Qualitative Spatial Reasoning about Relative Orientation

— A Question of Consistency —

Dominik Lücke

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Gutachter: Till Mossakowski
Reinhard Moratz

Zusammenfassung. Seit der Veröffentlichung von Allens Interval Algebra hat sich Qualitative Spatial Reasoning (Qualitatives Räumliches Schließen) zu einem ertragreichen Bereich in der Forschung über Künstliche Intelligenz entwickelte. Potentielle Anwendungsbereiche des Qualitativ Räumlichen Schließens zeigen sich im Bereich von Geoinformationssystemen (GIS) und im Bereich der Roboter-Navigation. Im Qualitativ Räumlichen Schließen wird von einer Beschreibung des Raumes durch mächtige mathematische Theorien (wie z.b. durch den Vektorraum $\mathbb{R}^2$) abstrahiert und statt dessen wird der Raum durch eine endliche, eher kleine, Anzahl von Relationen beschrieben, die gewisse Eigenschaften haben müssen. Oftmals wird dieser Ansatz als „adäquat im kognitiven Sinne“ angesehen. Ein großes Problem im Qualitativ Räumlichen Schließen ist es zu bestimmen, ob die Beschreibung einer räumlichen Gegebenheit, die in der Regel als sogenanntes *constraint network* angeben wird, konsistent ist. Wegen der Unendlichkeit der räumlichen Domäne (z.b. $\mathbb{R}^2$) können viele Methoden, die zum Lösen von *constraint satisfaction* Problemen entwickelt worden sind nicht angewendet werden, da diese auf Backtracking-Ansätzen über der Domäne beruhen. Eine Methode, und zwar die Methode der Pfadkonsistenz und ihre Verallgemeierung die Methode des Algebraischen Abschlusses, hat sich bis zu einem gewissen Grad als erfolgreich erwiesen. Der die Methode des Algebraische Abschlusses benötigt die Kompositionstabelle eines Kalküls. Leider ist die Berechnung dieser Tabelle für viele Kalküle eine alles andere als einfache Aufgabe. Der $\mathcal{DRA}_f$ Kalkül hat zum Beispiel 72 Basisrelationen und eine binäre Komposition, damit hat seine Kompositionstabelle 5184 Einträge, die alle aus Disjunktionen von bis zu 72 Basisrelationen bestehen können. Alle diese Einträge per Hand zu berechnen ist eine schwere und fehleranfällige, da ermüdende, Aufgabe. Die direkte Berechnung der Kompositionstabelle mit einem Computer unter Benutzung der Semantik des Kalküls in der Ebene ist auch nicht einfach, da diese Semantik auf nichtlinearen Ungleichungen basiert. Wir schlagen die Benutzung einer neunen Methode zur Berechnung der Kompositionstabellen der $\mathcal{DRA}$ Kalküle vor, einer Methode, die auf einer *Verdichteten Semantik* des Kalküls basiert. Die Domäne des Kalküls wird bei diesem Ansatz analysiert und so verdichtet, dass es ausreicht nur endlich viele Komfigurationen zu betrachten. Für die $\mathcal{DRA}$ Kalküle bedeutet dieses, dass nur Konfigurationen in der Ebene betrachtet werden müssen, die spezielle Charakteristika der Konfigurationen in drei Linien in der Ebene repräsentieren. Diese Methode hat sich als sehr effizient erwiesen, um das schwere Problem der Berechnung der Kompositionstabelle für die $\mathcal{DRA}$ Kalküle zu lösen. Ein anderer Ansatz eine Kompositionstabelle abzuleiten ist, sie sich von einem anderen Kalkül entlang eines geeigneten Morphimusses zu „borgen“. Wir untersuchen daher Morphismen zwischen Kalkülen. Mit der Berechnung der Kompositionstabelle enden unsere Probleme aber nicht, sie beginnen erst. Wir können nun zwar algebraisch abgeschlossene Verfeinerungen von Constraint Networks berechnen, aber was sagt uns deren Existenz oder Nicht-Existenz? Wir wissen, dass wenn keine solche Verfeinerung existiert, dann ist die Beschreibung inkonsistent. Existiert aber eine Verfeinerung, dann kann die Beschreibung konsistent, aber auch inkonsistent sein. Falls alle Beschreibungen mit einer algebraisch abgeschlossenen Verfeinerung konsistent sind, dann können wir uns freuen, da dann der Algebraische Abschluss Konsistenz entscheidet. Aber in den meisten Fällen ist es nicht so. Wir untersuchen das Verhalten von $\mathcal{LR}$, $\mathcal{DRA}_f$ und $\mathcal{DRA}_{fp}$ und zeigen, dass für alle diese Kalküle der algebraische Abschluss die Konsistenz nicht entscheidet. Für $\mathcal{LR}$ zeigen wir sogar, dass der algebraische Abschluss eine sehr schlechte Approximation der Konsistenz ist. Wir führen daher für $\mathcal{LR}$ eine neue Approximationsmethode der Konsistenz ein, die sich viel besser verhält als der Algebraische Abschluss. Diese neue Methode basiert auf Dreiecken.

Eine große Schwachstelle des Qualitativen Räumlichen Schließens ist die Existenz nur weniger Anwendungen. Ohne weiteres können wir diese auch nicht schließen, aber wir können Anwendungsgebiete untersuchen. Wir untersuchen das Verhalten von $\mathcal{DRA}$ und $\mathcal{OPRA}$ bei der Beschreibung von beziehungsweise Navigation durch Straßennetze, die als Constraint Networks basierend auf lokalen Beobachtungen beschrieben sind. Für $\mathcal{OPRA}$ untersuchen wir zudem noch das Verhalten eines Quotienten der Basisrelationen, der als „kognitiv adäquat“ bezeichnet wird. Wenn möglich verwenden wir echtes Kartenmaterial, das wir von OpenStreetMap beziehen.

Abstract. After the emergence of Allen's Interval Algebra Qualitative Spatial Reasoning has evolved into a fruitful field of research in artificial intelligence with possible applications in geographic information systems (GIS) and robot navigation. Qualitative Spatial Reasoning abstracts from the detailed metric description of space using rich mathematical theories and restricts its language to a finite, often rather small, set of relations that fulfill certain properties. This approach is often deemed to be "cognitively adequate". A major question in qualitative spatial reasoning is whether a description of a spatial situation given as a constraint network is consistent. The problem becomes a hard one since the domain of space (often $\mathbb{R}^2$) is infinite. In contrast many of the interesting problems for constraint satisfaction have a finite domain on which backtracking methods can be used. But because of the infinity of its domains these methods are generally not applicable to Qualitative Spatial Reasoning. Anyhow the method of path consistency or rather its generalization *algebraic closure* turned out to be helpful to a certain degree for many qualitative spatial calculi. The problem regarding this method is that it depends on the existence of a composition table, and calculating this table is not an easy task. For example the dipole calculus (operating on oriented dipoles) $\mathcal{DRA}_f$ has 72 base relations and binary composition, hence its composition table has 5184 entries. Finding all these entries by hand is a hard, long and error-prone task. Finding them using a computer is also not easy, since the semantics of $\mathcal{DRA}_f$ in the Euclidean Plane, its natural domain, rely on non-linear inequalities. This is not a special problem of the $\mathcal{DRA}_f$ calculus. In fact, all calculi dealing with relative orientation share the property of having semantics based on non-linear inequalities in the Euclidean plane. This not only makes it hard to find a composition table, it also makes it particularly hard to decide consistency for these calculi. As shown in [79] algebraic closure is always just an approximation to consistency for these calculi, but it is the only method that works fast. Methods like Gröbner reasoning can decide consistency for these calculi but only for small constraint networks. Still finding a composition table for $\mathcal{DRA}_f$ is a fruitful task, since we can use it analyze the properties of composition based reasoning for such a calculus and it is a starting point for the investigation of the quality of the approximation of consistency for this calculus. We utilize a new approach for calculating the composition table for $\mathcal{DRA}_f$ using *condensed semantics*, i.e. the domain of the calculus is compressed in such a way that only finitely many possible configurations need to be investigated. In fact, only the configurations need to be investigated that turn out to represent special characteristics for the placement of three lines in the plane. This method turns out to be highly efficient for calculating the composition table of the calculus. Another method of obtaining a composition table is "borrowing" it via a suitable morphism. Hence, we investigate morphisms between qualitative spatial calculi. Having the composition table is not the end but rather the beginning of the problem. With that table we can compute algebraically closed *refinements* of constraint networks, but how meaningful is this process? We know that all constraint networks for which such a refinement does not exist are inconsistent, but what about the rest? In fact, they may be consistent or not. If they are all consistent, then we can be happy, since algebraic closure would decide consistency for the calculus at hand. We investigate $\mathcal{LR}$, $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ and show that for all these calculi algebraic closure does not decide consistency. In fact, for the $\mathcal{LR}$ calculus algebraic closure is an extremely bad approximation of consistency. For this calculus we introduce a new method for the approximation of consistency based on triangles, that performs far better than algebraic closure.

A major weak spot of the field of Qualitative Spatial Reasoning is the area of applications. It is hard to refute the accusation of qualitative spatial calculi having only few applications so far. As a step into the direction of scrutinizing the applicability of these calculi, we examine the performance of $\mathcal{DRA}$ and $\mathcal{OPRA}$ in the issue of describing and navigating street networks based on local observations. Especially for $\mathcal{OPRA}$ we investigate a factorization of the base relations that is deemed "cognitively adequate". Whenever possible we use real-world data in these investigations obtained from OpenStreetMap.

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To my Grandmother

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CHAPTER 1

Introduction

Frayed ends of sanity
Hear them calling
Hear them calling me
—**Metallica**, *. . . And Justice for All.*

Since the introduction of Allen's Interval Algebra [1] qualitative temporal and spatial reasoning emerged to an important field of research in artificial intelligence. It is an aim of qualitative temporal and spatial reasoning to describe the inherently infinite domains of time ($\mathbb{R}$) and space ($\mathbb{R}^2$, $\mathbb{R}^3$) in a qualitative way, i.e. in a way that only uses finitely many distinctions, often formalized as relations between objects in space. These structures, called *qualitative temporal calculi* or *qualitative spatial calculi* introduce relations between "objects" in space or time. These relations often do not reflect all aspects of space and time, they only keep the aspects that are interesting for the calculus designer or the task at hand and disregard others. In fact, some kind of data reduction must take place when representing an infinite domain in a finite system. First let us discuss quantitative versus qualitative descriptions.

In history several different kinds to describe space and time have evolved. As a pair of different means of descriptions, we can name the qualitative versus quantitative dichotomy. Both approaches share the property of being based on a model of the real world. For the quantitative description of space and time, we need some adequate measurement units, which form the basis of this model. For time the historic definitions for the units were based on the sun, the moon and the rotation of the earth. In the course of the introduction of the `SI` units, less obvious and more precise definitions of many of these time units have been introduced.

In the domain of space there has been a similar development of measurement units in history. The original issue here was that measurement units were just standardized locally. If somebody said that a pole was six feet long in the grand duchy of Hesse, then this pole would be $150cm$ long, but a six feet long pole in the duchy of Nassau would be $300cm$ long. Even today several systems of measurement compete for dominance. There are the `SI` system, the Imperial system and the United States Customary Units system as the predominant ones.

We can conclude that measuring time and space is basically the comparison with a more or less artificial reference unit and these multiples of the reference unit can be used in rich mathematical theories to determine distances, areas, volumes and so on.

But do we really have to use a such an "artificial" system of units? In fact, we don't. It is even deemed to be quite "natural" or "cognitively adequate" to compare entities directly using some qualitative descriptions. Taking up the above example, we can just say that the $300cm$ long pole is longer than the $150cm$ pole, or we can say that it is "a lot longer" than the other pole. In this case we do not take the detour via the "artificially" standardized notion of meter. Or just think of everyday situations. Consider that you want to take a bath and make a pizza. Making the pizza takes $15mins$ and taking the bath $30mins$, hence you can say in a qualitative setting that the time interval of making the pizza is *shorter* than the time interval of taking a bath. With this information, you can derive the constraint that making a pizza while having a bath is impossible without leaving the bathtub given that you cannot reach the oven from the bathtub as it is the case in the layout of almost all apartments and houses. Qualitative knowledge is applied in everyday navigation tasks over and over again. Just consider that you come to a new town and are hungry. You ask the next person you see for the directions to a restaurant, and she will tell you something like:

Quote 1 (A way to a restaurant)**.** *To reach the restaurant, turn left at the bank, then go along the street and then turn right at the church, after a short walk along the street you will have reached the restaurant.*

Such a description does not use any quantitative scale, it is rather based on purely qualitative notions like “left”, “right” and “along”, which are given with respect to the orientation of the streets. In the same way a global sense of direction could haven been used, utilizing the compass directions “north”, “west”, “south” and “east”, but to use this, both persons need to have some information how they are oriented in this global reference frame.

As another example consider the navigation system in a car. When navigating from a place A to a place B a map is shown on its display providing all quantitative information about the route. But in most cases that information is far more than you really need. At bends the system provides you with the essential qualitative information as e.g. “turn left”, “turn right” and so on. In gyratory traffics sometimes a mix of qualitative and quantitative information is supplied as, “take the third turn to the right”.

Consider that you want to describe the layout of a room to another person. In most cases you will say something like: “The desk is in front of the windows and there is a locker to the left of it”. You do not provide any quantitative information in this way, but the recipient can get a rough understanding of the layout of that place quite easily. The actual layout is not uniquely defined with such a description, since the locker can be $1m$ or $5m$ away from the desk.

The distinctions of “front”, “left”, “back” and “right” have been formalized in Ligozat’s Flip-Flop calculus [37] and been refined in the $\mathcal{LR}$ calculus [73]. Further, a lot more calculi are based on these distinctions.

Even though qualitative spatial calculi are very “natural” and a nice finite representation, a big advantage of the quantitative representation remains: the rich mathematical theories that can be employed in the respective vector spaces. But still the quantitative representations are not as easily accessible to humans as the qualitative ones.

Some of these discussions are inspired by [30], but Hernández goes into further considerations that are of lesser interest for us.

Before we can give some examples for spatial relations, we need to discuss direction and orientation. Since this thesis is mainly on spatial reasoning, we will henceforth rather talk about spatial reasoning than temporal reasoning, even though both fields are related very closely.

1.1. Direction versus Orientation

For the calculi that are scrutinized in this thesis, it is of great interest to discuss the meanings of direction and orientation in geometry. Galton addresses the same topic in [23] in a similar way, even though his inspiration of the discussion is different from ours.

In geometry *direction* is the information about the relative position of one point (object) in space with respect to another one disregarding the distance information.

The reference frame that specifies the measurement of the direction may be absolute (like the compass scale) or relative (like the radians scale). As an example consider two cars driving along a street as given in Figure 1. We then can say that car 1 is

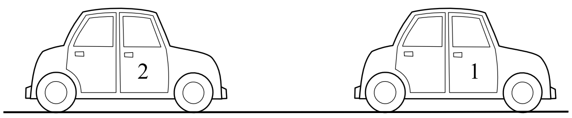


FIGURE 1. Car 1 *in front of* Car 2

situated in *front* with respect to car 2. But the same would be true in Figure 2. But

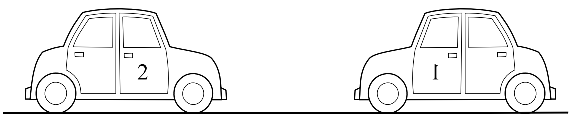


FIGURE 2. Car 1 *in front of* Car 2 again

what has changed between those two figures? In fact, another feature of objects in space has changed, the *orientation*.

Galton describes the *orientation* of an object as the direction into which it is pointing. Formally, we can describe *orientation* as the *rotation* of an object with respect to a reference frame that defines its initial orientation. To describe orientation, coordinate axes can be constructed at e.g. the center of gravity of the object and the orientation can be determined by the orientation of the local coordinate axes with respect to a reference frame.

Consider again Car 1 and Car 2 in Figure 1 and Figure 2 with Car 2 being the reference frame. We see that in Figure 1 Car 1 has the same orientation as the reference car, i.e. no rotation was applied, whereas in Figure 2 it has the opposite orientation.

In some cases talking about orientation and direction can become confusing. Consider a straight line in space with an orientation, then also each point on this line lies in a certain direction with respect to other points, even with respect to other points on the line itself. We might use the notions of orientation and direction for lines almost interchangeably, especially if we make the directions between the points noticeable.

1.2. Describing aspects of space

When navigating structured space like a building or streets in a city questions like "Where am I?", "Where do I have to go?" arise. Possible answers to the first question are , "At coordinate (x,y)", "A short way north of the church", "In Bremen", "Between Am Dom and Domshof" and many more. The first answer 'At coordinate (x,y)" is clearly a quantitative one. To provide it with any meaning a coordinate system is needed (as on a map). The second answer "A short way north of the church" still requires a possibility to determine what "north" means, e.g. a compass or some scouts' tricks. We further notice that different aspects of space are mixed in that answer, in fact distance and orientation. The answer "In Bremen" is very coarse and just says that the space covered by the person is contained in the space covered by Bremen. This is a purely qualitative description. But it contains no knowledge about orientations or directions.

Moreover, one is often interested in reaching a certain destination. If you are lucky you have a GPS-enabled device with routing software with you, then you only need to know the quantitative position of your destination. Otherwise you might ask a passerby. You might get route descriptions like: “To reach the bakery, turn left at the church, then turn left again” or “To reach the bakery turn west at the church and then turn south”. We observe that in both cases qualitative descriptions neglecting unnecessary details are given, but still we see a difference. In the first case all orientations are given locally at the crossings, whereas in the second case there is a superimposed global sense of orientation. One would need a compass to access this reference frame in practice.

How do we perceive the “left” and “right” on a street? This first leads to the

(a) Street as primitive (b) Points as primitives

Figure 3. Orientations of streets

question how do we perceive the street itself. Apart from one-way streets most streets are bi-directional. But when walking along a street this is of lesser interest for us, we are rather interested in the direction of the lane we are walking on, hence it is more precise to talk about “left” and “right” with respect to a lane than with respect to a street. For orientation at an abstract level, the width of a lane is also of lesser interest. We then can perceive a lane as a directed line segment as in Figure 3a. Such a line segment cannot have zero length, since then it loses its directional information. Alternatively we can perceive the street as the connection from point A to point B and hence being directed as in Figure 3b. In this case a line is not a primitive entity it is just given by the start and endpoints A and B whereas in the first case the line itself is the primitive entity. In fact, the difference between Figure 3a and Figure 3b is not a profound one. It rather lies in the interpretation of the perception of a one dimensional spatial entity which is a line. In the first case we directly perceive the line itself having a start- and endpoint. In the second case we perceive the start- and endpoint that are connected by a line. Both interpretations can be transformed into each other in geometry and are equivalent in that setting. A difference can be seen if we abstract from this geometric level and construct systems of abstract relations. Or more generally if we observe the setting from a higher, more abstract, level. In the case of Figure 3a we would compare the orientations and positions of lines and in the case of Figure 3b we compare the positions of points. In both cases, our representation has a direction and “left” and “right” are given with respect to that direction in a natural way. A third way is to make the perception of “left” and “right” dependent on the agent traveling along the street itself. Such an agent can be a car that has a defined front section. In this case we can abstract the agent as a point (on a lane) that has an orientation, called an *oriented point* as in Figure 4. “Left” and “right” are then perceived with respect to the natural orientation of the oriented point. We have to keep in mind that the models that we

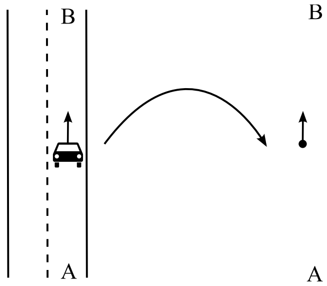


FIGURE 4. Orientations of streets

have derived for describing streets are suited to express *relative orientation*, but we did not consider distances etc. in their derivation, since that is no property of space that we want to be able to express. All we are interested in is the relative direction and orientation from one entity to the other which is based on the position and orientation of the entities in the plane.

We have identified three different kinds of *base entities* that we are going to use for describing relative orientation and direction in the plane:

- tuples of points;
- directed line segments;
- oriented points.

We do not claim to have an exhaustive description of all possible base entities, but these are the ones that we are going to work with. At a first glance, we can see that all of these base entities are related. Having a directed line segment A, we can express it as its start-point $\mathbf{s}_A$ and end-point $\mathbf{e}_A$. On the other hand, a tuple of points can be regarded as a line segment, refer to Figure 3 for a visualization. Further an oriented point can be perceived as a directed line segment A with $||A|| \longrightarrow 0$ and vice versa. We are going to call the directed line segments of non-zero length *dipoles* as in [49].

1.2.1. Calculus based on point tuples. Several calculi have been studied in literature whose local reference frame is defined by tuples of points. A simple example is Ligozat's Flip-Flop calculus [37] which has been refined to the $\mathcal{LR}$ calculus in [73]. We will work with the $\mathcal{LR}$ calculus in the course of our studies. Further examples of calculi of this kind are Freksa's single- and double-cross calculus [19]. The $\mathcal{TPCC}$ calculus [13, 50] is also based on point tuples, but additionally to orientation information, it represents some qualitative distance information.

For the $\mathcal{LR}$ calculus a reference frame is given by a tuple $\langle A, B \rangle$ of points. For $A \neq B$ this frame is a directed line segment from A to B written as $\overline{AB}$ and shown in Figure 5. For the sectioning of the Euclidean plane, we use the ray being collinear

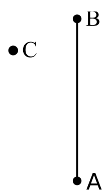


FIGURE 5. $\mathcal{LR}$ configuration

to $\overline{AB}$ and having the same direction. This ray divides the Euclidean plane into three sections: the half-plane to the left (l) of the ray; the half-plane to the right (r) of the ray; and the ray itself. The ray itself is divided into segments by A and B: The segment behind (b) A; the start point (s) A; the segment in-between (i) A and B; the end point (e) B and the segment in front (f) of B. Given a third point C, we can use the symbols introduced above as relation symbols R between reference frame being defined by A and B and C written as

$$(A\ B\ R\ C).$$

For Figure 5 we have the relation $(A\ B\ \mathrm{l}\ C)$. The seven relations above are the original Flip-Flop relations, which are lacking the case of $A = B$. Therefore in $\mathcal{LR}$ the relations *dou* for $A = B \neq C$ and *tri* for $A = B = C$ have been introduced.

1.2.2. Calculus based on dipoles. Early developments of a calculus based on relations between line segments have been conducted by Schlieder leading to his line segment calculus [69]. This calculus lacked the possibility to express polylines. Refinements by Moratz et al. [49] took care of that issue. Further refinements lead to the ability to express relations between line segments in any position and to describe qualitative angles between line segments. In the course of this work, we will use these latest refinements called $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ (the latter featuring qualitative angles).

The reference frame of these calculi is an directed line segment in the Euclidean plane with non-zero length. Such a segment has been called a dipole in [49]. A dipole A can be identified by the tuple $\langle \mathbf{s}_A, \mathbf{e}_A \rangle$ of its start and endpoint as shown in Figure 6. To introduce $\mathcal{DRA}_f$ relations we consider again Figure 6. We are

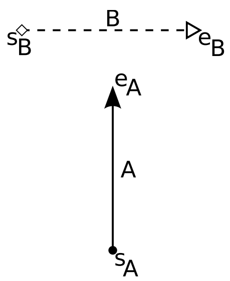


FIGURE 6. $\mathcal{DRA}$ configuration

interested in the $\mathcal{DRA}_f$ relation R with $(A\ R\ B)$. This relation R is in fact a concatenation of four $\mathcal{LR}$ relations (see Section 1.2.1) with $R = R_1 R_2 R_3 R_4$, where these $\mathcal{LR}$ relations are defined as:

$$(\mathbf{s}_A\ \mathbf{e}_A\ R_1\ \mathbf{s}_B)$$
$$(\mathbf{s}_A\ \mathbf{e}_A\ R_2\ \mathbf{e}_B)$$
$$(\mathbf{s}_B\ \mathbf{e}_B\ R_3\ \mathbf{s}_A)$$
$$(\mathbf{s}_B\ \mathbf{e}_B\ R_4\ \mathbf{e}_A)$$

Please note that the relations *dou* and *tri* cannot occur, since the dipoles have non-zero length. By this definition, Figure 6 shows the relation $(A\ \mathrm{lrrr}\ B)$. For this version of the dipole calculus, $\mathcal{DRA}_f$, 72 base relations exist. Distinguishing between qualitative angles just adds eight more relations, but we will defer the discussion of the calculus comprising this feature and being called $\mathcal{DRA}_{fp}$.

1.2.3. Calculus based on oriented points. A calculus based on oriented points has been described in [48] and is called $\mathcal{OPRA}$ or better $\mathcal{OPRA}_m$ calculus. The index m is a natural number with $m > 0$ and has impact on the reference frames. For our introductory discussion we will restrict ourselves to $\mathcal{OPRA}_2$. A general discussion will follow in Section 3.3.

Let an oriented point A be given, an $\mathcal{OPRA}_2$ reference frame is constructed as the line through the point being collinear to its direction, and a line trough A being perpendicular to the first one as in Figure 7. The plane is segmented into 4 linear

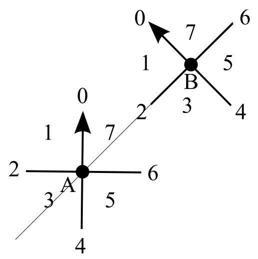


FIGURE 7. $\mathcal{OPRA}_2$ configuration

and 4 planar sectors as well as the point A itself. The sectors are assigned numbers from 0 to $7 = 4m - 1$ in a counterclockwise fashion. The distinction at the point A leads to a case distinction regarding the introduction of the relations. Details on how this works will follow. Let another oriented point B be given for which also a reference frame is constructed. The $\mathcal{OPRA}_2$ relation $A_2 \angle_i^j B$ is determined in the following way, if the positions of A and B differ: i is the sector of the frame of A in which B lies and j is the sector of the frame at B in which A lies. In Figure 7 we hence have the relation $A_2 \angle_7^2 B$. If the positions of A and B coincide, then we set i to s, a special symbol, and j is the sector of the frame of A which B points into.

1.3. What to do with these relations?

We have seen several sets of qualitative spatial relations in Section 1.2. And we can describe locations of objects in space with them in a qualitative way. But are we happy with that? In fact, we are not. The relations shown in Section 1.2 are atomic, the building blocks of the relations of a calculus. We call these atomic relations *base relations*. We can only use them to express precise knowledge, but often observations are not precise. Just consider that you observe several houses on a hill that is far away and try to tell all their relative positions. Such uncertainty is described via sets of base relations with disjunctive semantics. These sets are again perceived as relations and they are called *general relations*. Taking all possible of these sets, we obtain a power set Boolean algebra that, e.g., allows for the operations $\cup$ and $\cap$. With these operations we can already do some manipulations on the presented knowledge, but we need more.

Consider a configuration of three dipoles A, B and C as shown in Figure 8. Assume that we all know is that the relations

$$A \,\mathrm{rrrr}\, B \quad \text{and} \quad B \,\mathrm{slsr}\, C$$

hold, can we infer from that what is the relations from B to A and what is the relation from A to C? From the picture we can see that B rrrr A and A rrrl C hold. Are these all possible relations between A and C? In fact, the answer is no. Several relations can hold between A and C for the given qualitative case. Computing or deriving these coherences for each instance of a problem by hand is error prone and not very efficient. Computer-aided methods are a better way to go but in that case

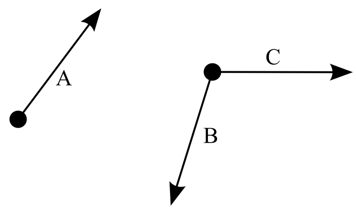


FIGURE 8. A simple dipole configuration

it has to be taken care of the facts that the method is sound and complete, that is roughly, that all and not too many of the possible relations between A and C are found. Finding a suited method for a calculus at hand is not always a simple task depending on its semantics. For calculi dealing with relative orientation a big issue are the non-linear semantics in the Euclidean plane that can be tackled with Gröbner reasoning but at a high price of doubly exponential running time which leads in practice to the issue that often the reasoner cannot decide if a solution is found in an acceptable time frame. In such cases a more optimized method is needed. In qualitative spatial reasoning operators have been introduced for the above mentioned problem. The change of perspective, i.e. deriving the relation from B to A from the relation from A to B in the above example, is covered by the *permutation* operators. Combining several relationships to derive a new one is called *composition*. The outcomes of permutation and composition are usually stored in tables indexed by the operands of the operations. The hard task of computing the compositions and permutations only needs to be performed once.

With the sets of relations and the operations on them, i.e. with our qualitative spatial calculi, we can perform reasoning tasks. The input problem for reasoning tasks is generally formulated as a *constraint satisfaction problem*, i.e. as a set of constraints given as relations between spatial variables called *constraint network*. Given such a constraint network, an important question is whether there is a valuation of the spatial variables such that all constraints (i.e. relations) in the network are fulfilled. A big issue in determining consistency is that for spatial as well as for temporal constraint satisfaction problems the domains are infinite, which makes backtracking techniques often used in the field of solving constraint satisfaction problems infeasible.

Hence, other methods have to be employed to tackle the problem of determining consistency. In qualitative spatial reasoning the method of *algebraic closure* has become a kind of standard approach. Algebraic closure determines if the relations in a constraint network agree on each other locally with respect to composition. The advantage of this method is its polynomial running time. But unfortunately this method is not the silver bullet. Basically algebraic closure is merely an approximation of consistency. If a constraint network has no algebraically closed refinement then it is not consistent, but if such a refinement exists, the constraint network may or may not be consistent. There are calculi for which subsets, so called tractable subsets, exist for which algebraic closure decides consistency, but finding them is a hard task (see [66]). For other calculi algebraic closure cannot decide consistency even in constraint networks in base relations, which are called *scenarios*. We will see such problems in Chapter 6. In such a case the developer of an application using the respective calculus has to decide if algebraic closure is "good enough" for the task of the application. Other reasoning methods have been developed for qualitative spatial reasoning like *neighborhood based reasoning* [17, 18] and reasoning based on *Gröbner bases* [80], but none of them has reached the prominence of algebraic closure so far. Especially reasoning based on Gröbner bases is an exponential time approach.

There is still a potentially big issue when trying to apply algebraic closure: the algebraic structure of the calculus at hand has to be known. Especially finding the composition table for many calculi is everything but an easy task. In fact, finding such a table by hand is more than tedious and error prone. Also approaches based on ad-hoc programs are often doomed to fail, since it is probable that there are important spatial configurations the designer of the program does not think of. An example for this is finding the composition table for $\mathcal{DRA}_f$ [49], where even the number of relations was faulty. The issue with $\mathcal{DRA}$ is that the semantics of the relations is based on non-linear inequalities which cannot be solved efficiently. In Section 4.1 we present a new approach to this problem that is both efficient and complete. In this approach the semantics of the calculus are compressed so far that we can derive all possible compositions in a finite setting.

For calculi which are obtained from other calculi via an equivalence relation on the relations, which are called *quotients* or *quotient calculi*, finding the composition table is rather easy, since it is already given via the respective *quotient homomorphism*. To understand this approach in detail a lot of mathematical machinery is needed which is introduced in Chapter 8. There we will additionally show that certain properties of calculi and constraint networks can be transported along certain homomorphisms (which do not have to be quotient homomorphisms). Investigating formal "links" between calculi seems to be highly interesting since it might be a tool to bring some order into the chaos of the plethora of qualitative spatial calculi, further it helps to modify calculi for a given task. Last but not least homomorphisms can be a tool for a "fusion" of several calculi with a formal basis. So far "fusions" have been formed on a rather informal level. Apart from Ligozat and Renz's work in [39] and our work there has not been much research into homomorphisms between spatial calculi, hence our results are rather the beginning of possible research.

For one calculus under our scrutiny, the $\mathcal{LR}$ calculus, algebraic closure turns out to be an extremely bad approximation for consistency. In fact it turned out to perform that badly, that we felt the need to search for a better approximation method for consistency based on triangles. In contrast to algebraic closure this approximation takes properties of the domain into consideration and performs surprisingly well. We discuss this approach in detail in Chapter 7.

A point of critics of qualitative spatial reasoning is that it is a nice toy to play with, but that there are only a few applications for it. Unfortunately these accusations are hard to refute. In Chapter 5 we investigate the feasibility of qualitative spatial reasoning in a navigation setting. The navigation in this case is based on incomplete knowledge.

Before we can start, we need to discuss the basics of qualitative spatial reasoning in Chapter 2. And in Chapter 3 we have a look at the qualitative spatial calculi discussed in this thesis in detail. We also give the formal semantics of them in their domain.

CHAPTER 2

Constraint Reasoning

Since the day that you were born the wheels are in motion
Turning ever faster — Play your part in the big machine
The stage is set, the road is chosen
You fate preordained
We are watching you — every step of the way

—**Arch Enemy**, *Revolution Begins.*

The domain of qualitative spatial and temporal calculi is normally infinite, i.e. the Euclidean Plane or the real numbers. This is why the standard methods for solving *constraint satisfaction problems* do not work for qualitative spatial reasoning, since these often rely on (exhaustive) backtracking search in finite domains. Other methods to determine consistency are needed, optimally ones that work on a purely symbolic level or can work with a finite subset of the domain.

2.1. Base and General Relations

Qualitative calculi are generally used to represent knowledge about an infinite domain using a finite set of *base relations*. These base relations partition the domain of the calculus. As an example, we remember the relations introduced in Section 1.2.1. Further with those relations, we can e.g. say that $(A\ B\ l\ C)$ knowing that the point lies in the left half-plane defined by the line segment $\overline{AB}$, but we know nothing about the distance of C to the line defined by $\overline{AB}$ nor about any concrete angle between the points. But we know, that by this statement, the point C has to be in the left half plane and nowhere else.

Commonly for the set of base relations the property of *jointly exhaustiveness and pairwise disjointness*, for short $JEPD$, is required to ensure that any constellation of domain objects is covered by exactly one base relation.

Definition 2 ($JEPD$)**.** Let $\mathcal{B} = \{B_1, \ldots, B_k\}$ be a set of n-ary relations over a domain $\mathcal{D}$. These relations are *jointly exhaustive and pairwise disjoint* ($JEPD$), if they satisfy the properties:

(1) $\forall i, j \in \{1, \ldots, k\}$ with $i \neq j$. $B_i \cap B_j = \varnothing$;
(2) $\mathcal{D}^n = \bigcup_{i \in \{1,\ldots,k\}} B_i$.

The base relations we were talking about so far express precise knowledge, they state that for an n-ary relation exactly one base relation R holds between objects $X_1, \ldots, X_n$ and no relation R' with $R' \neq R$ holds between them at the same time. But assume that the point C in Figure 5 is so close to the line $\overline{AB}$ and in front of B, that we cannot see if it lies to, say, the left of $\overline{AB}$ or in front of $\overline{AB}$. A way to express such uncertain knowledge are so called *general relations*. In the above example, we simply say $(A\,B\,(l\,f)\,C)$, meaning that the relation is l or f. Formally general relations are defined as:

Definition 3 (General Relation)**.** Let $\mathcal{B} = \{B_1, \ldots, B_k\}$ be a set of n-ary base relations over a domain $\mathcal{D}$. The set of *general relations* $\mathcal{R}_{\mathcal{B}}$ (or simply $\mathcal{R}$) is the powerset $\mathcal{P}(\mathcal{B})$ with the following semantics for each $R \in \mathcal{R}_{\mathcal{B}}$:

$$R(x_1, \ldots, x_n) \Leftrightarrow \exists B_i \in R.B_i(x_1, \ldots, x_n).$$

The set $\mathcal{P}(\mathcal{B})$ contains the empty set of relations $\varnothing$ and the set $\mathcal{B}$ itself. If $\mathcal{B}$ is a set of $JEPD$ base relations, then special names are assigned to the above two sets. The set $\varnothing$ is called the *impossible relation* and $\mathcal{B}$ is called the *universal relation*. This universal relation is the most uncertain relation possible and expresses the absolute lack of knowledge. Since reasoning in the domain itself is a very hard task (ref. to Section 2.5) in many cases, especially for relations being based on relative orientation, reasoning with qualitative knowledge takes place on a symbolic level on the relations from $\mathcal{R}$. But for this we need some special operators that allow us to manipulate the knowledge.

In qualitative spatial reasoning calculi based on relations of different arities have been introduced. Most prominent are calculi in binary or ternary relations. Especially for calculi in binary relations an elaborate theoretical background is available, whereas such a background is missing for calculi based on relations of higher arity. That is why we present a special section for calculi in binary relations.

2.2. Binary Relations

The theoretical background of algebraic structures and reasoning with binary relations is well understood. Most of the theory involved is based on non-associative algebras, which are a generalization of Tarski's relation algebras. In [39] Ligozat and Renz show that a non-associative algebra is the algebraic structure associated to a construction that imposes additional mild restrictions on a partition of a domain.

Definition 4 (Partition Scheme (see [39, 56]))**.** Let $\mathcal{B} = \{B_1, \ldots, B_k\}$ be a set of binary relations that partitions a non-empty domain $\mathcal{D}$ and fulfills the properties

(1) there is an $1 \leq i \leq k$ such that $B_\Delta = \{(x,x) \mid x \in \mathcal{D}\} = B_i$;
(2) $\forall i \in \{1, \ldots, k\}.\exists j \in \{1, \ldots, k\}.R_i^{\smile} = R_j$;

then $\mathcal{B}$ is called a *partition scheme* over $\mathcal{D}$.

The symbol $(\_)^{\smile}$ denotes the relational converse, which is to be defined formally. The additional restrictions (1) and (2) do little harm when observed from the view of qualitative spatial reasoning although there are some calculi lacking any or both of the properties. Freksa's double-cross calculus [19] is an example for a calculus lacking property (2). But normally calculi not fulfilling these properties have a problematic behavior with respect to reasoning. All binary calculi discussed in this thesis fulfill both properties. Restriction (1) states that the diagonal or identity relation has to be a base relation, this is in most cases desired by the calculus designer to be able to express equality in the calculus. The restriction (2) states that the converses of base relations need to be base relations. From the reasoning point of view this property is desirable, too, since when relying on this property the number of relations that need to be stored can be cut to half for binary relations.

In the next step we introduce the notion of a *non-associative algebra*, the algebraic structure associated to a partition scheme (see [39]). To complete the link between these structures, a meaning to the operations $\_;\_$, $(\_)^{\smile}$ and Δ has to be established. We will provide semantics in our setting after the definition of the structure.

Definition 5 (non-associative algebra (ref. to [45]))**.** A *non-associative (relation) algebra* or NAA for short is a tuple $\mathcal{R} = \langle R, +, -, \cdot, 0, 1, ;, ^{\smile}, \Delta\rangle$ such that:

(1) $\langle R, +, -, \cdot, 0, 1\rangle$ is a Boolean Algebra;

(2) Δ is a constant, $\breve{}$ a unary and ; a binary operation such that for any a, b, $c \in R$:

$$(a^{\smile})^{\smile} = a \qquad \Delta; a = a; \Delta = a$$
$$a; (b + c) = a; b + a; c \qquad (a + b)^{\smile} = a^{\smile} + b^{\smile}$$
$$(a - b)^{\smile} = a^{\smile} - b^{\smile} \qquad (a; b)^{\smile} = b^{\smile}; a^{\smile}$$
$$(a; b) \cdot c^{\smile} = 0 \iff (b; c) \cdot a^{\smile} = 0$$

If the operation ; is associative in an NAA, we obtain a relation algebra in Tarski's sense [74].

For a qualitative calculus, the Boolean algebra is the Powerset Boolean algebra $\mathcal{P}(\mathcal{B})$ with $\mathcal{B}$ being a set of $JEPD$ base relations. The operation + is then the union of sets ($\cup$), − is the binary minus operation of sets ($\setminus$), and $\cdot$ is the intersection operation ($\cap$). The constant 0 is the empty set and 1 is the set $\mathcal{B}$. The operations $\_;\_$, $(\_)^{\smile}$ and Δ cannot be defined in a universal way and need to be determined for each calculus individually. For qualitative calculi ; is commonly denoted by $\diamond$ and called *weak composition*. If ; is associative, then for qualitative calculi the operation is denoted by $\circ$ and called *strong composition*. Please note, that the weak composition is based on the strong operation as we will see in Definition 7.

Definition 6 (Relational Composition)**.** For two relations R and S over a domain $\mathcal{D}$ the operation $\circ$ is called *binary composition*, provided with the semantics of standard relational composition

$$R \circ S = \{\langle u, v\rangle \in \mathcal{D} \times \mathcal{D} \mid \exists w \in D.\langle u, w\rangle \in R \wedge \langle w, v\rangle \in S\}.$$

The composition as defined in Definition 6 is associative, and a non-associative algebra with such kind of composition is also a relation algebra. A common problem in qualitative reasoning with the relational composition is that for two relations R and S the composition $R \circ S$ cannot be expressed in the language of the calculus, i.e. $R \circ S$ equals no set in $\mathcal{P}(\mathcal{B})$ for the set $\mathcal{B}$ of base relations. This issue is cured by approximating the composition by the least superset of $R \circ S$ that is expressible in the language of the calculus. This approximating composition operation is called *weak composition* and denoted by $\diamond$. Weak composition is generally defined for base relations directly.

Definition 7 (Weak Composition(see [39]))**.** Let R and S be relations from a set of base relations $\mathcal{B}$. The *weak composition* $R \diamond S$ is then defined as:

$$\forall B_k \in \mathcal{B}.B_k \in R \diamond S \iff (R \circ S) \cap B_k \neq \varnothing$$

and can be rephrased as:

$$R \diamond S = \{B \in \mathcal{B} \mid R \circ S \cap B \neq \varnothing\}.$$

The results of weak composition for tuples of base relations are normally stored in a table indexed by these tuples. This table is called *composition table*. In general the semantics of the calculus at hand need to be employed to calculate the results of weak composition and hence building up a composition table is often not an easy task. This composition based approach has already been introduced by Allen with the "Transitivity Table" for his interval algebra in [1]. The weak composition on the base relations $\mathcal{B}$ is lifted to sets of base relations which form the general relations by Definition 3 as

$$R \diamond S = \bigcup \{B_R \diamond B_S \mid B_R \in R \wedge B_S \in S\}$$

for R and S in $\mathcal{R}_{\mathcal{B}}$, this implies by Definition 3 also that B_R and $B_S \in \mathcal{B}$ for which weak composition is defined. Weak composition is not necessarily associative.

The operation $\smile$ also needs to be derived from the semantics of each calculus. It is commonly called *converse*. Its behavior is derived from the semantics of the calculus at hand and stored in form of a table as in the case of the composition. On the level of relations the operation is defined as follows.

Definition 8 (Converse Operation)**.** Given a relation R, the *converse* $R^{\smile}$ is defined as

$$R^{\smile} = \{\langle y, x\rangle \in \mathcal{D} \times \mathcal{D} \mid \langle x, y\rangle \in R\}\,.$$

For many calculi the converse operation is strong, but not so for Freksa's Double Cross Calculus [19]. A weak operation must once more ensure that the outcomes of an operation are expressible in the language and can again only be defined with respect to a set of base relations.

Definition 9 (Weak Converse)**.** For a base relation R from a set of base relations $\mathcal{B}$ the operation $\frown$

$$R^{\frown} = \{B \in \mathcal{B} \mid R^{\smile} \cap B \neq \varnothing\}$$

is called *weak converse*.

As in the case of the weak composition, the weak converse can be lifted to the realm of sets to make it applicable for general relations. For a general relation R, we can get the weak converse from Definition 9 as

$$R^{\frown} = \bigcup \{B_R^{\frown} \mid B_R \in R \wedge B_R \in \mathcal{B}\}\,.$$

For a weak converse, several properties of a non-associative algebra can fail easily and need to be checked throughly. Without further investigation it is not even clear, if $(R^{\frown})^{\frown} = R$ holds for a calculus with a weak converse. For the strong converse in Definition 8, this property is always true.

The operation Δ is the *neutral element* of the composition. It is one of the base relations and can be found by inspection of the composition table. Further B_Δ is the relation

$$B_\Delta = \{\langle x, x\rangle \mid x \in \mathcal{D}\}\,,$$

the identity relation. Typically the diagonal relation B_Δ is also the neutral element of composition Δ. To see that the diagonal relation B_Δ is the neutral element of composition consider any base relation $B = \{\langle y, z\rangle \mid y, z \in \mathcal{D}\}$. We get

$$\begin{aligned} B \circ B_\Delta &= \{\langle y, z\rangle \mid \exists z.\langle y, z\rangle \in B \wedge \langle z, z\rangle \in B_\Delta\} \\ &= B \end{aligned}$$

via the definition of relational composition and see that for any B the relations B_Δ is also a neutral element of composition and hence we can infer $B_\Delta = \Delta$. The only base relation that has a non-empty intersection with B is B itself. For general relations we just need to lift these arguments to the level of sets of base relations. Showing the same for the left identity is analogous.

To establish the link between partition schemes and non-associative algebras, recall a proposition from [39].

Proposition 10 (Non-associative algebra (see [39])**.** *The algebraic structure associated to a partition scheme is a non-associative algebra.*

2.2.1. Weak representations.

An elegant way to characterize qualitative spatial calculi with binary relations is the notion of *weak representation*. This notion was introduced by Ligozat in [36] for relation algebras, but later extended to non-associative algebras in [39]. In fact, a weak representation can establish the link from a non-associative algebra to the underlying partition scheme, since it is a weak representation of the respective algebra.

Definition 11 (Weak representation (see [39]))**.** Let $\mathcal{R}$ be a non-associative algebra. A *weak representation* of $\mathcal{R}$ is a tuple $\langle \mathcal{D}, \varphi \rangle$ with $\mathcal{D}$ being a non-empty possibly infinite set and φ a map $\varphi : \mathcal{R} \longrightarrow \mathcal{P}(\mathcal{D} \times \mathcal{D})$ such that the properties

(1) φ is a homomorphism of Boolean algebras;
(2) $\varphi(\Delta) = \{(x, x) \in \mathcal{D} \times \mathcal{D}\}$;
(3) $\varphi(r^{\smile})$ is the transpose of $\varphi(r)$; and
(4) $\varphi(r; q) \supseteq \varphi(r) \circ \varphi(q)$

are fulfilled. A *weak representation* is a *representation* if the conditions

(5) φ is injective; and
(6) $\varphi(r; q) = \varphi(r) \circ \varphi(q)$

hold additionally.

Example 12 (Weak representation of $\mathcal{DRA}_{fp}$)**.** A natural weak representation for the $\mathcal{DRA}_{fp}$ calculus is the set of all dipoles, i.e. of all finite line segments of non-zero length, in the Euclidean Plane and the relations being defined with respect to the dipole semantics. Composition is relational composition and converse the relational converse.

With these definitions we can make clear the missing link between non-associative algebras and partition schemes. Consider a domain $\mathcal{D}$ and a partition scheme $\mathcal{B}$ over it. There needs to be a tuple $\langle \mathcal{D}, \varphi \rangle$ with $\varphi : \mathcal{R} \longrightarrow \mathcal{P}(\mathcal{D} \times \mathcal{D})$ where $\mathcal{R}$ is the non-associative algebra associated to $\mathcal{B}$. On the set of basic symbols r_i of the non-associative algebra φ is defined as

$$\varphi(r_i) = R_i \tag{1}$$

where R_i denotes the respective symbol in the partition scheme. This mapping is extended to the carrier set of $\mathcal{R}$ by a straight forward lifting to sets as

$$\varphi(r) = \sum_{r_i \in r} \varphi(r_i) \tag{2}$$

for $r \in \mathcal{R}$. Using this definition of φ Ligozat and Renz can prove the following proposition.

Proposition 13 (Weak representation (see [39]))**.** *Given a partition scheme on $\mathcal{D}$ with φ being defined as in equations* (1) *and* (2)*, then the pair $\langle \mathcal{D}, \varphi \rangle$ is a weak representation of $\mathcal{R}$.*

Further the weak representation associated to a partition scheme is a representation if and only if weak and strong composition coincide.

Using the notions of weak representations and non-associative algebras, an elegant definition of qualitative spatial calculi based on binary relations is possible.

Definition 14 (Qualitative Spatial Calculus (see [39])**.** A *qualitative spatial calculus* is a triple $(\mathcal{R}, \mathcal{D}, \varphi)$ where:

- $\mathcal{R}$ is a non-associative algebra; and
- $\langle \mathcal{D}, \varphi \rangle$ is a weak representation of $\mathcal{R}$.

Mossakowski et al. criticize in [56] the notion of weak representation as being too weak, since the abstract composition only provides little information about the concrete composition. The authors give several examples where this drawback leads to undesirable results, in the sense that the abstract composition in them turns out to be bigger than necessary and hence leading to an undesirable loss of information that might be curable. Basically Mossakowski et al. disagree with the slogan "A qualitative calculus is a weak representation" from [39] and they propose the notion of *semi-strong representation*.

Definition 15 (Semi-strong Representation (see [56]))**.** Given an atomic non-associative algebra $\mathcal{R}$, a weak representation $\varphi:\mathcal{R} \longrightarrow \mathcal{P}(\mathcal{D}\times\mathcal{D})$ is said to be *semi-strong* if for all $r,q \in \mathcal{R}$ the condition

$$r;q = \bigvee \{s \mid s \text{ atomic}, (\varphi(r)\circ\varphi(q)) \cap \varphi(s) \neq 0\}$$

holds.

But in the end even with this demand for a stronger notion for a formal definition of a qualitative spatial calculus, nothing is completely lost for Ligozat and Renz's definition. Since in [56] it is shown that the interesting weak representations that are induced by partition schemes are also semi-strong. The discussion of semi-strong representation again stresses the importance of the use of a partition scheme when defining a qualitative spatial calculus as a weak representation, since this also eases problems with possibly too weak composition operation in the non-associative algebra.

2.2.2. Binary constraint networks. Descriptions of (possibly not realizable) spatial configurations in qualitative spatial and temporal calculi are often expressed as constraint satisfaction problems. The instances of such a problem are described in the form of constraint networks. We will begin with an example of a constraint network. Consider the polygon in Figure 9. Each line segment of the polygon is interpreted as a dipole with the direction as in Figure 9. From the figure, we can

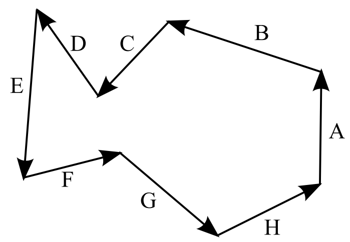


FIGURE 9. A polygon

derive the $\mathcal{DRA}_f$ relations

$(A \text{ ells } B)$ $(B \text{ ells } C)$ $(C \text{ errs } D)$ $(D \text{ ells } E)$ $(E \text{ ells } F)$ $(F \text{ errs } G)$
$(G \text{ ells } H)$ $(H \text{ ells } A)$

between consecutive dipoles. We observe that the relations between a lot of dipoles as e.g. between C and G are not listed, we assume that such relations are the universal ones. Then a common reasoning task is, to determine what refinements of the given set of constraints exist that can be realized (drawn) in the Euclidean plane.

Definition 16 (Constraint Network)**.** Let $\mathcal{R}$ be a non-associative algebra. A *constraint network* is a pair $\mathcal{N} = \langle N, \nu\rangle$, where N is a finite set of nodes or variables and ν a map $\nu : N \times N \longrightarrow \mathcal{R}$, i.e. for each pair $\langle x, y\rangle$ of nodes, which is also understood as an arc, $\nu(x,y)$ is the constraint on that arc.

If we are talking about a qualitative calculus with a set of base relations $\mathcal{B}$, we have $\mathcal{R} = \mathcal{R}_{\mathcal{B}}$ in Definition 16. Instead of writing $\nu(x,y) = R_1$, we also write $x\ R_1\ y$ or $(x\ R_1\ y)$ for x and y being variables and $R_1 \in \mathcal{R}_{\mathcal{B}}$.

Definition 17 (Atomic Constraint Network)**.** A constraint network $\mathcal{N} = \langle N, \nu\rangle$ over $\mathcal{R}_{\mathcal{B}}$ is called *atomic*, if $\nu(i,j) = b$ with $b \in \mathcal{B}$ for all $i,j \in N$. An atomic constraint network is also called a *scenario*.

In literature constraint networks are sometimes also called scenario, if for all i,j the mapping $\nu(i,j)$ yields values from the set of base relations or the universal relation, where especially the pairs of variables k,l with $\nu(k,l) = \mathcal{B}$ are left out in such a representation of constraint networks. We prefer to stick to the cleaner perspective on this of [39] as given in Definition 17. With our definition, the constraint network we gave for Figure 9 is not a scenario.

Definition 18 (Normalized Constraint Network)**.** A constraint network $\mathcal{N}$ is called *normalized* if for all $i,j \in N$ with $i = j$ the property $\nu(i,j) = \Delta$ and $\nu(i,j) = \nu(j,i)^{\smile}$ for $i \neq j$ is fulfilled.

In qualitative spatial reasoning it is assumed that each object is only in the identity relation to itself and such relations are commonly not noted in the constraint networks explicitly.

Definition 19 (Refinement (see [39]))**.** A constraint network $\mathcal{N}' = \langle N, \nu' \rangle$ is a *refinement* of $\mathcal{N} = \langle N, \nu \rangle$ if for all $i,j \in N$ the property $\nu'(i,j) \subseteq \nu(i,j)$ is fulfilled.

For given constraint networks the question arises if there is a valuation for all variables such that the constraint network can be realized in the plane. Further, getting rid of disjunctions that cannot lead to such a valuation is very desirable, i.e. we want a refinement of our constraint network that has such a valuation if and only if the the original network has one. We first introduce some notion that makes it easier to talk about constraint networks that have a valuation of all variables, such that it can be realized.

Definition 20 (Consistency)**.** A constraint network $\mathcal{N} = \langle N, \nu \rangle$ is called *consistent* if a valuation $\eta : N \longrightarrow \mathcal{D}$ exists such that all constraints are fulfilled.

Apart from consistency itself there are several interesting variations of this concept.

Definition 21 (n-consistency (see [20]))**.** A constraint network is called *n-consistent* ($n \in \mathbb{N}$) if every (partial) solution in $n-1$ variables can be extended to an n variable solution involving any further variable. A constraint network is called *strongly n-consistent*, if it is m-consistent for all $m \leq n$. A constraint network in n-variables is *globally consistent*, if it is strongly n-consistent.

A specialization of n-consistency is 3-*consistency* which is kin to *path consistency*. If the constraint network is understood as a graph, with the variables being the nodes and the constraints coloring the arcs, path consistency makes sure that for all triples of nodes (i,j,k) the constraints on the direct way from i to k are consistent with respect to composition on the way of length two from i to k via j.

Definition 22 (Path consistency (ref. to [47]))**.** A normalized constraint network $\mathcal{N}$ over a relation algebra $\mathcal{R}$ with $JEPD$ relations is called *path consistent* if for all $i,j,k \in N$ and $i\ R_1\ j$, $j\ R_2\ k$, and $i\ R_3\ k$ the property

$$R_3 \subseteq R_1 \circ R_2$$

holds for R_1, R_2 and $R_3 \in R$, the carrier set of $\mathcal{R}$.

Montanari has shown that if all paths of length two are (path) consistent, so are all path of any length [47]. Unfortunately path consistency does not guarantee the existence of a solution. Allen already used this method in [1] to determine consistency in his calculus, the Interval Algebra. This calculus turned out to be a lucky example, since algebraic closure decides consistency in scenarios and further tractable subsets exist.

Path consistency relies on the strong composition operation $\circ$. Unfortunately for many calculi composition is not strong and path consistency is not applicable to them. A similar, but weaker notion has been introduced, called *algebraic closure.*

Definition 23 (Algebraic closure)**.** A normalized constraint network $\mathcal{N}$ over a non-associative algebra $\mathcal{A}$ with $JEPD$ relations is called *algebraically closed* if for all $i, j, k \in N$ and $i\ R_1\ j$, $j\ R_2\ k$, and $i\ R_3\ k$ the property

$$R_3 \subseteq R_1 \diamond R_2$$

holds for R_1, R_2 and $R_3 \in A$, the carrier set of $\mathcal{A}$.

Algebraic closure can be enforced on any normalized constraint network by applying the operation

$$R_3 \leftarrow R_3 \cap R_1 \diamond R_2$$

until a fixed point is reached. If in this process $\varnothing$, the impossible relation, occurs, then there is no equivalent algebraically closed network to the given one and the given network is inconsistent. If the fixed point is reached (without any occurrence of $\varnothing$), then we get an equivalent algebraically closed network, but it still might be inconsistent. Algebraic closure is generally just an approximation to consistency, but for some calculi (as e.g. for Allen's Interval Algebra [1], $RCC8$ [61, 66]) it still can decide consistency and if only for atomic networks, but that has to be investigated for any calculus individually.

Formally, we can revisit the link between algebraic closure and consistency using weak representations. By the disjunctive semantics of general relations, a constraint network $\mathcal{N} = \langle N, \nu \rangle$ is consistent with respect to a qualitative calculus $(\mathcal{R}, \mathcal{D}, \varphi)$ if there is an atomic refinement $\mathcal{N}' = \langle N, \nu' \rangle$ of $\mathcal{N}$ that is consistent. If the refinement $\mathcal{N}' = \langle N, \nu' \rangle$ is atomic, normalized and algebraically closed, we consider its associated weak representation $\langle N, \rho \rangle$. Now we are able to strip down the consistency question to the question of the existence of a morphism, i.e. a structure-preserving mapping defined via identity and associativity properties. For more information on morphisms the interested reader is invited to refer to [44]. Consistency of our given network with respect to the weak representation $\langle \mathcal{D}, \varphi \rangle$ now means that there is a morphism $h : N \longrightarrow \mathcal{D}$, which is called an *instantiation*, that satisfies the property

$$\forall r \in \mathcal{R}.(i, j) \in \rho(r) \implies (h(i), h(j)) \in \varphi(r).$$

This leads to a definition of consistency relative to another weak representation, but we need to keep in mind that the consistency of Definition 20 is also relative, in fact relative to the domain where we take the values from. Let us recall the definition of consistency using weak representations from [39].

Definition 24 (Consistency (see [39]))**.** Let $\mathcal{N} = \langle N, \rho \rangle$ and $\mathbf{D} = \langle \mathcal{D}, \varphi \rangle$ be two weak representations of $\mathcal{R}$. Then $\mathcal{N}$ is *consistent* with respect to $\mathbf{D}$ if there is a morphism $h : N \longrightarrow \mathcal{D}$ such that the diagram in Figure 10 commutes.

