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Fractal Curvature Measures and Minkowski Content for Limit Sets of Conformal Function Systems

Sabrina Kombrink

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Gutachter: Prof. Dr. Marc Keßeböhmer (Universität Bremen)
Prof. Dr. Manfred Denker (The Pennsylvania State University, USA)

Abstract

We characterise fractal sets arising from conformal iterated function systems (cIFS) and conformal graph directed Markov systems (cGDMS) for which the Minkowski content and the fractal curvature measures, as introduced in [Win08], exist. With this, we generalise studies that have been carried out for invariant sets of iterated function systems consisting of similarities.

For self-conformal subsets of the d-dimensional Euclidean space we show, under certain geometric conditions, that the local average Minkowski content always exists and provide an explicit formula. If the system is non-lattice, we prove that also the local Minkowski content exists and coincides with its average version. From this general result we deduce new results for the subclass of self-similar sets, which significantly generalise the statements from [DKz$^+$10, LPW11]. For self-similar sets we additionally show that the fractal curvature measures exist in the non-lattice situation and that an average version exists for both lattice and non-lattice systems. With this, we provide a substantially different proof to those presented in [Win08, WZ10, Zäh11] and gain alternative useful formulae for the fractal curvature measures. Another important subclass of self-conformal sets is the class of conformal $\mathcal{C}^{1+\alpha}$-diffeomorphic images of self-similar sets, where $\alpha \in (0, 1]$. We show that the local Minkowski content of such an image exists, whenever the local Minkowski content of the considered self-similar set exists. This new result also yields nice relationships between the local Minkowski contents of the self-similar set and its image.

In contrast to the fact that the Minkowski content of a non-degenerate self-similar subset of $\mathbb{R}$ exists if and only if the system is non-lattice [LP93, Fal95, LvF06], we prove that there exist invariant sets of lattice cIFS for which the Minkowski content does exist. This surprising result is illustrated with examples and disproves Conjecture 4 of [Lap93] for self-conformal sets. We additionally show that even amongst the subclass of $\mathcal{C}^{1+\alpha}$-diffeomorphic images of self-similar sets, there exist lattice sets for which the Minkowski content exists. However, we prove that the fractal curvature measures of such non-degenerate sets in $\mathbb{R}$ exist if and only if the underlying system is non-lattice. The importance of this subclass is emphasised by the result that a lattice cIFS in $\mathbb{R}$, which consists of analytic maps, is automatically conjugate to a lattice system consisting of similarities. From this, we infer that the fractal curvature measures of a non-degenerate invariant set of a cIFS in $\mathbb{R}$ consisting of analytic maps exist if and only if the system is non-lattice.

The above-mentioned results for systems, whose invariant set is a subset of $\mathbb{R}$, are shown to be valid for the more general class of limit sets of cGDMS. Specifically, we show that the Minkowski content of a non-degenerate limit set of a cGDMS consisting of similarities exists if and only if the system is non-lattice, providing an important generalisation of the respective result for self-similar subsets of $\mathbb{R}$. Further, we obtain that limit sets of non-lattice cGDMS are Minkowski measurable and by this verify Conjecture 4 of [Lap93] for limit sets of Fuchsian groups of Schottky type, since they are always non-lattice (see for instance [Lal89]).

Contents

1 Introduction

A central theme in fractal geometry is to characterise the geometric structure of fractal objects. In this context, various different notions of dimension play a crucial role. However, sets of the same 'fractal' dimension, such as Minkowski or Hausdorff dimension, can differ significantly in their structure. Here, the Minkowski content and fractal curvature measures come into play to provide information on the texture of a set, beyond its dimension. Besides this geometric motivation, the intention behind introducing fractal curvature measures in [Win08] was to develop an alternative notion of curvature, since the classical notions do not seem to be appropriate for fractal sets. With this, the quest in geometric measure theory of extending the concept of curvature measures as far as possible is addressed.

In this thesis, we consider both of the above aspects by investigating the Minkowski content and fractal curvature measures of invariant sets arising from conformal iterated function systems, which we call self-conformal sets, and limit sets of conformal graph directed Markov systems. Through these investigations, we generalise studies which have been carried out for self-similar sets. This is of interest since fractal sets arising in geometry, for instance limit sets of Fuchsian groups, or in number theory, for instance sets defined by Diophantine inequalities, are not typically self-similar but are rather self-conformal. We see that the existence of the Minkowski content and the fractal curvature measures is dependant upon the underlying conformal system being lattice or non-lattice in the sense of [Lal89]. We discover that in this dependence lies a major difference between general self-conformal sets and the subclass of self-similar sets: Whereas the Minkowski content of a self-similar subset of $\mathbb{R}$ with zero Lebesgue measure, which satisfies the open set condition (OSC), exists if and only if the underlying system is non-lattice [LP93, Fal95, LvF06], we prove that the Minkowski content of a general self-conformal set exists in the non-lattice case and show that in the lattice situation both existence and non-existence of the Minkowski content is possible. In this introduction we present an outline of the original results, but beforehand, we give a more detailed description of the relevance of the Minkowski content and of studies which have already been carried out concerning its existence and evaluation.
After [Man95], it is well known that a Cantor dust on $[0,1]$ may achieve any given Hausdorff dimension within $(0,1)$ in many different ways and that the results need not look alike. As an example, consider the following two Cantor sets: To begin the construction of both, subdivide the unit interval $[0,1]$ into seven intervals of same lengths. For the first Cantor set C_1 keep the first, third, fifth and seventh interval from the left and repeat the same

procedure with the remaining intervals. For the second Cantor set C_2 keep at each step the two leftmost and the two rightmost intervals. Then the Minkowski as well as the Hausdorff dimension of C_1 and C_2 are equal, although the two sets differ significantly in their gap structure. The Minkowski content is capable of detecting this structural difference, as is discussed in [Man82, Man95], and was proposed therein as a measure of lacunarity for fractal sets. The word lacunarity originates from the latin word `lacuna` which means gap. According to [Man82], "a fractal is to be called lacunar if its gaps tend to be large, in the sense that they include large intervals (discs, or balls)." Thus, C_2 is more lacunar than C_1. This is also reflected by the fact that the average Minkowski content of C_1 is greater than that of C_2 (see Example 2.47). In this way, the Minkowski content can be viewed as a beneficial complement to the notion of dimension. Besides the geometric interpretation, results on the existence of the Minkowski content play an important role with respect to the Weyl-Berry conjecture concerning the asymptotic distribution of the eigenvalues of the Laplacian on sets with fractal boundaries. More precisely, the asymptotic second term is expressed in terms of the Minkowski dimension and the Minkowski content of the boundary of the set, whenever these quantities exist (see [Fal95, Section 4],[Lap91, LV96] and references therein). Another motivation for studying the Minkowski content of fractal sets arises from non-commutative geometry: In Connes' seminal book [Con94] the notion of a non-commutative fractal geometry is developed. There, it is shown that the natural analogue of the volume of a compact smooth Riemannian spin^c manifold for a fractal set in $\mathbb{R}$ is that of the Minkowski content. This idea is also reflected in the articles [GI03, Sam10, FS11].

There are various references available concerning the existence of the Minkowski content for self-similar sets and subsets of $\mathbb{R}$. A complete characterisation of Minkowski measurability of fractal strings has been obtained in [LP93, Fal95, LvF06]. In higher dimensional ambient spaces, using renewal theory, Gatzouras [Gat00] obtains Minkowski measurability of non-lattice self-similar sets satisfying the OSC. Further, Gatzouras shows that the average Minkowski content, which is defined via a logarithmic Cesàro average, exists for any self-similar set satisfying the OSC. By means of geometric zeta functions and their analytic properties alternative proofs and formulae to [Gat00] are provided in [DKz$^+$10, LPW11] under certain conditions on the geometric structure of the underlying set. Moreover, under these assumptions, it is shown that the Minkowski content does not exist in the lattice situation.

In the following, we present generalisations of the above-mentioned results for the much more general class of non-empty compact sets which arise through conformal systems. At first, we focus on self-conformal sets, which are the invariant sets of conformal iterated function systems satisfying the OSC. Such iterated function systems are abbreviated by cIFS and are introduced in [MU96, MU03]. A central role in our investigations is played by the bounded connected components of the complement of the self-conformal set, which we call gaps. For their rigorous definition, we fix a cIFS $\Phi = \{\phi_1, \ldots, \phi_N\}$, $N \geq 2$, with associated self-conformal set $F \subset \mathbb{R}^d$, which satisfies the OSC with a connected open set $O \subset \mathbb{R}^d$ satisfying $F \subset \overline{O}$ and $\partial O \subset F$. Here $\overline{O}$ and ∂O respectively denote the closure and the boundary of O. Since we are particularly interested in sets of a fractal nature, we focus on systems satisfying $\lambda_d(O \setminus \Phi\overline{O}) > 0$, where λ_d denotes the d-dimensional Lebesgue measure. In this case, the set $\overline{O} \setminus \Phi\overline{O}$ is assumed to possess a finite number of connected components, which we denote by $G^1, \ldots, G^Q$, for some $Q \in \mathbb{N}$, and call them the primary gaps of F. The primary gaps have a special geometric meaning, namely, letting G^i_ω denote the image of G^i under the map $\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_m}$ for $\omega = \omega_1 \cdots \omega_m \in \{1, \ldots, N\}^m =: \Sigma^m$, $m \in \mathbb{N}$, and setting $\Sigma^* := \{\varnothing\} \cup \bigcup_{m=1}^\infty \Sigma^m$ and $G^i_\varnothing := G^i$ for $i \in \{1, \ldots, Q\}$, the union $F \cup \bigcup_{\omega \in \Sigma^*} \bigcup_{i=1}^Q G^i_\omega$ provides a disjoint decomposition of $\overline{O}$. Our arguments suggest that it is natural to impose mild regularity conditions on the boundaries of O and $G^1, \ldots, G^Q$, namely that the upper Minkowski dimension of ∂O is strictly less than the Minkowski dimension δ of F and that a slightly weaker condition is satisfied for $\partial G^1, \ldots, \partial G^Q$. Note that the Minkowski dimension is proven to exist in [Bed88] for the sets we consider. With this background, we present a selection of our main results. Before stating the first one, we define $F_\varepsilon := \{x \in \mathbb{R}^d \mid \inf_{y \in F}|x - y| \leq \varepsilon\}$ for $\varepsilon > 0$, where $|x - y|$ denotes the Euclidean distance between $x, y \in \mathbb{R}^d$.

Theorem (Theorem 2.29, Remark 2.30)**.** *The average Minkowski content of F, which is defined to be $\widetilde{\mathcal{M}}(F) := \lim_{\varepsilon \to 0}|\ln \varepsilon|^{-1} \int_\varepsilon^1 T^{\delta-d-1}\lambda_d(F_T)\mathrm{d}T$, always exists and is equal to the well-defined positive and finite limit*

$$\frac{\delta}{H}\left(\lim_{m\to\infty} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G^i_\omega)\mathrm{d}T\right),$$

where the constant H is a measure theoretical entropy. Furthermore, in the non-lattice case, the Minkowski content $\mathcal{M}(F) := \lim_{\varepsilon\to 0} \varepsilon^{\delta-d}\lambda_d(F_\varepsilon)$ of F also exists and coincides with $\widetilde{\mathcal{M}}(F)$.

In the special case of F being a self-similar set the above formula simplifies to

$$\frac{\delta}{H}\int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G^i)\mathrm{d}T.$$

The preceding result emphasises the strong dependence of the Minkowski content on the structure of the gaps. However, often not only the global structure of a set is of interest, but also its local structure, since it contains more information on the texture of the set.

Therefore, we also consider a localised version of the Minkowski content. We discover that this localised version of the Minkowski content is a constant multiple of the δ-conformal measure ν associated with the cIFS Φ.

Theorem (Theorem 2.29)**.** *The local average Minkowski content of F, which is defined to be the weak limit $\widetilde{\mathcal{M}}(F,\cdot) := \text{w-lim}_{\varepsilon\to 0}|\ln\varepsilon|^{-1}\int_\varepsilon^1 T^{\delta-d-1}\lambda_d(F_T\cap\cdot)\mathrm{d}T$, exists and is equal to $\widetilde{\mathcal{M}}(F)\cdot\nu(\cdot)$. Further, in the non-lattice situation, the local Minkowski content $\mathcal{M}(F,\cdot) := \text{w-lim}_{\varepsilon\to 0}\,\varepsilon^{\delta-d}\lambda_d(F_\varepsilon\cap\cdot)$ also exists and is equal to $\widetilde{\mathcal{M}}(F)\cdot\nu(\cdot)$.*

The formula for the (average) Minkowski content in the first theorem is in the spirit of [DKz$^+$10, LPW11], where self-similar sets are considered. Our assumptions are significantly weaker. In particular, we do not require the condition of monophase main gaps and allow the boundaries of $G^1,\ldots,G^Q$ to be fractal. In this way, our results permit the study of a bigger variety of fractal sets, even in the self-similar setting. Our method of proof is based on analytic properties of the Perron-Frobenius operator, which are motivated by [Lal89].

An important subclass of self-conformal sets is formed by the class of conformal $\mathcal{C}^{1+\alpha}$-diffeomorphic images of self-similar sets, where $\alpha\in(0,1]$. In Theorem 2.41 we provide a neat relationship between the local Minkowski contents of a self-similar set $K\subset\mathbb{R}^d$ and its image $g(K)$. The statement is that $\widetilde{\mathcal{M}}(g(K),\cdot)$ is absolutely continuous with respect to the push-forward measure $g_\star\widetilde{\mathcal{M}}(K,\cdot)$ and their Radon-Nikodym derivative is given by

$$\frac{\mathrm{d}\widetilde{\mathcal{M}}(g(K),\cdot)}{\mathrm{d}\left(g_\star\widetilde{\mathcal{M}}(K,\cdot)\right)} = |g'\circ g^{-1}|^{\dim_M(K)},$$

where $\dim_M(K)$ denotes the Minkowski dimension of K. This yields the following nice relation between the respective Minkowski contents.

$$\mathcal{M}(g(K)) = \mathcal{M}(K)\cdot\int_K |g'|^{\dim_M(K)}\mathrm{d}\mu,$$

where μ denotes the $\dim_M(K)$-conformal measure associated with K (see Theorem 2.42). The class of diffeomorphic images of self-similar sets turns out to be important. We already alluded to the crucial difference between self-similar and general self-conformal subsets of $\mathbb{R}$, which is that self-similar subsets of $\mathbb{R}$ with zero Lebesgue measure satisfying the OSC are Minkowski measurable if and only if the underlying system is non-lattice, whereas we show that there exist self-conformal sets of zero Lebesgue measure arising from lattice cIFS which are Minkowski measurable. We provide a general condition under which the Minkowski content of a lattice self-conformal subset of $\mathbb{R}$ is proven to exist. This condition simplifies in the case that the self-conformal set can be obtained as a $\mathcal{C}^{1+\alpha}$-diffeomorphic image of a self-similar set, yielding that even amongst this class there exist Minkowski measurable sets arising from lattice systems. As a consequence, we gain the following for the Middle Third Cantor set, which is a self-similar set that is well known to be lattice.

Theorem (Example 2.45). *Let $C \subset \mathbb{R}$ denote the Middle Third Cantor set. Then, there exists a $\mathcal{C}^{1+\alpha}$-diffeomorphism g such that $g(C)$ is Minkowski measurable, whereas C is not Minkowski measurable.*

Note that this result together with Corollary 2.3 of [LP93] shows that there exist fractal strings having invariant sets of lattice cIFS with zero Lebesgue measure for boundary, for which the asymptotic second term of the eigenvalue counting function of the Laplacian is monotonic. In Conjecture 4 of [Lap93] it was conjectured that for 'approximately' self-similar sets monotonic behaviour of the asymptotic second term occurs if and only if the system is non-lattice. Conformal maps locally behave like similarities and thus the above theorem disproves the conjecture for self-conformal sets. Lattice cIFS which arise via conjugation of cIFS consisting of similarities play an important role in the general theory of lattice cIFS, which is stated in our next theorem.

Theorem (Theorem 2.46). *Suppose that $d = 1$ and assume that Φ consists of analytic maps and is lattice. Then there exists a self-similar set $K \subset \mathbb{R}$ and a map g which is analytic on an open neighbourhood of K, such that $F = g(K)$.*

An important generalisation of cIFS is given by conformal graph directed Markov systems (cGDMS). Such systems are presented in [MU03]. They allow us to study for example cIFS, cIFS with a transition rule, cIFS, where the open set O is not necessarily connected, Markov interval maps and Fuchsian groups of Schottky type, to name a few. Thus, all the results that we present for limit sets of cGDMS are automatically valid for self-conformal sets. In the special case that the cGDMS consists of similarities, we write sGDMS.

Theorem (Theorem 5.16). *Limit sets of sGDMS in $\mathbb{R}$ which have zero Lebesgue measure are Minkowski measurable if and only if the sGDMS is non-lattice.*

The preceding theorem provides a considerable generalisation of the respective dichotomy for self-similar sets given in [LP93, Fal95, LvF06]. We have already seen, that this dichotomy fails to hold for general self-conformal subsets of $\mathbb{R}$. We additionally show that there exist limit sets of lattice cGDMS, which are Minkowski measurable and are not self-conformal sets. We show that limit sets of non-lattice cGDMS on the other hand are always Minkowski measurable. This together with Corollary 2.3 of [LP93] verifies Conjecture 4 of [Lap93] for limit sets of Fuchsian groups of Schottky type, since such systems are non-lattice as is stated in [Lal89]. Further results concerning the Minkowski measurability of limit sets of general cGDMS (consisting of conformal maps) are presented in the more general framework of fractal curvature measures.

Notions of curvature are an important tool to describe the geometric structure of sets and have been introduced and intensively studied for broad classes of sets. Originally, the idea to characterise sets in terms of their curvature stems from the study of smooth manifolds as well as from the theory of convex bodies with sufficiently smooth boundaries. In his fundamental paper *Curvature Measures* [Fed59], Federer localises, extends and unifies the previously existing notions of curvature to sets of positive reach. This is where he introduces curvature measures, which can be viewed as a measure theoretical substitute for the notion of curvature for sets without a differentiable structure. Federer's curvature measures have been studied and generalised in various ways. An extension to finite unions of convex bodies is given in [Gro78] and [Sch80] and to finite unions of sets with positive reach in [Zäh84]. In [Win08], Winter extends the curvature measures to fractal sets in $\mathbb{R}^d$, which typically cannot be expressed as finite unions of sets with positive reach. These measures are referred to as fractal curvature measures and are defined to be weak limits of rescaled versions of the curvature measures introduced by Federer, Groemer and Schneider. Winter also examines conditions for their existence in the self-similar case. These considerations are generalised in [WZ10, Zäh11]. There, for a compact set $F \subset \mathbb{R}^d$ it is assumed that the closure of the complement $\overline{\mathbb{R}^d \setminus F_\varepsilon}$ of F_ε is a set of positive reach for Lebesgue-almost all $\varepsilon > 0$. Note that this condition is automatically satisfied if the ambient space is of dimension ≤ 3, see [Fu85], and that the condition is not needed if $k \in \{d-1, d\}$ for arbitrary $d \in \mathbb{N}$. Under this assumption, Federer's curvature measures of $\overline{\mathbb{R}^d \setminus F_\varepsilon}$ are determined for Lebesgue-almost every $\varepsilon > 0$ and are denoted by $C_0(\overline{\mathbb{R}^d \setminus F_\varepsilon}, \cdot), \dots, C_d(\overline{\mathbb{R}^d \setminus F_\varepsilon}, \cdot)$. This gives rise to the definition

$$C_k(F_\varepsilon, \cdot) := (-1)^{d-1-k} C_k(\overline{\mathbb{R}^d \setminus F_\varepsilon}, \cdot)$$

for $k \in \{0, \dots, d-1\}$ and $C_d(F_\varepsilon, \cdot) := \lambda_d(F_\varepsilon \cap \cdot)$. Note that these definitions are consistent in that the terms on the left and right hand sides coincide if F_ε and $\overline{\mathbb{R}^d \setminus F_\varepsilon}$ are both of positive reach. For an intuitive understanding, we remark that $C_{d-1}(F_\varepsilon, \cdot)$ coincides with half the surface measure on the boundary ∂F_ε of F_ε, that is, with $\lambda_{d-1}(\partial F_\varepsilon, \cdot)/2$. From the above introduced quantities, the fractal curvature measures arise as the essential weak limits $\varepsilon^{s_k(F)} C_k(F_\varepsilon, \cdot)$ as ε tends to zero, where $s_k(F)$ is an appropriate scaling exponent. Rescaling is necessary, because the term $C_d(F_\varepsilon, \mathbb{R}^d)$ typically tends to zero as ε tends to zero for a fractal set F, whereas $C_{d-1}(F_\varepsilon, \mathbb{R}^d)$ tends to infinity. Note that the limit $\varepsilon^{s_d(F)} C_d(F_\varepsilon, \cdot)$ coincides with the local Minkowski content in case of convergence, if $s_d(F) = \delta - d$. The central question arising in this context is to identify those sets for which these weak limits exist. In [Win08, WZ10, Zäh11] it has been shown that the fractal curvature measures exist for self-similar sets with positive Lebesgue measure as well as for self-similar sets which are non-lattice and satisfy the open set condition. We provide a substantially different proof of these existence results under our geometric conditions, where the mild regularity conditions on the boundaries of O and $G^1, \ldots, G^Q$ are substituted by mild regularity conditions involving the curvature measures. Our result is as follows.

Theorem (Theorem 2.37)**.** *Suppose that Φ consists of similarities. Then the weak limit $\text{w-lim}_{\varepsilon\to 0}|\ln\varepsilon|^{-1}\int_\varepsilon^1 T^{\delta-k-1}C_k(F_T,\cdot)\mathrm{d}T$ exists for $k\in\{0,\ldots,d\}$ and is equal to the finite signed Borel measure*

$$\frac{\delta}{H}\int_{-\infty}^{\infty}\mathrm{e}^{-T(\delta-k)}\sum_{i=1}^{Q}C_k(F_{\mathrm{e}^{-T}},G^i)\mathrm{d}T\cdot\nu(\cdot), \tag{1.1}$$

where H is a measure theoretical entropy. In the non-lattice situation the essential weak limit $\text{ess-w-lim}_{\varepsilon\to 0}\,\varepsilon^{\delta-k}C_k(F_\varepsilon,\cdot)$ also exists and coincides with the measure from Equation (1.1).

Equation (1.1) provides a significantly different and useful formula to those presented in [Win08, WZ10, Zäh11]. We strengthen the above result when $d=1$. Note that if the ambient space is of dimension one, then $C_0(F_\varepsilon,\cdot)=\lambda_0(\partial F_\varepsilon\cap\cdot)/2$ and $C_1(F_\varepsilon,\cdot)=\lambda_1(F_\varepsilon\cap\cdot)$.

Theorem (Theorem 2.31)**.** *Assume that $d=1$ so that $F\subset\mathbb{R}$. Then for $k\in\{0,1\}$ the weak limit $\widetilde{C}_k^f(F,\cdot):=\text{w-lim}_{\varepsilon\to 0}|\ln\varepsilon|^{-1}\int_\varepsilon^1 T^{\delta-1-k}C_k(F_T,\cdot)\mathrm{d}T$ exists and satisfies*

$$\widetilde{C}_0^f(F,\cdot)=\frac{2^{-\delta}c}{H}\cdot\nu(\cdot)\qquad\text{and}\qquad\widetilde{C}_1^f(F,\cdot)=\frac{2^{1-\delta}c}{(1-\delta)H}\cdot\nu(\cdot),$$

where the constant c is given by the well-defined positive and finite limit

$$c:=\lim_{m\to\infty}\sum_{\omega\in\Sigma^m}\sum_{i=1}^{Q}|G_\omega^i|^\delta.$$

Here $|G_\omega^i|$ denotes the length of the interval G_ω^i and H is a measure theoretical entropy. In the non-lattice case, additionally the following limits exist $C_k^f(F,\cdot):=\lim_{\varepsilon\to 0}\varepsilon^{\delta-k}C_k(F_\varepsilon,\cdot)$ and satisfy $C_k^f(F,\cdot)=\widetilde{C}_k^f(F,\cdot)$. Furthermore, if Φ consists of analytic maps and is lattice, then $C_k^f(F,\cdot)$ does not exist.

We provide an analogous statement to the above theorem for limit sets of cGDMS, for which the formula is more involved (see Theorem 5.13). For the subclass of $\mathcal{C}^{1+\alpha}$-diffeomorphic images of limit sets of sGDMS we obtain nice relationships between the fractal curvature measures of the image and the original set. Letting K denote the limit set of a lattice sGDMS and letting g be a $\mathcal{C}^{1+\alpha}$-diffeomorphism, we show that $\widetilde{C}_k^f(g(K),\cdot)$ is absolutely continuous with respect to the push-forward measure $g_\star\widetilde{C}_k^f(K,\cdot)$ and their Radon-Nikodym derivative is given by

$$\frac{\mathrm{d}\widetilde{C}_k^f(g(K),\cdot)}{\mathrm{d}\left(g_\star\widetilde{C}_k^f(K,\cdot)\right)}=|g'\circ g^{-1}|^{\dim_M(K)}.$$

In the non-lattice situation, we prove that $C_k^f(g(K), \cdot)$ exists and coincides with $\widetilde{C}_k^f(g(K), \cdot)$. In the lattice case, $C_k^f(g(K), \cdot)$ does not exist. This is a particularly interesting observation, since the Minkowski content of a $\mathcal{C}^{1+\alpha}$-diffeomorphic image of a limit set of a lattice sGDMS can exist and since $C_1^f(g(K), \mathbb{R})$, by definition, coincides with the Minkowski content of $g(K)$.

The thesis is organised in the following way.

In Chapter 2 we rigorously define the central notions such as the Minkowski content, the fractal curvature measures, conformal iterated function systems and self-conformal sets. We also provide a detailed description of the geometric conditions that we impose on the self-conformal sets. Furthermore, we provide the complete exposition of the original results concerning self-conformal sets. That is, we precisely state the results presented above and provide some further related statements. Our results are illustrated by some examples at the end of Chapter 2. In Chapter 3 we provide background material on the Perron-Frobenius theory and on volume functions of parallel sets. This background is central for proving our results in Chapter 4. In the first section of Chapter 4 we prove strong auxiliary key results, which enable us to unify the proofs of all the theorems from Chapter 2. The following sections of Chapter 4 contain the proofs. In Chapter 5, we define limit sets of conformal graph directed Markov systems, whose importance we illustrate by a collection of examples. Further, we provide an exposition of our results concerning such limit sets. Finally, a short appendix is attached, which gives supplementary material on measure theory.

To assist the reader, we provide an index, which contains all the relevant notions and nomenclature.

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2 Central Notions and Main Results

The aim of this chapter is to present original results concerning the Minkowski content and the fractal curvature measures of invariant sets of conformal iterated function systems.

We start by setting up the necessary definitions in Sections 2.1 to 2.3. More precisely, in Section 2.1 we introduce the (local) Minkowski content and the fractal curvature measures. In Section 2.2 we define conformal iterated function systems and associated important terminology. Section 2.3 is devoted to geometric conditions that we impose on the self-conformal sets and their geometric meaning. In Section 2.4, we exhibit our main results concerning the (local) Minkowski content and the fractal curvature measures for conformal iterated function systems and by this link Sections 2.1 to 2.3 together. Finally, in Section 2.5 our results are illustrated by a collection of examples.

2.1 (Local) Minkowski Content and Fractal Curvature Measures

In this section we provide the definitions and constructions of the (local) Minkowski content and the fractal curvature measures. We start by fixing the following notation.

Let $\mathbb{R}$ denote the set of real numbers, set $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$ and $\mathbb{R}_0^+ := \{x \in \mathbb{R} \mid x \geq 0\}$. We denote by $\mathbb{N}$ the set of natural numbers not containing zero and set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. For a compact subset Y of the d-dimensional Euclidean space $(\mathbb{R}^d, |\cdot|)$, where $d \in \mathbb{N}$, and for $\varepsilon > 0$ we define

$$Y_\varepsilon := \{x \in \mathbb{R}^d \mid \inf_{y \in Y} |x - y| \leq \varepsilon\}$$

to be the *ε-parallel neighbourhood* of Y and

$$Y_{<\varepsilon} := \{x \in \mathbb{R}^d \mid \inf_{y \in Y} |x - y| < \varepsilon\}$$

to be the *open ε-parallel neighbourhood* of Y. Further, we let λ_d denote the d-dimensional Lebesgue measure on $\mathbb{R}^d$.

A key concept in fractal geometry is provided by several different notions of dimension, such as the Hausdorff, packing and Minkowski dimensions, to name but a few. They characterise the scaling behaviour of the considered set and in a certain sense classify fractals as objects 'between' the Euclidean spaces (see [Man95]). The notion of dimension plays a vital role also in this thesis. Of particular interest for our purposes is the Minkowski dimension. The

Minkowski dimension coincides with the box-counting dimension (see Claim 3.1 in [Fal03]) and is defined as follows.

Definition 2.1 (Minkowski dimension)**.** For a non-empty compact set $Y \subset \mathbb{R}^d$ the *upper* and *lower Minkowski dimensions* are respectively defined to be

$$\overline{\dim}_M(Y) := d - \liminf_{\varepsilon \searrow 0} \frac{\ln \lambda_d(Y_\varepsilon)}{\ln \varepsilon} \qquad \text{and} \qquad \underline{\dim}_M(Y) := d - \limsup_{\varepsilon \searrow 0} \frac{\ln \lambda_d(Y_\varepsilon)}{\ln \varepsilon}.$$

In case that the upper and lower Minkowski dimensions coincide, we call the common value the *Minkowski dimension* of Y and denote it by $\dim_M(Y)$.

Definition 2.2 ((Average) Minkowski content, Minkowski measurable)**.** Let $Y \subset \mathbb{R}^d$ denote a set for which the Minkowski dimension $\dim_M(Y)$ exists. The *upper Minkowski content* $\overline{\mathcal{M}}(Y)$ and the *lower Minkowski content* $\underline{\mathcal{M}}(Y)$ of Y are defined by

$$\overline{\mathcal{M}}(Y) := \limsup_{\varepsilon \searrow 0} \varepsilon^{\dim_M(Y)-d} \lambda_d(Y_\varepsilon) \qquad \text{and} \qquad \underline{\mathcal{M}}(Y) := \liminf_{\varepsilon \searrow 0} \varepsilon^{\dim_M(Y)-d} \lambda_d(Y_\varepsilon).$$

If the upper and lower Minkowski contents coincide, we denote the common value by $\mathcal{M}(Y)$ and refer to it as the *Minkowski content* of Y. When the Minkowski content exists, is positive and finite, then we say that Y is *Minkowski measurable*. The *average Minkowski content* of Y is defined to be

$$\widetilde{\mathcal{M}}(Y) := \lim_{\varepsilon \searrow 0} |\ln \varepsilon|^{-1} \int_\varepsilon^1 T^{\dim_M(Y)-d-1} \lambda_d(Y_T) \mathrm{d}T,$$

provided that the limit exists.

Remark 2.3. The average Minkowski content is a logarithmic Cesàro average. If the Minkowski content of Y exists, then the average Minkowski content of Y also exists and $\mathcal{M}(Y) = \widetilde{\mathcal{M}}(Y)$.

Observe that, by definition, whenever it exists, the Minkowski content gives the asymptotic scaling factor between $\varepsilon^{\dim_M(Y)-d}$ and $\lambda_d(Y_\varepsilon)$. Thus, it detects the rate at which the d-dimensional volume of Y_ε shrinks as ε tends to zero and can be viewed as a beneficial complement to the notion of dimension. More precisely, two sets with the same Minkowski dimension which exhibit different shrinking rates can be distinguished by the value of their Minkowski contents. Examples of fractals with the same Minkowski dimension but different Minkowski contents are the two Cantor sets C_1 and C_2 from the introduction, which we return to at the beginning of Section 2.5. For further insight into this matter, we refer the reader to [Man82, Man95].

Moreover, the Minkowski content can be viewed as an analogue of the notion of length, area or volume (depending on the dimension) for fractional dimensional sets. This is motivated in [Fal97, p.45] in the following way: "In $\mathbb{R}^3$, if Y is a single point then Y_ε is a ball with $\lambda_3(Y_\varepsilon) = \frac{4}{3}\pi\varepsilon^3$, if Y is a segment of length l then Y_ε is 'sausage-like' with $\lambda_3(Y_\varepsilon) \sim \pi l \varepsilon^2$, and if Y is a flat set of area a then Y_ε is essentially a thickening of Y with $\lambda_3(Y_\varepsilon) \sim 2a\varepsilon$. In each case, $\lambda_3(Y_\varepsilon) \sim c\varepsilon^{3-\delta}$, where the integer δ is the dimension of Y, so that the exponent of ε is indicative of the dimension. The coefficient c of $\varepsilon^{3-\delta}$, known as the Minkowski content of Y, is a measure of the length, area or volume of the set as appropriate." Here, the meaning of $f(\varepsilon) \sim g(\varepsilon)$ for two functions $f, g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is that $\lim_{\varepsilon\to 0} f(\varepsilon)/g(\varepsilon) = 1$.

Often, not only the global structure of a set is of interest, but also its local structure, since it contains more information on the 'texture' of the set itself. This information is reflected by the local (average) Minkowski content, which gives a refinement of the (average) Minkowski content. For its definition we use terminology from measure theory for which we refer the reader to the appendix.

Definition 2.4 (Local (average) Minkowski content)**.** Let $Y \subset \mathbb{R}^d$ denote a non-empty compact set whose Minkowski dimension $\dim_M(Y)$ exists. The *local Minkowski content* $\mathcal{M}(Y, \cdot)$ of Y is defined to be the finite Borel measure

$$\mathcal{M}(Y, \cdot) := \operatorname*{w-lim}_{\varepsilon \searrow 0} \varepsilon^{\dim_M(Y)-d} \lambda_d(Y_\varepsilon \cap \cdot),$$

whenever this weak limit exists. The *local average Minkowski content* of Y is defined to be the finite Borel measure $\widetilde{\mathcal{M}}(Y, \cdot)$ which arises as the weak limit

$$\widetilde{\mathcal{M}}(Y, \cdot) := \operatorname*{w-lim}_{\varepsilon \searrow 0} |\ln \varepsilon|^{-1} \int_\varepsilon^1 T^{\dim_M(Y)-d-1} \lambda_d(Y_T \cap \cdot) \mathrm{d}T,$$

whenever this weak limit exists. Moreover, for a Borel set $B \in \mathfrak{B}(\mathbb{R}^d)$ we define

$$\overline{\mathcal{M}}(Y, B) := \limsup_{\varepsilon \searrow 0} \varepsilon^{\dim_M(Y)-d} \lambda_d(Y_\varepsilon \cap B) \qquad \text{and}$$
$$\underline{\mathcal{M}}(Y, B) := \liminf_{\varepsilon \searrow 0} \varepsilon^{\dim_M(Y)-d} \lambda_d(Y_\varepsilon \cap B),$$

where $\mathfrak{B}(\mathbb{R}^d)$ denotes the Borel σ-algebra on $\mathbb{R}^d$.

The local Minkowski content appears as one of the fractal curvature measures, which are defined by Winter in [Win08]. These measures provide a set of geometric characteristics for fractal sets and were introduced for extending the notion of curvature to the fractal setting. The definition of Winter's fractal curvature measures is based on the curvature measures which were introduced by Federer in [Fed59]. Federer's curvature measures localise, extend and unify the previously existing notions of curvature in differential and convex geometry. Before presenting Federer's curvature measures, we fix the following notation.

Figure 2.1: The set X is not of positive reach, since every point on the drawn line segments has two closest neighbours in X. The set Y is of positive finite reach. Its reach is equal to the radius r of the complementary circle.

The *reach* of a compact set $Y \subset \mathbb{R}^d$ is defined to be

$$\operatorname{reach}(Y) := \sup\{\varepsilon > 0 \mid \forall\, x \in Y_\varepsilon\ \exists!\ y \in Y : |x - y| = \varepsilon\},$$

where, following convention, $\sup(\varnothing) := -\infty$. A compact set $Y \subset \mathbb{R}^d$ is said to be of *positive reach* if $\operatorname{reach}(Y) > 0$. For a set Y of positive reach the *metric projection* pr_Y onto Y is defined to be the map

$$\operatorname{pr}_Y : Y_{<\operatorname{reach}(Y)} \to Y$$

which maps a point in $Y_{<\operatorname{reach}(Y)}$ to its unique closest neighbour in Y. Examples for sets of positive reach are non-empty, compact and convex sets. Note that these sets are even of infinite reach. An example for a set which is not of positive reach and an example for a set which is of positive finite reach are given in Figure 2.1. For sets of positive reach, the local Steiner formula shows that the volume of the ε-parallel neighbourhood can be expressed as a polynomial in ε. Federer's curvature measures arise as the coefficients of this polynomial expansion.

Theorem 2.5 (Local Steiner formula, [Fed59])**.** *Let $Y \subset \mathbb{R}^d$ be a compact set of positive reach. Then there exist uniquely defined signed Borel measures $C_0(Y, \cdot), \ldots, C_d(Y, \cdot)$ such that for every $B \in \mathfrak{B}(\mathbb{R}^d)$ and every $\varepsilon \in [0, \operatorname{reach}(Y))$ we have*

$$\lambda_d(Y_\varepsilon \cap \operatorname{pr}_Y^{-1}(B)) = \sum_{k=0}^{d} \varepsilon^{d-k} \kappa_{d-k} C_k(Y, B),$$

where κ_k denotes the k-dimensional volume of the k-dimensional unit ball.

Definition 2.6 (Curvature measure, total curvature)**.** For a compact set $Y \subset \mathbb{R}^d$ of positive reach and $k \in \{0, \ldots, d\}$ the *k-th curvature measure* of Y is defined to be the measure $C_k(Y, \cdot)$ from Theorem 2.5. The *k-th total curvature* of Y is defined to be the value $C_k(Y) := C_k(Y, \mathbb{R}^d)$.

The total curvatures are also known as *intrinsic volumes*, *Minkowski functionals* or *Quermassintegrale*. These are well-studied objects in classical convex geometry. Federer's curvature measures refine these notions for non-empty, compact, convex sets and can be viewed as a measure theoretical substitute of the notion of curvature for sets without a differentiable structure. Indeed, if the boundary of the convex set $Y \subset \mathbb{R}^d$ is twice continuously differentiable, then the curvature measures $C_0(Y,\cdot),\ldots,C_{d-1}(Y,\cdot)$ can be gained by integrating the elementary symmetric functions of principle curvature. That is

$$C_k(Y,B) = c(d,k) \cdot \int_{\partial Y \cap B} \sigma_{d-1-k} \mathrm{d}\mathcal{H}^{d-1}$$

for $k \in \{0,\ldots,d-1\}$, where $c(d,k)$ is a constant which only depends on d and k, $\mathcal{H}^{d-1}$ denotes the $(d-1)$-dimensional Hausdorff measure and σ_k is the k-th elementary symmetric function of principal curvature given by

$$\sigma_0 := 1 \quad \text{and} \quad \sigma_k := \sum_{1 \leq i_1 < \ldots < i_k \leq d-1} \kappa_{i_1} \cdot \ldots \cdot \kappa_{i_k} \quad \text{for } k \in \{1,\ldots,d-1\}.$$

Here, $\kappa_1,\ldots,\kappa_{d-1}$ denote the principal curvatures of Y. This connection is proven in [Sch93, p. 206] and justifies why Federer's measures are called curvature measures.

Some useful properties and geometric characterisations of the curvature measures are stated in the following proposition and can, for instance, be found in Chapters 5 and 6 of [Fed59]. Further properties, especially for convex sets, are given in [Sch78, SW92].

Proposition 2.7. *Let $Y \subset \mathbb{R}^d$ denote a compact set of positive reach, whose boundary is denoted by ∂Y and let $B \in \mathfrak{B}(\mathbb{R}^d)$ denote a bounded Borel set. Then the following hold.*

(i) $C_d(Y,B) = \lambda_d(Y \cap B)$.

(ii) $C_{d-1}(Y,B) = \lambda_{d-1}(\partial Y, B)/2$.

(iii) $C_0(Y)$ *is equal to the Euler-Poincaré characteristic of* Y.

(iv) $C_k(Y,B) = C_k(Y, B \cap \partial Y)$ *for* $k \in \{0,\ldots,d-1\}$, *which means that the k-th curvature measure of Y is concentrated on the boundary of Y for $k < d$.*

(v) Let $X, Y \subseteq \mathbb{R}^d$ be such that X, Y and $X \cap Y$ are of positive reach. If B is contained in the interior of X, then

$$C_k(X \cap Y, B) = C_k(Y,B) \qquad \text{for } k \in \{0,\ldots,d\}.$$

(vi) For $\beta > 0$ and $k \in \{0,\ldots,d\}$ we have

$$C_k(\beta Y, \beta B) = \beta^k \cdot C_k(Y,B).$$

Federer's curvature measures were studied and generalised in various ways. An extension to finite unions of non-empty compact convex sets is given in [Gro78, Sch80] and to finite unions of sets with positive reach in [Zäh84]. Unfortunately, fractal sets are commonly not representable as finite unions of sets of positive reach and thus Federer's curvature measures are a priori not defined for fractal sets. This is the reason why fractal curvature measures are introduced via a limiting procedure in [Win08]. There, fractal sets with polyconvex parallel sets are considered. We now present the more general approach for defining fractal curvature measures from [WZ10, Zäh11].

Let $\overline{X}$ denote the closure of a set $X \subset \mathbb{R}^d$ and in the following, let $Y \subset \mathbb{R}^d$ denote a compact set. A distance $\varepsilon > 0$ is called *regular* for Y, if ∂Y_ε is a Lipschitz manifold and the closure of the complement $\overline{\mathbb{R}^d \setminus Y_\varepsilon}$ of Y_ε is of positive reach. Thus, for regular $\varepsilon > 0$ the curvature measures of $\overline{\mathbb{R}^d \setminus Y_\varepsilon}$ are determined. This allows one to define the *k-th curvature measure* $C_k(Y_\varepsilon, \cdot)$ of Y_ε through

$$C_k(Y_\varepsilon, \cdot) := (-1)^{d-1-k} C_k(\overline{\mathbb{R}^d \setminus Y_\varepsilon}, \cdot) \tag{2.1}$$

for regular $\varepsilon > 0$ and $k \in \{0, \dots, d-1\}$. This definition is consistent, in that equality in the above equation is ensured, if both Y_ε and $\overline{R^d \setminus Y_\varepsilon}$ are of positive reach.

Remark 2.8. For regular $\varepsilon > 0$, Proposition 2.7(ii) implies that

$$C_{d-1}(\overline{\mathbb{R}^d \setminus Y_\varepsilon}, \cdot) = \lambda_{d-1}(\partial(\overline{\mathbb{R}^d \setminus Y_\varepsilon}) \cap \cdot)/2 = \lambda_{d-1}(\partial Y_\varepsilon \cap \cdot)/2.$$

For the sets Y, that we consider in the following, $\lambda_{d-1}(\partial Y_\varepsilon \cap \cdot)/2$ is a finite measure for all $\varepsilon > 0$. Therefore, this notion is used in any case. The family of curvature measures is completed by the volume measure

$$C_d(Y_\varepsilon, \cdot) := \lambda_d(Y_\varepsilon \cap \cdot)$$

for any $\varepsilon > 0$. Note that this definition is consistent with Proposition 2.7(i).

Under the assumption that Lebesgue-almost all $\varepsilon > 0$ are regular distances for a compact set Y, the fractal curvature measures of Y can be introduced via a limiting procedure of the curvature measures of Y_ε as ε tends to zero. Note, that the assumption of the regular distances having full measure is natural, since for any compact set $Y \subset \mathbb{R}^d$ in dimension $d \leq 3$, Lebesgue-almost all $\varepsilon > 0$ are regular distances for Y (see [Fu85]). However, this is no longer true in dimension $d \geq 4$ (see [Fer76]). It is nevertheless remarkable to note that in any dimension all sufficiently large distances, more precisely all $\varepsilon > \sqrt{d/(2d+2)}\operatorname{diam}(Y)$, are regular distances for Y (see [Fu85]). Here, $\operatorname{diam}(Y) := \sup_{x,y \in Y} |x - y|$ denotes the *diameter* of Y.

We call the Borel measure, which assigns measure zero to every Borel set *trivial*. Every other signed Borel measure is called *non-trivial*. For a function $f \colon U \to \mathbb{R}$ defined on

$U \subseteq X \in \{\mathbb{R}, \mathbb{R}_0^+\}$ with $\lambda_1(X \setminus U) = 0$, we say that $f(t)$ *essentially converges* as t tends to $x \in X$, if there exists a $y \in \mathbb{R}$ such that $\lim_{n\to\infty} f(u_n) = y$ for every sequence $(u_n)_{n\in\mathbb{N}}$ satisfying $u_n \in U$ and $\lim_{n\to\infty} u_n = x$. In this case, we write $\text{ess-lim}_{t\to x} f(t) = y$. The *essential limit superior* (ess-limsup), the *essential limit inferior* (ess-liminf) and the *essential weak limit* (ess-w-lim) are defined accordingly.

Definition 2.9 ((Average) fractal curvature measures, total (average) fractal curvature)**.** Let $Y \subset \mathbb{R}^d$ denote a non-empty compact set for which the Minkowski dimension $\dim_M(Y)$ exists and for which $\lambda_1(\mathbb{R}^+ \setminus \widetilde{U}_k) = 0$, where $\widetilde{U}_k := \{\varepsilon > 0 \mid \varepsilon \text{ is a regular distance for } Y\}$ if $d \geq 4$ and $k \in \{0, \ldots, d-2\}$, and $\widetilde{U}_k := \mathbb{R}^+$ otherwise. Fix $k \in \{0, \ldots, d\}$. Provided that the essential weak limit

$$C_k^f(Y, \cdot) := \underset{\varepsilon \searrow 0}{\text{ess-w-lim}}\, \varepsilon^{\dim_M(Y)-k} C_k(Y_\varepsilon, \cdot)$$

of the finite signed Borel measures $\varepsilon^{\dim_M(Y)-k} C_k(Y_\varepsilon, \cdot)$ exists and is non-trivial, where the essential limit is taken over $\varepsilon \in \widetilde{U}_k$, we call it the *k-th fractal curvature measure* of Y. Moreover, for a Borel set $B \in \mathfrak{B}(\mathbb{R}^d)$ we set

$$\overline{C}_k^f(Y, B) := \underset{\varepsilon \searrow 0}{\text{ess-limsup}}\, \varepsilon^{\dim_M(Y)-k} C_k(Y_\varepsilon, B) \quad \text{and}$$
$$\underline{C}_k^f(Y, B) := \underset{\varepsilon \searrow 0}{\text{ess-liminf}}\, \varepsilon^{\dim_M(Y)-k} C_k(Y_\varepsilon, B),$$

where the essential limits are taken over $\varepsilon \in \widetilde{U}_k$. Provided it exists and the limiting signed measure is non-trivial, the weak limit

$$\widetilde{C}_k^f(Y, \cdot) := \underset{\varepsilon \searrow 0}{\text{w-lim}} |\ln \varepsilon|^{-1} \int_\varepsilon^1 T^{\dim_M(Y)-k-1} C_k(Y_T, \cdot) \mathrm{d}T$$

is called the *k-th average fractal curvature measure* of Y. Finally,

$$C_k^f(Y) := C_k^f(Y, \mathbb{R}^d) \quad \text{and} \quad \widetilde{C}_k^f(Y) := \widetilde{C}_k^f(Y, \mathbb{R}^d)$$

are respectively called the *k-th total fractal curvature* and the *k-th total average fractal curvature* if they are non-zero.

Note that $C_d^f(Y, \cdot)$ and $\widetilde{C}_d^f(Y, \cdot)$ respectively coincide with $\mathcal{M}(Y, \cdot)$ and $\widetilde{\mathcal{M}}(Y, \cdot)$, whenever they exist and that $\widetilde{C}_k^f(Y, \cdot)$ is defined to be the weak limit of the logarithmic Cesàro-averages of $\varepsilon^{\dim_M(Y)-k} C_k(Y_\varepsilon, \cdot)$.

Remark 2.10. In [Win08], the fractal curvature measures of Y were actually introduced as the weak limits $\text{w-lim}_{\varepsilon\to 0} \varepsilon^{s_k(Y)} C_k(Y_\varepsilon, \cdot)$, where $s_k(Y)$ is a scaling exponent which is defined to be

$$s_k(Y) := \inf\{t > 0 \mid \varepsilon^t C_k^{\text{var}}(Y_\varepsilon) \to 0 \text{ as } \varepsilon \text{ tends to zero}\}.$$

Here, $C_k^{\text{var}}(Y_\varepsilon, \cdot)$ denotes the total variation measure of $C_k(Y_\varepsilon, \cdot)$ (see the appendix) and $C_k^{\text{var}}(Y_\varepsilon) := C_k^{\text{var}}(Y_\varepsilon, \mathbb{R}^d)$. In Proposition 2.2.10 of [Win08] it has been shown that $s_k(Y) = 0$ and that $C_k(Y) = \lim_{\varepsilon\to 0} C_k(Y_\varepsilon)$, whenever Y is a finite union of non-empty compact convex sets and satisfies $C_k(Y) \neq 0$. Thus, the definition is consistent with Federer's definition for such sets. In [Win08] it is moreover shown for certain self-similar sets Y that if $\widetilde{C}_k^f(Y) \neq 0$, then $s_k(Y) = \dim_M(Y) - k$. However, when $\widetilde{C}_k^f(Y) = 0$ both is possible, $s_k = \dim_M(Y) - k$ or $s_k < \dim_M(Y) - k$. Even for self-similar sets, it is not clear as to what the right scaling exponents should be if $\widetilde{C}_k^f(Y) = 0$. Thus, since we consider the more general self-conformal sets, we restricted ourselves to the situation that $\widetilde{C}_k^f(Y) \neq 0$ in Definition 2.9.

Before we give the precise definition of self-conformal sets in the next section, we conclude this section with statements on the relevance of the fractal curvature measures. The fractal curvature measures not only extend the notion of curvature to the fractal setting but also form a family of geometric characteristics for fractal sets. The d-th total fractal curvature can be interpreted as a 'fractal volume', the $(d-1)$-st total fractal curvature can be viewed as a 'fractal surface area' and the 0-th total fractal curvature can be interpreted as a 'fractal Euler number'. These interpretations are given in [Win08]. They are based on the properties in Proposition 2.7 and the geometric considerations from [Fal03, p.45] which we presented earlier in this section in the context of the Minkowski content. Studies on the geometric meaning of the total fractal curvatures have also been carried out in [Kom08], where an intuitive approach for understanding the geometric relevance is presented, which is illustrated by a series of examples.

2.2 Conformal Iterated Function Systems

The central objects of our study are self-conformal sets. They arise as the invariant sets of iterated function systems which consist of certain conformal maps. Such systems were intensively studied by Mauldin, Urbanski and their co-authors (see for example [MU96, MU03]). On our way to define these iterated function systems, we start by giving fundamental definitions through which we also fix our notation.

Definition 2.11 (Similarity, conformal map). A function $f\colon U \to \mathbb{R}^d$, which is defined on an open connected set $U \subseteq \mathbb{R}^d$ is called a *similarity*, if there exists an $r > 0$ such that $|f(x) - f(y)| = r|x - y|$ for all $x, y \in U$. We refer to r as the *similarity ratio* of f. A $\mathcal{C}^1$-diffeomorphism $f\colon U \to V$ between two open connected sets $U, V \subset \mathbb{R}^d$ is called *conformal* if its total derivative at every point of U is a similarity. In this case, we let $|f'(x)| \in \mathbb{R}$ denote the similarity ratio of the total derivative of f at $x \in U$ and call it the *length scaling ratio* of f at x.

