Wavelet and Fourier bases on Fractals

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Abstract. In this thesis we first develop a geometric framework for spectral pairs and for orthonormal families of complex exponential functions in $L^2(\mu)$, where μ is a given Borel probability measure compactly supported in $\mathbb{R}$. Secondly, we develop wavelet bases on L^2-spaces based on limit sets of different iteration systems.

In the framework of spectral pairs we consider families of exponential functions $(e_\lambda)_{\lambda\in\Gamma}$, $e_\lambda : x \mapsto e^{i2\pi\lambda x}$, satisfying $\Gamma - \Gamma = \mathbb{Z}$, and determine the L^2-spaces in which these functions are orthonormal or constitute a basis. We also consider invariant measures on Cantor sets and study for which measures we have a family of exponential functions $(e_\lambda)_{\lambda\in\Gamma}$ that is an orthonormal basis for the L^2-space with respect to this measure. For the case of Cantor sets the families of exponential functions are obtained via Hadamard matrices.

For the study of wavelet bases, we set up a multiresolution analysis on fractal sets derived from limit sets of Markov Interval Maps. For this we consider the $\mathbb{Z}$-convolution of a non-atomic measure supported on the limit set of such a system and give a thorough investigation of the space of square integrable functions with respect to this measure. We define an abstract multiresolution analysis, prove the existence of mother wavelets and then apply these abstract results to Markov Interval Maps. Even though, in our setting, the corresponding scaling operators are in general not unitary we are able to give a complete description of the multiresolution analysis in terms of multiwavelets.

We also set up a multiresolution analysis for enlarged fractals in $\mathbb{R}$ and $\mathbb{R}^2$, which are sets arising from fractals that are generated by iterated function systems, so that the enlarged fractals are dense in $\mathbb{R}$ or $\mathbb{R}^2$, respectively. The measure supported on the fractal is also extended to a measure on the enlarged fractal. We then construct a wavelet basis via multiresolution analysis on this L^2-space with respect to the measure having the enlarged fractal as the support, with the characteristic function of the original fractal as the father wavelet which gives us via the multiresolution analysis the wavelet basis for the L^2-space. In this construction we have two unitary operators. Finally, we also apply the wavelet bases on enlarged fractals in two dimensions to image compression.

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CHAPTER 1

Introduction

This thesis deals with the connections between the limit sets of different iteration systems of contractions and countable bases in L^2-spaces which are based on these limit sets. In Part 1, we consider countable Fourier bases of the form $x \mapsto e^{i2\pi\lambda x}$, $\lambda \in \Gamma \subset \mathbb{R}$, for L^2-spaces. We start with sets $\Gamma \subset \mathbb{R}$ satisfying $\Gamma - \Gamma = \mathbb{Z}$ and study in which L^2-spaces the family of exponential functions $x \mapsto e^{i2\pi\lambda x}$, $\lambda \in \Gamma$, is a basis. Then we consider L^2-spaces on Cantor sets equipped with the measure μ of maximal entropy. Cantor sets are given by an affine iterated function system that has the same scaling in all functions.

In the second part we consider wavelet bases on different L^2-spaces. We start with a general construction, where we consider an abstract multiresolution analysis, see Definition 1.2. We then apply these results to the limit sets of Markov Interval Maps and obtain the existence of a one-sided multiresolution analysis. Further, we consider a Markov measure and we obtain a two-sided multiresolution analysis. Then we turn to the limit sets of iterated function systems which are in the first step viewed as a special case of the limit sets of Markov Interval Maps. Afterward we construct enlarged fractals from these limit sets which are dense in $\mathbb{R}$ and consider a multiresolution analysis on the L^2-spaces with respect to measures that have the enlarged fractals as support.

In the third part we consider a construction of wavelet bases on enlarged fractals in two dimensions and apply these wavelet bases to image compressions.

We give for each part of the thesis a separate introduction and state the main results for each.

1.1. ... to Part 1

We study an approach to geometric measure theory based on Fourier techniques. While traditional Fourier analysis uses as a starting point the Lebesgue measure in one or several dimensions, there is an interest in developing transform theory for other measures. This is motivated in part by problems in geometric measure theory, with a recent emphasis on scale self-similarity.

In this context we consider Borel probability measures μ in $\mathbb{R}$ such that the Hilbert space $L^2(\mu)$ has a Fourier basis of complex exponential functions $(e_\lambda)_{\lambda\in\Gamma}$, $e_\lambda : x \mapsto e^{i2\pi\lambda x}$, $x \in \mathbb{R}$, for some countable set $\Gamma \subset \mathbb{R}$. If μ satisfies this property, the set of frequencies Γ in the orthonormal basis (ONB) is called a spectrum for μ and (μ, Γ) is called a spectral pair.

Historically, these questions arise from the study of spectral pairs in connection with the tiling of a space. This connection is stated in the Fuglede Conjecture from 1974, which says: A measurable set Ω in $\mathbb{R}^d$ is a spectral set if and only if it tiles $\mathbb{R}^d$ by translation. A measurable set $\Omega \subset \mathbb{R}^d$ is said to be spectral if the measure $\mu = \lambda|_\Omega$, where λ is the d-dimensional Lebesgue measure, is a spectral measure, see [**JP99**]. This conjecture is not completely solved; in fact there are counter examples for dimension $d \geq 5$. Despite the counter examples, the connection between spectral sets and tilings is strongly evident, especially in low dimensions. There are many positive results concerning the connection between spectral sets and tiles, see e.g. [**DJ07, DJ09b, IP98, JP92**]. The tiling spaces have connections to physics, for example in understanding the diffraction in molecular structures that form quasi-crystals.

Classical Fourier analysis considers the canonical Fourier duality of the torus $\mathbb{T}$ and its dual group $\mathbb{Z}$. One difference between the classical Fourier analysis on the one hand and spectral pairs (μ, Γ) on the other is the absence of a group structure in the context of general spectral pairs. In fact, for general spectral pairs, typically neither of the sets in the pair, the support of the measure μ, or its spectrum Γ, has a group structure.

Although fractals do not have a group structure or a Haar measure, it is possible to identify a substitute for the Haar measure. To each fractal set C associated to an iterated function system (IFS) there is a probability measure with support C that is, like a Haar measure, uniquely determined by an invariance property, see [**Hut81**] or Theorem A.9. This measure is typically singular with respect to the Lebesgue measure.

We consider a special class of measures which are supported on Cantor sets. More precisely, the Cantor sets are fractals that are given by an affine IFS (aIFS) with the same scale for all branches, that is $\left(\tau_b(x) = \frac{x+b}{R}\right)_{b\in B}$, $R \in \mathbb{N}$, $R \geq 2$, $B \subset \underline{R} := \{0, 1, \ldots, R-1\}$. The measure on such a Cantor set is the measure of maximal entropy which coincides with the $\frac{\log N}{\log R}$-Hausdorff measure restricted to the Cantor set C, where $N = \operatorname{card} B$. This measure also coincides with the measure obtained via Hutchinson's theorem, see Theorem A.9, with the weight $1/N$ on all subsets $\tau_b(C)$, $b \in B$.

The question of existence of a spectrum for a measure supported on a Cantor set is significant, even in simple examples. Jorgensen and Pedersen show in [**JP98a**] that the measure on the 1/4-Cantor set (given by the aIFS $\left(\tau_0(x) = \frac{x}{4}, \tau_1(x) = \frac{x+2}{4}\right)$) has a spectrum and in [**JP98b**] that the measure on the middle-third Cantor set (given by the aIFS $\left(\tau_0(x) = \frac{x}{3}, \tau_1(x) = \frac{x+2}{3}\right)$) has no spectrum. In fact, there are no more than two mutually orthonormal functions e_s, $s \in \mathbb{R}$, in the L^2-space of the measure of this set.

In the literature, predominantly one class of fractals is considered for the construction of spectral pairs, namely the Cantor sets and their measures of maximal entropy, see [**DHS09, DJ06, DJ07, DJ08, DJ09a, DJ09b, DHJ09, JP98b, JP98a, ŁW06**] et al. These measures have the advantage that their Fourier transforms can be explicitly written down as infinite products, which allow their zeros to be easily computed. In [**ŁW06**] Łaba and Wang obtain some general results for not necessarily self-similar measures. They show that a non-zero finite spectral Borel measure μ is either discrete or has no discrete part. Furthermore, if $\mu = \sum_{a\in A} p_a \delta_a$ is discrete, $p_a > 0$, then A is a finite set and μ assigns the same weight to each point of A. Further, they point out that in all known examples of spectral pairs the measures are either absolutely continuous with respect to the Lebesgue measure or they are purely singular.

There are other publications on spectral pairs in connection with different areas. Jorgensen and Pedersen, [**JP98a, JP98b**], found connections to Hardy spaces and other authors extended the research on spectral pairs to fractals with overlap. Some results for these fractals are stated in [**JKS07**].

There is also one approach that considers a "general" Fourier basis on not necessarily affine fractals, see [**BK10**]. This general Fourier basis is constructed via a homeomorphism between a Cantor set and the fractal under consideration. The Fourier basis for the Cantor set is then carried over to the other fractal. In this way, it is even possible to obtain such a general Fourier basis for the middle-third Cantor set by using a homeomorphism between the 1/4-Cantor set and the middle-third Cantor set.

Overview and main results of Part 1. In general the question of spectral pairs can be approached from two directions. We can either start with a measure μ and ask whether there exists a set Γ such that (μ, Γ) is a spectral pair or we can start with the countable set Γ and look for measures μ such that (μ, Γ) is a spectral pair.

Denote by $M^{\perp}(\Gamma)$ the set of all compactly supported probability measures μ such that the family $(e_\lambda)_{\lambda\in\Gamma}$ is an orthonormal family in $L^2(\mu)$. Analogously, $M^{OB}(\Gamma)$ is the set of all compactly supported probability measures μ such that (μ, Γ) is a spectral pair. We start with a thorough investigation of the spaces $M^{\perp}(\Gamma)$ and $M^{OB}(\Gamma)$ for arbitrary countable sets Γ. We refer the reader to Chapter 3 and Chapter 4 for further information.

In the next step we turn to specific sets Γ, namely those with the property $\Gamma - \Gamma = \mathbb{Z}$. For the corresponding sets $M^{\perp}(\Gamma)$ we obtain the following main results. The first result concerns the structure of the Fourier transform of measures in $M^{\perp}(\Gamma)$, see Theorem 5.3.

Theorem. *The Fourier transform of every $\mu \in M^{\perp}(\Gamma)$ factors as a product*

$$\widehat{\mu} = f \cdot \widehat{\lambda|_{[0,1]}},$$

where f extends to to an entire function on $\mathbb{C}$ of the form given in (5.0.3).

The function f in the theorem above is of the form $f(z) = e^{h(z)}\xi(z)$ with h and ξ suitable functions obtained via the Weierstrass Factorization Theorem. The last theorem already indicates a correspondence between the measures $\mu \in M^{\perp}(\Gamma)$ and the Lebesgue measure restricted to the unit interval. Our next result gives a connection between the sets $M^{OB}(\mathbb{Z})$ and $M^{\perp}(\mathbb{Z})$, see Theorem 5.7.

Theorem. *$M^{OB}(\mathbb{Z})$ is the set of extreme points of $M^{\perp}(\mathbb{Z})$, i.e. $M^{OB}(\mathbb{Z}) = \operatorname{ext}(M^{\perp}(\mathbb{Z}))$.*

The following theorem, see Theorem 5.9, completely answers the question of which countable sets $\Gamma \subset \mathbb{R}$ with $\Gamma - \Gamma = \mathbb{Z}$ admit a measure such that (μ, Γ) is a spectral pair.

Theorem. *For $\Gamma \subsetneq \mathbb{Z}$ with $\Gamma - \Gamma = \mathbb{Z}$, $M^{OB}(\Gamma) = \emptyset$.*

So spectral pairs exist for this class of sets only for $\Gamma = \mathbb{Z} + a$, $a \in \mathbb{R}$, and one natural element in $M^{OB}(\mathbb{Z})$ is $\lambda|_{[0,1]}$. Theorem 5.3 gives a connection between all elements in $M^{\perp}(\mathbb{Z})$ and the element $\lambda|_{[0,1]}$ via their Fourier transform.

One possible extension of the class of countable sets $\Gamma \subset \mathbb{R}$ with the above property is to the class $\{\Gamma \subset \mathbb{R} : \Gamma - \Gamma = k\mathbb{Z}, k \in \mathbb{R}\}$. By using the following proposition, compare Proposition 5.16, we obtain results analogous to those for the class $\{\Gamma \subset \mathbb{R} : \Gamma - \Gamma = \mathbb{Z}\}$.

Proposition. *$\mu \in M^{\perp}(k\mathbb{Z})$ if and only if there is $\nu \in M^{\perp}(\mathbb{Z})$ with $\widehat{\nu}(t) = \widehat{\mu}(kt)$ for all $t \in \mathbb{R}$.*

In Chapter 6 we turn the question around, that is, we start with a measure μ and ask when it is possible to find a set Γ such that (μ, Γ) is a spectral pair. We restrict ourselves to the consideration of invariant measures on Cantor sets and the spectra are obtained via complex Hadamard matrices. A complex Hadamard matrix is a unitary matrix given in terms of a scaling $R \geq 2$ and two sets $B, L \subset \mathbb{N}_0$ by $\frac{1}{\sqrt{N}}\left(e^{i2\pi blR^{-1}}\right)_{b\in B, l\in L}$, $N = \operatorname{card} B$. One result characterizes the existence of a Hadamard matrix in terms of the zeros of $\sum_{b\in B} e_b$, see Proposition 6.14. Let $Z\left(\sum_{b\in B} e_b\right)$ denote the set of zeros of $\sum_{b\in B} e_b$ in $\mathbb{R}$ and $\mathcal{Z}\left(\sum_{b\in B} e_b\right)$ denote the set of zeros in the unit interval.

Proposition. *Let $B \subset \underline{R}$, $0 \in B$, $R \in \mathbb{N}$, $R \geq 2$. There exists a Hadamard matrix $M_R(B, L)$, $L \subset \underline{R}$, $\operatorname{card} L = \operatorname{card} B$, if and only if there are $N-1$ elements $a_j \in \mathcal{Z}\left(\sum_{b\in B} e_b\right)$, $j \in \underline{N}\backslash\{0\}$, with $a_j - a_i \in Z\left(\sum_{b\in B} e_b\right)$, $i, j \in \underline{N}\backslash\{0\}$, $i \neq j$, and $Ra_j \in \mathbb{Z}$, $j \in \underline{N}\backslash\{0\}$.*

If we consider a specific class of Hadamard matrices, namely the Fourier matrices, we can even say precisely for which scalings we can obtain a Fourier matrix, see Lemma 6.22. Fourier matrices of size $(N \times N)$ are Hadamard matrices where the second row is given by $\left(1, \omega, \omega^2, \ldots, \omega^{N-1}\right)$, $\omega = e^{i2\pi/N}$.

Lemma. *Let $M_R(B, L)$, $R \geq 2$, $B, L \subset \mathbb{N}_0$, $\operatorname{card} B = \operatorname{card} L = N < \infty$, give an $(N \times N)$-Fourier matrix. Then the scaling R must be a multiple of N, i.e., $R = Nk$ for some $k \in \mathbb{N}$.*

The next result gives a partial answer to the question of when we can actually obtain a spectral pair directly from a Hadamard matrix, see Proposition 6.27. This proposition gives two cases in which we have a spectrum of an invariant measure μ directly from a Hadamard matrix.

Proposition. *Let $B \subset \underline{R}$, $0 \in B$, $R \in \mathbb{N}$, $R \geq 2$, and let $L \subset \underline{R}$, $\operatorname{card} L = \operatorname{card} B$, be such that $M_R(B, L)$ gives a Hadamard matrix.*

1. *If $\gcd\{b_1, \ldots, b_{N-1}\} = 1$, then $(\mu_B, \Gamma(L))$ is a spectral pair, where $\Gamma(L) := \left\{\sum_{i=0}^{n} l_i R^i : l_i \in L, n \in \mathbb{N}_0\right\}$. In particular, if $1 \in B$, then $(\mu_B, \Gamma(L))$ is always a spectral pair.*
2. *If $R \geq 2$ is even and $2^n \in B$ for some $n \in \mathbb{N}$, then $(\mu_B, \Gamma(L))$ is a spectral pair.*

This part of the thesis is organized as follows. In Chapter 2 we give the basic definitions, state results from the literature and fix the notation. In Chapter 3 we give results concerning the Banach algebra of measures with particular attention to measures that are absolutely continuous with respect to the Lebesgue measure, and in Chapter 4 we consider what can be deduced starting with a general spectral pair. Chapter 5 considers whether we can find for a set $\Gamma \subset \mathbb{R}$ with $\Gamma - \Gamma = \mathbb{Z}$ a probability measure μ such that $(e_\gamma)_{\gamma\in\Gamma}$ is an ONB in $L^2(\mu)$. In Chapter 6 we consider a specific class of fractals, precisely the Cantor sets, and for these fractals we obtain spectral pairs in one dimension via Hadamard matrices. We establish other properties of these spectral pairs. In Section 6.5, we state these result and give further results for the special class of (3×3)-Hadamard matrices. In Chapter 7, we give some ideas about possible further research and open problems.

Parts of the results stated here can be found in [**BJ11**] like most of Chapters 2, 3 and 4; of Chapter 5 there are parts like Theorem 5.3 and 5.9 published in [**BJ11**] and also results for the special case of (3×3)-Hadamard matrices of Chapter 6 are in [**BJ11**].

1.2. ... to Part 2

It is natural to consider wavelets in the context of fractals since both carry a self-similar structure; the fractal inherits it from the prescribed scaling of the iterated function system (IFS) while the wavelet satisfies a certain scaling identity.

The aim of wavelet analysis is to approximate functions by using superpositions from a wavelet basis. This basis is supposed to be orthonormal and derived from a finite set of functions, the so-called mother wavelets. To obtain such a basis we employ the multiresolution analysis (MRA) which uses a function, called father wavelet, that satisfies specific properties given below. Our main goal is therefore to set up an MRA in the non-linear situation. For this we generalize some ideas from [**DJ06, DMP08**], which are restricted to homogeneous linear cases with respect to the restriction of certain Hausdorff measures. These results have been extended in [**BK10**] to non-linear fractals with the measure of maximal entropy.

In the case of a fractal given by an IFS on $[0, 1]$ there are several approaches to constructing wavelet bases. All give bases on the L^2-space associated to suitable singular measures which are supported on enlarged versions of the original fractal. An enlarged fractal is derived from the original fractal by first mapping scaled copies of it to each gap interval and then taking the union of translates by $\mathbb{Z}$, thus defining a dense set in $\mathbb{R}$. In [**DJ06**], the authors construct a wavelet basis for fractals on self-similar Cantor sets, that is, sets that are given by affine IFS with the same scaling factor $1/N$, $N \geq 2$, for all $p \leq N$ branches. They consider the L^2-space with respect to μ, the δ-dimensional Hausdorff measure restricted to the enlarged fractal, where δ denotes the dimension of the Cantor set. In this situation the analysis depends on the two unitary operators U and T, where U denotes the *scaling operator* given by $Uf := \sqrt{p} f(N\cdot)$ and T denotes the *translation operator* given by $Tf := f(\cdot - 1)$ for $f \in L^2(\mu)$. Furthermore, a natural choice for a father wavelet φ is the characteristic function of the original fractal. The authors show that for a family of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(\mu)$ the following six conditions are satisfied, where cl stands for the closure.

- $\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$,
- $\mathrm{cl} \bigcup_{j\in\mathbb{Z}} V_j = L^2(\mu)$,
- $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
- $V_{j+1} = UV_j$, $j \in \mathbb{Z}$,
- $\{T^n\varphi : n \in \mathbb{Z}\}$ is an orthonormal basis for V_0,
- $U^{-1}TU = T^N$.

These observations allow the authors to construct a wavelet basis for $L^2(\mu)$ explicitly given via mother wavelets which are obtained from the father wavelet in terms of certain filter functions.

In [**BK10**] we generalized this approach by allowing conformal IFS satisfying the open set condition on $[0, 1]$. We chose the measure of maximal entropy supported on the fractal and this measure is extended to a measure μ supported on the enlarged fractal in $\mathbb{R}$. Then similarly as in [**DJ06**] we constructed the wavelet basis via MRA in terms of the unitary scaling operator U and the unitary translation operator T. Again via filter functions the mother wavelets ψ_i, $i \in \{1, \ldots, N-1\}$ were defined such that $\left\{U^n T^k \psi_i : n, k \in \mathbb{Z}, i \in \{1, \ldots, N-1\}\right\}$ provided an orthonormal basis of $L^2(\mu)$.

In the literature there are also constructions for limit sets of Markov Interval Maps. A Markov Interval Map (MIM) can be seen as a generalization of an IFS since it consists of contractions with an incidence matrix. To our knowledge there are at least two further approaches to construct wavelet bases on the limit sets of MIMs, namely [**MP09, KS10**], and there is one approach for the specific case of a β-transformation given in [**GP96**]. In [**MP09**] Marcolli and Paolucci consider the limit set X of an MIM inside the unit interval consisting of the inverse branches $\tau_i(x) = \frac{x+i}{N}$ for $i \in \underline{N} = \{0, \ldots, N-1\}$ with some transition rule encoded in an incidence matrix A. This limit set can be associated with a Cantor set inside the unit interval. The Cantor set is then equipped with the Hausdorff measure of the appropriate dimension δ. If all transitions were allowed, the limit set would coincide with a usual Cantor set given by an affine iterated function system. They then use the representation of the Cuntz-Krieger algebra $\mathcal{O}_A$, where A is the incidence matrix, for the construction of the orthonormal system of wavelets on $L^2\left(H^\delta|_X\right)$ and not a multiresolution analysis. Their proofs mainly rely on results in [**Bod07, Jon98**]. Finally, Marcolli and Paolucci give a possible application where they adapt the construction of a wavelet basis to graph wavelets for finite graphs with no sinks, which can be associated to Cuntz-Krieger algebras. These graph wavelets are a useful tool for spatial network traffic analysis, compare [**MP09, CK03**].

In [**KS10**] Keßeböhmer and Samuel construct a Haar basis analogous to the wavelet basis construction in [**DJ06**] for the middle-third Cantor set for a one-sided topologically exact sub-shift of finite type and with respect to a Gibbs measure μ_ϕ for a Hölder continuous potential ϕ. The construction is then used to obtain a spectral triple in the framework of non-commutative geometry.

The construction of wavelet bases in spaces other than $L^2(\mathbb{R}, \lambda)$, where λ is the Lebesgue measure on $\mathbb{R}$, may lead to a further understanding of non-commutative geometry in the sense that we can obtain a Fourier or wavelet basis for quasi-lattices or quasi-crystals.

As an essential non-linear example for the construction of a wavelet basis on limit sets of MIMs one can take the limit set of a Kleinian group together with the measure of maximal entropy or the Patterson-Sullivan measure, see Example 9.3.

As an example for wavelet bases for MIMs we apply the construction to a β-transformation, where $\beta = \frac{1+\sqrt{5}}{2}$ denotes the golden mean, see Example 1.1. In this way we obtain a wavelet basis for $L^2\left(\nu_\mathbb{Z}\right)$, where ν is the invariant measure for this transformation and $\nu_\mathbb{Z}$ arises from ν by translation by $\mathbb{Z}$, compare [**Rén57, Par60**]. This measure is absolutely continuous with respect to the Lebesgue measure. In [**GP96**], Gazeau and Patera construct a similar basis to ours for the β-transformation with respect to the Lebesgue measure on $\mathbb{R}$. Instead of a translation by the group $\mathbb{Z}$, they use a translation by so-called β-integers which are given by the β-adic expansions and are obtained by a greedy algorithm. There are some common features between our construction and the one in [**GP96**], in fact both give characteristic functions on intervals depending on powers of β. But since we consider different measures, we have different coefficients.

Overview and results of Part 2. Here, our aim is to extend the construction of wavelet bases with respect to fractal measures in different ways. We start with the extension to the construction of wavelet bases on the by $\mathbb{Z}$ translated limit set of a Markov Interval Map. A Markov Interval Map consists of a family $(B_i)_{i=0}^{N-1}$ of closed subintervals in $[0,1]$ with disjoint interior and a function $F : \bigcup_{i\in\underline{N}} B_i \to [0,1]$, such that $F|_{B_i}$ is expanding and C^1, $i \in \underline{N}$ and such that $F(B_i) \cap B_j \neq \emptyset$ implies $B_j \subset F(B_i)$. Its (fractal) *limit set* is given by $X := \bigcap_{n=0}^{\infty} F^{-n} J$, where $J := \bigcup_{i\in\underline{N}} B_i$. By considering its inverse branches $\tau_i := \left(F|_{B_i}\right)^{-1}$, $i \in \underline{N}$, we obtain a Graph Directed Markov System (see [**MU03**]) with incidence matrix $A = \left(A_{ij}\right)_{i,j\in\underline{N}}$, where $A_{ij} = 1$ if $F(B_i) \supset B_j$ and 0 otherwise. For the precise definition see Definition 9.1 and for an explicit example of an MIM see Example 1.1 where we consider the β-transformation. Up to a countable set where the coding map is finite-to-one, the limit set X is homeomorphic to the topological Markov chain $\Sigma_A := \{\omega = (\omega_0, \omega_1, \ldots) \in \underline{N}^\mathbb{N} : A_{\omega_i\omega_{i+1}} = 1 \text{ for all } i \geq 0\}$. For the definition of the canonical coding map π from Σ_A to X see (9.1.1).

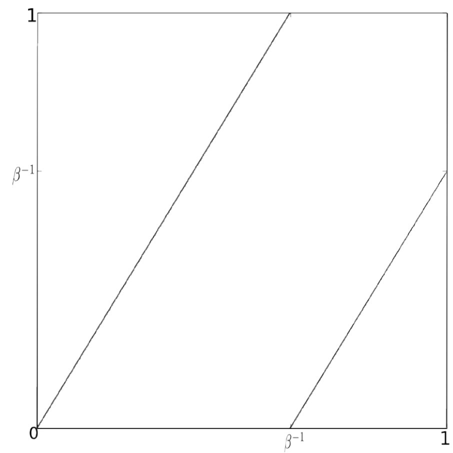


FIGURE 1.2.1. The graph of the β-transform.

Given a Markov measure $\widetilde{\nu}$ on the shift space Σ_A with a probability vector $(p_i)_{i\in\underline{N}}$ and stochastic matrix $(\pi_{ij})_{i,j\in\underline{N}}$, we consider the probability measure $\nu := \widetilde{\nu}\circ\pi^{-1}$, to which we also refer as a Markov measure. The $\mathbb{Z}$-convolution (by translations) of ν is given by

$$\nu_{\mathbb{Z}} := \sum_{k\in\mathbb{Z}} \nu(\cdot - k).$$

Similar to the construction in [**BK10**] we introduce the scaling operator

$$Uf(x) := \sum_{k\in\mathbb{Z}}\sum_{j\in\underline{N}}\sum_{i\in\underline{N}} \sqrt{\frac{p_i}{p_j\pi_{ji}}}\cdot \mathbb{1}_{[ji]}(x-k)\cdot f(\tau_j^{-1}(x-k)+j+Nk) \tag{1.2.1}$$

and the translation operator

$$Tf(x) := f(x-1) \tag{1.2.2}$$

for $f\in L^2(\nu_{\mathbb{Z}})$ and $x\in\mathbb{R}$, where $[ji]\subset\mathbb{R}$, $i,j\in\underline{N}$, denotes a cylinder set (see Section 9.1). It is important to note that in contrast to the construction of the scaling operator for IFS the operator U is in general not unitary. Nevertheless, we have the following properties (see Proposition 9.18).

Proposition. *Let $(\varphi_i)_{i\in\underline{N}}$ denote a family of father wavelets given by $\varphi_i := \sqrt{\nu([i])}^{-1}\mathbb{1}_{[i]}$, $i\in\underline{N}$. The translation operator T and the scaling operator U satisfy the following properties.*

(1) $TU = UT^N$,
(2) $\varphi_i = U\sum_{j\in\underline{N}}\sqrt{\pi_{ij}}T^i\varphi_j$, $i\in\underline{N}$,
(3) $\langle T^k\varphi_i | T^l\varphi_j\rangle = \delta_{(k,i),(l,j)}$, $k,l\in\mathbb{Z}$, $i,j\in\underline{N}$,
(4) $UU^* = I$,
(5) $U^*U = I$ *if and only if* $A_{ij}=1$ *for all* $i,j\in\underline{N}$.

For an explicit formula for U^* see (9.3.1). As an example for this setting we consider the β-transformation.

Example 1.1 (β-transformation)**.** Let $\beta := \frac{1+\sqrt{5}}{2}$ denote the golden mean. Then the β-transform is given by $F:[0,1]\to[0,1]$, $x\mapsto \beta x \bmod 1$ (see Figure 1.2.1 for the graph of F). This map can be considered as an MIM as follows. In this case we have $X := [0,1]$ and the inverse branches are $\tau_0(x) := \frac{x}{\beta}$, $x\in[0,1]$, and $\tau_1(x) := \frac{x+1}{\beta}$, $x\in[0,\beta-1]$. We may choose the two intervals $B_0 := [0,\beta-1]$ and $B_1 := [\beta-1,1]$ and the corresponding incidence matrix is then given by $A := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

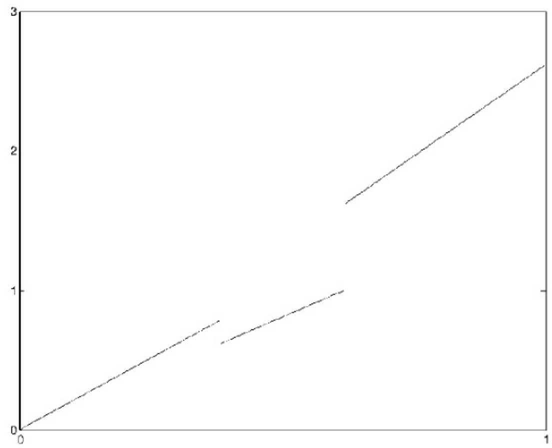


FIGURE 1.2.2. The graph of $U\left(\mathrm{id}_{[0,1]}\right)$.

From [**Rén57, Par60**] we know that there exists an invariant measure ν for the β-transformation absolutely continuous with respect to the Lebesgue measure restricted to $[0,1]$ with density h given by

$$h(x) := \begin{cases} \frac{5+3\sqrt{5}}{10}, & 0 \leq x < \frac{\sqrt{5}-1}{2}, \\ \frac{5+\sqrt{5}}{10}, & \frac{\sqrt{5}-1}{2} \leq x < 1. \end{cases}$$

The measure ν can be represented on Σ_A by a stationary Markov measure with the stochastic matrix

$$\Pi := \begin{pmatrix} \beta-1 & 2-\beta \\ 1 & 0 \end{pmatrix}$$

and probability vector $p := \left(\frac{\beta}{\sqrt{5}}, \frac{\beta-1}{\sqrt{5}}\right)$. The scaling operator U acting on $L^2\left(\nu_{\mathbb{Z}}\right)$ is then given for $x \in \mathbb{R}$ by

$$Uf(x) = \sum_{k\in\mathbb{Z}} \left(\sqrt{\beta}\mathbb{1}_{[0,\beta^{-2})}(x-k) + \mathbb{1}_{[\beta^{-2},\beta^{-1})}(x-k) + \beta \cdot \mathbb{1}_{[\beta^{-1},1)}(x-k)\right) \cdot f\left(\beta(x-k)+2k\right).$$

For the father wavelets we may choose $\varphi_0 = \left(\sqrt{5}/\beta\right)^{1/2} \mathbb{1}_{[0,\beta-1)}$ and $\varphi_1 = \left(\sqrt{5}\beta\right)^{1/2} \mathbb{1}_{[\beta-1,1)}$. The action of U is illustrated in Figure 1.2.2, where U is applied to the identity map $\mathrm{id}_{[0,1]} : x \mapsto x$, restricted to $[0,1]$. That is for $x \in [0,1]$ we have

$$U\left(\mathrm{id}_{[0,1]}\right)x = \left(\sqrt{\beta}\mathbb{1}_{[0,\beta^{-2})}(x) + \mathbb{1}_{[\beta^{-2},\beta^{-1})}(x) + \beta \cdot \mathbb{1}_{[\beta^{-1},1)}(x)\right)\beta x.$$

We further generalize our construction by considering non-atomic probability measures ν on X which we do not assume to be Markovian. In this case it is natural to consider more than one scaling operator U. More precisely, we consider a family of scaling operators $\left(U^{(n)}\right)_{n\in\mathbb{Z}}$ which allow us to construct an orthonormal wavelet basis. For this we define $U^{(0)} := I$, where I denotes the identity operator, and for $f \in L^2(\nu_{\mathbb{Z}})$ and $n \in \mathbb{N}$, $x \in \mathbb{R}$, we let

$$U^{(n)}f(x) := \sum_{k\in\mathbb{Z}} \sum_{\omega\in\Sigma_A^n} \sum_{j\in\underline{N}} \sqrt{\frac{\nu_{\mathbb{Z}}([j])}{\nu_{\mathbb{Z}}([\omega j])}} \mathbb{1}_{[\omega j]}(x-k) \cdot f\left(\tau_\omega^{-1}(x-k) + \sum_{i=0}^{n-1} \omega_{n-1-i}N^i + N^n k\right) \tag{1.2.3}$$

and

$$U^{(-n)}f(x) := \sum_{k\in\mathbb{Z}}\sum_{\omega\in\Sigma_A^n}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([j])}}\mathbb{1}_{[j]}\left(x-\sum_{i=0}^{n-1}\omega_{n-1-i}N^i-N^nk\right)$$

$$\cdot f\left(\tau_\omega\left(x-\sum_{i=0}^{n-1}\omega_{n-1-i}N^i-N^nk\right)+k\right). \tag{1.2.4}$$

It is straightforward to verify that if the measure ν is Markovian, then we have $U^{(n)}=U^n$ for $n\in\mathbb{N}_0$ and $U^{(-n)}=(U^*)^n$, $n\in\mathbb{N}$. More details are provided in Section 9.3. Furthermore, the operators $\left(U^{(n)}\right)_{n\in\mathbb{Z}}$ and T satisfy the following relations (see Proposition 9.9).

Proposition. *Let $(\varphi_j)_{j\in\underline{N}}$ denote the family of father wavelets given by $\varphi_i=(\nu_{\mathbb{Z}}([i]))^{-1/2}\,\mathbb{1}_{[i]}$, $i\in\underline{N}$. The translation operator T and the family of scaling operators $\left(U^{(n)}\right)_{n\in\mathbb{Z}}$ satisfy the following.*

(1) $TU^{(n)}=U^{(n)}T^{N^n}$, $n\in\mathbb{N}$,
(2) $U^{(-n)}T\varphi_j=T^{N^n}U^{(-n)}\varphi_j$, $n\in\mathbb{N}$, $j\in\underline{N}$,
(3) $\varphi_i=U^{(1)}T^i\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([ij])}{\nu_{\mathbb{Z}}([i])}}\varphi_j$, $i\in\underline{N}$,
(4) *if* $U^{(n)}T^k\varphi_i\neq 0$, *then* $\langle U^{(n)}T^k\varphi_i|U^{(n)}T^l\varphi_j\rangle=\delta_{(k,i),(l,j)}$, $n,k,l\in\mathbb{Z}$, $i,j\in\underline{N}$,
(5) $U^{(n)}U^{(-n)}=I$, $n\in\mathbb{N}$,
(6) *if* $U^{(n)}T^k\varphi_j\neq 0$, *then* $U^{(-n)}U^{(n)}T^k\varphi_j=T^k\varphi_j$, $n\in\mathbb{N}$, $k\in\mathbb{Z}$, $j\in\underline{N}$.

The properties of $\left(U^{(n)}\right)_{n\in\mathbb{Z}}$ and T lead us to the following abstract definition of a multiresolution analysis which involves more than one father wavelet. In the literature these functions are sometimes called multiwavelets (cf. [**Alp93**]). Let $\mathcal{B}$ be the Borel σ-algebra on $\mathbb{R}$.

Definition 1.2 (Abstract MRA)**.** Let μ be a non-atomic measure on $(\mathbb{R},\mathcal{B})$.

(1) Let $\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{Z}}$ and $\mathcal{T}$ be bounded linear operators on $L^2(\mu)$ such that $\mathcal{T}$ is unitary and $\mathcal{U}^{(0)}=I$. We say that $\left(\mu,\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{Z}},\mathcal{T}\right)$ allows a *two-sided multiresolution analysis (two-sided MRA)* if there exists a family $\{V_j:\ j\in\mathbb{Z}\}$ of closed subspaces of $L^2(\mu)$ and for some $N\in\mathbb{N}$ there exists a family of functions (called *father wavelets*) $\varphi_j\in L^2(\mu)$, $j\in\underline{N}$, with compact support, such that the following conditions are satisfied.
 (a) $\cdots\subset V_{-2}\subset V_{-1}\subset V_0\subset V_1\subset V_2\subset\cdots$,
 (b) $\operatorname{cl}\bigcup_{j\in\mathbb{Z}}V_j=L^2(\mu)$,
 (c) $\bigcap_{j\in\mathbb{Z}}V_j=\{0\}$,
 (d) $\left(\mathcal{U}^{(n)}\left\{\mathcal{T}^k\varphi_j:\ k\in\mathbb{Z},\ j\in\underline{N}\right\}\right)\setminus\{0\}$ is an orthonormal basis of V_n for all $n\in\mathbb{Z}$,
 (e) $\mathcal{U}^{(n)}\left\{\mathcal{T}^k\varphi_i:k\in\underline{N^n},i\in\underline{N}\right\}\subset\operatorname{span}\mathcal{U}^{(n+1)}\left\{\mathcal{T}^k\varphi_i:k\in\underline{N^{n+1}},i\in\underline{N}\right\}$, $n\in\mathbb{N}_0$, and $\mathcal{U}^{(-n)}\{\varphi_i:i\in\underline{N}\}\subset\operatorname{span}\mathcal{U}^{(-n+1)}\left\{\mathcal{T}^k\varphi_i:i\in\underline{N},k\in\underline{N}\right\}$, $n\in\mathbb{N}$,
 (f) $\mathcal{T}\,\mathcal{U}^{(n)}|_{V_0}=\mathcal{U}^{(n)}\,\mathcal{T}^{N^n}|_{V_0}$ and $\mathcal{U}^{(-n)}\,\mathcal{T}|_{V_0}=\mathcal{T}^{N^n}\,\mathcal{U}^{(-n)}|_{V_0}$, $n\in\mathbb{N}$.

(2) Let $\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{N}_0}$ and $\mathcal{T}$ be bounded linear operators on $L^2(\mu)$ such that $\mathcal{T}$ is unitary and $\mathcal{U}^{(0)}=I$. We say that $\left(\mu,\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{N}_0},\mathcal{T}\right)$ allows a *one-sided multiresolution analysis (one-sided MRA)* if there exists a family $\{V_j:\ j\in\mathbb{N}_0\}$ of closed subspaces of $L^2(\mu)$ and for some $N\in\mathbb{N}$ there exists a family of functions (called *father wavelets*) $\varphi_j\in L^2(\mu)$, $j\in\underline{N}$, with compact support, such that the following conditions are satisfied.
 (a) $V_0\subset V_1\subset V_2\subset\cdots$,
 (b) $\operatorname{cl}\bigcup_{j\in\mathbb{N}_0}V_j=L^2(\mu)$,
 (c) $\left(\mathcal{U}^{(n)}\left\{\mathcal{T}^k\varphi_j:\ k\in\mathbb{Z},\ j\in\underline{N}\right\}\right)\setminus\{0\}$ is an orthonormal basis of V_n for all $n\in\mathbb{N}_0$,

(d) $\mathcal{U}^{(n)}\left\{\mathcal{T}^k\varphi_i : k\in\underline{N^n}, i\in\underline{N}\right\}\subset \operatorname{span}\mathcal{U}^{(n+1)}\left\{\mathcal{T}^k\varphi_i : k\in\underline{N^{n+1}}, i\in\underline{N}\right\}$, $n\in\mathbb{N}_0$,
(e) $\mathcal{T}\,\mathcal{U}^{(n)}|_{V_0}=\mathcal{U}^{(n)}\,\mathcal{T}^{N^n}|_{V_0}$, $n\in\mathbb{N}$.

Our next theorem shows that in the setting of MIM as given above, $\left(\nu_{\mathbb{Z}},\left(U^{(n)}\right)_{n\in\mathbb{Z}},T\right)$ allows an abstract MRA as introduced above (see Theorem 9.11).

Theorem. *Let $\left(U^{(n)}\right)_{n\in\mathbb{N}_0}$ be given as in (1.2.3) and T given in (1.2.2). Then $\left(\nu_{\mathbb{Z}},\left(U^{(n)}\right)_{n\in\mathbb{N}_0},T\right)$ allows a one-sided MRA, where the father wavelets are taken to be $\varphi_i:=\left(\nu_{\mathbb{Z}}([i])\right)^{-1/2}\mathbb{1}_{[i]}$, $i\in\underline{N}$.*

For the abstract MRA we show that there always exists an orthonormal wavelet basis and we give a precise form of the basis (see Theorem 8.1).

Theorem. *Let μ be a non-atomic measure on $\mathbb{R}$, $\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{Z}}$ be a family of bounded linear operators on $L^2(\mu)$ and $\mathcal{T}$ be a unitary operator on $L^2(\mu)$. If $\left(\mu,\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{Z}},\mathcal{T}\right)$ allows a two-sided MRA with father wavelets φ_j, $j\in\underline{N}$, then there exist for every $n\in\mathbb{N}_0$ numbers $d_n\in\underline{N^{n+2}}$, $d_{-n}\in\underline{N^2}$, $q_n\in\underline{N^{n+1}}$, $q_{-n}\in\underline{N}$, with $d_n\geq q_n$, $d_{-n}\geq q_{-n}$, and two families of mother wavelets $\left(\psi_{n,l}:l\in\underline{d_n-q_n}\right)$, $\left(\psi_{-n,l}:l\in\underline{d_{-n}-q_{-n}}\right)$, $n\in\mathbb{N}_0$, such that the following set of functions defines an orthonormal basis for $L^2(\mu)$,*

$$\left\{\mathcal{T}^k\psi_{n,l}:\,n\in\mathbb{N}_0,\,l\in\underline{d_n-q_n},\,k\in\mathbb{Z}\right\}\cup\left\{\mathcal{T}^{N^nk}\psi_{-n,l}:\,n\in\mathbb{N},\,l\in\underline{d_{-n}-q_{-n}},\,k\in\mathbb{Z}\right\}.$$

Remark 1.3. We give a construction for the family of mother wavelets $\psi_{n,l}$ in Section 8.1. The mother wavelets $\psi_{n,l}$ are given as linear combinations of $\mathcal{U}^n\,\mathcal{T}^k\varphi_j$ with coefficients chosen appropriately. For each $n\in\mathbb{Z}$ we consider the linear subspaces $W_n:=V_{n+1}\ominus V_n$, where the closed subspaces V_n of $L^2(\mu)$ are as in Definition 1.2, together with the finite family of functions $\left(\psi_{n,l}:l\in\underline{d_n-q_n}\right)$. We show that for $n\geq 0$ and for $n<0$ the sets $\left\{\mathcal{T}^k\psi_{n,l}:k\in\mathbb{Z},l\in\underline{d_n-q_n}\right\}$ and $\left\{\mathcal{T}^{N^{|n|}k}\psi_{n,l}:k\in\mathbb{Z},l\in\underline{d_{-|n|}-q_{-|n|}}\right\}$, respectively, give an orthonormal basis of W_n.

An immediate consequence of the proof of Theorem 8.1 is the following corresponding result for the one-sided MRA (see Corollary 8.4).

Corollary. *Let μ be a non-atomic measure on $\mathbb{R}$, $\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{N}_0}$ a family of bounded linear operators on $L^2(\mu)$ and $\mathcal{T}$ a unitary operator on $L^2(\mu)$. If $\left(\mu,\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{N}_0},\mathcal{T}\right)$ allows a one-sided MRA with the father wavelets φ_j, $j\in\underline{N}$, then there exist for every $n\in\mathbb{N}_0$ numbers $d_n\in\underline{N^{n+2}}$, $q_n\in\underline{N^{n+1}}$ with $d_n\geq q_n$ and a family of mother wavelets $\left(\psi_{n,l}:l\in\underline{d_n-q_n}\right)$, $n\in\mathbb{N}_0$, such that the following set of functions defines an orthonormal basis for $L^2(\mu)$*

$$\left\{\mathcal{T}^k\psi_{n,l}:\,n\in\mathbb{N}_0,\,l\in\underline{d_n-q_n},\,k\in\mathbb{Z}\right\}\cup\left\{\mathcal{T}^k\varphi_i:k\in\mathbb{Z},n\in\underline{N}\right\}.$$

The construction for an MIM with an underlying Markov measure ν belongs to a specific class. In this class the scaling operators $\mathcal{U}^{(n)}$ can be represented multiplicatively. In our general framework we say that $\left(\mu,\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{N}_0},\mathcal{T}\right)$ is multiplicative if there exists a bounded linear operator $\mathcal{U}$ on $L^2(\mu)$ such that $\mathcal{U}^{(n)}=\mathcal{U}^n$ and $\mathcal{U}^{(-n)}=(\mathcal{U}^*)^n$ hold for all $n\in\mathbb{N}_0$. The results concerning the mother wavelets simplify in this case as a consequence of the following lemma (see Lemma 8.5).

Lemma. *Let us assume that $\left(\mu,\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{Z}},\mathcal{T}\right)$ allows a two-sided MRA with the closed subspaces V_n, $n\in\mathbb{Z}$, of $L^2(\mu)$ from Definition 1.2 and set $W_n:=V_{n+1}\ominus V_n$, $n\in\mathbb{Z}$.*

- *If there is a bounded linear operator $\mathcal{U}$ such that $\mathcal{U}^{(n)} = \mathcal{U}^n$ for all $n \in \mathbb{N}$, then $W_n = \mathcal{U}^n W_0$, $n \in \mathbb{N}$.*
- *If there is a bounded linear operator $\mathcal{U}$ such that $\mathcal{U}^{(-n)} = (\mathcal{U}^*)^n$ for all $n \in \mathbb{N}$, then $W_{-n} = (\mathcal{U}^*)^{n-1} W_{-1}$, $n \in \mathbb{N}$.*

Thus, we only have to find appropriate mother wavelets for W_0 and W_{-1} and obtain a wavelet basis by repeatedly applying $\mathcal{U}$. More precisely, this observation allows us to derive the following corollary from the Theorem 8.1 (see Corollary 8.8).

Corollary. *If $\left(\mu, \left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{N}_0}, \mathcal{T}\right)$ is multiplicative, then there exists an orthonormal basis of $L^2(\mu)$ of the form*

$$\left(\left\{\mathcal{U}^n \, \mathcal{T}^k \, \psi_l : n \in \mathbb{N}_0, k \in \mathbb{Z}, l \in \underline{d_0 - N}\right\} \cup \left\{(\mathcal{U}^*)^n \, \mathcal{T}^k \, \psi_{-,l} : n \in \mathbb{N}_0, k \in \mathbb{Z}, l \in \underline{d_{-1} - N}\right\}\right) \setminus \{0\},$$

where the functions ψ_l, $l \in \underline{d_0 - N}$, and $\psi_{-,l}$, $l \in \underline{d_{-1} - N}$, are given explicitly in Remark 8.7.

The above corollary applied to Example 1.1 with the β-transformation as the MIM leads to the following construction.

Example (Example 1.1 (continued))**.** The mother wavelet is given by

$$\psi = \left(\sqrt{5}(2-\beta)\right)^{1/2} \mathbb{1}_{[0,(\beta-1)^2)} - \left(\sqrt{5}\right)^{1/2} \mathbb{1}_{[(\beta-1)^2,\beta-1)}$$

and so a basis is given by

$$\left\{T^k \varphi_1 : \, k \in 2\mathbb{Z}+1\right\} \cup \left\{U^n T^k \psi : \, k \in D_n, \, n \in \mathbb{N}\right\} \cup \left\{(U^*)^n \, T^k \psi : \, k \in \mathbb{Z}, \, n \in \mathbb{N}\right\},$$

where

$$D_n := \left\{\sum_{j=0}^{n-1} k_j 2^j + 2^n l : (k_j)_{j\in\underline{n}} \in \{0,1\}^n, k_j \cdot k_{j-1} = 0, j \in \underline{n-1}, \, l \in \mathbb{Z}\right\}.$$

The proof that this indeed defines an orthonormal basis will be postponed to Section 9.3.2.

In the case of an MRA for an MIM with Markov measure ν we have in particular that $U^{(n)} = U^n$ and $U^{(-n)} = (U^*)^n$ and we even obtain a stronger correspondence between Markov measures for MIMs and a two-sided MRA (see Theorem 9.12).

Theorem. *$\left(\nu_{\mathbb{Z}}, \left(U^{(n)}\right)_{n\in\mathbb{Z}}, T\right)$ allows a two-sided MRA with respect to the father wavelets $\varphi_i := (\nu_{\mathbb{Z}}([i]))^{-1/2} \, \mathbb{1}_{[i]}$, $i \in \underline{N}$, if and only if the measure ν is Markovian.*

In the case of ν being a Markov measure we obtain an even stronger property than multiplicative: we have $\varphi_j \in \operatorname{span} U \left\{T^j \varphi_i : i \in \underline{N}\right\}$ for each $j \in \underline{N}$. We call an MRA with this property *translation complete*. We further investigate multiplicative MRA which are translation complete in Section 8.2. In this situation we derive a 0-1-valued transition matrix A given by $A_{ij} = 0$ if and only if $\mathcal{U} \, \mathcal{T}^i \, \varphi_j = 0$ and show that for an MIM the matrix coincides with the incidence matrix. This observation is used to construct the mother wavelets in a simpler way by using a unitary matrix for each father wavelet to obtain coefficients for the corresponding mother wavelets. We will use this approach to construct the mother wavelets for MIMs. The results so far can also be found in [**BK11**].

In the next step we consider an MIM with the measure $\nu_{\mathbb{Z}}$ and a different family of scaling operators $\left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}$ acting on $L^2(\nu_{\mathbb{Z}})$ which are given for $x \in \mathbb{R}$, $n \in \mathbb{N}$, by

$$\widetilde{U}^{(n)} f(x) := \sum_{k\in\mathbb{Z}} \sum_{\omega\in\Sigma_A^n} \sqrt{\frac{1}{\nu_{\mathbb{Z}}([\omega])}} \mathbb{1}_{[\omega]}(x-k) f\left(\tau_\omega^{-1}(x-k) + N^n k + \sum_{i=0}^{n-1} \omega_{n-1-i} N^i\right)$$

and for $n \in \mathbb{Z}$, $n < 0$, by

$$\widetilde{U}^{(-n)} f(x) := \sum_{k\in\mathbb{Z}} \sum_{\omega\in\Sigma_A^n} \sqrt{\nu_{\mathbb{Z}}([\omega])}\mathbb{1}_X\Big(x - \sum_{i=0}^{n-1}\omega_{n-1-i}N^i - N^n k\Big)$$
$$f\left(\tau_\omega(x - \sum_{i=0}^{n-1}\omega_{n-1-i}N^i - N^n k) + k\right).$$

This family of operators $\left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}$, the translation operator T defined above and the father wavelet $\varphi = \mathbb{1}_X$ satisfy the same properties that are given in Proposition 9.9 for $\left(U^{(n)}\right)_{n\in\mathbb{Z}}$, T and the family of father wavelets $\varphi_i = \frac{1}{\sqrt{\nu([i])}}\mathbb{1}_{[i]}$, $i \in \underline{N}$, only the scaling relation for the father wavelet takes the form $\varphi = \widetilde{U}^{(1)} \sum_{j\in\underline{N}} \sqrt{\nu_{\mathbb{Z}}([j])}T^j\varphi$. In this setting we obtain the following result concerning the existence of a two-sided MRA and the MIM being an IFS (see Theorem 9.28).

Theorem. $\left(\nu_{\mathbb{Z}}, \left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}, T\right)$ *allows a two-sided MRA with respect to the father wavelet* $\varphi := \mathbb{1}_X$*, if and only if the measure* ν *is a measure obtained by Hutchinson's theorem with a probability vector* $p = (p_0, \ldots, p_{N-1})$ *for an IFS. Furthermore, the family of scaling operators* $\left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}$ *is multiplicative, i.e.* $\widetilde{U}^{(n)} = \left(\widetilde{U}^{(1)}\right)^n$*, if and only if* ν *is a measure obtained by Hutchinson's theorem with a probability vector* $p = (p_0, \ldots, p_{N-1})$ *for an IFS.*

After establishing this correspondence between two-sided MRA with the family of scaling operators $\left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}$ and IFSs we turn to the construction of wavelet bases on enlarged fractals in analogy to [**BK10, DJ06**] and extend these constructions so that different measures are allowed. In this setting we have two unitary operators U and T and one father wavelet φ for the MRA and so the definition of the MRA can take a different form.

Remark 1.4. Let μ be a non-atomic measure on $(\mathbb{R}, \mathcal{B})$. Let $\mathcal{U}$ and $\mathcal{T}$ be unitary operators on $L^2(\mu)$. We say $(\mu, \mathcal{U}, \mathcal{T})$ allows a *two-sided multiresolution analysis (two-sided MRA)* if there exists a family $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mu)$ and a function (called a *father wavelet*) $\varphi \in L^2(\mu)$, with compact support, such that the following conditions are satisfied.

(1) $\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$,
(2) $\mathrm{cl}\bigcup_{j\in\mathbb{Z}} V_j = L^2(\mu)$,
(3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
(4) $\left\{\mathcal{T}^k \varphi : k \in \mathbb{Z}\right\}$ is an orthonormal basis of V_0,
(5) $\mathcal{U} V_j = V_{j+1}$ for all $j \in \mathbb{Z}$,
(6) $\mathcal{U}^{-1}\mathcal{T}\mathcal{U} = \mathcal{T}^N$ for some $N \in \mathbb{N}$.