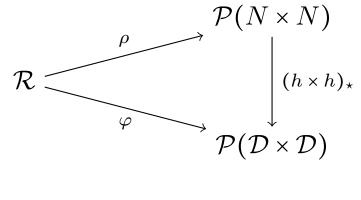


FIGURE 10. Consistency defined via weak representation

To sum it up for short, if we have a qualitative spatial calculus in binary relations that is based on a partition scheme and a constraint network on it, the network can be consistent if there is an algebraically closed refinement of it, but it is not consistent per se as soon as it is allows for such a refinement. This is the case if the valuation morphism h in Figure 10 does not exist. In the end, we have an efficient tool that outputs if a constraint network is "inconsistent" or "possibly consistent".

2.3. n-ary Relations

Even though there is no elaborated theory for them, as for calculi with binary relations, calculi with an arity of the relations different from two have been introduced. Especially calculi with ternary relations are quite prominent, as the $\mathcal{LR}$ calculus [73] and the double cross calculus [19]. Ad-hoc versions of compositions have been introduced that mostly simulated the binary compositions for calculi in binary relations in calculi with ternary relations leading to a high loss of information in reasoning.

For n-ary calculi a set of operations is defined in such a way that algebraic closure (see Definition 23) can be applied, but rather in an ad hoc fashion. A discussion of these operations can be found in [6]. In this section, we shall introduce relations for calculi of higher arity and the operations acting on them.

As a calculus on binary relations, one on higher arity relations is based on a set $\mathcal{B}$ of base relations (see Definition 2), which express precise knowledge about how n objects of an n-ary calculus are related. And again as before potentially imprecise knowledge is expressed by the elements of $\mathcal{P}(\mathcal{B}) = \mathcal{R}_{\mathcal{B}}$, the power-set of the base relations. For reasoning purposes the set $\mathcal{P}(\mathcal{B})$ has to be equipped with and closed under the operations $\cup$, $\cap$, $\varnothing$, fortunately it is. The power set $\mathcal{P}(\mathcal{B})$ also contains $\mathcal{B}$ and is equipped with the operation $\smallsetminus$. These operations are defined in a general way and no reference to the semantics of the calculus at hand is needed.

Two additional operations are needed for reasoning. Operations to "change perspective", and a weak composition operation denoted by $\diamond$. If composition is strong, the operation is often denoted by $\circ$. To change perspective on ternary relations Isli and Cohn [31] introduced the operations of permutation and rotation, which can be extended to n-ary versions in a straight forward way [6].

Definition 25 (Permutation & Rotation (see [6])). Let $R \in \mathcal{R}_{\mathcal{B}}$. The *permutation* $R^{\looparrowright}$ and the *rotation* $R^{\frown}$ of R are defined as

- $R^{\looparrowright} = \{(x_1, \dots, x_{n-2}, x_n, x_{n-1}) \mid (x_1, \dots, x_n) \in R\}$
- $R^{\frown} = \{(x_2, \dots, x_n, x_1) \mid (x_1, \dots, x_n) \in R\}$.

Now we have a look at the permutation $R^{\looparrowright}$ for binary relations, where we get

$$R^{\looparrowright} = \{\langle x_2, x_1\rangle \mid \langle x_1, x_2\rangle \in R\}\,.$$

which is equivalent to the converse operation $R^{\smile}$ (see Definition 8). Let us try the same for $R^{\frown}$

$$R^{\frown} = \{\langle x_2, x_1\rangle \mid \langle x_1, x_2\rangle \in R\}\,.$$

yielding again the definition of $R^{\smile}$. For higher arity calculi, say of arity n, all $n!$ permutations of the relation tuples can be obtained by combination of $(\_)^{\frown}$ and $(\_)^{\looparrowright}$.

For many calculi the operations $(\_)^{\frown}$ and $(\_)^{\looparrowright}$ yield base relations when applied to base relations. This is however not true for Freksa's double cross calculus [19].

For calculi in ternary relations special names for the permutations have been introduced, namely *homing*(HM), *shortcut*(SC) and *inverse*(INV) as well as *inverse homing*(HMI) and *inverse shortcut*(SCI).

Definition 26 (Permutations for ternary calculi)**.** Permutation operators for calculi in ternary relations are:

$$\begin{array}{rcl} INV(R) & := & \{(y,x,z) \mid (x,y,z) \in R\} = (\_)^{\frown} \circ (\_)^{\frown} \circ (\_)^{\looparrowright}(R) \\ SC(R) & := & \{(x,z,y) \mid (x,y,z) \in R\} = (\_)^{\looparrowright}(R) \\ SCI(R) & := & \{(z,x,y) \mid (x,y,z) \in R\} = (\_)^{\frown} \circ (\_)^{\frown}(R) \\ HM(R) & := & \{(y,z,x) \mid (x,y,z) \in R\} = (\_)^{\frown}(R) \\ HMI(R) & := & \{(z,y,x) \mid (x,y,z) \in R\} = (\_)^{\looparrowright} \circ (\_)^{\frown} \circ (\_)^{\frown}(R) \end{array}$$

If the calculus is closed under the operations INV, SC and HM, then HMI and SCI can be defined in terms of those operations. If desired, we can add an operation ID to the ones given above expressing the identity permutation.

Condotta et al. [6] propose to use an n-ary composition operation for a calculus in n-ary relations, which is defined as:

Definition 27 (n-ary relational composition (see [6]))**.** Let $R_1, R_2, \ldots, R_n \in \mathcal{R_B}$ be a sequence of n general relations of an n-ary qualitative calculus over the domain $\mathcal{D}$. Then the operation

$$\circ\,(R_1,\ldots,R_n) := \{(x_1,\ldots,x_n) \in \mathcal{D}^n \mid \exists u \in \mathcal{D}, (x_1,\ldots,x_{n-1},u) \in R_1, \\ (x_1,\ldots,x_{n-2},u,x_n) \in R_2,\ldots,(u,x_2,\ldots,x_n) \in R_n\}$$

is called *(strong) n-ary composition.*

If we take relations of arity 2 in definition Definition 27, we acquire the binary composition as in definition Definition 6.

2.3.1. Strong and Weak Operations. By the application of the operations of composition $\circ$, permutation $(\_)^{\looparrowright}$ and rotation $(\_)^{\frown}$, new relations are defined from old ones. As we have discussed for binary relations (see Definition 2.2), the problem arises whether they can be expressed in the language of the calculus or not. In fact, this is the question if the set of general relations of our calculus is *closed* under these operations. It turned out that the set relation of many calculi is not closed under at least one of these operations, which is in most cases the composition operation. There are even calculi for which no finite set of relations can exist that is closed under the composition operation, e.g. Freksa's Double Cross calculus [71].

In Section 2.2 we have discussed strong and weak operations in the special case of binary relations without introducing the notion formally. We will make up leeway.

Definition 28 (Strong Operation)**.** Let an n-ary qualitative calculus with relations $\mathcal{R_B}$ over a domain $\mathcal{D}$ and an m-ary operation $\phi : \mathcal{B}^m \longrightarrow \mathcal{P}(\mathcal{D}^n)$ be given. If the set of relations is closed under ϕ, i.e. $\forall \mathbf{B} \in \mathcal{B}^m \exists R' \in \mathcal{R_B}.\phi(\mathbf{B}) = \bigcup_{B \in R'} B$, then ϕ is called *strong.*

Qualitative spatial calculi are based on a finite set $\mathcal{B}$ of base relations and the set of general relations $\mathcal{P}(\mathcal{B})$ which as the powerset of $\mathcal{B}$ is also finite with a size of $2^{\|\mathcal{B}\|}$ elements. Hence, if an operation is not strong in the sense of Definition 28, we need to use an upper approximation of the true operation that is closed under the set of relations of the calculus at hand. We shall call such an derived operation *weak.*

Definition 29 (Weak Operation)**.** Given a qualitative calculus with n-ary relations $\mathcal{R_B}$ over a domain $\mathcal{D}$ and an m-ary operation $\phi : B^m \longrightarrow \mathcal{R_B}$, then the operator

$$\phi^\star : \mathcal{B}^m \longrightarrow \mathcal{R_B} \\ \phi^\star\,(B_1,\ldots,B_m) := \{R \in \mathcal{B} \mid R \cap \phi\,(B_1,\ldots,B_m) \neq \varnothing\}$$

is called a *weak operation,* namely the weak approximation of ϕ.

Observing the above definition of weak operations, we see that any calculus is closed under them. But this comes at a price: Since the weak operations are only approximations of the strong operations, applying them can lead to a loss of information. This loss of information has to be investigated and evaluated for the task at hand. In the best case the loss of information can be insignificant, in the worst case it can be critical. An operation and its weak approximation coincide if and only if the original operation is strong. In literature the weak composition operation is commonly denoted by $\diamond$. For weak permutations there is no commonly used operation symbol, since in many calculi the permutation turns out to be strong.

For some studies about the behavior of relations under composition, we need a special property of them, *convexity*.

Definition 30 (Convex Relations)**.** We call an m-ary relation R over $\mathbb{R}^n$ *convex*, if

$$\{y \mid R(x_1,\dots,x_{m-1},y), (x_1,\dots,x_{m-1},y) \in \mathbb{R}^n\}$$

is a convex subset of $\mathbb{R}^n$ for each $x_1,\dots,x_{m-1} \in \mathbb{R}^n$.

2.3.2. Traditional composition for $\mathcal{LR}$. Even though Condotta et al. propose in [6] to employ an n-ary composition operation for a calculus with n-ary relations, different (rather lossy) forms of composition operations have been used for different calculi. Prominent examples are the Flip-Flop calculus, $\mathcal{LR}$, and the Double Cross calculus, for which binary composition operations have been defined even though they are based on ternary relations. In the curse of this section be will further investigate $\mathcal{LR}$, since that calculus is of great interest for our work.

The composition operation of $\mathcal{LR}$ is the weakened version of the composition operation in the next definition.

Definition 31 (Traditional relational composition)**.** Let R_1^b and R_2^b be ternary base relations over a domain $\mathcal{D}$, then the operation

$$R_1^b \circ R_2^b = \left\{(x_1,x_2,x_3) \mid \exists u \in \mathcal{D}.\, (x_1,x_2,u) \in R_1^b \wedge (x_2,u,x_3) \in R_2^b\right\}$$

is called the *traditional relational composition* for the $\mathcal{LR}$ calculus.

Although the composition operation in Definition 31 theoretically works for all ternary calculi, we decide to talk about composition for the $\mathcal{LR}$ calculus, since for different ternary calculi binary composition operations based on different subsets of scenarios in four points have been employed.

In contrast to the composition operation from Definition 31, Condotta et al. [6] rather propose a ternary composition operation for calculi in ternary relations as a special case of the n-ary operation from Definition 27.

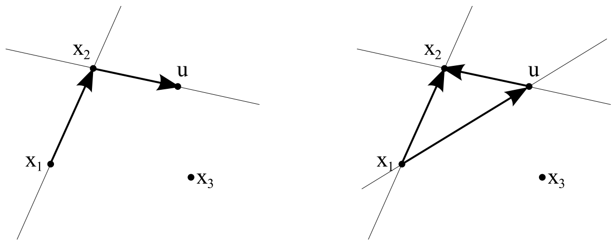
Definition 32 (Ternary relational composition (derived from [6]))**.** Let R_1^t, R_2^t, $R_3^t \in \mathcal{R}_\mathcal{B}$ be general relations of a ternary qualitative calculus over the domain $\mathcal{D}$. Then the operation

$$\circ\,(R_1^t,R_2^t,R_3^t) \coloneqq \{(x_1,x_2,x_3) \in \mathcal{D}^n \mid \exists u \in \mathcal{D}, (x_1,x_2,u) \in R_1^t, \\ (x_1,u,x_3) \in R_2^t, (u,x_2,x_3) \in R_3^t\}$$

is called *(strong) ternary composition.*

Let us have a closer look at the composition operations in Definition 31 and Definition 32. If we take the universal relation as R_2^t and set $R_2^b = INV(R_3^t)$ and $R_1^b = R_1^t$, we can directly recover the composition of Definition 31 from the composition in Definition 32.

But setting R_2^t to the universal relation is already a hint to the fact, that the composition from Definition 32 preserves more information than the composition according to Definition 31. In fact, if a caculus has strong permutations the composition according to Definition 32 is based on scenarios, while the one after Definition 32 is not (because of the universal relation). We have depicted this for some examples in Figure 11a and Figure 11b.



(a) Definition 31 (b) Definition 32

FIGURE 11. Examples of composition operations: (a) shows an example of the traditional composition as in Definition 31 and (b) show ternary composition as in Definition 32.

In those figures, the arrows depict the reference frames for $\mathcal{LR}$ for all relations involved in a composition operation. We can easily observe that in Figure 11a such a reference frame is "missing" wrt. Figure 11b, leaving the relation between x_1, x_2 and u undefined (which is synonymous to the universal relation).

Later, we will see that there is an utter weakness in the composition according to Definition 31 for $\mathcal{LR}$, but we will see that using ternary composition is no silver bullet either.

2.3.3. n-ary constraint networks. In Section 2.2.2 we have introduced constraint networks in binary constraints given by binary relations. In this section we are going to generalize the notions introduced there to n-ary relations.

Definition 33 (Constraint Network)**.** Let $\mathcal{R}_\mathcal{B}$ be a qualitative spatial calculus in n-ary relations. A *n-ary constraint network* is a pair $\mathcal{N} = \langle N, \nu \rangle$, where N is a finite set of nodes or variables and ν a map $\nu : N^n \longrightarrow \mathcal{R}$, i.e. for each tuple $(x_1, \ldots, x_n)$ of nodes, also known as an arc, $\nu(x_1, \ldots, x_n)$ is the constraint on that arc.

If we set $n = 2$ in Definition 33 we recover Definition 16 modulo the fact that non-associative algebras are only defined for binary relations. Instead of writing $\nu(x_1, \ldots, x_n) = R_1$ we shall also write $R_1(x_1, \ldots, x_n)$ or $(x_1 \ \ldots \ x_{n-1} \ R_1 \ x_n)$.

As for constraint networks for calculi in binary relations, we introduce some special kinds of constraint networks for calculi in ternary relations. These special constraint networks help us a lot when talking about reasoning.

First we introduce a form of constraint networks that is quite "basic" in the sense that only base relations are involved. In such a network there is no imprecise knowledge represented.

Definition 34 (n-ary Atomic Constraint Network)**.** A constraint network $\mathcal{N} = \langle N, \nu \rangle$ over an n-ary calculus $\mathcal{R}_\mathcal{B}$ is called *atomic*, if $\nu(x_1, \ldots, x_n) = b$ with $b \in \mathcal{B}$ for all $x_1, \ldots, x_n \in N$. An atomic constraint network is also called a *scenario*.

We introduce a notion of special constraint networks, where the permutation operations already agree with the given constraints. This notion eases talking about some reasoning aspects a lot.

Definition 35 (n-ary Normalized Constraint Network)**.** A constraint network $\mathcal{N}$ over an n-ary calculus is called *normalized* if for each vector of variables $(x_1, \ldots, x_n)$

with $x_1 = \dots = x_n$ the property $\nu(x_1, \dots, x_n) = B_\Delta$ holds and otherwise the relations between all permutations of $(x_1, \dots, x_n)$ agree with the permutation operations, that is

$$R^{\text{⊹}}(x_1, \dots, x_n, x_{n-1}) \iff R(x_1, \dots, x_{n-1}, x_n)$$
$$R^{\frown}(x_2, \dots, x_n, x_1) \iff R(x_1, \dots, x_{n-1}, x_n)$$

as well as compositions of $(\_)^{\text{⊹}}$ and $(\_)^{\frown}$ where the number of distinct compositions of these operations depends on the arity of the calculus.

In qualitative spatial reasoning it is assumed that each object is only in the identity relation to itself and such relations are not noted in the constraint networks explicitly.

The notions of consistency and n-consistency for n-ary constraint networks are the same as for binary ones. Please refer to Definition 20 and Definition 21 for them.

As for binary calculi (ref. to Definition 23), algebraic closure has been applied to calculi of higher arity without having the formal underpinnings of the non-associative algebras (ref. to Definition 5). In [6] a variant of algebraic closure for n-ary constraint networks is presented. We provide a better readable version of that.

Definition 36 (n-ary Algebraic closure)**.** A normalized constraint network $\mathcal{N}$ over an n-ary qualitative spatial calculus $\mathcal{R}_\mathcal{B}$ is called *algebraically closed* if for all $x_1, \dots, x_{n+1} \in N$ and

$$\begin{array}{l} R(x_1, \dots, x_n), \\ R_1(x_1, \dots, x_{n-1}, x_{n+1}), \\ R_2(x_1, \dots, x_{n-2}, x_{n+1}, x_n), \\ \dots, \\ R_n(x_{n+1}, x_2, \dots, x_n) \end{array}$$

the property

$$R \subseteq \diamond(R_1, \dots, R_n)$$

holds for R, R_1, …, $R_n \in \mathcal{N}$.

For algorithms that compute these kinds of algebraic closure the interested reader may refer to [6].

As for the case of composition, we need to consider the traditional composition operation as used for $\mathcal{LR}$ and double cross also for algebraic closure.

Definition 37 (Traditional ternary algebraic closure)**.** A normalized constraint network $\mathcal{N}$ over a ternary qualitative spatial calculus $\mathcal{R}_\mathcal{B}$ is called *algebraically closed* if for all $i, j, k, l \in N$ and $R_1(i, j, k)$, $R_2(i, j, l)$, and $R_3(j, l, k)$ the property

$$R_3 \subseteq R_1 \diamond R_2$$

holds for R_1, R_2 and $R_3 \in \mathcal{N}$.

As for the binary case it is per se unknown if algebraic closure decides consistency and this fact needs to be addressed for any calculus at hand.

2.4. Neighborhood-based Reasoning

Conceptual Neighborhoods were developed by Freksa in [17] as a complement to Allen's transitivity table (later known as composition table) based approach. Freksa claims that neighborhood based reasoning is cognitively more adequate and behaves more predictably when applied to the same scene at different levels of granularity than Allen's approach. In fact, Freksa applies his neighborhood based approach in [18] successfully to Allen's Interval Algebra [1] which base relations are shown in Figure 12.

For our work Neighborhood-based Reasoning is of lesser importance, since we want to supply results that can be used by developers interested in qualitative reasoning in an automated way. Even though the concept of Neighborhood-based Reasoning is not new, tool support for it is at most emerging, e.g. in the `SparQ` reasoner [78]. Because of this, we have to focus mainly on the well-traveled road of composition based reasoning as described in Section 2.2.2 and Section 2.3.3.

Conceptual neighborhoods have been initially developed for temporal reasoning in [17, 18]. Where one property of temporal reasoning is its one-dimensionality, i.e. it only operates on the set of reals. This is also reflected in the definition of Conceptual Neighborhood from [18].

Definition 38 (Conceptual Neighborhood (see [18]))**.** Two relations between pairs of events are *conceptual neighbors* if they can be directly transformed into one another by continuous deformation (i.e., shortening or lengthening) of the events.

For Allen's interval algebra the events are time intervals, i.e. intervals on the real line, and the direct transformation means that in the process of transforming relation R_1 into R_2 no other relation R_3 occurs.

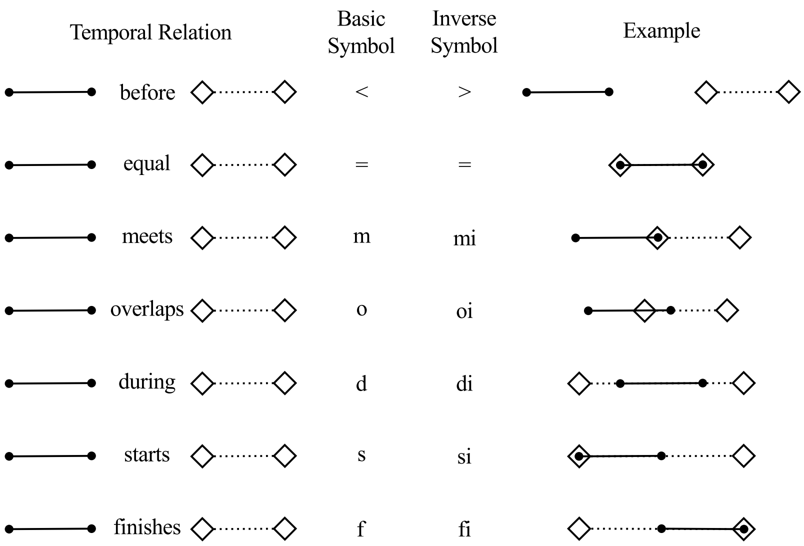


FIGURE 12. Allen's relations

Example 39 (Conceptual Neighbors < and m in Allen's calculus (see [18]))**.** Consider the base relations of Allen's interval algebra as shown in Figure 12. And consider two time intervals A and B, i.e. intervals on the real line, in the relation <

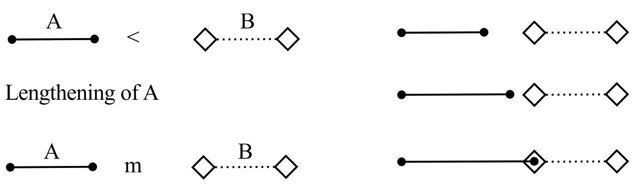


FIGURE 13. Example of Neighborhood in Allen's calculus

aka "before" as shown in Figure 13 on top. Now A is lengthened in the way that its endpoint is moved towards the start point of B as shown in the middle of Figure 13.

The relation remains < until the endpoint of A reaches the start point of B as in Figure 13 on the bottom. At that point the relation switches to m aka "meets" and we know that < and m are conceptual neighbors. If we move the endpoint of A further into the same direction (which is now towards the endpoint of B) the next relation we reach is o aka "overlaps". But < and o are not conceptual neighbors, since we always meet the relation m in the process of transforming < into o, since we always hit the borders of an interval before we can hit its interior. All this can be shown in a more general fashion using the semantics of the calculus, but we think that that is excessive at this point.

In [18] Freksa identified the neighborhood structure of Allen's Interval Algebra as shown in Figure 14.

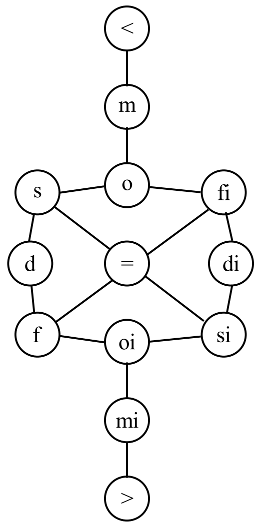


FIGURE 14. Neighborhood structure of Allen's Interval Algebra

In Figure 14 the nodes represent the relations of the calculus and are hence marked with their symbols. If any two nodes in the graph are linked by an arc, then they are neighbors. For reasoning a neighborhood oriented composition table is derived from this neighborhood structure, which has the property that the relations in the disjunctions on the right hand side of a composition always form a neighborhood. This table can then be used for reasoning. Further the level of granularity of the table can be altered easily and predictably by combining compositions of relations to compositions of neighborhoods. It is beyond the scope of this paper to explain the further construction of this neighborhood oriented composition table.

A crucial factor in the definition of the conceptual neighborhood is the *continuity* of the transformation. Continuity is a well-addressed issue in the literature [2, 9, 22, 23, 58]. Still in the case of space which continuous transformation to apply is not as clear as in the case of time, since we have more degrees of freedom. A discussion of this issue can be found in [12]. Dylla identifies translation[1], growing, shrinking and deformation as candidates for transformations that lead to neighborhood structures in $2D$ space.

[1]Called locomotion in [12].

2.5. Gröbner Reasoning

Gröbner reasoning can be used to solve systems of non-linear equations over the field of complex numbers. In [80] Wolter and Moshagen develop a method to test the solvability of systems of non-linear inequalities over the field of real numbers using Gröbner Reasoning. This method is implemented in `SparQ` [78]. It can be applied for all qualitative spatial calculi which relations possess semantics expressible as non-linear inequalities over the field of real numbers. For example $\mathcal{LR}$ and $\mathcal{DRA}$ have this property, the respective sets of equations are given in Section 3.1 and Section 3.2. We can already see that Gröbner reasoning is not performed on the level of the relations and that it takes properties of a specific domain into account. Basically it is a mighty tool, but unfortunately this might is reflected in its running time, which is basically doubly exponential.

In this section, we introduce Gröbner Reasoning and Wolter and Moshagen's method briefly, a more detailed description can be found in [80].

We are using methods from algebraic geometry to identify the solvability of systems of equations over a field. In fact these equations are sets of multivariate polynomials and the relations of a calculus must be describable as such in an adequate field.

A valuation of all variables of such a polynomial F with $F = 0$ is called a solution and the set of all solutions is called the *zero set*. A method for analyzing these zero sets is the use of Gröbner bases [4] which haven been introduced by Buchberger. The advantage of using them is the simplification of the view on polynomial equations that also simplifies the examination of the zero set.

To be able to talk about Gröbner bases, we need to remember some crucial definitions about constructions over polynomials. First, remember the definition of a *polynomial ring*.

Definition 40 (Polynomial Ring (see [80])). The set $F[x_1, \ldots, x_n]$ of all polynomials $f(x_1, \ldots, x_n)$ with coefficients from a field F is called a *polynomial ring*

$$F[x_1, \ldots, x_n] = \left\{ f = \sum_{\|\alpha\| \le k} a_\alpha \cdot x^\alpha, k \in \mathbb{N}^n, a_\alpha \in F \right\}$$

over F.

Each x^α with exponent $\alpha \in \mathbb{N}$ is called a *monomial*, e.g. $x^{(1,0,3)}$ stands for $x_1^1 \cdot x_2^0 \cdot x_3^3$. We introduce the notion of an ideal, which is formed by a common *zero set* of a set of polynomials.

Definition 41 (Ideal (see [80])). A subset I of a ring R is called *ideal* if the properties

(1) $0 \in I$;
(2) $\forall f, g \in I. f + g \in I$; and
(3) $\forall f \in I. \forall h \in R. h \cdot f \in I$

hold[2].

Wolter and Moshagen model the relations of a qualitative spatial calculus as polynomials and translate the constraint satisfaction problems to a finite basis. They then analyze the finitely generated ideal. To ease the effort of this inspection, they calculate and inspect the equivalent Gröbner bases instead of the original basis, but the ideal does not change in this process. The computation of the Gröbner bases depends on an ordering of the involved monomials.

[2]In fact, this is the definition of left ideal. But because of the commutativity of the ring of polynomials left, right and two-sided ideals coincide.

Definition 42 (Ordering of monomials (see [80]))**.** Let x^α be a monomial and $\alpha \in \mathbb{N}^n$ its exponent vector, then $\prec \subset \mathbb{N}^n \times \mathbb{N}^n$ is called a *monomoidal ordering* if the properties

(1) $\prec$ is total: $\forall \alpha, \beta \in \mathbb{N}^n . x^\alpha \prec x^\beta \vee x^\beta \prec x^\alpha$;
(2) $\prec$ is monotonous: $\forall \alpha, \beta, \gamma \in \mathbb{N}^n . x^\alpha \prec x^\beta \longrightarrow x^{\alpha+\gamma} \prec x^{\beta+\gamma}$; and
(3) $\prec$ is a well-ordering on $\mathbb{N}^n$, i.e. there exists a well-defined minimum wrt. $\prec$

are fulfilled.

Wolter and Moshagen discuss several examples of monomial orderings in [80], we will omit such an discussion, since it is out of the scope of this thesis. The interested reader is kindly referred to [80].

To apply Gröbner based methods we start with a set of polynomials that generate an ideal in a polynomial ring. Such an ideal is formed by the polynomials derived from a constraint satisfaction problem. A method to compute an equivalent basis of the ideal that satisfies certain properties, the Gröbner basis, is e.g. Buchberger's Algorithm [4]. Other methods for computing Gröbner bases are Faugère's F_4 [15] and F_5 [16] algorithms.

Definition 43 (Gröbner Basis)**.** Let F be a polynomial ring over a field and I be an ideal in this ring. A *Gröbner basis* G for I is characterized by any of the properties

- the ideal given by the leading terms of the polynomials in I is generated by the set of leading terms of the polynomials in G;
- for the leading term l_I of any polynomial in I there is a leading term l_G of a polynomial in of G such that l_I is divisible by l_G;
- multivariate division of any polynomial in F by G yields a unique remainder;
- multivariate division of any polynomial in I by G yields 0.

with respect to a monomial ordering $\prec$.

Example 44. [5] A small, nice and simple example regarding Gröbner bases can be found in [5]. Consider the system of equations

$$F = \left\{-2 \cdot y + x \cdot y, -x^2 + 2 \cdot y^2\right\}$$

where the equivalent Gröbner basis G is given as

$$G = \left\{-2 \cdot x^2 + x^3, -2 \cdot y + x \cdot y, -x^2 + 2 \cdot y^2\right\}$$

where the lexicographic order that ranks y higher than x is used for the ordering of the monomials. The Gröbner bases G has the same set of solutions as the original system of equations. Hence we can use G to determine the solutions of F, where the univariate term $-2 \cdot x^2 + x^3$ of G is of great help.

We also need to talk about a special form of a Gröbner basis, the *reduced Gröbner basis.*

Definition 45 (Reduced Gröbner basis)**.** A Gröbner basis G is called *reduced* if the properties

(1) the leading coefficient of all polynomials of G is 1; and
(2) no monomial in any element of G is in the ideal generated by the leading terms of the other elements of G

are fulfilled.

Gröbner bases an be used for a multitude of applications, as e.g. solving systems of polynomial equations. But we are not interested in solving systems of equations, all we are interested in the easier problem of determining is the mere existence of solutions. A theorem from [4] helps us with this.

Theorem 46 (Existence of solutions in $\mathbb{C}$ (see [4]))**.** *A polynomial ideal over the field $\mathbb{C}$ has an empty zero set if and only if the polynomial* 1 *is contained in the reduced Gröbner basis.*

Theorem 46 specifies a property of reduced Gröbner bases over the field of complex numbers $\mathbb{C}$. But we need to determine the existence of solutions in the field $\mathbb{R} \subset \mathbb{C}$, since many algebraic models of qualitative spatial calculi are defined over the field of reals.

Since $\mathbb{R}$ is a subset of $\mathbb{C}$, we know that if there are no solutions in $\mathbb{C}$, i.e. the reduced Gröbner basis contains the polynomial 1, we also know that there are no solutions in $\mathbb{R}$. But if there is a complex solutions by Theorem 46, a real solution may or may not exist. To determine the solvability in this case, Wolter and Moshagen introduce a set of rules. For the reason of completeness, we recall these rules:

(1) $x_i^k + c = 0 \Longrightarrow x_i = \sqrt[k]{-c}$

(2) $a_i x_i^k + a_j x_j^l = 0 \Longrightarrow x_i = -\sqrt[k]{\frac{-a_j}{a_i} x_j^l}$
Condition: l is divisible by k

(3) $\sum_{1 \le i \le k} a_i x_1^{2j_{1,i}} \cdot x_2^{2j_{2,i}} \cdot \dots \cdot x_n^{2j_{n,i}} = 0 \Longrightarrow \forall i \in \{1, \dots, k\} . \exists i_j . x_{i_j} = 0$
Condition: $\forall i, j \in \{1, \dots, k\} . \operatorname{sign}(a_i) = \operatorname{sign}(a_j)$

This set of rules is applied to the reduced Gröbner basis until none can be applied any more. If a rule yields an actual valuation for some variable, it is eliminated from the basis, as are variables by substitution (see rule (2)). If in this process a variable turns out to be necessarily non-real valued, the process stops and no real-valued solution can exist. Rule (3) gives alternative valuation branches for backtracking search. Wolter and Moshagen do not claim that these rules are complete, but they do claim that the rules work well for qualitative spatial reasoning.

The last issue that remains is that the algebraic modeling of spatial relations often yields set of non-linear inequalities rather than equations, that are needed. As an example of such a modeling refer to Section 3.1.2 for an algebraic modeling of $\mathcal{LR}$ via non-linear inequalities based on polynomials. To tackle this issue Wolter and Moshagen introduce an encoding of the non-linearities using so called *slack variables* s and v:

$$\begin{aligned}
f(x_1, \dots, x_n) < 0 &\rightsquigarrow f(x_1, \dots, x_n) + s^2 = 0 \wedge s^2 v^2 - 1 = 0 \\
f(x_1, \dots, x_n) \le 0 &\rightsquigarrow f(x_1, \dots, x_n) + s^2 = 0 \\
f(x_1, \dots, x_n) > 0 &\rightsquigarrow f(x_1, \dots, x_n) - s^2 = 0 \wedge s^2 v^2 - 1 = 0 \\
f(x_1, \dots, x_n) \ge 0 &\rightsquigarrow f(x_1, \dots, x_n) - s^2 = 0 \\
f(x_1, \dots, x_n) \ne 0 &\rightsquigarrow f(x_1, \dots, x_n) < 0 \vee f(x_1, \dots, x_n) > 0
\end{aligned}$$

One last effort has to be put into the handling of conjunctions and disjunctions. For example when verifying a composition table disjunctions are bound to appear on the right hand side of the composition. Conjunctions are a non-issue, since systems of equalities have conjunctive semantics. Disjunctions as $f(x_1, \dots, x_n) = 0 \vee g(x_1, \dots, x_n) = 0$ are modeled via a polynomial $f \cdot g$. Negations of $f(x_1, \dots, x_n) = 0$ lead to $f(x_1, \dots, x_n) \ne 0$.

We now have a complete toolkit for Gröbner reasoning in qualitative spatial reasoning. The main advantage of it is its might and generality. But the price to pay is high. First there is the doubly exponential running time of the transformation to Gröbner bases and the backtracking search induced by rule (3). The implementation of this approach in `SparQ` [78, 80] by Wolter and Moshagen can already verify a majority of entries in the composition table for several calculi, but it still often fails. This is a reason why we are searching for a more efficient and tailored solution for the $\mathcal{DRA}$ calculi, since for them the Gröbner approach still quite often fails.

Further, the high running time of the Gröbner approach still makes it questionable if it can be used on big constraint networks for actual reasoning.

CHAPTER 3

Qualitative Spatial Calculi

I was not made to smile
I was not built to lie
I ask my positron self
—**Voivod**, *Golem.*

After the introduction of the theory of qualitative spatial reasoning in Chapter 2 it is time for a detailed introduction of the calculi that we are going to use in this thesis. We are using a calculus in ternary relations, the $\mathcal{LR}$ calculus [72] and two families of calculi in binary relations, the $\mathcal{DRA}$ calculi [49, 52] and the $\mathcal{OPRA}$ calculi [54]. The $\mathcal{DRA}$ calculi allow for variations of the calculus that support the distinction of *qualitative angles* or not, and the $\mathcal{OPRA}$ calculi allow for different granularities of the base relations.

3.1. The $\mathcal{LR}$ Calculus

A calculus in our focus is the $\mathcal{LR}$-calculus [72], which is a refinement of Ligozat's Flip-Flop calculus [37]. The refinement was necessary, since equality of the two points of the reference frame was not expressible in the Flip-Flop calculus.

3.1.1. Basic Representation of $\mathcal{LR}$ Relations. The basic entity of *Flip-Flop* as well as of $\mathcal{LR}$ are points in the Euclidean plane. The local reference frame for a relation R between three points A, B and C written as $R(A,B,C)$ or $(A\ B\ \mathrm{R}\ C)$ in the $\mathcal{LR}$ calculus is formed by the ray l that contains A and B. Further, the

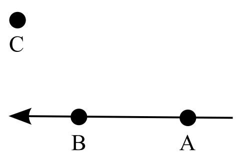


Figure 15. An $\mathcal{LR}$ base relation

equation $B > A$ has to be fulfilled with $>$ being the natural order on the ray. A possible instantiation of such a relation, in this case $(A\ B\ \mathrm{r}\ C)$, is shown in Figure 15. If $A = B$, we are in the special case that is handled by $\mathcal{LR}$ but not by Flip-Flop we will return to this case later. The Euclidean Plane is partitioned with respect to this ray for $A \neq B$. Special symbols are assigned to each of these partitions that provide the relation symbols R. The partitioning is as follows:

- The half-plane to the left of the ray providing $R = l$;
- the half-plane to the right of the ray providing $R = r$;
- the segment of the ray formed by all x with $x < A$ providing $R = b$;
- the point A providing $R = s$;
- the segment of the ray formed by all x with $A < x < B$ providing $R = i$;
- the point B providing $R = e$;
- the segment of the ray formed by all x with $B < x$ providing $R = f$.

The $\mathcal{LR}$ calculus adds two more relations (*dou*), if $A = B \neq C$ and (*tri*) if $A = B = C$. These are in fact the "holes" that are not covered by the Flip-Flop calculus. In Figure 15 the relation (A B r C) is shown. In Figure 16 instances for all $\mathcal{LR}$ base relations are shown.

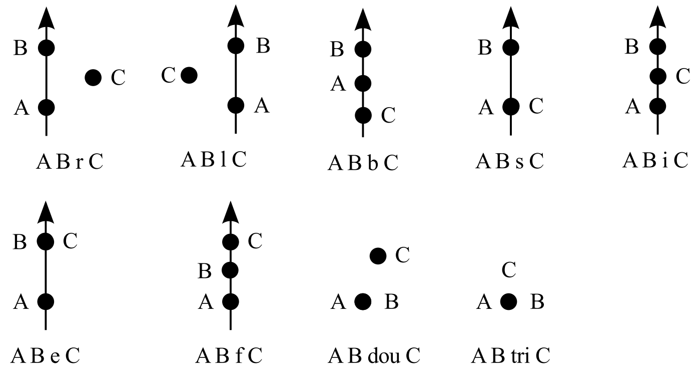


FIGURE 16. $\mathcal{LR}$ base relations

The $\mathcal{LR}$ calculus can be considered as the basis of other, more elaborate, calculi as the family of $\mathcal{DRA}$ calculi as well as Freksa's Double-Cross Calculus [19], which can be understood as a refinement of $\mathcal{LR}$.

In a first step, we consider the decision of consistency in the $\mathcal{LR}$ calculus as such, which already turns out to be a hard problem. But we are not shocked by this result, since solving consistency for arbitrary spatial CSPs often involves backtracking.

Theorem 1. *Deciding consistency of CSPs in $\mathcal{LR}$ is $\mathcal{NP}$-hard.*

SKETCH. In a straightforward adaption of the proof given in [70] for the $\mathcal{DCC}$ calculus, the $\mathcal{NP}$-hard problem NOT-ALL-EQUAL-3SAT can be reduced to equality of points. □

Algebraic closure usually is regarded the central tool for deciding consistency of qualitative CSPs. For the first qualitative calculi investigated (point calculus [75], Allen's interval algebra [1]) it turned out that algebraic closure decides consistency for the set of base relations, i.e. algebraic closure gives us a polynomial time decision procedure for consistency of qualitative CSPs when dealing with scenarios. This leads to the exponential time algorithm for deciding consistency of general CSPs using backtracking search to refine relations in the CSP to base relations [1]. Renz pioneered research on identifying larger sets for which algebraic closure decides consistency, thereby obtaining a practical decision procedure [66]. If however algebraic closure is too weak for deciding consistency of scenarios, no approaches are known for dealing with qualitative CSPs on the algebraic level. Unfortunately this is the case for the $\mathcal{LR}$-calculus as we shall see in Chapter 6.

3.1.2. Algebraic semantics for the $\mathcal{LR}$ calculus. The algebraic semantics of the $\mathcal{LR}$ calculus in the Euclidean Plane can be expressed in two ways. First we can use inequalities of a maximum degree of two to describe the relations, secondly we can make use of the atan2 function. The first approach yields descriptions that are well suited for Gröbner reasoning, whereas the second approach is better suited for implementations for qualification procedures for $\mathcal{LR}$, since many programming languages are equipped with an efficient implementation of the atan2 function.

For the first approach, we construct inequalities for each of the nine $\mathcal{LR}$ relations. Let three points A, B and C in the Euclidean Plane be given. We construct a local coordinate system at A with one of the axes pointing to B and the other having an offset of $-\frac{\pi}{2}$ to the former one. This provides us with an orthogonal coordinate system. The point C is situated in space with respect to this coordinate system. For $(A\ B\ \mathrm{r}\ C)$ this construction is shown in Figure 17. We introduce vectors

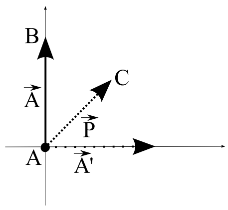


FIGURE 17. $\mathcal{LR}$ reference frame

$$\vec{A} = \begin{pmatrix} B_x - A_x \\ B_y - A_y \end{pmatrix} \qquad \vec{P} = \begin{pmatrix} C_x - A_x \\ C_y - A_y \end{pmatrix}$$

that have their common basis in our local coordinate system. We observe that $(A\ B\ \mathrm{l}\ C)$ is true if the angle from $\vec{A}$ to $\vec{P}$ lies in the interval $]0, \pi[$ and that $(A\ B\ \mathrm{r}\ C)$ is fulfilled if the same angle lies in the interval $]0, -\pi[$. For the angle having the values 0 or π, all points are collinear. Basically we then need an equation that lets us observe the sign of that angle. A standard approach was to get this angle ψ via the equation

$${}^t\vec{A} \cdot \vec{P} = ||\vec{A}|| \cdot ||\vec{P}|| \cdot \cos\psi$$

but unfortunately we would have to apply the inverse cosine function to obtain the angle. But $||\vec{A}|| \cdot ||\vec{P}|| \cdot \cos\psi$ is positive, if $\vec{P}$ points into the first or second quadrant with respect to $\vec{A}$ and negative if it points into the third or fourth. By the properties of the cosine function observing the normal vector of $\vec{A}$ (e.g. the normal pointing to the right of $\vec{A}$) yields the desired information already as the sign of the cosine. This normal vector $\vec{A}'$ is obtained as

$$\vec{A}' = \begin{pmatrix} B_y - A_y \\ A_x - B_x \end{pmatrix}$$

The angle from $\vec{A}'$ to $\vec{P}$ can be obtained as

$${}^t\vec{A}' \cdot \vec{P} = ||\vec{A}'|| \cdot ||\vec{P}|| \cdot \cos\gamma.$$

But all we are interested in is the sign of

$$||\vec{A}'|| \cdot ||\vec{P}|| \cdot \cos\gamma$$

which is basically the sign of $\cos\gamma$, since $||\vec{A}'|| \cdot ||\vec{P}||$ is always positive. We acquire this information from

$$\varphi \;=\; {}^t\vec{A}' \cdot \vec{P}$$

which is the scalar product of $\vec{A}'$ and $\vec{P}$. With the definitions of $\vec{A}'$ and $\vec{P}$, we get

$$\varphi \;=\; (B_y - A_y) \cdot (C_x - A_x) + (A_x - B_x) \cdot (C_y - A_y)$$

as our *characteristic* equation. By construction and definition of the $\mathcal{LR}$ relations, we obtain the case distinction under the assumption $A \neq B$

$$R = \begin{cases} l & \text{if } \varphi < 0 \\ r & \text{if } \varphi < 0 \\ b & \text{if } \varphi = 0 \wedge^t \vec{A} \cdot \vec{P} < 0 \\ s & \text{if } A = C \\ i & \text{if } \left[\begin{array}{l} \varphi = 0 \wedge^t \vec{A} \cdot \vec{P} > 0 \\ \|B - A\| > \|C - A\| \end{array} \right. \\ e & \text{if } B = C \\ f & \text{if } \left[\begin{array}{l} \varphi = 0 \wedge^t \vec{A} \cdot \vec{P} > 0 \\ \|B - A\| < \|C - A\| \end{array} \right. \end{cases}$$

The extension for $A = B$ is obvious and already contained in the definition of *dou* and *tri*.

The semantics of the relations can also be expressed in a more convenient and more regular way using atan2 and the function angle based on it:

$$\text{angle}(\vec{A}, \vec{B}) := \text{atan2}(\vec{A}_x \cdot \vec{B}_y - \vec{A}_y \cdot \vec{B}_x, \vec{A}_x \cdot \vec{B}_x + \vec{A}_y \cdot \vec{B}_y)$$

To determine an $\mathcal{LR}$-relation $(A\ B\ \mathrm{R}\ C)$ with $A \neq B$, we compute the angle from the vector $\vec{AB}$ to the vector $\vec{AC}$ as depicted in Figure 18a. Now we just need to

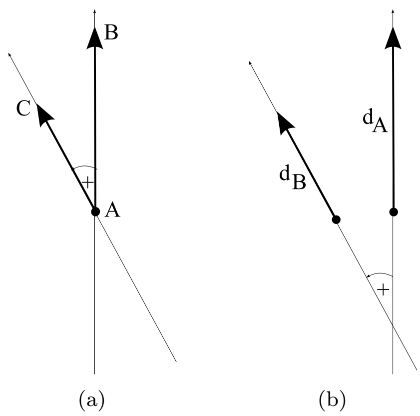


FIGURE 18. Computing $\mathcal{LR}$-relation (a) and qualitative angle (b) via vectors

substitute the $\mathcal{LR}$-relations for their definitions by the case distinction:

$$R = \begin{cases} l & \text{if } 0 < \text{angle}(AB, AC) < \pi \\ r & \text{if } -\pi < \text{angle}(ab, ac) < 0 \\ b & \text{if } \text{angle}(AB, AC) = \pi \vee \text{angle}(AB, AC) = -\pi \\ s & \text{if } \text{angle}(AB, AC) = 0 \wedge \|\vec{AC}\| = 0 \\ i & \text{if } \text{angle}(AB, AC) = 0 \wedge \|\vec{AC}\| < \|\vec{AB}\| \\ e & \text{if } \text{angle}(AB, AC) = 0 \wedge \|\vec{AC}\| = \|\vec{AB}\| \\ f & \text{if } \text{angle}(AB, AC) = 0 \wedge \|\vec{AC}\| > \|\vec{AB}\| \end{cases}$$

The conditions for s and e can be rewritten as $A = C$ and $B = C$ respectively. Again the extension for $A = B \neq C$ and $A = B = C$ is obvious.

We will see in Section 3.2.3 that we can reuse these definitions also for the semantics of the dipole calculi, we just need to apply them four times on the respective triples of points to obtain dipole relations.

3.2. The Dipole Calculi

After the introduction of a calculus in ternary relations which is based on configurations of points in the Euclidean plane, we are going to switch our attention to a calculus with more structured base entities. This calculus is based on directed lines of non-zero length which we also refer to as *dipoles*. The relations are formed by quadruples of $\mathcal{LR}$ relations between the start and end points of those lines in the simpler version of the calculus. In the more elaborate version we also account for the angles of intersection in a qualitative way, what we refer to as *qualitative angle*. Because of these different versions, we speak of the dipole calculi which are $\mathcal{DRA}_f$ (without distinction of angles of intersection) and $\mathcal{DRA}_{fp}$ (with the distinction of angles). Luckily, we can reuse a lot of the algebraic semantics introduced for $\mathcal{LR}$ for these calculi. $\mathcal{DRA}$ is an abbreviation for "Dipole Reasoning Algebra"[1].

3.2.1. Basic Representation of Dipole Relations. The basic entities we use are *dipoles*, i.e., oriented line segments of non-zero length formed by a pair of two points, a start point and an end point. Dipoles are denoted by $A, B, C, \ldots$, start points by $\mathbf{s}_A$ and end points by $\mathbf{e}_A$, respectively (see Figure 6). These dipoles are used for representing spatial objects with an intrinsic orientation. Objects exposing such an intrinsic orientation are quite prominent in the world as for example bikes, cars, motorbikes, ships and many other vehicles. Also furniture as desks, chairs and lockers comprise such an orientation. To a certain degree even animals and humans can comprise an intrinsic orientation. For example consider the car in Figure 19. The "natural" orientation of such a car is normally understood as the direction the car uses to travel into. We can describe it by an arrow at the centerline of the car

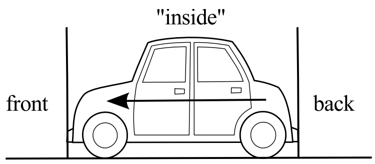


FIGURE 19. Car as oriented object

from the trunk to the hood. We can then assign several segments to this centerline. The segment "front" is everything in front of the front bumper, a segment at the end of the front bumper, a segment "inside" that is located in between the bumpers, the segment at the end of the rear bumper, and the segment "back" that describes everything behind the rear bumper. Apart from artificial things, also animals can be understood as oriented "objects". Consider the wolf in Figure 20. Again we

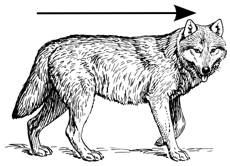


FIGURE 20. Wolf as oriented "object"

understand its natural orientation as equivalent to the direction it normally runs

[1]In literature $\mathcal{DRA}$ is often understood as an abbreviation for "Dipole Relation Algebra", but not all variants of this calculus form a relation algebra.

into. This orientation can be given as an arrow from the tail to the head of the animal[2].

To enable efficient reasoning, an attempt should be made to keep the number of different base relations relatively small, but on the other hand the set of base relations needs to be big enough to distinguish between the desired features. For this reason, we will restrict ourselves to using two-dimensional continuous space for now, in particular $\mathbb{R}^2$, and distinguish the location and orientation of different dipoles only according to a small set of seven different dipole–point relations. We distinguish between whether a point lies to the left, to the right, or at one of five qualitatively different locations on the straight line that passes through the corresponding dipole[3]. The corresponding regions are shown on Figure 21. These regions are very similar to the base frame of the Flip Flop [37] and the $\mathcal{LR}$ calculus [73]. We have discussed this calculus in Section 3.1. The $\mathcal{LR}$ calculus additionally features the relations dou and tri, which cannot occur as dipole–point relations, since a dipole has non-zero length and hence the reference points cannot coincide. In fact, we can recover the $\mathcal{LR}$ relations (apart from dou and tri) from the dipole–point relations, if we express the dipole as its start point paired with its endpoint.

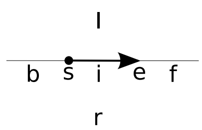


FIGURE 21. Dipole-point relations (= $\mathcal{LR}$ relations)

The dipole–point relations (ref. to Figure 21) describe cases where the point is: to the left of the dipole (l); to the right of the dipole (r); straight behind the dipole (b); at the start point of the dipole (s); inside the dipole (i); at the end of the dipole (e); or straight in front of the dipole (f). For example, in Figure 6, $\mathbf{s}_B$ lies to the left of A, expressed as $A\,\mathrm{l}\,\mathbf{s}_B$. Using these seven possible relations between a dipole and a point, the relations between two dipoles may be specified according to the following four relationships:

$$A\ \mathrm{R}_1\ \mathbf{s}_B \wedge A\ \mathrm{R}_2\ \mathbf{e}_B \wedge B\ \mathrm{R}_3\ \mathbf{s}_A \wedge B\ \mathrm{R}_4\ \mathbf{e}_A,$$

where $\mathrm{R_i} \in \{\mathrm{l, r, b, s, i, e, f}\}$ with $1 \leq i \leq 4$. Theoretically, this gives us 2401 relations, out of which 72 are geometrically possible. They constitute the dipole calculus $\mathcal{DRA}_f$ (f stands for fine grained) and they are listed in Figure 22. In the next subsection we present several versions of sets of dipole base relations also in an informal way. Then in Section 3.2.3 we define the dipole base relations in an algebraic way.

An embedding from Allen's interval algebra [1] into $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ exists. This calculus describes possible relations in linear flows of time on the real line. It has been introduced in Section 2.4 and its 13 base relations can be found in Figure 12.

[2]The author is aware of the fact that wolves and dogs like to chase their own tails from time to time what but these are not the situations interesting for our work.

[3]In his introduction of a set of qualitative spatial relations between oriented line segments, Schlieder [69] mainly focused on configurations in which no more than two end or start points were on the same straight line (e.g., all points were in general position). However, in many domains, we may wish to represent spatial arrangements in which more than two start or end points of dipoles are on a straight line.

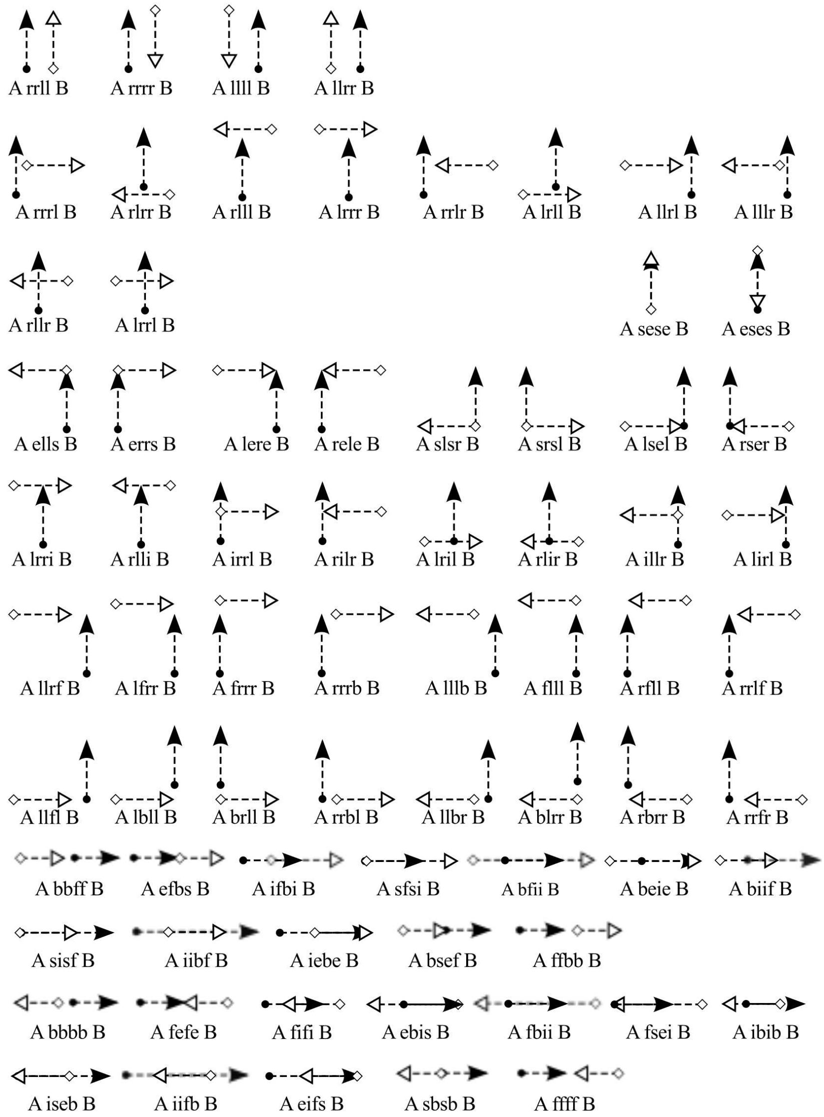


FIGURE 22. The 72 atomic relations of the $\mathcal{DRA}_f$ calculus. In the dipole calculus, orthogonality is not defined, although the graphical representation may suggest this.

Proposition 47. *Allen's interval algebra can be embedded into $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ by the following mapping of base relations:*[4]

$=$	$\mapsto$	sese			
b	$\mapsto$	ffbb	bi	$\mapsto$	bbff
m	$\mapsto$	efbs	mi	$\mapsto$	bsef
o	$\mapsto$	ifbi	oi	$\mapsto$	biif
d	$\mapsto$	bfii	di	$\mapsto$	iibf
s	$\mapsto$	sfsi	si	$\mapsto$	sisf
f	$\mapsto$	beie	fi	$\mapsto$	iebe

[4]Indeed, this yields a homomorphism of non-associative algebras. To show this fact, knowledge of the composition tables is mandatory, of course, as is the theory developed in Section 8.2.

Proof. See proof of Proposition 117. □

In cases stemming from the embedding of Allen's interval algebra, the dipoles lie on the same straight lines and have the same direction. $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ also contain 13 additional relations which correspond to the case with dipoles lying on a line but facing opposite directions.

3.2.2. Several Versions of Sets of Dipole Base Relations. In their paper on customizing spatial and temporal calculi, Renz and Schmid [65] investigated different methods for deriving variants of a given calculus that have a granularity better-suited for certain tasks. One of these methods uses only a subset of the original base relations as a new set of base relations. For example, Schlieder [69] introduced a set of base relations in which no more than two start or end points were on the same straight line. We call $\mathcal{DRA}_{lr}$ a calculus based on these base relations (where lr stands for left/right). The following base relations are part of $\mathcal{DRA}_{lr}$: rrrr, rrll, llrr, llll, rrrl, rrlr, rlrr, rllr, rlll, lrrr, lrrl, lrll, llrl, lllr. When inspecting the $\mathcal{DRA}_{lr}$ relations, we see that there is none that forms the two-sided identity of composition (for other versions of $\mathcal{DRA}$ this identity relation is sese), hence $\mathcal{DRA}_{lr}$ does not form a non-associative algebra.

Moratz et al. [49] introduced an extension of $\mathcal{DRA}_{lr}$ which adds relations for representing polygons and polylines. In this extension, two start or end points can share an identical location. In this calculus three points at different locations still cannot belong to the same straight line. Such a configuration of points is referred to as *general position* in geometry. This subset of $\mathcal{DRA}_f$ was named $\mathcal{DRA}_c$ (f refers to *fine*, c refers to *coarse*). The set of 24 base relations of $\mathcal{DRA}_c$ extends the base relations of $\mathcal{DRA}_{lr}$ with the following relations: ells, errs, lere, rele, slsr, srsl, lsel, rser, sese, eses.

The method of using only a subset of base relations reduces the number of base relations. Conversely, other methods extend the number of base relations. For example, Dylla and Moratz [14] have observed that $\mathcal{DRA}_f$ may not be sufficient for robot navigation tasks, because the dipole configurations that are pooled in certain base relations are too diverse. Thus, the representation has been extended with additional orientation knowledge and a more fine-grained $\mathcal{DRA}_{fp}$ calculus with additional orientation distinctions has been derived. It has slightly more base relations. In $\mathcal{DRA}_f$ it is impossible to express parallelism or angles of intersection of certain relations. Consider the three configurations in Figure 23. There, in all

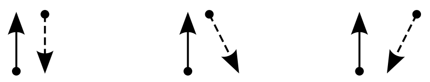


FIGURE 23. Different configurations with relation rrrr

three cases, the solid and dashed line are in the $\mathcal{DRA}_f$ relation rrrr, but we can see that the relative orientation of the dashed line with respect to the solid line is quite different. In the first case (on the left) the lines are anti-parallel, they never intersect in the Euclidean plane. In the second case the dashed line is pointing away from the solid one and in the third case it is pointing towards the solid one. Is maybe the configuration space for some relations in $\mathcal{DRA}_f$ still too big? Let us take a closer look.