Definition 2.12 (α-Hölder continuous, $\mathcal{C}^{1+\alpha}(U)$)**.** A map $f\colon X \to X'$ between two metric spaces (X, ϱ_X) and $(X', \varrho_{X'})$ is called *α-Hölder continuous* for $\alpha \in (0,1]$, if there exists a $c \in \mathbb{R}$ such that $\varrho_{X'}(f(x), f(y)) \leq c \cdot \varrho_X(x,y)^\alpha$ for all $x, y \in X$. In this case, we call α the *Hölder exponent* and c the *Hölder constant* of f. A $\mathcal{C}^1$-diffeomorphism $f\colon U \to V$ between two open connected subsets U, V of the Euclidean space $(\mathbb{R}^d, |\cdot|)$ belongs to $\mathcal{C}^{1+\alpha}(U)$ for $\alpha \in (0,1]$ if its total derivative Df is α-Hölder continuous; that is if there exists a $c \in \mathbb{R}$ such that $\|Df(x) - Df(y)\|_{\text{op}} \leq c \cdot |x-y|^\alpha$ for all $x, y \in U$. Here, $\|\cdot\|_{\text{op}}$ denotes the *operator norm*, which for a linear operator $A\colon \mathbb{R}^d \to \mathbb{R}^d$ is defined by $\|A\|_{\text{op}} := \sup_{x \in \mathbb{R}^d,\, |x|=1} |Ax|$.

Some well-known facts about conformal maps are compiled in the next proposition.

Proposition 2.13. *(i) For $d = 1$, f being conformal means that f is a strictly monotonic $\mathcal{C}^1$-diffeomorphism.*

(ii) For $d = 2$, conformal means holomorphic or anti-holomorphic with non-zero derivative.

(iii) For $d \geq 3$ every conformal map f defined on an open set U extends to the entire space $\mathbb{R}^d$ and takes the form

$$f = \beta A \circ i + b,$$

where $\beta > 0$, A is a linear isometry in $\mathbb{R}^d$, i is either an inversion with respect to a sphere or the identity map and $b \in \mathbb{R}^d$. This is known as Liouville's theorem (see for instance Chapter A.3 in [BP92]).

Remark 2.14. From Proposition 2.13 one can conclude that if $f \in \mathcal{C}^{1+\alpha}(U)$ is conformal, then the length scaling ratio $|f'|\colon U \to \mathbb{R}$ satisfies $||f'(x)| - |f'(y)|| \leq c|x-y|^\alpha$ for all $x, y \in U$ and some $c \in \mathbb{R}$ and thus is α-Hölder continuous.

Having considered conformal maps, we now turn to iterated function systems and show how their invariant sets can be encoded.

Definition 2.15 (Contraction, IFS)**.** Let (X, ϱ) denote a non-empty compact metric space. A function $\phi\colon X \to X$ is called a *contraction* if there exists a real number $r \in (0,1)$ such that $\varrho(\phi(x), \phi(y)) \leq r \cdot \varrho(x,y)$ for all $x, y \in X$. An *iterated function system* (*IFS*) acting on X is a collection of injective contractions $\Phi := \{\phi_i\colon X \to X \mid i \in \Sigma\}$, where Σ is a non-empty finite index-set containing at least two elements.

Theorem 2.16 (Hutchinson)**.** *For an IFS $\Phi := \{\phi_i\colon X \to X \mid i \in \Sigma\}$ acting on a metric space X, there exists a unique, non-empty and compact subset $F \subseteq X$, which is* invariant *under Φ, that is*

$$F = \bigcup_{i \in \Sigma} \phi_i F =: \Phi F.$$

The above theorem is a famous result in fractal geometry and can for instance be found in Theorem 9.1 of [Fal03]. The unique non-empty compact invariant set of an IFS can be encoded by the code space, which is introduced below.

The Code Space. For an IFS $\Phi := \{\phi_i \colon X \to X \mid i \in \Sigma\}$, we call $\Sigma =: \{1, \ldots, N\}$ the *alphabet*, where $N \geq 2$. We let Σ^n denote the set of words of length $n \in \mathbb{N}$ over Σ and let $\Sigma^* := \bigcup_{n \in \mathbb{N}_0} \Sigma^n$ denote the set of all finite words over Σ including the empty word $\varnothing$, where $\Sigma^0 := \{\varnothing\}$. For two finite words $u = u_1 \cdots u_n$, $\omega = \omega_1 \cdots \omega_m \in \Sigma^*$, we let $u\omega := u_1 \cdots u_n \omega_1 \cdots \omega_m \in \Sigma^*$ denote their concatenation. Likewise, we set $u\omega := u_1 \cdots u_n \omega_1 \omega_2 \cdots \in \Sigma^\infty$ for $u = u_1 \cdots u_n \in \Sigma^*$ and $\omega = \omega_1 \omega_2 \cdots \in \Sigma^\infty$. For $\omega = \omega_1 \cdots \omega_n \in \Sigma^*$ we set $\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_n}$ and define $\phi_\varnothing := \mathrm{id}_X$ to be the identity map on X. For a finite word $\omega \in \Sigma^*$, we let $n(\omega)$ denote its length, where $n(\varnothing) := 0$. Further, we call the set Σ^∞ of infinite words over Σ the *code space*. The code space Σ^∞ gives a coding of the unique non-empty compact invariant set of an IFS Φ which can be seen as follows. For $\omega = \omega_1 \omega_2 \cdots \in \Sigma^\infty$ and $n \in \mathbb{N}$ we denote the *initial word of length* n by $\omega|_n := \omega_1 \omega_2 \cdots \omega_n$. For each $\omega \in \Sigma^\infty$ the intersection $\bigcap_{n \in \mathbb{N}} \phi_{\omega|_n}(X)$ contains exactly one point $x_\omega \in X$ and gives rise to a continuous surjection $\pi \colon \Sigma^\infty \to F$, $\omega \mapsto x_\omega$ which we call the *code map*.

This notation allows us to introduce conformal iterated function systems and to describe their properties.

Definition 2.17 (cIFS)**.** Let X be a compact connected subset of the d-dimensional Euclidean Space $(\mathbb{R}^d, |\cdot|)$. An IFS $\Phi := \{\phi_i \colon X \to X \mid i \in \Sigma\}$ is said to be a *conformal iterated function system (cIFS)* acting on X, provided

(i) $\mathrm{int}_{\mathbb{R}^d}(X) \neq \varnothing$ and $\overline{\mathrm{int}_{\mathbb{R}^d}(X)} = X$, where $\mathrm{int}_{\mathbb{R}^d}(X)$ denotes the topological interior of X,

(ii) Φ satisfies the *open set condition (OSC)* with open set $O := \mathrm{int}_{\mathbb{R}^d}(X)$, that is $\phi_i(O) \subseteq O$ for every $i \in \Sigma$ and $\phi_i(O) \cap \phi_j(O) = \varnothing$ for distinct $i, j \in \Sigma$ and

(iii) there exists an open connected subset $V \supset X$ of $\mathbb{R}^d$ and an $\alpha \in (0, 1]$ such that for every $i \in \Sigma$ the map ϕ_i is conformal on V and belongs to $\mathcal{C}^{1+\alpha}(V)$.

Definition 2.18 (Self-conformal set, self-similar set)**.** We call the unique non-empty compact invariant set of a cIFS Φ, which exists by Theorem 2.16, the *self-conformal set* associated with Φ. If the maps $\phi_1, \ldots, \phi_N$ of the cIFS Φ are similarities, then the unique non-empty compact invariant set is called the *self-similar set* associated with Φ.

A crucial property of a cIFS with regard to our results is the property of being lattice or non-lattice. For defining these terms we now introduce the shift space and the geometric potential function.

The Shift Space. The *shift space* (Σ^∞, σ) is given by the code space Σ^∞ together with the *shift-map* σ which is defined to be the map $\sigma\colon \Sigma^* \cup \Sigma^\infty \to \Sigma^* \cup \Sigma^\infty$ given by $\sigma(\omega) := \varnothing$ for $\omega \in \{\varnothing\} \cup \Sigma^1$, $\sigma(\omega_1 \cdots \omega_n) := \omega_2 \cdots \omega_n \in \Sigma^{n-1}$ for $\omega_1 \cdots \omega_n \in \Sigma^n$, where $n \geq 2$ and $\sigma(\omega_1\omega_2\cdots) := \omega_2\omega_3\cdots \in \Sigma^\infty$ for $\omega_1\omega_2\cdots \in \Sigma^\infty$.

We equip Σ^∞ with the product topology of the discrete topologies on Σ and denote by $\mathcal{C}(\Sigma^\infty)$ the set of complex-valued continuous functions on Σ^∞.

Definition 2.19 (Cohomologous, (non-) lattice function)**.** Two functions $f_1, f_2 \in \mathcal{C}(\Sigma^\infty)$ are called *cohomologous*, if there exists a $\psi \in \mathcal{C}(\Sigma^\infty)$ such that $f_1 - f_2 = \psi - \psi \circ \sigma$. A function $f \in \mathcal{C}(\Sigma^\infty)$ is said to be *lattice*, if f is cohomologous to a function whose range is contained in a discrete subgroup of $\mathbb{R}$. Otherwise, we say that f is *non-lattice*.

The notion of being lattice or non-lattice can be carried over to a cIFS Φ and its self-conformal set F by considering the geometric potential function associated with Φ:

Definition 2.20 (Geometric potential function, (non-) lattice cIFS)**.** Fix a cIFS $\Phi := \{\phi_1, \ldots, \phi_N\}$. Denote by F the self-conformal set associated with Φ and let (Σ^∞, σ) be the associated shift space. Define the *geometric potential function* to be the map $\xi\colon \Sigma^\infty \to \mathbb{R}$ given by $\xi(\omega) := -\ln|\phi'_{\omega_1}(\pi\sigma\omega)|$ for $\omega = \omega_1\omega_2\cdots \in \Sigma^\infty$. If ξ is non-lattice, then we call Φ (and also F) *non-lattice*. On the other hand, if ξ is lattice, then we call Φ (and also F) *lattice*.

One of the key properties of a cIFS is the bounded distortion property. Our results require the following refinement of this property, which we could not find in this precise form in the literature. Therefore, we give a short proof.

Lemma 2.21 (Bounded distortion lemma)**.** *A cIFS $\Phi := \{\phi_1, \ldots, \phi_N\}$ acting on X satisfies the following* bounded distortion property (BDP)*. There exists a sequence $(\varrho_n)_{n\in\mathbb{N}}$ with $\varrho_n > 0$ for all $n \in \mathbb{N}$ and $\lim_{n\to\infty} \varrho_n = 1$ such that for all $\omega, u \in \Sigma^*$ and $x, y \in \phi_\omega X$ we have*

$$\varrho_{n(\omega)}^{-1} \leq \frac{|\phi'_u(x)|}{|\phi'_u(y)|} \leq \varrho_{n(\omega)}.$$

Proof. Fix $\omega \in \Sigma^n$ and let $x, y \in \phi_\omega X$ and $u = u_1 \cdots u_{n(u)} \in \Sigma^*$ be arbitrarily chosen. Noting that the length scaling ratio of a conformal map on a compact set is bounded away from zero, we can write

$$\frac{|\phi'_u(x)|}{|\phi'_u(y)|} \leq \exp\left(\sum_{k=1}^{n(u)} \underbrace{\big| \ln|\phi'_{u_k}(\phi_{\sigma^k u}(x))| - \ln|\phi'_{u_k}(\phi_{\sigma^k u}(y))| \big|}_{=: A_k} \right).$$

Since $|\phi_i'|$ moreover is α-Hölder continuous, it follows that $\ln|\phi_i'|$ is α-Hölder continuous for each $i \in \{1,\ldots,N\}$. Let c_i be the corresponding Hölder constant and set $c := \max_{i\in\{1,\ldots,N\}} c_i$. Further, let $r < 1$ be a common upper bound for the contraction ratios of the maps $\phi_1,\ldots,\phi_N$. Then we have

$$A_k \le c|\phi_{\sigma^k u}(x) - \phi_{\sigma^k u}(y)|^\alpha \le c \cdot \left(r^{n(u)-k}|x-y|\right)^\alpha$$

and thus

$$\sum_{k=1}^{n(u)} A_k \le \frac{c}{1-r^\alpha}|x-y|^\alpha \le \frac{c}{1-r^\alpha} \max_{\omega\in\Sigma^n} \sup_{x,y\in\phi_\omega X} |x-y|^\alpha =: \widetilde{\varrho}_n.$$

Since $\widetilde{\varrho}_n$ converges to 0 as $n \to \infty$, $\varrho_n := \exp(\widetilde{\varrho}_n)$ converges to 1 as $n \to \infty$. The estimate for the lower bound can be obtained by just interchanging the roles of x and y. □

Another central object in our studies is the δ-conformal measure, where δ denotes the Minkowski dimension of the underlying set. For its introduction, let Φ denote a cIFS acting on X and let F be the associated self-conformal set. It is well known that the Minkowski dimension $\delta := \dim_M(F)$ of F exists and is positive and finite (see Theorem 3.2). The unique probability measure ν supported on F which satisfies

$$\nu(\phi_i X \cap \phi_j X) = 0 \quad \text{and} \quad \nu(\phi_i B) = \int_B |\phi_i'|^\delta \mathrm{d}\nu \tag{2.2}$$

for all distinct $i,j \in \Sigma$ and for all Borel sets $B \subseteq X$, is called the *δ-conformal measure* associated with Φ. The statement on the uniqueness and existence is shown in [MU96] and goes back to the work of [Pat76, Sul79, DU91]. We remark that in [MU96] the cone condition is required to hold, since infinite cIFS, that is cIFS with a countable alphabet, are considered. For cIFS with a finite alphabet this condition is not necessary.

2.3 Geometric Conditions

In this section we introduce some geometric conditions which we impose on the self-conformal sets and comment on their meaning.

Throughout this section, we let $\Phi := \{\phi_1,\ldots,\phi_N\}$ denote a cIFS acting on a compact and connected set $X \subset \mathbb{R}^d$ with open set $O := \mathrm{int}_{\mathbb{R}^d}(X)$ and associated self-conformal set F. Further, we let $\delta := \dim_M(F)$ denote the Minkowski dimension of F and recall that $\overline{O} = X$.

The first set of conditions gives specifications on the geometric structure of the *exterior boundary* of the self-conformal set $F \subseteq X$, which is the part of the fractal which is accessible from the complement of X in $\mathbb{R}^d$ (see also Definition 2.24).

(COND 1) $F \subseteq \overline{O}$ and $\partial O \subseteq F$.

(COND 2) $\delta_O := \overline{\dim}_M(\partial O) < \delta$.

The first consequence of (COND 1) is the following. Writing $\Phi Y := \bigcup_{i=1}^{N} \phi_i Y$ for a subset $Y \subseteq X$, (COND 1) implies that $\partial O \setminus \Phi X \subseteq \partial O \setminus F = \varnothing$. This implies that either $\lambda_d(O \setminus \Phi X) > 0$ or $\lambda_d(X \setminus \Phi X) = 0$. These two cases respectively correspond to 'fractal' and full-dimensional self-conformal sets, as is stated in the following proposition, which results from combining Proposition 4.4 and Theorem 4.5 in [MU96] with Theorem 3.2.

Proposition 2.22. *If $\lambda_d(O \setminus \Phi X) > 0$, then $\lambda_d(F) = 0$ and $\delta < d$. Conversely, if $\lambda_d(X \setminus \Phi X) = 0$, then $\lambda_d(F) = \lambda_d(X) > 0$ and $\delta = d$.*

If $\lambda_d(O \setminus \Phi X) > 0$, then we call F *non-degenerate*. In Section 2.4 we are going to see that it is important to distinguish between these two cases for statements on the Minkowski measurability.

We now focus on the non-degenerate case, that is $\lambda_d(O \setminus \Phi X) > 0$, and by this impose a fractal structure on the invariant set F. Just like we assume a regularity condition to hold for the boundary of X (see (COND 2)), we assume regularity properties to be satisfied for the boundary of $X \setminus \Phi X$, too:

(COND 3) $X \setminus \Phi X$ possesses a finite number of connected components $G^1, \ldots, G^Q$ with $Q \in \mathbb{N}$.

(COND 4) There exists a $\delta_I < \delta$ for which $\mathrm{e}^{-t(\delta_I - d)} \lambda_d(F_{\mathrm{e}^{-t}} \cap G^i)$ is uniformly bounded from above for $t \in \mathbb{R}$ and $i \in \{1, \ldots, Q\}$.

We call $G^1, \ldots, G^Q$ the *primary gaps* of F. Their images under the maps ϕ_ω for $\omega \in \Sigma^*$ are called the *main gaps* of F and are denoted by $G^i_\omega := \phi_\omega G^i$ for $i \in \{1, \ldots, Q\}$.

When being interested in the k-th fractal curvature measure $C_k^f(F, \cdot)$ for $k \in \{0, \ldots, d\}$, we need conditions, which are similar to (COND 2) and (COND 4), to be satisfied for the total variation measure of the k-th fractal curvature measure. For presenting these conditions, we define

$$U := \{t \in \mathbb{R} \mid \mathrm{e}^{-t} \text{ is a regular distance for } F\}$$

and set $U_k := U$ if $d \geq 4$ and $k \in \{0, \ldots, d-2\}$, and $U_k := \mathbb{R}$ otherwise. Moreover, we let $C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}}, \cdot)$ denote the total variation measure of $C_k(F_{\mathrm{e}^{-t}}, \cdot)$ (see the appendix).

(COND 2') There exists a $\delta_O < \delta$ such that $\mathrm{e}^{-t(\delta_O - k)} C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}}, X_{\mathrm{e}^{-t}} \setminus X)$ is uniformly bounded from above for $t \in U_k$.

(COND 4') There exists a $\delta_I < \delta$ such that $\mathrm{e}^{-t(\delta_I - k)} C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}}, G^i)$ is uniformly bounded from above for $i \in \{1, \ldots, Q\}$ and $t \in U_k$.

Examples for systems which satisfy the above conditions include well-studied self-similar sets like the Sierpinski gasket or the Sierpinski carpet (see Section 2.5). An example of a more complicated self-similar set which satisfies our conditions and for which ∂O as well as ∂G^i are fractal is presented at the end of this section in Example 2.25. An example of a strictly self-conformal set satisfying our conditions and having fractal boundaries is given in Example 2.26. Before turning to these examples, we want to comment on the relevance of the geometric conditions (COND 1) to (COND 4), (COND 2') and (COND 4').

A pivotal object in studying the Minkowski content and the fractal curvature measures of F is the ε-parallel neighbourhood F_ε of F for $\varepsilon > 0$. The conditions (COND 1), (COND 3) and the OSC ensure that we can decompose $X_\varepsilon \supseteq F_\varepsilon$ for $\varepsilon > 0$ in the following way.

$$\begin{aligned} X_\varepsilon &= (X_\varepsilon \setminus X) \cup \bigcup_{n=0}^{\infty} \left(\Phi^n X \setminus \Phi^{n+1} X\right) \cup \bigcap_{n=0}^{\infty} \Phi^n X = (X_\varepsilon \setminus X) \cup \bigcup_{n=0}^{\infty} \Phi^n (X \setminus \Phi X) \cup F \\ &= (X_\varepsilon \setminus X) \cup \bigcup_{i=1}^{Q} \bigcup_{\omega \in \Sigma^*} G_\omega^i \cup F, \end{aligned} \tag{2.3}$$

where all the above unions are disjoint. Thus, the problem of characterising the structure of the set F_ε can be decomposed into studying the structure of F_ε inside the main gaps on the one hand and outside of the set X on the other hand. (COND 2), which states that the upper Minkowski dimension of ∂O is strictly less than the Minkowski dimension of F, ensures that $\varepsilon^{\delta - d} \lambda_d(F_\varepsilon \cap (X_\varepsilon \setminus X))$ converges to zero as $\varepsilon \to 0$ and (COND 4) is needed for investigating $\varepsilon^{\delta - d} \lambda_d(F_\varepsilon \cap \bigcup_{i=1}^{Q} \bigcup_{\omega \in \Sigma^*} G_\omega^i)$. Likewise, (COND 2') and (COND 4') are used for studying the terms $\varepsilon^{\delta - k} C_k(F_\varepsilon, X_\varepsilon \setminus X)$ and $\varepsilon^{\delta - k} C_k(F_\varepsilon, \bigcup_{i=1}^{Q} \bigcup_{\omega \in \Sigma^*} G_\omega^i)$ for $k \in \{0, \ldots, d\}$.

We remark that the above conditions are related to the conditions which are used in [DKz+10, LPW11] for obtaining results on the Minkowski measurability of self-similar sets. The assumptions in both these articles are especially given for self-similar sets and are satisfied only for a narrow class of self-conformal sets. In particular, besides (COND 1), (COND 3) and a slightly different version of (COND 2), it is assumed in [DKz+10, LPW11] that each of the main gaps $G_\omega^1, \ldots, G_\omega^Q$ for $\omega \in \Sigma^*$ is monophase, which is defined as follows.

Definition 2.23 (Monophase)**.** A non-empty and bounded open set $G \subset \mathbb{R}^d$ with *inradius* g, that is the radius of the largest d-dimensional ball which is inscribed in $\overline{G}$, is called *monophase* if there exist real numbers $\eta_0(G), \ldots, \eta_{d-1}(G)$ such that

$$\lambda_d\left(\overline{(\mathbb{R}^d \setminus G)}_\varepsilon \cap G\right) = \sum_{k=0}^{d-1} \eta_k(G) \cdot \varepsilon^{d-k} \qquad \text{for } \varepsilon \in (0, g]. \tag{2.4}$$

Thus, the main gaps of a self-conformal set $F \subset \mathbb{R}^d$ are monophase if the volume of the ε-parallel neighbourhood of F within each main gap can be expressed as a polynomial of degree d in ε for all sufficiently small $\varepsilon > 0$. Since the condition of monophase main gaps is relevant also for our investigations, we now allude to its geometric meaning.

It is easy to verify that open rectangles and triangles are examples for monophase subsets of $\mathbb{R}^2$. Thus, the Sierpinski gasket and the Sierpinski carpet are examples for self-similar sets with monophase main gaps. An example for a bounded open set which is not monophase is a half-circle. If we take the radius of the half-circle G to be one, then

$$\lambda_2\left(\overline{(\mathbb{R}^d \setminus G)}_\varepsilon \cap G\right) = \frac{\pi}{2}(2\varepsilon - \varepsilon^2) + (1-\varepsilon)^2 \cdot \arcsin\left(\frac{\varepsilon}{1-\varepsilon}\right) + \varepsilon\sqrt{1-2\varepsilon},$$

which cannot be expressed as a polynomial of degree two in ε.

We remark that the main gaps of a self-similar set are monophase if and only if its primary gaps are monophase, because of the following scaling property of the Lebesgue measure. Suppose that $G \subset \mathbb{R}^d$ satisfies Equation (2.4) with $\eta_0(G), \ldots, \eta_d(G) \in \mathbb{R}$ and $g > 0$. Take ϕ to be a similarity on $\mathbb{R}^d$ with similarity ratio $r > 0$. Then

$$\lambda_d\left(\overline{(\mathbb{R}^d \setminus \phi G)}_\varepsilon \cap \phi G\right) = r^d \cdot \lambda_d\left(\overline{(\mathbb{R}^d \setminus G)}_{\varepsilon/r} \cap G\right) = \sum_{k=0}^{d} \left(\eta_k(G) \cdot r^k\right) \cdot \varepsilon^{d-k} \quad \text{for } \varepsilon \in (0, rg].$$

Since conformal maps deform the structure of the underlying set, we do not necessarily have the dichotomy that the main gaps of a self-conformal set are monophase if and only if its primary gaps are. Thus, requiring that all the main gaps of a self-conformal set are monophase is a very restrictive assumption, whereas it is satisfied for many well-studied self-similar sets. (COND 4) is a substitute for the monophase condition. It even allows the boundary of the primary gaps to be fractal and thus permits the study of a bigger variety of fractal sets, even in the self-similar setting. Examples of such sets are presented now.

Examples

For introducing the set in our first example, we fix some terminology from [PW08].

Definition 2.24 (Exterior boundary, envelope)**.** Let $Y \subset \mathbb{R}^d$ denote a compact set. Then $\mathbb{R}^d \setminus Y$ has a unique unbounded component, which we denote by U. (For $d = 1$, there are actually two unbounded components in $\mathbb{R}^d \setminus Y$, if $+\infty$ and $-\infty$ are not identified. In this case, we let U be their union.) Then ∂U is called the *exterior boundary* of Y; it consists of that portion of Y which is accessible when approaching Y from infinity. The *envelope* of Y is the complement $\mathbb{R}^d \setminus U$ of U.

Example 2.25 (Self-similar set whose primary gaps have fractal boundary). We consider the self-similar set whose construction is depicted in Figure 2.2. Take $\widetilde{X}$ to be the equilateral triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$ and $(1/2, \sqrt{3}/2)$. Define 21 similarities on $\widetilde{X}$ all of which have similarity ratio $1/7$ and denote them by $\widetilde{\phi}_1, \ldots, \widetilde{\phi}_{21}$. They are visualised in Figure 2.2. Let $F \subset \widetilde{X}$ denote the self-similar set associated with $\widetilde{\Phi} := \{\widetilde{\phi}_1, \ldots, \widetilde{\phi}_{21}\}$. Take X to be the envelope of F (see Figure 2.3) and set $O := \text{int}_{\mathbb{R}^2} X$. Then $\Phi := \widetilde{\Phi}|_X$ is a cIFS with O as open set which satisfies (COND 1) and (COND 3). The set F possesses seven primary gaps $G^1, \ldots, G^7$ which are depicted in Figure 2.3. Conditions (COND 2) and (COND 4) are satisfied, since

$$\dim_M(F) = \frac{\ln 21}{\ln 7} \quad \text{and} \quad \dim_M(\partial X) = \frac{\ln 10}{\ln 7} = \dim_M(\partial G^i)$$

for $i \in \{1, \ldots, 7\}$. To see this, we remark that it is well known that for such a self-similar set the Minkowski dimension exists and coincides with the similarity dimension. The *similarity dimension* of a self-similar set which consists of N copies of size r of itself is given by the value $-\ln N/\ln r$. In our example, F consists of 21 copies of size $1/7$ of itself and thus, $\dim_M(F) = \ln 21/\ln 7$. For computing the Minkowski dimension of ∂X, we decompose ∂X into three parts, namely, we partition it at the points $(0,0)$, $(1,0)$ and $(1/2, \sqrt{3}/2)$. Each of these three parts consists of 10 copies of size $1/7$ of itself. Since the Minkowski dimension exists and the upper Minkowski dimension is stable with respect to finite unions (see [Fal03, Ch. 3.2]), that is $\overline{\dim}_M(X \cup Y) = \max\{\overline{\dim}_M(X), \overline{\dim}_M(Y)\}$, we obtain that $\dim_M(\partial X) = \ln 10/\ln 7$. Analogously, it can be seen that $\dim_M(\partial G^i) = \ln 10/\ln 7$ for each of the primary gaps $G^1, \ldots, G^7$ depicted in Figure 2.3.

In the next example we provide a construction of a self-conformal set which is not self-similar and satisfies our conditions.

Example 2.26 (Self-conformal set whose primary gaps have fractal boundary). An example for a strictly self-conformal set satisfying (COND 1) to (COND 4) can be obtained by applying the function

$$g\colon \mathbb{R}^2 \to \mathbb{R}^2,\ (x,y) \mapsto \big((x+1)^2 - (y+1)^2, 2(x+1)(y+1)\big)$$

to the self-similar set F from Example 2.25. This gives rise to a self-conformal system which is conjugate to a self-similar system. Such systems are examined in Section 2.4.3. A picture of the resulting self-conformal set is given in Figure 2.4.

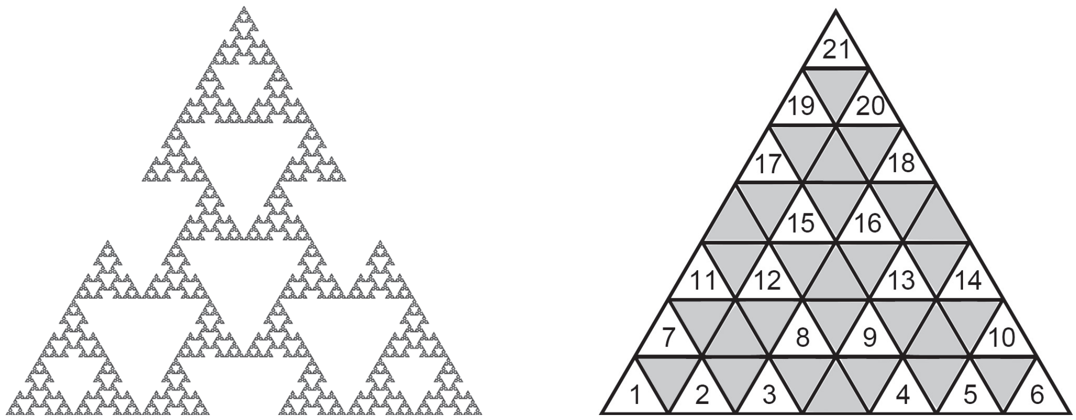


Figure 2.2: The self-similar set from Example 2.25 and its construction. Its exterior boundary as well as the boundaries of the primary gaps are fractal.

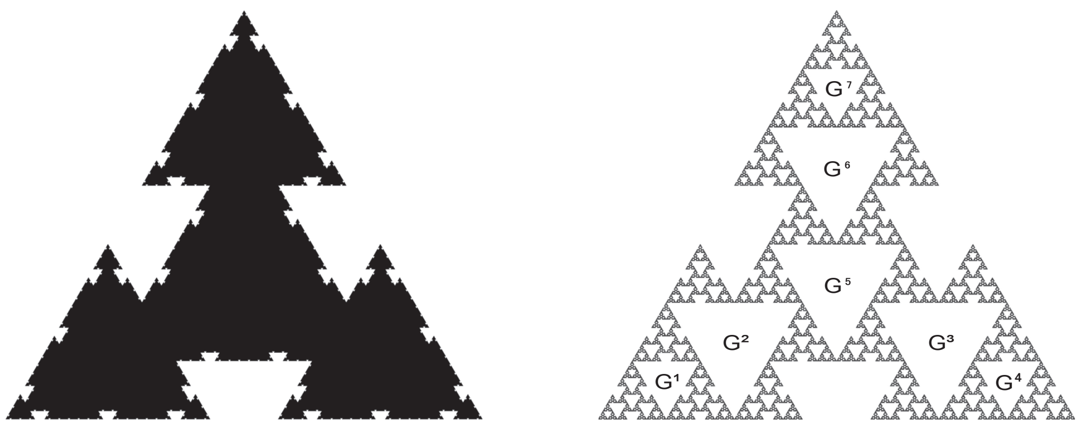


Figure 2.3: The envelope and the primary gaps $G^1, \dots, G^7$ of the self-similar set from Example 2.25.

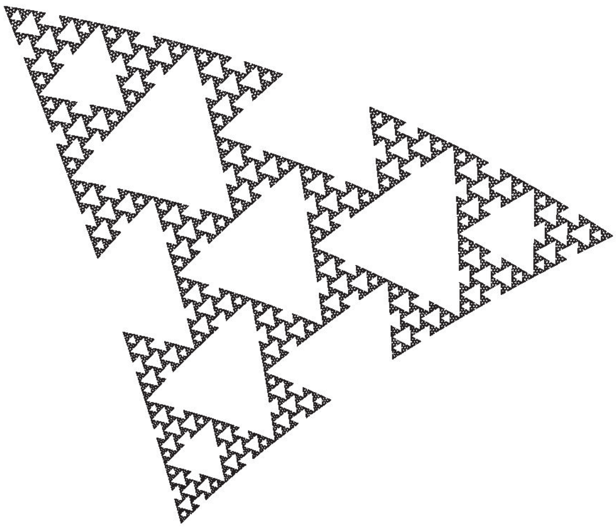


Figure 2.4: The strictly self-conformal set from Example 2.26. Its exterior boundary as well as the boundaries of the primary gaps are fractal.

2.4 Main Results for Conformal Iterated Function Systems

Now, we are ready to exhibit our main results concerning self-conformal sets. For ease of presentation, their proofs are provided in Chapter 4.

The results are subdivided into three categories. Firstly, in Section 2.4.1, we focus on the original results for general self-conformal sets. Secondly, in Section 2.4.2, we state the new results concerning fractal curvature measures of self-similar sets. Thirdly, concluding this section, in Section 2.4.3, we present our results for $\mathcal{C}^{1+\alpha}$-diffeomorphic images of self-similar sets, which form an important subclass of self-conformal sets.

2.4.1 Self-Conformal Sets

In presenting our results for general self-conformal sets, we firstly focus on the results concerning the (local) Minkowski measurability of self-conformal subsets of $\mathbb{R}^d$. These results extend the results which were obtained in [KK10] for self-conformal subsets of $\mathbb{R}$ to higher dimensional Euclidean spaces. Moreover, they extend the results which are given in [Gat00, Win08, DKz$^+$10, LPW11] for self-similar sets to the conformal setting. Secondly, we consider self-conformal subsets of $\mathbb{R}$, for which we provide stronger results concerning Minkowski measurability and also give results for the fractal curvature measures.

We impose the conditions (COND 1) to (COND 4) from Section 2.3. There, we have seen that for a cIFS Φ acting on X with associated open set $O := \mathrm{int}_{\mathbb{R}^d} X$ either $\lambda_d(O \setminus \Phi X) > 0$ or $\lambda_d(X \setminus \Phi X) = 0$. We now distinguish between these two cases and begin with the simpler one, namely with $\lambda_d(X \setminus \Phi X) = 0$. Here, Proposition 2.22 immediately yields the following result which we state without a proof.

Proposition 2.27. *Let F denote the self-conformal set associated with the cIFS Φ acting on X. Suppose that $\lambda_d(X \setminus \Phi X) = 0$ and that (COND 1) and (COND 2) are satisfied. Then the (local) Minkowski content of F exists and satisfies*

$$\mathcal{M}(F, \cdot) = \lambda_d(F \cap \cdot) \qquad \text{and} \qquad \mathcal{M}(F) = \lambda_d(F).$$

From now on, we suppose that $\lambda_d(O \setminus \Phi X) > 0$, imposing a fractal structure on the invariant set F. Here, the systems are more delicate and the results are more interesting. Throughout the remainder of this section we fix the following notation.

Notation 2.28. We let Φ denote a cIFS acting on a compact and connected set $X \subset \mathbb{R}^d$ with associated self-conformal set $F \subset \mathbb{R}^d$. The associated geometric potential function is denoted by $\xi \colon \Sigma^\infty \to \mathbb{R}$ and the Minkowski dimension of F by $\delta := \dim_M(F)$. Assuming (COND 3), we let $G^1, \ldots, G^Q$ denote the primary gaps of F and $G^1_\omega, \ldots, G^Q_\omega$ the associated main gaps for $\omega \in \Sigma^*$. Further, ν denotes the δ-conformal measure associated with Φ and $H_{\mu_{-\delta\xi}}$ denotes the measure theoretical entropy of the shift map σ with respect to the unique shift-invariant Gibbs measure $\mu_{-\delta\xi}$ for the potential function $-\delta\xi$ (see Equation (3.4)).

A key result in this thesis is the following theorem.

Theorem 2.29 (Self-conformal sets – local Minkowski content)**.** *Fix the notation from Notation 2.28. Suppose that $\lambda_d(O \setminus \Phi X) > 0$ and that (COND 1) to (COND 4) hold. Then we have the following.*

(i) The local average Minkowski content $\widetilde{\mathcal{M}}(F, \cdot)$ exists and is equal to the well-defined finite non-trivial measure

$$\frac{\delta}{H_{\mu_{-\delta\xi}}} \cdot \left(\lim_{m \to \infty} \sum_{\omega \in \Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta - d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G^i_\omega) \mathrm{d}T \right) \cdot \nu(\cdot).$$

(ii) If ξ is non-lattice, then $\mathcal{M}(F, \cdot)$ exists and is equal to $\widetilde{\mathcal{M}}(F, \cdot)$.

The proof of Theorem 2.29 is provided in Section 4.2.1.

Remark 2.30. Since $\mathcal{M}(F) = \mathcal{M}(F, \mathbb{R}^d)$, $\widetilde{\mathcal{M}}(F) = \widetilde{\mathcal{M}}(F, \mathbb{R}^d)$ and $\nu(\mathbb{R}^d) = 1$, the above theorem immediately yields that the average Minkowski content exists and is equal to $\widetilde{\mathcal{M}}(F, \mathbb{R}^d)$. Theorem 2.29 moreover gives that the Minkowski content $\mathcal{M}(F)$ of F exists if ξ is non-lattice, and that it is equal to the average Minkowski content in this case.

When the dimension of the underlying Euclidean space is one, we can strengthen the above results. In this case we obtain results on both the fractal curvature measures $C_0^f(F, \cdot)$, $C_1^f(F, \cdot)$ and statements about the existence or non-existence in the lattice situation. Note that if $d = 1$, then (COND 2) to (COND 4), (COND 2') and (COND 4') are always satisfied for a cIFS which satisfies (COND 1) (see Section 4.2.2, proof of Lemma 4.9).

Theorem 2.31 (Self-conformal subsets of $\mathbb{R}$ – fractal curvature measures)**.** *Fix the notation from Notation 2.28. Assume that $d = 1$ so that $X \subset \mathbb{R}$. Suppose that $\lambda_1(O \setminus \Phi X) > 0$ is satisfied and that (COND 1) holds. For a connected subset $G \subset \mathbb{R}$ let $|G|$ denote its length. Then for $k \in \{0, 1\}$ we have the following.*

(i) The average fractal curvature measures of F exist and satisfy

$$\widetilde{C}_0^f(F, \cdot) = \frac{2^{-\delta} c}{H_{\mu - \delta\xi}} \cdot \nu(\cdot) \qquad \text{and} \qquad \widetilde{C}_1^f(F, \cdot) = \frac{2^{1-\delta} c}{(1-\delta) H_{\mu - \delta\xi}} \cdot \nu(\cdot),$$

where the constant c is given by the well-defined positive and finite limit

$$c := \lim_{m \to \infty} \sum_{\omega \in \Sigma^m} \sum_{i=1}^{Q} |G_\omega^i|^\delta. \tag{2.5}$$

(ii) If ξ is non-lattice, then $C_k^f(F, \cdot)$ exists and equals $\widetilde{C}_k^f(F, \cdot)$.

(iii) If ξ is lattice and the system Φ consists of analytic maps, then $C_k^f(F, \cdot)$ does not exist.

The proof of Theorem 2.31 can be found in Section 4.2.2.

Remark 2.32. Theorem 2.31 yields that $s_k = \delta - k$ is the right choice for the scaling exponent, which we introduced in Remark 2.10.

An astonishing result on the Minkowski measurability for self-conformal subsets of $\mathbb{R}$ is presented in the next theorem. It states that a non-degenerate lattice self-conformal set can be Minkowski measurable. This contrasts the fact that a non-degenerate self-similar subset of $\mathbb{R}$ is Minkowski measurable if and only if it is non-lattice, which has been obtained in [LP93, Fal95, LvF06] (see also Theorem 2.39).

Theorem 2.33 (Self-conformal subsets of $\mathbb{R}$ – Minkowski content)**.** *Suppose that we are in the situation of Theorem 2.31. Then the following hold.*

(i) The average Minkowski content $\widetilde{\mathcal{M}}(F)$ *exists and is equal to the positive and finite value*

$$\frac{2^{1-\delta}c}{(1-\delta)H_{\mu_{-\delta\xi}}}$$

with c *as in Equation* (2.5)*.*

(ii) If ξ *is non-lattice, then the Minkowski content* $\mathcal{M}(F)$ *of* F *exists and coincides with* $\widetilde{\mathcal{M}}(F)$*.*

(iii) If ξ *is lattice, then we have that*

$$0 < \underline{\mathcal{M}}(F) \leq \overline{\mathcal{M}}(F) < \infty.$$

What is more, equality in the above equation can be attained. More precisely let $\zeta, \psi \in \mathcal{C}(\Sigma^{\infty})$ *denote the functions satisfying* $\xi - \zeta = \psi - \psi \circ \sigma$*, where the range of* ζ *is contained in a discrete subgroup of* $\mathbb{R}$*. Let* $a \in \mathbb{R}$ *denote the maximal real number for which* $\zeta(\Sigma^{\infty}) \subseteq a\mathbb{Z}$*. Further, denote by* $\nu_{-\delta\zeta}$ *the unique eigenmeasure with eigenvalue one of the dual of the Perron-Frobenius operator for the potential function* $-\delta\zeta$ *(see Section 3.1.2). If, for every* $t \in [0, a)$*, we have*

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\delta an}\nu_{-\delta\zeta}\circ\psi^{-1}([na, na+t)) = \frac{\mathrm{e}^{\delta t}-1}{\mathrm{e}^{\delta a}-1}\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\delta an}\nu_{-\delta\zeta}\circ\psi^{-1}([na,(n+1)a)), \quad (2.6)$$

then it follows that $\underline{\mathcal{M}}(F) = \overline{\mathcal{M}}(F)$*.*

Remark 2.34. The proof of Theorem 2.33(iii) shows that Equation (2.6) implies that also $\underline{C}_0^f(F) = \overline{C}_0^f(F)$.

Note that the sums occurring in Equation (2.6) are finite and that the proof of Theorem 2.33 is given in Section 4.2.3. An example, where Equation (2.6) is satisfied, is presented at the end of this section as Example 2.45. This example is particularly interesting as it sheds new light on a conjecture described in the following remark.

Remark 2.35 (On a conjecture by Lapidus from 1993)*.* Conjecture 3 in [Lap93] states that a non-degenerate self-similar set in $\mathbb{R}^d$ is Minkowski measurable if and only if it is non-lattice. This conjecture was proven to be correct in space dimension $d = 1$ in [LP93, Fal95, LvF06]. For higher dimensional spaces the part concerning the lattice situation is still an open problem. In the same paper, [Lap93], a similar conjecture is given for so-called 'approximately' self-similar sets. A precise definition of an 'approximately' self-similar set is not given. However, since conformal maps locally behave like similarities, we view self-conformal sets as being 'approximately' self-similar and remark that for such sets the preceding theorem in combination with Corollary 2.3 of [LP93] provides a negative answer to this conjecture (see also Example 2.45). Note that Theorem 2.33 combined with Corollary 2.3 of [LP93] in particular shows that there exist fractal strings with lattice self-conformal boundary for which the asymptotic second term of the eigenvalue counting function $N(\lambda)$ of the Laplacian (in the sense of [LP93]) is monotonic. We thank Lapidus for pointing the connection out to us. We will return to this conjecture at the very end of Chapter 5, where we study conformal graph directed Markov systems.

2.4.2 Self-Similar Sets

Self-similar sets form a special class of self-conformal sets. In the self-similar setting our methods of proof immediately allow to retrieve results on all the fractal curvature measures for higher dimensional Euclidean spaces. With this, we provide a substantially different proof and obtain alternative, useful formulae to [Win08, WZ10, Zäh11] and moreover extend the results given in [DKz$^+$10, LPW11].

Remark 2.36. The δ-conformal measure associated with a cIFS Φ consisting of similarities coincides with the normalised δ-dimensional Hausdorff-measure on the invariant set. Also, letting $r_1, \ldots, r_N$ denote the similarity ratios of the similarities $\phi_1, \ldots, \phi_N$, we have that $H_{\mu_{-\delta\xi}} = -\delta \sum_{i=1}^{N} \ln(r_i) r_i^{\delta}$.

Theorem 2.37 (Self-similar sets – fractal curvature measures)**.** *Fix the notation from Notation 2.28. Suppose that Φ consists of similarities, so that F is self-similar. Assume that $\lambda_d(O \setminus \Phi X) > 0$ and that (COND 1) to (COND 3), (COND 2') and (COND 4') are satisfied. If $d \geq 4$ and $k \leq d-2$ additionally assume that Lebesgue-almost all $\varepsilon > 0$ are regular distances for F. Then for $k \in \{0, \ldots, d\}$ the following hold.*

(i) $\widetilde{C}_k^f(F, \cdot)$ exists and is equal to the finite signed Borel measure

$$\frac{\delta}{H_{\mu_{-\delta\xi}}} \cdot \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} \sum_{i=1}^{Q} C_k(F_{\mathrm{e}^{-T}}, G^i) \mathrm{d}T \cdot \nu(\cdot).$$

(ii) If ξ is non-lattice, then $C_k^f(F, \cdot)$ exists and is equal to $\widetilde{C}_k^f(F, \cdot)$.

The proof of Theorem 2.37 is given in Section 4.3.1.

For self-similar sets studies on the existence of the fractal curvature measures have already been carried out. We now want to put the result of the above theorem into context with these results from the literature.

The existence of the limits $\widetilde{C}_k^f$ and C_k^f has first been investigated for self-similar sets in [Win08] under the assumption of polyconvex parallel sets. This means that the ε-parallel neighbourhoods of the underlying set F can be represented by a finite union of non-empty convex sets for all sufficiently small $\varepsilon > 0$. For $k = d$ this assumption is not necessary, as was gained in [Gat00]. The assumption has been eliminated in [WZ10, Zäh11] for $k = d-1$ and substituted by a regularity condition for $k \in \{0, \ldots, d-2\}$. The formulae for $\widetilde{C}_k^f(F, \cdot)$, and in the non-lattice situation also for $C_k^f(F, \cdot)$, given in [Win08, WZ10, Zäh11] depend on the overlap functions

$$R_k(\varepsilon) := C_k(F_\varepsilon) - \sum_{i=1}^{N} \mathbb{1}_{(0,r_i]}(\varepsilon) C_k((\phi_i F)_\varepsilon), \quad \varepsilon > 0,$$

where $r_1, \ldots, r_N$ denote the similarity ratios of $\phi_1, \ldots, \phi_N$. Their formula is

$$\frac{\delta}{H_{\mu_{-\delta\xi}}} \cdot \int_0^R T^{\delta-k-1} R_k(T) \mathrm{d}T \cdot \nu(\cdot), \tag{2.7}$$

where $R > \sqrt{2} \cdot \operatorname{diam}(F)$ is some real number and $\operatorname{diam}(F)$ denotes the diameter of F.

The formula from Equation (2.7) and the one from Theorem 2.37 differ quite significantly. While our formula is based on the structure of the primary gaps, the formula from Equation (2.7) is based on the structure of the overlaps $(\phi_i F)_\varepsilon \cap (\phi_j F)_\varepsilon$ for distinct $i, j \in \{1, \ldots, N\}$, this can be seen by an inclusion-exclusion argument. The total fractal curvatures have been evaluated in [Win08] for the Sierpinski gasket and the Sierpinski carpet using the formula from Equation (2.7). In Section 2.5 we are going to see that using the formula from Theorem 2.37 quickly yields the same results. Moreover, in Section 2.5 we present an example for a non-lattice self-similar set, for which the fractal curvature measures can be easily computed by using Theorem 2.37.

Now, we focus on the special case $k = d$ concerning the Minkowski content. The Minkowski content of self-similar subsets of $\mathbb{R}^d$ has first been investigated by Gatzouras in [Gat00]. He obtained existence of the Minkowski content in the non-lattice case and that the average Minkowski content always exists. Complementary to [Gat00], where renewal theory was used, the following studies have been carried out. By means of geometric zeta functions and their analytic properties, existence of the Minkowski content was shown for non-lattice self-similar sets in [DKz$^+$10, LPW11], where alternative formulae to the ones of [Gat00] are given. Recall from Section 2.3 that [DKz$^+$10, LPW11] impose geometric conditions and in particular require the fractal to possess monophase main gaps (see Definition 2.23). Remarkably, in these works non-existence of the Minkowski content in the lattice situation could be shown for non-degenerate self-similar sets satisfying these geometric conditions. Imposing the condition of monophase main gaps, we show that the results from [DKz$^+$10, LPW11] can be deduced from Theorem 2.37 and its proof. We additionally obtain results for the local Minkowski content.

Theorem 2.38 (Self-similar sets with monophase main gaps – local Minkowski content)**.** *Suppose that we are in the situation of Theorem 2.37 and that the Minkowski dimension*

δ of F satisfies $d-1<\delta<d$. Let g^i denote the inradius of G^i. Assume that for each $i\in\{1,\ldots,Q\}$ there exist $\eta_0(G^i),\ldots,\eta_{d-1}(G^i)\in\mathbb{R}$ such that

$$\lambda_d(F_{\mathrm{e}^{-t}}\cap G^i)=\sum_{j=0}^{d-1}\eta_j(G^i)\mathrm{e}^{-t(d-j)}\quad \text{for } t>-\ln(g^i) \tag{2.8}$$

and set $\eta_d(G^i):=-\lambda_d(G^i)$. Then in addition to the results from Theorem 2.37 the following hold.

(i) The formula for the local average Minkowski content (and in the non-lattice case also for the local Minkowski content) simplifies to

$$\frac{\delta}{H_{\mu_{-\delta\xi}}}\cdot\left(\sum_{i=1}^{Q}\sum_{j=0}^{d}\frac{\eta_j(G^i)(g^i)^{\delta-j}}{\delta-j}\right)\cdot\nu(\cdot).$$

(ii) If ξ is lattice, then the local Minkowski content of F does not exist. What is more, for every $\kappa\in\Sigma^*$ we have that

$$\underline{\mathcal{M}}(F,\phi_\kappa O)<\overline{\mathcal{M}}(F,\phi_\kappa O).$$

The proof of Theorem 2.38 is provided in Section 4.3.2.

We remark that Theorem 2.38 generalises the result from [DKz+10], since the authors assume that $O=\mathrm{int}_{\mathbb{R}^d}\langle F\rangle$, where $\langle F\rangle$ denotes the convex hull of F. This condition appears to be restrictive in the conformal setting. Theorem 2.38 in particular states that under its assumptions a non-degenerate self-similar set is Minkowski measurable if and only if its associated geometric potential function is non-lattice. This dichotomy is valid for any self-similar subset of $\mathbb{R}$ (see Remark 2.35). The validity of this dichotomy in $\mathbb{R}$ follows also from Theorem 2.38 because of the following. For $d=1$ the additional assumption of Theorem 2.38, namely, that there exists $\eta_0(G^i)\in\mathbb{R}$ such that Equation (2.8) holds, is always satisfied. This is the case, since for every non-empty open bounded and connected subset $G\subset\mathbb{R}$ we have that

$$\lambda_1(F_{\mathrm{e}^{-t}}\cap G)=\begin{cases}|G| & :t\leq-\ln(|G|/2),\\ 2\mathrm{e}^{-t} & :t>-\ln(|G|/2),\end{cases}$$

where $|G|$ denotes the length of G. The same dichotomy holds for the 0-th total fractal curvature.

Theorem 2.39 (Self-similar subsets of $\mathbb{R}$ – fractal curvature measures)**.** *Fix the notation from Notation 2.28. Suppose that Φ consists of similarities and that $X\subset\mathbb{R}$. Assume that $\lambda_1(O\setminus\Phi X)>0$ and that (COND 1) is satisfied. Then in addition to the results of Theorem 2.31 the following hold.*

(i) The formulae for the average fractal curvature measures (and in the non-lattice case also for the fractal curvature measures) simplify to

$$\widetilde{C}_0^f(F,\cdot) = \frac{2^{-\delta}}{H_{\mu_{-\delta\xi}}} \cdot \sum_{i=1}^{Q} |G^i|^\delta \cdot \nu(\cdot) \quad \text{and} \quad \widetilde{C}_1^f(F,\cdot) = \frac{2^{1-\delta}}{(1-\delta)H_{\mu_{-\delta\xi}}} \cdot \sum_{i=1}^{Q} |G^i|^\delta \cdot \nu(\cdot).$$

Note that both $\widetilde{C}_0^f(F,\cdot)$ and $\widetilde{C}_1^f(F,\cdot)$ are non-trivial.

(ii) If ξ is lattice, then the fractal curvature measures of F do not exist. Moreover, for $k \in \{0,1\}$ and every $B \in \mathfrak{B}(\mathbb{R})$ for which $B \cap F$ is non-empty and is equal to a finite union of sets of the form $\phi_\omega F$, where $\omega \in \Sigma^*$, and for which $F_\varepsilon \cap B = (F \cap B)_\varepsilon$ for all sufficiently small $\varepsilon > 0$ we have that

$$\underline{C}_k^f(F,B) < \overline{C}_k^f(F,B).$$

A proof is provided in Section 4.4.4.

2.4.3 $\mathcal{C}^{1+\alpha}$-Images of Self-Similar Sets

Another interesting class of self-conformal sets is the class of conformal $\mathcal{C}^{1+\alpha}$-images of self-similar sets, where $\alpha \in (0,1]$. This class was investigated by Freiberg and the author in [FK] with regard to (local) Minkowski measurability in $\mathbb{R}^d$. It was investigated also by Kesseböhmer and the author in [KK10] in view of the fractal curvature measures for subsets of $\mathbb{R}$. We start with the higher dimensional setting from [FK]. Here, we only state the results and refer to [FK] for their proofs.