Note that the condition (5) is equivalent to the condition $\varphi \in \mathrm{span}\,\mathcal{U}\left\{\mathcal{T}^k \varphi : k \in \mathbb{Z}\right\}$. In our setting of the MRA on enlarged fractals we even have that $\varphi \in \mathrm{span}\,\mathcal{U}\left\{\mathcal{T}^k \varphi : k \in \underline{N}\right\}$, which is equivalent to the condition (1e) of Definition 1.2.

We start with a fractal $C \subset [0,1]$ given by an IFS satisfying the open set condition (OSC) for $(0,1)$. In the first step the IFS is extended to one which has $[0,1]$ as the invariant set and that satisfies the OSC for $(0,1)$ by defining affine functions on the gaps. So the extended IFS consists of contractions $(\tau_i : i \in \underline{N})$, $N \in \mathbb{N}$. In a set A we encode those functions in the extended IFS that belong to the original IFS. The enlarged fractal is then defined by mapping scaled copies of the fractal into the gaps using the extended IFS and in the next step it is translated by $\mathbb{Z}$. So the enlarged fractal is defined to be

$$R := \bigcup_{k\in\mathbb{Z}} \left(\biguplus_{\omega\in\Sigma\cup\{\emptyset\}} \tau_\omega(C) \right) + k,$$

where $\Sigma = \{(i_0,\ldots,i_{k-1}) \in \underline{N}^k : k \in \mathbb{N}, i_{k-1} \notin A\}$. On the fractal we then consider a measure μ obtained by Hutchinson's theorem (see Theorem A.9) for the IFS with the weights $p_i \in (0,1)$, $i \in A$ and $\sum_{i\in A} p_i = 1$. This measure is also extended to a measure which has the enlarged fractal as the essential support. First this measure is defined on the unit interval by setting

$$\nu := \sum_{\omega\in\Sigma\cup\{\emptyset\}} \prod_{i=0}^{|\omega|-1} c_{\omega_i} \cdot \mu \circ \tau_\omega^{-1}$$

for some weights $c_i \in \mathbb{R}^+$, $i \in \underline{N}$, on the gaps. By translation it is then defined on $\mathbb{R}$ to be $\nu_\mathbb{Z}(\cdot) := \sum_{k\in\mathbb{Z}} \nu(\cdot - k)$. Then two unitary operators T and U acting on $L^2(\nu_\mathbb{Z})$ are defined by setting $Tf(\cdot) = f(\cdot - 1)$ and, for $x \in \mathbb{R}$,

$$\begin{aligned} Uf(x) = \sum_{k\in\mathbb{Z}} \Bigg(&\sum_{i\in\underline{N}\setminus A} \sqrt{c_i}^{-1} \cdot \mathbb{1}_{\tau_i([0,1))}(x-k) \cdot f\left(\tau_i^{-1}(x-k) + Nk + i\right) \\ &+ \sum_{i\in A} \left(\sqrt{c_i}^{-1} \cdot \mathbb{1}_{\tau_i([0,1)\setminus C)}(x-k) + \sqrt{p_i}^{-1} \cdot \mathbb{1}_{\tau_i(C)}(x-k) \right) \cdot f\left(\tau_i^{-1}(x-k) + Nk + i\right) \Bigg). \end{aligned}$$

So U is given in terms of the the extended IFS in analogy to the definition of the measure $\nu_\mathbb{Z}$. We then obtain the following theorem.

Theorem. *Let the father wavelet be $\varphi := \mathbb{1}_C$ and for $j \in \mathbb{Z}$ let*

$$V_j := \operatorname{cl}\operatorname{span}\left\{U^j T^k \varphi : k \in \mathbb{Z}\right\},$$

then $(\nu_\mathbb{Z}, U, T)$ allows a two-sided MRA with respect to φ and V_j, $j \in \mathbb{Z}$. In particular,

$$\operatorname{cl}\operatorname{span}\left\{U^n T^k \varphi : n \in \mathbb{Z}, k \in \mathbb{Z}\right\} = L^2(\nu_\mathbb{Z}).$$

We also study further the MRA, where the definition of the MRA uses a slightly different notion (see the definition in [**BK10**]). That is, we consider a measure $\nu_\mathbb{Z}$ on $(\mathbb{R},\mathcal{B})$ such that $\nu_\mathbb{Z}(A) = \nu_\mathbb{Z}(A+k)$, $A \in \mathcal{B}$, $k \in \mathbb{Z}$, and $\operatorname{cl}\left(\operatorname{supp}(\nu_\mathbb{Z}|_{[0,1]})\right) = [0,1]$. Furthermore the scaling operator U is given in terms of a scaling function. This function $\sigma : \mathbb{R} \to \mathbb{R}$ is a continuous, bijective, strictly increasing function with $\sigma' > 1$ such that for some fixed $N \in \mathbb{N}$ and $p \in \mathbb{N}$ we have

$$\begin{aligned} \sigma(x+k) &= \sigma(x) + Nk, \quad && x \in [0,1], \ k \in \mathbb{Z}, \\ \nu_\mathbb{Z}(\sigma(A)) &= p\nu_\mathbb{Z}(A), && A \in \mathcal{B}. \end{aligned}$$

Then we define the scaling operator U acting on $L^2(\nu_\mathbb{Z})$ by setting $Uf(\cdot) = \sqrt{p}f(\sigma(\cdot))$ and for all $j \in \mathbb{Z}$ the condition $UV_j = V_{j+1}$ is equivalent to $f \in V_j \iff f \circ \sigma \in V_{j+1}$. In this situation we only consider measures $\nu_\mathbb{Z}$ such that $\nu_\mathbb{Z}([0,1]) < \infty$, from which we deduce that $p = N$. For this case we can give different wavelet bases. To do this we use the classical MRA on $L^2(\mathbb{R},\lambda)$ with the scaling operator $\widetilde{U}f(\cdot) = \sqrt{N}f(N\cdot)$ and the translation operator $Tf(\cdot) = f(\cdot - 1)$, which is well known. Then there exists a homeomorphism ϕ between the two spaces $L^2(\nu_\mathbb{Z})$ and $L^2(\mathbb{R},\lambda)$ such that ϕ intertwines the functions σ and $x \mapsto Nx$. It also holds that $\nu_\mathbb{Z} = \lambda \circ \phi^{-1}$.

This part is organized as follows. We start with an abstract MRA in Chapter 8. In Section 8.2 we then consider the special case of multiplicative systems. In Section 8.3 we show how the condition of translation completeness simplifies the construction of the mother wavelets. Afterward, in Chapter 9, we apply the abstract MRA to the construction of a wavelet bases for Markov Interval Maps. There we start with a family of operators $\left(U^{(n)}\right)_{n\in\mathbb{Z}}$ acting on $L^2(\nu_\mathbb{Z})$ for an arbitrary non-atomic probability measure ν on the limit set of an MIM in the unit interval translated by $\mathbb{Z}$ and show that

$\left(\nu_{\mathbb{Z}}, \left(U^{(n)}\right)_{n\in\mathbb{N}_0}, T\right)$ always allows a one-sided MRA. If on the other hand $\left(\nu_{\mathbb{Z}}, \left(U^{(n)}\right)_{n\in\mathbb{Z}}, T\right)$ allows a two-sided MRA, we then prove that the measure ν is necessarily Markovian. The construction of the mother wavelets will be given explicitly. In Section 9.3 we give an explicit construction of the wavelet basis if the measure ν is Markovian. We also consider iterated function systems as a special case of MIMs.

In Chapter 10 we then turn to the construction of wavelet bases on so-called enlarged fractals in one dimension. We prove that if we consider a measure obtained by Hutchinson's theorem on the fractal and extend it to a measure on the enlarged fractal, then a two-sided MRA is allowed for $L^2(\nu_{\mathbb{Z}})$, where the father wavelet is the characteristic function of the fractal.

The results of Chapter 8 and Chapter 9 (except Section 9.4) can also be found in [**BK11**].

1.3. ... to Part 3

In Part 3 we turn to the construction of wavelet bases in dimensions higher than one, so that applications are possible, for instance to image compression. As for one dimensional wavelets and fractals, two dimensional versions also have properties in common like self-similarity. Furthermore, the wavelet analysis uses dilations and translations in the construction and many self-similar fractals also have dilations. Another interesting aspect is that both wavelets and fractals are used in image compression, where both have advantages and disadvantages, like blurring by zooming in, or long compression times. Because of these common features, we construct a common mathematical foundation.

The first approach in the literature for the construction of wavelet bases on fractals in two dimensions can be found in [**Str97**]. Strichartz constructs a wavelet basis consisting of piecewise linear functions on the Sierpinski Gasket itself. His construction uses triangulations of the Sierpinski Gasket and a one-sided MRA. More precisely, Strichartz's wavelets form a frame for the Sierpinski Gasket not an orthonormal basis since the functions are not orthogonal within each scale; only on different scales. This construction can also be applied to other connected fractals that are post-critically finite and also smoother wavelets can be constructed.

Another approach in the literature can be found in [**DMP08, D'A08**]. It also gives a construction of a wavelet basis on the Sierpinski Gasket, but this is a similar construction to the construction for the middle-third Cantor set given in [**DJ06**] and it considers an enlarged fractal for the Sierpinski Gasket. In [**DMP08, D'A08**] D'Andrea, Merrill and Packer consider the $\log(3)/\log(2)$-dimensional Hausdorff measure restricted to an enlarged fractal of the Sierpinski Gasket C. The enlarged fractal is defined as $R = \bigcup_{n\in\mathbb{N}} \bigcup_{(k,l)\in\mathbb{Z}^2} \mathbf{A}^n\left(C + (k,l)^t\right)$, where $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Their construction considers as the father wavelet the characteristic function on the Sierpinski Gasket and the unitary operators for the MRA are $Uf(\cdot) = \sqrt{3}f(\mathbf{A}\cdot)$ and $T^{(k,l)}f(\cdot) = f\left(\cdot - (k,l)^t\right)$, $(k,l) \in \mathbb{Z}^2$.

They also apply this wavelet basis to image compression. Their compression scheme is in analogy to the compression with the two dimensional Haar wavelet. We generalize this approach by considering different fractals and different measures obtained by Hutchinson's theorem on these fractals.

Overview and main results of Part 3. In Chapter 11 we construct two dimensional wavelet bases on enlarged fractals. This construction is analogous to that given earlier in dimension one. Nevertheless, there are more restrictions on possible fractals for the construction. The first restriction is that the fractal must lie inside a closed bounded set $D \subset \mathbb{R}^2$ such that the plane allows a tiling by the set D with $\mathbb{Z}^2$ and two vectors $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. Furthermore, there must exist an extension of the IFS of the form $\left(\tau_{(i,j)} : (i,j) \in \underline{N_1} \times \underline{N_2}\right)$ so that the extended IFS satisfies the open set condition for $\mathring{D}$ (the interior of D) and it has D as the invariant set. From the fractal we obtain an enlarged fractal by mapping scaled copies of the fractal under the extended IFS in its gaps and in the second step we translate this set by $\mathbb{Z}^2$ and the vectors $\overrightarrow{v_1}, \overrightarrow{v_2}$. On the fractal we consider a measure obtained by Hutchinson's theorem, which we then extend to one with the enlarged fractal as the support in an analogous way to the extension of the enlarged fractal. Then we construct a wavelet basis in the L^2-space with respect to the measure on the enlarged fractal. The construction is done via multiresolution analysis of the following form.

Remark 1.5. Let μ be a non-atomic measure on $\left(\mathbb{R}^2, \mathcal{B}\right)$. Let $\mathcal{U}$ and $\mathcal{T}$ be unitary operators on $L^2(\mu)$. We say $(\mu, \mathcal{U}, \mathcal{T})$ allows a *two-sided multiresolution analysis (two-sided MRA)* if there exists a family $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mu)$ and a function (called a *father wavelet*) $\varphi \in L^2(\mu)$, with compact support, such that the following conditions are satisfied.

(1) $\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$,
(2) $\operatorname{cl} \bigcup_{j\in\mathbb{Z}} V_j = L^2(\mu)$,
(3) $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$,
(4) $\left\{\mathcal{T}^{(k,l)} \varphi : (k,l) \in \mathbb{Z}^2\right\}$ is an orthonormal basis of V_0,
(5) $\mathcal{U} V_j = V_{j+1}$ for all $j \in \mathbb{Z}$,
(6) $\mathcal{U}^{-1} \mathcal{T}^{(k,l)} \mathcal{U} = \mathcal{T}^{(N_1 k, N_2 l)}$, $(k,l) \in \mathbb{Z}^2$, for some $N_1, N_2 \in \mathbb{N}$.

We consider a scaling operator U that is given in terms of the functions in the extended IFS. The translation operator T is defined by $T^{(k,l)} f(\cdot) = f(\cdot - k\overrightarrow{v_1} - \overrightarrow{v_2})$, $(k,l) \in \mathbb{Z}^2$. We consider as the father wavelet the characteristic function of the fractal. In this setting it allows a two-sided MRA. The wavelet basis is given in terms of mother wavelets which are constructed in terms of filter functions on the two-dimensional torus.

After giving this theoretical foundation, we apply the constructed wavelet bases to image compression, where we compare the results for different wavelet bases. The application to image compression follows along the lines of [**DMP08**]. As a result we observe that the structure of the underlying fractal is imposed on the compressed image. Furthermore, there is a correlation between the Hausdorff dimension of the underlying fractal and the compression results for the wavelet bases.

In image compression the actual choice of possible fractals is more limited than those that allow an MRA on their enlarged fractal, because the information about the operators U and T is not used; only the filter functions are used. So the fractal is assumed to lie in a rectangle and the IFS to consist of affine scalings that map the rectangle to equally sized copies of itself.

In Chapter 11, we construct in analogy to the one dimensional wavelet bases on enlarged fractals, the mathematical foundation for wavelet bases on enlarged fractals in two dimensions. In Chapter 12 we then start by explaining how the image compression takes place and in Section 12.2 we apply different wavelet bases to an image.

Remarks about the appendix. In the appendix there are introductions to the mathematical fields of fractal geometry, wavelet analysis and C^*-algebras and we also state various results that are used in the main part of the thesis. In Appendix B and C we give some connections to different mathematical fields for the wavelet bases on enlarged fractals as defined in Chapter 10 and Chapter 11. We give a connection to representations of the Cuntz algebra $\mathcal{O}_N$. More precisely, we consider the two representations $(Z_i)_{i\in\underline{N}}$ given by $Z_i|n\rangle = |Nn+i\rangle$, $i \in \underline{N}$, $n \in \mathbb{N}_0$, on $l^2(\mathbb{N}_0)$ and $(S_i)_{i\in\underline{N}}$ given by $(S_i f)(z) := m_i(z) f\left(z^N\right)$, $i \in \underline{N}$, $z \in \mathbb{T} := \{\omega \in \mathbb{C} : |\omega| = 1\}$, on $L^2(\mathbb{T}, \lambda)$, where m_i, $i \in \underline{N}$, are the filter functions obtained by the MRA, of the Cuntz algebra $\mathcal{O}_N$. Then we can write the scaling operator U of the MRA for enlarged fractals in one dimension in terms of these representations. More precisely, in Proposition B.5 we show that

$$U = \sum_{i\in\underline{N}} Z_i \otimes S_i^*,$$

where the correspondence $L^2(\nu_{\mathbb{Z}}) \backsimeq l^2(\mathbb{N}_0) \otimes L^2(\mathbb{T}, \lambda)$ is used (see Proposition B.3). An analogous interpretation of the operator U is also given for the two dimensional MRA on enlarged fractals.

In Appendix C we apply a direct limit approach, as considered in [**BLP$^+$10**], to the wavelet bases on enlarged fractals of Chapter 10 and Chapter 11. We consider in one dimension the isometry

$$S = S_{m_0} : f(z) \mapsto m_0(z) f\left(z^N\right), \ z \in \mathbb{T},$$

on $L^2(\mathbb{T}, \lambda)$, where m_0 is the low-pass filter obtained from the MRA on an enlarged fractal, and take the direct limit space $\left(\left(L^2(\mathbb{T}, \lambda)\right)_\infty, S_\infty, \varrho_\infty\right)$ of the system $\left(L^2(\mathbb{T}, \lambda), S\right)$, where ϱ_∞ is obtained from the unitary representation defined for $n \in \mathbb{Z}$ by $\left(\varrho_n f\right)(z) = z^n f(z)$, $f \in L^2(\mathbb{T}, \lambda)$, $z \in \mathbb{T}$. We show that this direct limit space $\left(\left(L^2(\mathbb{T}, \lambda)\right)_\infty, S_\infty, \varrho_\infty\right)$ is isomorphic to $\left(L^2(\nu_{\mathbb{Z}}), U, T\right)$ (see Corollary C.10). In the next step we construct an orthonormal basis for $\left(L^2(\mathbb{T}, \lambda)\right)_\infty$ by applying a theorem of [**BLP$^+$10**] to our setting and using the high-pass filter functions of the MRA. Finally, we show in Proposition C.15 that this orthonormal basis for $\left(L^2(\mathbb{T}, \lambda)\right)_\infty$ is mapped to the wavelet basis constructed in Chapter 10 of $L^2(\nu_{\mathbb{Z}})$ under application of a unique isometry $\mathcal{R}_\infty : \left(L^2(\mathbb{T}, \lambda)\right)_\infty \to L^2(\nu_{\mathbb{Z}})$ which exists by the direct limit theory. We also obtain the analogous results for the two dimensional wavelet bases on enlarged fractals. Our constructions fit nicely in this setting and in this way we can give partly different proofs.

In the nomenclature it is evident that some letters are used for different things, but we hope that it is always clear from the context which is meant. The entries in the nomenclature are divided by the parts they appear in or whether they are of a more general nature. Furthermore, we only put those defined objects in the nomenclature that appear in several sections.

1.4. A connection between Fourier bases and wavelet bases on fractals

We only show the connection between Fourier bases and wavelet bases on enlarged fractals for Cantor sets in one dimension since the construction of Fourier bases on fractals as considered in Part 1 is only done for Cantor sets. (Nevertheless, it is also possible to define a "general" Fourier bases on fractals given by an IFS and equipped with their measures of maximal entropy, see [**BK10**].) The connection is considered in [**DJ06, Jor06**] and we explain it on the example of the 1/4-Cantor set. We start by giving the precise setting and then give the corresponding result.

For further information concerning the example of the 1/4-Cantor set C compare Example 4.3. Recall that the IFS is $\left(\tau_0(x) = \frac{x}{4}, \tau_2(x) = \frac{x+2}{4}\right)$ and its invariant measure μ is $\mu = H^{1/2}|_C$, the 1/2-Hausdorff measure restricted to the Cantor set. We know that for $\Gamma = \left\{\sum_{i=0}^k l_i 4^i : l_i \in \{0,1\},\, k \in \mathbb{N}_0\right\}$ $(e_\lambda)_{\lambda \in \Gamma}$ is an ONB for $L^2(\mu)$

Now we turn to the construction of the wavelet basis. We consider the extended IFS $\left(\tau_i(x) = \frac{x+i}{4}\right)_{i \in \underline{4}}$ with the weights on the gaps given by $c_i = \frac{1}{2}$ for all $i \in \{0,1,2,3\}$. Consequently, the measure on the enlarged fractal is the 1/2-Hausdorff measure restricted to the enlarged fractal and the scaling operator U is given by $Uf(\cdot) = \sqrt{2} f(4 \cdot)$, $f \in L^2\left(H^{1/2}\right)$. The father wavelet φ is the characteristic function on the 1/4-Cantor set, $\varphi = \mathbb{1}_C$, and the corresponding filter functions are

$$\begin{aligned}
m_0(z) &= \frac{1}{\sqrt{2}}\left(1 + z^2\right), \\
m_1(z) &= z, \\
m_2(z) &= z^3, \\
m_3(z) &= \frac{1}{\sqrt{2}}\left(1 - z^2\right).
\end{aligned}$$

Thus, the mother wavelets are defined as $\psi_i = U m_i(T)$, $i \in \{1,2,3\}$, and the ONB for $L^2\left(H^{1/2}\right)$ is given by

$$\left\{U^n T^k \psi_i : n, k \in \mathbb{Z},\, i \in \{1,2,3\}\right\}.$$

We can rewrite this basis in the way of Proposition B.1 for $i \in \{0,1,2,3\}$ and $n \in \mathbb{N}_0$ as

$$\varphi_{4n+i} = U m_i(T) \varphi_n$$

and obtain the basis $\left\{T^k\varphi_n : k \in \mathbb{Z},\, n \in \mathbb{N}\right\}$ of $L^2(\nu_{\mathbb{Z}})$. From this sequence $(\varphi_n)_{n\in\mathbb{N}_0}$ we only consider the first four functions which are precisely φ, ψ_1, ψ_2, ψ_3, for the following result.

Theorem ([**Jor06**]). *Under the definitions above the family*

$$\{e_\lambda(t)\varphi_j(t-k) : \lambda \in \Gamma,\, j = 0, 3,\, k \in \mathbb{Z}\} \cup \{e_\lambda(t/4)\varphi_j(t-k) : \lambda \in \Gamma,\, j = 1, 2,\, k \in \mathbb{Z}\}$$

is an orthonormal basis in the Hilbert space $L^2\left(H^{1/2}\right)$.

Remark 1.6.

(1) This theorem holds also for different Cantor sets that allow a spectral pair. But it does not hold for the middle-third Cantor set, since there does not exist a spectrum for the measure $H^{\log(2)/\log(3)}$ restricted to the middle-third Cantor set.

(2) We can extend the result to other fractals that are homeomorphic to a Cantor set in the sense of [**BK10**]. In this case we do not have the "usual" Fourier basis but a general Fourier basis of the form $e_n \circ \phi^{-1}$, where ϕ is a suitable homeomorphism.

Part 1

Spectral pairs

CHAPTER 2

Basic definitions and some results from the literature

We start with the general definition of the spectrum and the spectral pair and afterward we state the definition that we mainly use.

Definition 2.1 ([**JP98b, JP99**]). Given (μ,ν), where μ, ν are two Borel probability measures in $\mathbb{R}^d$, $F_{\mu,\nu} : L^2(\mu) \to L^2(\nu)$ is defined by

$$(F_{\mu,\nu} f)(\xi) = \int_{\mathbb{R}^d} e_\xi(x) f(x) d\mu(x),$$

for $f \in L^2(\mu)$, $\xi \in \mathbb{R}^d$, $e_\xi(x) := e^{i2\pi\xi\cdot x}$. (μ,ν) is called a *spectral pair* if and only if $F_{\mu,\nu} : L^2(\mu) \to L^2(\nu)$ is unitary, i.e. isometric and onto.

For the spectral pairs there is the following simple equivalence.

Proposition 2.2 ([**JP98b**]). *Let μ and ν be positive Borel measures on $\mathbb{R}^d$. Then (μ,ν) is a spectral pair if and only if (ν,μ) is a spectral pair.*

Definition 2.3. Let μ and ν be positive Borel measures on $\mathbb{R}^d$. We say $\mu \in \{\nu\}^\perp$ if and only if $F_{\mu,\nu}$ is isometric.

Remark 2.4. If $F_{\mu,\nu}$ as in Definition 2.1 is an isometry, then we have that for all $f \in L^2(\mu)$

$$\|F_{\mu,\nu}(f)\|^2_{L^2(\nu)} = \|f\|^2_{L^2(\mu)}.$$

This is equivalent to the following: for all $f_1, f_2 \in L^2(\mu)$

$$\begin{aligned}&\langle F_{\mu,\nu} f_1 | F_{\mu,\nu} f_2\rangle_{L^2(\nu)}\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} e_\xi(x) f_1(x) \cdot \overline{e_\xi(y) f_2(y)} d\mu(y) d\mu(x) d\nu(\xi)\\ &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d} e_\xi(x-y) d\nu(\xi)\right) f_1(x)\overline{f_2(y)} d\mu(y) d\mu(x)\\ &= \langle f_1 | f_2\rangle_{L^2(\mu)}\end{aligned}$$

and consequently, for $\xi \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} e_\xi(x-y) d\nu(\xi) = \delta_{x,y}$$

for $(x,y) \in \mathrm{supp}(\mu) \times \mathrm{supp}(\mu)$. Consequently, $\mu \in \{\nu\}^\perp$ means for $\xi \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} e_\xi(x-y) d\nu(\xi) = \delta_{x,y}$$

for $(x,y) \in \mathrm{supp}(\mu) \times \mathrm{supp}(\mu)$. Furthermore, (μ,ν) being a spectral pair implies $\mu \in \{\nu\}^\perp$.

Now we specialize to the only case that we will consider, when $\nu = \delta_\Gamma$ is the counting measure supported on a countable set $\Gamma \subset \mathbb{R}$.

Proposition 2.5 ([**JP98b**]). *If $\nu = \delta_\Gamma$, $\Gamma \subset \mathbb{R}^d$, then $F_{\mu,\nu}$ is isometric if and only if $(e_\gamma)_{\gamma\in\Gamma}$ is orthogonal in $L^2(\mu)$.*

In this case we can state the definition for spectral pairs in the following way.

Definition 2.6. For $\gamma \in \mathbb{R}^d$, let $e_\gamma(x) = e^{2\pi i \gamma x}$, $x \in \mathbb{R}^d$. A probability measure μ on $\mathbb{R}^d$ is said to be a *spectral measure* if there exists a countable set $\Gamma \subset \mathbb{R}^d$ such that the family $(e_\gamma)_{\gamma\in\Gamma}$ forms an orthonormal basis for $L^2(\mu)$. In this case, the set Γ is called the *spectrum* of the measure μ and (μ, Γ) is called a *spectral pair*.

In Definition A.7 there is the general definition of an IFS. In this part we mainly consider measures that arise from affine IFS in $\mathbb{R}$ and so we also give the precise definition of these affine IFS.

Definition 2.7. Let $B \subset \mathbb{Z}$, $\operatorname{card} B =: N < \infty$, and $R \in \mathbb{N}$, $R \geq 2$. For each $b \in B$ we define the following affine maps on $\mathbb{R}$,

$$\tau_b(x) = \frac{x+b}{R}.$$

The family of functions $(\tau_b)_{b\in B}$ is called an *affine iterated function system* (aIFS).

Remark 2.8. We notice the following facts concerning aIFS.

(1) The functions $(\tau_b)_{b\in B}$ are similarities with similarity constant $1/R < 1$.
(2) The invariant sets for these aIFS satisfying the OSC for $(0, 1)$ are also called Cantor sets.

In Section 6 we only consider the following specific measures arising from Hutchinson's theorem, see Theorem A.9.

Definition 2.9. Let $B \subset \mathbb{Z}$, $\operatorname{card} B =: N < \infty$, and $R \in \mathbb{N}$, $R \geq 2$ and let $(\tau_b)_{b\in B}$ be a contractive iterated function system. The unique probability measure μ satisfying

$$\mu(E) = \frac{1}{N} \sum_{b\in B} \mu(\tau_b^{-1}(E)), \text{ for all Borel subsets } E,$$

is called the *invariant measur*e associated to the IFS $(\tau_b)_{b\in B}$.

Remark 2.10. This invariant measure is the measure of maximal entropy in the sense of a shift dynamical system.

For IFS the following notation will be prominent in this part of the thesis.

Definition 2.11. Let $B \subset \mathbb{N}_0$, $\operatorname{card} B < \infty$, and let $R \in \mathbb{N}$ be a scaling. Then we denote with $(\tau_b)_{b\in B}$ the aIFS consisting of $\tau_b(x) = \frac{x+b}{R}$, $b \in B$, and the corresponding invariant measure is denoted by μ_B.

We mainly consider one specific class of measures and sets that can give a spectral pair. In this class the measure is given as the invariant measure of an aIFS and the set $\Gamma(L)$ is given by $\Gamma(L) := \left\{\sum_{j=0}^k l_j R^j : l_j \in L, k \in \mathbb{N}_0\right\}$, where the set L satisfies the following relation.

Definition 2.12. Let $B, L \subset \mathbb{Z}$, $\operatorname{card} B = \operatorname{card} L =: N < \infty$, and let $R \in \mathbb{N}$, $R \geq 2$. Then (R, B, L) is called a *Hadamard triple* if the matrix

$$M_R(B, L) := \frac{1}{\sqrt{N}} \left(e^{2\pi i R^{-1} b\cdot l}\right)_{b\in B, l\in L}$$

is unitary. This matrix $M_R(B, L)$ is called the (complex) Hadamard matrix for (R, B, L) if it is unitary.

Remark 2.13. We later assume that $0 \in B$, $0 \in L$. Then the matrix is dephased in the sense of Section 6.1.

Now we turn to conditions under which we obtain a spectral pair. This will involve stating some results from the literature. We first fix some notation.

Definition 2.14. Let $\mathcal{M}$ be the set of all positive probability measures with compact support on $\mathbb{R}^d$ and let $\mu \in \mathcal{M}$. For $\Gamma \subset \mathbb{R}^d$, countable, and for $t \in \mathbb{R}^d$ let

$$S(\mu, \Gamma)(t) := \sum_{\gamma\in\Gamma} |\widehat{\mu}(t - \gamma)|^2,$$

where $\widehat{\mu}$ is the Fourier transform of the measure μ, defined for $t \in \mathbb{R}^d$ as

$$\widehat{\mu}(t) = \int_{\mathbb{R}^d} e^{i2\pi t\cdot x} d\mu(x).$$

Define for a countable set $\Gamma \subset \mathbb{R}^d$ and $A > 0$

$$M^A(\Gamma) := \left\{\mu \in \mathcal{M} : \, S(\mu, \Gamma)(t) \leq A \, \text{for all} \, t \in \mathbb{R}^d\right\},$$
$$M^{OB}(\Gamma) := \left\{\mu \in \mathcal{M} : \, S(\mu, \Gamma)(t) = 1 \, \text{for all} \, t \in \mathbb{R}^d\right\}$$

and in particular

$$M^{\perp}(\Gamma) := M^1(\Gamma) = \left\{\mu \in \mathcal{M} : \, S(\mu, \Gamma)(t) \leq 1 \, \text{for all} \, t \in \mathbb{R}^d\right\}.$$

Analogously define

$$M^{\perp}(\mu) := \left\{\Gamma \subset \mathbb{R} : \, S(\mu, \Gamma)(t) \leq 1 \, \text{for all} \, t \in \mathbb{R}^d\right\}$$

and

$$M^{OB}(\mu) := \left\{\Gamma \subset \mathbb{R} : \, S(\mu, \Gamma)(t) = 1 \, \text{for all} \, t \in \mathbb{R}^d\right\}.$$

Now we turn to results under which the existence of an orthonormal family or even an ONB of functions $(e_\gamma)_{\gamma\in\Gamma}$ is assured. We start with a formula for the Fourier transform of an invariant measure for an aIFS.

Lemma 2.15 (**[DJ09b]**). *Let $R \in \mathbb{N}$, $B \subset \underline{R}$, $0 \in B$, $\operatorname{card} B = N$, and let $(\tau_b)_{b\in B}$ be the aIFS. Let μ_B be the invariant measure for this aIFS $(\tau_b)_{b\in B}$. Then for $t \in \mathbb{R}$*

$$\widehat{\mu_B}(t) = \prod_{n=1}^{\infty} \frac{1}{N}\left(\sum_{b\in B} e_b\left(\frac{t}{R^n}\right)\right),$$

where $e_b(t) := e^{i2\pi tb}$. The infinite product is absolutely convergent.

Now we turn to results concerning the existence of an ONB in $L^2(\mu)$.

Proposition 2.16 (**[JP98a]**). *A set $\Gamma \subset \mathbb{R}$ is a spectrum for a probability measure μ if and only if for all $t \in \mathbb{R}$*

$$S(\mu, \Gamma)(t) = 1.$$

The following theorem gives results concerning the existence of orthonormal families of functions in $L^2(\mu)$.

Theorem 2.17 (**[DHJ09]**). *Let μ be a probability measure on $\mathbb{R}^d$ and $\Gamma \subset \mathbb{R}^d$, countable. The following are equivalent:*

(1) *The set $(e_\gamma)_{\gamma\in\Gamma}$ is orthonormal in $L^2(\mu)$.*
(2) *The function $S(\mu, \Gamma)$ satisfies the inequality $S(\mu, \Gamma)(t) \leq 1$ for all $t \in \mathbb{R}^d$.*

Furthermore, $(e_\gamma)_{\gamma\in\Gamma}$ is a maximal family of orthogonal exponentials if and only if $0 < S(\mu, \Gamma)(t) \leq 1$ for all $t \in \mathbb{R}^d$.

Remark 2.18. We notice the following connection to the notation fixed above.

(1) If Γ induces an orthonormal family or an orthonormal basis, then so does $\pm\Gamma + a$ for any $a \in \mathbb{R}^d$.
(2) Theorem 2.17 implies that for $\mu \in M^{\perp}(\Gamma)$ we have that $(e_\gamma)_{\gamma\in\Gamma}$ is an orthonormal family in $L^2(\mu)$, and for $\mu \in M^{OB}(\Gamma)$, that (μ, Γ) is a spectral pair, i.e. $(e_\gamma)_{\gamma\in\Gamma}$ is an ONB in $L^2(\mu)$.
(3) If a Hadamard matrix exists for (R, B, L), it follows that $(e_\gamma)_{\gamma\in\Gamma(L)}$ is an orthonormal family in $L^2(\mu_B)$ for $\Gamma(L) := \left\{\sum_{i=0}^{k} l_i R^i : \, l_i \in L, \, k \in \mathbb{N}_0\right\}$.

We can even say that in one dimension the existence of a Hadamard matrix ensures the existence of a spectrum.

Theorem 2.19 ([**DJ09b**]). *Let $B \subset \underline{R}$, $R \in \mathbb{N}$. If there is a Hadamard matrix for the Cantor set given by $(\tau_b)_{b\in B}$ in one dimension, then there exists a spectrum $\Gamma \subset \mathbb{R}$ for the corresponding invariant measure μ_B.*

This result does not give that the spectrum is always given by the set $\Gamma(L)$ defined above, where $M_R(B, L)$ is a Hadamard matrix. For the class of measures coming from a Hadamard matrix there is a more easily checked property in one dimension for ensuring that $(\mu_B, \Gamma(L))$ is a spectral pair. The property uses cycles.

Definition 2.20 ([**DJ06**]). Suppose two sets B and L, $N = \operatorname{card} B = \operatorname{card} L$, form a Hadamard matrix for the scaling R, then a *B-cycle* is a finite set $\{z_1, z_2, ..., z_{m+1}\} \subset \mathbb{T}$ such that $z_1 = e^{i2\pi\xi_1}, z_2 = e^{i2\pi\xi_2}, \ldots, z_{m+1} = e^{i2\pi\xi_{m+1}}$, with a pairing of points in B, say $b_1, b_2, ..., b_{m+1} \in B$, such that $z_i = e^{i2\pi\sigma_{-b_i}(\xi_{i+1})}$, $z_{m+1} = z_1$, where $\sigma_{-b_i}(x) = \frac{x-b_i}{R}$, and $|m_0(z_i)|^2 = N$, where $m_0(z) = \frac{1}{\sqrt{N}} \sum_{l\in L} z^l$. Equivalently, a B-cycle may be given by $\left\{\xi_1, ..., \xi_{m+1}\right\} \subset \mathbb{R}$ satisfying

$$\xi_{i+1} \equiv b_i + R\xi_i \bmod R\mathbb{Z}$$
$$(R^m - 1)\xi_1 \equiv b_m + Rb_{m-1} + \cdots + R^{m-1}b_1 \bmod R^m\mathbb{Z}$$

and $\left|m_0\left(e^{i2\pi\xi_j}\right)\right|^2 = N$ for $j = 1, \ldots, m+1$.

If the role of the two sets B and L is reversed, we talk of an L-cycle. Given the Hadamard property, in one dimension $d = 1$, the presence of cycles is the only obstruction to the corresponding pair $(\mu_B, \Gamma(L))$ being spectral.

We can check for a spectrum via cycles in the following way.

Theorem 2.21 ([**DJ06**]). *Let $R \in \mathbb{N}$, $R \geq 2$ be given. Let $B \subset \underline{R}$, and suppose there is a set $L \subset \mathbb{Z}$ such that $0 \in L$, $\operatorname{card} L = \operatorname{card} B = N$, and $M_R(B, L)$ is a Hadamard matrix. Then $(e_\lambda)_{\lambda\in\Gamma(L)}$ is an ONB for $L^2(\mu)$, where $\Gamma(L) := \left\{\sum_{i=0}^{k} l_i R^i : l_i \in L,\ k \in \mathbb{N}_0\right\}$ and μ_B is the invariant measure for the aIFS $(\tau_b)_{b\in B}$, if the only L-cycle is the singleton $\{1\} \subset \mathbb{T}$.*

Via Hadamard matrices we obtain a space duality between two aIFSs.

Proposition 2.22 ([**DJ09b**]). *Let μ_B be the invariant measure for $(\tau_b)_{b\in B}$ and let μ_L be the invariant measure for the dual system*

$$\left(\tau_l(x) := \frac{x+l}{R}\right)_{l\in L}.$$

Then $\Gamma(B)$ is orthogonal in $L^2(\mu_L)$, and $\Gamma(L)$ is orthogonal in $L^2(\mu_B)$, where $\Gamma(B)$ and $\Gamma(L)$ are given as in Theorem 2.21.

Notice that the last proposition does not give that when one pair $(\mu_B, \Gamma(L))$ is a spectral pair, the other $(\mu_L, \Gamma(B))$ is one as well.

Now we state some more general results regarding the measures μ and sets Γ we consider in our study. The results of [**DHJ09**] give a reason why we only consider the invariant measure for an aIFS and not a measure with different weights on the subsets. The other restriction that we impose is that the aIFS satisfies the open set condition (OSC), see Definition A.6. This is explained in the results of [**DHJ09**] which follow.

Theorem 2.23 ([**DHJ09**]). *Let an aIFS be given by $B \subset \mathbb{R}$, $\operatorname{card} B = N$, $0 \in B$, and $R \in \mathbb{R}$, $R > 1$ as $(\tau_b)_{b\in B}$. Let μ_B be its invariant measure. Suppose the invariant measure μ_B is spectral. Then there is no overlap.*

Proposition 2.24 ([**DHJ09**]). *Let an aIFS be given by $B \subset \mathbb{R}$, $\operatorname{card} B = N$, $0 \in B$, and $R \in \mathbb{R}$, $R > 1$ as $(\tau_b)_{b\in B}$ where there is no overlap. Let μ_p be the measure on the limit set of the aIFS given by Hutchinson's theorem (Theorem A.9) with weights $p_0, \ldots, p_{N-1} \in (0, 1)$. Suppose that μ_p is spectral. Then we have equal probabilities, $p_0 = \cdots = p_{N-1} = \frac{1}{N}$.*

The results of Łaba and Wang, [**ŁW06**], explain why we do not consider finite sets Γ as a spectrum.

Proposition 2.25 ([**ŁW06**]). *Let (μ, Γ) be a spectral pair. Then μ has an atom if and only if Γ is finite. In this case μ is purely atomic and all atoms have the same measure.*

There are two interesting results in [**DJ10**] concerning the measures with spectrum $\mathbb{Z}^d$ or measures such that $(e_\gamma)_{\gamma\in\mathbb{Z}^d}$ is orthonormal in $L^2(\mu)$.

Theorem 2.26 ([**DJ10**]). *Let μ be a Borel probability measure on $\mathbb{R}^d$. The following statements are equivalent:*

(1) *The set $(e_\gamma)_{\gamma\in\mathbb{Z}^d}$ forms an orthonormal set in $L^2(\mu)$.*
(2) *There exists a bounded measurable function $f \geq 0$ that satisfies*
$$\sum_{k\in\mathbb{Z}^d} f(x+k) = 1, \text{ for Lebesgue a.e. } x \in \mathbb{R}^d,$$
such that $d\mu = f d\lambda$.

The other result concerning spectral pairs with the spectrum $\mathbb{Z}^d$ requires the following definition.

Definition 2.27 ([**DJ10**]).

(1) Given a Borel measure μ on $\mathbb{R}^d$, a family of Borel subsets $(E_i)_{i\in J}$ is called a *partition of* μ if $\mu\left(\mathbb{R}^d \setminus \bigcup_{i\in J} E_i\right) = 0$ and $\mu(E_i \cap E_j) = 0$, $i \neq j$. We say that two Borel measures μ and μ' are *translation equivalent* if there exists a partition $(E_i)_{i\in J}$ and some integers $(k_i)_{i\in J}$ of μ such that $(E_i + k_i)_{i\in J}$ is a partition of μ', and the functions $E_i \ni x \mapsto x + k_i \in E_i + k_i$ map the measure μ into the measure μ'.
(2) A Borel subset E of $\mathbb{R}^d$ is called *translation congruent* to $Q = [0,1)^d$ if there exists a measurable partition $\left\{E_k : k \in \mathbb{Z}^d\right\}$ of Q and integers $l_k \in \mathbb{Z}$ such that
$$E = \bigcup_{k\in\mathbb{Z}^d} (E_k + l_k).$$

Theorem 2.28 ([**DJ10**]). *Let μ be a Borel probability measure on $\mathbb{R}^d$. Then μ has spectrum $\mathbb{Z}^d$ if and only if μ is the restriction of the Lebesgue measure to a set E which is translation congruent to Q.*

Another way to obtain from one spectral measure a new one is to consider translation congruent measures.

Proposition 2.29 ([**DJ10**]). *Let μ and μ' be two translation equivalent Borel probability measures on $\mathbb{R}^d$. If μ has spectrum Λ contained in $\mathbb{Z}^d$, then μ' is also a spectral measure with spectrum Λ.*

Remark 2.30. Notice that the results above, namely Theorem 2.26 and Theorem 2.28, do not require that the measures μ have compact support. But in the following we generally do impose this restriction.

CHAPTER 3

Transformation group

Let $\mathbb{B}$ be the Banach algebra of real signed Borel measures with finite variation equipped with the convolution and let $\mathcal{M}$ be the set of probability measures with compact support as before. The total variation is defined by

$$\|\nu\|_{tot} := \sup_{(E_i)_{i\in J}, E_i\subset\mathbb{R};\text{ partition}} \sum_{i\in J} |\nu(E_i)|,$$

where J is some index set. The convolution of two signed measures μ and ν is then defined as

$$\mu \star \nu := (\mu \otimes \nu) \circ s^{-1},$$

where $s : \mathbb{R}^2 \to \mathbb{R}$ is given by $(x, y) \mapsto x + y$ and $\mu \otimes \nu$ is the product measure. By **[Rud90]** it holds that

$$\|\nu_1 \star \nu_2\|_{tot} \leq \|\nu_1\|_{tot} \cdot \|\nu_2\|_{tot}.$$

We can write every element $\nu \in \mathbb{B}$ as $\nu = \nu^+ - \nu^-$ by the Jordan decomposition, where ν^+ and ν^- are positive measures. Besides $\mathbb{B}$ becomes a commutative Banach algebra under convolution with the identity δ_0, where δ_x, $x \in \mathbb{R}$, is the Dirac measure.

Now let $\mathbb{R}$ act on sets of measures in the following way: Define for $x \in \mathbb{R}$ and $\nu \in \mathbb{B}$:

$$\mathbf{T}_x\nu := \delta_x \star \nu.$$

Notice that $\mathbf{T}_{x+y} = \mathbf{T}_x\mathbf{T}_y$, i.e. $\delta_{x+y} \star \nu = \delta_x \star (\delta_y \star \nu)$. Furthermore, $\delta_0 \star \nu = \nu$.

Definition 3.1. A subset $\mathcal{K}$ of $\mathbb{B}$ is said to be $\mathbf{T}$-*translation invariant* if $\nu \in \mathcal{K}$ if and only if $\mathbf{T}_x\nu \in \mathcal{K}$ for all $x \in \mathbb{R}$.

In the following we consider specific subsets of $\mathbb{B}$.

Definition 3.2. Define $B_{abs} := \{\nu \in \mathcal{M} : \exists f \in L^1(\mathbb{R}, \lambda) \text{ such that } d\nu = f d\lambda\}$, λ is the Lebesgue measure.

Remark 3.3. We notice the following facts regarding B_{abs}.

(1) In B_{abs} the "abs" stands for absolutely continuous with respect to the Lebesgue measure.
(2) Recall that $\nu \in B_{abs}$ if and only if ν is absolutely continuous with respect to λ (denoted by $\nu \ll \lambda$) if and only if there exists a Radon-Nikodym derivative $\frac{d\nu}{d\lambda} = f$.
(3) By Theorem 2.26 we have that $M^\perp(\mathbb{Z}) \subset B_{abs}$.
(4) Notice that $\nu \in B_{abs}$ is equivalent to $\mathbf{T}_x\nu \in B_{abs}$. Consequently, B_{abs} is $\mathbf{T}$-translation invariant.

Proposition 3.4. *For each $\nu \in B_{abs}$, $(x, \nu) \mapsto \mathbf{T}_x\nu$, $x \in \mathbb{R}$, is continuous in x on B_{abs}. If $\nu \in \mathbb{B}\backslash B_{abs}$, then this map need not be continuous.*

Proof. Let $f \in L^1(\mathbb{R}, \lambda)$ set $\mathbf{T}_x f(\cdot) = f(\cdot - x)$, then $\mathbf{T}_x(f d\lambda) = f(\cdot - x)d\lambda$. Notice that if $d\nu = f d\lambda \in B_{abs}$ then

$$\|\nu - \mathbf{T}_x\nu\|_{tot} = \|f - f(\cdot - x)\|_{L^1(\mathbb{R},\lambda)} \to 0, \; x \to 0,$$

and consequently, we have the continuity for $\nu \in B_{abs}$.

If $\nu \in \mathbb{B}\backslash B_{abs}$ then the convergence need not hold because take for example $\nu = \delta_0$ then $\mathbf{T}_x\delta_0 = \delta_x$ and $\|\delta_0 - \delta_x\|_{tot} = \begin{cases} 0, & x = 0, \\ 1, & x \neq 0. \end{cases}$

Indeed if $x \neq 0$ and E is a Borel set, then

$$(\delta_0 - \delta_x)(E) = \begin{cases} 1, & \text{if } 0 \in E,\, x \notin E, \\ -1, & \text{if } 0 \notin E,\, x \in E, \\ 0, & \text{otherwise.} \end{cases}$$

$\square$

Lemma 3.5. *It holds that* $B_{abs} \subset \{\nu \in \mathcal{M} : \|\nu - \mathbf{T}_x\nu\|_{tot} \to 0,\, x \to 0\}$.

Proof. The proof follows from $\|\nu - \mathbf{T}_x\nu\|_{tot} = \|f - f(\cdot - x)\|_{L^1(\mathbb{R},\lambda)}$ for $\nu \in B_{abs}$, $d\nu = f d\lambda$. $\square$

Now we turn to the connection to spectral pairs.

Proposition 3.6. *For fixed countable set* $\Gamma \subset \mathbb{R}$*, both* $M^{\perp}(\Gamma)$ *and* $M^{OB}(\Gamma)$ *are* $\mathbf{T}$*-translation invariant.*

The proposition above follows immediately from the following lemma.

Lemma 3.7. $S(\mathbf{T}_x\mu, \Gamma) = S(\mu, \Gamma)$, $x \in \mathbb{R}$.

Proof. Notice that for $x, t \in \mathbb{R}$

$$\begin{aligned} \widehat{\mathbf{T}_x\mu}(t) &= \int e_t(y) d\mathbf{T}_x\mu(y) = \int e_t(y) d\mu(y - x) \\ &= \int e_t(y + x) d\mu(y) = e_t(x) \int e_t(y) d\mu(y) = e_t(x)\widehat{\mu}(t) \end{aligned}$$

and so we have for $t \in \mathbb{R}$

$$S(\mathbf{T}_x\mu, \Gamma)(t) = \sum_{\gamma \in \Gamma} |\widehat{\mathbf{T}_x\mu}(t - \gamma)|^2 = \sum_{\gamma \in \Gamma} |e_{t-\gamma}(x)|^2 \cdot |\widehat{\mu}(t - \gamma)|^2 = \sum_{\gamma \in \Gamma} |\widehat{\mu}(t - \gamma)|^2 = S(\mu, \Gamma)(t).$$

$\square$

Now we turn to further properties of the sets $M^A(\Gamma)$, $M^{\perp}(\Gamma)$ and $M^{OB}(\Gamma)$. For one of the following results regarding we have to drop the condition of compact support for the measures.

Proposition 3.8. *The sets* $M^A(\Gamma)$ *and* $M^{OB}(\Gamma)$ *have the following properties:*

1. *The sets* $M^A(\Gamma)$ *and* $M^{\perp}(\Gamma)$ *are convex.*
2. *The set* $M^{OB}(\Gamma)$ *is not convex.*
3. *Let* $\widetilde{\mathcal{M}}$ *be the set of all probability measures on* $\mathbb{R}$*, with not necessarily compact support. Let for* $A > 0$

$$\widetilde{M}^A(\Gamma) := \left\{ \mu \in \widetilde{\mathcal{M}} : \sum_{\gamma \in \Gamma} |\widehat{\mu}(t - \gamma)|^2 \leq A \text{ for all } t \in \mathbb{R} \right\}$$

and $\widetilde{M}^{\perp} = \widetilde{M}^1$*. Then* $\widetilde{M}^A(\Gamma)$ *and* $\widetilde{M}^{\perp}(\Gamma)$ *are closed.*

Proof. ad (1): Assume $\mu, \nu \in M^A(\Gamma)$ and $0 < \alpha < 1$. We have that

$$\left(\widehat{\alpha\mu + (1 - \alpha)\nu}\right) = \alpha\widehat{\mu} + (1 - \alpha)\widehat{\nu}$$

and so it follows that

$$\begin{aligned}
&\sum_{\gamma\in\Gamma} \left|\alpha\widehat{\mu}(t-\gamma)+(1-\alpha)\widehat{\nu}(t-\gamma)\right|^2\\
=&\sum_{\gamma\in\Gamma} \alpha^2\left|\widehat{\mu}(t-\gamma)\right|^2+(1-\alpha)^2\left|\widehat{\nu}(t-\gamma)\right|^2+\mathfrak{Re}\Big(2\alpha(1-\alpha)\overline{\widehat{\mu}(t-\gamma)}\widehat{\nu}(t-\gamma)\Big)\\
\le&\,\alpha^2 A+(1-\alpha)^2 A+\mathfrak{Re}\left(2\alpha(1-\alpha)\left(\sum_{\gamma\in\Gamma}|\widehat{\mu}(t-\gamma)|^2\sum_{\gamma\in\Gamma}|\widehat{\nu}(t-\gamma)|^2\right)^{1/2}\right)\\
\le&\, A.
\end{aligned}$$

Consequently, the set is convex. For $M^{\perp}(\Gamma)$ the proof follows in the same way with $A=1$.

ad (2): Assume that $\mu\in M^{OB}(\Gamma)$ and let $\nu=\delta_a\star\mu$, $a\in\mathbb{R}\backslash\{0\}$. Then it follows that $\nu\in M^{OB}(\Gamma)$ since for all $t\in\mathbb{R}$ we have

$$\sum_{\gamma\in\Gamma}|\widehat{\nu}(t-\gamma)|^2=\sum_{\gamma\in\Gamma}\underbrace{|e^{i2\pi(t-\gamma)a}|^2}_{=1}\cdot|\widehat{\mu}(t-\gamma)|^2=1.$$

Now we consider for $0<\alpha<1$ the measure $\alpha\mu+(1-\alpha)\nu$ and obtain

$$\begin{aligned}
&\sum_{\gamma\in\Gamma} \left|\alpha\widehat{\mu}(t-\gamma)+(1-\alpha)\widehat{\nu}(t-\gamma)\right|^2\\
=&\sum_{\gamma\in\Gamma} \alpha^2|\widehat{\mu}(t-\gamma)|^2+(1-\alpha)^2|\widehat{\nu}(t-\gamma)|^2+\mathfrak{Re}\Big(2\alpha(1-\alpha)\overline{\widehat{\mu}(t-\gamma)}\widehat{\nu}(t-\gamma)\Big)\\
=&\,\alpha^2+(1-\alpha)^2+\mathfrak{Re}\Big(2\alpha(1-\alpha)\sum_{\gamma\in\Gamma}\overline{\widehat{\mu}(t-\gamma)}\widehat{\nu}(t-\gamma)\Big).
\end{aligned}$$

For the expression above to be equal to 1 we must have that

$$\sum_{\gamma\in\Gamma}\mathfrak{Re}\big(\overline{\widehat{\mu}(t-\gamma)}\widehat{\nu}(t-\gamma)\big)=1.$$

But

$$\begin{aligned}
\sum_{\gamma\in\Gamma}\mathfrak{Re}\big(\overline{\widehat{\mu}(t-\gamma)}\widehat{\nu}(t-\gamma)\big)&=\sum_{\gamma\in\Gamma}|\widehat{\mu}(t-\gamma)|^2\cdot\mathfrak{Re}\left(e^{i2\pi(t-\gamma)a}\right)\\
&=\sum_{\gamma\in\Gamma}\cos\left(2\pi(t-\gamma)a\right)\cdot|\widehat{\mu}(t-\gamma)|^2\\
&<1,
\end{aligned}$$

since $\cos(2\pi(t-\gamma)a)$ is equal to 1 for only countably many $t\in\mathbb{R}$, namely $(t-\gamma)a\in\mathbb{Z}$, and for different $t\in\mathbb{R}$ we have $\cos(2\pi(t-\gamma)a)<1$.

ad (3): We assume that $\mu_n\in\widetilde{M}^A(\Gamma)$, $\mu_n\in\widetilde{M}^{\perp}(\Gamma)$, respectively, is a sequence of measures such that $\mu_n\to\mu$ weak-$*$. Consequently, we have that $\widehat{\mu_n}\to\widehat{\mu}$ by results in [**Bil79**]. Since $\sum_{\gamma\in\Gamma}|\widehat{\mu_n}(t-\gamma)|^2\le A$ for all $n\in\mathbb{N}$, we have by Fatou's lemma

$$\sum_{\gamma\in\Gamma}|\widehat{\mu_n}(t-\gamma)|^2\to\sum_{\gamma\in\Gamma}|\widehat{\mu}(t-\gamma)|^2\le A.$$

To see that μ need not have compact support compare Example 3.12. □

Now we give a more precise example for $\mu,\nu\in M^{OB}(\Gamma)$ and $\alpha\in(0,1)$ and

$$\sum_{\gamma\in\Gamma}\left|\alpha\widehat{\mu}(t-\gamma)+(1-\alpha)\widehat{\nu}(t-\gamma)\right|^2\le 1.$$

Example 3.9. Let us consider the set $\Gamma = \left\{\sum_{i=0}^{k} l_i 4^i : l_i \in \{0,1\},\, k \in \mathbb{N}_0\right\}$. Then the measure μ for the aIFS $\left(\tau_0(x) = \frac{x}{4}, \tau_1(x) = \frac{x+2}{4}\right)$ and the measure ν for the aIFS $\left(\sigma_0(x) = \frac{x-1}{4}, \sigma_1(x) = \frac{x+1}{4}\right)$ are in $M^{OB}(\Gamma)$, see **[JP98a]**. We have that $\mu = \delta_{1/4} \star \nu$. Furthermore for $0 < \alpha < 1$

$$\alpha\mu + (1-\alpha)\nu \notin M^{OB}(\Gamma),$$

since

$$\begin{aligned}\sum_{\gamma\in\Gamma} \left|\alpha\widehat{\mu}(t-\gamma) + (1-\alpha)\widehat{\nu}(t-\gamma)\right|^2 &= \sum_{\gamma\in\Gamma} \left|\alpha e^{i2\pi\frac{1}{4}(t-\gamma)}\widehat{\nu}(t-\gamma) + (1-\alpha)\widehat{\nu}(t-\gamma)\right|^2 \\ &= \sum_{\gamma\in\Gamma} \left|\alpha e^{i2\pi\frac{1}{4}(t-\gamma)} + (1-\alpha)\right|^2 \cdot \left|\widehat{\nu}(t-\gamma)\right|^2\end{aligned}$$

and

$$\left|\alpha e^{i2\pi\frac{1}{4}(t-\gamma)} + (1-\alpha)\right|^2 = 2\alpha^2 - 2\alpha + 1 + 2\left(\alpha^2-\alpha\right)\cos\left(\frac{\pi}{2}(t-\gamma)\right) < 1$$

for Lebesgue-a.e. $t \in \mathbb{R}$ and $\gamma \in \Gamma$.

