definition of an ISS-Lyapunov function can be obtained, replacing

(1.5) by

$$\nabla V(x) \cdot f(x,u) \leq \hat{\gamma}^{\text{ISS}}(|u|) - \tilde{\alpha}(V(x)),$$

where $\hat{\gamma}^{\text{ISS}} \in \mathcal{K},\ \tilde{\alpha} \in \mathcal{P}$, which is called the *dissipative Lyapunov form*, see [119], for example.

1.2 Interconnected systems

Many systems in applications are networks of subsystems, which are interconnected. This means that the evolution of a subsystem could depend on the states of other subsystems and external inputs. Analyzing such a network in view of stability the notion of ISS is useful, because it takes (internal and external) inputs of a system into account. The question is, under which condition the network possesses the ISS property and how it can be checked?

For the purpose of this work, we consider $n \geq 2$ interconnected subsystems of the form

$$\dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), u(t)),\ i = 1, \ldots, n, \tag{1.6}$$

where $n \in \mathbb{N}$, $x_i \in \mathbb{R}^{N_i},\ N_i \in \mathbb{N},\ u \in \mathrm{L}_\infty(\mathbb{R}_+, \mathbb{R}^m),\ f_i\ :\ \mathbb{R}^{\sum_{j=1}^n N_j + m} \rightarrow \mathbb{R}^{N_i}$. We assume that the function f_i satisfies the conditions in Theorem 1.0.4 to have existence and uniqueness of a solution of a subsystem for all $i = 1, \ldots, n$.

Without loss of generality we consider the same input function u for all subsystems in (1.6). One can use a projection P_i such that u_i is the input of the ith subsystem and $f_i(\ldots, u) = \tilde{f}_i(\ldots, P_i(u)) = \tilde{f}_i(\ldots, u_i)$ with $u = (u_1, \ldots, u_n)^T$, see also [25].

The ISS property for subsystems is the following: The i-th subsystem of (1.6) is called *LISS*, if there exist constants $\rho_i, \rho_{ij}, \rho^u > 0$ and functions $\gamma_{ij}^{\text{ISS}},\ \gamma_i^{\text{ISS}} \in \mathcal{K}_\infty$ and $\beta_i \in \mathcal{KL}$ such that for all initial values $\left|x_i^0\right| \leq \rho_i$, all inputs $\|x_j\|_{[0,\infty)} \leq \rho_{ij},\ \|u\| \leq \rho^u$ and all $t \in \mathbb{R}_+$ it holds

$$|x_i(t)| \leq \max\left\{\beta_i\left(\left|x_i^0\right|, t\right), \max_{j \neq i} \gamma_{ij}^{\text{ISS}}\left(\|x_j\|_{[0,t]}\right), \gamma_i^{\text{ISS}}(\|u\|)\right\}. \tag{1.7}$$

γ_{ij}^{ISS} are called gains. If $\rho_i = \rho_{ij} = \rho^u = \infty$, then the i-th subsystem of (1.6) is called ISS. By replacing (1.7) by

$$|x_i(t)| \leq \beta_i\left(\left|x_i^0\right|, t\right) + \sum_{j \neq i} \gamma_{ij}^{\text{ISS}}\left(\|x_j\|_{[0,t]}\right) + \gamma_i^{\text{ISS}}(\|u\|) \tag{1.8}$$

we get an equivalent formulation of ISS for subsystems. We refer to this as the *summation formulation* and (1.7) as the *maximum formulation* of ISS. Note that β_i and the gains in (1.7) and (1.8) are different in general, but we use the same notation for simplicity.

Note that in the ISS estimation (1.7) or (1.8) the internal and external inputs of a subsystem are taken into account. In contrast to the ISS estimation (1.2) or (1.3) for a single system, this results in adding the gains $\gamma_{ij}^{\text{ISS}}\left(\|x_j\|_{[0,t]}\right)$ to the ISS estimation of the ith subsystem, where the index j denotes the jth subsystem that is connected to the ith subsystem.

Also, an ISS-Lyapunov function for the ith subsystem can be given, where the subsystems have to be taken into account, which are connected to the ith subsystem. It reads as follows:

We assume that for each subsystem of the interconnected system (1.6) there exists a function $V_i : D_i \to \mathbb{R}_+$ with $D_i \subset \mathbb{R}^{N_i}$ open, which is locally Lipschitz continuous and positive definite. Then, the function V_i is called a *LISS-Lyapunov function of the i-th subsystem of* (1.6), if V_i satisfies the following two conditions:
There exist functions $\psi_{1i},\ \psi_{2i} \in \mathcal{K}_\infty$ such that

$$\psi_{1i}\left(|x_i|\right) \leq V_i(x_i) \leq \psi_{2i}\left(|x_i|\right),\ \forall\ x_i \in D_i \tag{1.9}$$

and there exist $\tilde{\gamma}_{ij}^{\mathrm{ISS}},\ \tilde{\gamma}_i^{\mathrm{ISS}} \in \mathcal{K}$, $\alpha_i \in \mathcal{P}$ and constants $\rho_i, \rho_{ij}, \rho^u > 0$ such that $B(0, \rho_i) \subset D_i$ and with $x = (x_1^T, \ldots, x_n^T)^T$ it holds

$$V_i(x_i) \geq \max\left\{\max_{j\neq i} \tilde{\gamma}_{ij}^{\mathrm{ISS}}\left(V_j(x_j)\right), \tilde{\gamma}_i^{\mathrm{ISS}}\left(|u|\right)\right\} \ \Rightarrow\ \nabla V_i(x_i)\cdot f_i(x,u) \leq -\alpha_i\left(V_i(x_i)\right) \tag{1.10}$$

for almost all $x_i \in B(0, \rho_i)$, $|x_j| \leq \rho_{ij}$, $|u| \leq \rho^u$. If $\rho_i = \rho_{ij} = \rho^u = \infty$, then V_i is called an *ISS-Lyapunov function of the i-th subsystem* of (1.6). Functions $\tilde{\gamma}_{ij}^{\mathrm{ISS}}$ are called *(L)ISS-Lyapunov gains*.

Note that an equivalent formulation of an ISS-Lyapunov function can be obtained, if we replace (1.10) by

$$V_i(x_i) \geq \sum_{j\neq i} \bar{\gamma}_{ij}^{\mathrm{ISS}}\left(V_j(x_j)\right) + \bar{\gamma}_i^{\mathrm{ISS}}\left(|u|\right) \Rightarrow\ \nabla V_i(x_i)\cdot f_i(x,u) \leq -\bar{\alpha}_i\left(V_i(x_i)\right), \tag{1.11}$$

where $\bar{\gamma}_{ij}^{\mathrm{ISS}}, \bar{\gamma}_i^{\mathrm{ISS}} \in \mathcal{K}$ and $\bar{\alpha}_i \in \mathcal{P}$.

We consider an interconnected system of the form (1.6) as one single system (1.1) with $x = \left(x_1^T, \ldots, x_n^T\right)^T,\ f(x,u) = \left(f_1(x,u)^T, \ldots, f_n(x,u)^T\right)^T$ and call it *overall or whole system*. It is not guaranteed that the overall system possesses the ISS property even if all subsystems are ISS. A well-developed condition to verify ISS and to construct a Lyapunov function for the whole system is a small-gain condition, see [25, 98, 28], for example. To this end, we collect all the gains $\tilde{\gamma}_{ij}^{\mathrm{ISS}}$ in a matrix, called gain-matrix $\Gamma := (\tilde{\gamma}_{ij}^{\mathrm{ISS}})_{n\times n},\ i,j = 1,\ldots,n,\ \tilde{\gamma}_{ii}^{\mathrm{ISS}} \equiv 0$, which defines a map $\Gamma :\ \mathbb{R}_+^n\ \to\ \mathbb{R}_+^n$ by

$$\Gamma\left(s\right) := \left(\max_j \tilde{\gamma}_{1j}^{\mathrm{ISS}}(s_j), \ldots, \max_j \tilde{\gamma}_{nj}^{\mathrm{ISS}}(s_j)\right)^T,\ s \in \mathbb{R}_+^n. \tag{1.12}$$

Note that the matrix Γ describes in particular the interconnection structure of the network. Moreover, it contains information about the mutual influence between the subsystems, which can be used to verify the (L)ISS property of networks.

If we use (1.11) instead of (1.10), we collect the gains in the matrix $\overline{\Gamma} := (\bar{\gamma}_{ij}^{\mathrm{ISS}})_{n\times n},\ i,j = 1,\ldots,n,\ \bar{\gamma}_{ii}^{\mathrm{ISS}} \equiv 0$, which defines a map $\overline{\Gamma} :\ \mathbb{R}_+^n\ \to\ \mathbb{R}_+^n$ by

$$\overline{\Gamma}\left(s\right) := \left(\sum_j \bar{\gamma}_{1j}^{\mathrm{ISS}}(s_j), \ldots, \sum_j \bar{\gamma}_{nj}^{\mathrm{ISS}}(s_j)\right)^T,\ s \in \mathbb{R}_+^n. \tag{1.13}$$

For the stability analysis of the whole system in view of LISS, we will use the following condition (see [27]): we say that a gain-matrix Γ satisfies the *local small-gain condition (LSGC) on* $[0, w^*]$, $w^* \in \mathbb{R}_+^n,\ w^* > 0$, provided that

$$\Gamma(w^*) < w^* \text{ and } \Gamma(s) \not\geq s,\ \forall s \in [0, w^*],\ s \neq 0. \tag{1.14}$$

Notation $\ngeq$ denotes that there is at least one component $i \in \{1, \ldots, n\}$ such that $\Gamma(s)_i < s_i$. In view of ISS, we say that Γ satisfies the *small-gain condition (SGC)* (see [98]) if

$$\Gamma(s) \ngeq s, \ \forall \ s \in \mathbb{R}_+^n \backslash \{0\} . \tag{1.15}$$

If we consider the summation formulation of ISS or ISS-Lyapunov functions, respectively, the SGC is of the form (see also [25])

$$\left(\overline{\Gamma} \circ D\right)(s) \ngeq s, \ \forall \ s \in \mathbb{R}_+^n \backslash \{0\} , \tag{1.16}$$

where $D : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is a diagonal operator defined by

$$D(s) := \begin{pmatrix} (\mathrm{Id} + \varpi)(s_1) \\ \vdots \\ (\mathrm{Id} + \varpi)(s_n) \end{pmatrix}, \ s \in \mathbb{R}_+^n, \ \varpi \in \mathcal{K}_\infty.$$

For simplicity, we will use Γ for a matrix defined by (1.12), using the maximum formulation or defined by (1.13), using the summation formulation. Note that by $\gamma_{ij}^{\mathrm{ISS}} \in \mathcal{K}_\infty \cup \{0\}$ and for $v, \ w \in \mathbb{R}_+^n$ we get

$$v \geq w \ \Rightarrow \ \Gamma(v) \geq \Gamma(w).$$

Remark 1.2.1. *The SGC (1.15) is equivalent to the cycle condition (see [98], Lemma 2.3.14 for details). A k-cycle in a matrix $\Gamma = (\gamma_{ij})_{n \times n}$ is a sequence of $\mathcal{K}_\infty$ functions $(\gamma_{i_0 i_1}, \gamma_{i_1 i_2}, \ldots, \gamma_{i_{k-1} i_k})$ of length k with $i_0 = i_k$. The cycle condition for a matrix Γ is that all k-cycles of Γ are contractions, i.e.,*

$$\gamma_{i_0 i_1} \circ \gamma_{i_1, i_2} \circ \ldots \circ \gamma_{i_{k-1}, i_k} < \mathrm{Id},$$

for all $i_0, \ldots, i_k \in \{1, \ldots, n\}$ with $i_0 = i_k$ and $k \leq n$. See [98] and [57] for further details.

To recall the Lyapunov versions of the small-gain theorem for the LISS and ISS property from [28] and [27], we need the following:

Definition 1.2.2. *A continuous path $\sigma \in \mathcal{K}_\infty^n$ is called an Ω-path with respect to Γ, if*

(i) for each i, the function σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;

(ii) for every compact set $P \subset (0, \infty)$ there are constants $0 < K_1 < K_2$ such that for all points of differentiability of σ_i^{-1} and $i = 1, \ldots, n$ we have

$$0 < K_1 \leq (\sigma_i^{-1})'(r) \leq K_2, \quad \forall r \in P; \tag{1.17}$$

(iii) it holds

$$\Gamma(\sigma(r)) < \sigma(r), \quad \forall r > 0. \tag{1.18}$$

More details about an Ω-path can be found in [98, 99, 28].

The following proposition is useful for the construction of an ISS-Lyapunov function for the whole system.

Proposition 1.2.3. *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be a gain-matrix. If Γ satisfies the small-gain condition (1.15), then there exists an Ω-path σ with respect to Γ. If Γ satisfies the SGC in the form* (1.16)*, then there exists an Ω-path σ, where $\Gamma(\sigma(r)) < \sigma(r)$ is replaced by* $(\Gamma \circ D)(\sigma(r)) < \sigma(r),\ \forall r > 0$.

The proof can be found in [28], Theorem 5.2, see also [98, 99], however only the existence is proved in these works. In [20], Proposition 3.4., it was shown how to construct a finite but arbitrary "long" path.

For the case that Γ satisfies the LSGC (1.14) a strictly increasing path $\sigma : [0,1] \to [0, w^*]$ exists, which satisfies $\Gamma(\sigma(r)) < \sigma(r),\ \forall r \in (0,1]$. σ is piecewise linear and satisfies $\sigma(0) = 0,\ \sigma(1) = w^*$, see Proposition 4.3 in [28], Proposition 5.2 in [99].

Now, we recall the main results of [27] and [28]. They show under which conditions the overall system possesses the (L)ISS property. Moreover, an explicit construction of the (L)ISS-Lyapunov function for the whole system is given.

Theorem 1.2.4. *Let V_i be an ISS-Lyapunov function for the i-th subsystem in* (1.6)*, for all $i = 1,\dots,n$. Let Γ be a gain-matrix and satisfies the SGC* (1.15)*. Then, the whole system of the form* (1.1) *is ISS and the ISS-Lyapunov function of the overall system is given by* $V(x) = \max_i \sigma_i^{-1}(V_i(x_i))$.

The proof can be found in [23], Theorem 6 or in a generalized form in [28], Corollary 5.5. A version using LISS is given by the following:

Theorem 1.2.5. *Assume that each subsystem of* (1.6) *admits an LISS-Lyapunov function and that the corresponding gain-matrix Γ satisfies the LSGC* (1.14)*. Then, the whole system of the form* (1.1) *is LISS and the LISS-Lyapunov function of the overall system is given by* $V(x) = \max_i \sigma_i^{-1}(V_i(x_i))$.

The proof can be found in [27], Theorem 5.5.

An approach for a numerical construction of LISS-Lyapunov functions can be found in [33].

The mentioned theorems provide tools how to check, if a network possesses the ISS property: we have to find ISS-Lyapunov functions and the corresponding gains for the subsystems. If the gains satisfy the small-gain condition, then the whole system is ISS.

In the following chapters, we use the mentioned tools for the stability analysis, observer design and control of interconnected systems. Moreover, tools for the stability analysis of networks of time-delay systems and of networks of impulsive systems with time-delays are derived. With all these notations and considerations of this chapter, we are able to formulate and prove the main results of this work in the next chapters.

Chapter 2

Input-to-state dynamical stability (ISDS)

In this chapter, the notion of input-to-state dynamical stability (ISDS) is described and as the main result of this chapter, we prove an ISDS-Lyapunov small-gain theorem.

The stability notion ISDS was introduced in [35], further investigated in [36] and some local properties studied in [40]. ISDS is equivalent to ISS, however, one advantage of ISDS over ISS is that the bound for the trajectories takes essentially only the recent values of the input u into account and in many cases it gives a better bound for trajectories due to the *memory fading effect* of the input u.

Similar to ISS systems, the ISDS property of system (1.1) is equivalent to the existence of an ISDS-Lyapunov function for system (1.1), see [36]. Also a 0-GAS small-gain theorem for two interconnected systems with the input $u = 0$ can be found in [36].

Another advantage of ISDS over ISS is that the gains in the trajectory based definition of ISDS are the same as in the definition of the ISDS-Lyapunov function, which is in general not true for ISS systems.

In this chapter, we extend the result for interconnected ISS systems to the case of ISDS systems. In particular, we provide a tool for the stability analysis of networks in view of ISDS. This is a small-gain theorem for $n \in \mathbb{N}$ interconnected ISDS systems of the form (1.6) with a construction of an ISDS-Lyapunov function as well as the rates and gains of the ISDS estimation for the entire system. Moreover, we derive decay rates of the trajectories of $n \in \mathbb{N}$ interconnected ISDS systems and the trajectory of the entire system with the external input $u = 0$. These results are compared to an example in [36] for $n = 2$ interconnected systems with $u = 0$.

The next section introduces the notion of ISDS for single systems of the form (1.1). Section 2.2 contains the main result of this chapter. Examples are given in Section 2.3.

2.1 ISDS for single systems

We consider systems of the form (1.1). For the ISDS property we define the class of functions $\mathcal{KLD}$ by

$$\mathcal{KLD} := \{\mu \in \mathcal{KL} \mid \mu(r, t+s) = \mu(\mu(r,t), s), \forall r, t, s \geq 0\}\,.$$

Remark 2.1.1. *The condition $\mu(r, t+s) = \mu(\mu(r,t), s)$ implies $\mu(r, 0) = r, \forall\, r \geq 0$. To show this, suppose that there exists $r \geq 0$ such that $\mu(r, 0) \neq r$. Then*

$$\mu(r, 0) = \mu(r, 0+0) = \mu(\mu(r,0), 0) \neq \mu(r, 0),$$

which is a contradiction. The last inequality follows from the strict monotonicity of μ with respect to the first argument. This shows the assertion.

The notion of ISDS was introduced in [35] and it is as follows:

Definition 2.1.2 (Input-to-state dynamical stability (ISDS))**.** *The system* (1.1) *is called input-to-state dynamically stable (ISDS), if there exist $\mu \in \mathcal{KLD}$, η, $\gamma^{ISDS} \in \mathcal{K}_\infty$ such that for all initial values x_0 and all inputs u it holds*

$$|x(t; x_0, u)| \leq \max\{\mu(\eta(|x_0|), t), \underset{\tau \in [0,t]}{\textit{ess sup}}\, \mu(\gamma^{ISDS}(|u(\tau)|), t-\tau)\} \tag{2.1}$$

for all $t \in \mathbb{R}_+$. μ is called decay rate*, η is called* overshoot gain *and γ^{ISDS} is called* robustness gain*.*

Remark 2.1.3. *One obtains an equivalent definition of ISDS if one replaces the Euclidean norm in* (2.1) *by any other norm. Moreover, it can be checked that all results in [36] and [35] hold true, if one uses a different norm instead of the Euclidean one.*

It was shown in [35], Proposition 3.4.4 (ii) that ISDS is equivalent to ISS in the maximum formulation (1.2). Note that in contrast to ISS, the ISDS property takes essentially only the recent values of the input u into account and past values of the input will be "forgotten" by time, which is also known as the *memory fading effect*. In particular, it follows immediately from (2.1):

Lemma 2.1.4. *If the system* (1.1) *is ISDS and $\underset{t\to\infty}{\textit{lim sup}}\, |u(t)| = 0$, then it holds*

$$\lim_{t\to\infty} |x(t; x_0, u)| = 0.$$

Proof. Since (1.1) is ISDS we have

$$\begin{aligned}
&|x(t; x_0, u)| \leq \max\{\mu(\eta(|x_0|), t), \underset{\tau \in [0,t]}{\text{ess sup}}\, \mu(\gamma^{\text{ISDS}}(|u(\tau)|), t-\tau)\} \\
=\ & \max\{\mu(\eta(|x_0|), t), \underset{\tau \in [0,\frac{t}{2}]}{\text{ess sup}}\, \mu(\gamma^{\text{ISDS}}(|u(\tau)|), t-\tau), \underset{\tau \in [\frac{t}{2},t]}{\text{ess sup}}\, \mu(\gamma^{\text{ISDS}}(|u(\tau)|), t-\tau)\} \\
\leq\ & \max\{\mu(\eta(|x_0|), t), \mu(\gamma^{\text{ISDS}}(\|u\|_{[0,\frac{t}{2}]}), \frac{t}{2}), \underset{\tau \in [\frac{t}{2},t]}{\text{ess sup}}\, \mu(\gamma^{\text{ISDS}}(|u(\tau)|), 0)\}.
\end{aligned}$$

It holds $\limsup_{t\to\infty} |u(t)| = 0$ and u is essentially bounded, i.e., there exists a $K \in \mathbb{R}_+$ such that $\|u\|_{[0,t]} \leq K$, for all $t > 0$. Furthermore, for all $\varepsilon > 0$ there exists a $T > 0$ such that for all $\tau \in \left[\frac{T}{2}, T\right]$ it holds $\operatorname{ess\,sup}_{\tau\in\left[\frac{T}{2},T\right]} \gamma^{\mathrm{ISDS}}(|u(\tau)|) < \varepsilon$. With these considerations, the $\mathcal{KLD}$-property of μ and Remark 2.1.1 we get

$$\begin{aligned}\lim_{t\to\infty} |x(t;x_0,u)| \leq & \lim_{t\to\infty} \max\{\mu(\eta(|x_0|),t), \mu(\gamma^{\mathrm{ISDS}}(\|u\|_{[0,\frac{t}{2}]}), \frac{t}{2}), \operatorname*{ess\,sup}_{\tau\in\left[\frac{t}{2},t\right]} \gamma^{\mathrm{ISDS}}(|u(\tau)|)\} \\ \leq & \max\{\lim_{t\to\infty} \mu(\gamma^{\mathrm{ISDS}}(K), \frac{t}{2}), \lim_{t\to\infty} \operatorname*{ess\,sup}_{\tau\in\left[\frac{t}{2},t\right]} \gamma^{\mathrm{ISDS}}(|u(\tau)|)\} = 0.\end{aligned}$$

□

In the rest of the thesis, we assume the functions μ, η and γ^{ISDS} to be C^∞ in $\mathbb{R}_+ \times \mathbb{R}$ or $\mathbb{R}_+$, respectively. This regularity assumption is not restrictive, because for non-smooth rates and gains one can find smooth functions arbitrarily close to the original ones, which was shown in [35], Appendix B.

As we know that Lyapunov functions are an important to tool to verify the ISS property of systems of the form (1.1), this is also the case for the ISDS property.

Definition 2.1.5 (ISDS-Lyapunov function)**.** *Given $\varepsilon > 0$, a function $V : \mathbb{R}^N \to \mathbb{R}_+$, which is locally Lipschitz continuous on $\mathbb{R}^N \setminus \{0\}$, is called an ISDS-Lyapunov function of the system* (1.1)*, if there exist $\eta, \gamma^{ISDS} \in \mathcal{K}_\infty$, $\mu \in \mathcal{KLD}$ such that it holds*

$$\frac{|x|}{1+\varepsilon} \leq V(x) \leq \eta\left(|x|\right), \ \forall x \in \mathbb{R}^N, \tag{2.2}$$

$$V(x) > \gamma^{ISDS}\left(|u|\right) \Rightarrow \nabla V(x) \cdot f(x,u) \leq -\left(1-\varepsilon\right) g\left(V(x)\right) \tag{2.3}$$

for almost all $x \in \mathbb{R}^N \setminus \{0\}$ and all u, where μ solves the equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\mu(r,t) = -g\left(\mu\left(r,t\right)\right), \quad r,t > 0 \tag{2.4}$$

for a locally Lipschitz continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$.

The equivalence of ISDS and the existence of a smooth ISDS-Lyapunov function was proved in [36]. Here, we use locally Lipschitz continuous Lyapunov functions, which are differentiable almost everywhere by Theorem of Rademacher (Theorem 1.0.3).

Theorem 2.1.6. *The system* (1.1) *is ISDS with $\mu \in \mathcal{KLD}$ and η, $\gamma^{ISDS} \in \mathcal{K}_\infty$, if and only if for each $\varepsilon > 0$ there exists an ISDS-Lyapunov function V.*

Proof. This follows by Theorem 4, Lemma 16 in [36] and Proposition 3.5.6 in [35]. □

Remark 2.1.7. *Note that for a system, which possesses the ISDS property, it holds that the decay rate μ and gains η, γ^{ISDS} in Definition 2.1.2 are exactly the same as in Definition 2.1.5. Recall that the gains of the definition of ISS (Definition 1.1.1) are different in general from the ISS-Lyapunov gains in Definition 1.1.3.*

In order to have ISDS-Lyapunov functions with more regularity, one can use Lemma 17 in [36], which shows that for a locally Lipschitz function V there exists a smooth function $\tilde{V}$ arbitrary close to V. To demonstrate the advantages of ISDS over ISS, we consider the following example:

Example 2.1.8. *Consider the system*

$$\dot{x}(t) = -x(t) + u(t), \tag{2.5}$$

$x \in \mathbb{R},\ t \in \mathbb{R}_+$ with a given initial value x_0. The input is chosen as

$$u(t) = \begin{cases} 4, & 0 \le t \le 10, \\ 0, & \textit{otherwise}. \end{cases}$$

From the general equation for the solution of linear systems, namely $x(t;x_0,u) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-s)}Bu(s)\mathrm{d}s$, we get with $t_0 = 0$

$$|x(t;x_0,u)| \le |x_0|\, e^{-t} + \|u\|_\infty\,,$$

which implies that the system (2.5) has the ISS property with $\beta(|x_0|\,,t) = |x_0|\, e^{-t}$ and $\gamma^{ISS}(\|u\|_\infty) = \|u\|_\infty$. The estimation is displayed in the Figure 2.1 with $x_0 = 0.1$.

To verify the ISDS property, we use ISDS-Lyapunov functions. We choose $V(x) = |x|$ as a candidate for the ISDS-Lyapunov function. For any $\varepsilon > 0$ and by the choice $\gamma^{ISDS}(|u|) := \frac{1}{\delta}|u|$, with given $0 < \delta < 1$ we obtain

$$(1+\varepsilon)\gamma^{ISDS}(|u|) \le V(x) \;\Rightarrow\; \nabla V(x)\cdot f(x,u) \le \tfrac{-1+\varepsilon^2+\delta-\delta\varepsilon}{1-\varepsilon^2}\,|x| \le -(1-\varepsilon)(1-\delta)\,|x|\,.$$

By $g(r) := (1-\delta)\,r$ we get $\mu(r,t) = e^{-(1-\delta)t}r$ (as solution of $\dot{\mu} = -g(\mu)$) and hence, the system (2.5) has the ISDS property.

Note that the choice δ close to 1 results in a sharp gain γ^{ISDS} but slow decay rate μ (Figure 2.2 with $\delta = \frac{99}{100}$ and $x_0 = 0.1$). In contrast, by a smaller choice δ this results in more conservative gain γ^{ISDS} but faster decay rate μ (Figure 2.3 with $\delta = \frac{3}{4}$ and $x_0 = 0.1$).

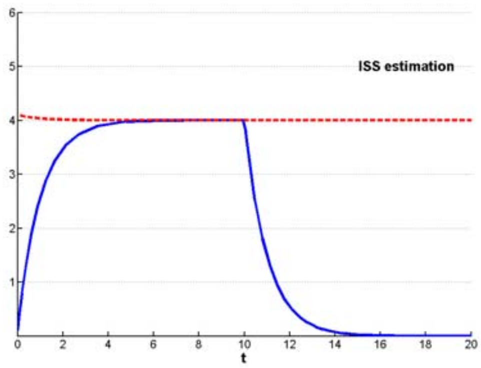

Figure 2.1: ISS estimation with $x_0 = 0.1$.

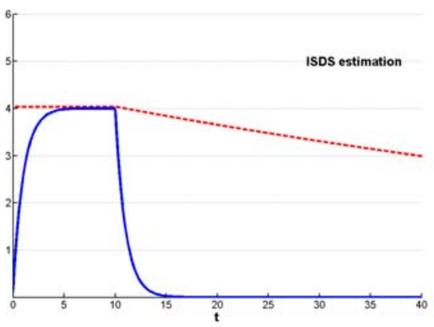

Figure 2.2: ISDS estimation with $\delta = \frac{99}{100}$, $x_0 = 0.1$.

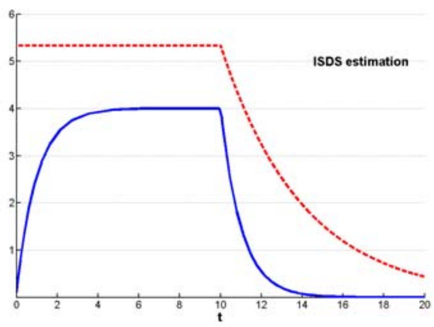

Figure 2.3: ISDS estimation with $\delta = \frac{3}{4}$, $x_0 = 0.1$.

From Figures 2.1-2.3, we perceive that the ISDS estimation tends to zero, if the input tends to zero in contrast to the ISS estimation. This property of the ISDS estimation is known as the memory fading effect.

In the next section, we provide an ISDS small-gain theorem for interconnected systems with a construction of an ISDS-Lyapunov function for the whole system.

2.2 ISDS for interconnected systems

We consider interconnected systems of the form (1.6). The ISDS property for subsystems reads as follows:

The *i-th subsystem of* (1.6) *is called ISDS*, if there exists a $\mathcal{KLD}$-function μ_i and functions η_i, γ_i^{ISDS} and $\gamma_{ij}^{\text{ISDS}} \in \mathcal{K}_\infty \cup \{0\}$, $i,j = 1,\ldots,n$ with $\gamma_{ii}^{\text{ISDS}} = 0$ such that the solution $x_i(t;x_i^0,u) = x_i(t)$ for all initial values x_i^0 and all inputs x_j, $j \neq i$, u satisfies

$$|x_i(t)| \leq \max\left\{\mu_i(\eta_i(|x_i^0|),t), \max_{j\neq i} \nu_{ij}(x_j,t), \nu_i(u,t)\right\} \tag{2.6}$$

for all $t \in \mathbb{R}_+$, where

$$\begin{aligned}\nu_i(u,t) &:= \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu_i(\gamma_i^{\text{ISDS}}(|u(\tau)|),t-\tau),\\ \nu_{ij}(x_j,t) &:= \sup_{\tau\in[0,t]} \mu_i(\gamma_{ij}^{\text{ISDS}}(|x_j(\tau)|),t-\tau)\end{aligned}$$

$i,j = 1,\ldots,n$. $\gamma_{ij}^{\text{ISDS}}$, γ_i^{ISDS} are called (nonlinear) robustness gains.

To show the ISDS property for networks, we need the gain-matrix Γ^{ISDS}, which is defined by $\Gamma^{\text{ISDS}} := \left(\gamma_{ij}^{\text{ISDS}}\right)_{n\times n}$ with $\gamma_{ii}^{\text{ISDS}} \equiv 0$, $i,j = 1,\ldots,n$ and defined by (1.12).

Definition 2.2.1. *For vector valued functions $x = (x_1^T,\ldots,x_n^T)^T : \mathbb{R}_+ \to \mathbb{R}^{\sum_{i=1}^n N_i}$ with $x_i : \mathbb{R}_+ \to \mathbb{R}^{N_i}$ and times $0 \leq t_1 \leq t_2$, $t \in \mathbb{R}_+$ we define*

$$|x(t)| := (|x_1(t)|,\ldots,|x_n(t)|)^T \in \mathbb{R}_+^n \text{ and } \|x\|_{[0,t]} \text{ accordingly.}$$

For $u \in \mathbb{R}^m$, $t \in \mathbb{R}_+$ and $s \in \mathbb{R}_+^n$ we define

$$\begin{aligned}\bar{\gamma}^{ISDS}(|u(t)|) &:= \left(\gamma_1^{ISDS}(|u(t)|), \ldots, \gamma_n^{ISDS}(|u(t)|)\right)^T \in \mathbb{R}_+^n,\\ \bar{\mu}(s,t) &:= \left(\mu_1(s_1,t), \ldots, \mu_n(s_n,t)\right)^T \in \mathbb{R}_+^n,\\ \bar{\eta}(s) &:= \left(\eta_1(s_1), \ldots, \eta_n(s_n)\right)^T \in \mathbb{R}_+^n.\end{aligned}$$

Now, we can rewrite condition (2.6) for all subsystems in a compact form

$$|x(t)| \leq \max\left\{\bar{\mu}\left(\bar{\eta}\left(|x^0|\right), t\right), \sup_{\tau\in[0,t]} \bar{\mu}\left(\Gamma^{\text{ISDS}}\left(|x(\tau)|\right), t-\tau\right), \operatorname*{ess\,sup}_{\tau\in[0,t]} \bar{\mu}(\bar{\gamma}^{\text{ISDS}}(|u(\tau)|), t-\tau)\right\} \tag{2.7}$$

for all $t \in \mathbb{R}_+$. Note that the maximum, the supremum and the essential supremum used in (2.7) for vectors are taken component-by-component. For the ISDS property, from (2.7), using the $\mathcal{KLD}$-property of μ and with $\Gamma^{\text{ISS}} := \Gamma^{\text{ISDS}}$, $\bar{\gamma}^{\text{ISS}} := \bar{\gamma}^{\text{ISDS}}$, $\bar{\beta}(r,t) := \bar{\mu}(\bar{\eta}(r),t)$ we get

$$|x(t)| \leq \max\left\{\bar{\beta}\left(|x^0|, t\right), \Gamma^{\text{ISS}}\left(\|x\|_{[0,t]}\right), \bar{\gamma}^{\text{ISS}}(\|u\|)\right\}.$$

This implies that each subsystem of (1.6) is ISS and provided that Γ^{ISDS} satisfies the SGC (1.15), also Γ^{ISS} satisfies the SGC (1.15), i.e., the interconnection is ISS and hence ISDS. However, we loose the quantitative information about the rate and gains of the ISDS estimation for the whole system in such a way.

In order to conserve the quantitative information of the ISDS rate and gains of the overall system, we utilize ISDS-Lyapunov functions. For subsystems of the form (1.6) they read as follows:

We assume that for each subsystem of (1.6) there exists a function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, which is locally Lipschitz continuous and positive definite. Given $\varepsilon_i > 0$, a function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, which is locally Lipschitz continuous on $\mathbb{R}^{N_i}\setminus\{0\}$ is an *ISDS-Lyapunov function of the i-th subsystem in* (1.6), if it satisfies:

(i) there exists a function $\eta_i \in \mathcal{K}_\infty$ such that for all $x_i \in \mathbb{R}^{N_i}$ it holds

$$\frac{|x_i|}{1+\varepsilon_i} \leq V_i(x_i) \leq \eta_i\left(|x_i|\right); \tag{2.8}$$

(ii) there exist functions $\mu_i \in \mathcal{KLD}$, $\gamma_i^{\text{ISDS}} \in \mathcal{K}_\infty \cup \{0\}$, $\gamma_{ij}^{\text{ISDS}} \in \mathcal{K}_\infty \cup \{0\}$, $j = 1, \ldots, n$, $i \neq j$ such that for almost all $x_i \in \mathbb{R}^{N_i}\setminus\{0\}$, all inputs x_j, $j \neq i$ and u it holds

$$V_i(x_i) > \max\{\gamma_i^{\text{ISDS}}\left(|u|\right), \max_{j\neq i} \gamma_{ij}^{\text{ISDS}}(V_j(x_j))\} \Rightarrow \nabla V_i(x_i) f_i(x,u) \leq -\left(1-\varepsilon_i\right) g_i(V_i(x_i)), \tag{2.9}$$

where $\mu_i \in \mathcal{KLD}$ solves the equation $\frac{\mathrm{d}}{\mathrm{d}t}\mu_i(r,t) = -g_i\left(\mu_i\left(r,t\right)\right)$, $r,t > 0$ for some locally Lipschitz continuous function $g_i : \mathbb{R}_+ \to \mathbb{R}_+$.

Now, we state the main result of this chapter, which provides a tool to check whether a network possesses the ISDS property. Moreover, the decay rate and the gains of the ISDS estimation for the network can be constructed explicitly.

Theorem 2.2.2. *Assume that each subsystem of* (1.6) *has the ISDS property. This means that for each subsystem and for each $\varepsilon_i > 0$ there exists an ISDS-Lyapunov function V_i, which satisfies (2.8) and (2.9). Let Γ^{ISDS} be given by* (1.12)*, satisfying the small-gain condition (1.15) and let $\sigma \in \mathcal{K}_\infty^n$ be an Ω-path from Proposition 1.2.3 with $\Gamma = \Gamma^{ISDS}$. Then, the whole system* (1.1) *has the ISDS property and its ISDS-Lyapunov function is given by*

$$V(x) = \psi^{-1}\left(\max_i \left\{\sigma_i^{-1}\left(V_i(x_i)\right)\right\}\right) \tag{2.10}$$

with rates and gains

$$\begin{aligned} g(r) &= (\psi^{-1})'\left(\psi(r)\right) \min_i \left\{(\sigma_i^{-1})'(\sigma_i(\psi(r)))g_i(\sigma_i(\psi(r)))\right\}, \\ \eta(r) &= \psi^{-1}(\max_i \left\{\sigma_i^{-1}(\eta_i(r))\right\}), \\ \gamma^{ISDS}(r) &= \psi^{-1}(\max_i \left\{\sigma_i^{-1}(\gamma_i^{ISDS}(r))\right\}), \end{aligned} \tag{2.11}$$

where $r > 0$ and $\psi\left(|x|\right) = \min_i \sigma_i^{-1}\left(\frac{|x|}{\sqrt{n}}\right)$.

The proof of Theorem 2.2.2 follows the idea of the proof of Theorem 5.3 in [28] and corresponding results in [24] with changes due to the construction of the gains and of the rate of the whole system.

Proof. Let $0 \neq x = \left(x_1^T, \dots, x_n^T\right)^T$. We define

$$\overline{V}(x) := \max_i \left\{\sigma_i^{-1}\left(V_i(x_i)\right)\right\}, \quad \bar{\eta}(|x|) := \max_i \left\{\sigma_i^{-1}(\eta_i(|x|))\right\}, \quad \psi\left(|x|\right) := \min_i \sigma_i^{-1}\left(\frac{|x|}{\sqrt{n}}\right),$$

where V_i satisfies (2.8) for $i = 1, \dots, n$. Note that $\sigma_i^{-1} \in \mathcal{K}_\infty$. Let j be such that $|x|_\infty = |x_j|_\infty$ for some $j \in \{1, \dots, n\}$, then it holds

$$\max_i \sigma_i^{-1}\left(\tfrac{|x_i|}{1+\varepsilon_i}\right) \geq \max_i \sigma_i^{-1}\left(\tfrac{|x_i|_\infty}{1+\varepsilon}\right) \geq \sigma_j^{-1}\left(\tfrac{|x_j|_\infty}{1+\varepsilon}\right) \geq \min_i \sigma_i^{-1}\left(\tfrac{|x|}{\sqrt{n}(1+\varepsilon)}\right) \tag{2.12}$$

where $\varepsilon := \max_i \varepsilon_i$ and we have

$$\psi\left(\frac{|x|}{1+\varepsilon}\right) \leq \overline{V}(x) \leq \bar{\eta}(|x|). \tag{2.13}$$

Note that $\overline{V}$ is locally Lipschitz continuous and hence it is differentiable almost everywhere. We define $I := \{i \in \{1, \dots, n\} |\ \overline{V}(x) = \left\{\sigma_i^{-1}\left(V_i(x_i)\right)\right\} \geq \max_{j, j\neq i}\{\sigma_j^{-1}\left(V_j(x_j)\right)\}\}$. Fix an $i \in I$. Let $\bar{\gamma}^{\text{ISDS}}(|u|) := \max_j \left\{\sigma_j^{-1}(\gamma_j^{\text{ISDS}}(|u|))\right\}$, $j = 1, \dots, n$. Assume $\overline{V}(x) > \bar{\gamma}^{\text{ISDS}}(|u|)$. Then,

$$V_i(x_i) = \sigma_i(\overline{V}(x)) > \sigma_i(\sigma_i^{-1}(\gamma_i^{\text{ISDS}}(|u|))) = \gamma_i^{\text{ISDS}}(|u|).$$

From (iii) in Definition 1.2.2 we have

$$V_i(x_i) = \sigma_i(\overline{V}(x)) > \max_{j\neq i} \gamma_{ij}^{\text{ISDS}}(\sigma_j(\overline{V}(x))) \geq \max_{j\neq i} \gamma_{ij}^{\text{ISDS}}(V_j(x_j)).$$

Thus, for almost all $x \in \mathbb{R}^N$ (2.9) implies

$$\nabla\overline{V}(x)f(x,u) \leq -(1-\varepsilon_i)\left(\sigma_i^{-1}\right)'(V_i(x_i))g_i(V_i(x_i)) = -(1-\varepsilon_i)\tilde{g}_i(\overline{V}(x)),$$

where $\tilde{g}_i(r) := \left(\sigma_i^{-1}\right)'(\sigma_i(r))g_i(\sigma_i(r))$ is positive definite and locally Lipschitz. As index i was arbitrary in these considerations, with $\bar{\gamma}^{\text{ISDS}}(|u|) = \max_j \left\{\sigma_j^{-1}(\gamma_j^{\text{ISDS}}(|u|))\right\}$ and $\bar{g}(r) := \min_i \tilde{g}_i(r)$, $\varepsilon = \max_i \varepsilon_i$ the condition (2.3) for the function $\overline{V}$ is satisfied. From (2.13) we get

$$\frac{|x|}{1+\varepsilon} \leq \psi^{-1}\left(\overline{V}(x)\right) \leq \psi^{-1}\left(\bar{\eta}\left(|x|\right)\right)$$

and we define $V(x) := \psi^{-1}\left(\overline{V}(x)\right)$ as the ISDS-Lyapunov function candidate of the whole system with $\eta(|x|) := \psi^{-1}\left(\bar{\eta}\left(|x|\right)\right)$. Note that $\psi^{-1} \in \mathcal{K}_\infty$ and $V(x)$ is locally Lipschitz continuous. By the previous calculations for $\overline{V}(x)$ it holds

$$V(x) > \psi^{-1}\left(\bar{\gamma}^{\text{ISDS}}\left(|u|\right)\right) =: \gamma^{\text{ISDS}}\left(|u|\right) \;\Rightarrow\; \dot{V}(x) \leq -(1-\varepsilon)g\left(V(x)\right), \text{ a.e.,}$$

where $g(r) := (\psi^{-1})'\left(\psi(r)\right)\bar{g}\left(\psi(r)\right)$ is locally Lipschitz continuous. Altogether, $V(x)$ satisfies (2.2) and (2.3). Hence, $V(x)$ is the ISDS-Lyapunov function of the whole system and by application of Proposition 2.1.6 the whole system has the ISDS property. □

This theorem provides a tool how to check, whether a network possesses the ISDS property: one has to find the ISDS-Lyapunov functions and gains of the subsystems and has to check, if the small-gain condition is satisfied. Moreover, the theorem gives an explicit construction of the ISDS-Lyapunov function and the corresponding rate and gains.

In the following, we present a corollary, which is similar to Theorem 10 in [36] for two coupled systems and covers $n \in \mathbb{N}$ coupled systems, where the rates and gains defined in Theorem 2.2.2 are used. Here, we get decay rates for the norm of the solution of the whole system and for each subsystem of n coupled systems with external input $u = 0$.

Corollary 2.2.3. *Consider the system* (1.6) *and assume that all subsystems have the ISDS property with decay rates μ_i and gains η_i, γ_i^{ISDS} and γ_{ij}^{ISDS}, $i,j = 1,\ldots,n$, $i \neq j$. If the small-gain condition (1.15) is satisfied, then the coupled system*

$$\dot{x} = \begin{pmatrix} \dot{x}_1 \\ \vdots \\ \dot{x}_n \end{pmatrix} = \begin{pmatrix} f_1(x_1,\ldots,x_n) \\ \vdots \\ f_n(x_1,\ldots,x_n) \end{pmatrix} = f(x) \tag{2.14}$$

is 0-GAS with

$$|x_j(t)| \leq |x(t)| \leq \mu\left(\psi^{-1}\left(\max_i\left\{\sigma_i^{-1}\left(\eta_i\left(\left|x^0\right|\right)\right)\right\}\right), t\right) \tag{2.15}$$

for $i,j = 1,\ldots,n$, all $t \in \mathbb{R}_+$, with functions μ, σ, ψ and η_i from Theorem 2.2.2.

Remark 2.2.4. *Note that for large n function ψ in (2.11) becomes "small" and hence the rates and gains defined by ψ^{-1} become "large" which is not desired in applications. To avoid this kind of conservativeness one can use the maximum norm $|x|_\infty$ for the states in the above definitions and in Theorem 2.2.2 and Corollary 2.2.3. This is possible as we have noted in Remark 2.1.3. In this case, the division by $\sqrt{n}$ in* (2.12) *can be avoided and we get* (2.11) *with $\psi\left(|x|_\infty\right) = \min_i \sigma_i^{-1}\left(|x|_\infty\right)$. This is used in our examples below.*

Unfortunately, we cannot compare directly the estimation of Theorem 10 in [36] with our estimation (2.15), since another approach for estimations of the trajectories for two coupled systems was used in [36]. The extension of this approach to $n > 2$ seems to be hardly possible. Our approach allows to consider n interconnected systems. In the first example of the next section, we compare our result for two coupled systems to the result in [36].