The large configuration space for the rrrr relation is visualized in Figure 24. The other analogous relations which are extremely coarse are llrr, rrll and llll. In

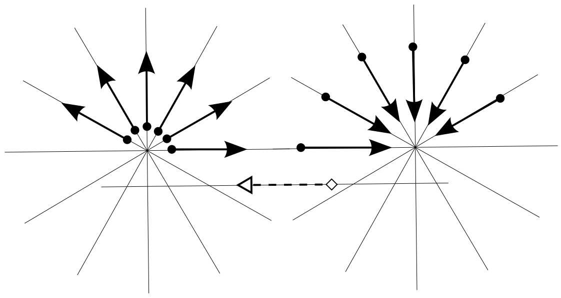


FIGURE 24. Pairs of dipoles subsumed by the same relation

many applications, this unwanted coarseness of four relations can lead to problems[5]. Therefore, an additional qualitative feature is introduced by considering the angle spanned by the two dipoles. This gives us an important additional distinguishing feature with four distinctive values. These qualitative distinctions are parallelism (P) or anti-parallelism (A) and mathematically positive and negative angles between dipoles A and B, leading to three refining relations for each of the four above-mentioned relations (Figure 25).

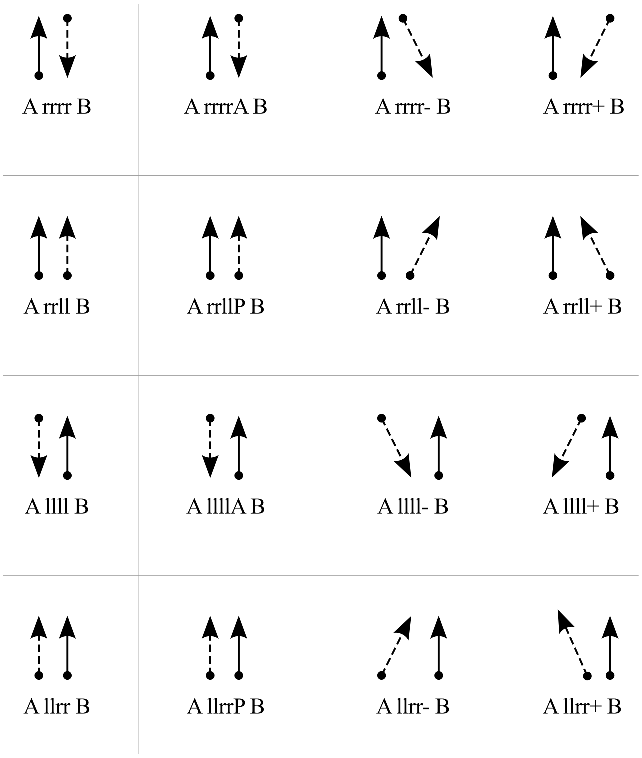


FIGURE 25. Refined base relations in $\mathcal{DRA}_{fp}$

We call this algebra $\mathcal{DRA}_{fp}$, as it is an extension of the fine-grained algebra $\mathcal{DRA}_f$ with additional distinguishing features due to “parallelism.” For the relations

[5]An investigation by Dylla and Moratz into cognitive robotics applications of dipole relations integrated in situation calculus [14] showed that the coarseness of $\mathcal{DRA}_f$ compared to $\mathcal{DRA}_{fp}$ would indeed lead to rather meandering paths for a spatial agent.

different from rrrr, llrr rrll and llll, the distinction between '+', '−', 'P' and 'A' is already determined by the original base relation and does not have to be mentioned explicitly. These base relations then have the same relation symbol as in $\mathcal{DRA}_f$.

The introduction of parallelism into dipole calculi not only has benefits in certain applications, the algebraic features also benefit from this extension (see Section 4.4.2).

The relations of $\mathcal{DRA}_{fp}$ can also be understood as a fusion of the $\mathcal{DRA}_f$ relations and the relations of the alignment calculus [12]. Dylla uses a similar approach in [12] to extend the $\mathcal{OPRA}$ calculus to $\mathcal{OPRA}^*$.

3.2.3. Formal Representation of Dipole Relations. The dipole relations have been introduced in an informal way in Section 3.2.1, but they can also be defined in an algebraic way. Every dipole D in the Euclidean plane $\mathbb{R}^2$ is an ordered pair of two points $\mathbf{s}_D$ and $\mathbf{e}_D$, each of them being represented by its Cartesian coordinates x and y, with $x, y \in \mathbb{R}$ and $\mathbf{s}_D \neq \mathbf{e}_D$:

$$D = (\mathbf{s}_D, \mathbf{e}_D), \qquad \mathbf{s}_D = ((\mathbf{s}_D)_x, (\mathbf{s}_D)_y), \qquad \mathbf{e}_D = ((\mathbf{e}_D)_x, (\mathbf{e}_D)_y)$$

To determine the $\mathcal{DRA}_f$ base relations, we remember the algebraic semantics of the $\mathcal{LR}$ relations in Section 3.1.2. Further we remember that a $\mathcal{DRA}_f$ relation $R_1R_2R_3R_4$ between two dipoles D_A and D_B is defined via dipole-point-relations as:

$$\begin{array}{l} D_A\ R_1\ s_{D_B} \\ D_A\ R_2\ e_{D_B} \\ D_B\ R_3\ s_{D_A} \\ D_B\ R_4\ e_{D_A} \end{array}$$

where we can further decompose the dipoles to their start and end points yielding the set of $\mathcal{LR}$ relations:

$$\begin{array}{l} R_1(s_{D_A}, e_{D_A}, s_{D_B}) \\ R_2(s_{D_A}, e_{D_A}, e_{D_B}) \\ R_3(s_{D_B}, e_{D_B}, s_{D_A}) \\ R_4(s_{D_B}, e_{D_B}, e_{D_A}) \end{array}$$

where each of the relations R_1, R_2, R_3 and $R4$ can be determined with the equations described in Section 3.1.2, apart from the fact that dou and tri cannot occur due to dipoles having non-zero length. We revisit this construction in Figure 26. The

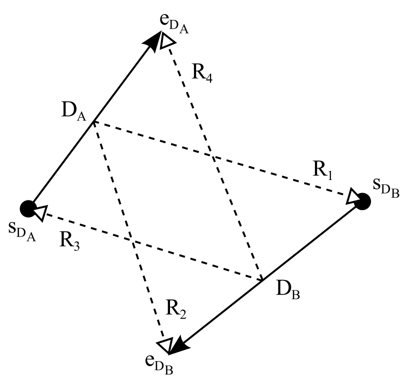


FIGURE 26. $\mathcal{DRA}_f$ relation expressed via $\mathcal{LR}$ relations

solid black arrows depict the dipoles in that figure, which can also be regarded as the line segment between their start and endpoints. The dashed arrows represent the $\mathcal{LR}$ relations R_1 to R_4 that make up the $\mathcal{DRA}_f$ relation. These arrows start at their reference frame given by the dipoles (in the picture we set this point to the middle of the line between the start and endpoints of the dipoles) and end at the respective point of the other dipole. Altogether, we need to apply the equations for $\mathcal{LR}$ relations that we constructed in Section 3.1.2 four times to calculate a $\mathcal{DRA}_f$ relation. But for computing the *qualitative angle* of a $\mathcal{DRA}_{fp}$ relation, we need a new equation that has not been used before.

For determining a $\mathcal{DRA}_{fp}$-relation, we additionally need to determine the angle of intersection of two dipoles D_A and D_B. We consider them as vectors $\vec{D}_A$ and $\vec{D}_B$ as shown in Figure 18b and we can get the quantitative angle via

$$\text{angle}(\vec{D}_A, \vec{D}_B)$$

where the function angle is defined in Section 3.1.2. By using the case-distinction

$$R_5 = \begin{cases} + & \text{if } 0 < \text{angle}(\vec{D}_A, \vec{D}_B) < \pi \\ - & \text{if } -\pi < \text{angle}(\vec{D}_A, \vec{D}_B) < 0 \\ A & \text{if } \text{angle}(\vec{D}_A, \vec{D}_B) = \pi \vee \text{angle}(\vec{D}_A, \vec{D}_B) = -\pi \\ P & \text{if } \text{angle}(\vec{D}_A, \vec{D}_B) = 0 \end{cases}$$

we retrieve the *qualitative angle* R_5, the abstraction of the relative orientation of the second dipole with respect to the first one, from the quantitative one. This component R_5 is concatenated to the right to the obtained $\mathcal{DRA}_f$ relation. If a $\mathcal{DRA}_{fp}$ relation only admits a unique fifth component, we normally do not explicitly note it.

3.3. The $\mathcal{OPRA}$ calculi

After the discussion of $\mathcal{LR}$, which is based on point triples in the plane, in Section 3.1 and of the family of the $\mathcal{DRA}$ calculi, which are based on dipoles in the plane, in Section 3.2 we focus on a family of calculi that is based on entities that lie in between, i.e. *oriented points*, which are, as the name suggests, points that are enriched with orientation information. But we also can perceive them as dipoles of infinitesimally small length. The calculi in question are called $\mathcal{OPRA}_m$, where $m \in \mathbb{N}$ is a parameter that determines the granularity of the base frames. As the dipole calculi the $\mathcal{OPRA}$ calculi feature binary relations. The abbreviation $\mathcal{OPRA}$ stands for "Oriented Point Reasoning Algebra"[6].

As briefly mentioned before the basic entity of the $\mathcal{OPRA}$ calculi is the *oriented point*, it is based on the perception of an agent moving through space that is going into a certain direction like a car. In fact the motivation for this abstraction is quite the same as for the dipole calculi in Section 3.2, the main difference is that now we disregard any possible lengths of objects. This disregard is not always a disadvantage, since we can lower the complexity of reasoning by limiting the number of base relations and observed from a certain distance objects look like points. Additionally to having coordinates in space (typically the Euclidean plane $\mathbb{R}^2$) *oriented points* are equipped with an orientation. This orientation can be given as an angle with respect to an axis. Using the Euclidean plane $\mathbb{R}^2$ is not mandatory for the definition of *oriented points*, in fact all we need is a dense and continuous vector space, but $\mathbb{R}^2$ is the most interesting space for qualitative spatial reasoning; that is why we stick to it for reasons of convenience. A configuration of oriented points is shown in Figure 27, where the location of the points is shown as the big black dots (●) and the orientations as the arrows starting at those dots.

Definition 48 (Oriented Point)**.** An *oriented point* is a tuple $\langle p, \varphi \rangle$, where p is a coordinate in $\mathbb{R}^2$ and φ an angle to an arbitrary but fixed axis.

[6]In literature $\mathcal{OPRA}$ is often referred to as "Oriented Point Relation Algebra" without having the knowledge if it is such kind of algebra, hence we prefer this notion.

We also can transcribe an oriented point as a tuple of points $\langle p_0, p_1 \rangle$, where p_0 is the anchor point, the point where the oriented point is located, and the vector from p_0 to p_1 encodes the orientation information. From this description, we can compute the angle φ to the axis easily. In fact this encoding of an oriented points gives us the link to dipoles, since p_0 can also be understood as the start point of a dipole and p_1 as its endpoint. This description contains more knowledge than is necessary for $\mathcal{OPRA}$ but if it is available, we can use it to derive our relations. By disregarding the lengths of the vectors, we arrive again at Definition 48.

FIGURE 27. Oriented points

The $\mathcal{OPRA}$ calculi comprise of relations between pairs of oriented points. These relations are of adjustable granularity. For any granularity $m > 0$ there is a version of the $\mathcal{OPRA}$ calculus denoted by $\mathcal{OPRA}_m$. For the introduction of relations the plane around each oriented point is sectioned by m lines with one of them having the same orientation as the oriented point itself. The angles between all lines are equal. The sectors, which are linear sectors (on the lines) and planar sectors (between two lines), are numbered from 0 to $4m - 1$ counterclockwise. The label 0 is assigned to orientation of the oriented point itself. Such a sectioning is shown in Figure 28; this

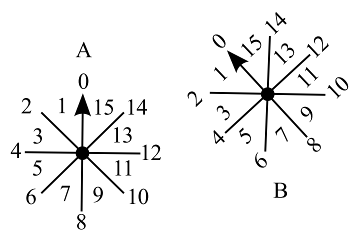


FIGURE 28. Oriented points with sectors

is in fact Figure 27 with the sectioning for $\mathcal{OPRA}_4$ being introduced. In fact, we introduce a set of angles

$$\bigcup_{0 \le i < 2m} \left\{ \left[i\frac{\pi}{m} \right], \left] i\frac{\pi}{m}, (i+1)\frac{\pi}{m} \right[\right\}$$

to partition the plane into the described sections. To introduce $\mathcal{OPRA}$ relations between two oriented points o and q, we need to distinguish between the two cases whether $\mathrm{pr}_1(o) = \mathrm{pr}_1(q)$ or not, where pr_1 is the projection to the first component of a tuple, i.e. we need to distinguish if both points have the same position in the plane.

An auxiliary construction to introduce $\mathcal{OPRA}_m$ relations are *half relations.*

Definition 49 (Half Relation)**.** For two oriented points o and q we call $o \rhd q$ the *half relation* from o to q.

If we want to annotate a sector i or granularity m to a half relation, we shall write $o_m \rhd_i q$. A half relation determines the number i of the sector around o where q lies in if $\mathrm{pr}_1(o) \neq \mathrm{pr}_1(q)$ and the sector around o which q points into if

$\mathrm{pr}_1(o) = \mathrm{pr}_1(q)$. E.g. in Figure 28 the oriented points B lies in sector 13 of A and we obtain the half relation $A_4 \rhd_{13} B$. And for A with respect to B we get $B_4 \rhd_3 A$.

For $\mathrm{pr}_1(o) \neq \mathrm{pr}_1(q)$ we obtain the $\mathcal{OPRA}_m$ relation from o to q as the product of $o_m \rhd_i q$ and $q_m \rhd_j o$, we will write this as $o_m \angle_i^j q$. And for $\mathrm{pr}_1(o) = \mathrm{pr}_1(q)$, we get the $\mathcal{OPRA}_m$ relation as the product of $o\ s\ q$ and $o_m \rhd_j q$ written as $o_m \angle_s^j q$, where s is a special symbol describing the coincidence of the position of points.

The semantics of half relations $o_m \rhd q$ with $\mathrm{pr}_1(o) \neq \mathrm{pr}_1(q)$ can be given with help of a factorization of the plane as

$$[i]_m = \begin{cases} \left]2\pi\frac{i-1}{4m}, 2\pi\frac{i+1}{4m}\right[& \text{if } i \text{ is odd} \\ \left[2\pi\frac{i}{4m}\right] & \text{if } i \text{ is even} \end{cases}$$

in the cyclic group $\mathbb{Z}_{4m}$ [54]. To compute the needed angles for given oriented points o and q for the half relations $o_m \rhd q$ we chose a third point z on the ray through o into the orientation of o. The orientation of z is arbitrary as well as its distance to o. We then form vectors using the positions of o, q and z:

$$\vec{a} = \begin{pmatrix} z_x - o_x \\ z_y - o_y \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} q_x - o_x \\ q_y - o_y \end{pmatrix}$$

For reasons of readability, we have omitted the pr_1 for the position. The angle ϕ' can now be determined via the atan2-function

$$\phi' = \mathrm{atan2}(\vec{a}_x\vec{b}_y - \vec{a}_y\vec{b}_x, \vec{a}_x\vec{b}_x + \vec{a}_y\vec{b}_y)$$

where $(\_)_x$ and $(\_)_y$ denote the projections to the respective coordinates. The angle ϕ' is normalized to the interval $[0, 2\pi]$ via

$$\phi = \begin{cases} \phi' + 2\pi & \text{if } \phi' < 0 \\ \phi' & \text{if } \phi' \geq 0 \end{cases}$$

to obtain the relation, we just need to look up the corresponding interval. For $\mathrm{pr}_1(o) = \mathrm{pr}_1(q)$ we just need to adapt the definition of the vectors, the rest stays the same. Let the oriented points o, q and z be defined as before. Additionally, we introduce a forth point y that lies on the ray through q having the same orientation as q. The distance from q to y is arbitrary as is the orientation of y. Using the position of these oriented points, we define the vectors:

$$\vec{a} = \begin{pmatrix} z_x - o_x \\ z_y - o_y \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} y_x - q_x \\ y_y - q_y \end{pmatrix}$$

Again we omitted the pr_1 in the definition. Now the angle between two vectors can be determined as before. In this case we could also simplify the formula further, since both vectors originate at the same position, but we forbear from doing so for the sake of uniformity.

The composition and converse tables for $\mathcal{OPRA}$ need to be calculated for any granularity separately, fortunately there is a quite efficient algorithm for this task [54].

CHAPTER 4

$\mathcal{DRA}$ Composition

Maybe plan a-1 should be
To make a b-line out of here
—**Voivod**, *Negatron.*

Creating a composition table for $\mathcal{DRA}_f$ and even determining the correct number of geometrically realizable base relations turned out to be a very hard and especially error prone task. In early publications even the number of base relations was wrong (ref. to [49]). Basically for a calculus as big and complex as $\mathcal{DRA}_f$ it is better to introduce an intermediate theory that allows for effective computation of the composition table and base relations by a machine than performing the task by hand to avoid errors that occur due to dwindling concentration of a human when working on the creation of such a table for a long time. Also using ad-hoc programs for the given task is often not very fruitful, since the designer of the program might forget to consider some important spatial configurations just too easily. It is our aim to present a theory that is as simple as possible and only needs finitely many representatives of the problem. Obtaining this for $\mathcal{DRA}_f$ is already asking a lot. The issue for $\mathcal{DRA}_{fp}$ is still a little more complex, but not too much.

4.1. A Condensed Semantics for the Dipole Calculus

This section appears in a similar way in [51] and [52]. Since the domains of most spatial calculi are infinite (e.g., the Euclidean plane), it is impossible just to enumerate all possible configurations relative to the composition operation when deriving a composition table[1]. Hence, the question remains how a composition table can be computed in an efficient and automatic way. To start, we tried generating the composition table of $\mathcal{DRA}_f$ directly, using the resulting quadratic inequalities as described in [49] and derived exhaustively on page 34. However, it turned out that it is infeasible to base the reasoning on these inequalities, even with the aid of interactive theorem provers such as Isabelle/HOL [60] and HOL-light [28] (the latter is dedicated to proving facts about real numbers) and Gröbner base reasoners (see Section 2.5)[2].

[1] It can be shown that the exhaustive inspection of a finite number of configurations in a finite grid would suffice to compute the composition table of the dipole calculi. The size of the grid needs to be double-exponential in the number of points [25], and therefore the number of grids to consider is triple-exponential. This is practically infeasible: even for three points, already $2^{2^{2^3}} \approx 10^{77}$ grids would need to be inspected.

[2] For the computation of the $\mathcal{DRA}_c$ composition table reported in [49], Gröbner base reasoning needed to be complemented by a grid method. In general, the research history of QSR about dipoles shows that it is necessary to use methods that yield more reliable results. The dipole composition on which we focus in this section involves configurations of three dipoles. However, even the much simpler question about a complete list of distinguishable dipole base relations characterized by certain properties (e.g., dipole to point relations) is not trivial. This question can be answered by configurations of just two dipoles and how to list them exhaustively. Deriving manually the 72 base relations of $\mathcal{DRA}_f$, or the 80 base relations of $\mathcal{DRA}_{fp}$, is an error-prone procedure. For this reason, the manually derived sets of base relations for the finer-grained dipole calculi described in [49, 69], as well as the composition tables, contained errors.

Therefore, we developed a qualitative abstraction which we call *condensed semantics*. It provides a level of abstraction from the metrics of the underlying space. We observe the Euclidean plane with respect to all possible line configurations that are distinguishable within the $\mathcal{DRA}$ calculi.

From a more formal point of view, a key insight is that two configurations are qualitatively different if they cannot be transformed into each other by maps that keep that part of the spatial structure invariant that is essential for the calculus. In our case, these maps are the (orientation preserving) affine bijections. A set of configurations that can be transformed into each other by appropriate maps is an *orbit* of a suitable automorphism group. Here, we use primarily the affine group $\mathbf{GA}(\mathbb{R}^2)$ and show in detail how this leads to qualitatively different spatial configurations. The results of this analysis can be mapped onto an effective method for computing the operation tables for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$.

4.1.1. Seven qualitatively different configurations. For the binary composition operation of $\mathcal{DRA}$ calculi, we have to consider all qualitatively different configurations of three lines. In order to formalize "qualitatively different configurations", we regard the $\mathcal{DRA}$ calculus as a first-order structure, with the Euclidean plane as its domain, together with all the base relations. Let us start with having a look at the automorphism groups for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$. Additionally we recall that an *automorphism* is an endomorphism that is also an isomorphism.

Recall that an affine map f from the Euclidean plane to itself is given by

$$f(x,y) = A\begin{pmatrix}x\\y\end{pmatrix} + (b_x, b_y)$$

where f is a bijection if and only if $\det(A)$ is non-zero.

Proposition 50 ($\mathcal{DRA}$ Automorphisms)**.** *All orientation preserving affine bijections are $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ automorphisms.*

(In [55], the converse is also shown.)

Proof. It suffices to show that orientation preserving affine bijections preserve the $\mathcal{LR}$ relations (which are the building parts of the $\mathcal{DRA}$ relations). Now, any orientation preserving affine bijection is a product of translations, rotations, scalings and shears. It is straightforward to see that these mappings preserve the $\mathcal{LR}$ relations. □

Automorphisms and their compositions form a group which acts on the set of points (and tuples of points, lines, etc.) by function application. Recall that if a group G acts on a set, an *orbit* consists of the set reachable from a fixed element by performing the action of all group elements: $G\,x = \{f(x) \mid f \in G\}$. The importance of this notion is the following:

> Two configurations which are qualitatively different belong to disjoint orbits of the automorphism group.

Note that while this is related to the theory of line arrangements [27], we here work in a slightly different setting. First, the theory of line arrangements uses a weaker notion of isomorphism than we do. Second, work about line arrangements mostly uses the projective plane where there are only two configurations of three lines, instead of the Euclidean plane where parallelism is possible. Here we are only interested in the Euclidean plane and have to distinguish the cases where two or more lines are parallel or even identical. The reason is that, e.g., $\mathcal{DRA}_{fp}$ distinguishes between $A\,\mathrm{llrrP}\,B$ (A and B point into the same direction and have distinct parallel carrier lines) and A and B being in some Allen relation (A and

B point into the same direction and have the same carrier line). Third, we also consider triples of lines (later on), not just sets of three lines.

Further note that Cristani's 2DSLA calculus [8], which can be used to reason about sets of lines, is too coarse for our purposes: our orbits (1) and (2) introduced below cannot be distinguished in 2DSLA.

We start with configurations consisting of from one up to three lines in the Euclidean plane, i.e., we consider the orbits of all sets $\{l_1, l_2, l_3\}$ where l_1, l_2 and l_3 are not necessarily distinct. We consider two such configurations to be isomorphic if they can be mapped into each other by an affine bijection. That is, we work with orbits in the group of *all* affine bijections (and not just the orientation preserving ones—orientations will come in at a later stage). This group is usually called the affine group of $\mathbb{R}^2$ and denoted by $\mathbf{GA}(\mathbb{R}^2)$.

A line in the Euclidean plane is given by the set of all points $\langle x, y\rangle$ for which $y = mx + b$. Given three lines $y = m_i x + b_i$ $(i = 1, 2, 3)$, we list their orbits by giving a defining property. In each case, it is fairly obvious that the defining property is preserved by affine bijections. Moreover, in each case, we show a *transformation property*, namely that given two instances of the defining properties, the first instance can be transformed into the second by an affine bijection. Together, this means that the defining property exactly specifies an orbit. The transformation property often follows from the following basic facts about affine bijections, see [21]:

(1) An *affine frame* [21] is for an affine space what a basis is for a vector space; in particular, any point of an affine space is a unique affine combination of points from the frame. An affine frame for an n-dimensional affine space consists of $n + 1$ points; in particular, an affine frame for the Euclidean plane is a point triple in general position. The importance of this notion in the present context is the following: An affine bijection is uniquely determined by its action on an affine frame, the result of which is given by another affine frame. Hence, given any two point triples in the Euclidean plane in general position, there is a unique affine bijection mapping the first point triple to the second.
(2) Affine maps transform lines into lines.
(3) Affine maps preserve parallelism of lines.

That is, it suffices to show that an instance of the defining property is determined by three points in general position and drawing lines and parallel lines.

We will consider the intersection of line i with line j $(i \neq j \in \{1, 2, 3\})$. This is given by the system of equations

$$\{y = m_i x + b_i,\ y = m_j x + b_j\}\,.$$

This does not cover the case of the line $x = 0$; however, without loss of generality, we can assume that this case does not occur: we always can apply an appropriate affine bijection mapping the three lines away from the line $x = 0$.

For $m_i \neq m_j$, the above system of equations has a unique solution:

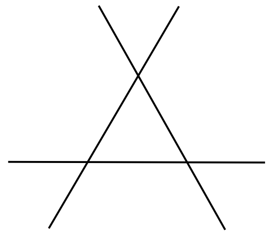
$$x = -\frac{b_i - b_j}{m_i - m_j},\ y = \frac{m_i b_j - m_j b_i}{m_i - m_j}.$$

For $m_i = m_j$, there is either is no solution ($b_i \neq b_j$; the lines are parallel), or there are infinitely many solutions ($b_i = b_j$; the lines are identical).

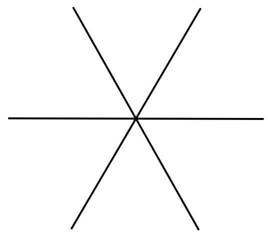
We can now distinguish seven cases:

(1) All m_i are distinct and the three systems of equations $\{y = m_i x + b_i,\ y = m_j x + b_j\}$ $(i \neq j \in \{1, 2, 3\})$ yield three different solutions. Geometrically, this means that all three lines intersect with three different intersection points. The transformation property follows

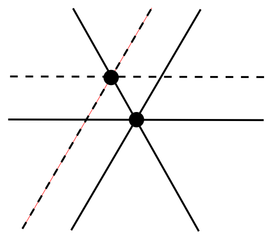
from the fact that the three intersection points determine the configuration. In the theory of line arrangements, this is called a *simple arrangement* [27].



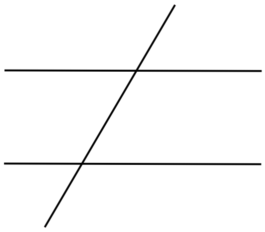
(2) All m_i are distinct and at least two of the three systems of equations $\{y = m_i x + b_i,\ y = m_j x + b_j\}$ $(i \neq j \in \{1, 2, 3\})$ have a common solution. Then, obviously, the single solution is common to all three equation systems. Geometrically, this means that all three lines intersect at the same point. In the theory of line arrangements, this is called a *trivial arrangement* [27].



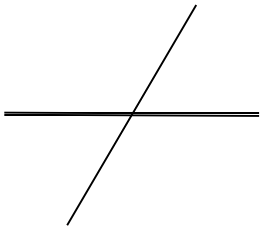
Take this point common point of intersection and a second point on one of the lines. By drawing parallels through this second point, we obtain two more points, one on each of the other two lines, such that the four points form a parallelogram. The transformation property now follows from the fact that any two non-degenerate parallelograms can be transformed into each other by an affine bijection.



(3) $m_i = m_j \neq m_k$ and $b_i \neq b_j$ for distinct $i, j, k \in \{1, 2, 3\}$. Geometrically, this means that two lines are parallel but not coincident, and the third line intersects them. Such a configuration is determined by three points: the points of intersection, plus a further point on one of the parallel lines. Hence, the transformation property follows.



(4) $m_i = m_j \neq m_k$ and $b_i = b_j$ for distinct $i, j, k \in \{1, 2, 3\}$. Geometrically, this means that two lines are equal and the third one intersects them. Again, such a configuration is determined by three points: the intersection point plus a further point on each of the (two) different lines. Hence, the transformation property follows.



(5) All m_i are equal, but the b_i are distinct. Geometrically, this means that all three lines are parallel, but not coincident. We cannot show the transformation property here, which means that this case comprises several orbits. Actually, we get one orbit for each distance ratio

$$\frac{b_1 - b_2}{b_1 - b_3}.$$

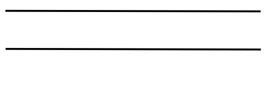
An affine bijection

$$f(x, y) = A\begin{pmatrix} x \\ y \end{pmatrix} + (b_x, b_y)$$

transforms a line $y = mx + b$ to $y = m'x + b'$, with $b' = c_1(m)b + c_2(m)$, where c_1 and c_2 depend nonlinearly on m. However, since $m = m_1 = m_2 = m_3$, this nonlinearity does not matter. This means that

$$\frac{b'_1 - b'_2}{b'_1 - b'_3} = \frac{c_1(m)b_1 - c_1(m)b_2}{c_1(m)b_1 - c_1(m)b_3} = \frac{b_1 - b_2}{b_1 - b_3},$$

i.e., the distance ratio is invariant under affine bijections (which is well-known in affine geometry). Given a fixed distance ratio, we can show the transformation property: three points suffice to determine two parallel lines, and the position of the third parallel line is then determined by the distance ratio. For a distance ratio 1, this configuration looks as follows:



Actually, for the qualitative relations between dipoles placed on parallel lines, their distance ratio does not matter. Hence, we will ignore distance ratios when computing the composition table below, and the fact that we get infinitely many orbits does not matter.

(6) All m_i are equal and two of the b_i are equal but different from the third. Geometrically, this means that two lines are coincident, and the third one is parallel but not coincident. Such a configuration is determined by three points: two points on the coincident lines and a third point on the third line. Hence, the transformation property follows.

(7) All m_i are equal, and the b_i are equal as well. This means that all three lines are equal. The transformation property is obvious.

Since we have exhaustively distinguished the various possible cases based on relations between the parameters m_i and b_i, this describes all possible orbits of three lines under the action of the group of affine bijections. Although we get infinitely many orbits for case (5), in contexts where the distance ratio introduced in case (5) does not matter, we will speak of seven qualitatively different configurations, and it is understood that the infinitely many orbits for case (5) are conceptually combined into one equivalence class of configurations.

Recall that we have considered *sets* of (up to) three lines. If we consider *triples* of lines instead, cases (3) to (6) split up into three sub-cases, because they feature distinguishable lines. We then get 15 different configurations, which we name 1, 2, 3a, 3b, 3c, 4a, 4b, 4c, 5a, 5b, 5c, 6a, 6b, 6c and 7. While 5a, 5b and 5c correspond to case (5) above and therefore comprise infinitely many orbits, the remaining configurations comprise a single orbit.

The next split appears at the point when we consider qualitatively different configurations of triples of undirected lines with respect to *orientation preserving* affine bijections. An affine map $f(x,y) = A\binom{x}{y} + (b_x, b_y)$ is *orientation preserving* if $\det(A)$ is positive. We now have to consider oriented affine frames. Let us call an affine frame (p_1, p_2, p_3) positively (+) oriented, if the angle $\angle p_1 p_2 p_3$ is positive, otherwise, it is negatively ($-$) oriented. Two given affine frames with the same orientation determine a unique orientation preserving affine bijection transforming the first one into the second. Thus, the orientation of the affine frame matters, and hence cases 1 and 2 above are split into two sub-cases each. For all the other cases, we have the freedom to choose the affine frames so that their orientations coincide. In the end, we get 17 different orbits of triples of oriented lines: 1+, 1-, 2+, 2-, 3a, 3b, 3c, 4a, 4b, 4c, 5a, 5b, 5c, 6a, 6b, 6c and 7. They are shown in Figure 29.

The structure of the orbits already gives us some insight into the nature of the dipole calculus. The fact that sub-case (1) corresponds to one orbit means that neither angles nor ratios of angles can be measured in the dipole calculus. By way of contrast, the presence of infinitely many orbits in sub-case (5) means that ratios of distances in a specific direction (not distances themselves) *can* be measured in the dipole calculus. Indeed, in $\mathcal{DRA}_{fp}$, it is even possible to replicate a given distance arbitrarily many times, as indicated in Figure 30. That is, $\mathcal{DRA}_{fp}$ can be used to generate a one-dimensional coordinate system. Note however that, due to the lack of well-defined angles, a two-dimensional coordinate system cannot be constructed. The ability to "count" in the $\mathcal{DRA}_{fp}$ calculus stems from the existence of relations

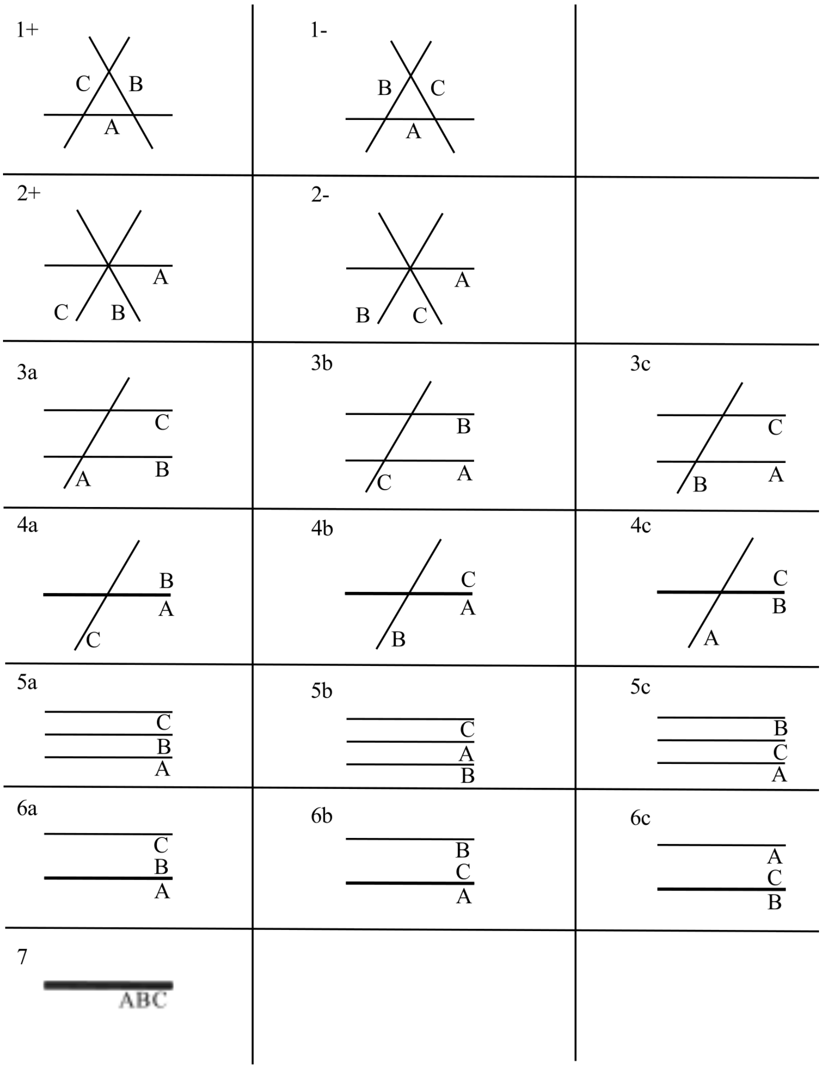


FIGURE 29. The 17 qualitatively different configurations of triples of oriented lines w.r.t. orientation preserving affine bijections

able to capture the feature of parallelism. Consider a sequence of parallelograms described in $\mathcal{DRA}_{fp}$ as

$$
\begin{array}{llll}
(A_i \text{ ells } B_i) & (A_i \text{ llrrP } C_i) & (A_i \text{ slsr } D_i) & (A_i \text{ lere } E_i) \\
(B_i \text{ lere } C_i) & (B_i \text{ llrrP } D_i) & (B_i \text{ lsel } E_i) & \\
(C_i \text{ rser } D_i) & (C_i \text{ srsl } E_i) & & \\
(D_i \text{ errs } E_i). & & &
\end{array}
$$

Such a sequence is depicted in Figure 30. The counting can be established by replicating such parallelograms by adding relations

$$(B_i \text{ sese } D_{i+1})$$

which claim that B_i and D_{i+1} of two consecutive parallelograms coincide. Such parallelograms can be constructed with all relations describing parallelism. The construction for anti-parallelism is a little more involved, in this case sequences of two parallelograms will be replicated.

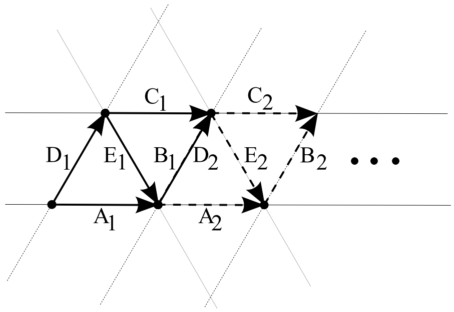


FIGURE 30. Example for replication of a given distance in $\mathcal{DRA}_{fp}$

During our investigations into the computation of the composition tables for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ based on condensed semantics two different methods to solve this task have been developed. The first method, presented in Section 4.2, comprises a very clear distinction between the different layers involved in this task. Unfortunately, it is a little complicated. In Section 4.3 a simpler method is presented. But this simplicity comes at the cost of a less clear distinction of the geometric and symbolic layers.

4.2. Computing the Composition Table via Condensed Semantics

This section covers our first method of computing composition tables and was published in [51]. The purpose of the condensed semantics is to provide a way of compressing the inherently infinite domain of a qualitative spatial calculus in such a way that only finitely many distinctions suffice to obtain the composition table for a calculus. Depending on the actual approach the computation of the composition tables can be performed on an abstract and purely symbolic level, not explicitly referring to geometric objects. Instead of reasoning over all possible triples of dipoles in the Euclidean plane, condensed semantics provides a notion of *qualitative composition configuration* (qcc, see Definition 52) that qualitatively describes the positions of three dipoles on their carrier lines. Roughly, a qcc provides the following information: (1) the configuration of the three carrier lines, in particular, how many intersection points they have, and (2) whether the start and end points of the dipoles lie between, on, or beyond the intersection points. A *geometric realization* of a qcc (Definition 56) is a triple of dipoles in the Euclidean plane that satisfies the constraints given by the qcc. This distinction between qcc and its realization provides us with a clean differentiation of the symbolic theory of gccs and their geometric semantics. Unfortunately, this comes at a price of increased complexity, a price we are willing to pay in this section. Later on in Section 4.3 we are going to present a simpler instance of condensed semantics that does not come with this clean distinction.

We will prove that each configuration of three dipoles in the Euclidean plane is the realization of a suitable qcc (Proposition 57), which means that qccs are complete in the sense that they cover all possible geometric configurations. Moreover, we will provide a means (via so-called classifiers) to compute from a qcc the relations between the three (abstract) dipoles, yielding an entry of the composition table. An important aspect is that this computation can be easily written down as a functional program that runs on a computer, such that there is not much distance between the mathematical formulation and the functional program. This greatly reduces the possibilities of errors due to incorrect implementation of the mathematical theory. Finally, we will prove that the dipole relations computed for a qcc coincide with those for any geometric realization (Theorem 68). Thus, using qccs for computing composition tables is sound. Altogether, qccs provide a sound and complete algorithmic way of obtaining composition tables.

The information about the configuration of the carrier lines of the three dipoles will be represented in a qcc using the seventeen different configurations (orbits) for triples of (undirected) lines for the automorphism group of orientation-preserving affine bijections that have been identified in the previous section (Figure 29). In order to be able to describe the qualitative positions of start and end points of dipoles, each of the three (abstract) lines l_A^a, l_B^a, l_C^a of a qcc carries two abstract segmentation points S_X and E_X ($X \in \{A, B, C\}$). $\mathbf{P} = \{S_A, S_B, S_C, E_A, E_B, E_C\}$ is the set of all abstract segmentation points. Note that the points of intersection of the three lines will be used as segmentation points whenever possible; however, in all cases except for the first two configurations (1+ and 1−), not all segmentation points can obtained in this way. The introduction of all segmentation points will be discussed later.

In a geometric realization of a qcc, the segmentation points lead to a segmentation of the lines. So, we introduce five abstract segments F, E, I, S, B (the letters are borrowed from the $\mathcal{LR}$ calculus, which has been introduced in Section 3.1, and share the same semantics as in that calculus). The set of abstract segments is denoted by $\mathcal{S}$. It is ordered in the following sequence:

$$F > E > I > S > B.$$

The geometric intuition behind this construction is shown in Figure 31.

FIGURE 31. Segmentation on the line.

Having this segmentation of line configurations, we can introduce qualitative configurations for *abstract dipoles* by qualitatively "locating" their start and end points based on the above segmentation. In the case that two or more points fall onto the same segment, information on the relative location of points within that segment is needed; this is provided by an ordering relation denoted by $<_p$.

By $\mathcal{D}$, we denote the set $\mathcal{S}\times\mathcal{S}\setminus\{(S,S),(E,E)\}$ (the exclusion of $\{(S,S),(E,E)\}$ is motivated by the fact that the start and end points of a dipole cannot coincide). For $dp \in \mathcal{D}$, by $st(dp)$ and $ed(dp)$, we denote the projections to the first and second components of each tuple, respectively. For convenience, we call the elements of the co-domains of st and ed *abstract points*.

Finally, we need information on the points of intersection of lines. Depending on orbit, there may be none, one, two or three such points. Hence, we introduce sets $\hat{\mathcal{S}}(i)$ with $i \in \{1+, 1-, 2+, 2-, 3a, 3b, 3c, 4a, 4b, 4c, 5a, 5b, 5c, 6a, 6b, 6c, 7\}$, which give names to each abstract point of intersection. These sets are defined as:

$$
\begin{array}{rcl@{\qquad\qquad}rcl}
\hat{\mathcal{S}}(1+) & := & \{\hat{s}_{AB}, \hat{s}_{BC}, \hat{s}_{AC}\} & \hat{\mathcal{S}}(4c) & := & \{\hat{s}_{ABC}\} \\
\hat{\mathcal{S}}(1-) & := & \{\hat{s}_{AB}, \hat{s}_{BC}, \hat{s}_{AC}\} & \hat{\mathcal{S}}(5a) & := & \varnothing \\
\hat{\mathcal{S}}(2+) & := & \{\hat{s}_{ABC}\} & \hat{\mathcal{S}}(5b) & := & \varnothing \\
\hat{\mathcal{S}}(2-) & := & \{\hat{s}_{ABC}\} & \hat{\mathcal{S}}(5c) & := & \varnothing \\
\hat{\mathcal{S}}(3a) & := & \{\hat{s}_{AB}, \hat{s}_{AC}\} & \hat{\mathcal{S}}(6a) & := & \varnothing \\
\hat{\mathcal{S}}(3b) & := & \{\hat{s}_{AC}, \hat{s}_{BC}\} & \hat{\mathcal{S}}(6b) & := & \varnothing \\
\hat{\mathcal{S}}(3c) & := & \{\hat{s}_{AB}, \hat{s}_{BC}\} & \hat{\mathcal{S}}(6c) & := & \varnothing \\
\hat{\mathcal{S}}(4a) & := & \{\hat{s}_{ABC}\} & \hat{\mathcal{S}}(7) & := & \varnothing \\
\hat{\mathcal{S}}(4b) & := & \{\hat{s}_{ABC}\} & & &
\end{array}
$$

where $\hat{s}_{XY}$ denotes the point of intersection of abstract lines l_X^a and l_Y^a and $\hat{s}_{XYZ}$ denotes the the point of intersection of the three abstract lines l_X^a, l_Y^a and l_Z^a. Although two identical lines intersect everywhere, we do not consider this here, because this does not provide a unique point of intersection.

In the geometric interpretation, we require segmentation points that coincide with points of intersection whenever possible, i.e. if a corresponding intersection point to a segmentation point exists, we map this segmentation point onto the intersection point. This coincidence is expressed via an *assignment mapping*, which is a partial mapping $a : \mathbf{P} \rightharpoonup \hat{\mathcal{S}}(i)$ subject to the following properties:

- if $a(S_X) = \hat{s}_y$, then y (as a string) contains X;
- if $a(E_X) = \hat{s}_y$, then y (as a string) contains X;
- if both $a(S_X)$ and $a(E_X)$ are defined, then $a(S_x) \neq a(E_x)$, for all $X \in \{A, B, C\}$;
- the domain of a has to be maximal.

The first two conditions express that each abstract segmentation point is mapped to the correspondingly named abstract point of intersection. The third condition requires that the abstract segmentation points of a line cannot be mapped to the same abstract point of intersection. The last condition ensures that abstract segmentation points are mapped to abstract points of intersection whenever possible.

Example 51. Consider a qcc with identifier $i = 3b$ be given, i.e. a qcc in configuration number $3b$ in Figure 29. A prototypical line configuration from its orbit is given in Figure 32. We can see that the lines A and C as well as B and C intersect, yielding two intersection points $\hat{s}_{AC}$ and $\hat{s}_{BC}$, which also form the set $\hat{\mathcal{S}}(3b)$. We have several degrees of freedom to assign the points S_X, E_X with $X \in \{A, B, C\}$ to our intersection points $\hat{s}_{AC}$ and $\hat{s}_{BC}$, but some of the choices will have effects on the orientation of the rays, that are obtained by giving an orientation to the lines in Figure 32 w.r.t. position of the points S_X and E_X. We can set $a(S_C) = \hat{s}_{AC}$, what demands that we assign $a(E_C) = \hat{s}_{BC}$, making the ray C pointing upwards and to the right in Figure 32, whereas the assignment $a(S_C) = \hat{s}_{BC}$ and $a(E_C) = \hat{s}_{AC}$ will make the same ray point downwards and to the left. We can set either $a(S_B) = \hat{s}_{BC}$ or $a(E_B) = \hat{s}_{BC}$, as well as be can assign either $a(S_A) = \hat{s}_{AC}$ or $a(E_A) = \hat{s}_{AC}$, this assignment is arbitrarily and without consequences so far. If we assign e.g. $a(S_A) = a(E_A) = \hat{s}_{AC}$, then the requirement if both $a(S_X)$ and $a(E_X)$ are defined, then $a(S_x) \neq a(E_x)$, for all $X \in \{A, B, C\}$ is violated. If the map is not maximal, we will have abstract intersection points that are not assigned to real intersection points breaking our further construction.

We now arrive at a formal definition:

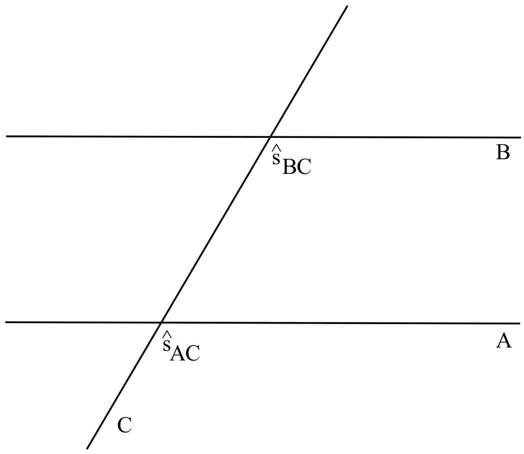


FIGURE 32. Mapping of intersection points

Definition 52 (Qualitative Composition Configuration)**.** A *qualitative composition configuration* (qcc) consists of:

- An identifier i from the set $\{1+, 1-, 2+, 2-, 3a, 3b, 3b, 4a, 4b, 4c, 5a, 5b, 5c, 6a, 6b, 6c, 7\}$ denoting one of the qualitatively different configurations of line triples as introduced in Section 4.1.1;
- An assignment mapping $a : \mathbf{P} \rightharpoonup \hat{\mathcal{S}}(i)$;
- A triple (dp_A, dp_B, dp_C) of elements from $\mathcal{D}$, where we call each such element an *abstract dipole*;
- A relation $<_p$ on all points, i.e. the start and end points of the abstract dipoles, which is compatible with $<$ (for the definition of $<$ refer to Figure 31).

As an additional ingredient we need the knowledge if an abstract dipole is pointing into the same direction as is the segmentation of its abstract carrier line or not. We use the order $<_p$ to obtain this information.

Definition 53 (Abstract direction)**.** For any abstract dipole dp, we say that $dir(dp) = +$ if and only if $ed(dp) >_p st(dp)$, otherwise $dir(dp) = -$.

4.2.1. Geometric Realization. In this section, we claim that each qcc has a realization. First of all, we need to define what such a realization is.

Definition 54 (Order on ray)**.** Given a ray l, for two points A and B, we say that $A <_r B$, if B lies further in the positive direction than A.

We construct a map on each ray that reflects the abstract segments shown in Figure 31 to provide a link between a qcc and a compatible line scenario.

Definition 55 (Segmentation map)**.** Given a ray X and two points $\tilde{S}_X$ and $\tilde{E}_X$ on it, the *segmentation map* $seg : X \longrightarrow \{\tilde{F}, \tilde{E}, \tilde{I}, \tilde{S}, \tilde{B}\}$ is defined as:

$$seg(x) = \begin{cases} \text{if } \tilde{S}_X <_r \tilde{E}_X & \begin{cases} \tilde{F} & \text{if } \tilde{E}_X <_r x \\ \tilde{E} & \text{if } \tilde{E}_X =_r x \\ \tilde{I} & \text{if } \tilde{x} <_r \tilde{E}_X \wedge \tilde{S}_X <_r x \\ \tilde{S} & \text{if } \tilde{S}_X =_r x \\ \tilde{B} & \text{if } x <_r \tilde{S}_X \end{cases} \\ \text{if } \tilde{E}_X <_r \tilde{S}_X & \begin{cases} \tilde{F} & \text{if } x <_r \tilde{E}_X \\ \tilde{E} & \text{if } x =_r \tilde{E}_X \\ \tilde{I} & \text{if } \tilde{E}_X <_r x \wedge x <_r \tilde{S}_X \\ \tilde{S} & \text{if } x =_r \tilde{S}_X \\ \tilde{B} & \text{if } \tilde{S}_X <_r x \end{cases} \end{cases}$$

for any point x on X.

When it is clear that we are talking about segments on an actual ray, we often omit the $\tilde{\_}$. A geometric realization of a qcc is a triple of dipoles in the Euclidean plane that satisfies the constraints given by the qcc. The geometric realization preserves the segmentation and order relations of the corresponding qcc. Also the geometric realization in our definition is augmented with information that is implicit in a potential metric specification of the configuration of the three dipoles. We now can give a formal definition for our concept of a geometric realization:

Definition 56 (Geometric Realization)**.** For any qcc Q a *geometric realization* $R(Q)$ consists of a triple of dipoles (d_A, d_B, d_C) in $\mathbb{R}^2$, three carrier rays l_A, l_B, l_C of the dipoles, and two points $\tilde{S}_X$ and $\tilde{E}_X$ on l_X for each $X \in \{A, B, C\}$, such that:

- (l_A, l_B, l_C) (more precisely, the corresponding triple of undirected lines) belongs to the configuration denoted by the identifier i of Q;
- the angle between l_A and the other two rays must lie in the interval $(\pi, 2\pi]$;
- for any $x, y \in \tilde{\mathbf{P}}$, if $a(p(x))$ and $a(p(y))$ are both defined and equal, then $x = y$ (where $p : \tilde{\mathbf{P}} = \{\tilde{S}_A, \tilde{S}_B, \tilde{S}_C, \tilde{E}_A, \tilde{E}_B, \tilde{E}_C\} \to \mathbf{P}$ be the obvious bijection);
- for all X, $st(dp_X) = seg(st(d_X))$ and $ed(dp_X) = seg(ed(d_X))$;
- for all points x and y on l_X, if $seg(x) < seg(y)$, then $x <_r y$;
- if $l_X = l_Y$, the order $<_p$ must be preserved for points $st(d_X)$, $ed(d_X)$, $st(d_Y)$, $ed(d_Y)$, in such a way that: if $st(dp_X) <_p st(dp_Y)$, then $st(d_Y) <_r st(d_X)$ and in the same manner between all other points.

Proposition 57. *Given three dipoles in $\mathbb{R}^2$, there is a qcc Q and a geometric realization of $R(Q)$ which uses these three dipoles.*

Proof. For this proof, we construct a qcc from a scenario of three dipoles in $\mathbb{R}^2$. Given three dipoles d_A, d_B, d_C in $\mathbb{R}^2$, we determine their carrier rays l_A, l_B, l_C in such a way that the angles between l_A and l_B as well as l_A and l_C lie in the interval $(\pi, 2\pi]$. We determine the identifier i of the configuration in Figure 29 which the the scenario belongs to. We determine the points of intersection of the rays and identify them with $\hat{s}_{XY}$ or $\hat{s}_{XYZ}$ in $\hat{\mathcal{S}}(i)$ for X, Y and $Z \in \{A, B, C\}$ denoting the abstract points of intersection. For all points Y in $\mathbf{P}$, for which a is undefined, the points $\hat{Y}$ are placed in such a way, that $\hat{Y} <_r \hat{E}_X$ (which is equivalent to $S_X < E_X$), if $\hat{Y} = \hat{S}_X$ for some $X \in \{A, B, C\}$ or $\hat{S}_X <_r \hat{Y}$ if $\hat{Y} = \hat{E}_X$. We identify the start and end points of the abstract dipoles dp_X namely $st(dp_X)$ and $ed(dp_X)$ according to the segmentation map on these rays. If two carrier rays coincide, we define the order $<_p$ w.r.t. $<_r$, otherwise it is arbitrary. This clearly gives a *qcc*.

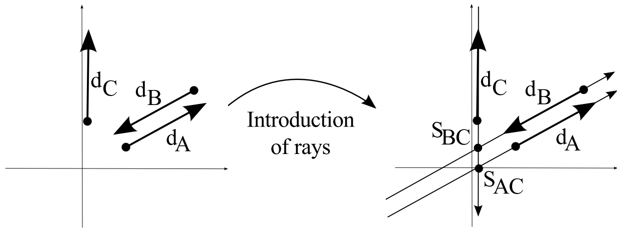


FIGURE 33. Construction of the qcc

An example of this construction is given in Fig. 33. On the left-hand-side of Fig. 33, there is a scenario with three dipoles, lying somewhere in $\mathbb{R}^2$. On the right hand side, rays and points of intersection are added. Comparison with orbits and placement of lines determine the identifier $3b$ for this scenario. The map a can be defined as

$$\begin{aligned} a(S_A) &= \hat{\mathcal{S}}_{AC} \\ a(S_B) &= \hat{\mathcal{S}}_{BC} \\ a(S_C) &= \hat{\mathcal{S}}_{BC} \\ a(E_C) &= \hat{\mathcal{S}}_{AC} \end{aligned}$$

where the assignment is undefined for E_A and E_B. This means that we can freely place E_A and E_B. We choose to place E_A at the start point of dipole d_A and E_B at the end point of dipole d_B. In this way, we get:

$$\begin{aligned} st(dp_A) &= E \\ ed(dp_A) &= F \\ st(dp_B) &= E \\ ed(dp_B) &= I \\ st(dp_C) &= B \\ ed(dp_C) &= B \end{aligned}$$

and

$$\begin{aligned} dir(dp_A) &= + \\ dir(dp_B) &= - \\ dir(dp_C) &= - \end{aligned}$$

In this case the assignment of $<_p$ is arbitrary.
This construction gives us the desired qcc and a realization of it. Basically any qcc has many realizations, but they can be transformed into each other by orientation preserving affine bijections as discussed in Section 4.1.1. □

4.2.2. Primitive Classifiers. We have established a theory that allows us to compute the composition table for $\mathcal{DRA}$, i.e. the relations in scenarios of three dipoles, in an effective way by the reduction of the number of configurations that we need to observe. In fact, we can even abstract from the geometric layer and work in the language of qccs. We can decompose the task of computing the relations between triples of dipoles into subtasks, since each $\mathcal{DRA}_f$ relation comprises four $\mathcal{LR}$ relations between a dipole and point as discussed in Section 3.2.3; these $\mathcal{LR}$ relations are obtained from a a sub-structure of a qualitative composition configuration using so-called *primitive classifiers*. The *basic classifiers* apply the *primitive classifiers* to the abstract dipoles in each qualitative composition configuration in an adequate manner to obtain $\mathcal{DRA}_f$ relations. For $\mathcal{DRA}_{fp}$ relations an extension of the *basic classifiers* is used in cases where the qualitative angle between several dipoles has to be determined. As discussed in Section 3.2.2 this is the case for the $\mathcal{DRA}_f$ relations rrrr, rrll, llrr, and llll which are split up into three $\mathcal{DRA}_{fp}$ relations each. Finally, the resulting data is collected in a composition table. In fact, by our approach a set of scenarios in three dipoles is generated, each standing guard for a possible outcome of composition. For each pair of relations R_1 and R_2 the composition $R_1 \diamond R_2$ is determined by uniting all those guards to sets, i.e. by determining all those guards and forming a disjunction of them.

Since primitive classifiers only operate on two dipoles, they do not need all information about three dipoles provided by a qcc, hence we introduce a substructure of a qcc that only contains the information needed by these qualifiers, a *primitive qualitative composition configuration.*

Definition 58 (Primitive Qualitative Composition Configuration)**.** A *primitive qualitative composition configuration* (pqcc) is a sub-configuration of a qualitative composition configuration (see Definition 52) containing two abstract dipoles dp_1 and dp_2 which are identified with two abstract dipoles from the set $\{dp_A, dp_B, dp_C\}$ such that $dp_1 \neq dp_2$. All information regarding the abstract dipoles of the qcc dp_1 and dp_2 are identified with is kept and all information about the third abstract dipole is discarded.

Primitive qualitative composition configurations can be identified via the set of injective but not surjective maps $pqcc : \{dp_1, dp_2\} \longrightarrow \{dp_A, dp_B, dp_C\}$ with $\{dp_A, dp_B, dp_C\}$ being the abstract dipoles of a qcc.

Primitive classifiers often turn out to be large case distinctions, that is why we need a kind of "compressed" notation to be able to present them on paper.

Notation 59. To simplify the explanation of large classifiers, we shall write:

$$f(x) \quad = \quad \begin{cases} cond_1 & \longrightarrow & value_1 \\ cond_2 & \longrightarrow & value_2 \end{cases}$$

instead of

$$f(x) \quad = \quad \begin{cases} value_1 & \text{if } cond_1 \\ value_2 & \text{if } cond_2. \end{cases}$$

If it is clear which function we are defining, we even omit the "$f(x) =$".

Given a primitive qualitative composition configuration Q, *primitive classifiers* map the qualitative locations of a dipole dp_1 and a point pt (which is the start or end point of the other dipole dp_2 in the pqcc) to a letter (or symbol) indicating the $\mathcal{LR}$ relation between the dipole and point.