Remark 3.10. It follows that for $\mu, \nu \in M^{OB}(\Gamma)$ we have $\alpha\mu + (1-\alpha)\nu \in M^{OB}(\Gamma)$, $0 < \alpha < 1$, $\mu \neq \nu$, if and only if $\sum_{\gamma\in\Gamma} \mathfrak{Re}\left(\widehat{\mu}(t-\gamma)\overline{\widehat{\nu}(t-\gamma)}\right) = 1$ for all $t \in \mathbb{R}$.

In most examples of spectral pairs, the measures are compactly supported. Now we give an example which shows that there are measures, not compactly supported, such that we have a spectral pair.

Example 3.11. We now give an example of a set, translation congruent to $[0,1]$, so that the restriction of the Lebesgue measure λ to this set has the spectrum $\mathbb{Z}$.

Consider the union $A := \bigcup_{n\geq 1}[n + \frac{1}{n+1}, n + \frac{1}{n}]$ and we take $\lambda|_A$. Now we have that the support is not compact. But on the other hand we have that $\lambda|_A \in M^{\perp}(\mathbb{Z})$ since

$$\begin{aligned}\widehat{\lambda|_A}(k) &= \int e^{i2\pi xk} d\lambda|_A(x) = \sum_{n\geq 1} \int_{n+\frac{1}{n+1}}^{n+\frac{1}{n}} e^{i2\pi xk} d\lambda(x) \\ &= \sum_{n\geq 1} \int_{\frac{1}{n+1}}^{\frac{1}{n}} e^{i2\pi xk} d\lambda(x) = \int_0^1 e^{i2\pi xk} d\lambda(x) \\ &= \delta_{0,k}.\end{aligned}$$

Furthermore, we have that $\lambda|_A \in M^{OB}(\mathbb{Z})$ by Theorem 2.28, since the set A is translation congruent to $[0,1)$.

We had to drop the hypothesis of the measure having compact support to get the third part of Proposition 3.8. Now we give an example of a sequence of measures μ_N, $N \in \mathbb{N}$, convergent to a measure μ, where each measure μ_N has compact support but the limit measure μ is not compactly supported.

Example 3.12. The example uses as the limit measure $\mu = \lambda|_A$ of Example 3.11. Define the measures $\mu_N = \lambda|_{A_N}$, where

$$A_N = \bigcup_{n=1}^{N} \left[n + \frac{1}{n+1}, n + \frac{1}{n}\right] \cup \left[N, N + \frac{1}{N+1}\right].$$

Consequently, we have that μ_N, for $N \in \mathbb{N}$, has compact support and $\mu_N \in M^{OB}(\mathbb{Z})$ by Theorem 2.28. We can show for $\mu = \lambda|_A$ that

$$\mu_N \to \mu, \;\text{ weak}*, \; N \to \infty.$$

The convergence can be seen by considering the Fourier transforms of the measures,

$$\widehat{\mu_N}(t) = \sum_{n=1}^{N} \frac{e^{i2\pi tn}\left(e^{i2\pi\frac{t}{n}} - e^{i2\pi\frac{t}{n+1}}\right)}{i2\pi t} + \frac{e^{i2\pi tN}\left(e^{i2\pi\frac{t}{N+1}} - 1\right)}{i2\pi t},$$

$$\widehat{\mu}(t) = \sum_{n=1}^{\infty} \frac{e^{i2\pi tn}\left(e^{i2\pi\frac{t}{n}} - e^{i2\pi\frac{t}{n+1}}\right)}{i2\pi t}.$$

It follows that $\widehat{\mu_N}(t) \to \widehat{\mu}(t)$, $N \to \infty$, for all $t \in \mathbb{R}$, since

$$\begin{aligned}
|\widehat{\mu}(t) - \widehat{\mu_N}(t)| &= \left| \sum_{n=N+1}^{\infty} \frac{e^{i2\pi tn}\left(e^{i2\pi\frac{t}{n}} - e^{i2\pi\frac{t}{n+1}}\right)}{i2\pi t} - \frac{e^{i2\pi tN}\left(e^{i2\pi\frac{t}{N+1}} - 1\right)}{i2\pi t} \right| \\
&\leq \sum_{n=N+1}^{\infty} \left| \frac{e^{i2\pi tn}\left(e^{i2\pi\frac{t}{n}} - e^{i2\pi\frac{t}{n+1}}\right)}{i2\pi t} \right| + \left| \frac{e^{i2\pi tN}\left(e^{i2\pi\frac{t}{N+1}} - 1\right)}{i2\pi t} \right| \\
&\leq \sum_{n=N+1}^{\infty} \left| \frac{1}{n+1} - \frac{1}{n} \right| + \frac{1}{N+1} \to 0,\ N \to \infty.
\end{aligned}$$

Consequently, the weak $*$-convergence for the measures follows.

Now we consider further the convolution operation on $\mathcal{M}$.

Proposition 3.13. *$M^{\perp}(\Gamma)$ is an ideal in $\mathcal{M}$.*

Remark 3.14. We use ideal as a set that is closed under convolution, not summation.

Proof. Recall that an ideal has to satisfy the following property. For $\mu \in M^{\perp}(\Gamma)$, $\nu \in \mathcal{M}$, we must have $\nu \star \mu \in M^{\perp}(\Gamma)$. This is obviously satisfied since $\widehat{\nu \star \mu}(t) = \widehat{\nu}(t) \cdot \widehat{\mu}(t)$, $t \in \mathbb{R}$, and thus, if $\mu \in M^{\perp}(\Gamma)$ we have $\widehat{\mu}(\beta - \gamma) = 0$ for $\beta, \gamma \in \Gamma$ and thus $\widehat{\mu \star \nu}(\beta - \gamma) = 0$. Obviously, we have $\nu \star \mu \in \mathcal{M}$. $\square$

Remark 3.15. $M^{\perp}(\Gamma)$ is not an ideal in $\mathbb{B}$; indeed $\mathbb{B}$ contains signed measures not just positive measures. The convolution of $\mu \in \mathcal{M}$ and $\nu \in \mathbb{B}$ need not even be in $\mathcal{M}$, which is one condition for being an element of $M^{\perp}(\Gamma)$.

Another fact to notice about the convolution of positive measures is that

$$\operatorname{supp}(\mu \star \nu) = \operatorname{supp}(\mu) + \operatorname{supp}(\nu).$$

But for signed measures $\nu = \nu^+ - \nu^-$, $\operatorname{supp}(\mu \star \nu^+)$ and $\operatorname{supp}(\mu \star \nu^-)$ do not have to be disjoint and so for convolutions with signed measures we only have $\operatorname{supp}(\mu \star \nu) \subset \operatorname{supp}(\mu) + \operatorname{supp}(\nu)$.

To see that $M^{OB}(\Gamma)$ need not be closed under convolution with elements in $\mathcal{M}$, consider e.g. $\mu = \left(\frac{1}{2}\delta_0 + \frac{1}{2}\delta_1\right) \star \lambda|_{[0,1]}$ then $\widehat{\mu}(t) = \frac{1}{2}\left(1 + e^{i2\pi t}\right) \cdot \widehat{\lambda|_{[0,1]}}$ and $|\frac{1}{2}\left(1 + e^{i2\pi t}\right)|^2 < 1$ for Lebesgue-a.e. $t \in \mathbb{R}$ and so $\sum_{\lambda \in \mathbb{Z}} |\widehat{\mu}(t - \gamma)|^2 < 1$ for Lebesgue-a.e. $t \in \mathbb{R}$. Thus, $\mu \notin M^{OB}(\mathbb{Z})$.

CHAPTER 4

Varying the measures in a spectral pair: Orthogonal measures

In this section we start with a spectral pair (μ, Γ) in $\mathbb{R}$ and consider what this tells us about $M^{\perp}(\Gamma)$. We have that $\widehat{\mu}(\beta-\gamma)=\delta_{\beta,\gamma}$ for $\beta,\gamma\in\Gamma$ and $\sum_{\gamma\in\Gamma}|\widehat{\mu}(t-\gamma)|^2=1$ for all $t\in\mathbb{R}$.

Definition 4.1. Let $Z(\widehat{\mu})$ be the *set of zeros* of $\widehat{\mu}$, i.e. $Z(\widehat{\mu})=\{t\in\mathbb{R}: \widehat{\mu}(t)=0\}$.

Remark 4.2. In general we have for spectral pairs (μ,Γ) that $Z(\widehat{\mu})\supsetneq(\Gamma-\Gamma)\setminus\{0\}$. In Chapter 6 we give a class of measures where $(\Gamma-\Gamma)\setminus\{0\}$ is always a real subset of $Z(\widehat{\mu})$.

The following example of the 1/4-Cantor set belongs to a class like the one considered in Chapter 6.

Example 4.3. For the invariant measure μ given by the aIFS $\left(\tau_0(x)=\frac{x}{4}, \tau_1(x)=\frac{x+2}{4}\right)$, we have that (μ,Γ) is a spectral pair, where

$$\Gamma=\left\{\sum_{i=0}^{k} l_i 4^i : l_i\in\{0,1\},\, k\in\mathbb{N}_0\right\}=\{0,1,4,5,16,17,20,21,\dots\}. \tag{4.0.1}$$

The Fourier transform of μ is

$$\widehat{\mu}(t)=\prod_{n=1}^{\infty}\frac{1}{2}\left(1+e_2\left(\frac{t}{4^n}\right)\right)=e^{i\frac{2\pi t}{3}}\prod_{n=1}^{\infty}\cos\left(\frac{2\pi t}{4^n}\right), \tag{4.0.2}$$

where $e_b(t)=e^{i2\pi bt}$, $b\in\mathbb{R}$, see [**JKS08, DJ09b**]. Then we have that

$$Z(\widehat{\mu})=\bigcup_{k=1}^{\infty}\bigcup_{n\in\mathbb{Z}}4^k\left(\left\{\frac{1}{4},\frac{3}{4}\right\}+n\right)\supsetneq(\Gamma-\Gamma)\setminus\{0\},$$

since $7\in Z\left(\widehat{\mu}\right)$ but $7\notin(\Gamma-\Gamma)$.

Proposition 4.4. *Let (μ,Γ) be a spectral pair. If $\nu=\xi\star\mu\in\mathcal{M}$, where ξ is a signed measure, then $(e_\gamma)_{\gamma\in\Gamma}$ is orthonormal in $L^2(\nu)$, i.e. $\nu\in M^{\perp}(\Gamma)$.*

Proof. We have that $\widehat{\mu}(\gamma)=0$ for all $\gamma\in(\Gamma-\Gamma)\setminus\{0\}$ and thus $\widehat{\nu}(\gamma)=\widehat{\xi}(\gamma)\cdot\widehat{\mu}(\gamma)=0$ for $\gamma\in(\Gamma-\Gamma)\setminus\{0\}$. ☐

Proposition 4.5. *Let (μ,Γ) be a spectral pair and let $\nu=\xi\star\mu\in\mathcal{M}$ with ξ a signed measure and $|\widehat{\xi}|=1$. Then $(e_\gamma)_{\gamma\in\Gamma}$ is orthonormal basis in $L^2(\nu)$, i.e. (ν,Γ) is a spectral pair.*

Proof. We have by Proposition 4.4 that $(e_\gamma)_{\gamma\in\Gamma}$ is orthonormal in $L^2(\nu)$ and we have that $\sum_{\gamma\in\Gamma}|\widehat{\mu}(t-\gamma)|^2=1$ for all $t\in\mathbb{R}$ and so

$$\sum_{\gamma\in\Gamma}|\widehat{\nu}(t-\gamma)|^2=\sum_{\gamma\in\Gamma}\underbrace{|\widehat{\xi}(t-\gamma)|^2}_{=1}\cdot|\widehat{\mu}(t-\gamma)|^2=1.$$

☐

Remark 4.6. If we consider in Proposition 4.5 only probability measures ξ instead of signed measures then $|\widehat{\xi}|=1$ ensures $\xi=\delta_a$ for some $a\in\mathbb{R}$, since for $t\in\mathbb{R}$

$$\left|\int_{\mathbb{R}}e^{i2\pi tx}d\xi(x)\right|=1 \iff \int_{\mathbb{R}}\int_{\mathbb{R}}\cos\left(2\pi t(x-y)\right)d\xi(x)d\xi(y)=1.$$

For signed measures ν with $|\widehat{\nu}| = 1$ it follows that $\widehat{\nu}(t) = e^{ig(t)}$ for some continuous function $g : \mathbb{R} \to \mathbb{R}$ with $g(0) = 0$.

Proposition 4.7. *Let (μ, Γ) and (ν, Δ) be spectral pairs, such that $\Gamma - \Gamma = \Delta - \Delta$.*

(1) *Then $\mu \in M^{\perp}(\Delta)$, $\nu \in M^{\perp}(\Gamma)$, $\mu \star \nu \in M^{\perp}(\Gamma)$ and $\mu \star \nu \in M^{\perp}(\Delta)$.*
(2) *If $\mu = \xi \star \nu$, where ξ is invertible and $|\widehat{\xi}| = 1$, we have $\mu \in M^{OB}(\Delta)$ and $\nu \in M^{OB}(\Gamma)$.*
(3) *Let (μ, Γ) be a spectral pair and let $\Delta \subset \mathbb{N}_0$ such that $\Gamma - \Gamma = \Delta - \Delta$. Then $(e_\gamma)_{\gamma \in \Delta}$ is orthonormal in $L^2(\mu)$.*

Remark 4.8. By an invertible measure, we mean a measure μ such that there exists a unique measure ξ so that for all measures ν we have $\xi \star (\mu \star \nu) = \nu$. We will write μ^{-1} for ξ.

PROOF. ad (1): Since we have $\widehat{\mu}(\gamma) = 0$ and $\widehat{\nu}(\gamma) = 0$ for $\gamma \in \Gamma - \Gamma = \Delta - \Delta$, $\gamma \neq 0$, it follows that $Z(\widehat{\mu}) \supset (\Delta - \Delta) \setminus \{0\}$ and $Z(\widehat{\nu}) \supset (\Gamma - \Gamma) \setminus \{0\}$. Thus $\mu \in M^{\perp}(\Delta)$ and $\nu \in M^{\perp}(\Gamma)$. The other two statements follow from Proposition 4.4.

ad (2): If $\mu = \xi \star \nu$, we have $\widehat{\mu} = \widehat{\xi} \cdot \widehat{\nu}$ and thus it follows that $Z(\widehat{\nu}) = Z(\widehat{\mu})$ since $Z(\widehat{\xi}) = \emptyset$. Furthermore,

$$1 = \sum_{\gamma \in \Gamma} |\widehat{\mu}(t - \gamma)|^2 = \sum_{\gamma \in \Gamma} |\widehat{\xi}(t - \gamma)|^2 \cdot |\widehat{\nu}(t - \gamma)|^2 = \sum_{\gamma \in \Gamma} |\widehat{\nu}(t - \gamma)|^2.$$

Thus, $\nu \in M^{OB}(\Gamma)$. Since ξ is invertible, we have that $\nu = \xi^{-1} \star \mu$. Thus, it follows that $\mu \in M^{OB}(\Delta)$.

ad (3): This result follows easily from the observations above. □

Now we turn to further properties of $M^{\perp}(\Gamma)$.

Proposition 4.9. *Let $\Gamma, \Delta \subset \mathbb{R}$, countable.*

(1) *If $\Gamma \subset \Delta$, then $M^{\perp}(\Delta) \subset M^{\perp}(\Gamma)$.*
(2) *Let $\nu \in M^{\perp}(\Delta)$, $\Delta \subset \mathbb{R}$ and $\Gamma \subset \mathbb{R}$ be such that $n \cdot (\Gamma - \Gamma) = \Delta - \Delta$ for some $n \in \mathbb{R}$. Then it holds for $\widehat{\tilde{\nu}} = \widehat{\nu}(n\cdot)$ that $\tilde{\nu} \in M^{\perp}(\Gamma)$.*

PROOF. ad (1): If $\Gamma \subset \Delta$, then for $\mu \in M^{\perp}(\Delta)$ we get

$$\sum_{\gamma \in \Gamma} |\widehat{\mu}(t - \gamma)|^2 \leq \sum_{\gamma \in \Delta} |\widehat{\mu}(t - \gamma)|^2 \leq 1.$$

ad (2): Easy observation. □

Remark 4.10. We further notice the following properties of spectral pairs and orthonormal families.

(1) If we have that $\Gamma \subsetneq \Delta$, then $M^{OB}(\Gamma) \cap M^{\perp}(\Delta) = \emptyset$, because if $\mu \in M^{OB}(\Gamma) \cap M^{\perp}(\Delta)$ then we have that $e_\beta \perp e_\gamma$ for $\beta \in \Delta \backslash \Gamma$ and all $\gamma \in \Gamma$ and consequently, $(e_\gamma)_{\gamma \in \Gamma}$ cannot be maximal. Thus, it is not an ONB in $L^2(\mu)$.
(2) In the setting of Proposition 4.7 (3) it is still unclear when $(e_\gamma)_{\gamma \in \Delta}$ is an ONB in $L^2(\mu)$. In general we know that (μ, Δ) does not give an spectral pair, compare $(\lambda|_{[0,1]}, \mathbb{Z})$ and $(\lambda|_{[0,1]}, \Gamma)$ in the next section (Example 5.10). But we have that (μ, Δ) is a spectral pair if $\Delta = \Gamma + a$ for any $a \in \mathbb{R}$.

The following example shows that if we start with a spectral pair (μ, Γ) and another set $\Delta \subsetneq \mathbb{Z}$ such that $\Gamma - \Gamma = \Delta - \Delta$, it does not follow that (μ, Δ) is a spectral pair, too.

Example 4.11. Consider the 1/4-Cantor set and the invariant measure μ given by the aIFS $\left(\tau_0(x) = \frac{x}{4}, \tau_1(x) = \frac{x+2}{4}\right)$ and the set Γ given as in (4.0.1). As the set Δ we take $\Delta = \Gamma \backslash \{0, 1\}$. Then we have that $\Gamma - \Gamma = \Delta - \Delta$ but (Δ, μ) is not a spectral pair since

$$\sum_{\gamma \in \Delta} |\widehat{\mu}(t + \gamma)|^2 = \sum_{\gamma \in \Gamma} |\widehat{\mu}(t + \gamma)|^2 - |\widehat{\mu}(t)|^2 - |\widehat{\mu}(t + 1)|^2 < \sum_{\gamma \in \Gamma} |\widehat{\mu}(t + \gamma)|^2 = 1,$$

if $|\widehat{\mu}(t)|^2 \neq 0$ or $|\widehat{\mu}(t+1)|^2 \neq 0$. Recall from Example 4.3 that $\widehat{\mu}$ takes the form in (4.0.2). Thus, the above fails to occur only if $\cos\left(\frac{2\pi t}{4^n}\right) = 0$ or $\cos\left(\frac{2\pi(t+1)}{4^n}\right) = 0$ for $n \in \mathbb{N}$. But this can only happen for a countable number of t. Consequently, we do not have that (μ, Δ) is a spectral pair.

Now we give ways to obtain another element in $M^{\perp}(\Gamma)$ if we already know that $\mu \in M^{\perp}(\Gamma)$, $\Gamma \subset \mathbb{Z}$.

Proposition 4.12. *Let μ be a probability measure on $\mathbb{R}$ with $\operatorname{supp}(\mu) \subset [0,1]$. Define $\mu_{\mathbb{Z}} = \sum_{k\in\mathbb{Z}} \mu(\cdot - k)$. Assume that the set $(e_\gamma)_{\gamma\in\Gamma}$, $\Gamma \subset \mathbb{Z}$, forms an orthonormal set in $L^2(\mu)$. Let f be a bounded measurable (with respect to $\mu_{\mathbb{Z}}$) function $f \geq 0$ that satisfies*

$$\sum_{k\in\mathbb{Z}} f(x+k) = 1, \text{ for } \mu_{\mathbb{Z}}\text{-a.e. } x \in \mathbb{R},$$

then the measure $\nu = f d\mu_{\mathbb{Z}}$ is in $M^{\perp}(\Gamma)$.

Proof. For $\gamma \in \Gamma - \Gamma$ we have

$$\int_{\mathbb{R}} e_\gamma d\nu = \int_{\mathbb{R}} e_\gamma f d\mu_{\mathbb{Z}} = \sum_{k\in\mathbb{Z}} \int_{\operatorname{supp}(\mu)} f(x+k) e_\gamma(x+k) d\mu_{\mathbb{Z}}(x)$$
$$= \int_{\operatorname{supp}(\mu)} e_\gamma(x) \sum_{k\in\mathbb{Z}} f(x+k) d\mu_{\mathbb{Z}}(x) = \int_{\operatorname{supp}(\mu)} e_\gamma(x) d\mu(x) = \delta_{\gamma,0}.$$

□

Remark 4.13. In the proposition above we do not have the other implication, i.e. from the measure $\nu \in M^{\perp}(\Gamma)$ it does not necessarily follow that there exists a function f with the properties above so that $d\nu = f d\mu_{\mathbb{Z}}$. We can see this if we consider the measures obtained by Hadamard matrices, see Chapter 6.

CHAPTER 5

Sets Γ forming a spectrum such that the set of differences is equal to $\mathbb{Z}$

In this chapter we consider a countable set $\Gamma \subset \mathbb{R}$ such that $\Gamma - \Gamma = \mathbb{Z}$ and we are looking for measures μ such that $(e_\gamma)_{\gamma\in\Gamma}$, is an orthonormal basis in $L^2(\mu)$, i.e. such that (μ, Γ) is a spectral pair.

Lemma 5.1. *Let $\Gamma \subset \mathbb{R}$ satisfy $\Gamma - \Gamma = \mathbb{Z}$. Then*

$$M^\perp(\Gamma) = M^\perp(\mathbb{Z}).$$

PROOF. Notice that $\langle e_\beta | e_\gamma \rangle = \delta_{\beta,\gamma}$ just depends on the difference $\gamma - \beta$. Consequently, the result follows. $\square$

Remark 5.2. Regarding the sets $M^{OB}(\mathbb{Z})$ and $M^\perp(\Gamma)$ we notice the following.

(1) It is well known that $\lambda|_{[0,1]} \in M^{OB}(\mathbb{Z})$.
(2) We easily see that $\mu = \nu \star \lambda|_{[0,1]}$ with $\mu \in \mathcal{M}$ is also in $M^\perp(\Gamma)$, $\Gamma - \Gamma = \mathbb{Z}$.
(3) If we consider $\Gamma \subset \mathbb{R}$ such that $\Gamma - \Gamma = \mathbb{Z}$, then it follows that for all $\gamma, \gamma' \in \Gamma$ we have $\gamma - \lfloor\gamma\rfloor = \gamma' - \lfloor\gamma'\rfloor$. Consequently, we get $\Gamma = \Gamma' + a$ for some $\Gamma' \subset \mathbb{Z}$ and $a \in \mathbb{R}$. Thus, we can assume $\Gamma \subset \mathbb{Z}$ without loss of generality.

Theorem 5.3. *For every $\mu \in M^\perp(\Gamma)$ its Fourier transform factors as a product*

$$\widehat{\mu} = f \cdot \widehat{\lambda|_{[0,1]}},$$

where f extends to to an entire function on $\mathbb{C}$ of the form given in (5.0.3).

For the proof of Theorem 5.3 we need the Weierstrass Factorization Theorem, which can be found in [**Rud87**].

Theorem ([**Rud87**], Weierstrass Factorization Theorem)**.** *Let f be an entire function, suppose $f(0) \neq 0$, and let $z_1, z_2, \dots$ be the zeros of f, listed according to their multiplicities. Then there exist an entire function g and a sequence $\{p_n\}_{n\in\mathbb{N}}$ of non-negative integers, such that*

$$f(z) = e^{g(z)} \prod_{n=1}^{\infty} E_{p_n}\left(\frac{z}{z_n}\right),$$

where $E_p(z) = (1-z)\exp\left\{z + \frac{z^2}{2} + \cdots + \frac{z^p}{p}\right\}$ for $p = 1, 2, \dots$ and $E_0(z) = 1 - z$.

PROOF OF THEOREM 5.3. Consider the Fourier transform of μ. We have that $\widehat{\mu}$ vanishes on $\mathbb{Z}$ since $\mu \in M^\perp(\Gamma)$.

The first step is to extend the functions $\widehat{\mu}$ and $\widehat{\lambda|_{[0,1]}}$ to entire functions on $\mathbb{C}$. For this notice that $\widehat{\mu}$ and $\widehat{\lambda|_{[0,1]}}$ are analytic on $\mathbb{T}$ since the measures have compact support. Then we define the extensions in the following way:

Consider $[-A, A]$ such that $\operatorname{supp}(\mu) \subset [-A, A]$ and consider $e^{i2\pi tx}$ in a power series:

$$\int e^{i2\pi tx} d\mu(x) = \int \sum_{n=0}^{\infty} \frac{(i2\pi t)^n}{n!} x^n d\mu(x) = \sum_{n=0}^{\infty} \frac{(i2\pi t)^n}{n!} \int x^n d\mu(x)$$

and $\left|\int x^n d\mu(x)\right| \leq \int |x|^n d\mu(x) \leq A^n$ and thus

$$\left|\int e^{i2\pi tx} d\mu(x)\right| \leq \sum_{n=0}^{\infty} \frac{|i2\pi t|^n \cdot A^n}{n!} \leq e^{2\pi|t|A}.$$

Consequently, we can extend the function to an entire function on $\mathbb{C}$. Thus, we get

$$\widehat{\mu}(z) = \int e^{i2\pi zx} d\mu(x) \text{ and } \widehat{\lambda|_{[0,1]}}(z) = \int e^{i2\pi zx} d\lambda|_{[0,1]}(x),\ z \in \mathbb{C}.$$

Furthermore, we have that for $z \in \mathbb{C}$

$$\widehat{\lambda|_{[0,1]}}(z) = \frac{1}{i2\pi z}\left(e^{i2\pi z} - 1\right) = e^{i\pi z}\frac{e^{i\pi z} - e^{-i\pi z}}{i2\pi z} = e^{i\pi z}\frac{\sin(z\pi)}{z\pi} = e^{i\pi z}\prod_{n=1}^{\infty}\left(1 - \frac{z^2}{n^2}\right)$$

by results of [**Rud87**] and the fact that $1 - \frac{z^2}{n^2} = \left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right)$.

Since we have that $\mathbb{Z}\backslash\{0\} \subset Z(\widehat{\mu})$ and $\widehat{\mu}(0) = 1$ we can apply the Weierstrass Factorization Theorem. For the zeros with $n \in \mathbb{Z}\backslash\{0\}$ we get for some $p_n \in \mathbb{N}_0$

$$E_{p_n}\left(\frac{z}{n}\right) = \left(1 - \frac{z}{n}\right) \cdot \exp\left\{\frac{z}{n} + \frac{z^2}{2n^2} + \cdots + \frac{z^{p_n}}{p_n n^{p_n}}\right\}.$$

Thus, we have all the factors $1 - \frac{z^2}{n^2}$ (since $n \in \mathbb{Z}$). Consequently, we get

$$\widehat{\mu}(z) = e^{h(z)} \cdot \xi(z) \cdot \widehat{\lambda|_{[0,1]}}(z),$$

where $h(z) = g(z) - i\pi z + \sum_{n\in\mathbb{Z}} f_n(z)$, g is the analytic function obtained by the Weierstrass Factorization Theorem and

$$f_n(z) = \frac{z}{n} + \frac{z^2}{2n^2} + \cdots + \frac{z^{p_n}}{p_n n^{p_n}},$$

and

$$\xi(z) = \prod_{n=1}^{\infty}\left(1 - \frac{z}{z_n}\right)\exp\left\{\frac{z}{z_n} + \frac{z^2}{2z_n^2} + \cdots + \frac{z^{p_n}}{p_n z_n^{p_n}}\right\},$$

where the z_n are the zeros which are not in $\mathbb{Z}$. Consequently, we can write

$$\widehat{\mu}(z) = f(z) \cdot \widehat{\lambda|_{[0,1]}}(z),$$

with

$$f(z) = e^{h(z)}\xi(z) \tag{5.0.3}$$

and f, the composition, product and sum of entire functions, is entire. □

Remark 5.4. We give some properties of the function f given in (5.0.3).

(1) $f : \mathbb{C} \to \mathbb{C}$ is an entire function.
(2) For f it holds that $f(-z) = \overline{f(z)}$, $z \in \mathbb{C}$.
(3) f does not have any purely imaginary zeros.
(4) $f(k) = (\widehat{\mu})'(k) \cdot k$ for $k \in \mathbb{Z}$ and $\left|(\widehat{\mu})'(k)\right| < \infty$, $(\widehat{\mu})'(t) = \int i2\pi x \cdot e^{i2\pi tx} d\mu(x)$.

Conjecture. *We conjecture that all measures in $M^{\perp}(\mathbb{Z})$ take the form $\nu \star \lambda|_{[0,1]}$, where ν is a signed measure.*

Remark 5.5. The conjecture can be differently stated as: The function f in (5.0.3) is the difference of two positive-definite functions. Consequently, it is the Fourier transform of a signed measure ν with $\nu \star \lambda|_{[0,1]}$ being a probability measure.

Example 5.6. Now we give an example for the correspondence between the characterization of $\mu \in M^{\perp}(\mathbb{Z})$ as $d\mu = f d\lambda$ for some function $f \geq 0$ with $\sum_{k\in\mathbb{Z}} f(x+k) = 1$ for Lebesgue-a.e. $x \in \mathbb{R}$ as given in Theorem 2.26 and $\widehat{\mu} = \widehat{\nu} \cdot \widehat{\lambda|_{[0,1]}}$ (in this case even $\mu = \nu \star \lambda|_{[0,1]}$) as given in Theorem 5.3. Consider the function $f(x) = (1-x)\cdot \mathbb{1}_{[0,1)}(x) + (x-1)\cdot \mathbb{1}_{[1,2)}(x)$. It follows that $f d\lambda \in M^{\perp}(\Gamma)$ for every set $\Gamma \subset \mathbb{Z}$ such that $\Gamma - \Gamma = \mathbb{Z}$. The corresponding measure ν such that $f d\lambda = d\left(\nu \star \lambda|_{[0,1]}\right)$ is $\nu = \delta_0 + \delta_1 - \lambda|_{[0,1]}$.

We are interested in the measures which give us a spectral pair. More precisely, in the set $M^{\perp}(\Gamma)$ with $\Gamma - \Gamma = \mathbb{Z}$ we look for the set $M^{OB}(\Gamma)$. A natural place to look for such measures is the set of extreme points of $M^{\perp}(\Gamma)$, since in Proposition 3.8 (2) we have seen that $M^{OB}(\Gamma)$ is not a convex set.

Theorem 5.7. *The elements in $M^{OB}(\mathbb{Z})$ are exactly the extreme points in $M^{\perp}(\mathbb{Z})$, i.e. $M^{OB}(\mathbb{Z}) = \operatorname{ext}(M^{\perp}(\mathbb{Z}))$.*

Proof. We prove first the inclusion $M^{OB}(\mathbb{Z}) \subset \operatorname{ext}\left(M^{\perp}(\mathbb{Z})\right)$ via the characterization of [**DJ10**], see Theorem 2.28. So assume that for $\mu \in M^{OB}(\mathbb{Z})$ and $\nu_1,\ \nu_2 \in M^{\perp}(\Gamma)$, $0 < \alpha < 1$,

$$\mu = \alpha\cdot\nu_1 + (1-\alpha)\cdot\nu_2.$$

By Theorem 2.26 we have that for $\nu_1,\ \nu_2$ there are functions $f_1,\ f_2 \geq 0$ such that for $j = 1, 2$,

$$\sum_{k\in\mathbb{Z}} f_j(y+k) = 1, \text{ for Lebesgue-a.e. } y \in \mathbb{R}.$$

For μ the corresponding function is $f = \mathbb{1}_A$, where A is a compact set that is translation congruent to the unit interval. Consequently, we have

$$f d\lambda = \alpha\cdot f_1 d\lambda + (1-\alpha)\cdot f_2 d\lambda = (\alpha\cdot f_1 + (1-\alpha)\cdot f_2)\, d\lambda.$$

Since the Radon-Nikodym derivative is Lebesgue-a.e. unique, we have $f = \alpha\cdot f_1 + (1-\alpha)\cdot f_2$. Furthermore, we have $\operatorname{supp} f = A$, and since $f_1,\ f_2 \geq 0$ it follows that $\operatorname{supp} f_j \subset A$ for $j = 1, 2$. Consequently in $\sum_{k\in\mathbb{Z}} f_j(y+k) = 1$, there is just one non-zero summand for Lebesgue-a.e. $y \in \mathbb{R}$ since there exists a partition of $[0,1]$ of sets $E_k \subset [0,1]$ such that $A = \bigcup_{k\in\mathbb{Z}} (E_k + l_k)$, $l_k \in \mathbb{Z}$. Thus for the $l \in \mathbb{Z}$ with $f_j(x+l) \neq 0$ it follows that $f_j(y+l) = 1$. Thus, it follows that $f_j(y) = \mathbb{1}_A(y)$ for Lebesgue-a.e. $y \in \mathbb{R}$, $j = 1, 2$. Hence, the result follows, i.e. μ is an extreme point in $M^{\perp}(\Gamma)$.

To prove the other inclusion, $\operatorname{ext}\left(M^{\perp}(\mathbb{Z})\right) \subset M^{OB}(\mathbb{Z})$, we fix $\mu \in \operatorname{ext}\left(M^{\perp}(\mathbb{Z})\right)$ and notice that the function f such that $f d\lambda = \mu \in M^{\perp}(\mathbb{Z})$ has compact support, so $\operatorname{supp}(f) \subset [-q, q]$ for some $q \in \mathbb{R}$. Furthermore if f is not a characteristic function of a set which is translation congruent to $[0,1]$, that means $f d\lambda = \mu$ for $\mu \notin M^{OB}(\mathbb{Z})$, then we have $\lambda\left(\operatorname{supp}(f)\right) = d > 1$.

Now we consider the positive function $(f-\varepsilon)^+$, defined for $x \in \mathbb{R}$ by

$$(f-\varepsilon)^+(x) := \begin{cases} (f-\varepsilon)(x), & \text{if } (f-\varepsilon)(x) \geq 0, \\ 0, & \text{otherwise,} \end{cases}$$

for some $\varepsilon > 0$. For these functions we have that $\lambda\left(\operatorname{supp}\left((f-\varepsilon)^+\right)\right) \nearrow d$ for $\varepsilon \to 0$ and for $\varepsilon_1 > \varepsilon_2$ $\operatorname{supp}\left((f-\varepsilon_1)^+\right) \subset \operatorname{supp}\left((f-\varepsilon_2)^+\right)$. Now we assume that for all $\varepsilon > 0$ and all partitions there exists an element $E_k \subset [0,1]$ of the partition such that for all $l \in \mathbb{Z}$ we have $(E_k + l) \not\subseteq \operatorname{supp}\left((f-\varepsilon)^+\right)$. This is equivalent to

$$\lambda\left((E_k + l) \cap \operatorname{supp}\left((f-\varepsilon)^+\right)\right) = 0$$

because if $\lambda\left((E_k + l) \cap \operatorname{supp}\left((f-\varepsilon)^+\right)\right) \neq 0$ and $E_k + l \not\subseteq \operatorname{supp}\left((f-\varepsilon)^+\right)$ we can consider a different partition that satisfies this property. Consequently, it follows that $f(x+l) < \varepsilon$ for Lebesgue-a.e. $x \in E_k$ and $l \in \mathbb{Z}$. Thus, we have to sum up $1/\varepsilon$ times these $x \in E_k$. This contradicts that f has compact support.

Thus, there exists $\varepsilon > 0$ with $\lambda\left(\operatorname{supp}\left((f-\varepsilon)^+\right)\right) > 1$ and there exists a partition $(E_k)_{k\in J}$ such that for all sets $E_k \subset [0,1]$ of the partition there exists an $l_k \in \mathbb{Z}$ with $(E_k + l_k) \subset \operatorname{supp}\left((f-\varepsilon)^+\right)$. Now we define $A = \bigcup_{k\in J} E_k + l_k$ and $f_1 = \mathbb{1}_A$. Define $f_2 = \frac{1}{1-\varepsilon}\left(f - \varepsilon \mathbb{1}_A\right)$. These two functions satisfy $\sum_{k\in\mathbb{Z}} f_j(x+k) = 1$ for almost all $x \in \mathbb{R}$. For f_1 it is obvious and for f_2 it follows from

$$\sum_{k\in\mathbb{Z}} \frac{1}{1-\varepsilon}\left(f(x+k) - \varepsilon \mathbb{1}_A(x+k)\right) = \frac{1}{1-\varepsilon}\left(\sum_{k\in\mathbb{Z}} f(x+k) - \varepsilon \sum_{k\in\mathbb{Z}} \mathbb{1}_A(x+k)\right) = 1.$$

Consequently, we can write any function with $\lambda(\operatorname{supp}(f)) = d > 1$ as a convex combination of two other functions, namely $f = \varepsilon f_1 + (1-\varepsilon) f_2$. Hence, $\operatorname{ext}\left(M^\perp(\mathbb{Z})\right) \subset M^{OB}(\mathbb{Z})$. □

Remark 5.8. If we consider $\widetilde{M}^\perp(\mathbb{Z})$, compare Proposition 3.8 (3), and

$$\widetilde{M}^{OB}(\mathbb{Z}) := \left\{\mu \in \widetilde{\mathcal{M}} : \sum_{\gamma\in\mathbb{Z}} |\widehat{\mu}(t-\gamma)|^2 = 1 \text{ for all } t \in \mathbb{R}\right\}$$

instead of $M^\perp(\mathbb{Z})$ and $M^{OB}(\mathbb{Z})$, then we can still prove $\widetilde{M}^{OB}(\mathbb{Z}) \subset \operatorname{ext}\left(\widetilde{M}^\perp(\mathbb{Z})\right)$ in the same way as we have proven $M^{OB}(\mathbb{Z}) \subset \operatorname{ext}\left(M^\perp(\mathbb{Z})\right)$. But the proof of the other inclusion does not work in the same way, consider e.g. the function

$$f(x) = \sum_{k\geq 1} \frac{1}{2^k} \sum_{n=0}^{2^k-1} \mathbb{1}_{\left[\frac{1}{2^k}, \frac{1}{2^{k+1}}\right]}(x-n), \ x \in \mathbb{R}.$$

Now we turn to further properties of the set $M^{OB}(\Gamma)$.

Theorem 5.9. *For $\Gamma \subsetneq \mathbb{Z}$ with $\Gamma - \Gamma = \mathbb{Z}$, $M^{OB}(\Gamma) = \emptyset$.*

Proof. First recall that $M^\perp(\Gamma) = M^\perp(\mathbb{Z})$. If $\mu \in M^{OB}(\Gamma)$ then consider $\beta \in \mathbb{Z}\backslash\Gamma$ and so $e_\beta \perp e_\gamma$ for all $\gamma \in \Gamma$. Consequently, $(e_\gamma)_{\gamma\in\Gamma}$ cannot be complete. Thus, $M^{OB}(\Gamma) = \emptyset$. □

Example 5.10. An example for a set $\Gamma \subset \mathbb{N}$ such that $\Gamma - \Gamma = \mathbb{Z}$ is

$$\Gamma = \left\{\sum_{i=0}^{k} l_i 3^i : l_i \in \{0,1\}, k \in \mathbb{N}_0\right\} = \{0,1,3,4,9,10,12,13,27,28,\dots\}.$$

Consequently this is an example for $M^{OB}(\Gamma) = \emptyset$. This set would be the natural choice for the middle-third Cantor set in analogy to the 1/4-Cantor set, compare Example 4.3. But we know that there is no spectral pair for the middle-third Cantor set, see **[JP98a]** and Example 6.18.

From now on we consider the special case of $\Gamma = \mathbb{Z}$.

Proposition 5.11. *If $\mu \in M^{OB}(\mathbb{Z})$ with $\mu = \mathbb{1}_A d\lambda$ and $A = \sum_{k=0}^{\infty} [a_k, b_k] + l_k$, $a_k, b_k \in [0,1]$, $l_k \in \mathbb{Z}$, then $\mu = \nu \star \lambda|_{[0,1]}$, where ν is given in (5.0.4).*

Proof. By translating A as necessary, which does not change the spectral properties of $\mu = \mathbb{1}_A d\lambda$, we may assume without loss of generality that $a_0 = 0$, $l_0 = 0$ and $A \subset \mathbb{R}^+$. Furthermore $l_{\max} := \max\{l_k : k \in \mathbb{N}_0\}$ exists since μ has compact support. Then with

$$\nu := \sum_{k=0}^{\infty} \sum_{j=l_k+1}^{l_{\max}-1} \left(\delta_{a_k+l_k+j} - \delta_{b_k+l_k+j}\right) + \delta_{l_{\max}} \tag{5.0.4}$$

we get $\mathbb{1}_A d\lambda = \nu \star \lambda|_{[0,1]}$. □

Remark 5.12. If in Proposition 5.11 (2) $A = \sum_{k=0}^{N} [a_k, b_k] + l_k$, $a_k, b_k \in [0,1]$, $l_k \in \mathbb{Z}$, $N \in \mathbb{N}$, then N must be even.

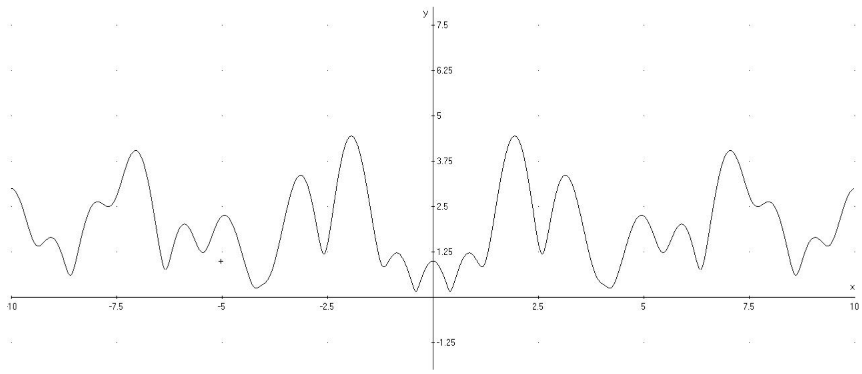


FIGURE 5.0.1. Display of the map $t \mapsto |\widehat{\nu}(t)|$.

Example 5.13. For the Example 3.11 we have that $\lambda|_A = \nu \star \lambda|_{[0,1]}$ with

$$\nu = \sum_{k=1}^{\infty} \left(\delta_{k+\frac{1}{k+1}} - \delta_{k+\frac{1}{k}} \right).$$

This example does not have compact support.

For an example with compact support we can consider $\lambda|_{[0,1/2]\cup[3/2,2]}$. Then ν takes the form

$$\nu = \delta_0 - \delta_{1/2} + \delta_1.$$

Example 5.14. As an example of a signed measure of the form $\nu = \sum_{k=0}^{N} (-1)^k \delta_{x_k}$, we consider $\nu = \delta_0 - \delta_{1/4} + \delta_{3/5} - \delta_{4/5} + \delta_1$. Then the map $t \mapsto |\widehat{\nu}(t)|$ looks as shown in Figure 5.0.1. Consequently, it does not satisfy $|\widehat{\nu}| = 1$. So there are signed measures ν such that $\nu \star \lambda|_{[0,1]} \in M^{OB}(\mathbb{Z})$ that does not satisfy $|\widehat{\nu}| = 1$.

5.1. Considerations about $M^{\perp}(\Gamma)$ for $\Gamma \subset \mathbb{R}$ satisfying $\Gamma - \Gamma = k\mathbb{Z}$, $k \in \mathbb{N}$

In this section we characterize the sets $M^{\perp}(\Gamma)$, $M^{OB}(\Gamma)$, where $\Gamma \subset \mathbb{R}$ with $\Gamma - \Gamma = k\mathbb{Z}$ for some $k \in \mathbb{R}^+$, $k \neq 0$. We obtain analogous results to those for $M^{\perp}(\mathbb{Z})$ and $M^{OB}(\mathbb{Z})$ by considering the relation between $M^{\perp}(\Gamma)$, $M^{OB}(\Gamma)$ and $M^{\perp}(\mathbb{Z})$, $M^{OB}(\mathbb{Z})$.

Remark 5.15. Before we start stating the results we notice the following.

(1) First notice that $M^{\perp}(\Gamma) = M^{\perp}(k\mathbb{Z})$ for $\Gamma \subset \mathbb{R}$ with $\Gamma - \Gamma = k\mathbb{Z}$ for some $k \in \mathbb{R}^+$, $k \neq 0$.
(2) For $\Gamma \subset \mathbb{R}$ with $\Gamma - \Gamma = k\mathbb{Z}$ for some $k \in \mathbb{R}^+$, $k \neq 0$, we notice that $\Gamma = \Gamma' + a$ for some set $\Gamma' \subset k\mathbb{Z}$ and $a \in \mathbb{R}$. Consequently, we can assume without loss of generality $\Gamma \subset k\mathbb{Z}$ with $\Gamma - \Gamma = k\mathbb{Z}$.

Proposition 5.16. *$\mu \in M^{\perp}(k\mathbb{Z})$ if and only if there is $\nu \in M^{\perp}(\mathbb{Z})$ with $\widehat{\nu}(t) = \widehat{\mu}(kt)$ for all $t \in \mathbb{R}$.*

PROOF. Let $\mu \in M^{\perp}(k\mathbb{Z})$ and take ν such that $\widehat{\nu}(t) = \widehat{\mu}(kt)$, $t \in \mathbb{R}$. Then ν is a positive measure since $\widehat{\nu}$ is a positive definite function and hence the Fourier transform of a positive measure. Furthermore, $\widehat{\nu}(0) = 1$ and so it is even a probability measure. For $l \in \mathbb{Z}$ it follows that $\widehat{\nu}(l) = \widehat{\mu}(kl) = \delta_{0,l}$. Consequently, $\nu \in M^{\perp}(\mathbb{Z})$.

On the other hand take $\nu \in M^{\perp}(\mathbb{Z})$ and consider a measure μ with $\widehat{\mu}(t) = \widehat{\nu}\left(\frac{t}{k}\right)$, $t \in \mathbb{R}$. Then the argument goes as above. $\square$

Corollary 5.17. *$\mu \in M^{OB}(k\mathbb{Z})$ if and only if there is $\nu \in M^{OB}(\mathbb{Z})$ with $\widehat{\nu}(t) = \widehat{\mu}(kt)$ for all $t \in \mathbb{R}$.*

PROOF. Let $\mu \in M^{OB}(k\mathbb{Z})$, then there exists a measure $\nu \in M^{\perp}(\mathbb{Z})$ with $\widehat{\nu}(t) = \widehat{\mu}\,(kt)$, $t \in \mathbb{R}$. It follows that

$$1 = \sum_{l \in k\mathbb{Z}} |\widehat{\mu}(t-l)|^2 = \sum_{l \in k\mathbb{Z}} \left| \widehat{\nu}\left(\frac{t-l}{k}\right) \right|^2 = \sum_{l \in \mathbb{Z}} \left| \widehat{\nu}\left(\frac{t}{k} - l\right) \right|^2$$

and thus $\nu \in M^{OB}(\mathbb{Z})$. Starting with $\nu \in M^{OB}(\mathbb{Z})$ we obtain in the same way that $\mu \in M^{OB}(k\mathbb{Z})$ with $\widehat{\mu}\,(t) = \widehat{\nu}\left(\frac{t}{k}\right)$, $t \in \mathbb{R}$. □

Remark 5.18. With Corollary 5.17 we can easily obtain an element in $M^{OB}(k\mathbb{Z})$ which is $k\lambda_{[0,1/k]}$ since

$$k\widehat{\lambda|_{[0,1/k]}}(t) = \int_0^{1/k} ke^{i2\pi tx} d\lambda(x) = \frac{k}{i2\pi t}\left(e^{i2\pi \frac{t}{k}} - 1\right) = \widehat{\lambda|_{[0,1]}}\left(\frac{t}{k}\right).$$

This element takes in the study of $M^{\perp}(k\mathbb{Z})$ the role that has the Lebesgue measure restricted to $[0,1]$ in the study of $M^{\perp}(\mathbb{Z})$.

Now we turn to the results about elements in $M^{\perp}(k\mathbb{Z})$ and $M^{OB}(k\mathbb{Z})$, which are analogue to those for $M^{\perp}(\mathbb{Z})$ and $M^{OB}(\mathbb{Z})$.

Corollary 5.19. *We obtain the following results for $M^{\perp}(k\mathbb{Z})$ and $M^{OB}(k\mathbb{Z})$, $k \in \mathbb{R}^+$, $k \neq 0$.*

(1) *Let μ be a Borel probability measure on $\mathbb{R}$. The following statements are equivalent:*
   (a) *The set $(e_\gamma)_{\gamma \in k\mathbb{Z}}$ forms an orthonormal set in $L^2(\mu)$, i.e. $\mu \in M^{\perp}(k\mathbb{Z})$.*
   (b) *There exists a bounded measurable function $f \geq 0$ that satisfies*
   $$\sum_{l \in \frac{1}{k}\mathbb{Z}} f(x+l) = k, \text{ for Lebesgue-a.e. } x \in \mathbb{R},$$
   such that $d\mu = f d\lambda$.

(2) *$\mu \in M^{OB}(k\mathbb{Z})$ if and only if $\mu = k\mathbb{1}_A d\lambda$, where kA is translation congruent to $[0,1]$.*

(3) *Let $\Gamma \subset \mathbb{R}$ with $\Gamma - \Gamma = k\mathbb{Z}$. For every $\mu \in M^{\perp}(\Gamma)$ its Fourier transform factors in the product*
$$\widehat{\mu} = k \cdot f \cdot \widehat{\lambda|_{[0,1/k]}},$$
where f extends to an entire function on $\mathbb{C}$.

(4) *The elements in $M^{OB}(k\mathbb{Z})$ are the extreme points in $M^{\perp}(k\mathbb{Z})$, i.e. $M^{OB}(k\mathbb{Z}) = \operatorname{ext}(M^{\perp}(k\mathbb{Z}))$.*

(5) *For $\Gamma \subsetneq k\mathbb{Z}$ with $\Gamma - \Gamma = k\mathbb{Z}$, $M^{OB}(\Gamma) = \emptyset$.*

PROOF. ad (1): To obtain this result we combine Theorem 2.26 and Proposition 5.16. Let $\mu \in M^{\perp}(k\mathbb{Z})$, then there is $\nu \in M^{\perp}(\mathbb{Z})$ with $\widehat{\nu}(t) = \widehat{\mu}(kt)$. For $\nu \in M^{\perp}(\mathbb{Z})$ we know that there exists a function $g \geq 0$ with $\sum_{l \in \mathbb{Z}} g(x+l) = 1$ for Lebesgue-a.e. $x \in \mathbb{R}$ and $\nu = g d\lambda$. Consequently, $\widehat{\nu}(t) = \widehat{g}(t) = \widehat{\mu}(kt)$ and so $\mu = f d\lambda$ with $f = kg\,(k\cdot)$ and f satisfies for Lebesgue-a.e $x \in \mathbb{R}$

$$\sum_{l \in \frac{1}{k}\mathbb{Z}} f(x+l) = \sum_{l \in \frac{1}{k}\mathbb{Z}} kg(k(x+l)) = k \sum_{m \in \mathbb{Z}} g(kx+m) = k.$$

The other directions follows analogously.

ad (2): To obtain this result we combine Theorem 2.28 and Proposition 5.16.

ad (3): To obtain this result we combine Theorem 5.3 and Proposition 5.16.

ad (4): To obtain this result we combine Theorem 5.7 and Proposition 5.16.

ad (5): To obtain this result we combine Theorem 5.9 and Proposition 5.16. □

CHAPTER 6

Construction of spectral pairs via $(N \times N)$-Hadamard matrices

In this chapter we consider one dimensional spectral pairs that we obtain via $(N \times N)$-Hadamard matrices. More precisely, we consider Cantor sets and the circumstances in which there is a spectrum or family of orthonormal functions of the corresponding invariant measure defined in Definition 2.9. The invariant measure for such a fractal is precisely the $\frac{\log N}{\log R}$-Hausdorff measure restricted to its invariant set.