2.3 Examples

To compare Theorem 10 in [36] with Corollary 2.2.3 for the case of two subsystems, we consider Example 12 given in [36].

Example 2.3.1. *Consider two interconnected systems*

$$\begin{aligned}\dot{x}_1(t) &= -x_1(t) + \tfrac{x_2^3(t)}{2},\\ \dot{x}_2(t) &= -x_2^3(t) + x_1(t).\end{aligned}$$

As in [36] we choose $V_i = |x_i|$ *and* $\gamma_1(r) = \frac{2}{3}r^3$, $\gamma_2(r) = \sqrt[3]{\frac{4}{3}r}$, $\eta_1, \eta_2 = \mathrm{Id}$, $g_1(r) = \frac{1}{4}r$, $g_2(r) = \frac{1}{4}r^3$. *It is easy to check that the small-gain condition is satisfied and an* Ω*-path can be chosen by* $\sigma_1(r) = \mathrm{Id}$, $\sigma_2(r) = \sqrt[3]{\frac{4.49}{3}r}$. *For* $x_1^0 = x_2^0 = 2$ *the solution* x *was calculated numerically. The plot of* $|x|_\infty$ *as well as its estimations by* (2.15) *and from [36] are shown on Figure 2.4. To compare our estimation with [36], we plot the ISDS estimation in Example 12 in [36] with respect to the maximum norm for states using Remark 11 in [36]. The solid (dashed) curve is the estimation of* $|x|_\infty$ *by Corollary 2.2.3 ([36]). Both estimations tend*

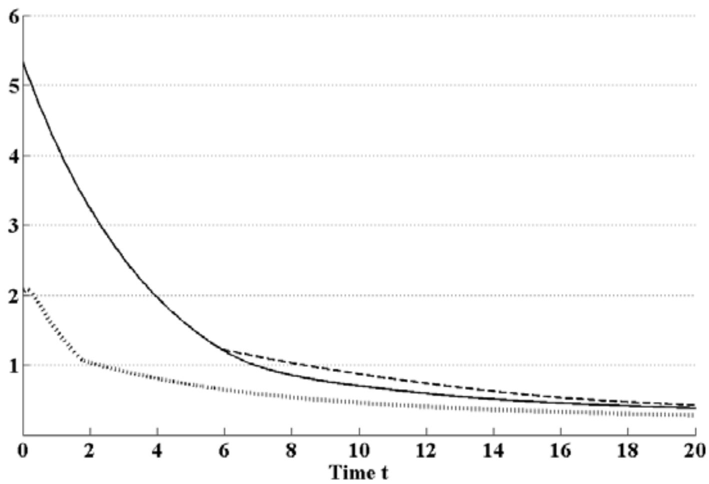

Figure 2.4: $|x|_\infty$ and estimations with help of Corollary 2.2.3 (solid curve) and Example 12 in [36] (dashed curve)

to zero as well as the trajectory and provide nearly the same estimate for the norm of the trajectory as it should be expected.

The advantage of our approach is that it can be applied for larger interconnections. The following example illustrates the application of Theorem 2.2.2 for a construction of an ISDS-Lyapunov function for the case $n \geq 2$.

Example 2.3.2. *Consider $n \in \mathbb{N}$ interconnected systems of the form*

$$\begin{aligned} \dot{x}_1(t) &= -a_1 x_1(t) + \sum_{j>1}^{n} \tfrac{1}{n} b_{1j} x_j^2(t) + \tfrac{1}{n} u(t), \\ \dot{x}_i(t) &= -a_i x_i(t) + \tfrac{1}{n} b_{i1} \sqrt{x_1(t)} + \sum_{j>1, j\neq i}^{n} \tfrac{1}{n} b_{ij} x_j(t) + \tfrac{1}{n} u(t), \ i = 2, \ldots, n, \end{aligned} \tag{2.16}$$

for $b_{ij} \in [0,1)$, $a_i = (1+\varepsilon_i)$, $\varepsilon_i \in (1,\infty)$ and any input $u \in \mathbb{R}^m$. We choose $V_i(x_i) = |x_i|_\infty$ as an ISDS-Lyapunov function candidate for the i-th subsystem, $i = 1, \ldots, n$ and define

$$\begin{aligned} \gamma_{1j}^{ISDS}(r) &:= b_{1j} r^2, \ j = 2, \ldots, n, \\ \gamma_{j1}^{ISDS}(r) &:= b_{j1} \sqrt{r}, \ j = 2, \ldots, n, \\ \gamma_{ij}^{ISDS}(r) &:= b_{ij} r, \ i,j = 2, \ldots, n, \ i \neq j, \\ \gamma_{i}^{ISDS}(r) &:= r, \ i = 1, \ldots, n, \end{aligned}$$

$\Gamma^{ISDS} := \left(\gamma_{ij}^{ISDS}\right)_{n\times n}$, $i,j = 1, \ldots, n$, $\gamma_{ii}^{ISDS} \equiv 0$, $\eta_i(r) := r$ and $\mu_i(r,t) = \mathrm{e}^{-\varepsilon_i t} r$ as solution of $\frac{\mathrm{d}}{\mathrm{d}t}\mu_i(r,t) = -g_i(\mu_i(r,t))$ with $g_i(r) := \varepsilon_i r$. We obtain that V_i is an ISDS-Lyapunov function of the i-th subsystem. To check whether the small-gain condition is satisfied, we use the cycle condition, which is satisfied (this can be easily verified).

We choose $\sigma(s) = (\sigma_1(s), \ldots, \sigma_n(s))^T$ with $\sigma_1(s) := s^2$ and $\sigma_j(s) := s$, $j = 2, \ldots, n$ for $s \in \mathbb{R}_+$, which is one of the possibilities of choosing σ. Then, σ is an Ω-path, which can be easily checked. In particular, σ satisfies $\Gamma^{ISDS}(\sigma(s)) < \sigma(s)$, $\forall s > 0$. Now, by application of Theorem 2.2.2 the whole system is ISDS and the ISDS-Lyapunov function is given by

$$V(x) = \psi^{-1}\left(\max_i \sigma_i^{-1}(|x_i|_\infty)\right)$$

with $\psi(r) = \min_i \sigma_i^{-1}(r) = \begin{cases} \sqrt{r}, & r \geq 1, \\ r, & r < 1 \end{cases}$. The gains and rates of the ISDS estimation and ISDS-Lyapunov function, respectively, are given by (2.11). Furthermore, if $u(t) \equiv 0$ then by Corollary 2.2.3 the whole system is 0-GAS and the decay rate is given by (2.15).

In the following, we illustrate the trajectory and the ISDS estimation for a system consisting of subsystems of the form (2.16) for $n = 3$. We choose $a_i = \frac{11}{10}$, $b_{ij} = \frac{1}{2}$, $i,j = 1,2,3$, $i \neq j$, $u(t) = e^{-t}$ as the input and the initial values $x_1^0 = 0.5$, $x_2^0 = 0.8$ and $x_3^0 = 1.2$. Then, we calculate the ISDS estimation of the whole system as described above and get

$$|x(t)|_\infty \leq \max\{\mu((x_3^0)^2, t), \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu(\sqrt{u(\tau)}, t-\tau)\}.$$

This estimation is displayed in the Figure 2.5 (dashed line). To verify whether the norm of the

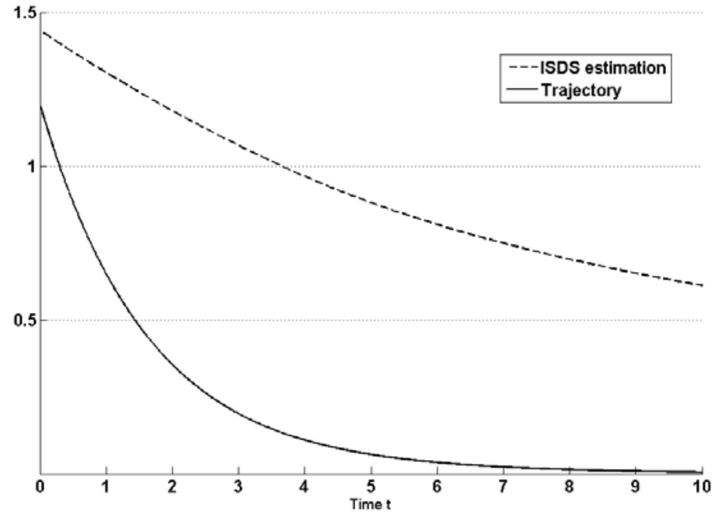

Figure 2.5: $|x|_\infty$ and ISDS estimation of the whole system consisting of $n = 3$ subsystems of the form (2.16).

trajectory of the whole system is below the ISDS estimation we solve the system of the form (2.16) for $n = 3$ *numerically. The norm of the resulting trajectory of the whole system is also displayed in the Figure 2.5. We see, if the input* $u(t)$ *tends to zero, the ISDS estimation tends to zero as well, whereas in the case of ISS this is not true. Also, the norm of the solution tends to zero and it is below the ISDS estimation.*

In the next chapter, the idea of ISDS and its advantages are transferred to observer design for single systems, for subsystems of interconnections and for whole networks. Furthermore, we combine the ISDS property with model predictive control in Chapter 6.

Chapter 3

Observer and quantized output feedback stabilization

In this chapter, we introduce the notion of quasi-ISDS reduced-order observers and use error Lyapunov functions to design such observers for single systems. Considering interconnected systems we design quasi-ISS/ISDS observers for each subsystem and the whole system under a small-gain condition. This is applied to stabilization of systems subject to quantization.

We consider systems of the form (1.1) with outputs

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= h(x), \end{aligned} \tag{3.1}$$

where $y \in \mathbb{R}^P$ is the output and function $h : \mathbb{R}^N \to \mathbb{R}^P$ is continuously differentiable with locally Lipschitz derivative (called a C^1_L function). In addition, it is assumed that $h(0) = 0$ holds.

In practice, observers are used for systems, where the state or parts of the state can not be measured due to uneconomic measurement costs or physical circumstances like high temperatures, where no measurement equipment is available, for example. They are also used in cases, where the output of a system is disturbed and for stabilization of a system, for example. There, a control law subject to stabilize a system is designed using the estimated state of the system generated by the observer based on the disturbed output. This can lead to an unbounded growth of the state estimation error and therefore to a design of a control law, which does not stabilizes the system.

A state observer for the system (3.1) is of the form

$$\begin{aligned} \dot{\hat{\xi}} &= F(\bar{y},\hat{\xi},u), \\ \hat{x} &= H(\bar{y},\hat{\xi},u), \end{aligned} \tag{3.2}$$

where $\hat{\xi} \in \mathbb{R}^L$ is the observer state, $\hat{x} \in \mathbb{R}^N$ is the estimate of the system state x and $\bar{y} \in \mathbb{R}^P$ is the measurement of y that may be disturbed by d: $\bar{y} = y + d$, where $d \in L_\infty(\mathbb{R}_+, \mathbb{R}^P)$. The function $F : \mathbb{R}^P \times \mathbb{R}^L \times \mathbb{R}^m \to \mathbb{R}^L$ is locally Lipschitz in $\bar{y}$ and $\hat{\xi}$ uniformly in u and function

$H : \mathbb{R}^P \times \mathbb{R}^L \times \mathbb{R}^m \to \mathbb{R}^N$ is a C_L^1 function. In addition, it is assumed that $F(0,0,0) = 0$ and $H(0,0,0) = 0$ holds.

We denote the state estimation error by

$$\tilde{x} = \hat{x} - x.$$

We are interested under which conditions the designed observer guarantees that the state estimation error is ISS or ISDS. The used stability properties for observers are based on ISS and ISDS, and are called quasi-ISS and quasi-ISDS, respectively.

Inspired by the work [110], where the notion of quasi-ISS reduced-order observers was introduced and the advantages of ISDS over ISS, investigated in Chapter 2, this motivates the introduction of the quasi-ISDS property for observers, where the approaches of reduced-order observers and the ISDS property are combined. The property has the advantage that the recent disturbance of the output of the system is taken into account. We investigate under which conditions a quasi-ISDS reduced-order observer can be designed for single nonlinear systems, where error Lyapunov functions (see [88, 60]) are used. The design of observers in the context of this thesis was investigated in [112, 60, 72, 61, 110], for example, and remarks on the equivalence of full order and reduced-order observers can be found in [111].

Considering interconnected systems it is desirable to have observers for each of the subsystems and the whole network. Here, we design quasi-ISS/ISDS reduced-order observers for each subsystem of an interconnected system, from which an observer for the whole system can be designed under a small-gain condition.

Furthermore, the problem of stabilization of systems is investigated and we apply the presented approach to quantized output feedback stabilization for single and interconnected systems. The goal of stabilizing a system is an important problem in applications. Many approaches were performed during the last years and the design of stabilizing feedback laws is a popular research area, which is linked up with many applications. The stabilization using output feedback quantization was investigated in [7, 70, 62, 63, 60, 71, 72, 110], for example. A quantizer is a device, which converts a real-valued signal into a piecewise constant signal, i.e., it maps $\mathbb{R}^P$ into a finite and discrete subset of $\mathbb{R}^P$. It may affect the process output or may also affect the control input.

Adapting the quantizer with a so-called zoom variable this leads to dynamic quantizers, which have the advantage that asymptotic stability for single and interconnected systems can be achieved under certain conditions.

This chapter is organized as follows: The notion of quasi-ISDS observers is introduced in Section 3.1, where the design of such an observer for single systems under the existence of an error ISDS-Lyapunov function is performed. Section 3.2 contains all the results for the quasi-ISS/ISDS observer design according to interconnected systems. The application of the results to quantized output feedback stabilization for single and interconnected systems can be found in Section 3.3.

3.1 Quasi-ISDS observer for single systems

In this section, we introduce quasi-ISDS observers and give a motivating example for the introduction. Then, we show that the reduced-order observer designed in [110], Theorem 1, is a quasi-ISDS observer provided that an error ISDS-Lyapunov function exists.

We recall the definition of quasi-ISS observers from [110], which guarantee that the norm of the state estimation error is bounded for all times.

Definition 3.1.1 (Quasi-ISS observer)**.** *The system* (3.2) *is called a quasi-ISS observer for the system* (3.1)*, if there exists a function* $\tilde{\beta} \in \mathcal{KL}$ *and for each* $K > 0$*, there exists a function* $\tilde{\gamma}_K^{ISS} \in \mathcal{K}_\infty$ *such that*

$$|\tilde{x}(t)| \leq \max\{\tilde{\beta}(|\tilde{x}_0|, t), \tilde{\gamma}_K^{ISS}(\|d_{[0,t]}\|)\}, \ \forall t \in \mathbb{R}_+$$

whenever $||u_{[0,t]}|| \leq K$ *and* $||x||_{[0,t]} \leq K$*.*

Modifying this definition by using the idea of the ISDS property to transfer the advantages of ISDS over ISS to observers we define quasi-ISDS observers:

Definition 3.1.2 (Quasi-ISDS observers)**.** *The system* (3.2) *is called a quasi-ISDS observer for the system* (3.1)*, if there exist functions* $\tilde{\mu} \in \mathcal{KLD}$*,* $\tilde{\eta} \in \mathcal{K}_\infty$ *and for each* $K > 0$ *a function* $\tilde{\gamma}_K^{ISDS} \in \mathcal{K}_\infty$ *such that*

$$|\tilde{x}(t)| \leq \max\{\tilde{\mu}(\tilde{\eta}(|\tilde{x}_0|), t), \operatorname*{\mathit{ess\ sup}}_{\tau \in [0,t]} \tilde{\mu}(\tilde{\gamma}_K^{ISDS}(|d(\tau)|), t - \tau)\}, \ \forall t \in \mathbb{R}_+$$

whenever $||u_{[0,t]}|| \leq K$ *and* $||x||_{[0,t]} \leq K$*.*

Recalling that ISDS possesses the memory fading effect, the motivation for the introduction of quasi-ISDS observers is the following: quasi-ISDS observers take the recent disturbance of the measurement into account, whereas a quasi-ISS observer takes into account the supremum norm of the disturbance. The advantage will be illustrated by the following example.

Example 3.1.3. *Consider the system as in Example 1 in [110]*

$$\begin{aligned} \dot{x} &= -x + x^2 u, \\ y &= x, \end{aligned} \tag{3.3}$$

where $\dot{\hat{x}} = -\hat{x} + y^2 u$ *is an observer. We consider the perturbed measurement* $\bar{y} = y + d$*, with* $d = e^{-t\frac{1}{10}}$*. Then, the error dynamics becomes*

$$\dot{\tilde{x}} = -\tilde{x} + 2xud + ud^2.$$

This system is ISS and ISDS from d *to* $\tilde{x}$ *when* u *and* x *are bounded. Let* $u \equiv 1$ *be constant, then the estimations of the error dynamics are displayed in the Figure 3.1 for* $x_0 = \tilde{x}_0 = 0.3$*. The ISS estimate is chosen equal to* 1*, since* $\tilde{\beta}(|\tilde{x}_0|, t) \leq \tilde{\gamma}_K^{ISS}(\|d_{[0,t]}\|)$ *for a sufficient function* $\tilde{\beta}$*,* $\|d\| = 1$*,* $\tilde{x}_0$ *small enough and with* $\tilde{\gamma}_K^{ISS} = \mathrm{Id}$*. The ISDS estimation follows by choosing*

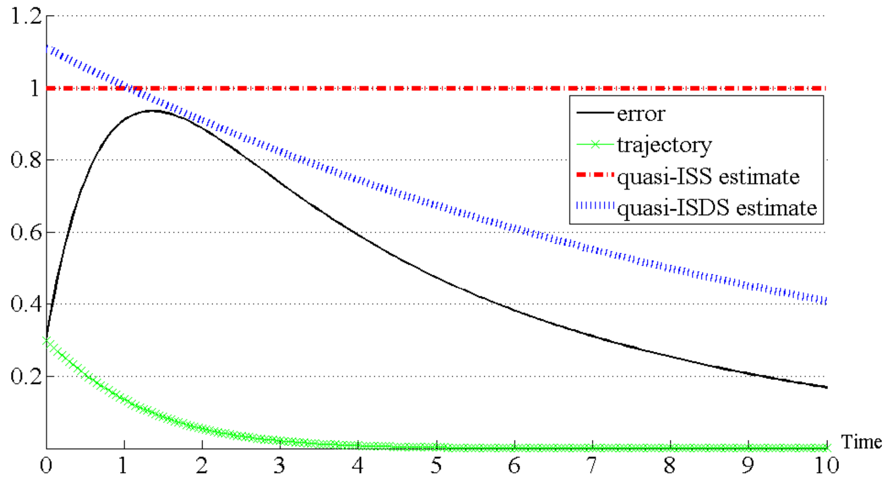

Figure 3.1: Displaying of the trajectory, error, quasi-ISS and quasi-ISDS estimate of the system (3.3).

$\tilde{\gamma}_K^{ISDS}(|d(\tau)|) = \frac{(d(\tau))^2}{1-\epsilon}$ and $\tilde{\mu}(r,t) = e^{-\epsilon(t)}r,\ r \geq 0$ and $\epsilon = 0.1$. Here, the quasi-ISS estimation takes the maximal value of d into account, whereas the quasi-ISDS estimation possesses the so-called memory-fading effect. Using a quasi-ISDS observer, it provides a better estimate of the norm of the state estimation error in contrast to the usage of a quasi-ISS observer.

In the following, we focus on the design of reduced-order observers. We assume that systems of the form (3.1) can be divided into one part, where the state can be measured and a second part, where the state can not be measured. The practical meaning is the following: for systems it can be uneconomic to measure all of the systems state, because the measurement equipment or the running costs for the measurement are very expensive, for example. Therefore, a part of the state is measured and the other part has to be estimated. Here, we use quasi-ISS/ISDS reduced-order observers for the state estimation, where only the part of the state is estimated that is not measured.

We assume that there exists a global coordinate change $z = \phi(x)$ such that the system (3.1) is globally diffeomorphic to a system with linear output of the form

$$\begin{aligned} \dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} &= \begin{bmatrix} \tilde{f}_1(z_1, z_2, u) \\ \tilde{f}_2(z_1, z_2, u) \end{bmatrix} = f(z,u), \\ y &= z_1, \end{aligned} \tag{3.4}$$

where $z_1 \in \mathbb{R}^P$ and $z_2 \in \mathbb{R}^{N-P}$.

For the construction of observers we need the following assumption, where we use reduced-order error Lyapunov functions. Error Lyapunov functions were first introduced in [88] and in [60] the equivalence of the existence of an error Lyapunov function and the existence of an observer was shown.

Assumption 3.1.4 (Error ISS-Lyapunov function)**.** *There exist a C^1_L function $l : \mathbb{R}^P \to \mathbb{R}^{N-P}$, a C^1 function $V : \mathbb{R}^{N-P} \to \mathbb{R}_+$, called an error ISS-Lyapunov function, and functions*

$\alpha_i \in \mathcal{K}_\infty,\ i = 1, \ldots, 4$ *such that for all* $e \in \mathbb{R}^{N-P}$, $z \in \mathbb{R}^N$ *and* $u \in \mathbb{R}^m$

$$\alpha_1(|e|) \leq V(e) \leq \alpha_2(|e|), \tag{3.5}$$

$$\left|\tfrac{\partial V(e)}{\partial e}\right| \leq \alpha_4(V(e)), \tag{3.6}$$

$$\begin{aligned}&\frac{\partial V(e)}{\partial e}\left([\tilde{f}_2(z_1, \tilde{e}, u) + \frac{\partial l(z_1)}{\partial z_1}\tilde{f}_1(z_1, \tilde{e}, u)] - [\tilde{f}_2(z_1, z_2, u) + \frac{\partial l(z_1)}{\partial z_1}\tilde{f}_1(z_1, z_2, u)]\right)\\ \leq\ & -\alpha_3(V(e)),\end{aligned} \tag{3.7}$$

$\tilde{e} := e + z_2$ *and there exists a function* $\alpha \in \mathcal{K}_\infty$ *such that*

$$\alpha(s)\alpha_4(s) \leq \alpha_3(s),\ \ s \in \mathbb{R}_+.$$

Remark 3.1.5. *Note that in [110] the properties of an error Lyapunov function are slightly different, namely for* $\tilde{\alpha}_3, \tilde{\alpha}_4 \in \mathcal{K}_\infty$

$$\begin{aligned}&\alpha_1(|e|) \leq V(e) \leq \alpha_2(|e|),\\ &\left|\tfrac{\partial V(e)}{\partial e}\right| \leq \tilde{\alpha}_4(|e|),\\ &\tfrac{\partial V(e)}{\partial e}\left([\tilde{f}_2(z_1, \tilde{e}, u) + \tfrac{\partial l(z_1)}{\partial z_1}\tilde{f}_1(z_1, \tilde{e}, u)] - [\tilde{f}_2(z_1, z_2, u) + \tfrac{\partial l(z_1)}{\partial z_1}\tilde{f}_1(z_1, z_2, u)]\right)\\ \leq &-\tilde{\alpha}_3(|e|),\end{aligned}$$

which are equivalent to (3.5), (3.6) *and* (3.7).

Now, the following lemma can be stated, which was proved in [110]. It shows, how a quasi-ISS observer for the system (3.4) can be designed, provided that an error ISS-Lyapunov function exists.

Lemma 3.1.6. *Under Assumption 3.1.4, the system*

$$\begin{aligned}\dot{\hat{\xi}} &= \tilde{f}_2(\bar{y}, \hat{\xi} - l(\bar{y}), u) + \frac{\partial l(\bar{y})}{\partial z_1}\tilde{f}_1(\bar{y}, \hat{\xi} - l(\bar{y}), u),\\ \hat{z}_1 &= \bar{y},\\ \hat{z}_2 &= \hat{\xi} - l(\bar{y})\end{aligned} \tag{3.8}$$

becomes a quasi-ISS reduced-order observer for the system (3.4)*, where* $\hat{\xi} \in \mathbb{R}^{N-P}$ *is the observer state and* $\hat{z}_1$*,* $\hat{z}_2$ *are the estimates of* z_1 *and* z_2*, respectively, and* $\bar{y} = y + d = z_1 + d$*, which is the measurement of* z_1 *disturbed by* d*.*

In order to use quasi-ISDS observers we adapt Assumption 3.1.4 according to the ISDS property:

Assumption 3.1.7 (Error ISDS-Lyapunov function)**.** *Let* $\varepsilon > 0$ *be given. There exist a* C^1_L *function* $l : \mathbb{R}^P \to \mathbb{R}^{N-P}$*, a* C^1 *function* $V : \mathbb{R}^{N-P} \to \mathbb{R}_+$*, called an error ISDS-Lyapunov function, functions* $\alpha, \bar{\eta} \in \mathcal{K}_\infty$ *and* $\bar{\mu} \in \mathcal{KLD}$ *such that for all* $e \in \mathbb{R}^{N-P}$*,* $z \in \mathbb{R}^N$ *and* $u \in \mathbb{R}^m$

$$\tfrac{|e|}{1+\varepsilon} \leq V(e) \leq \bar{\eta}(|e|), \tag{3.9}$$

$$\left|\tfrac{\partial V(e)}{\partial e}\right| \leq \alpha(V(e)), \tag{3.10}$$

$$\begin{aligned}&\frac{\partial V}{\partial e}(e)\left([\tilde{f}_2(z_1, \tilde{e}, u) + \frac{\partial l(z_1)}{\partial z_1}\tilde{f}_1(z_1, \tilde{e}, u)] - [\tilde{f}_2(z_1, z_2, u) + \frac{\partial l(z_1)}{\partial z_1}\tilde{f}_1(z_1, z_2, u)]\right)\\ \leq\ & -(1-\varepsilon)\, g(V(e)),\end{aligned} \tag{3.11}$$

$\tilde{e} := e + z_2$, where $\tilde{\mu}$ solves the equation $\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mu}(r,t) = -g\left(\bar{\mu}\left(r,t\right)\right), \quad r,t \geq 0$ for a locally Lipschitz continuous function $g : \mathbb{R}_+ \to \mathbb{R}_+$ and there exists a function $\bar{\alpha} \in \mathcal{K}_\infty$ such that

$$\bar{\alpha}(s)\alpha(s) \leq (1-\varepsilon)g(s), \; s \in \mathbb{R}_+. \tag{3.12}$$

The next theorem is a counterpart of Lemma 3.1.6. It provides a design of a quasi-ISDS reduced-order observer for the system (3.4) provided that an error ISDS-Lyapunov function exists.

Theorem 3.1.8. *Under Assumption 3.1.7 the system* (3.8) *becomes a quasi-ISDS reduced-order observer for the system* (3.4)*.*

The proof goes along the lines of the proof of Lemma 3.1.6 in [110] with corresponding changes according to Definition 3.1.2 and Assumption 3.1.7:

Proof. We define $\xi := z_2 + l(z_1)$ and convert the system (3.4) into

$$\begin{aligned}
\dot{z}_1 &= f_1(z_1, \xi - l(z_1), u),\\
y &= z_1,\\
\dot{\xi} &= f_2(z_1, \xi - l(z_1), u) + \frac{\partial l(z_1)}{\partial z_1} f_1(z_1, \xi - l(z_1), u) =: F(z_1, \xi, u).
\end{aligned}$$

With this F, the dynamic of the observer state in (3.8) can be written as $\dot{\hat{\xi}} = F(\bar{y}, \hat{\xi}, u)$, where $\bar{y} = y + d$ and d is the measurement disturbance. Let $e := \hat{\xi} - \xi$ and from Assumption 3.1.7 we obtain

$$\begin{aligned}
\dot{V}(e) = &\frac{\partial V(e)}{\partial e}\Big(f_2(z_1+d, \hat{\xi} - l(z_1+d), u) - f_2(z_1, \xi - l(z_1), u)\\
&\qquad + \tfrac{\partial l(z_1+d)}{\partial z_1} f_1(z_1+d, \hat{\xi} - l(z_1+d), u) - \tfrac{\partial l(z_1)}{\partial z_1} f_1(z_1, \xi - l(z_1), u)\Big)\\
= &\frac{\partial V(e)}{\partial e}\Big(f_2(z_1+d, \hat{\xi} - l(z_1+d), u) - f_2(z_1+d, \xi - l(z_1+d), u)\\
&\qquad + \tfrac{\partial l(z_1+d)}{\partial z_1} f_1(z_1+d, \hat{\xi} - l(z_1+d), u) - \tfrac{\partial l(z_1+d)}{\partial z_1} f_1(z_1+d, \xi - l(z_1+d), u)\Big)\\
&+ \frac{\partial V(e)}{\partial e}\left(F(\bar{y}, \xi, u) - F(y, \xi, u)\right)\\
\leq &-(1-\varepsilon)\, g(|e|) + \alpha(|e|)\left(F(\bar{y}, \xi, u) - F(y, \xi, u)\right).
\end{aligned}$$

In [32], Lemma A.14, it was shown that there exist a continuous positive function ρ and $\gamma \in \mathcal{K}$ such that

$$|F(\bar{y}, \xi, u) - F(y, \xi, u)| \leq \rho(y, \xi, u)\gamma(|d|)$$

and it follows for an arbitrary $\delta \in (0,1)$ with (3.12)

$$V(e) \geq \bar{\alpha}^{-1}\left(\frac{\rho(y, \xi, u)\gamma(|d|)}{1-\delta}\right) \;\Rightarrow\; \dot{V}(e) \leq -(1-\varepsilon)\bar{g}(V(e)),$$

where $\bar{g}(r) := \delta g(r),\ \forall r > 0$. By Theorem 3.5.8 in [35] and its proof this is equivalent to

$$|e(t)| \leq \max\left\{\bar{\mu}(\bar{\eta}(|e(0)|), t), \operatorname*{ess\,sup}_{\tau\in[0,t]} \bar{\mu}(\bar{\gamma}_K^{\mathrm{ISDS}}(|d(\tau)|), t-\tau)\right\} \tag{3.13}$$

under $||z||_{[0,t]} \leq K,\ ||u_{[0,t]}|| \leq K$, where $\bar{\gamma}_K^{\mathrm{ISDS}} \in \mathcal{K}_\infty$ is parametrized by K. Now, we have

$$\tilde{z} = \begin{pmatrix} \tilde{z}_1 \\ \tilde{z}_2 \end{pmatrix} := \begin{pmatrix} \hat{z}_1 - z_1 \\ \hat{z}_2 - z_2 \end{pmatrix} = \begin{pmatrix} d \\ e - (l(\bar{y}) - l(z_1)) \end{pmatrix}.$$

By $\theta_K \in \mathcal{K}$, parametrized by K such that $|l(z_1 + d) - l(z_1)| \leq \theta_K(|d|),\ |z_1| \leq K$ it follows

$$|\tilde{z}| \leq |e| + |d| + \theta_K(|d|) \text{ and } |e| \leq |\tilde{z}_2| + \theta_K(|d|). \tag{3.14}$$

Overall, combining (3.13) and (3.14) we have

$$\begin{aligned} |\tilde{z}| \leq&\ \max\left\{\bar{\mu}(\bar{\eta}(|e(0)|), t), \operatorname*{ess\,sup}_{\tau\in[0,t]} \bar{\mu}(\bar{\gamma}_K^{\mathrm{ISDS}}(|d(\tau)|), t-\tau)\right\} + |d(t)| + \theta_K(|d(t)|) \\ \leq&\ \max\left\{\bar{\mu}(\bar{\eta}(|e(0)|), t), \operatorname*{ess\,sup}_{\tau\in[0,t]} \bar{\mu}(\bar{\gamma}_K^{\mathrm{ISDS}}(|d(\tau)|), t-\tau)\right\} + \chi_K(|d(t)|) \end{aligned}$$

where $\chi_K(s) := s + \theta_K(s),\ s \geq 0$. Since $\bar{\mu}$ is a $\mathcal{KLD}$-function it follows

$$\begin{aligned} |\tilde{z}| \leq&\ \max\left\{2\bar{\mu}(\bar{\eta}(|e(0)|), t), \operatorname*{ess\,sup}_{\tau\in[0,t]} 2\bar{\mu}(\bar{\gamma}_K^{\mathrm{ISDS}}(|d(\tau)|), t-\tau), \operatorname*{ess\,sup}_{\tau\in[0,t]} 2\bar{\mu}(\chi_K(|d(\tau)|), t-\tau)\right\} \\ \leq&\ \max\left\{2\bar{\mu}(\bar{\eta}(|\tilde{z}_2(0)| + \theta_K(|d(0)|)), t), \operatorname*{ess\,sup}_{\tau\in[0,t]} 2\bar{\mu}(\check{\gamma}_K^{\mathrm{ISDS}}(|d(\tau)|), t-\tau)\right\}, \end{aligned}$$

where $\check{\gamma}_K^{\mathrm{ISDS}}(s) := \max\{\bar{\gamma}_K^{\mathrm{ISDS}}(s), \chi_K(s)\}$, and we used (3.14) and the inequality $\alpha(a+b) \leq \max\{\alpha(2a), \alpha(2b)\}$ for $\alpha \in \mathcal{K},\ a, b \geq 0$. Furthermore, we have

$$\begin{aligned} |\tilde{z}| \leq&\ \max\left\{2\bar{\mu}(\bar{\eta}(2\,|\tilde{z}_2(0)|), t), 2\bar{\mu}(\bar{\eta}(2\theta_K(|d(0)|)), t), \operatorname*{ess\,sup}_{\tau\in[0,t]} 2\bar{\mu}(\check{\gamma}_K^{\mathrm{ISDS}}(|d(\tau)|), t-\tau)\right\} \\ \leq&\ \max\Bigg\{2\bar{\mu}(\bar{\eta}(2\,|\tilde{z}_2(0)|), t), \operatorname*{ess\,sup}_{\tau\in[0,t]} 2\bar{\mu}(\bar{\eta}(2\theta_K(|d(\tau)|)), t-\tau), \\ &\qquad \operatorname*{ess\,sup}_{\tau\in[0,t]} 2\bar{\mu}(\check{\gamma}_K^{\mathrm{ISDS}}(|d(\tau)|), t-\tau)\Bigg\} \\ \leq&\ \max\left\{2\bar{\mu}(\bar{\eta}(2\,|\tilde{z}_2(0)|), t), \operatorname*{ess\,sup}_{\tau\in[0,t]} 2\bar{\mu}(\tilde{\gamma}_K^{\mathrm{ISDS}}(|d(\tau)|), t-\tau)\right\}, \end{aligned}$$

where $\tilde{\gamma}_K^{\mathrm{ISDS}}(s) := \max\{\bar{\eta}(2\theta_K(s)), \check{\gamma}_K^{\mathrm{ISDS}}(s)\}$. Finally, by definition of $\tilde{\mu}(r,t) := 2\bar{\mu}(r,t)$ and $\tilde{\eta}(s) := \bar{\eta}(2s)$ it follows

$$|\tilde{z}| \leq \max\left\{\tilde{\mu}(\tilde{\eta}(|\tilde{z}(0)|), t), \operatorname*{ess\,sup}_{\tau\in[0,t]} \tilde{\mu}(\tilde{\gamma}_K^{\mathrm{ISDS}}(|d(\tau)|), t-\tau)\right\},$$

for $||z||_{[0,t]} \leq K,\ ||u_{[0,t]}|| \leq K$, which proves the assertion. $\square$

Remark 3.1.9. *From Chapter 2 the decay rate and the gain of the definition of ISDS are the same as the ones using ISDS-Lyapunov functions. Note that this is not the case for the definition of a quasi-ISDS observer and using error ISDS-Lyapunov functions. It remains as an open topic to investigate if it is possible to use a different error ISDS-Lyapunov function, from which the information about the decay rate and the gains of the error ISDS-Lyapunov function can be preserved for the quasi-ISDS estimation.*

In the next section, we are going to extend the notion of quasi-ISS/ISDS observers for interconnected systems and provide tools to design such kind of observers.

3.2 Quasi-ISS and quasi-ISDS observer for interconnected systems

We consider $n \in \mathbb{N}$ interconnected systems of the form (1.6) with outputs

$$\begin{aligned} \dot{x}_i &= f_i(x_1, \dots, x_n, u_i), \\ y_i &= h_i(x_i), \end{aligned} \tag{3.15}$$

$i = 1, \dots, n$, where $u_i \in \mathbb{R}^{M_i}$ are control inputs, $y_i \in \mathbb{R}^{P_i}$ are the outputs and functions $h_i : \mathbb{R}^{N_i} \to \mathbb{R}^{P_i}$ are C_L^1 functions. In addition, it is assumed that $h_i(0) = 0$.

The state observer of the ith subsystem is of the form

$$\begin{aligned} \dot{\hat{\xi}}_i &= F_i(\bar{y}_1, \dots, \bar{y}_n, \hat{\xi}_1, \dots, \hat{\xi}_n, u_i), \\ \hat{x}_i &= H_i(\bar{y}_1, \dots, \bar{y}_n, \hat{\xi}_1, \dots, \hat{\xi}_n, u_i), \end{aligned} \tag{3.16}$$

$i = 1, \dots, n$, where $\hat{\xi}_i \in \mathbb{R}^{L_i}$ is the observer state of the ith subsystem, $\hat{x}_i \in \mathbb{R}^{N_i}$ is the estimate of the system state x_i and $\bar{y}_i \in \mathbb{R}^{P_i}$ is the measurement of y_i that may be disturbed by d_i: $\bar{y}_i = y_i + d_i$, $d_i \in L_\infty(\mathbb{R}_+, \mathbb{R}^{P_i})$. The function $F_i : \mathbb{R}^P \times \mathbb{R}^L \times \mathbb{R}^{M_i} \to \mathbb{R}^{L_i}$, $P = \sum_i P_i$, $L = \sum_i L_i$ is locally Lipschitz in $\bar{y} = (\bar{y}_1, \dots, \bar{y}_n)^T$ and $\hat{\xi} = (\hat{\xi}_1, \dots, \hat{\xi}_n)^T$ uniformly in u_i and function $H_i : \mathbb{R}^P \times \mathbb{R}^L \times \mathbb{R}^{M_i} \to \mathbb{R}^{N_i}$ is a C_L^1 function. In addition, it is assumed that $F_i(0,0,0) = 0$ and $H_i(0,0,0) = 0$ holds.

We denote the state estimation error of the ith subsystem by $\tilde{x}_i := \hat{x}_i - x_i$. It is influenced by the state estimation error as well as by the measurement error of the jth subsystem which is connected to the ith subsystem. Therefore, they have to be taken into account for the estimation of the norm of the state estimation error of the ith subsystem. The quasi-ISS/ISDS property for interconnected systems reads as follows:

1. The ith subsystem of (3.16) is a quasi-ISS observer for the ith subsystem of (3.15), if there exists a function $\tilde{\beta}_i \in \mathcal{KL}$ and for each $K_i > 0$ there exist functions $(\tilde{\gamma}_i^{K_i})^{\text{ISS}}$, $(\tilde{\gamma}_{ij}^{K_i})^{\text{ISS}} \in \mathcal{K}_\infty$, $j = 1, \dots, n$, $j \neq i$ such that

$$\begin{aligned} &|\tilde{x}_i(t)| \\ &\leq \max\left\{ \tilde{\beta}_i(|\tilde{x}_i^0|), t), \max_{j \neq i} (\tilde{\gamma}_{ij}^{K_i})^{\text{ISS}}(||\tilde{x}_j||_{[0,t]}), \max_{j \neq i} (\tilde{\gamma}_{ij}^{K_i})^{\text{ISS}}(||(d_j)_{[0,t]}||), (\tilde{\gamma}_i^{K_i})^{\text{ISS}}(||(d_i)_{[0,t]}||) \right\}, \end{aligned}$$

whenever $||(u_i)_{[0,t]}|| \leq K_i$ and $||x_j||_{[0,t]} \leq K_i$, $j = 1, \dots, n$.

2. The ith subsystem of (3.16) is a quasi-ISDS observer for the ith subsystem of (3.15), if there exist functions $\tilde{\mu}_i \in \mathcal{KLD}$, $\tilde{\eta}_i \in \mathcal{K}_\infty$ and for each $K_i > 0$ there exist functions $(\tilde{\gamma}_i^{K_i})^{\text{ISDS}}, (\tilde{\gamma}_{ij}^{K_i})^{\text{ISDS}} \in \mathcal{K}_\infty,\ j = 1, \dots, n,\ j \neq i$ such that

$$|\tilde{x}_i(t)| \leq \max\left\{\tilde{\mu}_i(\tilde{\eta}_i(|\tilde{x}_i^0|), t), \max_{j\neq i} \bar{\nu}_{ij}(\tilde{x}_j, t), \max_{j\neq i} \tilde{\nu}_{ij}(d_j, t), \tilde{\nu}_i(d_i, t)\right\}$$

whenever $||(u_i)_{[0,t]}|| \leq K_i$ and $||x_j||_{[0,t]} \leq K_i,\ j = 1, \dots, n$, where

$$\begin{aligned}
\bar{\nu}_{ij}(\tilde{x}_j, t) &:= \sup_{\tau\in[0,t]} \tilde{\mu}_i((\tilde{\gamma}_{ij}^{K_i})^{\text{ISDS}}(|\tilde{x}_j(\tau)|), t-\tau),\\
\tilde{\nu}_{ij}(d_j, t) &:= \operatorname*{ess\,sup}_{\tau\in[0,t]} \tilde{\mu}_i((\tilde{\gamma}_{ij}^{K_i})^{\text{ISDS}}(|d_j(\tau)|), t-\tau),\\
\tilde{\nu}_i(d_i, t) &:= \operatorname*{ess\,sup}_{\tau\in[0,t]} \tilde{\mu}_i((\tilde{\gamma}_{i}^{K_i})^{\text{ISDS}}(|d_i(\tau)|), t-\tau).
\end{aligned}$$

We assume that there exists a global coordinate change $z_i = \phi_i(x_i)$ with $x = (x_1^T, \dots, x_n^T)^T$ such that the ith subsystem of (3.15) is globally diffeomorphic to a system with linear output of the form

$$\begin{aligned}
\dot{z}_i &= \begin{bmatrix} \dot{z}_{1i} \\ \dot{z}_{2i} \end{bmatrix} = \begin{bmatrix} f_{1i}(z_{11}, \dots, z_{1n}, z_{21}, \dots, z_{2n}, u_i) \\ f_{2i}(z_{11}, \dots, z_{1n}, z_{21}, \dots, z_{2n}, u_i) \end{bmatrix} = f_i(z_1, \dots, z_n, u_i),\\
y_i &= z_{1i}
\end{aligned} \tag{3.17}$$

for $i = 1, \dots, n$, where $z_{1i} \in \mathbb{R}^{P_i}$ and $z_{2i} \in \mathbb{R}^{N_i - P_i}$.