We need three different types of primitive classifiers for our algorithm, corresponding to the three different possible configurations for the carrier lines of two given dipoles: 1) intersecting, 2) equal and 3) parallel. Moreover, the primitive classifiers assume that, in the geometric realization, the second ray points to the right w.r.t. the first one. If the second ray points to the left in the realization, it is sufficient to apply an operation that interchanges L with R on this classifier, in order to obtain the correct results. We will call this operation *com*.

The first type of primitive classifier is for primitive qualitative composition configurations that have two dipoles dp_1 and dp_2 with intersecting carrier rays in their realizations. The classifier itself only works on dp_1 and pt, where pt is either the start or end point of dp_2. A realization of the carrier rays of a pqcc of this kind is given in Figure 34 for the reader's convenience, the actual dipoles are omitted from the figure, since there are several possibilities to place them according to Figure 31

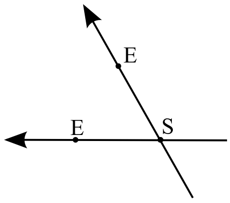


FIGURE 34. Line configuration for primitive Classifier

which is the basis for the definition of the primitive qualifiers. The line in Figure 31 can be mapped onto the rays in Figure 34 by identifying the points S and E in each case, this directly yields the segments for the several possibilities of placing the start and end points of the dipoles and then deriving the abstract dipoles.

The classifier takes an abstract dipole dp_1 and the start or end point of dp_2 called pt. Note that this includes information on whether dp_1 is pointing in the same direction as its ray ($dir(dp_1) = +$) or against it ($dir(dp_1) = -$). The classifier returns an $\mathcal{LR}$-relation determining the relation between dp_1 and pt (and hence between dp_1 and the start or end point of dp_2 which is part of the corresponding $\mathcal{DRA}$ relation). The classifier also needs two parameters $x, y \in \{S, E\}$. They indicate the role of the abstract intersection point for dp_1 and dp_2, resp., which is given by its preimage under the assignment map. For the case shown in Figure 34, we have $x = y = S$.

The classifier $cli_{x,y}(dp_1, pt)$ is given by (here cli is an abbreviation for "classifier on intersecting rays"):

$$
\begin{array}{lcl}
dir(dp_1) = + & \longrightarrow & \begin{cases} pt > y \longrightarrow R \\ pt = y \longrightarrow \begin{cases} st(dp_1) < x \wedge ed(dp_1) < x \longrightarrow F \\ st(dp_1) < x \wedge ed(dp_1) = x \longrightarrow E \\ st(dp_1) < x \wedge ed(dp_1) > x \longrightarrow I \\ st(dp_1) = x \wedge ed(dp_1) > x \longrightarrow S \\ st(dp_1) > x \wedge ed(dp_1) > x \longrightarrow B \end{cases} \\ pt < y \longrightarrow L \end{cases} \\
dir(dp_1) = - & \longrightarrow & \begin{cases} pt < y \longrightarrow R \\ pt = y \longrightarrow \begin{cases} st(dp_1) > x \wedge ed(dp_1) > x \longrightarrow F \\ st(dp_1) > x \wedge ed(dp_1) = x \longrightarrow E \\ st(dp_1) > x \wedge ed(dp_1) < x \longrightarrow I \\ st(dp_1) = x \wedge ed(dp_1) < x \longrightarrow S \\ st(dp_1) < x \wedge ed(dp_1) < x \longrightarrow B \end{cases} \\ pt > y \longrightarrow L \end{cases}
\end{array}
$$

We see that the table for $dir(dp_1) = -$ is exactly the complement of $dir(dp_1) = +$.

Secondly, we give a primitive classifier $cls(dp_1, pt)$ for two coinciding lines (cls is an abbreviation for "classifier for same line"), an example realization for such a case in shown in Figure 35. For $dir(dp_1) = +$, we get the cases:

FIGURE 35. Primitive classifier for same line.

$$
pt = F \longrightarrow \begin{cases}
st(dp_1) < F \wedge ed(dp_1) < F \longrightarrow F \\
st(dp_1) < F \wedge ed(dp_1) = F \longrightarrow \begin{cases} ed(dp_1) <_p pt \longrightarrow F \\ ed(dp_1) =_p pt \longrightarrow E \\ ed(dp_1) >_p pt \longrightarrow I \end{cases} \\
st(dp_1) = F \wedge ed(dp_1) = F \longrightarrow \begin{cases} st(dp_1) <_p pt \wedge ed(dp_1) <_p pt \longrightarrow F \\ st(dp_1) <_p pt \wedge ed(dp_1) =_p pt \longrightarrow E \\ st(dp_1) <_p pt \wedge ed(dp_1) >_p pt \longrightarrow I \\ st(dp_1) =_p pt \wedge ed(dp_1) >_p pt \longrightarrow S \\ st(dp_1) >_p pt \wedge ed(dp_1) >_p pt \longrightarrow B \end{cases}
\end{cases}
$$

$$
pt = E \longrightarrow \begin{cases}
st(dp_1) < E \wedge ed(dp_1) < E \longrightarrow F \\
st(dp_1) < E \wedge ed(dp_1) = E \longrightarrow E \\
st(dp_1) < E \wedge ed(dp_1) > E \longrightarrow I \\
st(dp_1) = E \wedge ed(dp_1) > E \longrightarrow S \\
st(dp_1) > E \wedge ed(dp_1) > E \longrightarrow B
\end{cases}
$$

$$
pt = I \longrightarrow \begin{cases}
st(dp_1) < I \wedge ed(dp_1) < I \longrightarrow F \\
st(dp_1) < I \wedge ed(dp_1) = I \longrightarrow \begin{cases} ed(dp_1) <_p pt \longrightarrow F \\ ed(dp_1) =_p pt \longrightarrow E \\ ed(dp_1) >_p pt \longrightarrow I \end{cases} \\
st(dp_1) < I \wedge ed(dp_1) > I \longrightarrow I \\
st(dp_1) = I \wedge ed(dp_1) = I \longrightarrow \begin{cases} st(dp_1) <_p pt \wedge ed(dp_1) <_p pt \longrightarrow F \\ st(dp_1) <_p pt \wedge ed(dp_1) =_p pt \longrightarrow E \\ st(dp_1) <_p pt \wedge ed(dp_1) >_p pt \longrightarrow I \\ st(dp_1) =_p pt \wedge ed(dp_1) >_p pt \longrightarrow S \\ st(dp_1) >_p pt \wedge ed(dp_1) >_p pt \longrightarrow B \end{cases} \\
st(dp_1) = I \wedge ed(dp_1) > I \longrightarrow \begin{cases} st(dp_1) <_p pt \longrightarrow I \\ st(dp_1) =_p pt \longrightarrow S \\ st(dp_1) >_p pt \longrightarrow B \end{cases} \\
st(dp_1) > I \wedge ed(dp_1) > I \longrightarrow B
\end{cases}
$$

$$
pt = S \longrightarrow \begin{cases}
st(dp_1) < S \wedge ed(dp_1) < S \longrightarrow F \\
st(dp_1) < S \wedge ed(dp_1) = S \longrightarrow E \\
st(dp_1) < S \wedge ed(dp_1) > S \longrightarrow I \\
st(dp_1) = S \wedge ed(dp_1) > S \longrightarrow S \\
st(dp_1) > S \wedge ed(dp_1) > S \longrightarrow B
\end{cases}
$$

$$
pt = B \longrightarrow \begin{cases}
st(dp_1) = B \wedge ed(dp_1) = B \longrightarrow \begin{cases} st(dp_1) <_p pt \wedge ed(dp_1) <_p pt \longrightarrow F \\ st(dp_1) <_p pt \wedge ed(dp_1) =_p pt \longrightarrow E \\ st(dp_1) <_p pt \wedge ed(dp_1) >_p pt \longrightarrow I \\ st(dp_1) =_p pt \wedge ed(dp_1) >_p pt \longrightarrow S \\ st(dp_1) >_p pt \wedge ed(dp_1) >_p pt \longrightarrow B \end{cases} \\
st(dp_1) = B \wedge ed(dp_1) > B \longrightarrow \begin{cases} st(dp_1) <_p pt \longrightarrow I \\ st(dp_1) =_p pt \longrightarrow S \\ st(dp_1) >_p pt \longrightarrow B \end{cases} \\
st(dp_1) > B \wedge ed(dp_1) > B \longrightarrow B
\end{cases}
$$

and for $dir(dp_1) = -$, we consider:

$$
pt = B \longrightarrow \begin{cases}
st(dp_1) > B \wedge ed(dp_1) > B \longrightarrow F \\
st(dp_1) > B \wedge ed(dp_1) = B \longrightarrow \begin{cases} ed(dp_1) <_p pt \longrightarrow I \\ ed(dp_1) =_p pt \longrightarrow E \\ ed(dp_1) >_p pt \longrightarrow F \end{cases} \\
st(dp_1) = B \wedge ed(dp_1) = B \longrightarrow \begin{cases} st(dp_1) <_p pt \wedge ed(dp_1) <_p pt \longrightarrow B \\ st(dp_1) =_p pt \wedge ed(dp_1) <_p pt \longrightarrow S \\ st(dp_1) >_p pt \wedge ed(dp_1) <_p pt \longrightarrow I \\ st(dp_1) >_p pt \wedge ed(dp_1) =_p pt \longrightarrow E \\ st(dp_1) >_p pt \wedge ed(dp_1) >_p pt \longrightarrow F \end{cases}
\end{cases}
$$

$$
pt = S \longrightarrow \begin{cases}
st(dp_1) > S \wedge ed(dp_1) > S \longrightarrow F \\
st(dp_1) > S \wedge ed(dp_1) = S \longrightarrow E \\
st(dp_1) > S \wedge ed(dp_1) < S \longrightarrow I \\
st(dp_1) = S \wedge ed(dp_1) < S \longrightarrow S \\
st(dp_1) < S \wedge ed(dp_1) < S \longrightarrow B
\end{cases}
$$

$$
pt = I \longrightarrow \begin{cases}
st(dp_1) > I \wedge ed(dp_1) > I \longrightarrow F \\
st(dp_1) > I \wedge ed(dp_1) = I \longrightarrow \begin{cases} ed(dp_1) >_p pt \longrightarrow F \\ ed(dp_1) =_p pt \longrightarrow E \\ ed(dp_1) <_p pt \longrightarrow I \end{cases} \\
st(dp_1) > I \wedge ed(dp_1) < I \longrightarrow I \\
st(dp_1) = I \wedge ed(dp_1) = I \longrightarrow \begin{cases} st(dp_1) >_p pt \wedge ed(dp_1) >_p pt \longrightarrow F \\ st(dp_1) >_p pt \wedge ed(dp_1) =_p pt \longrightarrow E \\ st(dp_1) >_p pt \wedge ed(dp_1) <_p pt \longrightarrow I \\ st(dp_1) =_p pt \wedge ed(dp_1) <_p pt \longrightarrow S \\ st(dp_1) <_p pt \wedge ed(dp_1) <_p pt \longrightarrow B \end{cases} \\
st(dp_1) = I \wedge ed(dp_1) < I \longrightarrow \begin{cases} st(dp_1) >_p pt \longrightarrow I \\ st(dp_1) =_p pt \longrightarrow S \\ st(dp_1) <_p pt \longrightarrow B \end{cases} \\
st(dp_1) < I \wedge ed(dp_1) < I \longrightarrow B
\end{cases}
$$

$$
pt = E \longrightarrow \begin{cases}
st(dp_1) > E \wedge ed(dp_1) > E \longrightarrow F \\
st(dp_1) > E \wedge ed(dp_1) = E \longrightarrow E \\
st(dp_1) > E \wedge ed(dp_1) < E \longrightarrow I \\
st(dp_1) = E \wedge ed(dp_1) < E \longrightarrow S \\
st(dp_1) < E \wedge ed(dp_1) < E \longrightarrow B
\end{cases}
$$

$$
pt = F \longrightarrow \begin{cases}
st(dp_1) = F \wedge ed(dp_1) = F \longrightarrow \begin{cases} st(dp_1) >_p pt \wedge ed(dp_1) >_p pt \longrightarrow F \\ st(dp_1) >_p pt \wedge ed(dp_1) =_p pt \longrightarrow E \\ st(dp_1) >_p pt \wedge ed(dp_1) <_p pt \longrightarrow I \\ st(dp_1) =_p pt \wedge ed(dp_1) <_p pt \longrightarrow S \\ st(dp_1) <_p pt \wedge ed(dp_1) <_p pt \longrightarrow B \end{cases} \\
st(dp_1) = F \wedge ed(dp_1) < F \longrightarrow \begin{cases} st(dp_1) >_p pt \longrightarrow I \\ st(dp_1) =_p pt \longrightarrow S \\ st(dp_1) <_p pt \longrightarrow B \end{cases} \\
st(dp_1) < F \wedge ed(dp_1) < F \longrightarrow B
\end{cases}
$$

All cases that are not listed in the above classifier are cases where the ordering $<_p$ is not compatible with the segmentation, and so they are impossible.[3]

This classifier looks are lot more complicated than cli, this is the case since we have to take the order $<_p$ into account what leads to a split up of several cases into sub-cases.

The third classifier is for parallel lines, i.e. a configuration like that in Figure 36. Let the lower line be the line the dipole lies on. The information about the line on which the dipole lies is handled by a basic classifier which uses this primitive classifier and exchanges L and R appropriately. Fortunately this classifier $clpar(dp_1, pt)$ is

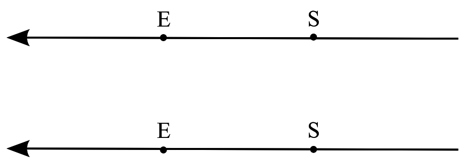


FIGURE 36. Primitive classifier for parallel lines.

simple, since we neither need to consider points of intersection nor any order. ($clpar$

[3] If the domain of definition is extended, a more compressed definition of the classifier is possible. However, then impossible cases cannot be detected anymore.

is an abbreviation for "classifier on parallel lines".):

$$dir(dp_1) = + \longrightarrow R$$
$$dir(dp_1) = - \longrightarrow L$$

This is a complete list of the primitive classifiers that are needed. Cases that are not covered so far can be derived from the existing classifiers via the operation *com*, how this is achieved will be shown in detail in Section 4.2.3.

4.2.3. Basic Classifiers. Based on the primitive classifiers introduced in Section 4.2.2, we construct the *basic classifiers* to determine the $\mathcal{DRA}$ relations in scenarios of three dipoles, each standing guard for an entry in the composition table. For $\mathcal{DRA}_f$, we always need exactly four primitive classifiers to determine the relation. For $\mathcal{DRA}_{fp}$, in some cases we need an additional fifth classifier to determine the qualitative angle. This fifth qualifier is needed in the cases where the $\mathcal{DRA}_f$ relations are split up into several $\mathcal{DRA}_{fp}$ relations, namely for rrrr, rrll, llrr, and llll. We first focus on the $\mathcal{DRA}_f$ case. Given a qcc, we apply four basic classifiers three times: namely (1) to the first and second abstract dipole, (2) to the second and third and (3) to the first and third. Thus, we obtain an entry in the composition table. For the definition of composition please refer to Section 2.2. Consider a qcc with $i = 1+$ and $a(S_A) = \hat{s}_{AB}$, $a(S_B) = \hat{s}_{AB}$ and $a(S_C) = \hat{s}_{AC}$. Such a configuration has a realization as in Figure 37. The dipole d_X lies on the ray l_X

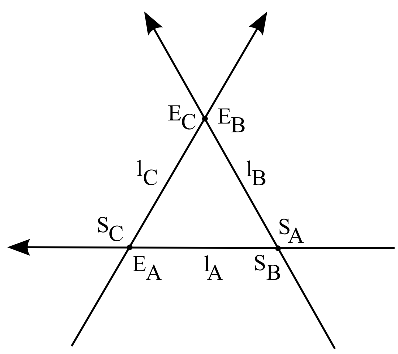


FIGURE 37. Line configuration for Basic Classifier

for $X \in \{A, B, C\}$. We now apply primitive classifiers to this scenario in the way defined in Section 3.2.1. Hence, we can construct the basic classifier for such a configuration as:

$$\begin{array}{rcl}
R(dp_A, st_B) & = & cli_{S,S}(dp_A, st_B) \\
R(dp_A, ed_B) & = & cli_{S,S}(dp_A, ed_B) \\
R(dp_B, st_A) & = & com \circ cli_{S,S}(dp_B, st_A) \\
R(dp_B, ed_A) & = & com \circ cli_{S,S}(dp_B, ed_A) \\
\\
R(dp_B, st_C) & = & cli_{E,E}(dp_B, st_C) \\
R(dp_B, ed_C) & = & cli_{E,E}(dp_B, ed_C) \\
R(dp_C, st_B) & = & com \circ cli_{E,E}(dp_C, st_B) \\
R(dp_C, st_B) & = & com \circ cli_{E,E}(dp_C, ed_B) \\
\\
R(dp_A, st_C) & = & cli_{E,S}(dp_A, st_C) \\
R(dp_A, ed_C) & = & cli_{E,S}(dp_A, ed_C) \\
R(dp_C, st_A) & = & com \circ cli_{S,E}(dp_C, st_A) \\
R(dp_C, ed_A) & = & com \circ cli_{S,E}(dp_C, ed_A)
\end{array}$$

And we obtain the relation between dp_A and dp_B: $R(dp_A, st_B)$ $R(dp_A, ed_B)$ $R(dp_B, st_A)$ $R(dp_B, ed_A)$. The first four lines constitute the part of the basic classifier for the relation between dp_A and dp_B. Since l_A and l_B, the carrier rays of dp_A and dp_B, intersect for both rays in the segment S the primitive classifier $cli_{S,S}$ is used. Further, since the l_B points to the right with respect to l_A no *com* operation is used in the first two lines. But l_A points to the left with respect to l_B, so an operation *com* is needed in the second pair of lines. The relations between dp_B and dp_C as well as between dp_A and dp_C are derived analogously. The basic classifiers depend on the configuration in which the qcc realization lies and on the angle between the rays in the realization. They are constructed for an angle between the rays in the interval $(\pi, 2\pi]$. If the angle is in the interval $(0, \pi]$, the $\mathcal{LR}$ relation between any line on the first ray and a point on the second just swaps. We capture this by introducing the operation *com* which is applied in this case. With it, we can limit the number of necessary primitive classifiers. The construction of the other basic classifiers is done analogously. In fact the construction of the $\mathcal{DRA}$ basic classifiers just follows the semantics of $\mathcal{DRA}$ relations arising from $\mathcal{LR}$ relations.

4.2.4. Extended Basic Classifiers for $\mathcal{DRA}_{fp}$. For $\mathcal{DRA}_{fp}$ basically the same classifiers as described for $\mathcal{DRA}_f$ in Section 4.2.3 are used. We simply extend them for the relations rrrr, rrll, llll and llrr to classify the information about qualitative angles of intersection. For this purpose, we have to have a look at the angles between dipoles in the realization of a given qcc. The qualitative angle between two dipoles d_A and d_B is called positive + (negative −) if the angle from the carrier ray of d_A called l_A to the carrier ray of d_B called l_B lies in the interval $(0, \pi)$ (resp. $(\pi, 2\pi)$). We provide an example of this before starting with the formalization. Consider the configuration of a $\mathcal{DRA}$ scenario in Figure 38a. In Figure 38b, the carrier rays are introduced and we can see that the angle clearly lies in the interval $(0, \pi)$ and hence the qualitative angle is positive. The definitions of parallel P and anti-parallel A are straightforward.

We need to revisit the mapping a of a qcc to be able to formalize our extended basic classifiers. The set $a^{-1}(\hat{s}_{xy})$ always contains exactly two elements, if $\hat{s}_{xy} \in \hat{\mathcal{S}}(i)$.

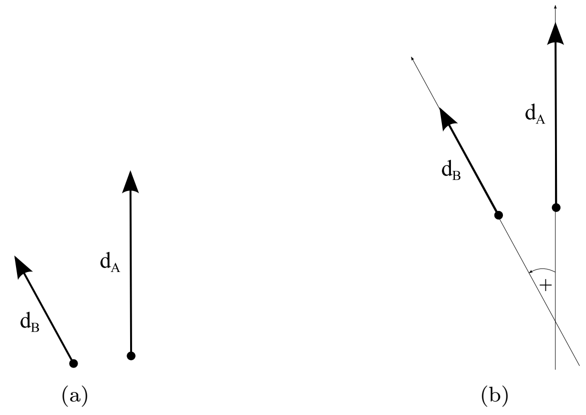


FIGURE 38. Scenario as representation of LLRR+.

To continue, we need functions $proj_x : \mathcal{P}(\mathbf{P}) \longrightarrow \mathcal{P}(\mathbf{P})$ defined as

$$proj_x = \{a \mid idx_x(a) = x\}$$

which form the set of all elements with index (idx) x. $\mathcal{P}$ denotes powerset formation. By the definition of a and the sets $\hat{\mathcal{S}}(i)$, these sets are always singletons, if $proj_x \circ a^{-1}$ is applied to an intersection point and if a^{-1} contains an element with index x, otherwise the set is empty. We shall write a_x^{-1} for $proj_x \circ a^{-1}$.

We observed that the qualitative angles between two dipoles can be classified very easily once the $\mathcal{DRA}_f$ relations between the dipoles d_A and d_B are known. All we need to do is to find out if the ray l_B intersects l_A in front of or behind d_A. In the language of qcc and abstract dipoles dp_A and dp_B, we can say that, if $a_A^{-1}(\hat{s}_{AB}) > ed(dp_A)$ for $dir(dp_A) = +$, or if $a_A^{-1}(\hat{s}_{AB}) < ed(dp_A)$ for $dir(dp_A) = -$, then the abstract point of intersection lies "in front of dp_A" and, if $st(dp_A) > a_A^{-1}(\hat{s}_{AB})$ for $dir(dp_A) = +$ or $a_A^{-1}(\hat{s}_{AB}) > st(dp_A)$ for $dir(dp_A) = -$, the abstract point of intersection lies "behind dp_A".

Proposition 60. *In a realization $R(Q)$ of a qcc Q, the carrier rays of any two dipoles d_1 and d_2 intersect in front of d_1 if and only if, in Q the property*

$$(a_1^{-1}(\hat{s}_{12}) > ed(dp_1) \wedge dir(dp_1) = +) \vee (a_1^{-1}(\hat{s}_{12}) < ed(dp_1) \wedge dir(dp_1) = -)$$

is fulfilled.

Proof. This is immediate by inspection of the property and respective scenarios. □

Proposition 61. *In a realization $R(Q)$ of a qcc Q, the carrier rays of any two dipoles d_1 and d_2 intersect behind d_1 if and only if, in Q the property*

$$(st(dp_1) > a_1^{-1}(\hat{s}_{12}) \wedge dir(dp_1) = +) \vee (a_1^{-1}\hat{s}_{12} > st(dp_1) \wedge dir(dp_1) = -)$$

is fulfilled.

Proof. This is immediate by inspection of the property and respective scenarios. □

The complete extension for the Basic Classifiers is given as:

$$\text{rrrr} \longrightarrow \begin{cases} a_A^{-1}(\hat{s}_{AB}) > ed(dp_A) \wedge dir(dp_A) = + & \longrightarrow & - \\ a_A^{-1}(\hat{s}_{AB}) < ed(dp_A) \wedge dir(dp_A) = - & \longrightarrow & - \\ st(dp_A) > a_A^{-1}(\hat{s}_{AB}) \wedge dir(dp_A) = + & \longrightarrow & + \\ a_A^{-1}(\hat{s}_{AB}) > st(dp_A) \wedge dir(dp_A) = - & \longrightarrow & + \end{cases}$$

$$\text{rrll} \longrightarrow \begin{cases} a_A^{-1}(\hat{s}_{AB}) > ed(dp_A) \wedge dir(dp_A) = + & \longrightarrow & + \\ a_A^{-1}(\hat{s}_{AB}) < ed(dp_A) \wedge dir(dp_A) = - & \longrightarrow & + \\ st(dp_A) > a_A^{-1}(\hat{s}_{AB}) \wedge dir(dp_A) = + & \longrightarrow & - \\ a_A^{-1}(\hat{s}_{AB}) > st(dp_A) \wedge dir(dp_A) = - & \longrightarrow & - \end{cases}$$

$$\text{llll} \longrightarrow \begin{cases} a_A^{-1}(\hat{s}_{AB}) > ed(dp_A) \wedge dir(dp_A) = + & \longrightarrow & + \\ a_A^{-1}(\hat{s}_{AB}) < ed(dp_A) \wedge dir(dp_A) = - & \longrightarrow & + \\ st(dp_A) > a_A^{-1}(\hat{s}_{AB}) \wedge dir(dp_A) = + & \longrightarrow & - \\ a_A^{-1}(\hat{s}_{AB}) > st(dp_A) \wedge dir(dp_A) = - & \longrightarrow & - \end{cases}$$

$$\text{llrr} \longrightarrow \begin{cases} a_A^{-1}(\hat{s}_{AB}) > ed(dp_A) \wedge dir(dp_A) = + & \longrightarrow & - \\ a_A^{-1}(\hat{s}_{AB}) < ed(dp_A) \wedge dir(dp_A) = - & \longrightarrow & - \\ st(dp_A) > a_A^{-1}(\hat{s}_{AB}) \wedge dir(dp_A) = + & \longrightarrow & + \\ a_A^{-1}(\hat{s}_{AB}) > st(dp_A) \wedge dir(dp_A) = - & \longrightarrow & + \end{cases}$$

Constructing the classifiers for qccs based on configurations with parallel lines is easy, depending on the $\mathcal{DRA}_f$ relations, the dipoles can either be parallel or anti-parallel in such cases, but never both of that for any $\mathcal{DRA}_f$ relation.

4.2.5. Correctness of the Construction. To establish the correctness of our constructions, we need to prove some lemmas first.

Lemma 62. *Given two intersecting lines, the $\mathcal{LR}$ relations between a dipole on the first line and a point on the second line are stable under the movement of the point along the line, unless it moves through the point of intersection of the two lines.*

Proof. By the definition of $\mathcal{LR}$ relations, the point can be in one of three different relative positions to the carrier ray of the dipole. The point can lie on either side of the point of intersection, yielding the relation l or r, or on the point of intersection itself, yielding exactly one relation on the line. □

Lemma 63. *Given a dipole and a point lying on its carrier line, the $\mathcal{LR}$ relations between the dipole and point are stable under the movement of the point along the line, unless it is moved over the start or end point of the dipole.*

Proof. Inspect the definition of $\mathcal{LR}$ relations on a line. In fact, this lemma just rephrases the definition of the $\mathcal{LR}$ relations on a line. □

Lemma 64. *For dipoles lying on intersecting rays, the $\mathcal{DRA}$ relations are stable under the movement of the start and end points of the dipoles along the rays, as long as the segments for the start and end points and the directions of the dipoles do not change.*

Proof. We observe that the segmentation is a stronger property than the one used in Lemma 62. For $\mathcal{DRA}_f$ relations it suffices to apply Lemma 62 four times. For $\mathcal{DRA}_{fp}$ relations, we also need to take the intersection property of Proposition 69 into account. □

Lemma 65. *For dipoles on the same line, the $\mathcal{DRA}$ relations are stable under the movement of the start and end points of the dipoles along the rays, so long as the relation $<_r$ does not change.*

Proof. Apply Lemma 63 four times. In fact, the order relation $<_r$ reflects the statement of Lemma 63. □

Lemma 66.

(1) *Transforming a given realization of a qcc along an orientation-preserving affine transformation preserves the segmentation map.*
(2) *If two dipoles are on the same line, affine transformations also preserve* $<_r$*.*

Proof.
1) According to Proposition 50, any orientation-preserving affine transformation preserves $\mathcal{LR}$ relations.
2) This follows from the preservation of length ratios by affine transformations, i.e. the length ratios between the start and end points of the dipoles and points S and E on the ray. □

Lemma 67. *Given a qcc, any two geometric realizations exhibit the same* $\mathcal{DRA}$ *relations among their dipoles.*

Proof. Let two geometric realizations R_1, R_2 of a qcc Q be given. Since the line triples of R_1 and R_2 belong to the same orbit, there is an orientation-preserving affine bijection f transforming the line triple of R_1 into the one of R_2. In case of configurations 5a, 5b and 5c, we assume that all distance ratios are adjusted to 1 in order to reach the same orbit. Note that this adjustment, although not an affine transformation, does not affect the relations between dipoles.

Since f maps R_1's line triple to R_2's line triple, it also maps the corresponding points of intersection to each other. For orbits 1+ and 1−, all segmentation points are points of intersection. Hence, f does not change the segments given by $r(x)$ in which the start and end points of the dipoles lie. For the rest of the argument, apply Lemma 64.

For cases 2+ and 2−, we just have a single point of intersection, but the relative directions of the rays are restricted by the definition of a realization and so is the location of all segmentation points w.r.t. the intersection point, as are the locations of the start and end points of the dipoles w.r.t. the segmentation points. For the rest of the argument, apply Lemma 64.

In cases 3*a*, 3*b* and 3*c*, we have two points of intersection and two segmentation points that are not points of intersection but, as before, the directions of the rays and the locations of all segmentation points are restricted and hence the locations of the start and end points of the dipoles, and again, we can apply Lemma 64.

In cases 4*a*, 4*b* and 4*c*, we have one point of intersection and three segmentation points that are not points of intersection. First, we can argue to restrict the location and direction. In the end, we can apply Lemma 65 and Lemma 64.

In cases 5*a*, 5*b* and 5*c*, we only have segmentation points that are not points of intersection, but all rays have the same directions and the relative orientations of segmentation points on the line are restricted. Hence, the directions of the dipoles do not change during the mapping and the relative direction between dipoles is all that is necessary to determine their $\mathcal{DRA}$ relations in the case of parallel dipoles.

The proof of cases 6*a*, 6*b* and 6*c* is similar to cases 4 and 5, with the argument based on Lemma 65 for dipoles on the same line, and the arguments of cases 5 for parallel lines.

For case 7, we need to apply Lemma 65.

For additional arguments for $\mathcal{DRA}_{fp}$-relations, please refer to the proof of Proposition 69.

□

Theorem 68 (Correctness of the Construction)**.** *Given a qcc Q and an arbitrary geometric realization $R(Q)$ of it, the $\mathcal{DRA}_f$ relation in $R(Q)$ is the same as the one computed by the basic classifiers on Q.*

Proof. Due to Lemma 67, we can focus on one geometric realization per qcc.

For this proof, we need to inspect once more the construction of the basic classifiers making use of the primitive classifiers. The actual values of a, dir and the start and end points of the abstract dipoles as well as the order $<_p$ are not directly used by basic classifiers[4]. They are passed through to primitive classifiers. The only information that is directly used in basic classifiers is the identifier i of the configuration.

We divide this proof in two steps. In the first step, we show that the primitive classifiers are correct and, in the second step, we do the same for basic classifiers. We will show a proof for the classifier $cli_{S,S}(dp_1, pt)$ and a pqcc with $dir(dp_1) = +$, $dp_1 = (I, I)$ and $pt = I$ for lines intersecting in segment S on both of them. A realization of this configuration is shown in Figure 39 and we can easily see that the

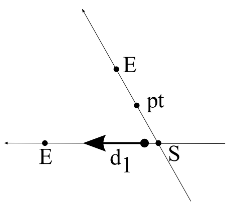


FIGURE 39. A realization

$\mathcal{LR}$ relation is d_1 r pt. By observing $cli_{S,S}(dp_1, pt)$, we see that we are in the case $dir(dp_1) = +$ and that $pt > S$ and so the primitive classifier also yields dp_1 r pt as expected. All other proofs for pqccs are done in an analogous way by inspection of the relations yielded by the primitive classifiers and their realizations. With primitive classifiers working correctly, we need to focus on the basic classifiers. Here, we will show this for the case $i = 1+$, all other cases are handled in an analogous fashion. First we take any realization of $i = 1+$ and add directions to the lines as described in the section about geometric realizations of qccs. For example, we can use the one depicted in Figure 40. In the next step, this realization is decomposed according

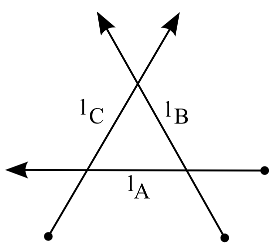


FIGURE 40. A realization for a qcc

to the definition of $\mathcal{DRA}_f$-relations and the basic classifiers as shown in Figure 41. The various parts of the decomposed line configuration need to be matched with the

[4]With the exception of the extended classifiers, but we will discuss this issue later

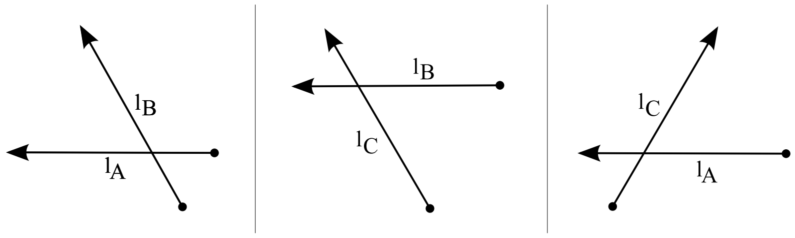


FIGURE 41. Decomposition of line configuration into three sub-configurations

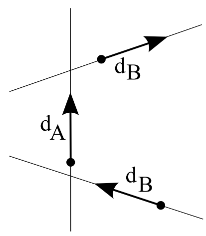
realization of the primitive classifier, here the realization of Figure 34. In our case, the classifier matches directly with the orientations from l_A to l_B, l_B to l_C and l_A to l_C. In the other cases, the angle between the lines may be inverted. Then, we need to swap r and l which is done by the operation *com*. Furthermore, we see that the lines l_C and l_B both intersect in segment E, whereas l_A and l_B intersect both in S. The intersection for l_A and l_C is E for l_A and S for l_C, we need to parameterize the respective primitive classifiers with that information. But in the end, our arguments yield exactly the basic classifier shown in Section 4.2.3. The arguments for the other 16 basic classifiers are analogous. □

After having shown that our construction of your basic classifiers is correct for $\mathcal{DRA}_f$, we need to prove the correctness of the extension for $\mathcal{DRA}_{fp}$.

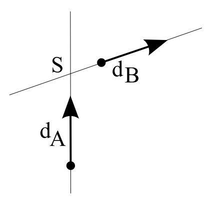
Proposition 69 (Correctness of extension for $\mathcal{DRA}_{fp}$)**.** *Given any qcc Q and its geometric realization R(Q), the extended basic classifiers determine the same $\mathcal{DRA}_{fp}$ relation as the realization exhibits.*

Proof. The $\mathcal{DRA}_f$ part of the relation is determined correctly due to Theorem 68. All we need to consider here are the "extended" relations.

We will give the proof for rrrr-, the proof for the other cases is analogous. Consider two dipoles d_A and d_B in an rrrr configuration on the rays l_A and l_B. There are two classes of qualitatively distinguishable configurations for d_A rrrr d_B in which the carrier rays intersect:



We can see that l_B intersects l_A either in front of or behind d_A. If the intersection point lies in front of d_A, we are in a situation like



where S is the intersection point. We can further see that the angle from l_A to l_B lies clearly in the interval $(\pi, 2\pi)$. Furthermore, l_B can be rotated in the whole interval $(\pi, 2\pi)$ without changing the $\mathcal{DRA}_f$ relation. Using this, we obtain the $\mathcal{DRA}_{fp}$-relation rrrr- between d_A and d_B if the point of intersection S lies in front of d_A. For any qcc belonging to such a scenario, the rest of the proof follows from Proposition 60 and Proposition 61 as well as the inspection of the extended classifiers:

$$\hat{S}_{AB} > ed(dp_A) \wedge dir(dp_A) = + \quad \longrightarrow \quad -$$
$$\hat{S}_{AB} < ed(dp_A) \wedge dir(dp_A) = - \quad \longrightarrow \quad -$$

But these also yield (dp_A rrrr- dp_B). By the same arguments, we show that d_A rrrr+ d_B if the point of intersection of l_A and l_B lies behind d_A. The proof for all other cases is analogous. □

Corollary 70. *The* 72 *relations in Figure 22 are those out of the* 2401 *formal combinations of four* $\mathcal{LR}$ *letters that are geometrically possible.*

Proof. By an exhaustive inspection of the primitive classifiers which occur in the basic classifiers for all pqccs. For the decomposition, we refer to the proof of Theorem 68. □

Theorem 71. *Given a qcc* Q *and an arbitrary geometric realization* $R(Q)$ *of it, the* $\mathcal{DRA}_{fp}$ *relation in* $R(Q)$ *is the same as that computed by the basic classifiers on* Q.

Proof. Follows from Theorem 68 and Proposition 69. □

4.2.6. Implementation of the Classification Procedure. Qualitative composition configurations can be naturally represented as a finite datatype. The classifiers are implemented as simple programs (mainly case distinctions) that operate on *qccs* in the sense of Definition 52. The classifiers are chosen with respect to the identifier i and the assignment mapping a of the *qcc*. In our particular implementation, we exploited some symmetries to limit the number of classifiers that we had to implement.

With the condensed semantics, we are able to compute the composition tables of the $\mathcal{DRA}$ calculi in an efficient way. In fact we have implemented the computation of composition tables for both $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ as Haskell programs, making use of Haskell's parallelism extensions. The Haskell implementations of the basic classifiers for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ are written in such a way that they share a library of primitive classifiers. In these programs, we further generate all qccs in an optimized way, i.e. we only generate the order $<_p$ if it is needed, and classify them with our basic classifiers. In the end, we collect our results into composition tables. For the case where three lines are collinear, we simply decided to enumerate all possible locations of points in a certain interval for reasons of simplicity, and this did not increase the overall runtime too much. According to the condensed semantics we would have to place 6 points into the respective intervals and make sure that the transitivity between the orders is fulfilled. Implementing this procedure is far more complicated than our chosen one, given that the abstract dipoles might point into different abstract directions.

The computation of the composition table for $\mathcal{DRA}_f$ takes less than one minute on a Notebook with an Intel Core 2 T7200 with 1.5 Gbyte of RAM, and the computation of the composition table for $\mathcal{DRA}_{fp}$ takes less than two minutes on the same computer. This is a great advancement compared to the enumeration of scenarios on a grid, which took several weeks to compute only an approximation to the composition table.

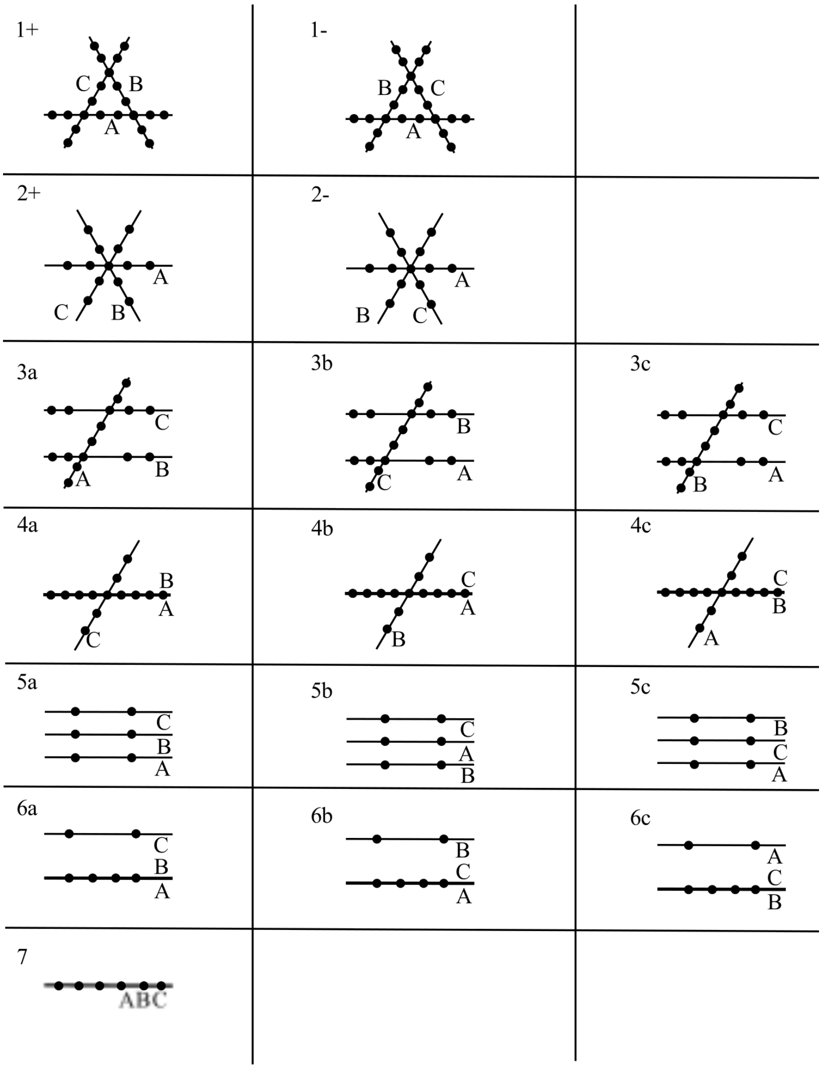


FIGURE 42. The 17 canonical configurations.

4.3. Simplified Computation of the Composition Table

This section has appeared as [52] in a very similar way. The purpose of condensed semantics is to provide a way of computing composition tables in a finite way. Therefore, we need to reduce the infinite space of possible dipole configurations to a finite one without losing any essential information. For each of the 17 oriented orbits in Figure 29 we introduce a *canonical configuration* in the Euclidean plane (depicted in Figure 42), i.e., a configuration with a suitable number of positions for the start and end points of the dipoles on each line that suffice to compute the composition table. The number of points needed is a function of the unoriented orbits, we call these points (that are displayed in Figure 42) *prototypical points*. We call a configuration from Figure 42 with three assigned dipoles a *prototypical configuration*. The computation of the composition table needs the orientation information in the configurations in order to be exhaustive.

The algorithm for computing the composition table is given in Algorithm 1. We place the configurations of Figure 42 into an arbitrary orthogonal coordinate

Algorithm 1 Composition Table

Computation of composition table for $\mathcal{DRA}$

Conf := the set of prototypical configurations from Figure 42
$R := \varnothing$
for all configurations $c \in Conf$ **do**
    for all dipoles $d_A = (\mathbf{s}_A, \mathbf{e}_A)$ of different prototypical points on A in c **do**
        for all dipoles $d_B = (\mathbf{s}_B, \mathbf{e}_B)$ of different prototypical points on B in c **do**
            for all dipoles $d_C = (\mathbf{s}_C, \mathbf{e}_C)$ of different prototypical points on C in c **do**
                compute the relations $d_A\ R_1\ d_B$, $d_B\ R_2\ d_C$, $d_A\ R_3\ d_C$ by the formula on page 34 and add the triple (R_1, R_2, R_3) to R.
            end for
        end for
    end for
end for
collect the triples in R in such a way that there is exactly one entry for every R_1 and R_2 having the union of all R_3 as third component

system. Each configuration provides a finite number of prototypical points with specific coordinates, which serve as start and end points of prototypical dipoles. For each triple of such prototypical dipoles we compute the $\mathcal{DRA}$-relations using the atan2-method as described in Section 3.2.3. Each triple that is obtained in this way corresponds to an entry in the composition table.

A program has been implemented in Java that uses Algorithm 1 and on a notebook with an Intel Core 2 T7200 with 1.5 Gbyte of RAM, the computation of the composition tables for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ takes approximately 14 seconds.

4.3.1. Soundness and Completeness of the Approach. By the soundness of Algorithm 1 we mean that the computed composition table contains enough entries to make it over-approximate geometric reality (i.e., no false conclusions can be drawn by qualitative reasoning). Conversely, completeness means that there are not more entries than necessary, that is, the composition table does not lead to overly weak conclusions. (However note that even in case of completeness it still can be the case that *algebraic closure* leads to overly weak conclusions, e.g., inconsistencies are not detected, see Section 6).

More specifically, completeness means that Algorithm 1 outputs only triples of dipole relations that are geometrically realizable, while soundness means that it outputs all such triples.[5] Soundness and completeness together imply that prototypical dipole triples are representative for all dipole triples, at least for what concerns dipole relations.

Proposition 72 (Completeness)**.** *Algorithm 1 is complete.*

Proof. Easy, since the triples of dipole relations are generated from prototypical dipole triples, which provide geometric realizations. □

Showing the soundness of Algorithm 1 is more involved. We need to identify a lower bound of points that is needed on our oriented orbits in Figure 29 with respect to the $\mathcal{DRA}$ semantics. We can identify those lower bounds for intersecting and collinear lines separately.

In a first step, we consider collinear lines. For soundness of the construction, we need to show that for two or three dipoles on the same line there is a lower bound

[5] Actually, the algorithm will output many triples more than once; these duplicates could be filtered out.

for the number of prototypical points needed to distinguish between the possible $\mathcal{DRA}$ relations on a line.

Consider any configuration of $n \in \{2,3\}$ collinear dipoles A, B (and C). We use an order induced by $\mathbf{e}_A < \mathbf{s}_A$ on the line, i.e., if B points into the same direction as A, we have $\mathbf{e}_B < \mathbf{s}_B$, otherwise $\mathbf{s}_B < \mathbf{e}_B$, and the same for C. This construction reflects the fact that dipoles always have non-zero length.

We translate the 13 Allen relations and the "opposite" Allen relations componentwise to our order for two dipoles A and B. For the first component this yields the transformation

$$\begin{aligned} A\, b\_\_\_\, B &\mapsto \mathbf{s}_A < \mathbf{s}_B \wedge \mathbf{e}_A < \mathbf{s}_B \\ A\, s\_\_\_\, B &\mapsto \mathbf{s}_A = \mathbf{s}_B \wedge \mathbf{e}_A < \mathbf{s}_B \\ A\, i\_\_\_\, B &\mapsto \mathbf{s}_B < \mathbf{s}_A \wedge \mathbf{e}_A < \mathbf{s}_B \\ A\, e\_\_\_\, B &\mapsto \mathbf{s}_B < \mathbf{s}_A \wedge \mathbf{e}_A = \mathbf{s}_B \\ A\, f\_\_\_\, B &\mapsto \mathbf{s}_B < \mathbf{s}_A \wedge \mathbf{s}_B < \mathbf{e}_A \end{aligned}$$

and likewise for the other components of the relations in question.

Example 73. Consider the relation $(A\ bbff\ B)$. Since both dipoles point into the same direction, we can derive $\mathbf{e}_A < \mathbf{s}_A \wedge \mathbf{e}_B < \mathbf{s}_B$. Now, we apply the translation rules for each component:

$$\begin{array}{ccccc} \mathbf{s}_A < \mathbf{s}_B & \wedge & \mathbf{e}_A < \mathbf{s}_B & \wedge \\ \mathbf{s}_A < \mathbf{e}_B & \wedge & \mathbf{e}_A < \mathbf{e}_B & \wedge \\ \mathbf{s}_A < \mathbf{e}_B & \wedge & \mathbf{s}_A < \mathbf{s}_B & \wedge \\ \mathbf{e}_A < \mathbf{s}_B & \wedge & \mathbf{e}_A < \mathbf{e}_B & \end{array}$$

we observe that the overall inequalities can be simplified to

$$\begin{array}{ccccc} \mathbf{e}_A < \mathbf{s}_A & \wedge & \mathbf{e}_B < \mathbf{s}_B & \wedge \\ \mathbf{s}_A < \mathbf{s}_B & \wedge & \mathbf{e}_A < \mathbf{s}_B & \wedge \\ \mathbf{s}_A < \mathbf{e}_B & \wedge & \mathbf{e}_A < \mathbf{e}_B & \end{array}$$

By transitivity of $<$, we can derive $\mathbf{e}_A < \mathbf{s}_A < \mathbf{e}_B < \mathbf{s}_B$. Hence we need at least four points in the plane to realize this dipole relation.

By an easy induction, we can show:

Lemma 74. *For n collinear dipoles in the Euclidean plane, $2 \cdot n$ points that can be the start and end points of the dipoles suffice to constitute all possible $\mathcal{DRA}$ relations between those dipoles.*

Corollary 75. *Realizing the relations between 1 (2, 3) collinear dipoles in the plane requires 2 (4, 6) prototypical points.*

After having considered the number of prototypical points needed for the realization of relations between collinear dipoles, we need to do the same for dipoles on intersecting carrier lines. For this purpose we need to consider the semantics of the $\mathcal{DRA}$ relations. The only case in which a point can lie on both intersecting lines is when it is positioned on the point of intersection. This is the only case where in this scenario relations can have a component from b, s, i, e, f, since these relations require one dipole being collinear with the start or end point of the other dipole. So we need to place a prototypical point onto the point of intersection. On each line on each side of the point of intersection, the rules for collinear lines are applied. Figure 43 shows the case for no collinear lines, that is only two intersection lines.

Lemma 76. *Transforming a scenario of dipoles along an orientation preserving affine transformation preserves the $\mathcal{DRA}_f$ ($\mathcal{DRA}_{fp}$) relations.*

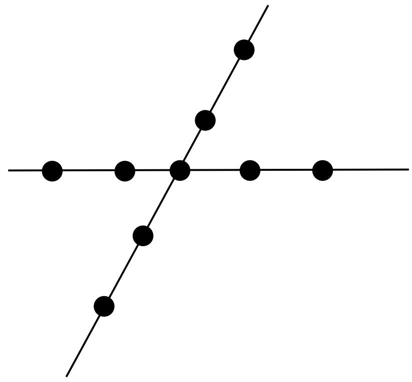


FIGURE 43. Intersecting carrier lines

Proof. This follows directly from Proposition 50. □

For the soundness proof, we need some preparatory lemmas.

Lemma 77. *Transforming a scenario in three dipoles along an orientation preserving affine transformation preserves betweenness of points.*

Proof. This follows from the proof of Proposition 50 for the $\mathcal{LR}$ relation i. □

Analogously to the qualitative angle in $\mathcal{DRA}_{fp}$, we define a *qualitative orientation* between two dipoles.

Definition 78. Given two non-parallel dipoles A and B, we say that the *qualitative orientation* from A to B is + if the angle from A to B is positive, otherwise it is −.

Lemma 79. *Given a fixed dipole B and a fixed intersection point S_{AB}, the relation A R B is determined by betweenness and equality among $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$, and the qualitative orientation between B and A provided that R does not involve parallelism or anti-parallelism.*

Proof. Let $A \mathrm{R} B$ and A' be such that $S_{AB} = S_{A'B}$ and that betweenness and equality among $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$, and $\{\mathbf{s}_{A'}, \mathbf{e}_{A'}, S_{AB}\}$ are the same, and the qualitative orientations from B to A and from B to A' are also the same. We introduce carrier rays for the dipoles called l_A, $l_{A'}$ and l_B. Without loss of generality, the rays point in the same direction as the dipoles, and hence reflect the qualitative orientation. The rays l_A and $l_{A'}$ are divided into three segments by S_{AB} and $S_{A'B}$ respectively. For l_A these are the segments with points $x <_r S_{AB}$, $x =_r S_{AB}$ and $S_{AB} <_r x$, and $l_{A'}$ is segmented in the same way, where we call the order $<_{r'}$. The relation R can be decomposed into the four $\mathcal{LR}$-relations

$$(A\,\mathrm{R}_1\,\mathbf{s}_B) \quad (A\,\mathrm{R}_2\,\mathbf{e}_B) \quad (B\,\mathrm{R}_3\,\mathbf{s}_A) \quad (B\,\mathrm{R}_4\,\mathbf{e}_A).$$

according to the semantics of $\mathcal{DRA}_f$ introduced in Section 3.2.3. First we will consider the relations R_3 and R_4. By definition of the $\mathcal{LR}$-relations, the relations between B and $\mathbf{s}_A$ or $\mathbf{e}_A$ change if the respective point is moved into a different segment, but since the betweenness and equality among $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$ are the same and the qualitative orientations also coincide, if $\mathbf{s}_A <_r S_{AB}$, so is $\mathbf{s}_{A'} <_r S_{A'B}$ and the same for the other segments and $\mathbf{e}_A$ and $\mathbf{e}_{A'}$. Hence, we obtain that R_3 = R'$_3$ and R_4 = R'$_4$. For R_1 and R_2, we use a similar argument with the roles of A' or A and B swapped. □

Proposition 80. *Algorithm 1 is sound.*

Proof. We will first give this proof for $\mathcal{DRA}_{fp}$ since soundness for $\mathcal{DRA}_f$ follows from soundness for $\mathcal{DRA}_{fp}$ by uniting particular relations. Given any triple of dipoles (d_A, d_B, d_C) in the Euclidean plane, we inspect their carrier lines (A, B, C) and the intersection points of the latter to identify their oriented orbit from Figure 29. As an example consider the configuration of dipoles in Figure 44 on the left hand side. The configuration on the right hand side shows the carrier lines and we can identify three different points of intersection. Together with the orientation of the lines, we see that this configuration lies in orbit 1−.

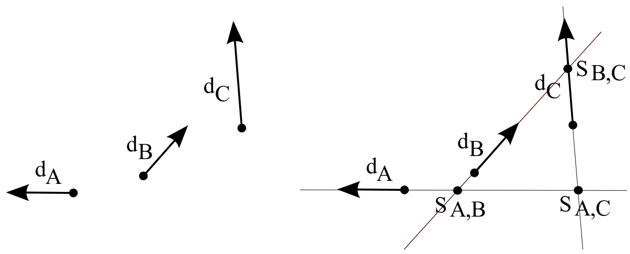


FIGURE 44. Introduction of carrier lines

We can identify the relations R_{AB}, R_{BC} and R_{AC} in that scenario. By S_{XY} we denote the point of intersection of the carrier lines X and Y. We call the lines in the corresponding configuration from Figure 42 A', B' and C', and we will find respective dipoles $d_{A'}$, $d_{B'}$ and $d_{C'}$ based on the prototypical points. We will show that the dipole relations for d_A, d_B, d_C and $d_{A'}$, $d_{B'}$, $d_{C'}$ are the same.

Note that on collinear lines the number of minimally needed points per section has been shown in Corollary 75. In any of the following cases of the proof, we need to consider all possible choices of points.

1+ & 1−) In this orbit, three distinct points of intersection exist, denoted by S_{AB}, S_{AC}, S_{BC} for the dipole configuration and $S_{A'B'}$, $S_{A'C'}$, $S_{B'C'}$ for the prototypical configuration. Since both triples A, B, C and A', B', C' are in the same oriented orbit, there is an orientation preserving affine bijection h between them, mapping A, B, C, S_{AB}, S_{AC} and S_{BC} to their primed variants. By Lemma 77 the point sets $\{\mathbf{s}_A, \mathbf{e}_A, S_{AC}\}$ and $\{\mathbf{s}_A, \mathbf{e}_A, S_{A'C'}\}$ are ordered in corresponding ways, and so are all other all other point sets involving the start and end points of dipoles and an intersection point. The points of the dipoles d_A, d_B and d_C are not necessarily mapped onto $d_{A'}$, $d_{B'}$ and $d_{C'}$, but the order between the start and end points and points of intersection is the same. By Lemma 79, only order and qualitative orientation has an influence on the $\mathcal{DRA}_{fp}$ relations at hand, so the mapped start and end points can be just moved onto the ones $d_{A'}$, $d_{B'}$ and $d_{C'}$ without changing the $\mathcal{DRA}_{fp}$ relations.

2+ & 2−) In this case all points of intersection coincide. This is the only difference between cases 1+ and 1−, but the argument stays the same.

3a & 3b & 3c) In this case we have parallel lines. First we consider case 3a. Here, the line A is intersected by the parallel lines B and C. Since A, B and C are in the same oriented orbit as A', B' and C', there is an orientation preserving affine transformation between them. By case 3 on page 49, choose the transformation h in such a way that it takes the affine frame $\{x_b, S_{AC}, S_{AB}\}$ to $\{x_{b'}, S_{A'C'}, S_{A'B'}\}$. The point x_b is chosen as $\mathbf{s}_b$ if $\mathbf{s}_b \neq S_{AB}$ and $\mathbf{e}_b$ otherwise. $x_{b'}$ is defined analogously. By Lemma 77 betweenness of $\{\mathbf{s}_A, \mathbf{e}_A, S_{AC}\}$ and $\{\mathbf{s}_C, \mathbf{e}_C, S_{AC}\}$ is preserved by h. The preservation of betweenness of the respective point-triples implies two possible orders. We introduce an order as in the proof of Lemma 79. The points $\mathbf{s}_X$ and $\mathbf{e}_X$ are not necessarily mapped onto $\mathbf{s}_{X'}$ and $\mathbf{e}_{X'}$ but the order with respect to the intersection points is the same. So the points in the image of h can be moved onto the respective prototypical points without affecting the dipole relations. On the parallel lines, the relation is preserved, since the direction of the dipoles is preserved (by the order with respect to the point of intersection). The preservation of betweenness is true for the triples $\{\mathbf{s}_A, \mathbf{e}_A, S_{AC}\}$, $\{\mathbf{s}_C, \mathbf{e}_C, S_{AC}\}$, $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$ and $\{\mathbf{s}_B, \mathbf{e}_B, S_{AB}\}$. By the preservation of the betweenness and orientation, the relations with respect to d_A are also the same. The argument for 3b and 3c is analogous.

4a & 4b & 4c) First we consider case 4a. We only have one point of intersection $S_{AB} = S_{AC} = S_{BC} = S$. There is an orientation preserving affine transformation h that takes the scenario to the instance shown in Figure 29, since both configurations are in the same orbit. We need to consider several triples of points on the line: $\{S, \mathbf{s}_A, \mathbf{e}_A\}$, $\{S, \mathbf{s}_B, \mathbf{e}_B\}$, $\{S, \mathbf{s}_A, \mathbf{s}_B\}$, $\{S, \mathbf{s}_A, \mathbf{e}_B\}$, $\{S, \mathbf{s}_B, \mathbf{e}_A\}$, and $\{S, \mathbf{e}_B, \mathbf{e}_A\}$ as well as $\{S, \mathbf{s}_C, \mathbf{e}_C\}$. By Lemma 77 the betweenness of the triples is preserved under h. From this betweenness of the triples and the orientation, we construct an order between the points. Since there are two possibilities to establish an order from the betweenness of a triple, we need to construct the order as in the proof of Lemma 79. If the dipoles on the coinciding line point into different directions, we can still construct an overall compatible order, by inverting one of the orders induced by A or B, which of them we invert is arbitrary. With this mapping we can determine a setup of prototypical points on this configuration and move the mapped points onto them. This does not change the involved $\mathcal{DRA}_{fp}$ relations, since the order of the points is preserved by this operation. Since the order of points on the line is not changed, we get in both cases the same relation and for the intersection we get the same relation by Lemma 79. The cases 4b and 4c are proved analogously.