Definition 2.12 gives the general definition of a Hadamard matrix. We will consider these in the sense of Theorem 2.21, i.e. we impose the restriction $0 \in L$. We further only consider cases with $0 \in B$.

Remark 6.1. The restriction $0 \in B$ is not a great restriction since if we consider an aIFS $\left(\tau_b(x) = \frac{x+b}{R}\right)_{b \in B}$, $R \in \mathbb{N}$, $B = \{b_1, \dots, b_N\} \subset \underline{R} \backslash \{0\}$ there is $x \in \mathbb{R}$ with $\tau_{b_1}(x) = x$. Furthermore, we have for the invariant set $C \subset [x, 1]$. Consequently, if we translate the invariant set by x to the left we obtain an invariant set $\widetilde{C} \subset [0, 1-x]$ for the aIFS $\left(\sigma_b(x) = \frac{x+b-b_1}{R}\right)_{b \in B}$. Furthermore, we obtain the invariant measure $\widetilde{\mu}$ for the translated fractal from the original invariant measure μ as $\widetilde{\mu} = \delta_{-x} \star \mu$ and we know that a convolution with δ_{-x} does not have any influence on the spectral properties.

In the next example we show that we cannot simply omit the property $0 \in L$.

Example 6.2. Recall the Example 4.3. There we have a Hadamard matrix for $R = 4$, $B = \{0, 2\}$ and $L_1 = \{0, 1\}$. Let us now consider $L_2 = \{4, 1\}$. It is easily checked that this also gives a Hadamard matrix $M_R(B, L_2)$. If we consider the corresponding sets Γ_1 and Γ_2 given as

$$\Gamma_1 = \left\{ \sum_{i=0}^{k} l_i 4^i \, : \, l_i \in L_1, \, k \in \mathbb{N}_0 \right\} = \{0, 1, 4, 5, 16, 17, 20, 21, 64, 65, \dots\},$$

$$\Gamma_2 = \left\{ \sum_{i=0}^{k} l_i 4^i \, : \, l_i \in L_2, \, k \in \mathbb{N}_0 \right\} = \{1, 4, 5, 8, 17, 20, 21, 24, 33, 36 \dots\}.$$

We can easily see that $\Gamma_2 \neq \Gamma_1 + k$ for all $k \in \mathbb{Z}$. Furthermore, Γ_2 does not even give an orthonormal set for μ since $32 \in \Gamma_2 - \Gamma_2$ and $\widehat{\mu}(32) = e^{i\frac{2\pi 32}{3}} \prod_{n=1}^{\infty} \cos\left(\frac{2\pi \cdot 32}{4^n}\right) \neq 0$.

6.1. Hadamard matrices

Before we start with the considerations about spectral pairs for Cantor sets, we give an introduction to Hadamard matrices based on **[TŻ06]**.

Definition 6.3. A square matrix $H = (H_{ij})_{i,j \in \underline{N}}$ of size N consisting of uni-modular entries, $|H_{ij}| = 1$, $i, j \in \underline{N}$, is called a Hadamard matrix if $HH^* = N \cdot I$, i.e. $\frac{1}{\sqrt{N}} H$ is unitary (I stands for the identity matrix). One distinguishes

(1) real Hadamard matrices, $H_{ij} \in \mathbb{R}$, for $i, j \in \underline{N}$,
(2) Hadamard matrices of Butson type $H(q, N)$, for which $(H_{ij})^q = 1$, $i, j \in \underline{N}$,
(3) complex Hadamard matrices, $H_{ij} \in \mathbb{C}$, $i, j \in \underline{N}$.

We will only consider complex Hadamard matrices.

Definition 6.4. We need two other general notions about Hadamard matrices.

(1) A complex Hadamard matrix is called dephased when the entries of the first row and column are equal to 1, i.e. $H_{0i} = H_{i0} = 1$ for all $i \in \underline{N}$.
(2) Two Hadamard matrices H_1 and H_2 are called equivalent if there exist diagonal unitary matrices D_1 and D_2 and permutation matrices P_1 and P_2 such that $H_1 = D_1 P_1 H_2 P_2 D_2$.

Lemma 6.5. *For a complex $(N \times N)$-Hadamard matrix H there exist uniquely determined diagonal unitary matrices $D_r = \operatorname{diag}\left(\overline{H}_{00}, \overline{H}_{10}, \ldots, \overline{H}_{(N-1)0}\right)$ and $D_c = \operatorname{diag}\left(1, H_{00}\overline{H}_{01}, \ldots, H_{00}\overline{H}_{0(N-1)}\right)$ such that $D_r H D_c$ is dephased.*

Remark 6.6. Some remarks about Hadamard matrices:

(1) Two Hadamard matrices with the same dephased form are equivalent.
(2) The Hadamard matrices are only completely classified for $N = 2, 3, 4, 5, 6$. For larger N we do not know all the possible forms of Hadamard matrices. Consequently, we cannot consider general Hadamard matrices but we restrict ourselves to the so-called Fourier matrices of size $(N \times N)$. These are the dephased Hadamard matrices with $(H_{1j})_{j\in\underline{N}} = \left(1, \omega, \omega^2, \ldots, \omega^{N-1}\right)$, $\omega = e^{i2\pi/N}$, as the second row.

Before we start with the spectral pairs obtained via Hadamard matrices we give the classification for $N = 4$.

Example 6.7. Let $N = 4$. All dephased Hadamard matrices have the form

$$\frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & e^{ia} & -1 & -e^{ia} \\ 1 & -1 & 1 & -1 \\ 1 & -e^{ia} & -1 & e^{ia} \end{pmatrix}, \; a \in [0, 2\pi).$$

In this way we can obtain e.g. the following spectral pair. Let $R = 8$, $B = \{0, 1, 4, 5\}$ and $L = \{0, 1, 4, 5\}$. Then we get

$$M_R(B, L) = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega & -1 & -\omega \\ 1 & -1 & 1 & -1 \\ 1 & -\omega & -1 & \omega \end{pmatrix}, \; \omega = e^{i2\pi/8}.$$

Furthermore, we have that $\Gamma(L)$ is even the spectrum for μ_B, see [**DJ10**].

6.2. Zeros of $\widehat{\mu}$

In this section we consider properties of an invariant measure μ_B of an aIFS $\left(\tau_b(x) = \frac{x+b}{R}\right)_{b\in B}$, $B \subset \underline{R}$, $0 \in B$, $\operatorname{card} B = N$, $R \geq 2$. The Fourier transform of μ_B takes the form:

$$\widehat{\mu_B}(t) = \prod_{n=1}^{\infty} \frac{1}{N}\left(\sum_{b\in B} e_b\left(\frac{t}{R^n}\right)\right), \; t \in \mathbb{R}.$$

The following result is in analogy to Lemma 11.2 of [**JKS08**], where it is shown for one specific kind of fractal.

Proposition 6.8. *Let $B \subset \underline{R}$, $R \in \mathbb{N}$, $R \geq 2$, and let μ_B be the invariant measure corresponding to the aIFS $(\tau_b)_{b\in B}$. Suppose $t \in \mathbb{R}$ is a fixed real number. There are $K \in \mathbb{N}$ and $c > 0$ such that*

$$\left|\prod_{n\geq K} \frac{1}{N}\left(\sum_{b\in B} e_b\left(\frac{t}{R^n}\right)\right)\right| \geq c.$$

In other words, if for some $t_0 \in \mathbb{R}$, $\widehat{\mu_B}(t_0) = 0$, then one of the factors of the product must be 0.

Proof. Let $t \in \mathbb{R}$ be fixed and notice that

$$\Big|\prod_{n\geq K}\frac{1}{N}\Big(\sum_{b\in B}e_b\Big(\frac{t}{R^n}\Big)\Big)\Big| = \prod_{n\geq K}\Big|\frac{1}{N}\Big(\sum_{b\in B}e_b\Big(\frac{t}{R^n}\Big)\Big)\Big| \geq \prod_{n\geq K}\Big|\mathfrak{Re}\Big(\frac{1}{N}\Big(\sum_{b\in B}e_b\Big(\frac{t}{R^n}\Big)\Big)\Big)\Big|.$$

We further consider the real part and show that $\prod_{n\geq K}\Big|\mathfrak{Re}\Big(\frac{1}{N}\Big(\sum_{b\in B}e_b\Big(\frac{t}{R^n}\Big)\Big)\Big)\Big| \geq c$. Furthermore, realize that for $c>0$

$$\prod_{n\geq K}\Big|\mathfrak{Re}\Big(\frac{1}{N}\Big(\sum_{b\in B}e_b\Big(\frac{t}{R^n}\Big)\Big)\Big)\Big| \geq c \iff \sum_{n\geq K}\ln\left(\Big|\mathfrak{Re}\Big(\frac{1}{N}\Big(\sum_{b\in B}e_b\Big(\frac{t}{R^n}\Big)\Big)\Big)\Big|\right) \geq \ln(c).$$

The Taylor expansion of $\frac{1}{N}\left(\sum_{b\in B}e_b\left(\frac{t}{R^n}\right)\right)$ around 0 gives

$$\begin{aligned}\frac{1}{N}\Big(\sum_{b\in B}e_b\Big(\frac{t}{R^n}\Big)\Big) &= \frac{1}{N}\sum_{b\in B}\Big(\sum_{k\geq 0}\frac{\left(2\pi i t R^{-n}\right)^k}{k!}\Big)\\ &= 1-\frac{1}{N}\sum_{b\in B}\sum_{k\geq 1}(-1)^{k+1}\frac{\left(2\pi tbR^{-n}\right)^{2k}}{(2k)!} - i\frac{1}{N}\sum_{b\in B}\sum_{k\geq 0}(-1)^{k+1}\frac{\left(2\pi tbR^{-n}\right)^{2k+1}}{(2k+1)!}.\end{aligned}$$

Thus, we have

$$\mathfrak{Re}\Big(\frac{1}{N}\Big(\sum_{b\in B}e_b\Big(\frac{t}{R^n}\Big)\Big)\Big) = 1-\frac{1}{N}\sum_{b\in B}\sum_{k\geq 1}(-1)^{k+1}\frac{\left(2\pi tbR^{-n}\right)^{2k}}{(2k)!}.$$

Now define for $n\in\mathbb{N}$, $t\in\mathbb{R}$,

$$\varepsilon_n(t) := \frac{1}{N}\sum_{b\in B}\sum_{k\geq 1}(-1)^{k+1}\frac{\left(2\pi tbR^{-n}\right)^{2k}}{(2k)!}.$$

We have that $\varepsilon_n(t)\geq 0$ since

$$-1 \leq \mathfrak{Re}\Big(\frac{1}{N}\Big(\sum_{b\in B}e_b\Big(\frac{t}{R^n}\Big)\Big)\Big) \leq 1 \iff -1\leq 1-\varepsilon_n(t)\leq 1 \iff 0\leq\varepsilon_n(t)\leq 2.$$

Now we choose $N_1\in\mathbb{N}$ such that for all $n\geq N_1$ we have

$$\varepsilon_n(t) \leq \frac{1}{N}\sum_{b\in B}\frac{\left(2\pi bR^{-n}\right)^2}{2}t^2$$

and choose $N_2\in\mathbb{N}$ such that for all $n\geq N_2$:

$$\frac{1}{N}\sum_{b\in B}\frac{\left(2\pi bR^{-n}\right)^2}{2}t^2 \leq 1.$$

Now we consider the Taylor expansion of $\ln\left|\mathfrak{Re}\left(\frac{1}{N}\left(\sum_{b\in B}e_b\left(\frac{t}{R^n}\right)\right)\right)\right|$. We have for $-1<\varepsilon_n(t)\leq 1$

$$\ln\left|\mathfrak{Re}\Big(\frac{1}{N}\Big(\sum_{b\in B}e_b\Big(\frac{t}{R^n}\Big)\Big)\Big)\right| = \ln\left(1-\varepsilon_n(t)\right) = -\sum_{k=1}^{\infty}\frac{\varepsilon_n(t)^k}{k}.$$

Now we choose $N_3\in\mathbb{N}$ such that for all $n\geq N_3$ we have

$$\sum_{k=1}^{\infty}\frac{\varepsilon_n(t)^k}{k} \leq 2\varepsilon_n(t).$$

This is possible since for $0<\varepsilon_n(t)\leq 1/2$ we have

$$\sum_{k=1}^{\infty}\frac{\varepsilon_n(t)^k}{k} \leq \sum_{k=1}^{\infty}\varepsilon_n(t)^k = \frac{\varepsilon_n(t)}{1-\varepsilon_n(t)} \leq 2\varepsilon_n(t).$$

Consequently, if we choose $K \geq \max\{N_1, N_2, N_3\}$, we have

$$\begin{aligned}\sum_{n\geq K} \ln \Big|\mathfrak{Re}\Big(\frac{1}{N}\Big(\sum_{b\in B} e_b\Big(\frac{t}{R^n}\Big)\Big)\Big)\Big| &= \sum_{n\geq K}\Big(-\sum_{k\geq 1}\frac{\varepsilon_n(t)^k}{k}\Big) \geq \sum_{n\geq K} -2\varepsilon_n(t)\\ &\geq -\sum_{n\geq K}\frac{1}{N}\sum_{b\in B}(2\pi b)^2 R^{-2n}t^2\\ &= -\frac{1}{N}\sum_{b\in B}(2\pi b)^2\sum_{n\geq K} R^{-2n}t^2.\end{aligned}$$

Now we define

$$c := \exp\Big(-\frac{1}{N}\sum_{b\in B}(2\pi b)^2\sum_{n\geq K}R^{-2n}t^2\Big) > 0$$

and thus we have

$$\Big|\prod_{n\geq K}\frac{1}{N}\sum_{b\in B}e_b\Big(\frac{t}{R^n}\Big)\Big| \geq \prod_{n\geq K}\Big|\mathfrak{Re}\Big(\frac{1}{N}\Big(\sum_{b\in B}e_b\Big(\frac{t}{R^n}\Big)\Big)\Big)\Big| \geq c.$$

□

Consequently, the only zeros of $\widehat{\mu_B}$ are those coming from zeros of the factors in the infinite product.

Lemma 6.9. *Let $B \subset \underline{R}$, $R \in \mathbb{N}$, $R \geq 2$, and let μ_B be the invariant measure corresponding to the aIFS $(\tau_b)_{b\in B}$. The zeros of $\widehat{\mu_B}$ are*

$$Z(\widehat{\mu_B}) = \bigcup_{l=1}^{\infty} R^l\Big(Z\Big(\sum_{b\in B}e_b\Big)\Big).$$

PROOF. With Proposition 6.8 we know that for $t_0 \in \mathbb{R}$ to be a zero of $\widehat{\mu_B}$, one of the factors $\frac{1}{N}\left(\sum_{b\in B} e_b\left(\frac{t_0}{R^n}\right)\right)$ must be equal to zero. □

Remark 6.10. We can write the elements of $Z\left(\sum_{b\in B} e_b\right)$ in terms of the ones in $\mathcal{Z}\left(\sum_{b\in B} e_b\right) := Z\left(\sum_{b\in B} e_b\right) \cap [0,1]$ as

$$Z\Big(\sum_{b\in B}e_b\Big) = \bigcup_{k\in\mathbb{Z}}\mathcal{Z}\Big(\sum_{b\in B}e_b\Big) + k.$$

As an easy observation we get the following proposition.

Proposition 6.11. *Let $B \subset \underline{R}$, $R \in \mathbb{N}$, $R \geq 2$, and let μ_B be the invariant measure for the aIFS $(\tau_b)_{b\in B}$. Then $(e_\gamma)_{\gamma\in\Gamma}$ are pairwise orthogonal in $L^2(\mu_B)$ if and only if*

$$\gamma - \gamma' \in Z(\widehat{\mu_B}),\ \textit{for all}\ \gamma, \gamma' \in \Gamma.$$

Consequently, for all $\gamma, \gamma' \in \Gamma$ there are $a \in \mathcal{Z}\left(\sum_{b\in B} e_b\right)$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$, with $\gamma - \gamma' = R^n(a+k)$.

PROOF. This follows from Proposition 6.8, Lemma 6.9 and Remark 6.10. □

Now we turn to the first correspondence between the existence of Hadamard matrices and the zeros of a measure μ_B.

Corollary 6.12. *Let $B \subset \underline{R}$, $R \in \mathbb{N}$, $R \geq 2$. If $\frac{1}{N}\sum_{b\in B} e_b$ does not have any zeros in $[0,1]$, then there does not exist any orthonormal functions e_γ, $\gamma \in \mathbb{R}$, in $L^2(\mu_B)$, where μ_B is the invariant measure for the aIFS $(\tau_b)_{b\in B}$. Furthermore, then there does not exist a Hadamard matrix.*

PROOF. From Proposition 6.11 we know that for $(e_\gamma)_{\gamma\in\Gamma}$, $\Gamma \subset \mathbb{R}$, to be orthonormal in $L^2(\mu_B)$ we must have $\gamma - \gamma' \in Z(\widehat{\mu_B})$ for all $\gamma, \gamma' \in \Gamma$ and $Z(\widehat{\mu_B}) = \bigcup_{l\in\mathbb{Z}} R^l \left(\bigcup_{k\in\mathbb{Z}} \mathcal{Z}\left(\sum_{b\in B} e_b\right) + k\right) = \emptyset$ if $\mathcal{Z}\left(\sum_{b\in B} e_b\right) = \emptyset$.

For the second part consider the Hadamard matrix $M_R(B, L)$ for some set $L \subset \mathbb{N}_0$ with $\operatorname{card} L = \operatorname{card} B$ and $0 \in L$. From the unitary condition of the Hadamard matrix it follows that $\sum_{b\in B} e_b\left(\frac{l-\widetilde{l}}{R}\right) = 0$ for $l, \widetilde{l} \in L$, $l \neq \widetilde{l}$, which is not possible if $Z\left(\sum_{b\in B} e_b\right) = \emptyset$. □

Remark 6.13. We have the following correspondence between the zeros of $\sum_{b\in B} e_b$ and the elements of the set $L = \{l_i : i \in \underline{N}\}$ of the Hadamard matrix. For any $l_i, l_j \in L$, $i \neq j$, we have $\frac{l_i - l_j}{R} \in Z\left(\sum_{b\in B} e_b\right)$ and since $0 \in L$ we also have $\frac{l_i}{R} \in Z\left(\sum_{b\in B} e_b\right)$, $i \in \underline{N}\backslash\{0\}$.

Proposition 6.14. *Let $B \subset \underline{R}$, $0 \in B$, $R \in \mathbb{N}$ and $R \geq 2$. There exists a Hadamard matrix $M_R(B, L)$, $L \subset \underline{R}$, $N = \operatorname{card} L = \operatorname{card} B$, if and only if there are at least $N - 1$ elements in $a_j \in \mathcal{Z}\left(\sum_{b\in B} e_b\right)$ with $a_j - a_i \in Z\left(\sum_{b\in B} e_b\right)$, $i, j \in \underline{N}$, $i \neq j$, and $Ra_j \in \mathbb{Z}$, $j \in \underline{N}$.*

PROOF. Assume that there exists a Hadamard matrix $M_R(B, L)$ then we know that for $l_i, l_j \in L$, $i \neq j$, $\frac{l_i - l_j}{R} \in Z\left(\sum_{b\in B} e_b\right)$ and since $0 \in L$ we even have $\frac{l}{R} \in \mathcal{Z}\left(\sum_{b\in B} e_b\right)$, $l \in L$, $l \neq 0$. Consequently, we have at least $N - 1$ elements $a_j = \frac{l_j}{R}$, $j \in \underline{N}\backslash\{0\}$ in $\mathcal{Z}\left(\sum_{b\in B} e_b\right)$ with $a_i - a_j \in Z\left(\sum_{b\in B} e_b\right)$ and $Ra_j \in \mathbb{Z}$.

For the other implication assume there are at least $N - 1$ elements in $a_j \in \mathcal{Z}\left(\sum_{b\in B} e_b\right)$ with $a_j - a_i \in Z\left(\sum_{b\in B} e_b\right)$, $i, j \in \underline{N}$, $i \neq j$, and $Ra_j \in \mathbb{Z}$, $j \in \underline{N}$. Define $l_j = Ra_j$ for $j \in \underline{N}\backslash\{0\}$ and $L = \{0, l_1, \ldots, l_{N-1}\} \subset \underline{R}$. Then $M_R(B, L)$ gives a Hadamard matrix. □

Corollary 6.15. *Let $B \subset \underline{R}$, $R \in \mathbb{N}$, $R \geq 2$, and let μ_B be the invariant measure corresponding to the aIFS $(\tau_b)_{b\in B}$. If there exists $a \in \mathcal{Z}\left(\sum_{b\in B} e_b\right)$ and some $m \in \mathbb{N}$ with $R^m a \in \mathbb{Z}$, then there exists a countable family of orthonormal functions, namely $(e_\gamma)_{\gamma\in\Gamma}$ with $\Gamma = \{R^{mn} a : n \in \mathbb{N}\}$ in $L^2(\mu_B)$.*

PROOF. This result follows analogous to Proposition 6.14. □

Example 6.16. Consider $B = \{0, 1, 2, 5\}$ and $R = 8$. We know $Z\left(\sum_{b\in B} e_b\right) = \left\{\frac{1}{2} + k : k \in \mathbb{Z}\right\}$ thus for $\Gamma = \left\{\frac{8^n}{2} : n \in \mathbb{N}\right\}$ we have that $(e_\gamma)_{\gamma\in\Gamma}$ is orthogonal in $L^2(\mu)$. But we do not have that (μ_B, Γ) is a spectral pair since $\sum_{\gamma\in\Gamma} |\widehat{\mu_B}(t - \gamma)|^2 \neq 1$ for a.e. $t \in \mathbb{R}$ by calculations.

Remark 6.17. Under the conditions of Corollary 6.15, if there exist at least two elements a_1, a_2 in $\mathcal{Z}\left(\sum_{b\in B} e_b\right)$ with $R^m a_j \in \mathbb{Z}$, $j = 1, 2$, $m \in \mathbb{N}$, and $a_1 - a_2 \in Z\left(\sum_{b\in B} e_b\right)$, then the family $(e_\lambda)_{\lambda\in\Gamma}$, $\Gamma = \{R^{mn} a : n \in \mathbb{N}\}$, is not maximal in $L^2(\mu_B)$.

If the positive integer m in $\Gamma = \{R^{mn} a : n \in \mathbb{N}\}$ of Corollary 6.15 is not $\min\{n \in \mathbb{N} : R^n a \in \mathbb{Z}\}$, then the family $(e_\gamma)_{\gamma\in\Gamma}$, $\Gamma = \{R^{mn} a : n \in \mathbb{N}\}$, is not maximal in $L^2(\mu_B)$.

Example 6.18. With the observations above we can see why we cannot find an ONB for the middle-third Cantor set. It is a well known result that for the middle-third Cantor set there do not exist more than two orthonormal functions e_γ, $\gamma \in \mathbb{R}$, see **[JP98a]**.

This Cantor set is given via $\left(\tau_0(x) = \frac{x}{3}, \tau_2(x) = \frac{x+2}{3}\right)$. Consequently, for its invariant measure μ we have

$$\widehat{\mu}(t) = \prod_{n=1}^{\infty} \frac{1}{2}\left(1 + e_2\left(\frac{t}{3^n}\right)\right) = e^{i\pi t} \prod_{n=1}^{\infty} \cos\left(\frac{2\pi t}{3^n}\right).$$

The zeros are

$$Z(\widehat{\mu}) = \bigcup_{n=1}^{\infty} \bigcup_{k\in\mathbb{Z}} \left(3^n\left(\frac{1}{4} + k\right)\right) = \left\{\frac{3^n}{4}(1 + 4\mathbb{Z}) : n \in \mathbb{N}\right\}.$$

Now assume that for some $x, y, z \in \mathbb{R}$ we have $x - y \in Z(\widehat{\mu})$ and $z - x \in Z(\widehat{\mu})$, i.e. $x - y = 3^{n_1}\left(\frac{1}{4} + k_1\right)$ and $z - x = 3^{n_2}\left(\frac{1}{4} + k_2\right)$. Consequently, $z - y = 3^{n_2}\left(\frac{1}{4} + k_2\right) + 3^{n_1}\left(\frac{1}{4} + k_1\right) \notin Z(\widehat{\mu})$. Thus, we cannot have more than two orthogonal elements in $(e_\gamma)_{\gamma\in\Gamma}$ for any set $\Gamma \subset \mathbb{R}$.

Now we turn to a more general case in which we only have finitely many orthonormal functions e_s, $s \in \mathbb{R}$, in $L^2(\mu)$.

Proposition 6.19. *Let $B \subset \underline{R}$, $R \in \mathbb{N}$, $R \geq 2$, and let μ_B be the invariant measure corresponding to the aIFS $(\tau_b)_{b\in B}$. If for all $a \in \mathcal{Z}\left(\sum_{b\in B} e_b\right)$ and for all $n \in \mathbb{N}$ we have $R^n a \notin \mathbb{Z}$, then there are only a finite number of orthonormal functions e_γ, $\gamma \in \mathbb{R}$, in $L^2(\mu_B)$.*

Proof. Assume that $(e_\gamma, e_{\gamma'})$ are orthonormal in $L^2(\mu_B)$. Then $\gamma - \gamma' = R^{n_1}(a_1 + k_1)$ for some $n_1 \in \mathbb{N}$, $a_1 \in \mathcal{Z}\left(\sum_{b\in B} e_b\right)$ and $k_1 \in \mathbb{Z}$. Consequently, any $\beta \in \mathbb{R}$ with e_β orthonormal to e_γ in $L^2(\mu_B)$ satisfies $\gamma' - \beta = R^{n_2}(a_2 + k_2)$, some $n_2 \in \mathbb{N}$, $a_2 \in \mathcal{Z}\left(\sum_{b\in B} e_b\right)$ and $k_2 \in \mathbb{Z}$. Furthermore, it follows that

$$\gamma' - \beta = R^{n_1}(a_1 + k_1) + R^{n_2}(a_2 + k_2) = R^{n_1}\left(a_1 + R^{n_2 - n_1} a_2 + l\right),$$

where we assume that $n_1 \geq n_2$ and $l \in \mathbb{Z}$. For $(e_\beta, e_{\gamma'})$ to be orthonormal we must have that $a_1 + R^{n_2-n_1} a_2 \in Z\left(\sum_{b\in B} e_b\right)$. But since there are only finitely many elements in $\mathcal{Z}\left(\sum_{b\in B} e_b\right)$ and for every $a \in \mathcal{Z}\left(\sum_{b\in B} e_b\right)$ we have $R^n a \notin \mathbb{Z}$ for every $n \in \mathbb{N}$, this can only happen for finitely many $\delta \in \mathbb{R}$.

□

Remark 6.20. One application of Proposition 6.19 is the following: Let $B = \underline{N}$ and $\gcd(R, N) = 1$, then there are only finitely many orthonormal functions e_γ, $\gamma \in \mathbb{R}$, in $L^2(\mu_B)$ since $\mathcal{Z}\left(\sum_{b\in B} e_b\right) = \left\{\frac{j}{N} : j \in \underline{N}\backslash\{0\}\right\}$ and $R^n \cdot \frac{j}{N} \notin \mathbb{Z}$ for any $n \in \mathbb{N}$, $j \in \underline{N}\backslash\{0\}$.

If we only assume $R \neq Nk$ instead of $\gcd(R, N) = 1$, it may be possible to find a countable family of orthogonal functions, consider e.g. $R = 6$, $N = 4$ and $B = \{0, 1, 2, 3\}$, then we have that $\Gamma \subset \mathbb{Z}$ with $\Gamma - \Gamma = \bigcup_{n\geq 1} R^n \frac{1}{2}$ gives a countably orthonormal family $(e_\gamma)_{\gamma\in\Gamma}$ in $L^2(\mu_B)$.

6.3. Classification of Cantor sets via existence of Fourier matrices

In the following we consider some cases in which we can say whether we have a spectral pair or only an orthonormal set. For this let $B \subset \mathbb{N}_0$ and $L \subset \mathbb{N}_0$ be finite sets such that $\operatorname{card} B = \operatorname{card} L = N$. We denote their elements as $B = \{0, b_1, \dots, b_{N-1}\}$ and $L = \{0, l_1, \dots, l_{N-1}\}$.

Remark 6.21. Notice that all the measures μ_B, $B \subset \underline{R}$, $R \geq 2$, obtained from Hutchinson's theorem for an IFS are mutually singular with respect to one another because they all have essentially disjoint support. This can be seen by first realizing that the measures μ_B are the $\frac{\log N}{\log R}$-dimensional Hausdorff measures restricted to the obtained fractal C_B, i.e. $\operatorname{supp}(\mu_B) = \left\{\sum_{i\geq 1} a_i R^{-i} : a_i \in B\right\}$. Furthermore, we have that $\operatorname{card}(B \cap B') \leq N-1$. More precisely it is either $B \cap B' = \{0\}$ or $B \cap B' = \{0, c_1, \dots, c_n\}$ for some $n \in \underline{N-1}\backslash\{0\}$, if B and B' have other elements in common. Consequently, we get that

$$\operatorname{supp}(\mu_B) \cap \operatorname{supp}(\mu_{B'}) = \left\{\sum_{i\geq 1} a_i R^{-i} : a_i \in B \cap B'\right\}$$

$$= \begin{cases} 0, & \operatorname{card}(B \cap B') = 1 \\ \left\{\sum_{i\geq 1} a_i R^{-i} : a_i \in B \cap B'\right\}, & \operatorname{card}(B \cap B') = n+1, \text{ some } n \in \underline{N-1}\backslash\{0\}. \end{cases}$$

For the first case we obviously have the statement. In the second case we can identify the set $\left\{\sum_{i\geq 1} a_i R^{-i} : a_i \in B \cap B'\right\}$ with the limit set of the aIFS

$$\left(\tau_c(x) = \frac{x + c}{R}\right)_{c\in B\cap B'}.$$

This limit set has the Hausdorff dimension $\frac{\log(n+1)}{\log R}$ and hence it has measure zero with respect to the $\frac{\log N}{\log R}$-Hausdorff measure.

Now we turn to our results for the existence of Fourier matrices. We start with the possible scaling for the aIFS.

Lemma 6.22. *Let $M_R(B, L)$, $R \geq 2$, $B, L \subset \mathbb{N}_0$, $\operatorname{card} B = \operatorname{card} L = N < \infty$, give an $(N \times N)$-Fourier matrix. Then the scaling R must be a multiple of N, i.e. $R = Nk$ for some $k \in \mathbb{N}$.*

PROOF. For the matrix $\frac{1}{\sqrt{N}} \left(e^{i2\pi R^{-1} bl} \right)_{b \in B, l \in L}$ to be a Fourier matrix, we must have for $j \in \underline{N} \backslash \{0\}$ that $e^{i2\pi R^{-1} b_1 l_j} = \omega^j$, $\omega = e^{i2\pi/N}$. Thus, this is only possible if we have that $R = Nk$ for some $k \in \mathbb{N}$ and k divides $b_1 l_j$ for all $j \in \underline{N} \backslash \{0\}$. $\square$

Analogously we can state the following result.

Lemma 6.23. *If $R \neq Nk$ for all $k \in \mathbb{N}$, $B \subset \mathbb{N}$, $\operatorname{card} B = N$, then there is no set $L \subset \mathbb{N}$, $\operatorname{card} L = N$, such that $M_R(B, L)$ is a Fourier matrix.*

PROOF. Compare the proof of Lemma 6.22. $\square$

Notice that we cannot obtain for every set $B \subset \mathbb{N}$ and $R \in \mathbb{N}$, $R \geq 2$, a set $L \subset \mathbb{N}$ such that $M_R(B, L)$ is a Hadamard matrix or even a Fourier matrix. In the next proposition we give some cases where we can obtain such a set L.

Proposition 6.24. *Let $R = Nk$, $k \in \mathbb{N}$. Then we have the following results.*

(1) *Let $L = k \cdot \underline{N}$ then there are at least k^{N-1} different sets $B \subset \underline{R}$ with $0 \in B$, $\operatorname{card} B = N$, such that $M_R(B, L)$ is a Hadamard matrix.*
(2) *For $L \neq k \cdot \underline{N}$, k prime, and $L \subset \underline{R}$, $\operatorname{card} L = N$, there is at most one set $B \subset \underline{R}$ such that $M_R(B, L)$ is a Fourier matrix.*

Remark 6.25. For (1) we can say more precisely that there are k^{N-1} sets such that we obtain a Fourier matrix.

PROOF. ad (1): We must have for some $s \in \pm\underline{N}$:

$$\frac{1}{N} \sum_{j \in \underline{N}} e^{i2\pi (N \cdot k)^{-1} b_j sk} = \frac{1}{N} \sum_{j \in \underline{N}} e^{i2\pi b_j s/N} = \delta_{s,0}.$$

Consequently, we can conclude that $b_j = Nn_j + j$ for $n_j \in \mathbb{N}_0$ and $j \in \underline{N}$, $b_0 = 0$. Hence with the condition $b_j \leq R - 1$, we have k^{N-1} options.

ad (2): From the Fourier condition it follows that we must have that

$$\frac{1}{N} \sum_{j \in \underline{N}} e^{i2\pi R^{-1} b_j (l_n - l_m)} = \delta_{n,m}.$$

Consequently, we sum up the Nth roots of unity. Since $R = Nk$ it follows that b_j must be a multiple of k. Thus, we can only have a Fourier matrix if $B = k\underline{N}$. $\square$

Remark 6.26. In the second part of the last proposition we do not always obtain a Fourier matrix. We only get one if the elements in L take the form $\{Nn_j + j : j \in \underline{N} \backslash \{0\}, \text{some } n_j \in \mathbb{N}\} \cup \{0\}$. In this case the sets B and L are interchanged with those in Proposition 6.24 (1).

Furthermore, if we assume $k = k_1 k_2$, $k_1, k_2 \geq 2$, then we can also consider sets of the form $B = k_1 \cdot \underline{N}$ and $L = k_2 \cdot \underline{N}$.

6.4. Classification of orthonormal families coming from Hadamard matrices

Now we turn to cases where we even get a spectral pair. In the following we always assume that we have a Hadamard matrix and we want to know when the corresponding measure μ_B and the set $\Gamma(L)$ actually give a spectral pair.

Proposition 6.27. *Let $B \subset \underline{R}$, $0 \in B$, $R \in \mathbb{N}$, $R \geq 2$, and let $L \subset \underline{R}$, $\operatorname{card} L = \operatorname{card} B = N$, be such that $M_R(B, L)$ gives a Hadamard matrix. Let μ_B be the invariant measure for the aIFS $(\tau_b)_{b \in B}$.*

(1) *If* $\gcd\{b_1, \ldots, b_{N-1}\} = 1$*, then* $(\mu_B, \Gamma(L))$ *is a spectral pair. In particular, if* $1 \in B$*, then* $(\mu_B, \Gamma(L))$ *is always a spectral pair.*
(2) *If* $R \geq 2$ *is even and* $2^n \in B$ *for some* $n \in \mathbb{N}$*, then* $(\mu_B, \Gamma(L))$ *is a spectral pair.*

Proof. ad (1): We assume that we have a non-singleton cycle, i.e. there is $x \in [0,1]$ with $\tau_\omega(x) = x$ for some $\omega = (a_1, \ldots, a_n)$, $a_j \in L$. Consequently, this property for x is equivalent to

$$\sum_{j=1}^{n} a_j R^{j-1} = (R^n - 1)x.$$

The other condition that x satisfies is

$$\frac{1}{N}\left|1 + e_{b_1}(x) + \cdots + e_{b_{N-1}}(x)\right|^2 = N.$$

Thus, we must have that all of $b_1 \cdot \frac{\sum_{j=1}^{n} a_j R^{j-1}}{(R^n-1)}, \ldots, b_{N-1} \cdot \frac{\sum_{j=1}^{n} a_j R^{j-1}}{(R^n-1)}$ are integers, which is not possible since even for the biggest possible $\sum_{j=1}^{n} a_j R^{j-1}$, i.e. $a_j = \max(L) \leq R-2$ for all j, we have that

$$\sum_{j=1}^{n} a_j R^{j-1} \leq (R-2)\frac{R^n - 1}{R - 1} < R^n - 1.$$

So we know that all b_i must have a common divisor with $R^n - 1$, which is not possible since we have $\gcd(b_1, \ldots, b_{N-1}) = 1$.

ad (2): The possible cycle points take the form $x = \frac{\sum_{j=1}^{n} a_j R^{j-1}}{R^n - 1}$ and $x < 1$. We have that $R^n - 1$ is odd. Thus, one divisor of b_j has to divide $R^n - 1$ for all $b_j \in B$. If $b_j = 2^n$ for some $n \in \mathbb{N}$, this is not possible. Hence we can only have the singleton cycle $\{0\}$. □

6.4.1. Connections between measures from Fourier matrices. In the following we only consider the case where we have a Fourier matrix with the set $L = k \cdot \underline{N}$. First we recall that for $M_R(B, L)$ to be a Fourier matrix we have that $B = \{Nn_j + j : \ j \in \underline{N}, n_j \in \mathbb{N}_0\} \subset \underline{R}$ and $R = kN$.

Lemma 6.28. *Let* $L = k \cdot \underline{N}$*. For all sets* $B \subset \underline{R}$*,* $0 \in B$*, such that* $M_R(B, L)$*,* $R = Nk$*, is a Fourier matrix, then* $\widehat{\mu_B}$ *is divisible by* $\widehat{\mu_{B_1}}$*, where* μ_{B_1} *is the invariant measure for the aIFS* $(\tau_b)_{b \in B_1}$ *with* $B_1 = \underline{N}$ *and* μ_B *is the invariant measure for the aIFS* $(\tau_b)_{b \in B}$*.*

Proof. First we have that

$$\widehat{\mu_B}(t) = \prod_{n=1}^{\infty} \frac{1}{N}\left(\sum_{b \in B} e_b\left(\frac{t}{R^n}\right)\right),$$

where $e_b(t) = e^{i2\pi bt}$. Thus, if we show that $1 + x^{b_1} + \cdots + x^{b_{N-1}}$ is divisible by $1 + x + \cdots + x^{N-1}$, we have the result. Now notice that we have $b_j = Nn_j + j$ for $j \in \underline{N}\backslash\{0\}$ and $n_j \in \mathbb{N}_0$ by the proof of Proposition 6.24 (1). The result will be shown by induction. The statement is obviously true when $n_1 = \cdots = n_{N-1} = 0$.

Assume it is already shown for $n_1, \ldots, n_{N-1}$. Now we consider the induction step $n_j \mapsto n_j + 1$ for some $j \in \underline{N-1}\backslash\{0\}$:

$$\begin{aligned} &1 + x^{Nn_1+1} + \cdots + x^{Nn_j+j+N} + \cdots + x^{Nn_{N-1}+N-1} \\ &= (1 + x + x^2 + \cdots + x^{N-1}) \cdot (x^{Nn_j+j+1} - x^{Nn_j+j}) \\ &\quad + 1 + x^{Nn_1+1} + \cdots + x^{Nn_{N-2}+N-2} + x^{Nn_{N-1}+N-1} \end{aligned}$$

and by induction hypothesis we have that $1 + x^{Nn_1+1} + \cdots + x^{Nn_{N-1}+N-1}$ is divisible by $1 + x + x^2 + \cdots + x^{N-1}$. □

Proposition 6.29. *Let* $L = k \cdot \underline{N}$ *and* $R = Nk$*. For all sets* $B \subset \underline{R}$ *satisfying* $0 \in B$ *such that* $M_R(B, L)$ *is a Fourier matrix, it follows that* $\mu_B = \nu \star \mu_{B_1}$*, where* ν *is a signed measure,* μ_B *is the invariant measure for the aIFS* $(\tau_b)_{b \in B}$ *and* μ_{B_1} *the one for* $(\tau_b)_{b \in B_1}$ *with* $B_1 = \underline{N}$*.*

PROOF. From Lemma 6.28 we already have that $\widehat{\mu_B} = g \cdot \widehat{\mu_{B_1}}$, where g is a specific function. Thus we have to show that g is the Fourier transform of a signed measure. From the proof of Lemma 6.28 we have that

$$g(t) = \prod_{l=1}^{\infty} \sum_{i=0}^{m} (-1)^i e_{j_i}\left(\frac{t}{R^l}\right),$$

where $j_0 = 0$ and $j_i \in \{1, \dots, \max(b_1, \dots, b_{N-1}) - N + 1\}$, $B = \{0, b_1, \dots, b_{N-1}\}$, $b_j = Nn_j + j$, $j \in \underline{N}\backslash\{0\}$, and $m \in \mathbb{N}$ depends on $n_1, \dots, n_{N-1}$.

Now let us consider $\nu(\cdot) = \star_{l=1}^{\infty} \sum_{i=0}^{m} (-1)^i \delta_{j_i}\left(\cdot R^l\right)$. Then we have that

$$\widehat{\nu}(\cdot) = \prod_{l=1}^{\infty} \sum_{i=0}^{m} (-1)^i e_{j_i}\left(\frac{\cdot}{R^l}\right).$$

Hence it remains to show that ν is indeed a signed measure with $\nu(\mathbb{R}) = 1$. It is obvious that $\nu(\emptyset) = 0$ and $\nu(\mathbb{R}) = 1$ since in every finite number s of steps we have that $\prod_{l=1}^{s} \sum_{i=0}^{m} (-1)^i \delta_{j_i}\left(R^l \cdot \mathbb{R}\right) = 1$. Furthermore, we have that $\nu\left(\biguplus_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \nu\left(A_n\right)$ for disjoint sets $\left(A_n\right)_{n\in\mathbb{N}}$ because it holds for the Dirac measure $\delta_j\left(\cdot R^l\right)$ and the sum of the Dirac measures. Consequently, it also holds for the convolution. $\square$

In the remaining part of this section we further consider properties of the ideal generated by μ_{B_1} with $B_1 = \underline{N}$. We use ideal as explained in Remark 3.14.

Proposition 6.30. *Let $L = k \cdot \underline{N}$, $R \in \mathbb{N}$, $R = Nk$, and let $H(L)$ denote the measures obtained from the Fourier matrices $M_R(B, L)$ for different sets $B \subset \mathbb{N}$. Then $H(L)$ is a subset of the* ideal (μ_{B_1}) *in $\mathbb{B}$, where μ_{B_1} is the invariant measure for the aIFS $(\tau_b)_{b\in B_1}$ with $B_1 = \underline{N}$.*

PROOF. Let $M_R(B, L)$ give a Fourier matrix, then $\widehat{\mu_B}$ is divisible by $\widehat{\mu_{B_1}}$ by Lemma 6.28. Thus we have that $\mu_B \in \mathrm{ideal}(\mu_{B_1})$. $\square$

Remark 6.31. Notice that $H(L) \subset M^{\perp}(\Gamma)$.

Proposition 6.32. *Let $B \subset \mathbb{N}$,* card $B = N < \infty$, *$R \in \mathbb{N}$, $R \geq 2$, and let μ_B be the invariant measure for the aIFS $(\tau_b)_{b\in B}$. Then $\delta_x \star \mu_B$ for $x \in \mathbb{R}$ is an extreme point in* $\mathrm{ideal}(\mu_B) \cap \mathcal{M}$.

PROOF. We assume that $\delta_x \star \mu_B = \alpha \cdot \nu_1 \star \mu_B + (1-\alpha) \cdot \nu_2 \star \mu_B$ for some $0 < \alpha < 1$, $\nu_1 \star \mu_B$, $\nu_2 \star \mu_B \in \mathcal{M}$ and ν_1, ν_2 are signed measures. First notice that we then have $\nu_1 = \nu_1^+ - \nu_1^-$, $\nu_2 = \nu_2^+ - \nu_2^-$ and thus

$$\delta_x \star \mu_B = \alpha \cdot \nu_1^+ \star \mu_B + (1-\alpha) \cdot \nu_2^+ \star \mu_B - \left(\alpha \cdot \nu_1^- \star \mu_B + (1-\alpha) \cdot \nu_2^- \star \mu_B\right).$$

Furthermore, we have that $\mathrm{supp}\left(\nu_j^- \star \mu_B\right) \subset \mathrm{supp}\left(\nu_j^+ \star \mu_B\right)$ and $\left(\mathrm{supp}\left(\nu_j^+ \star \mu_B\right) \setminus \mathrm{supp}\left(\nu_j^- \star \mu_B\right)\right) \subset \mathrm{supp}\left(\nu_j \star \mu_B\right) \subset \mathrm{supp}\left(\delta_x \star \mu_B\right)$ for $j = 1, 2$. We also have that $0 \in \mathrm{supp}\left(\mu_B\right)$ and $\mathrm{supp}\left(\delta_x \star \mu_B\right) \subset [x, b+x]$, where $b = \max\left(\mathrm{supp}(\mu_B)\right)$, and $\mathrm{supp}\left(\nu_j^+ \star \mu_B\right) = \mathrm{supp}\left(\nu_j^+\right) + \mathrm{supp}\left(\mu_B\right)$, $j = 1, 2$. Consequently, we have $\min\left(\mathrm{supp}\left(\nu_j^+\right)\right) \geq x$ and $\max\left(\mathrm{supp}\left(\nu_j^+\right)\right) \leq x$ for $j = 1, 2$. Thus, it follows that $\nu_j = \delta_x$ for $j = 1, 2$, and so $\delta_x \star \mu_B$ is an extreme point in $\mathrm{ideal}(\mu_B) \cap \mathcal{M}$. $\square$

Proposition 6.33. *For $B_1 = \underline{N}$, $R = Nk$, $k \in \mathbb{N}$, and μ_{B_1} being the invariant measure for the aIFS $(\tau_b)_{b\in B_1}$, we can find a measure ν such that $\widehat{\lambda|_{[0,1]}} = \widehat{\nu} \cdot \widehat{\mu_{B_1}}$.*

PROOF. First recall that $\lambda|_{[0,1]}$ is the invariant measure for the full aIFS, i.e. for $\left(\tau_i(x) = \frac{x+i}{R}\right)_{i\in\underline{R}}$. Consequently, we have that for $t \in \mathbb{R}$

$$\widehat{\lambda|_{[0,1]}}(t) = \prod_{n=1}^{\infty} \frac{1}{R}\left(\sum_{i\in\underline{R}} e_i\left(\frac{t}{R^n}\right)\right).$$

Consider for $t \in \mathbb{R}$

$$\widehat{\nu}(t) = \prod_{n=1}^{\infty} \frac{1}{k}\left(\sum_{a\in N\cdot\underline{k}} e_a\left(\frac{t}{R^n}\right)\right).$$

Then we have that $\widehat{\lambda|_{[0,1]}} = \widehat{\nu} \cdot \widehat{\mu_{B_1}}$ since

$$\widehat{\mu_{B_1}}(t) = \prod_{n=1}^{\infty} \frac{1}{N}\left(1 + e_1\left(\frac{t}{R^n}\right) + \cdots + e_{N-1}\left(\frac{t}{R^n}\right)\right).$$

Consequently, ν is the invariant measure for the aIFS $\left(\tau_a(x) = \frac{x+a}{R}\right)_{a \in N \cdot \underline{k}}$. $\square$

Proposition 6.34. *Let $B_1 = \underline{N}$, $R = Nk$, $k, N \in \mathbb{N}$, and μ_{B_1} being the invariant measure for the aIFS $(\tau_b)_{b \in B_1}$. Then* $\operatorname{ideal}\left(\lambda|_{[0,1]}\right) \subset \operatorname{ideal}\left(\mu_{B_1}\right)$ *in $\mathbb{B}$.*

Let $L = k \cdot \underline{N}$, $R = Nk$, $k, N \in \mathbb{N}$. Then for all sets $B \subset \underline{R}$ such that $0 \in B$ and $M_R(B, L)$ is a Fourier matrix, it follows that $\operatorname{ideal}(\mu_B) \subset \operatorname{ideal}(\mu_{B_1})$ *in $\mathbb{B}$, where μ_B is the corresponding invariant measure for the aIFS $(\tau_b)_{b \in B}$.*

Proof. For the first part we only need that $\lambda|_{[0,1]} \in \operatorname{ideal}(\mu_{B_1})$ and this follows from Proposition 6.33. The second part follows from Proposition 6.29. $\square$

Remark 6.35. For $N = 2, 3, 4, 5$ we can even say in general that for $R \neq Nk$ there does not exist any Hadamard matrix, not only Fourier matrix. Furthermore for $N = 4$ and $L = \{0, k, 2k, 3k\}$ we can only obtain a set B such that we get the Fourier matrix, not a different form of a Hadamard matrix.

6.5. Construction of (3×3)-Hadamard matrices

In this section we want to further explain the results of the previous sections in the case of (3×3)-Hadamard matrices. We only include the results that we find helpful or where the proofs are easier to understand. We also include some further results. There is only one dephased form of a (3×3)-Hadamard matrix, namely the form of the Fourier matrix. Consequently, we can give a complete characterization of the possible orthonormal families.

The unitary matrix of Definition 2.12 now takes the form for $B = \{0, b_1, b_2\}$ and $L = \{0, l_1, l_2\}$

$$\frac{1}{\sqrt{3}}\begin{pmatrix} 1 & 1 & 1 \\ 1 & e^{2\pi i R^{-1} b_1 l_1} & e^{2\pi i R^{-1} b_1 l_2} \\ 1 & e^{2\pi i R^{-1} b_2 l_1} & e^{2\pi i R^{-1} b_2 l_2} \end{pmatrix}.$$

For this matrix to be unitary it is enough to examine the exponents of the exponential functions because we have to consider the adding up of roots of unity and thus we get the following condition:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & R^{-1} b_1 l_1 & R^{-1} b_1 l_2 \\ 0 & R^{-1} b_2 l_1 & R^{-1} b_2 l_2 \end{pmatrix} \equiv \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1/3 & 2/3 \\ 0 & 2/3 & 1/3 \end{pmatrix} (\operatorname{mod} R).$$

In this case we get that $R = 3k$ for some $k \in \mathbb{N}$, see Lemma 6.22.

Proposition 6.36. *For $B = \{0, 1, 2\}$, $k \in \mathbb{N}$, $(\mu_B, \Gamma(L))$ is a spectral pair, where $\Gamma(L)$ is defined in terms of $L = \{0, k, 2k\}$ and μ_B is the invariant measure for the aIFS $\left(\tau_b(x) = \frac{x+b}{3k}\right)_{b \in \{0,1,2\}}$, i.e. $M_{3k}(B, L)$ is a Hadamard matrix.*

Remark 6.37. In general we cannot interchange B and L so that $(\mu_L, \Gamma(B))$ is a spectral pair whenever $(\mu_B, \Gamma(L))$ is. But if we consider $R = 6$, $L = \{0, 1, 5\}$ and $B = \{0, 2, 4\}$. Then we have a $M_R(B, L)$ Hadamard matrix and $(\mu_L, \Gamma(B))$ gives a spectral pair. We also have that $(\mu_B, \Gamma(L))$ is a spectral pair.

Example 6.38. In this example we want to show that for k odd we do not always have a spectral pair. Let us consider $k = 3$, i.e. $R = 9$, $L = \{0, 3, 6\}$ and $B = \{0, 4, 8\}$. We can easily check that L and B give a Hadamard matrix. Now we can consider $x = \frac{3}{4}$ and we have that $\tau_6\left(\frac{3}{4}\right) = \frac{3}{4}$. Furthermore, we have that

$$\frac{1}{3}\left|1 + e_4\left(\frac{3}{4}\right) + e_8\left(\frac{3}{4}\right)\right|^2 = \frac{1}{3}\left|1 + e_1(3) + e_1(6)\right|^2 = 3.$$

Consequently, in this example we can find a non-singleton cycle.

In analogy to Lemma 6.28 we can give an algorithm to calculate the signed measure ν that gives $\mu_B = \nu \star \mu_{B_1}$.

Remark 6.39. To find the signed measure ν that satisfies $\mu_B = \nu \star \mu_{B_1}$ for some set B we proceed inductively as follows.

Write $b_1 = 3n_1 + 1$ and $b_2 = 3n_2 + 2$ with $n_1, n_2 \in \mathbb{N}_0$. Then $\widehat{\mu_B}$ is the product of sums with $2 \cdot \min(n_1, n_2) + 1$ terms. We have to consider different cases.

Case 1: Let $n = 0$, $n_2 \geq 1$, then we get

$$1 + z + z^{3n_2+2} = \left(1 + z + z^2\right) \cdot (1 - z) + z\left(1 + z^{3n_2+1} + z^2\right).$$

Case 2: Let $n_2 = 0$, $n_1 \geq 1$, then we get

$$1 + z^{3n_1+1} + z^2 = \left(1 + z + z^2\right) \cdot (1 - z) + z^2\left(1 + z + z^{3(n_1-1)+2}\right).$$

Case 3: Let $n_2 \geq 1$, $n_1 \geq 1$, then we get

$$1 + z^{3n_1+1} + z^{3n_2+2} = \left(1 + z + z^2\right) \cdot (1 - z) + z^3\left(1 + z^{3(n_1-1)+1} + z^{3(n_2-1)+2}\right).$$

In the next step the same procedure is applied to the terms $1 + z^{3n_2+1} + z^2$, $1 + z + z^{3(n_1-1)+2}$ or $1 + z^{3(n_1-1)+1} + z^{3(n_2-1)+2}$.

In this case we give a precise formula for the m in the proof of Proposition 6.29.