For the design of a quasi-ISS/ISDS observer for each subsystem of (3.15) we assume the following:

Assumption 3.2.1. *1. For each $i = 1, \dots, n$ there exist a C_L^1 function $l_i : \mathbb{R}^{P_i} \to \mathbb{R}^{N_i - P_i}$, a C^1 function $V_i : \mathbb{R}^{N_i-P_i} \to \mathbb{R}_+$, functions $\alpha_{1i}, \alpha_{2i}, \alpha_{3i}, \alpha_{4i}, \gamma_{ij} \in \mathcal{K}_\infty$, $j = 1, \dots, n,\ j \neq i$ such that for all $e_i \in \mathbb{R}^{N_i-P_i},\ z_{1i} \in \mathbb{R}^{P_i},\ z_{2i} \in \mathbb{R}^{N_i-P_i}$ and $u_i \in \mathbb{R}^{M_i}$ it holds*

$$\begin{aligned}
\alpha_{1i}(|e_i|) \leq V_i(e_i) &\leq \alpha_{2i}(|e_i|),\\
\left|\tfrac{\partial V_i}{\partial e_i}(e_i)\right| &\leq \alpha_{4i}(V_i(e_i)),
\end{aligned}$$

and whenever $V_i(e_i) \geq \max_{j\neq i} \gamma_{ij}(V_j(e_j))$ holds, it follows

$$\begin{aligned}
&\frac{\partial V_i}{\partial e_i}(e_i)\left(\left[f_{2i}(z_1, e+z_2, u_i) + \frac{\partial l_i}{\partial z_{1i}}(z_{1i}) f_{1i}(z_1, e+z_2, u_i)\right]\right.\\
&\qquad\qquad \left. - \left[f_{2i}(z_1, z_2, u_i) + \frac{\partial l_i}{\partial z_{1i}}(z_{1i}) f_{1i}(z_1, z_2, u_i)\right]\right)\\
&\leq\ -\alpha_{3i}(V_i(e_i))
\end{aligned}$$

$z_1 = (z_{11}^T, \dots, z_{1n}^T)^T$, $z_2 = (z_{21}^T, \dots, z_{2n}^T)^T$, $e + z_2 = (e_1 + z_{21}, \dots, e_n + z_{2n})^T$ and there exists $\alpha_i \in \mathcal{K}_\infty$ such that

$$\alpha_i(s)\alpha_{4i}(s) \leq \alpha_{3i}(s),\ \ s \in \mathbb{R}_+.$$

2. *Let* ε_i *be given. For each* $i = 1, \ldots, n$ *there exist a* C^1_L *function* $l_i : \mathbb{R}^{P_i} \to \mathbb{R}^{N_i - P_i}$*, a* C^1 *function* $V_i : \mathbb{R}^{N_i - P_i} \to \mathbb{R}_+$*, functions* $\alpha_i,\ \bar{\eta}_i, \bar{\gamma}_{ij} \in \mathcal{K}_\infty,\ j = 1, \ldots, n,\ j \neq i$ *and* $\bar{\mu}_i \in \mathcal{KLD}$ *such that for all* $e_i \in \mathbb{R}^{N_i - P_i},\ z_{1i} \in \mathbb{R}^{P_i},\ z_{2i} \in \mathbb{R}^{N_i - P_i}$ *and* $u_i \in \mathbb{R}^{M_i}$ *it holds*

$$\frac{|e_i|}{1+\varepsilon_i} \leq V_i(e_i) \leq \bar{\eta}_i(|e_i|),$$
$$\left|\frac{\partial V_i}{\partial e_i}(e_i)\right| \leq \alpha_i(V_i(e_i)),$$

and whenever $V_i(e_i) > \max_{j \neq i} \bar{\gamma}_{ij}(V_j(e_j))$ *holds, it follows*

$$\begin{aligned}
&\frac{\partial V_i}{\partial e_i}(e_i)\left(\left[f_{2i}(z_1, e + z_2, u_i) + \frac{\partial l_i}{\partial z_{1i}}(z_{1i}) f_{1i}(z_1, e + z_2, u_i)\right]\right.\\
&\qquad\left. - \left[f_{2i}(z_1, z_2, u_i) + \frac{\partial l_i}{\partial z_{1i}}(z_1, z_2, u_i)\right]\right)\\
\leq\ & -(1 - \varepsilon_i) g_i(V(e_i)),
\end{aligned}$$

where $\bar{\mu}_i$ *solves the equation* $\frac{\mathrm{d}}{\mathrm{d}t}\bar{\mu}_i(r,t) = -g_i(\bar{\mu}_i(r,t)),\ r, t > 0$ *for a locally Lipschitz continuous function* $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ *and there exists* $\bar{\alpha}_i \in \mathcal{K}_\infty$ *such that*

$$\bar{\alpha}_i(s)\alpha_i(s) \leq (1 - \varepsilon_i) g_i(s),\ s \in \mathbb{R}_+.$$

The next theorem is a counterpart of Lemma 3.1.6 and Theorem 3.1.8 for the design of a quasi-ISS/ISDS reduced-order observer for a subsystem of an interconnected system.

Theorem 3.2.2. *1. Under Assumption 3.2.1, point 1., the system*

$$\begin{aligned}
\dot{\hat{\xi}}_i &= f_{2i}(\bar{y}_1, \ldots, \bar{y}_n, \hat{\xi}_1 - l_1(\bar{y}_1), \ldots, \hat{\xi}_n - l_n(\bar{y}_n), u_i)\\
&\quad + \frac{\partial l_i}{\partial z_{1i}}(\bar{y}_i) f_{1i}(\bar{y}_1, \ldots, \bar{y}_n, \hat{\xi}_1 - l_1(\bar{y}_1), \ldots, \hat{\xi}_n - l_n(\bar{y}_n), u_i)\\
\hat{z}_{1i} &= \bar{y}_i,\\
\hat{z}_{2i} &= \hat{\xi}_i - l_i(\bar{y}_i)
\end{aligned} \tag{3.18}$$

becomes a quasi-ISS reduced-order observer for the i*th subsystem of* (3.17).

2. *Under Assumption 3.2.1, point 2., the system* (3.18) *becomes a quasi-ISDS reduced-order observer for the* i*th subsystem of* (3.17).

Proof. The proof goes along the proof of Theorem 3.1.8 with changes according to the quasi-ISS/ISDS property for interconnected systems and Assumption 3.2.1.

We define $\xi_i := z_{2i} + l_i(z_{1i})$. Then,

$$\begin{aligned}
\dot{z}_{1i} &= f_{1i}(z_{11}, \ldots, z_{1n}, \xi_1 - l_1(z_{11}), \ldots, \xi_n - l_n(z_{1n}), u_i),\\
y_i &= z_{1i}\\
\dot{\xi}_i &= f_{2i}(z_{11}, \ldots, z_{1n}, \xi_1 - l_1(z_{11}), \ldots, \xi_n - l_n(z_{1n}), u_i)\\
&\quad + \frac{\partial l_i}{\partial z_{1i}}(z_{1i}) f_{1i}(z_{1i}, \ldots, z_{1n}, \xi_1 - l_1(z_{11}), \ldots, \xi_n - l_n(z_{1n}), u_i)\\
&=: F_i(z_{11}, \ldots, z_{1n}, \xi_1, \ldots, \xi_n, u_i).
\end{aligned}$$

The reduced-order observer (3.18) is written as $\dot{\hat{\xi}}_i = F_i(\bar{y}_1,\dots,\bar{y}_n,\hat{\xi}_1,\dots,\hat{\xi}_n,u_i)$. Let $e_i := \hat{\xi}_i - \xi_i$. We use the shorthand for $j = 1, 2$

$$\begin{aligned}\hat{f}^d_{ji} &= f_{ji}(y_1+d_1,\dots,y_n+d_n,\hat{\xi}_1 - l_1(y_1+d_1),\dots,\hat{\xi}_n - l_n(y_n+d_n),u_i),\\ f_{ji} &= f_{ji}(y_1,\dots,y_n,\xi_1 - l_1(y_1),\dots,\xi_n - l_n(y_n),u_i),\end{aligned}$$

then we have whenever $V_i(e_i) \geq \max_{j\neq i}\gamma_{ij}(V_j(e_j))$ holds, it follows

$$\begin{aligned}\dot{V}_i(e_i) =& \frac{\partial V_i}{\partial e_i}(e_i)\left(\left[\hat{f}^d_{2i} + \frac{\partial l_i}{\partial z_{1i}}(z_{1i}+d_i)\hat{f}^d_{1i}\right] - \left[f_{2i} + \frac{\partial l_i}{\partial z_{1i}}(z_{1i})f_{1i}\right]\right)\\ \leq & -\alpha_{3i}(V_i(e_i)) + \frac{\partial V_i}{\partial e_i}(e_i)\left(F_i(\bar{y}_1,\dots,\bar{y}_n,\xi_1,\dots,\xi_n,u_i) - F_i(y_1,\dots,y_n,\xi_1,\dots,\xi_n,u_i)\right)\\ \leq & -\alpha_{3i}(V_i(e_i)) + \alpha_{4i}(V_i(e_i))\rho_i(y_1,\dots,y_n,\xi_1,\dots,\xi_n,u_i)\max_j \gamma_i(|d_j|),\end{aligned}$$

where $\gamma_i \in \mathcal{K}$ and ρ_i is a continuous positive function such that

$$\begin{aligned}&|F_i(\bar{y}_1,\dots,\bar{y}_n,\xi_1,\dots,\xi_n,u_i) - F_i(y_1,\dots,y_n,\xi_1,\dots,\xi_n,u_i)|\\ &\leq \rho_i(y_1,\dots,y_n,\xi_1,\dots,\xi_n,u_i)\max_j \gamma_i(|d_j|)\end{aligned}$$

holds, whose existence can be shown using the results in the appendix in [32]:

$$\begin{aligned}&|F_i(\bar{y}_1,\dots,\bar{y}_n,\xi_1,\dots,\xi_n,u_i) - F_i(y_1,\dots,y_n,\xi_1,\dots,\xi_n,u_i)|\\ &\leq \rho_i(y_1,\dots,y_n,\xi_1,\dots,\xi_n,u_i)\tilde{\gamma}_i(|(d_1,\dots,d_n)^T|)\\ &\leq \rho_i(y_1,\dots,y_n,\xi_1,\dots,\xi_n,u_i)\tilde{\gamma}_i(\max_j \sqrt{n}|d_j|)\\ &=: \rho_i(y_1,\dots,y_n,\xi_1,\dots,\xi_n,u_i)\gamma_i(\max_j |d_j|).\end{aligned}$$

It follows that for an arbitrary $\delta_i \in (0,1)$, we have

$$\begin{aligned}&V_i(e_i) \geq \alpha_i^{-1}\left((1-\delta_i)^{-1}\rho_i(y_1,\dots,y_n,\xi_1,\dots,\xi_n,u_i)\max_j \gamma_i(|d_j|)\right)\\ \Rightarrow & \dot{V}_i \leq -\delta_i\alpha_{3i}(V_i(e_i)).\end{aligned}$$

Under the conditions that $||z_j||_{[0,t]} \leq K_i$ and $||(u_i)_{[0,t]}|| \leq K_i,\ j = 1,\dots,n$ it can be shown by standard arguments that there exist a function $\bar{\beta}_i \in \mathcal{KL}$, functions $\bar{\gamma}_i^{K_i}, \bar{\gamma}_{ij}^{K_i} \in \mathcal{K}_\infty,\ j = 1,\dots,n,\ j\neq i$ such that

$$|e_i(t)| \leq \max\left\{\bar{\beta}_i(|e_i^0|,t), \max_{j\neq i}\bar{\gamma}_{ij}^{K_i}(||(e_j)_{[0,t]}||), \max_j \bar{\gamma}_i^{K_i}(||(d_j)_{[0,t]}||)\right\},\ \forall t \in \mathbb{R}_+. \tag{3.19}$$

Recalling (3.18), we have that $|\tilde{z}_i| \leq |d_i| + |e_i| + \theta_{K_i}(|d_i|)$ and $|e_i| \leq |\tilde{z}_{2i}| + \theta_{K_i}(|d_i|)$, where $\theta_{K_i}(|d_i|)$ is a class-$\mathcal{K}$ function, parametrized by K_i such that $|l(z_{1i}+d_i) - l(z_{1i})| \leq \theta_{K_i}(|d_i|)$ when $|z_{1i}| \leq K_i$. Together with (3.19) we obtain

$$\begin{aligned}|\tilde{z}_i(t)| \leq & \max\left\{\bar{\beta}_i(|e_i^0|,t), \max_{j\neq i}\bar{\gamma}_{ij}^{K_i}(||(e_j)_{[0,t]}||), \max_j \bar{\gamma}_i^{K_i}(||(d_j)_{[0,t]}||)\right\} + \theta_{K_i}(|d_i(t)|) + |d_i(t)|\\ \leq & \max\Big\{3\bar{\beta}_i((|\tilde{z}_{2i}^0| + \theta_{K_i}(|d_i^0|)),t), 3\max_{j\neq i}\bar{\gamma}_{ij}^{K_i}(||\tilde{z}_{2j}||_{[0,t]} + \theta_{K_j}(||(d_j)_{[0,t]}||)),\\ & 3\max_{j\neq i}\bar{\gamma}_i^{K_i}(||(d_j)_{[0,t]}||), 3\chi_i^{K_i}(||(d_i)_{[0,t]}||)\Big\},\end{aligned}$$

where $\chi_i^{K_i}(r) := \max\{\bar{\gamma}_i^{K_i}(r), \theta_{K_i}(r), r\}$. By $\alpha(a+b) \leq \max\{\alpha(2a), \alpha(2b)\}$ for $\alpha \in \mathcal{K}$ we have that

$$\begin{aligned}
&|\tilde{z}_i(t)| \\
\leq\ & \max\Big\{3\bar{\beta}_i(2|\tilde{z}_{2i}^0|), t), 3\bar{\beta}_i(2\theta_{K_i}(|d_i^0|)), t), 3\max_{j\neq i}\bar{\gamma}_{ij}^{K_i}(2||\tilde{z}_{2j}||_{[0,t]}), \\
&\quad 3\max_{j\neq i}\bar{\gamma}_{ij}^{K_i}(2\theta_{K_j}(||(d_j)_{[0,t]}||)), 3\max_{j\neq i}\bar{\gamma}_i^{K_i}(||(d_j)_{[0,t]}||), 3\chi_i^{K_i}(||(d_i)_{[0,t]}||)\Big\} \\
\leq\ & \max\left\{\tilde{\beta}_i(|\tilde{z}_i^0|), t), \max_{j\neq i}(\tilde{\gamma}_{ij}^{K_i})^{\mathrm{ISS}}(||\tilde{z}_j||_{[0,t]}), \max_{j\neq i}(\tilde{\gamma}_{ij}^{K_i})^{\mathrm{ISS}}(||(d_j)_{[0,t]}||), (\tilde{\gamma}_i^{K_i})^{\mathrm{ISS}}(||(d_i)_{[0,t]}||)\right\},
\end{aligned}$$

$||z_j||_{[0,t]} \leq K_i$ and $||(u_i)_{[0,t]}|| \leq K_i$, $j = 1, \ldots, n$, where

$$\begin{aligned}
\tilde{\beta}_i(r,t) &:= 3\bar{\beta}_i(2r,t), \\
(\tilde{\gamma}_{ij}^{K_i})^{\mathrm{ISS}}(r) &:= \max\left\{3\bar{\gamma}_{ij}^{K_i}(2r), 3\bar{\gamma}_{ij}^{K_i}(2\theta_{K_j}(r), 3\bar{\gamma}_i^{K_i}(r)\right\}, \\
(\tilde{\gamma}_i^{K_i})^{\mathrm{ISS}}(r) &:= \max\left\{3\bar{\beta}_i(2\theta_{K_i}(r)), 0), 3\chi_i^{K_i}(r)\right\}.
\end{aligned}$$

This proves that the system (3.18) is a quasi-ISS reduced-order observer for the ith subsystem.

The proof for the quasi-ISDS reduced-order observer for the ith subsystem follows the same steps as for a quasi-ISS reduced-order observer. □

Now, if we define $P := \sum P_i$, $N := \sum N_i$, $m := \sum M_i$, $z := (z_1^T, \ldots, z_n^T)^T \in \mathbb{R}^N$, $z_1 := (z_{11}^T, \ldots, z_{1n}^T)^T \in \mathbb{R}^P$, $z_2 := (z_{21}^T, \ldots, z_{2n}^T)^T \in \mathbb{R}^{P-N}$, $u := (u_1^T, \ldots, u_n^T)^T \in \mathbb{R}^m$, $d = (d_1^T, \ldots, d_n^T)^T$ and $f := (f_1^T, \ldots, f_n^T)^T$, $\tilde{f}_1 := (f_{11}^T, \ldots, f_{1n}^T)^T$, $\tilde{f}_2 := (f_{21}^T, \ldots, f_{2n}^T)^T$, then the system (3.17) can be written as a system of the form (3.4).

Now, we investigate under which conditions a quasi-ISS/ISDS observer for the whole system can be designed.

We collect all gains $(\tilde{\gamma}_{ij}^{K_i})^{\mathrm{ISS}}$ in a gain-matrix $\widetilde{\Gamma}^{\mathrm{ISS}}$, which defines a map as in (1.12).We define $\widetilde{\Gamma}^{\mathrm{ISDS}}$ accordingly. With these considerations a quasi-ISS/ISDS reduced-order observer for the overall system can be designed.

Theorem 3.2.3. *Consider a system of the form* (3.17).

1. *Assume that Assumption 3.2.1, point 1. and Theorem 3.2.2, point 1. hold true for each $i = 1, \ldots, n$. If $\widetilde{\Gamma}^{ISS}$ satisfies the SGC (1.15), then the reduced-order error Lyapunov function V as in Assumption 3.1.4 is given by $V = \max_i\{\sigma_i^{-1}(V_i)\}$, where $\sigma = (\sigma_1, \ldots, \sigma_n)^T$ is an Ω-path from Proposition 1.2.3 and the quasi-ISS reduced-order observer for the overall system is given by*

$$\hat{\xi} = (\hat{\xi}_1^T, \ldots, \hat{\xi}_n^T)^T, \tag{3.20}$$

and

$$\begin{aligned}
\dot{\hat{\xi}} &= f_2(\bar{y}, \hat{\xi} - l(\bar{y}), u) + \frac{\partial l}{\partial z_1}(\bar{y}) f_1(\bar{y}, \hat{\xi}, u), \\
\hat{z}_1 &= \bar{y}, \\
\hat{z}_2 &= \hat{\xi} - l(\bar{y}).
\end{aligned} \tag{3.21}$$

2. *Assume that Assumption 3.2.1, point 2. and Theorem 3.2.2, point 2. hold true for each $i = 1, \ldots, n$. If $\widetilde{\Gamma}^{ISDS}$ satisfies the SGC (1.15), then the reduced-order error Lyapunov function V as in Assumption 3.1.7 is given by $V = \max_i\{\sigma_i^{-1}(V_i)\}$, where $\sigma = (\sigma_1, \ldots, \sigma_n)^T$ is an Ω-path from Proposition 1.2.3 and the quasi-ISDS reduced-order observer for the overall system is given by* (3.20) *and* (3.21).

Proof. Ad point 1.: We define the error Lyapunov function candidate of the whole system by $V(e) = \max_i\{\sigma_i^{-1}(V_i(e_i))\}$, $e = (e_1, \ldots, e_n)^T$. We have to verify the conditions from Assumption 3.1.4, from which with the help of Lemma 3.1.6 the observer defined by (3.20) becomes a quasi-ISS reduced-order observer for the whole system of the form (3.4).

Let $I := \{i \in \{1, \ldots, n\} | V(e) = \frac{1}{s_i} V_i(e_i) \geq \max_{j, j \neq i}\{\frac{1}{s_j}(V_j(e_j))\}\}$. Fix an $i \in I$. Let $e_i := \hat{\xi}_i - \xi_i$. From (1.18) it follows

$$V_i(e_i) = \sigma_i(V(e)) > \max_{j \neq i}(\tilde{\gamma}_{ij}^{K_i})^{\text{ISS}}(\sigma_i(V(e))) = \max_{j \neq i}(\tilde{\gamma}_{ij}^{K_i})^{\text{ISS}}(V_i(e_i))$$

We observe that there exist $c_1 > 0, c_2 > 0$ such that

$$V(e) \geq \min_i \sigma_i^{-1}(\alpha_{1i}(c_1|e|)) =: \alpha_1(|e|) \text{ and } V(e) \leq \max_i \sigma_i^{-1}(\alpha_{2i}(c_2|e|)) =: \alpha_2(|e|).$$

Now, with (1.17) it holds for each i

$$\left|\frac{\partial V(e)}{\partial e}\right| = \left|\frac{\partial \sigma_i^{-1}(V_i(e_i))}{\partial e_i}\right| = \left|(\sigma_i^{-1})'(V_i(e_i))\frac{\partial V_i(e_i)}{\partial e_i}\right| \leq K_2\alpha_{4i}(V_i(e_i)) \leq \alpha_4(V(e)),$$

where $\alpha_4(r) := \max_i K_2\alpha_{4i}(\sigma_i(r))$, K_2 is from (1.17) and (3.6) is satisfied.

Furthermore, we have for each i that it holds

$$\begin{aligned}
\dot{V}(e) &= (\sigma_i^{-1})'(V_i(e_i))\frac{\partial V_i(e_i)}{\partial e_i}\dot{e}_i \\
&= (\sigma_i^{-1})'(V_i(e_i))\frac{\partial V_i(e_i)}{\partial e_i}\Big([f_{2i}(z_1, e_1 + z_{21}, \ldots, e_n + z_{2n}, u_i) \\
&\qquad\qquad + \frac{\partial l_i}{\partial z_{1i}}(z_{1i})f_{1i}(z_1, e_1 + z_{21}, \ldots, e_n + z_{2n}, u_i)\Big] \\
&\qquad\qquad - \left[f_{2i}(z_1, z_2, u_i) + \frac{\partial l_i}{\partial z_{1i}}(z_{1i})f_{1i}(z_1, z_2, u_i)\right]\Big) \\
&\leq -K_1\alpha_{31}(V_i(e_i)) \leq -\alpha_3(V(e)),
\end{aligned}$$

where $\alpha_3(r) := \min_i K_1\alpha_{3i}(\sigma_i(r))$, K_1 is from (1.17) and (3.7) is satisfied. Finally, we choose a function $\alpha \in \mathcal{K}_\infty$ such that $\alpha(s)\alpha_4(s) \leq \alpha_3(s)$ and all the conditions from Assumption 3.1.4 are satisfied and by application of Lemma 3.1.6 a quasi-ISS reduced-order observer of the whole system can be designed.

Ad point 2.: We define

$$\begin{aligned}
\overline{V}(e) &:= \max_i\left\{\sigma_i^{-1}(V_i(e_i))\right\}, \\
\bar{\eta}(|e|) &:= \max_i\left\{\sigma_i^{-1}(\eta_i(|e|))\right\}, \\
\psi(|e|) &:= \min_i \sigma_i^{-1}\left(\frac{|e|}{\sqrt{n}}\right),
\end{aligned} \tag{3.22}$$

where V_i satisfies the conditions in Assumption 3.2.1, point 2. for $i = 1, \dots, n$. Let j be such that $|e|_\infty = |e_j|_\infty$ for some $j \in \{1, \dots, n\}$, then we have

$$\max_i \sigma_i^{-1}\left(\frac{|e_i|}{1+\varepsilon_i}\right) \geq \max_i \sigma_i^{-1}\left(\frac{|e_i|_\infty}{1+\varepsilon}\right) \geq \sigma_j^{-1}\left(\frac{|e_j|_\infty}{1+\varepsilon}\right) \geq \min_i \sigma_i^{-1}\left(\frac{|e|}{\sqrt{n}(1+\varepsilon)}\right) \tag{3.23}$$

where $\varepsilon := \max_i \varepsilon_i$ and we obtain

$$\psi\left(\frac{|e|}{1+\varepsilon}\right) \leq \overline{V}(e) \leq \eta(|e|). \tag{3.24}$$

Then, $\left|\frac{\partial \overline{V}(e)}{\partial e}\right| \leq \alpha(\overline{V}(e))$ holds with $\alpha(r) := \max_i K_2 \alpha_i(\sigma_i r)$, $r \geq 0$, where K_2 is from (1.17). Furthermore, we have

$$\dot{\overline{V}}(e) \leq -(1-\varepsilon)\bar{g}(\overline{V}(e))$$

with $\varepsilon = \max_i \varepsilon_i$ and $\bar{g}(r) := \min_i K_1 g_i(\sigma_i r)$, $r \geq 0$ is positive definite and locally Lipschitz, where K_1 is from (1.17). From (3.24) we get

$$\frac{|e|}{1+\varepsilon} \leq \psi^{-1}\left(\overline{V}(e)\right) \leq \psi^{-1}\left(\bar{\eta}\left(|e|\right)\right)$$

and we define $V(e) := \psi^{-1}\left(\overline{V}(e)\right)$ as the reduced-order error Lyapunov function candidate with $\eta\left(|e|\right) := \psi^{-1}\left(\bar{\eta}\left(|e|\right)\right)$. By the previous calculations for $\overline{V}(e)$ it holds

$$\dot{V}(e) \leq -(1-\varepsilon)g\left(V(e)\right),$$

where $g(r) := (\psi^{-1})'\left(\psi(r)\right)\bar{g}\left(\psi(r)\right)$ is locally Lipschitz continuous. Altogether, $V(e)$ satisfies all conditions in Assumption 3.1.7. Hence, $V(e)$ is the reduced-order error Lyapunov function of the whole system and by application of Theorem 3.1.8 a quasi-ISDS reduced-order observer of the whole system can be designed. □

Remark 3.2.4. *Note that for large n the function ψ in (3.22) becomes "small" and hence the rates and gains of the quasi-ISDS property defined by ψ^{-1} become "large", which is not desired in applications. To avoid this kind of conservativeness one can use the maximum norm $|\cdot|_\infty$ instead of the Euclidean one in the definitions above and in Theorem 3.2.3. In this case, the division by $\sqrt{n}$ in* (3.23) *can be avoided and we get* (3.22) *with $\psi\left(|e|_\infty\right) = \min_i \sigma_i^{-1}\left(|e|_\infty\right)$.*

3.3 Applications

In this section, we investigate the stabilization of single and interconnected systems subject to quantization. At first, we consider single systems and combine the quantized output feedback stabilization with the ISDS property as in Chapter V in [110] for the ISS property. Then, we consider interconnected systems and give a counterpart to Proposition 1 in [110] for such kind of systems. Furthermore, we investigate dynamic quantizers, where the quantizers can be adapted by a zooming variable. This leads to asymptotic stability of the overall closed-loop system.

We consider a single system of the form (3.4). By an output quantizer we mean a piecewise constant function $q : \mathbb{R}^P \to \mathcal{Q}$, where $\mathcal{Q}$ is a finite and discrete subset of $\mathbb{R}^P$. The quantization error is denoted by

$$d := q(y) - y. \tag{3.25}$$

We assume that there exists $M > 0$, the quantizer's range, and $\Delta > 0$, the error bound, such that $|y| \leq M$ implies $|d| \leq \Delta$. This property is referred to *saturation* in the literature, see [71], for example.

Now, suppose that Assumption 3.1.7 holds and a quasi-ISDS observer has been designed as in Lemma 3.1.8. With d as in (3.25) the observer acts on the quantized output measurements $\bar{y} = q(y)$. Furthermore, suppose that a controller is given in the form $u = k(z)$. We can now define a quantized output feedback law by

$$u = k(\hat{z}) = k(z + \tilde{z}),$$

where $\hat{z}$ is the state estimate generated by the observer and $\tilde{z} = \hat{z} - z$ is the state estimation error. We impose on the feedback law:

Assumption 3.3.1. *The system $\dot{z} = f(z, k(\hat{z})) = f(z, k(z + \tilde{z}))$ is ISDS , i.e.,*

$$|z(t)| \leq \max\left\{\hat{\mu}(\hat{\eta}(|z_0|), t), \underset{\tau\in[0,t]}{\mathit{ess\ sup}}\ \hat{\mu}(\hat{\gamma}^{ISDS}(|\tilde{z}(\tau)|), t-\tau)\right\} \tag{3.26}$$

for some $\hat{\mu} \in \mathcal{KLD}$, $\hat{\eta}$ and $\hat{\gamma}^{ISDS} \in \mathcal{K}_\infty$.

For a detailed discussion for the case of an ISS controller we refer to [72]. The overall closed-loop system obtained by combining the plant, represented in the form (3.4) after a suitable coordinate change, the observer and the control law can be written as

$$\begin{aligned}
\dot{z} &= \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} f_1(z_1, z_2, k(\hat{z})) \\ f_2(z_1, z_2, k(\hat{z})) \end{bmatrix}, \\
\hat{z} &= \begin{bmatrix} \hat{z}_1 \\ \hat{z}_2 \end{bmatrix} = \begin{bmatrix} q(z_1) \\ \hat{\xi} - l(q(z_1)) \end{bmatrix}, \\
\dot{\hat{\xi}} &= f_2(q(z_1), \hat{\xi} - l(q(z_1)), k(\hat{z})) + \frac{\partial l}{\partial z_1}(q(z_1)) f_1(q(z_1), \hat{\xi} - l(q(z_1)), k(\hat{z})).
\end{aligned}$$

We know from the equation (3.13) in the proof of Theorem 3.1.8 that for $e = \hat{\xi} - \xi$, where $\xi = z_2 + l(z_1)$, the bound

$$|e(t)| \leq \max\left\{\bar{\mu}(\bar{\eta}(|e_0|), t), \underset{\tau\in[0,t]}{\text{ess sup}}\ \bar{\mu}(\bar{\gamma}^{\text{ISDS}}(|d(\tau)|), t-\tau)\right\}$$

holds. Combining this with (3.26) and $|\tilde{z}| \leq |d| + |e| + \theta_K(|d|)$, $\theta_K \in \mathcal{K}$, we obtain the estimation

$$\left|\begin{pmatrix} z(t) \\ \hat{\xi}(t) \end{pmatrix}\right| \leq \max\left\{\mu\left(\eta\left(\left|\begin{pmatrix} z_0 \\ \hat{\xi}_0 \end{pmatrix}\right|\right), t\right), \nu(d, t)\right\}$$

for $\|z\|_{[0,t]} \leq K$, $\left\|u_{[0,t]}\right\| = \left\|k(\hat{z})_{[0,t]}\right\| \leq K$ and $\mu \in \mathcal{KLD}$, $\eta \in \mathcal{K}_\infty$, where $\nu(d,t) := \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu(\gamma^{\mathrm{ISDS}}(|d(\tau)|), t-\tau)$, $\gamma^{\mathrm{ISDS}} \in \mathcal{K}_\infty$. This follows according to the lines in the proof of Theorem 3.1.8.

Now, we consider an interconnected system of the form (3.15) or (3.17), respectively. The output quantizer of the ith subsystem is given by $q_i : \mathbb{R}^{P_i} \to \mathcal{Q}_i$, where $\mathcal{Q}_i$ is a finite subset of $\mathbb{R}^{P_i}$, the quantization error by $d_i := q_i(y_i) - y_i$, the quantizer's range $\tilde{M}_i > 0$ by $|y_i| \leq \tilde{M}_i$, which implies $|d_i| \leq \Delta_i$, where $\Delta_i > 0$ is the error bound. We suppose that Assumption 3.2.1 holds and an observer for the ith subsystem has been designed by Theorem 3.2.2, which acts on the quantized output measurements $\overline{y}_i = q_i(y_i)$.

Suppose that a controller of the ith subsystem is given by $u_i = k_i(z_i)$ and the quantized output feedback law is defined by

$$u_i := k_i(\hat{z}_i) = k_i(z_i + \tilde{z}_i),$$

where $\hat{z}_i$ is the state estimate generated by the observer and $\tilde{z}_i = \hat{z}_i - z_i$ is the state estimation error. In the rest of the section we suppose the following:

Assumption 3.3.2. *1. The ith subsystem $\dot{z}_i = f_i(y_1, \ldots, z_i, \ldots, y_n, k_i(\hat{z}_i))$ is ISS, i.e.,*

$$|z_i(t)| \leq \max\left\{\hat{\beta}_i(|z_i^0|, t), \max_{j\neq i}(\hat{\gamma}_{ij})^{ISS}(||(\tilde{z}_j)_{[0,t]}||), (\hat{\gamma}_i)^{ISS}(||(\tilde{z}_i)_{[0,t]}||)\right\}$$

for some $\hat{\beta}_i \in \mathcal{KL}$ and $(\hat{\gamma}_{ij})^{ISS}, (\hat{\gamma}_i)^{ISS} \in \mathcal{K}_\infty$.

2. The ith subsystem $\dot{z}_i = f_i(y_1, \ldots, z_i, \ldots, y_n, k_i(\hat{z}_i))$ is ISDS, i.e.,

$$|z_i(t)| \leq \max\left\{\hat{\mu}_i(\hat{\eta}_i(|z_i^0|), t), \max_{j\neq i} \hat{\nu}_{ij}(\tilde{z}_j, t), \hat{\nu}_i(\tilde{z}_i, t)\right\}$$

for some $\hat{\mu}_i \in \mathcal{KLD}$, $\hat{\eta} \in \mathcal{K}_\infty$ and

$$\hat{\nu}_{ij}(\tilde{z}_j, t) := \operatorname*{ess\,sup}_{\tau\in[0,t]} \hat{\mu}_i((\hat{\gamma}_{ij})^{ISDS}(|\tilde{z}_j(\tau)|), t-\tau),$$
$$\hat{\nu}_i(\tilde{z}_i, t) := \operatorname*{ess\,sup}_{\tau\in[0,t]} \hat{\mu}_i((\hat{\gamma}_i)^{ISDS}(|\tilde{z}_i(\tau)|), t-\tau),$$

where $(\hat{\gamma}_{ij})^{ISDS}, (\hat{\gamma}_i)^{ISDS} \in \mathcal{K}_\infty$.

It can be verified, if Assumption 3.3.2, point 1. holds for the ith subsystem of the overall closed-loop system obtained by combining the plant, the observer and the control law that it holds

$$\left|\begin{pmatrix} z_i(t) \\ \hat{\xi}_i(t) \end{pmatrix}\right| \leq \max\left\{\beta_i\left(\left|\begin{matrix} z_i^0 \\ \hat{\xi}_i^0 \end{matrix}\right|, t\right), \max_{j\neq i}(\gamma_{ij}^{K_i})^{\mathrm{ISS}}(||(d_j)_{[0,t]}||), (\gamma_i^{K_i})^{\mathrm{ISS}}(||(d_i)_{[0,t]}||)\right\}, \quad (3.27)$$

for $\beta_i \in \mathcal{KL}$, $(\gamma_{ij}^{K_i})^{\mathrm{ISS}}$ and $(\gamma_i^{K_i})^{\mathrm{ISS}} \in \mathcal{K}_\infty$, $||z_j||_{[0,t]} \leq K_i$ for $j = 1, \ldots, n$, and $||(u_i)_{[0,t]}|| = ||(k_i(\hat{z}_i))_{[0,t]}|| \leq K_i$.

If Assumption 3.3.2, point 2. holds for the ith subsystem of the overall closed-loop system obtained by combining the plant, the observer and the control law, then it holds

$$\left|\begin{pmatrix} z_i(t) \\ \hat{\xi}_i(t) \end{pmatrix}\right| \leq \max\left\{\mu_i\left(\eta_i\left(\left|\begin{matrix} z_i^0 \\ \hat{\xi}_i^0 \end{matrix}\right|\right), t\right), \max_{j\neq i} \nu_{ij}(d_j,t), \nu_i(d_i,t)\right\} \tag{3.28}$$

for $\mu_i \in \mathcal{KLD}$, $\eta_i, (\gamma_{ij}^{K_i})^{\text{ISDS}}, (\gamma_i^{K_i})^{\text{ISDS}} \in \mathcal{K}_\infty$, $||z_j||_{[0,t]} \leq K_i$, $j = 1,\ldots,n$ and $||(u_i)_{[0,t]}|| = ||(k_i(\hat{z}_i))_{[0,t]}|| \leq K_i$, where

$$\nu_{ij}(d_j,t) := \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu_i((\gamma_{ij}^{K_i})^{\text{ISDS}}(|d_j(\tau)|), t-\tau),$$
$$\nu_i(d_i,t) := \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu_i((\gamma_i^{K_i})^{\text{ISDS}}(|d_i(\tau)|), t-\tau).$$

Furthermore, we can show that if the small-gain condition (1.15) is satisfied for $\Gamma^{\text{ISS}} = ((\gamma_{ij}^{K_i})^{\text{ISS}})_{n\times n}$ or $\Gamma^{\text{ISDS}} = ((\gamma_{ij}^{K_i})^{\text{ISDS}})_{n\times n}$, respectively, which defines a map as in (1.12), then for the overall system it holds

1.

$$\left|\begin{pmatrix} z(t) \\ \hat{\xi}(t) \end{pmatrix}\right| \leq \max\left\{\beta\left(\left|\begin{matrix} z^0 \\ \hat{\xi}^0 \end{matrix}\right|, t\right), (\gamma^K)^{\text{ISS}}(||d_{[0,t]}||)\right\}, \tag{3.29}$$

2.

$$\left|\begin{pmatrix} z(t) \\ \hat{\xi}(t) \end{pmatrix}\right| \leq \max\left\{\mu\left(\eta\left(\left|\begin{matrix} z^0 \\ \hat{\xi}^0 \end{matrix}\right|\right), t\right), \operatorname*{ess\,sup}_{\tau\in[0,t]} \mu_i((\gamma^K)^{\text{ISDS}}(|d(\tau)|), t-\tau)\right\}. \tag{3.30}$$

Let $\kappa_i^l \in \mathcal{K}_\infty$ with the property $|l_i(z_{1i})| \leq \kappa_i^l(|z_{1i}|)$, $\forall z_{1i}$, $\kappa_i^u \in \mathcal{K}_\infty$ such that $|k_i(z_i)| \leq \kappa_i^u(|z_i|)$, $\forall z_i$ and define $K_i := \max\left\{\tilde{M}_i, \kappa_i^u(2\tilde{M}_i + \Delta_i + \kappa_i^l(\tilde{M}_i + \Delta_i))\right\}$.

With $z^1 = (z_{11}^T, \ldots, z_{1n}^T)^T$ we give a counterpart of Proposition 1 in [110] for interconnected systems, which provides estimations of the norm of the systems state and the observer state using quantized output feedbacks.

Proposition 3.3.3. *Let* $\gamma_{ij}^{K_i} \equiv (\gamma_{ij}^{K_i})^{ISDS} \equiv (\gamma_{ij}^{K_i})^{ISS}$ *and* $\gamma_i^{K_i} \equiv (\gamma_i^{K_i})^{ISDS} \equiv (\gamma_i^{K_i})^{ISDS}$.

1. *Assume,* $\max\{\max_{j\neq i} \gamma_{ij}^{K_i}(\Delta_j), \gamma_i^{K_i}(\Delta_i)\} \leq \tilde{M}_i$ *and*

$$\left|\begin{pmatrix} z_i^0 \\ \hat{\xi}_i^0 \end{pmatrix}\right| < E_i^0 \tag{3.31}$$

where $E_i^0 > 0$ *is such that* $\beta_i(E_i^0, 0) = \mu_i(\tilde{M}_i, 0) = \tilde{M}_i$. *Then, the corresponding solution of the* i*th subsystem of the overall closed-loop system satisfies*

$$\mathop{lim\ sup}_{t\to\infty} \left|\begin{pmatrix} z_i(t) \\ \hat{\xi}_i(t) \end{pmatrix}\right| \leq \max\left\{\max_{j\neq i} \gamma_{ij}^{K_i}(\Delta_j), \gamma_i^{K_i}(\Delta_i)\right\}. \tag{3.32}$$

2. *Assume that 1. holds for all* $i = 1, \ldots, n$. *Define* $M := \max \tilde{M}_i$, $\Delta := \max \Delta_i$, $K := \max K_i$ *and suppose that* $\Gamma = (\gamma_{ij}^{K_i})_{n\times n}$ *satisfies the small-gain condition* (1.15). *Then, the corresponding solution of the overall closed-loop system satisfies*

$$\limsup_{t\to\infty} \left| \begin{pmatrix} z(t) \\ \hat{\xi}(t) \end{pmatrix} \right| \leq \gamma^K(\Delta). \tag{3.33}$$

Proof. Ad point 1.: Note that it holds $|y_i(t)| = |z_{1i}(t)| \leq \tilde{M}_i$, $|q_i(z_{1i}(t)) - z_{1i}(t)| = |d_i(t)| \leq \Delta_i$. As long as it holds $\left| \begin{pmatrix} z_i(t) \\ \hat{\xi}_i(t) \end{pmatrix} \right| \leq \tilde{M}_i$, we have $|z_i(t)| \leq \tilde{M}_i \leq K_i$ and

$$\begin{aligned} |u_i(t)| &= |k_i(\hat{z}_i(t))| \leq \kappa_i^u(|\hat{z}_i(t)|) \leq \kappa_i^u(|q_i(z_{1i}(t))| + |\hat{\xi}_i(t)| + |l_i(q_i(z_{1i}(t)))|) \\ &\leq \kappa_i^u(\tilde{M}_i + \Delta_i + \tilde{M}_i + \kappa_i^l(\tilde{M}_i + \Delta_i)) \leq K. \end{aligned}$$

Define $T := \sup \left\{ t \geq 0 : \left| \begin{pmatrix} z_i(t) \\ \hat{\xi}_i(t) \end{pmatrix} \right| < \tilde{M}_i \right\} \leq \infty$. This is well-defined, since (3.31) and $\beta_i(E_i^0, 0) \geq E_i^0 = \mu_i(E_i^0, 0)$ holds. It follows that (3.27) or (3.28), respectively, is true for $t \in [0, T]$ and we obtain

$$\left| \begin{pmatrix} z_i(t) \\ \hat{\xi}_i(t) \end{pmatrix} \right| < \tilde{M}_i, \ \forall t \in [0, T] \tag{3.34}$$

using the requirements of this proposition. Now, assume that T is finite. Therefore, there must exists a T^* such that $\left| \begin{pmatrix} z_i(T^*) \\ \hat{\xi}_i(T^*) \end{pmatrix} \right| = \tilde{M}_i$. But from (3.27) or (3.28), respectively, and from the above calculations it holds $\left| \begin{pmatrix} z_i(T^*) \\ \hat{\xi}_i(T^*) \end{pmatrix} \right| < \tilde{M}_i$, which contradicts the assumption that T is finite. It follows that T is infinite and from the fact that z_i and $\hat{\xi}_i$ are continuous the estimation (3.34) holds for all $t \geq 0$.

Now, since $\beta_i \in \mathcal{KL}$ for every $\epsilon > 0$ there exists $T(\epsilon)$ such that

$$\beta_i \left(\left| \begin{pmatrix} z_i^0 \\ \hat{\xi}_i^0 \end{pmatrix} \right|, t \right) \leq \epsilon, \ \forall t \geq T(\epsilon)$$

and therefore

$$\left| \begin{pmatrix} z_i(t) \\ \hat{\xi}_i(t) \end{pmatrix} \right| \leq \max \left\{ \max_{j\neq i} \gamma_{ij}^{K_i}(\Delta_j), \gamma_i^{K_i}(\Delta_i) \right\}, \ \forall t \geq T(\epsilon),$$

which proves point 1.

Ad point 2.: This follows by the same steps as for the proof of point 1. using (3.29) or (3.30), respectively, under the small-gain condition. □

3.3.1 Dynamic quantizers

Now, we are going to improve the mentioned results in order to get smaller bounds (3.32) and (3.33) using a *dynamic quantizer*, see [70, 72]. Here, we obtain asymptotic convergence in (3.32) and (3.33) and we use the zooming-in strategy. We consider single systems of the form (3.4) and the dynamic quantizer

$$q_\lambda(y) := \lambda q(\tfrac{y}{\lambda}),$$

where $\lambda > 0$ is the zoom variable. The range of this quantizer is $M\lambda$ and $\Delta\lambda$ is the quantization error. If we increase λ this is referred to the zooming-out strategy and corresponds to a larger range and quantization error and by a decreasing λ we obtain a smaller range and smaller quantization error, referred to the zooming-in strategy. The parameter λ can be updated continuously, but the update of λ for discrete instants of time has some advantages, see [70]. Using a discrete time the dynamics of the system change suddenly and is referred to hybrid feedback stabilization, which was investigated in [70].