5a & 5b & 5c) We do not have any points of intersection in this case. Without loss of generality we assume the ratio of the distances between the parallel lines to be 1. First, we have a look at case 5a. We intersect the lines A, B and C with an additional line orthogonal to A in such a way that no intersection points are equal to any of the start or end points of any dipole. We call the points of intersection S_A, S_B and S_C. There is an orientation preserving affine transformation to the instance of 5a given in Figure 29, since both configurations are in the same orbit. The order of the triples $\{S_X, \mathbf{s}_X, \mathbf{e}_X\}$ with $X \in \{A, B, C\}$ is preserved. Again we can move the points in the image of h to the prototypical points and get in both cases the same relations, since they just depend on the order of the points. The cases 5b and 5c are treated analogously.

6a & 6b & 6c) We start with case 6a. We intersect the lines with a new one that is orthogonal to A and intersects all carrier lines in such a way that all points on A, B and C are on the same side of the new line. We call the points of intersection S_A, S_B and S_C. Again there is an orientation preserving affine transformation to the representative of the orbit. The respective orders of the start and endpoints of the dipoles and points of intersection are preserved. And the start and endpoints of the dipoles can be moved to the respective prototypical points without changing the dipole relation. As in case 5a the relations stay the same. Cases 6b and 6c are treated analogously.

7) We intersect the lines with a new one orthogonally (and do the same with the representative of the orbit) in such a way that the point is intersection is different from all start and end points of the dipoles and call this point of intersection S. There is an orientation preserving affine transformation h that

maps the configuration to the representative of the orbit, i.e. to the configuration in Figure 29. There are several triples of points on the line we need to consider: $\{S, \mathbf{s}_A, \mathbf{e}_A\}$, $\{S, \mathbf{s}_B, \mathbf{e}_B\}$, $\{S, \mathbf{s}_C, \mathbf{e}_C\}$, $\{S, \mathbf{s}_A, \mathbf{s}_B\}$, $\{S, \mathbf{s}_A, \mathbf{s}_C\}$, $\{S, \mathbf{s}_A, \mathbf{e}_B\}$, $\{S, \mathbf{s}_A, \mathbf{e}_C\}$, $\{S, \mathbf{s}_B, \mathbf{s}_C\}$, $\{S, \mathbf{s}_B, \mathbf{e}_C\}$, $\{S, \mathbf{e}_B, \mathbf{s}_C\}$, $\{S, \mathbf{e}_B, \mathbf{e}_C\}$, with an order that is constructed as in the proof of Lemma 79. As in step $4a$ & $4b$ & $4c$ we can make the orders compatible, if dipoles point into different directions. From the above list, we can infer that six prototypical points are needed to compute all dipole relations between three collinear dipoles. Again we can move the mapped points onto the prototypical ones without any harm, since only the relative ordering of the points matters by the definition of the $\mathcal{DRA}_{fp}$ relations.

For $\mathcal{DRA}_f$, we just take the union of the refined relations, i.e., we use the mapping

$$\begin{aligned} \{llll{+}, llll{-}, lllA\} &\mapsto llll \\ \{llrr{+}, llrr{-}, llrrP\} &\mapsto llrr \\ \{rrll{+}, rrll{-}, rrllP\} &\mapsto rrll \\ \{rrrr{+}, rrrr{-}, rrrrA\} &\mapsto rrrr. \end{aligned}$$

□

So far we have introduced two methods for the computation of the composition tables for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ based on condensed semantics. With them we are able to compute the actual tables and investigate formal properties of the composition operations. That investigation shall be conducted in Section 4.4.

4.4. Algebraic properties of composition

We now investigate several properties of the composition tables for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$. For both tables the properties

$$\begin{aligned} id^{\smile} &= id \\ (R^{\smile})^{\smile} &= R \\ id \circ R &= R \\ R \circ id &= R \\ (R_1 \circ R_2)^{\smile} &= R_2^{\smile} \circ R_1^{\smile} \\ R_1^{\smile} \in R_2 \circ R_3 &\iff R_3^{\smile} \in R_1 \circ R_2 \end{aligned}$$

hold with R, R_1, R_2, R_3 being any base relations and id the identical relation. These properties can be automatically tested by the qualitative reasoners `GQR` and `SparQ`. The other properties for a non-associative algebra follow trivially. Furthermore, we have tested the associativity of the composition. For $\mathcal{DRA}_f$, we have 373,248 triples of relations to consider of which 71,424 are not associative. So the composition of 19.14% of all possible triples of relations is not associative[6], e.g., associativity is violated in the compositions:

$$(\text{rrrl} \diamond \text{rrrl}) \diamond \text{llrl} \neq \text{rrrl} \diamond (\text{rrrl} \diamond \text{llrl}).$$

For $\mathcal{DRA}_{fp}$ all 512,000 triples of base relations are associative w.r.t. composition. Hence $\mathcal{DRA}_{fp}$ is a relation algebra.

[6] In the masters thesis of one of our students, a detailed analysis of a specific non-associative dipole configuration is presented [53]

4.4.1. Composition of $\mathcal{DRA}_f$ is weak. The failure of $\mathcal{DRA}_f$ to be associative implies that its composition is weak. We will still prove directly that $\mathcal{DRA}_f$ has weak composition by giving an example, since this is more illustrative:

Proposition 81. *The composition of $\mathcal{DRA}_f$ is weak.*

Proof. Consider the $\mathcal{DRA}_f$ composition A bfii $B \diamond B$ lllb $C \mapsto A$ llll C. We show that there are dipoles A and C such that there is no dipole B which reflects the composition. Consider dipoles A and C as shown in Figure 45. We observe that

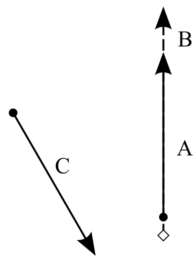


FIGURE 45. $\mathcal{DRA}_f$ weak composition

they are in the $\mathcal{DRA}_{fp}$ relation llll- with the dipole C pointing towards the line l_A dipole A lies on. Because of A bfii B, dipole B has to lie on l_A. But, since l_C, the carrier line of C, is a straight line and lines l_A and l_B lie in front of C with respect to the direction of the dipole, the endpoint of B cannot lie behind C. □

$\mathcal{DRA}_{fp}$ behaves differently, as shown in the next section.

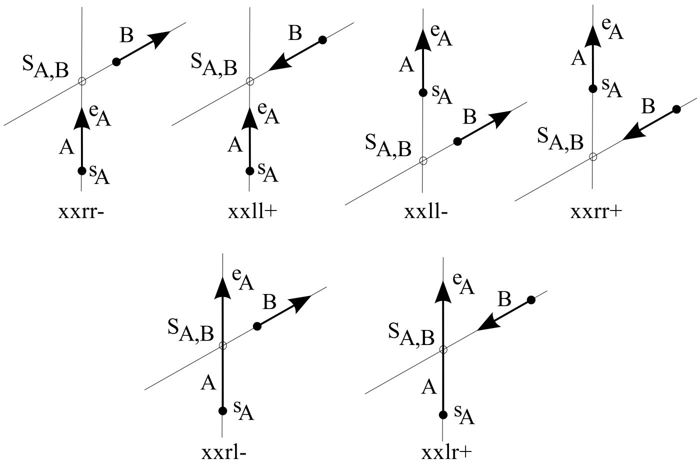
4.4.2. Strong Composition for $\mathcal{DRA}_{fp}$. We are now going to prove that $\mathcal{DRA}_{fp}$ has strong composition. The following lemma will be crucial; note that it does *not* hold for $\mathcal{DRA}_f$.

Lemma 82 (Orientation Lemma)**.** *For $\mathcal{DRA}_{fp}$ base relations R not involving parallelism or anti-parallelism, betweenness and equality among $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$ for given dipoles $A\,R\,B$ are independent of the choice of A and B, hence uniquely determined by R alone.*

Proof. Let $R = r_1r_2r_3r_4r_5$, where $r_5 \in \{+,-\}$ even if r_5 is omitted in the standard notation. Note that the assumption $r_5 \in \{+,-\}$ implies that S_{AB} is defined. If $r_3 \in \{b,s,i,e,f\}$, $\mathbf{e}_A \neq \mathbf{s}_A = S_{AB}$, and there is no betweenness. Analogously, $\mathbf{s}_A \neq \mathbf{e}_A = S_{AB}$ if $r_4 \in \{b,s,i,e,f\}$. The remaining possibilities for $r_3r_4r_5$ are:

(1) ll+, rr-: in these cases, $\mathbf{e}_A$ is between $\mathbf{s}_A$ and S_{AB};
(2) ll-, rr+: in these cases, $\mathbf{s}_A$ is between $\mathbf{e}_A$ and S_{AB};
(3) rl-, lr+: in these cases, S_{AB} is between $\mathbf{s}_A$ and $\mathbf{e}_A$.

Note that cases 1 and 2 cannot be distinguished in $\mathcal{DRA}_f$. In particular, the pictures for xxll+ and xxll- lead to the same $\mathcal{DRA}_f$ relation xxll, but for xxll+, $\mathbf{e}_A$ is between $\mathbf{s}_A$ and S_{AB}, while for xxll-, $\mathbf{s}_A$ is between $\mathbf{e}_A$ and S_{AB}.



□

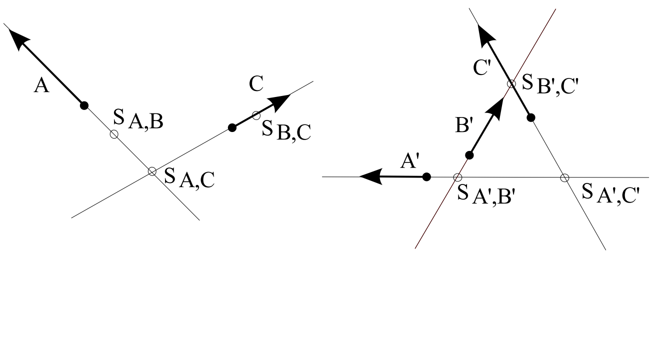
Corollary 83 (Orientation Corollary). *Let R be a $\mathcal{DRA}_{fp}$ base relation not involving parallelism or anti-parallelism. Let $A\,R\,B$ and $A'\,R\,B'$. Then, the map $\{\mathbf{s}_A \mapsto \mathbf{s}_{A'}; \mathbf{e}_A \mapsto \mathbf{e}_{A'}; S_{AB} \mapsto S_{A'B'}\}$ preserves betweenness and equality.*

Theorem 84 (Strong composition in $\mathcal{DRA}_{fp}$). *Composition in $\mathcal{DRA}_{fp}$ is strong.*

Proof. Obviously, strong composition $\circ$ is contained in weak composition $\diamond$. To show the converse, let $r_{ac} \in r_{ab} \diamond r_{bc}$ be an entry in the composition table, with r_{ac}, r_{ab} and r_{bc} being base relations. We need to show that $r_{ac} \in r_{ab} \circ r_{bc}$, i.e., that for any given dipoles A and C with $Ar_{ac}C$, there exists a dipole B with $Ar_{ab}B$ and $Br_{bc}C$.

Since $r_{ac} \in r_{ab} \diamond r_{bc}$, by definition of weak composition, we know that there are dipoles A', B' and C' with $A'r_{ab}B'$, $B'r_{bc}C'$ and $A'r_{ac}C'$. Given dipoles X and Y, let S_{XY} denote the point of intersection of the lines carrying X and Y; it is only defined if X and Y are not parallel. Consider now the three lines carrying A', B' and C', respectively. According to the results of Section 4.1.1, for the configuration of these three lines, there are seventeen qualitatively different cases 1+, 1−, 2+, 2−, 3a, 3b, 3c, 4a, 4b, 4c, 5a, 5b, 5c, 6a, 6b, 6c and 7:

(1) We consider cases 1+ and 1− simultaneously. Recall that all line configurations in the orbit 1+ have the same orientation, and the same holds for 1−. The three points of intersection $S_{A'B'}$, $S_{B'C'}$ and $S_{A'C'}$ exist and are different. Since $Ar_{ac}C$ and $A'r_{ac}C'$, by Lemma 82, the point sets $\{\mathbf{s}_A, \mathbf{e}_A, S_{AC}\}$ and $\{\mathbf{s}_{A'}, \mathbf{e}_{A'}, S_{A'C'}\}$ are ordered in corresponding ways on their carrier lines. Hence, it is possible to choose a point S_{AB} on the carrier line of A in such a way that the point sets $\{\mathbf{s}_A, \mathbf{e}_A, S_{AC}, S_{AB}\}$ and $\{\mathbf{s}_{A'}, \mathbf{e}_{A'}, S_{A'C'}, S_{A'B'}\}$ are ordered in corresponding ways on their lines. In a similar way (interchanging A and C), S_{BC} can be chosen.



Since both $\{S_{AB}, S_{AC}, S_{BC}\}$ and $\{S_{A'B'}, S_{A'C'}, S_{B'C'}\}$ are affine frames, there is a unique affine bijection $h{:}\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $h(S_{A'B'}) = S_{AB}$, $h(S_{A'C'}) = S_{AC}$ and $h(S_{B'C'}) = S_{BC}$. Since all line configurations in the orbit 1+ have the same orientation (and the same holds for 1−), h preserves orientation. Thus by Proposition 50 the $\mathcal{DRA}_{fp}$ relations are also preserved along h. Hence, by choosing $B = h(B')$, we get $h(A')r_{ab}B$ and $Br_{bc}h(C')$. Since the point sets $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$ and $\{\mathbf{s}_{A'}, \mathbf{e}_{A'}, S_{A'B'}\}$ are ordered in corresponding ways on their lines and h is an affine bijection, also $\{\mathbf{s}_A, \mathbf{e}_A, S_{AB}\}$ and $\{h(\mathbf{s}_{A'}), h(\mathbf{e}_{A'}), S_{AB}\}$ are ordered in corresponding ways on their lines, and moreover the qualitative orientation for A to B is the same as that from A' to B. Since also $S_{AB} = S_{A'B}$, by Lemma 79, from $h(A')r_{ab}B$ we thus get $Ar_{ab}B$. A similar argument shows that $Br_{bc}C$.

(2) We prove cases 2+ and 2− simultaneously. The three intersection points $S_{A'B'}$, $S_{B'C'}$ and $S_{A'C'}$ exist and coincide, i.e., $S_{A'B'} = S_{B'C'} = S_{A'C'} =: S'$. Let $S := S_{AC}$. Let x_A be $\mathbf{s}_A$ and $x_{A'}$ be $\mathbf{s}_{A'}$ if $\mathbf{s}_A \neq S$ (and therefore $\mathbf{s}_{A'} \neq S'$), otherwise, let x_A be $\mathbf{e}_A$ and $x_{A'}$ be $\mathbf{e}_{A'}$. x_C and $x_{C'}$ are chosen in a similar way. Since both $\{S, x_A, x_C\}$ and $\{S', x_{A'}, x_{C'}\}$ are affine frames, there is a unique affine bijection $h{:}\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with $h(S') = S$, $h(x_{A'}) = x_A$ and $h(x_{C'}) = x_C$. The rest of the argument is similar to case (1).

(3) (Two lines are parallel and intersect with the third one.) In the sequel, we will just specify how two affine frames are chosen; the rest of the argument (as well as the choice of points on the unprimed side in such a way that qualitative relations are preserved) is then similar to the previous cases.
Subcases (3*a*), (3*b*): The lines carrying A and C intersect. Choose x_A and $x_{A'}$ as in case (2), and chose an appropriate point S_{BC}. Then use the affine frames $\{x_A, S_{AC}, S_{BC}\}$ and $\{x_{A'}, S_{A'C'}, S_{B'C'}\}$.
Subcase (3*c*): The lines carrying A and C are parallel. Choose appropriate points S_{AB} and S_{BC} and use the affine frames $\{\mathbf{s}_A, S_{AB}, S_{BC}\}$ and $\{\mathbf{s}_{A'}, S_{A'B'}, S_{B'C'}\}$.

(4) (Two lines are identical and intersect with the third one.)
Subcases (4*a*) and (4*b*): The lines carrying A and C intersect. Choose x_A, $x_{A'}$, x_C and $x_{C'}$ as in case (2) and use the affine frames $\{S_{AC}, x_A, x_C\}$ and $\{S_{A'C'}, x_{A'}, x_{C'}\}$.
Subcase (4*c*): The lines carrying A and C are identical. This means that $S_{A'B'} = S_{A'C'} =: S'$. Choose an appropriate point S and x_A, $x_{A'}$ as in case (2). Moreover, in a similar way, choose $x_{B'} \neq S'$, and then some corresponding x_B being in the same $\mathcal{LR}$ relation to A as $x_{B'}$ has to A'. Then use the affine frames $\{S, x_A, x_B\}$ and $\{S, x_{A'}, x_{B'}\}$.

(5) (All three lines are distinct and parallel.) Subcases (5*a*), (5*b*) and (5*c*) can all be treated in the same way: Use the affine frames $\{\mathbf{s}_A, \mathbf{e}_A, \mathbf{s}_C\}$ and $\{\mathbf{s}_{A'}, \mathbf{e}_{A'}, \mathbf{s}_{C'}\}$. Note that the distance ratios may need to adjusted by a non-affine transformation which however preserves the dipole relations.

(6) (Two lines are identical and are parallel to the third one.)
Subcases (6*a*) and (6*b*): The lines carrying A and C are parallel. Proceed as in case (5).
Subcase (6*c*): The lines carrying A and C are identical. Choose some $\mathbf{s}_B$ in the same $\mathcal{LR}$ relation to A as $\mathbf{s}_{B'}$ is to A'. Then use the affine frames $\{\mathbf{s}_A, \mathbf{e}_A, \mathbf{s}_B\}$ and $\{\mathbf{s}_{A'}, \mathbf{e}_{A'}, \mathbf{s}_{B'}\}$.

(7) (All three lines are identical.) For this case, the result follows from the fact that Allen's interval algebra has strong composition (refer to [63]).

□

For the first time, we are able to present composition tables for $\mathcal{DRA}_f$ as well as for $\mathcal{DRA}_{fp}$ that are computed using a reliable framework and hence correct. Older versions of the $\mathcal{DRA}_f$ composition tables contained errors and there has never been a composition table for $\mathcal{DRA}_{fp}$ so far. Further we have investigated properties of the composition and it turned out that $\mathcal{DRA}_{fp}$, which is based on ideas from the application side, also has the property of having strong composition. After this work, we have to investigate the expressiveness of algebraic closure for $\mathcal{DRA}_f$ and especially for $\mathcal{DRA}_{fp}$. We will do this in Chapter 6. Further we will investigate a simple application for $\mathcal{DRA}$ in Chapter 5.

CHAPTER 5

Applications of Qualitative Spatial Reasoning

. . . and the road becomes my bride
I have stripped of all but pride
so in her I do confide
and she keeps me satisfied
gives me all I need.
—**Metallica**, *Wherever I may Roam.*

After the derivation of the composition table for the dipole calculus in Chapter 4 and the study of several properties of the composition operation, we come to a point where we want to investigate some simple possible applications for this calculus. In fact, we employ it to describe the layout of a simple street network using local information and deriving all other information with algebraic closure. Composition-based reasoning like algebraic closure is incomplete in most instances (as it is for the dipole calculus as will be shown in Chapter 6) but that does not directly imply that the information derived is not at least "good enough" for the task at hand.

After the investigation of the description of road networks using the dipole calculus, we are going to focus our attention on navigation tasks using a calculus that has been developed for agents traveling in the plane, the $\mathcal{OPRA}$ calculus, which has been introduced in Section 3.3 and been described in [54] in extenso. In this scenario of a navigation problem, we assume that one or more agents explore a street network that is unknown to them and make local observations at crossings. The knowledge obtained by all agents is shared and merged. Based on this, global knowledge about the explored street network is derived using the incomplete but efficient method of algebraic closure. Using this global knowledge navigation is performed using a least angle strategy. Preliminary results of this approach have already been published in [42] and it turned out that algebraic closure was good enough for the problems presented there.

5.1. Algebraic Closure Reasoning With the Dipole Calculus

The Achilles' heel of many qualitative spatial calculi and of qualitative spatial reasoning itself is the fact that very little investigations into applications have been conducted. In this section, which has been published in [52], we demonstrate with an example why the dipole calculus is a useful qualitative model for directional information. Moreover, the example shows that composition-based reasoning is useful for certain applications although it is incomplete. Our example uses the spatial knowledge expressed in $\mathcal{DRA}_{fp}$ for deductive reasoning based on constraint propagation (algebraic closure), resulting in the generation of useful indirect knowledge from partial observations in a spatial scenario. This is a direct application of the composition table which we generated based on our new condensed semantics for the dipole calculi (see Chapter 4).

In our sample application, a spatial agent (a simulated robot, cognitive simulation of a biological system, etc.) explores a spatial scenario. The agent collects local observations and wants to generate survey knowledge. Figure 46 shows our spatial environment. It consists of a street network in which some streets continue straight

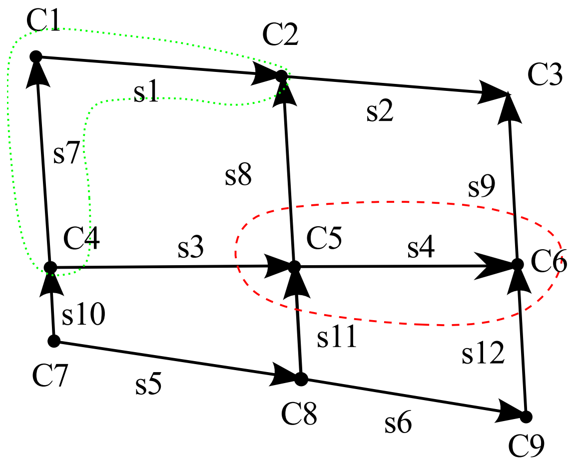


Figure 46. A street network and two local observations

after a crossing and some streets run parallel. These features are typical of real-world street networks. Spatial reasoning in our example uses constraint propagation (e.g., algebraic closure computation) to derive indirect constraints between the relative location of streets which are further apart and hence cannot be captured by the local observations between neighboring streets. The resulting survey knowledge can be used for several tasks including navigation tasks.

The environment is represented by streets s_i and crossings C_j. The streets and crossings have unique names (e.g., $s_1, \ldots, s_{12}$, and $C_1, \ldots, C_9$ in our example). The local observations are modeled in the following way, based on specific visibility rules simulating prototypical features of visual perception: Both at each crossing and at each straight street segment we have an observation. At each crossing the agent observes the neighboring crossings. At the middle of each straight street segment the agent can observe the direction of the outgoing streets at the adjacent crossings (but not at their other ends). Two specific examples of observations are marked in Figure 46. The observation "s_1 errs s_7" is marked in green at crossing C_1. The observation "s_8 rrllP s_9" is marked in red at street s_4.

These observations relate spatially neighboring streets to each other in a pairwise manner, using $\mathcal{DRA}_{fp}$ base relations, since the knowledge we can observe is precise with respect to the calculus. The agent has no additional knowledge about the specific environment. This is reflected in the constraint network by placing the universal relation between all pairs of streets where no observations could be make with respect to our observation scheme. This universal relation directly reflects the fact that no knowledge is available. In concrete constraint networks pairs of objects in the universal relation are often omitted, leading to the mantra: "No directly expressed knowledge means no knowledge". The global spatial world knowledge of the agent is expressed in the converse and composition tables of $\mathcal{DRA}_{fp}$.

The following sequence of partial observations could be the result of a tour made by the spatial agent exploring the street network of our example (see Figure 46):

`Observations at crossings`
C_1: (s_7 errs s_1)
C_2: (s_1 efbs s_2) (s_8 errs s_2) (s_1 rele s_8)
C_3: (s_2 rele s_9)
C_4: (s_{10} efbs s_7) (s_{10} errs s_3) (s_7 srsl s_3)
C_5: (s_3 efbs s_4) (s_{11} efbs s_8) (s_{11} errs s_4) (s_3 ells s_8)

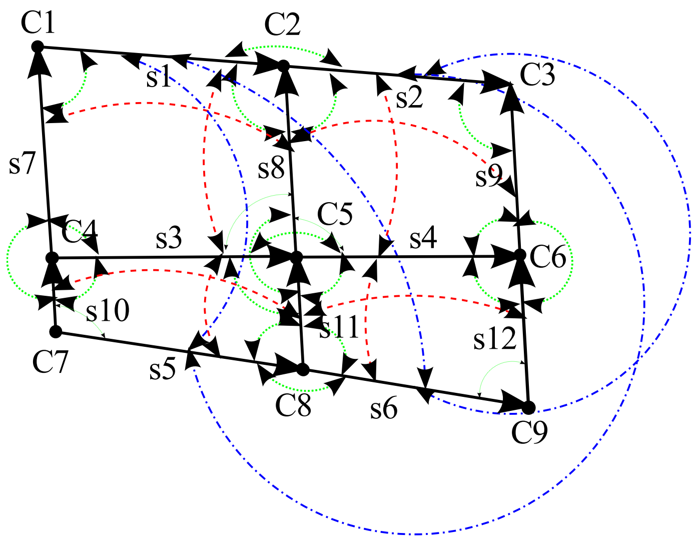


FIGURE 47. All observation and resulting uncertainty

```
    (s3  rele  s11) (s8  srsl  s4)
C6: (s12  efbs  s9) (s4  ells  s9) (s4  rele  s12)
C7: (s10  srsl  s5)
C8: (s5  efbs  s6) (s5  ells  s11) (s11  srsl  s6)
C9: (s6  ells  s12)

Observations at streets
s1: (s7  rrllP  s8)
s2: (s8  rrllP  s9)
s3: (s10  rrllP  s11)
s4: (s11  rrllP  s12)
s8: (s3  llrr-  s1)
s9: (s4  llrr-  s2)
s10:(s3  rrll-  s5)
s11:(s4  rrll-  s6)
```

The result of the algebraic closure computation/constraint propagation is a refined network with the same solution set. We employed the publicly available `SparQ` reasoning toolbox [78] supplied with our newly computed $\mathcal{DRA}_{fp}$ composition table to obtain the results. We have listed them in Section 5.1.1. Three different models are the only remaining consistent interpretations (see the Section 5.1.1 for a list of all the resulting data). The three different models agree on all but four relations. The solution set can be explained with the help of the diagram in Figure 47. We remember that the input crossing observations are marked with green arrows, the input street observations are marked with red arrows. The result shows that for all street pairs which could not be observed directly, the algebraic closure algorithm deduces a strong constraint, i.e., precise information: typically, the resulting spatial relation between street pairs comprises just one $\mathcal{DRA}_{fp}$ base relation. The exception consists of four relations between streets in which the three models differ (marked with dashed blue arrows in Figure 47). For these four relations, each model from the solution set agrees on the same $\mathcal{DRA}_f$ base relation for a given pair of dipoles, but the three consistent models differ on the finer granularity level of $\mathcal{DRA}_{fp}$ base relations. Since the refinement of one of these four underspecified relations into a single interpretation ($\mathcal{DRA}_{fp}$ base relation) as a logical consequence also assigns a single base relation to the other three relations, only three interpretations are valid models. The uncertainty/indeterminacy is the result of the specific street configuration in our example. The streets in a North–South direction are parallel, but the streets in an East–West direction are not parallel, resulting in fewer constraint propagation results. However, the small solution set of consistent models agrees on most of the relative position relations between street pairs and the differences between the models are small. In our judgment, this means that the system has generated the relevant survey knowledge about the whole street network from local observations.

5.1.1. Computation for the street network application with the `SparQ` toolbox. In this section, we demonstrate how to use the publicly available `SparQ` QSR toolbox [78] to compute the algebraic closure by constraint propagation for the street network example from the previous Section 5.1. For successful relative position reasoning, the `SparQ` tool has to be supplied with our newly computed $\mathcal{DRA}_{fp}$ composition table which we generated based on our new condensed semantics for the dipole calculi (see Section 4.1). This new composition table can be integrated into the `SparQ` toolbox quite easily.

The local street configuration observations by the spatial agent are listed in Section 5.1. The direct translation of these logical propositions into a `SparQ` spatial reasoning command looks as follows, where we suggest the interested reader to refer to the `SparQ` manual [78] for technical details:

```
sparq constraint-reasoning dra-fp path-consistency "( (s7 errs s1)
(s1 efbs s2) (s8 errs s2) (s1 rele s8) (s2 rele s9) (s10 efbs s7)
(s10 errs s3) (s7 srsl s3) (s3 efbs s4) (s11 efbs s8) (s11 errs s4)
(s3 ells s8) (s3 rele s11) (s8 srsl s4) (s12 efbs s9) (s4 ells s9)
(s4 rele s12) (s10 srsl s5) (s5 efbs s6) (s5 ells s11) (s11 srsl s6)
(s6 ells s12) (s7 rrllP s8) (s8 rrllP s9) (s10 rrllP s11)
(s11 rrllP s12) (s3 llrr- s1) (s4 llrr- s2) (s3 rrll- s5)
(s4 rrll- s6) )"
```
[1]

The result of this reasoning command is a refined network with the same solution set derived by the application of the algebraic closure/constraint propagation algorithm (see Chapter 2). `SparQ` omits the converses for a more compact presentation. For most calculi this is no issue, since often the converses of base relations are base relations.

```
Modified network.
((s5 (efbs) s6)(s12 (lsel) s6)(s12 (llfl) s5)(s11 (srsl) s6)
(s11 (lsel) s5)(s11 (rrllp) s12) (s4 (rrll-) s6)
(s4 (rrll-) s5)(s4 (rele) s12)(s4 (rser) s11)
(s3 (rrll-) s6)(s3 (rrll-) s5) (s3 (rfll) s12)(s3 (rele) s11)
(s3 (efbs) s4)(s10 (rrbl) s6)(s10 (srsl) s5)(s10 (rrllp) s12)
(s10 (rrllp) s11)(s10 (rrrb) s4)(s10 (errs) s3)
(s9 (lbll) s6)(s9 (llll-) s5)(s9 (bsef) s12)
(s9 (llrrp) s11)(s9 (lsel) s4)(s9 (llfl) s3)(s9 (llrrp) s10)
(s8 (brll) s6)(s8 (lbll) s5) (s8 (rrllp) s12)(s8 (bsef) s11)
(s8 (srsl) s4)(s8 (lsel) s3)(s8 (llrrp) s10)(s8 (rrllp) s9)
(s2 (rrll+ rrll- rrllp) s6)(s2 (rrll+ rrll- rrllp) s5)
```

[1] `SparQ` [78] does not accept line breaks which we have inserted here for better readability. All the data for this sample application including the new composition table can be obtained from the URL `http://www.informatik.uni-bremen.de/~till/Oslsa.tar.gz` (which also provides the composition table and other data for the `GQR` reasoning tool [24] `https://sfbtr8.informatik.uni-freiburg.de/R4LogoSpace/Resources/`).

```
(s2 (rrlf) s12)(s2 (rrfr) s11)(s2 (rrll+) s4) (s2 (rrll+) s3)
(s2 (rrrr+) s10)(s2 (rele) s9)(s2 (rser) s8)
(s1 (rrll+ rrll- rrllp) s6) (s1 (rrll+ rrll- rrllp) s5)
(s1 (rrll+) s12)(s1 (rrlf) s11)(s1 (rrll+) s4)(s1 (rrll+) s3)
(s1 (rrfr) s10)(s1 (rfll) s9)(s1 (rele) s8)(s1 (efbs) s2)
(s7 (rrll-) s6)(s7 (brll) s5) (s7 (rrllp) s12)(s7 (rrllp) s11)
(s7 (rrbl) s4)(s7 (srsl) s3)(s7 (bsef) s10)(s7 (rrllp) s9)
(s7 (rrllp) s8)(s7 (rrrb) s2)(s7 (errs) s1))
```

SparQ can output all algebraically closed scenarios (i.e., constraint networks in base relations). For this constraint network, only three slightly different algebraically closed scenarios exist. They differ only in the following three relation subsets:

(1) `(S2 (RRLLP) S6)(S2 (RRLLP) S5)`
`(S1 (RRLLP) S6)(S1 (RRLLP) S5)`
(2) `(S2 (RRLL-) S6)(S2 (RRLL-) S5)`
`(S1 (RRLL-) S6)(S1 (RRLL-) S5)`
(3) `(S2 (RRLL+) S6)(S2 (RRLL+) S5)`
`(S1 (RRLL+) S6)(S1 (RRLL+) S5)`

All the other relations were already assigned a single base relation in the refined network which is shown above as a result of the application of the algebraic closure algorithm. This result can be interpreted in terms of the goals of the reasoning task as done for $\mathcal{OPRA}$ in Section 5.2.5.

5.2. Streets to the $\mathcal{OPRA}$

In this section we investigate the usefulness of the $\mathcal{OPRA}$ calculi for a certain kind of navigation task. We investigate the problem of navigating with incomplete and hence possibly imprecise knowledge. To a certain extend some tasks performed in this application are quite similar to the application for the dipole calculus in Section 5.1, but there are some differences, mainly we do not just want to describe the street network, we also want to navigate in it. The starting point is the same. Again agents are exploring a street network as in Figure 48. These street networks might also be the layout of halls inside a building rendering the use of GPS inadequate. In fact, the use of positioning systems and maps is out of question for us at this point, since the problem of navigating with full knowledge of places is rather solved and not a problem for wayfinding algorithms. But the problem of navigating with imprecise knowledge, e.g. in a situation when you ask the next person you meet for directions, is important for us, especially since in many cases your will get qualitative descriptions of the way to go. Compared to Section 5.1 there are three major differences in the exploration phase. Firstly, we make use of the $\mathcal{OPRA}$ calculi which have no means to express parallelism. Secondly, observations are only make at crossings, this is partly due to the inability of $\mathcal{OPRA}$ to express parallelism and it is more natural for the task at hand. Thirdly, in $\mathcal{DRA}$ separate definitions of streets are not necessary, since they can be understood as dipoles. But in $\mathcal{OPRA}$ we need to define them. We can also use $\mathcal{OPRA}$ for this purpose. A legitimate question that might arise is why we do use $\mathcal{OPRA}$ and not only the "forward direction" of $\mathcal{OPRA}$. In fact, we want to make use of an existing calculus that has been studied before in many aspects and investigate its usefulness for a certain application. We do not want to forge new calculi directly tailored for our purpose and enlarge the big set of calculi available even more. Further, by only using the "forward direction" of $\mathcal{OPRA}$ the converses would be lost, which are necessary for using the state-of-the-art reasoners for calculating algebraic closure.

We have already described this approach in [42], but that paper has to be understood as containing preliminary but promising results. Since the paper has

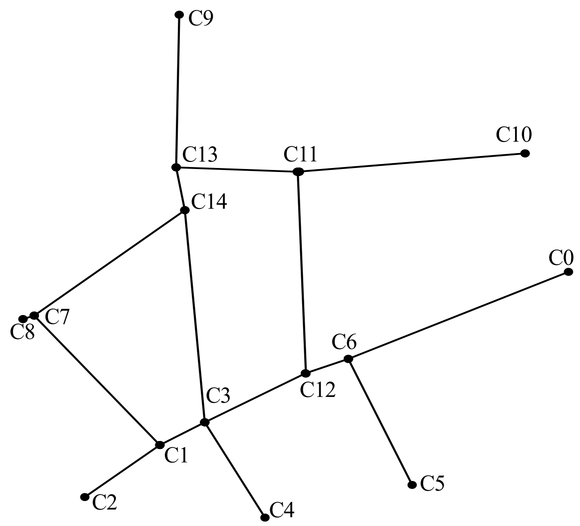


Figure 48. A small street network

been published we have refined our methodology of conducting and evaluating experiments in a profound way. The number of navigation experiments that we can conduct and evaluate has grown substantially in the meanwhile. In the paper we have showed results based on 66 navigation experiments, now conducting tens of thousands and more is no longer an issue for us. The only limiting factors are computation time and memory of a machine.

The work presented in this section is as the work in Section 5.1 a try to work on the main weakness of qualitative spatial reasoning: The little investigations into applications and especially into real world applications. Although a vast array of qualitative spatial calculi has been developed to describe certain aspects of space, little time has been invested to investigate their actual usefulness for applications. Again as in Section 5.1 the big question is if algebraic closure is "good enough" for our task at hand.

We investigate the applicability of $\mathcal{OPRA}$ calculi with reasonable granularity to navigation problems in street networks. For this task, we only rely on knowledge that a person or agent can observe at the *decision points*, i.e. the crossings, of a street network in a qualitative way. Technically the description of crossings is very similar to the one in Section 5.1, but the intention and the point of view is different, hence we describe the taking of observations with this new point of view. In Figure 49 such a crossing or decision point is shown. The person driving in the car knows where she comes from and can observe that the street with the pub is to the left, the one with the church is straight ahead and the one with the school is to the right. But she cannot observe where the airport at the other end of the city is with respect to the recently made observation. Further knowledge can be deducted from the observed facts, but that knowledge is only as good as is the reasoning for the calculus at hand. We have to ask the question, if this knowledge is "good enough". What helps us in this case is the fact that we are navigating in a grid that is pre-defined by the given street network. But there are still open questions, e.g.: is the "straight ahead" or "left" defined by $\mathcal{OPRA}$ the "straight ahead" or "left" as perceived by humans?

Research on *wayfinding choremes* by Klippel et al. [32–34] describes a representation of directions on decision points, e.g. crossings in street networks. The

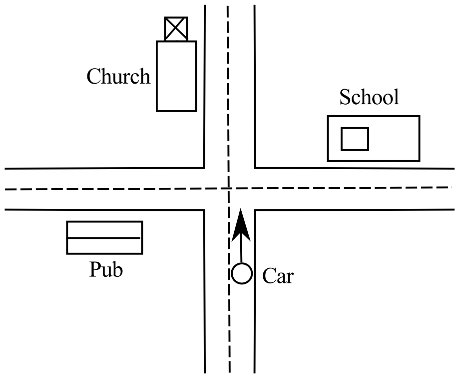


FIGURE 49. A crossing with landmarks

authors claim that their representation is *cognitively adequate*. Basically there are 7 choremes that describe turning situations at crossings as depicted in Figure 50. These choremes are ignorant of the situation of "going back", which is but formalized

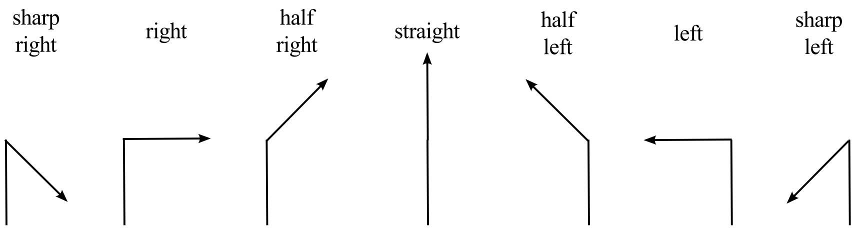


FIGURE 50. The seven wayfinding choremes

in $\mathcal{OPRA}$. Further, for our navigation task the situation of running into a dead end can always appear and we need the possibility of turning around directly and leaving that dead end. Modeling turning around as a sequence of motions might lead to reasoning and modeling difficulties that we want to avoid straight away. The derivation of these wayfinding choremes is based upon a sectorization of a circle as shown in Figure 51. With these sectors we would have the choice of directions from l, r, f, b in Figure 49, sharp or half turns do not occur there, since there are no streets into the according directions. This sectorization clearly has a "back" sector and is quite close to the definition of the $\mathcal{OPRA}$ relations. The main difference is the lack of relations formed by a line. The size of the sectors in Figure 51 is left undefined by Klippel. We are going to simulate these sectorizations by $\mathcal{OPRA}$ relations of adequate granularity. Where the choice of granularity is a trade-off between the minimum size of sectors and reasoning efficiency. Further, we need to keep in mind that with increasing granularity the composition tables for $\mathcal{OPRA}$ become very big and reasoning becomes very memory and time consuming. So far, it turned out that using $\mathcal{OPRA}_8$ is a good choice. The sectors for f, l, b and r can be encoded in such a way that they are still small enough and reasoning can still be performed in a reasonable time. We will use these Klippel's sectors encoded in $\mathcal{OPRA}$ to navigate our street network and see if the impact on the reasoning qualities is bad or not.

We apply our techniques for deriving observations in $\mathcal{OPRA}$ and in the representation of Klippel's sectors in $\mathcal{OPRA}$ to test data to gain knowledge about their fitness for navigation tasks in street networks. Since we believe that the best

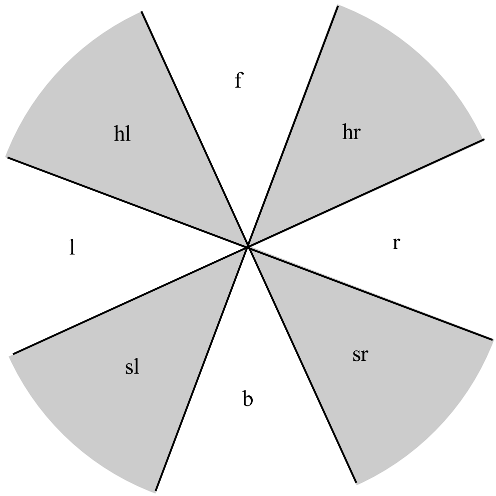


FIGURE 51. Sectors of a circle for wayfinding choremes

test data for street networks are real ones, we use descriptions of street networks compiled out of maps from OpenStreetMap[2].

For a description of the $\mathcal{OPRA}$ calculus please refer to Section 3.3. Especially the representation making use of the atan2 function is of great importance to us, since we can use it to compute the $\mathcal{OPRA}$ relations very efficiently. A description of the reasoning techniques used is given in Chapter 2.

5.2.1. Factorizing $\mathcal{OPRA}$ to Klippel's reference frame. Investigations of Klippel et al. [32–34] on a cognitively adequate representation of directions on a decision point lead to a reference frame as shown in Figure 52 for navigation tasks.

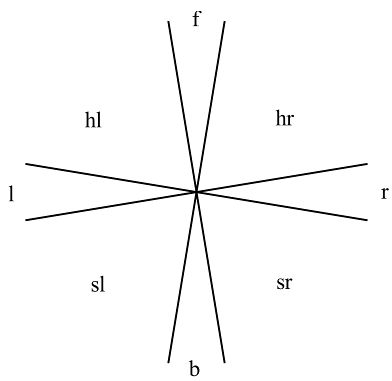


FIGURE 52. Klippel's frame

For the reference frame of this model the circle is divided into eight planar sectors:

[2]http://www.openstreetmap.org/

f	front
hl	half left
l	left
sl	sharp left
b	back
sr	sharp right
r	right
hr	half right

Nothing is said about the treatment of the borders of the sectors, i.e. about which sector the separating line belongs to, if it belongs to any. The sectors f, l, b and r are said to be "small", i.e. the angle from one border to another is small. Further, the model does not have to be symmetric. To simulate such a base frame in $\mathcal{OPRA}$ we need to make some assumptions that stem from the properties of these calculi. We need to encode Klippel's sectors as unions of planar and linear sectors in the $\mathcal{OPRA}$ frame. Further, we need well defined borders between the sectors, i.e. we need to decide which of Klippel's sectors will have linear $\mathcal{OPRA}$ sectors as borders. The treatment of symmetries only gets interesting in $\mathcal{OPRA}$ calculi with high granularity which have high demands on reasoning time and memory. Unfortunately we do not have computers available that can meet these demands.

We are mainly encoding Klippel's approach into $\mathcal{OPRA}_8$ (see Figure 53) to

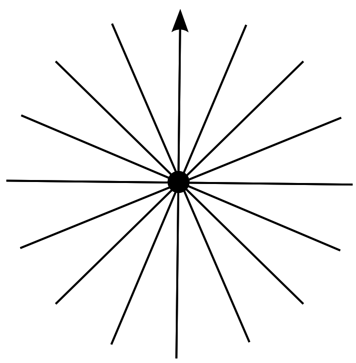


FIGURE 53. $\mathcal{OPRA}_8$

be able to define f, b, l and r sectors that are suitably small and to get constraint network sizes that still can be handled by algebraic reasoners. Unfortunately using $\mathcal{OPRA}_{16}$ turned out to be less fruitful, since `GQR` already needs $14Gb$ of memory to start up with the $\mathcal{OPRA}_{16}$ composition table and reasoning takes a painfully long time already on very small networks. It turned out that Klippel's relations can be modeled quite easily in $\mathcal{OPRA}$ calculi of a granularity being a power of two, since then, we can make use of the surrounding of the relations 0, m, $2 \cdot m$ and $3 \cdot m$ to model f, l, b and r respectively, further calculi of these granularities are available in both `SparQ` and `GQR`. An interesting investigation that is out of the scope of this work is developing an encoding of Klippel's frame in $\mathcal{OPRA}$ using a granularity containing uneven primes encoding some of Klippel's sectors more or less directly. For having suitably small sectors, we unite the $\mathcal{OPRA}_m$ ($m \in 2^n$ and $n > 2$) sectors via a mapping d as follows:

$$\begin{array}{rcl} f & \mapsto & \{0,1,2,4m-1,4m-2\} \\ l & \mapsto & \{m-2,m-1,m,m+1,m+2\} \\ b & \mapsto & \{2m-2,2m-1,2m,2m+1,2m+2\} \\ r & \mapsto & \{3m-2,3m-1,3m,3m+1,3m+2\} \end{array}$$

The Klippel sectors hl, sl, sr and hr are formed by the remaining $\mathcal{OPRA}$ sectors in the appropriate order. For $n = 2$, i.e. $\mathcal{OPRA}_4$, the sectors would overlap with this approach, hence $\mathcal{OPRA}_8$ is our first choice also with regard to the long reasoning time of $\mathcal{OPRA}_{16}$. We decided to add the border lines between f, b l and r and their respective neighbors to the relations 0, m, $2 \cdot m$ and $3 \cdot m$, since this still yields sectors for these relations for $\mathcal{OPRA}_m$ with $m \mapsto \infty$. In fact with this limit we would recover $\mathcal{OPRA}_2$ from the encoding to Klippel's approach. To apply this sectioning to $\mathcal{OPRA}_m$, for all sets $d_1, d_2 \in d(K)$ apply $d_1 \times d_2$ where K are Klippel's sectors und d the above mapping, and add the sets $\{s\} \times d_1$ to obtain the classes of $\mathcal{OPRA}_m$ relations that make use of Klippel's sectioning of the circle. We call the set of these classes D. From these sets of sectors we can easily define predicates $p_1 \dots p_n$ that are *true* if and only if a certain $\mathcal{OPRA}_m$ relation belongs to such a set.

Example 85. We encode Klippel's base frame into the base frame of $\mathcal{OPRA}_8$, which has the half relations $0 \dots 31$. With the above definitions we obtain the mapping

$$\begin{array}{rcl} f & \mapsto & \{30,31,0,1,2\} \\ hl & \mapsto & \{3,4,5\} \\ l & \mapsto & \{6,7,8,9,10\} \\ sl & \mapsto & \{11,12,13\} \\ b & \mapsto & \{14,15,16,17,18\} \\ sr & \mapsto & \{19,20,21\} \\ r & \mapsto & \{22,23,24,25,26\} \\ hr & \mapsto & \{27,28,29\} \end{array}$$

This mapping of Klippel's sectors to the sectors of $\mathcal{OPRA}_8$ is shown in Figure 54. Please note that the $\mathcal{OPRA}_8$ emulations of f, l, b and r are still quite big sectors spanning an angle of 22.5°. Another drawback is that all sectors are the same size. A big advantage is that reasoning with $\mathcal{OPRA}_8$ can still be performed in a sufficiently small time window.

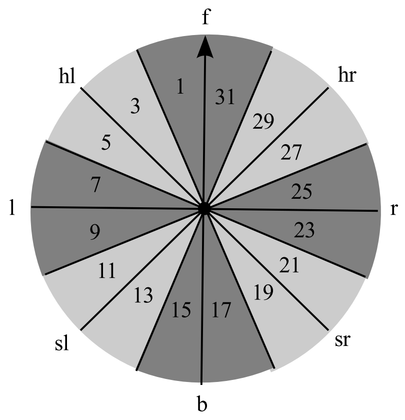


FIGURE 54. Mapping Klippel to $\mathcal{OPRA}$

Example 86. We can get smaller sectors by encoding Klippel's sectors into the sectors for $\mathcal{OPRA}_{16}$ as

$$\begin{array}{rcl} f & \mapsto & \{62, 63, 0, 1, 2\} \\ hl & \mapsto & \{3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\} \\ l & \mapsto & \{14, 15, 16, 17, 18\} \\ sl & \mapsto & \{19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29\} \\ b & \mapsto & \{30, 31, 32, 33, 34\} \\ sr & \mapsto & \{35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45\} \\ r & \mapsto & \{46, 47, 48, 49, 50\} \\ hr & \mapsto & \{51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61\} \end{array}$$

The sectors for f, l,b and r now have a size of 10.25° and the remaining sectors are bigger than them, what is closer to Klippel's intention. The issue with working with $\mathcal{OPRA}_{16}$ is already the sheer size of the composition table with 4160^2 entries and the long descriptions of constraint networks in 4160 base relations.

In the end we have a trade-off between staying close to Klippel's intentions, which can be done by a high granularity $\mathcal{OPRA}$ calculus and the possibility to perform reasoning over constraint networks. But for our task of navigation the reasoning results to not have to be perfect, they just need to be "good enough". Hence, we hope that on constraint networks of reasonable size $\mathcal{OPRA}_8$ does the job. It would also be nice to have high granularity $\mathcal{OPRA}$ calculi and efficient reasoning for them for having the possibility of being able to compare the impact of the size of f, l b and r in more detail. The higher the granularity of the $\mathcal{OPRA}$ calculus the smaller representations for f, l b and r we can define.

5.2.2. From observations to a constraint network. As stated, it is our aim to investigate navigation based on local observations in street networks using the $\mathcal{OPRA}$ calculus and a factorization of the $\mathcal{OPRA}$ base frame. A good source for realistic data about street networks is the world itself. We are using street networks that have been retrieved from `OpenStreetMap`[3], derive local observations on them and formalize these observations in $\mathcal{OPRA}$. We simplify the `OpenStreetMap` data in that sense that we abstract from bends in streets. Our streets are just straight lines.

[3]`http://www.openstreetmap.org/`

With algebraic reasoning global knowledge can be deducted. But this knowledge might be far too coarse. For algebraic reasoning we use the tools `GQR` [24] and `SparQ` [76, 78]. Using this overall knowledge, we navigate through the described street network using a least angle strategy with memory.

In the rest of this section we are using the street network in Figure 48 as the source for our examples. In our street networks, we label crossings by C_i with $i \in \mathbb{N}$. Please note that our definition of crossings at this point includes dead-ends, i.e. crossings with only one street. In our examples crossings are illustrated as big dots. Lines depict streets between crossings. We call crossings C_i and C_j with $i \neq j$ that are connected by a street *adjacent*.

5.2.2.1. *Local Observations.* It is our aim to navigate with knowledge that people or more abstract agents can observe at decision points, e.g. at crossings in a street network. When walking to a crossing, you are facing into a certain direction and hence have an orientation which can be used as the basis for a local reference frame. Further you can see which orientation the other streets at the crossing have with respect to your local reference frame. And of course you know that streets are streets with a crossing at both ends. You do not know what the situation at any other crossing looks like since your view is blocked by buildings. This is an abstraction from very short streets.

In the first step of the formalization of our local observations we derive oriented points from a given street network. For any point C_i in the network we determine the set A_i of adjacent points. For any $C \in A_i$ introduce the point $\langle C_i, C\rangle$ representing an oriented point. For the sake of brevity, we will also write C_iC for such a tuple. As described in Section 3.3 this representation of an oriented point still contains unnecessary information about the length of the vector from C_i to C, but this does no harm.

Example 87. Consider the network given in Figure 48 and the point C_6. The set of adjacent points to C_6 is $\{C_0, C_5, C_{12}\}$, we hence introduce the set of oriented points $\{\langle C_6, C_0\rangle, \langle C_6, C_5\rangle, \langle C_6, C_{12}\rangle\}$ or written in the short form $\{C_6C_0, C_6C_5, C_6C_{12}\}$.

In the second step, we define the streets. For each oriented point C_iC_j, we define the street via the $\mathcal{OPRA}_m$ relation $C_iC_{j\,m}\angle_0^0 C_jC_i$. The oriented point C_jC_i exists, since the streets in our network are not directed and hence if C_j is adjacent to C_i then C_i is adjacent to C_j.

Example 88. For a street as shown in Figure 55 we introduce the $\mathcal{OPRA}$ relations that define it. It is the street between the points C_6 and C_0, hence we have

FIGURE 55. A street

introduced the oriented points C_6C_0 and C_0C_6 in the previous step (see Example 87) at the respective locations which point to each other. So we introduce the relation $C_6C_{0\,m}\angle_0^0 C_0C_6$ that covers this fact and defines the street.

In the third step, we add the local observations. For each oriented point C_iC_j form the set $P_{i,j}$ of oriented points such that for each $p \in P_{i,j}$ the properties $pr_1(p) = C_i$ and $pr_2(p) \neq C_j$ hold. Here, pr_1 is the projection to the first component of a tuple and pr_2 to the second one. For each $p \in P_{i,j}$, we form the $\mathcal{OPRA}_m$ relation $C_iC_{j\,m}\angle_s^{C_iC_j \rhd p} p$. Since $C_i = pr_1(p)$, the first half-relation is clearly s, the computation of the second one will be explained in section Section 5.2.2.2. We can see from the above description that we are relating oriented points that share the same position (e.g. the crossing we are at) but have different orientations. We can think of that as taking the orientations of the different streets at a crossing with respect to a local reference orientation, i.e. the orientation we are facing when reaching the crossing.

Example 89. We again refer to Figure 48 and the oriented points introduced in Example 87 as well as the streets defined in Example 88. Consider the oriented point C_6C_0. For this point we derive $P_{i,j} = \{C_6C_5, C_6C_{12}\}$ from the set of all oriented points and by the description of the introduction of relations, we derive the relations

$$C_6C_{0m} \angle_s^{C_6C_0 \rhd C_6C_5} C_6C_5$$
$$C_6C_{0m} \angle_s^{C_6C_0 \rhd C_6C_{12}} C_6C_{12}$$

where $C_6C_0 \rhd C_6C_5$ and $C_6C_0 \rhd C_6C_{12}$ still need to be determined using the $\mathcal{OPRA}$ semantics.

In Algorithm 2 we show a slightly optimized version of the described algorithm where steps two and three are amalgamated.

Algorithm 2 Deriving Observations

1: $\mathcal{C}$ is the set of nodes of a street network
2: $\mathcal{S}$ the set of streets as tuples of start and end points
3: O is the set of oriented points
4: R is the set of relations
5: m is the granularity of the $\mathcal{OPRA}$ calculus
Require: $O = \varnothing$ **and** $R = \varnothing$ **and** $m > 0$
6: Require a correct description of a street network
Require: $\forall C \in \mathcal{C}. \exists s \in \mathcal{S}. C = \mathrm{pr}_1(s) \lor C = \mathrm{pr}_2(s)$
Require: $\forall s \in \mathcal{S}. \exists C_1 \in \mathcal{C}. \exists C_2 \in \mathcal{C}. s = \langle C_1, C_2 \rangle \land C_1 \neq C_2$
7: Introduction of oriented points
8: **for all** $C \in \mathcal{C}$ **do**
9:   **for all** $s \in \mathcal{S}$ **do**
10:     **if** $\mathrm{pr}_1(s) = C$ **then**
11:       $O := O \cup \{\langle C, \mathrm{pr}_2(s) \rangle\}$
12:     **end if**
13:   **end for**
14: **end for**
15: Definition of streets and local observations
16: **for all** $o \in O$ **do**
17:   $R := R \cup \left\{ o_m \angle_0^0 \langle \mathrm{pr}_2(o), \mathrm{pr}_1(o) \rangle \right\}$
18:   **for all** $p \in O$ **do**
19:     **if** $\mathrm{pr}_1(o) = \mathrm{pr}_1(p)$ **and** $\mathrm{pr}_2(o) \neq \mathrm{pr}_2(p)$ **then**
20:       $R := R \cup \{ o_m \angle_s^{o \rhd p} p \}$
21:     **end if**
22:   **end for**
23: **end for**
24: **return** R

If we are working with an approach as suggested by Klippel, we add another step that replaces the $\mathcal{OPRA}$-relations by sets of relations as described in Section 5.2.2.3.

5.2.2.2. *Deriving $\mathcal{OPRA}$-relations.* For our observations taken in Section 5.2.2.1, we need a way to derive $\mathcal{OPRA}$-relations from tuples of points (which we can also interpret as line segments). In particular we need this computation in Algorithm 2 Line 20, where $o \rhd p$ was not determined so far.

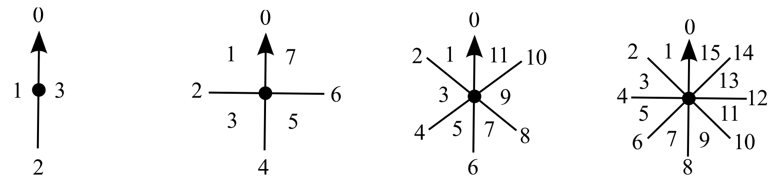
By scrutinizing the definitions of the $\mathcal{OPRA}_m$ relations, we see that for any $C_kC_{lm} \angle_i^j C_tC_v$, there is little dependence between the i and j. The only dependence that occurs between i and j is the fact if the two oriented points share the same position or not. We can distinguish this easily by determining if $C_k = C_t$ or not according to the $\mathcal{OPRA}$ semantics in Section 3.3. If $C_k = C_t$, we know that $i = s$ and can determine j as a half relation. If $C_k \neq C_t$, and we can determine both parts via half relations independently. We can apply Algorithm 3 perform this case distinction. In fact, due to the definition of our local observations (see Algorithm 2) we can even

Algorithm 3 Computing $\mathcal{OPRA}$-relations

```
C_k C_l oriented point
C_t C_v oriented point
m granularity of OPRA
Require: m > 0
if C_k = C_t then
    return m∠_s^{C_k C_l ⊳ C_t C_v}
else
    return m∠_{C_k C_l ⊳ C_t C_v}^{C_t C_v ⊳ C_k C_l}
end if
```

simplify this step, since we always know that the positions of the points coincide, so one half relation is always s and only the other one needs to be determined via the $\mathcal{OPRA}$ semantics. The $\mathcal{OPRA}$ relations in the step of the definition of streets are preordained by our approach (refer to Algorithm 2 Line 20). We still decided to show Algorithm 3 for the sake of completeness. In the remainder of this section we show how to use the $\mathcal{OPRA}$ semantics to compute the $\mathcal{OPRA}$ (half) relations practically.