Proposition 6.40. *Let $R = 3k$, $L = k \cdot \underline{N}$, $k \in \mathbb{N}$. For all sets B such that $M_R(B, L)$ is a Hadamard matrix, it follows that $\mu_B = \nu \star \mu_{B_1}$, where ν is a signed measure, $B_1 = \{0, 1, 2\}$ and μ_B is the invariant measure for the aIFS $(\tau_b)_{b \in B}$ and μ_{B_1} the one for $(\tau_b)_{b \in B_1}$.*

Proof. This proof is analogous to the one of Proposition 6.29 with $m = 2 \cdot \max(n_1, n_2)$, where $b_1 = 3n_1 + 1$ and $b_2 = 3n_2 + 2$. □

Now we state a result that we do not have for general Hadamard matrices. We first consider of the zeros of $\widehat{\mu_B}$. We have that $\widehat{\mu_B}$ has the form $\widehat{\mu_B}(t) = \prod_{n=1}^{\infty} f\left(\frac{t}{R^n}\right)$, $t \in \mathbb{R}$.

Lemma 6.41. *All the zeros of f are simple $(\operatorname{mod} R)$ if and only if all of $\widehat{\mu}$ are simple.*

Proof. "$\Rightarrow$": Assume that f has only simple zeros $(\operatorname{mod} R)$, i.e. let $a_1, \ldots, a_m \in \mathbb{R}$ be the zeros of f such that there does not exist $n \in \mathbb{N}$ with $R^n a_j = a_i$ for any $i, j \in \{1, \ldots, n\}$. It follows that

$$\{R^n a_j : n \in \mathbb{N},\, j \in \{1, \ldots, m\}\}$$

are the zeros of $\widehat{\mu}$ and these are simple.

"$\Leftarrow$": In this case we have that $\widehat{\mu}$ has simple zeros and we will prove by contradiction that f has only simple zeros. So assume that f has at least one zero that is not simple. Then we have that $R^n a_j$ is not a simple zero of $\widehat{\mu}$. This contradicts that $\widehat{\mu}$ has simple zeros. □

Proposition 6.42. *Let $B = \{0, b_1, b_2\}$ and $p_B(z) = 1 + z^{b_1} + z^{b_2}$, then $Z(p_B) \cap \mathbb{T}$, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, consists of simple zeros.*

Proof. If there is some z in $Z(p_B) \cap \mathbb{T}$ which is not simple, then

$$1 + z^{b_1} + z^{b_2} = 0,$$
$$b_1 z^{b_1} + b_2 z^{b_2} = 0.$$

Consequently, $z^{b_2} = \frac{b_1}{b_2 - b_1} \in \mathbb{T}$. This is just possible if $b_2 = 2b_1$ and this is in contradiction to $b_1 z^{b_1} + b_2 z^{b_2} = 0$. Thus, all the zeros of p_B in $\mathbb{T}$ are simple. □

From the Proposition 6.42 and the Lemma 6.41 it follows that $\widehat{\mu_B}$ for any B has only simple zeros, where μ_B is the invariant measure to the aIFS $(\tau_b)_{b \in B}$.

Example 6.43. Finally we give an example for the scaling $R = 6$, i.e. $k = 2$. We consider the set $\Gamma = \left\{\sum_{i=0}^{n} l_i 6^i : l_i \in \{0, 2, 4\}, n \in \mathbb{N}_0\right\}$, i.e. $L = \{0, 2, 4\}$. We have seen that for the possible choices of $B = \{0, b_1, b_2\} \subset \{0, \dots, 5\}$ we must have $b_1 = 3n_1 + 1$ and $b_2 = 3n_2 + 2$ for some $n_1, n_2 \in \mathbb{N}_0$. Thus, we obtain the possible sets $B_1 = \{0, 1, 2\}$, $B_2 = \{0, 2, 4\}$, $B_3 = \{0, 1, 5\}$ and $B_4 = \{0, 4, 5\}$. Furthermore, we know from Proposition 6.36 that (μ_{B_1}, Γ) is a spectral pair. We can also show that $\left(\mu_{B_j}, \Gamma\right)$ is a spectral pair for $j = 2, 3, 4$. To see this first notice that we must have for some $a_j \in L$ and $n \in \mathbb{N}$

$$b_1 \frac{\sum_{j=1}^{n} a_j R^{j-1}}{R^n - 1} \in \mathbb{Z} \quad \text{and} \quad b_2 \frac{\sum_{j=1}^{n} a_j R^{j-1}}{R^n - 1} \in \mathbb{Z},$$

where $b_1, b_2 \in B_i \backslash \{0\}$, $i = 2, 3, 4$. Notice that $R^n - 1$ is odd and $\sum_{j=1}^{n} a_j R^{j-1} < R^n - 1$. Thus, each b_i must be a divisor for $R^n - 1$. This is not possible if $b_i \in \{2, 4\}$. Hence we cannot have a non-trivial cycle for B_2 and B_4. For B_3 we obtain this conclusion since $1 \in B_3$.

Now we obtain the Fourier transform of the measures:

$$\widehat{\mu_{B_1}}(t) = \prod_{n=1}^{\infty} \frac{1}{3}\left(1 + e_1\left(\frac{t}{6^n}\right) + e_2\left(\frac{t}{6^n}\right)\right),$$

$$\widehat{\mu_{B_2}}(t) = \prod_{n=1}^{\infty} \frac{1}{3}\left(1 + e_2\left(\frac{t}{6^n}\right) + e_4\left(\frac{t}{6^n}\right)\right),$$

$$\widehat{\mu_{B_3}}(t) = \prod_{n=1}^{\infty} \frac{1}{3}\left(1 + e_1\left(\frac{t}{6^n}\right) + e_5\left(\frac{t}{6^n}\right)\right),$$

$$\widehat{\mu_{B_4}}(t) = \prod_{n=1}^{\infty} \frac{1}{3}\left(1 + e_4\left(\frac{t}{6^n}\right) + e_5\left(\frac{t}{6^n}\right)\right).$$

$\widehat{\mu_{B_1}}$ divides the other $\widehat{\mu_{B_j}}$, $j = 2, 3, 4$. So we can write $\widehat{\mu_{B_j}} = \widehat{\xi_j} \cdot \widehat{\mu_{B_1}}$ for $j = 2, 3, 4$ since

$$\begin{aligned}
1 + z^2 + z^4 &= (1 + z + z^2) \cdot (1 - z + z^2),\\
1 + z^1 + z^5 &= (1 + z + z^2) \cdot (1 - z^2 + z^3),\\
1 + z^4 + z^5 &= (1 + z + z^2) \cdot (1 - z + z^3).
\end{aligned}$$

For $\widehat{\mu_{B_2}} = \widehat{\xi_2} \cdot \widehat{\mu_{B_1}}$ we have for $t \in \mathbb{R}$

$$\widehat{\xi_2}(t) = \prod_{n=1}^{\infty}\left(1 - e_1\left(\frac{t}{6^n}\right) + e_2\left(\frac{t}{6^n}\right)\right).$$

Thus, we have that $\mu_{B_j} = \xi_j \star \mu_{B_1}$ for $j = 2, 3, 4$, where ξ_j is a signed measure.

Now we further consider the connection between μ_{B_1} and μ_{B_2}. For the support of these measures we have that

$$\operatorname{supp}(\mu_{B_1}) = \left\{\sum_{i=1}^{\infty} \alpha_i 6^{-i} : \alpha_i \in \{0, 1, 2\}\right\} \quad \text{and} \quad \operatorname{supp}(\mu_{B_2}) = \left\{\sum_{i=1}^{\infty} \alpha_i 6^{-i} : \alpha_i \in \{0, 2, 4\}\right\}.$$

So $\operatorname{supp}(\mu_{B_2}) = 2\operatorname{supp}(\mu_{B_1})$ and $\widehat{\mu_{B_2}}(t) = \widehat{\mu_{B_1}}(2t)$, $t \in \mathbb{R}$. Furthermore, we have that (μ_{B_1}, Γ) and $(\mu_{B_1}, 2\Gamma)$ are both spectral pairs since for $t \in \mathbb{R}$

$$1 = \sum_{\gamma \in \Gamma} |\widehat{\mu_{B_2}}(t + \gamma)|^2 = \sum_{\gamma \in \Gamma} |\widehat{\mu_{B_1}}(2(t + \gamma))|^2 = \sum_{\gamma \in \Gamma} |\widehat{\mu_{B_1}}(2t + 2\gamma)|^2.$$

In the next step we consider the zeros for $\widehat{\mu_{B_j}}$ for $j = 1, 2, 3, 4$: We get for $j = 1, 3, 4$

$$Z(\widehat{\mu_{B_j}}) = \bigcup_{k=1}^{\infty} \bigcup_{n \in \mathbb{Z}} 6^k \left(\left\{\frac{1}{3}, \frac{2}{3}\right\} + n\right) \quad \text{and} \quad Z(\widehat{\mu_{B_2}}) = \bigcup_{k=1}^{\infty} \bigcup_{n \in \mathbb{Z}} 6^k \left(\left\{\frac{1}{6}, \frac{2}{6}, \frac{4}{6}, \frac{5}{6}\right\} + n\right).$$

Consequently, we have that $Z(\widehat{\mu_{B_j}}) \not\supseteq \Gamma - \Gamma \backslash \{0\}$.

Furthermore, we have for $j = 1, 2$, that $\widehat{\lambda|_{[0,1]}}(t) = \widehat{\nu_j}(t) \cdot \widehat{\mu_{B_j}}(t)$, $t \in \mathbb{R}$, where

$$\widehat{\nu_1}(t) = \prod_{n=1}^{\infty} \frac{1}{2}\left(1 + e_3\left(\frac{t}{6^n}\right)\right) \quad \text{and} \quad \widehat{\nu_2}(t) = \prod_{n=1}^{\infty} \frac{1}{2}\left(1 + e_1\left(\frac{t}{6^n}\right)\right).$$

For $j = 3, 4$, there does not exist ν_j such that $\widehat{\lambda|_{[0,1]}}(t) = \widehat{\nu_j}(t) \cdot \widehat{\mu_{B_j}}(t)$, $t \in \mathbb{R}$. We have that

$$Z(\widehat{\nu_1}) = \bigcup_{k=1}^{\infty} \bigcup_{n \in \mathbb{Z}} 6^k \left(\left\{\frac{1}{6}, \frac{3}{6}, \frac{5}{6}\right\} + n\right) \quad \text{and} \quad Z(\widehat{\nu_2}) = \bigcup_{k=1}^{\infty} \bigcup_{n \in \mathbb{Z}} 6^k \left(\left\{\frac{3}{6}\right\} + n\right).$$

CHAPTER 7

Conclusion and outlook

There are still many open questions in connection with this work. The main restriction in the study of the geometry of spectral pairs is that we mainly consider fractals that are Cantor sets and not even fractals where the branches have different similarity coefficients. Even this slight generalization makes the whole study much more complicated. In [**BK10, Boh09**] we gave a way to construct a "general" Fourier basis on any kind of one dimensional fractal given via an IFS with increasing branches. This general Fourier basis was carried over from a Cantor set via a conjugating homeomorphism. In this way it is even possible to obtain a Fourier basis for the middle-third Cantor set.

Another open question is for which countable sets $\Gamma \subset \mathbb{Z}$ there exists a measure μ such that (μ, Γ) is a spectral pair. Here we gave a partial answer for the case $\Gamma - \Gamma = \mathbb{Z}$, but it is still open for sets Γ without this property. Another simpler open question is for which sets Γ has $M^{\perp}(\Gamma)$ at least two distinct elements, that are not obtained from the other by convolution with a Dirac measure δ_x, $x \in \mathbb{R}$.

For the single subsections there are still smaller open questions like for Chapter 3 whether there is $\nu \in \mathbb{B} \backslash B_{abs}$ such that $x \mapsto \mathbf{T}_x \nu$ is continuous and whether it follows from $\lim_{x \to 0} ||\nu - \mathbf{T}_x \nu||_{tot} = 0$ that $\nu \ll \lambda$. Furthermore, what is the subset $\left\{\nu \in \mathbb{B} : \lim_{x\to 0} ||\frac{1}{x}(\mathbf{T}_x \nu - \nu)||_{tot} = 0\right\}$?

Open questions regarding Chapter 4 are, for example, is the property (μ, Γ) and (ν, Δ) being spectral pairs and that $\Gamma - \Gamma = \Delta - \Delta$ enough to ensure (ν, Γ) and (μ, Δ) being spectral pairs? Or on the other hand is there a counter example?

Consequently, there is still a lot of work to study every aspect of spectral pairs even if we restrict ourselves to affine iterated function systems in one dimension.

We could although study higher dimensional fractals, which are given via aIFSs:

$$\left(\tau_b(x) = R^{-1}(x + b)\right)_{b \in B}$$

with R being a $(n \times n)$-matrix with eigenvalues λ, $|\lambda| > 1$, $B \subset \mathbb{N}^n$, $x \in \mathbb{R}^n$, such that it satisfies the OSC. For dimension 2 we can easily obtain a connection to one dimensional fractals in the following way.

Let C_1, C_2 be two Cantor sets such that $(e_\gamma)_{\gamma \in \Delta_1}$ and $(e_\gamma)_{\gamma \in \Delta_2}$ are orthonormal basis for $L^2(\mu_1)$, $L^2(\mu_2)$, respectively, where μ_1 and μ_2 are the invariant measures for the Cantor sets C_1, C_2. Then $(e_{\gamma'} \otimes e_\gamma)_{\gamma' \in \Delta_1, \gamma \in \Delta_2}$ is an orthonormal basis for $L^2(\mu_1 \otimes \mu_2)$.

We could although consider fractals that are not the tensor product of two one-dimensional ones. Then the study is more complicated. But it should still be possible to obtain analogous results to the one-dimensional ones.

Part 2

Wavelet bases on fractals in the line

CHAPTER 8

Abstract multiresolution analysis

In this chapter we prove the existence of a wavelet basis for an abstract MRA. Throughout this section we fix $\left(\mu, \left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{Z}}, \mathcal{T}\right)$ which allows a two-sided MRA as stated in Definition 1.2 and then we obtain the following result.

Theorem 8.1. *Let μ be a non-atomic measure on $\mathbb{R}$, $\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{Z}}$ be a family of bounded linear operators on $L^2(\mu)$ and $\mathcal{T}$ be a unitary operator on $L^2(\mu)$. If $\left(\mu, \left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{Z}}, \mathcal{T}\right)$ allows a two-sided MRA with father wavelets φ_j, $j \in \underline{N}$, then there exist for every $n \in \mathbb{N}_0$ numbers $d_n \in \underline{N^{n+2}}$, $d_{-n} \in \underline{N^2}$, $q_n \in \underline{N^{n+1}}$, $q_{-n} \in \underline{N}$, with $d_n \geq q_n$, $d_{-n} \geq q_{-n}$, and two families of mother wavelets $\left(\psi_{n,l} : l \in \underline{d_n - q_n}\right)$, $\left(\psi_{-n,l} : l \in \underline{d_{-n} - q_{-n}}\right)$, $n \in \mathbb{N}_0$, such that the following set of functions defines an orthonormal basis for $L^2(\mu)$*

$$\left\{\mathcal{T}^k \psi_{n,l} : n \in \mathbb{N}_0, l \in \underline{d_n - q_n}, k \in \mathbb{Z}\right\} \cup \left\{\mathcal{T}^{N^n k} \psi_{-n,l} : n \in \mathbb{N}, l \in \underline{d_{-n} - q_{-n}}, k \in \mathbb{Z}\right\}.$$

We turn to the proof of the theorem in Section 8.1, where we also give a explicit formula for the functions $\psi_{n,l}$, $\psi_{-n,l}$ compare (8.1.1) and (8.1.2).

For the construction of an ONB we cannot define the mother wavelets in terms of filter functions due to the fact that we have more than one father wavelet. Before we turn to the proof of Theorem 8.1 we notice that for $n \in \mathbb{N}$

$$\{(l,j) \in \underline{N^n} \times \underline{N}\} = \left\{\left(\left\lfloor \frac{k}{N} \right\rfloor, (k)_N\right) : k \in \underline{N^{n+1}}\right\}, \tag{8.0.1}$$

where $(m)_N := m \bmod N$ and $\lfloor x \rfloor = \max_{k\in\mathbb{Z}, k\leq x}(k)$ is the largest integer not exceeding $x \in \mathbb{R}$.

Clearly from the definition of the MRA, Definition 1.2 (1e), we also have the following:

(1) If for $n \in \mathbb{N}_0$, $k \in \underline{N^{n+1}}$, $\mathcal{U}^{(n)} \mathcal{T}^{\lfloor \frac{k}{N} \rfloor} \varphi_{(k)_N} \neq 0$, then there exist uniquely determined coefficients $\left(a_m^{n,k}\right)_{m\in\underline{N^{n+2}}} \in \mathbb{C}^{N^{n+2}}$ such that

$$\begin{gathered}\mathcal{U}^{(n)} \mathcal{T}^{\lfloor \frac{k}{N} \rfloor} \varphi_{(k)_N} = \mathcal{U}^{(n+1)} \textstyle\sum_{m\in\underline{N^{n+2}}} a_m^{n,k} \mathcal{T}^{\lfloor \frac{m}{N} \rfloor} \varphi_{(m)_N} \\ \text{and} \\ \left(a_m^{n,k} = 0,\ m \in \underline{N^{n+2}},\ \text{ if } \mathcal{U}^{(n+1)} \mathcal{T}^{\lfloor \frac{m}{N} \rfloor} \varphi_{(m)_N} = 0\right).\end{gathered} \tag{8.0.2}$$

(2) If $\mathcal{U}^{(-n)} \varphi_i \neq 0$, $n \in \mathbb{N}$, $i \in \underline{N}$, then there exist uniquely determined coefficients $\left(b_m^{n,i}\right)_{m\in\underline{N^2}} \in \mathbb{C}^{N^2}$ such that

$$\begin{gathered}\mathcal{U}^{(-n)} \varphi_i = \mathcal{U}^{(-n+1)} \textstyle\sum_{m\in\underline{N^2}} b_m^{n,i} \mathcal{T}^{\lfloor \frac{m}{N} \rfloor} \varphi_{(m)_N} \\ \text{and} \\ \left(b_m^{n,i} = 0,\ m \in \underline{N^2},\ \text{ if } \mathcal{U}^{(-n+1)} \mathcal{T}^{\lfloor \frac{m}{N} \rfloor} \varphi_{(m)_N} = 0\right).\end{gathered} \tag{8.0.3}$$

Remark 8.2. We only consider $\mathcal{U}^{(-n)} \varphi_i$ above, since $\mathcal{U}^{(-n)} \mathcal{T}^k \varphi_i = \mathcal{T}^{N^n k} \mathcal{U}^{(-n)} \varphi_i$ by (1f) of Definition 1.2.

Lemma 8.3. *The following statements hold for the coefficients* $\left(a_m^{n,k}\right)_{m\in\underline{N^{n+2}}}$, $k \in \underline{N^{n+1}}$, $n \in \mathbb{N}_0$, *and* $\left(b_m^{n,i}\right)_{m\in\underline{N^2}}$, $i \in \underline{N}$, $n \in \mathbb{N}$.

(1) *For fixed* $n \in \mathbb{N}_0$, *define*

$$Q_n := \left\{m \in \underline{N^{n+1}} : \mathcal{U}^{(n)}\,\mathcal{T}^{\lfloor \frac{m}{N} \rfloor}\,\varphi_{(m)_N} \neq 0\right\}. \tag{8.0.4}$$

Then the vectors $\left(a_m^{n,k}\right)_{m\in\underline{N^{n+2}}}$, $k \in Q_n$, *are orthonormal.*

(2) *For fixed* $n \in \mathbb{N}$, *define*

$$Q_{-n} := \left\{m \in \underline{N} : \mathcal{U}^{(-n)}\,\varphi_m \neq 0\right\}. \tag{8.0.5}$$

Then the vectors $\left(b_m^{n,i}\right)_{m\in\underline{N^2}}$, $i \in Q_{-n}$, *are orthonormal.*

Proof. ad (1): For fixed $n \in \mathbb{N}_0$, let $k, l \in Q_n$, then

$$\begin{aligned}
\delta_{k,l} &= \langle \mathcal{U}^{(n)}\,\mathcal{T}^{\lfloor \frac{k}{N} \rfloor}\,\varphi_{(k)_N} | \mathcal{U}^{(n)}\,\mathcal{T}^{\lfloor \frac{l}{N} \rfloor}\,\varphi_{(l)_N}\rangle \\
&= \Big\langle \mathcal{U}^{(n+1)} \sum_{m\in\underline{N^{n+2}}} a_m^{n,k}\,\mathcal{T}^{\lfloor \frac{m}{N} \rfloor}\,\varphi_{(m)_N} \Big| \mathcal{U}^{(n+1)} \sum_{m\in\underline{N^{n+2}}} a_m^{n,l}\,\mathcal{T}^{\lfloor \frac{m}{N} \rfloor}\,\varphi_{(m)_N} \Big\rangle \\
&= \sum_{m\in\underline{N^{n+2}}} a_m^{n,k}\overline{a}_m^{n,l}.
\end{aligned}$$

ad (2): This is a calculation similar to (1). □

8.1. Proof of Theorem 8.1

The aim is to prove the existence of a basis as given in Theorem 8.1. The proof is divided into two parts. First we construct coefficients such that the functions $\psi_{n,k}$ given in (8.1.1) and (8.1.2) give an orthonormal basis. In the second part we verify that these functions do indeed give an orthonormal basis. We prove these parts first for $n \in \mathbb{N}_0$ and then for $n \in \mathbb{Z}$, $n < 0$. The mother wavelets are defined for each scale $n \in \mathbb{Z}$ so that they, together with their translates, form a basis for $W_n = V_{n+1} \ominus V_n$, where V_n is given in Definition 1.2. Define for $n \in \mathbb{N}_0$

$$D_n := \left\{m \in \underline{N^{n+2}} : a_m^{n,k} \neq 0 \text{ for some } k \in Q_n\right\},$$
$$D_{-n} := \left\{m \in \underline{N^2} : b_m^{n,k} \neq 0 \text{ for some } k \in Q_{-n}\right\},$$

and $d_n := \operatorname{card} D_n$, $d_{-n} := \operatorname{card} D_{-n}$. Also define $q_n := \operatorname{card} Q_n$ and $q_{-n} := \operatorname{card} Q_{-n}$ for $n \in \mathbb{N}_0$ with Q_n and Q_{-n} given in (8.0.4) and (8.0.5) respectively.

- The mother wavelets for the subspaces W_n, $n \in \mathbb{N}_0$, of the MRA will take the form for $k \in \underline{d_n - q_n}$

$$\psi_{n,k} := \mathcal{U}^{(n+1)} \sum_{m\in\underline{N^{n+2}}} c_m^{n,k}\,\mathcal{T}^{\lfloor \frac{m}{N} \rfloor}\,\varphi_{(m)_N}, \tag{8.1.1}$$

  where the coefficients $c_m^{n,k} \in \mathbb{C}$ are given in (8.1.3).
- For the negative index subspaces W_{-n}, $n \in \mathbb{N}$, of $L^2(\mu)$ we define the mother wavelets in terms of the coefficients of the matrix in (8.1.4) for $n \in \mathbb{N}$ and $k \in \underline{d_{-n} - q_{-n}}$ by

$$\psi_{-n,k} := \mathcal{U}^{(-n+1)} \sum_{m\in\underline{N^2}} c_m^{-n,k}\,\mathcal{T}^{\lfloor \frac{m}{N} \rfloor}\,\varphi_{(m)_N}. \tag{8.1.2}$$

The coefficients $c_m^{n,k} \in \mathbb{C}$, $c_m^{-n,k} \in \mathbb{C}$, can be determined via the Gram-Schmidt process. Before we turn to the determination of the coefficients $c_m^{n,k} \in \mathbb{C}$, $c_m^{-n,k} \in \mathbb{C}$, we note that for all $n \in \mathbb{N}_0$ we have $d_n \geq q_n$ and $d_{-n} \geq q_{-n}$ by the properties (1d) and (1e) of Definition 1.2.

For the definition of the basis we fix $n \in \mathbb{N}_0$ and we construct an orthonormal basis for $\mathbb{C}^{d_n}$ in the following way. Consider the $(q_n \times d_n)$-matrix $\left(a_m^{n,k}\right)_{k\in Q_n, m\in D_n}$. Choose any $d_n - q_n$ vectors of length d_n that are orthonormal to the vectors $\left(a_m^{n,k}\right)_{m\in D_n}$, $k \in Q_n$, (e.g. by applying the Gram-Schmidt process to any linearly independent collection of $d_n - q_n$ vectors of length d_n). We denote these $d_n - q_n$ orthonormal vectors of length d_n by $\left(c_m^{n,i}\right)_{m\in D_n}$ for $i \in \underline{d_n - q_n}$. We extend each vector $\left(c_m^{n,i}\right)_{m\in D_n}$ to a vector of length N^{n+2} by $c_m^{n,i} = 0$ if $m \in \underline{N^{n+2}}\backslash D_n$ and we define matrices $\mathcal{C}_n := \left(c_m^{n,k}\right)_{k\in \underline{d_n-q_n}, m\in \underline{N^{n+2}}}$ of size $(d_n - q_n) \times N^{n+2}$ and $\mathcal{A}_n := \left(a_m^{n,k}\right)_{k\in Q_n, m\in \underline{N^{n+2}}}$ of size $q_n \times N^{n+2}$. So we obtain a matrix of size $d_n \times N^{n+2}$ by

$$\mathcal{M}_n := \begin{pmatrix} \mathcal{A}_n \\ \mathcal{C}_n \end{pmatrix}, \tag{8.1.3}$$

storing the coefficients for the mother wavelets in W_n.

Now we turn to the construction of the coefficients for $\psi_{-n,k}$, $n \in \mathbb{N}$, in (8.1.2). For each $n \in \mathbb{N}$, we define an orthonormal basis of $\mathbb{C}^{d_{-n}}$ in the following way. Consider the $(q_{-n} \times d_{-n})$-matrix $\left(b_m^{n,k}\right)_{k\in Q_{-n}, m\in D_{-n}}$. Now choose any collection of $d_{-n} - q_{-n}$ vectors which are orthonormal to the vectors $\left(b_m^{n,k}\right)_{m\in D_{-n}}$, $k \in Q_{-n}$, (e.g. by application of the Gram-Schmidt process to any linearly independent collection of $d_{-n} - q_{-n}$ vectors of length d_{-n}). In the last step we extend each vector $\left(c_m^{-n,i}\right)_{m\in D_{-n}}$ to a vector of length N^2 by defining $c_m^{-n,i} = 0$ if $m \in \underline{N^2}\backslash D_{-n}$. Now we define matrices $\mathcal{D}_n := \left(c_m^{-n,i}\right)_{i\in \underline{d_{-n}-q_{-n}}, m\in \underline{N^2}}$ and $\mathcal{B}_n := \left(b_m^{n,k}\right)_{k\in Q_{-n}, m\in \underline{N^2}}$ such that

$$\widetilde{\mathcal{M}}_n := \begin{pmatrix} \mathcal{B}_n \\ \mathcal{D}_n \end{pmatrix} \tag{8.1.4}$$

is a matrix of size $d_{-n} \times N^2$ storing the coefficients for the mother wavelets in W_{-n}.

In the next step we show that we do indeed obtain an orthonormal basis with the mother wavelets given in (8.1.1) and (8.1.2). First we prove this for $n \in \mathbb{N}_0$. Recall that $W_n = V_{n+1} \ominus V_n$ for $n \in \mathbb{N}_0$. Consequently, $\bigoplus_{n\in\mathbb{N}_0} W_n \oplus V_0 = L^2(\mu)$ since for every $n \in \mathbb{N}_0$ it follows iteratively that $V_{n+1} = \bigoplus_{k=0}^{n} W_k \oplus V_0$. Now we show that for fixed $n \in \mathbb{N}$, we have that $\left\{\mathcal{T}^l\, \psi_{n,k} : k \in \underline{d_n - q_n}, l \in \mathbb{Z}\right\}$ is an orthonormal basis of W_n. First we show the orthonormality.

To show the orthonormality of $\mathcal{T}^r\, \psi_{n,k}$ and $\mathcal{T}^s\, \psi_{n,l}$, $r, s \in \mathbb{Z}$, $k, l \in \underline{d_n - q_n}$, it is sufficient to consider $\mathcal{T}^r\, \psi_{n,k}$ and $\psi_{n,l}$ since the operator $\mathcal{T}$ is unitary. The orthonormality follows then from

$$\begin{aligned}
\langle \mathcal{T}^r\, \psi_{n,k} | \psi_{n,l} \rangle &= \Big\langle \sum_{m\in \underline{N^{n+2}}} c_m^{n,k}\, \mathcal{U}^{(n+1)}\, \mathcal{T}^{\lfloor \frac{m}{N} \rfloor + N^{n+1} r}\, \varphi_{(m)_N} \Big| \sum_{m\in \underline{N^{n+2}}} c_m^{n,l}\, \mathcal{U}^{(n+1)}\, \mathcal{T}^{\lfloor \frac{m}{N} \rfloor}\, \varphi_{(m)_N} \Big\rangle \\
&= \sum_{m\in \underline{N^{n+2}}} \sum_{s\in \underline{N^{n+2}}} c_m^{n,k} \overline{c}_s^{n,l} \Big\langle \mathcal{U}^{(n+1)}\, \mathcal{T}^{N^{n+1} r + \lfloor \frac{m}{N} \rfloor}\, \varphi_{(m)_N} \Big| \mathcal{U}^{(n+1)}\, \mathcal{T}^{\lfloor \frac{s}{N} \rfloor}\, \varphi_{(s)_N} \Big\rangle \\
&= \sum_{m\in \underline{N^{n+2}}} \sum_{s\in \underline{N^{n+2}}} c_m^{n,k} \overline{c}_s^{n,l} \cdot \delta_{(N^{n+1} r + \lfloor \frac{m}{N} \rfloor, (m)_N), (\lfloor \frac{s}{N} \rfloor, (s)_N)} \\
&= \delta_{r,0} \cdot \sum_{m\in \underline{N^{n+2}}} c_m^{n,k} \overline{c}_m^{n,l} \\
&= \delta_{r,0} \cdot \delta_{k,l}.
\end{aligned}$$

In the next step we consider a basis element of V_{n+1} of the form $\mathcal{U}^{(n+1)}\, \mathcal{T}^{\lfloor \frac{k}{N} \rfloor}\, \varphi_{(k)_N}$, $k \in \underline{N^{n+2}}$, and show that it is a linear combination of functions $\mathcal{U}^{(n)}\, \mathcal{T}^{\lfloor \frac{l}{N} \rfloor}\, \varphi_{(l)_N}$ and $\psi_{n,m}$, $l \in \underline{N^{n+1}}$, $m \in \underline{d_n - q_n}$. It is sufficient to consider only $k \in \underline{N^{n+2}}$ by Definition 1.2 (1e). If $\mathcal{U}^{(n+1)}\, \mathcal{T}^{\lfloor \frac{k}{N} \rfloor}\, \varphi_{(k)_N} = 0$ this is obvious.

If $\mathcal{U}^{(n+1)}\,\mathcal{T}^{\lfloor\frac{k}{N}\rfloor}\,\varphi_{(k)_N}\neq 0$, $k\in\underline{N^{n+2}}$, it can be written as the following linear combination:

$$\begin{aligned}
&\mathcal{U}^{(n+1)}\,\mathcal{T}^{\lfloor\frac{k}{N}\rfloor}\,\varphi_{(k)_N}\\
&=\mathcal{U}^{(n+1)}\left(\sum_{m\in\underline{N^{n+2}}}\underbrace{\left(\sum_{l\in Q_n}\overline{a}_k^{n,l}a_m^{n,l}+\sum_{l\in\underline{d_n-q_n}}\overline{c}_k^{n,l}c_m^{n,l}\right)}_{=\delta_{k,m}}\mathcal{T}^{\lfloor\frac{m}{N}\rfloor}\,\varphi_{(m)_N}\right)\\
&=\sum_{l\in Q_n}\overline{a}_k^{n,l}\,\mathcal{U}^{(n+1)}\sum_{m\in\underline{N^{n+2}}}a_m^{n,l}\,\mathcal{T}^{\lfloor\frac{m}{N}\rfloor}\,\varphi_{(m)_N}+\sum_{l\in\underline{d_n-q_n}}\overline{c}_k^{n,l}\,\mathcal{U}^{(n+1)}\sum_{m\in\underline{N^{n+2}}}c_m^{n,l}\,\mathcal{T}^{\lfloor\frac{m}{N}\rfloor}\,\varphi_{(m)_N}\\
&=\sum_{l\in Q_n}\overline{a}_k^{n,l}\,\mathcal{U}^{(n)}\,T^{\lfloor\frac{l}{N}\rfloor}\varphi_{(l)_N}+\sum_{l\in\underline{d_n-q_n}}\overline{c}_k^{n,l}\psi_{n,l}.
\end{aligned}$$

If we consider $\mathcal{T}^l\,\psi_{n,k}$ and $\mathcal{T}^r\,\psi_{m,s}$ for $n,m\in\mathbb{N}_0$, $n\neq m$, $l,r\in\mathbb{Z}$, $k\in\underline{d_n-q_n}$, $s\in\underline{d_m-q_m}$, the orthonormality follows from $\mathcal{T}^l\,\psi_{n,k}\in W_n$, $\mathcal{T}^r\,\psi_{m,s}\in W_m$ and by the definition of W_n, W_m.

Now we consider the closed subspaces V_n of $L^2(\mu)$ with $n<0$ and proof the corresponding results. We show that for fixed $n\in\mathbb{N}$ $\left\{\mathcal{T}^{N^nk}\,\psi_{-n,l}: l\in\underline{d_{-n}-q_{-n}},\, k\in\mathbb{Z}\right\}$ is an orthonormal basis of $W_{-n}=V_{-n+1}\ominus V_{-n}$. First we show that any function $\mathcal{U}^{(-n+1)}\,\varphi_j$ can be written as a linear combination of functions $\mathcal{U}^{(-n)}\,\varphi_i$ and $\psi_{-n,l}$, $i\in\underline{N}$, $l\in\underline{d_{-n}-q_{-n}}$. This linear combination is precisely

$$\begin{aligned}
&\mathcal{U}^{(-n+1)}\,\varphi_j\\
&=\mathcal{U}^{(-n+1)}\left(\sum_{m\in\underline{N^2}}\underbrace{\left(\sum_{l\in Q_{-n}}\overline{b}_j^{n,l}b_m^{n,l}+\sum_{l\in\underline{d_{-n}-q_{-n}}}\overline{c}_j^{-n,l}c_m^{-n,l}\right)}_{=\delta_{j,m}}\mathcal{T}^{\lfloor\frac{m}{N}\rfloor}\,\varphi_{(m)_N}\right)\\
&=\sum_{l\in Q_{-n}}\overline{b}_j^{n,l}\,\mathcal{U}^{(-n+1)}\sum_{m\in\underline{N^2}}b_m^{n,l}\,\mathcal{T}^{\lfloor\frac{m}{N}\rfloor}\,\varphi_{(m)_N}+\sum_{l\in\underline{d_{-n}-q_{-n}}}\overline{c}_j^{-n,l}\,\mathcal{U}^{(-n+1)}\sum_{m\in\underline{N^2}}c_m^{-n,l}\,\mathcal{T}^{\lfloor\frac{m}{N}\rfloor}\,\varphi_{(m)_N}\\
&=\sum_{l\in Q_{-n}}\overline{b}_j^{n,l}\,\mathcal{U}^{(-n)}\,\varphi_l+\sum_{l\in\underline{d_{-n}-q_{-n}}}\overline{c}_j^{-n,l}\psi_{-n,l}.
\end{aligned}$$

We have to show the orthonormality only for $\psi_{-n,l}$ and $\psi_{-n,k}$ since $\mathcal{T}$ is a unitary operator. For $\psi_{-n,l}$ and $\psi_{-n,k}$, $l,k\in\underline{d_{-n}-q_{-n}}$, the orthonormality follows from

$$\begin{aligned}
\langle\psi_{-n,l}|\psi_{-n,k}\rangle&=\left\langle\sum_{m\in\underline{N^2}}c_m^{-n,l}\,\mathcal{U}^{(-n+1)}\,\mathcal{T}^{\lfloor\frac{m}{N}\rfloor}\,\varphi_{(m)_N}\middle|\sum_{m\in\underline{N^2}}c_m^{-n,k}\,\mathcal{U}^{(-n+1)}\,\mathcal{T}^{\lfloor\frac{m}{N}\rfloor}\,\varphi_{(m)_N}\right\rangle\\
&=\sum_{m\in\underline{N^2}}c_m^{-n,l}\overline{c}_m^{-n,k}\\
&=\delta_{l,k}.
\end{aligned}$$

Furthermore, it follows that $L^2(\mu)=\bigoplus_{k\in\mathbb{Z}}W_k$, since we have shown before that $\bigoplus_{n\in\mathbb{N}_0}W_n\oplus V_0=L^2(\mu)$. We also have that $\psi_{n,k}$, $\psi_{-m,l}$, $m,n\in\mathbb{N}_0$, $k\in\underline{d_n-q_n}$, $l\in\underline{d_{-m}-q_{-m}}$, are orthonormal since $\psi_{n,k}\in W_n$, $\psi_{-m,l}\in W_{-m}$. Consequently, we have that

$$\left\{\mathcal{T}^l\,\psi_{n,k}: n\in\mathbb{Z}, k\in\underline{d_n-q_n}, l\in\mathbb{Z}\right\}$$

is an ONB of $L^2(\mu)$.

Corollary 8.4. *Let μ be a non-atomic measure on $\mathbb{R}$, $\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{N}_0}$ a family of bounded linear operators on $L^2(\mu)$ and $\mathcal{T}$ a unitary operator on $L^2(\mu)$. If $\left(\mu,\left(\mathcal{U}^{(n)}\right)_{n\in\mathbb{N}_0},\mathcal{T}\right)$ allows a one-sided MRA with* *the father wavelets φ_j, $j \in \underline{N}$, then there exists for every $n \in \mathbb{N}_0$ numbers $d_n \in \underline{N^{n+2}}$, $q_n \in \underline{N^{n+1}}$ with $d_n \geq q_n$ and a family of mother wavelets $\left(\psi_{n,l} : l \in \underline{d_n - q_n}\right)$, such that the following set of functions defines an orthonormal basis for $L^2(\mu)$*

$$\left\{\mathcal{T}^k \psi_{n,l} : n \in \mathbb{N}_0,\, l \in \underline{d_n - q_n},\, k \in \mathbb{Z}\right\} \cup \left\{\mathcal{T}^k \varphi_i : k \in \mathbb{Z}, i \in \underline{N}\right\}.$$

Proof. The proof follows from the first part of the proof of Theorem 8.1. In addition we have to show the orthonormality between $\psi_{n,k}$ and φ_i, which follows from the construction of the mother wavelets. $\square$

8.2. Abstract multiplicative multiresolution analysis

In this section we want to consider how the general results simplify if we impose the extra condition of a multiplicative MRA.

Recall from the introduction that in the case of Definition 1.2, we say that we have a multiplicative MRA if there exists an operator $\mathcal{U}$ such that $\mathcal{U}^{(n)} = \mathcal{U}^n$ for all $n \in \mathbb{N}$ and $\mathcal{U}^{(-n)} = (\mathcal{U}^*)^n$, $n \in \mathbb{N}$. We then say $\left(\mu, \left((\mathcal{U})^n, (\mathcal{U}^*)^n\right)_{n \in \mathbb{N}_0}, \mathcal{T}\right)$ allows a two-sided multiplicative MRA.

The key observation is contained in Lemma 8.5 which we prove first.

Lemma 8.5. *Let us assume that $\left(\mu, \left(\mathcal{U}^{(n)}\right)_{n \in \mathbb{Z}}, \mathcal{T}\right)$ allows a two-sided MRA with the closed subspaces V_n, $n \in \mathbb{Z}$, of $L^2(\mu)$ from Definition 1.2 and set $W_n := V_{n+1} \ominus V_n$, $n \in \mathbb{Z}$.*

- *If there is a bounded linear operator $\mathcal{U}$ such that $\mathcal{U}^{(n)} = \mathcal{U}^n$ for all $n \in \mathbb{N}$, then $W_n = \mathcal{U}^n W_0$, $n \in \mathbb{N}$.*
- *If there is a bounded linear operator $\mathcal{U}$ such that $\mathcal{U}^{(-n)} = (\mathcal{U}^*)^n$ for all $n \in \mathbb{N}$, then $W_{-n} = (\mathcal{U}^*)^{n-1} W_{-1}$, $n \in \mathbb{N}$.*

Proof. Recall that $\left\{\mathcal{T}^l \psi_{0,k} : k \in \underline{d_0 - N},\ l \in \mathbb{Z}\right\}$ is an orthonormal basis of W_0. We have $\psi_{0,k} = \sum_{m \in \underline{N^2}} c_m^{0,k}\, \mathcal{U} \mathcal{T}^{\lfloor \frac{m}{N} \rfloor} \varphi_{(m)_N}$ and we show that for fixed $n \in \mathbb{N}$, $\mathcal{U}^n W_0 = W_n$. First it follows that $\mathcal{U}^n \mathcal{T}^m \psi_{0,k} \in W_n \subset V_{n+1}$, $n \in \mathbb{N}$, $m \in \mathbb{Z}$, $k \in \underline{d_0 - N}$, since

$$\mathcal{U}^n \mathcal{T}^m \psi_{0,k} = \sum_{l \in \underline{N^2}} c_l^{0,k}\, \mathcal{U}^{n+1} \mathcal{T}^{\lfloor \frac{l}{N} \rfloor + Nm} \varphi_{(l)_N}$$

and for $i \in \underline{N}$, $r \in \mathbb{Z}$,

$$\begin{aligned}
\langle \mathcal{U}^n \mathcal{T}^m \psi_{0,k} | \mathcal{U}^n \mathcal{T}^r \varphi_i \rangle &= \langle \mathcal{U}^n \mathcal{T}^m \sum_{l \in \underline{N^2}} c_l^{0,k}\, \mathcal{U} \mathcal{T}^{\lfloor \frac{l}{N} \rfloor} \varphi_{(l)_N} | \mathcal{U}^n T^r \sum_{j \in \underline{N^2}} a_j^{0,i}\, \mathcal{U} \mathcal{T}^{\lfloor \frac{j}{N} \rfloor} \varphi_{(j)_N} \rangle \\
&= \langle \mathcal{U}^{n+1} \mathcal{T}^{Nm} \sum_{l \in \underline{N^2}} c_l^{0,k}\, \mathcal{T}^{\lfloor \frac{l}{N} \rfloor} \varphi_{(l)_N} | \mathcal{U}^{n+1} \mathcal{T}^{Nr} \sum_{j \in \underline{N^2}} a_j^{0,i}\, \mathcal{T}^{\lfloor \frac{j}{N} \rfloor} \varphi_{(j)_N} \rangle \\
&= \delta_{m,r} \cdot \sum_{l \in \underline{N^2}} c_l^{0,k} a_l^{0,i} = 0.
\end{aligned}$$

Consequently, $\mathcal{U}^n W_0 \subset W_n$. Now fix $m \in \mathbb{Z}$, $j \in \underline{N}$, $n \in \mathbb{N}_0$, and consider $\mathcal{U}^{n+1} \mathcal{T}^m \varphi_j \in V_{n+1}$. We show that this can be written as a linear combination of functions $\mathcal{U}^n \mathcal{T}^l \varphi_i$ and $\mathcal{U}^n \mathcal{T}^r \psi_{0,k}$, $l, r \in \mathbb{Z}$, $i \in \underline{N}$, $k \in \underline{d_0 - N}$, by considering the inner product. First we recall from the proof of Theorem 8.1 that $\mathcal{U} \mathcal{T}^{\lfloor \frac{k}{N} \rfloor} \varphi_{(k)_N} = \sum_{i \in \underline{N}} \overline{a}_k^{0,i} \varphi_i + \sum_{l \in \underline{d_0 - N}} \overline{c}_k^{0,l} \psi_{0,l}$ for $k \in \underline{N^2}$ and hence for $k \in \underline{N^2}$

$$1 = \left\langle \mathcal{U} \mathcal{T}^{\lfloor \frac{k}{N} \rfloor} \varphi_{(k)_N} \middle| \sum_{i \in \underline{N}} \overline{a}_k^{0,i} \varphi_i + \sum_{l \in \underline{d_0 - N}} \overline{c}_k^{0,l} \psi_{0,l} \right\rangle = \sum_{i \in \underline{N}} \overline{a}_k^{0,i} a_k^{0,i} + \sum_{l \in \underline{d_0 - N}} \overline{c}_k^{0,l} c_k^{0,l}.$$

It follows that for $m \in \underline{N^{n+2}}$ written as $m = k + N^2 k_1$, $k \in \underline{N^2}$, $k_1 \in \underline{N^n}$, we have $\lfloor \frac{m}{N} \rfloor = \lfloor \frac{k}{N} \rfloor + Nk_1$ and $(m)_N = (k)_N$, and so

$$\begin{aligned}
&\left\langle \mathcal{U}^{n+1}\, \mathcal{T}^{\lfloor \frac{k}{N} \rfloor + Nk_1}\, \varphi_{(k)_N} \middle| \mathcal{U}^n\, \mathcal{T}^{k_1} \sum_{i \in \underline{N}} \overline{a}_k^{0,i} \varphi_i + \mathcal{U}^n\, \mathcal{T}^{k_1} \sum_{l \in \underline{d_0 - N}} \overline{c}_k^{0,l} \psi_{0,l} \right\rangle \\
&= \left\langle \mathcal{U}^{n+1}\, \mathcal{T}^{\lfloor \frac{k}{N} \rfloor + Nk_1}\, \varphi_{(k)_N} \middle| \mathcal{U}^{n+1}\, \mathcal{T}^{Nk_1} \sum_{i \in \underline{N}} \overline{a}_k^{0,i} \sum_{l \in \underline{N^2}} a_l^{0,i}\, \mathcal{T}^{\lfloor \frac{l}{N} \rfloor}\, \varphi_{(l)_N} \right\rangle \\
&\quad + \left\langle \mathcal{U}^{n+1}\, \mathcal{T}^{\lfloor \frac{k}{N} \rfloor + Nk_1}\, \varphi_{(k)_N} \middle| \mathcal{U}^{n+1}\, \mathcal{T}^{Nk_1} \sum_{l \in \underline{d_0 - N}} \overline{c}_k^{0,l} \sum_{i \in \underline{N^2}} c_i^{0,l}\, \mathcal{T}^{\lfloor \frac{i}{N} \rfloor}\, \varphi_{(i)_N} \right\rangle \\
&= \sum_{i \in \underline{N}} \overline{a}_k^{0,i} a_k^{0,i} + \sum_{l \in \underline{d_0 - N}} \overline{c}_k^{0,l} c_k^{0,l} \\
&= 1.
\end{aligned}$$

Now we notice that we can write any element $k \in \mathbb{Z}$ as $k = k_0 + N^{n+2} l$ for some $k_0 \in \underline{N^{n+2}}$ and $l \in \mathbb{Z}$. Consequently, with $\mathcal{U}\,\mathcal{T}^N|_{V_0} = \mathcal{T}\,\mathcal{U}|_{V_0}$ we obtain the general result for $\mathcal{U}^{n+1}\,\mathcal{T}^k\,\varphi_j$, $k \in \mathbb{Z}$, $j \in \underline{N}$.

To obtain $W_{-n} = (\mathcal{U}^*)^n\, W_{-1}$, $W_{-1} = V_0 \ominus V_{-1}$, $n \in \mathbb{N}$, we can proceed as above. First we have from the proof of Theorem 8.1 that $\psi_{-1,k} = \sum_{l \in \underline{N^2}} c_l^{-1,k}\, \mathcal{T}^{\lfloor \frac{l}{N} \rfloor}\, \varphi_{(l)_N}$, $k \in \underline{d_{-1} - N}$, and that φ_j, $j \in \underline{N}$, can be represented as $\varphi_j = \sum_{l \in \underline{N}} \overline{b}_{0,j}^{-1,l}\, \mathcal{U}^*\, \varphi_l + \sum_{l \in \underline{d_{-1} - N}} \overline{c}_j^{-1,l} \psi_{-1,l}$. With these observations we obtain as above that for $m, r \in \mathbb{Z}$

$$\langle (\mathcal{U}^*)^n\, \mathcal{T}^m\, \varphi_j |\, (\mathcal{U}^*)^{n-1}\, \mathcal{T}^r\, \psi_{-1,k} \rangle = 0$$

and

$$\left\langle\, (\mathcal{U}^*)^{n-1}\, \varphi_j \middle|\, (\mathcal{U}^*)^n \sum_{l \in \underline{N}} \overline{b}_j^{-1,l} \varphi_l + (\mathcal{U}^*)^{n-1} \sum_{l \in \underline{d_{-1} - N}} \overline{c}_j^{-1,l} \psi_{-1,l} \right\rangle = 1.$$

□

Remark 8.6. If we have $\mathcal{U}\,\mathcal{U}^* = I$, then $W_0 = \mathcal{U}\,(W_{-1})$. Note that $\mathcal{U}$ is not necessarily injective on W_{-1}.

Now we turn to the mother wavelets for the multiplicative MRA.

Remark 8.7. If $\mathcal{U}^{(n)} = \mathcal{U}^n, \mathcal{U}^{(-n)} = (\mathcal{U}^*)^n$, then we only consider the mother wavelets for $k \in \underline{d_0 - N}$. So

$$\psi_{0,k} = \mathcal{U} \sum_{l \in \underline{N^2}} c_l^{0,k}\, \mathcal{T}^{\lfloor \frac{l}{N} \rfloor}\, \varphi_{(l)_N},$$

where the coefficients are from (8.0.2) and we define $\psi_k := \psi_{0,k}$.

For the negatively indexed part of the construction we write for $k \in \underline{d_{-1} - N}$

$$\psi_{-,k} = \sum_{l \in \underline{N^2}} c_l^{-1,k}\, \mathcal{T}^{\lfloor \frac{l}{N} \rfloor}\, \varphi_{(l)_N}.$$

Corollary 8.8. *If* $\left(\mu, \left(\mathcal{U}^{(n)}\right)_{n \in \mathbb{Z}}, \mathcal{T}\right)$ *is multiplicative (with the bounded linear operator* $\mathcal{U}$*), then there exists an orthonormal basis of* $L^2(\mu)$ *of the form*

$$\left(\left\{\mathcal{U}^n\, \mathcal{T}^k\, \psi_l : n \in \mathbb{N}_0, k \in \mathbb{Z}, l \in \underline{d_0 - N}\right\} \cup \left\{(\mathcal{U}^*)^n\, \mathcal{T}^k\, \psi_{-,l} : n \in \mathbb{N}_0, k \in \mathbb{Z}, l \in \underline{d_{-1} - N}\right\}\right) \setminus \{0\},$$

where the functions ψ_l*,* $l \in \underline{d_0 - N}$*, and* $\psi_{-,l}$*,* $l \in \underline{d_{-1} - N}$*, are given in Remark 8.7.*

8.3. Translation completeness

In the following we assume a stronger condition than (1e) of Definition 1.2, namely in a translation complete multiplicative MRA the father wavelets satisfy for $j \in \underline{N}$

$$\varphi_j \in \operatorname{span} \mathcal{U} \left\{ \mathcal{T}^j \varphi_i : i \in \underline{N} \right\}. \tag{8.3.1}$$

This condition implies that for φ_j, $j \in \underline{N}$, there exist complex numbers $a_i^{0,j}$, $i \in \underline{N}$, such that

$$\varphi_j = \sum_{i \in \underline{N}} a_i^{0,j} \, \mathcal{U} \, \mathcal{T}^j \, \varphi_i.$$

We would like to point out that this condition is also satisfied for the particular case of Markov Interval Maps with Markov measure where the father wavelets φ_j, $j \in \underline{N}$, are chosen to be the scaled characteristic functions on the cylinder sets $[j]$ (see Section 9).

The relation (8.0.2) takes for a multiplicative MRA the following form for $k \in \mathbb{Z}$, $j \in \underline{N}$, $n \in \mathbb{N}_0$,

$$\mathcal{U}^n \, \mathcal{T}^k \, \varphi_j = \sum_{l \in \underline{N^2}} a_l^{0,j} \, \mathcal{U}^{n+1} \, \mathcal{T}^{Nk + \lfloor \frac{l}{N} \rfloor} \, \varphi_{(l)_N}$$

and under condition (8.3.1) this simplifies to

$$\mathcal{U}^n \, \mathcal{T}^k \, \varphi_j = \sum_{i \in \underline{N}} a_i^{0,j} \, \mathcal{U}^{n+1} \, \mathcal{T}^{Nk+j} \, \varphi_i.$$

To simplify the notation we set $a_l^j := a_l^{0,j}$. We now show that condition (8.3.1) allows us to simplify the construction of the mother wavelets.

Lemma 8.9. *Under condition (8.3.1) one possible choice of the matrix $\mathcal{M}_0$ in (8.1.3) has a block structure consisting of N blocks.*

Proof. Define $Q^k := \{ j \in \underline{N} : \mathcal{U} \, \mathcal{T}^k \, \varphi_j \neq 0 \}$ and $q^k := \operatorname{card} Q^k$ for each $k \in \underline{N}$. Then $\left(a_j^k\right)_{j \in Q^k}$ is a vector of length q^k and we choose $q^k - 1$ orthonormal vectors to $\left(a_j^k\right)_{j \in Q^k}$ of length q^k (e.g. by applying the Gram-Schmidt process to any collection of linearly independent to $\left(a_j^k\right)_{j \in Q^k}$ family of $q^k - 1$ vectors of length q^k). We denote these vectors by $\left(c_j^{k,l}\right)_{j \in Q^k}$, $l \in \underline{q^k} \backslash \{0\}$. We extend each of the vectors $\left(c_j^{k,l}\right)_{j \in Q^k}$ to one $\left(c_j^{k,l}\right)_{j \in \underline{N}}$ of length N by defining $c_j^{k,l} = 0$ if $j \in \underline{N} \backslash Q^k$. Then

$$M_k := \begin{pmatrix} \left(a_j^k\right)_{j \in \underline{N}} \\ \left(c_j^{k,l}\right)_{l \in \underline{q^k} \backslash \{0\}, j \in \underline{N}} \end{pmatrix}$$

is a matrix of size $q^k \times N$.