Considering interconnected systems of the form (3.15) or (3.17), respectively, the dynamic quantizer of the ith subsystem is defined by $q_i^{\lambda_i}(y_i) := \lambda_i q_i(\frac{y_i}{\lambda_i})$, $\lambda_i > 0$ with range $\tilde{M}_i\lambda_i$ and quantization error $\Delta_i\lambda_i$. Note that to get contraction of the bound (3.32) it must hold that

$$\max\{\max_{j\neq i}\gamma_{ij}^{K_i}(\Delta_j),\gamma_i^{K_i}(\Delta_i)\} \leq E_i^0 \text{ and}$$
$$\beta_i(\max\{\max_{j\neq i}\gamma_{ij}^{K_i}(\Delta_j),\gamma_i^{K_i}(\Delta_i)\},0) = \mu_i(\max\{\max_{j\neq i}\gamma_{ij}^{K_i}(\Delta_j),\gamma_i^{K_i}(\Delta_i)\},0) < \tilde{M}_i.$$

Using this, we can find a $\lambda_i < 1$ such that

$$\beta_i(\max\{\max_{j\neq i}\gamma_{ij}^{K_i}(\Delta_j),\gamma_i^{K_i}(\Delta_i)\},0) = \mu_i(\max\{\max_{j\neq i}\gamma_{ij}^{K_i}(\Delta_j),\gamma_i^{K_i}(\Delta_i)\},0) < \tilde{M}_i\lambda_i.$$

Now, for $E_i^{\lambda_i} > 0$ such that $\beta_i(E_i^{\lambda_i},0) = \mu_i(\tilde{M}_i\lambda_i,0) = \tilde{M}_i\lambda_i$ there is a time $\tilde{t}$ for which $\left|\begin{pmatrix} z_i(\tilde{t}) \\ \hat{\xi}_i(\tilde{t}) \end{pmatrix}\right| < E_i^{\lambda_i}$. Define $K_i^{\lambda_i} := \max\left\{\tilde{M}_i\lambda_i, \kappa_i^u(2\tilde{M}_i\lambda_i + \Delta_i\lambda_i + \kappa_i^l(\tilde{M}_i\lambda_i + \Delta_i\lambda_i))\right\}$ and applying the same analysis as in Proposition 3.3.3, we obtain a smaller bound in (3.32):

$$\limsup_{t\to\infty}\left|\begin{pmatrix} z_i(t) \\ \hat{\xi}_i(t) \end{pmatrix}\right| \leq \max\{\max_{j\neq i}\gamma_{ij}^{K_i^{\lambda_i}}(\Delta_j\lambda_i),\gamma_i^{K_i^{\lambda_i}}(\Delta_i\lambda_i)\}. \tag{3.35}$$

Then, we can choose a smaller value of λ_i and repeat the procedure. Theoretically, we can decrease λ_i to 0 and we obtain asymptotic convergence of the ith subsystem. Practically, the choice of the size of λ_i depends on limitations, which determines the size of the bound (3.35), see [72]. The described procedure can be applied to the overall closed-loop system and we also obtain a smaller bound in (3.33) provided that $\Gamma = (\gamma_{ij}^{K_i^{\lambda_i}})_{n\times n}$ satisfies the small-gain condition. If we can decrease λ_i to 0, for all $i = 1,\ldots,n$, we obtain asymptotic convergence of the overall closed-loop system. We summarize the observations in the following corollary:

Corollary 3.3.4. *1. Under a dynamic quantizer of the form $q_i^{\lambda_i}(y_i) := \lambda_i q_i(\frac{y_i}{\lambda_i})$, $\lambda_i > 0$, it holds for the corresponding solution of the ith subsystem of the overall closed-loop system:*

$$\lambda_i \;\to\; 0, \quad\Rightarrow\quad \mathit{lim\,sup}_{t\to\infty}\left|\begin{pmatrix} z_i(t) \\ \hat{\xi}_i(t) \end{pmatrix}\right| \;\to\; 0.$$

2. *Assume that 1. holds for all* $i = 1, \ldots, n$. *Define* $\lambda := \max \lambda_i$. *If* $\Gamma = (\gamma_{ij}^{K_i^{\lambda_i}})_{n \times n}$ *satisfies the small-gain condition, then for the corresponding solution of the overall closed-loop system it holds:*

$$\lambda \to 0, \quad \Rightarrow \quad \mathit{lim\ sup}_{t \to \infty} \left| \begin{pmatrix} z(t) \\ \hat{\xi}(t) \end{pmatrix} \right| \to 0.$$

Proof. Ad point 1.: Applying Proposition 3.3.3, point 1. by using the dynamic quantizer $q_i^{\lambda_i}(y_i) = \lambda_i q_i(\frac{y_i}{\lambda_i})$, $\lambda_i > 0$, we obtain (3.35). If $\lambda_i \to 0$, the assertion follows.

Ad point 2.: This follows, using Proposition 3.3.3, point 2. □

Considering another type of systems, namely systems with time-delays, we provide tools to analyze such interconnections in view of ISS in the next chapter.

Chapter 4

ISS for time-delay systems

In this chapter, we state two main results, an ISS-Lyapunov-Razumikhin and an ISS-Lyapunov-Krasovskii small-gain theorem for general networks with time-delays. They provide tools how to check, whether a network possesses the ISS property. As an application, a scenario of a production network with transportations is analyzed in view of ISS.

Considering systems of the form (1.1) described by ODEs, the future state of the system is independent of the past states. However, in applications the future state of a system can depend on past values of the state, see [44]. The systems that include such kind of delays are called time-delay systems (TDS) and they have applications in many areas such as biology, economics, mechanics, physics and social sciences [5, 65], for example. In production networks, the time needed for transportation of parts from one location to another can be considered as time-delay, for example. According to [44, 48], the dynamics of TDS can be modeled using retarded functionals differential equations of the form

$$\begin{aligned} \dot{x}(t) &= f(x^t, u(t)),\ t \in \mathbb{R}_+, \\ x^0(\tau) &= \xi(\tau),\ \tau \in [-\theta, 0]\,, \end{aligned} \tag{4.1}$$

where $x \in \mathbb{R}^N$, $u \in \mathrm{L}_\infty(\mathbb{R}_+, \mathbb{R}^m)$ and "$\cdot$" represents the right-hand side derivative. θ is the maximum involved delay and $f : C\left([-\theta, 0]\,; \mathbb{R}^N\right) \times \mathbb{R}^m \to \mathbb{R}^N$ is locally Lipschitz continuous on any bounded set. This guarantees that the system (4.1) admits a unique solution on a maximal interval $[-\theta, b)$, $0 < b \leq +\infty$, which is locally absolutely continuous, see [44], Section 2.6. We denote the solution by $x(t; \xi, u)$ or $x(t)$ for short, satisfying the initial condition $x^0 \equiv \xi$ for any $\xi \in C([-\theta, 0], \mathbb{R}^N)$.

During the last fifty years, many interesting mathematical problems according to TDS were investigated and in particular a lot of results according to stability of TDS were obtained, see for example [44, 65, 42, 82]. A common tool for the stability analysis of TDS are Lyapunov functionals, introduced by Krasovskii in [67], which are a natural generalization of a Lyapunov function for systems without time-delays. The construction of such a functional is more challenging in contrast to the usage of a Lyapunov function. Therefore, another approach to investigate TDS in view of stability is a Lyapunov-Razumikhin function, see [44], Chapter 5.4.

Overviews of existing stability results for TDS can be found in [44, 42]. Investigating stability and the problem of stabilization of linear time-delay systems, one can find the existing results in [82]. Considering nonlinear continuous-time TDS with external inputs and the notion of ISS, it was shown in [121] that the existence of an ISS-Lyapunov-Razumikhin function is sufficient for the ISS property under the assumption that the internal Lyapunov gain is less than the identity function. For discrete-time TDS such type of a theorem was proved in [75]. Using Lyapunov-Krasovskii functionals, a theorem, which implies ISS for continuous-time TDS under the existence of an ISS-Lyapunov-Krasovskii functional, was proved in [87]. However, the necessity of the existence of such a Lyapunov function or functional for the ISS property is not proved yet.

Note that a TDS can be unstable even if the according delay-free system is 0-GAS or ISS, for example. To demonstrate this, we consider the following example:

Example 4.0.5. *Consider the system*

$$\begin{aligned}\dot{x}(t) &= -x(t-\tau),\\ x^0(s) &= \xi(s),\ \ s \in [-\tau, 0],\end{aligned}$$

where $x \in \mathbb{R}$, $\tau > 0$ is the delay and $\xi : [-\tau, 0] \to \mathbb{R}$ is the initial (continuous) function of the system. Obviously, the delay-free system $\dot{x} = -x$ is 0-GAS for any initial value $x^0 \in \mathbb{R}$. Also, for time-delays $\tau \leq 1.5$ the system is 0-GAS as shown in Figure 4.1 with $\xi \equiv 10$. If $\tau = 2$ and $\xi \equiv 10$ are chosen, then the system becomes unstable as it is displayed in Figure 4.2.

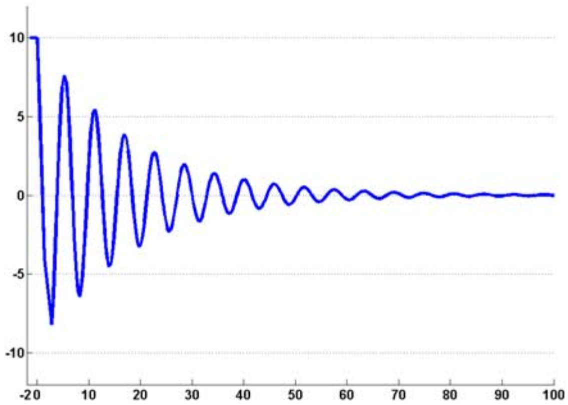

Figure 4.1: Systems behavior with $\tau = 1.5$.

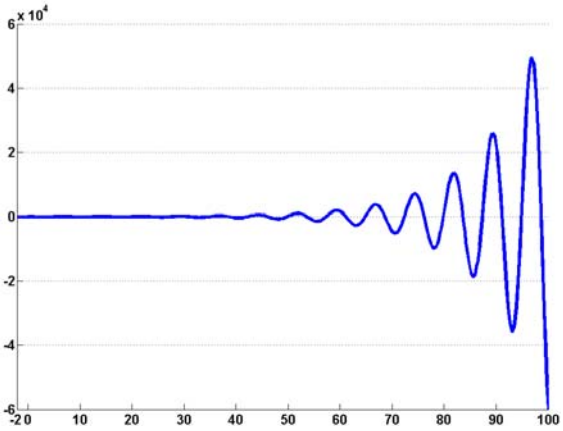

Figure 4.2: Systems behavior with $\tau = 2$.

We are interested in the ISS property for interconnections of systems with time-delays. In this chapter, we utilize on the one hand ISS-Lyapunov-Razumikhin functions and on the other hand ISS-Lyapunov-Krasovskii functionals to prove that a network of ISS systems with time-delays has the ISS property under a small-gain condition, provided that each subsystem has an ISS-Lyapunov-Razumikhin function or an ISS-Lyapunov-Krasovskii functional, respectively. To prove this, we construct the ISS-Lyapunov-Razumikhin function and ISS-Lyapunov-Razumikhin functional, respectively, and the corresponding gains of the whole system.

This chapter is organized as follows: In Section 4.1, we investigate the (L)ISS property of TDS and prove an ISS-Lyapunov-Razumikhin theorem. The ISS small-gain theorems for interconnected time-delay systems can be found in Section 4.2, where Subsection 4.2.1 contains the ISS-Lyapunov-Razumikhin type theorem and Subsection 4.2.2 the ISS-Lyapunov-Krasovskii type theorem. In Section 4.3, we apply the mentioned results to logistic systems, namely production networks. Finally, Section 7.3 concludes the chapter with a summary and an outlook.

4.1 ISS for single time-delay systems

The notion of ISS for TDS reads as follows:

Definition 4.1.1 ((L)ISS for TDS). *The system* (4.1) *is called LISS in maximum formulation, if there exist $\rho > 0, \rho_u > 0$, $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that for all initial functions ξ satisfying $\|\xi\|_{[-\theta,0]} \leq \rho$ and all inputs u satisfying $\|u\| \leq \rho_u$ it holds*

$$|x(t)| \leq \max\left\{\beta(\|\xi\|_{[-\theta,0]}, t), \gamma(\|u\|)\right\} \tag{4.2}$$

for all $t \in \mathbb{R}_+$. If we replace (4.2) *by*

$$|x(t)| \leq \beta(\|\xi\|_{[-\theta,0]}, t) + \gamma(\|u\|),$$

then the system (4.1) *is called ISS in summation formulation. If $\rho = \rho_u = \infty$, then the system* (4.1) *is called ISS (in maximum or summation formulation).*

To check, whether a TDS possesses the ISS property, we define (L)ISS-Lyapunov-Razumikhin functions, introduced in [121].

Definition 4.1.2 ((L)ISS-Lyapunov-Razumikhin function). *A locally Lipschitz continuous function $V : D \to \mathbb{R}_+$, with $D \subset \mathbb{R}^N$ open, is called a LISS-Lyapunov-Razumikhin function of the system* (4.1)*, if there exist $\rho > 0, \rho_u > 0$, $\psi_1, \psi_2 \in \mathcal{K}_\infty$, $\chi^d, \chi^u, \alpha \in \mathcal{K}$ such that $B(0,\rho) \subset D$ and the following conditions hold:*

$$\psi_1(|\phi(0)|) \leq V(\phi(0)) \leq \psi_2(|\phi(0)|),\ \forall \phi(0) \in D, \tag{4.3}$$

$$V(\phi(0)) \geq \chi^d\left(\left\|V^d(\phi)\right\|_{[-\theta,0]}\right) + \chi^u(|u|) \ \Rightarrow\ D^+V(\phi(0)) \leq -\alpha(V(\phi(0))) \tag{4.4}$$

for all $\|\phi\|_{[-\theta,0]} \leq \rho$, $\phi \in C\left([-\theta,0];\mathbb{R}^N\right)$ and all $|u| \leq \rho_u$, where $V^d : C\left([-\theta,0];\mathbb{R}^N\right) \to C\left([-\theta,0];\mathbb{R}_+\right)$ with $V^d(x^t)(\tau) := V(x(t+\tau))$, $\tau \in [-\theta,0]$. $D^+V(x(t))$ denotes the upper right-hand side derivative of V along the solution $x(t)$, which is defined as

$$D^+V(x(t)) := \limsup_{h \to 0^+} \tfrac{V(x(t+h)) - V(x(t))}{h}.$$

If $\rho = \rho_u = \infty$, then the function V is called an ISS-Lyapunov-Razumikhin function.

Note that we get an equivalent definition of a (L)ISS-Lyapunov-Razumikhin function, if we replace (4.4) by

$$V(\phi(0)) \geq \max\left\{\tilde{\chi}^d\left(\left\|V^d(\phi)\right\|_{[-\theta,0]}\right), \tilde{\chi}^u(|u|)\right\} \;\Rightarrow\; \mathrm{D}^+V(\phi(0)) \leq -\tilde{\alpha}(V(\phi(0))), \tag{4.5}$$

where $\tilde{\chi}^d, \tilde{\chi}^u, \tilde{\alpha} \in \mathcal{K}$ and χ^d, χ^u, α are different, in general. With this definition we state that the existence of a LISS-Lyapunov-Razumikhin function implies LISS:

Theorem 4.1.3. *If there exists a LISS-Lyapunov-Razumikhin function V for system* (4.1) *and $\chi^d(s) < s,\ s \in \mathbb{R}_+,\ s > 0$, then the system* (4.1) *is LISS in summation formulation. If an LISS-Lyapunov-Razumikhin function with* (4.5) *and $\tilde{\chi}^d(s) < s,\ s \in \mathbb{R}_+,\ s > 0$ is used, then the system* (4.1) *is LISS in maximum formulation.*

Proof. We use (4.5) and prove the LISS property in maximum formulation. The proof using the summation formulation follows similarly. We use the idea of the proof of Theorem 1 in [121] with according changes to the LISS property.

Using a standard comparison principle (see [74], Lemma 4.4 or [121], Lemma 1), we obtain from (4.5) that it holds

$$V(x(t)) \leq \max\left\{\tilde{\beta}(V(x_0), t), \tilde{\chi}^d\left(\sup_{s\leq t}\left\|V^d(x^s)\right\|_{[-\theta,0]}\right), \tilde{\chi}^u(\|u\|)\right\}, \tag{4.6}$$

$\tilde{\beta} \in \mathcal{KL}$. It holds

$$\sup_{s\leq t}\left\|V^d(x^s)\right\|_{[-\theta,0]} \leq \max\left\{\tilde{\beta}\left(\left\|V^d(x^0)\right\|_{[-\theta,0]}, 0\right), \sup_{s\leq t} V(x(s))\right\}, \tag{4.7}$$

taking the supremum over $[0,t]$ in (4.6) and inserting it into (4.7), we obtain, using $\tilde{\chi}^d(s) < s,\ s \in \mathbb{R}_+$ and the fact that for all $b_1, b_2 > 0$ from $b_1 \leq \max\{b_2, \tilde{\chi}^d(b_1)\}$ it follows $b_1 \leq b_2$:

$$\sup_{s\leq t}\left\|V^d(x^s)\right\|_{[-\theta,0]} \leq \max\left\{\tilde{\beta}\left(\left\|V^d(x^0)\right\|_{[-\theta,0]}, 0\right), \tilde{\chi}^d\left(\sup_{s\leq t}\left\|V^d(x^s)\right\|_{[-\theta,0]}\right), \tilde{\chi}^u(\|u\|)\right\}$$

and therefore

$$|x(t)| \leq \max\left\{\psi_1^{-1}(\tilde{\beta}(\psi_2(\|\xi\|_{[-\theta,0]}), 0)), \psi_1^{-1}(\tilde{\chi}^u(\|u\|))\right\},\ \forall t \geq 0,$$

for all $\|\xi\|_{[-\theta,0]} \leq \rho,\ \xi \in C\left([-\theta,0]\,;\mathbb{R}^N\right)$ and all $\|u\| \leq \rho_u$. Given $\varepsilon > 0$ and define $\kappa := \max\{\tilde{\beta}(\psi_2(\rho), 0), \tilde{\chi}^u(\rho_u)\}$. Note that it holds $\sup_{t\geq s}\left\|V^d(x^s)\right\|_{[-\theta,0]} \leq \kappa$, which implies $\|V^d(x^0)\|_{[-\theta,0]} \leq \kappa$. Let $\delta_2 > 0$ be such that $\tilde{\beta}(\kappa, \delta_2) \leq \psi_1(\varepsilon)$ and let $\delta_1 > \theta$. Then, by the estimate (4.6) we have

$$\begin{aligned}
\sup_{t\geq s\geq \delta_1+\delta_2}\left\|V^d(x^s)\right\|_{[-\theta,0]} &\leq \sup_{t\geq s+\delta_2} V(x(s)) \\
&\leq \max\left\{\psi_1(\varepsilon), \tilde{\chi}^d\left(\sup_{s\in[0,t]}\left\|V^d(x^s)\right\|_{[-\theta,0]}\right), \tilde{\chi}^u(\|u\|)\right\}.
\end{aligned}$$

It follows

$$\begin{aligned}\sup_{t\geq s\geq 2(\delta_1+\delta_2)}\left\|V^d(x^s)\right\|_{[-\theta,0]} &\leq \max\left\{\psi_1(\varepsilon),\tilde{\chi}^d\left(\sup_{s\in[\delta_1+\delta_2,t]}\left\|V^d(x^s)\right\|_{[-\theta,0]}\right),\tilde{\chi}^u(\|u\|)\right\}\\ &\leq \max\left\{\psi_1(\varepsilon),(\tilde{\chi}^d)^2\left(\sup_{s\in[0,t]}\left\|V^d(x^s)\right\|_{[-\theta,0]}\right),\tilde{\chi}^u(\|u\|)\right\}.\end{aligned}$$

Since $\tilde{\chi}^d < id$, there exists a number $\tilde{n}\in\mathbb{N}$, which depends on κ and ε such that

$$(\tilde{\chi}^d)^{\tilde{n}}(\kappa)\leq \max\left\{\psi_1(\varepsilon),\tilde{\chi}^u(\|u\|)\right\}.$$

By induction we conclude that

$$\sup_{t\geq s\geq \tilde{n}(\delta_1+\delta_2)}\left\|V^d(x^s)\right\|_{[-\theta,0]}\leq \max\left\{\psi_1(\varepsilon),\tilde{\chi}^u\gamma_u(\|u\|)\right\}$$

and finally we obtain

$$|x(t)|\leq \max\left\{\varepsilon,\psi_1^{-1}(\tilde{\chi}^u(\|u\|))\right\},\ \forall t\geq \tilde{n}(\delta_1+\delta_2) \tag{4.8}$$

for all $\|\xi\|_{[-\theta,0]}\leq\rho,\ \xi\in C\left([-\theta,0];\mathbb{R}^N\right)$ and all $\|u\|\leq\rho_u$.

Using the same technique for the construction of β as in the proof of Lemma 3.10 in [17], we obtain

$$|x(t)|\leq \max\left\{\beta(\|\xi\|_{[-\theta,0]},t),\gamma(\|u\|)\right\},$$

for all $\|\xi\|_{[-\theta,0]}\leq\rho$, and all $\|u\|\leq\rho_u$, where $\gamma(r):=\psi_1^{-1}(\tilde{\chi}^u(r)),\ r\geq 0$. $\square$

Note that in [121] a similar theorem as Theorem 4.1.3 for the ISS property was proved, but the ISS property defined in [121] differs from that in Definition 4.1.1. For systems without time-delays these definitions are equivalent, see [120]. Until now, it is an open problem whether these definitions are equivalent for time-delay systems. A crucial step needed for the proof is that the combination of global stability and the asymptotic gain property is equivalent to ISS, which is not proved yet, see [123, 122]. Nevertheless, the mentioned result in [121] is also valid using the ISS property in Definition 4.1.1.

Theorem 4.1.4. *If there exists an ISS-Lyapunov-Razumikhin function V for system* (4.1) *and $\chi^d(s)<s,\ s\in\mathbb{R}_+,\ s>0$, then the system* (4.1) *is ISS in summation formulation. If an ISS-Lyapunov-Razumikhin function with* (4.5) *is used, then the system* (4.1) *is ISS in maximum formulation, if $\tilde{\chi}^d(s)<s,\ s\in\mathbb{R}_+,\ s>0$.*

The proof can be found in [121] with changes according to the construction of the $\mathcal{KL}$-function β as in the proof of Theorem 4.1.3 and the proof of Lemma 3.10 in [17].

Another approach to check, whether a system of the form (4.1) has the ISS property was introduced in [87]. There, ISS-Lyapunov-Krasovskii functionals are used. Given a locally Lipschitz continuous functional $V: C\left([-\theta,0];\mathbb{R}^N\right)\to\mathbb{R}_+$, the upper right-hand side derivative

D^+V of the functional V along the solution $x(t;\xi,u)$ is defined according to [44], Chapter 5.2:

$$D^+V(\phi,u) := \limsup_{h\to 0^+} \tfrac{1}{h}\left(V\left(x^{t+h}\right) - V(\phi)\right), \tag{4.9}$$

where $x^{t+h} \in C\left([-\theta,0];\mathbb{R}^N\right)$ is generated by the solution $x(t;\phi,u)$ of $\dot{x}(t) = f(x^t,u(t)),\ t \in (t_0,t_0+h)$ with $x^{t_0} := \phi \in C\left([-\theta,0];\mathbb{R}^N\right)$.

Remark 4.1.5. *Note that in contrast to* (4.9)*, the definition of D^+V in [87] is slightly different, since the functional is assumed to be only continuous and in this case, D^+V can take infinite values. Nevertheless, the results in [87] also hold true, if the functional is chosen to be locally Lipschitz continuous, according to the results in [85], [86] and using* (4.9)*.*

By $\|\cdot\|_a$, we indicate any norm in $C\left([-\theta,0];\mathbb{R}^N\right)$ such that for some positive reals c_1, c_2 the following inequalities hold

$$c_1\,|\phi(0)| \le \|\phi\|_a \le c_2\,\|\phi\|_{[-\theta,0]}\,,\ \forall \phi \in C\left([-\theta,0];\mathbb{R}^N\right).$$

Definition 4.1.6 (ISS-Lyapunov-Krasovskii functional)**.** *A locally Lipschitz continuous functional $V : C\left([-\theta,0];\mathbb{R}^N\right) \to \mathbb{R}_+$ is called an ISS-Lyapunov-Krasovskii functional for the system* (4.1)*, if there exist functions $\psi_1,\psi_2 \in \mathcal{K}_\infty$ and functions $\chi,\alpha \in \mathcal{K}$ such that*

$$\psi_1(|\phi(0)|) \le V(\phi) \le \psi_2(\|\phi\|_a)\,, \tag{4.10}$$

$$V(\phi) \ge \chi(|u|) \ \Rightarrow\ D^+V(\phi,u) \le -\alpha(V(\phi))\,, \tag{4.11}$$

for all $\phi \in C\left([-\theta,0];\mathbb{R}^N\right),\ u \in L_\infty(\mathbb{R}_+,\mathbb{R}^m)$.

The next theorem is a counterpart to Theorem 4.1.4 with according changes to Lyapunov-Krasovskii functionals.

Theorem 4.1.7. *If there exists an ISS-Lyapunov-Krasovskii functional V for the system* (4.1)*, then the system* (4.1) *has the ISS property.*

Proof. This follows by Theorem 3.1 in [87] with functions ρ and α_3 used there defining $\rho := \psi_2^{-1} \circ \chi$ and

$$\mathrm{D}^+V(\phi,u) \le -\alpha_3(||\phi||_a) \le -\alpha(V(\phi)),$$

where $\alpha := \alpha_3 \circ \psi_2^{-1}$ and the functional is chosen locally Lipschitz continuous according to the results in [85], [86]. □

Remark 4.1.8. *We conjecture that an LISS version of Theorem 4.1.7 can be stated. The proof will follow the lines of the one of Theorem 3.1 in [87] with according changes to the LISS property. We skip the formulation of the theorem and its proof.*

Remark 4.1.9. *By an ISS-Lyapunov-Razumikhin function and an ISS Lyapunov-Krasovskii functional, there exists two tools to check, whether a TDS has the ISS property. For some systems, the usage of a Lyapunov-Krasovskii functional is more challenging in contrast to the usage of a Lyapunov-Razumikhin function, because the construction of such a functional is not always an easy task. An approach to construct Lyapunov-Krasovskii functionals based on decomposition of a TDS can be found in [51]. More examples can be found in [87, 44].*

In the next section, we consider interconnected TDS and develop tools to check, whether a network of TDS has the ISS property.

4.2 ISS for interconnected time-delay systems

In this section, we provide ISS small-gain theorems for interconnected TDS using ISS-Lyapunov-Razumikhin functions and ISS-Lyapunov-Krasovskii functionals.

We consider $n \in \mathbb{N}$ interconnected TDS of the form

$$\dot{x}_i(t) = f_i\left(x_1^t, \ldots, x_n^t, u(t)\right), \; i = 1, \ldots, n, \tag{4.12}$$

where $x_i^t \in C\left([-\theta, 0]\,; \mathbb{R}^{N_i}\right)$, $x_i^t(\tau) := x_i(t+\tau)$, $\tau \in [-\theta, 0]$, $x_i \in \mathbb{R}^{N_i}$ and $u \in \mathrm{L}_\infty(\mathbb{R}_+, \mathbb{R}^m)$. θ denotes the maximal involved delay and x_j^t, $j \neq i$ can be interpreted as internal inputs of the ith subsystem. The functionals $f_i : C\left([-\theta, 0]\,; \mathbb{R}^{N_1}\right) \times \ldots \times C\left([-\theta, 0]\,; \mathbb{R}^{N_n}\right) \times \mathbb{R}^m \to \mathbb{R}^{N_i}$ are locally Lipschitz continuous on any bounded set. We denote the solution of a subsystem by $x_i(t; \xi_i, u)$ or $x_i(t)$ for short, satisfying the initial condition $x_i^0 \equiv \xi_i$ for any $\xi_i \in C([-\theta, 0], \mathbb{R}^{N_i})$.

The (L)ISS property for a subsystem of (4.12) reads as follows: The *i-th subsystem of* (4.12) *is LISS*, if there exist $\beta_i \in \mathcal{KL}$, $\gamma_{ij}^d, \gamma_i^u \in \mathcal{K}_\infty \cup \{0\}$, $j = 1, \ldots, n$, $j \neq i$ and $\rho_i^j > 0, \rho_i^u > 0$, $j = 1, \ldots, n$ such that for all $\|\xi_i\|_{[-\theta,0]} \leq \rho_i^i$, $\|x_j\|_{[-\theta,\infty)} \leq \rho_i^j$, $j \neq i$, $\|u\| \leq \rho_i^u$ and for all $t \in \mathbb{R}_+$ it holds

$$|x_i(t)| \leq \max\left\{\beta_i(\|\xi_i\|_{[-\theta,0]}, t), \max_{j, j\neq i} \gamma_{ij}^d(\|x_j\|_{[-\theta,t]}), \gamma_i^u(\|u\|)\right\}. \tag{4.13}$$

If $\rho_i^j = \rho_i^u = \infty$, then the i-th subsystem of (4.12) is called ISS. This is referred to (L)ISS in maximum formulation. We get an equivalent formulation, if we replace (4.13) by

$$|x_i(t)| \leq \beta_i(\|\xi_i\|_{[-\theta,0]}, t) + \sum_{j, j\neq i} \gamma_{ij}^d(\|x_j\|_{[-\theta,t]}) + \gamma_i^u(\|u\|),$$

where we use the same function β_i and same gains for simplicity.

If we define $N := \sum N_i$, $x := (x_1^T, \ldots, x_n^T)^T$ and $f := (f_1^T, \ldots, \; f_n^T)^T$, then (4.12) can be written as a system of the form (4.1), which we call the whole system. We investigate under which conditions the whole system has the ISS property and utilize Lyapunov-Razumikhin functions as well as Lyapunov-Krasovskii functionals.

4.2.1 Lyapunov-Razumikhin approach

A locally Lipschitz continuous function $V_i : D_i \to \mathbb{R}_+$, with $D_i \subset \mathbb{R}^{N_i}$ open, is an *LISS-Lyapunov-Razumikhin function for the i-th subsystem of* (4.12), if there exist functions $V_j,\ j = 1, \dots, n$, which are continuous, positive definite and locally Lipschitz continuous on $\mathbb{R}^{N_j} \supset D_j \backslash \{0\}$, $\rho_i^j > 0, \rho_i^u > 0$, functions $\psi_{1i}, \psi_{2i} \in \mathcal{K}_\infty$, $\tilde{\chi}_i^u \in \mathcal{K} \cup \{0\}$, $\tilde{\chi}_{ij}^d \in \mathcal{K}_\infty \cup \{0\}$, $\tilde{\alpha}_i \in \mathcal{K}$, $j = 1, \dots, n$, such that $B(0, \rho_i^i) \subset D_i$ and the following conditions hold:

$$\psi_{1i}(|\phi_i(0)|) \leq V_i(\phi_i(0)) \leq \psi_{2i}(|\phi_i(0)|),\ \forall \phi_i(0) \in D_i, \tag{4.14}$$

$$\begin{aligned} &V_i(\phi_i(0)) \geq \max\left\{ \max_j \tilde{\chi}_{ij}^d \left(\left\| V_j^d(\phi_j) \right\|_{[-\theta,0]} \right), \tilde{\chi}_i^u(|u|) \right\} \\ \Rightarrow\ &\mathrm{D}^+ V_i(\phi_i(0)) \leq -\tilde{\alpha}_i(V_i(\phi_i(0))), \end{aligned} \tag{4.15}$$

for all $\|\phi_j\|_{[-\theta,0]} \leq \rho_i^j,\ \phi_j \in C\left([-\theta, 0]\,; \mathbb{R}^{N_j}\right),\ j = 1, \dots, n$ and all $|u| \leq \rho_i^u$. (4.15) can be replaced by

$$V_i(\phi_i(0)) \geq \sum_j \chi_{ij}^d \left(\left\| V_j^d(\phi_j) \right\|_{[-\theta,0]} \right) + \chi_i^u(|u|) \ \Rightarrow\ \mathrm{D}^+ V_i(\phi_i(0)) \leq -\alpha_i(V_i(\phi_i(0))), \tag{4.16}$$

where $\chi_{ij}^d, \chi_i^u, \alpha_i \in \mathcal{K}$ and $\tilde{\chi}_{ij}^d, \tilde{\chi}_i^u, \tilde{\alpha}_i$ are different, in general. Without loss of generality, we use χ_{ij}^d and χ_i^u for the summation and maximum formulation. If $\rho_i^j = \rho_i^u = \infty$, then V_i is called an ISS-Lyapunov-Razumikhin function for the i-th subsystem of (4.12). We collect all the gains in a gain-matrix $\overline{\Gamma} := (\chi_{ij}^d)_{n\times n},\ i, j = 1, \dots, n$, which defines a map $\overline{\Gamma} : \mathbb{R}_+^n \to \mathbb{R}_+^n$ by

$$\overline{\Gamma}(s) = \left(\max_j \chi_{1j}^d(s_j), \dots, \max_j \chi_{nj}^d(s_j) \right)^T,\ s \in \mathbb{R}_+^n$$

using (4.15) and considering (4.16) by

$$\overline{\Gamma}(s) = \left(\sum_j \chi_{1j}^d(s_j), \dots, \sum_j \chi_{nj}^d(s_j) \right)^T,\ s \in \mathbb{R}_+^n. \tag{4.17}$$

Recall that we get for $v, w \in \mathbb{R}_+^n$: $v \geq w \ \Rightarrow\ \overline{\Gamma}(v) \geq \overline{\Gamma}(w)$. In contrast to interconnected delay-free systems the gain-matrix $\overline{\Gamma}$ using the Razumikhin approach has not necessarily zero entries on the main diagonal.

To check, whether a network of TDS possesses the ISS property, we state the following theorem. It provides a tool utilizing ISS-Lyapunov-Razumikhin functions to verify ISS of a network under a small-gain condition.

Theorem 4.2.1. *(ISS-Lyapunov-Razumikhin theorem for general networks with time-delays) Consider the interconnected system* (4.12). *Each subsystem has an ISS-Lyapunov-Razumikhin function V_i with summation formulation* (4.16). *If the corresponding gain-matrix $\overline{\Gamma}$, given by* (4.17), *satisfies the small-gain condition* (1.16), *then the function*

$$V(x) = \max_i \{\sigma_i^{-1}(V_i(x_i))\}$$

is the ISS-Lyapunov-Razumikhin function for the whole system of the form (4.1)*, which is ISS in summation formulation, where* $\sigma = (\sigma_1, \ldots, \sigma_n)^T$ *is an* Ω*-path as in Definition 1.2.2. The gains are given by*

$$\begin{aligned}\chi^d(r) &:= \max_{ij} \sigma_j^{-1}((\chi_{ij}^d)^{-1}((\mathrm{Id} + \tfrac{\varpi}{2})^{-1})(\chi_{ij}^d(\sigma_j(r)))),\\ \chi^u(r) &:= \max_i \lambda^{-1}(\chi_i^u(r))\end{aligned}$$

for $r \geq 0$*, where* $\lambda(r) := \min_k \lambda_k(r),\ \lambda_k(r) := \frac{\varpi}{2}(\sum \chi_{kj}^d(\sigma_k(r)))$.

Remark 4.2.2. *If we consider in Theorem 4.2.1 ISS-Lyapunov-Razumikhin functions* V_i *with maximum formulation* (4.15) *and* $\overline{\Gamma}$ *satisfies the SGC* (1.15)*, then the whole system is ISS in maximum formulation and the Lyapunov gains are given by*

$$\begin{aligned}\chi^d(r) &:= \max_{i,j} \sigma_i^{-1}(\chi_{ij}^d(\sigma_j(r))),\\ \chi^u(r) &:= \max_i \sigma_i^{-1}(\chi_i^u(r)).\end{aligned}$$

Proof. All subsystems of (4.12) have an ISS-Lyapunov-Razumikhin function $V_i,\ i = 1, \ldots, n$, i.e., V_i satisfies (4.14) and (4.16). From the small-gain condition (1.16) for $\overline{\Gamma}$, given by (4.17), it follows that there exists an Ω-path $\sigma = (\sigma_1, \ldots, \sigma_n)^T$ as in Definition 1.2.2 and Proposition 1.2.3. Note that $\sigma_i^{-1} \in \mathcal{K}_\infty,\ i = 1, \ldots, n$. Let $0 \neq x = (x_1^T, \ldots, x_n^T)^T$. We define

$$V(x) := \max_i \{\sigma_i^{-1}(V_i(x_i))\}$$

as the ISS-Lyapunov-Razumikhin function candidate for the overall system. Note that V is locally Lipschitz continuous. V satisfies (4.3), which can be easily checked defining $\psi_1(r) := \min_i \sigma_i^{-1}(\psi_{1i}(\frac{r}{\sqrt{n}}))$, $\psi_2(r) := \max_i \sigma_i^{-1}(\psi_{2i}(r))$, $r > 0$ and using the condition (4.14).

Let $I := \{i \in \{1, \ldots, n\} |\ V(x) = \sigma_i^{-1}(V_i(x_i)) \geq \max_{j, j\neq i}\{\sigma_j^{-1}(V_j(x_j))\}\}$. Fix an $i \in I$. We define $\chi^d(r) := \max_{ij} \sigma_j^{-1}((\chi_{ij}^d)^{-1}((\mathrm{Id} + \frac{\varpi}{2})^{-1})(\chi_{ij}^d(\sigma_j(r))))$, $\chi^u(r) := \max_i \lambda^{-1}(\chi_i^u(r))$, $r > 0$, where $\lambda(r) := \min_k \lambda_k(r),\ \lambda_k(r) := \frac{\varpi}{2}(\sum \chi_{kj}^d(\sigma_k(r)))$ and assume

$$V(x(t)) \geq \chi^d\left(\left\|V^d(x^t)\right\|_{[-\theta,0]}\right) + \chi^u(|u(t)|).$$

Note that $\chi^d(r) < r$. It follows from $\left(\overline{\Gamma} \circ D\right)(\sigma(r)) < \sigma(r),\ \forall r > 0$ that it holds

$$\begin{aligned}V_i(x_i(t)) = \sigma_i(V(x(t))) &> (\mathrm{Id} + \varpi) \sum_{j=1}^n \chi_{ij}^d(\sigma_j(V(x(t))))\\ &\geq \sum_{j=1}^n \chi_{ij}^d\left(\left\|V_j^d(x_j^t)\right\|_{[-\theta,0]}\right) + \chi_i^u(|u(t)|).\end{aligned}$$

From (4.16) we obtain

$$\begin{aligned}\mathrm{D}^+V(x(t)) = \mathrm{D}^+\sigma_i^{-1}(V_i(x_i(t))) &= (\sigma_i^{-1})'(V_i(x_i(t)))\mathrm{D}^+V_i(x_i(t))\\ &\leq -(\sigma_i^{-1})'(V_i(x_i(t)))\alpha_i(V_i(x_i(t))) = -\bar{\alpha}_i(V(x(t))),\end{aligned}$$

where $\bar{\alpha}_i(r) := (\sigma_i^{-1})'(\sigma_i(r))\alpha_i(\sigma_i(r)),\ r > 0$. By definition of $\alpha := \min_i \bar{\alpha}_i$, the function V satisfies (4.4). All conditions of Definition 4.1.2 are satisfied and V is the ISS-Lyapunov-Razumikhin function of the whole system of the form (4.1). By Theorem 4.1.4, the whole system is ISS in summation formulation. Using the maximum formulation in (4.15), (4.5) and of the ISS property, the proof follows the same steps as for the summation formulation. □

For many applications, an LISS small-gain theorem for TDS using Lyapunov-Razumikhin functions will be beneficial due to restrictions on the initial value or the input function in practice. This is provided in the next corollary.

Corollary 4.2.3. *Consider the interconnected system* (4.12) *and assume that each subsystem has a LISS-Lyapunov-Razumikhin function* $V_i(x_i)$ *in summation or maximum formulation, for all* $\|\phi_j\|_{[-\theta,0]} \leq \rho_i^j,\ \phi_j \in C\left([-\theta,0];\mathbb{R}^{N_j}\right)$ *with* $\phi_j(0) = x_j,\ j = 1,\ldots,n$ *and all* $|u| \leq \rho_i^u$*. If the corresponding gain-matrix* $\overline{\Gamma}$ *satisfies the LSGC* (1.14)*, then the function* V *as defined in Theorem 4.2.1 is the LISS-Lyapunov-Razumikhin function for the whole system with* $\phi = (\phi_1,\ldots,\phi_n)^T,\ \|\phi_i\|_{[-\theta,0]} \leq \tilde{\rho}_i,\ \phi_i \in C\left([-\theta,0];\mathbb{R}^{N_i}\right)$*, where* $\tilde{\rho}_i := \min\{w_i^*, \psi_{i2}^{-1}(w_i^*), \min_j \rho_i^j\}$ *such that* $\rho := \min \tilde{\rho}_i$ *and* $\rho_u := \min \rho_i^u$*. Furthermore, the whole system is LISS in summation or maximum formulation, respectively.*

Proof. Since $\overline{\Gamma}$ satisfies the LSGC (1.14), we know that there exists a strictly increasing path $\sigma : [0,1] \to [0,w^*]$, which satisfies $\overline{\Gamma}(\sigma(r)) < \sigma(r),\ \forall r \in (0,1]$ and $\sigma(0) = 0,\ \sigma(1) = w^*$. The LISS-Lyapunov-Razumikhin function candidate $V(x) := \max_i\{\sigma_i^{-1}(V_i(x_i))\}$ is well defined, because $\|\phi_i\|_{[-\theta,0]} \leq \psi_{i2}^{-1}(w_i^*),\ \phi_i \in C\left([-\theta,0];\mathbb{R}^{N_i}\right)$ with $\phi_i(0) = x_i$ implies $V_i(x_i) \leq w_i^*$.

Following the same steps as in the proof of Theorem 4.2.1, we conclude that $V(x)$ is the LISS-Lyapunov-Razumikhin function for the whole system. □

In the next subsection, we prove a small-gain theorem using Lyapunov-Krasovskii functionals for subsystems to check, whether a network of TDS possesses the ISS property.

4.2.2 Lyapunov-Krasovskii approach

The Krasovskii functionals for subsystems are as follows:

A locally Lipschitz continuous functional $V_i : C([-\theta,0];\mathbb{R}^{N_i}) \to \mathbb{R}_+$ is an *ISS-Lyapunov-Krasovskii functional of the i-th subsystem* of (4.12), if there exist functionals $V_j,\ j = 1,\ldots,n$, which are positive definite and locally Lipschitz continuous on $C([-\theta,0];\mathbb{R}^{N_j})$, functions $\psi_{1i}, \psi_{2i} \in \mathcal{K}_\infty,\ \tilde{\chi}_{ij}, \tilde{\chi}_i \in \mathcal{K} \cup \{0\},\ \tilde{\alpha}_i \in \mathcal{K},\ j = 1,\ldots,n,\ i \neq j$ such that

$$\psi_{1i}\left(|\phi_i(0)|\right) \leq V_i(\phi_i) \leq \psi_{2i}\left(\|\phi_i\|_a\right),\ \forall \phi_i \in C\left([-\theta,0],\mathbb{R}^{N_i}\right) \tag{4.18}$$

$$V_i\left(\phi_i\right) \geq \max\left\{\max_{j,j\neq i} \tilde{\chi}_{ij}(V_j(\phi_j)), \tilde{\chi}_i\left(|u|\right)\right\} \Rightarrow D^+V_i\left(\phi_i,u\right) \leq -\tilde{\alpha}_i\left(V_i\left(\phi_i\right)\right), \tag{4.19}$$

for all $\phi_i \in C\left([-\theta,0],\mathbb{R}^{N_i}\right)$, $u \in \mathrm{L}_\infty(\mathbb{R}_+,\mathbb{R}^m)$, $i = 1,\ldots,n$. We get an equivalent formulation, if we replace (4.19) by

$$V_i\left(\phi_i\right) \geq \sum_{j,j\neq i} \chi_{ij}(V_j(\phi_j)) + \chi_i\left(|u|\right) \Rightarrow \mathrm{D}^+V_i\left(\phi_i,u\right) \leq -\alpha_i\left(V_i\left(\phi_i\right)\right), \tag{4.20}$$

where the gains are different in general. Without loss of generality, we use χ_{ij} and χ_i for the summation and maximum formulation. The gain-matrix is defined by $\Gamma := (\chi_{ij})_{n\times n}$, $\chi_{ii} \equiv 0$, $i = 1, \dots, n$ and the map $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is defined by (1.12) using (4.19) and considering (4.20) it is defined by (1.13).

The next theorem is the second main result of this chapter. It is a counterpart of Theorem 4.2.1 and it provides a tool how to verify the ISS property for networks of TDS using ISS-Lyapunov-Krasovskii functionals.

Theorem 4.2.4. *(ISS-Lyapunov-Krasovskii theorem for general networks with time-delays) Consider an interconnected system of the form* (4.12)*. Assume that each subsystem has an ISS-Lyapunov-Krasovskii functional V_i, which satisfies the conditions* (4.18) *and* (4.20)*, $i = 1, \dots, n$. If the corresponding gain-matrix Γ satisfies the small-gain condition* (1.16)*, then*

$$V(\phi) := \max_i \{\sigma_i^{-1}(V_i(\phi_i))\}$$

is the ISS-Lyapunov-Krasovskii functional for the whole system of the form (4.1)*, which is ISS in summation formulation, where $\sigma = (\sigma_1, \dots, \sigma_n)^T$ is an Ω-path as in Definition 1.2.2 and $\phi = (\phi_i, \dots, \phi_n)^T \in C([-\theta, 0]; \mathbb{R}^N)$. The Lyapunov gain is given by $\chi(r) := \max_i \lambda^{-1}(\chi_i(r))$ with $\lambda := \min_{k=1,\dots,n} \lambda_k$, $\lambda_k(r) := \varpi(\sum_{j=1,k\neq j}^n \chi_{kj}(\sigma_j(r)))$.*

Remark 4.2.5. *If we consider in Theorem 4.2.4 ISS-Lyapunov-Krasovskii functionals V_i with maximum formulation* (4.19) *and Γ satisfies the SGC* (1.15)*, then the whole system is ISS in maximum formulation and the Lyapunov gain is given by $\chi(r) := \max_i \sigma_i^{-1}(\chi_i(r))$.*

Proof. All subsystems of (4.12) have an ISS-Lyapunov-Krasovskii functional V_i, $i = 1, \dots, n$, i.e., V_i satisfies (4.18) and (4.20). From the small-gain condition (1.16) for Γ, there exists an Ω-path $\sigma = (\sigma_1, \dots, \sigma_n)^T$. Let $0 \neq x^t = ((x_1^t)^T, \dots, (x_n^t)^T)^T \in C([-\theta, 0]; \mathbb{R}^N)$. We define

$$V(x^t) := \max_i \{\sigma_i^{-1}(V_i(x_i^t))\}$$

as the ISS-Lyapunov-Krasovskii functional candidate of the whole system. Note that V is locally Lipschitz. V satisfies (4.10), by definition of $\psi_1(r) := \min_i \sigma_i^{-1}(\psi_{1i}(\frac{r}{\sqrt{n}}))$, $\psi_2(r) := \max_i \sigma_i^{-1}(\psi_{2i}(r))$, $r > 0$ and using (4.18).