To determine the $\mathcal{OPRA}_m$ half relations between oriented points C_kC_l and C_tC_v, we determine sectors of the unit circle[4] in the Euclidean plane that correspond to those relations with respect to the $\mathcal{OPRA}_m$ semantics. Then, we compute the angle from C_kC_l to C_tC_v and determine into which sector this angle belongs. This directly yields the half relation. In Figure 56 such sectors are shown for $\mathcal{OPRA}_1$ to $\mathcal{OPRA}_4$. By inspecting the definition of $\mathcal{OPRA}$ relations, we also see that half



(a) (b) (c) (d)

FIGURE 56. Sectors of the circle for $\mathcal{OPRA}_1$ (a), $\mathcal{OPRA}_2$ (b), $\mathcal{OPRA}_3$ (c), and $\mathcal{OPRA}_4$ (d)

relations with an even identifier are linear relations, while the ones with an odd identifier are planar relations.

The sectioning for $\mathcal{OPRA}_m$ is is done by identifying an angle interval with every element of the cyclic group $\mathbb{Z}_{4m}$ as

$$[i]_m = \begin{cases} \left]2\pi\frac{i-1}{4m}, 2\pi\frac{i+1}{4m}\right[& \text{if } i \text{ is odd} \\ \left\{2\pi\frac{i}{4m}\right\} & \text{if } i \text{ is even} \end{cases}$$

[4]In fact the radius of the circle does not matter, since we are disregarding lengths.

Please note that these intervals are normalized to the representation of angles in the interval $[0, 2\pi[$. For an efficient implementation one can create a look-up-table with the borders of the respective intervals and the respective values for i.

In Section 3.3 algebraic semantics for $\mathcal{OPRA}$ haven been introduced. To determine the relations between points C_kC_l and C_tC_v, we pre-calculate the angle intervals $[i]_m$ for $\mathcal{OPRA}_m$ with m being the desired arity of the calculus and store them. Then we determine the angles between the points C_kC_l and C_tC_v and determine the $\mathcal{OPRA}_m$ relations according to the $\mathcal{OPRA}$ semantics using the stored angle intervals as a look-up table.

All pairs of oriented points that are neither used for the definition of streets or for local observations are in the universal relation, since we cannot observe any knowledge about them in our model. It is customary not to make these universal relations explicit in qualitative spatial reasoning. Reasoners like `SparQ` [78] and `GQR` [24] follow this custom and threat "missing" relations between objects as the universal relation.

So far we can translate our local observations to $\mathcal{OPRA}$ relations. We still need to take care of the fact how to translate them to relations in the base frame suggested by Klippel [32–34].

5.2.2.3. *Factorizing the $\mathcal{OPRA}$-relations.* Additionally to investigating navigation with $\mathcal{OPRA}$ relations, we also want to emulate relations as proposed by Klippel [32–34] in $\mathcal{OPRA}_m$. For this reason, we use 72 unary predicates p_i with $1 \leq i \leq 72$ that partition the set of the $\mathcal{OPRA}_m$ base relations. The number 72 is determined by the 8 sectors in Klippel's base frame. If an $\mathcal{OPRA}_m$ base relation $o_m \angle_s^t q$ has been determined between o and q with Algorithm 3, we form the new relation

$$o\left\{r \mid p_i(r) = p_i({}_m\angle_s^t) \text{ for } 1 \leq i \leq n \text{ and all base relations } r\right\} q$$

that is in most cases not a base relation. We do this for all pairs of oriented points that haven been introduced in Section 5.2.2.2. All other pairs are in the universal relation anyways. For this factorization adjacent sectors will be united to a single

Algorithm 4 Klippel-Factorization

R set of determined $\mathcal{OPRA}_m$ relations
p_i with $1 \leq i \leq 72$ set of predicates indicating the classes
R' set of output relations
Require: $R' = \varnothing$
for all $o_m \angle_s^t q \in R$ **do**
  $R_{\text{tmp}} = \varnothing$
  for all ${}_m\angle_x^y$ being a $\mathcal{OPRA}_m$ base relation **do**
    prop = *true*
    for $1 \leq i \leq n$ **do**
      prop := prop $\wedge\ p_i({}_m\angle_s^t) = p_i({}_m\angle_x^y)$
    end for
    if prop **then**
      $R_{\text{tmp}} := R_{\text{tmp}} \cup \{{}_m\angle_x^y\}$
    end if
  end for
  $R' := R' \cup \{o\ R_{\text{tmp}}\ q\}$
end for
return R'

relation, but the operation involved works for all kinds of predicates, even tough the usefulness might be questionable in many cases. Please note that the output of Algorithm 4 is again an $\mathcal{OPRA}$ constraint network, only the base relations in the local observations (and definition of streets) are substituted by general relations that reflect Klippel's reference frame. Further calculations like the application of algebraic closure can be performed in $\mathcal{OPRA}$ this way. Another way to get observations in Klippel's reference frame is applying the factorization directly when calculating the half relations in Algorithm 2.

5.2.3. Navigation. Having obtained a description of a street network as an $\mathcal{OPRA}$ constraint network, we are able to apply algebraic closure on it to obtain a refined constraint network. Since we are starting from consistent descriptions, we do not have to fear that algebraic closure detects inconsistencies. In fact, in the descriptions from Section 5.2.2.2 and Section 5.2.2.3 many universal relations are contained, since we only made local observations. E.g. the relation between $C_{13}C_9$ and C_6C_5 from Figure 48 is universal, since these oriented points cannot be observed together locally at any crossing. Algebraic closure only approximates consistency for $\mathcal{OPRA}$, hence our refined constraint networks might be too big, but this is no big issue for our navigation task, it might just lead to detours. We just hope that the results of algebraic closure are "good enough" for good navigation, we want to navigate on routes that are not much longer than the optimal ones and in any case we want to have shorter routes than a random walk.

Starting from a refined constraint network of a street network, we want to navigate through it (hopefully without taking too many detours). We are going to apply a least angle strategy with memory for navigation with imprecise and maybe faulty data. We can base the navigation on half relations, though the full relations were needed for the reasoning task, since the half relations do not allow for any converse operation. For navigation the converses are no longer needed. If $C_kC_l\, m \angle_i^j C_tC_v$, then C_tC_v is in sector i of C_kC_l with granularity m. The way backwards is of no interest for forward navigation. Based on this we introduce weights on half $\mathcal{OPRA}$ relations for the implementation of a least angle strategy with memory. Going straight towards the destination and taking slight deviations from the straight line to the destination is normally good for such a navigation task, taking big deviations from the straight line and going away from the destination is bad. We can assign the weights $w(i)$ to $\mathcal{OPRA}_m$ half relations i as

$$w(i) = \begin{cases} i & \text{if } 0 \le i \le 2m \\ 4m - i & \text{if } 2m < i < 4m \end{cases}$$

this yields a weight distribution that assigns the lowest weights to going forward and making slight bends. To apply a least angle strategy, we then need to look for the directions with the lowest weight.

Example 90. Consider again the sectors for $\mathcal{OPRA}_4$ in Figure 56d. Applying weights with respect to our formula yields the distribution

$$\begin{array}{rr} w(0) = 0 & w(5) = w(11) = 5 \\ w(1) = w(15) = 1 & w(6) = w(10) = 6 \\ w(2) = w(14) = 2 & w(7) = w(9) = 7 \\ w(3) = w(13) = 3 & w(8) = 8 \\ w(4) = w(12) = 4 & \end{array}$$

which is depicted in Figure 57. We can observe that going straight towards the destination or taking only slight deviations has small weights whereas going away from the destination and taking big deviations leads to high weights.

In the navigation task, we start at a point $from$ and want to reach a point to. The current point is cur initialized by $from$. These are points that represent

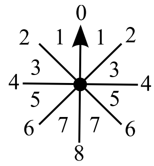


FIGURE 57. $\mathcal{OPRA}_4$ weight distribution

crossings in the street scenario, not oriented points. We determine the set of all $\mathcal{OPRA}$ relations $o_m \angle_i^j p$ with $\text{pr}_1(o) = cur$ and $\text{pr}_1(p) = to$. We then form the half relations $o \rhd to$ as

$$o \rhd to = \sum_{\text{pr}_1(p)=to} \left\{ o_m \rhd_i to \mid o_m \angle_i^j p \right\}$$

We then normalize the weights as

$$w = \frac{\sum_{i=o \rhd to} w(i)}{\|o \rhd to\|} \cdot penalty$$

where *penalty* is a property of $\text{pr}_2(o)$ that is initialized with 1 and incremented by 1 each time $\text{pr}_2(o)$ is visited on a path. This is introduced to make loops bad ways to go and to get out of dead ends, this *penalty* is the memory added to the least angle strategy. We now take all o with the minimum w, if there is more than one, we chose by fortune. $\text{pr}_2(o)$ becomes our new point *cur* and its penalty is increased since it is visited. We repeat this, until *to* is reached. The algorithm for navigation is shown in Line 5.

The assignment of weights is shown in Algorithm 6. Please note that we have used disjoint unions of the half relation symbols in Line 5, since those lead to better navigation results in our experiments, even for low granularities.

With the description of the navigation procedure, we have shown all of the methods that are used in our navigation investigation. We will go on to describe our actual navigation environment.

5.2.4. Environment for Experiments. We have implemented the algorithms and techniques described in Section 5.2.2 in a `Java` program providing a graphical user interface and a command line interface for batch processing. In the first part of this section we are going to describe the graphical user interface. The graphical mode is invoked by the command

```
java -jar Streets.jar
```

without any further parameters. The main window with a loaded street network is shown in Figure 58. Data regarding street networks can be input in a simple language describing nodes (aka. crossings) and streets between them. The language can be described as:

```
SCEN     := ([NODE], [STREET])*
NODE     := Node [Name] ([Decimal], [Decimal]);
STREET   := Street [Name] ([NodeName], [NodeName]);
NodeName := Name
Name     := [A-Za-z0-9]+
Decimal  := -?[0-9]+(\.[0-9]+)?
```

with the side condition that for every `[NodeName]` a `NODE` must exist with `[Name] = [NodeName]`, for every `NODE` with the name `[Name]` a street must exist with `[NodeName] = [Name]`. With these restrictions, all streets have a start and

Algorithm 5 Navigation

$from$ start point
to end point
for all points o: $o.penalty := 1$
$cur := from$
$ROUTE := cur$
while $cur \neq to$ **do**
  $R := \varnothing$
  $W := \varnothing$
  for all p with $\mathrm{pr}_1(p) = to$ **do**
    for all o with $\mathrm{pr}_1(o) = cur$ **do**
      if R contains a relation $o \rhd p$ **then**
        $R := (R \setminus o \rhd p) \uplus o(\rhd \uplus o_m \rhd_i)p$ if $o_m \angle_i^j p$
      else
        $R := R \uplus o_m \rhd_i p$ if $o_m \angle_i^j p$
      end if
    end for
  end for
  for all $r \in R$ **do**
    $W := W \cup (r, \mathrm{weight}(r))$
  end for
  $cand := r \in R$ with $w(r) = min$
  $next :=$ random element from cand
  increase $\mathrm{pr}_2(next).penalty$
  $cur := \mathrm{pr}_2(next)$
  $ROUTE := ROUTE \circ cur$
end while
return $ROUTE$

Algorithm 6 Weight assignment: weight

$o \rhd p$ is given
$W := 0$
for all $r \in \rhd$ **do**
  if $0 \leq r \leq 2m$ **then**
    $W := W + r$
  else
    $W := W + 4m - r$
  end if
end for
$W := \frac{W}{\|\rhd\|} \cdot \mathrm{pr}_2(o).penalty$
return W

end node and all nodes are reachable. Further the positions of all nodes must be distinct. Simple networks can be described in this language quite easily as e.g.

```
Node C1 (0,1);
Node C2 (0,2);
Node C3 (0,3);
Node C4 (2,2);
Street W1 (C1, C2);
Street W2 (C2, C3);
Street W3 (C2,C4);
```

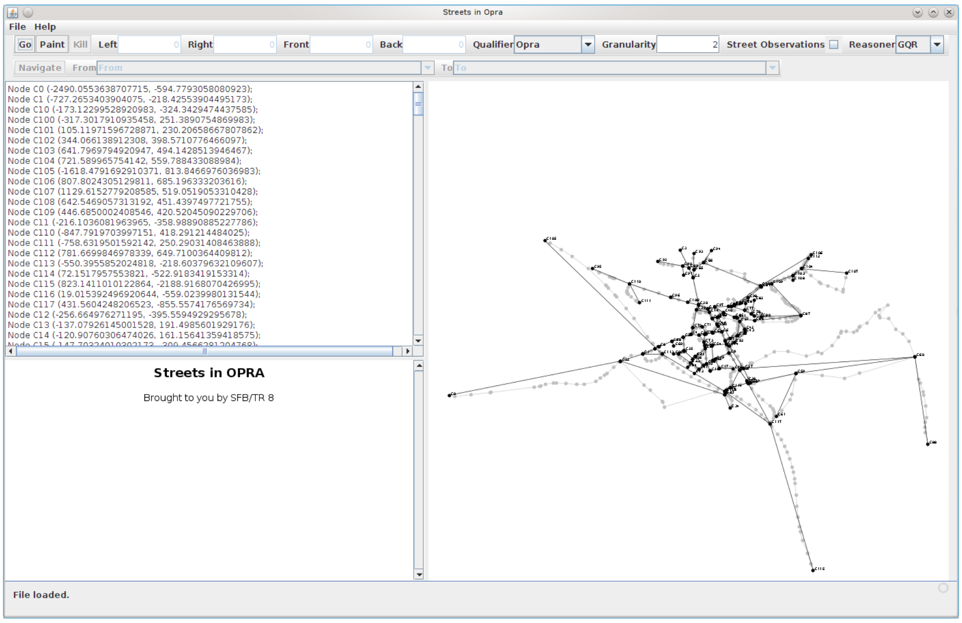


FIGURE 58. The environment

This simple example is a network describing a T rotated by 90° counter clockwise. This language can be input directly into the top left text-field of the demonstration environment. Still transforming and typing descriptions as shown on the right hand side in Figure 58 is quite cumbersome, that is why our environment supports import of `OpenStreetMap`[5] `XML` data. Which is transformed to our simple language and then displayed in the top left text-field, where it can be edited by the user if desired. This representation is more compact than the original `OpenStreetMap` representation leaving out all data that is unnecessary for our purpose. The only issue regarding importing `OpenStreetMap` data is the representation of coordinates. `OpenStreetMap` uses latitude and longitude, but for $\mathcal{OPRA}$ we need coordinates in the Euclidean plane. We use the Mercator transformation with the earth radius as described by `WGS84` to derive our coordinates in the plane.

For the qualification stage we supply several qualifiers

Entry	Description	Options
$\mathcal{OPRA}$	$\mathcal{OPRA}$ calculus	Granularity
Klippel8	$\mathcal{OPRA}_8$ with $left = right = front = back = 2$	none
Klippel16	$\mathcal{OPRA}_{16}$ with $left = right = front = back = 2$	none
Blurred $\mathcal{OPRA}$	$\mathcal{OPRA}$ with adjustable sectors	Granularity, $left$, $right$, $front$, $back$

that are all based on the $\mathcal{OPRA}$ calculus. For the qualifiers "Klippel8", "Klippel16" and "Blurred $\mathcal{OPRA}$" several sectors of the $\mathcal{OPRA}$ base frame are united to a single sector as described in Section 5.2.1. The sectors $left$, $right$, $front$, and $back$ are here described in the number of $\mathcal{OPRA}$ relations to both sides around m, $3m$, 0 and $2m$ respectively that form the above Klippel sectors. Of course such sectors cannot overlap and the program will complain if they do.

[5] `http://www.openstreetmap.org/`

By the checkbox “Blur Streets” factorization for the definitions of streets can be turned off, even if it is turned on for observations.

The currently supported reasoners are `GQR` [24] and `SparQ` [76, 78]. The reasoner `GQR` is mostly faster on big constraint networks, but it does not support all granularities for $\mathcal{OPRA}$, whereas `SparQ` does.

If you press the “Go” button, the local observations (Section 5.2.2.1) will be calculated and $\mathcal{OPRA}$ relations will be derived (Section 5.2.2.2). If needed the $\mathcal{OPRA}$ relations will be factorized according to Kippel’s reference frame (Section 5.2.2.3). The output of this process is shown in the bottom left window of the demonstrator. Then the reasoner will be invoked and the constraint network will be refined. This step can take a long time, especially for higher granularity versions of $\mathcal{OPRA}$ and/or big constraint networks.

After this step, the navigation button and the choosers for the start and end point will be available and can be used for navigation (Section 5.2.3) through the network. Calculated routes will be highlighted in the graphics window on the right and printed out in the bottom left window together with shortest path and random walk results.

The graphical mode is quite good for smaller routing tasks and to understand navigation tasks but it turned out to be less suited for tasks in bigger networks with a higher granularity of the $\mathcal{OPRA}$ calculus, since in such a case the calculation of the algebraic closure can already take days on a very fast computer. Further the output of such a task is more than the `Swing` windows of the graphical user interface can handle (in fact many state-of-the-art text editors choke on reading this much of data). For this reason we implemented a command line interface with batch navigation. Calling

```
java -jar Streets.jar -h
```

will bring up the help message for the command line mode

```
Starting up...
---------------------------------------------------
Processors:       4
Operating System: Linux 2.6.38-10-generic
Java Runtime:     Java(TM) SE Runtime Environment
Java Version:     1.6.0_26
Heapsize:         1779 Mb
---------------------------------------------------

Help for the commandline interface for Streets to the Opra
java -jar Streets.jar -q <qualifier> [Options] <file.osm>
Options are: -g -s -l -r -f -b

-g <int>        Granularity
-s <0|1>        Apply uncertainty to definition of streets
-f <int>        Uncertainty at front
-l <int>        Uncertainty at left
-b <int>        Uncertainty at back
-r <int>        Uncertainty at right

Supported qualifiers: Opra "Blurred Opra" Klippel8 Klippel16

Options for qualifiers:
Opra:           -g
Blurred Opra:   -g -s -f -l -b -r
```

```
Klippel8:       -s
Klippel16:      -s
The specified options are mandatory
```

together with a bit of system information. The options that can be set are equivalent to the settings of the graphical user interface. The command line interface exclusively works on data from `OpenStreetMap` and performs batch navigation, i.e. it performs three navigation tasks between any pair of distinct points in the street network. We decided to run any task three times to rule out the effect of random choices a little. In the command line mode any output is written to a `XHTML` file that can be viewed in any web-browser. Further a small tool is available that converts the output to `.csv` or `.ods`.

5.2.5. Experiments. Finding good data for experiments with navigation based on local observations is a hard task. A big issue is that the maximal size of a street network that we can use for navigation is limited by the number of nodes and by the granularity of the underlying $\mathcal{OPRA}$ calculus. The time needed for applying the algebraic closure algorithm rises steeply with any of these two parameters growing. As a rule of thumb we can say that we can e.g. handle street networks with around 120 to 170 points with $\mathcal{OPRA}_8$ in a reasonable time (2 to 4 hours) when computing algebraic closure with `GQR`. On the other hand a network in 170 points in our representation (the reduction of data is described in Section 5.2.2) does not cover big areas in most cases. For example the network shown in Figure 59 that is derived from the data on `OpenStreetMap` (latitude 51.8241200 longitude 9.3117500) for a village with about 1400 inhabitants already has 117 points even with our simplifications. Large cities like Paris or Moscow of course have many more points in our representation and cannot be handled efficiently with the algebraic reasoners so far. This is partly given already due to the sheer size of the cities and partly because of their street networks with many crossings. For Moscow already the inspection of the highways in the city reveals a large and dense network of interconnected streets. But on the other hand, we want to observe navigations along paths of very differing lengths, including very long paths to be able to judge the navigation properties of our networks based on local observations under very differing circumstances. For long paths networks of a sufficiently big size are needed. Unfortunately this problem grows even bigger by the fact that the closer we get to boundary of our street network the worse our local observations and refinements will be. For any point in the middle of a street network (as in the inner circle in Figure 59) there are many points around it in all directions with observations being made, putting this point into its place in a qualitative sense. In the middle circle the observations around a certain point already get sparser and information about the points gets less certain, this gets worse in the outer circle. Outside of the outer circle information about the points is very bad. In fact, it turned out that navigating into the dead ends at the boundary of the map is very alluring, since their position with respect to other points is not very restricted. For meaningful experiments about the navigation performance, we need street networks that are big enough to provide an area in the center for which enough information can be derived.

Our test data has hence to consist of street networks that are small enough to be manageable with qualitative reasoners and that are big enough to yield enough information. For the first requirement networks in no more than 20 points would be nice, for the second one the whole world, since then there would be no boundary problem.

Another requirement on the test data is the use of street networks of diverse style. It is e.g. interesting to use layouts of planned and grown cities and villages. Further gyratory traffics (e.g. at Place-Charles-de-Gaulle) are of increased interest.

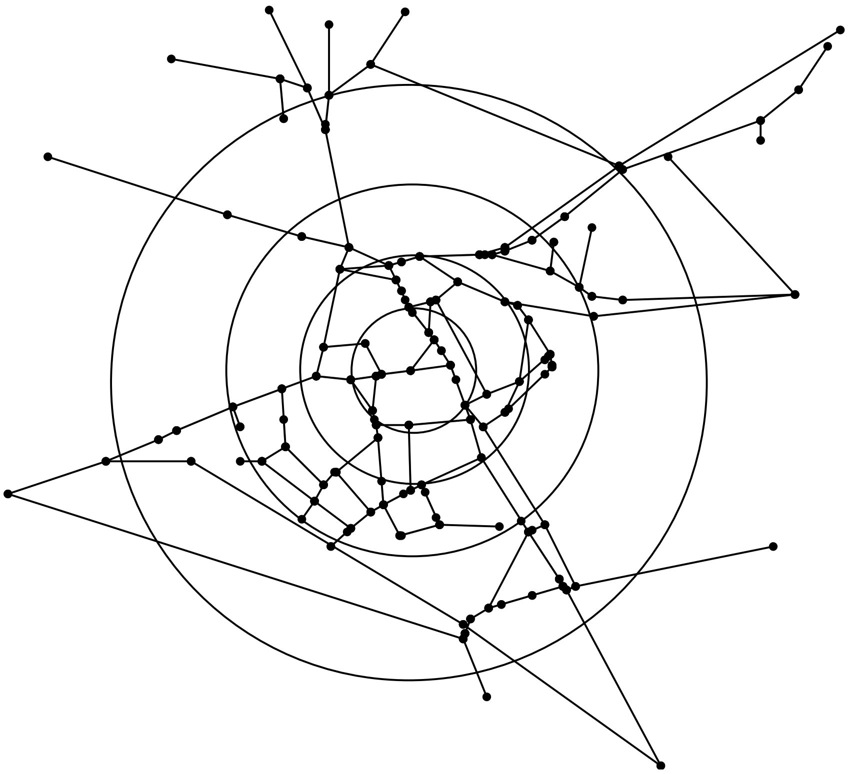


FIGURE 59. A street network of a village

In our experiments we decided to count the number of visited nodes needed for navigation and shortest path as well as random walk. We think that the restriction on counting the number of visited nodes (which is equivalent to the visited path segments) is still meaningful in our experiments since the lengths of the involved path segments are rather short. When navigating along long road like highways computing the real distance would be needed for deriving meaningful results, but due to the technical restrictions this is out of our scope so far.

In [42] we presented first navigation results based on the Fürstenau scenario shown in Figure 59. This scenario was a first choice for our experiments since it is of manageable size and has a more or less square layout. Further the author spent a lot of his life in that place, so the navigation results could be simply verified. In [42] we were only able to show the results of 66 more or less randomly chosen navigation experiments since the tools for handling big numbers of them were still under development. We conducted the experiments using a factorization of $\mathcal{OPRA}_8$ to Klippel's reference frame (called Klippel$_8$) in the table yielding an average length of paths of 16.0. Navigation with $\mathcal{OPRA}_8$ yielded an average path length of 16.2 and shortest path (using the length of the segments) had an average path length of 13.2 while uniform shortest path (having all path length set to 1) yielded 11.0 segments in average. Random walk had an average access time of 718.0. Since our approach works on incomplete data, the mean deviation is higher than of the shortest path approaches, which use all data available in the street network.

	Klippel$_8$	$\mathcal{OPRA}_8$	shortest path	uniform shortest path	random walk
average	16.0	16.2	13.2	11.0	718.0
deviation	9.3	9.3	5.1	3.5	519.9

In these experiments our approach based on incomplete knowledge performs quite well when compared to the approaches that make use of full knowledge. Of course we do not expect to be better than the optimal approach, but we are quite close to it. On the other hand the Fürstenau scenario is quite friendly to our approach especially when doing the navigation in the middle of it. But with our evolved tools, we are now able to present further navigation results.

After these first promising results, we investigated the navigation properties of our approach in more depth. In fact, we implemented a batch navigation algorithm that performed navigations between all points (or crossings) in a street network in different positions. Each of such a navigation is performed three times to rule out the effects of the random choices a little. It is observable at least if a bad random choice leads to a long way or if long ways are a real issue of the approach in special instances of the navigation problem. This approach of experimentation generates a lot of data, normally too much to be handled by a human. For example in the Fürstenau scenario (Figure 59), $13,806$ different navigations are performed each of them three times. This leads to $41,418$ sets of data for each calculus used for the experiment. That is why the results of these experiments are compiled into a spreadsheet semi-automatically to allow us to handle and analyze these amounts of data.

For the following experiments we used another distinction in the factorization of $\mathcal{OPRA}$ (in this case $\mathcal{OPRA}_8$) to Klippel's reference frame. We did one experiment factorizing only the local observations denoted by Klippel$_8$ $s = 0$ and one experiment factorizing the local observations and the definitions of streets denoted by Klippel$_8$ $s = 1$ in the table. The other columns contain the same kind of data as in the previous experiment. A full analysis of the navigation approach in the Fürstenau scenario reveals the following results:

	Klippel$_8$ $s = 0$	Klippel$_8$ $s = 1$	$\mathcal{OPRA}_8$	shortest path	uniform shortest path	random walk
average	13.22	34.57	13.19	10.13	8.99	551.53
deviation	10.93	36.99	9.27	4.44	3.47	416.47

Again we can see that our navigation strategy performs quite well when using Klippel$_8$ and only factorizing the local observations and $\mathcal{OPRA}_8$ when compared to the shortest paths. Of course our strategy again has a higher mean deviation than shortest path. The interesting point is that Klippel$_8$ $s = 0$ and $\mathcal{OPRA}_8$ almost perform in the same way. So far Klippel's factorization does not seem to have a bad impact on the navigation quality. On the other hand using Klippel's reference frame encoded in $\mathcal{OPRA}$ also for the definitions of streets seems to be a very bad idea, we then lose information too quickly and go very long ways. When looking at the single results for this approach, very short navigations are still quite good, but already a little longer ones for the other approaches become very long in this case.

The next experiment with our navigation approach is at the close surroundings of the Place-Charles-de-Gaulle in Paris, France shown in Figure 60. The core of this scenario is a big gyratory traffic. In theory a drawback of our approach and the standard shortest path algorithms is the fact that they are ignorant of priority lanes, so both approaches tend to enter and exit the gyratory traffic more often than they

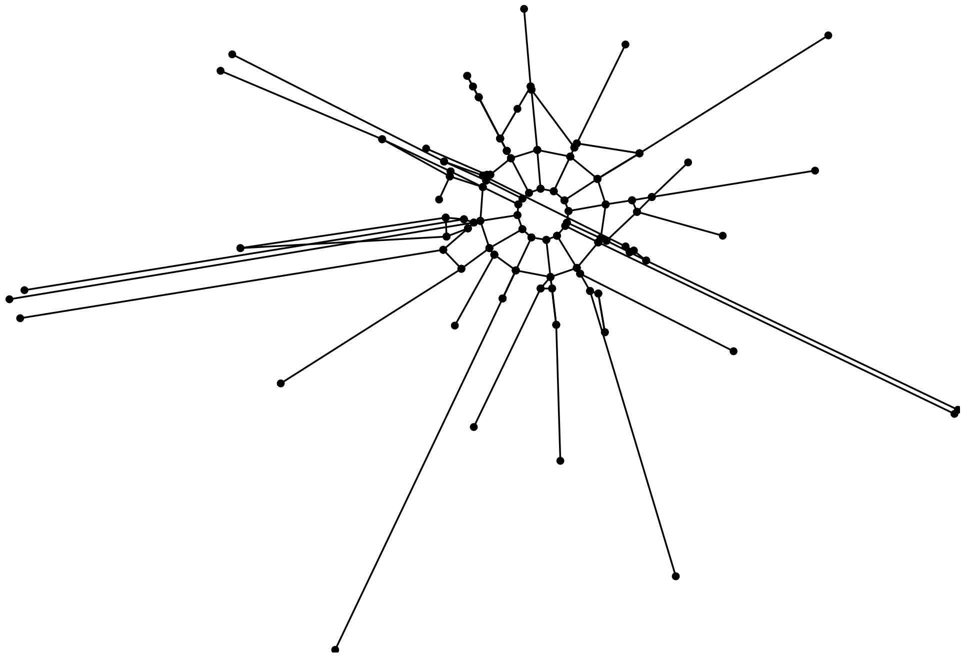


FIGURE 60. Place-Charles-de-Gaulle with some surroundings

have to if there are enough alternative routes to go which is basically not the case in the Place-Charles-de-Gaulle scenario. With all data reduction applied the scenario under scrutiny still has 90 nodes or crossings and we made 24570 navigations in it for any of the approaches. The average values and the standard deviation are shown in the following table:

	Klippel$_8$ $s = 0$	Klippel$_8$ $s = 1$	$\mathcal{OPRA}_8$	shortest path	uniform shortest path	random walk
average	10.74	20.37	9.90	7.96	7.43	312.15
deviation	8.42	21.97	5.79	3.21	2.69	195.21

We can see that Klippel$_8$ $s = 0$ and $\mathcal{OPRA}_8$ again roughly perform in the same way and are again close to the shortest path approaches. And as before Klippel$_8$ $s = 1$ performs a lot worse. The Place-Charles-de-Gaulle is again a friendly scenario for our approach since it is square, compact and has only a few dead ends. Further the local turning situations as grasped on the map are rather simple.

After the observation of navigations in scenarios where our approach performed rather well, we also want to show that this approach can perform not so good depending on the scenario. We use a map of Sinaia[6], a cozy Romanian mountain village that is at the same time some kind of hell for our navigation approach. The map is shown in Figure 61 with the usual simplifications already been applied. The long straight lines are serpentines in reality but we can abstract from that for now. The street network has 123 crossings and 45757 navigations have been performed. The results of the experiments are shown in the following table:

[6]Special thanks to Mihai Codescu for his suggestion to try this village's street network for navigation.

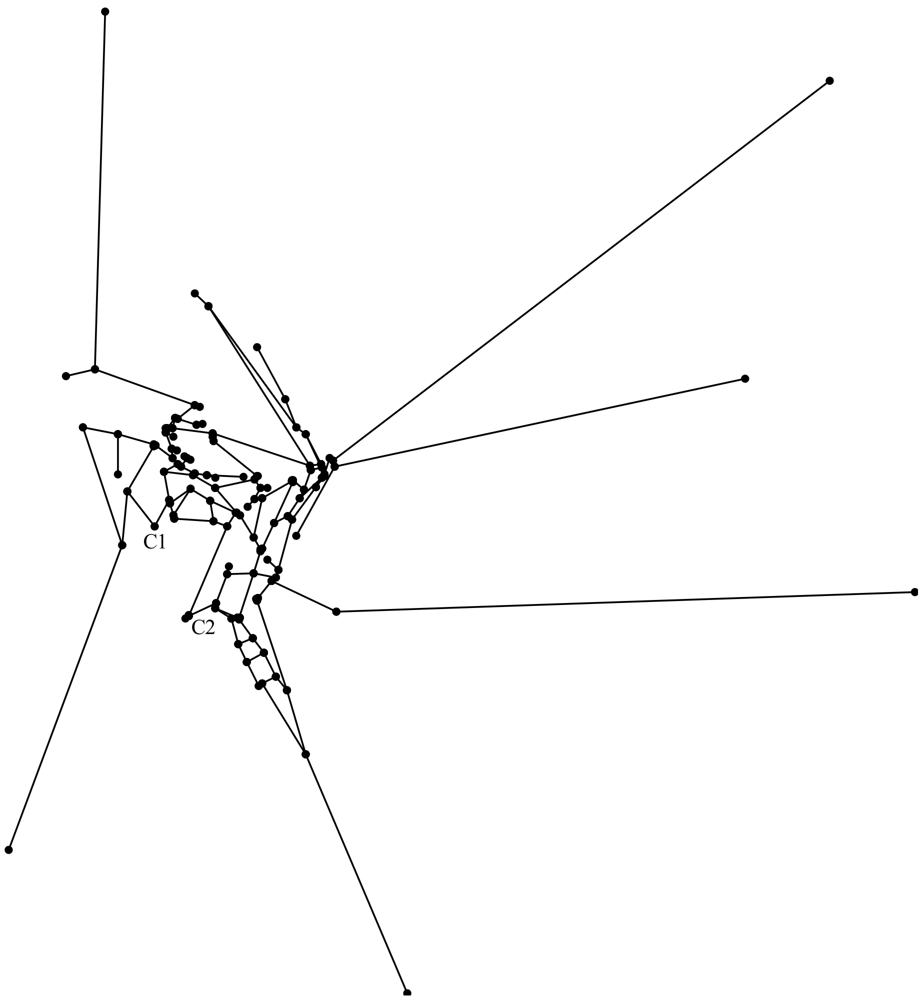


Figure 61. Sinaia, Romania

	Klippel$_8$ $s = 0$	Klippel$_8$ $s = 1$	$\mathcal{OPRA}_8$	shortest path	uniform shortest path	random walk
average	21.64	45.67	17.90	10.87	9.62	621.59
deviation	30.67	50.61	21.75	4.95	3.87	390.95

We see that our navigation approach did not perform so well in this scenario. All $\mathcal{OPRA}$-based approaches almost took twice the steps the shortest path approaches needed. Further the deviation of the navigation is high in all cases. Due to its location in the mountains Sinaia has some properties in its street network that our navigation algorithm does not like: in fact there are honey pots for making mistakes in the network. The street network contains many dead ends that our navigation approach based on $\mathcal{OPRA}$ likes to run into. Further the turning situations at crossings are often triangle shaped. This leads to more possibilities to take a wrong turn if you only have limited knowledge. The street network contains crossings between which a way can only be found if you go into the direction away from your destination, for example if you want to reach C_2 them from C_1 in Figure 61. For an approach based on limited knowledge the navigation still performs quite well, but not as well as on a network that is "friendly" to it.

After examining the street network for Sinaia that was not well suited for our navigation approach, we want to have a look at a network that is presumably better suited for navigation based on $\mathcal{OPRA}$ once more. We are going to have a look at a street network in chess-board style. For this purpose use a part of Valetta, Malta in the vicinity of St.-Dominic-Street[7]. The map of the area is shown in Figure 62. The street network has 111 nodes or crossings with many of them having

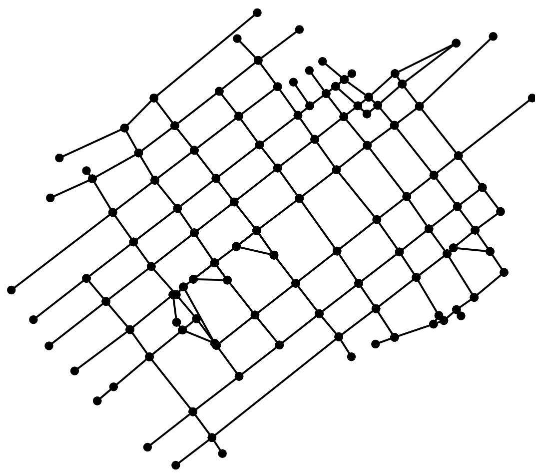


FIGURE 62. Valetta, Malta

almost 90° angles between the meeting streets. We anticipate that this is a scenario where the factorization of $\mathcal{OPRA}_8$ without factorization of the street definitions (Klippel$_8$ $s = 0$) can show its strength. $\mathcal{OPRA}_8$ itself is supposed to perform worse, since the described angles are mostly just close to 90° but not exactly 90° leading to planar instead of the desired linear relations. We have performed 37296 navigation experiments with an aggregation shown in the following table:

	Klippel$_8$ $s = 0$	Klippel$_8$ $s = 1$	$\mathcal{OPRA}_8$	shortest path	uniform shortest path	random walk
average	8.50	23.42	8.72	8.62	7.65	356.98
deviation	3.79	27.00	4.13	3.83	2.96	231.06

As expected Klippel$_8$ $s = 0$ performs very well, interestingly it performs better than shortest path and close to uniform shortest path in the means of visited nodes. As uniform shortest path Klippel$_8$ $s = 0$ does not really see the lengths of the street segments and so shortest path might visit more segments for a path that is shorter overall. But in the end counting the nodes visited is still an appropriate measure for our coarse navigation in small street networks. $\mathcal{OPRA}_8$ performs a little worse because of the above observations. Applying the factorization to the definition of streets turns out to be a bad idea as in all other cases.

It turned out to be a very hard task to find street networks on `OpenStreetMap` that are suited for investigating the properties of the limited knowledge navigation approach. The networks have to be small enough to be experimented on in an acceptable time frame on a normal computer, further they need to be small enough

[7]This are was chosen for its suitability for navigation not for the author's vanity.

to fit into memory. Finding such networks is simple, but they need to be suited to make expressive observations, too. And this is the hard part. The bugs in many tools around `OpenStreetMap` did not make the job easier as did the missing possibility to retrieve maps from `OpenStreetMap` at certain levels of granularity. For example the author still considers to try the navigation on the network of highways in Moscow as very interesting but is lacking the supercomputer to handle that amount of data. We have spent months to find well suited street networks and optimize the navigation software in such a way that it could handle them efficiently working around several limitations of `JAVA` that made handling the large output of the experiments impossible. Another big drawback is that the algebraic reasoners can only perform algebraic closure in a single thread leading to very long computation times for that step.

As a conclusion we can say that the navigation approach based on limited knowledge using $\mathcal{OPRA}$ constraint networks performs astonishingly well on small constraint networks. But the topology of the networks is a factor that has a high impact on the overall navigation quality. There are some honey pots like dead ends that lead the navigation algorithm to wrong paths, but due to its simple memory it neither gets stuck in dead ends nor in circles. The destination is always reached and often in not many more steps than shortest path. Unfortunately the encoding in $\mathcal{OPRA}$ generates many oriented points (basically around three to four) per node in the street network blowing up the memory consumption and reasoning time, but on the other hand $\mathcal{OPRA}$ is very well suited for a moving agent with its own orientation.

CHAPTER 6

Limits of Algebraic Closure

Abolish the rules made of stone
—**Slayer**, *Raining Blood.*

Algebraic closure, an adaption of Montanari's path-consistency [47] to constraint networks in relations equipped with non-associative composition, was considered as the standard technique to examine constraint networks over qualitative spatial calculi for a long time. A composition based technique to reason about such calculi has already been proposed by Allen for his interval algebra [1]. Several factors make algebraic closure an alluring method for investigating constraint networks:

- algebraic closure has polynomial running time;
- algebraic closure is not dependent on the domain of the calculi;
- algebraic closure only needs the converse and composition tables of a calculus;
- several (optimized) implementations are available in state-of-the-art reasoning tools.

But algebraic closure is often not the way to go for solving consistency problems for qualitative spatial reasoning. In general, if no algebraically closed refinement of a constraint network exists, then the network is inconsistent, but if such a refinement exists, then the constraint network may or may not be consistent. For example, for the $\mathcal{RCC}$ calculi [61] so called tractable subsets, i.e. subsets of the set of general relations such that algebraic closure decides consistency have been identified in [66]. For the very simple $\mathcal{LR}$ calculus it was believed [73] for a long time that algebraic closure decides consistency. We will show that that is not even true for simple scenarios in Section 6.1. For $\mathcal{LR}$ we additionally investigate prominent notions of consistency from the field of CSP solving that also turn out to be no solution for the initial problem. Further we will show that for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ algebraic closure also fails on scenarios Section 6.2. These results can also be derived using [79] but we consider that providing actual counterexamples is a more vivid way to go and provides more insight into the calculus to the interested reader.

6.1. Limits for $\mathcal{LR}$

In this section we investigate if algebraic closure decides consistency for the very simple $\mathcal{LR}$ calculus. This section was first released as a part of a conference paper [41] and was the starting point for a lot of our research. We remember the $\mathcal{LR}$ calculus from Section 3.1 and the definition of the reasoning technique from Section 2.3. For a long time it was believed that algebraic closure decides consistency for this simple calculus, we will prove this wrong. Further we made some shattering observations about algebraic closure with the traditional composition for the $\mathcal{LR}$ calculus, i.e. we show that for some kinds of scenarios algebraic closure decides absolutely nothing.

Proposition 91. *All scenarios only containing constraints in the relations l and r are algebraically closed with respect to the $\mathcal{LR}$-calculus with traditional composition.*

Proof. We have a look at the permutations of $\mathcal{LR}$ and see that

operation	operand	result
INV	l	r
	r	l
SC	l	r
	r	l
HM	l	l
	r	r

the set of $\{l, r\}$ is closed under all permutations. A look at the traditional binary composition table of $\mathcal{LR}$, as for example provided as part of `SparQ` [78], reveals that all compositions containing only l and r on their left hand side, always have the set $\{l, r\}$ as a subset of their right hand side:

operand 1	operand 2	result
l	l	$\{b, s, i, l, r\}$
l	r	$\{f, l, r\}$
r	l	$\{f, l, r\}$
r	r	$\{b, s, i, l, r\}$

But with this we can conclude, that

$$R_{i,k} \diamond R_{k,j} \cap R_{i,j} \neq \emptyset$$

for all i, k, j, given that $R_{n,m} \in \{l, r\}$ for all m, n. □

If algebraic closure would decide consistency for the $\mathcal{LR}$ calculus it would be implied by Proposition 91 that any scenario in the relations l and r would be consistent. It is very doubtful that that is true. But Proposition 91 simplifies our search for scenarios that are algebraically closed but inconsistent a lot. All we need to do is find a scenario in the relations l and r that is inconsistent, since it is algebraically closed by Proposition 91. We use the scenario

$$\begin{aligned} \texttt{SCEN} \;:=\; & \{(A\,B\;r\;C), (A\,E\;r\;D), (D\,B\;r\;A), \\ & (D\,C\;r\;A), (D\,C\;r\;B), (D\,E\;r\;B), \\ & (D\,E\;l\;C), (E\,B\;r\;A), (E\,C\;r\;A), \\ & (E\,C\;r\;B)\} \end{aligned}$$

that is algebraically closed by Proposition 91 and show that it is inconsistent. We will prove it inconsistent by showing that any projection of this scenario to the natural domain, the Euclidean plane of $\mathbb{R}^2$, of the $\mathcal{LR}$-calculus yields a contradiction, i.e. that the scenario cannot be realized at all.

Scenarios of the $\mathcal{LR}$-calculus are invariant with respect to the affine transformations of translation, rotation and scaling. This means that we can fix two points to arbitrary values, in `SCEN` we chose to set D to $(0, 0)$ and B to $(0, 1)$. With this we obtain the inequalities (see Section 3.1.2):

$$\begin{array}{llll} (3) & A_x \cdot E_y < A_y \cdot E_x & (6) & C_x < 0 \\ (4) & C_x \cdot A_y < C_y \cdot A_x & (7) & E_x < 0 \\ (5) & E_y \cdot C_x < E_x \cdot C_y & (8) & 0 < A_x \end{array}$$

In fact, the fixing of these two points just simplifies the proof of the non-existence of solutions to the system of inequalities. Leaving out this step leads to a system of inequalities that is harder to handle but not solvable either.

In fact more inequalities than shown above are derivable, but already these ones suffice to show that they are not jointly satisfiable. Hence, we conclude:

Theorem 92. *Traditional algebraic closure does not decide consistency of scenarios for the $\mathcal{LR}$-calculus.*

Proof. We consider the algebraically closed $\mathcal{LR}$ scenario `SCEN` and the inequalities (3) to (8) that we derived when projecting it to $\mathbb{R}^2$, the intended domain of $\mathcal{LR}$. From inequalities (3), (8), (6), (7) and (5) we obtain

$$\frac{E_x \cdot C_y}{C_x} \quad < \quad E_y \quad < \quad \frac{A_y \cdot E_x}{A_x}$$

and again using inequalities (8), (6) and (7) we get

$$C_y \cdot A_x \quad < \quad C_x \cdot A_y$$

contradicting (4). Hence our scenario is not consistent. □

As discussed in Section 2.3 ternary composition is more natural for ternary calculi than binary composition. Therefore we examined the ternary composition table of the $\mathcal{LR}$-calculus[1] and unfortunately have to conclude:

Theorem 93. *Algebraic closure with respect to ternary composition does not decide consistency of scenarios for the $\mathcal{LR}$-calculus.*

Proof. Let us have a closer look at the ternary composition operation with respect to the relations contained in `SCEN`, namely the relation l and r. Recall that the set $\{l, r\}$ of $\mathcal{LR}$-relations is closed under all permutation operations. So we only need to consider the fragment of the composition table with triples over l, r:

$$\begin{array}{rclrcl} \diamond(r,r,r) & = & \{r\}, & \diamond(r,r,l) & = & \{b,r,l\}, \\ \diamond(r,l,r) & = & \{f,r,l\}, & \diamond(r,l,l) & = & \{i,r,l\}, \\ \diamond(l,r,r) & = & \{i,r,l\}, & \diamond(l,r,l) & = & \{f,r,l\}, \\ \diamond(l,l,r) & = & \{b,r,l\}, & \diamond(l,l,l) & = & \{l\}. \end{array}$$

We see that any composition that contains r as well as l in the triple on the left-hand side yields a superset of $\{r, l\}$ on the right-hand side. So all composable triples that have both l and r on their left hand side cannot yield an empty set while applying algebraic closure. So, we have to investigate how the compositions $\diamond(l,l,l)$ and $\diamond(r,r,r)$ are used when enforcing algebraic closure. Enumerating all composable triples $(X_1\, X_2\, r_1\, X_4), (X_1\, X_4\, r_2\, X_3), (X_4\, X_2\, r_3\, X_3)$ and their respective refinement relation $(X_1\, X_2\, r_f\, X_3)$ yields a list with 18 entries shown in Appendix A. All of those entries list l as refinement relation whenever composing $\diamond(l,l,l)$ and analogously for r. Thus, no refinement containing the empty relations occurs and the given scenario is algebraically closed with respect to ternary composition. But in Theorem 93 it has already been shown that `SCEN` is inconsistent. □

We believe that advancing to even higher arity composition will not provide us with a sound algebraic closure algorithm. It turns out, however, that moving to a certain level of k-consistency does indeed make a change. We will investigate this use of other notions of consistency in Section 6.1.1.

Remark 94. Of course it is theoretically possible to solve these systems of inequalities by quantifier elimination, or by the more optimized Cylindrical Algebraic Decomposition (CAD). Unfortunately the CAD algorithm has a double exponential worst case running time [35] (even though this can be reduced to polynomial running time with a optimal choice of the involved projections). Our experiments with CAD tools unfortunately were quite disillusioning, since those tools choked on our problems mainly because of the large number of involved variables (consider that

[1] A ternary composition table for $\mathcal{LR}$ is available via the qualitative reasoner `SparQ` [78].

each point in our scenarios introduces 2 variables in our systems of inequalities). Unfortunately, these CAD tools did not support the use of custom projections. We believe that using CAD with projections tailored for qualitative spatial reasoning this approach might be fruitful. In fact, an investigation into the usability of CAD for qualitative reasoning is currently ongoing research by Lee and Wolter [35].

6.1.1. Deciding Global Consistency. In this section we will generalize a technique from [31] and we will show that this generalization decides global consistency for arbitrary CSPs over m-ary convex relations over a domain $\mathbb{R}^n$. The resulting theorem transfers Theorem 5 of [68] from classical constraint satisfaction to qualitative spatio-temporal reasoning.

As a starting point we need Helly's theorem about convex regions of space. Additionally from Definition 30 we remember what convexity means for our spatial relations.

Theorem 95 (Helly's Theorem [29])**.** *Let S be a set of convex regions of the n-dimensional space $\mathbb{R}^n$. If every $n+1$ elements in S have a nonempty intersection then the intersection of all elements of S is nonempty.*

Making use of Helly's Theorem, we can provide a property for CSPs over convex relations $\mathbb{R}^n$ for being globally consistent that depends on the arity of the relations and dimension of the space, thus simplifying the decision of global consistency especially for big CSPs. In fact we will show that a CSP over m-ary convex relations over a domain $\mathbb{R}^n$ is globally consistent if it is strongly $((m-1)\cdot(n+1)+1)$-consistent. This is already an alleviation, since we only need to check strong $((m-1)\cdot(n+1)+1)$-consistency for all $((m-1)\cdot(n+1)+1)$-point sub-scenarios. We already want to stress that global consistency is a pretty strong notion and we still need to investigate how it relates to consistency for our calculi in question.

Theorem 96. *A CSP over m-ary convex relations over a domain $\mathbb{R}^n$ is globally consistent, i.e. k-consistent for all $k \in \mathbb{N}$, if and only if it is strongly $((m-1)\cdot(n+1)+1)$-consistent.*

Proof.

In the first step of this proof consider an arbitrary CSP over convex m-ary relations that is strongly $((m-1)\cdot(n+1)+1)$-consistent. By induction on k, which is the number of variables that can be instantiated in a strongly consistent way, we show that it is $k+1$ consistent for an arbitrary k. Assume that for each tuple $(X_1,\dots,X_k)$ of these variables a consistent valuation $(z_1,\dots,z_k)$ exists. For this purpose we define sets

$$p_s\left((z_{i_1},\dots,z_{i_{m-1}}), R_{i_1,\dots,i_s,k+1,i_{s+1},\dots,i_{m-1}}\right) = \{z \mid R_{i_1,\dots,i_s,k+1,i_{s+1},\dots,i_{m-1}}(z_{i_1},\dots,z_{i_s},z,z_{i_{s+1}},\dots,z_{i_{m-1}})\}$$

with $1 \le i_j \le k$ and $1 \le s \le m-1$ and the $R_{i_1,\dots,i_s,k+1,i_{s+1},\dots,i_{m-1}}$ are constraints of the CSP. If for a particular sequence of indexes no constraint is given, then the universal relation is taken as the constraint, expressing that no knowledge is available for this sequence. Fortunately, the universal relation leads to no restrictions regarding the valuation of variables. By prerequisite, these are sets of convex regions of the particular space defined by the assignment of the variables $(X_1,\dots,X_k) \mapsto (z_1,\dots,z_k)$ and the particular relation $R_{i_1,\dots,i_s,k+1,i_{s+1},\dots,i_{m-1}}$. Let

$$\mathbf{P} = \{p_s\left((z_{i_1},\dots,z_{i_{m-1}}), R_{i_1,\dots,i_s,k+1,i_{s+1},\dots,i_{m-1}}\right) \mid 1 \le s \le m-1 \wedge 1 \le i_j \le k\}$$

be the set of all such convex regions. Observe that $n+1$ tuples of elements of $\mathbf{P}$ are induced by constraints containing up to $(m-1)\cdot(n+1)$ different variables. By strong $((m-1)\cdot(n+1)+1)$-consistency we know that the intersection of all these regions is non-empty. The application of Helly's Theorem yields

$$\bigcap_{p\in\mathbf{P}} p \neq \varnothing.$$

Hence a valuation for $k+1$ variables exists. The second step of this proof is trivial, since global consistency implies k-consistency for all $k \in \mathbb{N}$ by definition. □

In [63, Proposition 1] it was shown that whether composition is weak or strong is independent of the property of algebraic closure to decide consistency. However, in some cases, these two properties *are* related:

Theorem 97. *In a binary calculus over the real line that*

(1) *has only* 2*-consistent relations*
(2) *and has strong binary composition*

algebraic closure decides consistency of CSPs over convex base relations.

Proof. By Theorem 96 we know that strong 3-consistency decides global consistency. Since composition is strong, algebraic closure decides 3-consistency and, since we have 2 consistency, it decides strong 3-consistency too. Thus algebraically closed scenarios are either inconsistent (containing the empty relation) or globally consistent. Put differently, global consistency and consistency coincide. □

Making use of Theorem 96 we can derive concrete levels of k-consistency for spatial calculi that are needed to decide global consistency for their convex relations. We can do that for example for $\mathcal{LR}$ and $\mathcal{DCC}$, Freksa's double cross calculus [19], whose natural domains are $\mathbb{R}^2$.

Corollary 98. *For CSPs over convex* $\{\mathcal{LR}, \mathcal{DCC}\}$*-relations strong* 7*-consistency decides global consistency.*

Proof. Follows directly from Theorem 96 for both calculi. □

Having an approach to decide global consistency for convex relations in $\mathcal{LR}$ and $\mathcal{DCC}$ we are also interested in its rough complexity. We are interested in the fact if this global consistency can be decided in polynomial time for our CSPs with infinite domains.

Corollary 99. *Global consistency of scenarios in convex* $\{\mathcal{LR}, \mathcal{DCC}\}$*-relations is polynomially decidable.*

Proof. Compute the set of strongly 7-consistent 7-point scenarios in constant time (e.g. using quantifier elimination[2]). The given scenario is strongly 7-consistent if and only if all 7-point sub-scenarios are contained in the set of strongly 7-consistent 7-point scenarios. By Theorem 96 this decides global consistency. □

Unfortunately consistency and global consistency are not equivalent in the $\mathcal{LR}$-calculus and hence not for $\mathcal{DCC}$. The following counterexample can be used for $\mathcal{DCC}$ as well using convex general relations instead of base relations.

Proposition 100. *For the* $\mathcal{LR}$*-calculus not every consistent scenario is globally consistent.*

[2]Here we just want to state the computation is possible, we do not claim to suggest a practical method though.

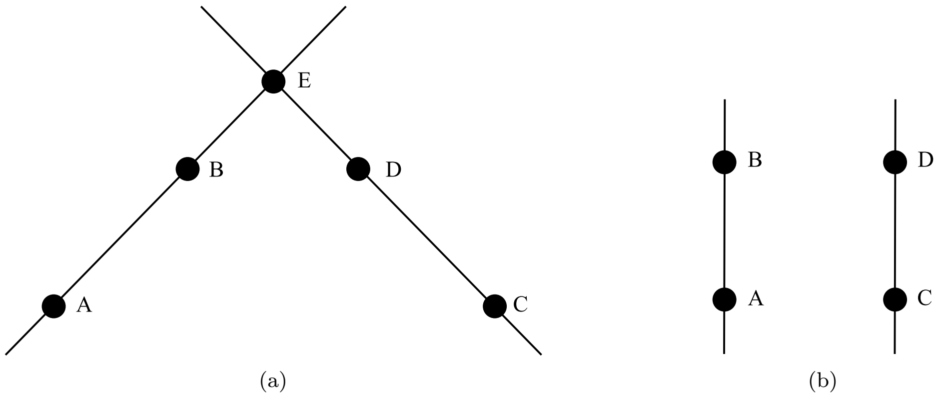


FIGURE 63. Illustration for Proposition 100

PROOF. Consider the consistent scenario

$$\{(A\,B\;r\;C),\quad (A\,B\;r\;D),\quad (C\,D\;l\;A)$$
$$(C\,D\;l\;B),\quad (A\,B\;f\;E),\quad (C\,D\;f\;E)\}$$

which has a realization as shown in Figure 63a, the lines $\overline{AB}$ and $\overline{CD}$ intersect. Now consider the sub-CSP in the variables A, B, C, and D with the a possible solution shown in Figure 63b. We see that the lines $\overline{AB}$ and $\overline{CD}$ are parallel, but the constraints $(A\,B\;f\;E)$ and $(C\,D\;f\;E)$ demand that the point E is on the line $\overline{AB}$ as well as on the line $\overline{CD}$, hence the given scenario is not 5-consistent, and so it is not globally consistent. □

We have investigated the question if algebraic closure decides consistency for $\mathcal{LR}$ with binary and ternary composition operations. In both cases we were able to show that algebraic closure fails to decide consistency. Due to the properties of algebraic closure the set of algebraically closed scenarios contains all consistent scenarios but also inconsistent ones. Tries with other notions of consistency from the field of CSP solving, namely global consistency, yields a set of globally consistent scenarios but it does not contain all consistent scenarios. Some are consistent but not globally consistent. This leaves us with two polynomial techniques that approximate consistency in $\mathcal{LR}$. But one technique is too weak to decide while the other one is too strong. Further the counterexamples needed to show this were frighteningly simple. We will not give up at this point and try and find another polynomial technique that approximates $\mathcal{LR}$ consistency quite well in Section 7.

After we have given a negative answer to the question if algebraic closure decides consistency for $\mathcal{LR}$, we move on and investigate the same question for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ in Section 6.2.

6.2. Limits for $\mathcal{DRA}$

Similar sections to this one can be found in [51] and [52]. In Section 6.1 we have shown that algebraic closure does not decide consistency for $\mathcal{LR}$. Since we now have computed the composition tables for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ in Chapter 4 and investigated the usefulness of $\mathcal{DRA}_{fp}$ for a sample application in Section 5.1 it is the time to investigate if algebraic closure decides consistency for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ especially since we have shown in Section 4.4.2 that composition for $\mathcal{DRA}_{fp}$ is strong and hence associative. Further in contrast to $\mathcal{LR}$ the $\mathcal{DRA}$ calculi are binary and all the theory concerning non-associative algebras as shown in Section 2.2

applies to them. But in spite of all these positive sounding hints, in the end we show that algebraic closure does not decide consistency for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$.

6.2.1. Limits of Algebraic Closure. We now consider the question whether algebraic closure decides consistency for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$. This section was part of the journal paper [52]. Remember the `LISP`-like style of `SparQ` [78] to describe constraint networks. A relation between two dipoles A R B is written in this notation as (A R B). This notation allows for a very compact representation of scenarios and arbitrary constraint networks, further the interested reader can copy and paste all of our networks directly into the `SparQ` qualitative reasoner [78] and verify our claims.

With the help of the embedding of Allen's interval algebra into $\mathcal{DRA}_f$ (see Proposition 47), we can show that algebraic closure decides the consistency of $\mathcal{DRA}_f$ scenarios that only involve images of relations of the interval algebra. Moreover, for calculi such as $\mathcal{RCC}8$ [62], the interval algebra [59], etc., (maximal) *tractable subsets* (see [64]) have been determined, i.e., sets of relations for which algebraic closure decides consistency also of non-atomic constraint networks involving these relations. We then also obtain that algebraic closure in $\mathcal{DRA}_f$ decides the consistency of any constraint network involving (the image of) a maximal tractable subset of the interval algebra. Similar remarks apply to $\mathcal{DRA}_{fp}$.