The matrix $\widehat{\mathcal{M}}_0 = (h_{ij})_{i \in \underline{q_1}, j \in \underline{N^2}}$ given with the blocks M_k, $k \in \underline{N}$, which are defined for $k = 0$ by

$$(h_{ij})_{i \in \underline{q^0}, j \in \underline{N}} = M_0,$$

and for $k \in \underline{N} \backslash \{0\}$ by

$$(h_{ij})_{i \in \underline{\sum_{l=0}^{k} q^l} \backslash \underline{\sum_{l=0}^{k-1} q^l}, j \in \underline{(k+1)N} \backslash \underline{kN}} = M_k$$

and otherwise zeros satisfies the conditions imposed on $\mathcal{M}_0$ in (8.1.3), i.e. if we restrict the columns to those in D_1 give an ONB of $\mathbb{C}^{d_0}$, and $\widehat{\mathcal{M}_0}$ is of size $q_0 \times N^2$ since $\sum_{k \in \underline{N}} q^k = q_0$. We notice that $\widehat{\mathcal{M}}_0$ is ordered in a different way than $\mathcal{M}_0$, since the rows $\left(a_j^k\right)_{j \in Q^k}$ are not grouped in $\mathcal{M}_0$. $\square$

Remark 8.10.

(1) If $\mathcal{U}^{(n)} = \mathcal{U}^n$, $\mathcal{U}^{(-n)} = (\mathcal{U}^*)^n$ and (8.3.1), the mother wavelets take the simpler form for $k = 0$, $l \in \underline{q^0}\backslash\{0\}$ and for $k \in \underline{N}\backslash\{0\}$, $l \in \underline{\sum_{i=0}^{k} q^i}\backslash\underline{\sum_{i=0}^{k-1} q^i}$, as
$$\psi_l = \sum_{j\in\underline{N}} c_j^{k,l}\, \mathcal{U}\, \mathcal{T}^k\, \varphi_j,$$
where the coefficients are as constructed in Lemma 8.9. For negative indexed part we define for $k \in \underline{d_{-1} - N}$
$$\psi_{-,k} = \sum_{l\in\underline{N^2}} c_l^{-1,k}\, \mathcal{T}^{\lfloor \frac{l}{N} \rfloor}\, \varphi_{(l)_N}.$$

(2) Under the condition (8.3.1), or the slightly weaker statement
$$\mathcal{U}^{(n)}\, \mathcal{T}^{\lfloor \frac{k}{N} \rfloor}\, \varphi_{(k)_N} = \sum_{i\in\underline{N}} a_i^{n,k}\, \mathcal{U}^{(n+1)}\, \mathcal{T}^{N\lfloor \frac{k}{N} \rfloor + (k)_N}\, \varphi_i \tag{8.3.2}$$
we can obtain the coefficients for the mother wavelets by constructing for each $k \in \underline{N^n}$ with $\mathcal{U}^{(n)}\, \mathcal{T}^k\, \varphi_j \neq 0$ for at least one $j \in \underline{N}$ a matrix of size $q^{n,k} \times q^{n,k}$, where $q^{n,k} := \mathrm{card}\{j \in \underline{N} : \mathcal{U}^{(n)}\, \mathcal{T}^k\, \varphi_j \neq 0\}$ instead of one unitary matrix of size $d_n \times d_n$. In this way we need at most N^n matrices on the scale $n \in \mathbb{N}$.

Now we turn to a correspondence to the construction of a wavelet basis for an MIM. The next proposition shows how the incidence matrix of an MIM plays a role in the MRA.

Proposition 8.11. *In the case of* $\mathcal{U}^{(n)} = (\mathcal{U})^n$, $n \in \mathbb{N}_0$, *(8.3.1) and if it further holds that* $a_i^{0,j} \neq 0$ *if and only if* $\mathcal{U}\, \mathcal{T}^j\, \varphi_i \neq 0$, $i, j \in \underline{N}$, *then we have for* $n \in \mathbb{N}$, $k \in \mathbb{Z}$, $\mathcal{U}^n\, \mathcal{T}^k\, \varphi_j \neq 0$ *if and only if for all* $i = 0, \ldots, n-2$, $\mathcal{U}\, \mathcal{T}^{k_{i+1}}\, \varphi_{k_i} \neq 0$ *and* $\mathcal{U}\, \mathcal{T}^{k_0}\, \varphi_j \neq 0$, *where* $k = \sum_{i=0}^{n-1} k_i N^i + lN^n$, $k_i \in \underline{N}$, $i \in \underline{n}$, *and* $l \in \mathbb{Z}$.

PROOF. We prove this for $k = k_0 + Nk_1$, $k_0, k_1 \in \underline{N}$. The general result follows iteratively. Notice that $\mathcal{U}^2\, \mathcal{T}^{k_0+Nk_1}\, \varphi_j = \mathcal{U}\, \mathcal{T}^{k_1} \left(\mathcal{U}\, \mathcal{T}^{k_0}\, \varphi_j\right)$. Consequently, from $\mathcal{U}^2\, \mathcal{T}^{k_0+Nk_1}\, \varphi_j \neq 0$ it follows that $\mathcal{U}\, \mathcal{T}^{k_0}\, \varphi_j \neq 0$. Furthermore, we have that
$$\mathcal{U}\, \mathcal{T}^{k_1}\, \varphi_{k_0} = \mathcal{U}\, \mathcal{T}^{k_1} \sum_{i\in\underline{N}} a_i^{k_0}\, \mathcal{U}\, \mathcal{T}^{k_0}\, \varphi_i = \mathcal{U}^2\, \mathcal{T}^{Nk_1+k_0} \sum_{i\in\underline{N}} a_i^{k_0} \varphi_i \neq 0$$
if $\mathcal{U}^2\, \mathcal{T}^{k_0+Nk_1}\, \varphi_j \neq 0$.

If we assume that $\mathcal{U}\, \mathcal{T}^{k_1}\, \varphi_{k_0} \neq 0$ and $\mathcal{U}\, \mathcal{T}^{k_0}\, \varphi_j \neq 0$ then
$$\begin{aligned}
\mathcal{U}^2\, \mathcal{T}^{Nk_1+k_0}\, \varphi_j &= \mathcal{U}\, \mathcal{T}^{k_1}\, \mathcal{U}\, \mathcal{T}^{k_0}\, \varphi_j = \left(a_j^{k_0}\right)^{-1} \mathcal{U}\, \mathcal{T}^{k_1} \left(\varphi_{k_0} - \sum_{i\in\underline{N}\backslash\{j\}} a_i^{k_0}\, \mathcal{U}\, \mathcal{T}^{k_0}\, \varphi_i\right) \\
&= \left(a_j^{k_0}\right)^{-1} \left(\mathcal{U}\, \mathcal{T}^{k_1}\, \varphi_{k_0} - \sum_{i\in\underline{N}\backslash\{j\}} a_i^{k_0}\, \mathcal{U}^2\, \mathcal{T}^{Nk_1+k_0}\, \varphi_i\right) \neq 0,
\end{aligned}$$
since
$$\|\mathcal{U}\, \mathcal{T}^{k_1}\, \varphi_{k_0} - \sum_{i\in\underline{N}\backslash\{j\}} a_i^{k_0}\, \mathcal{U}^2\, \mathcal{T}^{Nk_1+k_0}\, \varphi_i\|^2 = 1 - \sum_{i\in\underline{N}\backslash\{j\}} |a_i^{k_0}|^2 = |a_j^{k_0}|^2 \neq 0.$$
$\square$

Remark 8.12. The same result can be shown if for all $n \in \mathbb{N}$, $k \in \underline{N^n}$ and $j \in \underline{N}$ there is $c \in \mathbb{R}$ that may depend on n, k, j, for which we have $\mathcal{U}^{(n)}\, \mathcal{T}^k\, \varphi_j = c \left(\mathcal{U}^{(1)}\right)^n \mathcal{T}^k\, \varphi_j$ and (8.3.1).

Under the conditions of Proposition 8.11 we can give a $(N \times N)$ matrix A which coincides with the incidence matrix in the case of MIM given by $A = (A_{ij})_{i,j \in \underline{N}}$ with

$$A_{ij} := \begin{cases} 0, & \text{if } \mathcal{U}\mathcal{T}^i \varphi_j = 0, \\ 1, & \text{otherwise.} \end{cases}$$

CHAPTER 9

Applications to Markov Interval Maps

9.1. Markov Interval Maps

In this section we give some basic definitions and notations. We consider limit sets of one-dimensional Markov Interval Maps.

Definition 9.1. Let $(B_i)_{i\in\underline{N}}$ be closed intervals in $[0,1]$ with disjoint interior and define $J := \bigcup_{i\in\underline{N}} B_i$. Suppose a function $F : J \to [0,1]$ is expanding and C^1 on each B_i, $i \in \underline{N}$, such that if $F(B_i) \cap B_j \neq \emptyset$ then $B_j \subset F(B_i)$ for $i, j \in \underline{N}$. Then we call the system $\left((B_i)_{i\in\underline{N}}, F\right)$ a Markov Interval Map (MIM). Its limit set is defined to be the set $X := \bigcap_{n=0}^{\infty} F^{-n} J$.

Remark 9.2.

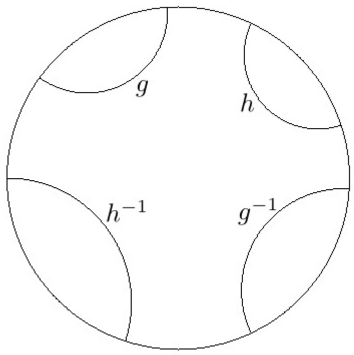
1. We define the inverse branches $\tau_i := \left(F|_{B_i}\right)^{-1}$, $i \in \underline{N}$. The family $(\tau_i)_{i\in\underline{N}}$ is called a one-dimensional Graph Directed Markov System (GDMS) with the incidence matrix $A = (A_{ij})_{i,j\in\underline{N}}$ which is given by
$$A_{ij} := \begin{cases} 1, & \text{if } B_j \subset F(B_i) \\ 0, & \text{otherwise,} \end{cases}$$
and it follows that $F(B_i) = \bigcup_{j\in\underline{N}\colon A_{ij}=1} B_j$.
2. If $F(B_i) = [0,1]$ for each $i \in \underline{N}$, then τ_i, $i \in \underline{N}$, (given in (1)) corresponds to an iterated function system (IFS).

Example 9.3. An example is a convex, co-compact Kleinian group, as shown in Figure 9.1.1a, where the limit set is the set that is obtained by successive application of these four maps, where the composition of g_i and $g_i^{-1}$are forbidden. We can associate the limit set in hyperbolic space to the limit set in $[0,1]$ of the corresponding Bowen-Series map to the maps g_i and g_i^{-1}, which gives rise to a Markov Interval Map, compare Figure 9.1.1b. A typical measure to be studied would be the measure of maximal entropy or the conformal measure (of maximal dimension).

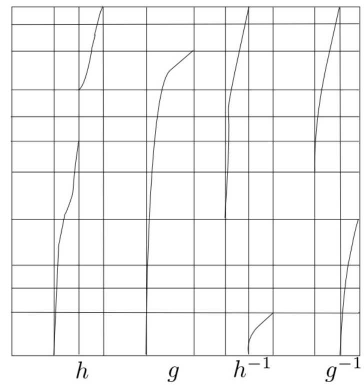
Next we consider the corresponding shift space with the alphabet $\underline{N} = \{0, \ldots, N-1\}$. The limit set X is then homeomorphic (mod ν) to the set of all admissible words
$$\Sigma_A := \{\omega = (\omega_0, \omega_1, \ldots) \in \underline{N}^{\mathbb{N}} : A_{\omega_i\omega_{i+1}} = 1 \text{ for all } i \geq 0\}.$$
A homeomorphism $\pi : \Sigma_A \to X$ can be given by fixing any point $x \in X$ and defining π by the rule
$$\omega \mapsto \lim_{n\to\infty} \tau_{\omega_0} \circ \cdots \circ \tau_{\omega_n}(x), \tag{9.1.1}$$
The map is independent of the particular choice of $x \in X$ and π is called the coding map.

Remark 9.4. We define the cylinder sets for $\omega_0, \ldots, \omega_k \in \underline{N}$, $k \in \mathbb{N}_0$, by
$$[\omega_0 \ldots \omega_k] := \left\{(\omega_0', \omega_1', \ldots) \in \Sigma_A : \omega_i = \omega_i',\, i \in \{0, \ldots, k\}\right\}.$$
If for some $i \in \{0, \ldots, k-1\}$ we have $A_{\omega_i\omega_{i+1}} = 0$ then $[\omega_0 \ldots \omega_k] = \emptyset$.

For $i \in \underline{N}$, the sets B_i and $F(B_i)$ are homeomorphic (mod ν) to the sets
$$\pi^{-1}(B_i) = [i]$$



(A) A fundamental domain of the action of $\langle g, h\rangle$ on the Poincaré disc model.



(B) The corresponding Bowen-Series map.

FIGURE 9.1.1. Example of a Fuchsian group.

and

$$\pi^{-1}\left(F(B_i)\right) = \{\omega = (\omega_0, \omega_1, \dots) \in \Sigma_A : A_{i\omega_0} = 1\}$$

in the shift space, respectively. The map F is conjugated via π to the shift dynamic $\theta : \Sigma_A \to \Sigma_A$, $\theta\left(\omega_0, \omega_1, \dots\right) = \left(\omega_1, \omega_2, \dots\right)$ and consequently, the functions τ_i correspond to the inverse branches of the shift function, i.e. $\tau_i \circ \pi\left(\omega_0, \omega_1, \dots\right) = \pi\left(i, \omega_0, \omega_1, \dots\right)$, for $\omega \in \pi^{-1}\left(F(B_i)\right)$, $i \in \underline{N}$.

Furthermore, let us fix the following notation.

- $\Sigma_A^n := \left\{\omega = (\omega_0, \dots, \omega_{n-1}) \in \underline{N}^n : A_{\omega_i\omega_{i+1}} = 1 \text{ for all } i \in \{0, \dots, n-1\}\right\}$ defines the set of admissible words of length $n \in \mathbb{N}$.
- Σ_A^* stands for all finite words, i.e. $\Sigma_A^* = \bigcup_{n\geq 1} \Sigma_A^n$.
- For $\omega \in \Sigma_A^n$ we define $\tau_\omega := \tau_{\omega_0} \circ \tau_{\omega_1} \circ \cdots \circ \tau_{\omega_{n-1}}$.
- For $\omega \in \Sigma_A^n$, $\tau \in \Sigma_A^m$ we define their concatenation

  $$\omega\tau := (\omega_0, \dots, \omega_{n-1}, \tau_0, \dots, \tau_{m-1})$$

  which is an element of Σ_A^{n+m} whenever $A_{\omega_{n-1}\tau_0} = 1$.

On the shift space Σ_A we consider the product topology on $\underline{N}^{\mathbb{N}}$ and we consider the Borel σ-algebra $\mathcal{B}$ on Σ_A which is generated by the open sets in the product topology.

As a measure on X we could consider, for instance, the pullbacks under π of Gibbs measures on Σ_A (for definitions see e.g. [**KS10**]).

Now we define the appropriate space for which we want to construct a wavelet basis.

Definition 9.5. Let $\widetilde{\nu}$ be a probability measure on $(\Sigma_A, \mathcal{B})$ and $\nu = \widetilde{\nu} \circ \pi^{-1}$. Define the *enlarged fractal* by

$$R = \bigcup_{k\in\mathbb{Z}} X + k$$

and define the $\mathbb{Z}$-convolution $\nu_{\mathbb{Z}}$ of the measure ν for a Borel set B in $\mathbb{R}$ by

$$\nu_{\mathbb{Z}}(B) = \sum_{k\in\mathbb{Z}} \nu(B - k),$$

which clearly is an invariant measure under $\mathbb{Z}$-translation.

Remark 9.6. One example is the space $L^2\left(\Sigma_A, \mu_\phi\right)$, where Σ_A denotes a one-sided topologically exact sub-shift of finite type. An important class of measures on Σ_A are given by invariant Gibbs measures with respect to a Hölder continuous potentials $\phi \in \mathcal{C}(\Sigma_A, \mathbb{R})$, denoted by μ_ϕ, compare [**KS10**]. μ_ϕ corresponds to the measure $\widetilde{\nu}$ in Definition 9.5.

In the following we use the convention $0^{-1} \cdot \mathbb{1}_\emptyset = 0$. For simplicity we let $[\omega_0 \dots \omega_{n-1}]$ also denote the set $\tau_{\omega_0} \circ \dots \circ \tau_{\omega_{n-1}}(X)$ using the identification by π. Furthermore, here the measure ν supported on $[0,1]$ always corresponds to a measure $\widetilde{\nu}$ on Σ_A by $\nu = \widetilde{\nu} \circ \pi^{-1}$ and $\nu_\mathbb{Z}$ denotes the measure obtained from ν by $\mathbb{Z}$-convolution.

9.2. Multiresolution analysis for MIMs

Now we apply the results of Chapter 8 to Markov Interval Maps. More precisely, we construct a wavelet basis on the L^2-space of a limit set of a Markov Interval Map translated by $\mathbb{Z}$ with respect to a measure. First we consider the case where we do not have any relation between $\nu_\mathbb{Z}([ij])$ and $\nu_\mathbb{Z}([i])$, $\nu_\mathbb{Z}([j])$, $i,j \in \underline{N}$. In this case we cannot define only one scaling operator U, but on each scale $n \in \mathbb{Z}$ we consider a different operator $U^{(n)}$. Consequently, we obtain a family of operators $\left(U^{(n)}\right)_{n\in\mathbb{Z}}$. For this we define $U^{(0)} := I$ and for $f \in L^2(\nu_\mathbb{Z})$, $x \in \mathbb{R}$ and $n \in \mathbb{N}$ we let

$$U^{(n)} f(x) := \sum_{k\in\mathbb{Z}} \sum_{\omega\in\Sigma_A^n} \sum_{j\in\underline{N}} \sqrt{\frac{\nu_\mathbb{Z}([j])}{\nu_\mathbb{Z}([\omega j])}} \mathbb{1}_{[\omega j]}(x-k) \cdot f\left(\tau_\omega^{-1}(x-k) + \sum_{i=0}^{n-1} \omega_{n-1-i} N^i + N^n k\right)$$

and

$$\begin{aligned} U^{(-n)} f(x) := \sum_{k\in\mathbb{Z}} \sum_{\omega\in\Sigma_A^n} \sum_{j\in\underline{N}} \sqrt{\frac{\nu_\mathbb{Z}([\omega j])}{\nu_\mathbb{Z}([j])}} \mathbb{1}_{[j]} &\left(x - \sum_{i=0}^{n-1} \omega_{n-1-i} N^i - N^n k\right) \\ &\cdot f\left(\tau_\omega\left(x - \sum_{i=0}^{n-1} \omega_{n-1-i} N^i - N^n k\right) + k\right). \end{aligned}$$

The unitary translation operator T acting on $L^2(\nu_\mathbb{Z})$ is defined by

$$Tf(\cdot) := f(\cdot - 1).$$

Remark 9.7.

(1) Notice that in general we have $U^{(1)}U^{(1)} \neq U^{(2)}$ since for $i,j,k \in \underline{N}$ the multiplicative constant $\sqrt{\frac{\nu_\mathbb{Z}([i])}{\nu_\mathbb{Z}([ki])} \frac{\nu_\mathbb{Z}([j])}{\nu_\mathbb{Z}([ij])}}$ for $U^{(1)}U^{(1)}$ and $\sqrt{\frac{\nu_\mathbb{Z}([j])}{\nu_\mathbb{Z}([kij])}}$ for $U^{(2)}$ on the cylinder sets may differ.
(2) The operator T is unitary.
(3) The operators $\left(U^{(n)}\right)_{n\in\mathbb{Z}}$ are well defined, namely for $f \in L^2(\nu_\mathbb{Z})$ we have $U^{(n)} f \in L^2(\nu_\mathbb{Z})$.

Define the N father wavelets as $\varphi_i := (\mu([i]))^{-1/2} \mathbb{1}_{[i]}$ for $i \in \underline{N}$.

Remark 9.8. Notice that for $\omega \in \Sigma_A^n$, $j \in \underline{N}$ and $k \in \mathbb{Z}$ with $k = \sum_{i=0}^{n-1} \omega_{n-1-i} N^i + N^n l$, $l \in \mathbb{Z}$, we have

$$U^{(n)} T^k \varphi_j \quad = \quad \begin{cases} 0, & \text{if } A_{\omega_{n-1} j} = 0, \\ (\nu_\mathbb{Z}([\omega j]))^{-1/2} \, T^l \mathbb{1}_{[\omega j]}, & \text{otherwise.} \end{cases} \tag{9.2.1}$$

Now we turn to the proof of the properties of $\left(U^{(n)}\right)_{n\in\mathbb{Z}}$ and T stated in Proposition 9.9.

Proposition 9.9. *Let $(\varphi_j)_{j\in\underline{N}}$ denote the family of father wavelets given by $\varphi_i = (\nu_\mathbb{Z}([i]))^{-1/2} \mathbb{1}_{[i]}$, $i \in \underline{N}$. The translation operator T and the family of scaling operators $\left(U^{(n)}\right)_{n\in\mathbb{Z}}$ satisfy the following.*

(1) $TU^{(n)} = U^{(n)} T^{N^n}$, $n \in \mathbb{N}$,
(2) $U^{(-n)} T \varphi_j = T^{N^n} U^{(-n)} \varphi_j$, $n \in \mathbb{N}$, $j \in \underline{N}$,
(3) $\varphi_i = U^{(1)} T^i \sum_{j\in\underline{N}} \sqrt{\frac{\nu_\mathbb{Z}([ij])}{\nu_\mathbb{Z}([i])}} \varphi_j$, $i \in \underline{N}$,
(4) *if* $U^{(n)} T^k \varphi_i| \neq 0$, *then* $\langle U^{(n)} T^k \varphi_i | U^{(n)} T^l \varphi_j \rangle = \delta_{(k,i),(l,j)}$, $n,k,l \in \mathbb{Z}$, $i,j \in \underline{N}$,

(5) $U^{(n)}U^{(-n)} = I$, $n \in \mathbb{N}$,
(6) *if* $U^{(n)}T^k\varphi_j \neq 0$, *then* $U^{(-n)}U^{(n)}T^k\varphi_j = T^k\varphi_j$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$, $j \in \underline{N}$.

PROOF. ad (1): Let $n \in \mathbb{N}$, $f \in L^2(\nu_{\mathbb{Z}})$, $x \in \mathbb{R}$, then

$$\begin{aligned}
&TU^{(n)}f(x)\\
=&\sum_{k\in\mathbb{Z}}\sum_{\omega\in\Sigma_A^n}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([j])}{\nu_{\mathbb{Z}}([\omega j])}}\mathbb{1}_{[\omega j]}(x-1-k)\cdot f\left(\tau_\omega^{-1}(x-1-k)+\sum_{i=0}^{n-1}\omega_{n-1-i}N^i+N^nk\right)\\
=&\sum_{l\in\mathbb{Z}}\sum_{\omega\in\Sigma_A^n}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([j])}{\nu_{\mathbb{Z}}([\omega j])}}\mathbb{1}_{[\omega j]}(x-l)\cdot f\left(\tau_\omega^{-1}(x-l)+\sum_{i=0}^{n-1}\omega_{n-1-i}N^i+N^nl-N^n\right)\\
=&U^{(n)}T^{N^n}f(x).
\end{aligned}$$

ad (3): Let $i \in \underline{N}$, $x \in \mathbb{R}$, then

$$\begin{aligned}
\varphi_i(x) &= (\nu_{\mathbb{Z}}([i]))^{-1/2}\sum_{j\in\underline{N}}\mathbb{1}_{[ij]}(x)\\
&=\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([ij])}{\nu_{\mathbb{Z}}([i])}\frac{\nu_{\mathbb{Z}}([j])}{\nu_{\mathbb{Z}}([ij])}}\cdot(\mu([j]))^{-1/2}\,\mathbb{1}_{[ij]}(x)\\
&=U^{(1)}T^i\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([ij])}{\nu_{\mathbb{Z}}([i])}}\varphi_j(x).
\end{aligned}$$

ad (2): Notice that for $n \in \mathbb{N}$, $l \in \underline{N}$, $k \in \mathbb{Z}$,

$$U^{(-n)}\varphi_l(x) = \sum_{\omega\in\Sigma_A^n:\omega_0=l}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([l])}}\varphi_j\left(x-\sum_{i=0}^{n-1}\omega_{n-1-i}N^i\right)$$

and

$$U^{(-n)}T^k\varphi_l(x) = \sum_{\omega\in\Sigma_A^n:\omega_0=l}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([l])}}\varphi_j\left(x-\sum_{i=0}^{n-1}\omega_{n-1-i}N^i-N^nk\right).$$

Consequently, $T^{N^nk}U^{(-n)}\varphi_j = U^{(-n)}T^k\varphi_j$ for all $k \in \mathbb{Z}$, $n \in \mathbb{N}$, $j \in \underline{N}$.

ad (4): Let $n \in \mathbb{N}$ and $k = \sum_{i=0}^{n-1}\omega_{n-1-i}N^i + N^nk_1$, $\omega \in \Sigma_A^n$, $k_1 \in \mathbb{Z}$, and $l = \sum_{i=0}^{n-1}\tilde{\omega}_{n-1-i}N^i + N^nl_1$, $\tilde{\omega} \in \Sigma_A^n$, $l_1 \in \mathbb{Z}$ and $A_{\omega_{n-1}i} = 1$, $A_{\tilde{\omega}_{n-1}j} = 1$ for $i, j \in \underline{N}$ then

$$\langle U^{(n)}T^k\varphi_i|U^{(n)}T^l\varphi_j\rangle = \left\langle (\nu_{\mathbb{Z}}([\omega i]))^{-1/2}\,T^{k_1}\mathbb{1}_{[\omega i]}\,\middle|\,(\nu_{\mathbb{Z}}([\tilde{\omega}j]))^{-1/2}\,T^{l_1}\mathbb{1}_{[\tilde{\omega}j]}\right\rangle = \delta_{k_1,l_1}\delta_{(\omega,i),(\tilde{\omega},j)}.$$

Otherwise, we have $U^{(n)}T^k\varphi_i = 0$ or $U^{(n)}T^l\varphi_j = 0$.

Furthermore for $n \in \mathbb{N}$, $k, j \in \mathbb{Z}$, $i, m \in \underline{N}$, we have

$$\begin{aligned}
&\langle U^{(-n)}T^k\varphi_i|U^{(-n)}T^l\varphi_m\rangle\\
=&\left\langle\sum_{\omega\in\Sigma_A^n:\omega_0=i}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([i])}}T^{\sum_{i=0}^{n-1}\omega_{n-1-i}N^i+N^nk}\varphi_j\,\middle|\,\sum_{\omega\in\Sigma_A^n:\omega_0=m}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([m])}}T^{\sum_{i=0}^{n-1}\omega_{n-1-i}N^i+N^nl}\varphi_j\right\rangle\\
=&\delta_{k,l}\cdot\delta_{i,m}\cdot\sum_{\omega\in\Sigma_A^n:\omega_0=i}\sum_{j\in\underline{N}}\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([i])}\\
=&\delta_{(k,i),(l,m)},
\end{aligned}$$

where we used in the second equality that $\langle T^k\varphi_j|T^l\varphi_i\rangle = \delta_{(k,j),(l,i)}$.

ad (5): Let $n \in \mathbb{N}$, $f \in L^2(\nu_{\mathbb{Z}})$, $x \in \mathbb{R}$, then

$$\begin{aligned}
&U^{(n)}U^{(-n)}f(x)\\
&=\sum_{l\in\mathbb{Z}}\sum_{\widetilde{\omega}\in\Sigma_A^n}\sum_{r\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([r])}{\nu_{\mathbb{Z}}([\widetilde{\omega}r])}}\mathbb{1}_{[\widetilde{\omega}r]}(x-l)\sum_{k\in\mathbb{Z}}\sum_{\omega\in\Sigma_A^n}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([j])}}\\
&\quad\mathbb{1}_{[j]}\left(\tau_{\widetilde{\omega}}^{-1}(x-l)+\sum_{i=0}^{n-1}\widetilde{\omega}_{n-1-i}N^i+N^n l-\sum_{i=0}^{n-1}\omega_{n-1-i}N^i-N^n k\right)\\
&\quad f\left(\tau_\omega\left(\tau_{\widetilde{\omega}}^{-1}(x-l)+\sum_{i=0}^{n-1}\widetilde{\omega}_{n-1-i}N^i+N^n l-\sum_{i=0}^{n-1}\omega_{n-1-i}N^i-N^n k\right)+k\right)\\
&=\sum_{k\in\mathbb{Z}}\sum_{\omega\in\Sigma_A^n}\sum_{j\in\underline{N}}\mathbb{1}_{[\omega j]}(x-k)\cdot f(x)\\
&=f(x),
\end{aligned}$$

where we used in the third equality that $i=r$, $\omega=\widetilde{\omega}$ and $k=l$ since otherwise it is zero.

ad (6): For $n\in\mathbb{N}$, $k\in\mathbb{Z}$, $j\in\underline{N}$, with $U^{(n)}T^k\varphi_j\neq 0$, there is $\omega\in\Sigma_A^n$, $l\in\mathbb{Z}$, with $k=\sum_{i=0}^{n-1}\omega_{n-1-i}N^i+N^n l$ and so

$$\begin{aligned}
U^{(-n)}U^{(n)}T^k\varphi_j &= U^{(-n)}\left((\nu_{\mathbb{Z}}([\omega j]))^{-1/2}\,T^l\mathbb{1}_{[\omega j]}\right)=T^{N^n l}U^{(-n)}\left((\nu_{\mathbb{Z}}([\omega j]))^{-1/2}\,\mathbb{1}_{[\omega j]}\right)\\
&=T^{N^n l}T^{\sum_{i=0}^{n-1}\omega_{n-1-i}N^i}\,(\nu_{\mathbb{Z}}([j]))^{-1/2}\,\mathbb{1}_{[j]}=T^k\varphi_j.
\end{aligned}$$

□

Remark 9.10. We further notice that for $n\in\mathbb{N}$, $f\in L^2(\nu_{\mathbb{Z}})$, $x\in\mathbb{R}$, we have

$$U^{(-n)}U^{(n)}f(x)=\sum_{k\in\mathbb{Z}}\sum_{\omega j\in\Sigma_A^{n+1}}\mathbb{1}_{[j]}\left(x-\sum_{i=0}^{n-1}\omega_{n-1-i}N^i-N^n k\right)\cdot f(x),$$

and consequently, in general we do not have $U^{(-n)}U^{(n)}=I$.

Theorem 9.11. *Let $\left(U^{(n)}\right)_{n\in\mathbb{N}_0}$ be given as in (1.2.3). Then $\left(\nu_{\mathbb{Z}},\left(U^{(n)}\right)_{n\in\mathbb{N}_0},T\right)$ allows a one-sided MRA with respect to the family of father wavelets $\varphi_i:=(\nu_{\mathbb{Z}}([i]))^{-1/2}\,\mathbb{1}_{[i]}$, $i\in\underline{N}$.*

Proof. We verify that the properties (2a) to (2e) of Definition 1.2 are satisfied with the father wavelets $\varphi_i=(\nu([i])^{-1/2}\,\mathbb{1}_{[i]}$, $i\in\underline{N}$. We define the closed subspaces of $L^2(\nu_{\mathbb{Z}})$ for $j\in\mathbb{N}$ as

$$V_j:=\operatorname{cl}\operatorname{span}\left\{U^{(j)}T^k\varphi_i:\,k\in\mathbb{Z},\,i\in\underline{N}\right\}.$$

ad (2c): By the definition of V_j we obviously have that $\left\{U^{(j)}T^k\varphi_i:k\in\mathbb{Z},i\in\underline{N}\right\}$ spans V_j, $j\in\mathbb{Z}$. The orthonormality follows from Proposition 9.9 (4).

ad (2d): We notice that if $k\in\mathbb{Z}$ takes the form $k=\sum_{i=0}^{n-1}\omega_{n-1-i}N^i+N^n l$ for some $\omega\in\Sigma_A^n$, $l\in\mathbb{Z}$ and $j\in\underline{N}$, we have

$$U^{(n)}T^k\varphi_j=\sum_{i\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega ji])}{\nu_{\mathbb{Z}}([\omega j])}}U^{(n+1)}T^{Nk+j}\varphi_i.$$

If there is no $\omega\in\Sigma_A^n$, $l\in\mathbb{Z}$, such that $k=\sum_{i=0}^{n-1}\omega_{n-1-i}N^i+N^n l$, then $U^{(n)}T^k\varphi_j=0$.

ad (2a): Notice that for $n\in\mathbb{N}$, $k\in\mathbb{Z}$ and $i\in\underline{N}$, we obtain with Proposition 9.9 (2) and (3) that

$$U^{(n)}T^k\varphi_i=U^{(n)}T^kU^{(1)}T^i\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([i])}{\nu_{\mathbb{Z}}([ij])}}\varphi_j=U^{(n)}U^{(1)}T^{Nk+i}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([i])}{\nu_{\mathbb{Z}}([ij])}}\varphi_j.$$

By Remark 9.7 (1) the inclusion $V_n\subset V_{n+1}$ follows.

ad (2b): First we notice that X is either totally disconnected or we can consider X as an interval in $[0,1]$. Furthermore, every characteristic function on a cylinder $[\omega]\subset\Sigma_A$ can be obtained by $U^{(n)}T^k\varphi_j$, $n\in\mathbb{N}_0$, $k\in\mathbb{Z}$, $j\in\underline{N}$. Thus, we are left to show that $\left\{T^k\mathbb{1}_{[\omega]}:\,k\in\mathbb{Z},\,\omega\in\Sigma_A^*\right\}$ is dense in $L^2(\nu_{\mathbb{Z}})$.

If X is totally disconnected, it follows by the Stone-Weierstrass Theorem that

$$\left\{T^k\mathbb{1}_{[\omega]}:\,k\in\mathbb{Z},\,\omega\in\Sigma_A^*\right\}$$

is dense in $\mathcal{C}(R,\mathbb{C})$, where R is the enlarged fractal, see e.g. **[KSS07]**. Besides it is well known that $\mathcal{C}(R,\mathbb{C})$ is dense in $L^2(\nu_{\mathbb{Z}})$ and so $\operatorname{cl}\operatorname{span}\left\{T^k\mathbb{1}_{[\omega]}:\,k\in\mathbb{Z},\,\omega\in\Sigma_A^*\right\}=L^2(\nu_{\mathbb{Z}})$.

If $X=[a,b]$, notice that every interval $J\subset[0,1]$ can be approximated by $\tau_\omega(X)$, $\omega\in\Sigma_A^*$. Hence $\tau_\omega(X)$, $\omega\in\Sigma_A^*$, generates the Borel σ-algebra $\mathcal{B}$ in $\mathbb{R}$, thus every element $A\in\mathcal{B}$ can be approximated by elements of $\{\tau_\omega(X):\omega\in\Sigma_A^*\}$. Consequently, every elementary function can be approximated by functions $\mathbb{1}_{\tau_\omega(X)}$ and so all functions in $L^2(\nu_{\mathbb{Z}})$ can be approximated by elements of

$$\left\{T^k\mathbb{1}_{[\omega]}:\,\omega\in\Sigma_A^*,k\in\mathbb{Z}\right\}=\left\{U^{(n)}T^l\varphi_i:\,n\in\mathbb{N}_0,\,l\in\mathbb{Z},\,i\in\underline{N}\right\}.$$

Consequently, $\operatorname{cl}\bigcup_{n\in\mathbb{N}_0}V_n=L^2(\nu_{\mathbb{Z}})$ and so $\operatorname{cl}\bigcup_{n\in\mathbb{Z}}V_n=L^2(\nu_{\mathbb{Z}})$.

ad (2e): This follows from Proposition 9.9 (1) and (2). □

Next we give the connection between a two-sided MRA and a Markov measure. The if direction of the following theorem will be shown in Section 9.3.

Theorem 9.12. $\left(\nu_{\mathbb{Z}},\left(U^{(n)}\right)_{n\in\mathbb{Z}},T\right)$ *allows a two-sided MRA with respect to the family of father wavelets* $\varphi_i:=(\nu_{\mathbb{Z}}([i]))^{-1/2}\,\mathbb{1}_{[i]}$, $i\in\underline{N}$, *if and only if the measure* ν *is Markovian.*

Proof of "$\Longrightarrow$". We assume that $\left(\nu_{\mathbb{Z}},\left(U^{(n)}\right)_{n\in\mathbb{Z}},T\right)$ allows a two-sided MRA with the father wavelets $\varphi_i=(\nu_{\mathbb{Z}}([i])^{-1/2}\,\mathbb{1}_{[i]}$, $i\in\underline{N}$. Then in particular, it holds by (2d) of Definition 1.2 that for $n\in\mathbb{N}$

$$U^{(-n)}\left\{\varphi_i:i\in\underline{N}\right\}\subset\operatorname{span}U^{(-n+1)}\left\{T^k\varphi_i:i\in\underline{N},k\in\underline{N}\right\}.$$

We further notice that for $n\in\mathbb{N}$, $k,i\in\underline{N}$,

$$U^{(-n)}\varphi_k=\sum_{\omega\in\Sigma_A^n:\omega_0=k}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([k])}}T^{\sum_{l=0}^{n-1}\omega_{n-1-l}N^l}\varphi_j$$

and

$$U^{(-n+1)}T^k\varphi_i=\sum_{\omega\in\Sigma_A^{n-1}:\omega_0=i}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([i])}}T^{\sum_{l=0}^{n-2}\omega_{n-2-l}N^l+N^{n-1}k}\varphi_j.$$

From the explicit formula of $U^{(-n)}\varphi_k$ and $U^{(-n+1)}T^m\varphi_i$, $n\in\mathbb{N}$, $k,m,i\in\underline{N}$, we have that $\langle U^{(-n)}\varphi_k|U^{(-n+1)}T^m\varphi_i\rangle\neq0$ only if $m=k$ since

$$\begin{aligned}&\langle U^{(-n)}\varphi_k|U^{(-n+1)}T^m\varphi_i\rangle\\&=\left\langle\sum_{\omega\in\Sigma_A^n:\omega_0=k}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([k])}}T^{\sum_{l=0}^{n-1}\omega_{n-1-l}N^l}\varphi_j\,\middle|\,\sum_{\omega\in\Sigma_A^{n-1}:\omega_0=i}\sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([i])}}T^{\sum_{l=0}^{n-2}\omega_{n-2-l}N^l+N^{n-1}m}\varphi_j\right\rangle\\&=\sum_{\omega\in\Sigma_A^n:\omega_0=k}\sum_{j_1\in\underline{N}}\sum_{\widetilde{\omega}\in\Sigma_A^{n-1}:\widetilde{\omega}_0=i}\sum_{j_2\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j_1])}{\nu_{\mathbb{Z}}([k])}\frac{\nu_{\mathbb{Z}}([\widetilde{\omega}j_2])}{\nu_{\mathbb{Z}}([i])}}\left\langle T^{\sum_{l=0}^{n-1}\omega_{n-1-l}N^l}\varphi_{j_1}\,\middle|\,T^{\sum_{l=0}^{n-2}\widetilde{\omega}_{n-2-l}N^l+N^{n-1}m}\varphi_{j_2}\right\rangle\\&=\delta_{m,k}\cdot\sum_{j\in\underline{N}}\sum_{\omega\in\Sigma_A^{n-1}:\omega_0=i}\sqrt{\frac{\nu_{\mathbb{Z}}([k\omega j])}{\nu_{\mathbb{Z}}([k])}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([i])}},\end{aligned}$$

where we used in the third equality the property of Proposition 9.9 (4), namely $\langle T^k\varphi_{j_1}|T^l\varphi_{j_2}\rangle=\delta_{(k,j_1),(l,j_2)}$ for any $k,l\in\mathbb{Z}$ and $j_1,j_2\in\underline{N}$.

As a consequence of (2c) and (2d) of Definition 1.2 and the observation above it follows that for every $n \in \mathbb{N}$, $k \in \underline{N}$, there exist unique $\left(\alpha_i^{n,k}\right)_{i\in\underline{N}} \in \mathbb{C}^N$ such that

$$U^{(-n)}\varphi_k = \sum_{i\in\underline{N}} \alpha_i^{n,k} U^{(-n+1)} T^k \varphi_i = \sum_{i\in\underline{N}} \alpha_i^{n,k} \sum_{\omega\in\Sigma_A^{n-1}:\omega_0=i} \sum_{j\in\underline{N}} \sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([i])}} T^{\sum_{l=0}^{n-2}\omega_{n-2-l}N^l + N^{n-1}k}\varphi_j.$$

On the other hand, from the precise form of $U^{(-n)}\varphi_k$ it follows that

$$\begin{aligned} U^{(-n)}\varphi_k &= \sum_{\omega\in\Sigma_A^n:\omega_0=k} \sum_{j\in\underline{N}} \sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([k])}} T^{\sum_{l=0}^{n-1}\omega_{n-1-l}N^l}\varphi_j \\ &= \sum_{\omega\in\Sigma_A^{n-1}} \sum_{j\in\underline{N}} \sqrt{\frac{\nu_{\mathbb{Z}}([k\omega j])}{\nu_{\mathbb{Z}}([k])}} T^{\sum_{l=0}^{n-2}\omega_{n-2-l}N^l + N^{n-1}k}\varphi_j \\ &= \sum_{i\in\underline{N}} \sum_{\omega\in\Sigma_A^{n-1}:\omega_0=i} \sum_{j\in\underline{N}} \sqrt{\frac{\nu_{\mathbb{Z}}([k\omega j])}{\nu_{\mathbb{Z}}([k])}} T^{\sum_{l=0}^{n-2}\omega_{n-2-l}N^l + N^{n-1}k}\varphi_j. \end{aligned}$$

By comparing the coefficients it follows that for every $\omega \in \Sigma_A^{n-1}$, $\omega_0 = i$, we have $\alpha_i^{n,k}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([i])}} = \sqrt{\frac{\nu_{\mathbb{Z}}([k\omega j])}{\nu_{\mathbb{Z}}([k])}}$. Consequently, $\alpha_i^{n,k} \in \mathbb{R}^+$ and

$$\nu_{\mathbb{Z}}([k\omega j]) = \nu_{\mathbb{Z}}([\omega j])\left(\alpha_i^{n,k}\right)^2 \frac{\nu_{\mathbb{Z}}([k])}{\nu_{\mathbb{Z}}([i])}.$$

Now it remains to be shown that $\alpha_i^{n,k}$ are independent of $n \in \mathbb{N}$. For $n \in \mathbb{N}$, $\omega \in \Sigma_A^n$ with $\omega_0 = i$, and $k \in \underline{N}$ it follows that

$$\nu_{\mathbb{Z}}([k\omega]) = \sum_{j\in\underline{N}} \nu_{\mathbb{Z}}([k\omega j]) = \sum_{j\in\underline{N}} \nu_{\mathbb{Z}}([\omega j]) \frac{\left(\alpha_i^{n,k}\right)^2 \nu_{\mathbb{Z}}([k])}{\nu_{\mathbb{Z}}([i])} = \nu_{\mathbb{Z}}([\omega]) \frac{\left(\alpha_i^{n,k}\right)^2 \nu_{\mathbb{Z}}([k])}{\nu_{\mathbb{Z}}([i])}.$$

On the other hand we can write $\omega \in \Sigma_A^n$ with $\omega_0 = i$ as $\omega = \widetilde{\omega}\omega_{n-1}$ for a suitable $\widetilde{\omega} \in \Sigma_A^{n-1}$, $\widetilde{\omega}_0 = i$, and so

$$\nu_{\mathbb{Z}}([k\omega]) = \nu_{\mathbb{Z}}([k\widetilde{\omega}\omega_{n-1}]) = \nu_{\mathbb{Z}}([\widetilde{\omega}\omega_{n-1}]) \frac{\left(\alpha_i^{n-1,k}\right)^2 \nu_{\mathbb{Z}}([k])}{\nu_{\mathbb{Z}}([i])} = \nu_{\mathbb{Z}}([\omega]) \frac{\left(\alpha_i^{n-1,k}\right)^2 \nu_{\mathbb{Z}}([k])}{\nu_{\mathbb{Z}}([i])}.$$

Thus, for $k, i \in \underline{N}$ we have $\alpha_i^{n-1,k} = \alpha_i^{n,k}$ and so iteratively $\alpha_i^{n,k} = \alpha_i^{m,k}$ for all $n, m \in \mathbb{N}$. In the following we write α_i^k for $\alpha_i^{n,k}$.

So we have $\nu_{\mathbb{Z}}([k\omega j]) = \left(\alpha_{\omega_0}^k\right)^2 \nu_{\mathbb{Z}}([k])/\nu_{\mathbb{Z}}([\omega_0]) \cdot \nu_{\mathbb{Z}}([\omega j])$ for all $\omega \in \Sigma_A^*$, $j, k \in \underline{N}$. From this property we conclude the Markov relation since for any $k, i \in \underline{N}$

$$\nu([ki]) = \sum_{j\in\underline{N}} \nu([kij]) = \sum_{j\in\underline{N}} \frac{\left(\alpha_i^k\right)^2 \nu_{\mathbb{Z}}([k])}{\nu_{\mathbb{Z}}([i])} \nu([ij]) = \left(\alpha_i^k\right)^2 \nu_{\mathbb{Z}}([k]).$$

Define $\pi_{ki} := \left(\alpha_i^k\right)^2$, then π_{ki} is an incidence probability. Therefore the measure ν is Markovian as desired.

The reversed implication will be shown in Section 9.3. $\square$

9.2.1. Mother wavelets for MIMs. In this section we are in the case of Remark 8.10 (2) and so we consider for each father wavelet φ_i, $i \in \underline{N}$, a matrix of coefficients; more precisely on each scale we have to consider for each element of the alphabet $\underline{N}$ a matrix of coefficients. We slightly change the notation from $c_j^{n,k,l}$ to $c_j^{\omega,l}$ for $\omega \in \Sigma_A^n$, since the information about n and k is encoded in ω (n is given by the length of the word ω and $k = \sum_{i=0}^{n-1} \omega_{n-1-i} N^i$).

For $\omega \in \Sigma_A^{n+1}$ we need a matrix of size $q^{\omega_n} \times q^{\omega_n}$, where $q^{\omega_n} = \operatorname{card}\{j \in \underline{N} : A_{\omega_n j} = 1\}$. First we determine $c_j^{\omega,k} \in \mathbb{C}$, $j \in \underline{N}$, $k \in \underline{q^{\omega_n}} \backslash \{0\}$, such that the $(q^{\omega_n} \times q^{\omega_n})$-matrix

$$M_\omega := \begin{pmatrix} \left(\sqrt{\nu_{\mathbb{Z}}([\omega j])}\right)_{j \in D_{\omega_n}} \\ \left(A_{\omega_n j} c_j^{\omega,k}\right)_{k \in \underline{q^{\omega_n}} \backslash \{0\}, j \in D_{\omega_n}} \end{pmatrix},$$

where $D_{\omega_n} = \{j \in \underline{N} : A_{\omega_n j} = 1\}$ is unitary. This is done as explained above.

We define for $\omega \in \Sigma_A^{n+1}$, $k = \sum_{i=0}^{n} \omega_{n-i} N^i$ the basis functions by declaring for $l \in \underline{q^{\omega_n}} \backslash \{0\}$ that

$$\psi^{\omega,l} = U^{(n)} T^k \sum_{j \in \underline{N}} A_{\omega_n j} c_j^{\omega,l} \varphi_j.$$

These functions can be written differently for $\omega \in \Sigma_A^{n+1}$, $k \in \underline{q^{\omega_n}} \backslash \{0\}$, as

$$\psi^{\omega,k} = \sum_{j \in \underline{N}} A_{\omega_n j} c_j^{\omega,k} \cdot \left(\nu_{\mathbb{Z}}([\omega j])\right)^{-1/2} \cdot \mathbb{1}_{[\omega j]}.$$

Theorem 8.1 and Theorem 9.11 imply the following.

Corollary 9.13. *The set*

$$\left\{T^l \psi^{\omega,k} : l \in \mathbb{Z}, \omega \in \Sigma_A^*, k \in \{1, \ldots, q^{\omega_{|\omega|-1}} - 1\}\right\} \cup \left\{T^l \varphi_j : l \in \mathbb{Z}, j \in \underline{N}\right\}$$

is an orthonormal basis for $L^2(\nu_{\mathbb{Z}})$.

Remark 9.14. In fact, the proofs of Theorem 8.1 and Theorem 9.11 show that we have for $n \in \mathbb{N}$

$$\operatorname{cl\,span}\left\{T^l \psi^{\omega,k} : l \in \mathbb{Z}, \omega \in \Sigma_A^n, k \in \{1, \ldots, q^{\omega_{n-1}} - 1\}\right\} = V_n \ominus V_{n-1}.$$

9.3. MRA for Markov measures

In this section we construct a wavelet basis on the limit set translated by $\mathbb{Z}$ under the hypothesis that the underlying measure ν is Markovian. For this fix a probability vector $p = (p_0, p_1, \ldots, p_{N-1})$ and a $(N \times N)$ stochastic matrix $\Pi = (\pi_{jk})_{j,k \in \underline{N}}$ such that for $\omega \in \Sigma_A^n$ we have

$$\nu([\omega]) = p_{\omega_0} \prod_{i=0}^{n-2} \pi_{\omega_i \omega_{i+1}}.$$

Furthermore, we have that $\pi_{jk} = 0$ if $A_{jk} = 0$.

The construction is a special case of the one in the last section. Therefore, we omit some proofs here and mainly state the results, so that the special features of this case become clear.

Recall the definition of the scaling operator acting on $L^2(\nu_{\mathbb{Z}})$ given in (1.2.1):

$$Uf(x) := \sum_{k \in \mathbb{Z}} \sum_{j \in \underline{N}} \sum_{i \in \underline{N}} \sqrt{\frac{p_i}{p_j \pi_{ji}}} \cdot \mathbb{1}_{[ji]}(x-k) \cdot f(\tau_j^{-1}(x-k) + j + Nk).$$

and the translation operator is as defined in (1.2.2).

In this construction we only have to define one operator U since we obtain $U^{(n)} = U^n$ for all $n \in \mathbb{N}_0$. Another main difference is that we do not need one matrix for every $\omega \in \Sigma_A^*$ to obtain the mother wavelets, but we only need matrices for $\omega \in \Sigma_A^1 = \underline{N}$. So we do not need more than N^2 matrices. This follows from Lemma 8.5.

The setting is as defined in Section 9.1. Set $U := U^{(1)}$and so it takes the form in (1.2.1). By the Markov property we have $\frac{\nu_{\mathbb{Z}}([i])}{\nu_{\mathbb{Z}}([ji])} = \frac{p_i}{p_j \pi_{ji}}$ and hence one easily verifies that $U^{(n)} = U^n$. Also notice that U is not unitary unless we have that $A_{ij} = 1$ for all $i, j \in \underline{N}$.

Now we turn to the form of U^*.

Lemma 9.15. *U^* has the form for $f \in L^2(\nu_{\mathbb{Z}})$, $x \in \mathbb{R}$,*

$$U^* f(x) = \sum_{k\in\mathbb{Z}} \sum_{j\in\underline{N}} \sum_{i\in\underline{N}} \sqrt{\frac{p_j \pi_{ji}}{p_i}} \cdot \mathbb{1}_{[i]}(x - j - Nk) \cdot f(\tau_j(x - j - Nk) + k). \tag{9.3.1}$$

Remark 9.16. Notice that $U^* = U^{(-1)}$ and $(U^*)^n = U^{(-n)}$.

Proof. To prove that U^* has the form above we use the $\mathbb{Z}$-translation invariance of the measure $\nu_{\mathbb{Z}}$ and the fact that $\frac{d(\nu_{\mathbb{Z}} \circ \tau_j)}{d\nu_{\mathbb{Z}}} = \frac{p_j \pi_{ji}}{p_i}$ on $[i]$. We obtain this Radon-Nikodym derivative since for a cylinder set $[\omega]$, $\omega \in \Sigma_A^n$, $n \in \mathbb{N}$, we have

$$\nu_{\mathbb{Z}}\left(\tau_j([\omega])\right) = p_j \pi_{j\omega_0} \prod_{i=0}^{n} \pi_{\omega_i \omega_{i+1}}$$

and $\nu_{\mathbb{Z}}([\omega]) = p_{\omega_0} \prod_{i=0}^{n} \pi_{\omega_i \omega_{i+1}}$.

Consequently, we obtain for $f, g \in L^2(\nu_{\mathbb{Z}})$ that

$$\begin{aligned}
&\langle Uf|g\rangle \\
&= \int \sum_{k\in\mathbb{Z}} \sum_{j\in\underline{N}} \sum_{i\in\underline{N}} \sqrt{\frac{p_i}{p_j \pi_{ji}}} \cdot \mathbb{1}_{[ji]}(x-k) \cdot f(\tau_j^{-1}(x-k) + j + Nk) \cdot \overline{g(x)} d\nu_{\mathbb{Z}}(x) \\
&= \int \sum_{k\in\mathbb{Z}} \sum_{j\in\underline{N}} \sum_{i\in\underline{N}} \sqrt{\frac{p_i}{p_j \pi_{ji}}} \cdot \mathbb{1}_{[ji]}(x) \cdot f(\tau_j^{-1}(x) + j + Nk) \cdot \overline{g(x+k)} d\nu_{\mathbb{Z}}(x) \\
&= \int \sum_{k\in\mathbb{Z}} \sum_{j\in\underline{N}} \sum_{i\in\underline{N}} \sqrt{\frac{p_i}{p_j \pi_{ji}}} \cdot \mathbb{1}_{[ji]}(\tau_j(x)) \cdot f(x + j + Nk) \cdot \overline{g(\tau_j(x)+k)} d\nu_{\mathbb{Z}}(\tau_j(x)) \\
&= \int \sum_{k\in\mathbb{Z}} \sum_{j\in\underline{N}} \sum_{i\in\underline{N}} \sqrt{\frac{p_i}{p_j \pi_{ji}}} \cdot \mathbb{1}_{[ji]}(\tau_j(x)) \cdot f(x + j + Nk) \cdot \overline{g(\tau_j(x)+k)} \cdot \frac{p_j \pi_{ji}}{p_i} \cdot d\nu_{\mathbb{Z}}(x) \\
&= \int f(x) \sum_{k\in\mathbb{Z}} \sum_{j\in\underline{N}} \sum_{i\in\underline{N}} \sqrt{\frac{p_j \pi_{ji}}{p_i}} \cdot \mathbb{1}_{[i]}(x - j - Nk) \cdot \overline{g(\tau_j(x - j - Nk) + k)} d\nu_{\mathbb{Z}}(x) \\
&= \langle f|U^* g\rangle,
\end{aligned}$$

with $U^* g$ as in (9.3.1). ☐

Now we turn to the definition of the father wavelets which we will use in the MRA. Define the N father wavelets as $\varphi_i := (\nu_{\mathbb{Z}}([i]))^{-1/2}\, \mathbb{1}_{[i]}$ for $i \in \underline{N}$.