Let $I := \{i \in \{1, \dots, n\} |\ V(x^t) = \{\sigma_i^{-1}\left(V_i(x_i^t)\right)\} \geq \max_{j,j\neq i}\{\sigma_j^{-1}(V_j(x_j^t))\}\}$. Fix an $i \in I$. From $(\Gamma \circ D)(\sigma(r)) < \sigma(r)$, for all $r > 0$ we get

$$\begin{aligned} &\sigma_i(r) > (\mathrm{Id} + \varpi)\left(\sum_{j,j\neq i} \chi_{ij}(\sigma_j(r))\right) \\ \Leftrightarrow\ &\sigma_i(r) - \sum_{j,j\neq i} \chi_{ij}(\sigma_j(r)) > \varpi\left(\sum_{j=1,i\neq j}^{n} \chi_{ij}(\sigma_j(r))\right) =: \lambda_i(r). \end{aligned}$$

Define $\lambda := \min_i \lambda_i$ and assume $V(x^t) \geq \lambda^{-1}(\chi_i(|u|))$. Then, it follows

$$\lambda(V(x^t)) \geq \chi_i(|u|) \ \Rightarrow\ \sigma_i(V(x^t)) - \sum_{j,j\neq i} \chi_{ij}(\sigma_j(V(x^t))) > \chi_i(|u|)$$

and we get

$$V_i(x_i^t) = \sigma_i(V(x^t)) > \chi_i(|u|) + \sum_{j,j\neq i} \chi_{ij}(\sigma_j(V(x^t))) = \chi_i(|u|) + \sum_{j,j\neq i} \chi_{ij}(V_j(x_j^t)).$$

From (4.20) we obtain

$$\mathrm{D}^+V(x^t,u) = (\sigma_i^{-1})'(V_i(x_i^t))\mathrm{D}^+V_i(x_i^t,u) \leq -(\sigma_i^{-1})'(V_i(x_i^t))\alpha_i(V_i(x_i^t)) = -\bar{\alpha}_i(V(x^t)),$$

where $\bar{\alpha}_i(r) := (\sigma_i^{-1})'(\sigma_i(r))\alpha_i(\sigma_i(r)),\ r \geq 0$. By definition of $\chi := \max_i \lambda^{-1}\chi_i$ and $\alpha := \min_i \tilde{\alpha}_i$, the function V satisfies (4.11). All conditions of Definition 4.1.6 are satisfied and V is the ISS-Lyapunov-Krasovskii functional of the whole system of the form (4.1). By Theorem 4.1.7 the whole system is ISS in summation formulation. The case in the maximum formulations follows the same steps as for the summation formulation. □

4.3 Applications in logistics

In this section, we present an application of the Lyapunov-Razumikhin approach, namely the application of Theorem 4.2.1, for the investigation of LISS of logistics networks.

A typical example of a logistic network is a production network. Such an interconnected system describes a company or cross-company owned network with geographically dispersed plants [126], which are connected by transport routes. By an increasing number of plants and logistic objects, a central control of the network becomes challenging. Autonomous control can help to handle such complex networks, see [104, 107, 106, 105], for example. Autonomous control policies allow objects of a network, such as vehicles, containers or machines, for example, to decide and to execute their own decisions based on some given rules and available local information to route themselves through a network. It can be seen as a paradigm shift from centralized to decentralized control in logistics.

Due to economic circumstances, such as high inventory costs, for example, an unbounded growth of the number of logistic objects such as parts or orders, is undesired in logistic networks. The stability analysis of logistic networks allows to get knowledge of the dynamical properties of a network and to design stable networks to avoid negative economic outcomes.

For production networks without taking transportations into account, there exist modeling approaches and the corresponding stability analysis for certain scenarios, see [15, 103, 12, 13], for example. In [103], a dual approach for the identification of stability regions of production networks was presented: based on the mathematical stability analysis, the identified stability regions were refined by an engineering simulation approach. The benefit of the dual approach is that it saves much time in contrast to a pure simulation approach.

Considering production networks with transportations, the modeling was performed in [16, 12, 102, 14], for example, which we use in the next subsection. Transportations can be modeled using TDS. In the next subsection, we provide a modeling approach to describe a certain scenario of a production network. We show, how a stability analysis can be performed using Theorem 4.2.1. The autonomous control method, which we use in the next subsection, was modeled in [12, 13, 102]. The impact of the presence of delays and autonomous control methods was investigated in [13], where also breakdowns of machines or plants were considered and analyzed in view of stability.

4.3.1 A certain scenario

Consider a production network that processes iron ore and that consists of four plants: one iron ore mine, one iron steel manufacturer, one car manufacturer and one tool manufacturer. $x_i \in \mathbb{R}_+$ is measured as the amount of iron ore within the ith location, i.e., for one car a certain amount of iron ore is needed.

The mine, location 1, produces iron ore and sends it to the steel manufacturer, location 2. The produced steel is send to the car or tool manufacturer, locations 3 and 4, according to an autonomous control method. From there, one third each of the production of cars and tools are send to the mine, which need them for production. The reflux of cars and tools can be interpreted as a model for a recycling process. All plants can send and get some material to/from plants or customers outside the network. The input of each location is denoted by u_i. The scenario is displayed in Figure 4.3.

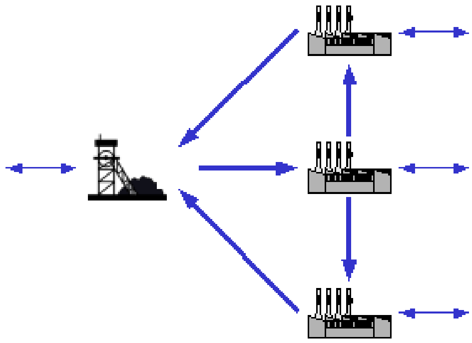

Figure 4.3: Scenario of a production network.

The production rates $p_i(x_i) \in \mathcal{K}$ of the locations are chosen as $p_i(x_i) = \alpha_i\sqrt{x_i},\ \alpha_i \in \mathbb{R}_+$, which means in practice that the plants can increase their production arbitrarily. The processed material is send to other subsystems of the network with the rate $c_{ij}(x(t))p_i(x_i(t))$, where $c_{ij}(x) \in \mathbb{R}_+,\ i \neq j$ are some positive distribution coefficients, or to customers outside the network.

We interpret the constant distribution coefficients as central planning and variable distribution coefficients can be used for some autonomous control method, e.g., the queue length estimator (QLE), see [106], for example. The QLE policy enables parts in a production system to estimate the waiting and processing times of different alternative processing resources. It uses exclusively local information to evaluate the states of the alternatives. The distribution rates representing the QLE are given by

$$c_{ji}(x(t-\tau_{ji})) := \frac{\frac{1}{x_i(t-\tau_{ji})+\varepsilon}}{\sum_k \frac{1}{x_k(t-\tau_{ji})+\varepsilon}},$$

where k is the index of the subsystems which get material from subsystem j. It holds $0 \leq c_{ji} \leq 1$. $\varepsilon > 0$ is inserted to let the fractions be well-defined. The interpretation of c_{ji} is the following: if the queue length of the ith subsystem is small, then more material will be send to subsystem i in contrast to the case where x_i is large and c_{ji} is small.

The time needed for the transportation of material from the jth to the ith location is denoted by $\tau_{ji} \in \mathbb{R}_+$. Then, the dynamics of the ith subsystem can be described by retarded differential equations (4.12) as

$$\begin{aligned}
\dot{x}_1(t) &= \tfrac{1}{3}\alpha_3\sqrt{x_3(t-\tau_{31})} + \tfrac{1}{3}\alpha_4\sqrt{x_4(t-\tau_{41})} + u_1(t) - \alpha_1\sqrt{x_1(t)},\\
\dot{x}_2(t) &= \tfrac{6}{10}\alpha_1\sqrt{x_1(t-\tau_{12})} + u_2(t) - \alpha_2\sqrt{x_2(t)},\\
\dot{x}_3(t) &= \frac{\frac{1}{x_3(t-\tau_{23})+\varepsilon}}{\frac{1}{x_3(t-\tau_{23})+x_4(t-\tau_{23})+\varepsilon}}\tfrac{6}{10}\alpha_2\sqrt{x_2(t-\tau_{23})} + u_3(t) - \alpha_3\sqrt{x_3(t)},\\
\dot{x}_4(t) &= \frac{\frac{1}{x_4(t-\tau_{24})+\varepsilon}}{\frac{1}{x_3(t-\tau_{24})+x_4(t-\tau_{24})+\varepsilon}}\tfrac{6}{10}\alpha_2\sqrt{x_2(t-\tau_{24})} + u_4(t) - \alpha_4\sqrt{x_4(t)},
\end{aligned}$$

where $x_i(\tau) \in \mathbb{R}_+,\ \tau \in [-\theta, 0],\ \theta := \max\{\tau_{31}, \tau_{12}, \tau_{23}, \tau_{24}\}$ are given.

To apply Theorem 4.2.1, we have to find LISS-Lyapunov-Razumikhin functions and to check, whether the small-gain condition is satisfied.

We choose $V_i(x_i) = |x_i| = x_i$ as a LISS-Lyapunov-Razumikhin function candidate for the ith subsystem. Obviously, $V_i(x_i)$ satisfies the condition (4.14). To prove that the condition (4.15) holds, we choose the functions $\tilde{\chi}_{ij}^d$ and $\tilde{\chi}_i^u$ as

$$\begin{aligned}
\tilde{\chi}_{1j}^d(s) &:= \left(\tfrac{\alpha_j}{\alpha_1(1-\varepsilon_1)}\right)^2 s,\ j = 3, 4,\\
\tilde{\chi}_{kl}^d(s) &:= \left(\tfrac{9\alpha_l}{10\alpha_k(1-\varepsilon_k)}\right)^2 s,\ kl \in \{21, 32, 42\},\\
\tilde{\chi}_i^u(s) &:= \left(\frac{\frac{1}{3}}{\alpha_i(1-\varepsilon_i)} s\right)^2,\ i = 1, 2, 3, 4,
\end{aligned}$$

with $0 < \varepsilon_i < 1$. We investigate the first subsystem and it holds

$$\begin{aligned}
x_1 &\geq \tilde{\chi}_{1j}^d\left(||V_j^d(x_j^t)||_{[-\theta,0]}\right) \Rightarrow \tfrac{1}{3}\alpha_j\sqrt{||V_j^d(x_j^t)||_{[-\theta,0]}} \leq \tfrac{1}{3}\alpha_1(1-\varepsilon_1)\sqrt{x_1},\\
x_1 &\geq \tilde{\chi}_1^u(|u_1|) \Rightarrow |u_1| \leq \tfrac{1}{3}\alpha_1(1-\varepsilon_1)\sqrt{x_1}.
\end{aligned}$$

Using $0 \leq c_{ji} \leq 1$ it follows

$$\begin{aligned}
V_1(x_1(t)) &\geq \max\{\max_j \tilde{\chi}_{1j}^d(||V_j^d(x_j^t)||_{[-\theta,0]}), \tilde{\chi}_1^u(|u_1(t)|)\} \Rightarrow\\
D^+V_1(x_1(t)) &= \tfrac{1}{3}\alpha_3\sqrt{x_3(t-\tau_{31})} + \tfrac{1}{3}\alpha_4\sqrt{x_4(t-\tau_{41})} + u_1(t) - \alpha_1\sqrt{x_1(t)}\\
&\leq -\varepsilon_1\alpha_1\sqrt{x_1(t)} =: -\tilde{\alpha}_1(V_1(x_1(t))).
\end{aligned}$$

We conclude that V_1 is the LISS-Lyapunov-Razumikhin function for the first subsystem. By similar calculations, we conclude that V_i are the LISS-Lyapunov-Razumikhin functions for the subsystems. To verify whether the small-gain condition is satisfied, we use the cycle condition (see Remark 1.2.1) and it holds with the choice $\varepsilon_i = \frac{1}{100}$

$$\tilde{\chi}_{21}^d(\tilde{\chi}_{32}^d(\tilde{\chi}_{13}^d(s))) = (\tfrac{9\alpha_1}{10\alpha_2(1-\varepsilon_2)})^2(\tfrac{9\alpha_2}{10\alpha_3(1-\varepsilon_3)})^2(\tfrac{\alpha_3}{\alpha_1(1-\varepsilon_1)})^2 s < s,$$

and $\tilde{\chi}^d_{21}(\tilde{\chi}^d_{42}(\tilde{\chi}^d_{14}(s))) < s$ by similar calculations. Thus, the cycle condition is satisfied and the small gain condition also. By Theorem 4.2.1, the whole network is LISS for all $x_i(\tau) \in \mathbb{R}_+,\ \tau \in [-\theta, 0]$ and all inputs $u_i \in \mathbb{R}_+$.

In the next chapter, we transfer the tools of this chapter to impulsive systems with time-delays and their interconnection.

Chapter 5

ISS for impulsive systems with time-delays

Another type of systems are impulsive systems. They combine continuous and discontinuous dynamics in one system, see [3, 43], for example. The continuous dynamics is typically described by differential equations and the discontinuous behavior consists of instantaneous state jumps that occur at given time instants, also referred to as impulses. Impulsive systems are closely related to hybrid systems [43] and switched systems [113] and have a wide range of applications in network control, engineering, biological or economical systems, see [3, 124, 43, 113], for example.

An impulsive system is of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)),\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t) &= g(x^-(t), u^-(t)),\ t = t_k,\ k \in \mathbb{N}, \end{aligned} \tag{5.1}$$

where $t_0 \leq t \in \mathbb{R}_+,\ x \in \mathbb{R}^N,\ u \in \mathrm{L}_\infty(\mathbb{R}_+, \mathbb{R}^m)$ and $\{t_1, t_2, t_3, \ldots\}$ is a strictly increasing sequence of impulse times in (t_0, ∞) for some initial time $t_0 < t_1$. The impulse times are independent from the state of the system. The generalization that includes state-dependent resetting of the systems state is known as hybrid systems, see for example [43, 100, 34, 66]. Impulsive systems can be viewed as a subclass of hybrid systems as proposed in [100].

The set of impulse times is assumed to be either finite or infinite and unbounded, and impulse times t_k have no finite accumulation point. Given a sequence $\{t_k\}$ and a pair of times $s,\ t$ satisfying $t_0 \leq s < t,\ N(t, s)$ denotes the number of impulse times t_k in the semi-open interval $(s, t]$.

Furthermore, $f : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^N$, $g : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}^N$, where we assume that f is locally Lipschitz. All signals (x and inputs u) are assumed to be right-continuous and to have left limits at all times and we denote $x^- := \lim_{s \nearrow t} x(s)$.

These requirements assure that a unique solution of the impulsive system (5.1) exists, which is absolutely continuous between impulses, see [43], Chapter 2.2. We denote the solution by $x(t; x_0, u)$ or $x(t)$ for short for any initial value $x(t_0) = x_0$.

Investigating the stability of impulsive systems, it turns out that a condition on the frequency of the impulse times is helpful, if one of the continuous or discrete dynamics destabilizes the system. To clarify this, we provide the following example.

Example 5.0.1. *We consider the system*

$$\dot{x}(t) = -\tfrac{2}{5}x(t), \;\; t \neq t_k, \;\; k \in \mathbb{N},$$
$$x(t) = 2x^-(t), \;\; t = t_k, \;\; k \in \mathbb{N},$$

$x \in \mathbb{R}$ and choose $x(0) = 1$. Note that the continuous dynamics is 0-GAS, but the discrete dynamics destabilizes the system. Let the impulse times be given by $t_k = 2k$, $k = 1, 2, \ldots$. Then, we observe the stable behavior of the system displayed in Figure 5.1. If the impulse times occur more frequently, for example $t_k = k$, $k = 1, 2, \ldots$, then the trajectory tends to infinity, as shown in Figure 5.2.

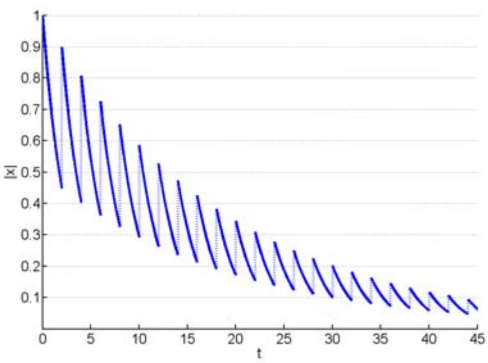

Figure 5.1: Systems behavior with $t_k = 2k$.

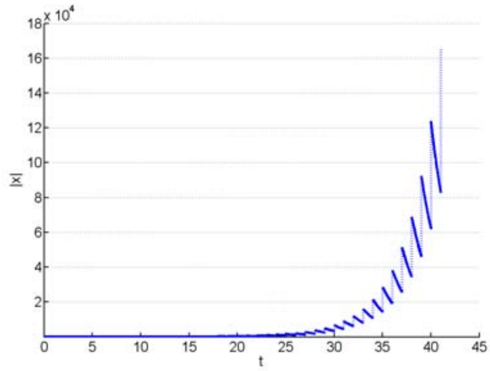

Figure 5.2: Systems behavior with $t_k = k$.

An example with unstable continuous dynamics and stable discrete dynamics can be formulated similarly. The example illustrates the importance of the frequency of the impulse times and motivates the introduction of a dwell-time condition to check, whether a system is stable, as in [46, 45, 10], for example.

In this chapter, we study the ISS property of impulsive systems with time-delays and their interconnections using exponential ISS-Lyapunov-Razumikhin functions and exponential ISS-Lyapunov-Krasovskii functionals. The ISS property and the iISS property for single systems without time-delays were studied in [45] and in [10, 76] for single non-autonomous time-delay systems. There, sufficient conditions, which assure ISS and iISS of an impulsive system, were derived using exponential ISS-Lyapunov(-Razumikhin) functions, where in [76] multiple Krasovskii functionals are used. A recent paper investigates the ISS property of discrete-time impulsive systems with time-delays using a Razumikhin approach, see [128].

In [45], the average dwell-time condition was used, whereas in [10] a fixed dwell-time condition was utilized. The average dwell-time condition was introduced in [46] for switched systems. This condition considers the average of impulses over an interval, whereas the fixed dwell-time condition considers the (minimal or maximal) interval between two impulses.

We provide a Lyapunov-Krasovskii type theorem and a Lyapunov-Razumikhin type ISS theorem using the average dwell-time condition for single impulsive systems with time-delays. The proofs use the idea of the proof of [45], [121] and [120]. For the Razumikhin type ISS theorem we require as an additional condition that the Lyapunov gain fulfills a small-gain condition. In contrast to the Razumikhin type theorem from [10], we consider autonomous time-delay systems and the average dwell-time condition. Our theorem allows to verify the ISS property for larger classes of impulse time sequences, however, we have used an additional technical condition on the Lyapunov gain in our proofs.

Considering interconnected impulsive systems, our main goal is to find sufficient conditions which assure ISS of such interconnections without time-delays. To this end, we use the approach used for continuous networks. We prove that under a small-gain condition with linear gains and a dwell-time condition a network of impulsive subsystems possesses the ISS property. We construct the exponential ISS-Lyapunov-Razumikhin and ISS-Lyapunov-Krasovskii function(al)s and the corresponding gains of the whole system.

In Section 5.1, single impulsive systems with time-delays are considered. Subsection 5.1.2 presents the Lyapunov-Krasovskii approach and Subsection 5.1.1 presents the Lyapunov-Razumikhin approach. The ISS property for interconnections of impulsive systems with time-delays is investigated in Section 5.2. The tools to check whether a network possesses the ISS property can be found in Subsection 5.2.1 for the Lyapunov-Razumikhin approach and in Subsection 5.2.2 for the Lyapunov-Krasovskii approach. An example is given in Section 5.3.

5.1 Single impulsive systems with time-delays

We consider single impulsive system with time-delays of the form

$$\begin{aligned} \dot{x}(t) &= f(x^t, u(t)),\ t \neq t_k,\ k \in \mathbb{N}, \\ x(t) &= g((x^t)^-, u^-(t)),\ t = t_k,\ k \in \mathbb{N}, \end{aligned} \tag{5.2}$$

where we make the same assumptions as in the delay-free case, where $f : PC\left([-\theta,0]\,;\mathbb{R}^N\right) \times \mathbb{R}^m \to \mathbb{R}^N$ is locally Lipschitz, $g : PC\left([-\theta,0]\,;\mathbb{R}^N\right) \times \mathbb{R}^m \to \mathbb{R}^N$ and we denote $(x^t)^- := \lim_{s\nearrow t} x^s$.

We assume that the regularity conditions (see e.g., [4]) for the existence and uniqueness of a solution of system (5.2) are satisfied. We denote the solution by $x(t;\xi,u)$ or $x(t)$ for short for any $\xi \in PC([-\theta,0],\mathbb{R}^N)$ that exists in a maximal interval $[-\theta,b)$, $0 < b \leq +\infty$, satisfying the initial condition $x^{t_0} = \xi$. The solution is piecewise right-continuous for all $t \geq t_0$ and it is continuous at each $t \neq t_k, t \geq t_0$.

The ISS property is redefined with respect to impulsive systems with time-delays, see [10]:

Definition 5.1.1 (ISS for impulsive systems with time-delays)**.** *Suppose that a sequence $\{t_k\}$ is given. We call the system* (5.2) *or* (5.19) *ISS, if there exist functions $\beta \in \mathcal{KL}$, $\gamma_u \in \mathcal{K}_\infty$, such that for every initial condition $\xi \in PC([-\theta,0],\mathbb{R}^N)$ and every input u it holds*

$$|x(t)| \leq \max\{\beta(\|\xi\|_{[-\theta,0]}, t-t_0), \gamma_u(\|u\|_{[t_0,t]})\},\ \forall t \geq t_0. \tag{5.3}$$

The impulsive system (5.2) *or* (5.19) *is uniformly ISS over a given class $\mathcal{S}$ of admissible sequences of impulse times, if* (5.3) *holds for every sequence in $\mathcal{S}$, with functions β and γ_u, which are independent on the choice of the sequence.*

Remark 5.1.2. *Note that we get an equivalent definition of ISS, if we use the summation formulation instead of the maximum formulation in Definition 5.1.1. In the following, we only use the maximum formulation. The results using the summation formulation follow equivalently.*

We use the tools, introduced in Chapter 4, namely Lyapunov-Krasovskii functionals and Lyapunov-Razumikhin functions, to check whether an impulsive time-delay system has the ISS property.

5.1.1 The Lyapunov-Razumikhin methodology

At first, we give the definition of an exponential ISS-Lyapunov-Razumikhin function:

Definition 5.1.3 (Exponential ISS-Lyapunov-Razumikhin functions)**.** *A function $V : \mathbb{R}^N \to \mathbb{R}_+$ is called an exponential ISS-Lyapunov-Razumikhin function for the system* (5.2) *with rate coefficients $c, d \in \mathbb{R}$, if V is locally Lipschitz continuous and there exist functions ψ_1, ψ_2, γ_d, $\gamma_u \in \mathcal{K}_\infty$ such that*

$$\psi_1(|\phi(0)|) \leq V(\phi(0)) \leq \psi_2(|\phi(0)|), \ \forall \phi(0) \in \mathbb{R}^N$$

and whenever $V(\phi(0)) \geq \max\{\gamma_d(\|V^d(\phi)\|_{[-\theta,0]}), \gamma_u(|u|)\}$ holds, it follows

$$D^+V(\phi(0)) \leq -cV(\phi(0)) \text{ and} \tag{5.4}$$

$$V(g(\phi,u)) \leq e^{-d}V(\phi(0)), \tag{5.5}$$

for all $\phi \in PC\left([-\theta,0]\,;\mathbb{R}^N\right)$ and $u \in \mathbb{R}^m$, where $V^d : PC\left([-\theta,0]\,;\mathbb{R}^N\right) \to PC\left([-\theta,0]\,;\mathbb{R}_+\right)$ is defined by $V^d(x^t)(\tau) := V(x(t+\tau)),\ \tau \in [-\theta,0]$.

Roughly speaking, the condition (5.4) states that if c is positive, then the function V decreases along the solution $x(t)$ at t. On the other hand, if $c < 0$ then the function V can increase along the solution $x(t)$ at t. Condition (5.5) states that if d is positive, then the jump (impulse) decreases the magnitude of V. On the other hand, if $d < 0$ then the jump (impulse) increase the magnitude of V.

Remark 5.1.4. *Note that in [10] the conditions* (5.4) *and* (5.5) *are in the dissipative form. By Proposition 2.6 in [8], the conditions in dissipative form are equivalent to the conditions in implication form, used in Definition 5.1.3, where the coefficients c, d are different in general.*

For the main result of this subsection, we need the following:

Definition 5.1.5. *Assume that a sequence $\{t_k\}$ is given. We call the system* (5.2) *or* (5.19) *globally stable (GS), if there exist functions $\varphi, \gamma \in \mathcal{K}_\infty$ such that for all $\xi \in PC([-\theta,0],\mathbb{R}^N)$ and all u it holds*

$$|x(t)| \leq \max\{\varphi(\|\xi\|_{[-\theta,0]}), \gamma(\|u\|_{[t_0,t]})\},\ \forall t \geq t_0. \tag{5.6}$$

The impulsive system (5.2) *or* (5.19) *is uniformly GS over a given class $\mathcal{S}$ of admissible sequences of impulse times, if* (5.6) *holds for every sequence in $\mathcal{S}$, with functions φ and γ_u, which are independent on the choice of the sequence.*

To prove a Razumikhin type theorem for the verification of ISS for impulsive time-delay systems, we need the following characterization of the uniform ISS property:

Lemma 5.1.6. *The system* (5.2) *or* (5.19) *is uniformly ISS over $\mathcal{S}$, if and only if it is*

- *uniformly GS over $\mathcal{S}$ and*
- *there exists $\gamma \in \mathcal{K}$ such that for each $\epsilon > 0$, $\eta_x \in \mathbb{R}_+$, $\eta_u \in \mathbb{R}_+$ there exists $T \geq 0$ (which does not depend on the choice of the impulse time sequence from $\mathcal{S}$) such that $\|\xi\|_{[-\theta,0]} \leq \eta_x$ and $\|u\|_{[t_0,\infty)} \leq \eta_u$ imply $|x(t)| \leq \max\{\epsilon, \gamma(\|u\|_{[t_0,t]})\}$, for all $t \geq T + t_0$.*

The proof can be found in [17]. Now, we prove as a main result of this section the following theorem. It provides a tool to check whether a single impulsive system with time-delays possesses the ISS property using exponential Lyapunov-Razumikhin functions.

Theorem 5.1.7 (Lyapunov-Razumikhin type theorem)**.** *Let V be an exponential ISS-Lyapunov-Razumikhin function for the system* (5.2) *with $c, d \in \mathbb{R}$, $d \neq 0$. For arbitrary constants $\mu, \lambda > 0$, let $\mathcal{S}[\mu, \lambda]$ denote the class of impulse time sequences $\{t_k\}$ satisfying the average dwell-time condition*

$$-dN(t,s) - (c - \lambda)(t - s) \leq \mu, \ \forall t \geq s \geq t_0. \tag{5.7}$$

If γ_d satisfies $\gamma_d(r) < e^{-\mu-|d|}r$, $r > 0$, then the system (5.2) *is uniformly ISS over $\mathcal{S}[\mu, \lambda]$.*

For $d = 0$ the jumps do not influence the stability of the system and the system will be ISS, if the corresponding continuous dynamics has the ISS property. This case was investigated more detailed in [45], Section 6.

Note that the condition (5.7) guarantees stability of an impulsive system even if the continuous or discontinuous behavior is unstable. For example, if the continuous behavior is unstable, which means $c < 0$, then this condition assumes that the discontinuous behavior has to stabilize the system ($d > 0$) and the jumps have to occur often enough. Conversely, if the discontinuous behavior is unstable ($d < 0$) and the continuous behavior is stable ($c > 0$) then the jumps have to occur rarely, which stabilizes the system.

Proof. From (5.4) we have for any two consecutive impulses t_{k-1}, t_k, for all $t \in (t_{k-1}, t_k)$

$$V(x(t)) \geq \max\{\gamma_d(\|V^d(x^t)\|_{[-\theta,0]}), \gamma_u(|u(t)|)\} \Rightarrow \mathrm{D}^+V(x(t)) \leq -cV(x(t)) \tag{5.8}$$

and similarly with (5.5) for every impulse time t_k

$$\begin{aligned} &V(x(t_k)) \geq \max\{\gamma_d(\|V^d((x^{t_k})^-)\|_{[-\theta,0]}), \gamma_u(|u^-(t_k)|)\} \\ &\Rightarrow V(g((x^{t_k})^-, u^-(t_k))) \leq e^{-d}V(x(t_k)). \end{aligned} \tag{5.9}$$

Because of the right-continuity of x and u, there exists a sequence of times $t_0 := \tilde{t}_0 < \bar{t}_1 < \tilde{t}_1 < \bar{t}_2 < \tilde{t}_2 < \dots$ such that for $i = 0, 1, \dots$ we have

$$V(x(t)) \geq \max\left\{\gamma_d\left(\sup_{r\in[t_0,t]}\left\|V^d(x^r)\right\|_{[-\theta,0]}\right), \gamma_u(\|u\|_{[t_0,t]})\right\}, \ \forall t \in [\tilde{t}_i, \bar{t}_{i+1}) \tag{5.10}$$

and for all $i = 1, 2, \dots$ it holds

$$V(x(t)) \leq \max\left\{\gamma_d\left(\sup_{r\in[t_0,t]}\left\|V^d(x^r)\right\|_{[-\theta,0]}\right), \gamma_u(\|u\|_{[t_0,t]})\right\}, \ \forall t \in [\bar{t}_i, \tilde{t}_i), \tag{5.11}$$

where this sequence breaks the interval $[t_0, \infty)$ into a disjoint union of subintervals. Suppose $t_0 < \bar{t}_1$, so that $[t_0, \bar{t}_1)$ is nonempty. Otherwise we can continue the proof in the line below (5.13). Between any two consecutive impulses $t_{k-1}, t_k \in [t_0, \bar{t}_1]$ from (5.10) and (5.8) we have $\mathrm{D}^+V(x(t))) \leq -cV(x(t))$, for all $t \in (t_{k-1}, t_k)$ and therefore

$$V(x^-(t_k)) \leq e^{-c(t_k - t_{k-1})}V(x(t_{k-1})).$$

From (5.9) and (5.10), we have $V(x(t_k)) \leq e^{-d}V(x^-(t_k))$. Combining this, it follows

$$V(x(t_k)) \leq e^{-d-c(t_k-t_{k-1})}V(x(t_{k-1}))$$

and by the iteration over the $N(t, t_0)$ impulses on $[t_0, t]$, we obtain the bound

$$V(x(t)) \leq e^{-dN(t,t_0)-c(t-t_0)}V(x(t_0)), \ \forall t \in [t_0, \bar{t}_1]. \tag{5.12}$$

Using the dwell-time condition (5.7), we get

$$V(x(t)) \leq e^{\mu-\lambda(t-t_0)}V(x(t_0)), \ \forall t \in [t_0, \bar{t}_1]. \tag{5.13}$$

For any subinterval of the form $[\bar{t}_i, \tilde{t}_i), \ i = 1, 2, \dots$, we have (5.11) as a bound for $V(x(t))$. Now, consider two cases.

Let $\tilde{t}_i$ be not an impulse time, then (5.11) is a bound for $t = \tilde{t}_i$. Consider the subinterval $[\tilde{t}_i, \bar{t}_{i+1})$. Repeating the argument used to establish (5.13), with $\tilde{t}_i$ in place of t_0 and using (5.11) with $t = \tilde{t}_i$ we get for all $t \in (\tilde{t}_i, \bar{t}_{i+1}]$

$$V(x(t)) \leq e^{\mu-\lambda(t-\tilde{t}_i)}V(x(\tilde{t}_i)) \leq e^{\mu}\max\left\{\gamma_d\left(\sup_{r\in\left[t_0,\tilde{t}_i\right]}\left\|V^d(x^r)\right\|_{[-\theta,0]}\right), \gamma_u(\|u\|_{[t_0,\tilde{t}_i]})\right\}.$$

Now, let $\tilde{t}_i$ be an impulse time. Then, we have

$$V(x(\tilde{t}_i)) \leq e^{-d}\max\left\{\gamma_d\left(\sup_{r\in\left[t_0,\tilde{t}_i\right]}\left\|V^d(x^r)\right\|_{[-\theta,0]}\right), \gamma_u(\|u\|_{[t_0,\tilde{t}_i]})\right\} \tag{5.14}$$

and in either case

$$V(x(t)) \leq e^{|d|}\max\left\{\gamma_d\left(\sup_{r\in[t_0,t]}\left\|V^d(x^r)\right\|_{[-\theta,0]}\right), \gamma_u(\|u\|_{[t_0,t]})\right\}, \ \forall t \in [\bar{t}_i, \tilde{t}_i]. \tag{5.15}$$

Repeating the argument used to establish (5.13), with $\tilde{t}_i$ in place of t_0 and using (5.15) with $t=\tilde{t}_i$ we get

$$V(x(t)) \leq e^{\mu-\lambda(t-\tilde{t}_i)}V(x(\tilde{t}_i)) \leq e^{\mu+|d|}\max\left\{\gamma_d\left(\sup_{r\in[t_0,\tilde{t}_i]}\left\|V^d(x^r)\right\|_{[-\theta,0]}\right), \gamma_u(\|u\|_{[t_0,\tilde{t}_i]})\right\},$$

for all $t\in(\tilde{t}_i,\bar{t}_{i+1}]$, $i\geq 1$. Overall, we obtain for all $t\geq t_0$

$$V(x(t)) \leq \max\left\{e^{\mu-\lambda(t-t_0)}V(x(t_0)), e^{\mu+|d|}\gamma_d\left(\sup_{r\in[t_0,t]}\left\|V^d(x^r)\right\|_{[-\theta,0]}\right), e^{\mu+|d|}\gamma_u(\|u\|_{[t_0,t]})\right\}. \tag{5.16}$$

Now, it holds

$$\sup_{t\geq s\geq t_0}\left\|V^d(x^s)\right\|_{[-\theta,0]} \leq \max\left\{\left\|V^d(x^{t_0})\right\|_{[-\theta,0]}, \sup_{t\geq s\geq t_0}V(x(s))\right\}. \tag{5.17}$$

We take the supremum over $[t_0,t]$ in (5.16) and insert it into (5.17). Then, using $\gamma_d(r) < e^{-\mu-|d|}r$ and the fact that for all $b_1,b_2>0$ from $b_1\leq\max\{b_2, e^{\mu+|d|}\gamma_d(b_1)\}$ it follows $b_1\leq b_2$, we obtain

$$\sup_{t\geq s\geq t_0}\|V^d(x^s)\|_{[-\theta,0]} \leq \max\left\{e^{\mu+|d|}\psi_2(\|\xi\|_{[-\theta,0]}), e^{\mu+|d|}\gamma_u(\|u\|_{[t_0,t]})\right\}$$

and therefore

$$|x(t)| \leq \max\left\{\psi_1^{-1}(e^{\mu+|d|}\psi_2(\|\xi\|_{[-\theta,0]})), \psi_1^{-1}(e^{\mu+|d|}\gamma_u(\|u\|_{[t_0,t]}))\right\},\ \forall t\geq t_0,$$

which means that the system (5.2) is uniformly GS over $\mathcal{S}[\mu,\lambda]$.

Note that $\tilde{\varphi}(\cdot) := \psi_1^{-1}(e^{\mu+|d|}\psi_2(\cdot))$ is a $\mathcal{K}_\infty$-function. Now, for given $\epsilon,\eta_x,\eta_u>0$ such that $\|\xi\|_{[-\theta,0]}\leq\eta_x$, $\|u\|_{[t_0,\infty)}\leq\eta_u$ let $\kappa := \max\left\{e^{\mu+|d|}\psi_2(\eta_x), e^{\mu+|d|}\gamma_u(\eta_u)\right\}$. It holds $\sup_{t\geq s\geq t_0}\|V^d(x^s)\|_{[-\theta,0]}\leq\kappa$. Let $\rho_2>0$ be such that $e^{-\lambda\rho_2}\kappa\leq\psi_1(\epsilon)$ and let $\rho_1>\theta$. Then, by the estimate (5.16) we have

$$\begin{aligned}\sup_{t\geq s\geq t_0+\rho_1+\rho_2}\|V^d(x^s)\|_{[-\theta,0]} &\leq \sup_{t\geq s\geq t_0+\rho_2}V(x(s))\\ &\leq \max\left\{\psi_1(\epsilon), e^{\mu+|d|}\gamma_d\left(\sup_{r\in[t_0,t]}\left\|V^d(x^r)\right\|_{[-\theta,0]}\right), e^{\mu+|d|}\gamma_u(\|u\|_{[t_0,t]})\right\}.\end{aligned}$$

Replacing t_0 by $t_0+\rho_1+\rho_2$ in the previous inequality we obtain

$$\begin{aligned}&\sup_{t\geq s\geq t_0+2(\rho_1+\rho_2)}\|V^d(x^s)\|_{[-\theta,0]}\\ &\leq \max\left\{\psi_1(\epsilon), e^{\mu+|d|}\gamma_d\left(\sup_{r\in[t_0+\rho_1+\rho_2,t]}\left\|V^d(x^r)\right\|_{[-\theta,0]}\right), e^{\mu+|d|}\gamma_u(\|u\|_{[t_0+\rho_1+\rho_2,t]})\right\}\\ &\leq \max\left\{\psi_1(\epsilon), (e^{\mu+|d|}\gamma_d)^2\left(\sup_{r\in[t_0,t]}\left\|V^d(x^r)\right\|_{[-\theta,0]}\right), e^{\mu+|d|}\gamma_u(\|u\|_{[t_0,t]})\right\}.\end{aligned}$$

Since $e^{\mu+|d|}\gamma_d < Id$, there exists a number $\tilde{n} \in \mathbb{N}$, which depends on κ and ϵ such that

$$(e^{\mu+|d|}\gamma_d)^{\tilde{n}}(\kappa) := \underbrace{(e^{\mu+|d|}\gamma_d) \circ \ldots \circ (e^{\mu+|d|}\gamma_d)}_{\tilde{n} \text{ times}}(\kappa) \leq \max\left\{\psi_1(\epsilon), e^{\mu+|d|}\gamma_u(\|u\|_{[t_0,t]})\right\}.$$

By induction, we conclude that

$$\sup_{t \geq s \geq t_0+\tilde{n}(\rho_1+\rho_2)} \|V^d(x^s)\|_{[-\theta,0]} \leq \max\left\{\psi_1(\epsilon), e^{\mu+|d|}\gamma_u(\|u\|_{[t_0,t]})\right\},$$

and finally, we obtain

$$|x(t)| \leq \max\left\{\epsilon, \psi_1^{-1}(e^{\mu+|d|}\gamma_u(\|u\|_{[t_0,t]}))\right\}, \ \forall t \geq t_0 + \tilde{n}(\rho_1+\rho_2). \tag{5.18}$$

Thus, the system (5.2) satisfies the second property from Lemma 5.1.6, which implies that a system of the form (5.2) is uniformly ISS over $\mathcal{S}[\mu, \lambda]$. □

Note that the condition $\gamma_d(r) < e^{-\mu-|d|}r, \ r > 0$ means that the gain is connected with the average dwell-time condition (5.7). As we will see, this is also the case for interconnected impulsive time-delay systems. Also note that the condition $\gamma_d(r) < e^{-\mu-|d|}r, \ r > 0$ leads to some kind of conservativeness in contrast to the condition $\gamma_d(r) < r, \ r > 0$ for time-delay systems without impulses. To relax this condition, one can get rid of the term $e^{-|d|}$ in the previous condition, which we quote in the following theorem.

Theorem 5.1.8. *Consider all assumptions from Theorem 5.1.7, with γ_d satisfying $\gamma_d(r) < e^{-\mu}r, \ r > 0$. Then, the system (5.2) is uniformly ISS over $\mathcal{S}[\mu, \lambda]$.*

The proof with the technique to get rid of the term $e^{-|d|}$ can be found in [17]. The idea is that in the proof of Theorem 5.1.7, namely in equation (5.14), the set of time sequences $\mathcal{S}^*[\mu, \lambda]$ is considered for which the average dwell-time condition holds with the number of jumps in the compact interval $[s, t]$, defined by $N^*(t, s)$, instead of the number of jumps $N(t, s)$ in the interval (s,t]. It can be shown that $\mathcal{S}^*[\mu, \lambda] = \mathcal{S}[\mu, \lambda]$ and that the estimations around (5.14) hold true with an additional multiplier e^d in (5.14), such that $e^d e^{-d} = 1$ and the term $e^{-|d|}$ is not necessary in the estimations of the proof of Theorem 5.1.7.

Remark 5.1.9. *Another Razumikhin type theorem (for non-autonomous systems) has been proposed in [10]. There, a so-called fixed dwell-time condition was used to characterize the class of impulse time sequences, over which the system is uniformly ISS. In contrast to Theorems 1 and 2 from [10], we prove the Razumikhin type theorem (Theorem 5.1.7) over the class of sequences, which satisfy the average dwell-time condition. This class is larger than the class of sequences, which satisfy the fixed dwell-time condition. However, the small-gain condition, that we have used in this thesis, $\gamma_d(r) < e^{-\mu}r, \ r > 0$ or $\gamma_d(r) < e^{-\mu-|d|}r, \ r > 0$, respectively, is stronger than that from [10].*

In the next subsection, we give a counterpart to Theorem 5.1.7 or Theorem 5.1.8, respectively, using exponential Lyapunov-Krasovskii functionals.

5.1.2 The Lyapunov-Krasovskii methodology

We consider another type of impulsive systems with time-delays of the form

$$\begin{aligned} \dot{x}(t) &= f(x^t, u(t)),\ t \neq t_k,\ k \in \mathbb{N}, \\ x^t &= g((x^t)^-, u(t)),\ t = t_k,\ k \in \mathbb{N}, \end{aligned} \tag{5.19}$$

where we make the same assumptions as before and the functional g is now a map from $PC\left([-\theta,0]\,;\mathbb{R}^N\right) \times \mathbb{R}^m$ into $PC\left([-\theta,0]\,;\mathbb{R}^N\right)$.

According to [48], Section 2, the initial state and the input together determine the evolution of a system according to the right-hand side of the differential equation. Therefore, for time-delay systems, we denote the state by the function $x^t \in PC([-\theta,0],\mathbb{R}^N)$ and we change the discontinuous behavior in (5.19): in contrast to the system (5.2), at an impulse time t_k not only the point $x(t_k)$ "jumps", but all the states x^t in the interval $(t_k-\theta,t_k]$. Due to this change, the Lyapunov-Razumikhin approach cannot be applied. In this case, we propose to use Lyapunov-Krasovskii functionals for the stability analysis of systems of the form (5.19).

Another approach using Lyapunov functionals can be found in [109]. There, Lyapunov functionals for systems of the form (5.2) with zero input are used for stabilization results of impulsive systems, where the definition of such a functional is different to the approach presented here according to impulse times.

Definition 5.1.10 (Exponential ISS-Lyapunov-Krasovskii functionals). *A functional $V : PC\left([-\theta,0]\,;\mathbb{R}^N\right) \to \mathbb{R}_+$ is called an exponential ISS-Lyapunov-Krasovskii functional with rate coefficients $c,d \in \mathbb{R}$ for the system* (5.19)*, if V is locally Lipschitz continuous, there exist $\psi_1,\psi_2 \in \mathcal{K}_\infty$ such that*

$$\psi_1(|\phi(0)|) \leq V(\phi) \leq \psi_2(|\phi|_a),\ \forall \phi \in PC\left([-\theta,0]\,;\mathbb{R}^N\right) \tag{5.20}$$

and there exists a function $\gamma \in \mathcal{K}$ such that whenever $V(\phi) \geq \gamma(|u|)$ holds, it follows

$$D^+V(\phi,u) \leq -cV(\phi) \text{ and} \tag{5.21}$$

$$V(g(\phi,u)) \leq e^{-d}V(\phi), \tag{5.22}$$

for all $\phi \in PC\left([-\theta,0]\,;\mathbb{R}^N\right)$ and $u \in \mathbb{R}^m$.

Note that the rate coefficients c,d are not required to be non-negative. The following result is a counterpart of Theorem 1 in [45] and Theorems 1 and 2 in [10] for impulsive systems with time-delays using the Lyapunov-Krasovskii approach.