We say that an IFS $\Phi := \{\phi_1, \dots, \phi_N\}$ with invariant set F satisfies the *strong separation condition (SSC)*, if $\phi_i F \cap \phi_j F = \varnothing$ for all distinct $i, j \in \{1, \dots, N\}$. The precise setting from [FK] is the following.

Setting 2.40. Let K denote the invariant set of an IFS $R := \{R_1, \dots, R_N\}$ which consists of similarities and satisfies the SSC. Let $\mathcal{U}$ denote an open domain containing the (1/2)-parallel neighbourhood of K and introduce a conformal $\mathcal{C}^{1+\alpha}(\mathcal{U})$-diffeomorphism $g\colon \mathcal{U} \to \mathbb{R}^d$, where $\alpha \in (0,1]$. We set $F := g(K)$ and note that F satisfies

$$F = \bigcup_{i=1}^{N} gR_i g^{-1}(F).$$

The maps $\phi_i := gR_i g^{-1}$, $i \in \{1, \dots, N\}$, are not necessarily contractions. However, the α-Hölder continuity of the length scaling ratio $|g'|$ (see Remark 2.14) implies that an iterate $\widetilde{\Phi}$ of the system $\Phi := \{\phi_1, \dots, \phi_N\}$ consists solely of contractions. Indeed, $\widetilde{\Phi}$ is an IFS and F is its unique non-empty compact invariant set. Note that the IFS $\widetilde{\Phi}$ satisfies the SSC, since g is a diffeomorphism. Moreover, since g is bi-Lipschitz the Minkowski dimensions of K and F coincide. We denote the common value $\dim_M(K) = \dim_M(F)$ by δ. The function g is called a *conjugacy* between the systems R and Φ.

Theorem 2.41 ($\mathcal{C}^{1+\alpha}$-images – local Minkowski content, [FK])**.** *With the notation of Setting 2.40 the following hold.*

(i) The local average Minkowski contents of K and F always exist. Moreover, $\widetilde{\mathcal{M}}(F,\cdot)$ is absolutely continuous with respect to the push-forward measure $g_\star\widetilde{\mathcal{M}}(K,\cdot)$ (see the appendix) and their Radon-Nikodym derivative is

$$\frac{\mathrm{d}\widetilde{\mathcal{M}}(F,\cdot)}{\mathrm{d}\left(g_\star\widetilde{\mathcal{M}}(K,\cdot)\right)} = |g' \circ g^{-1}|^\delta.$$

(ii) If the local Minkowski content of K exists, then the local Minkowski content of F exists. Moreover, $\mathcal{M}(K,\cdot) = \widetilde{\mathcal{M}}(F,\cdot)$ and $\mathcal{M}(F,\cdot) = \widetilde{\mathcal{M}}(F,\cdot)$.

Theorem 2.42 ($\mathcal{C}^{1+\alpha}$-images – Minkowski content, [FK])**.** *Fix the notation from Setting 2.40. Let ν denote the δ-conformal measure associated with K. Then the following hold.*

(i) The average Minkowski contents of K and F always exist and are positive and finite. Moreover, they satisfy the relation

$$\widetilde{\mathcal{M}}(F) = \widetilde{\mathcal{M}}(K) \cdot \int_K |g'|^\delta \mathrm{d}\nu.$$

(ii) F is Minkowski measurable if K is Minkowski measurable. In this case we have $\mathcal{M}(K) = \widetilde{\mathcal{M}}(K)$ and $\mathcal{M}(F) = \widetilde{\mathcal{M}}(F)$.

The above theorems in tandem with the results from Section 2.4.2 and [Gat00] imply that F is Minkowski measurable if it is non-lattice. It is worth to notify that the geometric conditions (COND 1) to (COND 4) are not required in [FK] with regard to [Gat00]. Further, it is important to remark that the converse of Theorem 2.42(ii) is not true, that is, F can be Minkowski measurable if K is not. This is going to be alluded to in the following, where we concentrate on $\mathcal{C}^{1+\alpha}$-images of self-similar subsets of $\mathbb{R}$. Here, the assumption of SSC can be substituted by the weaker OSC assumption and a relation between the fractal curvature measures of K and F is obtained. The forthcoming results are going to be proven in Section 4.4.

Theorem 2.43 ($\mathcal{C}^{1+\alpha}$-images in $\mathbb{R}$ – fractal curvature measures)**.** *Let R denote a cIFS acting on $X \subset \mathbb{R}$ which consists of similarities and let K denote its invariant set. By*

$\delta := \dim_M(K)$ *we denote the Minkowski dimension of* K *and by* $\mathcal{U} \supset X$ *a connected open neighbourhood of* X *in* $\mathbb{R}$. *Let* $g\colon \mathcal{U} \to \mathbb{R}$ *be a* $\mathcal{C}^{1+\alpha}(\mathcal{U})$ *map, for which* $|g'|$ *is bounded away from zero, where* $\alpha \in (0,1]$. *Assume that* $\lambda_1(X \setminus RX) > 0$ *and set* $F := g(K)$. *Then the following hold.*

(i) The average fractal curvature measures of both K *and* F *exist. Moreover,* $\widetilde{C}_k^f(F,\cdot)$ *is absolutely continuous with respect to the push-forward measure* $g_\star\widetilde{C}_k^f(K,\cdot)$ *for* $k \in \{0,1\}$ *and their Radon-Nikodym derivative is given by*

$$\frac{\mathrm{d}\widetilde{C}_k^f(F,\cdot)}{\mathrm{d}\left(g_\star\widetilde{C}_k^f(K,\cdot)\right)} = |g' \circ g^{-1}|^\delta.$$

(ii) If R *is non-lattice, then the fractal curvature measures of both* K *and* F *exist. Further,* $C_k^f(F,\cdot)$ *is absolutely continuous with respect to the push-forward measure* $g_\star C_k^f(K,\cdot)$ *for* $k \in \{0,1\}$ *with Radon-Nikodym derivative*

$$\frac{\mathrm{d}C_k^f(F,\cdot)}{\mathrm{d}\left(g_\star C_k^f(K,\cdot)\right)} = |g' \circ g^{-1}|^\delta.$$

(iii) If R *is lattice, then neither the 0-th nor the 1-st fractal curvature measure of* F *exists.*

The proof of Theorem 2.43 can be found in Section 4.4.1.

The above theorem in particular states that the fractal curvature measures of a $\mathcal{C}^{1+\alpha}$-image of a non-degenerate self-similar set $K \subset \mathbb{R}$ exist if and only if K is non-lattice. On the contrary, the dichotomy lattice versus non-lattice does not carry over to the Minkowski content. This is stated in the next corollary and shows that even amongst the class of $\mathcal{C}^{1+\alpha}$-diffeomorphic images of self-similar sets, there exist Minkowski measurable lattice sets. We have already seen that this is possible for general self-conformal sets in Section 2.4.1.

Corollary 2.44 ($\mathcal{C}^{1+\alpha}$-images in $\mathbb{R}$ – Minkowski content)**.** *Suppose that we are in the situation of Theorem 2.43. Let* ν *denote the* δ*-conformal measure associated with* K. *Then we have the following.*

(i) The average Minkowski contents of both K *and* F *exist, are positive and finite and satisfy*

$$\widetilde{\mathcal{M}}(F) = \widetilde{\mathcal{M}}(K) \cdot \int_K |g'|^\delta \mathrm{d}\nu.$$

(ii) If R *is non-lattice, then the Minkowski contents of both* K *and* F *exist, are positive and finite and satisfy*

$$\mathcal{M}(F) = \mathcal{M}(K) \cdot \int_K |g'|^\delta \mathrm{d}\nu.$$

(iii) If R is lattice, then the Minkowski content of K does not exist, whereas the Minkowski content of F might or might not exist. More precisely, assume that $K \subseteq [0,1]$ and that the geometric potential function ξ associated with R is lattice. Let $a > 0$ be maximal such that the range of ξ is contained in $a\mathbb{Z}$. Define $\widetilde{g}\colon \mathbb{R} \to \mathbb{R}$, $\widetilde{g}(x) := \nu((-\infty, x])$ to be the distribution function of ν. For $n \in \mathbb{N}$ define the function $g_n\colon [-1,\infty) \to \mathbb{R}$ by

$$g_n(x) := \int_{-1}^{x} \left(\widetilde{g}(r)(\mathrm{e}^{\delta a n} - 1) + 1\right)^{-1/\delta} \mathrm{d}r$$

and set $F_n := g_n(K)$. Then for every $n \in \mathbb{N}$ we have $\underline{\mathcal{M}}(F_n) = \overline{\mathcal{M}}(F_n)$.

Corollary 2.44 is proven in Section 4.4.2.

From the condition in the above corollary, we now construct explicit examples of non-degenerate lattice Minkowski measurable self-conformal sets and thus add to the discussion given in Remark 2.35.

Example 2.45. Let $K \subseteq [0,1]$ be the Middle Third Cantor Set and let ν denote the $\ln 2/\ln 3$-conformal measure associated with K. Let $\widetilde{g}\colon \mathbb{R} \to \mathbb{R}$ denote the Devil's Staircase Function defined by $\widetilde{g}(r) := \nu((-\infty, r])$. Define the function $g\colon [-1,\infty) \to \mathbb{R}$ by

$$g(x) := \int_{-1}^{x} (\widetilde{g}(y) + 1)^{-\ln 3/\ln 2} \mathrm{d}y$$

and set $F := g(K)$. Then we have $\underline{\mathcal{M}}(F) = \overline{\mathcal{M}}(F)$, although $\underline{\mathcal{M}}(K) < \overline{\mathcal{M}}(K)$. This is a consequence of Theorem 2.39 and Corollary 2.44.

Lattice cIFS which arise via a $\mathcal{C}^{1+\alpha}$ conjugation of IFS consisting of similarities play an important role in the general theory of lattice cIFS. Namely, if a lattice cIFS is analytic, then it is automatically conjugate to a lattice system consisting of similarities.

Theorem 2.46 (Analytic lattice cIFS)**.** *Let Φ be a lattice cIFS acting on $X \subset \mathbb{R}$ and consisting of analytic maps. Let F denote the associated self-conformal set. Then there exist a self-similar set $K \subset \mathbb{R}$ and a map g which is analytic on an open neighbourhood of K such that $F = g(K)$.*

The above result is of interest, since it allows us to carry the obtained results for $\mathcal{C}^{1+\alpha}$-images of lattice self-similar sets over to general self-conformal sets. It will be proven in Section 4.4.3

2.5 Examples

We start this section by investigating the two Cantor sets C_1 and C_2 from the introduction. Further, we are going to apply the results from the preceding section to some well-known fractal sets, namely the Sierpinski gasket and the Sierpinski carpet. Both these sets arise from lattice systems. Therefore, at the end of this section, we additionally consider a set arising from a non-lattice system, for which the computations become especially easy with the formula from Theorem 2.37.

Example 2.47. Recall the construction of the two Cantor sets C_1 and C_2 from the introduction. The set C_1 is the invariant set of the iterated function system $\Phi := \{\phi_1, \dots, \phi_4\}$, where $\phi_i(x) = x/7 + 2(i-1)/7$ for $i \in \{1, \dots, 4\}$. It can be easily verified that the IFS Φ satisfies the prerequisites of Theorem 2.39 and that the Minkowski (Hausdorff or similarity) dimension of C_1 is equal to $\delta := \dim_M(C_1) = \ln 4/\ln 7$. An application of Theorem 2.39 yields that

$$\widetilde{\mathcal{M}}(C_1) = \frac{3}{2} \cdot \frac{2^{-\delta}}{(1-\delta)\ln 4}.$$

The Cantor set C_2 is the invariant set of the IFS $\Psi := \{\psi_1, \dots, \psi_4\}$, where $\psi_1(x) = x/7$, $\psi_2(x) = x/7 + 1/7$, $\psi_3(x) = x/7 + 5/7$ and $\psi_4(x) = x/7 + 6/7$. Its Minkowski dimension is also equal to $\ln 4/\ln 7 = \delta$. Here Theorem 2.39 yields

$$\widetilde{\mathcal{M}}(C_2) = \frac{3^\delta}{2} \cdot \frac{2^{-\delta}}{(1-\delta)\ln 4}.$$

Thus, $\widetilde{\mathcal{M}}(C_1) > \widetilde{\mathcal{M}}(C_2)$, which reflects the capability of the average Minkowski content to distinguish between sets of the same Minkowski (Hausdorff or similarity) dimension.

Example 2.48 (The Sierpinski gasket). The Sierpinski gasket F is visualised in Figure 2.5. It is the invariant set of the IFS $\Phi = \{\phi_1, \phi_2, \phi_3\}$ which is constructed as follows. Let $X \subset \mathbb{R}^2$ denote the equilateral triangle with edges $(0,0), (1,0)$ and $(1/2, \sqrt{3}/2)$. Then $\phi_1, \phi_2, \phi_3 \colon X \to X$ are defined by $\phi_1(x) := x/2$, $\phi_2(x) := x/2 + (1/2, 0)$ and $\phi_3(x) := x/2 + (1/4, \sqrt{3}/4)$.

The geometric potential function ξ associated with Φ only takes the value $\ln 2$ and thus is lattice. Therefore, we evaluate the total average fractal curvatures. Indeed, it is not

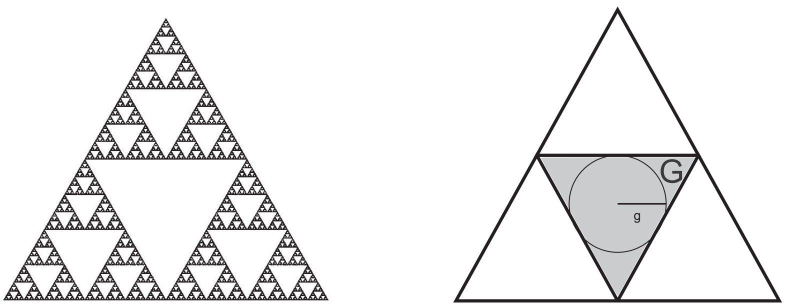


Figure 2.5: The Sierpinski gasket with primary gap G whose inradius is g.

difficult to see that the total fractal curvatures of the Sierpinski gasket do not exist. In case $k = 2$ this follows from Theorem 2.38. It is well known that the Minkowski dimension of F is equal to $\ln 3/\ln 2$. The Sierpinski gasket possesses exactly one primary gap G, which is the equilateral triangle with edges $(1/2, 0)$, $(1/4, \sqrt{3}/4)$ and $(3/4, \sqrt{3}/4)$. The inradius g of G is $g = \sqrt{3}/12$. Moreover, from the definition of the curvature measures in Equation (2.1) and from Proposition 2.7 it follows that

$$C_0(F_{e^{-T}}, G) = \begin{cases} 0 & : T \leq -\ln g, \\ -1 & : T > -\ln g, \end{cases}$$

$$C_1(F_{e^{-T}}, G) = \begin{cases} 0 & : T \leq -\ln g, \\ 3(1/4 - \sqrt{3}e^{-T}) & : T > -\ln g, \end{cases}$$

$$C_2(F_{e^{-T}}, G) = \begin{cases} \sqrt{3}/16 & : T \leq -\ln g, \\ 3e^{-T}(1/2 - \sqrt{3}e^{-T}) & : T > -\ln g. \end{cases}$$

Using that $\delta^{-1} H_{\mu_{-\delta\xi}} = \int \ln 2 \, d\mu_{-\delta\xi} = \ln 2$ (see Equation (3.4)) and evaluating the integrals from the formula in Theorem 2.37 we directly obtain

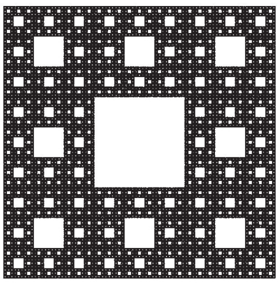
$$\widetilde{C}_0^f(F) = -\frac{1}{\ln 3} \cdot g^\delta \approx -0.0423,$$

$$\widetilde{C}_1^f(F) = \frac{3}{4 \cdot \ln(3/2)} \cdot g^{\delta-1} - \frac{3\sqrt{3}}{\ln 3} \cdot g^\delta \approx 0.3761 \quad \text{and}$$

$$\widetilde{C}_2^f(F) = \frac{\sqrt{3}}{16 \cdot \ln(4/3)} \cdot g^{\delta-2} + \frac{3}{2 \cdot \ln(3/2)} \cdot g^{\delta-1} - \frac{3\sqrt{3}}{\ln 3} \cdot g^\delta \approx 1.8126.$$

The same values have been gained in [Win08] by using the formula which we presented in Equation (2.7).

Example 2.49 (The Sierpinski carpet). The Sierpinski carpet F is depicted in Figure 2.6. It is generated by the following IFS Φ. Let $X := [0, 1]^2$ denote the unit square in $\mathbb{R}^2$. For



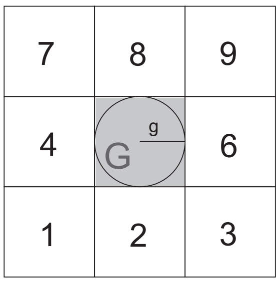


Figure 2.6: The Sierpinski carpet and its primary gap G whose inradius is g.

$x \in \mathbb{R}$ set $\lceil x \rceil := \min\{z \in \mathbb{Z} \mid x \leq z\}$. Define $\phi_j \colon X \to X$ for $j \in \{1, \ldots, 9\}$ by

$$\phi_j(x) := \frac{x}{3} + \begin{pmatrix} (j-1)(\mathrm{mod}3) \\ \lceil j/3 \rceil - 1 \end{pmatrix}$$

and set $\Phi := \{\phi_j \mid j \in \{1, \ldots, 9\} \setminus \{5\}\}$. The geometric potential function ξ associated with Φ only takes the value $\ln 3$. Therefore, ξ is lattice. Like for the Sierpinski gasket it is not hard to see that the total fractal curvatures do not exist. Therefore, we evaluate the total average fractal curvatures. The Sierpinski carpet possesses exactly one primary gap, which we denote by G. The primary gap G is the square with vertices $(1/3, 1/3)$, $(2/3, 1/3)$, $(2/3, 2/3)$ and $(1/3, 2/3)$. Its inradius g is equal to $g = 1/6$. Moreover, the Minkowski dimension of F is $\delta := \dim_M(F) = \ln 8/\ln 3$. From the definition of the curvature measures in Equation (2.1) and from Proposition 2.7 it follows that

$$C_0(F_{\mathrm{e}^{-T}}, G) = \begin{cases} 0 & : T \leq -\ln g, \\ -1 & : T > -\ln g, \end{cases}$$

$$C_1(F_{\mathrm{e}^{-T}}, G) = \begin{cases} 0 & : T \leq -\ln g, \\ 2(1/3 - 2\mathrm{e}^{-T}) & : T > -\ln g, \end{cases} \quad \text{and}$$

$$C_2(F_{\mathrm{e}^{-T}}, G) = \begin{cases} 1/9 & : T \leq -\ln g, \\ 4\mathrm{e}^{-T}(1/3 - \mathrm{e}^{-T}) & : T > -\ln g. \end{cases}$$

The measure theoretical entropy of the shift map with respect to the measure $\mu_{-\delta\xi}$ satisfies $\delta^{-1} H_{\mu_{-\delta\xi}} = \int \ln 3 \mathrm{d}\mu_{-\delta\xi} = \ln 3$ (see Equation (3.4)). Evaluating the integrals from the formula in Theorem 2.37 we obtain

$$\widetilde{C}_0^f(F) = -\frac{1}{\ln 8} \cdot g^\delta \approx -0.0162,$$

$$\widetilde{C}_1^f(F) = \frac{2}{3 \cdot \ln(8/3)} \cdot g^{\delta-1} - \frac{4}{\ln 8} \cdot g^\delta \approx 0.0725 \quad \text{and}$$

$$\widetilde{C}_2^f(F) = \frac{1}{9 \cdot \ln(9/8)} \cdot g^{\delta-2} + \frac{4}{3 \cdot \ln(8/3)} \cdot g^{\delta-1} - \frac{4}{\ln 8} \cdot g^\delta \approx 1.3529.$$

Also these values have been obtained in [Win08] by means of the formula from Equation (2.7).

Example 2.50 (Non-lattice self-similar set). In this example, we consider the self-similar set F whose construction via the IFS Φ is visualised in Figure 2.7. The IFS Φ which is depicted in Figure 2.7 consists of ten similarities $\phi_1, \ldots, \phi_{10}$ acting on the unit square $X := [0, 1]^2$. The similarity ratios of ϕ_1 and ϕ_2 are $1/2$. The similarity ratios of ϕ_3 and ϕ_4 are $1/3$ and the similarity ratios of ϕ_5 to ϕ_{10} are $1/6$. Thus, the geometric potential function ξ associated with Φ takes the values $\ln 2$, $\ln 3$ and $\ln 6$. As $\ln 2$ and $\ln 3$ are not contained in

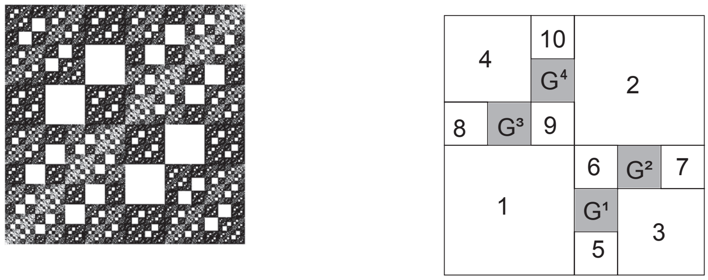


Figure 2.7: The self-similar set which is examined in Example 2.50, its construction and its primary gaps $G^1, \ldots, G^4$.

a discrete subgroup of $\mathbb{R}$, ξ is non-lattice. Hence, the fractal curvature measures do exist according to Theorem 2.37. The invariant set F possesses four primary gaps, which we denote by $G^1, \ldots, G^4$. Each of the primary gaps is a square with side length 1/6. Thus, the inradius of G^i is $g^i = 1/12$ for each $i \in \{1, \ldots, 4\}$. From the definition of the curvature measures in Equation (2.1) and from Proposition 2.7 it follows that for $i \in \{1, \ldots, 4\}$ we have that

$$C_0(F_{\mathrm{e}^{-T}}, G^i) = \begin{cases} 0 & : T \leq -\ln g^i, \\ -1 & : T > -\ln g^i, \end{cases}$$

$$C_1(F_{\mathrm{e}^{-T}}, G^i) = \begin{cases} 0 & : T \leq -\ln g^i, \\ 2(1/6 - 2\mathrm{e}^{-T}) & : T > -\ln g^i, \end{cases} \quad \text{and}$$

$$C_2(F_{\mathrm{e}^{-T}}, G^i) = \begin{cases} 1/36 & : T \leq -\ln g^i, \\ 4\mathrm{e}^{-T}(1/6 - \mathrm{e}^{-T}) & : T > -\ln g^i. \end{cases}$$

By the Moran-Hutchinson formula (see for instance Theorem 9.3 in [Fal03]) the Minkowski dimension of F is the unique solution δ of the equation

$$2^{1-\delta} + 2 \cdot 3^{-\delta} + 6^{1-\delta} = 1.$$

It approximately is $\delta \approx 1.8835$. Equation (3.4) yields $\delta^{-1} H_{\mu_{-\delta\xi}} = 2^{1-\delta} \ln 2 + 2 \cdot 3^{-\delta} \ln 3 + 6^{1-\delta} \ln 6 \approx 1.0212$. Evaluating the integrals from the formula in Theorem 2.37 we obtain

$$C_0^f(F) = -\frac{4}{H_{\mu_{-\delta\xi}}} \cdot (g^1)^\delta \approx -0.0193,$$

$$C_1^f(F) = \frac{4\delta}{3(\delta-1)H_{\mu_{-\delta\xi}}} \cdot (g^1)^{\delta-1} - \frac{16}{H_{\mu_{-\delta\xi}}} \cdot (g^1)^\delta \approx 0.0874 \quad \text{and}$$

$$C_2^f(F) = \frac{\delta}{9(2-\delta)H_{\mu_{-\delta\xi}}} \cdot (g^1)^{\delta-2} + \frac{8\delta}{3(\delta-1)H_{\mu_{-\delta\xi}}} \cdot (g^1)^{\delta-1} - \frac{16}{H_{\mu_{-\delta\xi}}} \cdot (g^1)^\delta \approx 1.4992.$$

3 Preliminaries

In this chapter, we present useful tools and necessary background for proving the results from Chapter 2. We moreover give further explanation on the constants which occur in our theorems.

We start in Section 3.1 with providing crucial material from the Perron-Frobenius theory and in Section 3.2 we present two results concerning volume functions of parallel sets.

3.1 Perron-Frobenius Theory

In order to provide the necessary background to define the constants in our main statements and also to set up the tools needed in the proofs, we now recall some facts from the Perron-Frobenius theory.

We begin in Section 3.1.1 by introducing important notions. In Section 3.1.2 we present a central result from the Perron-Frobenius theory and in Section 3.1.3 we study analytic properties of the complex Perron-Frobenius operator which are crucial for the proofs in Chapter 4.

3.1.1 The Topological Pressure Function and Hölder Continuity

The aim of this subsection is to establish important terminology which is required in the following two subsections. A good reference for the exposition provided below is [Bow08].

The Perron-Frobenius theory is a theory on the shift space (Σ^∞, σ), which we introduced in Section 2.2. Recall from Section 2.2 that we equip Σ^∞ with the product topology of the discrete topologies on Σ and let $\mathcal{C}(\Sigma^\infty)$ denote the set of continuous complex-valued functions on Σ^∞.

A central notion in this theory is that of the topological pressure function. For its introduction take $f \in \mathcal{C}(\Sigma^\infty)$, $n \in \mathbb{N}_0$ and let $S_n f$ denote the *n-th Birkhoff sum*, which is defined to be

$$S_n f := \sum_{k=0}^{n-1} f \circ \sigma^k \quad \text{for } n \in \mathbb{N} \qquad \text{and} \qquad S_0 f := 0.$$

Definition 3.1 (Topological pressure function)**.** The *topological pressure function* is defined by

$$P\colon \mathcal{C}(\Sigma^\infty) \to \mathbb{R}, \qquad P(f) := \lim_{n\to\infty} n^{-1} \ln \sum_{\omega\in\Sigma^n} \exp \sup_{u\in[\omega]} S_n f(u)$$

for $f \in \mathcal{C}(\Sigma^\infty)$, where $[\omega] := \{u \in \Sigma^\infty \mid u_i = \omega_i \text{ for } 1 \le i \le n(\omega)\}$ denotes the *ω-cylinder set* for $\omega \in \Sigma^*$, in particular $[\varnothing] = \Sigma^\infty$.

A major attribute of the topological pressure function is that it is linked to the Minkowski dimension of the invariant set of a cIFS.

Theorem 3.2. *The Minkowski as well as the Hausdorff dimension of the self-conformal set associated with the cIFS Φ is equal to the unique positive real number $t > 0$ for which $P(-t\xi) = 0$, where ξ denotes the geometric potential function associated with Φ.*

The above theorem is provided in [Bed88]. There, in particular, the statement concerning the Minkowski dimension was obtained. The statement concerning the Hausdorff dimension goes back to [Bow79, Rue82].

Also linked to the topological pressure function is the notion of a Gibbs measure for a potential function. Gibbs measures carry the nice property that the measure of a cylinder set of length n is comparable to the n-th Birkhoff sum of the potential function. More precisely, a finite Borel measure μ on the code space Σ^∞ is said to be a *Gibbs measure* for $f \in \mathcal{C}(\Sigma^\infty)$ if there exists a constant $c > 0$ such that

$$c^{-1} \le \frac{\mu([x|_n])}{\exp(S_n f(x) - n \cdot P(f))} \le c \tag{3.1}$$

for every $x \in \Sigma^\infty$ and $n \in \mathbb{N}$.

Another central notion in the following is the class of Hölder continuous functions on Σ^∞. It forms an important subclass of the class of continuous functions $\mathcal{C}(\Sigma^\infty)$ and is defined below.

Definition 3.3 (Hölder continuous on Σ^∞)**.** For $f \in \mathcal{C}(\Sigma^\infty)$, $\alpha \in (0,1)$ and $n \in \mathbb{N}_0$ define

$$\begin{aligned} \mathrm{var}_n(f) &:= \sup\{|f(\omega) - f(u)| \mid \omega, u \in \Sigma^\infty \text{ and } \omega_i = u_i \text{ for all } i \in \{1,\dots,n\}\}, \\ |f|_\alpha &:= \sup_{n\ge 0} \frac{\mathrm{var}_n(f)}{\alpha^n} \qquad \text{and} \\ \mathcal{F}_\alpha(\Sigma^\infty) &:= \{f \in \mathcal{C}(\Sigma^\infty) \mid |f|_\alpha < \infty\}. \end{aligned}$$

Elements of $\mathcal{F}_\alpha(\Sigma^\infty)$ are called *α-Hölder continuous* functions on Σ^∞.

In Section 3.1.3 we are going to make use of the fact that the space $\mathcal{F}_\alpha(\Sigma^\infty)$ endowed with the norm $\|\cdot\|_\alpha := |\cdot|_\alpha + \|\cdot\|_\infty$, where $\|\cdot\|_\infty$ denotes the supremum-norm, is a Banach space.

Remark 3.4. The geometric potential function ξ associated with a cIFS $\Phi := \{\phi_1, \ldots, \phi_N\}$ satisfies $\xi \in \mathcal{F}_{\widetilde{\alpha}}(\Sigma^\infty)$ for some $\widetilde{\alpha} \in (0,1)$. To see this, we let $r < 1$ denote a common upper bound for the contraction ratios of $\phi_1, \ldots, \phi_N$. Because of the α-Hölder continuity of $|\phi_1'|, \ldots, |\phi_N'|$ and the fact that $|\phi_1'| \ldots, |\phi_N'|$ are bounded away from zero we obtain that there exists a constant $c \in \mathbb{R}$ such that for every $n \in \mathbb{N}$ we have $\mathrm{var}_n(\xi) \leq c r^{\alpha(n-1)}$ and $\mathrm{var}_0(\xi) < \infty$. Thus, $\xi \in \mathcal{F}_{\widetilde{\alpha}}(\Sigma^\infty)$, where $\widetilde{\alpha} := r^\alpha \in (0,1)$.

3.1.2 Ruelle's Perron-Frobenius Theorem

This subsection is devoted to important results in the Perron-Frobenius theory. They originate mainly from [Rue68] and have been further developed for example in [Wal01, Bow08]. Let us start with introducing the central object of these studies.

Definition 3.5 (Perron-Frobenius operator)**.** For $f \in \mathcal{C}(\Sigma^\infty)$ define the *Perron-Frobenius operator* $\mathcal{L}_f \colon \mathcal{C}(\Sigma^\infty) \to \mathcal{C}(\Sigma^\infty)$ by

$$\mathcal{L}_f\psi(x) := \sum_{y\colon \sigma y = x} \mathrm{e}^{f(y)}\psi(y) \tag{3.2}$$

for $x \in \Sigma^\infty$ and let $\mathcal{L}_f^*$ be the dual of $\mathcal{L}_f$ acting on the set of Borel probability measures on Σ^∞.

By Theorem 2.16 and Corollary 2.17 of [Wal01] and Theorem 1.7 of [Bow08], for each real-valued Hölder continuous $f \in \mathcal{F}_\alpha(\Sigma^\infty)$, some $\alpha \in (0,1)$, there exists a unique Borel probability measure ν_f on Σ^∞ such that $\mathcal{L}_f^*\nu_f = \gamma_f\nu_f$ for some $\gamma_f > 0$. Moreover, γ_f is uniquely determined by this equation and satisfies $\gamma_f = \exp(P(f))$, where P denotes the topological pressure function.

Further, there exists a unique strictly positive eigenfunction $h_f \in \mathcal{C}(\Sigma^\infty)$ of $\mathcal{L}_f$ satisfying $\mathcal{L}_f h_f = \gamma_f h_f$. We take h_f to be normalised so that $\int h_f \mathrm{d}\nu_f = 1$. By μ_f we denote the σ-invariant probability measure defined by $\frac{\mathrm{d}\mu_f}{\mathrm{d}\nu_f} = h_f$. This is the unique σ-invariant Gibbs measure for the potential function f. Additionally, under some normalisation assumptions we have convergence of the iterates of the Perron-Frobenius operator to the projection onto its eigenfunction h_f. To be more precise we have

$$\lim_{m\to\infty} \|\gamma_f^{-m}\mathcal{L}_f^m\psi - \int\psi \mathrm{d}\nu_f \cdot h_f\|_\infty = 0 \quad \forall\ \psi \in \mathcal{C}(\Sigma^\infty). \tag{3.3}$$

The definition of the topological pressure function and the relation $\gamma_f = \exp(P(f))$ imply the following.

Proposition 3.6. *Let $f \in \mathcal{C}(\Sigma^\infty)$ be positive and real-valued. Then $z \mapsto P(zf)$ and $z \mapsto \gamma_{zf}$ are strictly monotonically increasing functions for real z.*

With the notation established above, we now introduce the measure theoretical entropy, which occurs in our formulae in Section 2.4. Note, that the geometric potential function ξ of a cIFS Φ belongs to $\in \mathcal{F}_\alpha(\Sigma^\infty)$ by Remark 3.4. It can be shown that the *measure theoretical entropy* $H_{\mu_{-\delta\xi}}$ of the shift-map σ with respect to $\mu_{-\delta\xi}$ is given by

$$H_{\mu_{-\delta\xi}} = \delta \int_{\Sigma^\infty} \xi \mathrm{d}\mu_{-\delta\xi}, \tag{3.4}$$

where δ denotes the Minkowski dimension of the self-conformal set associated with Φ. This observation follows for example from the variational principle (see Theorem 1.22 of [Bow08]) and Theorem 3.2.

An interesting property of $h_{-\delta\xi}$ is given in Section 6.1 of [MU03]:

Theorem 3.7 ([MU03])**.** *Suppose that the contractions $\phi_1, \ldots, \phi_N$ of a cIFS acting on $X \subset \mathbb{R}$ are real-analytic on an open neighbourhood of X in $\mathbb{R}$. Let ξ denote the geometric potential function associated with $\Phi := \{\phi_1, \ldots, \phi_N\}$ and let δ denote the Minkowski dimension of the invariant set of Φ. Then $h_{-\delta\xi}$ has a real-analytic extension on an open connected neighbourhood of X in $\mathbb{R}$.*

The above theorem follows from Corollary 6.1.4 of [MU03]. Note that there cIFS with a countable alphabet are considered and it is additionally required that there exists an open connected set $U \subset \mathbb{R}^2$ containing X such that all elements of the cIFS extend to analytic functions on U and such that U is invariant under all elements of the cIFS. This condition is always satisfied for systems with a finite alphabet.

3.1.3 Analytic Properties of the Perron-Frobenius Operator

This subsection is concerned with analytic properties of the operator-valued function $z \mapsto (I - \mathcal{L}_{zf})^{-1}$, where $f \in \mathcal{F}_\alpha(\Sigma^\infty)$ is fixed and $z \in \mathbb{C}$.

Let us start by giving the precise definition of what it means for an operator-valued function to be holomorphic. We let $\mathcal{B}(\mathcal{F}_\alpha(\Sigma^\infty))$ denote the set of all bounded linear operators on $\mathcal{F}_\alpha(\Sigma^\infty)$ to $\mathcal{F}_\alpha(\Sigma^\infty)$. Note that since $(\mathcal{F}_\alpha(\Sigma^\infty), \|\cdot\|_\alpha)$ is a Banach space, also $(\mathcal{B}(\mathcal{F}_\alpha(\Sigma^\infty)), \|\cdot\|_{\mathrm{op}})$ is a Banach space (see for example [Kat95, p.150]). Here $\|\cdot\|_{\mathrm{op}}$ denotes the operator norm, which for $A \in \mathcal{B}(\mathcal{F}_\alpha(\Sigma^\infty))$ is defined through $\|A\|_{\mathrm{op}} := \sup_{g \in \mathcal{F}_\alpha(\Sigma^\infty),\, \|g\|_\alpha = 1} \|Ag\|_\alpha$.

Definition 3.8 (Operator-valued holomorphic function)**.** An operator-valued function $f\colon D \to \mathcal{B}(\mathcal{F}_\alpha(\Sigma^\infty))$ defined on an open domain $D \subset \mathbb{C}$ is called *holomorphic*, if for all $z \in D$ there exists an $l(z) \in \mathcal{B}(\mathcal{F}_\alpha(\Sigma^\infty))$ such that

$$\lim_{h \to 0} \|h^{-1}(f(z+h) - f(z)) - l(z)\|_{\mathrm{op}} = 0.$$

For more insights on these notions, we refer to [Kat95], especially Chapters III.3 and VII.1 in there. Following convention, we interchangeably use the terms holomorphic and analytic. A central role in studying the analytic properties of $z \mapsto (I - \mathcal{L}_{zf})^{-1}$ is played by the spectrum of the Perron-Frobenius operator. The Perron-Frobenius operator $\mathcal{L}_{zf}$ is a bounded linear operator on the Banach space $(\mathcal{F}_\alpha(\Sigma^\infty), \| \cdot \|_\alpha)$. Its *spectrum* is the set of all complex numbers λ such that $\mathcal{L}_{zf} - \lambda I$ is not invertible. The *spectral radius* $\text{spr}(\mathcal{L}_{zf})$ of $\mathcal{L}_{zf}$ is the radius of the smallest closed disc with centre at the origin which contains the spectrum of $\mathcal{L}_{zf}$. The *spectral radius formula* states that

$$\lim_{n\to\infty} \|\mathcal{L}_{zf}^n\|_{\text{op}}^{1/n} = \text{spr}(\mathcal{L}_{zf}). \tag{3.5}$$

The spectrum of the complex Perron-Frobenius operator has been studied by Parry and Pollicott. The following statement, which characterises the spectrum of $\mathcal{L}_{zf}$ for non-real $z \in \mathbb{C}$, originates from [Pol84]. Here, we present it in the form of Theorem B of [Lal89]. For this, let $\Re(z)$ and $\Im(z)$ denote the real and imaginary parts of $z \in \mathbb{C}$.

Theorem 3.9 ([Pol84]). *Take $f \in \mathcal{F}_\alpha(\Sigma^\infty)$ for $\alpha \in (0,1)$ and suppose that $z \in \mathbb{C} \setminus \mathbb{R}$.*

(i) If for some $a \in \mathbb{R}$ the function $(\Im(z)f - a)/(2\pi)$ is cohomologous to an integer-valued function, then $\mathrm{e}^{ia}\gamma_{\Re(z)f}$ is a simple eigenvalue of $\mathcal{L}_{zf}$, and the rest of the spectrum is contained in a disc centred at zero of radius strictly less than $\gamma_{\Re(z)f}$.

(ii) Otherwise, the entire spectrum of $\mathcal{L}_{zf}$ is contained in a disc centred at zero of radius strictly less than $\gamma_{\Re(z)f}$.

Now, we present a couple of useful results from [Lal89] concerning holomorphic properties of the operator-valued function $z \mapsto (I - \mathcal{L}_{zf})^{-1}$ for $z \in \mathbb{C}$.

Proposition 3.10 (Propositions 7.1 and 7.2 in [Lal89]). *Let $f \in \mathcal{F}_\alpha(\Sigma^\infty)$ denote a real-valued α-Hölder continuous function for $\alpha \in (0,1)$ and let $-\delta$ denote the unique real zero of $z \mapsto P(zf)$. Then the following hold.*

(i) The function $z \mapsto (I - \mathcal{L}_{zf})^{-1}$ is holomorphic in the half-plane $\Re(z) < -\delta$.

(ii) The function $z \mapsto (I - \mathcal{L}_{zf})^{-1}$ has a simple pole at $z = -\delta$. In particular, for each $g \in \mathcal{F}_\alpha(\Sigma^\infty)$,

$$(I - \mathcal{L}_{zf})^{-1} g = \gamma_{zf}(1 - \gamma_{zf})^{-1} \int g \,\mathrm{d}\nu_{zf} \cdot h_{zf} + (I - \mathcal{L}''_{zf})^{-1} g \tag{3.6}$$

for z in some punctured neighbourhood of $z = -\delta$, where

$$\begin{aligned} \mathcal{L}''_{zf} &:= \mathcal{L}_{zf} - \mathcal{L}'_{zf} && \text{and } \mathcal{L}'_{zf} \text{ is defined through} \\ \mathcal{L}'_{zf} g &:= \gamma_{zf} \int g \,\mathrm{d}\nu_{zf} \cdot h_{zf} && \text{for } g \in \mathcal{F}_\alpha(\Sigma^\infty) \text{ and some } \alpha \in (0,1). \end{aligned}$$

Moreover, $z \mapsto (I - \mathcal{L}''_{zf})^{-1}$ *is a holomorphic operator-valued function in a neighbourhood of* $z = -\delta$*.*

We remark that the factor γ_{zf} of the first summand on the right hand side of Equation (3.6) is missing in [Lal89]. This however does not affect the results, since the z-value of interest is $z = -\delta$, where $\gamma_{zf} = 1$.

We are going to be interested in the residue of $z \mapsto (I - \mathcal{L}_{zf})^{-1}$ at the simple pole $z = -\delta$. For this, we use that the topological pressure function $z \mapsto P(zf)$ is real-analytic for $z \in \mathbb{R}$ and real-valued $f \in \mathcal{F}_\alpha(\Sigma^\infty)$ and that it satisfies

$$\frac{\mathrm{d}}{\mathrm{d}z} P(zf) = \int f \mathrm{d}\mu_{zf}, \quad z \in \mathbb{R}. \tag{3.7}$$

The analyticity of the topological pressure function can be proven with methods of analytic perturbation theory as presented in [Kat95]. This method of proof is due to Ruelle [Rue78]. Combined with Equation (3.7), Proposition 3.10(ii) yields the following corollary, since $\gamma_{zf} = \exp(P(zf))$ and since $z \mapsto \gamma_{zf}$, $z \mapsto \int g \mathrm{d}\nu_{zf}$ and $z \mapsto h_{zf}$ are continuous at $z = -\delta$.

Corollary 3.11. *Assume that* f *is real-valued and take* $g \in \mathcal{F}_\alpha(\Sigma^\infty)$*. Then the residue of* $(I - \mathcal{L}_{zf})^{-1} g(x)$ *at* $z = -\delta$*, where* $x \in \Sigma^\infty$*, is*

$$-\frac{\int g \mathrm{d}\nu_{-\delta f}}{\int f \mathrm{d}\mu_{-\delta f}} h_{-\delta f}(x).$$

We remark that the equation from Corollary 3.11 and the respective equation in [Lal89, p.25] differ by sign.

We end this section by presenting two statements from [Lal89], which address the analyticity of $z \mapsto (I - \mathcal{L}_{zf})^{-1}$ on the line $\Re(z) = -\delta$. These two statements show that the analytic properties highly depend on f being lattice or non-lattice.

Proposition 3.12 (Proposition 7.3 in [Lal89])**.** *If* f *is non-lattice then the function* $z \mapsto (I - \mathcal{L}_{zf})^{-1}$ *is holomorphic in a neighbourhood of every* z *on the line* $\Re(z) = -\delta$ *except for* $z = -\delta$*.*

Proposition 3.13 (Proposition 7.4 in [Lal89])**.** *If* f *is integer-valued but not cohomologuous to any function valued in a proper subgroup of the integers, then* $z \mapsto (I - \mathcal{L}_{zf})^{-1}$ *is* $2\pi\mathbf{i}$*-periodic, and holomorphic at every* z *on the line* $\Re(z) = -\delta$ *such that* $\Im(z)/(2\pi)$ *is not an integer.*

3.2 Volume Functions of Parallel Sets

In this short section, we present two tools which provide relationships between the 0-th and the 1-st total fractal curvatures of subsets of $\mathbb{R}$. These statements are needed for proving the statements concerning the lattice situation.

The first tool is a special case of Corollary 3.2 of [RW10] and allows one to deduce the existence of the Minkowski content from the existence of the 0-th total fractal curvature.

Theorem 3.14 (Rataj, Winter)**.** *Let $Y \subset \mathbb{R}$ be a non-empty and compact set for which the Minkowski dimension $\delta := \dim_M(Y)$ exists and which is such that $\lambda^1(Y) = 0$. Then*

$$\liminf_{\varepsilon \searrow 0} \frac{\varepsilon^\delta \lambda^0(\partial Y_\varepsilon)}{1-\delta} \leq \liminf_{\varepsilon \searrow 0} \varepsilon^{\delta-1} \lambda^1(Y_\varepsilon) \leq \limsup_{\varepsilon \searrow 0} \varepsilon^{\delta-1} \lambda^1(Y_\varepsilon) \leq \limsup_{\varepsilon \searrow 0} \frac{\varepsilon^\delta \lambda^0(\partial Y_\varepsilon)}{1-\delta}.$$

The proof is based on an interesting relationship between the derivative $\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda^1(F_\varepsilon)$ which exists Lebesgue-almost everywhere and the quantity $\lambda^0(\partial F_\varepsilon)$ which was established in [Sta76] for arbitrary bounded subsets of $\mathbb{R}^d$ and builds on the work of [Kne51]. As this relationship is of use also for us, we state it in the form of Corollary 2.5 in [RW10].

Proposition 3.15 (Stachó)**.** *Let $Y \subset \mathbb{R}$ be compact. Then the function $\varepsilon \mapsto \lambda^1(Y_\varepsilon)$ is differentiable for all but a countable number of $\varepsilon > 0$ with differential*

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\lambda^1(Y_\varepsilon) = \lambda^0(\partial Y_\varepsilon).$$

4 Proofs

In this chapter we present the proofs of our main theorems. These theorems are concerned with different structures of the underlying set F and thus are subdivided into three categories just like in Chapter 2.

The common theme of all the theorems is that they make a statement about the existence of the (essential-) weak-limit

$$\text{(ess-)}\ \underset{\varepsilon\searrow 0}{\text{w-lim}}\, \varepsilon^{\delta-k} C_k(F_\varepsilon, \cdot)$$

and the weak limit

$$\underset{\varepsilon\searrow 0}{\text{w-lim}} |\ln \varepsilon|^{-1} \int_\varepsilon^1 \mathrm{e}^{-T(\delta-k)} C_k(F_{\mathrm{e}^{-T}}, \cdot)\mathrm{d}T$$

for some or all $k \in \{0, \ldots, d\}$.

We begin in Section 4.1 by giving an outline of the structure of the proofs of the theorems and by presenting key results which will be used in the proofs. In Section 4.2 these key results are used for proving the theorems concerning self-conformal sets. In Section 4.3 the self-similar setting is addressed and the final Section 4.4 covers the situation of $\mathcal{C}^{1+\alpha}$-images of self-similar sets.

4.1 Key Tools and Outline of the Proofs

The common factor of the theorems from Chapter 2 is that the underlying set F is an invariant set of a cIFS Φ acting on a non-empty, compact connected set $X \subset \mathbb{R}^d$ such that $\lambda_d(O \setminus \Phi X)$ is strictly positive and such that the geometric conditions from Section 2.3 hold. (Note that (COND 2) to (COND 4), (COND 2') and (COND 4') are always satisfied in space dimension one as we will see later in the proofs.) Recall from Definition 2.17 that a cIFS F acting on X satisfies the OSC with open set $O := \text{int}_{\mathbb{R}^d} X$. Moreover, recall that the geometric conditions from Section 2.3 imply that $X \setminus \Phi X$ possesses finitely many connected components, which are denoted by $G^1, \ldots, G^Q$ and called the primary gaps of F. Their images under the map ϕ_ω are denoted by $G^1_\omega, \ldots, G^Q_\omega$ for $\omega \in \Sigma^*$.

The proofs of our theorems are rather technical. Thus, to gain a better overview of the structure of the proofs and in order to show the relevance of the forthcoming lemmas, we now present an outline of the proofs. For ease of presentation, we restrict ourselves to the non-lattice situation here and remark that the fundamental idea behind the proofs on the average parts is the same.

(I) We introduce the set-class

$$\mathcal{E}_F := \{\phi_\omega O \mid \omega \in \Sigma^*\} \cup \mathcal{K}_F, \quad \text{where} \tag{4.1}$$

$$\mathcal{K}_F := \Big\{ K \in \mathfrak{B}(\mathbb{R}^d) \,\Big|\, \exists n \in \mathbb{N} \colon K \subseteq \mathbb{R}^d \setminus \textstyle\bigcup_{\omega\in\Sigma^n} \phi_\omega O \Big\} \tag{4.2}$$

and show that $\mathcal{E}_F$ is an intersection stable generator for $\mathfrak{B}(\mathbb{R}^d)$. This is done in Lemma 4.4.

(II) Using Step (I) together with Prohorov's Theorem (Theorem A.7) and the fact that two signed Borel measures which coincide on an intersection stable generator for $\mathfrak{B}(\mathbb{R}^d)$ coincide on $\mathfrak{B}(\mathbb{R}^d)$ (see Proposition A.4) we show the following. If there exists a signed Borel measure μ such that $\mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, B)$ (essentially) converges to $\mu(B)$ for every $B \in \mathcal{E}_F$ as $t \to \infty$ on the domain of definition, then (ess-) $\text{w-lim}_{t\to\infty} \mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, \cdot) = \mu(\cdot)$. This is shown in Lemma 4.6.

In order to show the existence of μ as in Step (II) we distinguish between the cases $B \in \mathcal{K}_F$ and $B \in \mathcal{E}_F \setminus \mathcal{K}_F$.

(III) For $B \in \mathcal{K}_F$ we show that (ess-) $\lim_{t\to\infty} \mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, B) = 0$.
For $B = \phi_\kappa O \in \mathcal{E}_F \setminus \mathcal{K}_F$, where $\kappa \in \Sigma^*$, we show that

$$C_k(F_{\mathrm{e}^{-t}}, B) = \sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x) C_k\Bigg(F_{\mathrm{e}^{-t}}, \bigcup_{i=1}^{Q} G^i_{u\omega} \cap B\Bigg) + \mathfrak{o}(\mathrm{e}^{t(\delta-k)}) \tag{4.3}$$

for $t \in U_k$, where $U_k := \{t \in \mathbb{R} \mid \mathrm{e}^{-t} \text{ is a regular distance for } F\}$ if $d \geq 4$ and $k \in \{0, \dots, d-2\}$ and $U_k := \mathbb{R}$ else (see Lemma 4.5).

Here $\mathfrak{o}$ denotes the *Landau symbol*, which is defined as follows. For a function $f \colon U \to \mathbb{R}$ defined on $U \subseteq \mathbb{R}$ with $\lambda_1(\mathbb{R}\setminus U) = 0$ and $g \colon \mathbb{R} \to \mathbb{R}$ we write $f = \mathfrak{o}(g)$ if $\text{ess-}\lim_{t\to\infty} f(t)/g(t) = 0$, where the essential limit is taken over U. Since F has finitely many primary gaps, we assume without loss of generality that F possesses exactly one primary gap which we denote by G. Its main gaps are denoted by G_ω for $\omega \in \Sigma^*$.

In view of Equation (4.3) we introduce functions

$$f_{k,\omega}(t) := C_k(F_{\mathrm{e}^{-t}}, G_\omega)$$

for $\omega \in \Sigma^*$ whose domain of definition depends on the respective setting of the theorem (see Lemmas 4.8, 4.9 and 4.13).

(IV) We show that the expression on the right hand side of Equation (4.3) can be approximated by

$$\underbrace{\sum_{\omega\in\Sigma^m}\sum_{n=0}^{\infty}\sum_{u\in\Sigma^n}\mathbb{1}_{[\kappa]}(u\omega x)\mathrm{e}^{-kS_n\xi(u\omega x)}f_{k,\omega}(t-S_n\xi(u\omega x))}_{=:N_{k,m,\kappa}(t)}+\mathfrak{o}(\mathrm{e}^{t(\delta-k)}) \tag{4.4}$$

and determine the (essential) asymptotic as $t\to\infty$ for the functions $N_{k,m,\kappa}$ (see Key Lemma 4.2).

Note that in the self-similar case the terms in Equations (4.3) and (4.4) are equal.