However, the situation changes if we move to the full calculus. The scenario consistency problem for the $\mathcal{DRA}_f$ calculus is already $\mathcal{NP}$-hard, see [79], and hence algebraic closure (which is polynomial) does not decide scenario consistency in this case (assuming $\mathcal{P} \neq \mathcal{NP}$). This means that there are essentially no tractable subsets.

To illustrate the failure of algebraic closure to decide consistency, we now construct constraint networks which are geometrically unrealizable but still algebraically closed. We do this by constructing constraint networks that are consistent and algebraically closed, and then we will change a relation in such a way that they remain algebraically closed but become inconsistent. We follow the approach of [67] in using a simple geometric shape for which scenarios exist, where algebraic closure fails to decide consistency. In our case, the basic shape is a convex hexagon for constructing the counterexample for $\mathcal{DRA}_f$ (see Figure 64).

First we show that algebraic closure does not decide consistency for $\mathcal{DRA}_f$. Consider a convex hexagon consisting of the dipoles A, B, C, D, E and F. Such an object is described as

$$(A \text{ errs } B)(B \text{ errs } C)(C \text{ errs } D)(D \text{ errs } E)(E \text{ errs } F)(F \text{ errs } A)(F \text{ rrrr } C)$$
$$(A \text{ rrrr } D)(B \text{ rrrr } E)(A \text{ rrrr } C)\,(F \text{ rrrr } D)(B \text{ rrrr } D)(A \text{ rrrr } E)(C \text{ rrrr } E)$$
$$(B \text{ rrrr } F)$$

where the relations *rrrr* make sure that none of the dipoles intersect and together with the components r of the relations *errs* ensure convexity, since they enforce an angle between 0 and π between the respective first and second dipole, i.e., the endpoint of consecutive dipoles always lies to the right of the preceding dipole. Such an object is depicted in Figure 64. Any object inside the hexagon lies to the right of all the dipoles, otherwise it is on the border or outside. To this scenario we add dipoles G and H inside the hexagon

$$(F \text{ rrrl } H)(C \text{ rrlr } G)(H \text{ efbs } G)$$

that are collinear and the end point of H is the start point of G. This gives us the overall constraint network

$$(A \text{ errs } B)(B \text{ errs } C)(C \text{ errs } D)(D \text{ errs } E)\,(E \text{ errs } F)(F \text{ errs } A)(F \text{ rrrr } C)$$
$$(A \text{ rrrr } D)(B \text{ rrrr } E)(A \text{ rrrr } C)\,(F \text{ rrrr } D)\,(B \text{ rrrr } D)(A \text{ rrrr } E)(C \text{ rrrr } E)$$
$$(B \text{ rrrr } F)(F \text{ rrrl } H)(C \text{ rrlr } G)\,(H \text{ efbs } G)$$

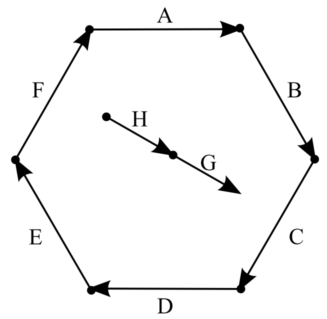


FIGURE 64. Convex hexagon

We construct an inconsistency by postulating that H (i.e., its start and end point) lies to the left of E, meaning that it lies outside the hexagon by introducing the constraint (E llrr H). By applying algebraic closure, we get a refinement of our network that does not contain the empty set and hence no inconsistency is detected:

(D (llrr rrrr) H)(F rrrl G) (F rrrl H)(E llrr G) (E llrr H)
(D (llrr rrrr) G) (H efbs G)(D rrrr F)(D errs E)(C rrlr G)
(A (llll rrll) H) (C rrrr F) (C rrrr E)(C errs D)(A rrrr D)
(B (llll rrll) H) (B rrrr F) (B rrrr E)(B rrrr D)(B errs C)
(A (llll rrll) G) (C rrlr H) (A rser F)(A rrrr E) (A rrrr C)
(B (llll rrll) G) (A errs B) (E errs F)

But H has to lie to the left of E, meaning outside the convex hexagon and inside of it at the same time. This is impossible in the Euclidean plane. In fact, we can construct similar inconsistencies for several dipoles, just check the above constraint network for the relation llrr. Hence, algebraic closure does not decide consistency for the $\mathcal{DRA}_f$ calculus. In a next step, let us try how algebraic closure of this network performs using $\mathcal{DRA}_{fp}$.

The given inconsistent constraint network can be extended to a $\mathcal{DRA}_{fp}$ constraint network in a straightforward manner by replacing rrrr by {rrrr+, rrrr-, rrrrA} and llrr by {llrr+, llrr-, llrrP}. Algebraic closure with $\mathcal{DRA}_{fp}$ then detects the inconsistency in the network. To stress this, we drop the constraint (E(llrr+, llrr-, llrrP)H) and observe that the relation between E and H is refined to

(E(rrrr+ rrrr- rrrrA)H).

This constraint has no component that demands H being outside of the hexagon, which led to the failure in the $\mathcal{DRA}_f$ case. We can double check this problem by doing the same for $\mathcal{DRA}_f$. We take the constraint network for $\mathcal{DRA}_f$ and drop the constraint (E llrr H). But this time the refined relation between E and H is

(E(rrrr llrr)H)

containing the base relation llrr that leads to the inconsistency.

We have found an example that shows that algebraic closure for $\mathcal{DRA}_{fp}$ finds inconsistencies in constraint networks where it fails for $\mathcal{DRA}_f$. This leads to the question: Can we show using a simple scenario that algebraic closure does not decide consistency for $\mathcal{DRA}_{fp}$, too? The answer to this question is "yes".

To construct a counterexample for $\mathcal{DRA}_{fp}$, we begin with a point configuration with nine points A, B, …, I as in Figure 65. This configuration corresponds to a Pappus configuration [7]. A Pappus configuration has nine points and nine straight lines. Eight collinearities of point triples: GHI, ABC, ADH, AEI, BDG, BFI, CEG, CFH enforce the collinearity of the ninth point triple DEF (by Pappus'

Hexagon Theorem [7]). We can reconstruct this arrangement with dipoles and add an inconsistency with Pappus' Hexagon Theorem which is not detected with the algebraic closure for $\mathcal{DRA}_{fp}$.

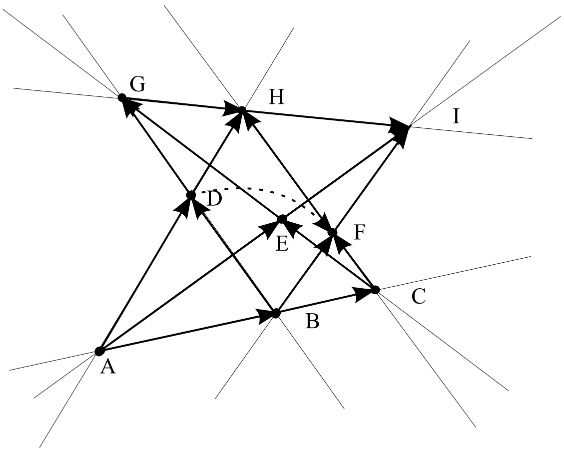


FIGURE 65. Construction of the counterexample as a Pappus configuration.

The configuration from Figure 65 can be described as a constraint network in the following way:

$(GH \text{ efbs } HI)(AB \text{ efbs } BC)\,(AD \text{ efbs } DH)(AE \text{ efbs } EI)\,(BD \text{ efbs } DG)$
$(BF \text{ efbs } FI)\,(CE \text{ efbs } EG)\,(CF \text{ efbs } FH)\,(DG \text{ errs } GH)(DG \text{ rele } EG)$
$(FI \text{ lere } HI)\,(FI \text{ lere } EI)\,(AD \text{ srsl } AE)\,(AD \text{ srsl } AB)\,(CF \text{ lsel } BC)$
$(CF \text{ slsr } CE)\,(GH \text{ rele } DH)(GH \text{ rele } FH)\,(AB \text{ ells } BF)\,(AB \text{ ells } BD)$
$(AD \text{ ells } DG)\,(AD \text{ rele } BD)\,(CF \text{ errs } FI)\,(CF \text{ lere } BF)\,(AE \text{ rele } CE)$

a dipole XY in this description is the dipole from point X to point Y. We observe that by Pappus' Hexagon Theorem the points D, E and F are collinear. We now add a constraint

$$(AE \text{ (lrrr lrrl) } DF)$$

that states that the carrier lines of AE and DF intersect between A and E or in front of AE, but not in E. But since D, E and F are collinear, the only possible intersection point is E, a contradiction. Any scenario based on this constraint network cannot be consistent. But applying algebraic closure with $\mathcal{DRA}_{fp}$ yields a refinement and dozens of possible scenarios, e.g.,

$(FH\ \mathrm{rser}\ DF)$	$(CF\ \mathrm{rele}\ DF)$	$(CF\ \mathrm{efbs}\ FH)$	$(EG\ \mathrm{rrrr{+}}\ DF)$
$(EG\ \mathrm{rrll{-}}\ FH)$	$(EG\ \mathrm{brll}\ CF)$	$(CE\ \mathrm{rrlr}\ DF)$	$(CE\ \mathrm{rrbl}\ FH)$
$(CE\ \mathrm{srsl}\ CF)$	$(CE\ \mathrm{efbs}\ EG)$	$(FI\ \mathrm{rser}\ DF)$	$(FI\ \mathrm{slsr}\ FH)$
$(FI\ \mathrm{rser}\ CF)$	$(FI\ \mathrm{llrr{+}}\ EG)$	$(FI\ \mathrm{rlrr}\ CE)$	$(BF\ \mathrm{rele}\ DF)$
$(BF\ \mathrm{ells}\ FH)$	$(BF\ \mathrm{rele}\ CF)$	$(BF\ \mathrm{lllr}\ EG)$	$(BF\ \mathrm{rllr}\ CE)$
$(BF\ \mathrm{efbs}\ FI)$	$(DG\ \mathrm{rrrr{+}}\ DF)$	$(DG\ \mathrm{rrllP}\ FH)$	$(DG\ \mathrm{rrllP}\ CF)$
$(DG\ \mathrm{rele}\ EG)$	$(DG\ \mathrm{rrlf}\ CE)$	$(DG\ \mathrm{rrll{-}}\ FI)$	$(DG\ \mathrm{brll}\ BF)$
$(BD\ \mathrm{rrlr}\ DF)$	$(BD\ \mathrm{rrllP}\ FH)$	$(BD\ \mathrm{rrllP}\ CF)$	$(BD\ \mathrm{rfll}\ EG)$
$(BD\ \mathrm{rrll{+}}\ CE)$	$(BD\ \mathrm{rrbl}\ FI)$	$(BD\ \mathrm{srsl}\ BF)$	$(BD\ \mathrm{efbs}\ DG)$
$(EI\ \mathrm{lrrr}\ DF)$	$(EI\ \mathrm{rllr}\ FH)$	$(EI\ \mathrm{rrlr}\ CF)$	$(EI\ \mathrm{slsr}\ EG)$
$(EI\ \mathrm{rser}\ CE)$	$(EI\ \mathrm{rele}\ FI)$	$(EI\ \mathrm{rrlf}\ BF)$	$(EI\ \mathrm{llrr{+}}\ DG)$
$(EI\ \mathrm{rlrr}\ BD)$	$(AE\ \mathrm{lrrr}\ DF)$	$(AE\ \mathrm{rlll}\ FH)$	$(AE\ \mathrm{rrll{+}}\ CF)$
$(AE\ \mathrm{ells}\ EG)$	$(AE\ \mathrm{rele}\ CE)$	$(AE\ \mathrm{rfll}\ FI)$	$(AE\ \mathrm{rrll{+}}\ BF)$
$(AE\ \mathrm{lllr}\ DG)$	$(AE\ \mathrm{rllr}\ BD)$	$(AE\ \mathrm{efbs}\ EI)$	$(DH\ \mathrm{rrrrA}\ DF)$
$(DH\ \mathrm{rele}\ FH)$	$(DH\ \mathrm{rrlf}\ CF)$	$(DH\ \mathrm{rll}\ EG)$	$(DH\ \mathrm{rrlr}\ CE)$
$(DH\ \mathrm{rrll{+}}\ FI)$	$(DH\ \mathrm{rrll{+}}\ BF)$	$(DH\ \mathrm{slsr}\ DG)$	$(DH\ \mathrm{rser}\ BD)$
$(DH\ \mathrm{rrll{-}}\ EI)$	$(DH\ \mathrm{brll}\ AE)$	$(AD\ \mathrm{rrrrA}\ DF)$	$(AD\ \mathrm{rfll}\ FH)$
$(AD\ \mathrm{rrll{+}}\ CF)$	$(AD\ \mathrm{rlll}\ EG)$	$(AD\ \mathrm{rrll{+}}\ CE)$	$(AD\ \mathrm{rrll{+}}\ FI)$
$(AD\ \mathrm{rrll{+}}\ BF)$	$(AD\ \mathrm{ells}\ DG)$	$(AD\ \mathrm{rele}\ BD)$	$(AD\ \mathrm{rrbl}\ EI)$
$(AD\ \mathrm{srsl}\ AE)$	$(AD\ \mathrm{efbs}\ DH)$	$(BC\ \mathrm{llll{-}}\ DF)$	$(BC\ \mathrm{lllb}\ FH)$
$(BC\ \mathrm{ells}\ CF)$	$(BC\ \mathrm{lllb}\ EG)$	$(BC\ \mathrm{ells}\ CE)$	$(BC\ \mathrm{llbr}\ FI)$
$(BC\ \mathrm{slsr}\ BF)$	$(BC\ \mathrm{llbr}\ DG)$	$(BC\ \mathrm{slsr}\ BD)$	$(BC\ \mathrm{llrr{+}}\ EI)$
$(BC\ \mathrm{blrr}\ AE)$	$(BC\ \mathrm{llrr{+}}\ DH)$	$(BC\ \mathrm{blrr}\ AD)$	$(AB\ \mathrm{llrl}\ DF)$
$(AB\ \mathrm{llll{+}}\ FH)$	$(AB\ \mathrm{flll}\ CF)$	$(AB\ \mathrm{llll{+}}\ EG)$	$(AB\ \mathrm{flll}\ CE)$
$(AB\ \mathrm{lllb}\ FI)$	$(AB\ \mathrm{ells}\ BF)$	$(AB\ \mathrm{lllb}\ DG)$	$(AB\ \mathrm{ells}\ BD)$
$(AB\ \mathrm{llbr}\ EI)$	$(AB\ \mathrm{slsr}\ AE)$	$(AB\ \mathrm{llbr}\ DH)$	$(AB\ \mathrm{slsr}\ AD)$
$(AB\ \mathrm{efbs}\ BC)$	$(HI\ \mathrm{lrrr}\ DF)$	$(HI\ \mathrm{rser}\ FH)$	$(HI\ \mathrm{rrfr}\ CF)$
$(HI\ \mathrm{rbrr}\ EG)$	$(HI\ \mathrm{rrrr{+}}\ CE)$	$(HI\ \mathrm{rele}\ FI)$	$(HI\ \mathrm{rrlf}\ BF)$
$(HI\ \mathrm{rbrr}\ DG)$	$(HI\ \mathrm{rrrr{+}}\ BD)$	$(HI\ \mathrm{rele}\ EI)$	$(HI\ \mathrm{rrlf}\ AE)$
$(HI\ \mathrm{rser}\ DH)$	$(HI\ \mathrm{rrfr}\ AD)$	$(HI\ \mathrm{rrllP}\ BC)$	$(HI\ \mathrm{rrllP}\ AB)$
$(GH\ \mathrm{lrrr}\ DF)$	$(GH\ \mathrm{rele}\ FH)$	$(GH\ \mathrm{rrlf}\ CF)$	$(GH\ \mathrm{rser}\ EG)$
$(GH\ \mathrm{rrfr}\ CE)$	$(GH\ \mathrm{rfll}\ FI)$	$(GH\ \mathrm{rrll{+}}\ BF)$	$(GH\ \mathrm{rser}\ DG)$
$(GH\ \mathrm{rrfr}\ BD)$	$(GH\ \mathrm{rfll}\ EI)$	$(GH\ \mathrm{rrll{+}}\ AE)$	$(GH\ \mathrm{rele}\ DH)$
$(GH\ \mathrm{rrlf}\ AD)$	$(GH\ \mathrm{rrllP}\ BC)$	$(GH\ \mathrm{rrllP}\ AB)$	$(GH\ \mathrm{efbs}\ HI)$

Hence, for $\mathcal{DRA}_{fp}$ algebraic closure does not decide consistency even for scenarios. This counterexample can also be used for $\mathcal{DRA}_f$, but the former one is simpler and shows differences in the reasoning effectiveness of algebraic closure for $\mathcal{DRA}_{fp}$ and $\mathcal{DRA}_f$.

6.3. Aftermath

We have investigated the effectiveness of algebraic closure to decide consistency for $\mathcal{LR}$, $\mathcal{DRA}_f$, and $\mathcal{DRA}_{fp}$ using simple geometric configurations. Applying the results from [79] makes it clear directly that algebraic closure cannot decide consistency for any of these calculi as long as $\mathcal{P} \neq \mathcal{NP}$ but going that way does not allow for any insights into the structure of configurations for which algebraic closure fails. Further we could not try the counterexamples with similar calculi and compare the behavior on them. For these reasons we decided to go the hard and direct way of finding counterexamples using geometric shapes. Interestingly $\mathcal{DRA}_{fp}$ manages to find the inconsistencies in the counterexample for $\mathcal{DRA}_f$ using relational algebraic reasoning. But even for that calculus it is possible to find a rather simple example that makes algebraic closure fail. So far we can say that algebraic closure is just an approximation of consistency for all of these calculi and for $\mathcal{LR}$ it is even a very bad approximation for binary traditional algebraic closure due to Proposition 91.

CHAPTER 7

A new Approximation of Consistency for $\mathcal{LR}$

Is it black or is it white
Let's find another compromise
—**Wolfsheim**, *The Sparrows and the Nightingales.*

Parts of this section have been published as conference paper [40]. In [41] and Section 6.1 we have shown that algebraic closure does not decide consistency for the $\mathcal{LR}$-calculus, i.e., not every algebraically closed scenario is consistent. Further in [79] it was shown that there cannot be a polynomial time algorithm that can decide consistency in $\mathcal{LR}$ as long as $\mathcal{P} \neq \mathcal{NP}$. In fact the results of [41] and Section 6.1 revealed more than the fact that algebraic closure is just an approximation to consistency for $\mathcal{LR}$. By Proposition 91 it is show that traditional algebraic closure decides absolutely nothing for a wide range of constraint networks for $\mathcal{LR}$. We think we can improve on this and develop a better approximation technique for consistency in $\mathcal{LR}$. We consider such a technique as valuable since we have shown in Chapter 5 that an approximation of consistency just has to be "good enough" for the task at hand. A technique that does not decide anything cannot be "good enough", but let us work on that. We suggest a technique for approximating consistency of $\mathcal{LR}$ scenarios based on the linear properties of triangles in the Euclidean plane.

7.1. Triangle Consistency: An Approximation of Consistency for $\mathcal{LR}$

Our new approach is inspired by the simple observation that if given n points in the Euclidean plane and all $\frac{n \cdot (n-1)}{2}$ undirected lines connecting each point with every other point in such a configuration, the connecting lines between arbitrary 3 points form a (possibly degenerated, if all points lie on the same line) triangle. Because of that for any such triple of points and their connecting lines, all well-known properties of triangles need to be fulfilled. "Left" and "Right" can be distinguished by the orientation (read sectors of a circle) of the involved angles (ref. to. Figure 66). For the scenarios in the relations of l and r algebraic closure performs very badly for $\mathcal{LR}$ as shown in Proposition 91 and therefore scenarios in these relations were the starting point for our research. Further with that approach we did not have to deal with any degenerate triangles. These first tries worked out well and we decided to cover all relations of the calculus facing the problems that arise from different relative directions between the different points in degenerate triangles. The approach restricted to l and r has been published in [40]. Here we present the extension to all $\mathcal{LR}$ relations.

In our new approach, we translate any $\mathcal{LR}$-scenario (which has to contain base relations between all permutations of all triples of distinct points) into a set of inequalities over triangles in the Euclidean plane. We normalize all angles to the interval $(-\pi, \pi]$ to simplify later calculations. Let 3 points A, B and C in relation $(A\,B\;r\;C)$ be given. For such a relation, we get a scenario in the plane as in Figure 66. We can derive that the angle from C to B at A, which we call CAB, is in the open interval $(0, \pi)$. The derivation for $(A\,B\;l\;C)$ is similar and yields

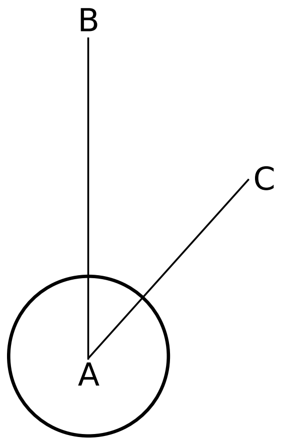


FIGURE 66. $(A\,B\;r\;C)$ in the plane

angles in the interval $(-\pi, 0)$. Similar arguments apply for the rest of the relations. Further, by a simple geometrical argument, we can show that:

Lemma 101. *For a non-degenerate triangle in the Euclidean plane with points A, B, C, if any of the angles BAC, ACB and CBA is in the interval $(0, \pi)$, so are all of the others. The same is true for angles from the interval $(-\pi, 0)$.*

With this and the well-known properties of triangles in the plane, we can derive our system of inequalities `INEQNB(`BAC`)` for any arbitrary angle BAC which is depicted in Figure 67.

In Figure 67 no inequalities for the relations s, e, *dou* and *tri* have been listed, examining these relations is just a comparison for equality of points. Further points in those relations can be substituted in a step preceding the translation of the relations to equalities without modifying the overall satisfiability of the network. Inconsistencies might already become apparent in that substitution step.

The vigilant reader might discover that the system of inequalities `INEQNB(`BAC`)` in Figure 67, especially the inequalities for degenerate triangles, looks somewhat "asymmetrical" but that is already due to our design of pinning $-\pi$ and π to π and to the restriction of the calculation to the interval $(-\pi, \pi]$. This is a simplification in the further calculations since by design all angular values that can occur are different.

To the inequalities `INEQNB(`BAC`)` from Figure 67, we add the ones derived in Lemma 101

$$0 < BAC < \pi \Leftrightarrow 0 < ACB < \pi \Leftrightarrow 0 < CAB < \pi.$$

With them, we obtain the set `INEQN(`BAC`)` for any angle BAC. Such sets of inequalities can be generated for each triple of points in an $\mathcal{LR}$-scenario. By `INEQN` we denote the set of all inequalities for an $\mathcal{LR}$ scenario.

Definition 102. We call an $\mathcal{LR}$-scenario *triangle consistent*, if there is at least one solution for all of its inequalities `INEQN`.

The "compression" of knowledge is done at this point by not considering any lengths of lines. Considering them would yield non-linear inequalities that cannot be solved efficiently. We want to use as little knowledge in this approach as possible to make it computationally efficient:

Theorem 103. *Systems of linear inequalities can be decided in polynomial time.*

PROOF. This follows directly from [3]. □

`Distinction of relations`	
$0 < BAC < \pi$	if $(A\,B\,l\,C)$
$-\pi < BAC < 0$	if $(A\,B\,r\,C)$
$\left.\begin{array}{l} BAC = \pi \\ CBA = 0 \\ ACB = 0 \end{array}\right\}$	if $(A\,B\,b\,C)$
$\left.\begin{array}{l} BAC = 0 \\ CBA = \pi \\ ACB = 0 \end{array}\right\}$	if $(A\,B\,f\,C)$
$\left.\begin{array}{l} BAC = 0 \\ CBA = 0 \\ ACB = \pi \end{array}\right\}$	if $(A\,B\,i\,C)$
`Opposite angles`	
$BAC = -CAB$	if $(A\,B\,l\,C) \vee (A\,B\,r\,C)$
$BAC = CAB$	if $(A\,B\,b\,C) \vee (A\,B\,f\,C) \vee (A\,B\,i\,C)$
`Sum of angles`	
$BAC + CBA + ACB = \pi$	if $(A\,B\,l\,C) \vee (A\,B\,b\,C) \vee (A\,B\,f\,C) \vee (A\,B\,i\,C)$
$BAC + CBA + ACB = -\pi$	if $(A\,B\,r\,C)$
`Adjacent angles`	
$BAC + CAD = BAD + 2 \cdot \pi$	if $\left\{\begin{array}{l} (A\,B\,l\,C) \wedge (A\,C\,l\,D) \wedge (A\,B\,r\,D) \vee \\ (A\,B\,b\,C) \wedge (A\,C\,b\,D) \wedge (A\,B\,f\,D) \vee \\ (A\,B\,b\,C) \wedge (A\,C\,b\,D) \wedge (A\,B\,i\,D) \vee \\ (A\,B\,b\,C) \wedge (A\,C\,l\,D) \wedge (A\,B\,r\,D) \vee \\ (A\,B\,l\,C) \wedge (A\,C\,b\,D) \wedge (A\,B\,r\,D) \end{array}\right.$
$BAC + CAD = BAD - 2 \cdot \pi$	if $\left\{\begin{array}{l} (A\,B\,r\,C) \wedge (A\,C\,r\,D) \wedge (A\,B\,l\,D) \vee \\ (A\,B\,r\,C) \wedge (A\,C\,r\,D) \wedge (A\,B\,b\,D) \end{array}\right.$
$BAC + CAD = BAD$	otherwise

FIGURE 67. `INEQNB`(BAC)

Triangle consistency yields a decision procedure consisting of just two steps:

(1) Translate the $\mathcal{LR}$-scenario to a system of linear inequalities (this is just a substitution in the number of relations contained in the scenario and can clearly be performed in linear time),
(2) Check the solvability of the system with a standard polynomial algorithm. (In fact, also `simplex` can be applied in this step, it has exponential worst-case running time, but often performs very well.)

With Theorem 103 we obtain:

Proposition 104. *Triangle consistency has polynomial running time.*

Each geometric realization of an $\mathcal{LR}$ scenario obviously leads to a system of angles for the involved point triples; it is easy to show that such a system of angles is a solution for the inequalities `INEQN`. We thus arrive at:

Proposition 105. *Consistency implies triangle consistency.*

By Proposition 104 and the fact that deciding consistency is NP-hard due to [79], we obtain that under the assumption $\mathrm{P} \neq \mathrm{NP}$, the converse implication (triangle consistency implies consistency) does not hold. What we still need to find out is how well this approximation works and our experiments so far were very promising.

7.2. Experiments

We have conducted intensive experiments evaluating the quality of our approach of triangle consistency. A big issue about doing experiments with relative orientation calculi is the infinite size of their domain. It is basically impossible to list all possible quantitative positions of points in the Euclidean plane. But for restricted numbers of points an upper bound for the number of qualitatively different configurations can be derived. This still leaves the problem that for a certain number of points (6 in our case) the memory consumption of a program listing those different configurations becomes excessive, as well as the running time.

We have implemented a prototype solver in Haskell for deciding whether a given $\mathcal{LR}$-scenario is triangle consistent. This tool can generate all $\mathcal{LR}$-scenarios in n-points and calculate the corresponding set of inequalities `INEQN`. As the reasoning engine this tool uses the `Yices` SMT-solver [11]. We decided to use an external possibly not completely optimal reasoning engine for the prototype to overcome the issue of programming bugs and intensive debugging. However, the performance of the actual equation solver dominates the running time. So far when comparing to other approaches like Gröbner reasoning the running time of our solver was never the problem. To obtain a set of consistent scenarios, we have written a Haskell program that enumerates n-point $\mathcal{LR}$-scenarios using a grid of $m \times m$ points. This program starts with a specified value of m_0 and calculates all possible $\mathcal{LR}$-scenarios, then it increases the current bound and calculates again. This is continued until the user requests to terminate it. If the list of scenarios from run l and $l+1$ differ, the new list of scenarios is displayed, otherwise a message, that the scenarios are the same. To list all 5 point scenarios completely, it turned out, that a grid of 8×8 points already is sufficient, which can be shown by an involved geometric argument. Unfortunately, for scenarios in 6 points we neither had the computation power nor the memory to enumerate all possible configurations. In fact, the memory usage grows double exponentially with the number of points involved as the running time increases in a triple exponential way.

We also have used the Gröbner reasoner that is available in `SparQ` [77]. It led to the same set of consistent scenarios as our reasoner if the Göbner reasoner managed to get to a decision. However, the exponential runtime of both the grid method and the Gröbner reasoner prevented a computation of all consistent 6 point scenarios. Trying to enumerate them by hand is in vain anyways. For algebraic closure, we used the tool `SparQ` [77]. To examine scenarios with more than 5 points, we implemented a Haskell program that constructs algebraically closed scenarios in up to n points (where n is a given parameter). We let the program run for several days with the parameter n set up to 9 (examining thousands of scenarios). Unfortunately in many cases the Gröbner reasoner was not able to decide consistency, but if it was it gave the same answer as triangle consistency. Candidates for failure of the triangle consistency remained problems where the Gröbner reasoner could not decide consistency, but triangle consistency was fulfilled. We double checked dozens of such scenarios by hand, and could deem all of them being consistent. This gives a first hint at a good approximation, but we cannot be too trustful yet. But checking all scenarios of this kind by hand is in vain, again.

Since we were able to enumerate all consistent all 5-point scenarios in the relations l and r[1], we can quantify how well our method performs on them. We could identify 1955 consistent $\mathcal{LR}$-scenarios (s_{con}) of this kind, algebraic closure with binary composition yields 3095 scenarios (s_{bin}) while ternary algebraic closure leads to 2355 scenarios (s_{ter}). We found exactly 1955 triangle consistent scenarios (s_{Δ}), and when inspecting them, we found out that $s_{con} = s_{\Delta}$. These numbers are

[1]Smaller scenarios are not really interesting, since we can already detect inconsistencies with algebraic closure with ternary composition for 4 point scenarios, all smaller ones do not need additional consideration, since they are at most base relations. The 14 scenarios in 4 points are detected by our method.

depicted in Figure 68. Please note that the numbers of the scenarios are given modulo swapping of the names of the points. As noted above, we could not identify

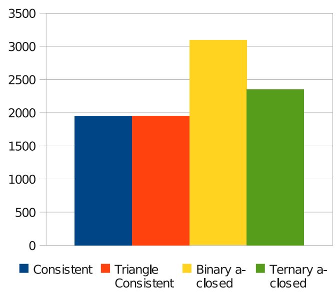


FIGURE 68. Comparison of algebraic closure with triangle consistency

all consistent 6 point scenarios, mainly because of hardware restrictions. But we were still be able to identify a subset of consistent $\mathcal{LR}$-scenarios. So far we were able to classify 429449 scenarios with our approach, of which 4698 are deemed triangle consistent. Not all of them are in the list of our pre-calculated scenarios, but we have to remember that that list is incomplete. We took samples from the list of triangle consistent scenarios that were not in the pre-calculated list, and it turned out that we could find a realization for all of them. Many of those samples had a cloud of points lying close together, with at least one point lying very far away from the others. This is a case that could not be found with our grid method in limits that yield a feasible computation time.

Algebraic closure (binary as well as ternary) failed badly when we had scenarios with one more point than the number of points captured by the composition table; this is fortunately not true for triangle consistency.

All of these experimental results imply that triangle consistency approximates consistency better than binary as well as ternary algebraic closure. Compared to Tarski's quantifier elimination and Gröbner Reasoning, it has a much better running time, since it is a polynomial time algorithm. Since we are only using a prototype implementation of our algorithm and since the used Gröbner reasoner can detect patterns in the equations that imply that it cannot decide consistency, quantifying the running time of the programs is in vain. Indeed, the Gröbner reasoner integrated into `SparQ` often fails to determine consistency for scenarios if the number of points grows. Sometimes it even gives up for quite small scenarios. For scenarios in more than nine or ten points the Gröbner reasoner gives up more often than it can decide. By contrast, triangle consistency works well with scenarios consisting of dozens of points, which is a size that is quite realistic for e.g. robot navigation applications.

So far triangle consistency performed very well in our experiments but by [79] it can only be an approximation to consistency. Hence, we still need to construct a scenario that is triangle consistent but not consistent. As such a scenario we take again the Pappus configuration from Figure 65 which we could use to construct an inconsistent $\mathcal{DRA}_{fp}$ scenario that is algebraically closed. Since the semantics of $\mathcal{LR}$ and $\mathcal{DRA}$ (see Section 3.1.2 and Section 3.2.3) are very similar and since Pappus' Hexagon Theorem depends on the side lengths of the triangles the configuration can be composed into, we think that trying the Pappus' configuration as a counterexample is quite natural. The Pappus configuration from Figure 65 can be described in $\mathcal{LR}$ as the "Pappus scenario" `PSCEN` where only one representative for any of the six permutations is given for any triple of points. The permutations that are not given can be calculated using the permutation table of $\mathcal{LR}$ without loss of information:

$$\begin{array}{lll} \texttt{PSCEN} & := & \{(A\,B\;f\;C)(A\,B\;l\;D)\;(A\,B\;l\;E)\;(A\,B\;l\;F)\;(A\,B\;l\;G)\;(A\,B\;l\;H) \\ & & (A\,B\;l\;I)\;\;(A\,C\;l\;D)\;(A\,C\;l\;E)\;(A\,C\;l\;F)\;(A\,C\;l\;G)\;(A\,C\;l\;H) \\ & & (A\,C\;l\;I)\;\;(A\,D\;f\;H)(A\,D\;l\;G)\;(A\,D\;r\;I)\;(A\,D\;r\;F)(A\,D\;r\;E) \\ & & (A\,E\;f\;I)\;\;(A\,E\;l\;G)\;(A\,E\;r\;F)\;(A\,H\;r\;I)\;(A\,H\;r\;F)(A\,H\;r\;E) \\ & & (B\,D\;f\;G)\;(B\,D\;r\;I)\;(B\,D\;r\;H)(B\,D\;r\;E)(B\,D\;r\;C)(B\,F\;f\;I) \\ & & (B\,F\;l\;H)\;(B\,F\;l\;G)\;(B\,F\;l\;E)\;(B\,F\;l\;D)(B\,F\;r\;C)(B\,G\;r\;E) \\ & & (B\,G\;r\;C)\;(B\,I\;r\;C)\;(B\,I\;l\;H)\;(B\,I\;l\;E)\;(C\,E\;f\;G)(C\,E\;l\;D) \\ & & (C\,E\;l\;B)\;(C\,E\;r\;F)\;(C\,E\;r\;H)(C\,E\;r\;I)\;(C\,H\;l\;B)(C\,I\;l\;D) \\ & & (D\,F\;l\;G)\;(D\,F\;l\;H)(D\,F\;l\;I)\;(D\,F\;r\;C)(D\,G\;r\;C)(D\,G\;r\;E) \\ & & (D\,H\;r\;C)\;(D\,H\;r\;E)(E\,I\;l\;D)\;(F\,I\;l\;A)\;(F\,I\;r\;C)\;(F\,I\;l\;E) \\ & & (G\,F\;r\;A)\;(G\,F\;r\;C)(G\,F\;r\;E)(G\,H\;f\;I)(G\,H\;r\;F)(G\,H\;r\;E) \\ & & (G\,H\;r\;D)\;(G\,H\;r\;C)(G\,H\;r\;B)(G\,H\;r\;A)(G\,I\;r\;F)\;(G\,I\;r\;E) \\ & & (G\,I\;r\;D)\;(G\,I\;r\;C)\;(G\,I\;r\;B)\;(G\,I\;r\;A)\;(H\,B\;l\;E)(H\,F\;f\;C) \\ & & (H\,F\;r\;E)\;(H\,I\;r\;C)\;(H\,I\;r\;D)\;(H\,I\;r\;E)\;(H\,I\;r\;F) \\ & & (D\,F\;i\;E)\} \end{array}$$

Because of Pappus' Hexagon Theorem [7] the relation $(D\,F\;i\;E)$ must hold in this scenario to make it consistent. For our counterexample we exchange this relation with $(D\,F\;r\;E)$ violating Pappus' Hexagon Theorem since the point D, E and F cannot lie on a line anymore with that constraint we call this inconsistent scenario `ISCEN`. Formally `ISCEN` is defined as

$$\texttt{ISCEN} := (\texttt{PSCEN} \setminus \{(D\,F\;i\;E)\}) \cup \{(D\,F\;r\;E)\}\,.$$

The scenario `ISCEN` is algebraically closed and triangle consistent, but inconsistent. Algebraic closure can be enforced using **`SparQ`** [78] while triangle consistency can be verified using our prototype tool or by translating the relations to inequalities and solving the system of linear inequalities by any means. Hence we have found our separating example between consistency and triangle consistency.

In our experiments triangle consistency performed very well as an approximation of consistency for $\mathcal{LR}$. It performed that well that finding a separating example between consistency and triangle consistency turned out to be a very hard task partly due to its good performance partly since it is new and not as well understood as algebraic closure. Our main issue in evaluating this approach is that no "benchmarking" data is available that is there are not many big scenarios with a known state of consistency and there are no suited tools available we could use in our experiments. For smaller scenarios the Gröbner reasoner turned out to be very valuable but when trying to examine big ones, the Gröbner reasoner failed to decide in most cases in acceptable time or just gave up. Lee and Wolter are working on a reasoner based on cylindrical algebraic decomposition and we hope that that reasoner will enable us to gain even deeper insights into our approximation for big scenarios and constraint networks. Cylindrical algebraic decomposition can solve the systems of non-linear (in)equalities that form the semantics of many qualitative spatial calculi, but depending on the particular projection the running time can be high and reasoning might take too long for on-line use. This is where a good approximation, i.e. an approximation that is "good enough" for the task at hand, with a low running time enters the stage as for $\mathcal{LR}$ our triangle consistency.

CHAPTER 8

Homomorphisms and Quotient Calculi

Everyday I look into the mirror
Staring back I look less familiar
—**Slayer**, *Seven Faces.*

This chapter is based on still unpublished work [57]. In Proposition 47 we have given an embedding of Allen's Interval Algebra $\mathcal{IA}$ [1] into $\mathcal{DRA}$. At that point we did not establish the theory for such an embedding of a qualitative spatial calculus, but we will make up leeway in this chapter by investigating homomorphisms between qualitative spatial calculi. Homomorphisms between calculi are interesting since they can be used to derive a calculus from another one making it easy to adapt a calculus to a certain application scenario or to compare properties of calculi.

8.1. Relation Algebras for Spatial Reasoning Revisited

From Section 2.1 we recall the theory behind binary qualitative spatial calculi, especially the definition of a non-associative algebra (see Definition 5). The elements of such an algebra will be called (abstract) relations. We are mainly interested in finite non-associative algebras that are *atomic*, which means that there is a set of pairwise disjoint minimal relations, called base relations, and all relations can be obtained as unions of these. Then, the following fact is well-known and easy to prove:

Proposition 106 ([10])**.** *An atomic non-associative algebra is uniquely determined by its set of base-relations, together with the converses and compositions of base-relations. (Note that the composition of two base-relations is in general not a base-relation.)*

When discussing qualitative spatial calculi as non-associative algebras a good point to start with are partition schemes which were introduced in Definition 4. Partition schemes capture the $JPED$ property by their definition and enforce the existence of the diagonal relation and converses. In the remainder of this chapter we shall often denote partition schemes as

$$\mathcal{D} \times \mathcal{D} = \bigcup_{1 \le i \le k} B_i$$

with $\mathcal{D}$ being a non-empty domain and $\mathcal{B} = \{B_1, \ldots, B_k\}$ the set of the relations of the partition scheme over $\mathcal{D}$. Sometimes be will index the relations by a set I.

We remember from Section 2.2 that we can obtain a non-associative algebra given the following steps:

Proposition 107 ([39, 56])**.** *Given a partition scheme*

$$\mathcal{D} \times \mathcal{D} = \bigcup_{i \in I} B_i$$

we obtain a non-associative algebra as follows: the Boolean algebra component is $\mathcal{P}(I)$*. The converse is given by pointwise application of* $(\_)^{\smile}$*; one relation* B_i *is the diagonal which we denote by* i_0*. Composition is given by weak composition as* *defined in Section 2.2. We abuse the notation slightly, since strictly speaking* $(\_)^{\smile}$ *and* $\diamond$ *act on the* B_i *and not the the* $i \in I$.

We now introduce several qualitative calculi by just giving their domain $\mathcal{D}$ and their set of base relations; diagonal and converse are clear. Obtaining the correct (weak) composition operation unfortunately still stays a problem in practice.

Example 108. The most prominent temporal calculus is Allen's interval algebra [1], which describes possible relations between intervals in linear flows of time. An interval is a pair (s, t) of real numbers such that $s < t$. The 13 base relations between such intervals are depicted in Figure 12 to be found in Section 2.4.

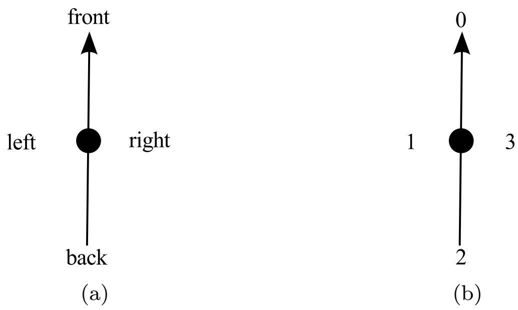


FIGURE 69. Named and numbered $\mathcal{OPRA}_1$ relations

Example 109. The $\mathcal{OPRA}_1^\star$ calculus [12], a refinement of the $\mathcal{OPRA}_1$ calculus as defined in Section 3.3, is based on the domain of oriented points in Euclidean plane. We remember that an oriented points is a point in the plane described by its coordinates and enriched with orientation information. Figure 69a depicts an oriented point and the corresponding division of the plane, the base frame, into the regions front, left, right and back and same (the latter stands for the point itself) which correspond to the classical numbered $\mathcal{OPRA}_1$ relations shown in Figure 69b in the obvious way. The relation between two oriented points in different positions is determined by considering the position of the second w.r.t. the first and vice versa. Using front, left, right and back, this leads to 16 base relations: FRONTfront, FRONTleft, FRONTright, FRONTback, LEFTfront, LEFTleft, LEFTright, LEFTback, RIGHTfront, RIGHTleft, RIGHTright, RIGHTback, BACKfront, BACKleft, BACKright, and BACKback. Moreover, if the positions both points coincide, we can compare there orientations. This leads to the relations SAMEfront, SAMEleft, SAMEright and SAMEback.

Finally, the relations RIGHTright, RIGHTleft, LEFTleft, and LEFTright are refined by marking them with letters '+' or '−', 'P' or 'A', according to whether the two orientations of the oriented points are positive, negative, parallel or anti-parallel, as in Figure 70. In that figure the dots denote the position of the oriented points while the arrows visualize their orientations. Note that this figure is very similar to Figure 25 as is the process of deriving $\mathcal{DRA}_{fp}$ from $\mathcal{DRA}_f$ to the process of deriving $\mathcal{OPRA}_1^\star$ from $\mathcal{OPRA}_1$. LEFTleft is refined into LEFTleftA, LEFTleft+ and LEFTleft-. LEFTright is refined into LEFTrightP, LEFTright+ and LEFTright-. RIGHTright is refined into RIGHTrightA, RIGHTright+ and RIGHTright-. RIGHTleft is refined into RIGHTleftP, RIGHTleft+ and RIGHTleft-. The remaining four options LEFTleftP, LEFTrightA, RIGHTrightP and RIGHTleftA are geometrically impossible. In fact the distinction between parallelism and anti-parallelism is already determined by the underlying relations. In an adventurous step one could even unite the two symbols A and P into a single one. But we prefer to use both symbols to stay compatible to [12]. Altogether, we obtain a set of 28 base relations.

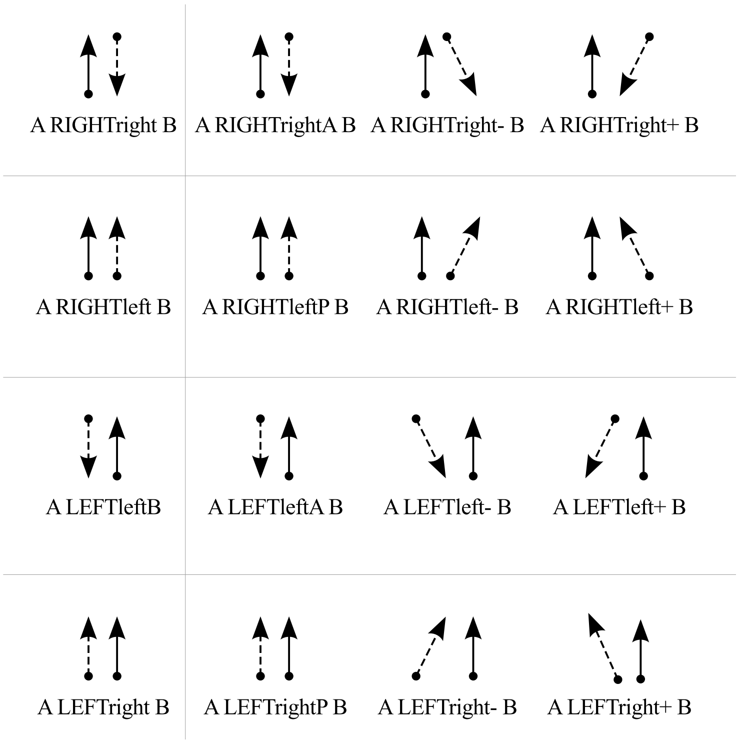


FIGURE 70. Refined base relations in a named $\mathcal{OPRA}_1^\star$

Example 110. We have introduced the dipole calculi $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ in Section 3.2. We remember that these calculi are based on binary relations over dipoles, i.e. directed line segments of non-zero length, in the Euclidean plane. $\mathcal{DRA}_f$ is a non-associative algebra with 72 base relations, and $\mathcal{DRA}_{fp}$, a refinement of $\mathcal{DRA}_f$ similar to the one in Example 109, is even a relation algebra (see Theorem 84) with 80 base relations. For both calculi sese is the identity relation.

8.2. Homomorphisms and Weak Representations

After the introduction of non-associative algebras in Section 2.2 and after having regarded several calculi as such in Section 8.1 we are now at the point to scrutinize them formally with respect to our theory. Firstly, we consider the notions of several kinds of homomorphisms in non-associative algebras.

For non-associative algebras, we define lax homomorphisms, which allow for both the embedding of a calculus into its domain, as well as relating several calculi to each other.

Definition 111 (Lax homomorphism)**.** Given non-associative algebras A and B, a *lax homomorphism* is a homomorphism $h\colon A \longrightarrow B$ on the underlying Boolean algebras such that:

- $h(\Delta_A) \geq \Delta_B$
- $h(a^\smile) = h(a)^\smile$ for all $a \in A$

- $h(a \diamond b) \geq h(a) \diamond h(b)$ for all $a, b \in A$.

Dually to lax homomorphisms, we can define oplax homomorphisms, which enable us to define projections from one calculus to another. The terminology is motivated by that for monoidal functors.

Definition 112 (Oplax homomorphism)**.** Given non-associative algebras A and B, an *oplax homomorphism* is a homomorphism $h: A \longrightarrow B$ on the underlying Boolean algebras such that:

- $h(\Delta_A) \leq \Delta_B$
- $h(a^{\smile}) = h(a)^{\smile}$ for all $a \in A$
- $h(a \diamond b) \leq h(a) \diamond h(b)$ for all $a, b \in A$

A *proper homomorphism* (sometimes just called a homomorphism) of non-associative algebras is a homomorphism that is lax and oplax at the same time; the above inequalities then turn into equations.

A homomorphism between complete atomic non-associative algebras can be given by its action on base-relations; it is extended to general relations by

$$h(\bigvee_{i \in I} B_i) = \bigvee_{i \in I} h(B_i),$$

where $\bigvee$ is arbitrary (possibly infinite) join and the B_i denote base relations. In the sequel, we will always define homomorphisms in this way.

A first important application of homomorphisms is their use in the definition of qualitative calculus. Ligozat and Renz [39] define a qualitative calculus in terms of a so-called *weak representation* [38] which definition we took up in Definition 11. Now we reformulate this definition in terms of our newly introduced homomorphisms:

Definition 113 (Weak representation)**.** A *weak representation* (see also Definition 11) is an identity-preserving (i.e. $\varphi(\Delta_A) = \Delta_B$) lax homomorphism φ from a (finite atomic) non-associative algebra into the relation algebra of a domain $\mathcal{D}$. The latter is given by the canonical relation algebra on the powerset $\mathcal{P}(\mathcal{D} \times \mathcal{D})$, where identity, converse and composition (as well as the Boolean algebra operations) are given by their set-theoretic interpretations.

Example 114. Let $\mathbb{D}$ be the set of all dipoles over $\mathbb{R}^2$. Then the weak representation of $\mathcal{DRA}_{fp}$ is the lax homomorphism $\varphi_f: \mathcal{DRA}_{fp} \longrightarrow \mathcal{P}(\mathbb{D} \times \mathbb{D})$ given by

$$\varphi_f(b) = b.$$

Here, the left b is an element of the abstract relation algebra, while the right b is the set-theoretic extension as a relation. Since we have chosen to use set-theoretic relations themselves as elements of the relation algebra, here both are the same.

The following is straightforward:

Proposition 115. *A calculus has a strong composition if and only if its weak representation is a proper homomorphism.*

Proof. Since a weak representation is identity-preserving, being proper means that $\varphi(R_1 \diamond R_2) = \varphi(R_1) \circ \varphi(R_2)$, which is exactly the strength of the composition. □

The following is straightforward [38]:

Proposition 116. *A weak representation φ is injective if and only if $\varphi(b) \neq \varnothing$ for each base-relation b.*

The second main application of homomorphisms is relating different calculi. For example, the algebra over Allen's interval relations [1] can be embedded into

$\mathcal{DRA}_{fp}$ via a homomorphism. Now our theory developed allows us to prove the facts stated in Proposition 47, but we shall repeat the mapping from back there in Proposition 117.

Proposition 117. *A homomorphism from Allen's interval algebra to $\mathcal{DRA}_{fp}$ exists and is given by the following mapping of base-relations.*

=	↦	sese			
b	↦	ffbb	bi	↦	bbff
m	↦	efbs	mi	↦	bsef
o	↦	ifbi	oi	↦	biif
d	↦	bfii	di	↦	iibf
s	↦	sfsi	si	↦	sisf
f	↦	beie	fi	↦	iebe

Proof. The identity relation = is clearly mapped to the identity relation sese. For the composition and converse properties, we just inspect the composition and converse tables for the two calculi. The mapping of the base-relation is then lifted directly to a mapping of all relations, where the map is applied component-wise on the relations. Using the laws of non-associative algebras, the homomorphism property of these relations follows from that of the base-relations. □

In cases stemming from the embedding of Allen's Interval Algebra, the dipoles lie on the same straight lines and have the same direction. $\mathcal{DRA}_{fp}$ also contains 13 additional relations which correspond to the case with dipoles lying on a line but facing opposite directions.

As we shall see, it is very useful to extend the notion of homomorphisms to weak representations:

Definition 118. Given weak representations $\varphi: A \to \mathcal{P}(\mathcal{D}\times\mathcal{D})$ and $\psi: B \to \mathcal{P}(\mathcal{V}\times\mathcal{V})$, a *lax (oplax, proper) homomorphism of weak representations* $(h,i): \varphi \to \psi$ is given by

- a proper homomorphism of non-associative algebras $h: A \to B$, and
- a map $i: \mathcal{D} \to \mathcal{V}$, such that the diagram

$$\begin{array}{ccc} A & \xrightarrow{\varphi} & \mathcal{P}(\mathcal{D}\times\mathcal{D}) \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle \mathcal{P}(i\times i)} \\ B & \xrightarrow[\psi]{} & \mathcal{P}(\mathcal{V}\times\mathcal{V}) \end{array}$$

commutes laxly (respectively oplaxly, properly). Here, lax commutation means that for all $R \in A$, $\psi(h(R)) \subseteq \mathcal{P}(i\times i)(\varphi(R))$, oplax commutation means the same with $\supseteq$, and proper commutation with =. Note that $\mathcal{P}(i\times i)$ is the obvious extension of i to a function between relation algebras; further note that (unless i is bijective) this is not even a homomorphism of Boolean algebras (it may fail to preserve top, intersections and complements), although it satisfies the oplaxness property (and the laxness property if i is injective).[1]

[1]The reader with background in category theory may notice that the categorically more natural formulation would use the contravariant powerset functor, which yields homomorphisms of Boolean algebras, see also [56]. However, the present formulation fits better with the examples.

Ligozat [38] defines a more special notion of morphism between weak representations; it corresponds to our notion of oplax homomorphism of weak representations where the component h is the identity.

Example 119. The homomorphism from Proposition 117 can be extended to an oplax homomorphism of weak representations by letting i be the embedding of time intervals to dipoles on the x-axis.

Definition 120. A homomorphism of non-associative algebras is said to be a *quotient homomorphism*[2] if it is proper and surjective. A (lax, oplax or proper) homomorphism of weak representations is a quotient homomorphism if it is surjective in both components.

The easiest way to form a quotient of a weak representation is via an equivalence relation on the domain:

Definition 121. Given a weak representation $\varphi: A \to \mathcal{P}(\mathcal{D} \times \mathcal{D})$ and an equivalence relation $\sim$ on $\mathcal{D}$, we obtain the *quotient representation* $\varphi/\!\sim$ as follows:

$$\begin{array}{ccc} A & \xrightarrow{\quad\varphi\quad} & \mathcal{P}(\mathcal{D} \times \mathcal{D}) \\ \Big\downarrow q_A & & \Big\downarrow \mathcal{P}(q \times q) \\ A/\!\sim_A & \xrightarrow[\varphi/\sim]{\quad\quad} & \mathcal{P}(\mathcal{D}/\!\sim \times \mathcal{D}/\!\sim) \end{array}$$

- Let $q: \mathcal{D} \to \mathcal{D}/\!\sim$ be the factorization of $\mathcal{D}$ by $\sim$;
- q extends to relations: $\mathcal{P}(q \times q): \mathcal{P}(\mathcal{D} \times \mathcal{D}) \to \mathcal{P}(\mathcal{D}/\!\sim \times \mathcal{D}/\!\sim)$;
- let $\sim_A$ be the congruence relation on A generated by
  $$\mathcal{P}(q \times q)(\varphi(b_1)) \cap \mathcal{P}(q \times q)(\varphi(b_2)) \neq \emptyset \;\Rightarrow\; b_1 \sim_A b_2$$
  for base-relations $b_1, b_2 \in A$. The relation $\sim$ is called *regular w.r.t.* φ if $\sim_A$ is the kernel of $\mathcal{P}(q \times q) \circ \varphi$ (i.e. the set of all pairs made equal by $\mathcal{P}(q \times q) \circ \varphi$);
- let $q_A: A \to A/\!\sim_A$ be the quotient of A by $\sim_A$ in the sense of universal algebra [26], which uses proper homomorphisms; hence, q_A is a proper homomorphism;
- finally, the function $\varphi/\!\sim$ is defined as
  $$\varphi/\!\sim(R) = \mathcal{P}(q \times q)(\varphi(q_A^{-1}(R))).$$

Now we revisit the definition of quotient in Definition 121 from the point of view from homomorphisms of non-associative algebras:

Proposition 122. *The function $\varphi/\!\sim$ defined in Definition 121 is a lax homomorphism of non-associative algebras.*

Proof. To show this, notice that an equivalent definition works on the base-relations of $A/\!\sim_A$:

$$\varphi/\!\sim(R) = \bigcup_{b \in R} \mathcal{P}(q \times q)(\varphi(q_A^{-1}(b))).$$

It is straightforward to show that bottom and joins are preserved; since q is surjective, also top is preserved.
Concerning meets, since general relations in $A/\!\sim_A$ can be considered to be sets of

[2] For more information on this subject, refer to a textbook on universal algebra, e.g. [26]. Maddux [46] introduces a notion of quotient that is not helpful here.

base-relations, it suffices to show that $b_1 \wedge b_2 = 0$ implies $\mathcal{P}(q \times q)(\varphi(q_A^{-1}(b_1))) \cap \mathcal{P}(q \times q)(\varphi(q_A^{-1}(b_2))) = \varnothing$. Assume to the contrary that $\mathcal{P}(q \times q)(\varphi(q_A^{-1}(b_1))) \cap \mathcal{P}(q \times q)(\varphi(q_A^{-1}(b_2))) \neq \varnothing$. Then already $\mathcal{P}(q \times q)(\varphi(b_1')) \cap \mathcal{P}(q \times q)(\varphi(b_2')) \neq \varnothing$ for base-relations $b_i' \in q_A^{-1}(b_i)$, $i = 1, 2$. But then $b_1' \sim_A b_2'$, hence $q_A(b_1') = q_A(b_2') \leq b_1 \wedge b_2$, contradicting $b_1 \wedge b_2 = 0$. Preservation of complement follows from this.
Using properness of the quotient, it is then easily shown that the relation algebra part of the lax homomorphism property carries over from φ to $\varphi/\!\!\sim$: Concerning composition, by surjectivity of q_A, we know that any given relations $R_1, R_2 \in A/\!\!\sim_A$ are of the form $R_1 = q_A(S_1)$ and $R_2 = q_A(S_2)$. Hence, $\varphi/\!\!\sim (R_1 \diamond R_2) = \varphi/\!\!\sim (q_A(S_1) \diamond q_A(S_2)) = \varphi/\!\!\sim (q_A(S_1 \diamond S_2)) = \mathcal{P}(q \times q)(\varphi(S_1 \diamond S_2)) \geq \mathcal{P}(q \times q)(\varphi(S_1) \diamond \varphi(S_2)) = \mathcal{P}(q \times q)(\varphi(S_1)) \diamond \mathcal{P}(q \times q)(\varphi(S_2)) = \varphi/\!\!\sim (q_A(S_1)) \diamond \varphi/\!\!\sim (q_A(S_2)) = \varphi/\!\!\sim (R_1) \diamond \varphi/\!\!\sim (R_2)$. The inequality for the identity is shown similarly. □

Proposition 123. $(q_A, q){:}\, \varphi \to \varphi/\!\!\sim$ *is an oplax quotient homomorphism of weak representations. If* $\sim$ *is regular w.r.t.* φ*, then the quotient homomorphism is proper and satisfies the following universal property: if* $(q_B, i){:}\, \varphi \to \psi$ *is another oplax homomorphism of weak representations with* ψ *injective and* $\sim \subseteq ker(i)$*, then there is a unique oplax homomorphism of weak representations* $(h, k){:}\, \varphi/\!\!\sim \to \psi$ *with* $(q_B, i) = (h, k) \circ (q_A, q)$.