Theorem 5.1.11 (Lyapunov-Krasovskii type theorem). *Let V be an exponential ISS-Lyapunov-Krasovskii functional for the system* (5.19) *with $c,d \in \mathbb{R}$, $d \neq 0$. For arbitrary constants $\mu,\ \lambda \in \mathbb{R}_+$, let $\mathcal{S}[\mu,\lambda]$ denote the class of impulse time sequences $\{t_k\}$ satisfying the dwell-time condition* (5.7)*. Then, the system* (5.19) *is uniformly ISS over $\mathcal{S}[\mu,\lambda]$.*

Proof. From (5.21) we have for any two consecutive impulses t_{k-1},t_k, for all $t \in (t_{k-1},t_k)$

$$V(x^t) \geq \gamma(|u(t)|) \Rightarrow \mathrm{D}^+V(x^t,u(t)) \leq -cV(x^t) \tag{5.23}$$

and similarly with (5.22) for every impulse time t_k

$$V(x^{t_k}) \geq \gamma(|u^-(t_k)|) \Rightarrow V(g((x^{t_k})^-, u^-(t_k))) \leq e^{-d}V(x^{t_k}). \tag{5.24}$$

Because of the right-continuity of x and u there exists a sequence of times $t_0 := \tilde{t}_0 < \bar{t}_1 < \tilde{t}_1 < \bar{t}_2 < \tilde{t}_2 < \dots$ such that we have

$$V(x^t) \geq \gamma(\|u\|_{[t_0,t]}),\ \forall t \in [\tilde{t}_i, \bar{t}_{i+1}),\ i = 0, 1, \dots, \tag{5.25}$$

$$V(x^t) \leq \gamma(\|u\|_{[t_0,t]}),\ \forall t \in [\bar{t}_i, \tilde{t}_i),\ i = 1, 2, \dots, \tag{5.26}$$

where this sequence breaks the interval $[t_0, \infty)$ into a disjoint union of subintervals. Suppose $t_0 < \bar{t}_1$, so that $[t_0, \bar{t}_1)$ is nonempty. Otherwise we can continue the proof in the line below (5.27). Between any two consecutive impulses $t_{k-1}, t_k \in (t_0, \bar{t}_1]$ with (5.25) and (5.23) we have $\mathrm{D}^+V(x^t, u(t)) \leq -cV(x^t)$, for all $t \in (t_{k-1}, t_k)$ and therefore

$$V((x^{t_k})^-) \leq e^{-c(t_k - t_{k-1})}V(x^{t_{k-1}}).$$

From (5.24) and (5.25) we have $V(x^{t_k}) \leq e^{-d}V((x^{t_k})^-)$. Combining this, it follows

$$V(x^{t_k}) \leq e^{-d}e^{-c(t_k - t_{k-1})}V(x^{t_{k-1}})$$

and by iteration over the $N(t, t_0)$ impulses on $(t_0, t]$ we obtain the bound

$$V(x^t) \leq e^{-dN(t,t_0) - c(t-t_0)}V(\xi),\ \forall t \in (t_0, \bar{t}_1].$$

Using the dwell-time condition (5.7), we get

$$V(x^t) \leq e^{\mu - \lambda(t-t_0)}V(\xi),\ \forall t \in (t_0, \bar{t}_1]. \tag{5.27}$$

Now, on any subinterval of the form $[\bar{t}_i, \tilde{t}_i)$, we already have (5.26) as a bound. If $\tilde{t}_i$ is not an impulse time, then (5.26) is a bound for $t = \tilde{t}_i$. If $\tilde{t}_i$ is an impulse time, then we have

$$V(x^{\tilde{t}_i}) \leq e^{-d}\gamma(\|u\|_{[t_0,\tilde{t}_i]})$$

and in either case

$$V(x^t) \leq e^{|d|}\gamma(\|u\|_{[t_0,t]}),\ \forall t \in [\bar{t}_i, \tilde{t}_i],\ i \geq 1, \tag{5.28}$$

where this bound holds for all $t \geq \bar{t}_i$, if $\tilde{t}_i = \infty$. Now, consider any subinterval of the form $[\tilde{t}_i, \bar{t}_{i+1}),\ i \geq 1$. Repeating the argument used to establish (5.27) with $\tilde{t}_i$ in place of t_0 and using (5.28) with $t = \tilde{t}_i$, we get

$$V(x^t) \leq e^{\mu - \lambda(t - \tilde{t}_i)}V(x^{\tilde{t}_i}) \leq e^{\mu + |d|}\gamma(\|u\|_{[t_0,\tilde{t}_i]})$$

for all $t \in (\tilde{t}_i, \bar{t}_{i+1}],\ i \geq 1$. Combining this with (5.27) and (5.28), we obtain

$$V(x^t) \leq \max\{e^{\mu - \lambda(t-t_0)}V(\xi), e^{\mu+|d|}\gamma(\|u\|_{[t_0,t]})\},\ \forall t \geq t_0.$$

By definition of $\beta(r, t-t_0) := \psi_1^{-1}(e^{\mu-\lambda(t-t_0)}\psi_2(\tilde{c}r))$ and $\gamma_u(r) := \psi_1^{-1}(e^{\mu+|d|}\gamma(r))$ the uniform ISS property follows from (5.20). Note that β and γ_u do not depend on the particular choice of the time sequence and therefore uniformity is clear. □

In the next section, we investigate interconnected impulsive time-delay systems in view of stability.

5.2 Networks of impulsive systems with time-delays

We consider n interconnected impulsive systems with time-delays of the form

$$\begin{aligned} \dot{x}_i(t) &= f_i(x_1^t, \ldots, x_n^t, u_i(t)),\ t \neq t_k, \\ x_i(t) &= g_i((x_1^t)^-, \ldots, (x_n^t)^-, u_i^-(t)),\ t = t_k, \end{aligned} \tag{5.29}$$

where the same assumptions on the system as in the delay-free case are considered with the following differences: We denote $x_i^t(\tau) := x_i(t+\tau),\ \tau \in [-\theta, 0]$ and $(x_i^t)^-(\tau) := x_i^-(t+\tau) := \lim_{s \nearrow t} x_i(s+\tau),\ \tau \in [-\theta, 0]$. Furthermore, $f_i : PC([-\theta,0], \mathbb{R}^{N_1}) \times \ldots \times PC([-\theta,0], \mathbb{R}^{N_n}) \times \mathbb{R}^{M_i} \to \mathbb{R}^{N_i}$, and $g_i : PC([-\theta,0], \mathbb{R}^{N_1}) \times \ldots \times PC([-\theta,0], \mathbb{R}^{N_n}) \times \mathbb{R}^{M_i} \to \mathbb{R}^{N_i}$, where we assume that $f_i,\ i = 1, \ldots, n$ are locally Lipschitz.

If we define $N := \sum_i N_i,\ m := \sum_i M_i,\ x = (x_1^T, \ldots, x_n^T)^T,\ u = (u_1^T, \ldots, u_n^T)^T,\ f = (f_1^T, \ldots, f_n^T)^T$ and $g = (g_1^T, \ldots, g_n^T)^T$, then (5.29) becomes a system of the form (5.2). The ISS property for systems with several inputs and time-delays is as follows:

Suppose that a sequence $\{t_k\}$ is given. The ith subsystem of (5.29) is *ISS*, if there exist $\beta_i \in \mathcal{KL},\ \gamma_{ij}, \gamma_i^u \in \mathcal{K}_\infty \cup \{0\}$ such that for every initial condition ξ_i and every input u_i it holds

$$|x_i(t)| \leq \max\{\beta_i(\|\xi_i\|_{[-\theta,0]}, t - t_0), \max_{j, j \neq i} \gamma_{ij}(\|x_j\|_{[t_0-\theta, t]}), \gamma_i^u(\|u_i\|_{[t_0,t]})\} \tag{5.30}$$

for all $t \geq t_0$. The ith subsystem of (5.29) is *uniformly ISS* over a given class $\mathcal{S}$ of admissible sequences of impulse times, if (5.30) holds for every sequence in $\mathcal{S}$, with functions $\beta_i,\ \gamma_{ij}$ and γ_i^u that are independent of the choice of the sequence.

In the following, we present tools to analyze systems of the form (5.29) in view of ISS: exponential ISS-Lyapunov-Razumikhin functions and exponential ISS-Lyapunov-Krasovskii functionals for the subsystems.

5.2.1 The Lyapunov-Razumikhin approach

Assume that for each subsystem of the interconnected system (5.29) there is a given function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, which is continuous, positive definite and locally Lipschitz continuous on $\mathbb{R}^{N_i} \backslash \{0\}$. For $i = 1, \ldots, n$ the function V_i is an *exponential ISS-Lyapunov-Razumikhin function of the ith subsystem of (5.29)*, if there exist $\psi_{1i}, \psi_{2i} \in \mathcal{K}_\infty,\ \gamma_i^u \in \mathcal{K} \cup \{0\},\ \gamma_{ij} \in \mathcal{K}_\infty \cup \{0\},\ j = 1, \ldots, n$ and scalars $c_i, d_i \in \mathbb{R}$, such that

$$\psi_{1i}(|\phi_i(0)|) \leq V_i(\phi_i(0)) \leq \psi_{2i}(|\phi_i(0)|),\ \forall \phi_i(0) \in \mathbb{R}^{N_i} \tag{5.31}$$

and whenever $V_i(\phi_i(0)) \geq \max\{\max_j \gamma_{ij}(\|V_j^d(\phi_j)\|_{[-\theta,0]}), \gamma_i^u(|u_i|)\}$ holds, it follows

$$\mathrm{D}^+ V_i(\phi_i(0)) \leq -c_i V_i(\phi_i(0)) \tag{5.32}$$

for all $\phi = (\phi_1^T, \ldots, \phi_n^T)^T \in PC([-\theta,0], \mathbb{R}^N)$ and $u_i \in \mathbb{R}^{M_i}$, where $V_j^d : PC\left([-\theta,0]; \mathbb{R}^{N_j}\right) \to PC\left([-\theta,0]; \mathbb{R}_+\right)$ with $V_j^d(x_j^t)(\tau) := V_j(x_j(t+\tau)),\ \tau \in [-\theta, 0]$ and it holds

$$V_i(g_i(\phi_1, \ldots, \phi_n, u_i)) \leq \max\{e^{-d_i} V_i(\phi_i(0)), \max_j \gamma_{ij}(\|V_j^d(\phi_j)\|_{[-\theta,0]}), \gamma_i^u(|u_i|)\}, \tag{5.33}$$

for all $\phi \in PC([-\theta,0],\mathbb{R}^N)$ and $u_i \in \mathbb{R}^{M_i}$. Another formulation can be obtained by replacing (5.33) by

$$V_i(\phi_i(0)) \geq \max\{\max_j \tilde{\gamma}_{ij}(\|V_j^d(\phi_j)\|_{[-\theta,0]}), \tilde{\gamma}_i^u(|u_i|)\} \;\Rightarrow\; V_i(g_i(\phi,u_i)) \leq e^{-d_i} V_i(\phi_i(0)),$$

where $\tilde{\gamma}_{ij}, \tilde{\gamma}_i^u \in \mathcal{K}_\infty$.

In the following, we assume that the gains γ_{ij} are linear and we denote throughout the chapter $\gamma_{ij}(r) = \gamma_{ij} r$, $\gamma_{ij}, r \geq 0$. All the gains are collected in a matrix $\Gamma := (\gamma_{ij})_{n\times n}$, $i,j = 1,\dots,n$, which defines a map as in (1.12).

Since the gains are linear, the small-gain condition (1.15) is equivalent to

$$\Delta(\Gamma) < 1, \tag{5.34}$$

where Δ is the spectral radius of Γ, see [24, 98]. Note that $\Delta(\Gamma) < 1$ implies that there exists a vector $s \in \mathbb{R}^n$, $s > 0$ such that

$$\Gamma(s) < s. \tag{5.35}$$

Now, we state one of the main results: the ISS small-gain theorem for interconnected impulsive systems with time-delays and linear gains. We construct the Lyapunov-Razumikhin function and the gain of the overall system under a small-gain condition, which is here of the form

$$\Gamma(s) \not\geq \min\{e^{-\mu}, e^{-d-\mu}\}s, \forall s \in \mathbb{R}^n_+ \setminus \{0\} \;\Leftrightarrow\; \exists s \in \mathbb{R}^n_+ \setminus \{0\} : \; \Gamma(s) < \min\{e^{-\mu}, e^{-d-\mu}\}s, \tag{5.36}$$

where μ is from the dwell-time condition (5.7) and $d := \min_i d_i$.

Theorem 5.2.1. *Assume that each subsystem of (5.29) has an exponential ISS-Lyapunov-Razumikhin function with $c_i, d_i \in \mathbb{R}$, $d_i \neq 0$ and gains γ_i^u, γ_{ij}, where γ_{ij} are linear. Define $c := \min_i c_i$ and $d := \min_i d_i$. If for some $\mu > 0$ the operator Γ satisfies the small-gain condition* (5.36)*, then for all $\lambda > 0$ the whole system* (5.2) *is uniformly ISS over $\mathcal{S}[\mu,\lambda]$ and the exponential ISS-Lyapunov-Razumikhin function is given by*

$$V(x) := \max_i \{\tfrac{1}{s_i} V_i(x_i)\}, \tag{5.37}$$

where $s = (s_1,\dots,s_n)^T$ is from (5.36)*. The gains are given by*

$$\gamma_d(r) := \max\{e^d, 1\} \max_{k,j} \frac{1}{s_k} \gamma_{kj} s_j r,$$
$$\gamma_u(r) := \max\{e^d, 1\} \max_i \tfrac{1}{s_i} \gamma_i^u(r).$$

Proof. Let $0 \neq x = (x_1^T,\dots,x_n^T)^T$. We define $V(x)$ as in (5.37) and show that V is the exponential ISS-Lyapunov-Razumikhin function for the overall system. Note that V is locally Lipschitz continuous and satisfies (5.31), which can be easily checked. Let $I := \{i \in \{1,\dots,n\} |\; V(x) = \frac{1}{s_i} V_i(x_i) \geq \max_{j, j\neq i}\{\frac{1}{s_j}\,(V_j(x_j))\}\}$. Fix an $i \in I$.

We define the gains $\bar{\gamma}_d(r) := \max_{k,j} \frac{1}{s_k}\gamma_{kj}s_j r,\ \bar{\gamma}_u(r) := \max_i \frac{1}{s_i}\gamma_i^u(r),\ r>0$ and assume $V(x(t)) \geq \max\{\bar{\gamma}_d(\|V^d(x^t)\|_{[-\theta,0]}), \bar{\gamma}_u(|u(t)|)\}$. It follows

$$\begin{aligned} V_i(x_i(t)) &\geq s_i \max\{\max_{k,j} \frac{1}{s_k}\gamma_{kj}s_j\|V^d(x^t)\|_{[-\theta,0]}, \max_i \tfrac{1}{s_i}\gamma_i^u(|u(t)|)\} \\ &\geq \max\{\max_j \gamma_{ij}\|V_j^d(x_j^t)\|_{[-\theta,0]}, \gamma_i^u(|u_i(t)|)\}. \end{aligned}$$

Then, from (5.32) we obtain

$$\mathrm{D}^+V(x(t)) = \mathrm{D}^+\tfrac{1}{s_i}V_i(x_i(t)) \leq -\tfrac{1}{s_i}c_iV_i(x_i(t)) = -c_iV(x(t)).$$

We have shown that for $c = \min_i c_i$ the function V satisfies (4.5) with $\bar{\gamma}_d,\ \bar{\gamma}_u$.

Defining $d := \min_i d_i$ and using (5.33) it holds

$$\begin{aligned} V(g(x^t,u(t))) &= \max_i\{\tfrac{1}{s_i}V_i(g_i(x_1^t,\dots,x_n^t,u_i(t)))\} \\ &\leq \max_i\{\tfrac{1}{s_i}\max\{e^{-d_i}V_i(x_i(t)), \max_j \gamma_{ij}\|V_j^d(x_j^t)\|_{[-\theta,0]}, \gamma_i^u(|u_i(t)|)\}\} \\ &\leq \max_{i,j\ j\neq i}\{\tfrac{1}{s_i}e^{-d_i}s_iV(x(t)), \tfrac{1}{s_i}\gamma_{ij}s_j\|V^d(x^t)\|_{[-\theta,0]}, \tfrac{1}{s_i}\gamma_i^u(|u_i(t)|)\} \\ &\leq \max\{e^{-d}V(x(t)), \bar{\gamma}_d(\|V^d(x^t)\|_{[-\theta,0]}), \bar{\gamma}_u(|u(t)|)\}. \end{aligned}$$

Defining $\tilde{\gamma}_d(r) := e^d\bar{\gamma}_d(r),\ \tilde{\gamma}_u(r) := e^d\bar{\gamma}_u(r)$ and assuming that it holds $V(x(t)) \geq \max\{\tilde{\gamma}_d(\|V^d(x^t)\|_{[-\theta,0]}), \tilde{\gamma}_u(|u(t)|)\}$, it follows

$$\begin{aligned} V(g(x^t,u(t))) &\leq \max\{e^{-d}V(x(t)), \bar{\gamma}_d(\|V^d(x^t)\|_{[-\theta,0]}), \bar{\gamma}_u(|u(t)|)\} \\ &= \max\{e^{-d}V(x(t)), e^{-d}\tilde{\gamma}_d(\|V^d(x^t)\|_{[-\theta,0]}), e^{-d}\tilde{\gamma}_u(|u(t)|)\} \\ &\leq e^{-d}V(x(t)), \end{aligned}$$

i.e., V satisfies the condition (5.5) with $\tilde{\gamma}_d,\ \tilde{\gamma}_u$. Now, define $\gamma_d(r) := \max\{\bar{\gamma}_d(r), \tilde{\gamma}_d(r)\}$ and $\gamma_u(r) := \max\{\bar{\gamma}_u(r), \tilde{\gamma}_u(r)\}$. Then, V satisfies (4.5) and (5.5) with $\gamma_d,\ \gamma_u$.

By (5.36) it holds

$$\gamma_d(r) = \max\{\bar{\gamma}_d(r), e^d\bar{\gamma}_d(r)\} < \max\{\min\{e^{-\mu}, e^{-d-\mu}\}, \min\{e^{d-\mu}, e^{-\mu}\}\}r = e^{-\mu}r.$$

All conditions of Definition 5.1.3 are satisfied and V is the exponential ISS-Lyapunov-Razumikhin function of the whole system of the form (5.2). We can apply Theorem 5.1.8 and the whole system is uniformly ISS over $\mathcal{S}[\mu,\lambda]$. □

5.2.2 The Lyapunov-Krasovskii approach

Let us consider n interconnected impulsive subsystems of the form

$$\begin{aligned} \dot{x}_i(t) &= f_i(x_1^t,\dots,x_n^t,u_i(t)),\ t \neq t_k, \\ x_i^t &= g_i((x_1^t)^-,\dots,(x_n^t)^-,u_i^-(t)),\ t = t_k, \end{aligned} \tag{5.38}$$

$k \in \mathbb{N},\ i = 1,\dots,n$, where we make the same assumptions as in the previous subsections and the functionals g_i are now maps from $PC\left([-\theta,0];\mathbb{R}^{N_1}\right) \times \ldots \times PC\left([-\theta,0];\mathbb{R}^{N_n}\right) \times \mathbb{R}^{M_i}$ into $PC\left([-\theta,0];\mathbb{R}^{N_i}\right)$.

Note that the ISS property for systems of the form (5.38) is the same as in (5.30). If we define $N,\ m,\ x,\ u,\ f$ and g as before, then (5.38) becomes a system of the form (5.19). Exponential ISS-Lyapunov-Krasovskii functionals of systems with several inputs are as follows:

Assume that for each subsystem of the interconnected system (5.38) there is a given functional $V_i : PC\left([-\theta,0]\,;\mathbb{R}^{N_i}\right) \to \mathbb{R}_+$, which is locally Lipschitz continuous and positive definite. For $i=1,\dots,n$ the functional V_i is an *exponential ISS-Lyapunov-Krasovskii functional of the ith subsystem of* (5.38), if there exist $\psi_{1i},\psi_{2i}\in\mathcal{K}_\infty,\ \gamma_i\in\mathcal{K}\cup\{0\},\ \gamma_{ij}\in\mathcal{K}_\infty\cup\{0\},\ \gamma_{ii}\equiv 0,\ i,j=1,\dots,n$ and scalars $c_i,d_i\in\mathbb{R}$ such that

$$\psi_{1i}(|\phi_i(0)|)\le V_i(\phi_i)\le\psi_{2i}(|\phi_i|_a),\ \forall\phi_i\in PC\left([-\theta,0]\,;\mathbb{R}^{N_i}\right)$$

and whenever $V_i(\phi_i)\ge\max\{\max_{j,j\neq i}\gamma_{ij}(V_j(\phi_j)),\gamma_i(|u_i|)\}$ holds, it follows

$$\mathrm{D}^+V_i(\phi_i,u_i)\le -c_iV_i(\phi_i) \tag{5.39}$$

and

$$V_i(g_i(\phi,u_i))\le\max\{e^{-d_i}V_i(\phi_i),\max_{j,j\neq i}\gamma_{ij}(V_j(\phi_j)),\gamma_i(|u_i|)\}, \tag{5.40}$$

for all $\phi_i\in PC\left([-\theta,0]\,;\mathbb{R}^{N_i}\right),\ \phi=(\phi_1^T,\dots,\phi_n^T)^T$ and $u_i\in\mathbb{R}^{M_i}$. A different formulation can be obtained by replacing (5.40) by

$$V_i(\phi_i)\ge\max\{\max_{j,j\neq i}\tilde\gamma_{ij}(V_j(\phi_j)),\tilde\gamma_i(|u_i|)\}\ \Rightarrow\ V_i(g_i(\phi,u_i))\le e^{-d_i}V_i(\phi_i),$$

where $\tilde\gamma_{ij},\tilde\gamma_i\in\mathcal{K}_\infty$.

Furthermore, we define the gain-matrix $\Gamma=(\gamma_{ij})_{n\times n}$ with $\gamma_{ii}\equiv 0$.

The next result is an ISS small-gain theorem for impulsive systems with time-delays using the Lyapunov-Krasovskii methodology. This theorem allows to construct an exponential ISS-Lyapunov-Krasovskii functional and the corresponding gain for the whole interconnection under a dwell-time and a small-gain condition.

Theorem 5.2.2. *Assume that each subsystem of* (5.38) *has an exponential ISS-Lyapunov-Krasovskii functional V_i with corresponding gains γ_i,γ_{ij}, where γ_{ij} are linear, and rate coefficients $c_i,\ d_i,\ d_i\neq 0$. Define $c:=\min_i c_i$ and $d:=\min_{i,j,\ j\neq i}\{d_i,-\ln(\frac{s_j}{s_i}\gamma_{ij})\}$. If Γ satisfies the small-gain condition* (1.15)*, then the impulsive system* (5.19) *is uniformly ISS over $\mathcal{S}[\mu,\lambda]$, $\mu,\lambda>0$ and the exponential ISS-Lyapunov-Krasovskii functional is given by*

$$V(\phi):=\max_i\{\tfrac{1}{s_i}V_i(\phi_i)\}, \tag{5.41}$$

where $s=(s_1,\dots,s_n)^T$ is from (5.35)*, $\phi\in PC\left([-\theta,0]\,;\mathbb{R}^N\right)$. The gain is given by $\gamma(r):=\max\{e^d,1\}\max_i\frac{1}{s_i}\gamma_i(r)$.*

Proof. Let $0\neq x^t=((x_1^t)^T,\dots,(x_n^t)^T)^T$ and let V be defined by $V(x^t):=\max_i\{\frac{1}{s_i}(V_i(x_i^t))\}$. Define the index set $I:=\{i\in\{1,\dots,n\}|\ \frac{1}{s_i}V_i(\phi_i)\ge\max_{j,j\neq i}\{\frac{1}{s_j}\left(V_j(\phi_j)\right)\}\}$, where $\phi_i\in PC\left([-\theta,0]\,;\mathbb{R}^{N_i}\right),\ \phi=(\phi_1^T,\dots,\phi_n^T)^T$. Fix an $i\in I$.

It can be easily checked that the condition (5.20) is satisfied.

We define $\bar{\gamma}(r) := \max_i \frac{1}{s_i}\gamma_i(r)$, $r > 0$ and assume $V(x^t) \geq \bar{\gamma}(|u(t)|)$. It follows from (5.35) that it holds

$$\begin{aligned} V_i(x_i^t) = s_i V(x^t) &\geq \max\{\max_j \gamma_{ij} s_j V(x^t), s_i\bar{\gamma}(|u(t)|)\} \\ &\geq \max\{\max_j \gamma_{ij} V_j(x_j^t), \gamma_i(|u_i(t)|)\}. \end{aligned}$$

Then, from (5.39) we obtain

$$\mathrm{D}^+V(x^t, u(t)) = \tfrac{1}{s_i} D^+V_i(x_i^t, u_i(t)) \leq -\tfrac{1}{s_i} c_i V_i(x_i^t) = -c_i V(x^t).$$

By definition of $c = \min_i c_i$, the function V satisfies (5.21) with $\bar{\gamma}$.

With $d := \min_{i,j,\ j\neq i}\{d_i, -\ln(\frac{s_j}{s_i}\gamma_{ij})\}$ and using (5.40) it holds

$$\begin{aligned} V(g(x^t, u(t))) &= \max_i\{\tfrac{1}{s_i} V_i(g_i(x_1^t, \dots, x_n^t, u_i(t)))\} \\ &\leq \max_i\{\tfrac{1}{s_i}\max\{e^{-d_i} V_i(x_i^t), \max_{j,j\neq i}\gamma_{ij}V_j(x_j^t), \gamma_i(|u_i(t)|)\}\} \\ &\leq \max_{i,j\ j\neq i}\{\tfrac{1}{s_i}e^{-d_i}s_iV(x^t), \tfrac{1}{s_i}\gamma_{ij}s_jV(x^t), \tfrac{1}{s_i}\gamma_i(|u_i(t)|)\} \\ &\leq \max\{e^{-d}V(x^t), \bar{\gamma}(|u(t)|)\}. \end{aligned}$$

Define $\tilde{\gamma}(r) := e^d\bar{\gamma}(r)$. If it holds $V(x^t) \geq \tilde{\gamma}(|u(t)|)$, it follows

$$V(g(x^t, u(t))) \leq \max\{e^{-d}V(x^t), \bar{\gamma}(|u(t)|)\} = \max\{e^{-d}V(x^t), e^{-d}\tilde{\gamma}(|u(t)|)\} \leq e^{-d}V(x^t)$$

for all x^t and u and V satisfies (5.22) with $\tilde{\gamma}$. Define $\gamma(r) := \max\{e^d, 1\}\max_i \frac{1}{s_i}\gamma_i(r)$, then we conclude that V is an exponential ISS-Lyapunov-Krasovskii functional for the overall system of the form (5.19) with rate coefficients c, d and gain γ. We can apply Theorem 5.1.11 and the overall system is uniformly ISS over $S[\mu, \lambda]$, $\mu, \lambda > 0$. □

In the next section, we give an example of a networked control system and we apply Theorem 5.2.1 to check whether the system has the ISS property.

5.3 Example

Networked control systems (NCS) are spatially distributed systems, where sensors, actuators and controllers communicate through a shared band-limited digital communication network. They are applied in mobile sensor networks, remote surgery, haptics collaboration over the Internet or automated highway systems and unmanned vehicles, see [47] and the references therein.

Stability analysis for such kinds of networks were performed in [125], [83] and [45], for example. Here, we consider a class of NCS given by an interconnection of linear systems with time-delays. The ith subsystem is described as follows:

$$\begin{aligned} \dot{x}_i(t) &= -a_i x_i(t) + \sum_{j,j\neq i} a_{ij} x_j(t-\tau_{ij}) + b_i\nu_i(t), \\ y_i(t) &= x_i(t) + \mu_i(t), \end{aligned} \tag{5.42}$$

where $x_i \in \mathbb{R}$, $i = 1, \ldots, n$, $\tau_{ij} \in [0, \theta]$ is a time-delay of the input from the other subsystems with maximum involved delay $\theta > 0$ and $\nu_i \in \mathrm{L}_\infty(\mathbb{R}_+, \mathbb{R})$ is an input disturbance. $y_i \in \mathbb{R}$ is the measurement of the system, disturbed by μ_i and $a_{ij}, a_i, b_i > 0$ are some parameters to describe the interconnection. $\{t_1, t_2, \ldots\}$ is a sequence of time instances at which measurements of x_i are sent. It is allowed to send only one measurement per each time instant. Between the sending of new measurements the estimate $\hat{x}_i$ of x_i is given by

$$\dot{\hat{x}}_i(t) = -a_i \hat{x}_i(t) + \sum_{j, j \neq i} a_{ij} \hat{x}_j(t - \tau_{ij}), \ t \notin \{t_1, t_2, \ldots\}. \tag{5.43}$$

At time t_k the node i_k gets access to the measurement y_{i_k} of x_{i_k} and all other nodes stay unchanged:

$$\hat{x}_i(t_k) = \begin{cases} y_{i_k}^-(t_k), & i = i_k, \\ \hat{x}_i^-(t_k), & i \neq i_k. \end{cases}$$

The estimation error of the ith subsystem is defined as in Section 3.2 by $e_i := \hat{x}_i - x_i$. The dynamics of e_i can be then given by the following impulsive system:

$$\dot{e}_i(t) = -a_i e_i(t) + \sum_{j, j \neq i} a_{ij} e_j(t - \tau_{ij}) - b_i \nu_i(t), \ t \neq t_k, k \in \mathbb{N}, \tag{5.44}$$

$$e_i(t_k) = \begin{cases} \mu_{i_k}^-(t_k), & i = i_k, \\ e_i^-(t_k), & i \neq i_k. \end{cases} \tag{5.45}$$

The decision which measurement of a subsystem will be sent, is performed using some protocol, see [83], for example.

To verify that the error dynamics of the whole interconnected system (5.44), (5.45) has the uniformly ISS property, we show that there exists an exponential ISS-Lyapunov-Razumikhin function for each subsystem and that the small-gain condition (5.36) and the dwell-time condition (5.7) are satisfied.

At first, we will define an ISS-Lyapunov-Razumikhin function candidate for each subsystem. Consider the function $V_i(e_i) := |e_i|$. If $t = t_k$, then $V_i(g_i(e_i)) \leq \max\{|e_i|, |\mu_i|\} = \max\{e^{-d_i} V_i(e_i), |\mu_i|\}$ with $d_i = 0$. Consider now the case $t \neq t_k$. If

$$|e_i| \geq \max\left\{ \max_{j, j \neq i} n \frac{a_{ij}}{a_i - \epsilon_i} \max_{t - \theta \leq s \leq t} V_j(e_j(s)), n \frac{b_i |\nu_i|}{a_i - \epsilon_i} \right\}, \ \epsilon_i \in [0, a_i),$$

then we have

$$\begin{aligned} D^+ V_i(e_i) =& (-a_i e_i + \sum_{j, j \neq i} a_{ij} e_j(t - \tau_{ij}) - b_i \nu_i) \cdot \operatorname{sign} e_i \\ \leq& - a_i |e_i| + \sum_{j, j \neq i} a_{ij} |e_j(t - \tau_{ij})| + b_i |\nu_i| \leq -a_i |e_i| + (a_i - \epsilon_i)|e_i| \\ =& - \epsilon_i |e_i| = -\epsilon_i V_i(e_i) =: -c_i V_i(e_i) \end{aligned}$$

Thus, the function $V_i(e_i) = |e_i|$ is an exponential ISS-Lyapunov-Razumikhin function for the ith subsystem with $c_i = \epsilon_i$, $d_i = 0$, $\gamma_{ij}(r) = n \frac{|a_{ij}|}{a_i - \epsilon_i} r$ and $\gamma_i(r) = \max\{1, n \frac{|b_i|}{a_i - \epsilon_i}\} r$.

To prove ISS of the whole error system, we need to check the dwell-time condition (5.7) and the small-gain condition (5.36), see Theorem 5.2.1. Let us check condition (5.7). We have $d = \min_i d_i = 0$, $c = \min_i c_i = \min_i \epsilon_i > 0$. Taking $0 < \lambda \leq c$ and any $\mu > 0$, the dwell-time condition is satisfied for any $t \geq s \geq 0$ and time sequence $\{t_k\}$:

$$-dN(t,s) - (c-\lambda)(t-s) = -(c-\lambda)(t-s) \leq 0 < \mu.$$

The fulfillment of the small gain condition (5.36) can be checked by slightly modifying the cycle condition: for all $(k_1, ..., k_p) \in \{1, ..., n\}^p$, where $k_1 = k_p$, it holds

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ ... \circ \gamma_{k_{p-1}k_p} < e^{-\mu}\,\mathrm{Id},$$

where in this example we can choose μ arbitrarily small. Let us check this condition for the following parameters: $n = 3$, $b_i = 1$, $\nu_i \equiv 2$, $\epsilon_i = 0.1$, $i = 1,2,3$; $\tau_{ij} = 0.03$, $i,j = 1,2,3$, $i \neq j$; $e(s) = (0.9; 0.3; 0.6)^T$, $s \in [-\theta, 0]$; $a_1 = 1$, $a_2 = 2$, $a_3 = 0.5$,

$$A := (a_{ij})_{3\times3} = \begin{pmatrix} 0 & 0.25 & 0.25 \\ 0.7 & 0 & 0.65 \\ 0.15 & 0.1 & 0 \end{pmatrix}.$$

The system uses a TOD-like protocol [83] and it sends measurements at time instants $t_k = 0.1k, k \in \mathbb{N}$.

The gain matrix Γ is then given by

$$\Gamma := (\gamma_{ij})_{3\times3} = \begin{pmatrix} 0 & 0.8333 & 0.8333 \\ 1.1053 & 0 & 1.0263 \\ 1.1250 & 0.7500 & 0 \end{pmatrix}.$$

It is easy to check that all the cycles are less than the identity function multiplied by e^{μ}, because μ can be chosen arbitrarily small. Thus, the cycle condition is satisfied and by application of Theorem 5.2.1 the error system (5.44), (5.45) is uniformly ISS. The trajectory of the Euclidean norm of the error is given in Figure 5.3.

It remains as an open research topic, not analyzed in this thesis, to investigate the influence of the input disturbances to the error systems in more detail as well as to investigate the influence to the error system of the usage of different protocols (see [83]).

In the next chapter, we investigate on the one hand the ISDS property of MPC for single and interconnected systems, and on the other hand we analyze the ISS property of MPC for single and interconnected TDS. The tools, presented in Chapter 2 and Chapter 4 are used for MPC.

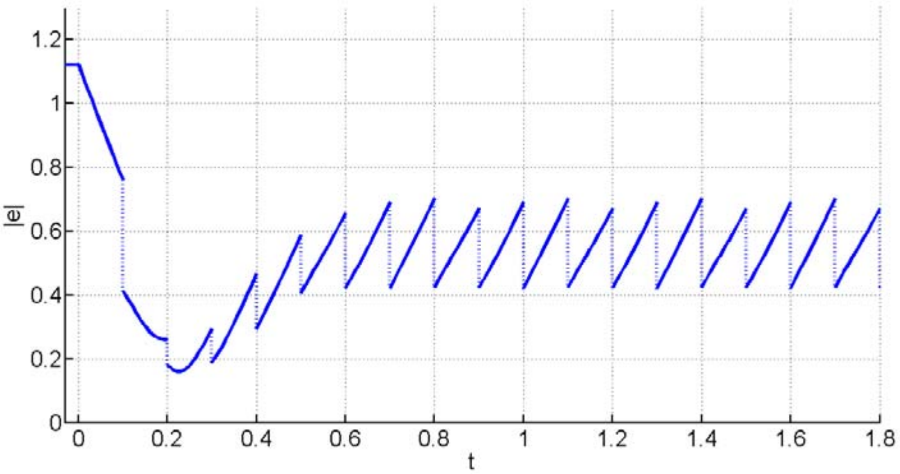

Figure 5.3: The trajectory of the Euclidean norm of the error of the networked control system.

Chapter 6

Model predictive control

In this chapter, we introduce the ISDS property for MPC of single and interconnected systems and we introduce the ISS property for MPC of single and interconnected TDS. We derive conditions which assure ISDS and ISS, respectively, for such systems using an MPC scheme and using the results of Chapter 2 and Chapter 4, respectively.

The approach of MPC started in the late 1970s and spread out in the 1990s by an increasing usage of automation processes in the industry. It has a wide range of applications, see the survey papers [89, 90].

The aim of MPC is to control a system to follow a certain trajectory or to steer the solution of a system into an equilibrium point under constraints and unknown disturbances. Additionally, the control should be optimal in view of defined goals, e.g., optimal regarding time or effort. We consider systems of the form (1.1) with disturbances,

$$\dot{x}(t) = f(x(t), w(t), u(t)), \tag{6.1}$$

where $w \in \mathcal{W} \subseteq L_\infty(\mathbb{R}_+, \mathbb{R}^P)$ is the unknown disturbance and $\mathcal{W}$ is a compact and convex set containing the origin. The input u is a measurable and essentially bounded control subject to input constraints $u \in \mathcal{U}$, where $\mathcal{U} \subseteq \mathbb{R}^m$ is a compact and convex set containing the origin in its interior. The function f has to satisfy the same conditions as in Chapter 1 to assure that a unique solution exists, which is denoted by $x(t; x_0, w, u)$ or $x(t)$ in short.

Note that the majority of the works regarding MPC consider discrete-time systems. One can derive a discrete-time system from a continuous-time system using sampled-data systems, see [84], for example.

The principle of MPC is the following: at sampling time $t = k\Delta,\ k \in \mathbb{N},\ \Delta > 0$, the current state of a system of the form (6.1) is measured, which is $x(t)$. This is used as an initial value to predict the trajectory of the system until the time $t + T$, where $T > 0$ is the finite prediction horizon, with an arbitrary control u. Let us first assume $w \equiv 0$. Then, the cost function defined by

$$J(x(t), u; t, T) := \int_t^{t+T} l(x(t'), u(t'))\mathrm{d}t' \tag{6.2}$$

will be minimized with respect to the control u subject to constraints $x \in \mathcal{X},\ u \in \mathcal{U}$, where $\mathcal{X} \subseteq \mathbb{R}^N$ is a compact and convex set containing the origin in its interior. The function $l : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}_+$ is the stage cost and it is a penalty of the distance of the state from the equilibrium point $f(0,0,0) = 0$ and a penalty of the control effort. A popular choice of the stage cost is $l(x,u) = |x|^2 + c_q|u|^2$, for example, where $c_q > 0$ is a weighting coefficient of the control. This control strategy is referred to as open-loop MPC [79, 92, 84].

A suitable approach to stabilize a system with implemented control obtained by an MPC scheme is that the terminal predicted state of the system should satisfy a terminal constraint, namely $x(t+T) \in \Omega$, where $\Omega \in \mathbb{R}^N$ is the terminal region. Then, we add the additional term $V_f(x(t+T))$ to the cost function of the MPC scheme, where $V_f : \Omega \to \mathbb{R}_+$.

The solution of this optimization problem is a control in feedback form $u^*(t) := \pi(t, x(t))$ and $u^*(t'),\ t' \in [t, t+\Delta]$ is applied to the system. We assume that the feedback $\pi \in \Pi$ is essentially bounded, locally Lipschitz in x and measurable in t. The set $\Pi \subseteq \mathbb{R}^m$ is assumed to be compact and convex containing the origin in its interior. At the sampling time $t+\Delta$, the state of the system will be measured and the procedure starts again moving the prediction horizon T. Overall, an optimal control $u^*(t)$ for $t \in \mathbb{R}^+$ is obtained.

A control law in feedback form or closed-loop control is a function $\pi : \mathbb{R}_+ \times \mathbb{R}^N \to \Pi$ and it is applied to the system by $u(\cdot) = \pi(\cdot, x(\cdot))$.

Illustrations and more details of this procedure as well as an overview about MPC can be found in the books [78, 9, 38] and the PhD theses [92, 84, 68], for example.

We are interested in stability of MPC. It was shown in [91] that the application of the control obtained by an MPC scheme to a system does not guarantee that a system without disturbances is asymptotically stable. For stability of a system in applications it is desired to analyze under which conditions stability of a system can be achieved using an MPC scheme. An overview about existing results regarding stability and MPC for systems without disturbances can be found in [81] and recent results are included in [92, 84, 68, 38]. To design stabilizing MPC controllers for nonlinear systems, a general framework can be found in [31].

On the one hand, to assure stability, there exist MPC schemes with stabilizing constraints, see [78, 31, 9, 92, 68], for example. On the other hand conditions were derived in [37, 39, 41] to assure asymptotic stability of unconstrained nonlinear MPC schemes.

Taking the unknown disturbance $w \in \mathcal{W}$ into account, MPC schemes which guarantee ISS were developed. First results can be found in [80] regarding ISS for MPC of nonlinear discrete-time systems. Furthermore, results using the ISS property with initial states from a compact set, namely regional-ISS, are given in [79, 92]. In [73, 69], an MPC scheme which guarantees ISS using the so-called *min-max approach* was given. The approach uses a closed-loop formulation of the optimization problem to compensate the effect of the unknown disturbance. It takes the worst case into account, i.e., it minimizes the cost function with respect to the control law π and maximizes the cost function with respect to the disturbance w:

$$\min_{\pi} \max_{w} J(x(t), \pi, w; t, T) := \min_{\pi} \max_{w} \int_t^{t+T} (l(x(t'), \pi(t', x(t')), w(t')))\mathrm{d}t'.$$

The usage of this approach is motivated by the circumstance that the solution of the open-loop MPC strategy can be extremely conservative and it can provide a small robust output admissible set (see [93], Definition 11, for example).

Stable MPC schemes for interconnected systems were investigated in [93, 92, 41], where in [93, 92] conditions to assure ISS of the whole system were derived and in [41] asymptotically stable MPC schemes without terminal constraints were provided. Note that in [41], the subsystems are not directly connected, but they exchange information over the network to control themselves according to state constraints.

However, for networks there exist several approaches for MPC schemes due to the interconnected structure of the system. This means that large-scale systems can be difficult to control with a centralized control structure, due to computational complexity or due to communication bandwidth limitations, for example, see [101]. A review of architectures for distributed and hierarchical MPC can be found in [101].

Considering TDS and MPC, recent results for asymptotically stable MPC schemes of single systems can be found in [29, 96, 95]. In these works, continuous-time TDS were investigated and conditions were derived, which guarantee asymptotic stability of a TDS using a Lyapunov-Krasovskii approach. Moreover, with the help of Lyapunov-Razumikhin arguments it was shown, how to determine the terminal cost and terminal region, and to compute a locally stabilizing controller.

For the implementation of MPC schemes in applications, numerical algorithms were developed, see [84, 68, 38] for example.

This chapter provides two new directions in MPC: at first, we combine the ISDS property with MPC for single and interconnected systems. Then, ISS of MPC for TDS is investigated. Conditions are derived such that single systems and whole networks with an optimal control obtained by an MPC scheme have the ISDS or ISS property, respectively. The result of Chapter 2, the ISDS small-gain theorem for networks, Theorem 2.2.2, is applied as well as the result of Chapter 4, namely Theorem 4.2.4.

The ISS property for MPC schemes of TDS has not been investigated so far as well as MPC for interconnected TDS. The approach of ISS for TDS provides the calculation of an optimal control for TDS with unknown disturbances, which was not done before. We use the min-max closed-loop MPC strategy and Lyapunov-Krasovskii arguments.

The advantage of the usage of ISDS over ISS for MPC is that the ISDS estimation takes only recent values of the disturbance into account due to the memory fading effect, see Chapter 2. In particular, if the disturbance tends to zero, then the ISDS estimation tends to zero. Moreover, the decay rate can be derived using ISDS-Lyapunov functions.

This information can be useful for applications of MPC. For example, consider two airplanes which are flying to each other. To avoid a collision at a certain time $\widetilde{T}$ we use MPC to calculate optimal controls for both planes under constraints and unknown disturbances. The disturbances are turbulences caused by winds, which influence the altitude of both planes. The constraint is that the altitude of the planes at time $\widetilde{T}$ are not equal taking the disturbances into account. Therefore, the ISDS or ISS estimation are used for checking the constraint. Assume that at the time $\tilde{t}$, where $\tilde{t}$ is a time before the time $\widetilde{T}$, the disturbances, i.e., the winds, are large. Closer to $\widetilde{T}$ we assume that the disturbances tend to zero. In practice, the information about this circumstance can be obtained from the weather forecast. Then, the following observations can be made:

The ISS estimation takes the supremum of the disturbances into account and for large disturbances the estimation is also large. Hence, the control taking into account the collision constraint using the ISS estimation, is conservative and more effort for the control is used than it is needed. In contrast to this, the control calculated by an MPC scheme under the collision constraint using the ISDS estimation does not need so much effort as the control using the ISS estimation, because we assume that the disturbances tend to zero. By the control with less effort, jet fuel could be saved, for example.