(V) From the knowledge of the asymptotic of $N_{k,m,\kappa}$ we deduce the existence of a signed Borel measure μ such that

$$\text{(ess-)}\lim_{t\to\infty}\mathrm{e}^{-t(\delta-k)}C_k(F_{\mathrm{e}^{-t}},B)=\mu(B)$$

for all $B\in\mathcal{E}_F$ and determine the limiting signed Borel measure μ. This is done in the respective proofs of the theorems.

Combining Steps (I) to (V) shows the desired statement.

Now we present the precise statements and proofs of the auxiliary results and start with the most important one, namely with Key Lemma 4.2. This result is formulated in a general way so that it can be applied in all the cases we consider. In this way it unifies the proofs of all the results from Chapter 2.

4.1.1 Key Lemma 4.2 with Proof

We first fix some notation. We let $\lfloor x\rfloor$ denote the integer part of $x\in\mathbb{R}$, that is the largest integer $z\in\mathbb{Z}$ satisfying $z\le x$. Moreover, $\{x\}:=x-\lfloor x\rfloor\in[0,1)$ is the fractional part of x.

Definition 4.1 (essentially asymptotic)**.** Let $U\subseteq\mathbb{R}$ denote a set which satisfies $\lambda_1(\mathbb{R}\setminus U)=0$. We say that a function $f\colon U\to\mathbb{R}$ is *essentially asymptotic* to a function $g\colon\mathbb{R}\to\mathbb{R}$ on U as $t\to\infty$, if

$$\operatorname*{ess-lim}_{t\to\infty}f(t)/g(t)=1,$$

where the essential limit is taken over U. For such f,g and U we write $f(t)\sim^U g(t)$ as $t\to\infty$.

Key lemma 4.2. *Let Φ denote a cIFS acting on a compact connected subset X of $\mathbb{R}^d$ with associated self-conformal set $F\subset\mathbb{R}^d$. Let $\delta:=\dim_M(F)>0$ be the Minkowski dimension* *of F and denote by $\xi\colon \Sigma^\infty \to \mathbb{R}$ the geometric potential function. Let $\nu_{-\delta\xi}$ and $\mu_{-\delta\xi}$ be the measures defined in Ruelle's Perron-Frobenius Theorem (see Section 3.1.2) and denote by $h_{-\delta\xi} = \frac{\mathrm{d}\mu_{-\delta\xi}}{\mathrm{d}\nu_{-\delta\xi}}$ their Radon-Nikodym derivative. For $k \in \mathbb{N}$, $m \in \mathbb{N}$ and $\omega \in \Sigma^m$ let $f_{k,\omega}\colon U \to \mathbb{R}$ be functions satisfying the following.*

(A) The domain of definition $U \subseteq \mathbb{R}$ fulfils $\lambda_1(\mathbb{R} \setminus U) = 0$ and for every $t \in U$ there exists an $\varepsilon_0 > 0$ such that $t - \varepsilon \in U$ for every $\varepsilon \in [0, \varepsilon_0)$.

(B) The function $f_{k,\omega}$ is left-continuous on U, meaning that $\lim_{\substack{t\to x\\ t<x}} f_{k,\omega}(t) = f_{k,\omega}(x)$ for every $x \in U$.

(C) The integral

$$\int_{-\infty}^{\infty} \mathrm{e}^{-t(\delta-k)} |f_{k,\omega}(t)| \mathrm{d}t$$

exists.

(D) For $x \in \Sigma^\infty$, $\kappa \in \Sigma^*$ and $m \in \mathbb{N}$ the series

$$N_{k,m,\kappa}(t,x) := \sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x) \mathrm{e}^{-kS_n\xi(u\omega x)} f_{k,\omega}(t - S_n\xi(u\omega x))$$

converges absolutely for t in the domain of definition V of $N_{k,m,\kappa}(\cdot, x)$. Note that Item (A) implies that $\lambda_1(\mathbb{R} \setminus V) = 0$.

(E) One of the following is valid for a fixed $m \in \mathbb{N}$.

(a) There exists a $t^* > 0$ such that $f_{k,\omega}(t) = 0$ for all $t < t^*$ and $\omega \in \Sigma^m$ or

(b) $N_{k,m,\kappa}(\cdot, x)$ is non-increasing.

(F) There exists a $\mathfrak{C} \in \mathbb{R}$ such that $\mathrm{e}^{-t(\delta-k)} N^{abs}_{k,m}(t,x) \le \mathfrak{C}$ for all $x \in \Sigma^\infty$ and $t \in V$, where

$$N^{abs}_{k,m}(t,x) := \sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \sum_{u\in\Sigma^n} \mathrm{e}^{-kS_n\xi(u\omega x)} |f_{k,\omega}(t - S_n\xi(u\omega x))|.$$

(G) There exist $t^* \in \mathbb{R}$ (which coincides with the t^* from Item (E)a when in the situation Item (E)a) and $\mathfrak{C}' \ge 0$ such that for all $t < t^*$ we have

$$N^{abs}_{k,m}(t,x) \le \mathfrak{C}'$$

for all $m \in \mathbb{N}$ and $x \in \Sigma^\infty$. If $k \le \delta$, then we require $\mathfrak{C}' = 0$.

Then the following hold.

(i) *For $x \in \Sigma^\infty$ we always have that*

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T = \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} f_{k,\omega}(T)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \cdot h_{-\delta\xi}(\omega x).$$

(ii) *If ξ is non-lattice, then*

$$N_{k,m,\kappa}(t,x) \sim^V \underbrace{\sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} f_{k,\omega}(T)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} h_{-\delta\xi}(\omega x)}_{=:D_{k,m,\kappa}(x)} \cdot \mathrm{e}^{t(\delta-k)}$$

as $t\to\infty$, for all $x\in\Sigma^\infty$, where V is as in Item (D).

(iii) *Assume that ξ is lattice. Let $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ denote functions such that $\xi - \zeta = \psi - \psi\circ\sigma$ and such that the range of ζ is contained in a discrete subgroup of $\mathbb{R}$. Let $a > 0$ be maximal satisfying $\zeta(\Sigma^\infty) \subseteq a\mathbb{Z}$. Assume that (a) $U = \mathbb{R}$ (which implies $V = \mathbb{R}$) or that (b) $\psi \equiv 0$. In case (a), set $\widetilde{V} := \mathbb{R}$ and in case (b) let $\widetilde{V}$ denote the domain of definition of $t \mapsto \sum_{l=-\infty}^{\infty} \mathrm{e}^{-al(\delta-k)} f_{k,\omega}(al + a\{t/a\})$. (Note that Item (A) implies that $\lambda_1(\mathbb{R}\setminus\widetilde{V}) = 0$ in both cases.) Then*

$$N_{k,m,\kappa}(t,x) \sim^{V\cap\widetilde{V}} a \sum_{\omega\in\Sigma^m} \mathrm{e}^{a\left\lfloor \frac{t+\psi(\omega x)}{a}\right\rfloor(\delta-k)} \mathrm{e}^{k\psi(\omega x)} \frac{h_{-\delta\zeta}(\omega x)}{\int \zeta \mathrm{d}\mu_{-\delta\zeta}} \cdot \int_{\Sigma^\infty} \mathrm{e}^{-k\psi(y)} \mathbb{1}_{[\kappa]}(y) \sum_{l=-\infty}^{\infty} \mathrm{e}^{-al(\delta-k)} f_{k,\omega}\left(al + a\left\{\frac{t+\psi(\omega x)}{a}\right\} - \psi(y)\right) \mathrm{d}\nu_{-\delta\zeta}(y)$$

as $t\to\infty$ along $\widetilde{V}$ uniformly for $x\in\Sigma^\infty$.

Note that Items (A) to (G) of Key Lemma 4.2 are satisfied for the functions given by $f_{k,\omega}(t) := C_k(F_{\mathrm{e}^{-t}}, G_\omega)$, where F is as in our theorems. This is shown in Lemmas 4.8, 4.9 and 4.13.

The proofs of the three parts of Key Lemma 4.2 are very different and so we split the proof and start by considering Item (ii). Before carrying out this proof, we present a useful smoothing argument for showing the asymptotic (see [Lal89] pp. 27 ff.). For a probability density $\Pi\colon \mathbb{R}\to\mathbb{R}$ we consider its Fourier-Laplace transform given by

$$\widehat{\Pi}(\mathbf{i}\theta) := \int_{-\infty}^{\infty} \mathrm{e}^{\mathbf{i}\theta t}\Pi(t)\mathrm{d}t$$

and introduce the following class of probability densities.

$$\begin{aligned}\mathfrak{P} := \{\Pi\colon \mathbb{R} \to \mathbb{R} \mid \Pi \text{ is a probability density, } \Pi(t) = \Pi(-t) \text{ for } t \in \mathbb{R} \text{ and} \\ \widehat{\Pi}(\mathbf{i}\theta) \text{ is non-negative, } \mathcal{C}^\infty \text{ and has compact support}\}.\end{aligned}$$

Note that the function $\widehat{\Pi}\colon \mathbb{R} \to \mathbb{C}$ given by

$$\widehat{\Pi}(\mathbf{i}\theta) := \begin{cases} \exp\left(\frac{-\theta^2}{1-\theta^2}\right) & : |\theta| \leq 1, \\ 0 & : \text{else} \end{cases} \tag{4.5}$$

defines an even probability density $\Pi\colon \mathbb{R} \to \mathbb{R}$ which lies in $\mathfrak{P}$. That Π is a probability density is due to Bochner's theorem (see for instance [Kle08, Satz 15.29]). Thus, $\mathfrak{P}$ is non-empty. For the following, fix Π as such. As Π is an even probability density we know that for all $\varepsilon > 0$ there exists a $\tau > 0$ such that

$$\int_{-\tau}^{\tau} \Pi(t)\mathrm{d}t \geq 1 - \varepsilon.$$

For each $\varepsilon > 0$ fix such a $\tau = \tau(\varepsilon)$. Thus, Π_ε which for $\varepsilon > 0$ is defined by

$$\Pi_\varepsilon(t) := \frac{\tau(\varepsilon)}{\varepsilon} \cdot \Pi\left(t \cdot \tau(\varepsilon)/\varepsilon\right) \tag{4.6}$$

satisfies

$$\int_{-\varepsilon}^{\varepsilon} \Pi_\varepsilon(t)\mathrm{d}t = \int_{-\tau(\varepsilon)}^{\tau(\varepsilon)} \Pi(t)\mathrm{d}t \geq 1 - \varepsilon.$$

Moreover, it can be easily verified that $\Pi_\varepsilon \in \mathfrak{P}$ for all $\varepsilon > 0$. The smoothing argument goes as follows.

Lemma 4.3. *In order to prove Key Lemma 4.2(ii) it suffices to prove that for all $x \in \Sigma^\infty$ we have that*

$$\lim_{r\to\infty} \int_{-\infty}^{\infty} \left(\Pi_\varepsilon(r-T) + \Pi_\varepsilon(-r-T)\right) \mathrm{e}^{-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T = D_{k,m,\kappa}(x) \tag{4.7}$$

uniformly for $\varepsilon \in (0,1]$.

Proof. In case $N_{k,m,\kappa}(t,x)$ is a monotonic function in t, this statement follows directly from the proof of Lemma 8.2 of [Lal89]. In order to cover also functions which are not necessarily monotonic, we are going to adapt the methods of proof used in [Lal89]. Thus, in the following we suppose that we are in case (E)a, namely that there exists a $t^* > 0$ such that $f_{k,\omega}(t) = 0$ for all $t < t^*$ and $\omega \in \Sigma^m$.

Firstly, for $r \in \mathbb{R}$ and $\varepsilon > 0$, Item (F) implies that

$$\begin{aligned}&\left|\int_{-\infty}^{\infty} \Pi_\varepsilon(r-T)\mathrm{e}^{-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T - \int_{r-\varepsilon}^{r+\varepsilon} \Pi_\varepsilon(r-T)\mathrm{e}^{-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T\right| \\ &\quad \leq \mathfrak{C} \cdot \varepsilon,\end{aligned} \tag{4.8}$$

which tends to 0 as $\varepsilon \to 0$ uniformly for $x \in \Sigma^\infty$. Secondly, using Item (G) and that $\mathfrak{C}' = 0$ for $k \leq \delta$ we have that

$$\lim_{r\to\infty}\lim_{\varepsilon\searrow 0}\left|\int_{-r}^{-r+2\varepsilon}\Pi_\varepsilon(-r+\varepsilon-T)\mathrm{e}^{-T(\delta-k)}N_{k,m,\kappa}(T,x)\mathrm{d}T\right| \leq \lim_{r\to\infty}\lim_{\varepsilon\searrow 0}\mathfrak{C}'\mathrm{e}^{(r-2\varepsilon)(\delta-k)} = 0. \tag{4.9}$$

Thirdly, we observe that with V as in Item (D) we have that

$$\begin{aligned}\inf_{\substack{\widetilde{\varepsilon}\in[0,2\varepsilon]\\ r-\widetilde{\varepsilon}\in V}} &\mathrm{e}^{-(r-\widetilde{\varepsilon})(\delta-k)}N_{k,m,\kappa}(r-\widetilde{\varepsilon},x)\cdot(1-\varepsilon)\\ &\leq \int_{r-2\varepsilon}^{r}\Pi_\varepsilon(r-\varepsilon-T)\mathrm{e}^{-T(\delta-k)}N_{k,m,\kappa}(T,x)\mathrm{d}T\\ &\leq \sup_{\substack{\widetilde{\varepsilon}\in[0,2\varepsilon]\\ r-\widetilde{\varepsilon}\in V}}\mathrm{e}^{-(r-\widetilde{\varepsilon})(\delta-k)}N_{k,m,\kappa}(r-\widetilde{\varepsilon},x).\end{aligned} \tag{4.10}$$

For a fixed $r \in V$, Item (E)a implies that the series which defines $N_{k,m,\kappa}(r-\widetilde{\varepsilon},x)$ is a finite sum, since ξ is positive. Moreover, the number of summands is bounded uniformly for $x \in \Sigma^\infty$ and $\widetilde{\varepsilon} \in [0,2\varepsilon]$ for which $r-\widetilde{\varepsilon} \in V$. Since according to Item (B) the functions $f_{k,\omega}$ are left-continuous on U it follows that

$$\begin{aligned}\lim_{\varepsilon\searrow 0}\inf_{\substack{\widetilde{\varepsilon}\in[0,2\varepsilon]\\ r-\widetilde{\varepsilon}\in V}}\mathrm{e}^{-(r-\widetilde{\varepsilon})(\delta-k)}N_{k,m,\kappa}(r-\widetilde{\varepsilon},x) &= \lim_{\varepsilon\searrow 0}\sup_{\substack{\widetilde{\varepsilon}\in[0,2\varepsilon]\\ r-\widetilde{\varepsilon}\in V}}\mathrm{e}^{-(r-\widetilde{\varepsilon})(\delta-k)}N_{k,m,\kappa}(r-\widetilde{\varepsilon},x)\\ &= \mathrm{e}^{-r(\delta-k)}N_{k,m,\kappa}(r,x).\end{aligned} \tag{4.11}$$

Using Equations (4.8) to (4.11) and that by hypothesis the convergence in Equation (4.7) is uniform in ε, we conclude that

$$\begin{aligned}D_{k,m,\kappa}(x) &= \lim_{\varepsilon\searrow 0}\lim_{r\to\infty}\int_{-\infty}^{\infty}\left(\Pi_\varepsilon(r-T)+\Pi_\varepsilon(-r-T)\right)\mathrm{e}^{-T(\delta-k)}N_{k,m,\kappa}(T,x)\mathrm{d}T\\ &= \lim_{r\to\infty}\lim_{\varepsilon\searrow 0}\int_{-\infty}^{\infty}\left(\Pi_\varepsilon(r-T)+\Pi_\varepsilon(-r-T)\right)\mathrm{e}^{-T(\delta-k)}N_{k,m,\kappa}(T,x)\mathrm{d}T\\ &= \lim_{r\to\infty}\lim_{\varepsilon\searrow 0}\left(\int_{r-2\varepsilon}^{r}\Pi_\varepsilon(r-\varepsilon-T)\mathrm{e}^{-T(\delta-k)}N_{k,m,\kappa}(T,x)\mathrm{d}T\right.\\ &\qquad\left.+\int_{-r}^{-r+2\varepsilon}\Pi_\varepsilon(-r+\varepsilon-T)\mathrm{e}^{-T(\delta-k)}N_{k,m,\kappa}(T,x)\mathrm{d}T\right)\\ &= \operatorname*{ess-lim}_{r\to\infty,\, r\in V}\mathrm{e}^{-r(\delta-k)}N_{k,m,\kappa}(r,x).\end{aligned}$$

□

Proof of Key Lemma 4.2(ii). In this proof we use some of the tools presented in the proof of Theorem 1 of [Lal89], which we repeat here for the convenience of the reader. For $z \in \mathbb{C}$,

$\kappa \in \Sigma^*$ and $x \in \Sigma^\infty$ denote the Fourier-Laplace transform by

$$L_{k,m,\kappa}(z,x) := \int_{-\infty}^{\infty} \mathrm{e}^{zT-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T. \tag{4.12}$$

Items (F) and (G) imply that the function $L_{k,m,\kappa}(\cdot,x)$ is well-defined and analytic on

$$\mathcal{Z}_k := \begin{cases} \{z \in \mathbb{C} \mid \Re(z) < 0\} & : k \le \delta, \\ \{z \in \mathbb{C} \mid \delta - k < \Re(z) < 0\} & : k > \delta. \end{cases}$$

Here, $\Re(z)$ denotes the real part of $z \in \mathbb{C}$. What is more, for small enough $\varepsilon > 0$, Items (F) and (G) imply that $L_{k,m,\kappa}(\cdot,x)$ converges absolutely and uniformly on

$$\mathcal{Z}_k(\varepsilon) := \begin{cases} \{z \in \mathbb{C} \mid \Re(z) < -\varepsilon\} & : k \le \delta, \\ \{z \in \mathbb{C} \mid \delta - k + \varepsilon < \Re(z) < -\varepsilon\} & : k > \delta. \end{cases}$$

Now, in every such region, using Item (D) as well as the monotone and dominated convergence theorems, we obtain the following.

$$\begin{aligned} &L_{k,m,\kappa}(z,x) \\ &= \sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x)\mathrm{e}^{-kS_n\xi(u\omega x)} \int_{-\infty}^{\infty} \mathrm{e}^{zT-T(\delta-k)} f_{k,\omega}(T - S_n\xi(u\omega x))\mathrm{d}T \\ &= \sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x)\mathrm{e}^{-kS_n\xi(u\omega x)} \int_{-\infty}^{\infty} \mathrm{e}^{zT-T(\delta-k)} f_{k,\omega}(T)\mathrm{d}T \cdot \mathrm{e}^{(z-\delta+k)S_n\xi(u\omega x)} \\ &= \sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \mathcal{L}^n_{(z-\delta)\xi} \mathbb{1}_{[\kappa]}(\omega x) \cdot \int_{-\infty}^{\infty} \mathrm{e}^{zT-T(\delta-k)} f_{k,\omega}(T)\mathrm{d}T. \end{aligned}$$

By Theorem 3.9(ii), the spectral radius formula (see Equation (3.5)) and the fact that $\gamma_{-\delta\xi} = 1$ (see Theorem 3.2), the series $\sum_{n=0}^{\infty} \mathcal{L}^n_{(z-\delta)\xi} \mathbb{1}_{[\kappa]}(\omega x)$ converges for $\Re(z) < 0$ and we obtain

$$L_{k,m,\kappa}(z,x) = \sum_{\omega\in\Sigma^m} (I - \mathcal{L}_{(z-\delta)\xi})^{-1} \mathbb{1}_{[\kappa]}(\omega x) \cdot \int_{-\infty}^{\infty} \mathrm{e}^{zT-T(\delta-k)} f_{k,\omega}(T)\mathrm{d}T.$$

Moreover, by Proposition 3.12, the operator-valued function $z \mapsto (I - \mathcal{L}_{z\xi})^{-1}$ is holomorphic at every z on the line $\Re(z) = -\delta$ except for $z = -\delta$, which is a simple pole by Proposition 3.10. According to Corollary 3.11 the residue of $(I - \mathcal{L}_{z\xi})^{-1}\mathbb{1}_{[\kappa]}(\omega x)$ at $z = -\delta$ is

$$-\frac{\int \mathbb{1}_{[\kappa]} \mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \cdot h_{-\delta\xi}(\omega x) =: C_\kappa(\omega x). \tag{4.13}$$

Hence, the residue of $L_{k,m,\kappa}(z,x)$ at $z = 0$ is

$$\sum_{\omega\in\Sigma^m} C_\kappa(\omega x) \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} f_{k,\omega}(T)\mathrm{d}T.$$

Thus, $L_{k,m,\kappa}(z,x)$ has the following representation.

$$L_{k,m,\kappa}(z,x) = q_{k,m,\kappa}(z,x) + \sum_{\omega\in\Sigma^m} \frac{C_\kappa(\omega x)}{z} \cdot \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} f_{k,\omega}(T)\mathrm{d}T, \tag{4.14}$$

where $q_{k,m,\kappa}(\cdot,x)\colon \mathbb{C} \to \mathbb{C}$ is holomorphic in a region containing the closed half-plane $\Re(z) \leq 0$ if $k \leq \delta$ and the strip $\{z \in \mathbb{C} \mid \delta - k + \varepsilon \leq \Re(z) \leq 0\}$ if $k > \delta$, where $\varepsilon > 0$ is sufficiently small. Items (F) and (G) and Lebesgue's dominated convergence theorem imply for every $\varepsilon \in (0,1]$ that

$$\begin{aligned}&\int_{-\infty}^{\infty} (\Pi_\varepsilon(r-T) + \Pi_\varepsilon(-r-T))\,\mathrm{e}^{-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T\\ &\quad= \lim_{s\searrow 0}\int_{-\infty}^{\infty} (\Pi_\varepsilon(r-T) + \Pi_\varepsilon(-r-T))\,\mathrm{e}^{-T(\delta-k+s)} N_{k,m,\kappa}(T,x)\mathrm{d}T. \end{aligned}\tag{4.15}$$

Using the inverse Fourier-Laplace transform $\Pi_\varepsilon(t) = \int_{-\infty}^{\infty} \mathrm{e}^{-\mathbf{i}\theta t}\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\mathrm{d}\theta/(2\pi)$ and that the integral on the left hand side of Equation (4.15) exists, we can convert the integral from the right hand side for sufficiently small $s > 0$ as follows.

$$\begin{aligned}&\int_{-\infty}^{\infty} (\Pi_\varepsilon(r-T) + \Pi_\varepsilon(-r-T))\,\mathrm{e}^{-sT-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T\\
&\quad= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{\mathbf{i}\theta T}\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\left(\mathrm{e}^{-\mathbf{i}\theta r} + \mathrm{e}^{\mathbf{i}\theta r}\right)\frac{\mathrm{d}\theta}{2\pi}\mathrm{e}^{-sT-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T\\
&\quad= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{e}^{(\mathbf{i}\theta-s)T-T(\delta-k)} N_{k,m,\kappa}(T,x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)2\cos(\theta r)\frac{\mathrm{d}\theta}{2\pi}\mathrm{d}T\\
&\quad= \int_{-\infty}^{\infty} L_{k,m,\kappa}(\mathbf{i}\theta - s, x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)2\cos(\theta r)\frac{\mathrm{d}\theta}{2\pi}\\
&\quad\overset{(4.14)}{=} \int_{-\infty}^{\infty}\left(q_{k,m,\kappa}(\mathbf{i}\theta - s, x) - \sum_{\omega\in\Sigma^m}\int_{-\infty}^{\infty}\mathrm{e}^{-T(\delta-k)} f_{k,\omega}(T)\mathrm{d}T\cdot\frac{C_\kappa(\omega x)(\mathbf{i}\theta+s)}{\theta^2+s^2}\right)\\
&\qquad\qquad\cdot\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)2\cos(\theta r)\frac{\mathrm{d}\theta}{2\pi}.\end{aligned}\tag{4.16}$$

The measures given by $\frac{s}{\pi(\theta^2+s^2)}\mathrm{d}\theta$ converge weakly to the Dirac point-mass at zero as $s \to 0$. Moreover, we can ignore the imaginary part on the right hand side of Equation (4.16) since the left hand side is real. Using that $\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)$ is real and that $\widehat{\Pi_\varepsilon}(0) = 1$ for all $\varepsilon \in (0,1]$ we therefore obtain

$$\begin{aligned}&\lim_{s\searrow 0}\int_{-\infty}^{\infty} (\Pi_\varepsilon(r-T) + \Pi_\varepsilon(-r-T))\,\mathrm{e}^{-sT-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T\\
&\quad= \Re\left(\int_{-\infty}^{\infty} q_{k,m,\kappa}(\mathbf{i}\theta, x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\cos(\theta r)\frac{\mathrm{d}\theta}{\pi}\right) - \sum_{\omega\in\Sigma^m}\int_{-\infty}^{\infty}\mathrm{e}^{-T(\delta-k)} f_{k,\omega}(T)\mathrm{d}T\cdot C_\kappa(\omega x).\end{aligned}\tag{4.17}$$

Combining Lemma 4.3 with Equations (4.13), (4.15) and (4.17) it follows that all that remains to show is that the first integral on the right hand side of Equation (4.17) converges to zero as $r \to \infty$ uniformly for $\varepsilon \in (0,1]$. Recall that $\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)$ is $\mathcal{C}^\infty$ and has compact support; assume that it is contained in $[-S,S]$. Also, recall that $q_{k,m,\kappa}(\cdot,x)$ is analytic in a neighbourhood of $[-\mathbf{i}S,\mathbf{i}S]$ and continuous in x. The Cauchy integral formula for derivatives implies that $\frac{\mathrm{d}}{\mathrm{d}z}q_{k,m,\kappa}(z,x)|_{z=\mathbf{i}\theta}$ is uniformly continuous in θ and hence bounded on $[-S,S]\times\Sigma^\infty$. Thus, $\frac{\mathrm{d}}{\mathrm{d}z}q_{k,m,\kappa}(z,x)$ is bounded on $[-\mathbf{i}S,\mathbf{i}S]\times\Sigma^\infty$. Integrating by parts now implies

$$\begin{aligned}\int_{-S}^{S} q_{k,m,\kappa}(\mathbf{i}\theta,x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\cos(\theta r)\frac{\mathrm{d}\theta}{\pi}\\ = q_{k,m,\kappa}(\mathbf{i}\theta,x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\frac{\sin(\theta r)}{\pi r}\Big|_{\theta=-S}^{S} - \int_{-S}^{S}\frac{\mathrm{d}}{\mathrm{d}\theta}\Big(q_{k,m,\kappa}(\mathbf{i}\theta,x)\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)\Big)\frac{\sin(\theta r)}{\pi r}\mathrm{d}\theta. \end{aligned} \tag{4.18}$$

As the support of $\widehat{\Pi_\varepsilon}$ is contained in $[-\mathbf{i}S,\mathbf{i}S]$ and $\widehat{\Pi_\varepsilon}$ is $\mathcal{C}^\infty$, the first term on the right hand side of Equation (4.18) equals zero for all $r>0$. For the second term on the right hand side of Equation (4.18) we remark that $\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)=\widehat{\Pi}(\mathbf{i}\theta\varepsilon/\tau(\varepsilon))$. Therefore, the definition of $\widehat{\Pi}$ in Equation (4.5) implies that $\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)$, as well as, $\frac{\mathrm{d}}{\mathrm{d}\theta}\widehat{\Pi_\varepsilon}(\mathbf{i}\theta)$ is uniformly bounded for $\varepsilon\in(0,1]$. This shows that the second term on the right hand side of Equation (4.18) converges to zero uniformly for $\varepsilon\in(0,1]$ as $r\to\infty$. Hence the first integral on the right hand side of Equation (4.17) converges to zero as $r\to\infty$ uniformly for $\varepsilon\in(0,1]$. □

Proof of Key Lemma 4.2(iii). Items (F) and (G) imply that for a fixed $\beta\in[0,a)$, $m\in\mathbb{N}$, $\omega\in\Sigma^m$, $\kappa\in\Sigma^*$ and $x\in\Sigma^\infty$ for which $al+\beta-\psi(\omega x)\in V$ for all $l\in\mathbb{Z}$, the function $\widehat{N}^\beta_{k,\omega,\kappa}(\cdot,x)$ given by

$$\widehat{N}^\beta_{k,\omega,\kappa}(z,x) := \sum_{l=-\infty}^{\infty} \mathrm{e}^{lz} N_{k,\omega,\kappa}(al+\beta-\psi(\omega x),x) \tag{4.19}$$

is well-defined and analytic for $z\in\widetilde{\mathcal{Z}}_k$, where

$$\widetilde{\mathcal{Z}}_k := \begin{cases}\{z\in\mathbb{C}\mid \Re(z)<a(k-\delta)\} & : k\le\delta,\\ \{z\in\mathbb{C}\mid 0<\Re(z)<a(k-\delta)\} & : k>\delta\end{cases}$$

and

$$N_{k,\omega}(t,x) := \sum_{n=0}^{\infty}\sum_{u\in\Sigma^n}\mathbb{1}_{[\kappa]}(u\omega x)\mathrm{e}^{-kS_n\xi(u\omega x)}f_{k,\omega}(t-S_n\xi(u\omega x))$$

for $t\in V$. Furthermore, Items (F) and (G) imply that we can make the following conversions for $z\in\widetilde{\mathcal{Z}}_k$, for which β is additionally assumed to satisfy $al+\beta-\psi(u\omega x)\in U$ for all $l\in\mathbb{Z}$ and $u\in\Sigma^*$. Note that $S_n\xi=S_n\zeta+\psi-\psi\circ\sigma^n$ and recall that $S_n\zeta\in a\mathbb{Z}$ for all $n\in\mathbb{N}$.

$$
\begin{aligned}
&\widehat{N}^{\beta}_{k,\omega,\kappa}(z,x)\\
&= \sum_{n=0}^{\infty}\sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x)\mathrm{e}^{-kS_n\xi(u\omega x)} \sum_{l=-\infty}^{\infty} \mathrm{e}^{lz} f_{k,\omega}(al+\beta-\psi(\omega x)-S_n\xi(u\omega x))\\
&= \sum_{n=0}^{\infty}\sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x)\mathrm{e}^{-kS_n\xi(u\omega x)} \sum_{l=-\infty}^{\infty} \mathrm{e}^{(l+a^{-1}S_n\zeta(u\omega x))z} f_{k,\omega}(al+\beta-\psi(u\omega x))\\
&= \sum_{n=0}^{\infty}\sum_{u\in\Sigma^n} \mathrm{e}^{(a^{-1}z-k)S_n\zeta(u\omega x)}\mathrm{e}^{-k(\psi(u\omega x)-\psi(\omega x))}\mathbb{1}_{[\kappa]}(u\omega x) \sum_{l=-\infty}^{\infty} \mathrm{e}^{lz} f_{k,\omega}(al+\beta-\psi(u\omega x))\\
&= \mathrm{e}^{k\psi(\omega x)}\sum_{n=0}^{\infty}\mathcal{L}^n_{(a^{-1}z-k)\zeta}\left(\mathrm{e}^{-k\psi}\mathbb{1}_{[\kappa]}\sum_{l=-\infty}^{\infty}\mathrm{e}^{lz}f_{k,\omega}(al+\beta-\psi)\right)(\omega x)
\end{aligned}
$$

For $z\in\widetilde{\mathcal{Z}}_k$ we have that $\Re(a^{-1}z-k)<-\delta$. Thus, by Theorem 3.2 and Proposition 3.6 we know that $\gamma_{\Re(a^{-1}z-k)\zeta}<1$. Theorem 3.9 (both parts) and the spectral radius formula (Equation (3.5)) then imply for every $z\in\widetilde{\mathcal{Z}}_k$ that

$$
\widehat{N}^{\beta}_{k,\omega,\kappa}(z,x)=\mathrm{e}^{k\psi(\omega x)}(I-\mathcal{L}_{(a^{-1}z-k)\zeta})^{-1}\left(\mathrm{e}^{-k\psi}\mathbb{1}_{[\kappa]}\sum_{l=-\infty}^{\infty}\mathrm{e}^{lz}f_{k,\omega}(al+\beta-\psi)\right)(\omega x).
$$

Since $a^{-1}\zeta$ is integer-valued but not cohomologous to any function valued in a proper subgroup of the integers, we can apply Proposition 3.13. Thus, $z\mapsto(I-\mathcal{L}_{z\zeta})^{-1}$ is $2\pi a^{-1}\mathbf{i}$-periodic and holomorphic at every z on the line $\Re(z)=-\delta$ such that $\Im(z)/(2\pi a^{-1})$ is not an integer, where $\Im(z)$ denotes the imaginary part of $z\in\mathbb{C}$. Therefore, $z\mapsto(I-\mathcal{L}_{(a^{-1}z-k)\zeta})^{-1}$ has got an isolated singularity at $z=a(k-\delta)$ and is holomorphic at each $z=a(k-\delta)+\mathbf{i}\theta$ for $0<|\theta|\le\pi a^{-1}$. By Proposition 3.10 the singularity of $\widehat{N}^{\beta}_{k,\omega,\kappa}(z,x)$ at $z=a(k-\delta)$ is

$$
\frac{\gamma_{(a^{-1}z-k)\zeta}}{1-\gamma_{(a^{-1}z-k)\zeta}}\underbrace{\int_{\Sigma^\infty}\mathrm{e}^{-k\psi}\mathbb{1}_{[\kappa]}\sum_{l=-\infty}^{\infty}\mathrm{e}^{lz}f_{k,\omega}(al+\beta-\psi)\mathrm{d}\nu_{(a^{-1}z-k)\zeta}\cdot h_{(a^{-1}z-k)\zeta}(\omega x)}_{=:E_z(\omega x)}.
$$

Since $z\mapsto E_z(\omega x)$ and $z\mapsto\gamma_{(a^{-1}z-k)\zeta}$ are continuous in $z=a(k-\delta)$, the singularity is a simple pole with residue

$$
\frac{E_{a(k-\delta)}(\omega x)}{-\frac{\mathrm{d}}{\mathrm{d}z}\gamma_{(a^{-1}z-k)\zeta}|_{z=a(k-\delta)}}=a\frac{E_{a(k-\delta)}(\omega x)}{-\int_{\Sigma^\infty}\zeta\mathrm{d}\mu_{-\delta\zeta}},
$$

which follows from Equation (3.7). It follows that $\widehat{N}^{\beta}_{k,\omega,\kappa}(z,x)$ is meromorphic in $\widetilde{\widetilde{\mathcal{Z}}}_k(\varepsilon)$ for some $\varepsilon>0$, where

$$
\widetilde{\widetilde{\mathcal{Z}}}_k(\varepsilon):=\left(\widetilde{\mathcal{Z}}_k\cup\{z\in\mathbb{C}\mid a(k-\delta)\le\Re(z)<a(k-\delta)+\varepsilon\}\right)\cap\{z\in\mathbb{C}\mid 0\le\Im(z)\le\pi a^{-1}\}
$$

and that the only singularity in this region is a simple pole at $z = a(k-\delta)$ with residue

$$-a\mathrm{e}^{k\psi(\omega x)} \int_{\Sigma^\infty} \mathrm{e}^{-k\psi} \mathbb{1}_{[\kappa]} \sum_{l=-\infty}^{\infty} \mathrm{e}^{-al(\delta-k)} f_{k,\omega}(al+\beta-\psi) \mathrm{d}\nu_{-\delta\zeta} \cdot \frac{h_{-\delta\zeta}(\omega x)}{\int_{\Sigma^\infty} \zeta \mathrm{d}\mu_{-\delta\zeta}} =: C_{\omega,\kappa}(x).$$

Additionally,

$$\sum_{l=-\infty}^{-1} \mathrm{e}^{-al(\delta-k)} N_{k,\omega,\kappa}(al+\beta-\psi(\omega x), x)$$

is finite by Item (G). Recalling the definition of $\widehat{N}^{\beta}_{k,\omega,\kappa}$ from Equation (4.19) we thus conclude that

$$\sum_{l=0}^{\infty} \mathrm{e}^{lz} N_{k,\omega,\kappa}(al+\beta-\psi(\omega x), x) - \frac{C_{\omega,\kappa}(x)}{z-a(k-\delta)}$$

is holomorphic in $\widetilde{\widetilde{\mathcal{Z}}}_k(\varepsilon)$ for some $\varepsilon > 0$. We observe that $z \mapsto (\mathrm{e}^{z+a(\delta-k)}-1)/(z+a(\delta-k))$ is holomorphic in $\mathbb{C}$. Making the transform of variables $\widetilde{z} := \mathrm{e}^{z+a(\delta-k)}$ we thus obtain that

$$\sum_{l=0}^{\infty} \widetilde{z}^l \mathrm{e}^{-al(\delta-k)} N_{k,\omega,\kappa}(al+\beta-\psi(\omega x), x) - \frac{C_{\omega,\kappa}(x)}{\widetilde{z}-1}$$

is holomorphic in $\{\mathrm{e}^{z+a(\delta-k)} \mid z \in \widetilde{\widetilde{\mathcal{Z}}}_k(\varepsilon)\}$. This implies that

$$L_{k,\omega,\kappa}(\widetilde{z}, x) := \sum_{l=0}^{\infty} \widetilde{z}^l \left(\mathrm{e}^{-al(\delta-k)} N_{k,\omega,\kappa}(al+\beta-\psi(\omega x), x) + C_{\omega,\kappa}(x) \right)$$

is holomorphic in $\{\widetilde{z} \mid |\widetilde{z}| < \mathrm{e}^{\varepsilon}\}$. Since $L_{k,\omega,\kappa}(\cdot, x)$ is holomorphic in $\{\widetilde{z} \mid |\widetilde{z}| < \mathrm{e}^{\varepsilon}\}$ and $\mathrm{e}^{\varepsilon} > 1$, the coefficient sequence of the power series of $L_{k,\omega,\kappa}(\cdot, x)$ converges to zero exponentially fast, more precisely,

$$\mathrm{e}^{-an(\delta-k)} N_{k,\omega,\kappa}(an+\beta-\psi(\omega x), x) + C_{\omega,\kappa}(x) = \mathfrak{o}((1+(\mathrm{e}^{\varepsilon}-1)/2)^{-n})$$

as $n \to \infty$ ($n \in \mathbb{N}$). Thus, for $x \in \Sigma^\infty$ we have that

$$\begin{aligned} &N_{k,\omega,\kappa}(t,x) \\ &\quad = N_{k,\omega,\kappa}\Bigg(a \underbrace{\left\lfloor \frac{t+\psi(\omega x)}{a} \right\rfloor}_{=:n} + \underbrace{a \left\{ \frac{t+\psi(\omega x)}{a} \right\}}_{=:\beta} - \psi(\omega x), x \Bigg) \\ &\quad \sim^V a\mathrm{e}^{a\left\lfloor \frac{t+\psi(\omega x)}{a} \right\rfloor (\delta-k)} \mathrm{e}^{k\psi(\omega x)} \frac{h_{-\delta\zeta}(\omega x)}{\int \zeta \mathrm{d}\mu_{-\delta\zeta}} \\ &\qquad \cdot \int_{\Sigma^\infty} \mathrm{e}^{-k\psi(y)} \mathbb{1}_{[\kappa]}(y) \sum_{l=-\infty}^{\infty} \mathrm{e}^{-al(\delta-k)} f_{k,\omega}\left(al + a\left\{ \frac{t+\psi(\omega x)}{a} \right\} - \psi(y) \right) \mathrm{d}\nu_{-\delta\zeta}(y) \end{aligned}$$

as $t \to \infty$, which finishes the proof. □

Proof of Key Lemma 4.2(i). First note that if $f \sim^{V} g$ as $t \to \infty$, then the existence of $\lim_{t\to\infty} t^{-1}\int_0^t g(T)\mathrm{d}T$ implies the existence of $\lim_{t\to\infty} t^{-1}\int_0^t f(T)\mathrm{d}T$ and moreover that $\lim_{t\to\infty} t^{-1}\int_0^t f(T)\mathrm{d}T = \lim_{t\to\infty} t^{-1}\int_0^t g(T)\mathrm{d}T$.

If ξ is non-lattice, Key Lemma 4.2(i) thus is a straightforward consequence of Key Lemma 4.2(ii). Therefore, we now consider the case that ξ is lattice. Clearly,

$$\begin{aligned}&\int_0^t \mathrm{e}^{-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T\\ &\quad= \int_0^{a\lfloor a^{-1}t\rfloor} \mathrm{e}^{-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T + \int_{a\lfloor a^{-1}t\rfloor}^{t} \mathrm{e}^{-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T. \qquad (4.20)\end{aligned}$$

Due to Item (F) the second summand on the right hand side of Equation (4.20) is absolutely bounded by $\mathfrak{C}\cdot(t-a\lfloor a^{-1}t\rfloor) \leq \mathfrak{C}\cdot a$. For the first summand we use that by Item (iii) we have that $N_{k,m,\kappa}(t,x) \sim^{V\cap\widetilde{V}} aM_{k,m,\kappa}(t,x)$ as $t\to\infty$, where

$$M_{k,m,\kappa}(t,x) := \sum_{\omega\in\Sigma^m} \mathrm{e}^{k\psi(\omega x)} \frac{h_{-\delta\zeta}(\omega x)}{\int \zeta\mu_{-\delta\zeta}} \cdot M^{\omega}_{k,\kappa}(t,x)$$

and for $\omega \in \Sigma^*$ and $t \in \widetilde{V}$

$$\begin{aligned}&M^{\omega}_{k,\kappa}(t,x)\cdot \mathrm{e}^{-a\left\lfloor\frac{t+\psi(\omega x)}{a}\right\rfloor(\delta-k)}\\ &\quad:= \int_{\Sigma^\infty} \mathrm{e}^{-k\psi(y)}\mathbb{1}_{[\kappa]}(y)\sum_{l=-\infty}^{\infty} \mathrm{e}^{-al(\delta-k)} f_{k,\omega}\left(al + a\left\{\frac{t+\psi(\omega x)}{a}\right\} - \psi(y)\right)\mathrm{d}\nu_{-\delta\zeta}(y).\end{aligned}$$

It is easy to verify that $t\mapsto \mathrm{e}^{-t(\delta-k)}M_{k,m,\kappa}(t,x)$ is periodic with period a. Thus,

$$\begin{aligned}&\lim_{t\to\infty} t^{-1}\int_0^t \mathrm{e}^{-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T\\ &\quad= \lim_{t\to\infty} \frac{a\lfloor a^{-1}t\rfloor}{t}\cdot (a\lfloor a^{-1}t\rfloor)^{-1}\int_0^{a\lfloor a^{-1}t\rfloor} \mathrm{e}^{-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T\\ &\quad= \lim_{t\to\infty} (a\lfloor a^{-1}t\rfloor)^{-1}\lfloor a^{-1}t\rfloor \int_0^a \mathrm{e}^{-T(\delta-k)} aM_{k,m,\kappa}(T,x)\mathrm{d}T\\ &\quad= \int_0^a \mathrm{e}^{-T(\delta-k)} M_{k,m,\kappa}(T,x)\mathrm{d}T.\end{aligned}$$

Consider

$$\begin{aligned}
&\int_0^a \mathrm{e}^{-T(\delta-k)} M_{k,\kappa}^{\omega}(T,x)\mathrm{d}T \\
&= \int_0^a \mathrm{e}^{-T(\delta-k)} \mathrm{e}^{a\left\lfloor \frac{T+\psi(\omega x)}{a}\right\rfloor(\delta-k)} \\
&\qquad \cdot \int_{\Sigma^\infty} \mathrm{e}^{-k\psi(y)} \mathbb{1}_{[\kappa]}(y) \sum_{l=-\infty}^{\infty} \mathrm{e}^{-al(\delta-k)} f_{k,\omega}\left(al + a\Big\{\frac{T+\psi(\omega x)}{a}\Big\} - \psi(y)\right) \mathrm{d}\nu_{-\delta\zeta}(y)\mathrm{d}T \\
&= \int_0^a \mathrm{e}^{(-T+\psi(\omega x))(\delta-k)} \int_{\Sigma^\infty} \mathrm{e}^{-k\psi(y)} \mathbb{1}_{[\kappa]}(y) \sum_{l=-\infty}^{\infty} \mathrm{e}^{-al(\delta-k)} f_{k,\omega}(al + T - \psi(y))\mathrm{d}\nu_{-\delta\zeta}(y)\mathrm{d}T \\
&= \mathrm{e}^{\psi(\omega x)(\delta-k)} \int_{\Sigma^\infty} \mathrm{e}^{-k\psi(y)} \mathbb{1}_{[\kappa]}(y) \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} f_{k,\omega}(T-\psi(y))\mathrm{d}T\mathrm{d}\nu_{-\delta\zeta}(y) \\
&= \mathrm{e}^{\psi(\omega x)(\delta-k)} \int_{\Sigma^\infty} \mathrm{e}^{-\delta\psi(y)} \mathbb{1}_{[\kappa]}(y) \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} f_{k,\omega}(T)\mathrm{d}T\mathrm{d}\nu_{-\delta\zeta}(y) \\
&= \mathrm{e}^{\psi(\omega x)(\delta-k)} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} f_{k,\omega}(T)\mathrm{d}T \cdot \int_{\Sigma^\infty} \mathbb{1}_{[\kappa]}(y)\mathrm{e}^{-\delta\psi(y)}\mathrm{d}\nu_{-\delta\zeta}(y).
\end{aligned}$$

This implies

$$\begin{aligned}
&\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-k)} N_{k,m,\kappa}(T,x)\mathrm{d}T \\
&\quad = \sum_{\omega\in\Sigma^m} \mathrm{e}^{\delta\psi(\omega x)} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} f_{k,\omega}(T)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}(y)\mathrm{e}^{-\delta\psi(y)}\mathrm{d}\nu_{-\delta\zeta}(y)}{\int \zeta \mathrm{d}\mu_{-\delta\zeta}} \cdot h_{-\delta\zeta}(\omega x).
\end{aligned}$$

Noting that $\mathrm{e}^{-\delta\psi}\mathrm{d}\nu_{-\delta\zeta} = \mathrm{d}\nu_{-\delta\xi}$, $\int_{\Sigma^\infty} \zeta\mathrm{d}\mu_{-\delta\zeta} = \int_{\Sigma^\infty} \xi\mathrm{d}\mu_{-\delta\xi}$ and $\mathrm{e}^{\delta\psi}h_{-\delta\zeta} = h_{-\delta\xi}$, the statement of Key Lemma 4.2(i) follows. □

4.1.2 Lemmas 4.4 to 4.6 with Proofs

As described in the beginning of this section, we make use of the fact that two signed Borel measures which coincide on an intersection stable generator for $\mathfrak{B}(\mathbb{R}^d)$ coincide on $\mathfrak{B}(\mathbb{R}^d)$ (see Proposition A.4) in order to show convergence of the signed Borel measures $\mathrm{e}^{-t(\delta-k)}C_k(F_{\mathrm{e}^{-t}},\cdot)$. For the application of Proposition A.4 we use the set-class $\mathcal{E}_F$, which we introduced in Equation (4.1), and now show that $\mathcal{E}_F$ forms an intersection stable generator for the Borel σ-algebra $\mathfrak{B}(\mathbb{R}^d)$ of $\mathbb{R}^d$ (see Definition A.1). We remark that similar ideas were used in [Win08] for the self-similar setting. The following lemma is a modification of Lemma 6.1.1 of [Win08].

Lemma 4.4. *$\mathcal{E}_F$ is an intersection stable generator for $\mathfrak{B}(\mathbb{R}^d)$.*

Proof. Since O is the open set satisfying the OSC for Φ, it is clear to see that $\mathcal{E}_F$ is intersection stable. Moreover, letting $\sigma(\mathcal{E}_F)$ denote the σ-algebra generated by $\mathcal{E}_F$ we have that $\sigma(\mathcal{E}_F) \subseteq \mathfrak{B}(\mathbb{R}^d)$. Thus, in order to show that $\sigma(\mathcal{E}_F) = \mathfrak{B}(\mathbb{R}^d)$ we will now show $\mathfrak{B}(\mathbb{R}^d) \subseteq \sigma(\mathcal{E}_F)$. The idea of how to prove this is to show that every open set in $\mathbb{R}^d$ is a countable union of sets in $\mathcal{E}_F$.

Let $B \subseteq \mathbb{R}^d$ denote an open set and take $x \in B$. Then there exists some $r > 0$ such that $B_r(x) \subset B$, where $B_r(x)$ denotes the open ball with radius r centred at x. Define

$$n(r) := \min\{n \in \mathbb{N} \mid \operatorname{diam}(\phi_\omega O) \leq r \text{ for all } \omega \in \Sigma^n\}.$$

Set $\Sigma_x(r) := \{\omega \in \Sigma^{n(r)} \mid x \in \overline{\phi_\omega O}\}$. By definition, $\operatorname{diam}(\phi_\omega O) \leq r$ for all $\omega \in \Sigma_x(r)$ and thus $\phi_\omega O \subset B_r(x)$ for these ω. Moreover, for each $\omega \in \Sigma^{n(r)} \setminus \Sigma_x(r)$, $\phi_\omega O$ has positive distance to x. Therefore, we can find some positive constant $0 < c \leq r$ such that $|x - \overline{\phi_\omega O}| > c$ for all $\omega \in \Sigma^{n(r)} \setminus \Sigma_x(r)$. Set $D_x := B_c(x) \setminus \bigcup_{\omega \in \Sigma_x(r)} \phi_\omega O$. Then D_x is a subset of $\mathbb{R}^d \setminus \bigcup_{\omega \in \Sigma^{n(r)}} \phi_\omega O$ and thus an element of $\mathcal{K}_F$ (for the definition of $\mathcal{K}_F$ see Equation (4.2)). Set $E_x := D_x \cup \bigcup_{\omega \in \Sigma_x(r)} \phi_\omega O$. By construction, E_x is a finite union of sets from $\mathcal{E}_F$ and $E_x \subseteq B_r(x) \subset B$. On the other hand, the family $\{E_x \mid x \in B\}$ forms an open cover of B, which by the Lindelöf Theorem possesses a countable open subcover. More precisely, there exist $x_1, x_2, \ldots$ in B such that $B = \bigcup_{i \in \mathbb{N}} E_{x_i}$. Since each set E_{x_i} is a finite union of sets from $\mathcal{E}_F$, the proof is complete. □

Now, we turn to Step (III) from the beginning of Section 4.1.

Lemma 4.5. *Let Φ denote a cIFS acting on X and suppose that $\lambda_d(X \setminus \Phi X) > 0$. Denote by $F \subset \mathbb{R}^d$ the self-conformal set associated with Φ and let $\delta > 0$ be its Minkowski dimension. Assume (COND 1) to (COND 3) and let $G^1_\omega, \ldots, G^Q_\omega$ denote the main gaps of F for $\omega \in \Sigma^*$. If $d \geq 4$ and $k \in \{0, \ldots, d-2\}$ suppose that Lebesgue-almost all distances are regular for F and write $U := \{t \in \mathbb{R} \mid \mathrm{e}^{-t}$ is a regular distance for $F\}$. Further, assume that there exists a $\delta_I < \delta$ such that $C^{\mathrm{var}}_k(F_{\mathrm{e}^{-t}}, G^i_\omega) \cdot \mathrm{e}^{-t(\delta_I - k)}$ is uniformly bounded from above by a constant $\widetilde{\mathfrak{C}} \in \mathbb{R}$ for $i \in \{1, \ldots, Q\}$, $\omega \in \Sigma^*$ and $t \in U_k$, where $U_k := U$ if $d \geq 4$ and $k \in \{0, \ldots, d-2\}$ and $U_k := \mathbb{R}$ else. Then the following hold.*

(i) If $B \in \mathcal{K}_F$, then $C_k(F_{\mathrm{e}^{-t}}, B) = \mathfrak{o}(\mathrm{e}^{t(\delta-k)})$ as $t \to \infty$ on U_k.

(ii) If $B = \phi_\kappa O \in \mathcal{E}_F \setminus \mathcal{K}_F$ for some $\kappa \in \Sigma^*$, then for all $m \in \mathbb{N}$ and $x \in \Sigma^\infty$ we have

$$C_k(F_{\mathrm{e}^{-t}}, B) = \sum_{\omega \in \Sigma^m} \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x) C_k\left(F_{\mathrm{e}^{-t}}, \bigcup_{i=1}^{Q} G^i_{u\omega}\right) + \mathfrak{o}(\mathrm{e}^{t(\delta-k)})$$

as $t \to \infty$ on U_k.