Remark 9.17. Notice that the family of father wavelets $(\varphi_i)_{i\in\underline{N}}$ is orthonormal by definition.

Now we turn to the properties of the operators U and T given in the next proposition.

Proposition 9.18. *Let $(\varphi_i)_{i\in\underline{N}}$ denote the family of father wavelets given by $\varphi_i = \sqrt{\nu([i])}^{-1} \mathbb{1}_{[i]}$, $i \in \underline{N}$. The translation operator T and the scaling operator U satisfy the following properties.*

(1) $TU = UT^N$,
(2) $\varphi_i = U \sum_{j\in\underline{N}} \sqrt{\pi_{ij}} T^i \varphi_j$, $i \in \underline{N}$,
(3) $\langle T^k \varphi_i | T^l \varphi_j\rangle = \delta_{(k,i),(l,j)}$, $k, l \in \mathbb{Z}$, $i, j \in \underline{N}$,
(4) $UU^* = I$,
(5) $U^*U = I$ *if and only if* $A_{ij} = 1$ *for all* $i, j \in \underline{N}$.

Proof. Each of (1), (2), (3) and (4) follows directly from Proposition 9.9.

ad (5): This is analogous to the proof of (4) or Proposition 9.9 (5). We obtain for $f \in L^2(\nu_{\mathbb{Z}})$, $x \in \mathbb{R}$,

$$U^*Uf(x) = \sum_{k\in\mathbb{Z}} \sum_{j\in\underline{N}} A_{ji} \mathbb{1}_{[i]}(x - j - Nk) \cdot f(x)$$

and $\sum_{k\in\mathbb{Z}} \sum_{j\in\underline{N}} \sum_{i\in\underline{N}} A_{ji}\mathbb{1}_{[i]}(x - j - Nk) = 1$ for all $x \in \mathbb{R}$ if and only if $A_{ji} = 1$ for all $i, j \in \underline{N}$. □

Now we turn to the proof of the backward direction of Theorem 9.12. So we show that $(\nu_{\mathbb{Z}}, U, T)$ allows a two-sided MRA. Some of the properties follow directly from the proof of Theorem 9.11.

Proof of Theorem 9.12 ”$\Longleftarrow$”. We establish the properties (1a) to (1f) of Definition 1.2. The property (1b) follows from Theorem 9.11.

ad (1e): For $n \in \mathbb{N}_0$ it follows directly by Theorem 9.11. For $n \in \mathbb{Z}$, $n < 0$, $x \in \mathbb{R}$, $k \in \underline{N}$, it follows by

$$\begin{aligned}
&(U^*)^{|n|}\,\varphi_k(x)\\
&= \sum_{\omega\in\Sigma_A^{|n|}:\omega_0=k} \sum_{i\in\underline{N}} \sqrt{\prod_{l=1}^{|n|-2} \pi_{\omega_l\omega_{l+1}} \cdot \pi_{k\omega_1}\pi_{\omega_{|n|-1}i}} \cdot \varphi_i\left(x - \sum_{l=0}^{|n|-1} \omega_{|n|-1-l}N^l\right)\\
&= \sum_{j\in\underline{N}} \sqrt{\pi_{kj}} \left(\sum_{\omega\in\Sigma_A^{|n|-1}:\omega_0=j} \sum_{i\in\underline{N}} \sqrt{\prod_{l=1}^{|n|-3} \pi_{\omega_l\omega_{l+1}} \cdot \pi_{j\omega_1}\pi_{\omega_{|n|-2}i}} \cdot \varphi_i\left(x - \sum_{l=0}^{|n|-2} \omega_{|n|-2-l}N^l - kN^{|n|-1}\right)\right)\\
&= \sum_{j\in\underline{N}} \sqrt{\pi_{kj}}\,(U^*)^{|n|-1}\,T^k\varphi_j(x).
\end{aligned}$$

ad (1a): For $n \in \mathbb{N}_0$ it follows directly from Theorem 9.11. For $n \in \mathbb{Z}$, $n < 0$, $k \in \underline{N}$, it follows from the calculation

$$(U^*)^{|n|}\,\varphi_k = \sum_{j\in\underline{N}} \sqrt{\pi_{kj}}\,(U^*)^{|n|-1}\,T^k\varphi_j.$$

ad (1c): We have that $\bigcap_{n\in\mathbb{Z}} V_n = \{0\}$, because the support of $(U^*)^n\,\varphi_j$, $j \in \underline{N}$, increases with $n \in \mathbb{N}$. More precisely, for $j \in \underline{N}$

$$\nu_{\mathbb{Z}}\left(\mathrm{supp}\left((U^*)^n\,\varphi_j\right)\right) = \sum_{i\in\underline{N}} \nu_{\mathbb{Z}}\left([i]\right)\left(\mathrm{card}\left\{\omega \in \Sigma_A^{n+1} : \omega_0 = j, \omega_n = i\right\}\right).$$

Consequently, $\{0\} = \bigcap_{j\in\mathbb{Z}} V_j$ since any function $f \in \bigcap_{j\in\mathbb{Z}} V_j$ must be constant on $\mathrm{supp}\left((U^*)^n\,\varphi_j\right)$ for every $n \in \mathbb{N}$ and for some $j \in \underline{N}$.

ad (1d): This property follows directly from the definition of the spaces V_j and Proposition 9.9 (4) with the observation that $U^{(n)} = U^n$ and $U^{(-n)} = (U^*)^n$, $n \in \mathbb{N}_0$.

ad (1f): This property follows from Proposition 9.18 (4) and (5). □

Remark 9.19. Now we give some remarks concerning the father wavelets.

(1) The relation for the functions φ_i, $i \in \underline{N}$, can also be written as

$$(\varphi_j)^t_{j\in\underline{N}} = \sum_{l\in\underline{N}} M_l\left(UT^l\varphi_j\right)^t_{j\in\underline{N}},$$

where the M_l are $(N\times N)$-matrices with $(M_l)_{n,k} = \begin{cases}\sqrt{\pi_{lk}}, & n = l,\\ 0, & \text{otherwise,}\end{cases}$ for $n, k \in \underline{N}$.

(2) Notice that for $k \in \mathbb{Z}$ we can write $k = a_0 + Nl$, where $a_0 \in \underline{N}$ and some $l \in \mathbb{Z}$, i.e. k is in the N-adic expansion. Then we obtain

$$UT^k\varphi_j \quad = \quad \begin{cases} 0, & \text{if } A_{a_0 j} = 0, \\ \left(p_{a_0} \cdot \pi_{a_0 j}\right)^{-1/2} T^l \mathbb{1}_{[a_0 j]}, & \text{otherwise.} \end{cases}$$

(3) Notice that in $\left\{U^n T^k \varphi_i : n \in \mathbb{N},\, k \in \mathbb{Z},\, i \in \underline{N}\right\}$ some functions are constantly zero. These functions are precisely those where for $k \in \mathbb{Z}$ written in the N-adic expansion, $k = \sum_{j=0}^{n-1} k_{n-1-j} N^i + lN^n$, $k_j \in \underline{N}$, $l \in \mathbb{Z}$, either $A_{k_j k_{j+1}} = 0$ for some $j \in \{0, \ldots, n-2\}$ or $A_{k_{n-1} i} = 0$.

(4) We would like to point out some interesting connections to C^*-algebras of Cuntz-Krieger type, [**KSS07**]. We start by further considering the scaling operator U for the MRA in the setting of an MIM with the incidence matrix A and Markov measure ν. We can also write the operator U in a different way using the representation of a Cuntz-Krieger algebra. For this we consider the partial isometries S_i given for $i \in \underline{N}$, $f \in L^2(\nu)$, $x \in \operatorname{supp}(\nu)$ by

$$S_i f(x) = \left(\nu([i])\right)^{-1/2} \mathbb{1}_{[i]}(x) f(\tau_i^{-1}(x)).$$

It has been shown in [**KSS07**] that this gives a representation of the Cuntz-Krieger algebra $\mathcal{O}_A$ by bounded operators acting on $L^2(\nu)$, that is the S_i, $i \in \underline{N}$, are partial isometries and satisfy

$$S_i^* S_i = \sum_{j \in \underline{N}} A_{ij} S_j S_j^*,$$
$$I = \sum_{i \in \underline{N}} S_i S_i^*.$$

The scaling operator U acting on $L^2\left(\nu_{\mathbb{Z}}\right)$ can then alternatively be written in terms of the partial isometries as

$$U = \sum_{k \in \mathbb{Z}} \sum_{j \in \underline{N}} \sum_{i \in \underline{N}} \sqrt{\frac{p_i}{\pi_{ji}}} T^k S_j \mathbb{1}_{[i]} T^{-(j+Nk)},$$

where we notice that $S_j \mathbb{1}_{[i]}$, $j, i \in \underline{N}$, acts on $L^2\left(\nu_{\mathbb{Z}}\right)$. We can also write U^* in terms of the partial isometries S_i, $i \in \underline{N}$. In this way we obtain

$$U^* = \sum_{k \in \mathbb{Z}} \sum_{j \in \underline{N}} \sum_{i \in \underline{N}} \sqrt{\frac{\pi_{ji}}{p_i}} T^{j+Nk} \mathbb{1}_{[i]} S_j^* T^{-k}.$$

The spaces V_n, $n \in \mathbb{N}_0$, as defined in the proof of Theorem 9.12 can also be written in terms of the partial isometries S_i, $i \in \underline{N}$, that is for $n \in \mathbb{N}$ a basis of V_n is given by

(5)
$$\left\{ \sqrt{\frac{p_i}{\pi_{\omega_{n-1} i}}} T^l S_\omega \varphi_i : l \in \mathbb{Z}, \omega \in \Sigma_A^n,\, i \in \underline{N} \right\},$$

where for $\omega \in \Sigma_A^n$ we have $S_\omega = S_{\omega_0} S_{\omega_1} \cdots S_{\omega_{n-1}}$, $\omega = (\omega_0, \ldots, \omega_{n-1})$.

9.3.1. Mother wavelets for Markov measures. The construction of the mother wavelets simplifies in this setting because we only have to consider mother wavelets for one scale and obtain the other by iterative application of the operators U and T by Lemma 8.5. The mother wavelets are constructed via N matrices as given in Lemma 8.9 and so the mother wavelets are defined for $k \in \underline{N}$ and $l \in \underline{q^k} \backslash \{0\}$, by

$$\psi^{k,l} = UT^k \sum_{j \in \underline{N}} A_{kj} c_j^{k,l} \varphi_j$$

for coefficients $c_j^{k,l} \in \mathbb{C}$ as in Lemma 8.9.

Remark 9.20.

(1) The number of mother wavelets we obtain is $\sum_{k\in\underline{N}} q^k \leq N^2$. In the case of N^2 mother wavelets we are in the case of fractals given by an IFS.
(2) Notice that $\sum_{l=1}^{q^k-1} A_{ki}A_{kj}c_i^{k,l}\overline{c}_j^{k,l} + \sqrt{\pi_{ki}}\sqrt{\pi_{kj}} = \delta_{i,j}$.
(3) Alternatively we can define the mother wavelets as the elements of the vector

$$\left(\psi^{k,l}\right)^t_{l\in\{1,\dots,q^k-1\}} = \left(\left(A_{kj}c_j^{k,l}\right)_{l\in\underline{q^k}\setminus\{0\},j\in\underline{N}}\right)\left(UT^k\varphi_j\right)^t_{j\in\underline{N}}.$$

(4) Here we can see that we only need mother wavelets for W_0 since

$$\sum_{j\in\underline{N}} A_{kj}c_j^{k,i}\left(\nu_{\mathbb{Z}}([\omega j])\right)^{1/2} = \sqrt{p_{\omega_0}\prod_{i=1}^{n-2}\pi_{i(i+1)}}\sum_{j\in\underline{N}} A_{kj}c_j^{k,i}\sqrt{\pi_{\omega_{n-1}j}} = 0,$$

which was the crucial condition in the case of the last section.

Corollary 9.21. *The set*

$$\begin{aligned}&\left\{U^nT^m\psi^{k,l} : n\in\mathbb{N}_0,\, m\in D_{n,k},\, k\in\underline{N},\, l\in\underline{q^k}\setminus\{0\}\right\}\\ \cup&\left\{\left(U^*\right)^n T^m\psi^{k,l} : n\in\mathbb{N},\, m\in\mathbb{Z},\, k\in\underline{N},\, l\in\underline{q^k}\setminus\{0\}\right\}\\ \cup&\left\{\left(U^*\right)^n T^k\varphi_j : n\in\mathbb{N},\, k\in N\mathbb{Z}+l,\, j,l\in\underline{N},\, A_{jl}=0\right\}\end{aligned}$$

is an ONB for $L^2(\nu_{\mathbb{Z}})$, where

$$D_{n,k} = \left\{m\in\mathbb{Z} : m = \sum_{i=0}^{n-1}\omega_{n-1-i}N^i + N^n l, \omega_i\in\underline{N},\, (\omega_0,\dots,\omega_{n-1})\in\Sigma_A^n \text{ and } A_{\omega_0 k}=1,\, l\in\mathbb{Z}\right\}.$$

Remark 9.22.

(1) Because $UW_{-1} = W_0$ we only have to add those functions $T^k\varphi_j$, $k\in\mathbb{Z}$, $j\in\underline{N}$, that satisfy $UT^k\varphi_j = 0$ to the basis of $U^*(W_0)$ to obtain a basis of W_{-1}.
(2) Notice that

$$\psi^{k,l} = UT^k\sum_{i\in\underline{N}} A_{ki}c_i^{k,l}\varphi_i = \sum_{i\in\underline{N}} A_{ki}c_i^{k,l}\cdot\left(p_k\cdot\pi_{ki}\right)^{-1/2}\cdot\mathbb{1}_{[ki]}.$$

9.3.2. Examples. In the construction of [**MP09**] only Cantor sets with an incidence matrix are considered, i.e. the inverse maps of the MIM have the form $\left(\tau_i(x) = \frac{x+i}{N}\right)_{i\in\underline{N}}$, and there exists a incidence matrix A. The limit set has then the Hausdorff dimension $\delta = \dim_H(X) = \frac{\log r(A)}{\log N}$, where $r(A)$ is the spectral radius of A. So we consider the δ-dimensional Hausdorff measure μ restricted to the by $\mathbb{Z}$ translated set X. It follows that $p_j = \mu([j])$ and $\pi_{ij} = \frac{N^{-2\delta}p_j}{p_i}$. Consequently, in this case we can rewrite our conditions for obtaining the coefficients of the mother wavelets in a simpler way. More precisely, for $k\in\underline{N}$ instead of

$$\sum_{j\in\underline{N}} A_{kj}c_j^{k,i}\sqrt{\pi_{kj}} = 0$$

we obtain the condition

$$\sum_{j\in\underline{N}} A_{kj}c_j^{k,i}\sqrt{p_j} = 0.$$

Although the basis in [**MP09**] is only given in terms of the representation of a Cuntz-Krieger algebra we can now give a scaling operator U in the sense of (1.2.1) for this case. More precisely, we obtain

$$Uf(x) = N^\delta\sum_{k\in\mathbb{Z}}\sum_{j\in\underline{N}}\mathbb{1}_{[j]}(x-k)\cdot f(\tau_j^{-1}(x-k)+j+Nk).$$

Proof of Example 1.1: Recall that the β-transformation is given by $F:[0,1]\to[0,1]$, $x\mapsto\beta x \mod 1$. We have $X=[0,1]$ and the inverse branches are $\tau_0(x) = \frac{x}{\beta}$, $x\in[0,1]$, and $\tau_1(x) = \frac{x+1}{\beta}$,

$x \in [0, \beta-1]$. We may choose the two intervals $B_0 = [0, \beta-1]$ and $B_1 = [\beta-1, 1]$ and the corresponding incidence matrix is then given by $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. This map clearly belongs to the class of Markov measures with the stochastic matrix

$$\Pi := \begin{pmatrix} \beta - 1 & 2 - \beta \\ 1 & 0 \end{pmatrix}$$

and probability vector $p := \left(\frac{\beta}{\sqrt{5}}, \frac{\beta-1}{\sqrt{5}} \right)$. Recall that the scaling operator U acting on $L^2(\nu_{\mathbb{Z}})$ is given for $x \in \mathbb{R}$ by

$$Uf(x) = \sum_{k \in \mathbb{Z}} \left(\sqrt{\beta} \mathbb{1}_{[0,\beta^{-2})}(x-k) + \mathbb{1}_{[\beta^{-2},\beta^{-1})}(x-k) + \beta \cdot \mathbb{1}_{[\beta^{-1},1)}(x-k) \right) \cdot f(\beta(x-k)+2k).$$

For the father wavelets we choose $\varphi_0 = \left(\sqrt{5}/\beta\right)^{1/2} \mathbb{1}_{[0,\beta-1)}$ and $\varphi_1 = \left(\sqrt{5}\beta\right)^{1/2} \mathbb{1}_{[\beta-1,1)}$.

Consequently, (μ, U, T) allows a two-sided MRA. We can construct the mother wavelets along the lines of Section 9.3. Since in this case $q^0 = 2$ and $q^1 = 1$ we only have to construct coefficients for φ_0 to obtain the mother wavelets. These coefficients are given in the following unitary matrix

$$\begin{pmatrix} \sqrt{\beta-1} & \sqrt{2-\beta} \\ \sqrt{2-\beta} & -\sqrt{\beta-1} \end{pmatrix}.$$

Thus, the mother wavelet is $\psi = U\left(\sqrt{2-\beta}\varphi_0 - \sqrt{\beta-1}\varphi_1\right)$. To obtain the basis we further notice that $UT\varphi_1 = 0$ and so we have to keep $T^k\varphi_1$, $k \in 2\mathbb{Z}+1$ in the basis.

9.4. Different approach for general measures on MIMs

In this section we give the construction of a wavelet basis via MRA for an MIM with respect to a general non-atomic probability measure ν with respect to a different family of scaling operators $\left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}$ than the one given in (1.2.3) and only one father wavelet $\varphi := \mathbb{1}_X$, where X is the limit set of the MIM.

We define a family of scaling operators $\left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}$ for $f \in L^2(\nu_{\mathbb{Z}})$, $x \in \mathbb{R}$, $n \in \mathbb{N}$, as

$$\widetilde{U}^{(n)} f(x) := \sum_{k\in\mathbb{Z}} \sum_{\omega \in \Sigma_A^n} \sqrt{\frac{1}{\nu_{\mathbb{Z}}([\omega])}} \mathbb{1}_{[\omega]}(x-k) f\left(\tau_\omega^{-1}(x-k) + N^n k + \sum_{i=0}^{n-1} \omega_{n-1-i} N^i \right)$$

and as

$$\widetilde{U}^{(-n)} f(x) := \sum_{k\in\mathbb{Z}} \sum_{\omega \in \Sigma_A^n} \sqrt{\nu_{\mathbb{Z}}([\omega])} \mathbb{1}_X (x - \sum_{i=0}^{n-1} \omega_{n-1-i} N^i - N^n k)$$
$$f\left(\tau_\omega (x - \sum_{i=0}^{n-1} \omega_{n-1-i} N^i - N^n k) + k \right).$$

Remark 9.23. For $n \in \mathbb{N}_0$, $k \in \mathbb{Z}$ with $k = \sum_{i=0}^{n-1} k_{n-1-i} N^i + N^n l$, $(k_i)_{i\in\underline{N}} \in \underline{N}^n$, $l \in \mathbb{Z}$, we have

$$\widetilde{U}^{(n)} T^k \varphi = \sqrt{\frac{1}{\nu_{\mathbb{Z}}([\omega])}} T^l \mathbb{1}_{[\omega]},$$

where $\omega = (k_0, \ldots, k_{n-1})$. For $n \in \mathbb{Z}$, $n < 0$, we have

$$\widetilde{U}^{(n)} \varphi = \sum_{\omega \in \Sigma_A^{|n|}} \sqrt{\nu_{\mathbb{Z}}([\omega])} T^{\sum_{i=0}^{|n|-1} \omega_{|n|-1-i} N^i} \varphi. \tag{9.4.1}$$

Then the family of operators $\left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}$, the operator T and the father wavelet φ satisfy the following properties.

Proposition 9.24. *$\widetilde{U}^{(n)}$, T and φ satisfy the following relations.*

1. $T\widetilde{U}^{(n)} = \widetilde{U}^{(n)}T^{N^n}$, $n \in \mathbb{N}$,
2. $\widetilde{U}^{(-n)}T\varphi = T^{N^n}\widetilde{U}^{(-n)}\varphi$, $n \in \mathbb{N}$,
3. $\varphi = \widetilde{U}^{(1)}\sum_{j\in\underline{N}}\sqrt{\nu_{\mathbb{Z}}([j])}\,T^j\varphi$,
4. *if* $\widetilde{U}^{(n)}T^k\varphi \neq 0$, *then* $\langle \widetilde{U}^{(n)}T^k\varphi|\widetilde{U}^{(n)}T^l\varphi\rangle = \delta_{k,l}$, $n,k,l \in \mathbb{Z}$,
5. $\widetilde{U}^{(n)}\widetilde{U}^{(-n)} = I$, $n \in \mathbb{N}$,
6. *if* $\widetilde{U}^{(n)}T^k\varphi \neq 0$, *then* $\widetilde{U}^{(-n)}U^{(n)}T^k\varphi = T^k\varphi$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$.

Proof. This proof goes as the one of Proposition 9.9 with the difference that we take $\widetilde{U}^{(n)}$ instead of $U^{(n)}$, $n \in \mathbb{Z}$, and for the family of father wavelets $(\varphi_i)_{i\in\underline{N}}$ we consider the only father wavelet φ. So we only show the property (3) and we obtain for $x \in \mathbb{R}$

$$\varphi(x) = \sum_{j\in\underline{N}} \mathbb{1}_{[j]}(x) = \sum_{j\in\underline{N}} \varphi\left(\tau_j^{-1}(x)\right) = \widetilde{U}^{(1)}\sum_{j\in\underline{N}}\sqrt{\nu_{\mathbb{Z}}([j])}\,T^j\varphi(x).$$

□

Remark 9.25. For $n \in \mathbb{N}_0$, $k \in \underline{N^n}$ with $k = \sum_{i=0}^{n-1}\omega_{n-1-i}N^i$, $\omega \in \Sigma_A^n$, we have

$$\widetilde{U}^{(n)}T^k\varphi = \sum_{j\in\underline{N}}\sqrt{\frac{\nu_{\mathbb{Z}}([\omega j])}{\nu_{\mathbb{Z}}([\omega])}}\,\widetilde{U}^{(n+1)}T^{k+N^n j}\varphi. \tag{9.4.2}$$

Corollary 9.26. $\left(\nu_{\mathbb{Z}}, \left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{N}_0}, T\right)$ *allows a one-sided MRA with respect to the father wavelet* $\varphi := \mathbb{1}_X$.

Remark 9.27. The correspondence between the one-sided MRA with respect to $\left(U^{(n)}\right)_{n\in\mathbb{N}_0}$ and $\left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{N}_0}$ is as follows: for $k \in \mathbb{Z}$, $k = \sum_{i=0}^{n-1}k_{n-1-i}N^i + N^n l$, $(k_i)_{i\in\underline{N}} \in \underline{N}^n$, $l \in \mathbb{Z}$, we have

$$\widetilde{U}^{(n)}T^k\varphi = U^{(n-1)}T^{\sum_{i=1}^{n-1}k_{n-1-i}N^{i-1}+N^{n-1}l}\varphi_{k_0}.$$

Consequently, the closed subspaces V_j, $j \in \mathbb{N}_0$, given in the proof of Theorem 9.12, and $\widetilde{V}_j := \operatorname{cl}\operatorname{span}\left\{\widetilde{U}^{(j)}T^k\varphi : k \in \mathbb{Z}\right\}$, $j \in \mathbb{N}_0$, of $L^2(\nu_{\mathbb{Z}})$ are related for each $j \in \mathbb{N}_0$ by $\widetilde{V}_{j+1} = V_j$.

Proof of Corollary 9.26. This result follows from Theorem 9.11 and Remark 9.27, since for the MRA we define the closed subspaces of $L^2(\nu_{\mathbb{Z}})$ as $\widetilde{V}_j := \operatorname{cl}\operatorname{span}\left\{\widetilde{U}^{(j)}T^k\varphi : k \in \mathbb{Z}\right\}$, $j \in \mathbb{N}_0$. Consequently, the properties (2a) and (2b) of Definition 1.2 follow directly from Theorem 9.11. The properties (2d) and (2c) follow from Proposition 9.24. The property (2d) follows from (9.4.2). □

Theorem 9.28. $\left(\nu_{\mathbb{Z}}, \left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}, T\right)$ *allows a two-sided MRA with respect to the father wavelet* $\varphi := \mathbb{1}_X$, *if and only if the incidence matrix A consists only of ones and the measure ν is a measure obtained by Hutchinson's theorem with a probability vector $p = (p_0, \dots, p_{N-1})$. Furthermore,* $\widetilde{U}^{(n)} = \left(\widetilde{U}^{(1)}\right)^n$ *and* $\widetilde{U}^{(-n)} = \left(\widetilde{U}^{(-1)}\right)^n$ *for all $n \in \mathbb{N}$, if and only if ν is a measure obtained by Hutchinson's theorem with a probability vector $p = (p_0, \dots, p_{N-1})$.*

Remark 9.29. The theorem above gives that the MIM is an IFS if for this family of scaling operators $\left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}$ a two-sided MRA holds for this setting. In addition to be multiplicative, we also have that $\widetilde{U}^{(1)}$ is a unitary operator if a two-sided MRA is allowed.

Proof. Assume that $\left(\nu_{\mathbb{Z}}, \left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}, T\right)$ allows a two-sided MRA with the father wavelet $\varphi = \mathbb{1}_X$, then for all $n \in \mathbb{N}$

$$\widetilde{U}^{(-n)}\varphi \subset \operatorname{span}\widetilde{U}^{(-n+1)}\left\{T^k\varphi : k \in \underline{N}\right\}.$$

Therefore there are $c_k^n \in \mathbb{C}$, $k \in \underline{N}$, such that

$$\widetilde{U}^{(-n)}\varphi = \sum_{k\in\underline{N}} c_k^n \widetilde{U}^{(-n+1)}T^k\varphi = \sum_{k\in\underline{N}} c_k^n \sum_{\omega\in\Sigma_A^{n-1}} \sqrt{\nu_{\mathbb{Z}}([\omega])}T^{\sum_{i=0}^{n-2}\omega_{n-2-i}N^i+N^{n-1}k}\varphi.$$

On the other hand, we have

$$U^{(-n)}\varphi = \sum_{\omega\in\Sigma_A^n} \sqrt{\nu_{\mathbb{Z}}([\omega])}T^{\sum_{i=0}^{n-1}\omega_{n-1-i}N^i}\varphi = \sum_{j\in\underline{N}}\sum_{\omega\in\Sigma_A^{n-1}} \sqrt{\nu_{\mathbb{Z}}([j\omega])}T^{\sum_{i=0}^{n-2}\omega_{n-2-i}N^i+N^{n-1}j}\varphi.$$

Consequently, it follows for all $\omega \in \Sigma_A^{n-1}$ and $j \in \underline{N}$ that $\sqrt{\nu_{\mathbb{Z}}([j\omega])} = c_j^n\sqrt{\nu_{\mathbb{Z}}([\omega])}$ and so $c_j^n \in \mathbb{R}^+$. c_j^n is independent of $n \in \mathbb{N}$ since for $\omega \in \Sigma_A^n$ and $j \in \underline{N}$ we have

$$\nu_{\mathbb{Z}}([j\omega]) = \sum_{k\in\underline{N}} \nu_{\mathbb{Z}}([j\omega k]) = \left(c_j^{n+1}\right)^2 \sum_{k\in\underline{N}} \nu_{\mathbb{Z}}([\omega k]) = \left(c_j^{n+1}\right)^2 \nu_{\mathbb{Z}}([\omega]).$$

Thus, $c_j^n = c_j^{n+1}$ for all $n \in \mathbb{N}$, $j \in \underline{N}$. We denote $c_j = c_j^n$ for $j \in \underline{N}$ and these satisfy $\sum_{j\in\underline{N}} (c_j)^2 = 1$ since $\nu_{\mathbb{Z}}([j]) = \sum_{k\in\underline{N}} \nu_{\mathbb{Z}}([jk]) = (c_j)^2 \sum_{k\in\underline{N}} \nu_{\mathbb{Z}}([k]) = (c_j)^2$ and $\sum_{j\in\underline{N}} \nu_{\mathbb{Z}}([j]) = 1$. So it follows that for $\omega \in \Sigma_A^n$

$$\nu_{\mathbb{Z}}([\omega]) = \prod_{i=0}^{n-1} (c_{\omega_i})^2$$

and so ν must be the probability measure obtained by Hutchinson's theorem for an IFS with the probability vector $p = \left((c_0)^2, \ldots, (c_{N-1})^2\right)$ and the incidence matrix A of the MIM consists only of ones.

Furthermore, the family of operators $\left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}$ is multiplicative since for $f \in L^2(\nu_{\mathbb{Z}})$, $x \in \mathbb{R}$,

$$\widetilde{U}^{(2)}f(x) \quad = \sum_{k\in\mathbb{Z}}\sum_{j_1\in\underline{N}}\sum_{j_2\in\underline{N}} c_{j_1}^{-1}c_{j_2}^{-1}\mathbb{1}_{[j_1j_2]}(x-k)f\left(\tau_{j_2}^{-1}\left(\tau_{j_1}^{-1}(x-k)\right) + N^2k + j_1N + j_2\right)$$

and

$$\begin{aligned}\widetilde{U}^{(1)}\widetilde{U}^{(1)}f(x) &= \widetilde{U}^{(1)}\left(\sum_{k\in\mathbb{Z}}\sum_{j_1\in\underline{N}} c_{j_1}^{-1}\mathbb{1}_{[j_1]}(x-k)f\left(\tau_{j_1}^{-1}(x-k) + Nk + j_1\right)\right)\\ &= \sum_{l\in\mathbb{Z}}\sum_{j_2\in\underline{N}} c_{j_2}^{-1}\mathbb{1}_{[j_2]}(x-l)\sum_{k\in\mathbb{Z}}\sum_{j_1\in\underline{N}} c_{j_1}^{-1}\mathbb{1}_{[j_1]}(\tau_{j_2}^{-1}(x-l) + Nl + j_2 - k)\\ &\quad f\left(\tau_{j_1}^{-1}\left(\tau_{j_2}^{-1}(x-l) + Nl + j_2 - k\right) + Nk + j_1\right)\\ &= \sum_{l\in\mathbb{Z}}\sum_{j_2\in\underline{N}}\sum_{j_1\in\underline{N}} c_{j_2}^{-1}c_{j_1}^{-1}\mathbb{1}_{[j_2j_1]}(x-l)f\left(\tau_{j_1}^{-1}\left(\tau_{j_2}^{-1}(x-l)\right) + N^2l + Nj_2 + j_1\right).\end{aligned}$$

So $\widetilde{U}^{(2)} = \widetilde{U}^{(1)}\widetilde{U}^{(1)}$. Iteratively, we can show the result for all $n \in \mathbb{N}_0$. To obtain the result for $n \in \mathbb{Z}$, $n < 0$, we notice that $U^{(-1)} = \left(U^{(1)}\right)^*$ and then we can prove the result in a way similar to the above.

Consequently, now we have that if there exists a two-sided MRA then the measure ν is a measure obtained by Hutchinson's theorem with a probability vector $p = (p_0, \ldots, p_{N-1})$ for an IFS and furthermore the MRA is multiplicative. To show the backward direction we can assume that the measure ν is a measure obtained by Hutchinson's theorem for an IFS and hence for the scaling operators $\left(\widetilde{U}^{(n)}\right)_{n\in\mathbb{Z}}$ we have $\widetilde{U}^{(n)} = \widetilde{U}^n$, $n \in \mathbb{Z}$, with $\widetilde{U} = \widetilde{U}^{(1)}$ and $\widetilde{U}$ being unitary, which can be easily verified. Then the proof goes analogous to the proof of the backward direction of Theorem 9.12, where we use the operator $\widetilde{U}$ and φ instead of U and $(\varphi_i)_{i\in\underline{N}}$ and define the closed subspaces of $L^2(\nu_{\mathbb{Z}})$ as $\widetilde{V}_j = \operatorname{cl\,span}\left\{\widetilde{U}^jT^k\varphi : k \in \mathbb{Z}\right\}$, $j \in \mathbb{Z}$. □

9.4.1. Mother wavelets. Now we turn to the mother wavelets that we obtain via this MRA. We have for $n \in \mathbb{N}_0$, $k = \sum_{i=0}^{n-1} \omega_{n-1-i} N^i$, that

$$\widetilde{U}^{(n)} T^k \varphi = \sum_{j \in \underline{N}} \sqrt{\frac{\nu_{\mathbb{Z}}([j\omega])}{\nu_{\mathbb{Z}}([\omega])}} \widetilde{U}^{(n+1)} T^{j+Nk} \varphi.$$

So for every scale $n \in \mathbb{N}_0$ we need for $\omega \in \Sigma_A^n$ at most a matrix of size $(N \times N)$. Define for $\omega \in \Sigma_A^n$, $n \in \mathbb{N}$,

$$D_{\omega_{n-1}} = \left\{ j \in \underline{N} : A_{j\omega_{n-1}} = 1 \right\}$$

and $q^\omega := \operatorname{card} D_{\omega_{n-1}}$. Then we choose $q^{\omega_{n-1}} - 1$ vectors of length $q^{\omega_{n-1}}$ which are orthonormal to $\left(\sqrt{\frac{\nu_{\mathbb{Z}}([j\omega])}{\nu_{\mathbb{Z}}([\omega])}} \right)_{j \in D_{\omega_{n-1}}}$. We denote the vectors by $\left(c_j^{\omega,i} \right)_{j \in D_{\omega_{n-1}}}$, $i \in \underline{q^{\omega_{n-1}}} \setminus \{0\}$. We extend the vectors $\left(c_j^{\omega,i} \right)_{j \in D_{\omega_{n-1}}}$ to some of length N via $c_j^{\omega,i} = 0$ if $j \in \underline{N} \setminus D_{\omega_{n-1}}$. We can denote this extension as $A_{\omega_{n-1} j} c_j^{\omega,i}$ since $A_{\omega_{n-1} j} = 0$ if and only if $j \in \underline{N} \setminus D_{\omega_{n-1}}$ and if $j \in D_{\omega_{n-1}}$, $A_{\omega_{n-1} j} = 1$.

The mother wavelets are defined for $\omega \in \Sigma_A^n$, $n \in \mathbb{N}_0$, $i \in \underline{q^{\omega_{n-1}}} \setminus \{0\}$, as

$$\psi_{\omega,i} = \sum_{j \in \underline{N}} c_j^{\omega,i} \widetilde{U}^{(n)} T^{j+Nk} \varphi,$$

where $k = \sum_{i=0}^{n-1} \omega_{n-1-i} N^i \in \underline{N^n}$, $\omega \in \Sigma_A^n$, and they take the explicit form

$$\psi_{\omega,i} = \sum_{j \in \underline{N}} A_{\omega_{n-1} j} c_j^{\omega,i} \left(\nu_{\mathbb{Z}}([\omega j]) \right)^{-1/2} \mathbb{1}_{[\omega j]}.$$

Corollary 9.30. *The set*

$$\left\{ T^l \psi_{\omega,i} : \omega \in \Sigma_A^n, n \in \mathbb{N}, i \in \underline{q^{\omega_{n-1}}} \setminus \{0\}, l \in \mathbb{Z} \right\} \cup \left\{ T^l \varphi : l \in \mathbb{Z} \right\}$$

is an ONB of $L^2(\nu_{\mathbb{Z}})$.

Proof. This result follows from Theorem 8.1 with Theorem 9.28. □

Remark 9.31.

(1) If the measure ν is Markovian, then for each scale $n \in \mathbb{N}_0$ we can consider the same matrices of coefficients, since for $k \in \underline{N^n}$ with $k = \sum_{i=0}^{n-1} \omega_{n-1-i} N^i$, $\omega \in \Sigma_A^n$, we have

$$\widetilde{U}^{(n)} T^k \varphi = \sum_{j \in \underline{N}} \sqrt{\frac{\nu_{\mathbb{Z}}([j\omega])}{\nu_{\mathbb{Z}}([\omega])}} \widetilde{U}^{(n+1)} T^{j+Nk} \varphi = \sum_{j \in \underline{N}} \sqrt{\frac{p_j \pi_{j\omega_0}}{p_{\omega_0}}} \widetilde{U}^{(n+1)} T^{j+Nk} \varphi.$$

(2) If we compare the mother wavelets for the construction with respect to the family of scaling operators $\left(U^{(n)} \right)_{n \in \mathbb{N}_0}$, compare Section 9.2, with the one with respect to $\left(\widetilde{U}^{(n)} \right)_{n \in \mathbb{N}_0}$, then we notice that we can consider the same coefficient matrices for both construction, since we already know that $\widetilde{V}_{j+1} = V_j$ for $j \in \mathbb{N}_0$ and furthermore for $k \in \underline{N}$ we have

$$\varphi_k = \widetilde{U}^{(1)} T^k \varphi = \sum_{i \in \underline{N}} \sqrt{\frac{\nu_{\mathbb{Z}}([ki])}{\nu_{\mathbb{Z}}([k])}} \widetilde{U}^{(2)} T^{k+Ni} \varphi = \sum_{i \in \underline{N}} \sqrt{\frac{\nu_{\mathbb{Z}}([ki])}{\nu_{\mathbb{Z}}([k])}} U^{(1)} T^k \varphi_i.$$

Consequently, we have the same initial vector for the construction of the mother wavelets for $\widetilde{W}_1 := \widetilde{V}_2 \ominus \widetilde{V}_1$ and $W_0 = V_1 \ominus V_0$. Analogously we obtain for $j \in \underline{N}$, $k = \sum_{i=0}^{n-1} \omega_{n-1-i} N^i$,

$\omega j \in \Sigma_A^{n+1}$, that

$$U^{(n)} T^k \varphi_j = \sum_{i \in \underline{N}} \sqrt{\frac{\nu_{\mathbb{Z}}([\omega j i])}{\nu_{\mathbb{Z}}([\omega j])}} U^{(n+1)} T^{Nk+j} \varphi_i,$$

$$\widetilde{U}^{(n+1)} T^{Nk+j} \varphi = \sum_{i \in \underline{N}} \sqrt{\frac{\nu_{\mathbb{Z}}([\omega j i])}{\nu_{\mathbb{Z}}([\omega j])}} \widetilde{U}^{(n+2)} T^{Nk+j+N^{n+1} i} \varphi.$$

It also follows that the same choice of coefficients for the mother wavelets for $\widetilde{W}_n := \widetilde{V}_{n+1} \ominus \widetilde{V}_n$ and W_{n-1}, $n \in \mathbb{N}$, is possible, and in this case the mother wavelets for the two constructions coincide.

9.4.2. Application to IFSs. Now we give the construction of a wavelet basis on a fractal that is translated by $\mathbb{Z}$ with a general non-atomic measure. This construction is a special case of the one in Section 9.4. Here the incidence matrix A consists only of ones.

First we clarify the notation. Let $\mathcal{S} := (\tau_i : [0,1] \to [0,1] : i \in \underline{N})$ be an IFS. Then there exists an invariant set C for $\mathcal{S}$ by Theorem A.8. On C we fix a measure ν, e.g. a Gibbs measure. We consider for ν the convolution with $\mathbb{Z}$, more precisely $\nu_{\mathbb{Z}} = \sum_{k \in \mathbb{Z}} \nu(\cdot - k)$, which has the support $R = \bigcup_{k \in \mathbb{Z}} C + k$. Furthermore, in the corresponding shift space let $\Sigma^n = \underline{N}^n$ denote the set of words of length $n \in \mathbb{N}$. We denote $\omega \in \Sigma^n$ as $\omega = (\omega_0 \ldots \omega_{n-1})$, $\omega_i \in \underline{N}$. The set of all finite words is denoted by $\Sigma^* = \bigcup_{n \in \mathbb{N}} \Sigma^n$. Between the shift space and the limit set C there is a coding map π (see (9.1.1)) such that we can identify $\tau_\omega(C)$ with $[\omega]$. For further information see Section 9.1.

Now we want to construct a wavelet basis on $L^2(\nu_{\mathbb{Z}})$, where the measure $\nu_{\mathbb{Z}}$ is supported on R. We obtain the wavelet basis via a one-sided MRA, compare Definition 1.2 for the precise definition.

In this case the scaling operators $\left(U^{(n)}\right)_{n \in \mathbb{N}_0}$, $U^{(0)} = I$, have a simpler form: they are given for $n \in \mathbb{N}$, $x \in \mathbb{R}$, $f \in L^2(\nu_{\mathbb{Z}})$, by

$$U^{(n)} f(x) = \sum_{k \in \mathbb{Z}} \sum_{\omega \in \Sigma^n} \sqrt{\frac{1}{\nu_{\mathbb{Z}}([\omega])}} \mathbb{1}_{[\omega]}(x-k) f\left(\tau_\omega^{-1}(x-k) + N^n k + \sum_{i=0}^{n-1} \omega_{n-1-i} N^i\right). \tag{9.4.3}$$

The translation operator T acting on $L^2(\nu_{\mathbb{Z}})$ is again defined by setting $Tf(\cdot) = f(\cdot - 1)$. The operators $\left(U^{(n)}\right)_{n \in \mathbb{N}_0}$, T and the father wavelet $\varphi = \mathbb{1}_C$ satisfy the properties in Proposition 9.24 (1), (3), (4).

Remark 9.32. Notice that we could also consider a family of father wavelets $(\varphi_i)_{i \in \underline{N}}$, $\varphi_i = (\nu([i]))^{-1/2} \mathbb{1}_{[i]}$, as for an MIM and then we would obtain an analogous wavelet basis.

Corollary 9.33. *Let $\left(U^{(n)}\right)_{n \in \mathbb{N}_0}$ be given as in (9.4.3). Then $\left(\nu_{\mathbb{Z}}, \left(U^{(n)}\right)_{n \in \mathbb{N}_0}, T\right)$ allows a one-sided MRA with respect to the father wavelet $\varphi = \mathbb{1}_C$.*

Remark 9.34 (Mother wavelets)**.** The construction of the mother wavelets is as described in Section 9.4.1 with the only difference that for all $j \in \underline{N}$ we have $D_j = \underline{N}$ and so $q^j = N$. Consequently, for every scale $n \in \mathbb{N}_0$ and every $\omega \in \Sigma_A^n$ we need an $(N \times N)$-matrix with coefficients for the mother wavelets.

CHAPTER 10

Construction of wavelet bases on enlarged fractals in $\mathbb{R}$

10.1. Setting for wavelet bases on enlarged fractals

We start by clarifying the setting for the construction of the wavelet bases on enlarged fractal. The following starting point is as in [**BK10**]. We start with an IFS

$$\mathcal{S} := \big(\sigma_i : [0,1] \to [0,1] : i \in \underline{p} = \{0, \dots, p-1\}\big)$$

consisting of p injective contractions σ_i which are uniformly Lipschitz with Lipschitz-constant $0 < c_{\mathcal{S}} < 1$, i.e. $|\sigma_i(x) - \sigma_i(y)| \leq c_{\mathcal{S}}|x-y|$, $x, y \in [0,1]$, $i \in \underline{p}$. Here we always assume that all contractions have the same orientation (in fact are increasing) and that the IFS satisfies the OSC for $(0,1)$, i.e. $\bigcup_{i\in\underline{p}} \sigma_i((0,1)) \subset (0,1)$ and $\sigma_i((0,1)) \cap \sigma_j((0,1)) = \emptyset$, $i, j \in \underline{p}$, $i \neq j$. It is well known that there exists a unique non-empty compact set $C \subset [0,1]$ such that $C = \bigcup_{i\in\underline{p}} \sigma_i(C)$. This set C will be called the *limit set* (or the fractal) of $\mathcal{S}$, compare Theorem A.8. Furthermore, let $\mathcal{B}$ denote the Borel σ-algebra in $\mathbb{R}$.

Remark 10.1.

(1) We will always assume that the IFS $(\sigma_i)_{i\in\underline{p}}$ is arranged in ascending order, that is $\sigma_i([0,1])$ lies to the left of $\sigma_{i+1}([0,1])$ for all $i \in \underline{p-1}$.
(2) The restriction that all functions in the IFS are increasing is imposed on the setting for simplicity in notation.

Fact 10.2. *It is always possible to extend the IFS $\mathcal{S}$ by linear contractions to obtain an IFS*

$$\mathbb{S} := (\tau_i : [0,1] \to [0,1] : i \in \underline{N})$$

which leaves no gaps. More precisely, there exists a number $N \in \{p, \dots, 2p+1\}$ and a set $A \subset \underline{N}$ such that

(1) $\{\tau_j : j \in A\} = \big\{\sigma_i : i \in \underline{p}\big\}$,
(2) $\tau_0(0) = 0$, $\tau_{N-1}(1) = 1$ *and* $\tau_i(1) = \tau_{i+1}(0)$,
(3) $\forall i \in \underline{N} \setminus A$*:* $\tau_i : [0,1] \to [0,1]$ *is an affine increasing contraction and either* $i = N-1$ *or* $i+1 \in A$.

Remark 10.3.

(1) The extended IFS $\mathbb{S}$ satisfies also the OSC for $(0,1)$ and it has $[0,1]$ as the invariant set.
(2) Note that it is not essential to choose the "gap filling functions" τ_i, $i \in \underline{N}\setminus A$, to be affine. Our analysis would work for any set of contracting injections as long as (1), (2) in Fact 10.2 and the condition that for all $i \in \underline{N}\setminus A$ either $i = N-1$ or $i+1 \in A$ are satisfied. Nevertheless, the particular choice has an influence on the enlarged fractal R defined in Definition 10.4 and the measure $\nu_{\mathbb{Z}}$ on R given in Definition 10.6.
(3) We could also drop condition Fact 10.2 (3) and have even more freedom in the extension of the IFS. Then we could even vary in the number of gap-filling functions, e.g. if we consider the IFS $\big(\sigma_0(x) = \frac{x}{4}, \sigma_1(x) = \frac{x+3}{4}\big)$ then we could consider this either as an IFS with one gap (i.e. $\big[\frac{1}{4}, \frac{3}{4}\big]$) or as an IFS with two gaps (i.e. $\big[\frac{1}{4}, \frac{2}{4}\big]$ and $\big[\frac{2}{4}, \frac{3}{4}\big]$).

Now we turn to the definition for the enlarged fractal which is as in [**BK10**]. The enlarged fractal is first defined in the unit interval $[0,1]$ and in the next step it is translated to the whole real line.

Definition 10.4. Define the *enlarged fractal* in $[0,1]$ as

$$R_{[0,1]} := \biguplus_{\omega\in\Sigma\cup\{\emptyset\}} \tau_\omega(C),$$

where $\Sigma := \{(i_0,\dots,i_{k-1}) \in \underline{N}^k : k \in \mathbb{N},\, i_{k-1} \notin A\}$ and for $\omega = (i_0,\dots,i_{k-1})$ $\tau_\omega = \tau_{i_0} \circ \dots \circ \tau_{i_{k-1}}$. The *enlarged fractal* in $\mathbb{R}$ is defined as

$$R := \bigcup_{k\in\mathbb{Z}} R_{[0,1]} + k.$$

Remark 10.5.

(1) This enlarged fractal in $[0,1]$ can be understood in the way that scaled copies of the original fractal C are mapped in the gaps of this fractal. In this way we get a dense set in $[0,1]$. Afterward this set is just translated in $\mathbb{R}$.
(2) In the shift space for the alphabet $\underline{N}$ we have that the enlarged fractal in $[0,1]$, $R_{[0,1]}$, is homeomorphic to the set

$$\left\{\omega \in \underline{N}^{\mathbb{N}} : \exists n \in \mathbb{N}\ \theta^n(\omega) \in \Sigma^A\right\},$$

where θ is the shift map and $\Sigma^A = A^{\mathbb{N}}$, i.e. we take all the infinite words that are eventually in the original fractal. The homeomorphism between these two spaces is the coding map π (see (9.1.1) with the incidence matrix consisting only of ones). Furthermore, we denote with Σ^* the set of all words of finite length, i.e. $\Sigma^* = \bigcup_{n\in\mathbb{N}} \underline{N}^n$.

Now we turn to the measure. On the limit set C we consider a measure that we obtain by Hutchinson's theorem, see Theorem A.8. So we consider weights $p_i > 0$ on $\tau_i(C)$ for $i \in A$ such that $\sum_{i\in A} p_i = 1$, and from Hutchinson's theorem we know that there exists a unique probability measure μ on C satisfying

$$\mu = \sum_{i\in A} p_i \cdot \mu \circ \tau_i^{-1}.$$

For the definition of the measure on the enlarged fractal we have to fix weights on the gaps, i.e. we take weights $0 < c_i$ on $\tau_i([0,1])$ for $i \in \underline{N}\backslash A$ and $0 < c_i$ on $\tau_i([0,1]\backslash C)$ for $i \in A$. Then the measure on the enlarged fractal is, like the enlarged fractal itself, defined initially on $[0,1]$ and then translated to $\mathbb{R}$.

Definition 10.6. Let a set function on $[0,1]$ be defined by

$$\nu := \sum_{\omega\in\Sigma\cup\{\emptyset\}} \left(\prod_{i=0}^{|\omega|-1} c_{\omega_i}\right) \cdot \mu \circ \tau_\omega^{-1}$$

for weights $c_i \in \mathbb{R}^+$, $i \in \underline{N}$, and on $\mathbb{R}$ by $\nu_{\mathbb{Z}}(\cdot) := \sum_{k\in\mathbb{Z}} \nu(\cdot - k)$.

Proposition 10.7. *The set functions ν and $\nu_{\mathbb{Z}}$ are measures.*

Proof. The proof follows along the lines of the corresponding proof in [**Boh09**]. It is obvious that $\nu \geq 0$ and $\nu(\emptyset) = 0$. Thus, it remains to show that ν is σ-additive. So take $(A_n)_{n\in\mathbb{N}}$, $A_n \in \mathcal{B}$, disjoint. Notice that $\mu \circ \tau_\omega^{-1}\left(\biguplus_{n\in\mathbb{N}} A_n\right) = \sum_{n\in\mathbb{N}} \mu \circ \tau_\omega^{-1}(A_n)$ and so

$$\begin{aligned}
\nu\left(\biguplus_{n} A_n\right) &= \mu\left(\biguplus_{n\in\mathbb{N}} A_n\right) + \sum_{\omega\in\Sigma}\left(\prod_{i=0}^{|\omega|-1} c_{\omega_i}\right) \mu \circ \tau_\omega^{-1}\left(\biguplus_{n\in\mathbb{N}} A_n\right) \\
&= \sum_{n\in\mathbb{N}} \mu\left(A_n\right) + \sum_{n\in\mathbb{N}}\sum_{\omega\in\Sigma}\left(\prod_{i=0}^{|\omega|-1} c_{\omega_i}\right) \mu \circ \tau_\omega^{-1}\left(A_n\right) \\
&= \sum_{n\in\mathbb{N}}\sum_{\omega\in\Sigma}\left(\prod_{i=0}^{|\omega|-1} c_{\omega_i}\right)\left(\mu + \mu \circ \tau_\omega^{-1}\right)\left(A_n\right) \\
&= \sum_{n\in\mathbb{N}} \nu(A_n).
\end{aligned}$$

□

Lemma 10.8. *The measure ν satisfies the following relation:*

$$\nu = \sum_{i\in\underline{N}} c_i \cdot \check{\nu} \circ \tau_i^{-1} + \sum_{i\in\underline{N}\setminus A} c_i \cdot \mu \circ \tau_i^{-1} + \sum_{i\in A} p_i \cdot \mu \circ \tau_i,$$

where $\check{\nu} = \sum_{\omega\in\Sigma}\prod_{i=0}^{|\omega|-1} c_{\omega_i} \cdot \mu \circ \tau_\omega^{-1} = \nu - \mu$.

Proof. Rewrite the definition of ν as

$$\begin{aligned}
\nu = \sum_{\omega\in\Sigma\cup\{\emptyset\}}\left(\prod_{i=0}^{|\omega|-1} c_{\omega_i}\right) \cdot \mu \circ \tau_\omega^{-1} &= \sum_{\omega\in\Sigma:|\omega|\geq 2}\left(\prod_{i=0}^{|\omega|-1} c_{\omega_i}\right) \cdot \mu \circ \tau_\omega^{-1} + \sum_{i\in\underline{N}\setminus A} c_i \cdot \mu \circ \tau_i^{-1} + \mu \\
&= \sum_{i\in\underline{N}} c_i \cdot \check{\nu} \circ \tau_i^{-1} + \sum_{i\in\underline{N}\setminus A} c_i \cdot \mu \circ \tau_i^{-1} + \sum_{i\in A} p_i \cdot \mu \circ \tau_i^{-1}.
\end{aligned}$$

□

Remark 10.9. Recall to say that a measure μ on $\mathbb{R}$ is locally finite means that for every $x \in \mathbb{R}$ there is an open neighborhood of x with finite μ-measure. In this case it is enough to show this condition for intervals and hence it is sufficient to show it for the unit interval $[0, 1]$.