This chapter is organized as follows: Section 6.1 introduces the ISDS property for MPC of nonlinear systems. A result with respect to ISDS for MPC of single systems is provided in Subsection 6.1.1 and a result for interconnected systems is included in Subsection 6.1.2. In Section 6.2, time-delay systems and ISS for MPC are considered, where Subsection 6.2.1 investigates the ISS property for MPC of single systems and Subsection 6.2.2 analyzes ISS for MPC of networks.

6.1 ISDS and MPC

In this section, we combine ISDS and MPC for nonlinear single and interconnected systems. Conditions are derived, which assure ISDS of a system obtained by application of the control to the system (6.1), calculated by an MPC scheme.

6.1.1 Single systems

We consider systems of the form (6.1) and we use the min-max approach to calculate an optimal control: to compensate the effect of the disturbance w, we apply a *feedback control law* $\pi(t, x(t))$ to the system. An optimal control law is obtained by solving the finite horizon optimal control problem (FHOCP), which consists of minimization of the cost function J with respect to $\pi(t, x(t))$ and maximization of the cost function J with respect to the disturbance w:

Definition 6.1.1 (Finite horizon optimal control problem (FHOCP)). *Let* $1 > \varepsilon > 0$ *be given. Let* $T > 0$ *be the prediction horizon and* $u(t) = \pi(t, x(t))$ *be a feedback control law. The finite horizon optimal control problem for a system of the form* (6.1) *is formulated as*

$$\begin{aligned} &\min_{\pi} \max_{w} J(\bar{x}_0, \pi, w; t, T) \\ := \; &\min_{\pi} \max_{w} \; (1-\varepsilon) \int_t^{t+T} (l(x(t'), \pi(t', x(t'))) - l_w(w(t')))\mathrm{d}t' + V_f(x(t+T)) \end{aligned}$$

subject to

$$\begin{aligned}
&\dot{x}(t') = f(x(t'), w(t'), u(t')),\ x(t) = \bar{x}_0,\ t' \in [t, t+T],\\
&x \in \mathcal{X},\\
&w \in \mathcal{W},\\
&\pi \in \Pi,\\
&x(t+T) \in \Omega \subseteq \mathbb{R}^N,
\end{aligned}$$

where $\bar{x}_0 \in \mathbb{R}^N$ is the initial value of the system at time t, the terminal region Ω is a compact and convex set with the origin in its interior and $\pi(t, x(t))$ is essentially bounded, locally Lipschitz in x and measurable in t. $l - l_w$ is the stage cost, where $l : \mathbb{R}^N \times \mathbb{R}^m \to \mathbb{R}_+$ penalizes the distance of the state from the equilibrium point 0 of the system and it penalizes the control effort. $l_w : \mathbb{R}^P \to \mathbb{R}_+$ penalizes the disturbance, which influences the systems behavior. l and l_w are locally Lipschitz continuous with $l(0,0) = 0$, $l_w(0) = 0$, and $V_f : \Omega \to \mathbb{R}_+$ is the terminal penalty.

The FHOCP will be solved at the sampling instants $t = k\Delta,\ k \in \mathbb{N},\ \Delta \in \mathbb{R}_+$. The optimal solution is denoted by $\pi^*(t', x(t'); t, T)$ and $w^*(t')$, $t' \in [t, t+T]$. The optimal cost function is denoted by $J^*(\bar{x}_0, \pi^*, w^*; t, T)$. The control input to the system (6.1) is defined in the usual receding horizon fashion as

$$u(t') = \pi^*(t', x(t'); t, T),\ t' \in [t, t+\Delta].$$

In the following, we need some definitions:

Definition 6.1.2.
- *A feedback control π is called a feasible solution of the FHOCP at time t, if for a given initial value $\bar{x}_0$ at time t the feedback $\pi(t', x(t'))$, $t' \in [t, t+T]$ controls the state of the system (6.1) into Ω at time $t+T$, i.e., $x(t+T) \in \Omega$, for all $w \in \mathcal{W}$.*
- *A set $\Omega \subseteq \mathbb{R}^N$ is called positively invariant, if for all $x_0 \in \Omega$ a feedback control π keeps the trajectory of the system (6.1) in Ω, i.e.,*

$$x(t; x_0, w, \pi) \in \Omega,\ \forall t \in (0, \infty),$$

for all $w \in \mathcal{W}$.

To prove that the system (6.1) with the control obtained by solving the FHOCP has the ISDS property, we need the following:

Assumption 6.1.3. *1. There exist functions $\alpha_l, \alpha_w \in \mathcal{K}_\infty$, where α_l is locally Lipschitz continuous such that*

$$\begin{aligned}
l(x, \pi) &\geq \alpha_l(|x|),\ x \in \mathcal{X},\ \pi \in \Pi,\\
l_w(w) &\leq \alpha_w(|w|),\ w \in \mathcal{W}.
\end{aligned}$$

2. *The FHOCP in Definition 6.1.1 admits a feasible solution at the initial time* $t = 0$*.*

3. *There exists a controller* $u(t) = \pi(t, x(t))$ *such that the system* (6.1) *has the ISDS property.*

4. *For each* $1 > \varepsilon > 0$ *there exists a locally Lipschitz continuous function* $V_f(x)$ *such that the terminal region* Ω *is a positively invariant set and we have*

$$V_f(x) \leq \eta(|x|), \ \forall x \in \Omega, \tag{6.3}$$

$$\dot{V}_f(x) \leq -(1-\varepsilon) l(x,\pi) + (1-\varepsilon) l_w(w), \quad f.a.a. \ x \in \Omega, \tag{6.4}$$

where $\eta \in \mathcal{K}_\infty$*,* $w \in \mathcal{W}$ *and* $\dot{V}_f$ *denotes the derivative of* V_f *along the solution of system* (6.1) *with the control* $u \equiv \pi$ *from point 3. of this assumption.*

5. *For each sufficiently small* $\varepsilon > 0$ *it holds*

$$(1-\varepsilon) \int_t^{t+T} l(x(t'), \pi(t', x(t'))) \mathrm{d}t' \geq \tfrac{|x(t)|}{1+\varepsilon}. \tag{6.5}$$

6. *The optimal cost function* $J^*(\bar{x}_0, \pi^*, w^*; t, T)$ *is locally Lipschitz continuous.*

Remark 6.1.4. *In [92], it is discussed that a different stage cost, for example by the definition of* $l_s := l - l_w$*, can be used for the FHOCP. In view of stability, the stage cost* l_s *has to fulfill some additional assumptions, see [92], Chapter 3.4.*

Remark 6.1.5. *The assumption* (6.5) *is needed to assure that the cost function satisfies the lower estimation in* (2.2)*. However, we did not investigated whether this condition is restrictive or not. In case of discrete-time systems and the according cost function, the assumption* (6.5) *is not necessary, see the proofs in [80, 79, 92, 73, 69].*

The following theorem establishes ISDS of the system (6.1), using the optimal control input $u \equiv \pi^*$ obtained from solving the FHOCP.

Theorem 6.1.6. *Consider a system of the form* (6.1)*. Under Assumption 6.1.3, the system resulting from the application of the predictive control strategy to the system, namely* $\dot{x}(t) = f(x(t), w(t), \pi^*(t, x(t))), \ t \in \mathbb{R}_+, \ x(0) = x_0$*, possesses the ISDS property.*

Remark 6.1.7. *Note that the gains and the decay rate of the definition of the ISDS property, Definition 2.1.2, can be calculated using Assumption 6.1.3, as it is partially displayed in the following proof.*

Proof. We show that the optimal cost function $J^*(\bar{x}_0, \pi^*, w^*; t, T) =: V(\bar{x}_0)$ is an ISDS-Lyapunov function, following the steps:

- the control problem admits a feasible solution π for all times $t > 0$,
- $J^*(\bar{x}_0, \pi^*, w^*; t, T)$ satisfies the conditions (2.2) and (2.3).

Then, by application of Theorem 2.1.6, the ISDS property follows.

Let us proof feasibility: we suppose that a feasible solution $\tilde{\pi}(t', x(t'))$, $t' \in [t, t+T]$ at time t exists. For $\Delta > 0$, we construct a control by

$$\hat{\pi}(t', x(t')) = \begin{cases} \tilde{\pi}(t', x(t')), & t' \in [t+\Delta, t+T], \\ \pi(t', x(t')), & t' \in (t+T, t+T+\Delta], \end{cases} \tag{6.6}$$

where π is the controller from Assumption 6.1.3, point 3. Since $\tilde{\pi}$ controls $x(t+\Delta)$ into $x(t+T) \in \Omega$ and Ω is a positively invariant set, $\pi(t', x(t'))$ keeps the systems trajectory in Ω for $t+T < t' \leq t+T+\Delta$ under the constraints of the FHOCP. This means that from the existence of a feasible solution for the time t, we have a feasible solution for the time $t+\Delta$. Since, we assume that a feasible solution for the FHOCP at the time $t=0$ exists (Assumption 6.1.3, point 2.), it follows that a feasible solution exists for every $t > 0$.

We replace $\tilde{\pi}$ in (6.6) by π^*. Then, it follows from (6.4) that it holds

$$\begin{aligned}
&J^*(\bar{x}_0, \pi^*, w^*; t, T+\Delta) \\
&\leq J(\bar{x}_0, \hat{\pi}, w^*; t, T+\Delta) \\
&= (1-\varepsilon)\int_t^{t+T} (l(x(t'), \pi^*(t', x(t'); t, T)) - l_w(w^*(t')))\mathrm{d}t' \\
&\quad + (1-\varepsilon)\int_{t+T}^{t+T+\Delta} (l(x(t'), \pi(t', x(t'))) - l_w(w^*(t')))\mathrm{d}t' \\
&\quad + V_f(x(t+T+\Delta)) \\
&= J^*(\bar{x}_0, \pi^*, w^*; t, T) - V_f(x(t+T)) + V_f(x(t+T+\Delta)) \\
&\quad + (1-\varepsilon)\int_{t+T}^{t+T+\Delta} (l(x(t'), \pi(t', x(t'))) - l_w(w^*(t')))\mathrm{d}t' \\
&\leq J^*(\bar{x}_0, \pi^*, w^*; t, T).
\end{aligned}$$

From this and with (6.3) it holds

$$J^*(\bar{x}_0, \pi^*, w^*; t, T) \leq J^*(\bar{x}_0, \pi^*, w^*; t, 0) = V_f(\bar{x}_0) \leq \eta(|\bar{x}_0|).$$

Now, with Assumption 6.1.3, point 5., we have

$$V(\bar{x}_0) \geq J(\bar{x}_0, \pi^*, 0; t, T) \geq (1-\varepsilon)\int_t^{t+T} l(x(t'), \pi^*(t', x(t')))\mathrm{d}t' \geq \tfrac{|\bar{x}_0|}{1+\varepsilon}.$$

This shows that J^* satisfies (2.2). Now, denote $\tilde{x}_0 := x(t+h)$. From $J^*(\bar{x}_0, \pi^*, w^*; t, T+\Delta) \leq J^*(\bar{x}_0, \pi^*, w^*; t, T)$ we get

$$\begin{aligned}
&(1-\varepsilon)\int_t^{t+h} (l(x(t'), \pi^*(t', x(t'); t, T)) - l_w(w^*(t')))\mathrm{d}t' + J^*(\tilde{x}_0, \pi^*, w^*; t+h, T+\Delta-h) \\
&\leq (1-\varepsilon)\int_t^{t+h} (l(x(t'), \pi^*(t', x(t'); t, T)) - l_w(w^*(t')))\mathrm{d}t' + J^*(\tilde{x}_0, \pi^*, w^*; t+h, T-h),
\end{aligned}$$

and therefore

$$J^*(\tilde{x}_0, \pi^*, w^*; t+h, T+\Delta-h) \leq J^*(\tilde{x}_0, \pi^*, w^*; t+h, T-h). \tag{6.7}$$

Now, we show that J^* satisfies the condition (2.3). Note that by Assumption 6.1.3, point 6., J^* is locally Lipschitz continuous. With (6.7) it holds

$$\begin{aligned}
&J^*(\bar{x}_0, \pi^*, w^*; t, T) \\
&= (1-\varepsilon)\int_t^{t+h} (l(x(t'), \pi^*(t', x(t'); t, T)) - l_w(w^*(t')))\mathrm{d}t' + J^*(\tilde{x}_0, \pi^*, w^*; t+h, T-h) \\
&\geq (1-\varepsilon)\int_t^{t+h} (l(x(t'), \pi^*(t', x(t'); t, T)) - l_w(w^*(t')))\mathrm{d}t' + J^*(\tilde{x}_0, \pi^*, w^*; t+h, T).
\end{aligned}$$

This leads to

$$\begin{aligned}
&\frac{J^*(\tilde{x}_0, \pi^*, w^*; t+h, T) - J^*(\bar{x}_0, \pi^*, w^*; t, T)}{h} \\
\leq\ & -\frac{1}{h}(1-\varepsilon)\int_t^{t+h} (l(x(t'), \pi^*(t', x(t'); t, T)) - l_w(w^*(t')))\mathrm{d}t'.
\end{aligned}$$

For $h \to 0$ and using the first point of Assumption 6.1.3 we obtain

$$\dot{V}(\bar{x}_0) \leq -(1-\varepsilon)\alpha_l(|\bar{x}_0|) + (1-\varepsilon)\alpha_w(|w^*|), \quad \text{f.a.a. } \bar{x}_0 \in \mathcal{X},\ \forall w \in \mathcal{W}.$$

By definition of $\gamma(r) := \eta(\alpha_l^{-1}(2\alpha_w(r)))$ and $g(r) := \frac{1}{2}\alpha_l(\eta^{-1}(r)),\ r \geq 0$ this implies

$$V(\bar{x}_0) > \gamma(|w^*|) \ \Rightarrow\ \dot{V}(\bar{x}_0) \leq -(1-\varepsilon)g(V(\bar{x}_0)),$$

where the function g is locally Lipschitz continuous. We conclude that J^* is an ISDS-Lyapunov function for the system

$$\dot{x}(t) = f(x(t), w(t), \pi^*(t, x(t)))$$

and by application of Theorem 2.1.6 the system has the ISDS property. $\square$

In the next subsection, we transform the analysis of ISDS for MPC of single systems to interconnected systems.

6.1.2 Interconnected systems

We consider interconnected systems with disturbances of the form

$$\dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), w_i(t), u_i(t)),\ i = 1, \ldots, n, \tag{6.8}$$

where $u_i \in \mathbb{R}^{M_i}$, measurable and essentially bounded, are the control inputs and $w_i \in \mathbb{R}^{P_i}$ are the unknown disturbances. We assume that the states, disturbances and inputs fulfill the constraints

$$x_i \in \mathcal{X}_i,\ w_i \in \mathcal{W}_i,\ u_i \in \mathcal{U}_i,\ i = 1, \ldots, n,$$

where $\mathcal{X}_i \subseteq \mathbb{R}^{N_i},\ \mathcal{W}_i \subseteq \mathrm{L}_\infty(\mathbb{R}_+, \mathbb{R}^{P_i})$ and $\mathcal{U}_i \subseteq \mathbb{R}^{M_i}$ are compact and convex sets containing the origin in their interior.

Now, we are going to determine an MPC scheme for interconnected systems. An overview of existing distributed and hierarchical MPC schemes can be found in [101]. The used scheme in this thesis is inspired by the min-max approach for single systems as in Definition 6.1.1, see [73, 69].

At first, we determine the cost function of the ith subsystem by

$$\begin{aligned}&J_i(\bar{x}_i^0,(x_j)_{j\neq i},\pi_i,w_i;t,T)\\ :=&\ (1-\varepsilon_i)\int_t^{t+T}(l_i(x_i(t'),\pi_i(t',x(t')))-(l_w)_i(w_i(t'))-\sum_{j\neq i}l_{ij}(x_j(t'))\mathrm{d}t'+(V_f)_i(x_i(t+T)),\end{aligned}$$

where $1>\varepsilon_i>0$, $\bar{x}_i^0\in\mathcal{X}_i$ is the initial value of the ith subsystem at time t and $\pi_i\in\Pi_i$ is a feedback, essentially bounded, locally Lipschitz in x and measurable in t, where $\Pi_i\subseteq\mathbb{R}^{M_i}$ is a compact and convex set containing the origin in its interior. $l_i-(l_w)_i-\sum l_{ij}$ is the stage cost, where $l_i:\mathbb{R}^{N_i}\times\mathbb{R}^{M_i}\to\mathbb{R}_+$. $(l_w)_i:\mathbb{R}^{P_i}\to\mathbb{R}_+$ penalizes the disturbance and $l_{ij}:\mathbb{R}^{N_j}\to\mathbb{R}_+$ penalizes the internal input for all $j=1,\dots,n,\ j\neq i$. $l_i,\ (l_w)_i$ and l_{ij} are locally Lipschitz continuous functions with $l_i(0,0)=0,\ (l_w)_i(0)=0,\ l_{ij}(0)=0$, and $(V_f)_i:\Omega_i\to\mathbb{R}_+$ is the terminal penalty of the ith subsystem, $\Omega_i\subseteq\mathbb{R}^{N_i}$.

In contrast to single systems we add the terms $l_{ij}(x_j),\ j\neq i$ to the cost function due to the interconnected structure of the subsystems. Here, two problems arise: the formulation of an optimal control problem for each subsystem and the calculation/determination of the internal inputs $x_j,\ j\neq i$.

We conserve the minimization of J_i with respect to π_i and the maximization of J_i with respect to w_i as in Definition 6.1.1 for single systems. In the spirit of ISS/ISDS, which treat the internal inputs as "disturbances", we maximize the cost function with respect to $x_j,\ j\neq i$ (worst-case approach). Since we assume that $x_j\in\mathcal{X}_j$, we get an optimal solution $\pi_i^*,w_i^*,x_j^*,\ j\neq i$ of the control problem.

The drawbacks of this approach are that, on the one hand, we do not use the systems equations (6.8) to predict $x_j,\ j\neq i$ and, on the other hand, the computation of the optimal solution could be numerically inefficient, especially if the number of subsystems n is "huge" or/and the sets $\mathcal{X}_i$ are "large". Moreover, taking into account the worst-case approach, the maximization over x_j, the obtained optimal control π_i^* for each subsystem could be extremely conservative, which leads to extremely conservative ISS or ISDS estimations.

To avoid these drawbacks of the maximization of J_i with respect to $x_j,\ j\neq i$, one could use the systems equations (6.8) to predict $x_j,\ j\neq i$ instead.

A numerically efficient way to calculate the optimal solutions π_i^*,w_i^* of the subsystems is a parallel calculation. Due to interconnected structure of the system the information about systems states of the subsystems should be exchanged. But this exchange of information causes that an optimal solution π_i^*,w_i^* could not be calculated. To the best of our knowledge, no theorem is proved that provides the existence of an optimal solution of the optimal control problem using such a parallel strategy. We conclude that a parallel calculation can not help in our case.

Another approach of an MPC scheme for networks is inspired by the hierarchical MPC scheme in [97]. One could use the predictions of the internal inputs $x_j,\ j\neq i$ as follows: at sampling time $t=k\Delta,\ k\in\mathbb{N},\ \Delta>0$ all subsystems calculate the optimal solution iteratively.

This means that for the calculation the optimal solution for the ith subsystem, the currently "optimized" trajectories of the subsystems $1, \ldots, i-1$ will be used, denoted by $x_p^{opt,k\Delta},\ p = 1, \ldots, i-1$, and the "optimal" trajectories of the subsystems $i+1, \ldots, n$ of the optimization at sampling time $t = (k-1)\Delta$ will be used, denoted by $x_p^{opt,(k-1)\Delta},\ p = i+1, \ldots, n$.

The advantage of this approach would be that the optimal solution is not that much conservative as the min-max approach and the calculation of the optimal solution could be performed in a numerically efficient way, due to the usage of the model to predict the "optimal" trajectories and that the maximization over $x_j,\ j \neq i$ will be avoided. The drawback is that the optimal cost function of each subsystem depends on the trajectories $x_j^{opt,\cdot},\ j \neq i$ using this hierarchical approach. Then, to the best of our knowledge, it is not possible to show that the optimal cost functions are ISDS-Lyapunov functions of the subsystems, which is a crucial step for proving ISDS of a subsystem or the whole network, because no helpful estimations for the Lyapunov function properties can be performed due to the dependence of the optimal cost functions of the trajectories $x_j^{opt,\cdot},\ j \neq i$.

The FHOCP for the ith subsystem reads as follows:

$$\min_{\pi_i} \max_{w_i} \max_{(x_j)_{j \neq i}} J_i(\bar{x}_i^0, (x_j)_{j \neq i}, \pi_i, w_i; t, T)$$

subject to

$$\begin{aligned}
&\dot{x}_i(t') = f_i(x_1(t'), \ldots, x_n(t'), w_i(t'), u_i(t')),\ t' \in [t, t+T],\\
&x_i(t) = \bar{x}_i^0,\\
&x_j \in \mathcal{X}_j,\ j = 1, \ldots, n,\\
&w_i \in \mathcal{W}_i,\\
&\pi_i \in \Pi_i,\\
&x_i(t+T) \in \Omega_i \subseteq \mathbb{R}^{N_i},
\end{aligned}$$

where the terminal region Ω_i is a compact and convex set with the origin in its interior.

The resulting optimal control of each subsystem is a feedback control law, i.e., $u_i^*(t) = \pi_i^*(t, x(t))$, where $x = (x_1^T, \ldots, x_n^T)^T \in \mathbb{R}^N$, $N = \sum_i N_i$ and $\pi_i^*(t, x^{*i}(t))$ is essentially bounded, locally Lipschitz in x and measurable in t, for all $i = 1, \ldots, n$.

To show that each subsystem and the whole system have the ISDS property using the mentioned distributed MPC scheme, we suppose for the ith subsystem of (6.8):

Assumption 6.1.8. *1. There exist functions $\alpha_i^l, \alpha_i^w, \alpha_{ij} \in \mathcal{K}_\infty,\ j = 1, \ldots, n,\ j \neq i$ such that*

$$\begin{aligned}
l_i(x_i, \pi_i) &\geq \alpha_i^l(|x_i|),\ x_i \in \mathcal{X}_i,\ \pi_i \in \Pi_i,\\
(l_w)_i(w_i) &\leq \alpha_i^w(|w_i|),\ w_i \in \mathcal{W}_i,\\
l_{ij}(x_j) &\leq \alpha_{ij}(V_j(x_j)),\ x_j \in \mathcal{X}_j,\ j = 1, \ldots, n,\ j \neq i.
\end{aligned}$$

2. The FHOCP admits a feasible solution at the initial time $t = 0$.

3. *There exists a controller* $u_i(t) = \pi_i(t, x(t))$ *such that the* i*th subsystem of* (6.8) *has the ISDS property.*

4. *For each* $1 > \varepsilon_i > 0$ *there exists a locally Lipschitz continuous function* $(V_f)_i(x_i)$ *such that the terminal region* Ω_i *is a positively invariant set and we have*

$$(V_f)_i(x_i) \leq \eta_i(|x_i|), \ \forall x_i \in \Omega_i,$$
$$(\dot{V}_f)_i(x_i) \leq -(1-\varepsilon_i) l_i(x_i, \pi_i) + (1-\varepsilon_i)(l_w)_i(w_i) + (1-\varepsilon_i) \sum_{j \neq i} l_{ij}(x_j),$$

for almost all $x_i \in \Omega_i$*, where* $\eta_i \in \mathcal{K}_\infty$*,* $w_i \in \mathcal{W}_i$ *and* $(\dot{V}_f)_i$ *denotes the derivative of* $(V_f)_i$ *along the solution of the* i*th subsystem of* (6.8) *with the control* $u_i \equiv \pi_i$ *from point 3. of this assumption.*

5. *For each sufficiently small* $\varepsilon_i > 0$ *it holds*

$$(1-\varepsilon_i) \int_t^{t+T} l_i(x_i(t'), \pi_i^*(t', x(t'))) - \sum_{j \neq i} l_{ij}(x_j(t')) \mathrm{d}t' \geq \tfrac{|x(t)|}{1+\varepsilon_i} \tag{6.9}$$

6. *The optimal cost function* $J_i^*(\bar{x}_i^0, (x_j)^*_{j \neq i}, \pi_i^*, w_i^*; t, T)$ *is locally Lipschitz continuous.*

Now, we can state that each subsystem possesses the ISDS property using the mentioned MPC scheme.

Theorem 6.1.9. *Consider an interconnected system of the form* (6.8)*. Let Assumption 6.1.8 be satisfied for each subsystem. Then, each subsystem resulting from the application of the control obtained by the FHOCP for each subsystem to the system, namely*

$$\dot{x}_i(t) = f_i(x_1(t), \ldots, x_n(t), w_i(t), \pi_i^*(t, x(t))), \ t \in \mathbb{R}_+, \ x_i^0 = x_i(0),$$

possesses the ISDS property.

Proof. Consider the ith subsystem. We show that the optimal cost function $V_i(\bar{x}_i^0) := J_i^*(\bar{x}_i^0, (x_j)^*_{j \neq i}, \pi_i^*, w_i^*; t, T)$ is an ISDS-Lyapunov function for the ith subsystem. We abbreviate $x_j = (x_j)^*_{j \neq i}$.

By following the steps of the proof of Theorem 6.1.6, we conclude that there exists a feasible solution for all times $t > 0$ and that by (6.9) the functional $V_i(\bar{x}_i^0)$ satisfies the condition

$$\tfrac{|\bar{x}_i^0|}{(1+\varepsilon_i)} \leq V_i(\bar{x}_i^0) \leq \eta_i(|\bar{x}_i^0|),$$

using $|\bar{x}_0| \geq |\bar{x}_i^0|$. Note that by Assumption 6.1.8, point 6., J_i^* is locally Lipschitz continuous. We have that it holds

$$\dot{V}_i(\bar{x}_i^0) \leq -(1-\varepsilon_i)\alpha_i^l(\eta_i^{-1}(V_i(\bar{x}_i^0))) + (1-\varepsilon_i)\alpha_i^w(|w_i^*|) + (1-\varepsilon_i)\sum_{j \neq i} \alpha_{ij}(V_j((\bar{x}_j^0)))$$

and equivalently

$$\dot{V}_i(\bar{x}_i^0) \leq -(1-\varepsilon_i)\alpha_i^l(\eta_i^{-1}(V_i(\bar{x}_i^0))) + (1-\varepsilon_i)\max\{n\alpha_i^w(|w_i^*|), \max_{j \neq i} n\alpha_{ij}(V_j((\bar{x}_j^0)))\},$$

which implies

$$V_i(\bar{x}_i^0) > \max\{\gamma_i(|w_i^*|), \max_{j\neq i} \gamma_{ij}(V_j((\bar{x}_i^0)))\} \Rightarrow \dot{V}_i(\bar{x}_i^0) \leq -(1-\varepsilon_i) g_i(V_i(\bar{x}_i^0)),$$

for almost all $\bar{x}_i^0 \in \mathcal{X}_i$ and all $w_i^* \in \mathcal{W}_i$, where $\gamma_i(r) := \eta_i((\alpha_i^l)^{-1}(2n\alpha_i^w(r)))$, $\gamma_{ij}(r) := \eta_i((\alpha_i^l)^{-1}(2n\alpha_{ij}(r)))$ and $g_i(r) := \frac{1}{2}\alpha_i^l(\eta_i^{-1}(r))$, where g_i is locally Lipschitz continuous.

Since, i can be chosen arbitrarily, we conclude that each subsystem has an ISDS-Lyapunov function. It follows that each subsystem has the ISDS property. □

To investigate whether the whole system has the ISDS property, we collect all functions γ_{ij} in a matrix $\Gamma := (\gamma_{ij})_{n\times n}$, $\gamma_{ii} \equiv 0$, which defines a map as in (1.12).

Using the small-gain condition for Γ, the ISDS property for the whole system can be guaranteed:

Corollary 6.1.10. *Consider an interconnected system of the form* (6.8)*. Let Assumption 6.1.8 be satisfied for each subsystem. If Γ satisfies the small-gain condition* (1.15)*, then the whole system possesses the ISDS property.*

Proof. Each subsystem has an ISDS-Lyapunov function with gains γ_{ij}. This follows from Theorem 6.1.9. The matrix Γ satisfies the SGC and all assumptions of Theorem 2.2.2 are satisfied. It follows that with $x = (x_1^T, \ldots, x_n^T)^T$, $w = (w_1^T, \ldots, w_n^T)^T$ and $\pi^*(\cdot, x(\cdot)) = ((\pi_1^*(\cdot, x(\cdot)))^T, \ldots, (\pi_n^*(\cdot, x(\cdot)))^T)^T$, the whole system of the form

$$\dot{x}(t) = f(x(t), w(t), \pi^*(t, x(t)))$$

has the ISDS property. □

In the next section, we investigate the ISS property for MPC of TDS.

6.2 ISS and MPC of time-delay systems

Now, we introduce the ISS property for MPC of TDS. We derive conditions to assure that a single system, a subsystem of a network and the whole system possess the ISS property applying the control obtained by an MPC scheme for TDS.

6.2.1 Single systems

We consider systems of the form (4.1) with disturbances,

$$\begin{aligned} \dot{x}(t) &= f(x^t, w(t), u(t)),\ t \in \mathbb{R}_+, \\ x_0(\tau) &= \xi(\tau),\ \tau \in [-\theta, 0], \end{aligned} \tag{6.10}$$

where $w \in \mathcal{W} \subseteq L_\infty(\mathbb{R}_+, \mathbb{R}^P)$ is the unknown disturbance and $\mathcal{W}$ is a compact and convex set containing the origin. The input u is an essentially bounded and measurable control subject to input constraints $u \in \mathcal{U}$, where $\mathcal{U} \subseteq \mathbb{R}^{\Updownarrow}$ is a compact and convex set containing the origin in its interior. The function f has to satisfy the same conditions as in Chapter 4 to assure that a unique solution exists, which is denoted by $x(t;\xi,w,u)$ or $x(t)$ in short.

The aim is to find an (optimal) control u such that the system (6.10) has the ISS property.

Due to the presence of disturbances, we apply a feedback control structure, which compensates the effect of the disturbance. This means that we apply a *feedback control law* $\pi(t,x^t)$ to the system. In the rest of this section, we assume that $\pi(t,x^t)\in\Pi$ is essentially bounded, locally Lipschitz in x^t and measurable in t. The set $\Pi\subseteq\mathbb{R}^m$ is assumed to be compact and convex containing the origin in its interior. We obtain an MPC control law by solving the control problem:

Definition 6.2.1 (Finite horizon optimal control problem with time-delays (FHOCPTD))**.** *Let T be the prediction horizon and $\pi(t,x^t)$ be a feedback control law. The finite horizon optimal control problem with time-delays for a system of the form* (6.10) *is formulated as*

$$\min_{\pi}\max_{w} J(\bar{\xi},\pi,w;t,T):=\min_{\pi}\max_{w}\int_t^{t+T}(l(x(t'),\pi(t',x^{t'}))-l_w(w(t')))\mathrm{d}t'+V_f(x^{t+T})$$

subject to

$$\begin{aligned}
&\dot{x}(t')=f(x^{t'},w(t'),u(t')),\ t'\in[t,t+T],\\
&x(t+\tau)=\bar{\xi}(\tau),\ \tau\in[-\theta,0],\\
&x^{t'}\in\mathcal{X},\\
&w\in\mathcal{W},\\
&\pi\in\Pi,\\
&x^{t+T}\in\Omega\subseteq C([-\theta,0],\mathbb{R}^N),
\end{aligned}$$

where $\bar{\xi}\in C([-\theta,0],\mathbb{R}^N)$ is the initial function of the system at time t, the terminal region Ω and the state constraint set $\mathcal{X}\subseteq C([-\theta,0],\mathbb{R}^N)$ are compact and convex sets with the origin in their interior. $l-l_w$ is the stage cost, where $l:\mathbb{R}^N\times\mathbb{R}^m\to\mathbb{R}_+$ and $l_w:\mathbb{R}^P\to\mathbb{R}_+$ are locally Lipschitz continuous with $l(0,0)=0,\ l_w(0)=0$, and $V_f:\Omega\to\mathbb{R}_+$ is the terminal penalty.

The control problem will be solved at the sampling instants $t=k\Delta,\ k\in\mathbb{N},\ \Delta\in\mathbb{R}_+$. The optimal solution is denoted by $\pi^*(t',x^{t'};t,T)$ and $w^*(t')$, $t'\in[t,t+T]$ and the optimal cost functional is denoted by $J^*(\bar{\xi},\pi^*,w^*;t,T)$. The control input to the system (6.10) is defined in the usual receding horizon fashion as

$$u(t')=\pi^*(t',x^{t'};t,T),\ t'\in[t,t+\Delta].$$

Definition 6.2.2.

- *A feedback control π is called a feasible solution of the FHOCPTD at time t, if for a given initial function $\bar{\xi}$ at time t the feedback $\pi(t',x^{t'}),\ t'\in[t,t+T]$ controls the state of the system* (6.10) *into Ω at time $t+T$, i.e., $x^{t+T}\in\Omega$, for all $w\in\mathcal{W}$.*

- *A set $\Omega \subseteq C([-\theta,0],\mathbb{R}^N)$ is called positively invariant, if for all initial functions $\bar{\xi} \in \Omega$ a feedback control π keeps the trajectory of the system* (6.10) *in Ω, i.e.,*

$$x^t \in \Omega, \ \forall t \in (0,\infty),$$

for all $w \in \mathcal{W}$.

For the goal of this section, establishing ISS of TDS with the help of MPC, we need the following:

Assumption 6.2.3. *1. There exist functions $\alpha_l, \alpha_w \in \mathcal{K}_\infty$ such that*

$$l(\phi(0),\pi) \geq \alpha_l(|\phi|_a), \ \phi \in \mathcal{X}, \ \pi \in \Pi,$$
$$l_w(w) \leq \alpha_w(|w|), \ w \in \mathcal{W}.$$

2. The FHOCPTD in Definition 6.2.1 admits a feasible solution at the initial time $t = 0$.

3. There exists a controller $u(t) = \pi(t, x^t)$ such that the system (6.10) *has the ISS property.*

4. There exists a locally Lipschitz continuous functional $V_f(\phi)$ such that the terminal region Ω is a positively invariant set and for all $\phi \in \Omega$ we have

$$V_f(\phi) \leq \psi_2(|\phi|_a), \tag{6.11}$$
$$D^+V_f\,(\phi, w) \leq -l(\phi(0),\pi) + l_w(w), \tag{6.12}$$

where $\psi_2 \in \mathcal{K}_\infty$, $w \in \mathcal{W}$ and D^+V_f denotes the upper right-hand side derivate of the functional V along the solution of (6.10) *with the control $u \equiv \pi$ from point 3. of this assumption.*

5. There exists a $\mathcal{K}_\infty$ function ψ_1 such that for all $t > 0$ it holds

$$\int_t^{t+T} l(x(t'),\pi(t',x^{t'}))\mathrm{d}t' \geq \psi_1(|\bar{\xi}(0)|), \ \bar{\xi}(0) = x(t). \tag{6.13}$$

6. The optimal cost functional $J^*(\bar{\xi},\pi^*,w^*;t,T)$ is locally Lipschitz continuous.

Now, we can state a theorem that assures ISS of MPC for a single time-delay system with disturbances.

Theorem 6.2.4. *Let Assumption 6.2.3 be satisfied. Then, the system resulting from the application of the predictive control strategy to the system, namely $\dot{x}(t) = f(x^t, w(t), \pi^*(t,x^t))$, $t \in \mathbb{R}_+, \ x_0(\tau) = \xi(\tau), \ \tau \in [-\theta, 0]$, possesses the ISS property.*

Proof. The proof goes along the lines of the proof of Theorem 6.1.6 with according changes to time-delays and functionals, i.e., we show that the optimal cost functional $V(\bar{\xi}) := J^*(\bar{\xi},\pi^*,w^*;t,T)$ is an ISS-Lyapunov-Krasovskii functional.

For a feasible solution for all times $t > 0$, we suppose that a feasible solution $\pi(t', x^{t'}),\ t' \in [t, t+T]$ at time t exists. We construct a control by

$$\hat{\pi}(t', x^{t'}) = \begin{cases} \tilde{\pi}(t', x^{t'}), & t' \in [t+\Delta, t+T], \\ \pi(t', x^{t'}), & t' \in (t+T, t+T+\Delta], \end{cases} \tag{6.14}$$

where π is the controller from Assumption 6.2.3, point 3, and $\Delta > 0$. $\tilde{\pi}$ steers $x^{t+\Delta}$ into $x^{t+T} \in \Omega$ and Ω is a positively invariant set. This means that $\pi(t', x^{t'})$ keeps the system trajectory in Ω for $t + T < t' \leq t + T + \Delta$ under the constraints of the FHOCPTD. This implies that from the existence of a feasible solution for the time t, we have a feasible solution for the time $t + \Delta$. From Assumption 6.2.3, point 2., there exists a feasible solution for the FHOCPTD at the time $t = 0$ and it follows that a feasible solution exists for every $t > 0$.

Replacing $\tilde{\pi}$ in (6.14) by π^*, it follows from (6.12) that it holds

$$\begin{aligned}
&J^*(\bar{\xi}, \pi^*, w^*; t, T+\Delta) \\
&\leq J(\bar{\xi}, \hat{\pi}, w^*; t, T+\Delta) \\
&= \int_t^{t+T} (l(x(t'), \pi^*(t', x^{t'}; t, T)) - l_w(w^*(t')))\mathrm{d}t' + \int_{t+T}^{t+T+\Delta} (l(x(t'), \pi(t', x^{t'})) - l_w(w^*(t')))\mathrm{d}t' \\
&\quad + V_f(x^{t+T+\Delta}) \\
&= J^*(\bar{\xi}, \pi^*, w^*; t, T) - V_f(x^{t+T}) + V_f(x^{t+T+\Delta}) + \int_{t+T}^{t+T+\Delta} (l(x(t'), \pi(t', x^{t'})) - l_w(w^*(t')))\mathrm{d}t' \\
&\leq J^*(\bar{\xi}, \pi^*, w^*; t, T)
\end{aligned}$$

and with (6.11) this implies

$$J^*(\bar{\xi}, \pi^*, w^*; t, T) \leq J^*(\bar{\xi}, \pi^*, w^*; t, 0) = V_f(\bar{\xi}) \leq \psi_2(|\bar{\xi}|_a).$$

For the lower bound, it holds

$$V(\bar{\xi}) \geq J(\bar{\xi}, \pi^*, 0; t, T) \geq \int_t^{t+T} l(x(t'), \pi^*(t', x^{t'}))\mathrm{d}t'$$

and by (6.13) we have $V(\bar{\xi}) \geq \psi_1(|\bar{\xi}(0)|)$. This shows that J^* satisfies (4.10).

Now, we use the notation $x^t(\tau) := \bar{\xi}(\tau),\ \tau \in [-\theta + t, t]$. With $J^*(x^t, \pi^*, w^*; t, T+\Delta) \leq J^*(x^t, \pi^*, w^*; t, T)$ we have

$$\begin{aligned}
&\int_t^{t+h} (l(x(t'), \pi^*(t', x^{t'}; t, T)) - l_w(w^*(t')))\mathrm{d}t' + J^*(x^{t+h}, \pi^*, w^*; t+h, T+\Delta-h) \\
\leq &\int_t^{t+h} (l(x(t'), \pi^*(t', x^{t'}; t, T)) - l_w(w^*(t')))\mathrm{d}t' + J^*(x^{t+h}, \pi^*, w^*; t+h, T-h).
\end{aligned}$$

This implies

$$J^*(x^{t+h}, \pi^*, w^*; t+h, T+\Delta-h) \leq J^*(x^{t+h}, \pi^*, w^*; t+h, T-h). \tag{6.15}$$

Note that by Assumption 6.2.3, point 6., J^* is locally Lipschitz continuous. With (6.15) it holds

$$\begin{aligned} & J^*(x^t,\pi^*,w^*;t,T) \\ = & \int_t^{t+h} (l(x(t'),\pi^*(t',x^{t'};t,T)) - l_w(w^*(t')))\mathrm{d}t' + J^*(x^{t+h},\pi^*,w^*;t+h,T-h) \\ \geq & \int_t^{t+h} (l(x(t'),\pi^*(t',x^{t'};t,T)) - l_w(w^*(t')))\mathrm{d}t' + J^*(x^{t+h},\pi^*,w^*;t+h,T), \end{aligned}$$

which leads to

$$\begin{aligned} & \frac{J^*(x^{t+h},\pi^*,w^*;t+h,T) - J^*(x^t,\pi^*,w^*;t,T)}{h} \\ \leq & -\frac{1}{h}\int_t^{t+h} (l(x(t'),\pi^*(t',x^{t'};t,T)) - l_w(w^*(t')))\mathrm{d}t'. \end{aligned}$$

Let $h \to 0^+$ and using the first point of Assumption 6.2.3 we get

$$D^+V(x^t,w^*) \leq -\alpha_l(|x^t|_a) + \alpha_w(|w^*|).$$

By definition of $\chi(r) := \psi_2(\alpha_l^{-1}(2\alpha_w(r)))$ and $\alpha(r) := \frac{1}{2}\alpha_l(\psi_2^{-1}(r)),\ r \geq 0$ this implies

$$V(x^t) \geq \chi(|w^*|) \ \Rightarrow \ D^+V(x^t,w^*) \leq -\alpha(V(x^t)),$$

i.e., J^* satisfies the condition (4.11).

We conclude that J^* is an ISS-Lyapunov-Krasovskii functional for the system

$$\dot{x}(t) = f(x^t, w(t), \pi^*(t,x^t))$$

and by application of Theorem 4.1.7 the system has the ISS property. □

Now, we consider interconnections of TDS and provide conditions such that the whole network with an optimal control obtained from an MPC scheme has the ISS property.

6.2.2 Interconnected systems

We consider interconnected systems with time-delays and disturbances of the form

$$\dot{x}_i(t) = \tilde{f}_i\left(x_1^t,\dots,x_n^t,w_i(t),u_i(t)\right),\ i=1,\dots,n, \tag{6.16}$$

where $u_i \in \mathbb{R}^{M_i}$ are the essentially bounded and measurable control inputs and $w_i \in \mathbb{R}^{P_i}$ are the unknown disturbances. We assume that the states, disturbances and inputs fulfill the constraints

$$x_i \in \mathcal{X}_i,\ w_i \in \mathcal{W}_i,\ u_i \in \mathcal{U}_i,\ i=1,\dots,n,$$

where $\mathcal{X}_i \subseteq C([-\theta,0],\mathbb{R}^{N_i}),\ \mathcal{W}_i \subseteq \mathrm{L}_\infty(\mathbb{R}_+,\mathbb{R}^{P_i})$ and $\mathcal{U}_i \subseteq \mathbb{R}^{M_i}$ are compact and convex sets containing the origin in their interior.

We assume the same MPC strategy for interconnected TDS as in Subsection 6.1.2. The FHOCPTD for the ith subsystem of (6.16) reads as

$$\begin{aligned}
&\min_{\pi_i}\max_{w_i}\max_{(x_j)_{j\neq i}} J_i(\bar{\xi}_i,(x_j)_{j\neq i},\pi_i,w_i;t,T)\\
:=&\min_{\pi_i}\max_{w_i}\max_{(x_j)_{j\neq i}} \int_t^{t+T}(l_i(x_i(t'),\pi_i(t',x_i^{t'}))-(l_w)_i(w_i(t'))-\sum_{j\neq i} l_{ij}(x_j(t')))\mathrm{d}t'+(V_f)_i(x_i^{t+T})
\end{aligned}$$

subject to

$$\begin{aligned}
&\dot{x}_i(t')=f_i(x_1^{t'},\dots,x_n^{t'},w_i(t'),u_i(t')),\ t'\in[t,t+T],\\
&x_i(t+\tau)=\bar{\xi}_i(\tau),\ \tau\in[-\theta,0],\\
&x_j\in\mathcal{X}_j,\ j=1,\dots,n,\\
&w_i\in\mathcal{W}_i,\\
&\pi_i\in\Pi_i,\\
&x_i^{t+T}\in\Omega_i\subseteq C([-\theta,0],\mathbb{R}^{N_i}),
\end{aligned}$$

where $\bar{\xi}_i\in\mathcal{X}_i$ is the initial function of the ith subsystem at time t, the terminal region Ω is a compact and convex set with the origin in its interior. $\pi_i(t,x^t)$ is essentially bounded, locally Lipschitz in x and measurable in t and $\Pi_i\subseteq\mathbb{R}^{M_i}$ is a compact and convex sets containing the origin in its interior. $l_i-(l_w)_i-\sum l_{ij}$ is the stage cost, where $l_i:\mathbb{R}^{N_i}\times\mathbb{R}^{M_i}\to\mathbb{R}_+$. $(l_w)_i:\mathbb{R}^{P_i}\to\mathbb{R}_+$ penalizes the disturbance and $l_{ij}:\mathbb{R}^{N_j}\to\mathbb{R}_+$ penalizes the internal input for all $j=1,\dots,n,\ j\neq i$. l_i, $(l_w)_i$ and l_{ij} are locally Lipschitz continuous functions with $l_i(0,0)=0,\ (l_w)_i(0)=0,\ l_{ij}(0)=0$, and $(V_f)_i:\Omega_i\to\mathbb{R}_+$ is the terminal penalty of the ith subsystem.