Both Items (i) and (ii) remain valid when substituting the total variation measure C^{var}_k in for C_k.

Proof. From Items (i) and (iv) of Proposition 2.7 it follows that $C_k(F_{e^{-t}}, B) = C_k(F_{e^{-t}}, B \cap D)$, whenever $D \in \mathfrak{B}(\mathbb{R}^d)$ contains $F_{e^{-t}}$. Now, because $F_{e^{-t}}$ is contained in the disjoint union $(\overline{O}_{e^{-t}} \setminus \overline{O}) \cup \bigcup_{\omega\in\Sigma^*, i\in\{1,\ldots,Q\}} G^i_\omega \cup F$ for any $t \in \mathbb{R}$, the following holds for any $t \in U_k$ and $B \in \mathcal{E}_F$.

$$C_k(F_{e^{-t}}, B) = C_k(F_{e^{-t}}, \overline{O}_{e^{-t}} \setminus \overline{O} \cap B) + C_k(F_{e^{-t}}, \bigcup_{\substack{\omega\in\Sigma^* \\ i\in\{1,\ldots,Q\}}} G^i_\omega \cap B) + C_k(F_{e^{-t}}, F \cap B). \quad (4.21)$$

The first summand on the right hand side of Equation (4.21) is $\mathfrak{o}(e^{t(\delta-k)})$ since (COND 2') implies that there exists a $\mathfrak{C}' \in \mathbb{R}$ such that

$$e^{-t(\delta-k)} C_k^{\text{var}}(F_{e^{-t}}, \overline{O}_{e^{-t}} \setminus \overline{O} \cap B) \leq \mathfrak{C}' e^{-t(\delta-\delta_O)} \to 0$$

as $t \to \infty$ on U_k. Also, the third summand on the right hand side of Equation (4.21) is $\mathfrak{o}(e^{t(\delta-k)})$. To see this, we distinguish between two cases. If $k = d$, then $0 \leq C_k(F_{e^{-t}}, F \cap B) \leq \lambda_d(F) = 0$ by Propositions 2.7 and 2.22. If $k < d$, then $C_k^{\text{var}}(F_{e^{-t}}, \cdot)$ is concentrated on the boundary $\partial F_{e^{-t}}$ of $F_{e^{-t}}$ (see Proposition 2.7(iv)), which implies that $C_k^{\text{var}}(F_{e^{-t}}, F \cap B) = 0$, since $\partial F_{e^{-t}} \cap F = \varnothing$ for all $t \in \mathbb{R}$. Therefore, it only remains to consider the second summand on the right hand side of Equation (4.21). To this end assume without loss of generality that F possesses exactly one primary gap which we denote by G.

ad (i): For $B \in \mathcal{K}_F$ there exists an $n \in \mathbb{N}$ such that $B \cap \phi_\omega O = \varnothing$ for all $\omega \in \Sigma^n$. Since $G_\omega \subseteq \phi_\omega O$ for every $\omega \in \Sigma^*$ we conclude that $\bigcup_{m=n}^\infty \bigcup_{\omega\in\Sigma^m} G_\omega \cap B = \varnothing$ and hence that

$$C_k^{\text{var}}(F_{e^{-t}}, \bigcup_{\omega\in\Sigma^*} G_\omega \cap B) = \sum_{m=0}^{n-1} \sum_{\omega\in\Sigma^m} C_k^{\text{var}}(F_{e^{-t}}, G_\omega \cap B). \quad (4.22)$$

By the hypothesis we have

$$e^{-t(\delta-k)} C_k^{\text{var}}(F_{e^{-t}}, G_\omega \cap B) \leq \widetilde{\mathfrak{C}} e^{-t(\delta-\delta_I)} \to 0$$

as $t \to \infty$ on U_k, which together with Equation (4.22) shows the assertion of Item (i).

ad (ii): For $B = \phi_\kappa O \in \mathcal{E}_F \setminus \mathcal{K}_F$ and for an arbitrary $m \in \mathbb{N}$ write

$$C_k(F_{e^{-t}}, \bigcup_{\omega\in\Sigma^*} G_\omega \cap B)$$
$$= \sum_{n=0}^{m-1} \sum_{\omega\in\Sigma^n} C_k(F_{e^{-t}}, G_\omega \cap B) + \sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \sum_{u\in\Sigma^n} C_k(F_{e^{-t}}, G_{u\omega} \cap B).$$

As in the proof of Item (i) it can be easily seen that the first summand on the right hand side of the preceding equation is $\mathfrak{o}(\mathrm{e}^{t(\delta-k)})$. Further, we have either $G_{u\omega} \subseteq \phi_\kappa O$ or $G_{u\omega} \cap \phi_\kappa O = \varnothing$, the former of which holds if and only if $u\omega x \in [\kappa]$ for some and thus any $x \in \Sigma^\infty$. This implies $C_k(F_{\mathrm{e}^{-t}}, G_{u\omega} \cap \phi_\kappa O) = \mathbb{1}_{[\kappa]}(u\omega x) \cdot C_k(F_{\mathrm{e}^{-t}}, G_{u\omega})$ which shows Item (ii).

That the same is true for the total variation measure can be easily verified in exactly the same way. □

With regard to Step (II) from the beginning of Section 4.1 we now show that Prohorov's theorem can be applied in order to obtain the desired convergence.

Lemma 4.6. *Suppose that we are in the situation of Lemma 4.5. Moreover, assume that* $\mathrm{e}^{-t(\delta-k)} C_k^{var}(F_{\mathrm{e}^{-t}})$ *is uniformly bounded for* $t \in (1,\infty) \cap U_k$. *Then we have the following.*

(i) If there exists a signed Borel measure μ *such that* $\operatorname{ess-lim}_{t\to\infty} \mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, B) = \mu(B)$ *for every* $B \in \mathcal{E}_F$, *where* $\mathcal{E}_F$ *is as defined in Equation* (4.1) *and the essential limit is taken over* U_k, *then*

$$\operatorname*{ess-w-lim}_{t\to\infty} \mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, \cdot) = \mu(\cdot).$$

(ii) If there exists a signed Borel measure μ *such that for every* $B \in \mathcal{E}_F$, $\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-k)} C_k(F_{\mathrm{e}^{-T}}, B)\mathrm{d}T = \mu(B)$, *then*

$$\operatorname*{w-lim}_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-k)} C_k(F_{\mathrm{e}^{-T}}, \cdot)\mathrm{d}T = \mu(\cdot).$$

Proof. We start with proving Item (i). Define

$$\mathcal{P} := \{\mu_t(\cdot) := \mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, \cdot) \mid t \in (1,\infty) \cap U_k\}.$$

By the hypothesis and Proposition 2.7, the family $\mathcal{P}$ is uniformly tight and totally bounded. Let $(T_n)_{n\in\mathbb{N}}$ denote a sequence in $(0,1)$ which converges to zero and for which $T_n^{-1} \in U_k$ for $n \in \mathbb{N}$. Then by Prohorov's Theorem (Theorem A.7) there exists a subsequence $(T_{n_k})_{k\in\mathbb{N}}$ and a finite signed Borel measure $\widetilde{\mu}$ such that $(\mu_{T_{n_k}^{-1}})_{k\in\mathbb{N}}$ converges weakly to $\widetilde{\mu}$. However, for $B \in \mathcal{E}_F$ the essential limit $\operatorname{ess-lim}_{t\to\infty} \mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, B)$ exists by assumption and coincides with $\mu(B)$. Lemma 4.4 states that $\mathcal{E}_F$ is an intersection stable generator for $\mathfrak{B}(\mathbb{R}^d)$. Thus, by Proposition A.4 we conclude that $\widetilde{\mu}$ coincides with the measure μ for every such sequence $(T_{n_k})_{k\in\mathbb{N}}$.

Item (ii) can be shown in the same way by taking

$$\mathcal{P} := \left\{ \mu_t(\cdot) := t^{-1} \int_0^t \mathrm{e}^{-T(\delta-k)} C_k(F_{\mathrm{e}^{-T}}, \cdot)\mathrm{d}T \;\middle|\; t \in (1,\infty) \right\}.$$

□

The tools that we established above, now allow us to prove our main results. We begin with the results for general self-conformal sets in the the next section.

4.2 Self-Conformal Sets – Proofs of Theorems 2.29, 2.31 and 2.33

In this section we prove Theorems 2.29, 2.31 and 2.33, which are concerned with the existence of the local Minkowski content and the fractal curvature measures of self-conformal sets.

For showing the existence of the limits in Theorems 2.29 and 2.31 we require the following approximation argument.

Lemma 4.7. *For an arbitrary $x \in \Sigma^\infty$ and $\Upsilon \in \mathbb{R}$ the following hold.*

(i) *(a) Suppose that we are in the situation of Theorem 2.29. Then having $\Upsilon \leq \varrho_m^{2d-\delta} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega^i)\mathrm{d}T \cdot h_{-\delta\xi}(\omega x)$ for all sufficiently large $m \in \mathbb{N}$ implies*

$$\Upsilon \leq \liminf_{m\to\infty} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega^i)\mathrm{d}T.$$

(b) Suppose that we are in the situation of Theorem 2.31. Then having $\Upsilon \leq \varrho_m^{\delta} \sum_{\omega\in\Sigma^m} \sum_{i=1}^{Q} |G_\omega^i|^\delta h_{-\delta\xi}(\omega x)$ for all sufficiently large $m \in \mathbb{N}$ implies

$$\Upsilon \leq \liminf_{m\to\infty} \sum_{i=1}^{Q} \sum_{\omega\in\Sigma^m} |G_\omega^i|^\delta.$$

(ii) *(a) Suppose that we are in the situation of Theorem 2.29. Then having $\Upsilon \geq \varrho_m^{-2d+\delta} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega^i)\mathrm{d}T \cdot h_{-\delta\xi}(\omega x)$ for all sufficiently large $m \in \mathbb{N}$ implies*

$$\Upsilon \geq \limsup_{m\to\infty} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega^i)\mathrm{d}T.$$

(b) Suppose that we are in the situation of Theorem 2.31. Then having $\Upsilon \geq \varrho_m^{-\delta} \sum_{\omega\in\Sigma^m} \sum_{i=1}^{Q} |G_\omega^i|^\delta h_{-\delta\xi}(\omega x)$ for all sufficiently large $m \in \mathbb{N}$ implies

$$\Upsilon \geq \limsup_{m\to\infty} \sum_{i=1}^{Q} \sum_{\omega\in\Sigma^m} |G_\omega^i|^\delta.$$

Proof. We are first going to approximate the eigenfunction $h_{-\delta\xi}$ of the Perron-Frobenius operator $\mathcal{L}_{-\delta\xi}$. For that we claim that $\mathcal{L}^n_{-\delta\xi}1(x) = \sum_{u\in\Sigma^n}|\phi'_u(\pi x)|^\delta$ for each $x \in \Sigma^\infty$ and $n \in \mathbb{N}$, where 1 is the constant one-function. This can be easily seen by induction. Since $\mathcal{L}^n_{-\delta\xi}1$ converges uniformly to the eigenfunction $h_{-\delta\xi}$ as $n \to \infty$ (see Equation (3.3)) we have that

$$\forall \varepsilon > 0\ \exists M \in \mathbb{N} \colon \forall n \geq M,\ \forall x \in \Sigma^\infty \colon \left| \sum_{u\in\Sigma^n} |\phi'_u(\pi x)|^\delta - h_{-\delta\xi}(x) \right| < \varepsilon. \tag{4.23}$$

Furthermore, according to the BDP (see Lemma 2.21) we know that

$$\forall \varepsilon' > 0\ \exists M' \in \mathbb{N} \colon \forall m \geq M' \colon |\varrho_m - 1| < \varepsilon'. \tag{4.24}$$

Without loss of generality we assume that F possesses exactly one primary gap G with corresponding main gaps G_ω for $\omega \in \Sigma^*$.

ad (i)a: Note that for $u \in \Sigma^*$, $m \in \mathbb{N}$, $\omega \in \Sigma^m$, $x \in \Sigma^\infty$ and $T \in \mathbb{R}$ we have that

$$\begin{aligned} |\phi'_u(\pi\omega x)|^d \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega) &\leq \varrho_m^d \lambda_d(\phi_u(F_{\mathrm{e}^{-T}}) \cap G_{u\omega}) \\ &\leq \varrho_m^d \lambda_d((\phi_u(F)_{\mathrm{e}^{-T}\varrho_m|\phi'_u(\pi\omega x)|} \cap G_{u\omega}). \end{aligned}$$

Moreover, $(\phi_u F)_{\mathrm{e}^{-T}} \cap G_{u\omega} = F_{\mathrm{e}^{-T}} \cap G_{u\omega}$ for all $T \in \mathbb{R}$, since $G_{u\omega} \subseteq \phi_u O$. Thus, Equations (4.23) and (4.24) imply the following for all $n \geq M$ and $m \geq M'$.

$$\begin{aligned} \Upsilon &\leq \varrho_m^{2d-\delta} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega)\mathrm{d}T \cdot h_{-\delta\xi}(\omega x) \\ &\leq \varrho_m^{3d-\delta} \sum_{\omega\in\Sigma^m} \sum_{u\in\Sigma^n} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} |\phi'_u(\pi\omega x)|^{\delta-d} \lambda_d(F_{\mathrm{e}^{-T}|\phi'_u(\pi\omega x)|\varrho_m} \cap G_{u\omega})\mathrm{d}T \\ &\quad + \varepsilon\varrho_m^{2d-\delta} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega)\mathrm{d}T \\ &= \varrho_m^{4d-2\delta} \sum_{\omega\in\Sigma^m} \sum_{u\in\Sigma^n} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_{u\omega})\mathrm{d}T \\ &\quad + \varepsilon\varrho_m^{2d-\delta} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega)\mathrm{d}T \\ &\leq (1+\varepsilon')^{4d-2\delta} \sum_{\omega\in\Sigma^m} \sum_{u\in\Sigma^n} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_{u\omega})\mathrm{d}T \\ &\quad + \varepsilon(1+\varepsilon')^{2d-\delta} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega)\mathrm{d}T \\ &=: A_{m,n}. \end{aligned}$$

Hence, for all $\varepsilon, \varepsilon' > 0$

$$\begin{aligned}\Upsilon &\leq \liminf_{m\to\infty} \liminf_{n\to\infty} A_{m,n} \\ &\leq (1+\varepsilon')^{4d-2\delta} \liminf_{m\to\infty} \liminf_{n\to\infty} \sum_{\omega\in\Sigma^m} \sum_{u\in\Sigma^n} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_{u\omega}) \mathrm{d}T \\ &\quad + \varepsilon(1+\varepsilon')^{2d-\delta} \limsup_{m\to\infty} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_{\omega}) \mathrm{d}T. \qquad (4.25)\end{aligned}$$

The BDP implies that the limits of the sums in Equation (4.25) are finite, which is shown in the following. Recall that we let $\|\phi'_\omega\|_\infty := \sup_{x\in X} |\phi'_\omega(x)|$ denote the supremum-norm of ϕ'_ω. Note that $\lambda_d(F_{\mathrm{e}^{-T}} \cap G) = \lambda_d(G)$ for $T \leq -\ln \mathrm{diam} G$ and that by (COND 4) there exists a constant $\widetilde{\mathfrak{C}} \in \mathbb{R}$ and a $\delta_I < \delta$ such that $\mathrm{e}^{-t(\delta_I-d)} \lambda_d(F_{\mathrm{e}^{-t}} \cap G) \leq \widetilde{\mathfrak{C}}$ for all $t \in \mathbb{R}$.

$$\begin{aligned}&\sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega) \mathrm{d}T \\ &\leq \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \|\phi'_\omega\|_\infty^d \lambda_d(F_{\mathrm{e}^{-T}\varrho_0/\|\phi'_\omega\|_\infty} \cap G) \mathrm{d}T \\ &= \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \|\phi'_\omega\|_\infty^\delta \varrho_0^{d-\delta} \lambda_d(F_{\mathrm{e}^{-T}} \cap G) \mathrm{d}T \\ &= \sum_{\omega\in\Sigma^m} \left(\int_{-\infty}^{-\ln \mathrm{diam} G} \mathrm{e}^{-T(\delta-d)} \|\phi'_\omega\|_\infty^\delta \varrho_0^{d-\delta} \lambda_d(G) \mathrm{d}T \right. \\ &\qquad \left. + \int_{-\ln \mathrm{diam} G}^{\infty} \widetilde{\mathfrak{C}} \mathrm{e}^{-T(\delta-\delta_I)} \|\phi'_\omega\|_\infty^\delta \varrho_0^{d-\delta} \mathrm{d}T \right) \\ &= \varrho_0^{d-\delta} \left(\frac{\lambda_d(G)}{d-\delta} (\mathrm{diam} G)^{\delta-d} + \frac{\widetilde{\mathfrak{C}}}{\delta-\delta_I} (\mathrm{diam} G)^{\delta-\delta_I} \right) \sum_{\omega\in\Sigma^m} \|\phi'_\omega\|_\infty^\delta.\end{aligned}$$

In Lemma 4.2.12 in [MU03] it is shown that $\sum_{\omega\in\Sigma^m} \|\phi'_\omega\|_\infty^\delta =: a_m$ defines a bounded sequence $(a_m)_{m\in\mathbb{N}}$. Therefore, letting ε and ε' tend to zero, Equation (4.25) implies the assertion of Item (i)a.

ad (i)b: Here, we use the same methods which we applied for showing Item (i)a. For

$n \geq M$ and $m \geq M'$ we have

$$\begin{aligned}
\Upsilon &\leq \varrho_m^\delta \sum_{\omega \in \Sigma^m} |G_\omega|^\delta h_{-\delta\xi}(\omega x) \\
&\leq \varrho_m^\delta \sum_{\omega \in \Sigma^m} |G_\omega|^\delta \left(\sum_{u \in \Sigma^n} |\phi_u'(\pi\omega x)|^\delta + \varepsilon \right) \\
&\leq \varrho_m^{2\delta} \sum_{\omega \in \Sigma^m} \sum_{u \in \Sigma^n} |G_{u\omega}|^\delta + \varepsilon \varrho_m^\delta \sum_{\omega \in \Sigma^m} \|\phi_\omega'\|_\infty^\delta \cdot |G|^\delta \\
&\leq (1+\varepsilon')^{2\delta} \sum_{\omega \in \Sigma^m} \sum_{u \in \Sigma^n} |G_{u\omega}|^\delta + \varepsilon(1+\varepsilon')^\delta \sum_{\omega \in \Sigma^m} \|\phi_\omega'\|_\infty^\delta \cdot |G|^\delta \\
&=: A_{m,n}.
\end{aligned}$$

For all $\varepsilon, \varepsilon' > 0$ we have

$$\begin{aligned}
\Upsilon &\leq \liminf_{m\to\infty} \liminf_{n\to\infty} A_{m,n} \\
&\leq (1+\varepsilon')^{2\delta} \liminf_{m\to\infty} \liminf_{n\to\infty} \sum_{\omega \in \Sigma^m} \sum_{u \in \Sigma^n} |G_{u\omega}|^\delta \\
&\quad + \varepsilon(1+\varepsilon')^\delta \limsup_{m\to\infty} \sum_{\omega \in \Sigma^m} \|\phi_\omega'\|_\infty^\delta \cdot |G|^\delta.
\end{aligned}$$

As in the proof of Item (i)a, we conclude that the assertion is implied by taking the limits as ε and ε' tend to zero.

The same arguments can be used to show Item (ii). □

4.2.1 Proof of Theorem 2.29

In proving Theorem 2.29, we are going to follow the structure presented in the beginning of Section 4.1. In particular, we need to verify the prerequisites of the Lemmas from Section 4.1. That we can apply Key Lemma 4.2 is shown next.

Lemma 4.8. *Suppose that we are in the situation of Theorem 2.29. Then Items (A) to (G) of Key Lemma 4.2 are satisfied for $f_{d,\omega} \colon \mathbb{R} \to \mathbb{R}$ given by*

$$f_{d,\omega}(t) := \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-t}} \cap G_\omega^i),$$

where $\omega \in \Sigma^m$ for some fixed $m \in \mathbb{N}$.

Proof. Without loss of generality we assume that F possesses exactly one primary gap which we denote by G. The associated main gaps are denoted by G_ω for $\omega \in \Sigma^*$. Set $g_\omega := \mathrm{diam}(G_\omega)/2$. For $t < -\ln g_\omega$ we have $f_{d,\omega}(t) = \lambda_d(G_\omega)$.

ad (A): $f_{d,\omega}$ is defined on the whole real line.

ad (B): $f_{d,\omega}$ is continuous which results by combining Theorem 1 with Lemma 5 from [Sta76].

ad (C): By (COND 4), $\lambda_d(F_{\mathrm{e}^{-t}} \cap G)\mathrm{e}^{-t(\delta_I - d)}$ is uniformly bounded by a constant $\widetilde{\mathfrak{C}}$. Thus

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{-t(\delta-d)} |f_{d,\omega}(t)| \mathrm{d}t &\leq \int_{-\infty}^{-\ln g_\omega} \mathrm{e}^{-t(\delta-d)} \lambda_d(G_\omega) \mathrm{d}t + \int_{-\ln g_\omega}^{\infty} \widetilde{\mathfrak{C}} \mathrm{e}^{-t(\delta-\delta_I)} \mathrm{d}t \\ &= \frac{1}{d-\delta} g_\omega^{\delta-d} \lambda_d(G_\omega) + \frac{1}{\delta-\delta_I} \widetilde{\mathfrak{C}} g_\omega^{\delta-\delta_I} \\ &< \infty.\end{aligned}$$

ad (D): Note that by the definition of the geometric potential function ξ we have $\mathrm{e}^{-dS_n\xi(u\omega x)} = |\phi_u'(\pi\omega x)|^d$ for every $n \in \mathbb{N}$, $u \in \Sigma^n$, $\omega \in \Sigma^*$ and $x \in \Sigma^\infty$. Thus,

$$\begin{aligned}N_{d,m}^{\mathrm{abs}}(t,x) &:= \sum_{\omega \in \Sigma^m} \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \mathrm{e}^{-dS_n\xi(u\omega x)} \lambda_d(F_{\mathrm{e}^{-t+S_n\xi(u\omega x)}} \cap G_\omega) \\ &\leq \sum_{\omega \in \Sigma^m} \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \varrho_m^d \lambda_d(F_{\mathrm{e}^{-t}\varrho_m} \cap G_{u\omega}) \\ &\leq \sum_{\omega \in \Sigma^m} \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \varrho_m^d \lambda_d(G_{u\omega}) \leq \varrho_m^d \lambda_d(O).\end{aligned}$$

Here, we have used the fact that $(\phi_u F)_\varepsilon \cap G_{u\omega} = F_\varepsilon \cap G_{u\omega}$ for all $\varepsilon > 0$ and u, ω as above.

ad (E): Item (E)b is satisfied, since $t \mapsto \lambda_d(F_{\mathrm{e}^{-t+S_n\xi(u\omega x)}} \cap G_\omega)$ is non-increasing for every $n \in \mathbb{N}$, $u \in \Sigma^n$, $\omega \in \Sigma^*$ and $x \in \Sigma^\infty$.

ad (F): First, define

$$N_d^\omega(t,x) := \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \mathrm{e}^{-dS_n\xi(ux)} \lambda_d(F_{\mathrm{e}^{-t+S_n\xi(ux)}} \cap G_\omega)$$

for $\omega \in \Sigma^m$, $t \in \mathbb{R}$ and $x \in \Sigma^\infty$ and note that N_d^ω satisfies a renewal type equation:

$$\sum_{y\colon \sigma y = x} N_d^\omega(t - \xi(y), y) \mathrm{e}^{-d\xi(y)} = N_d^\omega(t,x) - \lambda_d(F_{\mathrm{e}^{-t}} \cap G_\omega). \tag{4.26}$$

Thus, the function M_d^ω given by $M_d^\omega(t,x) := \mathrm{e}^{-t(\delta-d)} N_d^\omega(t,x)/h_{-\delta\xi}(x)$ satisfies

$$\begin{aligned}M_d^\omega(t,x) =& \mathrm{e}^{-t(\delta-d)} \lambda_d(F_{\mathrm{e}^{-t}} \cap G_\omega)/h_{-\delta\xi}(x) \\ &+ \sum_{y\colon \sigma y = x} M_d^\omega(t - \xi(y), y) \mathrm{e}^{-\delta\xi(y)} \frac{h_{-\delta\xi}(y)}{h_{-\delta\xi}(x)}.\end{aligned} \tag{4.27}$$

For showing the assertion of Item (F) we now show that the function M_d^ω is uniformly bounded for $t \in \mathbb{R}$ and $x \in \Sigma^\infty$. This finishes the proof of Item (F), since $h_{-\delta\xi}$ is continuous and thus bounded on Σ^∞. Define $\underline{h}_{-\delta\xi} := \inf_{x\in\Sigma^\infty} h_{-\delta\xi}(x)$. Since $h_{-\delta\xi}$ is strictly positive and Σ^∞ is compact, $\underline{h}_{-\delta\xi} > 0$. Further define

$$\overline{M}_d^\omega(t) := \sup_{t'<t,\, x\in\Sigma^\infty} M_d^\omega(t', x)$$

and note that $\overline{M}_d^\omega(-\ln g_\omega)$ is finite:

$$\begin{aligned}
\overline{M}_d^\omega(-\ln g_\omega) &\le \frac{g_\omega^{\delta-d}}{\underline{h}_{-\delta\xi}} \cdot \sup_{t'<-\ln g_\omega,\, x\in\Sigma^\infty} N_d^\omega(t', x) \\
&= \frac{g_\omega^{\delta-d}}{\underline{h}_{-\delta\xi}} \cdot \sup_{x\in\Sigma^\infty} \sum_{n=0}^\infty \sum_{u\in\Sigma^n} \mathrm{e}^{-dS_n\xi(ux)} \lambda_d(G_\omega) \\
&= \frac{g_\omega^{\delta-d}}{\underline{h}_{-\delta\xi}} \cdot \sup_{x\in\Sigma^\infty} \sum_{n=0}^\infty \mathcal{L}_{-d\xi}^n 1(x) \lambda_d(G_\omega) < \infty.
\end{aligned}$$

This bound follows from the spectral radius formula (see Equation (3.5)) and the fact that the spectral radius of $\mathcal{L}_{-d\xi}$ satisfies $\gamma_{-d\xi} < \gamma_{-\delta\xi} = 1$ (see Theorems 3.2 and 3.9 and Proposition 3.6). Now consider the case $t > -\ln g_\omega$ and choose $\varepsilon > 0$ such that $\xi(x) > \varepsilon$ for all $x \in \Sigma^\infty$. Such an ε exists, since the maps $\phi_1, \ldots, \phi_N$ are defined on a compact set and are contractions and differentiable which implies that their derivative is uniformly bounded away from 1. By (COND 4), there exists a $\delta_I < \delta$ and a $\widetilde{\mathfrak{C}} \in \mathbb{R}$ such that $\mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}} \cap G) \le \widetilde{\mathfrak{C}}\mathrm{e}^{-t(\delta-\delta_I)}$. Thus, for $\omega \in \Sigma^*$ we have that

$$\begin{aligned}
\mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}} \cap G_\omega) &= \mathrm{e}^{-t(\delta-d)}\lambda_d((\phi_\omega F)_{\mathrm{e}^{-t}} \cap G_\omega) \\
&\le \underbrace{\varrho_0^{d-\delta_I} \|\phi'_\omega\|_\infty^{\delta_I} \widetilde{\mathfrak{C}}}_{=:\overline{\mathfrak{C}}} \cdot \mathrm{e}^{-t(\delta-\delta_I)}.
\end{aligned} \tag{4.28}$$

Since $h_{-\delta\xi}$ is the eigenfunction with eigenvalue $\gamma_{-\delta\xi} = 1$ of $\mathcal{L}_{-\delta\xi}$, we have that $1 = \sum_{y\colon \sigma y = x} \mathrm{e}^{-\delta\xi(y)} h_{-\delta\xi}(y)/h_{-\delta\xi}(x)$ for all $x \in \Sigma^\infty$. Since furthermore $\overline{M}_d^\omega(t-\varepsilon) \ge \overline{M}_d^\omega(t-\xi(x))$ for all $x \in \Sigma^\infty$ we conclude that

$$\overline{M}_d^\omega(t-\varepsilon) \ge \sum_{y\colon \sigma y = x} \overline{M}_d^\omega(t-\xi(y)) \cdot \mathrm{e}^{-\delta\xi(y)} \frac{h_{-\delta\xi}(y)}{h_{-\delta\xi}(x)} \tag{4.29}$$

for all $x \in \Sigma^\infty$. Together with Equation (4.27) this implies that

$$\begin{aligned}
&\sup_{\substack{t'\in[t-\varepsilon,t)\\ x\in\Sigma^\infty}} M_d^\omega(t',x) - \overline{M}_d^\omega(t-\varepsilon)\\
&\overset{(4.29)}{\leq} \sup_{\substack{t'\in[t-\varepsilon,t)\\ x\in\Sigma^\infty}} M_d^\omega(t',x) - \sup_{x\in\Sigma^\infty} \sum_{y\colon \sigma y = x} \overline{M}_d^\omega(t-\xi(y)) \cdot \mathrm{e}^{-\delta\xi(y)} \cdot \frac{h_{-\delta\xi}(y)}{h_{-\delta\xi}(x)}\\
&\leq \sup_{\substack{t'\in[t-\varepsilon,t)\\ x\in\Sigma^\infty}} M_d^\omega(t',x) - \sup_{\substack{t'\in[t-\varepsilon,t)\\ x\in\Sigma^\infty}} \sum_{y\colon \sigma y = x} M_d^\omega(t'-\xi(y),y) \cdot \mathrm{e}^{-\delta\xi(y)} \cdot \frac{h_{-\delta\xi}(y)}{h_{-\delta\xi}(x)}\\
&\leq \sup_{\substack{t'\in[t-\varepsilon,t)\\ x\in\Sigma^\infty}} \left(M_d^\omega(t',x) - \sum_{y\colon \sigma y = x} M_d^\omega(t'-\xi(y),y) \cdot \mathrm{e}^{-\delta\xi(y)} \cdot \frac{h_{-\delta\xi}(y)}{h_{-\delta\xi}(x)} \right)\\
&\overset{(4.27)}{=} \sup_{\substack{t'\in[t-\varepsilon,t)\\ x\in\Sigma^\infty}} \mathrm{e}^{-t'(\delta-d)} \lambda_d(F_{\mathrm{e}^{-t'}} \cap G_\omega)/h_{-\delta\xi}(x)\\
&\overset{(4.28)}{\leq} \overline{\mathfrak{C}} \cdot \mathrm{e}^{-(t-\varepsilon)(\delta-\delta_I)}/\underline{h}_{-\delta\xi}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\overline{M}_d^\omega(t) &= \overline{M}_d^\omega(t-\varepsilon) + \max\left\{0, \sup_{\substack{t'\in[t-\varepsilon,t)\\ x\in\Sigma^\infty}} M_d^\omega(t',x) - \overline{M}_d^\omega(t-\varepsilon)\right\}\\
&\leq \overline{M}_d^\omega(t-\varepsilon) + \overline{\mathfrak{C}}\mathrm{e}^{-(t-\varepsilon)(\delta-\delta_I)}/\underline{h}_{-\delta\xi}.
\end{aligned}$$

Hence, for arbitrary $n \in \mathbb{N}$ and $t \geq -\ln g_\omega$ we have

$$\begin{aligned}
\overline{M}_d^\omega(t+n\varepsilon) &\leq \frac{\overline{\mathfrak{C}}}{\underline{h}_{-\delta\xi}} \cdot \sum_{j=0}^{n-1} \mathrm{e}^{-j\varepsilon(\delta-\delta_I)} \mathrm{e}^{-t(\delta-\delta_I)} + \overline{M}_d^\omega(t)\\
&\leq \frac{\overline{\mathfrak{C}}\mathrm{e}^{-t(\delta-\delta_I)}}{\underline{h}_{-\delta\xi}(1-\mathrm{e}^{-\varepsilon(\delta-\delta_I)})} + \overline{M}_d^\omega(t).
\end{aligned}$$

This now implies

$$\sup_{n\in\mathbb{N}} \overline{M}_d^\omega(-\ln g_\omega + n\varepsilon) \leq \frac{\overline{\mathfrak{C}} \cdot g_\omega^{\delta-\delta_I}}{\underline{h}_{-\delta\xi} \cdot (1-\mathrm{e}^{-\varepsilon(\delta-\delta_I)})} + \overline{M}_d^\omega(-\ln g_\omega) < \infty,$$

proving the assertion. The inspiration for the construction of the functions M_d^ω comes from [Lal89].

ad (G): Since $\lambda_d(F_{\mathrm{e}^{-t}} \cap G_\omega) = \lambda_d(G_\omega)$ for all $t < -\ln(g_\omega)$ and by the gibbs property of the measure $\mu_{-d\xi}$ (see Equation (3.1)) it follows for all $t < -\ln \operatorname{diam} G$ that

$$\begin{aligned} N_{d,m}^{\mathrm{abs}}(t,x) &\leq \sum_{\omega\in\Sigma^m}\sum_{n=0}^{\infty}\sum_{u\in\Sigma^n} \mathrm{e}^{-dS_n\xi(u\omega x)}\lambda_d(G_\omega) \\ &\leq \sum_{\omega\in\Sigma^m}\sum_{n=0}^{\infty}\sum_{u\in\Sigma^n} c\mu_{-d\xi}([u])\mathrm{e}^{nP(-d\xi)}\lambda_d(G_\omega) \\ &= \sum_{\omega\in\Sigma^m}\sum_{n=0}^{\infty} c\mathrm{e}^{nP(-d\xi)}\lambda_d(G_\omega) \\ &\leq c\lambda_d(O)/(1-\mathrm{e}^{P(-d\xi)}) =: \mathfrak{C} < \infty, \end{aligned}$$

where c denotes the constant in the Gibbs property of $\mu_{-\delta\xi}$. Here, we have used that $P(-d\xi) < P(-\delta\xi) = 0$, which follows from Theorem 3.2 and Proposition 3.6.

□

We now turn to proving the two items of Theorem 2.29 and start with Item (ii) which is concerned with the non-lattice case.

Proof of Theorem 2.29(ii). Recall that the main ideas of this proof are presented in Steps (I) to (V) in the beginning of Section 4.1.

We start with showing that $\mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}} \cap B)$ converges to the well-defined limit

$$\lim_{m\to\infty}\sum_{\omega\in\Sigma^m}\int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)}\sum_{i=1}^{Q}\lambda_d(F_{\mathrm{e}^{-T}}\cap G_\omega^i)\mathrm{d}T\frac{\nu(B)}{\int\xi\mathrm{d}\mu_{-\delta\xi}} \tag{4.30}$$

as $t\to\infty$ for every $B\in\mathcal{E}_F$, where $\mathcal{E}_F$ is as defined in Equation (4.1) and by this specify the measure μ from Step (II). Then an application of Lemma 4.6 finishes the proof.

For showing that $\mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}}\cap B)$ converges to the limit in Equation (4.30) for $B\in\mathcal{E}_F$, we want to apply Lemma 4.5. (This is Step (III).) In Equation (4.28) we have shown that the fact that there exists a $\delta_I < \delta$ such that $\mathrm{e}^{-t(\delta_I-d)}\cdot\lambda_d(F_{\mathrm{e}^{-t}}\cap G^i)$ is uniformly bounded by some constant $\widetilde{\mathfrak{C}}$ for $t\in\mathbb{R}$ and every $i\in\{1,\ldots,Q\}$ implies that $\mathrm{e}^{-t(\delta_I-d)}\cdot\lambda_d(F_{\mathrm{e}^{-t}}\cap G_\omega^i)$ is also uniformly bounded for $t\in\mathbb{R}$, $\omega\in\Sigma^*$ and $i\in\{1,\ldots,Q\}$. Therefore, Lemma 4.5 is applicable. We distinguish between the two cases $B\in\mathcal{K}_F$ and $B\in\mathcal{E}_F\setminus\mathcal{K}_F$.

For $B\in\mathcal{K}_F$, Lemma 4.5 implies that $\lim_{t\to\infty}\mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}}\cap B) = 0 = \nu(B)$. The case $B\in\mathcal{E}_F\setminus\mathcal{K}_F$ requires some more work. For $B\in\mathcal{E}_F\setminus\mathcal{K}_F$ there exists a $\kappa\in\Sigma^*$ such that $B=\phi_\kappa O$. By Lemma 4.5(ii) we know that for all $m\in\mathbb{N}$ and $x\in\Sigma^\infty$ we have that

$$\lambda_d(F_{\mathrm{e}^{-t}}\cap\phi_\kappa O) = \sum_{\omega\in\Sigma^m}\sum_{n=0}^{\infty}\sum_{u\in\Sigma^n}\mathbb{1}_{[\kappa]}(u\omega x)\lambda_d(F_{\mathrm{e}^{-t}}\cap\textstyle\bigcup_{i=1}^{Q}G_{u\omega}^i) + \mathfrak{o}(\mathrm{e}^{t(\delta-d)}) \tag{4.31}$$

as $t \to \infty$. Now we come to Step (IV) and approximate the first term on the right hand side of Equation (4.31).

Since $F_{\mathrm{e}^{-t}} \cap G^i_{u\omega} = (\phi_u F)_{\mathrm{e}^{-t}} \cap G^i_{u\omega}$, the BDP implies the following for an arbitrary $x \in \Sigma^\infty$, $i \in \{1, \dots, Q\}$, $\omega \in \Sigma^m$ and $u \in \Sigma^*$.

$$\lambda_d(F_{\mathrm{e}^{-t}} \cap G^i_{u\omega}) \leq |\phi'_u(\pi\omega x)|^d \varrho_m^d \lambda_d(F_{\mathrm{e}^{-t}\varrho_m/|\phi'_u(\pi\omega x)|} \cap G^i_\omega) \qquad \text{and}$$
$$\lambda_d(F_{\mathrm{e}^{-t}} \cap G^i_{u\omega}) \geq |\phi'_u(\pi\omega x)|^d \varrho_m^{-d} \lambda_d(F_{\mathrm{e}^{-t}\varrho_m^{-1}/|\phi'_u(\pi\omega x)|} \cap G^i_\omega).$$

Thus, using that $|\phi'_u(\pi\omega x)| = \mathrm{e}^{-S_n\xi(u\omega x)}$ for $n \in \mathbb{N}$, $u \in \Sigma^n$, $\omega \in \Sigma^*$ and $x \in \Sigma^\infty$ by definition of the geometric potential function in Definition 2.20 we altogether obtain the following for an arbitrary $x \in \Sigma^\infty$

$$\begin{aligned} &\lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\kappa O) \\ &\leq \sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x) \mathrm{e}^{-dS_n\xi(u\omega x)} \varrho_m^d \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-(t-\ln\varrho_m)+S_n\xi(u\omega x)}} \cap G^i_\omega) + \mathfrak{o}(\mathrm{e}^{t(\delta-d)}) \\ &= \varrho_m^d N_{d,m,\kappa}(t - \ln\varrho_m, x) + \mathfrak{o}(\mathrm{e}^{t(\delta-d)}) \end{aligned} \tag{4.32}$$

as $t \to \infty$, where $N_{d,m,\kappa}$ is as defined in Key Lemma 4.2 with $f_{d,\omega}(t) := \sum_{i=1}^Q \lambda_d(F_{\mathrm{e}^{-t}} \cap G^i_\omega)$. Analogously, a lower bound for $\lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\kappa O)$ is given by

$$\lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\kappa O) \geq \varrho_m^{-d} N_{d,m}(t + \ln\varrho_m, x) + \mathfrak{o}(\mathrm{e}^{t(\delta-d)}).$$

Lemma 4.8 allows us to apply Key Lemma 4.2(ii) and we obtain for all $m \in \mathbb{N}$ and $x \in \Sigma^\infty$ that

$$\begin{aligned} &N_{d,m,\kappa}(t - \ln\varrho_m, x) \\ &\sim^{\mathbb{R}} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G^i_\omega) \mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]} \mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} h_{-\delta\xi}(\omega x) \cdot \mathrm{e}^{t(\delta-d)} \varrho_m^{d-\delta} \end{aligned} \tag{4.33}$$

as $t \to \infty$. Thus, combining Equations (4.32) and (4.33) we conclude that

$$\begin{aligned} &\limsup_{t\to\infty} \mathrm{e}^{-t(\delta-d)} \lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\kappa O) \\ &\leq \varrho_m^{2d-\delta} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G^i_\omega) \mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]} \mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} h_{-\delta\xi}(\omega x) \end{aligned}$$

and analogously that

$$\begin{aligned} &\liminf_{t\to\infty} \mathrm{e}^{-t(\delta-d)} \lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\kappa O) \\ &\geq \varrho_m^{-2d+\delta} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G^i_\omega) \mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]} \mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} h_{-\delta\xi}(\omega x) \end{aligned}$$

hold for all $m \in \mathbb{N}$ and $x \in \Sigma^\infty$. An application of Lemma 4.7 now gives

$$\begin{aligned}
&\limsup_{m\to\infty} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega^i)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \\
&\quad \le \liminf_{t\to\infty} \mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\kappa O) \\
&\quad \le \limsup_{t\to\infty} \mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\kappa O) \\
&\quad \le \liminf_{m\to\infty} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega^i)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}}
\end{aligned}$$

which shows that all the above limits exist. Moreover,

$$\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi} = \nu(\phi_\kappa O), \tag{4.34}$$

which can be seen as follows. (COND 2) states that the upper Minkowski dimension of ∂O is strictly less than δ. Therefore, the δ-dimensional Hausdorff measure $\mathcal{H}^\delta$ of ∂O is zero. It follows that $\nu(\partial O) = 0$, since ν is equivalent to $\mathcal{H}^\delta$ by Corollary 4.18 in [MU96]. This shows that for $B \in \mathcal{E}_F \setminus \mathcal{K}_F$ the expression $\mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}} \cap B)$ also converges to the well-defined limit from Equation (4.30). This furthermore implies that $\mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}})$ is uniformly bounded for $t \in (1,\infty)$. Therefore, we can apply Lemma 4.6 which finishes the proof. □

Proof of Theorem 2.29(i). The main ideas of this proof are presented in Steps (I) to (V) in the beginning of Section 4.1. We start with showing that $t^{-1}\int_0^t \mathrm{e}^{-T(\delta-d)}\lambda_d(F_{\mathrm{e}^{-T}} \cap B)\mathrm{d}T$ converges to the well-defined limit

$$\lim_{m\to\infty} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega^i)\mathrm{d}T \cdot \frac{\nu(B)}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \tag{4.35}$$

as $t \to \infty$ for every $B \in \mathcal{E}_F$, where $\mathcal{E}_F$ is defined in Equation (4.1) and by this specify the measure μ from Step (II).

Firstly, take $B \in \mathcal{K}_F$ (see Equation (4.2) for the definition of $\mathcal{K}_F$). In Equation (4.28) we have shown that the fact that there exists a $\delta_I < \delta$ such that $\mathrm{e}^{-t(\delta_I-d)} \cdot \lambda_d(F_{\mathrm{e}^{-t}} \cap G^i)$ is uniformly bounded by some constant $\widetilde{\mathfrak{C}}$ for $t \in \mathbb{R}$ and $i \in \{1,\dots,Q\}$ implies that $\mathrm{e}^{-t(\delta_I-d)}\lambda_d(F_{\mathrm{e}^{-t}} \cap G_\omega^i)$ is also uniformly bounded for $t \in \mathbb{R}$, $\omega \in \Sigma^*$ and $i \in \{1,\dots,Q\}$. Thus, we can apply Lemma 4.5. By Lemma 4.5(i) we know that $\lambda_d(F_{\mathrm{e}^{-t}} \cap B) = \mathfrak{o}(\mathrm{e}^{t(\delta-d)})$ as $t \to \infty$. Therefore, for all $\varepsilon > 0$ there exists a $\widetilde{T} \in \mathbb{R}$ such that for all $t \ge \widetilde{T}$ we have that $\mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}} \cap B) \leq \varepsilon$. Thus, for $t \geq \widetilde{T}$ we have

$$\begin{aligned}
&t^{-1}\int_0^t \mathrm{e}^{-T(\delta-d)}\lambda_d(F_{\mathrm{e}^{-T}} \cap B)\mathrm{d}T \\
&\quad = t^{-1}\left(\int_0^{\widetilde{T}} \mathrm{e}^{-T(\delta-d)}\lambda_d(F_{\mathrm{e}^{-T}} \cap B)\mathrm{d}T + \int_{\widetilde{T}}^t \mathrm{e}^{-T(\delta-d)}\lambda_d(F_{\mathrm{e}^{-T}} \cap B)\mathrm{d}T\right) \\
&\quad \leq t^{-1}\left(\int_0^{\widetilde{T}} \mathrm{e}^{-T(\delta-d)}\lambda_d(F_{\mathrm{e}^{-T}} \cap B)\mathrm{d}T + \varepsilon(t-\widetilde{T})\right) \to \varepsilon
\end{aligned}$$

as $t \to \infty$. Hence,

$$\lim_{t\to\infty} t^{-1}\int_0^t \mathrm{e}^{-T(\delta-d)}\lambda_d(F_{\mathrm{e}^{-T}} \cap B)\mathrm{d}T = 0 = \nu(B). \tag{4.36}$$

Now, take $B = \phi_\kappa O \in \mathcal{E}_F \setminus \mathcal{K}_F$ for some $\kappa \in \Sigma^*$. By Lemma 4.5(ii) we have for all $m \in \mathbb{N}$ and all $x \in \Sigma^\infty$ that

$$\lambda_d(F_{\mathrm{e}^{-t}} \cap B) = \sum_{\omega\in\Sigma^m}\sum_{n=0}^{\infty}\sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x)\lambda_d(F_{\mathrm{e}^{-t}} \cap \textstyle\bigcup_{i=1}^Q G^i_{u\omega}) + \mathfrak{o}(\mathrm{e}^{t(\delta-d)}) \tag{4.37}$$

as $t \to \infty$. The BDP implies that for $m \in \mathbb{N}$, $\omega \in \Sigma^m$, $i \in \{1,\dots,Q\}$ and $x \in \Sigma^\infty$ we have that

$$\begin{aligned}
\lambda_d(F_{\mathrm{e}^{-t}} \cap G^i_{u\omega}) &= \lambda_d((\phi_u F)_{\mathrm{e}^{-t}} \cap \phi_u(G^i_\omega)) \\
&\leq \varrho_m^d |\phi_u'(\pi\omega x)|^d \lambda_d(F_{\mathrm{e}^{-t}\varrho_m/|\phi_u'(\pi\omega x)|} \cap G^i_\omega).
\end{aligned} \tag{4.38}$$

Setting $f_{d,\omega}(t) := \sum_{i=1}^Q \lambda_d(F_{\mathrm{e}^{-t}} \cap G^i_\omega)$ and recalling the definition of $N_{d,m,\kappa}$ from Key Lemma 4.2 we conclude from Equation (4.37) that

$$\lambda_d(F_{\mathrm{e}^{-t}} \cap B) \leq \varrho_m^d N_{d,m,\kappa}(t - \ln\varrho_m, x) + \mathfrak{o}(\mathrm{e}^{t(\delta-d)}). \tag{4.39}$$

Applying the same argument as in the case of $B \in \mathcal{K}_F$ to the second summand on the right hand side of Equation (4.39) now shows that for an arbitrary $x \in \Sigma^\infty$ we have

$$\begin{aligned}
&\limsup_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap B)\mathrm{d}T \\
&\quad \le \limsup_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-d)} \varrho_m^d N_{d,m,\kappa}(T - \ln \varrho_m, x)\mathrm{d}T \\
&\quad = \limsup_{t\to\infty} t^{-1} \int_{-\ln \varrho_m}^{t-\ln \varrho_m} \mathrm{e}^{-T(\delta-d)} N_{d,m,\kappa}(T, x)\mathrm{d}T \cdot \varrho_m^{2d-\delta} \\
&\quad = \limsup_{t\to\infty} \Bigg(t^{-1} \int_{-\ln \varrho_m}^{0} \mathrm{e}^{-T(\delta-d)} N_{d,m,\kappa}(T, x)\mathrm{d}T + \\
&\qquad\qquad \frac{t - \ln \varrho_m}{t} \cdot \frac{1}{t - \ln \varrho_m} \int_0^{t-\ln \varrho_m} \mathrm{e}^{-T(\delta-d)} N_{d,m,\kappa}(T, x)\mathrm{d}T \Bigg) \cdot \varrho_m^{2d-\delta} \\
&\quad = \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega^i)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \cdot h_{-\delta\xi}(\omega x)\varrho_m^{2d-\delta},
\end{aligned}$$

where the last equality is a consequence of Key Lemma 4.2(i), which we can apply because of Lemma 4.8. Analogously one can show that

$$\begin{aligned}
&\liminf_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap B)\mathrm{d}T \\
&\quad \ge \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega^i)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \cdot h_{-\delta\xi}(\omega x)\varrho_m^{-2d+\delta}.
\end{aligned}$$

Hence, Lemma 4.7 implies

$$\begin{aligned}
&\limsup_{m\to\infty} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega^i)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \\
&\quad \le \liminf_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap B)\mathrm{d}T \\
&\quad \le \limsup_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap B)\mathrm{d}T \\
&\quad \le \liminf_{m\to\infty} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega^i)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}}
\end{aligned}$$

which shows that all the above limits exist and are equal. Altogether, we therefore obtain for an arbitrary $B \in \mathcal{E}_F$ that

$$\begin{aligned}
&\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-d)} \lambda_d(F_{\mathrm{e}^{-T}} \cap B)\mathrm{d}T \\
&\quad = \lim_{m\to\infty} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-T}} \cap G_\omega^i)\mathrm{d}T \cdot \frac{\nu(B)}{\int \xi \mathrm{d}\mu_{-\delta\xi}},
\end{aligned}$$

since $\int \mathbb{1}_{[\kappa]} \mathrm{d}\nu_{-\delta\xi} = \nu(\phi_\kappa O)$ was shown in Equation (4.34). The rest follows from Lemma 4.6, as in the end of the proof of Theorem 2.29(ii) we have shown that $\mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}})$ is uniformly bounded for $t \in (1,\infty)$. □

4.2.2 Proof of Theorem 2.31

In the following we provide the proof of Theorem 2.31. It results by applying Key Lemma 4.2. That the prerequisites of Key Lemma 4.2 are satisfied is shown in the next lemma.

Lemma 4.9. *Assume that we are in the situation of Theorem 2.31. Then Items (A) to (G) of Key Lemma 4.2 are satisfied for $f_{k,\omega}\colon \mathbb{R} \to \mathbb{R}$ given by*

$$f_{k,\omega}(t) := \sum_{i=1}^{Q} C_k(F_{\mathrm{e}^{-t}}, G_\omega^i),$$

where $k \in \{0,1\}$ and $\omega \in \Sigma^m$ for some fixed $m \in \mathbb{N}$.

Proof. We first show that the prerequisites of Theorem 2.31 imply (COND 2) to (COND 4) when $d = 1$. Since $X \setminus \Phi X$ has at most $N-1$ connected components, the number of primary gaps of F is finite, implying (COND 3). Thus, we can assume without loss of generality that F possesses exactly one primary gap which we denote by G. We use the interpretations $C_1(F_{\mathrm{e}^{-t}}, G) = \lambda_1(F_{\mathrm{e}^{-t}} \cap G)$ and $C_0(F_{\mathrm{e}^{-t}}, G) = \lambda_0(\partial F_{\mathrm{e}^{-t}} \cap G)/2$ from Proposition 2.7. Then

$$\lambda_1(F_{\mathrm{e}^{-t}} \cap G) = \begin{cases} |G| & : t \leq -\ln(|G|/2) \\ 2\mathrm{e}^{-t} & : t > -\ln(|G|/2) \end{cases}$$

which implies that $\lambda_1(F_{\mathrm{e}^{-t}} \cap G) \cdot \mathrm{e}^{-t(\delta_I - 1)}$ is uniformly bounded in t for any $\delta_I \in [0,1]$. Similarly, $\lambda_1(F_{\mathrm{e}^{-t}} \cap \overline{O}_{\mathrm{e}^{-t}} \setminus \overline{O}) = 2\mathrm{e}^{-t}$ shows that $\lambda_1(F_{\mathrm{e}^{-t}} \cap \overline{O}_{\mathrm{e}^{-t}} \setminus \overline{O}) \cdot \mathrm{e}^{-t(\delta_O - 1)}$ is uniformly bounded in $t \in \mathbb{R}$ for $\delta_O = 0$. Therefore, (COND 2) and (COND 4) also hold. Hence, for $k = 1$ the assertion of this lemma follows from Theorem 2.29.