Proof. The oplax homomorphism property for (q_A, q) is $\mathcal{P}(q \times q) \circ \varphi \subseteq \varphi/\!\!\sim \circ q_A$, which by definition of $\varphi/\!\!\sim$ amounts to

$$\mathcal{P}(q \times q) \circ \varphi \subseteq \mathcal{P}(q \times q) \circ \varphi \circ q_A^{-1} \circ q_A,$$

which follows from surjectivity of q. Regularity of $\sim$ w.r.t. φ means that $\sim_A$ is the kernel of $\mathcal{P}(q \times q) \circ \varphi$, which turns the above inequality into an equality. Concerning the universal property, let $(q_B, i){:}\, \varphi \to \psi$ with the mentioned properties be given. Since $\sim \subseteq ker(i)$, there is a unique function $k{:}\, \mathcal{D}/\!\!\sim \to \mathcal{V}$ with $i = k \circ q$. The homomorphism h we are looking for is determined uniquely by $h(q_A(b)) = q_B(b)$; this also ensures the proper homomorphism property. All that remains to be shown is well-definedness. Suppose that $b_1 \sim_A b_2$. By regularity, $\mathcal{P}(q \times q)(\varphi(b_1)) = \mathcal{P}(q \times q)(\varphi(b_2))$. Hence also $\mathcal{P}(i \times i)(\varphi(b_1)) = \mathcal{P}(i \times i)(\varphi(b_2))$ and $\psi(q_B(b_1)) = \psi(q_B(b_2))$. By injectivity of ψ, we get $q_B(b_1) = q_B(b_2)$. □

After having introduced a lot of formal machinery, we are going to have a look at a first example that uses it now.

Example 124. $\mathcal{OPRA}_1^\star$ is a quotient of $\mathcal{DRA}_{fp}$. At the level of non-associative algebras, the quotient is given by the table in Figure 71. At the level of domains, it acts as follows: Given dipoles $d_1, d_2 \in \mathbb{D}$, let $d_1 \sim d_2$ denote that d_1 and d_2 have the same start point and point into the same direction. (This is regular w.r.t. φ_f.) Then $\mathbb{D}/\!\!\sim$ is the domain $\mathbb{OP}$ of oriented points in $\mathbb{R}^2$. See Figure 72.

This way of constructing $\mathcal{OPRA}_1^\star$ by a quotient gives us their converse and composition tables for no extra effort; we can obtain them by applying the respective congruences to the tables for $\mathcal{DRA}_{fp}$, respectively. Moreover, the next result shows that we also can use quotients to transfer an important property of calculi.

Proposition 125. *Quotient homomorphisms of weak representations that are bijective in the second component preserve strength of composition.*

Proof. Let $(h, i){:}\, \varphi \to \psi$ with $\varphi{:}\, A \to \mathcal{P}(\mathcal{D} \times \mathcal{D})$ and $\psi{:}\, B \to \mathcal{P}(\mathcal{V} \times \mathcal{V})$ be a quotient homomorphism of weak representations. According to Proposition 115, the strength of the composition is equivalent to φ (respectively ψ) being a proper homomorphism. We assume that φ is a proper homomorphism and need to show that ψ is proper

{llllA}	↦	LEFTleftA
{llll+, lllb+, lllr+}	↦	LEFTleft+
{lrll, lbll}	↦	LEFTleft−
{ffff, eses, fefe, fifi, ibib, fbii, fsei, ebis, iifb, eifs, iseb}	↦	FRONTfront
{bbbb}	↦	BACKback
{llbr}	↦	LEFTback
{llfl, lril, lsel}	↦	LEFTfront
{llrrP}	↦	LEFTrightP
{llrr+}	↦	LEFTright+
{llrf, llrl, llrr−, lfrr, lrrr, lere, lirl, lrri, lrrl}	↦	LEFTright−
{rrrrA}	↦	RIGHTrightA
{rrrr+, rbrr, rlrr}	↦	RIGHTright+
{rrrr−, rrrl, rrrb}	↦	RIGHTright−
{rrllP}	↦	RIGHTleftP
{rrll+, rrlr, rrlf, rlll, rfll, rllr, rele, rlli, rilr}	↦	RIGHTleft+
{rrll−}	↦	RIGHTleft−
{rrbl}	↦	RIGHTback
{rrfr, rser, rlir}	↦	RIGHTfront
{ffbb, efbs, ifbi, iibf, iebe}	↦	FRONTback
{frrr, errs, irrl}	↦	FRONTright
{flll, ells, illr}	↦	FRONTleft
{blrr}	↦	BACKright
{brll}	↦	BACKleft
{bbff, bfii, beie, bsef, biif}	↦	BACKfront
{slsr}	↦	SAMEleft
{sese, sfsi, sisf}	↦	SAMEfront
{sbsb}	↦	SAMEback
{srsl}	↦	SAMEright

FIGURE 71. Mapping from $\mathcal{DRA}_{fp}$ to $\mathcal{OPRA}_1^\star$ relations

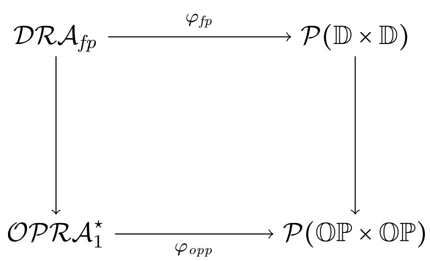


FIGURE 72. Quotient homomorphism of weak representations from $\mathcal{DRA}_{fp}$ to $\mathcal{OPRA}_1^\star$

as well. We also know that h and $\mathcal{P}(i \times i)$ are proper. Let R_2, S_2 be two abstract relations in B. By surjectivity of h, there are abstract relations $R_1, S_1 \in A$ with $h(R_1) = R_2$ and $h(S_1) = S_2$. Now $\psi(R_2 \diamond S_2) = \psi(h(R_1) \diamond h(S_1)) = \psi(h(R_1 \diamond S_1)) = \mathcal{P}(i \times i)(\varphi(R_1 \diamond S_1)) = \mathcal{P}(i \times i)(\varphi(R_1)) \diamond \mathcal{P}(i \times i)(\varphi(S_1)) = \psi(h(R_1)) \diamond \psi(h(S_1)) = \psi(R_2) \diamond \psi(S_2)$, hence ψ is proper. □

The method how to compute the composition table for $\mathcal{OPRA}_1^\star$ directly is described in [12] and a reference composition table is provided with the tool `SparQ` [78]. In the course of checking the isomorphism properties between the quotient of $\mathcal{DRA}_{fp}$ and $\mathcal{OPRA}_1^\star$, we discovered errors in 197 entries of the composition table of $\mathcal{OPRA}_1^\star$ as it was shipped with the qualitative reasoner `SparQ` [78]. This emphasizes our point how important it is to develop a sound mathematical theory to compute a composition table and to stay as close as possible with the implementation to the theory as done for $\mathcal{DRA}$ in Chapter 4. Further it shows how important it is to have means to verify composition tables directly or if possible via quotients. In the original composition table for $\mathcal{OPRA}_1^\star$ it was claimed that

$$\begin{aligned}\text{SAMEright} \diamond \text{RIGHTrightA} \quad \Longrightarrow \quad & \{\text{LEFTright+}, \text{LEFTrightP}, \text{LEFTright-},\\ & \text{BACKright}, \text{RIGHTright+},\\ & \text{RIGHTrightA}, \text{RIGHTright-}\}\end{aligned}$$

So the abstract composition SAMEright ⋄ RIGHTrightA contains the base-relation LEFTrightP, which however is not supported geometrically. Consider three oriented points o_A, o_B and o_C with o_A SAMEright o_B and o_B RIGHTrightA o_C, as depicted

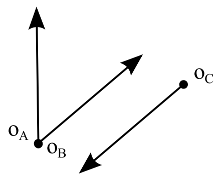


FIGURE 73. $\mathcal{OPRA}_1^\star$ configuration

in Figure 73. For the relation o_A LEFTrightP o_C to hold, the carrier rays of o_A and o_C need to be parallel, but because of o_B RIGHTrightA o_C, the carrier rays of o_B and o_C and hence also those of o_A and o_B need to be parallel as well. Since the start point of o_A and o_B coincide, this can only be achieved, if o_A and o_B are collinear, which is a contradiction to o_A SAMEright o_B.

Altogether, we get the following chain of calculi (weak representations) and homomorphisms:

$$\mathcal{IA} \xrightarrow{\text{oplax}} \mathcal{DRA}_{fp} \xrightarrow{\text{oplax quotient}} \mathcal{OPRA}_1^\star$$

8.3. Constraint Reasoning

We remember the definition of constraint-networks from Definition 16 and atomic constraint network from Definition 17.

Given a constraint network $\mathcal{N}$, e.g. over $\mathcal{DRA}$, an important reasoning problem is to decide whether $\mathcal{N}$ is *consistent*, i.e., whether there is an assignment of all variables of $\mathcal{N}$ with dipoles such that all constraints are satisfied (a *solution*). We call this problem DSAT. DSAT is a Constraint Satisfaction Problem (CSP) [43]. We rely on relation algebraic methods to check consistency, namely the path consistency algorithm [47]. For non-associative algebras, the abstract composition of relations need not coincide with the (associative) set-theoretic composition. Hence, in this case, the standard path-consistency algorithm does not necessarily lead to path consistent networks, but only to algebraically closed ones [63]. Algebraic closure has been defined in Definition 23 and been explained in the text following that definition. We want to stress again that in general, algebraic closure is only a one-sided approximation of consistency: if algebraic closure detects an inconsistency, then we are sure that the constraint network is inconsistent; however, algebraic closure may fail to detect some inconsistencies: an algebraically closed network is not necessarily consistent. For some calculi, like Allen's interval algebra, algebraic closure is known to exactly decide consistency of scenarios, for others it does not, see [63], where it is also shown that this question is completely orthogonal to the question whether the composition is strong. Examples of calculi where algebraic closure does not decide consistency even in scenarios are $\mathcal{DRA}_f$ (ref. to Section 6.2), $\mathcal{DRA}_{fp}$ (ref. to Section 6.2) and $\mathcal{LR}$ (ref. to Section 6.1).

Constraint networks can be translated along homomorphisms of non-associative algebras as follows: Given $h: A \longrightarrow B$ and $\nu: N \times N \longrightarrow A$, let $h(\nu)$ be the composition $h \circ \nu$. Fortunately, it turns out that oplax homomorphisms preserve algebraic closure.

Proposition 126. *Given non-associative algebras A and B, an oplax homomorphism $h: A \longrightarrow B$ preserves algebraic closure. If h is injective, it also reflects algebraic closure.*

Proof. Since an oplax homomorphism is a homomorphism between Boolean algebras, it preserves the order. So for any three relations for $X_1 \mathrm{R}_1 X_2,\ X_2 \mathrm{R}_2 X_3,\ X_1 \mathrm{R}_3 X_3$ in the algebraically closed constraint network over A, with

$$R_3 \leq R_1 \diamond R_2$$

the preservation of the order implies:

$$h(R_3) \leq h(R_1 \diamond R_2).$$

Applying the oplaxness property yields:

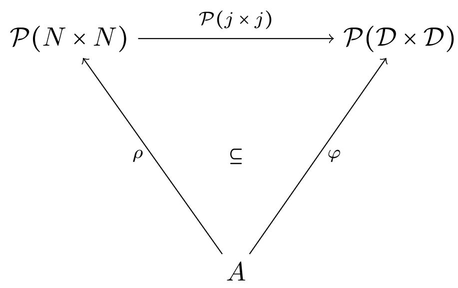
$$h(R_3) \leq h(R_1) \diamond h(R_2).$$

and hence the image of the constraint network under h is also algebraically closed. If h is injective, it reflects equations and inequalities, and the converse implication follows. □

Given a scenario $\nu: N \times N \to A$, following [63], we can reorganize it as a function $\rho: A \to \mathcal{P}(N \times N)$ by defining $\rho(b) = \{(X, Y) \in N \times N \mid \nu(X, Y) = b\}$ for base relations b and extending this to all relations using joins as usual. Note that ρ is a weak representation if and only if the scenario is algebraically closed and normalized.

For atomic homomorphisms (i.e. those mapping atoms to atoms), the translation of constraint networks can be lifted to scenarios represented as $\rho: A \to \mathcal{P}(N \times N)$ using the above correspondence, we then obtain $h(\rho): B \to \mathcal{P}(N \times N)$.

Definition 127 (Solution)**.** Given a scenario $\rho: A \to \mathcal{P}(N \times N)$, a *solution* for ρ in a weak representation $\varphi: A \to \mathcal{P}(\mathcal{D} \times \mathcal{D})$ is a function $j: N \to \mathcal{D}$ such that for all $R \in A$ the property $\mathcal{P}(j \times j)(\rho(R)) \subseteq \varphi(R)$ which is equivalent to $\mathcal{P}(j \times j) \circ \rho \subseteq \varphi$ holds. For better lucidity, we depict this property as a diagram:



Proposition 128. *Oplax homomorphisms of weak representations preserve solutions for scenarios.*

Proof. Let weak representations $\varphi: A \to \mathcal{P}(\mathcal{D} \times \mathcal{D})$ and $\psi: B \to \mathcal{P}(\mathcal{V} \times \mathcal{V})$ and an oplax homomorphism of weak representations $(h, i): \varphi \to \psi$ be given.

A given solution $j: N \to \mathcal{D}$ for a scenario ρ in φ is defined by $\mathcal{P}(j \times j) \circ \rho \subseteq \varphi$. From this and the oplax commutation property $\mathcal{P}(i \times i) \circ \varphi \subseteq \psi \circ h$ we infer $\mathcal{P}(i \circ j \times i \circ j) \circ \rho \subseteq \psi \circ h$, which implies that $i \circ j$ is a solution for $h(\rho)$. □

An important question for a calculus (= weak representation) is whether algebraic closure decides consistency of scenarios, see [63]. We will now prove that this property is preserved under certain homomorphisms.

Proposition 129. *Atomic oplax homomorphisms (h, i) of weak representations with h being injective preserve the property that algebraic closure decides scenario-consistency to the image of h.*

Proof. Let weak representations $\varphi: A \to \mathcal{P}(\mathcal{D} \times \mathcal{D})$ and $\psi: B \to \mathcal{P}(\mathcal{V} \times \mathcal{V})$ and an atomic oplax homomorphism of weak representations $(h, i): \varphi \to \psi$ be given. Further assume that for φ, algebraic closure decides consistency of scenarios.

Any scenario in the image of h can be written as $h(\rho): B \to \mathcal{P}(N \times N)$. If $h(\rho)$ is algebraically closed, by Proposition 126, this carries over to ρ. Hence, by the assumption, ρ is consistent, i.e. has a solution. By Proposition 128, $h(\rho)$ is consistent as well. □

Note that the converse directly always holds: any consistent scenario is algebraically closed. After having defined and proved the formal heavy machinery in this section, we shall use it on several calculi in the next one.

8.4. Application to Specific Calculi

The general scenario consistency problem for the $\mathcal{DRA}_{fp}$ calculus is $\mathcal{NP}$-hard, see [79]. However, for specific scenarios, we can do better: We can apply Proposition 129 to the homomorphism from interval algebra to $\mathcal{DRA}_{fp}$ (see Example 119) and obtain:

Corollary 130. *Algebraic closure decides consistency of $\mathcal{DRA}_{fp}$ scenarios that involve the interval algebra relations only.*

Hence, consistency of such scenarios can be decided in polynomial time (in spite of the NP-hardness of the general scenario consistency problem).

For calculi such as $\mathcal{RCC}8$, interval algebra etc., (maximal) *tractable subsets* have been determined, i.e. sets of relations for which algebraic closure decides consistency also of non-atomic constraint networks involving these relations. We then also obtain that algebraic closure in $\mathcal{DRA}_{fp}$ decides consistency of any constraint network involving (the homomorphic image of) a maximal tractable subset of the interval algebra only.

We also obtain a result about strength of composition:

Corollary 131. *Composition in* $\mathcal{OPRA}_1^\star$ *is strong.*

Proof. Composition in $\mathcal{DRA}_{fp}$ is known to be strong, see Theorem 84. By Example 124 and Proposition 125, strength of composition carries over to $\mathcal{OPRA}_1^\star$. □

8.5. Aftermath

With purely algebraic methods, we can lift the properties "strength of composition" and "algebraic closure decides consistency" along homomorphisms of qualitative calculi. The latter is particularly important, because algebraic closure is a polynomial-time method, whereas qualitative constraint satisfaction problems in some cases turn out to be $\mathcal{NP}$-hard, even for scenarios of base relations.

This study only is an initial step in the application of universal algebraic methods to qualitative constraint reasoning. In recent years, there has been an explosion of qualitative constraint calculi, and in many cases, the questions whether composition is strong or algebraic closure decides consistency has not been examined yet. We are currently starting a more systematic study of possible homomorphisms between qualitative calculi, which will lead to more examples, and possibly also to more theoretical investigations.

CHAPTER 9

Epilogue

At the end of this thesis we shall briefly repeat the main results of the chapters once more in Section 9.1. The impatient reader can use this section to get an overview over the results that are most interesting for them. After that we conclude the thesis with Section 9.2.

9.1. Overview

After having introduced the topic of qualitative spatial reasoning in general as well as several concrete calculi on a coarse level, we have given a detailed introduction into qualitative spatial reasoning in Chapter 2. We have divided that section into reasoning about binary calculi for which an elaborate formal foundation is available and calculi of higher arity (mostly 3) for which no such foundations are known. The methods presented in that chapter are the state-of-the-art approaches while much effort is put into finding better methods like reasoning based on algebraic geometry (e.g. `CAD`-based reasoning or our triangle-based algorithm in Chapter 7). After the introduction of the theory of qualitative spatial reasoning, we have introduced several calculi in Chapter 3. They are the $\mathcal{LR}$ calculus in Section 3.1, the $\mathcal{DRA}$ calculus in the versions $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ in Section 3.2, and the $\mathcal{OPRA}$ calculus in Section 3.3. All of these calculi share the fact that the the relations are determined with respect to base frames that are local to the spatial objects, i.e. we deal with relative orientation, and there is no global base frame, i.e. no global sense of orientation and direction. A commonality that these relative orientation calculi as we call them share is that their semantics is based on non-linear inequalities in the (Euclidean) plane. This property makes dealing with these calculi a very hard task, since unlike linear inequalities, non-linear inequalities cannot be handled efficiently in an direct approach. Approaches like Gröbner Reasoning extended with features to handle inequalities (see Section 2.5) can theoretically handle such systems of non-linear inequalities but at a cost of doubly-exponential running time. In reality Gröbner reasoners comprising these features as the one implemented in `SparQ` [78] unfortunately fail to come to a decision in an acceptable time frame, whereas finding a suitable ordering of the monomials that has a great impact on the fact if a decision can be made and how long it takes to make a decision is a big problem inherent to that approach. `CAD` (cylindrical algebraic decomposition) can also solve such systems of inequalities in theory with an doubly-exponential worst case running time. The crucial problem with `CAD` is finding a suitable projection function which has a big impact on the performance of decision making or finding a solution. Finding an optimal or at least suitable projection for `CAD` in a qualitative spatial reasoning environment is definitely not a simple task and unfortunately out of the scope of this thesis but this issue is being investigated by Lee and Wolter.

These problems with handling the semantics of relative orientation calculi lead to the unpicturesque situation that for example for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ since the appearance of the $\mathcal{DRA}$ paper in [49] in the year 2000 for about 10 years no correct composition table was available. In Chapter 4 for the first time ever we have calculated composition tables for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ based on condensed semantics, i.e. with a method based on oriented orbits in the affine group of the Euclidean plane. In fact, we have computed those tables twice, in the first approach, we have used a mathematically very clear method in Section 4.2. Unfortunately that method is quite lengthy to describe and hard to grasp. Therefore we have done the same task again using a simplified method in Section 4.3 that is quite easy to understand but does not distinguish between the several layers (symbolical layer of relations and geometric layer) as clearly as the more complicated method. We do like both of them and that is why we present both of them. It is left to the reader to like or dislike any of those methods, especially since both of them are based on the same theory, proved correct and they produce the same composition table. For scenarios in up to eight points the problem of orientability of matroids can be used to decide them, that is this method can theoretically be used for computing the composition table for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ which is based on scenarios in six points. We do not want to doom any of these possible methods, but we think that our approach of condensed semantics is a good way to go, especially since it can be extended to scenarios in arbitrarily many points and for orbits in many affine spaces. A problem with condensed semantics is that at the current state of the approach a lot of things have to be done by hand. The classes of the prototypical configurations in the plane with respect to the semantics of the calculus have to be known. For composition scenarios in many points this is a big problem, since the number of such configurations can rise steeply, but it stays manageable for scenarios in up to eight points for sure. A possible perspective for future research is developing a method to determine consistency in scenarios or even constraint networks directly based on condensed semantics using the properties of the orbits of a suitable automorphism group.

In Section 4.4.2 we have discovered that $\mathcal{DRA}_{fp}$ has strong composition, a feature that is not common to many qualitative spatial calculi. In fact the extension of $\mathcal{DRA}_f$, which has weak composition, to $\mathcal{DRA}_{fp}$ which is based on the intention to make $\mathcal{DRA}$ more suitable for robot navigation also gives $\mathcal{DRA}_{fp}$ this nice theoretical property. But we have to see that this nice feature does not help in any case with solving the consistency problem of constraint networks in $\mathcal{DRA}_{fp}$ using algebraic closure. In literature about qualitative spatial reasoning the notions of path consistency and algebraic closure are often mixed up, but for $\mathcal{DRA}_{fp}$ they really coincide.

In Chapter 5 we have investigated the applicability of $\mathcal{DRA}_{fp}$ for the description of street scenarios based on local observations. One can assume in such a situation that agents (for example robots) are exploring a street network that is a priori unknown to them. At certain points they can take observations, but their visibility range is limited. All local observations are then united to a constraint network together with pairs of dipoles nothing is known about. Then algebraic closure is used to derive the global knowledge as well as possible. $\mathcal{DRA}_{fp}$ is quite well suited for such an approach since it can grasp the feature of parallelism in such networks. But this experiment is just a small one to depict the applicability of $\mathcal{DRA}_{fp}$ for the task of such a description.

In the remainder of Chapter 5, that is in Section 5.2, we have investigated an application scenario based on the $\mathcal{OPRA}$ calculus [54] in more detail. The $\mathcal{OPRA}$ calculus has been chosen for this task since it is very well suited to model the perception at a decision point, i.e. a crossing, from the view of an agent (e.g. a human or a robot). In fact for this task using only half of the $\mathcal{OPRA}$ relations would suffice, but using only those half relations as we have chosen to call them in Definition 49 would get us into trouble when trying to apply algebraic closure, since we would eliminate all converse relations by that step. And algebraic closure is a technique that we use in our approach. Our navigation approach is based on a description of a street network about which only things are known that can be derived with local observations. This is again a situation in which agents (e.g. robots or humans) explore a street network and can make local observations at each decision point, i.e. at each crossing. Because of the limited range of sight and the building density non-local observations cannot be made in this model. These local observations are made using the $\mathcal{OPRA}$ calculus and all made observations are compiled into a constraint network. All pairs of crossings for which no observations could be made are added to the constraint network with the universal relation between them. In the semantics of a qualitative spatial calculus this means that nothing about the relations between those pairs of points is known. Additionally to making our local observations using the $\mathcal{OPRA}$ base relations directly, we encoded a base frame based on Klippel's research [32–34], which is deemed "cognitively adequate", using $\mathcal{OPRA}$ relations. The main feature of this base frame in contrast to $\mathcal{OPRA}$ is that the relations front, left, back and right are no linear relations, they are planar, which is deemed to be closer to human perception. In our experiments we used maps retrieved from `OpenStreetMap`[1] to retrieve constraint networks with local observations on the crossings. This approach enables us to conduct experiments on street networks from very different places. The derived constraint networks are refined using algebraic closure using the tool `GQR` [24]. `SparQ` [78] turned out to be too slow to handle the constraint networks that were already very big when derived from maps of the size of a village. These refined constraint networks are used for navigation using a least angle strategy with memory. Adding memory about visited nodes (or crossings) to the least angle strategy is necessary to get out of dead ends. For this navigation task the question regarding algebraic closure is not if the refined networks were consistent, the question is if the the information contained in the refined networks is "good enough" for the task at hand. To investigate this, we have conducted about $150,000$ navigation experiments and it has turned out that on many map layouts our navigation approach performs very well, but on the other hand there are features of street networks that lead our navigation approach to perform less well. One known issue is the problem of the less constrained areas at the boundary of the maps. A very evil feature, one could say a honey pot, for our navigation approach are dead ends. Sine the navigation is only performed on a local least angle strategy, dead ends cannot be detected until it is too late. Even though there are some drawbacks, we were impressed how well our navigation approach performed when compared to shortest path strategies, especially if you consider that our approach is based on imprecise knowledge. Roughly we work on a level of knowledge that is comparable to a route description given to you by a person at the side of the road who you ask for directions. In the end, we can say that $\mathcal{OPRA}$ is suitable for navigation based on imprecise knowledge. Moreover, the navigation algorithm still has a lot of space for tinkering like adding a detection feature for dead ends to further improve the performance. But it has not been our aim to construct an optimal navigation algorithm based on $\mathcal{OPRA}$, it has been our aim to investigate the applicability of $\mathcal{OPRA}$ for navigation tasks at all which are based on imprecise knowledge. We want to stress once more that it is not our aim to find an acceptable navigation solution for `OpenStreetMap`, since problems of that kind are rather solved ones. We want to investigate the applicability of $\mathcal{OPRA}$ for some kind of "natural language" descriptions of street scenarios and want to see how well we can do when navigating in them. Using `OpenStreetMap` is only a means of obtaining a big repository of street networks for experimental purposes.

[1] `http://www.openstreetmap.org/`

After the more applied considerations in Chapter 5 we have again focused on more theoretical reflections in Chapter 6. In fact, we focus on the question if algebraic closure decides consistency for $\mathcal{LR}$, $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$. Using the results of [79] this question can be answered negatively in an easy fashion. But the results represented in Section 6.1 formed the motivation for Wolter and Lee to investigate the issue that led to the results in [79]. Further we believe that investigating the calculus directly to construct counterexamples might give deeper insights into when we may rely on algebraic closure and when not. In Section 6.1 we investigated the algebraic closure problem for $\mathcal{LR}$ with binary and ternary composition. Already our first results in Proposition 91 were staggering, since we have discovered that all scenarios in $\mathcal{LR}$ in the relations l and r were algebraically closed with respect to binary composition. All we have had to do now was finding an arbitrary network of this type and show that it is not consistent. Constructing and finding such a scenario has not been too hard since a scenario in five points has already been sufficient. Proving the inconsistency using algebraic methods turns out to be not so easy, but manageable. With these results we can claim that algebraic closure for $\mathcal{LR}$ with respect to binary composition over scenarios in l and r does not provide any information about consistency at all, rendering this technique a very bad approximation of consistency for this calculus. For a calculus in ternary relations it is more natural to use ternary composition according to [6]. We have calculated the ternary composition table for $\mathcal{LR}$ and have investigated the same scenario as for binary composition with respect to ternary composition, also with the result that it is deemed algebraically closed even though it is inconsistent. A difference between binary and ternary composition is that ternary composition does not exhibit the weakness of deeming all scenarios in l and r consistent. In the remainder of Section 6.1 we have investigated other notions of consistency at least for convex $\mathcal{LR}$ relations. In fact, we could identify a special notion of k-consistency with k depending on the dimension of the space and on the arity of the relations that ensures global consistency. Further investigations have showed that global consistency is a too strong notion to decide consistency. So far there is no efficient decision method that can decide consistency in $\mathcal{LR}$ scenarios. Gröbner reasoning (as implemented in `SparQ` [78]) is a decision method, but it is not very efficient. Research in `CAD` might reveal interesting results. In the remainder of Chapter 6 we have investigated algebraic closure for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$. For $\mathcal{DRA}_f$ we have constructed a hexagon with a line consisting of two dipoles in the middle. As an inconsistency we postulate that one of these dipoles has to lie inside and outside the hexagon. Algebraic closure with $\mathcal{DRA}_f$ does not detect this inconsistency. The scenario can be refined to $\mathcal{DRA}_{fp}$ in several ways and for any of them the inconsistency is detected when applying algebraic closure with respect to $\mathcal{DRA}_{fp}$. Finding a counterexample for $\mathcal{DRA}_{fp}$ has turned out to be a hard task. In the end using the Pappus' Configuration (see Figure 65) and adding a constraint that contradicts Pappus' Hexagon Theorem has turned out to be the last resort for a constraint network that is algebraically closed but inconsistent. Due to the nature of the constraints, that is no refined relations of $\mathcal{DRA}_{fp}$ were needed in the constraint network, this counterexample can also be used for $\mathcal{DRA}_f$. And again we see that by using simple geometrical shapes, we can show that algebraic closure does not decide consistency for calculi dealing with relative base frames.

After we have seen that algebraic closure is a very bad approximation to consistency for the $\mathcal{LR}$ calculus, in Chapter 7 we have developed a new polynomial-time approximation method to consistency for the $\mathcal{LR}$ calculus. Because of [79] it is in vain to try and find a polynomial time decision method, since that should show $\mathcal{P} = \mathcal{NP}$. In contrast to algebraic closure, our new method takes properties of the domain of the calculus into consideration directly. In fact, linear properties of triangles in the plane are used, since any $\mathcal{LR}$ base relation can be considered as such a triangle. Any $\mathcal{LR}$ scenario can be translated into a system of linear inequalities using this method and if such a system has a solution the corresponding scenario is called *triangle consistent*. In our many experiments with triangle consistency, we came to the conclusion that it is a quite good approximation of consistency and it really turned out to be hard to find a counterexample. In the end, after having found the counterexample for $\mathcal{DRA}_{fp}$, we have used the same configuration for $\mathcal{LR}$ using triangle consistency and it has turned out that the Pappus' configuration is a separating example between consistency and triangle consistency, too. We think that triangle consistency fails on this scenario since any information about the side lengths of the triangles is discarded. But including these side lengths into the system of inequalities would make it non-linear and we would no longer have a polynomial time approximation.

Chapter 8 is a very theoretical one once more. Roughly speaking in that chapter we have investigated several kinds of homomorphisms of qualitative spatial calculi in binary relations. In more detail, we have defined *lax*, *oplax* and *proper* homomorphisms between non-associative algebras as well as between weak representations. Using the notion of these homomorphisms, we also have defined the notion of quotient (see Definition 121). This notions turns out to be quite fruitful, since it can be used to define a new calculus from an existing one by neglecting certain features of a calculus. For example $\mathcal{OPRA}_1^\star$ can be derived as a quotient of $\mathcal{DRA}_{fp}$ as we have seen in Example 124. Practically speaking $\mathcal{OPRA}_1^\star$ is obtained from $\mathcal{DRA}_{fp}$ by neglecting the "lengths" of the dipoles. Further in Proposition 125 we have been able to prove that strength of composition is preserved by quotient homomorphisms of weak representations that are bijective in the second component. We also have been able to prove results for homomorphisms that are applicable to constraint reasoning. In Proposition 126 we have proved that algebraic closure is preserved by oplax homomorphisms of non-associative algebras. With this result, we can use information about algebraically closed constraint networks in $\mathcal{DRA}_{fp}$ and transform it along the proper homomorphism of non-associative algebras (the level of domains is not interesting for algebraic reasoning and hence we can neglect that part of the homomorphism at this point) from $\mathcal{DRA}_{fp}$ to $\mathcal{OPRA}_1^\star$ and use the results for the target calculus. This does not sound promising at first, but it enables us to use benchmarking problems for $\mathcal{DRA}_{fp}$ also for $\mathcal{OPRA}_1^\star$. For concrete applications it is of course better to perform the reasoning $\mathcal{OPRA}_1^\star$ directly. Further we have been able to show in Proposition 128 that oplax homomorphisms of weak representation preserve *solutions* of scenarios. We conclude the chapter by using our results on specific calculi in Section 8.4. All in all we have had to see that investigating homomorphisms between qualitative spatial calculi is a very fruitful field that has not yet been investigated in detail, since in many cases only single calculi were in consideration or rather informal "fusions" of several calculi. It would be interesting to consider such "fusions" as for example $\mathcal{OPRA}_1^\star$ [12] via homomorphisms. We can think at this point of product injections concatenated with quotients. How much benefit the results regarding algebraic reasoning might have for the future is currently a little questionable since the belief in the method of algebraic closure to determine the consistency of constraint networks is shattered and a search for new methods has begun.

9.2. General conclusions

A general and repeating theme of this thesis is whether a spatial constraint network is consistent. Consistency, the question whether a constraint network or scenario has a realization in its natural domain, can be answered quite easily using algebraic closure for Allen's Interval Algebra [1] and subsets of $\mathcal{RCC}$ [66]. However, for other calculi, especially those being based on local reference frames like $\mathcal{LR}$, $\mathcal{OPRA}$, and $\mathcal{DRA}$, this question is not answered easily because of the non-linear semantics of these calculi in their domain. But if algebraic closure does not decide consistency even for scenarios in a calculus at hand, is this already a reason to drop this method completely? We always have to keep in mind that algebraic closure is a method that has polynomial running time and can handle quite large constraint networks. For such a case we can reformulate our question. It is no longer the question whether algebraic closure decides consistency, it is now the question whether the results of algebraic closure for the calculus at hand are good enough for the given task. It is the question whether we can live with suboptimal results where not all inconsistent constraint networks can be detected. In Chapter 5 we have done such investigations mainly for $\mathcal{OPRA}$ and we have had to come to the conclusion that algebraic closure performs well enough for the given navigation task.

Since we are dealing with the question of consistency of constraint networks it is almost natural to ask why we do not make use of other methods to determine consistency from the field of constraint satisfaction. But our constraint networks have a property that not many methods from the field of constraint satisfaction can deal with. In constraint satisfaction mostly constraint networks over a finite domain are considered and methods for determining consistency of constraint networks exploit this finiteness often by a kind of backtracking approach. But our domains are infinite. Often the domain is the Euclidean plane $\mathbb{R}^2$. So far the only method that is also applicable to qualitative spatial reasoning is path-consistency [47] and its generalization of algebraic closure. It is applicable since it only works on a symbolic level without making use of the domain directly. But unfortunately consistency of constraint networks and scenarios of calculi like $\mathcal{LR}$ and $\mathcal{DRA}$ cannot be decided using algebraic closure. In Chapter 6 we have given geometric examples that show that algebraic closure and consistency do not coincide for these calculi. On a less vivid level the results of [79] state the same. So far the challenge is to find decision procedures and approximation procedures for consistency of constraint networks for calculi like $\mathcal{DRA}$, $\mathcal{LR}$ and $\mathcal{OPRA}$ that perform sufficiently well. Gröbner reasoning as implemented in `SparQ` [78] is a decision method, but its time complexity makes it only applicable to small constraint networks. A promising approach for the decision of consistency is `CAD` based reasoning, but a lot of efforts still have to be put into investigating it. And after all the problem of exponential time complexity remains. Algebraic closure is losing its reputation as a good procedure to approximate or decide consistency in the community of qualitative spatial reasoning and the search for new and better methods has started. In Chapter 7 we have participated in this search. In fact, we have contributed to finding a better approximation to consistency for the $\mathcal{LR}$ calculus, since algebraic closure turns out to be a very bad approximation to consistency for this calculus. Our new approach takes properties of the domain into account as do Gröbner reasoning and `CAD`. Maybe including properties of the domain or the algebraic semantics of the calculi into the reasoning process is the future. Or maybe there is another approach that is the "silver bullet" just nobody had the idea what approach that might be yet.

Investigating the properties of composition based reasoning for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ was already a problem since for about ten years no correct composition table was available for these calculi. For $\mathcal{DRA}_f$ it has been tried to create composition tables using an ad hoc `C` program and verifying them by hand, but this approach was everything but fruitful. Creating a composition table for a calculus with more than a few base relations is a very hard task. Using ad hoc programs often leads to the problem of forgotten cases regarding the geometric properties of the calculus, and hence erroneous tables. Verifying such a table by hand is not a good idea either, since such a task is boring and tiresome and hence concentrating dwindles just too easily. Fortunately in Chapter 4 we have been able to construct a composition table in a formally backed-up and computer-aided approach based on condensed semantics. The condensed semantics have helped us to strip down the infinite domain to only a finite number of spatial configurations that we have had to consider. Condensed semantics can be adapted to other calculi dealing with orientation by examining the configurations of objects in space that matter for the particular calculus. Because of Chapter 6 and [79] we know that algebraic closure does not decide consistency for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$, but still finding a composition table is not in vain. For tasks as discussed in Chapter 5 algebraic closure does not decide consistency, but the results of that procedure are good enough and that is often all we are asking for. This becomes even more apparent when we consider that no practically usable procedure to determine consistency for $\mathcal{DRA}_f$ and $\mathcal{DRA}_{fp}$ is available. Answering the question whether a $\mathcal{DRA}_f$ or $\mathcal{DRA}_{fp}$ scenario is consistent is still a hard task as it is for many other calculi of the same kind.

The final contribution of this thesis is an investigation into homomorphisms between qualitative spatial calculi. Based on the work of Ligozat and Renz in [39] we consider homomorphisms in the abstract field of non-associative algebras which neglect any information about the domain of a calculus as well as homomorphisms in the field of weak representations where the canonical relation algebra of the domain is an important part. We use homomorphisms to embed calculi into others and to obtain calculi as quotients. Further we show that certain properties of calculi and of constraint networks can be transported along several kinds of homomorphisms.

Consistency and several approximations to consistency are of great and growing interest in qualitative spatial reasoning and its applications once more. For long years it was believed that algebraic closure was *the* reasoning technique for qualitative spatial calculi. With showing that algebraic closure does not even decide consistency for the simple $\mathcal{LR}$ calculus, we have opened this field once more and already papers have spawned that are follow-up on this consistency problem like [79]. The main contributions of this thesis are the investigation of several types of "consistency" for qualitative spatial calculi and the investigation into what algebraic closure can still do for us.

Appendix A

Table of composable l/r triples

$$\begin{array}{ccccc} (A\,C\,l\,B) & (A\,B\,l\,D) & (B\,C\,l\,D) & & (A\,C\,l\,D) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (A\,C\,l\,E) & (A\,E\,l\,D) & (E\,C\,l\,D) & & (A\,C\,l\,D) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (A\,C\,l\,B) & (A\,B\,l\,E) & (B\,C\,l\,E) & & (A\,C\,l\,E) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (E\,A\,l\,B) & (E\,B\,l\,C) & (B\,A\,l\,C) & & (E\,A\,l\,C) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (C\,D\,l\,B) & (C\,B\,l\,A) & (B\,D\,l\,A) & & (C\,D\,l\,A) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (C\,D\,l\,E) & (C\,E\,l\,A) & (E\,D\,l\,A) & & (C\,D\,l\,A) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (C\,E\,l\,B) & (C\,B\,l\,A) & (B\,D\,l\,A) & & (C\,E\,l\,A) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (E\,C\,r\,B) & (E\,B\,r\,A) & (B\,C\,r\,A) & & (E\,C\,r\,A) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ r, & r, & r) & \cap & \{r\} = \{r\} \end{array}$$

$$\begin{array}{ccccc} (D\,A\,l\,B) & (D\,B\,l\,C) & (B\,A\,l\,C) & & (D\,A\,l\,C) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (D\,A\,l\,E) & (D\,E\,l\,C) & (E\,A\,l\,C) & & (D\,A\,l\,C) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ l, & l, & l) & \cap & \{l\} = \{l\} \end{array}$$

$$\begin{array}{ccccc} (A\,D\,r\,B) & (A\,B\,r\,C) & (B\,D\,r\,C) & & (A\,D\,r\,C) \\ \downarrow & \downarrow & \downarrow & & \downarrow \\ \diamond(\ r, & r, & r) & \cap & \{r\} = \{r\} \end{array}$$

$$\begin{array}{ccccc} (A\,D\,r\,E) & (A\,E\,r\,C) & (E\,D\,r\,C) & & (A\,D\,r\,C) \end{array}$$

$$
\begin{array}{llllll}
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
(A\,E\,r\,B) & & (A\,B\,r\,C) & (B\,E\,r\,C) & & (A\,E\,r\,C) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
(C\,A\,r\,B) & & (C\,B\,r\,E) & (B\,A\,r\,E) & & (C\,A\,r\,E) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
(C\,A\,r\,E) & & (C\,E\,r\,D) & (E\,A\,r\,D) & & (C\,A\,r\,D) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
(C\,A\,r\,B) & & (C\,B\,r\,D) & (B\,A\,r\,D) & & (C\,A\,r\,D) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
(D\,C\,r\,B) & & (D\,B\,r\,A) & (B\,C\,r\,A) & & (D\,C\,r\,A) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\} \\
\hline
(D\,C\,r\,E) & & (D\,E\,r\,A) & (E\,C\,r\,A) & & (D\,C\,r\,A) \\
 & \downarrow & \downarrow & \downarrow & & \downarrow \\
\diamond(& r, & r, & r) & \cap & \{r\} = \{r\}
\end{array}
$$

Bibliography

[1] J. F. Allen. Maintaining knowledge about temporal intervals. *Communications of the ACM*, pages 832–843, 1983.

[2] B. Bennett and A. P. Galton. A unifying semantics for time and events. *Artificial Intelligence*, 153(1–2):13–48, 2004.

[3] A. R. Bradley and Z. Manna. *The Calculus of Computation*. Springer, 2007.

[4] B. Buchberger. Gröbner bases: An algorithmic approach in polynomial ideal theory. In N. K. Bose, editor, *Multidimensional Systems Theory — Progress, Directions and Open Problems in Multidimensional Systems*, chapter 6, pages 184–232. Reidel Publishing Company, Dordrecht – Boston – Lancaster, 1985.

[5] B. Buchberger. Gröbner bases: A short introduction for systems theorists. In R. Moreno-Diaz, B. Buchberger, and J.L. Freire, editors, *Proc. of EUROCAST 2001*, volume 2178 of *Lecture Notes in Comput. Sci.*, pages 1–19, Berlin, 2001. Springer Verlag.

[6] J.-F. Condotta, M. Saade, and G. Ligozat. A Generic Toolkit for n-ary Qualitative Temporal and Spatial Calculi. In *Proc. of* 13*th International Symposium on Temporal Representation and Reasoning (TIME 2006)*, pages 78–86. IEEE Computer Society, 2006.

[7] H. S. M. Coxeter. Self-dual configurations and regular graphs. *Bull. Amer. Math. Soc.*, 56:413–455, 1950.

[8] M. Cristani. Reasoning about Qualitative Relations between Straight Lines. Technical report, University of Verona, 2003.

[9] E. Davis. Continuous shape transformation and metrics of shape. *Fundamenta Informaticae*, 46:31–54, 2001.

[10] I. Düntsch. Relation algebras and their application in temporal and spatial reasoning. *Artif. Intell. Rev.*, 23(4):315–357, 2005.

[11] B. Dutertre and L. de Moura. The Yices SMT solver. Tool paper at `http://yices.csl.sri.com/tool-paper.pdf`, August 2006.

[12] F. Dylla. *An Agent Control Perspective on Qualitative Spatial Reasoning — Towards More Intuitive Spatial Agent Development*. PhD thesis, University of Bremen, 2008. Published by Akademische Verlagsgesellschaft Aka GmbH.

[13] F. Dylla and R. Moratz. Empirical complexity issues of practical qualitative spatial reasoning about relative position. In *Workshop on Spatial and Temporal Reasoning at ECAI*, 2004.

[14] F. Dylla and R. Moratz. Exploiting Qualitative Spatial Neighborhoods in the Situation Calculus. In C. Freksa, M. Knauff, B. Krieg-Brückner, B. Nebel, and T. Barkowsky, editors, *Proc. of Spatial Cognition 2004*, pages 304–322, 2005.

[15] J.-C. Faugère. A new efficient algorithm for computing Gröbner bases (F_4). *Journal of Pure and Applied Algebra*, 139(1):61–88, 1999.

[16] J.-C. Faugère. A new efficient algorithm for computing Gröbner bases without reduction to zero (F_5). In *Proc. of International Symposium on Symbolic and Algebraic Computation (ISSAC)*, pages 75–83. ACM Press, 2002.

[17] C. Freksa. Temporal reasoning based on semi-intervals. Technical Report ICSI TR-90-016, International Computer Science Institute, Berkeley, 1990.

[18] C. Freksa. Conceptual Neighborhood and its role in temporal and spatial reasoning. In M. Singh and L. Traveé-Massuyès, editors, *Proc. of IMACS Workshop on Decision Support Systems and Qualitative Reasoning*, pages 181–187, Amsterdam, 1991. North-Holland.

[19] C. Freksa. Using orientation information for qualitative spatial reasoning. In A. U. Frank, I. Campari, and U. Formentini, editors, *Theories and methods of spatio-temporal reasoning in geographic space*, volume 639 of *Lecture Notes in Comput. Sci.*, pages 162–178. Springer, 1992.

[20] E. C. Freuder. Synthesizing constraint expression. *Comm. ACM*, 21:958–966, 1978.

[21] J. H. Gallier. *Curves and surfaces in geometric modeling: theory and algorithms*. Morgan Kaufmann, 2000.

[22] A. Galton. Continuous motion in discrete space. In A. Cohn, F. Giunchiglia, and B. Selman, editors, *Proc. of KR '00*, pages 26–37, San Francisco, 2000. Morgan Kaufman.

[23] A. Galton. *Qualitative Spatial Change*. Oxford University Press, 2000.

[24] Z. Gantner, M. Westphal, and S. Wölfl. GQR - A Fast Reasoner for Binary Qualitative Constraint Calculi. In *Proc. of the AAAI-08 Workshop on Spatial and Temporal Reasoning*, 2008.

[25] J. E. Goodman, R. Pollack, and B. Sturmfels. Coordinate representation of order types requires exponential storage. In *STOC '89: Proceedings of the twenty-first annual ACM symposium on Theory of computing*, pages 405–410, New York, NY, USA, 1989. ACM. ISBN 0-89791-307-8. doi: http://doi.acm.org/10.1145/73007.73046.

[26] G. Grätzer. *Universal Algebra*. Springer-Verlag, New York, NY, second edition, 1979.

[27] B. Grünbaum. *Arrangements and Spreads*, volume 10 of *Regional Conference Series in Mathematics*. Providence, R.I.: American Mathematical Society, 1972.

[28] J. Harrison. HOL Light: An Overview. In S. Berghofer, T. Nipkow, C. Urban, and M. Wenzel, editors, *Proc. of TPHOLs-09*, volume 5674 of *Lecture Notes in Comput. Sci.*, pages 60–66, 2009.

[29] E. Helly. Über Mengen konvexer Körper mit gemeinschaftlichen Punkten. *Jber. Deutsch. Math. Verein*, 32:175–176, 1923.

[30] D. Hernández. *Qualitative Representation of Spatial Knowledge*, volume 804 of *Lecture Notes in Artificial Intelligence*. Springer, 1994.

[31] A. Isli and A. G. Cohn. A new approach to cyclic ordering of 2D orientations using ternary relation algebras. *Artificial Intelligence*, 122(1-2):137–187, 2000.

[32] A. Klippel. Wayfinding choremes - conceptualizing wayfinding and route direction elements. *KI*, 18(1):63–, 2004.

[33] A. Klippel and D. R. Montello. Linguistic and Nonlinguistic Turn Direction Concepts. In S. Winter, M. Duckham, L. Kulik, and B. Kuipers, editors, *Proc. of COSIT 2007*, volume 4736 of *Lecture Notes in Comput. Sci.*, pages 354–372. Springer, 2007.

[34] A. Klippel, H. Tappe, L. Kulik, and P. U. Lee. Wayfinding choremes–a language for modeling conceptual route knowledge. *Journal of Visual Languages and Computing*, 16 (4):311 – 329, 2005. ISSN 1045-926X. doi: DOI:10.1016/j.jvlc.2004.11.004. Perception and ontologies in visual, virtual and geographic space.

[35] J. H. Lee and D. Wolter. A new perspective on reasoning with qualitative spatial knowledge. In J. Renz, A. G. Cohn, and S. Wölfl, editors, *Proc. of IJCAI-2011 Workshop Benchmarks and Applications of Spatial Reasoning*, 2011.

[36] G. Ligozat. Weak Representations of Interval Algebras. In *Proc. of AAAI'90*, pages 715–720, 1990.

[37] G. Ligozat. Qualitative triangulation for spatial reasoning. In A. U. Frank and I. Campari, editors, *Proc. International Conference on Spatial Information Theory.*, volume 716 of *Lecture Notes in Comput. Sci.*, pages 54–68. Springer, 1993.

[38] G. Ligozat. Categorical Methods in Qualitative Reasoning: The Case for Weak Representations. In A. G. Cohn and D. M. Mark, editors, *Proc. of COSIT*, volume 3693 of *Lecture Notes in Comput. Sci.*, pages 265–282. Springer, 2005.

[39] G. Ligozat and J. Renz. What Is a Qualitative Calculus? A General Framework. In C. Zhang, H. W. Guesgen, and W.-K. Yeap, editors, *Proc. of PRICAI*, volume 3157 of *Lecture Notes in Computer Science*, pages 53–64. Springer, 2004.

[40] D. Lücke and T. Mossakowski. A much better polynomial time approximation of consistency in the $\mathcal{LR}$ calculus. In A. Gomez-Perez and T. Agotnes, editors, *Proceedings of the 5th Starting AI Researcher Symposium (STAIRS 2010)*, volume 222 of *Frontiers in Artificial Intelligence and Applications*, pages 175–185, Amsterdam,

2010. IOS Press.

[41] D. Lücke, T. Mossakowski, and D. Wolter. Qualitative reasoning about convex relations. In C. Freksa, N. S. Newcombe, P. Gärdenfors, and S. Wölfl, editors, *Proc. of Spatial Cognition-08*, volume 5248 of *Lecture Notes in Comput. Sci.*, pages 426–440. Springer, 2008.

[42] D. Lücke, T. Mossakowski, and R. Moratz. Streets to the $\mathcal{OPRA}$ — finding your destination with imprecise knowledge. In *Proc. of Benchmarks and Applications of Spatial Reasoning*, pages 25–32. IJCAI, 2011.

[43] A. K. Mackworth. Consistency in Networks of Relations. *Artif. Intell.*, 8:99–118, 1977.

[44] S. MacLane. *Categories for the Working Mathematician*. Graduate Texts in Mathematics. Springer, 2nd edition, 1998.

[45] R. Maddux. Some varieties containing relation algebras. *Transactions of American Mathematical Society*, 272(2):501–526, 1982.

[46] R. Maddux. *Relation Algebras*. Stud. Logic Found. Math. Elsevier Science, 2006.

[47] U. Montanari. Networks of constraints: Fundamental properties and applications to picture processing. *Inform. Sci.*, 7:95–132, 1974.

[48] R. Moratz. Representing Relative Direction as a Binary Relation of Oriented Points. In G. Brewka, S. Coradeschi, A. Perini, and P. Traverso, editors, *Proc. of ECAI-06*, pages 407–411. IOS Press, 2006.

[49] R. Moratz, J. Renz, and D. Wolter. Qualitative Spatial Reasoning about Line Segments. In *Proc. of ECAI 2000*, pages 234–238, 2000.

[50] R. Moratz, B. Nebel, and C. Freksa. Qualitative spatial reasoning about relative position: The tradeoff between strong formal properties and successful reasoning about route graphs. In C. Freksa, W. Brauer, C. Habel, and K.F. Wender, editors, *Spatial Cognition III*, volume 2685, pages 385 – 400. Springer, 2003.

[51] R. Moratz, D. Lücke, and T. Mossakowski. Oriented straight line segment algebra: Qualitative spatial reasoning about oriented objects. *CoRR*, abs/0912.5533, 2009. URL `http://arxiv.org/abs/0912.5533`.

[52] R. Moratz, D. Lücke, and T. Mossakowski. A condensed semantics for qualitative spatial reasoning about oriented straight line segments. *Artificial Intelligence*, 175: 2099–2127, October–November 2011.

[53] F. Mossakowski. Algebraische Eigenschaften qualitativer Constraint-Kalküle. Master's thesis, Universität Bremen, 2007.

[54] T. Mossakowski and R. Moratz. Qualitative Reasoning about Relative Direction on Adjustable Levels of Granularity. *Journal of Artificial Intelligence*, to appear.

[55] T. Mossakowski and S. Wölfl. An algebraic charaterisation of qualitative spatial and temporal calculi. unpublished results, 2009.

[56] T. Mossakowski, L. Schröder, and S. Wölfl. A categorical perspective on qualitative constraint calculi. In Stefan Wölfl and Till Mossakowski, editors, *Qualitative Constraint Calculi - Application and Integration. Workshop at KI 2006*, pages 28–39, 2006.

[57] T. Mossakowski, D. Lücke, and A. Scholz. Universal algebraic methods applied to qualitative constraint calculi. unpublished results, 2011.

[58] P. Muller. A qualitative theory of motion based on spatio-temporal primitives. In A. Cohn, L. Schubert, and S. C. Shapiro, editors, *Proc. of KR '98*, pages 131–141, San Francisco, 1998. Morgan Kaufman.

[59] B. Nebel and H.-J. Bürckert. Reasoning about temporal relations: A maximal tractable subclass of allen's interval algebra. *Journal of the ACM*, 42:43–66, 1995.

[60] T. Nipkow, L. C. M. Paulson, and M. Wenzel. *Isabelle/HOL — A Proof Assistant for Higher-Order Logic*. Springer, 2002.

[61] D. A. Randell, Z. Cui, and A. G. Cohn. A spatial logic based on regions and connection. In *Proc. of Conf. on Knowledge Representation and Reasoning*, pages 165–176. Morgan Kaufmann, 1992.

[62] J. Renz. *Qualitative Spatial Reasoning with Topological Information*. Springer, 2002.

[63] J. Renz and G. Ligozat. Weak Composition for Qualitative Spatial and Temporal Reasoning. In P. van Beek, editor, *Principles and Practice of Constraint Programming - CP 2005*, volume 3709 of *Lecture Notes in Comput. Sci.*, pages 534–548. Springer, 2005.

[64] J. Renz and B. Nebel. Qualitative Spatial Reasoning Using Constraint Calculi. In M. Aiello, I. Pratt-Hartmann, and J. van Benthem, editors, *Handbook of Spatial Logics*, pages 161–215. Springer, 2007.

[65] J. Renz and F. Schmid. Customizing Qualitative Spatial and Temporal Calculi. In M. A. Orgun and J. Thornton, editors, *Proc. of Australian Conference on Artificial Intelligence*, volume 4830 of *Lecture Notes in Comput. Sci.*, pages 293–304. Springer, 2007.

[66] Jochen Renz. *Qualitative Spatial Reasoning with Topological Information*, volume LNCS 2293. Springer, Berlin, 2002.

[67] R. Röhrig. Representation and processing of qualitative orientation knowledge. In G. Brewka, C. Habel, and B. Nebel, editors, *Proc. of KI-97*, volume 1303 of *Lecture Notes in Artificial Intelligence*, pages 219–230. Springer, 1997.

[68] D. Sam-Haroud and B. Faltings. Consistency techniques for continuous constraints. *Constraints*, 1:85–118, 1996.

[69] C. Schlieder. Reasoning about Ordering. In A. Frank and W. Kuhn, editors, *Spatial Information Theory: a theoretical basis for GIS*, volume 988 of *Lecture Notes in Comput. Sci.*, pages 341–349, Berlin, 1995.

[70] A. Scivos. Einführung in eine Theorie der ternären RST-Kalküle für qualitatives räumliches Schließen. Master's thesis, Universität Freiburg, April 2000. in German.

[71] A. Scivos and B. Nebel. Double-crossing: Decidability and computational complexity of a qualitative calculus for navigation. In D. R. Montello, editor, *Spatial Information Theory: Foundations of Geographic Information Science, COSIT*, volume 2205 of *Lecture Notes in Computer Science*, pages 431–446. Springer, 2001.

[72] A. Scivos and B. Nebel. The finest of its class: The natural, point-based ternary calculus $\mathcal{LR}$ for qualitative spatial reasoning. In *Spatial Cognition*, pages 283–303, 2004.

[73] A. Scivos and B. Nebel. The finest of its class: The natural point-based ternary calculus for qualitative spatial reasoning. In C. Freksa, M. Knauff, B. Krieg Brückner, B. Nebel, and T.Barkowski, editors, *Spatial Cognition*, volume 3343 of *Lecture Notes in Comput. Sci.*, pages 283–303. Springer, 2004.

[74] A. Tarksi. On the Calculus of Relations. *Journal of Symbolic Logic*, 6(3):73–89, 1941.

[75] M. B. Vilain, H. A. Kautz, and P. G. van Beek. Constraint propagation algorithms for temporal reasoning: A revised report. In *Readings in Qualitative Reasoning about Physical Systems*. Morgan Kaufmann, 1989.

[76] J. O. Wallgrün, L. Frommberger, D. Wolter, F. Dylla, and C. Freksa. Qualitative Spatial Representation and Reasoning in the SparQ-Toolbox. In T. Barkowsky, M. Knauff, G. Ligozat, and D. R. Montello, editors, *Spatial Cognition*, volume 4387 of *Lecture Notes in Comput. Sci.*, pages 39–58. Springer, 2006.

[77] J. O. Wallgrün, L. Frommberger, D. Wolter, F. Dylla, and C. Freksa. A toolbox for qualitative spatial representation and reasoning. In T. Barkowsky, M. Knauff, G. Ligozat, and D. Montello, editors, *Spatial Cognition V: Reasoning, Action*, number 4387 in Lecture Notes in Computer Science, pages 39–58, 2007.

[78] J. O. Wallgrün, L. Frommberger, F. Dylla, and D. Wolter. SparQ User Manual V0.7. User manual, University of Bremen, January 2009.

[79] D. Wolter and J. H. Lee. On Qualitative Reasoning about Relative Point Position. *Artificial Intelligence*, 174:1498–1507, 2010.

[80] D. Wolter and L. Moshagen. Algebraic methods for analyzing qualitative spatio-temporal calculi. In *Proc. of ECAI-Workskop Spatial and Temporal Reasoning*, 2008.

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