We obtain an optimal solution $\pi_i^*,\ (x_j)_{j\neq i}^*,\ w_i^*$, where the control of each subsystem is a feedback control law, which depends on the current states of the whole system, i.e., $u_i(t)=\pi_i^*(t,x^t)$, where $x^t=((x_1^t)^T,\dots,(x_n^t)^T)^T\in C([-\theta,0],\mathbb{R}^N)$, $N=\sum_i N_i$.

For the ith subsystem of (6.16) we suppose:

Assumption 6.2.5. *1. There exist functions $\alpha_i^l,\alpha_i^w,\alpha_{ij}\in\mathcal{K}_\infty$, $j=1,\dots,n,\ j\neq i$ such that*

$$\begin{aligned}
l_i(\phi_i(0),\pi_i)&\geq\alpha_i^l(|\phi_i|_a),\ \phi_i\in C([-\theta,0],\mathbb{R}^{N_i}),\ \pi_i\in\Pi_i,\\
(l_w)_i(w_i)&\leq\alpha_i^w(|w_i|),\ w_i\in\mathcal{W}_i,\\
l_{ij}(\phi_j(0))&\leq\alpha_{ij}(V_j(\phi_j)),\ \phi_j\in C([-\theta,0],\mathbb{R}^{N_j}),\ j=1,\dots,n,\ j\neq i.
\end{aligned}$$

2. *The FHOCPTD admits a feasible solution at the initial time $t=0$.*

3. *There exists a controller $u_i(t)=\pi_i(t,x^t)$ such that the ith subsystem of* (6.16) *has the ISS property.*

4. *There exists a locally Lipschitz continuous functional $(V_f)_i(\phi_i)$ such that the terminal region Ω_i is a positively invariant set and for all $\phi_i \in \Omega_i$ we have*

$$(V_f)_i(\phi_i) \leq \psi_{2i}(|\phi_i|_a),$$
$$D^+(V_f)_i(\phi_i, w_i) \leq -l_i(\phi_i(0), \pi_i) + (l_w)_i(w_i) + \textstyle\sum_{j\neq i} l_{ij}(\phi_j(0)),$$

where $\psi_{2i} \in \mathcal{K}_\infty$, $\phi_j \in C([-\theta, 0], \mathbb{R}^{N_j})$, $j = 1, \ldots, n$ and $w_i \in \mathcal{W}_i$. $D^+(V_f)_i$ denotes the upper right-hand side derivate of the functional $(V_f)_i$ along the solution of the ith subsystem of (6.16) *with the control $u_i \equiv \pi_i$ from point 3. of this assumption.*

5. *For each i, there exists a $\mathcal{K}_\infty$ function ψ_{1i} such that for all $t > 0$ it holds*

$$\int_t^{t+T} l_i(x_i(t'), \pi_i(t', x^{t'}))\mathrm{d}t' \geq \psi_{1i}(|\bar{\xi}(0)|),\ \bar{\xi}(0) = x(t). \tag{6.17}$$

6. *The optimal cost functional $J_i^*(\bar{\xi}_i, (x_j)^*_{j\neq i}, \pi_i^*, w_i^*; t, T)$ is locally Lipschitz continuous.*

Now, we state that each subsystem of (6.16) has the ISS property by application of the optimal control obtained by the FHOCPTD.

Theorem 6.2.6. *Consider an interconnected system of the form* (6.16)*. Let Assumption 6.2.5 be satisfied for each subsystem. Then, each subsystem resulting from the application of the predictive control strategy to the system, namely $\dot{x}_i(t) = f_i(x_1^t, \ldots, x_n^t, w_i(t), \pi_i^*(t, x^t))$, $t \in \mathbb{R}_+$, $x_i^0(\tau) = \xi_i(\tau)$, $\tau \in [-\theta, 0]$, possesses the ISS property.*

Proof. Consider the ith subsystem. We show, that the optimal cost functional $V_i(\bar{\xi}_i) := J_i^*(\bar{\xi}_i, (x_j)^*_{j\neq i}, \pi_i^*, w_i^*; t, T)$ is an ISS-Lyapunov-Krasovskii functional for the ith subsystem. We abbreviate $x_j^t = ((x_j)^t_{j\neq i})^*$.

Following the lines of the proof of Theorem 6.2.4, we have that there exists a feasible solution of the ith subsystem for all times $t > 0$ and that the functional $V_i(\bar{\xi}_i)$ satisfies the condition

$$\psi_{1i}(|\bar{\xi}_i(0)|) \leq V_i(\bar{\xi}_i) \leq \psi_{2i}(|\bar{\xi}_i|_a),$$

using (6.17) and $|\bar{\xi}(0)| \geq |\bar{\xi}_i(0)|$. Note that by Assumption 6.2.5, point 6., J_i^* is locally Lipschitz continuous. We arrive that it holds

$$D^+V_i(x_i^t, w_i^*) \leq -\alpha_i^l(\psi_{2i}^{-1}(V_i(x_i^t))) + \alpha_i^w(|w_i^*|) + \sum_{j\neq i} \alpha_{ij}(V_j(x_j^t)).$$

This is equivalent to

$$D^+V_i(x_i^t, w_i^*) \leq -\alpha_i^l(\psi_{2i}^{-1}(V_i(x_i^t))) + \max\{n\alpha_i^w(|w_i^*|), \max_{j\neq i} n\alpha_{ij}(V_j(x_j^t))\},$$

which implies

$$V_i(x_i^t) \geq \max\{\tilde{\chi}_i(|w_i^*|), \max_{j\neq i} \tilde{\chi}_{ij}(V_j(x_j^t))\} \;\Rightarrow\; D^+V_i(x_i^t, w_i^*) \leq -\bar{\alpha}_i^l(V_i(x_i^t)),$$

where

$$\tilde{\chi}_i(r) := \psi_{2i}((\alpha_i^l)^{-1}(2n\alpha_i^w(r))),$$
$$\tilde{\chi}_{ij}(r) := \psi_{2i}((\alpha_i^l)^{-1}(2n\alpha_{ij}(r))),$$
$$\bar{\alpha}_i^l(r) := \tfrac{1}{2}\alpha_i^l(\psi_{2i}^{-1}(r)).$$

This can be shown for each subsystem and we conclude that each subsystem has an ISS-Lyapunov-Krasovskii functional. It follows that the ith subsystem is ISS in maximum formulation. □

We collect all functions $\tilde{\chi}_{ij}$ in a matrix $\Gamma := (\tilde{\chi}_{ij})_{n\times n}$, $\tilde{\chi}_{ii} \equiv 0$, which defines a map as in (1.12).

Using the small-gain condition for Γ, it follows from Theorem 6.2.6:

Corollary 6.2.7. *Consider an interconnected system of the form* (6.16). *Let Assumption 6.2.5 be satisfied for each subsystem. If* Γ *satisfies the small-gain condition* (1.15)*, then the whole system possesses the ISS property.*

Proof. We know from Theorem 6.2.6 that each subsystem of (6.16) has an ISS-Lyapunov-Krasovskii functional with gains $\tilde{\chi}_{ij}$. Since, the matrix Γ satisfies the SGC, all assumptions of Theorem 4.2.4 are satisfied and by Remark 4.2.5 the whole system of the form

$$\dot{x}(t) = f(x^t, w(t), \pi^*(t, x^t))$$

is ISS in maximum formulation, where $x^t = ((x_1^t)^T, \dots, (x_n^t)^T)^T$, $w = (w_1^T, \dots, w_n^T)^T$ and $\pi^*(t, x^t) = ((\pi_1^*(t, x^t))^T, \dots, (\pi_n^*(t, x^t))^T)^T$. □

The next chapter summarizes the thesis and open questions for future research activities are listed.

Chapter 7

Summary and Outlook

We have provided several tools for different kinds of systems to analyze them in view of stability, to design observers and to control them. This thesis has extended the analysis toolbox of nonlinear single and interconnected systems using Lyapunov methods.

The results can be used for a wide range of applications from different areas. Moreover, the theory presented here can be seen as a starting point for further investigations.

In this chapter, we summarize all main results of this thesis and propose several topics for future research activities as well as open questions.

7.1 ISDS

Considering networks of interconnected ISDS subsystems, we have shown that they possess the ISDS property, if the small-gain condition (1.15) is satisfied. In this case, we have provided explicit expressions for an ISDS-Lyapunov function and the corresponding rates and gains of the entire interconnection, which is Theorem 2.2.2. As an application of this result, we have investigated a system of interconnections with zero external input and we have derived decay rates of the subsystems and the entire system, see Corollary 2.2.3. An example with two systems taken from [36] compares the resulting estimates of the norm of a trajectory obtained by [36] and by (2.15), see Example 2.3.1. Another example (Example 2.3.2) with n interconnected ISDS systems has illustrated the application of our main result.

The ISDS property with its advantages over ISS is not deeply investigated and used in applications and only few works exist in the research literature according to this topic until now. The advantages of ISDS could be used in practice for a wide range of applications to obtain (economic) benefits by the analysis of dynamical systems. For example, in production networks, ISDS can help to design systems avoiding overdimensioned production lines or warehouses.

Possible research activities regarding ISDS can be, for example: the investigation of a local variant of ISDS (see [40]) for networks and applications to production networks, for example. A small-gain theorem using LISDS-Lyapunov functions of the subsystems similar to Theorem 2.1.6 can be proved and the question arises, if ρ, ρ_u in the definition of local ISDS

(LISDS) according to LISS are equal to ρ, ρ_u using LISDS-Lyapunov functions.

Similar to iISS (see [116, 1]), an integral variant of ISDS, namely iISDS, could be investigated. Since, the set of iISS systems is larger than the set of ISS systems, this is also expected for iISDS systems in comparison to ISDS systems. The benefits of iISDS over ISDS or ISS for applications and the characterization of iISDS by Lyapunov functions should be analyzed.

As ISS was investigated for networks of discrete-time systems in [53, 54], this could also be done for the ISDS property and similar theorems presented here, can be adapted for such class of systems. This can be used for MPC, for example. Furthermore, the ISDS property can be defined for time-delay systems and the influence of the presence of delays to the decay rate could be analyzed. Also, all the theory according to ISDS could be developed for TDS. This would increase the range of possible applications of ISDS.

We have transfered one of the advantage of ISDS over ISS, namely the memory fading effect, to observer design:

7.2 Observer and quantized output feedback stabilization

We have introduced the quasi-ISDS property for observers (Definition 3.1.2), which main advantage over the quasi-ISS property is the memory fading effect due to measurement disturbances. This was demonstrated in Example 3.1.3. A quasi-ISDS observer was designed (Theorem 3.1.8) using error ISDS-Lyapunov functions. Considering networks, we have shown how to design quasi-ISS/ISDS reduced-order observers for subsystems of interconnected systems in Theorem 3.2.2. They were used to design a quasi-ISS/ISDS reduced-order observer for the overall system under a small-gain condition (Theorem 3.2.3).

As an application, we have shown that quantized output feedback stabilization for a subsystem is achievable, under the assumptions that the subsystem possesses a quasi-ISS/ISDS reduced-order observer and a state feedback controller providing ISS/ISDS with respect to measurement errors (Proposition 3.3.3, point 1.). If this holds for all subsystems of the large-scale system and the small-gain condition is satisfied, then quantized output feedback stabilization is also achievable for the overall system (Proposition 3.3.3, point 2.). The obtained bounds can be improved by using dynamic quantizers. We have shown that asymptotic convergence can be achieved for each subsystem and for the overall system provided that a small-gain condition is satisfied.

In future, it would be interesting to study the design of nonlinear output feedback control or nonlinear observers to satisfy the small-gain condition. The application of the results in this thesis to the design of dynamic quantized interconnected control systems could be investigated. Also, the design of the observers introduced in this thesis for systems with time-delays is of interest for future research activities as far as time-delays occur in many real-world applications.

Moreover, it could be investigated, if the decay rate and the gains of the quasi-ISDS property can be directly obtained by the usage of a different error ISDS-Lyapunov function in opposite to the one used in Assumption 3.1.4 and adapting/improving the proof of

Theorem 3.1.8.

We have investigated another type of systems:

7.3 ISS for TDS

For networks of time-delay systems, we have proved two theorems: an (L)ISS-Lyapunov-Razumikhin and an ISS-Lyapunov-Krasovskii small-gain theorem. They provide a tool how to check whether a network of TDS has the (L)ISS property, using a small-gain condition and (L)ISS-Lyapunov-Razumikhin functions or ISS-Lyapunov-Krasovskii functionals, respectively. Furthermore, we have shown how to construct the (L)ISS-Lyapunov-Razumikhin function, the ISS-Lyapunov-Krasovskii functional and the corresponding gains of the whole system, see Theorem 4.2.1 and Theorem 4.2.4.

As an application of the results, we have considered a detailed scenario of a production network and we have analyzed it, using the presented tools in this chapter, see Section 4.3.

Further research activities could be the definition of ISDS for TDS and the investigation of its characterization by ISDS-Lyapunov-Razumikhin functions and ISDS-Lyapunov-Krasovskii functionals. The influence of the decay rate of the ISDS estimation by time-delays could be analyzed. The presented results of this work can be further extended and an ISDS small-gain theorem for networks of TDS could be proved.

For systems without time-delays, a converse ISS-Lyapunov theorem was proved in [119, 74]. This is not done yet for TDS using ISS-Lyapunov-Razumikhin functions or ISS-Lyapunov-Krasovskii functionals, respectively, and remains as an open question. Another topic to be analyzed is the equivalence of ISS and 0-GAS+GS, see [120] for systems without time-delays. Considering time-delays, this is not proved yet and new techniques should be developed for the proof. For further details, see [123, 122], for example.

In practice, considering production networks it can happen that machines within a plant will break down, for example. Using autonomous control methods, the network is stable despite the breakdowns. Due to the idea of autonomous control, it is not necessary to monitor the production process such that breakdowns will not discovered or not discovered very fast. By an increasing number of breakdowns, the performance of the network will decrease, which causes economic drawbacks. To overcome these negative outcomes, one can use a fault detection approach (see [129, 52, 2, 6], for example) to detect breakdowns in a network using autonomous control methods. An observer-based approach for the fault detection can be used (see [130], for example) as well as autonomous control methods. Once a fault in the network is observed/detected, a message to the mechanics of the plant will be sent immediately such that the breakdown could be repaired. In practice, this would help to treat autonomously controlled networks with time-delays and breakdowns or other disturbances and would help to avoid negative economic effects.

We have used the presented tools in this chapter for the stability analysis of impulsive systems with time-delays:

7.4 ISS for impulsive systems with time-delays

Considering impulsive systems with time-delays, we have introduced the Lyapunov-Krasovskii and the Lyapunov-Razumikhin methodology for establishing ISS of single impulsive systems, see Theorem 5.1.7 and Theorem 5.1.11. Then, we have investigated networks of impulsive subsystems with time-delays. We have proved ISS small-gain theorems, which guarantees that the whole network has the ISS property under a small-gain condition with linear gains and a dwell-time condition using the Lyapunov-Razumikhin (Theorem 5.2.1) and Lyapunov-Krasovskii (Theorem 5.2.2) approach. To prove this, we have constructed the Lyapunov function(al)s and the corresponding gains of the whole system. These theorems provide tools to check, whether a network of impulsive time-delay systems possesses the ISS property.

It seems that the usage of general Lyapunov functions instead of exponential Lyapunov functions for single systems could be possible, if the dwell-time condition is formulated in a different way. Then, for an interconnection one can also use general Lyapunov functions and general gains instead of only linear gains. This should be investigated more detailed.

Considering interconnected impulsive systems, there is a relationship between the choice of the gains satisfying the small-gain condition and the dwell-time condition. This is an interesting topic for a more detailed investigation.

It could be investigated, how to develop tools for the stability analysis of interconnections, if the impulse sequences of subsystems are different. Also, a proof with nonlinear gains instead of linear ones could be performed.

Impulsive systems using a dynamical dwell-time condition, presented in [127] for switched systems, could be investigated. There, the dwell-time condition is formulated using Lyapunov functions proving asymptotic stability. This approach can be adapted to impulsive systems and the ISS property, where the benefits of that approach could be investigated.

Since impulsive systems are closely connected to hybrid systems, time-delays could be introduced to hybrid systems and such interconnections. Also, the ISDS property for impulsive systems could be investigated. According to [45, 10], one can analyze the iISS property for interconnected impulsive systems with and without time-delays.

The analysis of the ISDS property have motivated the investigation of ISDS for MPC of single systems and networks. Moreover, the tool of a Lyapunov-Krasovskii functional have been used for the analysis of ISS for MPC of single systems and networks with time-delays:

7.5 MPC

We have combined the ISDS property with MPC for nonlinear continuous-time systems with disturbances. For single systems, we have derived conditions such that by application of the control obtained by an MPC scheme to the system, it has the ISDS property, see Theorem 6.1.6. Considering interconnected systems, we have proved that each subsystem possesses the ISDS property using the control of the proposed MPC scheme, which is Theorem 6.1.9. Using a small-gain condition, we have shown in Corollary 6.1.10 that the whole network has the ISDS property. For the proof, we have used one of the results of this thesis, namely Theorem 2.2.2.

Considering single systems with time-delays, we have proved in Theorem 6.2.4 that a TDS has the ISS property using the control obtained by an MPC scheme, where we have used ISS-Lyapunov-Krasovskii functionals. For interconnected TDS, we have established a theorem, which guarantees that each closed subsystem obtained by application of the control obtained by a min-max MPC scheme has the ISS property, see Theorem 6.2.6. From this result and using Theorem 4.2.4, we have shown that the whole network with time-delays has the ISS property under a small-gain condition, see Corollary 6.2.7.

Note that the results presented here are first steps of the approaches of ISDS for MPC and ISS for MPC of TDS. More detailed studies should be done in these directions, especially in applications of these approaches.

An interesting topic for investigations is the usage of an MPC scheme for interconnected systems, see the discussion in Subsection 6.1.2. It should be investigated if and how a different MPC scheme which guarantees ISS or ISDS, respectively, can be formulated such that the drawbacks of the proposed min-max approach will be eliminated.

Besides the proposed closed-loop MPC scheme, conditions for open-loop MPC schemes to ISDS or ISS of TDS, respectively, could be derived. The differences of both schemes should be analyzed and applied in practice.

Since we have focused on theoretical results regarding MPC, it remains to develop numerical algorithms for the implementation of the proposed schemes, as in [84, 38], for example. It could be analyzed, if and how other existing algorithms could be used or how they should be adapted for implementation for the results presented in this thesis.

Finally, one can investigate networks, where the subsystems are not directly interconnected but they are able to exchange information to control themselves in dependence of the other subsystems to fulfill possible constraints. For this case, a distributed MPC scheme as in [97, 41] should be used to calculate an optimal control for the subsystems.

Bibliography

[1] D. Angeli, E. D. Sontag, and Y. Wang. A characterization of integral input-to-state stability. *IEEE Trans. Automat. Control*, 45(6):1082–1097, 2000.

[2] L. Bai, Z. Tian, and S. Shi. Robust fault detection for a class of nonlinear time-delay systems. *J. Franklin Inst.*, 344(6):873–888, 2007.

[3] D. D. Bainov and P. S. Simeonov. *Systems with Impulse Effect: Stability, Theory and Applications*. Ellis Horwood Limited, 1989.

[4] G. Ballinger and X. Liu. Existence and uniqueness results for impulsive delay differential equations. *Dynam. Contin. Discrete Impuls. Systems*, 5(1-4):579–591, 1999. Differential equations and dynamical systems (Waterloo, ON, 1997).

[5] R. Bellman and J. M. Danskin. A survey of the mathematical theory of time lag, retarded control, and hereditary processes. R-256. The Rand Corporation, 1954.

[6] J. Bokor and Z. Szabo. Fault detection and isolation in nonlinear systems. *Annual Reviews in Control*, 33(2):113–123, 2009.

[7] R. W. Brockett and D. Liberzon. Quantized feedback stabilization of linear systems. *IEEE Trans. Automat. Control*, 45(7):1279–1289, 2000.

[8] C. Cai and A. R. Teel. Characterizations of input-to-state stability for hybrid systems. *Systems & Control Letters*, 58(1):47–53, 2009.

[9] E. F. Camacho and C. Bordons Alba. *Model Predictive Control*. Springer-Verlag, 2004.

[10] W.-H. Chen and W. X. Zheng. Input-to-state stability and integral input-to-state stability of nonlinear impulsive systems with delays. *Automatica J. IFAC*, 45(6):1481–1488, 2009.

[11] S. Dashkovskiy, D. Efimov, and E. D. Sontag. Input to state stability and allied system properties. *Automation and Remote Control*, 72(8):1579–1614, 2011.

[12] S. Dashkovskiy, M. Görges, M. Kosmykov, A. Mironchenko, and L. Naujok. Modeling and stability analysis of autonomously controlled production networks. *Logistics Research*, 3(2):145–157, 2011.

[13] S. Dashkovskiy, M. Görges, and L. Naujok. Autonomous control methods in logistics - a mathematical perspective. *Applied Mathematical Modelling*, 36(7):2947–2960, 2012.

[14] S. Dashkovskiy, M. Görges, and L. Naujok. Stability analysis scheme for autonomously controlled production networks with transportations. In *Proc. of the 3rd International Conference on Dynamics in Logistics, LDIC 2012, Bremen, Germany*, 2012. To appear.

[15] S. Dashkovskiy, M. Görges, and L. Naujok. Local input-to-state stability of production networks. In H.-J. Kreowski, B. Scholz-Reiter, and K.-D. Thoben, editors, *Dynamics in Logistics*, pages 79–89. Springer Berlin Heidelberg, 2011.

[16] S. Dashkovskiy, H. R. Karimi, M. Kosmykov, A. Mironchenko, and L. Naujok. Application of the LISS Lyapunov-Krasovskii small-gain theorem to autonomously controlled production networks with time-delays. In *Proceedings of SysTol, Nice, France, Oct. 6-8*, pages 765–770, 2010.

[17] S. Dashkovskiy, M. Kosmykov, A. Mironchenko, and L. Naujok. Stability of interconnected impulsive systems with and without time-delays using Lyapunov methods. *Nonlinear Analysis: Hybrid Systems*, 6(3):899–915, 2012.

[18] S. Dashkovskiy, M. Kosmykov, and L. Naujok. ISS of interconnected impulsive systems with and without time-delays. In *Proceedings of the 8th IFAC NOLCOS, Bologna, Italy, Sep. 01-03, 2010*, pages 831–836, 2010.

[19] S. Dashkovskiy, M. Kosmykov, and F. Wirth. A small gain condition for interconnections of ISS systems with mixed ISS characterizations. *IEEE Transactions on Automatic Control*, 55(6):1247–1258, 2011.

[20] S. Dashkovskiy and L. Naujok. ISDS small-gain theorem and construction of ISDS Lyapunov functions for interconnected systems. *Systems & Control Letters*, 59(5):299–304, 2010.

[21] S. Dashkovskiy and L. Naujok. Lyapunov-Razumikhin and Lyapunov-Krasovskii theorems for interconnected ISS time-delay systems. In *Proceedings of the 19th MTNS 2010, Budapest, Hungary, July 5-9*, pages 1179–1184, 2010.

[22] S. Dashkovskiy and L. Naujok. Quasi-ISS and Quasi-ISDS Reduced-Order Observers for interconnected Systems. In *Proceedings of the 49th IEEE Conference on Decision and Control, Dec. 15.-17., Atlanta, USA*, pages 5732–5737, 2010.

[23] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. Construction of ISS Lyapunov functions for networks. Technical Report 06-06, Zentrum für Technomathematik, University of Bremen, Juli 2006.

[24] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS Lyapunov function for networks of ISS systems. In *Proceedings of the 17th International Symposium on Mathematical Theory of Networks and Systems (MTNS), Kyoto, Japan, July 24-28*, pages 77–82, 2006.

[25] S. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. An ISS small gain theorem for general networks. *Math. Control Signals Systems*, 19(2):93–122, 2007.

[26] S. N. Dashkovskiy and L. Naujok. Input-to-state dynamical stability of interconnected systems. In *Proceedings of the 48th IEEE Conference on Decision and Control, Shanghai, China, Dec. 16-18*, pages 1411–1416, 2009.

[27] S. N. Dashkovskiy and B. S. Rüffer. Local ISS of large-scale interconnections and estimates for stability regions. *Systems & Control Letters*, 59(3-4):241–247, 2010.

[28] S. N. Dashkovskiy, B. S. Rüffer, and F. R. Wirth. Small Gain Theorems for Large Scale Systems and Construction of ISS Lyapunov Functions. *SIAM Journal on Control and Optimization*, 48(6):4089–4118, 2010.

[29] R. M. Esfanjani, M. Reble, U. Münz, S. K. Y. Nikravesh, and F. Allgöwer. Model predictive control of constrained nonlinear time-delay systems. In *Proceedings of the 48th IEEE Conference on Decision and Control*, pages 1324–1329, 2009.

[30] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag, 1969.

[31] F. A. C. C. Fontes. A general framework to design stabilizing nonlinear model predictive controllers. *Systems & Control Letters*, 42(2):127–143, 2001.

[32] R. A. Freeman and P. V. Kokotović. *Robust nonlinear control design*. Systems & Control: Foundations & Applications. Birkhäuser Boston Inc., Boston, MA, 1996. State-space and Lyapunov techniques.

[33] R. Geiselhart and F. Wirth. Numerical Construction of LISS Lyapunov Functions under a Small Gain Condition. *Mathematics of Control, Signals, and Systems (MCSS)*, 24(1):3–32, 2012.

[34] R. Goebel, R. Sanfelice, and A. Teel. Hybrid dynamical systems. *IEEE Control Systems Magazine*, pages 28–93, 2009.

[35] L. Grüne. *Asymptotic behavior of dynamical and control systems under perturbation and discretization*, volume 1783 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 2002.

[36] L. Grüne. Input-to-state dynamical stability and its Lyapunov function characterization. *IEEE Trans. Automat. Control*, 47(9):1499–1504, 2002.

[37] L. Grüne. Analysis and design of unconstrained nonlinear MPC schemes for finite and infinite dimensional systems. *SIAM J. Control Optim.*, 48(2):1206–1228, 2009.

[38] L. Grüne and J. Pannek. *Nonlinear Model Predictive Control*. Springer-Verlag, London, 2011.

[39] L. Grüne, J. Pannek, M. Seehafer, and K. Worthmann. Analysis of Unconstrained Nonlinear MPC Schemes with Time Varying Control Horizon. *SIAM J. Control Optim.*, 48(8):4938–4962, 2010.

[40] L. Grüne and P. Saint-Pierre. An invariance kernel representation of ISDS Lyapunov functions. *Systems & Control Letters*, 55(9):736–745, 2006.

[41] L. Grüne and K. Worthmann. A distributed NMPC scheme without stabilizing terminal constraints. In R. Johansson and A. Rantzer, editors, *Distributed Decision Making and Control*, pages 259–285. Springer Verlag, 2010.

[42] K. Gu, V. L. Kharitonov, and J. Chen. *Stability of Time-Delay Systems*. Birkhäuser, 2003.

[43] W. M. Haddad, V. S. Chellaboina, and S. G. Nersesov. *Impulsive and hybrid dynamical systems*. Princeton Series in Applied Mathematics. Princeton University Press, Princeton, NJ, 2006.

[44] J. K. Hale and S. M. Verduyn Lunel. *Introduction to functional-differential equations*, volume 99 of *Applied Mathematical Sciences*. Springer-Verlag, New York, 1993.

[45] J. P. Hespanha, D. Liberzon, and A. R. Teel. Lyapunov conditions for input-to-state stability of impulsive systems. *Automatica*, 44(11):2735–2744, 2008.

[46] J. P. Hespanha and A. S. Morse. Stabilization of nonholonomic integrators via logic-based switching. *Automatica J. IFAC*, 35(3):385–393, 1999.

[47] J.P. Hespanha, P. Naghshtabrizi, and Yonggang Xu. A survey of recent results in networked control systems. *Proceedings of the IEEE*, 95(1):138–162, 2007.

[48] D. Hinrichsen and A. J. Pritchard. *Mathematical systems theory I*, volume 48 of *Texts in Applied Mathematics*. Springer-Verlag, Berlin, 2005. Modelling, state space analysis, stability and robustness.

[49] H. Ito. State-dependent scaling problems and stability of interconnected iISS and ISS systems. *IEEE Trans. Automat. Control*, 51(10):1626–1643, 2006.

[50] H. Ito, S. Dashkovskiy, and F. Wirth. On a small gain theorem for networks of iISS systems. In *Proceedings of the 48th IEEE Conference on Decision and Control, Shanghai, China, Dec. 16-18*, pages 4210–4215, 2009.

[51] H. Ito, P. Pepe, and Z.-P. Jiang. Construction of Lyapunov-Krasovskii functionals for interconnection of retarded dynamic and static systems via a small-gain condition. In *Proc. of CDC/CCC 2009, The Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference*, pages 1310–1316, Shanghai, P.R. China, 2009.

[52] C. Jiang and D. H. Zhou. Fault detection and identification for uncertain linear time-delay systems. *Computers & Chemical Engineering*, 30(2):228–242, 2005.

[53] Z.-P. Jiang, Y. Lin, and Y. Wang. Nonlinear small-gain theorems for discrete-time feedback systems and applications. *Automatica J. IFAC*, 40(12):2129–2136, 2005.

[54] Z.-P. Jiang, Y. Lin, and Y. Wang. Nonlinear Small-Gain Theorems for Discrete-Time Large-Scale Systems. In *Proceedings of the 27th Chinese Control Conference, Kunming, Yunnan, China, July 16-18, 2008*, pages 704–708, 2008.

[55] Z.-P. Jiang, I. M. Y. Mareels, and Y. Wang. A Lyapunov formulation of the nonlinear small-gain theorem for interconnected ISS systems. *Automatica J. IFAC*, 32(8):1211–1215, 1996.

[56] Z.-P. Jiang, A. R. Teel, and L. Praly. Small-gain theorem for ISS systems and applications. *Math. Control Signals Systems*, 7(2):95–120, 1994.

[57] Z.-P. Jiang and Y. Wang. A generalization of the nonlinear small-gain theorem for large-scale complex systems. In *Proceedings of the 7th World Congress on Intelligent Control and Automation (WCICA)*, pages 1188–1193, June 2008.

[58] I. Karafyllis and Z.-P. Jiang. A small-gain theorem for a wide class of feedback systems with control applications. *SIAM J. Control Optim.*, 46(4):1483–1517 (electronic), 2007.

[59] I. Karafyllis and Z.-P. Jiang. A vector small-gain theorem for general non-linear control systems. *IMA Journal of Mathematical Control and Information*, 28(3):309–344, 2011.

[60] I. Karafyllis and C. Kravaris. Robust output feedback stabilization and nonlinear observer design. *Systems & Control Letters*, 54(10):925–938, 2005.

[61] I. Karafyllis and C. Kravaris. From continuous-time design to sampled-data design of observers. *IEEE Trans. Automat. Control*, 54(9):2169–2174, 2009.

[62] D. Karagiannis, A. Astolfi, and R. Ortega. Two results for adaptive output feedback stabilization of nonlinear systems. *Automatica J. IFAC*, 39(5):857–866, 2003.

[63] D. Karagiannis, Z. P. Jiang, R. Ortega, and A. Astolfi. Output-feedback stabilization of a class of uncertain non-minimum-phase nonlinear systems. *Automatica J. IFAC*, 41(9):1609–1615, 2005.

[64] H. K. Khalil. *Nonlinear systems.* Prentice Hall, New York, 2002.

[65] V. Kolmanovskii and A. Myshkis. *Introduction to the theory and applications of functional-differential equations*, volume 463 of *Mathematics and its Applications.* Kluwer Academic Publishers, Dordrecht, 1999.

[66] M. Kosmykov. *Hybrid dynamics in large-scale logistics networks.* PhD thesis, University of Bremen, Germany, 2011.

[67] N. Krasovskii. *Stability in Motion.* Moscow, 1959. Translation, Stanford University Press, 1963.

[68] M. Lazar. *Predictive Control Algorithms for Nonlinear Systems.* PhD thesis, Technical University of Iasi, Romania, 2009.

[69] M. Lazar, D. M. de la Pena, W. P. M. H. Heemels, and T. Alamo. On input-to-state stability of min-max nonlinear model predictive control. *Systems & Control Letters*, 57(1):39–48, 2008.

[70] D. Liberzon. *Switching in Systems and Control.* Birkhäuser, Boston, 2003.

[71] D. Liberzon. Quantization, Time Delays, and Nonlinear Stabilization. *IEEE Transactions on Automatic Control*, 51(7):1190–1195, 2006.

[72] D. Liberzon. Observer-based quantized output feedback control of nonlinear systems. In *Proceedings of the 17th IFAC World Congress, Seoul, Korea, July 6-11*, pages 8039–8043, 2008.

[73] D. Limon, T. Alamo, F. Salas, and E. F. Camacho. Input-to-state stability of min-max MPC controllers for nonlinear systems with bounded uncertainties. *Automatica*, 42:797–803, 2006.

[74] Y. Lin, E. D. Sontag, and Y. Wang. A smooth converse Lyapunov theorem for robust stability. *SIAM J. Control Optim.*, 34(1):124–160, 1996.

[75] B. Liu and D. J. Hill. Input-to-state stability for discrete time-delay systems via the Razumikhin technique. *Systems & Control Letters*, 58(8):567–575, 2009.

[76] J. Liu, X. Liu, and W.-C. Xie. Input-to-state stability of impulsive and switching hybrid systems with time-delay. *Automatica J. IFAC*, 47:899–908, 2011.

[77] T. Liu, D. J. Hill, and Z.-P. Jiang. Lyapunov formulation of ISS cyclic-small-gain in continuous-time dynamical networks. *Automatica J. IFAC*, 47(9):2088–2093, 2011.

[78] J. Maciejowski. *Predictive Control with Constraints.* Prentice-Hall, 2001.

[79] L. Magni, D. M. Raimondo, and R. Scattolini. Regional Input-to-State Stability for Nonlinear Model Predicitve Control. *IEEE Trans. Automat. Control*, 51(9):1548–1553, 2006.

[80] D. L. Marruedo, T. Alamo, and E. F. Camacho. Input-to-state stable MPC for constrained discrete-time nonlinear systems with bounded additive uncertainties. In *Proceedings of the 41st IEEE Conference on Decision and Control, Las Vegas, Nevada, USA, December*, pages 4619–4624, 2002.

[81] D. Q. Mayne, J. B. Rawlings, C. V. Rao, and P. O. M. Scokaert. Constrained model predictive control: Stability and optimality. *Automatica*, 36(6):789–814, 2000.

[82] W. Michiels and S.-I. Niculescu. *Stability and Stabilization of Time-Delay Systems.* SIAM, 2007.

[83] D. Nesic and A.R. Teel. Input-to-state stability of networked control systems. *Automatica*, 40(12):2121–2128, 2004.

[84] J. Pannek. *Receding Horizon Control: A Suboptimality-based Approach.* PhD thesis, University of Bayreuth, Germany, 2009.

[85] P. Pepe. On Liapunov-Krasovskii functionals under Carathéodory conditions. *Automatica J. IFAC*, 43(4):701–706, 2007.

[86] P. Pepe. The problem of the absolute continuity for Lyapunov-Krasovskii functionals. *IEEE Trans. Automat. Control*, 52(5):953–957, 2007.

[87] P. Pepe and Z.-P. Jiang. A Lyapunov-Krasovskii methodology for ISS and iISS of time-delay systems. *Systems & Control Letters*, 55(12):1006–1014, 2006.

[88] L. Praly. On Observers with State Independent Error Lyapunov Function. In *Proceedings of the 5th IFAC Symposium NOLCOS'01*, pages 1425–1430, July 2001.

[89] S. J. Qin and T. A. Badgwell. An Overview of Nonlinear Model Predictive Control Applications. In F. Allgöwer and A. Zheng, editors, *Nonlinear Predictive Control*, pages 369–392. Birkhäuser, Berlin, 2000.

[90] S. J. Qin and T. A. Badgwell. A survey of industrial model predictive control technology. *Control Engineering Practice 11*, pages 733–764, 2003.

[91] T. Raff, S. Huber, Z. K. Nagy, and F. Allgöwer. Nonlinear model predicitve control of a four tank system: An experimental stability study. In *Proceedings of IEEE Conference on Control in Applications*, pages 237–242, 2006.

[92] D. M. Raimondo. *Nonlinear Model Predictive Control.* PhD thesis, University of Pavia, Italy, 2008.

[93] D. M. Raimondo, L. Magni, and R. Scattolini. Decentralized MPC of nonlinear systems: an input-to-state stability approach. *Internat. J. Robust Nonlinear Control*, 17(17):1651–1667, 2007.

[94] M. Reble and F. Allgöwer. General Design Parameters of Model Predictive Control for Nonlinear Time-Delay Systems. In *Proceedings of the 49th IEEE Conference on Decision and Control, Atlanta, Georgia, USA, Dec. 15-17*, pages 176–181, 2010.

[95] M. Reble, F. D. Brunner, and F. Allgöwer. Model Predictive Control for Nonlinear Time-Delay Systems without Terminal Constraint. In *Proceedings of the 18th IFAC World Congress, Milano, Italy, Aug. 28 - Sep. 02*, pages 9254–9259, 2011.

[96] M. Reble, R. M. Esfanjani, S. K. Y. Nikravesh, and F. Allgöwer. Model predictive control of constrained non-linear time-delay systems. *IMA Journal of Mathematical Control and Information*, 2010.

[97] A. Richards and J. P. How. Robust distributed model predictive control. *International Journal of Control*, 80(9):1517–1531, 2007.

[98] B. Rüffer. *Monotone dynamical systems, graphs, and stability of large-scale interconnected systems*. PhD thesis, University of Bremen, Germany, 2007.

[99] B. S. Rüffer. Monotone inequalities, dynamical systems, and paths in the positive orthant of Euclidean n-space. *Positivity*, 14(2):257–283, 2010.

[100] R. G. Sanfelice. *Robust Hybrid Control Systems*. PhD thesis, University of California, Santa Barbara, 2007.

[101] R. Scattolini. Architectures for distributed and hierarchical Model Predictive Control - A review. *Journal of Process Control*, 19:723–731, 2009.

[102] B. Scholz-Reiter, S. Dashkovskiy, M. Görges, T. Jagalski, and L. Naujok. Autonomous Decision Policies for Networks of Production Systems. In D. Armbruster and K. G. Kempf, editors, *Decision Policies for Production Networks*, pages 79–89. Springer Verlag, 2012.

[103] B. Scholz-Reiter, S. Dashkovskiy, M. Görges, and L. Naujok. Stability analysis of autonomously controlled production networks. *International Journal of Production Research*, 49(16):4857–4877, 2011.

[104] B. Scholz-Reiter, Ch. de Beer, M. Freitag, and T. Jagalski. Bio-inspired and pheromone-based shop-floor control. *International Journal of Computer Integrated Manufacturing*, 21(2):201–205, 2008.

[105] B. Scholz-Reiter, M. Görges, T. Jagalski, and L. Naujok. Modelling and analysis of an autonomous control method based on bacterial chemotaxis. In *Proceedings of the 43rd CIRP International Conference on Manufacturing Systems (ICMS 2010)*, pages 699–706, Neuer Wissenschaftlicher Verlag, Wien, 2010.

[106] B. Scholz-Reiter, M. Görges, and T. Philipp. Autonomously controlled production systems - Influence of autonomous control level on logistic performance. *CIRP Annals - Manufacturing Technology*, 58(1):395–398, 2009.

[107] B. Scholz-Reiter, T. Jagalski, and J. Bendul. Autonomous control of a shop floor based on bee's foraging behaviour. In *Proceedings of the 1st International Conference on Dynamics in logistics, LDIC 2007*, pages 415–423. Haasis, H.-D. and Kreowski, H.-J. and Scholz-Reiter, B. (eds.), Springer, Berlin, Heidelberg, 2008.

[108] R. Sepulchre, M. Janković, and P. V. Kokotović. *Constructive nonlinear control*. Communications and Control Engineering Series. Springer-Verlag, Berlin, 1997.

[109] J. Shen, Z. Luo, and X. Liu. Impulsive Stabilization of Functional Differential Equations via Liapunov Functionals. *Journal of Mathematical Analysis and Applications*, 240(1-5):1–15, 1999.

[110] H. Shim, D. Liberzon, and J.-S. Kim. Quasi-ISS Reduced-Order Observers and Quantized Output Feedback. In *Proceedings of the 48th IEEE Conference on Decision and Control, Shanghai, China, Dec. 16-18*, pages 6680–6685, 2009.

[111] H. Shim and L. Praly. Remarks on Equivalence between Full Order and Reduced Order Nonlinear Observers. In *Proceedings of the 42nd IEEE CDC, Maui, Hawaii, USA, December*, pages 5837–5840, 2003.

[112] H. Shim, J. H. Seo, and A. R. Teel. Nonlinear observer design via passivation of error dynamics. *Automatica J. IFAC*, 39:885–892, 2003.

[113] R. Shorten, F. Wirth, O. Mason, K. Wulff, and C. King. Stability criteria for switched and hybrid systems. *SIAM Rev.*, 49(4):545–592, 2007.

[114] E. D. Sontag. Smooth stabilization implies coprime factorization. *IEEE Trans. Automat. Control*, 34(4):435–443, 1989.

[115] E. D. Sontag. Further facts about input to state stabilization. *IEEE Trans. Automat. Control*, 35(4):473–476, 1990.

[116] E. D. Sontag. Comments on integral variants of ISS. *Systems & Control Letters*, 34(1-2):93–100, 1998.

[117] E. D. Sontag. *Mathematical control theory*, volume 6 of *Texts in Applied Mathematics*. Springer-Verlag, New York, second edition, 1998. Deterministic finite-dimensional systems.

[118] E. D. Sontag. Input to state stability: Basic concepts and results. In *Nonlinear and Optimal Control Theory*, pages 163–220. Springer, 2006.

[119] E. D. Sontag and Y. Wang. On characterizations of the input-to-state stability property. *Systems & Control Letters*, 24(5):351–359, 1995.

[120] E. D. Sontag and Y. Wang. New characterizations of input-to-state stability. *IEEE Trans. Automat. Control*, 41(9):1283–1294, 1996.

[121] A. R. Teel. Connections between Razumikhin-type theorems and the ISS nonlinear small gain theorem. *IEEE Trans. Automat. Control*, 43(7):960–964, 1998.

[122] S. Tiwari and Y. Wang. Razumikhin-type Small-Gain Theorems for Large-Scale Systems with Delay. In *Proceedings of the 49th IEEE CDC, Atlanta, GA, USA, Dec. 15-17*, pages 7407–7412, 2010.

[123] S. Tiwari, Y. Wang, and Z.-P. Jiang. A Nonlinear Small-Gain Theorem for Large-Scale Time Delay Systems. In *Proceedings of the 48th IEEE CDC, Shanghai, China, Dec. 16-18*, pages 7204–7209, 2009.

[124] A. van der Schaft and H. Schumacher. *An introduction to hybrid dynamical systems*, volume 251 of *Lecture Notes in Control and Information Sciences*. Springer-Verlag London Ltd., London, 2000.

[125] G. C. Walsh, H. Ye, L. G. Bushnell, J. Nilsson, and B. Bernhardsson. Stability analysis of networked control systems. *Automatica*, 34:57–64, 1999.

[126] H.-P. Wiendahl and S. Lutz. Production in Networks. *Annals of the CIRP- Manufacturing Technology*, 51(2):1–14, 2002.

[127] Z. R. Xiang and W. M. Xiang. Stability analysis of switched systems under dynamical dwell time control approach. *Internat. J. Systems Sci.*, 40(4):347–355, 2009.

[128] S. Yang, B. Shi, and S. Hao. Input-to-state stability for discrete-time nonlinear impulsive systems with delays. *International Journal of Robust and Nonlinear Control*, 2012. To appear.

[129] M. Zhong, S. X. Ding, J. Lam, and H. Wang. An LMI approach to design robust fault detection filter for uncertain LTI systems. *Automatica J. IFAC*, 39(3):543–550, 2003.

[130] F. Zhu and F. Cen. Full-order observer-based actuator fault detection and reduced-order observer-based fault reconstruction for a class of uncertain nonlinear systems. *Journal of Process Control*, 20(10):1141–1149, 2010.

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