This leaves to consider the case $k = 0$.

ad (A): $f_{0,\omega}$ is defined on the whole real line.

ad (B): Note that $\lambda_0(\partial F_{\mathrm{e}^{-t}} \cap G_\omega)/2 = \mathbb{1}_{(-\ln(|G_\omega|/2),\infty)}(t)$. Thus, $f_{0,\omega}$ is left-continuous.

ad (C): The equality $\lambda_0(\partial F_{\mathrm{e}^{-t}} \cap G_\omega)/2 = \mathbb{1}_{(-\ln(|G_\omega|/2),\infty)}(t)$ implies that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-\delta t} |f_{0,\omega}(t)| \mathrm{d}t = \frac{1}{\delta}\left(\frac{|G_\omega|}{2}\right)^{\delta} < \infty.$$

ad (D): Since $f_{0,\omega}$ is defined on the whole real line for every $\omega \in \Sigma^m$, we have $V = \mathbb{R}$.

$$\begin{aligned} N_{0,m}^{\mathrm{abs}}(t,x) &= \sum_{\omega\in\Sigma^m}\sum_{n=0}^{\infty}\sum_{u\in\Sigma^n} |f_{0,\omega}(t - S_n\xi(u\omega x))| \\ &= \sum_{\omega\in\Sigma^m}\sum_{n=0}^{\infty}\sum_{u\in\Sigma^n} \mathbb{1}_{(-\ln(|G_\omega|/2),\infty)}(t - S_n\xi(u\omega x)). \end{aligned}$$

As we assume that the derivatives ϕ_i' are bounded away from one for $i \in \{1,\dots,N\}$ and as $\xi(u\omega x) := -\ln|\phi_{u_1}'(\pi\sigma u\omega x)|$, the sequence $(S_n\xi(u\omega x))_{n\in\mathbb{N}}$ is strictly increasing and unbounded. Therefore, for a fixed $t \in \mathbb{R}$ the above series actually is a finite sum and hence $N_{0,m}^{\mathrm{abs}}(t,x)$ is finite for every $t \in \mathbb{R}$ and $x \in \Sigma^\infty$.

ad (E): Item (E)a is satisfied with $t^* := -\ln(|G|/2)$.

ad (F): The proof of this part can be carried out in analogy to the proof of the corresponding part in Lemma 4.8: Setting $N_0^\omega(t,x) := \sum_{n=0}^\infty \sum_{u\in\Sigma^n} f_{0,\omega}(t - S_n\xi(ux))$ for $x \in \Sigma^\infty$ gives the respective function which satisfies a renewal type equation

$$\sum_{y\colon \sigma y = x} N_0^\omega(t - \xi(y), y) = N_0^\omega(t,x) - f_{0,\omega}(t)$$

for $x \in \Sigma^\infty$. Setting $M_0^\omega(t,x) := \mathrm{e}^{-t\delta} N_0^\omega(t,x)/h_{-\delta\xi}(x)$ as in the proof of Lemma 4.8(F), we obtain $\overline{M}_0^\omega(-\ln(|G_\omega|/2)) = 0$. The rest follows through in the same way.

ad (G): Since $f_{0,\omega} = 0$ for $t \le -\ln(|G|/2)$ for all $m \in \mathbb{N}$ and all $\omega \in \Sigma^m$, we have $N_{0,m}^{\mathrm{abs}}(t,x) = 0$ for $t \le t^* := -\ln(|G|/2)$. Note that the here defined t^* coincides with the t^* from Item (E)a.

□

Now, we turn to proving the three parts of Theorem 2.31. Item (iii) of Theorem 2.31 follows from the results which we obtain for $\mathcal{C}^{1+\alpha}$-diffeomorphic images of self-similar sets, and which are proven in Section 4.4.

Proof of Theorem 2.31(iii). This statement follows from Theorem 2.43(iii) together with Theorem 2.46. Both these theorems are proven in Section 4.4. □

Before providing the proofs of Items (i) and (ii) of Theorem 2.31, we present global statements which we use in the proofs of both parts.

Remark 4.10. In the beginning of the proof of Lemma 4.9 we have seen that the prerequisites of Theorem 2.31 imply (COND 2) to (COND 4). Therefore, the results in Items (i) and (ii) of Theorem 2.31 concerning the case $k=1$ follow from the respective results in Theorem 2.29 by using that

$$C_1(F_{\mathrm{e}^{-t}}, G^i_\omega) = \begin{cases} |G^i_\omega| & : t \leq -\ln(|G^i_\omega|/2) \\ 2\mathrm{e}^{-t} & : t > -\ln(|G^i_\omega|/2), \end{cases} \tag{4.40}$$

which follows from the geometric interpretation for $C_k(F_{\mathrm{e}^{-t}}, \cdot)$ from Proposition 2.7.

Thus, we concentrate on the case $k=0$ and remark that the idea of proof is the same as the one for Theorem 2.29: We first consider the limiting behaviour of $\mathrm{e}^{-t\delta}C_0(F_{\mathrm{e}^{-t}}, B)$ for $B \in \mathcal{E}_F$ and then apply Lemma 4.6 to obtain the results concerning the measures. Without loss of generality we assume that F possesses exactly one primary gap which we denote by G.

Remark 4.11. We can apply Lemma 4.5 since firstly, $C_0^{\mathrm{var}}(F_{\mathrm{e}^{-t}}, \overline{O}_{\mathrm{e}^{-t}} \setminus \overline{O})\mathrm{e}^{-t\delta_O} = \mathrm{e}^{-t\delta_0}$ is uniformly bounded in t for $\delta_O = 0$ and secondly,

$$C_0^{\mathrm{var}}(F_{\mathrm{e}^{-t}}, G_\omega) \cdot \mathrm{e}^{-t\delta_I} = \mathbb{1}_{(-\ln(|G_\omega|/2),\infty)}(t) \cdot \mathrm{e}^{-t\delta_I}$$

is bounded for all $\delta_I \in [0,\delta)$ and all $\omega \in \Sigma^*$ by 1.

An application of Lemma 4.5 implies that

$$\lim_{t\to\infty} \mathrm{e}^{-t\delta} C_0(F_{\mathrm{e}^{-t}}, B) = 0 = \nu(B) \quad \text{for } B \in \mathcal{K}_F. \tag{4.41}$$

Furthermore, for $B = \phi_\kappa O \in \mathcal{E}_F \setminus \mathcal{K}_F$, where $\kappa \in \Sigma^*$, Lemma 4.5 implies that for all $m \in \mathbb{N}$ and all $x \in \Sigma^\infty$ we have

$$C_0(F_{\mathrm{e}^{-t}}, B) = \sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x) \cdot C_0(F_{\mathrm{e}^{-t}}, G_{u\omega}) + \mathfrak{o}(\mathrm{e}^{t\delta}).$$

Setting $f_{0,\omega}(t) := C_0(F_{\mathrm{e}^{-t}}, G_\omega) = \mathbb{1}_{(-\ln(|G_\omega|/2),\infty)}(t)$ as in Lemma 4.9 and using that $|G_{u\omega}| \leq \varrho_m |\phi'_u(\pi\omega x)| \cdot |G_\omega|$ holds for all $u \in \Sigma^*$, $\omega \in \Sigma^m$ we obtain that

$$\begin{aligned} C_0(F_{\mathrm{e}^{-t}}, G_{u\omega}) &= \mathbb{1}_{(-\ln(|G_{u\omega}|/2),\infty)}(t) \leq \mathbb{1}_{(-\ln(\varrho_m|\phi'_u(\pi\omega x)|\cdot|G_\omega|/2),\infty)}(t) \\ &= C_0(F_{\mathrm{e}^{-t-\ln\varrho_m+S_n\xi(u\omega x)}}, G_\omega) = f_{0,\omega}(t+\ln\varrho_m - S_n\xi(u\omega x)) \end{aligned} \tag{4.42}$$

and analogously that

$$C_0(F_{\mathrm{e}^{-t}}, G_{u\omega}) \geq f_{0,\omega}(t - \ln\varrho_m - S_n\xi(u\omega x)) \tag{4.43}$$

hold for all $x \in \Sigma^\infty$. Therefore, recalling the definition of $N_{0,m,\kappa}$ from Key Lemma 4.2 with this $f_{0,\omega}$ we conclude that

$$C_0(F_{\mathrm{e}^{-t}}, \phi_\kappa O) \leq N_{0,m,\kappa}(t+\ln\varrho_m) + \mathfrak{o}(\mathrm{e}^{t\delta}) \quad \text{and} \tag{4.44}$$

$$C_0(F_{\mathrm{e}^{-t}}, \phi_\kappa O) \geq N_{0,m,\kappa}(t-\ln\varrho_m) + \mathfrak{o}(\mathrm{e}^{t\delta}) \tag{4.45}$$

as $t \to \infty$ for $\kappa \in \Sigma^*$.

Proof of Theorem 2.31(ii). For $k = 1$ the statement follows from Remark 4.10. Thus, we assume $k = 0$ from now on. By Equation (4.41) it suffices to consider the case $B = \phi_\kappa O \in \mathcal{E}_F \setminus \mathcal{K}_F$ for some $\kappa \in \Sigma^*$. Without loss of generality we assume that F posesses exactly one primary gap which we denote by G. The associated main gaps are denoted by G_ω for $\omega \in \Sigma^*$. We define $f_{0,\omega}(t) := C_0(F_{\mathrm{e}^{-t}}, G_\omega)$ for $\omega \in \Sigma^m$, $t \in \mathbb{R}$ and some fixed $m \in \mathbb{N}$. Due to Lemma 4.9 we can apply Key Lemma 4.2(ii) to these $f_{0,\omega}$ and obtain that $N_{0,m,\kappa}$ satisfies

$$N_{0,m,\kappa}(t,x) \sim^{\mathbb{R}} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T\delta} C_0(F_{\mathrm{e}^{-T}}, G_\omega)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} h_{-\delta\xi}(\omega x)\mathrm{e}^{t\delta} \tag{4.46}$$

as $t \to \infty$. Evaluating the integral in Equation (4.46) by using that $C_0(F_{\mathrm{e}^{-T}}, G_\omega) = \mathbb{1}_{(-\ln(|G_\omega|/2),\infty)}(T)$ gives the asymptotic

$$N_{0,m,\kappa}(t,x) \sim^{\mathbb{R}} \sum_{\omega\in\Sigma^m} \frac{2^{-\delta}|G_\omega|^\delta}{\delta} \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} h_{-\delta\xi}(\omega x)\mathrm{e}^{t\delta}$$

as $t \to \infty$. Thus, Equations (4.44) and (4.45) imply that

$$\limsup_{t\to\infty} \mathrm{e}^{-t\delta} C_0(F_{\mathrm{e}^{-t}}, \phi_\kappa O) \le \varrho_m^\delta \sum_{\omega\in\Sigma^m} \frac{2^{-\delta}|G_\omega|^\delta}{\delta} \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} h_{-\delta\xi}(\omega x) \quad \text{and}$$
$$\liminf_{t\to\infty} \mathrm{e}^{-t\delta} C_0(F_{\mathrm{e}^{-t}}, \phi_\kappa O) \ge \varrho_m^{-\delta} \sum_{\omega\in\Sigma^m} \frac{2^{-\delta}|G_\omega|^\delta}{\delta} \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} h_{-\delta\xi}(\omega x)$$

hold for all $m \in \mathbb{N}$. Applying Lemma 4.7 we hence obtain

$$\begin{aligned}\limsup_{m\to\infty} \sum_{\omega\in\Sigma^m} \frac{2^{-\delta}|G_\omega|^\delta}{\delta} \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} &\le \liminf_{t\to\infty} \mathrm{e}^{-t\delta} C_0(F_{\mathrm{e}^{-t}}, \phi_\kappa O)\\ &\le \limsup_{t\to\infty} \mathrm{e}^{-t\delta} C_0(F_{\mathrm{e}^{-t}}, \phi_\kappa O)\\ &\le \liminf_{m\to\infty} \sum_{\omega\in\Sigma^m} \frac{2^{-\delta}|G_\omega|^\delta}{\delta} \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}},\end{aligned}$$

which shows that all the above limits exist and are equal. Since $\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi} = \nu(\phi_\kappa O)$ (see Equation (4.34)) this shows on the one hand that $\mathrm{e}^{-t\delta} C_0^{\mathrm{var}}(F_{\mathrm{e}^{-t}})$ is uniformly bounded for $t \in (1,\infty)$ and on the other hand together with Equation (4.41) that

$$\lim_{t\to\infty} \mathrm{e}^{-t\delta} C_0(F_{\mathrm{e}^{-t}}, B) = \lim_{m\to\infty} \sum_{\omega\in\Sigma^m} |G_\omega|^\delta \frac{2^{-\delta}}{H_{\mu_{-\delta\xi}}} \cdot \nu(B)$$

for all $B \in \mathcal{E}_F$. Thus, an application of Lemma 4.6 implies the assertion. □

Proof of Theorem 2.31(i). For $k = 1$ the statement follows from Remark 4.10. For $k = 0$ the proof is carried out in a similar vein as the proof of Theorem 2.29(i).

Without loss of generality, we assume that F possesses exactly one primary gap, which we denote by G. We start by showing that $t^{-1} \int_0^t \mathrm{e}^{-\delta T} C_0(F_{\mathrm{e}^{-T}}, B)\mathrm{d}T$ converges to the well-defined limit

$$\lim_{m\to\infty} \sum_{\omega\in\Sigma^m} \frac{2^{-\delta}|G_\omega|^\delta}{H_{\mu_{-\delta\xi}}} \cdot \nu(B)$$

as $t \to \infty$ for every $B \in \mathcal{E}_F$, where $\mathcal{E}_F$ is defined as in Equation (4.1). Due to Remark 4.11 we may apply Lemma 4.5. Lemma 4.5(i) states that $C_0(F_{\mathrm{e}^{-t}}, B) = \mathfrak{o}(\mathrm{e}^{\delta t})$ as $t \to \infty$ for $B \in \mathcal{K}_F$. The same arguments as in the proof of Theorem 2.29(i) imply that

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-\delta T} C_0(F_{\mathrm{e}^{-T}}, B)\mathrm{d}T = 0 = \nu(B)$$

for $B \in \mathcal{K}_F$ (see Equation (4.36)). Now, take $B = \phi_\kappa O \in \mathcal{E}_F \setminus \mathcal{K}_F$ for some $\kappa \in \Sigma^*$. Lemma 4.5(ii) states that the following holds for all $m \in \mathbb{N}$ and $x \in \Sigma^\infty$.

$$C_0(F_{\mathrm{e}^{-t}}, \phi_\kappa O) = \sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x) C_0(F_{\mathrm{e}^{-t}}, G_{u\omega}) + \mathfrak{o}(\mathrm{e}^{\delta t}) \tag{4.47}$$

as $t \to \infty$. As seen in Equations (4.42) and (4.43) we have for $m \in \mathbb{N}$, $\omega \in \Sigma^m$, $u \in \Sigma^n$ and $x \in \Sigma^\infty$ that

$$f_{0,\omega}(t - \ln \varrho_m - S_n\xi(u\omega x)) \le C_0(F_{\mathrm{e}^{-t}}, G_{u\omega}) \le f_{0,\omega}(t + \ln \varrho_m - S_n\xi(u\omega x)) \tag{4.48}$$

with $f_{0,\omega}(t) := C_0(F_{\mathrm{e}^{-t}}, G_\omega)$ as in Lemma 4.9. Recalling the definition of $N_{0,m,\kappa}$ from Key Lemma 4.2, we conclude from Equations (4.47) and (4.48) that

$$N_{0,m,\kappa}(t - \ln \varrho_m, x) + \mathfrak{o}(\mathrm{e}^{t\delta}) \le C_0(F_{\mathrm{e}^{-t}}, \phi_\kappa O) \le N_{0,m,\kappa}(t + \ln \varrho_m, x) + \mathfrak{o}(\mathrm{e}^{t\delta})$$

as $t \to \infty$. Lemma 4.9 allows an application of Key Lemma 4.2, yielding

$$\begin{aligned}
&\varrho_m^{-\delta} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T\delta} C_0(F_{\mathrm{e}^{-T}}, G_\omega)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \cdot h_{-\delta\xi}(\omega x) \\
&\le \liminf_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T\delta} C_0(F_{\mathrm{e}^{-T}}, B)\mathrm{d}T \\
&\le \limsup_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T\delta} C_0(F_{\mathrm{e}^{-T}}, B)\mathrm{d}T \\
&\le \varrho_m^{\delta} \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T\delta} C_0(F_{\mathrm{e}^{-T}}, G_\omega)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \cdot h_{-\delta\xi}(\omega x).
\end{aligned}$$

Recalling that $C_0(F_{\mathrm{e}^{-T}}, G_\omega) = \mathbb{1}_{(-\ln(|G_\omega|/2),\infty)}(T)$ for $\omega \in \Sigma^*$ by Proposition 2.7, we have that

$$\int_{-\infty}^{\infty} \mathrm{e}^{-T\delta} C_0(F_{\mathrm{e}^{-T}}, G_\omega)\mathrm{d}T = 2^{-\delta}|G_\omega|^\delta/\delta.$$

Hence an application of Lemma 4.7 gives

$$\begin{aligned}\limsup_{m\to\infty}\sum_{\omega\in\Sigma^m}|G_\omega|^\delta 2^{-\delta}\frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{H_{\mu_{-\delta\xi}}} &\leq \liminf_{t\to\infty} t^{-1}\int_0^t \mathrm{e}^{-\delta T}C_0(F_{\mathrm{e}^{-T}},B)\mathrm{d}T\\ &\leq \limsup_{t\to\infty} t^{-1}\int_0^t \mathrm{e}^{-\delta T}C_0(F_{\mathrm{e}^{-T}},B)\mathrm{d}T\\ &\leq \liminf_{m\to\infty}\sum_{\omega\in\Sigma^m}|G_\omega|^\delta 2^{-\delta}\frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{H_{\mu_{-\delta\xi}}}.\end{aligned}$$

Together with Equations (4.34) and (4.41) the assertion now follows from Lemma 4.6, since it was shown in the end of the proof of Theorem 2.31(ii) that $\mathrm{e}^{-\delta t}C_0^{\mathrm{var}}(F_{\mathrm{e}^{-t}})$ is uniformly bounded for $t \in (1,\infty)$. □

4.2.3 Proof of Theorem 2.33

Items (i) and (ii) of Theorem 2.33 follow from the respective items in Theorem 2.31. Item (iii) is the key part of Theorem 2.33. For its proof we use the following lemma.

Lemma 4.12. *Assume that we are in the situation of Theorem 2.31 and that ξ is lattice. Let $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ denote functions satisfying $\xi - \zeta = \psi - \psi\circ\sigma$ and $\zeta(\Sigma^\infty) \subseteq a\mathbb{Z}$, where $a > 0$ is maximal with this property. For $x \in \Sigma^\infty$ define the function $\eta\colon \mathbb{R}\to\mathbb{R}$ by*

$$\eta(t) := \int_{\Sigma^\infty} \mathrm{e}^{-a\delta\lfloor a^{-1}(\psi(y)-t)\rfloor}\mathrm{d}\nu_{-\delta\zeta}(y)\cdot \mathrm{e}^{-\delta t}.$$

Then the following are equivalent.

(i) $\lim_{t\to\infty}\eta(t)$ *exists.*

(ii) η *is constant.*

(iii) $\sum_{n\in\mathbb{Z}}\mathrm{e}^{-a\delta n}\nu_{-\delta\zeta}\circ\psi^{-1}([na,na+t)) = \frac{\mathrm{e}^{\delta t}-1}{\mathrm{e}^{at}-1}\sum_{n\in\mathbb{Z}}\mathrm{e}^{-a\delta n}\nu_{-\delta\zeta}\circ\psi^{-1}([na,(n+1)a))$

for all $t \in [0,a)$.

Proof. Clearly, η is periodic with period a. Therefore, (i) and (ii) are equivalent. Further,

$$\begin{aligned}\eta(t) &= \int_{\Sigma^\infty} \mathrm{e}^{-a\delta\lfloor a^{-1}(\psi(y)-t)\rfloor} \mathrm{d}\nu_{-\delta\zeta}(y) \cdot \mathrm{e}^{-\delta t} \\ &= \sum_{n\in\mathbb{Z}} \int_{na}^{(n+1)a} \mathrm{e}^{-a\delta\lfloor a^{-1}(y-t)\rfloor} \mathrm{d}\nu_{-\delta\zeta} \circ \psi^{-1}(y) \cdot \mathrm{e}^{-\delta t} \\ &= \sum_{n\in\mathbb{Z}} \mathrm{e}^{-a\delta n} \Big(\nu_{-\delta\zeta} \circ \psi^{-1}([na, na + a\{a^{-1}t\}]) \cdot (1 - \mathrm{e}^{-a\delta}) \\ &\qquad + \nu_{-\delta\zeta} \circ \psi^{-1}([na, (n+1)a)) \mathrm{e}^{-a\delta} \Big) \mathrm{e}^{a\delta\{-a^{-1}t\}}.\end{aligned}$$

Thus, η is constant if and only if there exists a $c \in \mathbb{R}$ such that for all $t \in [0, a)$ we have

$$\begin{aligned}&\sum_{n\in\mathbb{Z}} \mathrm{e}^{-a\delta n} \nu_{-\delta\zeta} \circ \psi^{-1}([na, na + t)) \\ &\quad = \left(\mathrm{e}^{\delta t} c - \sum_{n\in\mathbb{Z}} \mathrm{e}^{-a\delta n} \nu_{-\delta\zeta} \circ \psi^{-1}([na, (n+1)a)) \mathrm{e}^{-a\delta} \right) (1 - \mathrm{e}^{-a\delta})^{-1}.\end{aligned}$$

Taking the limit as $t \to a$ we obtain that necessarily

$$c = \mathrm{e}^{-a\delta} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-a\delta n} \nu_{-\delta\zeta} \circ \psi^{-1}([na, (n+1)a))$$

which proves the statement. □

Proof of Theorem 2.33. Items (i) and (ii) are immediate consequences of the respective items in Theorem 2.29. Thus, it remains to show Item (iii). For this, we are going to apply Theorem 3.14 and therefore start by considering the 0-th fractal curvature measure. Without loss of generality, we assume that F possesses exactly one primary gap, which we denote by G. Recall that $[\varnothing] = \Sigma^\infty$ from Definition 3.1. In Equations (4.44) and (4.45) we have seen that

$$C_0(F_{\mathrm{e}^{-t}}, O) \leq N_{0,m,\varnothing}(t + \ln \varrho_m) + \mathfrak{o}(\mathrm{e}^{t\delta}) \quad \text{and} \tag{4.49}$$

$$C_0(F_{\mathrm{e}^{-t}}, O) \geq N_{0,m,\varnothing}(t - \ln \varrho_m) + \mathfrak{o}(\mathrm{e}^{t\delta}) \tag{4.50}$$

as $t \to \infty$, where $f_{0,\omega} \colon \mathbb{R} \to \mathbb{R}$ is defined by $f_{0,\omega}(t) := C_0(F_{\mathrm{e}^{-t}}, G_\omega)$. Lemma 4.9 allows us to apply Key Lemma 4.2. Since ξ is lattice by assumption, there exist $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ such that $\xi - \zeta = \psi - \psi \circ \sigma$ and such that the range of ζ is contained in a discrete subgroup of $\mathbb{R}$. We let $a > 0$ denote the maximal real number for which $\zeta(\Sigma^\infty) \subseteq a\mathbb{Z}$. Then Key

Lemma 4.2(iii) yields

$$N_{0,m,\varnothing}(t,x) \sim^{\mathbb{R}} a \sum_{\omega\in\Sigma^m} \mathrm{e}^{a\delta\left\lfloor\frac{t+\psi(\omega x)}{a}\right\rfloor} \cdot \frac{h_{-\delta\zeta}(\omega x)}{\int \zeta \mathrm{d}\mu_{-\delta\zeta}} \cdot \underbrace{\int_{\Sigma^\infty} \sum_{l=-\infty}^{\infty} \mathrm{e}^{-a\delta l} f_{0,\omega}\left(al + a\left\{\frac{t+\psi(\omega x)}{a}\right\} - \psi(y)\right) \mathrm{d}\nu_{-\delta\zeta}(y)}_{=:A_{0,\omega}(x)} \tag{4.51}$$

as $t\to\infty$. The equality $C_0(F_{\mathrm{e}^{-t}}, G_\omega) = \mathbb{1}_{(-\ln(|G_\omega|/2),\infty)}(t)$ implies that

$$f_{0,\omega}\left(al + a\left\{\frac{t+\psi(\omega x)}{a}\right\} - \psi(y)\right) = \mathbb{1}_{[l^*(y),\infty)}(l),$$

where

$$l^*(y) := \left\lfloor \frac{-\ln(|G_\omega|/2)+\psi(y)}{a} - \left\{\frac{t+\psi(\omega x)}{a}\right\}\right\rfloor + 1.$$

Assuming that m is large enough so that $l^*(y)$ is positive, we hence obtain

$$\begin{aligned} A_{0,\omega}(x) &= \int_{\Sigma^\infty} \sum_{l=l^*(y)}^{\infty} \mathrm{e}^{-a\delta l} \mathrm{d}\nu_{-\delta\zeta}(y) \\ &= \int_{\Sigma^\infty} \frac{\mathrm{e}^{-a\delta l^*(y)}}{1-\mathrm{e}^{-a\delta}} \mathrm{d}\nu_{-\delta\zeta}(y) \\ &= (\mathrm{e}^{a\delta}-1)^{-1} \int_{\Sigma^\infty} \mathrm{e}^{-a\delta\left\lfloor\frac{-\ln(|G_\omega|/2)+\psi(y)}{a} - \left\{\frac{t+\psi(\omega x)}{a}\right\}\right\rfloor} \mathrm{d}\nu_{-\delta\zeta}(y). \end{aligned}$$

Thus, with η as defined in Lemma 4.12 we have

$$\begin{aligned} &N_{0,m,\varnothing}(t,x) \\ &\sim^{\mathbb{R}} a \sum_{\omega\in\Sigma^m} \frac{h_{-\delta\zeta}(\omega x)}{\int \zeta \mathrm{d}\mu_{-\delta\zeta}} (\mathrm{e}^{a\delta}-1)^{-1} \int_{\Sigma^\infty} \mathrm{e}^{-a\delta\left\lfloor\frac{-\ln(|G_\omega|/2)+\psi(y)-t-\psi(\omega x)}{a}\right\rfloor} \mathrm{d}\nu_{-\delta\zeta}(y) \\ &= a(\mathrm{e}^{a\delta}-1)^{-1} \sum_{\omega\in\Sigma^m} \frac{h_{-\delta\zeta}(\omega x)}{\int \zeta \mathrm{d}\mu_{-\delta\zeta}} \eta(t+\ln(|G_\omega|/2)+\psi(\omega x)) \cdot \mathrm{e}^{\delta t} \left(\frac{|G_\omega|}{2}\right)^{\delta} \mathrm{e}^{\delta\psi(\omega x)}. \end{aligned}$$

The prerequisites of Theorem 2.33(iii) in tandem with Lemma 4.12 imply that $\eta(t) = \eta(0)$ for all $t \in \mathbb{R}$. Using that $h_{-\delta\xi} = \mathrm{e}^{\delta\psi} \cdot h_{-\delta\zeta}$ and that $C_0(F_{\mathrm{e}^{-t}}) = C_0(F_{\mathrm{e}^{-t}}, O) + 1$ it thus follows from Equations (4.49) and (4.50) that for all $m \in \mathbb{N}$

$$\begin{aligned} &a(\mathrm{e}^{a\delta}-1)^{-1} \sum_{\omega\in\Sigma^m} \frac{h_{-\delta\xi}(\omega x)}{\int \zeta \mathrm{d}\mu_{-\delta\zeta}} \eta(0) \left(\frac{|G_\omega|}{2}\right)^{\delta} \varrho_m^{-\delta} \\ &\le \liminf_{t\to\infty} \mathrm{e}^{-\delta t} C_0(F_{\mathrm{e}^{-t}}) \\ &\le \limsup_{t\to\infty} \mathrm{e}^{-\delta t} C_0(F_{\mathrm{e}^{-t}}) \\ &\le a(\mathrm{e}^{a\delta}-1)^{-1} \sum_{\omega\in\Sigma^m} \frac{h_{-\delta\xi}(\omega x)}{\int \zeta \mathrm{d}\mu_{-\delta\zeta}} \eta(0) \left(\frac{|G_\omega|}{2}\right)^{\delta} \varrho_m^{\delta}. \end{aligned}$$

An application of Lemma 4.7 shows that $\lim_{t\to\infty} \mathrm{e}^{-\delta t} C_0(F_{\mathrm{e}^{-t}})$ exists. Theorem 3.14 implies that $\lim_{t\to\infty} \mathrm{e}^{-t(\delta-1)} C_1(F_{\mathrm{e}^{-t}})$ also exists, showing the assertion. □

4.3 Self-Similar Sets – Proofs of Theorems 2.37 and 2.38

Also in the self-similar situation, we want to apply Key Lemma 4.2. Therefore, we firstly show that its prerequisites are satisfied for the setting of Section 2.4.2.

Lemma 4.13. *Assume that we are in the situation of Theorem 2.37. Set*

$$U := \{t \in \mathbb{R} \mid \mathrm{e}^{-t} \textit{ is a regular distance for } F\},$$

$U_k := U$ *for* $k \in \{0, \ldots, d-2\}$ *if* $d \geq 4$ *and* $U_k := \mathbb{R}$ *else. Then for* $k \in \{0, \ldots, d\}$ *and* $\omega \in \Sigma^*$ *the functions* $f_{k,\omega} \colon U_k \to \mathbb{R}$ *given by*

$$f_{k,\omega}(t) := \sum_{i=1}^{Q} C_k(F_{\mathrm{e}^{-t}}, G_\omega^i),$$

satisfy Items (A) to (G) of Key Lemma 4.2.

Proof. Without loss of generality we assume that F possesses exactly one primary gap which we denote by G. The case $k = d$ has been treated in the more general situation of self-conformal sets in Lemma 4.8. Thus, we only consider the case $k < d$ here.

ad (A): For $d \leq 3$ it has been shown in [Fu85] that Lebesgue-almost all distances are regular, which implies $\lambda_1(\mathbb{R} \setminus U) = 0$. For $d \geq 4$ and $k \in \{d-1, d\}$, the functions $t \mapsto C_k(F_{\mathrm{e}^{-t}}, G_\omega)$ are defined everywhere (see Remark 2.8) for $\omega \in \Sigma^*$. For $d \geq 4$ and $k \leq d-2$ the assumptions of Theorem 2.37 imply $\lambda_1(\mathbb{R} \setminus U) = 0$. The property that for every $t \in U$ there exists an $\varepsilon_0 > 0$ such that $t - \varepsilon \in U$ for every $0 \leq \varepsilon \leq \varepsilon_0$ is proven in Proposition 1 of [RZ03].

ad (B): See Lemma 2.3.4 in [Zäh11].

ad (C): Let r_ω denote the contraction ratio of ϕ_ω (that is $r_\omega := |\phi'_\omega(x)|$ for an arbitrary $x \in X$) and suppose that $t \in U$ is such that $t + \ln r_\omega \in U$. By definition G_ω is contained in the interior of $(\phi_\omega \overline{O})_{\mathrm{e}^{-t}}$ for all $t \in \mathbb{R}$. Since $F_{\mathrm{e}^{-t}} \cap G_\omega = (\phi_\omega F)_{\mathrm{e}^{-t}} \cap G_\omega$ for all $t \in \mathbb{R}$ we know by the locality and homogeneity properties of curvature measures (see Proposition 2.7(v),(vi)) that

$$C_k(F_{\mathrm{e}^{-t}}, G_\omega) = C_k((\phi_\omega F)_{\mathrm{e}^{-t}}, G_\omega) = r_\omega^k C_k(F_{\mathrm{e}^{-t}/r_\omega}, G)$$

for all $t \in U_k$ for which also $t + \ln r_\omega \in U_k$. Thus,

$$f_{k,\omega}(t) = r_\omega^k C_k(F_{\mathrm{e}^{-t}/r_\omega}, G)$$

for such t. If $\mathrm{e}^{-t}/r_\omega \geq \mathrm{diam}(G)/2$, then $F_{\mathrm{e}^{-t}/r_\omega} = \overline{O}_{\mathrm{e}^{-t}/r_\omega}$ and, as $C_k(F_{\mathrm{e}^{-t}/r_\omega}, \cdot)$ is concentrated on the boundary of $F_{\mathrm{e}^{-t}/r_\omega}$ for $k < d$ (see Proposition 2.7(iv)), we conclude $C_k(F_{\mathrm{e}^{-t}/r_\omega}, G) = 0$. Since, by assumption, $\mathrm{e}^{-t(\delta_I - k)} C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}}, G)$ is uniformly bounded by some constant $\mathfrak{C} > 0$ we thus conclude (for $k < d$) that

$$\begin{aligned}\int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} |f_{k,\omega}(T)| \mathrm{d}T &\leq \int_{-\ln(r_\omega \mathrm{diam}(G)/2)}^{\infty} \mathrm{e}^{-T(\delta-k)} r_\omega^k C_k^{\mathrm{var}}(F_{\mathrm{e}^{-T}/r_\omega}, G) \mathrm{d}T \\ &= \int_{-\ln(\mathrm{diam}(G)/2)}^{\infty} \mathrm{e}^{-T(\delta-k)} r_\omega^\delta C_k^{\mathrm{var}}(F_{\mathrm{e}^{-T}}, G) \mathrm{d}T \\ &\leq r_\omega^\delta \int_{-\ln(\mathrm{diam}(G)/2)}^{\infty} \mathfrak{C} \mathrm{e}^{-T(\delta-\delta_I)} \mathrm{d}T \\ &= \frac{\mathfrak{C} r_\omega^\delta}{\delta - \delta_I} \left(\frac{\mathrm{diam}(G)}{2} \right)^{\delta-\delta_I} < \infty.\end{aligned}$$

ad (E): As we only consider the case $k < d$ we know by Proposition 2.7(iv) that $C_k(F_{\mathrm{e}^{-t}}, \cdot)$ is concentrated on the boundary of $F_{\mathrm{e}^{-t}}$. Thus $C_k(F_{\mathrm{e}^{-t}}, G_\omega) = 0$ for $\mathrm{e}^{-t} \geq \mathrm{diam}(G_\omega)/2$. Hence Item (E)a obtains with $t^* := -\ln(g_m/2)$, where $g_m := \max_{\omega \in \Sigma^m} \mathrm{diam}(G_\omega)$.

ad (D): The statement of (D) follows from the observation that

$$N_{k,m}^{\mathrm{abs}}(t, x) := \sum_{\omega \in \Sigma^m} \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \mathrm{e}^{-kS_n\xi(u\omega x)} |f_{k,\omega}(t - S_n\xi(u\omega x))|$$

is a finite sum for every t in the domain of definition of $N_{k,m}^{\mathrm{abs}}(\cdot, x)$. This is the case, since $f_{k,\omega}(t - S_n\xi(u\omega x)) = 0$, whenever $t - S_n\xi(u\omega x) \leq t^*$ (see ad (E)) and there are only finitely many $u \in \Sigma^*$ for which $t - S_n\xi(u\omega x) > t^*$ for a given t and a given $\omega \in \Sigma^m$.

ad (F): The proof of this part will be carried out in analogy to the proof of Lemma 4.8(F). Therefore, we try to keep it short here. For an $\omega \in \Sigma^m$, $x \in \Sigma^\infty$ and $t \in V$ define

$$N_k^\omega(t, x) := \sum_{n=0}^{\infty} \sum_{u \in \Sigma^n} \mathrm{e}^{-kS_n\xi(ux)} C_k(F_{\mathrm{e}^{-t+S_n\xi(ux)}}, G_\omega).$$

Then N_k^ω satisfies a type of renewal equation:

$$\sum_{y \colon \sigma y = x} N_k^\omega(t - \xi(y), y) \mathrm{e}^{-k\xi(y)} = N_k^\omega(t, x) - C_k(F_{\mathrm{e}^{-t}}, G_\omega).$$

Thus, the function M_k^ω given by $M_k^\omega(t,x) := \mathrm{e}^{-t(\delta-k)} N_k^\omega(t,x)/h_{-\delta\xi}(x)$ satisfies

$$\begin{aligned} &M_k^\omega(t,x) \\ &= \mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, G_\omega)/h_{-\delta\xi}(x) + \sum_{y\colon \sigma y = x} \mathrm{e}^{-\delta\xi(y)} M_k^\omega(t-\xi(y),y) \cdot \frac{h_{-\delta\xi}(y)}{h_{-\delta\xi}(x)}. \end{aligned}$$

We now show that M_k^ω is uniformly bounded for $t \in V$ and $x \in \Sigma^\infty$. Set $\underline{h}_{-\delta\xi} := \inf_{x\in\Sigma^\infty} h_{-\delta\xi}(x)$ and recall that $\underline{h}_{-\delta\xi} > 0$. Further, define

$$\overline{M}_k^\omega(t) := \sup_{\substack{t'\in V,\, t'<t \\ x\in\Sigma^\infty}} M_k^\omega(t',x).$$

Since $N_k^\omega(t,x) = 0$ for $t < -\ln(\mathrm{diam} G_\omega)$ we have that

$$\overline{M}_k^\omega(-\ln(\mathrm{diam} G_\omega)) = 0.$$

The rest of the proof now follows through in exactly the same way as in Lemma 4.8(F).

ad (G): The arguments in ad (E) give that $N_{k,m}^{\mathrm{abs}}(t,x) = 0$ for all $t < t^*$ with the there defined t^*.

□

4.3.1 Proof of Theorem 2.37

Proof of Theorem 2.37. We simultaneously prove Items (i) and (ii). We point out that the main ideas of the proof are presented as Steps (I) to (V) in the beginning of Section 4.1. Set

$$U := \{t \in \mathbb{R} \mid \mathrm{e}^{-t} \text{ is a regular distance for } F\},$$

$U_k := U$ for $k \in \{0,\dots,d-2\}$ if $d \geq 4$ and $U_k := \mathbb{R}$ else. For $\omega \in \Sigma^*$ we let r_ω denote the contraction ratio of ϕ_ω, that is $r_\omega := |\phi_\omega'(x)|$ for an arbitrary $x \in X$. We start by showing that $\mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, B)$ (resp. $t^{-1}\int_0^t \mathrm{e}^{-T(\delta-k)} C_k(F_{\mathrm{e}^{-T}}, B)\mathrm{d}T$) converges essentially to

$$\int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} \sum_{i=1}^{Q} C_k(F_{\mathrm{e}^{-T}}, G^i)\mathrm{d}T \cdot \frac{\nu(B)}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \tag{4.52}$$

for every $B \in \mathcal{E}_F$, where the essential limit as $t \to \infty$ is taken over U_k. Then we apply Lemma 4.6.

By (COND 4'), there exists some $\mathfrak{C} \in \mathbb{R}$ for which $C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}}, G^i)\mathrm{e}^{-t(\delta_I-k)} \leq \mathfrak{C}$ for all $t \in U_k$ and $i \in \{1,\dots,Q\}$. By using the Jordan decompositions of the signed Borel measures $C_k(F_{\mathrm{e}^{-t}}, \cdot)$ and $C_k(F_{\mathrm{e}^{-t}/r_\omega}, \cdot)$ one can show that this implies that

$$\mathrm{e}^{-t(\delta_I-k)} C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}}, G_\omega^i) \leq r_\omega^{\delta_I} \cdot \mathfrak{C} \tag{4.53}$$

holds for all $t \in U_k$, $\omega \in \Sigma^*$ and $i \in \{1,\dots,Q\}$. Thus, we can apply Lemma 4.5. For $B \in \mathcal{K}_F$ Lemma 4.5(i) yields

$$\operatorname*{ess-lim}_{t\to\infty,\, t\in U_k} \mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, B) = 0 = \nu(B). \tag{4.54}$$

Moreover, $C_k(F_{\mathrm{e}^{-t}}, B) = \mathfrak{o}(\mathrm{e}^{t(\delta-k)})$ implies that for all $\varepsilon > 0$ there exists a $\widetilde{T} \in \mathbb{R}$ such that for all $t \geq \widetilde{T}$ we have $|\mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, B)| < \varepsilon$. Thus,

$$\begin{aligned}
&\lim_{t\to\infty} \left| t^{-1} \int_0^t \mathrm{e}^{-T(\delta-k)} C_k(F_{\mathrm{e}^{-T}}, B) \mathrm{d}T \right| \\
&\quad \leq \lim_{t\to\infty} \left(\left| t^{-1} \int_0^{\widetilde{T}} \mathrm{e}^{-T(\delta-k)} C_k(F_{\mathrm{e}^{-T}}, B) \mathrm{d}T \right| + t^{-1} \int_{\widetilde{T}}^t \varepsilon \mathrm{d}T \right) = \varepsilon,
\end{aligned}$$

which implies that

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-k)} C_k(F_{\mathrm{e}^{-T}}, B) \mathrm{d}T = 0 = \nu(B).$$

For $B = \phi_\kappa O \in \mathcal{E}_F \setminus \mathcal{K}_F$, where $\kappa \in \Sigma^*$, we use the locality and the homogeneity properties of the curvature measures (see Proposition 2.7(v) and (vi)) to obtain that

$$C_k(F_{\mathrm{e}^{-t}}, G^i_{u\omega}) = C_k((\phi_u F)_{\mathrm{e}^{-t}}, \phi_u(G^i_\omega)) = r_u^k C_k(F_{\mathrm{e}^{-t}/r_u}, G^i_\omega)$$

holds for all $u, \omega \in \Sigma^*$, $i \in \{1,\dots,Q\}$ and $t \in U_k$ for which $t + \ln(r_u) \in U_k$. Note that $r_u = \mathrm{e}^{-S_n\xi(u\omega x)}$ for arbitrary $\omega \in \Sigma^m$, $x \in \Sigma^\infty$ and $u \in \Sigma^n$. Together with Lemma 4.5(ii) this shows for an arbitrary $m \in \mathbb{N}$ that

$$\begin{aligned}
C_k(F_{\mathrm{e}^{-t}}, \phi_\kappa O) &= \sum_{\omega\in\Sigma^m} \sum_{n=0}^\infty \sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x) C_k\left(F_{\mathrm{e}^{-t}}, \bigcup_{i=1}^Q G^i_{u\omega}\right) + \mathfrak{o}(\mathrm{e}^{t(\delta-k)}) \\
&= \sum_{\omega\in\Sigma^m} \sum_{n=0}^\infty \sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa]}(u\omega x) \mathrm{e}^{-kS_n\xi(u\omega x)} \sum_{i=1}^Q C_k(F_{\mathrm{e}^{-t+S_n\xi(u\omega x)}}, G^i_\omega) + \mathfrak{o}(\mathrm{e}^{t(\delta-k)}) \\
&= N_{k,m,\kappa}(t,x) + \mathfrak{o}(\mathrm{e}^{t(\delta-k)})
\end{aligned} \tag{4.55}$$

as $t \to \infty$, where $N_{k,m,\kappa}$ is defined as in Key Lemma 4.2 with $f_{k,\omega}$ defined as in Lemma 4.13. Due to Lemma 4.13 we can apply Key Lemma 4.2(i) and obtain

$$\begin{aligned}
&\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-k)} C_k(F_{\mathrm{e}^{-T}}, \phi_\kappa O) \mathrm{d}T \\
&\quad = \sum_{\omega\in\Sigma^m} \int_{-\infty}^\infty \mathrm{e}^{-T(\delta-k)} \sum_{i=1}^Q C_k(F_{\mathrm{e}^{-T}}, G^i_\omega) \mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]} \mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} h_{-\delta\xi}(\omega x),
\end{aligned}$$

since $\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-k)} f(T) \mathrm{d}T = 0$ for any $f(t)$ which is $\mathfrak{o}(\mathrm{e}^{t(\delta-k)})$ (see Equation (4.36)).

In the non-lattice case, Key Lemma 4.2(ii) yields

$$\begin{aligned}
&\operatorname*{ess\text{-}lim}_{t\to\infty,\, t\in U_k} \mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, \phi_\kappa O)\\
&\quad = \sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} \sum_{i=1}^{Q} C_k(F_{\mathrm{e}^{-T}}, G_\omega^i)\mathrm{d}T \cdot \frac{\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi}}{\int \xi \mathrm{d}\mu_{-\delta\xi}} h_{-\delta\xi}(\omega x).
\end{aligned}$$

Note that $C_k(F_{\mathrm{e}^{-t}}, G_\omega^i) = C_k((\phi_\omega F)_{\mathrm{e}^{-t}}, G_\omega^i) = r_\omega^k C_k(F_{\mathrm{e}^{-t}/r_\omega}, G^i)$ for $t \in U_k$ for which $t + \ln r_\omega \in U_k$ implies that

$$\begin{aligned}
\sum_{\omega\in\Sigma^m} \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} \sum_{i=1}^{Q} C_k(F_{\mathrm{e}^{-T}}, G_\omega^i)\mathrm{d}T &= \sum_{\omega\in\Sigma^m} r_\omega^\delta \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} \sum_{i=1}^{Q} C_k(F_{\mathrm{e}^{-T}}, G^i)\mathrm{d}T\\
&= \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} \sum_{i=1}^{Q} C_k(F_{\mathrm{e}^{-T}}, G^i)\mathrm{d}T. \qquad (4.56)
\end{aligned}$$

Using that $h_{-\delta\xi} \equiv 1$, since ξ is the geometric potential function to a self-similar system and that $\int \mathbb{1}_{[\kappa]}\mathrm{d}\nu_{-\delta\xi} = \nu(\phi_\kappa O)$ (see Equation (4.34)), we hence conclude the following.

(i) We always have

$$\lim_{t\to\infty} t^{-1} \int_0^t \mathrm{e}^{-T(\delta-k)} C_k(F_{\mathrm{e}^{-T}}, \phi_\kappa O)\mathrm{d}T = \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} \sum_{i=1}^{Q} C_k(F_{\mathrm{e}^{-T}}, G^i)\mathrm{d}T \cdot \frac{\nu(\phi_\kappa O)}{\int \xi \mathrm{d}\mu_{-\delta\xi}}.$$

(ii) If ξ is non-lattice, then

$$\operatorname*{ess\text{-}lim}_{t\to\infty,\, t\in U_k} \mathrm{e}^{-t(\delta-k)} C_k(F_{\mathrm{e}^{-t}}, \phi_\kappa O) = \int_{-\infty}^{\infty} \mathrm{e}^{-T(\delta-k)} \sum_{i=1}^{Q} C_k(F_{\mathrm{e}^{-T}}, G^i)\mathrm{d}T \cdot \frac{\nu(\phi_\kappa O)}{\int \xi \mathrm{d}\mu_{-\delta\xi}}.$$

Thus, in both cases, we have convergence to the term in Equation (4.52) for all $B \in \mathcal{E}_F$. By Lemma 4.13(C) the term in Equation (4.52) is finite. An application of Lemma 4.6 finishes the proof. Therefore, all that remains to be shown is, that $\mathrm{e}^{-t(\delta-k)} C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}})$ is uniformly bounded for $t \in (1,\infty) \cap U_k$. This is shown in the following.

That $C_d^{\mathrm{var}}(F_{\mathrm{e}^{-t}})\mathrm{e}^{-t(\delta-d)} = \lambda_d(F_{\mathrm{e}^{-t}})\mathrm{e}^{-t(\delta-d)}$ is uniformly bounded for $t \in (1,\infty)$ has been shown in the proof of Theorem 2.29 even for self-conformal subsets of $\mathbb{R}^d$. Therefore, in the following, we assume that $k < d$. From Equation (2.3) we deduce

$$C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}}) = C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}}, X_{\mathrm{e}^{-t}} \setminus X) + C_k^{\mathrm{var}}\Bigg(F_{\mathrm{e}^{-t}}, \bigcup_{\substack{\omega\in\Sigma^* \\ i\in\{1,\ldots,Q\}}} G_\omega^i\Bigg) + C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}}, F). \qquad (4.57)$$

By (COND 2'), the first summand on the right hand side of Equation (4.57) is $\mathfrak{o}(\mathrm{e}^{t(\delta-k)})$ as $t \to \infty$. Also, the third summand on the right hand side of Equation (4.57) is $\mathfrak{o}(\mathrm{e}^{t(\delta-k)})$, since

$C_k(F_{\mathrm{e}^{-t}}, \cdot)$ is concentrated on the boundary of $F_{\mathrm{e}^{-t}}$ (see Proposition 2.7) and $F_{\mathrm{e}^{-t}} \cap F = \varnothing$ for all $t \in \mathbb{R}$. Therefore, it remains to consider the second summand on the right hand side of Equation (4.57).

Since $C_k(F_{\mathrm{e}^{-t}}, \cdot)$ is concentrated on the boundary of $F_{\mathrm{e}^{-t}}$, we have that $C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}}, G_\omega) = 0$, whenever $\mathrm{e}^{-t} \geq \mathrm{diam}(G_\omega)/2 = r_\omega \mathrm{diam}(G)/2 \geq r_{\min}^{n(\omega)} \mathrm{diam}(G)/2$, where $r_{\min} := \min_{i \in \{1,\ldots,N\}} r_i$. Thus, setting

$$A(t) := -(t + \ln(\mathrm{diam} G/2))/\ln r_{\min}$$

the following holds.

$$\begin{aligned}
&\mathrm{e}^{-t(\delta-k)} C_k^{\mathrm{var}}\Bigg(F_{\mathrm{e}^{-t}}, \bigcup_{\substack{\omega\in\Sigma^* \\ i\in\{1,\ldots,Q\}}} G_\omega^i\Bigg) \\
&\quad= \sum_{i=1}^{Q} \sum_{n=0}^{\lfloor A(t)\rfloor} \sum_{\omega\in\Sigma^n} \mathrm{e}^{-t(\delta_I - k)} C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}}, G_\omega^i)\mathrm{e}^{-t(\delta-\delta_I)} \\
&\quad\overset{(4.53)}{\leq} \sum_{i=1}^{Q} \sum_{n=0}^{\lfloor A(t)\rfloor} \sum_{\omega\in\Sigma^n} r_\omega^{\delta_I} \mathfrak{C} \mathrm{e}^{-t(\delta-\delta_I)} \\
&\quad\leq \sum_{i=1}^{Q} \sum_{n=0}^{\lfloor A(t)\rfloor} \underbrace{\sum_{\omega\in\Sigma^n} r_\omega^{\delta}}_{=1} (r_{\min}^n)^{\delta_I-\delta} \mathfrak{C} \mathrm{e}^{-t(\delta-\delta_I)} \\
&\quad= Q\mathfrak{C}\mathrm{e}^{-t(\delta-\delta_I)} \frac{1-(r_{\min}^{\delta_I-\delta})^{\lfloor A(t)\rfloor+1}}{1-r_{\min}^{\delta_I-\delta}} \\
&\quad= Q\mathfrak{C}(r_{\min}^{\delta_I-\delta}-1)^{-1}\left(\left(\frac{\mathrm{diam} G}{2}\right)^{\delta-\delta_I} r_{\min}^{(\delta_I-\delta)(1-\{-\frac{t+\ln(\mathrm{diam}G/2)}{\ln r_{\min}}\})} - \mathrm{e}^{-t(\delta-\delta_I)}\right).
\end{aligned}$$

Hence, $\mathrm{e}^{-t(\delta-k)} C_k^{\mathrm{var}}(F_{\mathrm{e}^{-t}})$ is uniformly bounded for $t \in (1,\infty) \cap U_k$, which allows us to apply Lemma 4.6. □

4.3.2 Proof of Theorem 2.38

Proof of Theorem 2.38. Noting that $C_d(F_{\mathrm{e}^{-t}}, G^i) = \lambda_d(F_{\mathrm{e}^{-t}} \cap G^i) = \lambda_d(G^i)$ for $t \leq -\ln(g^i)$ and every $i \in \{1,\ldots,Q\}$, Item (i) is an immediate consequence of Theorem 2.37(i). For proving Item (ii) we make use of some of the steps in the proof of Theorem 2.37. Firstly, for $B \in \mathcal{K}_F$, we have

$$\lim_{t\to\infty} \mathrm{e}^{-t(\delta-d)} \lambda_d(F_{\mathrm{e}^{-t}}, B) = 0 = \nu(B)$$

by Equation (4.54). Secondly, for $B = \phi_\kappa O \in \mathcal{E}_F \setminus \mathcal{K}_F$, where $\kappa \in \Sigma^*$, we have that

$$\lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\kappa O) = N_{d,m,\kappa}(t,x) + \mathfrak{o}(\mathrm{e}^{t(\delta-d)}) \tag{4.58}$$

holds for an arbitrary $m \in \mathbb{N}$ by Equation (4.55). Here, $N_{d,m,\kappa}$ is defined as in Key Lemma 4.2 with $f_{d,\omega}\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f_{d,\omega}(t) := \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-t}} \cap G_\omega^i). \tag{4.59}$$

In Lemma 4.13 we have shown that we may apply Key Lemma 4.2. By assumption, the geometric potential function ξ is lattice. Therefore, there exist $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ satisfying $\xi - \zeta = \psi - \psi \circ \sigma$, where the range of ζ is contained in a discrete subgroup of $\mathbb{R}$. It follows that the range of ξ is contained in a discrete subgroup of $\mathbb{R}$ because of the following. Firstly, ξ is constant on cylinder sets of length one, as it is associated with a cIFS consisting of similarities and secondly, every cylinder set of length one contains a word $x \in \Sigma^\infty$ satisfying $x = \sigma x$. Thus, the equation $\xi - \zeta = \psi - \psi \circ \sigma$ is satisfied for $\zeta = \xi$ and $\psi \equiv 0$. We let $a > 0$ denote the maximal real number for which $\xi(\Sigma^\infty) \subseteq a\mathbb{Z}$ holds. Further, we remark that $h_{-\delta\xi} \equiv 1$. Then, combining Key Lemma 4.2(iii) with Equations (4.58) and (4.59) we obtain the following.

$$\begin{aligned}
&\mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\kappa O)\\
&\sim^{\mathbb{R}} \mathrm{e}^{-t(\delta-d)} a \sum_{\omega\in\Sigma^m} \mathrm{e}^{a\lfloor a^{-1}t\rfloor(\delta-d)} \frac{\nu_{-\delta\xi}([\kappa])}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \sum_{l=-\infty}^{\infty} \mathrm{e}^{-al(\delta-d)} \sum_{i=1}^{Q} \lambda_d(F_{\mathrm{e}^{-al-a\{a^{-1}t\}}} \cap G_\omega^i) + \mathfrak{o}(1).
\end{aligned}$$

For $i \in \{1, \ldots, Q\}$ define $L_\omega^i(t) := -a^{-1}(\ln(g^i) + \ln r_\omega) - \{a^{-1}t\}$ and assume that m is large enough so that $L_\omega^i(t) > 0$ for all $i \in \{1, \ldots, Q\}$, $\omega \in \Sigma^m$ and $t \in \mathbb{R}$. Then using that $F_{\mathrm{e}^{-t}} \supset G^i$ for $t \leq -\ln(g^i)$ we have

$$\begin{aligned}
&\sum_{l=-\infty}^{\infty} \mathrm{e}^{-al(\delta-d)}\lambda_d(F_{\mathrm{e}^{-al-a\{a^{-1}t\}}} \cap G_\omega^i) = \sum_{l=-\infty}^{\infty} \mathrm{e}^{-al(\delta-d)}\lambda_d((\phi_\omega F)_{\mathrm{e}^{-al-a\{a^{-1}t\}}} \cap G_\omega^i)\\
&= r_\omega^d \sum_{l=-\infty}^{\infty} \mathrm{e}^{-al(\delta-d)}\lambda_d(F_{\mathrm{e}^{-al-a\{a^{-1}t\}-\ln r_\omega}} \cap G^i)\\
&= r_\omega^d \left(\sum_{l=-\infty}^{\lfloor L_\omega^i(t)\rfloor} \mathrm{e}^{-al(\delta-d)}\lambda_d(G^i) + \sum_{l=\lfloor L_\omega^i(t)\rfloor+1}^{\infty} \mathrm{e}^{-al(\delta-d)} \sum_{j=0}^{d-1} \eta_j(G^i)\mathrm{e}^{-(al+a\{a^{-1}t\}+\ln r_\omega)(d-j)} \right)\\
&= -r_\omega^d\lambda_d(G^i)\frac{\mathrm{e}^{-a\lfloor L_\omega^i(t)\rfloor(\delta-d)}}{\mathrm{e}^{a(\delta-d)}-1} + \sum_{j=0}^{d-1} r_\omega^j \eta_j(G^i)\frac{\mathrm{e}^{-a\lfloor L_\omega^i(t)\rfloor(\delta-j)}}{\mathrm{e}^{a(\delta-j)}-1}\mathrm{e}^{-a\{a^{-1}t\}(d-j)}\\
&= \sum_{j=0}^{d} r_\omega^j \eta_j(G^i)\frac{\mathrm{e}^{-a\lfloor L_\omega^i(t)\rfloor(\delta-j)}}{\mathrm{e}^{a(\delta-j)}-1}\mathrm{e}^{-a\{a^{-1}t\}(d-j)}.
\end{aligned}$$

Now, recall that $a^{-1}\ln r_\omega \in \mathbb{Z}$ for all $\omega \in \Sigma^*$. Therefore, $\lfloor L_\omega^i(t)\rfloor = -a^{-1}\ln r_\omega + \lfloor -a^{-1}\ln(g^i) - \{a^{-1}t\}\rfloor$. It follows that

$$\begin{aligned}
&\mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\kappa O)\\
&\sim^{\mathbb{R}} \mathrm{e}^{-a\{a^{-1}t\}(\delta-d)} a \frac{\nu_{-\delta\xi}([\kappa])}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \sum_{\omega\in\Sigma^m}\sum_{j=0}^{d}\sum_{i=1}^{Q} r_\omega^j \eta_j(G^i)\frac{\mathrm{e}^{-a\lfloor L_\omega^i(t)\rfloor(\delta-j)}}{\mathrm{e}^{a(\delta-j)}-1}\mathrm{e}^{-a\{a^{-1}t\}(d-j)} + \mathfrak{o}(1)\\
&= \frac{a\nu_{-\delta\xi}([\kappa])}{\int \xi \mathrm{d}\mu_{-\delta\xi}} \underbrace{\sum_{\omega\in\Sigma^m} r_\omega^\delta}_{=1} \sum_{j=0}^{d}\sum_{i=1}^{Q} \frac{\eta_j(G^i)}{\mathrm{e}^{a(\delta-j)}-1}\mathrm{e}^{-a\lfloor -a^{-1}\ln(g^i)-\{a^{-1}t\}\rfloor(\delta-j)}\mathrm{e}^{-a\{a^{-1}t\}(\delta-j)} + \mathfrak{o}(1)\\
&=: q(t)
\end{aligned}$$

This is a periodic function in t with period a, which is non-constant because of the following. The function $\chi_j\colon [0,a) \to \mathbb{R}$,

$$\chi(t) := \sum_{i=1}^{Q} \frac{\eta_j(G^i)}{\mathrm{e}^{a(\delta-j)}-1}\mathrm{e}^{-a\lfloor -a^{-1}\ln(g^i)-\{a^{-1}t\}\rfloor(\delta-j)}$$

is piecewise constant with at most Q points of discontinuity in $[0,a)$ for each $j \in \{1,\ldots,d\}$. Moreover, the points of discontinuity coincide for all $j \in \{0,\ldots,d\}$. Thus, there exists a non-empty interval $I \subseteq [0,a)$ on which each χ_j is constant. On this interval I, the function q can be viewed as a polynomial in $\mathrm{e}^{-a\{a^{-1}t\}} = \mathrm{e}^{-t}$. Since the monomials are linearly independent, it follows that q is non-constant, as we can assume without loss of generality that there exist i,j such that $\eta_j(G^i) \neq 0$. Thus, $\mathrm{e}^{-t(\delta-d)}\lambda_d(F_{\mathrm{e}^{-t}} \cap \phi_\kappa O)$ as a function in t is asymptotic to a periodic non-constant function. This shows the statement. □

4.4 $\mathcal{C}^{1+\alpha}$-Images of Self-Similar Sets – Proofs of Theorems 2.39, 2.43 and 2.46 and Corollary 2.44

In this section we provide the proofs of the results concerning $\mathcal{C}^{1+\alpha}$-diffeomorphic images of self-similar sets. We start with proving Theorem 2.43 and Corollary 2.44. Since $\mathcal{C}^{1+\alpha}$-diffeomorphic images of self-similar sets are special self-conformal sets, Items (i) and (ii) of Theorem 2.43 and Corollary 2.44 follow from the respective items in Theorem 2.31 by using the special structure of $\mathcal{C}^{1+\alpha}$-diffeomorphic images in the following way.

We let F denote an image of a self-similar set $K \subset \mathbb{R}$ under a conformal map $g \in \mathcal{C}^{1+\alpha}(\mathcal{U})$, where $\alpha > 0$ and $\mathcal{U}$ is a convex open neighbourhood of K. In both Theorem 2.43 and Corollary 2.44 we assume that $|g'|$ is bounded away from 0 on its domain of definition. Thus, g is bi-Lipschitz and therefore the Minkowski dimension of F coincides with the

Minkowski dimension of K (see for instance Corollary 2.4 of [Fal03]). We denote the common value by δ.

The similarities generating K are denoted by $R_1, \ldots, R_N$ and we set $\phi_i := g \circ R_i \circ g^{-1}$ for each $i \in \Sigma$. Recall from the comment in Setting 2.40 that the maps ϕ_i are not necessarily contractions, but that an iterate $\widetilde{\Phi}$ of the system $\Phi := \{\phi_1, \ldots, \phi_N\}$ consists solely of contractions, that $F := g(K)$ is the invariant set of $\widetilde{\Phi}$ and that F is thus self-conformal. We respectively denote by π_K and π_F the code maps from Σ^∞ to K and F and recall that ν denotes the δ-conformal measure associated with R. If we further let $r_1, \ldots, r_N$ denote the respective similarity ratios of $R_1, \ldots, R_N$, then we have the following list of observations.

(a) ϕ_i is differentiable for every $i \in \Sigma$ with differential

$$\phi_i'(y) = \frac{g'(R_i \circ g^{-1}(y))}{g'(g^{-1}(y))} \cdot r_i,$$

where $y \in Y$ and Y is the non-empty compact interval which each ϕ_i is defined on.

(b) The geometric potential function ξ_K associated with K is given by $\xi_K(\omega) = -\ln r_{\omega_1}$, for $\omega = \omega_1\omega_2 \cdots \in \Sigma^\infty$. The geometric potential function ξ_F associated with F is given by $\xi_F(\omega) = -\ln|g'(g^{-1}(\pi_F\omega))| + \ln|g'(g^{-1}(\pi_F\sigma\omega))| - \ln r_{\omega_1}$. Thus ξ_K is non-lattice, if and only if ξ_F is non-lattice.

(c) The unique σ-invariant Gibbs measures for the potential functions $-\delta\xi_F$ and $-\delta\xi_K$ satisfy $\mu_{-\delta\xi_F} = \mu_{-\delta\xi_K}$.

(d) From Items (b) and (c) we obtain

$$H_{-\delta\xi_F} = \int_{\Sigma^\infty} \xi_F \mathrm{d}\mu_{-\delta\xi_F} = -\sum_{i\in\Sigma} \ln r_i \cdot r_i^\delta = \int_{\Sigma^\infty} \xi_K \mathrm{d}\mu_{-\delta\xi_K} = H_{-\delta\xi_K}.$$

Further, let $\widetilde{G}^1, \ldots, \widetilde{G}^Q$ and $\widetilde{G}^1_\omega, \ldots, \widetilde{G}^Q_\omega$ denote the primary and main gaps of K for $\omega \in \Sigma^*$ and let $G^1, \ldots, G^Q$ and $G^1_\omega, \ldots, G^Q_\omega$ respectively denote the primary and main gaps of F. Then

(e) $G^i_\omega = g(\widetilde{G}^i_\omega)$ for $i \in \{1, \ldots, Q\}$ and $\omega \in \Sigma^*$. Since furthermore $|\widetilde{G}^i_\omega| = r_\omega|\widetilde{G}^i|$, we have

$$\lim_{n\to\infty} \sum_{i=1}^{Q} \sum_{\omega\in\Sigma^n} |G^i_\omega|^\delta = \lim_{n\to\infty} \sum_{i=1}^{Q} \sum_{\omega\in\Sigma^n} \left(r_\omega|\widetilde{G}^i| \cdot |g'(x_\omega)|\right)^\delta = \sum_{i=1}^{Q} |\widetilde{G}^i|^\delta \int_K |g'|^\delta \mathrm{d}\nu,$$

where $x_\omega \in R_\omega X$ is arbitrary for each $\omega \in \Sigma^*$. Note that the above line can be rigorously proven by using the Bounded Distortion Lemma (see Lemma 2.21).

(f) The δ-conformal measure ν_F associated with F and the push-forward measure of the δ-conformal measure ν associated with K are absolutely continuous with Radon-Nikodym derivative

$$\frac{\mathrm{d}\nu_F}{\mathrm{d}\,(g_\star\nu)} = |g' \circ g^{-1}|^\delta \left(\int_K |g'|^\delta \mathrm{d}\nu \right)^{-1}.$$

From these observations we can infer the statements concerning the average and the non-lattice case of Theorem 2.43 and Corollary 2.44. This is done in the following subsections.

4.4.1 Proof of Theorem 2.43

We prove the three parts (i) to (iii) of Theorem 2.43 separately.

Proof of Theorem 2.43(i),(ii). Using the notation that we set up in the beginning of Section 4.4 and Items (a) to (f) from there, an application of Items (i) and (ii) of Theorem 2.31 to Φ and of Theorem 2.39 to R proves Items (i) and (ii) of Theorem 2.43. □

Now, we turn to the lattice case. In order to show the statements on the non-existence, we use the following lemma.

Lemma 4.14. *Let F denote a self-conformal subset of $\mathbb{R}$ associated with the cIFS $\Phi := \{\phi_1, \dots, \phi_N\}$. Let $\delta := \dim_M(F)$ denote the Minkowski dimension of F and let $B \subseteq \mathbb{R}$ denote a Borel set for which $F_{\mathrm{e}^{-t}} \cap B = (F \cap B)_{\mathrm{e}^{-t}}$ for all sufficiently large $t > 0$. Assume that there exists a positive, bounded, periodic and Borel-measurable function $q \colon \mathbb{R}^+ \to \mathbb{R}^+$ which has the following properties.*

(i) q is not equal to an almost everywhere constant function.

(ii) There exist sequences $(a_m)_{m\in\mathbb{N}}$ and $(c_m)_{m\in\mathbb{N}}$, where $a_m, c_m > 0$ for all $m \in \mathbb{N}$ and $a_m \to 1$ as $m \to \infty$ such that the following property is satisfied. For all $\varepsilon > 0$ and $m \in \mathbb{N}$ there exists an $M \in \mathbb{N}$ such that for all $t \geq M$

$$\begin{aligned}(1-\varepsilon)a_m^{-\delta} q(t - \ln a_m) - c_m \mathrm{e}^{-\delta t} &\leq \mathrm{e}^{-\delta t} \lambda_0(\partial F_{\mathrm{e}^{-t}} \cap B) \\ &\leq (1+\varepsilon) a_m^{\delta} q(t + \ln a_m) + c_m \mathrm{e}^{-\delta t}. \qquad (4.60)\end{aligned}$$

Then for $k \in \{0, 1\}$ we have that

$$\underline{C}_k^f(F, B) < \overline{C}_k^f(F, B).$$

Proof. We first cover the case $k = 0$. Since q is positive and not equal to an almost everywhere constant function, there exist $\widetilde{t}_1, \widetilde{t}_2 > 0$ such that $R := q(\widetilde{t}_2)/q(\widetilde{t}_1) > 1$. Choose $m \in \mathbb{N}$ so that $a_m^{2\delta} < \sqrt{R}$ and choose $\varepsilon > 0$ such that $(1+\varepsilon)/(1-\varepsilon) < \sqrt{R}$. Then $\widetilde{R} := (1-\varepsilon)a_m^{-\delta}q(\widetilde{t}_2) - (1+\varepsilon)a_m^{\delta}q(\widetilde{t}_1) > 0$. By Item (ii) we can find an $M \in \mathbb{N}$ for these ε and m such that Equation (4.60) is satisfied for all $t \geq M$. Because of the periodicity of q we can find $t_1, t_2 \geq M$ such that $q(\widetilde{t}_1) = q(t_1 + \ln a_m)$ and $q(\widetilde{t}_2) = q(t_2 - \ln a_m)$. Moreover, we can assume that t_1, t_2 are so large that $c_m \mathrm{e}^{-\delta t_1} + c_m \mathrm{e}^{-\delta t_2} \leq \widetilde{R}/2$. Then

$$\begin{aligned}\mathrm{e}^{-\delta t_1}\lambda_0(\partial F_{\mathrm{e}^{-t_1}} \cap B) &\leq (1+\varepsilon)a_m^{\delta} q(t_1 + \ln a_m) + c_m \mathrm{e}^{-\delta t_1} \\ &\leq (1-\varepsilon)a_m^{-\delta} q(t_2 - \ln a_m) - \widetilde{R}/2 - c_m \mathrm{e}^{-\delta t_2} \\ &< \mathrm{e}^{-\delta t_2}\lambda_0(\partial F_{\mathrm{e}^{-t_2}} \cap B).\end{aligned}$$

Because of the periodicity of q this proves the case $k = 0$.

For $k = 1$ observe that the function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ defined by

$$g(t) := \int_0^\infty q(s+t)\mathrm{e}^{(\delta-1)s}\mathrm{d}s$$

is periodic. Also, g is not a constant function. Since if it was, then $0 = g(0) - g(t)$ for all $t \geq 0$. This would imply $\int_t^\infty q(s)\mathrm{e}^{(\delta-1)s}\mathrm{d}s = \mathrm{e}^{(\delta-1)t}\int_0^\infty q(s)\mathrm{e}^{(\delta-1)s}\mathrm{d}s$ for all $t \geq 0$. Differentiating with respect to t would imply that q itself is constant almost everywhere which is a contradiction. Using that $F_{\mathrm{e}^{-t}} \cap B = (F \cap B)_{\mathrm{e}^{-t}}$ for sufficiently large $t > 0$ and Stachó's Theorem (Proposition 3.15), we obtain for sufficiently large $t \geq 0$ that

$$\begin{aligned}\mathrm{e}^{-t(\delta-1)}\lambda_1(F_{\mathrm{e}^{-t}} \cap B) &= \mathrm{e}^{-t(\delta-1)}\int_t^\infty \lambda_0(\partial F_{\mathrm{e}^{-s}} \cap B)\mathrm{e}^{-s}\mathrm{d}s \\ &\leq \mathrm{e}^{-t(\delta-1)}(1+\varepsilon)a_m^{\delta}\int_t^\infty q(s+\ln a_m)\mathrm{e}^{s(\delta-1)}\mathrm{d}s + c_m\mathrm{e}^{-t\delta} \\ &= (1+\varepsilon)a_m^{\delta} g(t+\ln a_m) + c_m\mathrm{e}^{-\delta t}.\end{aligned}$$

Analogously, we obtain

$$\mathrm{e}^{-t(\delta-1)}\lambda_1(F_{\mathrm{e}^{-t}} \cap B) \geq (1-\varepsilon)a_m^{-\delta} g(t-\ln a_m) - c_m\mathrm{e}^{-\delta t}.$$

Therefore, the same arguments which were used in the proof of the case $k = 0$ imply that

$$\liminf_{\varepsilon \searrow 0} \varepsilon^{\delta-1}\lambda_1(F_\varepsilon \cap B) < \limsup_{\varepsilon \searrow 0} \varepsilon^{\delta-1}\lambda_1(F_\varepsilon \cap B).$$

□

Proof of Theorem 2.43(iii). We want to apply Lemma 4.14 in order to show that there exists a Borel set $B \subseteq \mathbb{R}$ for which $\underline{C}_k^f(F,B) < \overline{C}_k^f(F,B)$ for $k \in \{0,1\}$ from which we then deduce that the fractal curvature measures do not exist. For applying Lemma 4.14 we first introduce a family Δ of non-empty Borel subsets of Σ^∞, where Σ^∞ denotes the code space associated with R. For every $\kappa \in \Delta$ we then construct a pair $(B(\kappa), q_\kappa)$ which consists of a non-empty Borel set $B(\kappa) \subseteq \mathbb{R}$ satisfying $F_{\mathrm{e}^{-t}} \cap B(\kappa) = (F \cap B(\kappa))_{\mathrm{e}^{-t}}$ for all sufficiently large $t > 0$ and a positive bounded periodic Borel-measurable function $q_\kappa \colon \mathbb{R}^+ \to \mathbb{R}^+$, such that Lemma 4.14(ii) is satisfied for $B = B(\kappa)$ and $q = q_\kappa$. Then, we show that there always exists a $\kappa \in \Delta$ for which q_κ is not equal to an almost everywhere constant function, verifying Lemma 4.14(i).

We let $R_1, \ldots, R_N$ denote the similarities which the cIFS R consists of, that is $R =: \{R_1, \ldots, R_N\}$ and let $r_1, \ldots, r_N$ denote their respective similarity ratios, that is $r_i := R_i'(x)$ for any $x \in X$. Note that g is a bijective function by definition. For $i \in \{1, \ldots, N\}$ define $\phi_i := g \circ R_i \circ g^{-1}$ and set $\Phi := \{\phi_1, \ldots, \phi_N\}$. From the fact that $R_1, \ldots, R_N$ are contractions and g' is Hölder continuous and bounded away from zero, one can deduce that there exists an iterate $\widetilde{\Phi}$ of Φ which solely consists of contractions. Without loss of generality we assume that $\phi_1, \ldots, \phi_N$ are contractions themselves. Then Φ is a cIFS with open set $\mathrm{int}(gX)$ and bounded distortion constants $\varrho_m = 1 + \max_{\omega \in \Sigma^m} c|R_\omega X|^\alpha / k_g$, where $k_g > 0$ is such that $|g'| \geq k_g$ on $\mathcal{U}$ and c is a constant depending on the Hölder constant of g. Clearly, $\varrho_m \to 1$ as $m \to \infty$. Moreover, $F := g(K)$ is its associated self-conformal set, since $\bigcup_{i=1}^N \phi_i F = \bigcup_{i=1}^N g R_i g^{-1} g(K) = \bigcup_{i=1}^N g R_i(K) = F$.

Let us begin by introducing the family Δ. Recall that $\langle Y \rangle$ denotes the convex hull of a compact set $Y \subset \mathbb{R}$. Fix an $n \in \mathbb{N}_0$ and define

$$\Delta_n := \Big\{ \bigcup_{j=1}^{l} [\kappa^{(j)}] \;\Big|\; \kappa^{(j)} \in \Sigma^n,\, l \in \{1, \ldots, N^n\},\, \bigcup_{j=1}^{l} \langle \phi_{\kappa^{(j)}} F \rangle \text{ is an interval,} \\ \bigcup_{i=1}^{l} \phi_{\kappa^{(i)}} F \cap \phi_\omega F = \varnothing \text{ for every } \omega \in \Sigma^n \setminus \{\kappa^{(1)}, \ldots, \kappa^{(l)}\} \Big\}.$$

(Note that if the strong separation condition was satisfied, then $\Delta_n = \{[\omega] \mid \omega \in \Sigma^n\}$.) We remark that the condition $\lambda_1(X \setminus \Phi X) > 0$ implies that $\kappa \subsetneq \Sigma^\infty$ for every $\kappa \in \Delta_n$, whenever $n \in \mathbb{N}$. Further, note that $\Delta_n \neq \varnothing$ for all $n \in \mathbb{N}$ because of the OSC and set $\Delta := \bigcup_{n \in \mathbb{N}_0} \Delta_n$. Now, fix an $n \in \mathbb{N}_0$ and a $\kappa = \bigcup_{j=1}^l [\kappa^{(j)}] \in \Delta_n$ and choose $\theta > 0$ such that $\bigcup_{j=1}^l \langle \phi_{\kappa^{(j)}} F \rangle_{3\theta} \cap \phi_\omega F = \varnothing$ for every $\omega \in \Sigma^n \setminus \{\kappa^{(1)}, \ldots, \kappa^{(l)}\}$. Then $B(\kappa) := \bigcup_{j=1}^l \langle \phi_{\kappa^{(j)}} F \rangle_\theta$ is a non-empty Borel subset of $\mathbb{R}$ satisfying $F_\varepsilon \cap B(\kappa) = (F \cap B(\kappa))_\varepsilon$ for all $\varepsilon < \theta$.

Denote by $G^1, \ldots, G^Q$ the primary gaps of F and by $G_\omega^1, \ldots, G_\omega^Q$ the associated main gaps for $\omega \in \Sigma^*$. For constructing the function q_κ fix an $m \in \mathbb{N}$ and choose $M \in \mathbb{N}$ so that $\mathrm{e}^{-M} < \theta$ and that for every $\omega \in \Sigma^m$ all main gaps $G_\omega^1, \ldots, G_\omega^Q$ which lie in $B(\kappa)$ are of length greater than $2\mathrm{e}^{-M}$. Then for all $T \geq M$ we have

$$\begin{aligned}\lambda_0\left(\partial F_{\mathrm{e}^{-T}} \cap B(\kappa)\right)/2 &= \sum_{i=1}^{Q} \#\{\omega \in \Sigma^* \mid G_\omega^i \subseteq B(\kappa),\ |G_\omega^i| > 2\mathrm{e}^{-T}\} + 1 \\ &\leq \sum_{i=1}^{Q} \sum_{\omega \in \Sigma^m} \Xi_\omega^i(\mathrm{e}^{-T}) + \underbrace{\sum_{j=1}^{m-n} Q \cdot N^{j-1} + 1}_{=:c_m},\end{aligned}$$

where we agree that $\sum_{j=1}^{m-n} Q \cdot N^{j-1} = 0$ if $m - n < 1$ and where

$$\Xi_\omega^i(\mathrm{e}^{-T}) := \#\{u \in \Sigma^* \mid G_{u\omega}^i \subseteq B(\kappa),\ |G_{u\omega}^i| > 2\mathrm{e}^{-T}\}$$

for $\omega \in \Sigma^*$. Likewise

$$\lambda_0\left(\partial F_{\mathrm{e}^{-T}} \cap B(\kappa)\right)/2 \geq \sum_{i=1}^{Q} \sum_{\omega \in \Sigma^m} \Xi_\omega^i(\mathrm{e}^{-T}).$$

We let ξ and ζ respectively denote the geometric potential functions associated with Φ and R. Moreover, we let π_F and π_K respectively denote the code maps from Σ^∞ to F and K. They satisfy $\pi_F = g\pi_K$. For $x \in \Sigma^\infty$ we have the following relation.

$$\begin{aligned}\xi(x) &= -\ln|\phi'_{x_1}(\pi_F \sigma x)| \\ &= -\ln|g'(R_{x_1} g^{-1} \pi_F \sigma x)| - \ln|R'_{x_1}(g^{-1}\pi_F \sigma x)| + \ln|g'(g^{-1}\pi_F \sigma x)| \\ &= -\ln|g'(\pi_K x)| + \zeta(x) + \ln|g'(\pi_K \sigma x)|.\end{aligned}$$

Therefore, $\psi\colon \Sigma^\infty \to \mathbb{R}$ given by $\psi(x) := -\ln|g'(\pi_K x)|$ defines a continuous function which satisfies

$$\xi - \zeta = \psi - \psi \circ \sigma.$$

Recall that c_g denotes the Hölder constant of g' and that $k_g > 0$ is such that $|g'| \geq k_g$ on $\mathcal{U}$. Also, g satisfies a bounded distortion property, since we have for all $x, y \in \langle R_\omega K\rangle$, where $\omega \in \Sigma^n$, and $n \in \mathbb{N}$ that

$$\left|\frac{g'(x)}{g'(y)}\right| \leq \left|\frac{g'(x) - g'(y)}{g'(y)}\right| + 1 \leq \frac{c_g|x-y|^\alpha}{k_g} + 1 \leq \max_{\omega \in \Sigma^n} \frac{c_g|\langle R_\omega K\rangle|^\alpha}{k_g} + 1 =: p_n \tag{4.61}$$

and clearly, $p_n \to 1$ as $n \to \infty$.

Denote by $\widetilde{G}^i$ the primary gaps of K and by $\widetilde{G}_\omega^i$ the main gaps of $R_\omega K$, where $i \in \{1, \dots, N\}$ and $\omega \in \Sigma^*$. Thus, for an arbitrary $x \in \Sigma^\infty$, $i \in \{1, \dots, Q\}$, $\omega \in \Sigma^m$ and $u \in \Sigma^n$ we have that

$$\begin{aligned}|G_{u\omega}^i| = |g\widetilde{G}_{u\omega}^i| &\leq p_m \cdot |g'(R_{u\omega}\pi_K x)| \cdot |R_{u\omega}\widetilde{G}^i| \\ &= \exp\left(\ln p_m - \psi(u\omega x) - S_n\zeta(u\omega x) + \ln|R_\omega \widetilde{G}^i|\right) \\ &= \exp\left(\ln p_m - \psi(\omega x) - S_n\xi(u\omega x) + \ln|R_\omega \widetilde{G}^i|\right)\end{aligned}$$

Therefore, for $x \in \Sigma^\infty$ and $\omega \in \Sigma^m$

$$\Xi^i_\omega(\mathrm{e}^{-T}) \leq \#\{u \in \Sigma^* \mid G^i_{u\omega} \subseteq B(\kappa), \\ \ln(2/p_m) + \psi(\omega x) - \ln|R_\omega \widetilde{G}^i| < T - S_n\xi(u\omega x)\}.$$

By construction κ is a finite union of cylinder sets. Moreover, it is easy to verify that Items (A) to (G) of Key Lemma 4.2 are satisfied for $f_{0,\omega} \colon \mathbb{R} \to \mathbb{R}$ defined by $f_{0,\omega}(t) := \sum_{i=1}^Q \mathbb{1}_{(\psi(\omega x)+\ln(2/p_m)-\ln|R_\omega\widetilde{G}^i|,\infty)}(t)$, where Item (F) can be checked in the same way as in Lemma 4.9(F).

Recall the definition of $N_{0,m,\kappa^{(j)}}$ from Key Lemma 4.2 with this $f_{0,\omega}$. By hypothesis, ζ is lattice. Since it is the geometric potential function associated with an IFS which consists of similarities, ζ is thus contained in a discrete subgroup of $\mathbb{R}$ (see proof of Theorem 2.38). Let $a > 0$ denote the maximal real number for which $\zeta(\Sigma^\infty) \subseteq a\mathbb{Z}$. Then Key Lemma 4.2(iii) gives

$$\begin{aligned}
&\lambda_0(F_{\mathrm{e}^{-T}} \cap B(\kappa))/2 - c_m \\
&\leq \sum_{j=1}^{l} \sum_{\omega\in\Sigma^m} \sum_{n=0}^{\infty} \sum_{u\in\Sigma^n} \mathbb{1}_{[\kappa^{(j)}]}(u\omega x) f_{0,\omega}(T - S_n\xi(u\omega x)) \\
&= \sum_{j=1}^{l} N_{0,m,\kappa^{(j)}}(T,x) \\
&\sim^{\mathbb{R}} \sum_{j=1}^{l} a \sum_{\omega\in\Sigma^m} \mathrm{e}^{a\delta\left\lfloor \frac{T+\psi(\omega x)}{a}\right\rfloor} \frac{h_{-\delta\zeta}(\omega x)}{\int \zeta \mathrm{d}\mu_{-\delta\zeta}} \\
&\qquad \int_{\Sigma^\infty} \mathbb{1}_{[\kappa^{(j)}]}(y) \sum_{z=-\infty}^{\infty} \mathrm{e}^{-a\delta z} f_{0,\omega}\left(az + a\left\{\frac{T+\psi(\omega x)}{a}\right\} - \psi(y)\right) \mathrm{d}\nu_{-\delta\zeta}(y) \qquad (4.62)
\end{aligned}$$

as $T \to \infty$. For $i \in \{1,\ldots,Q\}$, $\omega \in \Sigma^m$ and $x, y \in \Sigma^\infty$ define

$$A^i_\omega(y) := \frac{\ln(2/p_m) + \psi(\omega x) - \ln|R_\omega\widetilde{G}^i| + \psi(y)}{a} - \left\{\frac{T+\psi(\omega x)}{a}\right\}.$$

Then the definition of $f_{0,\omega}$ implies

$$\begin{aligned}
\sum_{z=-\infty}^{\infty} \mathrm{e}^{-a\delta z} f_{0,\omega}\left(az + a\left\{\frac{T+\psi(\omega x)}{a}\right\} - \psi(y)\right) &= \sum_{i=1}^{Q} \sum_{z=\lfloor A^i_\omega(y)\rfloor+1}^{\infty} \mathrm{e}^{-a\delta z} \\
&= \sum_{i=1}^{Q} (\mathrm{e}^{a\delta} - 1)^{-1} \mathrm{e}^{-a\delta\lfloor A^i_\omega(y)\rfloor}.
\end{aligned}$$

Moreover, note that $h_{-\delta\zeta} \equiv 1$. Thus, the term on the right hand side of Equation (4.62) is equal to

$$\frac{a(\mathrm{e}^{a\delta}-1)^{-1}}{\int\zeta\mathrm{d}\mu_{-\delta\zeta}}\sum_{i=1}^{Q}\sum_{\omega\in\Sigma^m}\mathrm{e}^{a\delta\left\lfloor\frac{T+\psi(\omega x)}{a}\right\rfloor}\int_{\Sigma^\infty}\mathbb{1}_\kappa(y)\mathrm{e}^{-a\delta\left\lfloor\frac{\ln(2/p_m)+\psi(\omega x)-\ln|R_\omega\widetilde{G}^i|+\psi(y)}{a}-\left\{\frac{T+\psi(\omega x)}{a}\right\}\right\rfloor}\mathrm{d}\nu_{-\delta\zeta}(y)$$
$$=\frac{a(\mathrm{e}^{a\delta}-1)^{-1}}{\int\zeta\mathrm{d}\mu_{-\delta\zeta}}\sum_{i=1}^{Q}\underbrace{\sum_{\omega\in\Sigma^m}r_\omega^\delta}_{=1}\int_{\Sigma^\infty}\mathbb{1}_\kappa(y)\mathrm{e}^{-a\delta\left\lfloor\frac{\ln(2/p_m)-\ln|\widetilde{G}^i|+\psi(y)-T}{a}\right\rfloor}\mathrm{d}\nu_{-\delta\zeta}(y),$$

where we used the fact that $r_\omega\in a\mathbb{Z}$ for $\omega\in\Sigma^*$. Define the function $q_\kappa\colon\mathbb{R}^+\to\mathbb{R}^+$ by

$$q_\kappa(T):=\mathrm{e}^{-\delta T}\frac{2a(\mathrm{e}^{a\delta}-1)^{-1}}{\int\zeta\mathrm{d}\mu_{-\delta\zeta}}\sum_{i=1}^{Q}\int_{\Sigma^\infty}\mathbb{1}_\kappa(y)\mathrm{e}^{-a\delta\left\lfloor\frac{\ln 2-\ln|\widetilde{G}^i|+\psi(y)-T}{a}\right\rfloor}\mathrm{d}\nu_{-\delta\zeta}(y). \tag{4.63}$$

Altogether, for all $\varepsilon>0$ there exists an $M'\geq M$ such that for all $t\geq M'$ we have

$$\mathrm{e}^{-\delta T}\lambda_0(\partial F_{\mathrm{e}^{-T}}\cap B)\leq(1+\varepsilon)p_m^\delta q_\kappa(T+\ln p_m)+2c_m\mathrm{e}^{-\delta T}$$

and likewise

$$\mathrm{e}^{-\delta T}\lambda_0(\partial F_{\mathrm{e}^{-T}}\cap B)\geq(1-\varepsilon)p_m^{-\delta}q_\kappa(T-\ln p_m).$$

Clearly, q_κ is periodic with period a. Thus, Lemma 4.14(ii) is satisfied for $B=B(\kappa)$ and $q=q_\kappa$. Thus, in order to apply Lemma 4.14 it remains to prove the validity of Lemma 4.14(i), that is that there exists a $\kappa\in\Delta$ for which q_κ is not equal to an almost everywhere constant function. For this, it suffices to consider the function $\widetilde{q}_\kappa\colon\mathbb{R}^+\to\mathbb{R}^+$ given by

$$\widetilde{q}_\kappa(t):=\mathrm{e}^{-\delta t}\sum_{i=1}^{Q}\int_{\Sigma^\infty}\mathbb{1}_\kappa(y)\mathrm{e}^{-a\delta\left\lfloor\frac{-\ln|\widetilde{G}^i|+\psi(y)-t}{a}\right\rfloor}\mathrm{d}\nu_{-\delta\zeta}(y).$$

Set $\underline{\beta}:=\min\{\{a^{-1}\ln|\widetilde{G}^i|\}\mid i=1,\ldots,Q\}$ and $\overline{\beta}:=\max\{\{a^{-1}\ln|\widetilde{G}^i|\}\mid i=1,\ldots,Q\}$. We first assume that $\underline{\beta}>0$ and consider the following four cases.

<u>Case 1:</u> $\underline{D}:=\{y\in\Sigma^\infty\mid\{a^{-1}\psi(y)\}<\underline{\beta}\}\neq\varnothing$.
Since $\psi\in\mathcal{C}(\Sigma^\infty)$ and thus $\underline{D}$ is open, there exists a $\kappa\in\Delta$ such that $\kappa\subseteq\underline{D}$. For $n\in\mathbb{N}$ and $r\in(0,1-\overline{\beta})$ define $T_n(r):=a(n+r)$. Then

$$\widetilde{q}_\kappa(T_n(r))=\mathrm{e}^{-\delta ar}\sum_{i=1}^{Q}\int_{\Sigma^\infty}\mathbb{1}_\kappa(y)\mathrm{e}^{-\delta a\left\lfloor\frac{\psi(y)-\ln|\widetilde{G}^i|}{a}\right\rfloor}\mathrm{d}\nu_{-\delta\zeta}(y).$$

This shows that $\widetilde{q}_\kappa$ is strictly decreasing on $(an,a(n+1-\overline{\beta}))$ for every $n\in\mathbb{N}$. Therefore, q_κ is not equal to an almost everywhere constant function.

<u>Case 2:</u> $\overline{D}:=\{y\in\Sigma^\infty\mid\{a^{-1}\psi(y)\}>\overline{\beta}\}\neq\varnothing$.
Like in Case 1, there exists a $\kappa\in\Delta$ such that $\kappa\subseteq\overline{D}$. For $n\in\mathbb{N}$ and $r\in(0,\underline{\beta})$ set

$T_n(r) := a(n-r)$. Then

$$\widetilde{q}_\kappa(T_n(r)) = \mathrm{e}^{\delta ar} \sum_{i=1}^{Q} \int_{\Sigma^\infty} \mathbb{1}_\kappa(y) \mathrm{e}^{-\delta a \left\lfloor \frac{\psi(y) - \ln|\widetilde{G}^i|}{a} \right\rfloor} \mathrm{d}\nu_{-\delta\zeta}(y).$$

This shows that $\widetilde{q}_\kappa$ is strictly decreasing on $(a(n-\underline{\beta}), an)$ for every $n \in \mathbb{N}$. Therefore, q_κ is not equal to an almost everywhere constant function.

For the remaining cases we let $n^* \in \mathbb{N}_0$ be maximal such that $\underline{\beta} + n^*(1-\overline{\beta}) \leq \overline{\beta}$.

<u>Case 3:</u> There exists an $n' \in \{0, \ldots, n^*\}$ such that

$$D_{n'} := \{y \in \Sigma^\infty \mid \underline{\beta} + n'(1-\overline{\beta}) < \{a^{-1}\psi(y)\} < \underline{\beta} + (n'+1)(1-\overline{\beta})\} \neq \varnothing.$$

As in the above cases, there exists a $\kappa \in \Delta$ such that $\kappa \subseteq D_{n'}$. For $n \in \mathbb{N}$ and $r \in (0, \underline{\beta})$ set $T_n^{n'}(r) := a(n - \overline{\beta} + \underline{\beta} + n'(1-\overline{\beta}) - r)$. Then

$$\widetilde{q}_\kappa(T_n^{n'}(r)) = \mathrm{e}^{\delta ar} \mathrm{e}^{\delta a(\overline{\beta} - \underline{\beta} - n'(1-\overline{\beta}))} \sum_{i=1}^{Q} \int_{\Sigma^\infty} \mathbb{1}_\kappa(y) \mathrm{e}^{-\delta a \left\lfloor \frac{\psi(y)}{a} \right\rfloor} \mathrm{d}\nu_{-\delta\zeta}(y).$$

This shows that $\widetilde{q}_\kappa$ is strictly decreasing on $(a(n-\overline{\beta}+q(1-\overline{\beta})), a(n-\overline{\beta}+\underline{\beta}+q(1-\overline{\beta})))$. Therefore, q_κ is not equal to an almost everywhere constant function.

If we are not in any of the above three cases, then we have the following.

<u>Case 4:</u> $\{y \in \Sigma^\infty \mid \{a^{-1}\psi(y)\} \subseteq \{\underline{\beta} + n'(1-\overline{\beta}) \mid n' \in \{0, \ldots, n^*\}\}\} = \Sigma^\infty$.

Define $s_i := \max(\{\underline{\beta} + n'(1-\overline{\beta}) - \{a^{-1}\ln|\widetilde{G}^i|\} < 0 \mid n' \in \{0, \ldots, n^*\}\} \cup \{1\})$ and $s := \max\{s_1, \ldots, s_N, 1 - \overline{\beta} + \underline{\beta}\}$. For $n \in \mathbb{N}$ and $r \in (0, s/2)$ define $T_n(r) := a(n+r)$. Then

$$\widetilde{q}_\varnothing(T_n(r)) = \mathrm{e}^{-\delta ar} \sum_{i=1}^{Q} \int_{\Sigma^\infty} \mathrm{e}^{-\delta a \left\lfloor \frac{\psi(y) - \ln|\widetilde{G}^i|}{a} \right\rfloor} \mathrm{d}\nu_{-\delta\zeta}(y).$$

This shows that $\widetilde{q}_\varnothing$ is strictly decreasing on $(an, a(n+s/2))$. Therefore, $q_\varnothing$ is not equal to an almost everywhere constant function.

If $\underline{\beta} = 0$, then the same methods can be applied after shifting the origin by $(1-\overline{\beta})/2$ to the left.

Thus, we can apply Lemma 4.14 in all four cases and obtain that there always exists a Borel set $B(\kappa)$ such that $\underline{C}_k^f(F, B(\kappa)) < \overline{C}_k^f(F, B(\kappa))$ for $k \in \{0, 1\}$.

In order to deduce that the fractal curvature measures do not exist, construct a function $\eta\colon \mathbb{R} \to [0,1]$ which is continuous, equal to 1 on $B(\kappa)$ and equal to 0 on $\mathbb{R} \setminus B(\kappa)_\theta$. Then $\liminf_{\varepsilon\to 0} \int \eta \varepsilon^\delta \mathrm{d}\lambda^0(\partial F_\varepsilon \cap \cdot)/2 = \underline{C}_0^f(F, B(\kappa)) < \overline{C}_0^f(F, B(\kappa)) = \limsup_{\varepsilon\to 0} \int \eta \varepsilon^\delta \mathrm{d}\lambda^0(\partial F_\varepsilon \cap \cdot)/2$. Thus, the 0-th fractal curvature measure does not exist. Using the same function η it follows analogously, that the 1-st fractal curvature measure does not exist, which completes the proof. □

4.4.2 Proof of Corollary 2.44

Proof of Corollary 2.44. Items (i) and (ii) of Corollary 2.44 are immediate consequences of Theorem 2.43. Corollary 2.44(iii) is going to be deduced from Theorem 2.33(iii). We let π_K and π_{F_n} respectively denote the code maps from Σ^∞ to K and F_n and observe that $\pi_K = g_n^{-1} \circ \pi_{F_n}$. Further, we let ξ_n denote the geometric potential function associated with F_n. By Property (b) from the beginning of Section 4.4 we see that $\xi_n - \xi_K = \psi - \psi \circ \sigma$, where $\psi := -\ln|g_n' \circ \pi_K|$. By definition we have that $g_n'(x) = \left(\widetilde{g}(x)(\mathrm{e}^{\delta an} - 1) + 1\right)^{-1/\delta}$ for $x \in [-1, \infty)$. Thus, $\psi(\Sigma^\infty) = -\ln|g_n' \circ \pi_K(\Sigma^\infty)| \subseteq [0, an]$. We now show that Condition (2.6) from Theorem 2.33(iii) is satisfied.

$$\begin{aligned}
\sum_{z\in\mathbb{Z}} \mathrm{e}^{-\delta az} \nu_{-\delta\xi_K} \circ \psi^{-1}([za, za+t)) &= \sum_{i=0}^{n} \mathrm{e}^{-\delta ai} \nu_{-\delta\xi_K} \circ \psi^{-1}([ai, ai+t)) \\
&= \sum_{i=0}^{n} \mathrm{e}^{-\delta ai} \nu \circ \widetilde{g}^{-1} \left(\left[\frac{\mathrm{e}^{\delta ai} - 1}{\mathrm{e}^{\delta an} - 1}, \frac{\mathrm{e}^{\delta ai + \delta t} - 1}{\mathrm{e}^{\delta an} - 1} \right) \right) = \sum_{i=0}^{n} \frac{\mathrm{e}^{\delta t} - 1}{\mathrm{e}^{\delta an} - 1} \\
&= \frac{\mathrm{e}^{\delta t} - 1}{\mathrm{e}^{\delta a} - 1} \sum_{i=0}^{n} \mathrm{e}^{-\delta ai} \nu \circ \widetilde{g}^{-1} \left(\left[\frac{\mathrm{e}^{\delta ai} - 1}{\mathrm{e}^{\delta an} - 1}, \frac{\mathrm{e}^{\delta a(i+1)} - 1}{\mathrm{e}^{\delta an} - 1} \right) \right) \\
&= \frac{\mathrm{e}^{\delta t} - 1}{\mathrm{e}^{\delta a} - 1} \sum_{i=0}^{n} \mathrm{e}^{-\delta ai} \nu_{-\delta\xi_K} \circ \psi^{-1}([ai, a(i+1)))
\end{aligned}$$

holds for all $t \in [0, a)$ which completes the proof. □

4.4.3 Proof of Theorem 2.46

Here, we show that every analytic lattice cIFS is conjugate to a cIFS consisting of similarities and by this prove Theorem 2.46.

Proof of Theorem 2.46. For ease of notation, we assume without loss of generality that $\{0,1\} \subset F \subset [0,1]$. Let ξ denote the geometric potential function associated with Φ and let π denote the code map from Σ^∞ to F. By Theorem 3.7 the eigenfunction $h_{-\delta\xi}$ of the Perron-Frobenius operator $\mathcal{L}_{-\delta\xi}$ possesses a real-analytic extension to an open neighbourhood of X in $\mathbb{R}$. Denote this extension by h and define $\widetilde{\psi} := \delta^{-1} \ln h$. Since ξ is lattice, there exist $\zeta, \psi \in \mathcal{C}(\Sigma^\infty)$ such that

$$\xi - \zeta = \psi - \psi \circ \sigma$$

and such that ζ is a function whose range is contained in a discrete subgroup of $\mathbb{R}$. Let $a > 0$ be the maximal real number for which $\zeta(\Sigma^\infty) \subseteq a\mathbb{Z}$. The function $\widetilde{\psi}$ satisfies the equation

$$\widetilde{\psi} \circ \pi = \psi + \delta^{-1} \ln h_{-\delta\zeta},$$

since h satisfies

$$h \circ \pi = h_{-\delta\xi} = \frac{\mathrm{d}\mu_{-\delta\xi}}{\mathrm{d}\nu_{-\delta\xi}} = \frac{\mathrm{d}\mu_{-\delta\zeta}}{\mathrm{e}^{-\delta\psi}\mathrm{d}\nu_{-\delta\zeta}} = \mathrm{e}^{\delta\psi} h_{-\delta\zeta}.$$

We define the function $\widetilde{g}\colon [0,1] \to \mathbb{R}$ by $\widetilde{g}(x) := \int_0^x \mathrm{e}^{\widetilde{\psi}(y)}\mathrm{d}y/A$ for $x \in [0,1]$, where $A := \int_0^1 \mathrm{e}^{\widetilde{\psi}(y)}\mathrm{d}y$. As $\widetilde{\psi}$ is analytic, the Fundamental Theorem of Calculus implies that $\widetilde{\psi} - \ln A = \ln \widetilde{g}'$. Moreover, the analyticity of $\widetilde{\psi}$ implies that $\widetilde{\psi}$ is bounded on $[0,1]$. Therefore, $\widetilde{g}'$ is bounded away from both 0 and ∞ and thus $\widetilde{g}$ is invertible. Note that $\widetilde{g}([0,1]) = [0,1]$, set $g := \widetilde{g}^{-1}\colon [0,1] \to [0,1]$ and extend g to an analytic function on an open neighbourhood $\mathcal{U}$ of $[0,1]$ such that $|g'| > 0$ on $\mathcal{U}$. Define

$$R_i := g^{-1} \circ \phi_i \circ g \quad \text{for } i \in \{1, \ldots, N\} \quad \text{and} \quad K := g^{-1}(F) \subseteq [0,1].$$

Then setting $\pi_K := g^{-1} \circ \pi$, we have for $x \in \Sigma^\infty$ that

$$\begin{aligned} -\ln|R'_{x_1}(\pi_K \sigma x)| &= -\ln \widetilde{g}'(\phi_{x_1} g \pi_K \sigma x) - \ln|\phi'_{x_1}(g\pi_K \sigma x)| + \ln \widetilde{g}'(g \pi_K \sigma x) \\ &= -\widetilde{\psi}(\pi x) + \ln A + \xi(x) + \widetilde{\psi}(\pi\sigma x) - \ln A \\ &= -\psi(x) - \delta^{-1}\ln(h_{-\delta\zeta}(x)) + \xi(x) + \psi(\sigma x) + \delta^{-1}\ln(h_{-\delta\zeta}(\sigma x)) \\ &= \zeta(x) - \delta^{-1}\ln\left(\frac{h_{-\delta\zeta}(x)}{h_{-\delta\zeta}(\sigma x)}\right). \end{aligned}$$

Since the range of ζ is contained in the group $a\mathbb{Z}$ and ξ and ψ are bounded on Σ^∞, ζ in fact takes a finite number of values. Moreover, ζ is continuous which implies that there exists an $N \in \mathbb{N}$ such that ζ is constant on each $[\omega]$ for $\omega \in \Sigma^N$. This cleary implies that $\mathcal{L}^n_{-\delta\zeta}1$ is constant on $[\omega]$ for all $\omega \in \Sigma^N$ and all $n \in \mathbb{N}$, where 1 denotes the constant one-function. Thus, Equation (3.3) implies that also $h_{-\delta\zeta}$ is constant on cylinder sets of length N. This can be seen by considering $|h_{-\delta\zeta}(x) - h_{-\delta\zeta}(y)|$ for x, y lying in the same cylinder of length N and applying the triangle inequality. Therefore, $x \mapsto -\ln|R'_{x_1}(\pi_K \sigma x)|$ is constant on cylinder sets of length $N+1$. Hence, for $\omega \in \Sigma^{N+1}$ and $i \in \{1, \ldots, N\}$ there exists a $c^i_\omega \in \mathbb{R}$ such that $R'_i(\pi_K x) = c^i_\omega$ for all $x \in [\omega]$. Since for each $\omega \in \Sigma^{N+1}$ the set $\{\pi_K x \mid x \in [\omega]\}$ has got accumulation points and is compact and the map R'_i is analytic by construction, it follows that R'_i is constant on its domain of definition. Therefore, the maps $R_1, \ldots, R_N$ are similarities. From the fact that $\phi_1, \ldots, \phi_N$ are contractions and g' is differentiable and bounded away from 0, one can deduce that there exists an iterate $\widetilde{R}$ of $R := \{R_1, \ldots, R_N\}$ which solely consists of contractions. The system $\widetilde{R}$ satisfies the OSC with open set $(0,1) = g^{-1}(0,1)$. Therefore, the unique non-empty compact invariant set of $\widetilde{R}$ is a self-similar set. It coincides with $K := g^{-1}(F)$, since $R_i(g^{-1}F) = g^{-1}\phi_i g(g^{-1}F) = g^{-1}F$ for each $i \in \{1, \ldots, N\}$. □

4.4.4 Proof of Theorem 2.39

Clearly, every self-similar set belongs to the class of $\mathcal{C}^{1+\alpha}$-diffeomophic images of self-similar sets. Thus, we infer Theorem 2.39 from the more general proofs presented in this subsection.

Proof of Theorem 2.39. We let r_ω denote the similarity ratio of ϕ_ω. Using that $|G^i_\omega| = r_\omega \cdot |G^i|$, Item (i) is a direct consequence of Theorem 2.31(i). Since self-similar sets form a sub-class of $\mathcal{C}^{1+\alpha}$-diffeomorphic images of self-similar sets, the statement from Item (ii) concerning the fractal curvature measures follows from the stronger result in Theorem 2.43(iii). Thus, all that remains to be shown is the statement $\underline{C}^f_k(F,B) < \overline{C}^f_k(F,B)$. With regard to the proofs of Theorem 2.43 and Lemma 4.14, we only need to show that the function q_κ from Equation (4.63) is non-constant. In the situation of a self-similar set, q_κ simplifies to the following.

$$q_\kappa(T) = \mathrm{e}^{-\delta T} \frac{2a\nu_{-\delta\xi}([\kappa])}{\int \xi \mathrm{d}\mu_{-\delta\xi}(\mathrm{e}^{a\delta} - 1)} \sum_{i=1}^{Q} \mathrm{e}^{-a\delta \left\lfloor \frac{\ln 2 - \ln|G^i| - T}{a} \right\rfloor}.$$

Since q_κ is a periodic function with period a which is non-constant the statement follows. $\square$

5 Extensions to Conformal Graph Directed Markov Systems

In this chapter we provide a preview to work that has been carried out by the author regarding conformal graph directed Markov systems (cGDMS), which is soon to appear in [KK11]. Such systems form an interesting extension to the systems which were discussed in Chapter 2. Here, we exhibit the main results of [KK11] and illustrate their importance through a collection of examples. The examples moreover serve to clarify the more involved definition of the primary gaps. We remark that the results are new, even for systems which consist of similarities.

We start this chapter in Section 5.1 by introducing cGDMS and presenting important examples. Then, in Section 5.2, we provide statements on the existence of the fractal curvature measures and the Minkowski content of limit sets of cGDMS and evaluate the Minkowski content for the examples from Section 5.1.

5.1 Conformal Graph Directed Markov Systems

A core text concerning conformal graph directed Markov systems (cGDMS) is [MU03]. The class of cGDMS generalises the notion of cIFS and gives rise to a broader collection of fractal sets. Before formally defining cGDMS, we first introduce some necessary notions.

Definition 5.1 (Directed multigraph)**.** A *directed multigraph* (V, E, i, t) consists of a finite set of vertices V, a finite set of directed edges E and functions $i, t\colon E \to V$ which determine the initial and terminal vertex of an edge. The edge $e \in E$ goes from $i(e)$ to $t(e)$. Thus, the *initial* and *terminal vertices* of e are $i(e)$ and $t(e)$ respectively.

Definition 5.2 (Incidence matrix)**.** Given a directed multigraph (V, E, i, t), an $(\#E) \times (\#E)$-matrix A with entries in $\{0, 1\}$ is called an *incidence matrix*. It determines which edges may follow a given edge via $A_{e,e'} = 1$ if and only if $t(e) = i(e')$ for edges $e, e' \in E$. The incidence matrix is called *aperiodic and irreducible* if there exists an $n \in \mathbb{N}$ such that the entries of the n-th iterate $A^n =: (A^{(n)}_{e,e'})_{e,e' \in E}$ are positive, that is $A^{(n)}_{e,e'} > 0$ for all $e, e' \in E$.

Definition 5.3 (GDMS)**.** A *graph directed Markov system (GDMS)* consists of a directed multigraph (V, E, i, t), an incidence matrix A, a set of non-empty compact metric spaces

$\{X_v\}_{v\in V}$, $r \in (0,1)$ and for every edge $e \in E$ an injective contraction $\phi_e \colon X_{t(e)} \to X_{i(e)}$ with contraction ratio less than or equal to r. Briefly, the set $\Phi := \{\phi_e \colon X_{t(e)} \to X_{i(e)}\}_{e\in E}$ is called a GDMS.

In Section 5.2 we consider fractal subsets of the real line. Therefore, we restrict the definition of a cGDMS to the one-dimensional Euclidean space $(\mathbb{R}, |\cdot|)$. For a subset $Y \subset \mathbb{R}$ we write $\operatorname{int}(Y) := \operatorname{int}_{\mathbb{R}}(Y)$ for the interior of Y.

Definition 5.4 (cGDMS)**.** We call a GDMS a *conformal graph directed Markov system (cGDMS)* if

(i) for every vertex $v \in V$, $X_v \subset \mathbb{R}$ is a compact and connected, $X_v = \overline{\operatorname{int}(X_v)}$ and $\operatorname{int}(X_v) \cap \operatorname{int}(X_{v'}) = \varnothing$ for distinct $v, v' \in V$,

(ii) the *open set condition (OSC)* is satisfied, in the sense that, for all $e \neq e' \in E$ we have

$$\phi_e(\operatorname{int}(X_{t(e)})) \cap \phi_{e'}(\operatorname{int}(X_{t(e')})) = \varnothing \qquad \text{and}$$

(iii) if for every vertex $v \in V$ there exists an open connected set $W_v \supset X_v$ such that for every $e \in E$ with $t(e) = v$ the map ϕ_e extends to a $\mathcal{C}^{1+\alpha}$-diffeomorphism from W_v into $W_{i(e)}$, whose derivative ϕ_e' is bounded away from zero on X_v, where $\alpha \in (0,1]$.

We consider also the special case of cGDMS where the contractions $\{\phi_e\}_{e\in E}$ are similarities.

Definition 5.5 (sGDMS)**.** A cGDMS whose maps $\{\phi_e\}_{e\in E}$ are similarities is referred to as *sGDMS*.

Remark 5.6. Our definition of a cGDMS differs slightly from the definition given in [MU03]. Firstly, an incidence matrix in [MU03] is defined via the property that $A_{e,e'} = 1$ implies $t(e) = i(e')$. Secondly, the condition that $\operatorname{int}(X_v) \cap \operatorname{int}(X_{v'}) = \varnothing$ for distinct $v, v' \in V$ is not required in the definition of a cGDMS in [MU03]. Thirdly, we require the contractions ϕ_e for $e \in E$ to extend to $\mathcal{C}^{1+\alpha}$-diffeomorphisms with derivatives bounded away from zero, whereas in [MU03] the contractions need to extend to $\mathcal{C}^1$-diffeomorphisms and are required to satisfy a bounded distortion property. However, disregarding the third difference, a cGDMS in the sense of [MU03] can always be represented by a cGDMS in our sense, namely by substituting $\{\phi_e(X_{t(e)})\}_{e\in E}$ in for the sets $\{X_v\}_{v\in V}$ and defining the edges accordingly. Conversely, every cGDMS in our sense is a cGDMS in the sense of [MU03].

For defining a limit set of a cGDMS, we fix a cGDMS with the notation from Definitions 5.3 and 5.4. The set of *infinite admissible words* given by the incidence matrix A is defined to be

$$E_A^\infty := \{\omega = \omega_1\omega_2\cdots \in E^{\mathbb{N}} \mid A_{\omega_n,\omega_{n+1}} = 1 \text{ for all } n \in \mathbb{N}\}. \tag{5.1}$$

The set of subwords of length $n \in \mathbb{N}$ is denoted by E_A^n and the set of all finite subwords including the empty word $\varnothing$ by E_A^*. For $\omega \in E_A^*$ we let $n(\omega)$ denote its length, where $n(\varnothing) := 0$, define $\phi_\varnothing$ to be the identity map on $\bigcup_{v \in V} X_v$ and for $\omega \in \Sigma^* \setminus \{\varnothing\}$ set

$$\phi_\omega := \phi_{\omega_1} \circ \cdots \circ \phi_{\omega_{n(\omega)}} \colon X_{t(\omega_{n(\omega)})} \to X_{i(\omega_1)},$$

where we let ω_i denote the i-th letter of the word ω, for $i \in \{1, \ldots, n(\omega)\}$, that is $\omega = \omega_1 \cdots \omega_{n(\omega)}$. As in Chapter 2 the *initial word of length* $n \in \mathbb{N}$ of $\omega = \omega_1 \omega_2 \ldots \in E_A^\infty$ is defined to be $\omega|_n := \omega_1 \cdots \omega_n$.

For $\omega \in E_A^\infty$ the sets $\{\phi_{\omega|_n}(X_{t(\omega_n)})\}_{n \in \mathbb{N}}$ form a descending sequence of non-empty compact sets and therefore $\bigcap_{n \in \mathbb{N}} \phi_{\omega|_n}(X_{t(\omega_n)}) \neq \varnothing$. Recall from Definition 5.3 that $r \in (0,1)$ denotes an upper bound for the contraction ratios of the functions ϕ_e for $e \in E$. Since $\operatorname{diam}(\phi_{\omega|_n}(X_{t(\omega_n)})) \leq r^n \operatorname{diam}(X_{t(\omega_n)}) \leq r^n \max\{\operatorname{diam}(X_v) \mid v \in V\}$ for every $n \in \mathbb{N}$, the intersection

$$\bigcap_{n \in \mathbb{N}} \phi_{\omega|_n}(X_{t(\omega_n)})$$

is a singleton and we denote its only element by $\pi(\omega)$. In analogy to the situation for cIFS the projection $\pi \colon E_A^\infty \to \bigcup_{v \in V} X_v$ is called the *code map*.

Definition 5.7 (Limit set of a cGDMS)**.** The *limit set* of a cGDMS is defined to be

$$F := \pi(E_A^\infty).$$

Limit sets of cGDMS often have a fractal structure. They include self-conformal sets as well as self-similar sets. In order to show the significance of cGDMS we are now going to present three important classes of sets which can be obtained as limit sets of cGDMS.

Conformal Iterated Function Systems with Incidence Matrix.

Assume that the cIFS $\Psi := \{\psi_1, \ldots, \psi_N\}$ is equipped with an $N \times N$ incidence matrix $A' := (A'_{i,j})_{i,j \in \{1,\ldots,N\}}$ with entries $0, 1$ which determines which functions may follow a given function, that is $A'_{i,j} = 1$ if and only if $\psi_i \circ \psi_j$ is allowed. The system (Ψ, A') can be represented by a cGDMS by setting $V := \{1, \ldots, N\}$, $E := \{1, \ldots, M\}$, where $M := \sum_{i,j=1}^N A'_{i,j}$ and where for all $v, v' \in V$ with $A'_{v,v'} = 1$ there exists an edge $e \in E$ such that $i(e) = v$ and $t(e) = v'$.

Example 5.8. For $i \in \{1,2,3\}$ define $\psi_i \colon [0,1] \to [0,1]$ by setting $\psi_1(x) := x/4$, $\psi_2(x) := x/4 + 3/8$ and $\psi_3(x) := x/4 + 3/4$ and set

$$A' := \begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix}.$$

A corresponding sGDMS is given by $V := \{1,2,3\}$, $E := \{1,\dots,6\}$,

$$i(e) := \begin{cases} 1 & : e \in \{1,2\} \\ 2 & : e = 3 \\ 3 & : e \in \{4,5,6\} \end{cases}, \quad t(e) := \begin{cases} 1 & : e \in \{1,4\} \\ 2 & : e = 5 \\ 3 & : e \in \{2,3,6\} \end{cases}, \quad A := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

$X_v := \psi_v([0,1])$ for $v \in V$, $r = 1/4$ and

$$\phi_1\colon X_1 \xrightarrow{\psi_1} X_1, \qquad \phi_3\colon X_3 \xrightarrow{\psi_2} X_2, \qquad \phi_5\colon X_2 \xrightarrow{\psi_3} X_3,$$
$$\phi_2\colon X_3 \xrightarrow{\psi_1} X_1, \qquad \phi_4\colon X_1 \xrightarrow{\psi_3} X_3, \qquad \phi_6\colon X_3 \xrightarrow{\psi_3} X_3.$$

An illustration for this system is provided in Figure 5.2.

Conformal iterated function systems with disconnected open set.

By definition, a cIFS acting on X needs to satisfy the OSC with open set $\mathrm{int}(X)$. If we allow the OSC to be satisfied with a different open set, then the system can still be represented by a cGDMS.

Example 5.9. For $i \in \{1,2,3\}$ define $\psi_i\colon [0,1] \to [0,1]$ by $\psi_1(x) := x/3$, $\psi_2(x) := x/3 + 2/3$ and $\psi_3(x) := x/9 + 1/9$ and set $\Psi := \{\psi_1, \psi_2, \psi_3\}$. Then Ψ is not a cIFS since the open set condition is not satisfied with $(0,1)$ as open set. However, Ψ can be represented by a sGDMS as follows. Set $V := \{1,2\}$, $E := \{1,\dots,6\}$,

$$i(e) := \begin{cases} 1 & : e \in \{1,\dots,4\} \\ 2 & : e \in \{5,6\} \end{cases}, \quad t(e) := \begin{cases} 1 & : e \in \{1,3,5\} \\ 2 & : e \in \{2,4,6\} \end{cases}, \quad A := \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix},$$

$X_v := \psi_v([0,1])$ for $v \in \{1,2\}$, $r = 1/3$ and

$$\phi_1\colon X_1 \xrightarrow{\psi_1} X_1, \qquad \phi_3\colon X_1 \xrightarrow{\psi_3} X_1, \qquad \phi_5\colon X_1 \xrightarrow{\psi_2} X_2,$$
$$\phi_2\colon X_2 \xrightarrow{\psi_1} X_1, \qquad \phi_4\colon X_2 \xrightarrow{\psi_3} X_1, \qquad \phi_6\colon X_2 \xrightarrow{\psi_2} X_2.$$

An illustration for this example is given in Figure 5.3.

Markov Interval Maps.

For closed intervals $X_1, \ldots, X_N$ in $[0,1]$ with disjoint interior, $N \geq 2$, and $X := \bigcup_{i=1}^{N} X_i$ we call a map $f : X \to [0,1]$ a *Markov interval map* if

(i) $f|_{X_i}$ is expanding and there exists a $C^{1+\alpha}$-continuation to a neighbourhood of X_i and

(ii) if $f(X_i) \cap X_j \neq \varnothing$, then $X_j \subset f(X_i)$ for $i, j \in \{1, \ldots, N\}$.

For a representation by a cGDMS, set $V := \{1, \ldots, N\}$ and for $v \in V$ define $f_v := \{v' \in V \mid X_{v'} \subseteq f(X_v)\}$. For every pair (v, v'), where $v \in V$ and $v' \in f_v$, introduce an edge $e = e(v, v')$ with $i(e) = v$ and $t(e) = v'$. Set $E := \{e(v, v') \mid v \in V, v' \in f_v\}$ and define $\phi_e \colon X_{t(e)} \to X_{i(e)}$ by $\phi_e := (f|_{X_{i(e)}})^{-1}|_{X_{t(e)}}$.

Example 5.10. Set $X_1 := [0, 1/4]$, $X_2 := [1/4, 1/2]$, $X_3 := [2/3, 1]$ and let the Markov interval map $f \colon \bigcup_{i=1}^{3} X_i \to [0,1]$ be given by $f|_{X_1}(x) := 5x/2$, $f|_{X_2}(x) := 3x - 1/2$ and $f|_{X_3}(x) := 3x - 2$.

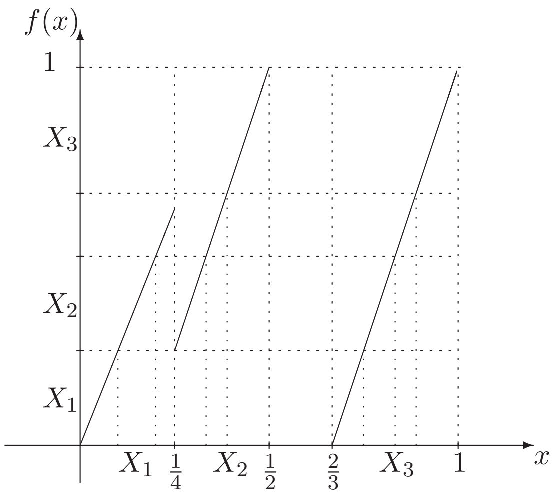


Figure 5.1: Graph of the Markov Interval Map f from Example 5.10.

A corresponding sGDMS is given by $V := \{1, 2, 3\}$, $E := \{1, \dots, 7\}$,

$$i(e) := \begin{cases} 1 & : e \in \{1,2\} \\ 2 & : e \in \{3,4\} \\ 3 & : e \in \{5,6,7\} \end{cases}, \quad t(e) := \begin{cases} 1 & : e \in \{1,5\} \\ 2 & : e \in \{2,3,6\} \\ 3 & : e \in \{4,7\} \end{cases}, \quad A := \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix},$$

$r = 3/4$ and

$$\phi_1\colon X_1 \xrightarrow{(f|_{X_1})^{-1}} X_1, \quad \phi_3\colon X_2 \xrightarrow{(f|_{X_2})^{-1}} X_2, \quad \phi_5\colon X_1 \xrightarrow{(f|_{X_3})^{-1}} X_3,$$
$$\phi_2\colon X_2 \xrightarrow{(f|_{X_1})^{-1}} X_1, \quad \phi_4\colon X_3 \xrightarrow{(f|_{X_2})^{-1}} X_2, \quad \phi_6\colon X_2 \xrightarrow{(f|_{X_3})^{-1}} X_3, \quad \phi_7\colon X_3 \xrightarrow{(f|_{X_3})^{-1}} X_3.$$

The graph of the Markov interval map f is presented in Figure 5.1. An illustration of how the limit set is obtained is given in Figure 5.4.

5.2 Results for Conformal Graph Directed Markov Systems

In this section, we will characterise limit sets of cGDMS with aperiodic irreducible incidence matrices for which the (average) fractal curvature measures and the (average) Minkowski content exist. We concentrate on the results and the examples which we introduced in the preceding section and refer to [KK11] for the proofs. The original results presented below are generalisations of the results for self-conformal subsets of $\mathbb{R}$ from Chapter 2 and are new even in the setting of sGDMS.

It follows from Theorems 4.2.9, 4.2.11 and 4.2.13 in [MU03] that the Minkowski dimension of a limit set of a cGDMS always exists. Moreover, we can show that such a limit set either is a non-empty compact interval or has zero one-dimensional Lebesgue measure. We now distinguish between these two cases.

Proposition 5.11. *If $Y \subset \mathbb{R}$ is a non-empty compact interval, then the 1-st fractal curvature measure exists and satisfies*

$$C_1^f(Y, \cdot) = \lambda^1(Y \cap \cdot).$$

Moreover, taking Winter's definition of the fractal curvature measures (see Remark 2.10), we have that $s_0(Y) = 0$ and that $\text{w-lim}_{\varepsilon \to 0} \varepsilon^{s_0(Y)} C_0(Y_\varepsilon, \cdot) = \lambda^0(\partial Y \cap \cdot)/2$.

Let us now focus on limit sets with zero one-dimensional Lebesgue measure. For stating our results we fix a cGDMS (V, E, i, t, A) and assume that the incidence matrix A is aperiodic and irreducible (see Definition 5.2). Let $\{X_v\}_{v\in V}$ denote the associated non-empty compact connected subsets of $\mathbb{R}$ and let $\Phi := \{\phi_e\colon X_{t(e)} \to X_{i(e)}\}_{e\in E}$ denote the set of injective contractions whose contraction ratios are bounded by some $r \in (0, 1)$. Further, let F denote the unique limit set and let $\delta := \dim_M(F)$ be its Minkowski dimension.

Note that the notions from Chapters 2 and 3 concerning the full shift space (Σ^∞, σ) can be analogously introduced for the code space E_A^∞. In particular, the *geometric potential function* $\xi\colon E_A^\infty \to \mathbb{R}$ is given by $\xi(\omega) := -\ln|\phi'_{\omega_1}(\pi(\sigma\omega))|$ for $\omega = \omega_1\omega_2\cdots \in E_A^\infty$, where σ denotes the *shift-map* on E_A^∞, which is defined in the same manner as in Section 2.2. Moreover, the results presented in Section 3.1.2 concerning Ruelle's Perron-Frobenius theorem are valid also for E_A^∞ (see [Bow08]). Specifically, there exists a unique σ-invariant Gibbs measure $\mu_{-\delta\xi}$ for the potential function $-\delta\xi$. Further, the measure theoretical entropy $H_{\mu_{-\delta\xi}}$ of σ with respect to $\mu_{-\delta\xi}$ is defined as in Equation (3.4).

Now, we present necessary notions which are specific for cGDMS. The unique probability measure ν supported on F, which for all distinct $e, e' \in E$ satisfies

$$\nu\left(\phi_e(X_{t(e)}) \cap \phi_{e'}(X_{t(e')})\right) = 0 \quad \text{and} \quad \nu(\phi_e B) = \int_B |\phi'_e|^\delta \mathrm{d}\nu \tag{5.2}$$

for all Borel sets $B \subseteq X_{t(e)}$ is called the *δ-conformal measure* associated with Φ. The statement on the uniqueness and existence for cGDMS is provided in Theorem 4.2.9 of [MU03] and goes back to the work of [Pat76, Sul79, DU91]. For a vertex $v \in V$ we denote the set of edges whose initial and respectively terminal vertex is v by

$$I_v := \{e \in E \mid i(e) = v\} \quad \text{and} \quad T_v := \{e \in E \mid t(e) = v\}.$$

Moreover, for $n \in \mathbb{N}$ we set

$$\begin{aligned} I_v^n &:= \{\omega \in E_A^n \mid i(\omega_1) = v\}, & T_v^n &:= \{\omega \in E_A^n \mid t(\omega_n) = v\}, \\ I_v^* &:= \textstyle\bigcup_{n\in\mathbb{N}} I_v^n, & T_v^* &:= \textstyle\bigcup_{n\in\mathbb{N}} T_v^n \quad \text{and} \\ I_v^\infty &:= \{\omega \in E_A^\infty \mid i(\omega_1) = v\}. \end{aligned}$$

For a finite word $\omega \in E_A^*$ the *ω-cylinder set* is defined to be

$$[\omega] := \{u \in E_A^\infty \mid u_i = \omega_i \text{ for } i \in \{1, \ldots, n(\omega)\}\}, \quad \text{in particular} \quad [\varnothing] = E_A^\infty.$$

As in the situation of self-conformal sets, another central role is played by the primary and main gaps of F. Like for self-conformal sets, these are certain intervals in the complement of the limit set. Such a definition for limit sets of cGDMS is more involved and is given as follows. Recall that $\langle Y\rangle$ denotes the convex hull of a set $Y \subset \mathbb{R}$. For $v \in V$ we define

$$G^v := \left\langle \textstyle\bigcup_{e\in I_v} \pi[e]\right\rangle \setminus \textstyle\bigcup_{e\in I_v} \langle \pi[e]\rangle \tag{5.3}$$

and denote by n_v the number of connected components of G^v. In [KK11] we show that $\bigcup_{v\in V} G^v \neq \varnothing$ if $\lambda^1(F)=0$, hence, $\sum_{v\in V} n_v \geq 1$. If $G^v \neq \varnothing$, we denote the connected components of G^v by $G^{v,j}$, where j ranges over $\{1,\ldots,n_v\}$ and call them the *primary gaps* of F. For every $\omega \in T_v^*$ we define $G_\omega^{v,j} := \phi_\omega(G^{v,j})$ and call these sets the *main gaps* of F.

Having introduced these notions, we are now able to present our results and fix the following notation.

Notation 5.12. We let $\Phi := \{\phi_e\}_{e\in E}$ denote a cGDMS with aperiodic irreducible incidence matrix and let F denote its limit set. We set δ equal to the Minkowski dimension of F and let ξ denote the geometric potential function associated with Φ. Further, denote by $H_{-\delta\xi}$ the measure theoretical entropy of the shift map with respect to the unique shift-invariant Gibbs measure $\mu_{-\delta\xi}$ for the potential function $-\delta\xi$ (see Section 3.1.2).

Theorem 5.13 (cGDMS – fractal curvature measures)**.** *Fix the notation from Notation 5.12 and assume that $\lambda^1(F)=0$. Then the following hold.*

(i) The average fractal curvature measures always exist and are both constant multiples of the δ-conformal measure ν associated with F, that is

$$\widetilde{C}_0^f(F,\cdot) = \frac{2^{-\delta}c}{H_{\mu_{-\delta\xi}}}\cdot\nu(\cdot) \quad \textit{and} \quad \widetilde{C}_1^f(F,\cdot) = \frac{2^{1-\delta}c}{(1-\delta)H_{\mu_{-\delta\xi}}}\cdot\nu(\cdot),$$

where the constant c is given by the well-defined positive and finite limit

$$c := \lim_{m\to\infty}\sum_{v\in V}\sum_{j=1}^{n_v}\sum_{\omega\in T_v^m}|G_\omega^{v,j}|^\delta. \tag{5.4}$$

(ii) If ξ is non-lattice, then both the 0-th and 1-st fractal curvature measures exist and satisfy $C_k^f(F,\cdot)=\widetilde{C}_k^f(F,\cdot)$, for $k\in\{0,1\}$.

(iii) If ξ is lattice, then there exists a constant $\overline{c}\in\mathbb{R}$ such that $\overline{C}_k^f(F,B)\leq\overline{c}$ for every Borel set $B\subseteq\mathbb{R}$ and $k\in\{0,1\}$. Additionally, $\underline{C}_k^f(F,\mathbb{R})$ is positive for $k\in\{0,1\}$.

Using the definition of the Minkowski content, we see that the existence of the fractal curvature measures immediately implies the existence of the Minkowski content. Thus, the Minkowski content of F exists, if ξ is non-lattice. As in the situation of self-conformal sets, the lattice case is quite interesting with regard to the Minkowski content. A sufficient condition under which the Minkowski content exists is given in Part (iii) of the next theorem. Parts (i) and (ii) of the following theorem are immediate consequences of Theorem 5.13. For stating the result, we equip $E^{\mathbb{N}}$ with the product topology of the discrete topologies on E and equip the set of infinite admissible words $E_A^\infty \subset E^{\mathbb{N}}$ with the subspace topology.

This is the weakest topology with respect to which the canonical projections onto the coordinates are continuous. The space of real-valued continuous functions on E_A^∞ is denoted by $\mathcal{C}(E_A^\infty)$.

Theorem 5.14 (cGDMS – Minkowski content)**.** *Under the conditions of Theorem 5.13 and letting c denote the constant given in Equation* (5.4)*, the following hold.*

(i) The average Minkowski content exists and is equal to

$$\widetilde{\mathcal{M}}(F) = \frac{2^{1-\delta}c}{(1-\delta)H_{\mu_{-\delta\xi}}}.$$

(ii) If ξ is non-lattice, then the Minkowski content $\mathcal{M}(F)$ of F exists and coincides with $\widetilde{\mathcal{M}}(F)$.

(iii) If ξ is lattice, then we have that

$$0 < \underline{\mathcal{M}}(F) \leq \overline{\mathcal{M}}(F) < \infty.$$

Further, equality in the above equation can be attained. More precisely let $\zeta, \psi \in \mathcal{C}(E_A^\infty)$ denote the functions satisfying $\xi - \zeta = \psi - \psi \circ \sigma$, where the range of ζ is contained in a discrete subgroup of $\mathbb{R}$ and $a \in \mathbb{R}$ is maximal such that $\zeta(E_A^\infty) \subseteq a\mathbb{Z}$. Moreover, let $\nu_{-\delta\zeta}$ denote the unique eigenmeasure with eigenvalue one of the dual for the Perron-Frobenius operator for the potential function $-\delta\zeta$ (see Section 3.1.2). If for every $t \in [0, a)$ we have that

$$\sum_{n\in\mathbb{Z}} \mathrm{e}^{-\delta a n} \nu_{-\delta\zeta} \circ \psi^{-1}([na, na+t)) = \frac{\mathrm{e}^{\delta t}-1}{\mathrm{e}^{\delta a}-1} \sum_{n\in\mathbb{Z}} \mathrm{e}^{-\delta a n} \nu_{-\delta\zeta} \circ \psi^{-1}([na, (n+1)a)), \quad (5.5)$$

then $\underline{\mathcal{M}}(F) = \overline{\mathcal{M}}(F)$.

Remark 5.15. (i) The sums occurring in Equation (5.5) are finite.

(ii) Condition (5.5) not only implies the existence of the Minkowski content but also that $\underline{C}_0^f(F, \mathbb{R}) = \overline{C}_0^f(F, \mathbb{R})$.

We have already presented an example for a self-conformal set, which satisfies Condition (5.5) in Chapter 2. An example of a limit set of a lattice cGDMS, which satisfies Condition (5.5), thus is Minkowski measurable, and which is not a self-conformal set will be presented in Example 5.20. Observe that, in the special case, when the maps $\{\phi_e\}_{e\in E}$ of the cGDMS are similarities, Condition (5.5) cannot be satisfied. In this case it even turns out, that the limit set F is Minkowski measurable if and only if the system is non-lattice. This provides an important extension to the result for self-similar sets given in [LP93, Fal95, LvF06] and is reflected in the following theorem.

Theorem 5.16 (sGDMS – fractal curvature measures)**.** *Fix the notation from Notation 5.12. Suppose that ϕ_e is a similarity for each $e \in E$, so that Φ is an sGDMS. Assume that $\lambda^1(F) = 0$ and let $h_{-\delta\xi}$ denote the unique strictly positive eigenfunction with eigenvalue one of the Perron-Frobenius operator for the potential function $-\delta\xi$ (see Section 3.1.2). Then, additionally to the statements of Theorem 5.13, the following hold.*

(i) The constant c from Equation (5.4) *simplifies to*

$$c = \sum_{v \in V} \sum_{j=1}^{n_v} h_{-\delta\xi}(x^v) |G^{v,j}|^{\delta},$$

where $x^v \in I_v^{\infty}$ is arbitrary for $v \in V$.

(ii) If ξ is lattice, then the following holds. For $k \in \{0,1\}$ and for every Borel set $B \subseteq \mathbb{R}$ for which $F \cap B$ is non-empty and is equal to a finite union of sets of the form $\pi[\omega]$, where $\omega \in E_A^*$, and for which $F_\varepsilon \cap B = (F \cap B)_\varepsilon$ for all sufficiently small $\varepsilon > 0$ we have that

$$0 < \underline{C}_k^f(F, B) < \overline{C}_k^f(F, B) < \infty.$$

The statement that the limit set of an sGDMS is Minkowski measurable if and only if it is non-lattice is a speciality for GDMS consisting of similarities. We have already seen in Theorem 5.14 that this dichotomy is not valid for limit sets of general cGDMS. Below, we will see that this dichotomy already fails to hold for the subclass of $\mathcal{C}^{1+\alpha}$-diffeomorphic images of limit sets of sGDMS. However, here there is a dichotomy for the fractal curvature measures. That is, the fractal curvature measures exist if and only if the underlying system is non-lattice. This is stated in the next theorem, where we moreover give a relationship between the (average) fractal curvature measures of the limit set of the sGDMS and of its $\mathcal{C}^{1+\alpha}$-diffeomorphic image.

Theorem 5.17 ($\mathcal{C}^{1+\alpha}$-images – fractal curvature measures)**.** *Let $K \subset \mathbb{R}$ denote the limit set of the sGDMS Φ with aperiodic irreducible incidence matrix and let δ denote its Minkowski dimension. Set $X := \langle \bigcup_{v \in V} X_v \rangle$ and let $\mathcal{U} \supset X$ be a connected open neighbourhood of X in $\mathbb{R}$. Define $g \colon \mathcal{U} \to \mathbb{R}$ to be a $\mathcal{C}^{1+\alpha}(\mathcal{U})$ map, for which $|g'|$ is bounded away from zero and $\alpha \in (0,1]$. Assume that $\lambda_1(K) = 0$ and set $F := g(K)$.*

(i) The average fractal curvature measures of both K and F exist. Moreover, $\widetilde{C}_k^f(F, \cdot)$ is absolutely continuous with respect to the push-forward measure $g_\star \widetilde{C}_k^f(K, \cdot)$ for $k \in \{0,1\}$. Their Radon-Nikodym derivative is given by

$$\frac{\mathrm{d}\widetilde{C}_k^f(F, \cdot)}{\mathrm{d}\left(g_\star \widetilde{C}_k^f(K, \cdot)\right)} = |g' \circ g^{-1}|^{\delta}.$$

(We refer the reader to the appendix for the definition of the push-forward measure.)

(ii) If Φ is non-lattice, then the fractal curvature measures of both K and F exist and coincide with the respective average fractal curvature measures.

(iii) If Φ is lattice, then neither the 0-th nor the 1-st fractal curvature measure of K and F exist.

Theorem 5.18 ($\mathcal{C}^{1+\alpha}$-images – Minkowski content)**.** *Suppose that we are in the situation of Theorem 5.17. Let ν denote the δ-conformal measure associated with K. Then we have the following.*

(i) The average Minkowski content of both K and F exist and are related by

$$\widetilde{\mathcal{M}}(F) = \widetilde{\mathcal{M}}(K) \cdot \int |g'|^{\delta} \mathrm{d}\nu.$$

(ii) If Φ is non-lattice, then the Minkowski contents of both K and F exist and coincide with the respective average Minkowski contents.

(iii) Assume that $K \subseteq [0,1]$ and that the geometric potential function ξ associated with Φ is lattice. Let $a > 0$ be maximal such that the range of ξ is contained in $a\mathbb{Z}$. Define $\widetilde{g}\colon \mathbb{R} \to \mathbb{R}$, $\widetilde{g}(x) := \nu((-\infty, x])$ to be the distribution function of ν. For $n \in \mathbb{N}$ define the function $g_n\colon [-1,\infty) \to \mathbb{R}$ by

$$g_n(x) := \int_{-1}^{x} \left(\widetilde{g}(r)(\mathrm{e}^{\delta a n} - 1) + 1\right)^{-1/\delta} \mathrm{d}r$$

and set $F_n := g_n(K)$. Then for every $n \in \mathbb{N}$ we have $\underline{\mathcal{M}}(F_n) = \overline{\mathcal{M}}(F_n)$.

For the reason why $\xi \subset a\mathbb{Z}$ for some $a > 0$ we refer to the proof of Theorem 2.38 and remark that Items (i) and (ii) of the above theorem are direct consequences of the respective items in Theorem 5.17.

Remark 5.19. The sets F_n constructed in Theorem 5.18 are not only Minkowski measurable but also satisfy $\underline{C}_0^f(F_n) = \overline{C}_0^f(F_n)$.

We now illustrate the above results by calculating the Minkowski content of the examples which were presented at the end of the preceding section.

Example 5.8 continued. For determining the Minkowski content of the limit set F of the cGDMS from Example 5.8, we apply Theorem 5.16 and thus need to find the primary gaps. Observe that

$$\langle\pi[1]\rangle = [0, 1/16], \quad \langle\pi[2]\rangle = [3/16, 1/4], \quad \langle\pi[3]\rangle = [9/16, 5/8],$$
$$\langle\pi[4]\rangle = [3/4, 13/16], \quad \langle\pi[5]\rangle = [57/64, 29/32] \quad \text{and} \quad \langle\pi[6]\rangle = [15/16, 1].$$

Thus,

$$G^1 = \underbrace{\left(\frac{1}{16}, \frac{3}{16}\right)}_{=:G^{1,1}}, \ G^2 = \varnothing \quad \text{and} \quad G^3 = \underbrace{\left(\frac{13}{16}, \frac{57}{64}\right)}_{=:G^{3,1}} \cup \underbrace{\left(\frac{29}{32}, \frac{15}{16}\right)}_{=:G^{3,2}}.$$

The primary gaps $G^{1,1}, G^{3,1}$ and $G^{3,2}$ are illustrated in Figure 5.2.

Another ingredient in the formula of Theorem 5.16 is the eigenfunction $h_{-\delta\xi}$ of the Perron-Frobenius operator $\mathcal{L}_{-\delta\xi}$ (see Section 3.1.2), where δ denotes the Minkowski dimension of F and ξ is the geometric potential function associated with Φ. In order to determine $h_{-\delta\xi}$, we first determine the measure $\nu_{-\delta\xi}$. This is done by solving the linear system of equations which arises by combining the following three ingredients. *(i)* For $e \in E$ the defining equation for $\nu_{-\delta\xi}$ implies that $\nu_{-\delta\xi}([ee']) = 4^{-\delta} \cdot \nu_{-\delta\xi}([e'])$ for every $e' \in T_{i(e)}$. *(ii)* $\nu_{-\delta\xi}([e]) = \sum_{e' \in T_{i(e)}} \nu_{-\delta\xi}([ee'])$ and *(iii)* $\sum_{e \in E} \nu_{-\delta\xi}([e]) = 1$. The resulting measure $\nu_{-\delta\xi}$ satisfies

$$\begin{aligned}
&\nu_{-\delta\xi}([1]) = \nu_{-\delta\xi}([4]) = (3 \cdot 4^{\delta} - 4^{-\delta})^{-1}, \\
&\nu_{-\delta\xi}([2]) = \nu_{-\delta\xi}([3]) = \nu_{-\delta\xi}([6]) = (4^{\delta} - 1) \cdot \nu_{-\delta\xi}([1]) \quad \text{and} \\
&\nu_{-\delta\xi}([5]) = (1 - 4^{-\delta}) \cdot \nu_{-\delta\xi}([1]).
\end{aligned}$$

For finding $h_{-\delta\xi}$, we use the approximation argument from Equation (3.3). We let 1 denote the constant one-function on Σ^{∞}. Since $\mathcal{L}^n_{-\delta\xi}1(x) = \sum_{\omega \in T^n_{i(x)}} r^{\delta}_{\omega} = \mathcal{L}^n_{-\delta\xi}1(y)$ for all $x, y \in X_v$, where $v \in V$ is arbitrary, it follows that $h_{-\delta\xi}$ is constant on one-cylinders. Using that the eigenvalue $\gamma_{-\delta\xi}$ is equal to one, that $\mathcal{L}_{-\delta\xi} h_{-\delta\xi} = \gamma_{-\delta\xi} h_{-\delta\xi}$ and that $\int h_{-\delta\xi} \mathrm{d}\nu_{-\delta\xi} = 1$, it follows that

$$\begin{aligned}
h_{-\delta\xi}(x^1) &= \tfrac{3-4^{-2\delta}}{-2\cdot 4^{-\delta}+6-4^{\delta}} && \text{for } x^1 \in I^{\infty}_1, \\
h_{-\delta\xi}(x^2) &= (1-4^{-\delta}) \cdot h_{-\delta\xi}(x^1) && \text{for } x^2 \in I^{\infty}_2 \quad \text{and} \\
h_{-\delta\xi}(x^3) &= (4^{\delta}-1) \cdot h_{-\delta\xi}(x^1) && \text{for } x^3 \in I^{\infty}_3.
\end{aligned}$$

From the above evaluations we additionally infer that the Minkowski dimension δ is the unique positive root of $4^{-\delta} - 4^{-2\delta} + 2 - 4^{\delta}$. Clearly, $H_{\mu_{-\delta\xi}} = \delta \ln 4$. Thus, altogether we

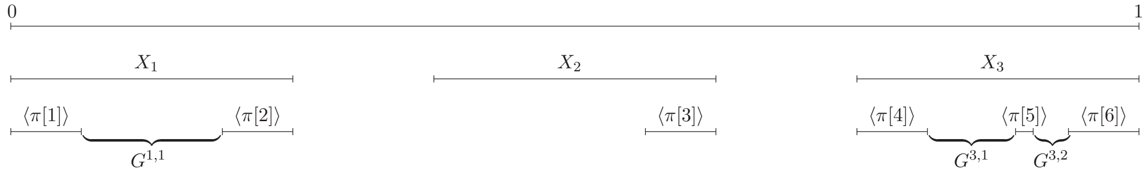


Figure 5.2: Construction of the primary gaps of the cGDMS from Example 5.8.

obtain from Theorem 5.16 that

$$\widetilde{\mathcal{M}}(F) = \frac{2^{1-\delta}}{(1-\delta)\delta \cdot \ln 4} \cdot \frac{3-4^{-2\delta}}{-2\cdot 4^{-\delta}+6-4^{\delta}} \cdot \left(\left(\frac{1}{8}\right)^{\delta} + (4^{\delta}-1) \cdot \left(\left(\frac{5}{64}\right)^{\delta} + \left(\frac{1}{32}\right)^{\delta} \right) \right).$$

Example 5.9 continued. For the system which we introduced in Example 5.9, the construction of the primary gaps is illustrated in Figure 5.3. Here, $G^{1,1} = (4/27, 5/27)$ and $G^{2,1} = (7/9, 8/9)$. The eigenfunction $h_{-\delta\xi}$ of the Perron-Frobenius operator $\mathcal{L}_{-\delta\xi}$ with eigenvalue 1 is equal to the constant one function 1 because of the following. Firstly, $\mathcal{L}_{-\delta\xi}1 \equiv 2/3^{\delta} + 1/9^{\delta}$ and secondly, $1 = 2/3^{\delta} + 1/9^{\delta}$ which can be concluded from the fact that $0 = P(-\delta\xi)$, where P denotes the topological pressure function. Thus,

$$\widetilde{\mathcal{M}}(F) = \frac{2^{1-\delta}\cdot(27^{-\delta}+9^{-\delta})}{(1-\delta)H_{\mu_{-\delta\xi}}}.$$

Figure 5.3: Primary gaps of the limit set of the cGDMS from Example 5.9.

Let us now turn to the last example from the end of the previous section. For this example, we limit ourselves to illustrating the primary gaps, since presenting the complete calculations would not give any further insights.

Example 5.10 continued. For the limit set of the sGDMS from Example 5.10, the convex hulls of the projections of the cylinder sets are given by

$$\begin{aligned} &\langle\pi[1]\rangle = [0, 2/25], && \langle\pi[3]\rangle = [1/4, 1/3], && \langle\pi[5]\rangle = [2/3, 11/15], \\ &\langle\pi[2]\rangle = [1/10, 1/5], && \langle\pi[4]\rangle = [7/18, 1/2], && \langle\pi[6]\rangle = [3/4, 5/6], && \langle\pi[7]\rangle = [8/9, 1]. \end{aligned}$$

Thus, the primary gaps are

$$G^{1,1} = (2/25, 1/10),\ G^{2,1} = (1/3, 7/18),\ G^{3,1} = (11/15, 3/4) \quad \text{and} \quad G^{3,2} = (5/6, 8/9).$$

They are illustrated in Figure 5.4. This cGDMS is non-lattice and hence its Minkowski content exists.

We end this chapter with concluding remarks addressing Conjecture 4 from [Lap93] which we commented on in Remark 2.35. With Theorem 5.16 we have seen that the Minkowski content of a limit set of an sGDMS in $\mathbb{R}$ exists if and only if the sGDMS is non-lattice.

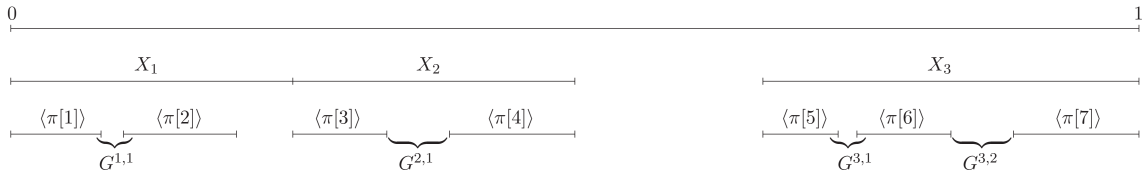


Figure 5.4: Primary gaps of the limit set of the cGDMS given in Example 5.10.

Theorem 5.14 shows that the Minkowski content of the limit set of a non-lattice cGDMS in $\mathbb{R}$ exists. It furthermore provides a condition, which implies the existence of the Minkowski content in the lattice case. How this condition simplifies for cGDMS which arise via $\mathcal{C}^{1+\alpha}$ conjugation of sGDMS is stated in Theorem 5.18. An explicit example which satisfies this condition is given below.

Example 5.20. Let $K \subseteq [0,1]$ denote the limit set of the sGDMS given in Example 5.8. Let δ denote its Minkowski dimension and let ν denote the associated δ-conformal measure. Further, let $\widetilde{g}\colon \mathbb{R} \to \mathbb{R}$ denote the distribution function of ν, that is $\widetilde{g}(x) := \nu((-\infty, x])$. Define the function $g\colon [-1,\infty) \to \mathbb{R}$ by

$$g(x) := \int_{-1}^{x} (\widetilde{g}(r)+1)^{-1/\delta} \mathrm{d}r$$

and set $F := g(K)$. Then we have $\underline{\mathcal{M}}(F) = \overline{\mathcal{M}}(F)$, although $\underline{\mathcal{M}}(K) < \overline{\mathcal{M}}(K)$. This is a consequence of Theorems 5.16 and 5.18.

Thus, altogether, we obtain that for limit sets of cGDMS the dichotomy that the Minkowski content exists if and only if the cGDMS is non-lattice is not true in general. In this context it is worth to point out that we obtain such a dichotomy statement for the fractal curvature measures of $\mathcal{C}^{1+\alpha}$-diffeomorphic images of limit sets of sGDMS in Theorem 5.17.

Finally, note that limit sets of Fuchsian groups of Schottky type (see for instance [Nic89] for a definition) can be represented as limit sets of cGDMS. In Part II of [Lal89] it is stated that such objects are non-lattice. Combined with Corollary 2.3 of [LP93], Theorem 5.14 thus verifies Conjecture 4 of [Lap93] for limit sets of Fuchsian groups of Schottky type.

A Measure Theory

Here, we collect some facts and notions from measure theory, which are needed throughout this thesis. Good references are [Bog07, Els05].

We let $\mathfrak{B}(\mathbb{R}^d)$ denote the *Borel σ-algebra* on $\mathbb{R}^d$, that is the σ-algebra generated by open sets in $\mathbb{R}^d$.

Definition A.1 (Intersection stable generator)**.** An *intersection stable generator* of $\mathfrak{B}(\mathbb{R}^d)$ is a collection of sets $\mathcal{E} \subset \mathfrak{B}(\mathbb{R}^d)$ such that the smallest σ-algebra containing $\mathcal{E}$ coincides with $\mathfrak{B}(\mathbb{R}^d)$ and such that the intersection of any two elements of $\mathcal{E}$ again is an element of $\mathcal{E}$.

Definition A.2 (Signed Borel measure)**.** A *signed Borel measure* is a σ-additive set function $\mu\colon \mathfrak{B}(\mathbb{R}^d) \to \mathbb{R} \cup \{\pm\infty\}$ for which $\mu(\varnothing) = 0$. If a signed Borel measure in non-negative, then it is a Borel measure.

For a signed Borel measure μ on $\mathfrak{B}(\mathbb{R}^d)$ the *Hahn decomposition theorem* provides a disjoint decomposition of $\mathbb{R}^d$ into two sets $\mathcal{P}, \mathcal{N} \in \mathfrak{B}(\mathbb{R}^d)$ for which $\mu(A) \geq 0$ for all Borel sets $A \subseteq \mathcal{P}$ and $\mu(A) \leq 0$ for all Borel sets $A \subseteq \mathcal{N}$. Here, the sets $\mathcal{P}$ and $\mathcal{N}$ are unique up to sets of μ-measure zero.

Definition A.3 (Variation measures, Jordan decomposition)**.** For a signed Borel measure μ we define the set functions $\mu^+, \mu^-, \mu^{\mathrm{var}}\colon \mathfrak{B}(\mathbb{R}^d) \to \mathbb{R}_0^+ \cup \{\infty\}$ through

$$\mu^+(B) := \mu(B \cap \mathcal{P}), \quad \mu^-(B) := -\mu(B \cap \mathcal{N}) \quad \text{and} \quad \mu^{\mathrm{var}}(B) := \mu^+(B) + \mu^-(B)$$

for $B \in \mathfrak{B}(\mathbb{R}^d)$, where the sets $\mathcal{P}, \mathcal{N} \in \mathfrak{B}(\mathbb{R}^d)$ are given by the Hahn decomposition theorem. μ^+, μ^- and μ^{var} are respectively called the *positive*, *negative* and *total variation measures* of μ. The decomposition $\mu = \mu^+ - \mu^-$ is known as the *Jordan decomposition* of the signed Borel measure μ.

Proposition A.4 (Theorem 6.1.2 in [Win08])**.** *Let μ_1 and μ_2 be signed Borel measures on $\mathfrak{B}(\mathbb{R}^d)$ whose total variation measures are finite. Let $\mathcal{E}$ denote an intersection stable generator of $\mathfrak{B}(\mathbb{R}^d)$ such that $\mu_1(E) = \mu_2(E)$ for all $E \in \mathcal{E}$. Then μ_1 and μ_2 agree on the whole of $\mathfrak{B}(\mathbb{R}^d)$.*

Definition A.5 (Weak convergence)**.** A family $(\mu_n)_{n\in\mathbb{N}}$ of signed Borel measures on $(\mathbb{R}^d, \mathfrak{B}(\mathbb{R}^d))$ is called *weakly convergent* to a signed Borel measure μ as $n \to \infty$, if for every bounded continuous real-valued function f on $\mathbb{R}^d$ one has

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x) \mathrm{d}\mu_n(x) = \int_{\mathbb{R}^d} f(x) \mathrm{d}\mu.$$

The definition of the integral with respect to a signed Borel measure is deduced from the definition of the integral with respect to a non-negative measure, by using the Jordan decomposition of the signed Borel measure. If $I \subset \mathbb{R}$ is an index set having x as a limit point, then the family of signed measures $(\mu_\varepsilon)_{\varepsilon\in I}$ is said to be *weakly convergent* to μ as $\varepsilon \to x$, if $(\mu_{\varepsilon_n})_{n\in\mathbb{N}}$ is weakly convergent to μ for every sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $\varepsilon_n \in I$ and $\lim_{n\to\infty} \varepsilon_n = x$.

Definition A.6 (Uniformly tight, totally bounded)**.** A family $\mathfrak{P}$ of signed Borel measures on $\mathbb{R}^d$ is called *uniformly tight*, if for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \in \mathfrak{B}(\mathbb{R}^d)$ such that $\mu^{\mathrm{var}}(\mathbb{R}^d \setminus K_\varepsilon) < \varepsilon$ for all $\mu \in \mathfrak{P}$. The family $\mathfrak{P}$ is called *totally bounded* if there exists a $c \in \mathbb{R}$ such that $\mu^{\mathrm{var}}(\mathbb{R}^d) \leq c$ for all $\mu \in \mathfrak{P}$.

Theorem A.7 (Prohorov, (see for instance Theorem 8.6.7 in [Bog07]))**.** *Let $\mathfrak{P}$ denote a uniformly tight and totally bounded family of signed Borel measures on $\mathfrak{B}(\mathbb{R}^d)$. Then every sequence in $\mathfrak{P}$ contains a weakly convergent subsequence.*

Definition A.8 (Push-forward measure)**.** Let $(U, \mathfrak{B}_1)$ and $(V, \mathfrak{B}_2)$ denote two measurable spaces. Let $T \colon U \to V$ be measurable and let μ denote a signed Borel measure on $\mathfrak{B}_1$. The push-forward measure of μ is defined to be the signed Borel measure $T_\star\mu \colon \mathfrak{B}_2 \to \mathbb{R}$ given by $T_\star\mu(B) = \mu \circ T^{-1}(B)$ for $B \in \mathfrak{B}_2$.

Index

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