Proposition 10.10. *The measure $\nu_{\mathbb{Z}}$ is a locally finite measure if and only if*

$$\sum_{i\in\underline{N}} c_i < 1.$$

In particular, $\nu_{\mathbb{Z}}$ is not locally finite if $c_i = p_i$ for $i \in A$.

Proof. If $\nu_{\mathbb{Z}}$ is a locally finite measure, we have that $\nu_{\mathbb{Z}}([0, 1]) < \infty$. This gives

$$\nu([0,1]) = \sum_{\omega\in\Sigma\cup\{\emptyset\}}\left(\prod_{i=0}^{|\omega|-1} c_{\omega_i}\right) \cdot \mu\left(\tau_\omega^{-1}([0,1])\right) = \sum_{\omega\in\Sigma\cup\{\emptyset\}}\prod_{i=0}^{|\omega|-1} c_{\omega_i},$$

since $\mu\left(\tau_\omega^{-1}([0, 1])\right) = 1$ for all $\omega \in \Sigma \cup \{\emptyset\}$. Furthermore, we get

$$\sum_{\omega\in\Sigma\cup\{\emptyset\}}\prod_{i=0}^{|\omega|-1} c_{\omega_i} = 1 + \sum_{i\in\underline{N}\setminus A} c_i\left(\sum_{n=0}^{\infty}\left(\sum_{j\in\underline{N}} c_j\right)^n\right)$$

and this sum converges only if $\sum_{j\in\underline{N}} c_j < 1$.

On the other hand if we have that $\sum_{j\in\underline{N}} c_j < 1$, it follows that

$$\begin{aligned}\nu_{\mathbb{Z}}([0,1]) &= \sum_{\omega\in\Sigma\cup\{\emptyset\}} \prod_{i=0}^{|\omega|-1} c_{\omega_i} \cdot \mu\left(\tau_\omega^{-1}([0,1])\right) = \sum_{\omega\in\Sigma\cup\{\emptyset\}} \prod_{i=0}^{|\omega|-1} c_{\omega_i} \\ &= 1 + \sum_{i\in\underline{N}\setminus A} c_i \left(\sum_{n=0}^{\infty}\left(\sum_{j\in\underline{N}} c_j\right)^n\right) < \infty\end{aligned}$$

and thus, $\nu_{\mathbb{Z}}$ is locally finite.

In the case $c_j = p_j$ for $j \in A$ we have

$$\sum_{j\in\underline{N}} c_j = \sum_{j\in A} p_j + \sum_{j\in\underline{N}\setminus A} c_j$$

and $\sum_{j\in A} p_j = 1$. Thus, $\sum_{j\in\underline{N}} c_j > 1$. □

Remark 10.11.

(1) From Proposition 10.10 it follows that we cannot obtain a probability measure ν from μ on $R_{[0,1]}$ unless we rescale the measure μ, since μ is already a probability measure.
(2) In the case of [**BK10, DJ06**] the authors consider the measure of maximal entropy on the fractal, that is $p_i = \frac{1}{p}$ for $i \in A$. The authors extend this measure in the way explained above with the weights $c_i = \frac{1}{p}$ for all $i \in \underline{N}$. Thus the measure is not locally finite. This observation is in correspondence to the construction in [**DJ06**]. Since there the resulting measure $\nu_{\mathbb{Z}}$ is the $\log(2)/\log(3)$-dimensional Hausdorff measure restricted to the enlarged fractal for the middle-third Cantor set. It is also well known that $H^s([0,1]) = \infty$ for $s < 1$, where H^s stands for the s-dimensional Hausdorff measure, and it also holds that $H^s(R \cap [0,1]) = \infty$, where R is the enlarged fractal since $R \cap [0,1]$ is dense in $[0,1]$.

Lemma 10.12. *We have for $B \in \mathcal{B}$, $B \subset [0,1]$ that $\nu(\tau_j(B)) = c_j \cdot \nu(B)$, $j \in \underline{N}\setminus A$, and $\nu(\tau_j(B)) = c_j \cdot \nu(B) + (p_j - c_j)\cdot\mu(B)$, $j \in A$. In particular, $\nu(\tau_j(C)) = c_j$ for $j \in \underline{N}\setminus A$ and $\nu(\tau_i(C)) = p_i$ for $i \in A$.*

Proof. For $j \in A$ we get

$$\begin{aligned}\nu(\tau_j(B)) &= \sum_{\omega\in\Sigma\cup\{\emptyset\}} \left(\prod_{i=0}^{|\omega|-1} c_{\omega_i}\right)\cdot\mu\left(\tau_\omega^{-1}(\tau_j(B))\right) \\ &= \mu(\tau_j(B)) + \underbrace{\sum_{i\in\underline{N}\setminus A} c_i\cdot\mu\left(\tau_i^{-1}(\tau_j(B))\right)}_{=0} + \sum_{\omega\in\Sigma:\,|\omega|>1}\left(\prod_{i=0}^{|\omega|-1} c_{\omega_i}\right)\cdot\mu\left(\tau_\omega^{-1}(\tau_j(B))\right) \\ &= p_j\cdot\mu(B) + c_j\cdot\sum_{\omega\in\Sigma}\left(\prod_{i=0}^{|\omega|-1} c_{\omega_i}\right)\cdot\mu\left(\tau_\omega^{-1}(B)\right) \\ &= c_j\cdot\nu(B) + (p_j - c_j)\cdot\mu(B).\end{aligned}$$

For $j \in \underline{N}\backslash A$ we get

$$\begin{aligned}\nu(\tau_j(B)) &= \sum_{\omega\in\Sigma\cup\{\emptyset\}} \prod_{i=0}^{|\omega|-1} c_{\omega_i}\cdot\mu\left(\tau_\omega^{-1}(\tau_j(B))\right)\\ &= \underbrace{\mu(\tau_j(B))}_{=0} + \sum_{i\in\underline{N}\backslash A} c_i\cdot\mu\left(\tau_i^{-1}(\tau_j(B))\right) + \sum_{\omega\in\Sigma\colon|\omega|>1}\prod_{i=0}^{|\omega|-1} c_{\omega_i}\cdot\mu\left(\tau_\omega^{-1}(\tau_j(B))\right)\\ &= c_j\cdot\mu(B) + c_j\cdot\sum_{\omega\in\Sigma}\prod_{i=0}^{|\omega|-1} c_{\omega_i}\cdot\mu\left(\tau_\omega^{-1}(B)\right)\\ &= c_j\cdot\nu(B).\end{aligned}$$

For the last part of the statement we notice that $\nu(C)=\mu(C)=1$ and so the result follows. $\square$

10.1.1. Definition of the operators for the MRA. In this section we define two unitary operators acting on $L^2(\nu_{\mathbb{Z}})$ for the MRA and give some properties of these. The scaling operator is furthermore set in relation to representations of the Cuntz algebra $\mathcal{O}_N$ obtained by an IFS.

Definition 10.13. Define the operators T and U acting on $L^2(\nu_{\mathbb{Z}})$ for $f\in L^2(\nu_{\mathbb{Z}})$, $x\in\mathbb{R}$, as

$$Tf(x) := f(x-1)$$

and

$$\begin{aligned}Uf(x) = &\sum_{k\in\mathbb{Z}}\Bigg(\sum_{i\in\underline{N}\backslash A}\sqrt{c_i}^{-1}\cdot\mathbb{1}_{\tau_i([0,1))}(x-k)\cdot f\left(\tau_i^{-1}(x-k)+Nk+i\right)\\ &+\sum_{i\in A}\left(\sqrt{c_i}^{-1}\cdot\mathbb{1}_{\tau_i([0,1)\backslash C)}(x-k)+\sqrt{p_i}^{-1}\cdot\mathbb{1}_{\tau_i(C)}(x-k)\right)\cdot f\left(\tau_i^{-1}(x-k)+Nk+i\right)\Bigg).\end{aligned}$$

T is called the *translation operator* and U is called the *scaling operator*.

Example 10.14. To explain further how the operator U acts on functions in $L^2(\nu_{\mathbb{Z}})$ we visualize its action for the example of the 1/4-Cantor set given by $\left(\tau_0(x)=\frac{x}{4},\tau_2(x)=\frac{x+3}{4}\right)$. This IFS is extended with the function $\tau_1(x)=\frac{x+1/2}{2}$. We consider first the measure on the Cantor set given for the weights $(1/2,1/2)$ by Hutchinson's theorem and for the extension we consider $c_0=c_1=c_2=\frac{1}{2}$. Then we consider the measure given with the weights $p_0=\frac{1}{4}$ and $p_2=\frac{3}{4}$ by Hutchinson's theorem and the extension $c_0=p_0$, $c_2=p_2$ and $c_1=1$. For these two settings we apply the corresponding scaling operators U to the map $x\mapsto x^2$ on $[0,1]$, which are shown in Figure 10.1.1(A) and Figure 10.1.1(B). Notice that this map $x\mapsto x^2$ does not belong to $L^2(\nu_{\mathbb{Z}})$ for any of the two settings since in both cases we have $\nu_{\mathbb{Z}}([0,1])=\infty$.

Remark 10.15.

(1) By the definition of the operator U we can see that U has the same structure as the measure $\nu_{\mathbb{Z}}$, compare the structure of ν given in Lemma 10.8.

(2) Note that we can write the operator U in terms of the representation of a Cuntz algebra. It is well known that for an IFS $(\tau_i)_{i\in A}$ we get the following representation of the Cuntz algebra $\mathcal{O}_p$ on $L^2(\mu)$: for $i\in A$, $f\in L^2(\mu)$, $x\in\operatorname{supp}(\mu)$, set

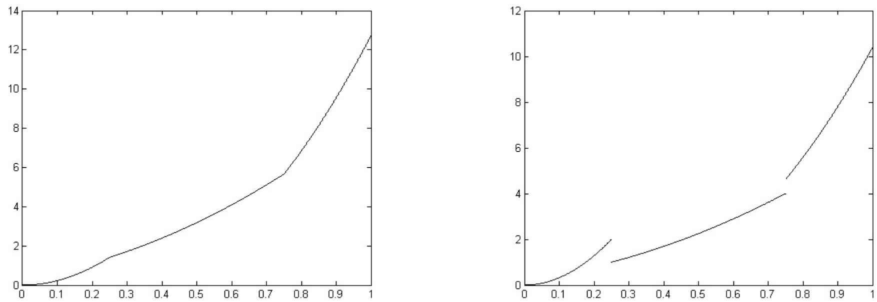
$$S_i f(x) := \sqrt{p_i}^{-1}\cdot\mathbb{1}_{\tau_i(C)}(x)\cdot f(\tau_i^{-1}(x)).$$

Furthermore define for $i\in\underline{N}\backslash A$, $f\in L^2(\nu)$, $x\in\operatorname{supp}(\nu)$,

$$\widetilde{S}_i f(x) := \sqrt{c_i}^{-1}\cdot\mathbb{1}_{\tau_i([0,1])}(x)\cdot f(\tau_i^{-1}(x))$$

and for $i\in A$, $f\in L^2(\nu)$, $x\in\operatorname{supp}(\nu)$,

$$\widetilde{S}_i f(x) := \sqrt{c_i}^{-1}\cdot\mathbb{1}_{\tau_i([0,1]\backslash C)}(x)\cdot f(\tau_i^{-1}(x)).$$



(A) The action of U for $p_0 = p_2 = c_0 = c_1 = c_2 = \frac{1}{2}$.

(B) The action of U for $c_0 = p_0 = \frac{1}{4}$, $c_2 = p_2 = \frac{3}{4}$ and $c_1 = 1$.

FIGURE 10.1.1. Application of the operator U to the map $x \mapsto x^2$, $x \in [0,1]$.

Notice that if for $i \in \underline{N}$ we have that $\widetilde{S}_i f(x) = \sqrt{c_i}^{-1} \cdot \mathbb{1}_{\tau_i([0,1])}(x) \cdot f(\tau_i^{-1}(x))$ and $\sum_{i \in \underline{N}} c_i = 1$, $\left(\widetilde{S}_i\right)_{i \in \underline{N}}$ would be a representation of the Cuntz algebra $\mathcal{O}_N$. So this is only a representation of the Cuntz algebra $\mathcal{O}_N$ if the IFS and the extended IFS coincide. Furthermore, notice that since in the definition of S_i, $i \in A$, and $\widetilde{S}_i$, $i \in \underline{N}$, the characteristic function on $\tau_i(C)$ and $\tau_i([0,1])$ or $\tau_i([0,1]) \backslash C$, respectively, are included we can consider functions $f \in L^2(\nu_{\mathbb{Z}})$ with support exceeding $\operatorname{supp}(\mu)$ and $\operatorname{supp}(\nu)$, respectively. Consequently, we can consider the composition of $S_i^* \widetilde{S}_j$, $i \in A$, $j \in \underline{N}$, and we obtain that $S_i^* \widetilde{S}_j = 0$ for all $i \in A$, $j \in \underline{N}$, and furthermore

$$\widetilde{S}_i^* \widetilde{S}_j = \begin{cases} \delta_{i,j} \cdot \mathbb{1}_{[0,1]} \cdot I, & i,j \in \underline{N} \backslash A, \\ \delta_{i,j} \cdot \mathbb{1}_{[0,1] \backslash C} \cdot I, & i,j \in A, \\ 0, & i \in \underline{N} \backslash A,\ j \in A, \\ 0, & j \in \underline{N} \backslash A,\ i \in A. \end{cases}$$

Then we have that U can be written for $f \in L^2(\nu_{\mathbb{Z}})$ as

$$Uf = \sum_{k \in \mathbb{Z}} \left(\sum_{i \in \underline{N} \backslash A} T^k \widetilde{S}_i T^{-(Nk+i)} f + \sum_{i \in A} T^k \left(\widetilde{S}_i + S_i \right) T^{-(Nk+i)} f \right). \tag{10.1.1}$$

Now we turn to the properties of the operators U and T. We start with their unitarity and the precise form of U^{-1}.

Corollary 10.16. *The operators T and U are unitary in $L^2(\nu_{\mathbb{Z}})$ and the inverse of U is given for $f \in L^2(\nu_{\mathbb{Z}})$, $x \in \mathbb{R}$, by*

$$\begin{aligned} U^{-1} f(x) := & \sum_{k \in \mathbb{Z}} \sum_{i \in \underline{N} \backslash A} \left(\sqrt{c_i} \cdot \mathbb{1}_{[0,1)}(x - i - Nk) \cdot f\left(\tau_i(x - i - Nk) + k\right) \right) \\ & + \sum_{i \in A} \left(\sqrt{c_i} \cdot \mathbb{1}_{[0,1) \backslash C}(x - i - Nk) + \sqrt{p_i} \cdot \mathbb{1}_C(x - i - Nk) \right) \cdot f\left(\tau_i(x - i - Nk) + k\right) \end{aligned} \tag{10.1.2}$$

or equivalently for $f \in L^2(\nu_{\mathbb{Z}})$

$$U^{-1} f = \sum_{k \in \mathbb{Z}} \left(\sum_{i \in \underline{N} \backslash A} T^{i+Nk} \widetilde{S}_i^* T^{-k} f + \sum_{i \in A} T^{i+Nk} \left(\widetilde{S}_i^* + S_i^* \right) T^{-k} f \right). \tag{10.1.3}$$

PROOF. T is a unitary operator since $\nu_{\mathbb{Z}}$ is translation invariant under $\mathbb{Z}$ by definition.

Furthermore, it can be easily verified that the two formulas in (10.1.2) and (10.1.3) define the same operator. To show that U^{-1} has this form we use the notation of (10.1.3). Notice that we only have to consider $\widetilde{S}_i^*\widetilde{S}_i$, $i \in \underline{N}$, and $S_i^*S_i$, $i \in A$, in the following equality by Remark 11.14 (2). Let $f \in L^2(\nu_{\mathbb{Z}})$ and $x \in \mathbb{R}$, then

$$\begin{aligned}
&U^{-1}Uf(x)\\
&=\sum_{l\in\mathbb{Z}}\Bigg(\sum_{j\in\underline{N}\setminus A} T^{j+Nl}\widetilde{S}_j^*T^{-l}\Big(\sum_{k\in\mathbb{Z}}\sum_{i\in\underline{N}\setminus A} T^k\widetilde{S}_iT^{-(Nk+i)}f(x) + +\sum_{i\in A}T^k\left(\widetilde{S}_i+S_i\right)T^{-(Nk+i)}f(x)\Big)\\
&\qquad +\sum_{j\in A}T^{j+Nl}\left(\widetilde{S}_j^*+S_j^*\right)T^{-l}\Big(\sum_{k\in\mathbb{Z}}\sum_{i\in\underline{N}\setminus A}T^k\widetilde{S}_iT^{-(Nk+i)}f(x)+\sum_{i\in A}T^k\left(\widetilde{S}_i+S_i\right)T^{-(Nk+i)}f(x)\Big)\Bigg)\\
&=\sum_{l\in\mathbb{Z}}\Big(\sum_{i\in\underline{N}\setminus A}T^{i+Nl}\widetilde{S}_i^*\widetilde{S}_iT^{Nl-i}f(x)+\sum_{i\in A}T^{i+Nl}\left(\widetilde{S}_i^*\widetilde{S}_i+S_i^*S_i\right)T^{-(Nl+i)}f(x)\Big)\\
&=\sum_{l\in\mathbb{Z}}\Big(\sum_{i\in\underline{N}\setminus A}T^{i+Nl}\left(\mathbb{1}_{[0,1]}(x)T^{Nl-i}f(x)\right)+\sum_{i\in A}T^{i+Nl}\left(\mathbb{1}_{[0,1]}(x)T^{-(Nl+i)}f(x)\right)\Big)\\
&=\sum_{l\in\mathbb{Z}}\sum_{i\in\underline{N}}\mathbb{1}_{[0,1]}(x-i-Nl)f(x)\\
&=f(x),
\end{aligned}$$

where we used in the second equality that only the summands with $k = l$ and $j = i$ are non-zero.

Now it remains to be shown that U is indeed unitary. For this we again use the representation of U^{-1} in (10.1.3). Let $f, g \in L^2(\nu_{\mathbb{Z}})$, then

$$\begin{aligned}
\langle U^{-1}f|g\rangle &= \Big\langle\sum_{k\in\mathbb{Z}}\Big(\sum_{i\in\underline{N}\setminus A}T^{i+Nk}\widetilde{S}_i^*T^{-k}f+\sum_{i\in A}T^{i+Nk}\left(\widetilde{S}_i^*+S_i^*\right)T^{-k}f\Big)\Big|g\Big\rangle\\
&=\sum_{k\in\mathbb{Z}}\Big(\sum_{i\in\underline{N}\setminus A}\langle T^{i+Nk}\widetilde{S}_i^*T^{-k}f\big|g\rangle+\sum_{i\in A}\langle T^{i+Nk}\widetilde{S}_i^*T^{-k}f\big|g\rangle+\langle T^{i+Nk}S_i^*T^{-k}f\big|g\rangle\Big)\\
&=\sum_{k\in\mathbb{Z}}\Big(\sum_{i\in\underline{N}\setminus A}\langle f\big|T^k\widetilde{S}_iT^{-(i+Nk)}g\rangle+\sum_{i\in A}\Big(\langle f\big|T^k\widetilde{S}_iT^{-(i+Nk)}g\rangle+\langle f\big|T^kS_iT^{-(i+Nk)}g\rangle\Big)\Big)\\
&=\langle f|Ug\rangle
\end{aligned}$$

with U having the form of (10.1.1). $\square$

Now we turn to the definition of the father wavelet φ for the MRA. In the following we set $\varphi := \mathbb{1}_C$ and for $z \in \mathbb{T}$ let

$$m_0(z) := \sum_{i\in A}\sqrt{p_i}\cdot z^i.$$

We call the function $m_0 : \mathbb{T} \to \mathbb{C}$ the low-pass filter. Then we have the following relations between the operators U, T and the filter function m_0, where we consider m_0 to be applied to the operator T in z.

Remark 10.17. Often in the literature, compare e.g. [**BJ02**], a low-pass filter m_0 satisfies $m_0(1) = \sqrt{N}$. Notice that although we call m_0 low-pass filter, it does not satisfy this condition since $m_0(1) = \sum_{i\in A}\sqrt{p_i} \neq \sqrt{N}$.

Proposition 10.18. *The operators T and U satisfy the following.*

(1) $U^{-1}TU = T^N$,
(2) $U^{-1}\varphi = m_0(T)\varphi$,
(3) $\langle T^k\varphi|T^l\varphi\rangle = \delta_{k,l},\ k,l \in \mathbb{Z}$.

PROOF. ad (3): This follows directly from $\nu_{\mathbb{Z}}\left((C+k)\cap(C+l)\right)=\delta_{k,l},\ k,l\in\mathbb{Z}$, which is true since the intersection is for $k\neq l$ at most one point and the measure $\nu_{\mathbb{Z}}$ is non-atomic.

ad (1): Let $f\in L^2(\nu_{\mathbb{Z}})$, then

$$
\begin{aligned}
&U^{-1}TUf\\
=&\sum_{k\in\mathbb{Z}}\Bigg(\sum_{i\in\underline{N}\setminus A}T^{i+Nk}\widetilde{S}_i^*T^{-k}\Big(\sum_{l\in\mathbb{Z}}\Big(\sum_{j\in\underline{N}\setminus A}T^{l+1}\widetilde{S}_jT^{-(Nl+j)}f+\sum_{j\in A}T^{l+1}\left(\widetilde{S}_j+S_j\right)T^{-(Nl+j)}f\Big)\Big)\\
&+\sum_{i\in A}T^{i+Nk}\left(\widetilde{S}_i^*+S_i^*\right)T^{-k}\Big(\sum_{l\in\mathbb{Z}}\Big(\sum_{j\in\underline{N}\setminus A}T^{l+1}\widetilde{S}_jT^{-(Nl+j)}f+\sum_{j\in A}T^{l+1}\left(\widetilde{S}_j+S_j\right)T^{-(Nl+j)}f\Big)\Big)\Bigg)\\
=&\sum_{k\in\mathbb{Z}}\Bigg(\sum_{i\in\underline{N}\setminus A}T^{i+Nk}\widetilde{S}_i^*T^{-k}\left(T^k\widetilde{S}_iT^{-(N(k-1)+i)}f\right)\\
&+\sum_{i\in A}T^{i+Nk}\left(\widetilde{S}_i^*+S_i^*\right)T^{-k}\left(T^k\left(\widetilde{S}_i+S_i\right)T^{-(N(k-1)+i)}f\right)\Bigg)\\
=&\sum_{k\in\mathbb{Z}}\Big(\sum_{i\in\underline{N}\setminus A}T^{i+Nk}\left(\mathbb{1}_{[0,1]}Tf\right)+\sum_{i\in A}T^{i+Nk}\left(\mathbb{1}_{[0,1]\setminus C}+\mathbb{1}_C\right)\left(T^{-(N(k-1)+i)}f\right)\Big)\\
=&\sum_{k\in\mathbb{Z}}\Big(\sum_{i\in\underline{N}\setminus A}\mathbb{1}_{[0,1]}(\cdot-i-Nk)+\sum_{i\in A}\left(\mathbb{1}_{[0,1]\setminus C}(\cdot-i-Nk)+\mathbb{1}_C(\cdot-i-Nk)\right)\Big)f(\cdot-N)\\
=&T^Nf,
\end{aligned}
$$

where in the second equality we used that only the summands for $k=l+1$ and $i=j$ are non-zero.

ad (2): Let $x\in\mathbb{R}$, then

$$
\begin{aligned}
U^{-1}\varphi(x)=&\sum_{k\in\mathbb{Z}}\sum_{i\in\underline{N}\setminus A}\left(\sqrt{c_i}\cdot\mathbb{1}_{[0,1)}(x-i-Nk)\cdot\varphi\left(\tau_i(x-i-Nk)+k\right)\right)\\
&+\sum_{i\in A}\left(\sqrt{c_i}\cdot\mathbb{1}_{[0,1)\setminus C}(x-i-Nk)+\sqrt{p_i}\cdot\mathbb{1}_C(x-i-Nk)\right)\cdot\varphi\left(\tau_i(x-i-Nk)+k\right)\\
=&\sum_{i\in A}\sqrt{p_i}\cdot\mathbb{1}_C(x-i)\varphi\left(\tau_i(x-i)\right)\\
=&\sum_{i\in A}\sqrt{p_i}\cdot\mathbb{1}_C(x-i).
\end{aligned}
$$

□

10.2. Construction of the wavelet basis via MRA

Now we turn to the construction of a wavelet basis of $L^2(\nu_{\mathbb{Z}})$. This construction goes along the lines of the MRA, explained in [**BK10**]. Before we state the theorem we further explore how the operators U and T act on the father wavelet φ.

Now we turn to the form of $U^nT^k\varphi,\ n,k\in\mathbb{Z}$. We have that for $n\in\mathbb{N},\ k\in\mathbb{Z}$ with $k=\sum_{i=0}^{n-1}k_iN^{n-1-i}+N^nl,\ k_i\in\underline{N},\ l\in\mathbb{Z}$,

$$U^nT^k\varphi=T^lc\mathbb{1}_{\tau_\omega(C)},$$

where $\omega=(k_0,k_1,\dots,k_{n-1})$ and $c\in\mathbb{R}^+$ depends on all corresponding weights p_{k_j} and c_{k_j} for $j\in\{0,\dots,n-1\}$ that come from the application of the operator U.

For $U^nT^k\varphi,\ n\in\mathbb{Z},\ n<0,\ k\in\mathbb{Z}$, we obtain the following form:

$$U^nT^k\varphi=\sum_{(j_0,\dots,j_{|n|-1})\in A^{|n|}}\left(\prod_{i=0}^{|n|-1}\sqrt{p_{j_i}}\right)T^{\sum_{i=0}^{|n|-1}j_iN^i+N^{|n|}k}\varphi.$$

Remark 10.19. If we consider the action of U and T on the function φ in the shift space, we can see that the operators U and T act together like the shift function on Σ^A, i.e. U^nT^k, $n \in \mathbb{N}$, $k \in \underline{N^n}$, indicates how many and which elements of the alphabet $\underline{N}$ are set in front of Σ^A. So we obtain elements in $\left\{\omega \in \underline{N}^{\mathbb{N}} : \exists n \in \mathbb{N}\ \theta^n(\omega) \in \Sigma^A\right\}$, where θ is the shift map.

Now we come to the main theorem of this section.

Theorem 10.20. *Let the father wavelet be $\varphi := \mathbb{1}_C$ and for $j \in \mathbb{Z}$ let*

$$V_j := \operatorname{cl}\operatorname{span}\left\{U^jT^k\varphi : k \in \mathbb{Z}\right\}.$$

Then $(\nu_{\mathbb{Z}}, U, T)$ allows a two-sided MRA with respect to the father wavelet φ and the subspaces V_j, $j \in \mathbb{Z}$. In particular, it holds that

$$\operatorname{cl}\operatorname{span}\left\{U^nT^k\varphi : n \in \mathbb{Z},\, k \in \mathbb{Z}\right\} = L^2(\nu_{\mathbb{Z}}).$$

Proof. We recall that we have to prove six properties, compare Remark 1.4.

ad (4): This follows directly from Proposition 10.18 (3) and the definition of V_0.

ad (5): This follows directly from the definition of V_j for $j \in \mathbb{Z}$.

ad (6): This was shown in Proposition 10.18 (1).

ad (1): We have that $U^{-1}TU = T^N$ and $\varphi = Um_0(T)\varphi$, compare Proposition 10.18. From this it follows that for $j, k \in \mathbb{Z}$

$$U^jT^k\varphi = U^jT^kUm_0(T)\varphi = U^{j+1}T^{Nk}m_0(T)\varphi$$

and thus $V_j \subset V_{j+1}$.

ad (2): First we consider the case where C is totally disconnected. This proof is similar to the one in [**BLP**$^+$**10**]. We define a function σ for $x \in \mathbb{R}$ by

$$\sigma(x) := \sum_{k\in\mathbb{Z}}\sum_{i\in\underline{N}} \mathbb{1}_{[\tau_i(0),\tau_i(1))}(x-k)\left(\tau_i^{-1}(x-k) + i + Nk\right)$$

and for $x \in \mathbb{R}$ its inverse is

$$\sigma^{-1}(x) = \sum_{k\in\mathbb{Z}}\sum_{i\in\underline{N}} \mathbb{1}_{[i,i+1)}(x-Nk)\left(\tau_i(x-Nk-i)+k\right).$$

This function is bijective on $\mathbb{R}$. With this function we can write the enlarged fractal R as

$$R = \bigcup_{k\in\mathbb{Z}}\left(\bigcup_{\omega\in\Sigma\cup\{\emptyset\}} \tau_\omega(C)\right) + k = \bigcup_{n\in\mathbb{N}} \sigma^{-n}\left(\biguplus_{k\in\mathbb{Z}} C + k\right).$$

We have that it is an increasing union in n of disjoint unions. We claim that it suffices to approximate functions f which have support in $\sigma^{-N}(C+K)$ for $N \geq 0$ and $K \in \mathbb{Z}$. To see why, consider an arbitrary non-negative function $f \in L^2(\nu_{\mathbb{Z}})$ and define for $n \in \mathbb{N}$

$$g_n := f \cdot \mathbb{1}_{\sigma^{-n}(\bigcup_{k\in\mathbb{Z}} C+k)},$$

and for $k \in \mathbb{N}$

$$g_n^k := f \cdot \mathbb{1}_{\sigma^{-n}(\bigcup_{j=-k}^{k} C+j)}.$$

We have that $g_n^k \nearrow g_n$ for $k \to \infty$ and $g_n \nearrow f$ for $n \to \infty$. We additionally have $|g_n^k| \leq f$ for all $n \in \mathbb{N}$, $k \in \mathbb{N}$. Consequently, by the dominated convergence theorem applied twice we obtain that the function f can be approximated by functions with support contained in sets $\sigma^{-n}(C+k)$ for $n \in \mathbb{N}_0$ and $k \in \mathbb{Z}$. For general $f \in L^2(\nu_{\mathbb{Z}})$, we can write $f = f^+ - f^-$ with uniquely defined $f^+, f^- \geq 0$, and apply the above argument to f^+ and f^-.

Now we realize that $T^{-K}U^{-N}f$ has support in C if $f \in L^2(\nu_{\mathbb{Z}})$ satisfies $\operatorname{supp}(f) \subset \sigma^{-N}(C+K)$. We consider the sets $\sigma^{-n}(C+k)$, $n \in \mathbb{N}$, $k \in \mathbb{Z}$, which are contained in C. For each $n \geq 0$, there are exactly p^n sets of the form $\sigma^{-n}(C+k)$, $k \in \mathbb{Z}$, which are contained in C. The sets $\sigma^{-n}(C+k)$, $k \in \mathbb{Z}$, are disjoint and satisfy

$$\sigma^{-n}(C+k) = \bigcup_{j\in A} \sigma^{-(n+1)}(C+j+Nk).$$

Thus, two sets $\sigma^{-n}(C+k) \subset C$, $n \in \mathbb{N}_0$, $k \in \mathbb{Z}$, are either disjoint or one is contained in the other, and it can be easily checked that

$$B := \operatorname{span}\left\{\mathbb{1}_{\sigma^{-n}(C+k)} : n \geq 0,\, k \in \mathbb{Z},\, \sigma^{-n}(C+k) \subset C\right\}$$

is a $*$-subalgebra of $\mathcal{C}(C,\mathbb{C})$. Furthermore B separates points of C since if we consider two distinct points $x, y \in C$ then we can always find two sets $\tau_\omega(C)$ and $\tau_{\widetilde{\omega}}(C)$, $\omega, \widetilde{\omega} \in \Sigma^*$, such that $\tau_\omega(C) \cap \tau_{\widetilde{\omega}}(C) = \emptyset$ and $x \in \tau_\omega(C)$, $y \in \tau_{\widetilde{\omega}}(C)$. So B is uniformly dense in $\mathcal{C}(C,\mathbb{C})$ by the Stone-Weierstrass theorem.

By [**Kig01**], μ is a regular Borel measure on C. It now follows that $\mathcal{C}(C,\mathbb{C})$ is dense in $L^2(\mu)$. Hence, we can find for any $\varepsilon > 0$ a function g in

$$\operatorname{span}\left\{\mathbb{1}_{\sigma^{-n}(C+k)} : n, k \in \mathbb{Z}\right\} = \operatorname{span}\left\{U^n T^k \varphi : n, k \in \mathbb{Z}\right\}$$

such that $\|T^{-K}U^N f - g\| < \varepsilon$. We know that T^K and U^{-N} are unitary and so

$$\|f - U^N T^K g\| = \|T^{-K}U^{-N}\left(f - U^N T^K g\right)\| = \|T^{-K}U^{-N} f - g\| < \varepsilon.$$

Since

$$U^N T^K \left(U^n T^k \varphi\right) = U^{N+n} T^{N^n K + k}\varphi,$$

we see that $U^N T^K g$ has the required form.

In the case $C = [0,1]$ we have that $\mu = \nu$ and we notice that every set $[a,b] \subset [0,1]$ can be approximated by $\tau_\omega(C)$, $\omega \in \Sigma^*$. Hence every element $A \in \mathcal{B}$ can be approximated by elements of $\{\tau_\omega(C) : \omega \in \Sigma^*\}$. Consequently, every elementary function can be approximated by functions $\mathbb{1}_{\tau_\omega(C)}$ and so all functions in $L^2(\mu)$ can be approximated by elements of $\left\{T^k \mathbb{1}_{\tau_\omega(C)} : \omega \in \Sigma^*\right\} = \left\{U^n T^l \varphi : n \in \mathbb{N}_0,\, l \in \mathbb{Z}\right\}$.

ad (3): Clearly $0 \in \bigcap_{j\in\mathbb{Z}} V_j$. Now take $f \in \bigcap_{j\in\mathbb{Z}} V_j$. Then $f \in V_j$ for all $j \in \mathbb{Z}$. Notice that if $0 \neq f \in V_{j_0}$ for some $j_0 \in \mathbb{Z}$ it follows that for some $k \in \mathbb{Z}$, $c \neq 0$, $f|_{\sigma^{-j_0}(C+k)} = c\mathbb{1}_{\sigma^{-j_0}(C+k)}$ and since $(V_j)_{j\leq j_0}$ is a nested sequence it follows that for every $j \leq j_0$ there exists exactly one $k_j \in \mathbb{Z}$ such that $f|_{\sigma^{-j}(C+k_j)} = c\mathbb{1}_{\sigma^{-j}(C+k_j)}$. Consequently, f takes the value c on the nested union $\bigcup_{j\leq j_0} \sigma^{-j}(C+k_j)$. Since this union has infinite measure, f must be constantly 0. □

10.2.1. Filter functions and mother wavelets. The mother wavelets are constructed in terms of filter functions. For the definition of the mother wavelets we need $N-1$ filter functions $m_k : \mathbb{T} \to \mathbb{C}$, $k \in \underline{N}\backslash\{0\}$. These have the form $m_k : z \mapsto \sum_{j\in\underline{N}} a_j^k \cdot z^j$, $k \in \underline{N}\backslash\{0\}$, for some $a_j^k \in \mathbb{C}$, such that

$$M(z) := \frac{1}{\sqrt{N}}\left(m_j(\rho^l z)\right)_{j,l\in\underline{N}}, \tag{10.2.1}$$

where $\rho = e^{2\pi i/N}$, is unitary for almost all $z \in \mathbb{T}$. The condition above on the filter functions can be found in [**BK10, DJ06**].

Remark 10.21. In the case of a multifractal measure, i.e. a measure with different weights, it is more difficult to give a precise definition of the filter functions than if we consider the measure of maximal entropy, see [**BK10**]. In the case of the measure of maximal entropy the first $N-p$ high-pass filters, $m_1, \ldots m_{N-p}$, on $\mathbb{T}$ can be defined as

$$m_{i+1} : z \mapsto z^{d_i},\; i \in \underline{N-p-1},$$

where $d_i \in \mathcal{G} := \{j \in \underline{N}\backslash A\} = \left\{d_j : j \in \underline{N-p}\right\}$, and the last $p-1$ filters are defined with $A = \{b_0, \ldots, b_{p-1}\}$ by

$$m_{N-p+k} : z \mapsto \frac{1}{\sqrt{p}}\sum_{j\in\underline{p}} \eta^{kj} z^{b_j},\text{ for } k \in \underline{p}\backslash\{0\},\; \eta = e^{2\pi i/p}.$$

Lemma 10.22. *The unitarity of the matrix $M(z)$ in (10.2.1) is equivalent to*

$$\textstyle\sum_{i\in A} a_i^k \cdot \sqrt{p_i} \quad = 0 \quad \text{and} \quad \sum_{i\in \underline{N}} a_i^k \overline{a}_i^l = \delta_{k,l},\ \ k,l \in \underline{N}. \tag{10.2.2}$$

Remark 10.23. The condition in (10.2.2) is equivalent to the matrix

$$\begin{pmatrix} \left(a_i^0\right)_{i\in\underline{N}} \\ \left(a_i^k\right)_{i\in\underline{N},k\in\underline{N}\setminus\{0\}} \end{pmatrix},$$

with $a_i^0 = \sqrt{p_i}$, $i \in A$, and $a_i^0 = 0$, $i \in \underline{N}\setminus A$, being unitary. Consequently, this is analogous to the construction of the mother wavelets as considered in Section 8.1. So for further information regarding the construction of the filter functions see Section 8.1.

Proof of Lemma 10.22. Firstly, we know from [**Boh09, DJ06**] that the unitarity of the matrix is equivalent to

$$\frac{1}{N}\sum_{\omega^N=z} m_i(\omega)\overline{m_j(\omega)} = \delta_{i,j}$$

holding for all $i,j \in \underline{N}$. Secondly, we have from [**Boh09, DJ06**] that for any functions $f_1(z) = \sum_{i\in\mathbb{Z}} \alpha_i z^i$ and $f_2(z) = \sum_{i\in\mathbb{Z}} \beta_i z^i$ on $\mathbb{T}$ with $\alpha_i, \beta_i \in \mathbb{C}$

$$\frac{1}{N}\sum_{\omega^N=z} f_1(\omega)\overline{f_2(\omega)} = \sum_{i\in\mathbb{Z}} \alpha_i\overline{\beta_i}.$$

If we apply these results to the filter functions m_i, $i \in \underline{N}$, we get for $f_1 = m_j$ for $j \in \underline{N}\setminus\{0\}$ and $f_2 = m_0$ that

$$\sum_{i\in A} a_i^j \cdot \sqrt{p_i} = 0$$

and for $f_1 = m_j$ and $f_2 = m_k$ for $j,k \in \underline{N}\setminus\{0\}$ that

$$\sum_{i\in\underline{N}} a_i^j \overline{a}_i^k = \delta_{j,k}.$$

On the other hand it follows from $\sum_{i\in\underline{N}} a_i^k\overline{a}_i^l = \delta_{k,l}$, $k,l \in \underline{N}$, that for $i,j \in \underline{N}\setminus\{0\}$

$$\begin{aligned}\frac{1}{N}\sum_{\omega^N=z} m_i(\omega)\overline{m_j(\omega)} &= \frac{1}{N}\sum_{\omega^N=z}\sum_{k\in\underline{N}} a_k^i\omega^k \sum_{l\in\underline{N}} \overline{a}_l^j\omega^{-l} = \sum_{k\in\underline{N}}\sum_{l\in\underline{N}} a_k^i\overline{a}_l^j \cdot \underbrace{\frac{1}{N}\sum_{\omega^N=z}\omega^{k-l}}_{=\delta_{k,l}} \\ &= \sum_{k\in\underline{N}} a_k^i\overline{a}_k^j = \delta_{i,j}.\end{aligned}$$

For m_0 and m_j, $j \in \underline{N}$, we can show the condition analogously with $\sum_{i\in A} a_i^j\sqrt{p_i} = 0$ and so the matrix in (10.2.1) is unitary. □

Example 10.24. Now we give an example with a definition of filter functions. We consider the middle-third Cantor set with the weights $p_0 = \frac{1}{4}$ and $p_1 = \frac{3}{4}$. Then we get as the low-pass filter:

$$m_0(z) = \frac{1}{2} + \frac{\sqrt{3}}{2}z^2. \tag{10.2.3}$$

Now we give two possible choices for the two high-pass filter functions. Our first choice is for $z \in \mathbb{T}$

$$\begin{aligned} m_1^{(1)}(z) &= \sqrt{\tfrac{3}{2}}\left(\tfrac{1}{2} + \tfrac{1}{\sqrt{3}}z - \sqrt{\tfrac{1}{12}}z^2\right), \\ m_2^{(1)}(z) &= \sqrt{\tfrac{3}{2}}\left(-\tfrac{1}{2} + \tfrac{1}{\sqrt{3}}z + \sqrt{\tfrac{1}{12}}z^2\right), \end{aligned} \tag{10.2.4}$$

and another possible choice for the filter functions is for $z \in \mathbb{T}$

$$m_1^{(2)}(z) = z,$$
$$m_2^{(2)}(z) = -\frac{\sqrt{3}}{2} + \frac{1}{2}z^2.$$

Remark 10.25. In the previous example we notice that the filter functions do not depend on the weights c_i, $i \in \underline{N}$, i.e. on the weights on the gaps and on $\tau_i\left([0,1]\backslash C\right)$. They just depend on the original weights of the fractal. The fact that there is not a unique choice of the filter functions can be easily seen from the construction since any orthonormal vectors to the vector given by the low-pass filter can be taken. Via the Gram-Schmidt process we can also see this since we can start with any linearly independent vectors and different choices made at this step may give different families of mother wavelets.

Now we turn to the definition of the mother wavelets and to the orthonormal basis for $L^2(\nu_{\mathbb{Z}})$.

Corollary 10.26. *Define the mother wavelets as*

$$\psi_i := Um_i\left(T\right)\varphi,\ i \in \underline{N}\backslash\{0\}.$$

The set

$$\left\{U^nT^k\psi_i : i \in \underline{N}\backslash\{0\},\, n,k \in \mathbb{Z}\right\}$$

is an ONB for $L^2(\nu_{\mathbb{Z}})$.

This corollary follows directly from Theorem 8.1 and Theorem 10.20 and so we do not give any proof.

Remark 10.27.

(1) Notice that the mother wavelets ψ_j, $j \in \underline{N}\backslash\{0\}$, take the following form:

$$\psi_j = \sum_{i\in\underline{N}\backslash A} a_i^j \cdot \sqrt{c_i}^{-1}\mathbb{1}_{\tau_i(C)} + \sum_{i\in A} a_i^j \cdot \sqrt{p_i}^{-1}\mathbb{1}_{\tau_i(C)}.$$

So they are weighted sums of characteristic functions which are not supported on $\tau_i([0,1]\backslash C)$, $i \in \underline{N}$.

(2) We can write the functions $U^nT^k\varphi$ also in terms of the mother wavelets and $U^{n-1}T^l\varphi$. We illustrate this for $n=1$ and $k \in \underline{N}$. Then we have for $k \in A$

$$UT^k\varphi = \left(p_k\right)^{1/2}\varphi + \sum_{j\in\underline{N}\backslash\{0\}} \overline{a}_k^j\psi_j$$

and for $k \in \underline{N}\backslash A$

$$UT^k\varphi = \sum_{j\in\underline{N}\backslash\{0\}} \overline{a}_k^j\psi_j.$$

We apply the construction of wavelet bases on enlarged fractals to two examples. We start with a continuation of Example 10.24. The second example considers a Julia set in one dimension.

Example (Example 10.24 continued)**.** Recall the setting of Example 10.24. For the middle-third Cantor set with the weights $p_0 = \frac{1}{4}$ and $p_1 = \frac{3}{4}$ we have the low-pass filter as given in (10.2.3) and a possible choice of high-pass filters is given in (10.2.4). For these filter function the corresponding mother wavelets are

$$\psi_1(x) = \sqrt{\frac{3}{2}}\left(\mathbb{1}_{\tau_0(C)}(x) + \frac{1}{\sqrt{3}}\cdot\sqrt{c_1}^{-1}\mathbb{1}_{\tau_1(C)}(x) - \frac{1}{3}\cdot\mathbb{1}_{\tau_2(C)}\right),$$
$$\psi_2(z) = \sqrt{\frac{3}{2}}\left(-\mathbb{1}_{\tau_0(C)}(x) + \frac{1}{\sqrt{3}}\cdot\sqrt{c_1}^{-1}\mathbb{1}_{\tau_1(C)}(x) + \frac{1}{3}\cdot\mathbb{1}_{\tau_2(C)}\right),$$

where c_1 is the weight given to $\tau_1([0,1])$. Here we can see that in the definition of the mother wavelets the weights on the gaps $\tau_i([0,1])$, $i \in \underline{N}\backslash A$, have an influence on the orthonormality of the basis.

We give one example of this construction applied to a member of an interesting class of fractals, namely Julia sets. It is not possible to apply this wavelet construction to all Julia sets.

Example 10.28. We consider an example of a Cantor like Julia set, compare [**Bea91**]. The Julia set is given by the inverse map of the IFS. Let $F : x \mapsto 2x - 1/x$ on $J = [-1, 1]$ with a pole at $x = 0$. This function F is strictly increasing on each of the intervals $I_0 = [-1, -1/2]$ and $I_1 = [1/2, 1]$. Consequently, the two inverse branches $\sigma_0 := \left(F|_{[-1,-1/2]}\right)^{-1}$ and $\sigma_1 := \left(F|_{[0,1/2]}\right)^{-1}$ satisfy $\sigma_0(J) = I_0$ and $\sigma_1(J) = I_1$ and since $|F'(x)| > 2$ on I_0 and I_1 it follows that $|\sigma_0'(x)| < 1/2$ and $|\sigma_1'(x)| < 1/2$. Thus, these are contractions and we can consider the limit set C for σ_0 and σ_1. Furthermore, there exists an invariant probability measure on any Julia set by results of [**Bea91, Bro65**]. This measure will be denoted by μ and it satisfies $\mu = \frac{1}{2}\left(\mu \circ \sigma_0^{-1} + \mu \circ \sigma_1^{-1}\right)$.

Now we define the extended IFS. We extend the IFS (σ_0, σ_1) to the IFS $\mathbb{S} = (\tau_0 = \sigma_0,\ \tau_1,\ \tau_2 = \sigma_1)$ with the function $\tau_1 : x \mapsto \frac{x}{2}$, $x \in J$. For this we can now construct the enlarged fractal and the measure $\nu_{\mathbb{Z}}$ on it as given in Definition 10.4 and Definition 10.6 with one small change. Namely, the translation has to be done by 2 instead of 1, since in this example we start with the interval $[-1, 1]$; an interval of length two. Consequently,

$$R := R_{[0,1]} + 2\mathbb{Z} = \bigcup_{l\in\mathbb{Z}} R_{[0,1]} + 2l,$$

and $\nu_{\mathbb{Z}} : \mathcal{B} \to \mathbb{R}_0^+$, $B \mapsto \sum_{k\in\mathbb{Z}} \nu(B + 2k)$. For the definition of the wavelet basis we have to induce the same change in the definition of the operators U and T, i.e. $Tf(x) = f(x - 2)$ and for $c_0 = c_2 = \frac{1}{2}$, $c_1 \in \mathbb{R}^+$, $f \in L^2(\nu_{\mathbb{Z}})$, $x \in \mathbb{R}$,

$$\begin{aligned} Uf(x) = \sum_{k\in\mathbb{Z}} & \sqrt{2} \cdot \mathbb{1}_{[-1,-1/2)}(x-k) \cdot f\left(\tau_0^{-1}(x-k) + 6k\right) \\ & + \sqrt{c_1}^{-1} \cdot \mathbb{1}_{[-1/2,1/2)}(x-k) f\left(\tau_1^{-1}(x-k) + 6k + 2\right) \\ & + \sqrt{2} \cdot \mathbb{1}_{[1/2,1)}(x-k) \cdot f\left(\tau_2^{-1}(x-k) + 6k + 4\right). \end{aligned}$$

The wavelet basis can now be constructed in the way explained above with the father wavelet $\varphi = \mathbb{1}_C$. The corresponding low-pass filter is $m_0(z) = \frac{1}{\sqrt{2}}(1 + z^2)$. Possible high-pass filters are $m_1(z) = z$ and $m_2(z) = \frac{1}{\sqrt{2}}(1 - z^2)$. The corresponding mother wavelets are $\psi_1 = \frac{1}{\sqrt{c_1}} \mathbb{1}_{\tau_1(C)}$ and $\psi_2 = \mathbb{1}_{\tau_0(C)} - \mathbb{1}_{\tau_2(C)}$. So the set

$$\left\{U^n T^k \psi_i : n, k \in \mathbb{Z}, i = 1, 2\right\}$$

is an ONB for $L^2(\nu_{\mathbb{Z}})$.

10.3. Further results for MRA for the measure of maximal entropy

In this section we want to further consider the MRA in the way we defined it in [**BK10**]. We consider a non-atomic measure $\nu_{\mathbb{Z}}$ on $(\mathbb{R}, \mathcal{B})$ such that $\nu_{\mathbb{Z}}(A) = \nu_{\mathbb{Z}}(A + k)$, $A \in \mathcal{B}$, $k \in \mathbb{Z}$, and $\mathrm{cl}\left(\mathrm{supp}(\nu_{\mathbb{Z}}|_{[0,1]})\right) = [0, 1]$. Furthermore the scaling operator U is given in terms of a scaling function σ. This function $\sigma : \mathbb{R} \to \mathbb{R}$ is a continuous, bijective, expanding function with $|\sigma(x) - \sigma(y)| > c \cdot |x - y|$ for all $x, y \in \mathbb{R}$, $c > 1$, such that for some fixed $N \in \mathbb{N}$ and $p \in \mathbb{N}$

$$\begin{aligned} \sigma(x + k) &= \sigma(x) + Nk, && x \in [0, 1], \ k \in \mathbb{Z}, \\ \nu_{\mathbb{Z}}(\sigma(A)) &= p\nu_{\mathbb{Z}}(A), && A \in \mathcal{B}. \end{aligned} \tag{10.3.1}$$

Then we define the scaling operator U acting on $L^2(\nu_{\mathbb{Z}})$ as $Uf(\cdot) = \sqrt{p} f(\sigma(\cdot))$ and we can rewrite the property $UV_j = V_{j+1}$ of the MRA as $f \in V_j \iff f \circ \sigma \in V_{j+1}$, $j \in \mathbb{Z}$.

For this setting we deduce properties of the measure $\nu_{\mathbb{Z}}$ and the scaling function σ. We assume that $\nu_{\mathbb{Z}}$ is locally finite, i.e. we assume $\nu_{\mathbb{Z}}([0, 1]) < \infty$.

Lemma 10.29. *If $\nu_{\mathbb{Z}}([0, 1]) < \infty$ it follows that $p = N$ in (10.3.1).*

PROOF. We have that $\nu_{\mathbb{Z}} \circ \sigma = p \cdot \nu_{\mathbb{Z}}$ and $\sigma(x+k) = x+Nk$. Consequently, we have $\sigma([0,1]) = [0,N]$ and so

$$p \cdot \nu_{\mathbb{Z}}([0,1]) = \nu_{\mathbb{Z}}(\sigma([0,1])) = \nu_{\mathbb{Z}}([0,N]) = N \cdot \nu_{\mathbb{Z}}([0,1]),$$

since $\nu_{\mathbb{Z}}$ is translation invariant under $\mathbb{Z}$. Thus, $p = N$. $\square$

Remark 10.30.

(1) Without loss of generality we set $\nu_{\mathbb{Z}}([0,1]) = 1$.
(2) Notice that we always have $\nu_{\mathbb{Z}}(\sigma^{-1}([k,k+1])) = N^{-1}$ for $k \in \underline{N}$. Thus, if we consider $\tau_k(x) = \sigma^{-1}(x+k)$, $x \in [0,1]$, $k \in \underline{N}$, as an IFS with the invariant set $[0,1]$ and if $\operatorname{supp}\left(\nu_{\mathbb{Z}}|_{[0,1]}\right) = [0,1]$ then the measure $\nu_{\mathbb{Z}}|_{[0,1]}$ coincides with the measure of maximal entropy, since $\nu_{\mathbb{Z}}|_{[0,1]} = \frac{1}{N}\sum_{k\in\underline{N}} \nu_{\mathbb{Z}}|_{[0,1]} \circ \tau_k^{-1}$.

Proposition 10.31. *The measure $\nu_{\mathbb{Z}}$ is unique with the properties in (10.3.1).*

Remark 10.32. The assertion of uniqueness is that if m is any measure satisfying the property (10.3.1), we must have $\operatorname{supp}(m|_{[0,1]}) = \operatorname{supp}(\nu_{\mathbb{Z}}|_{[0,1]})$ up to a set of measure zero.

PROOF. We can prove the uniqueness via Hutchinson's theorem, compare Theorem A.9. First notice that the measure $\nu_{\mathbb{Z}}$ is translation invariant under $\mathbb{Z}$. Consequently, it is enough to consider the measure on $[0,1]$. To see the uniqueness consider an IFS obtained from σ^{-1} where the functions are given by $\tau_k(x) = \sigma^{-1}(x+k)$, $x \in [0,1]$, $k \in \underline{N}$. Since σ is expanding the τ_k are contractions. This IFS satisfies the OSC for $(0,1)$ and its invariant set is $[0,1]$. Hence, there is a unique invariant measure ν on $[0,1]$ such that

$$\nu = \frac{1}{N}\sum_{k\in\underline{N}} \nu \circ \tau_k^{-1}.$$

Now define $\widetilde{\nu_{\mathbb{Z}}} : A \mapsto \sum_{k\in\mathbb{Z}} \nu(A-k)$. We will show that $\widetilde{\nu_{\mathbb{Z}}}$ satisfies the condition $\widetilde{\nu_{\mathbb{Z}}} \circ \sigma = N \cdot \widetilde{\nu_{\mathbb{Z}}}$. First notice that $\tau_k^{-1}(x) = \sigma(x) - k$ and

$$\widetilde{\nu_{\mathbb{Z}}}(\sigma(A)) = \sum_{k\in\mathbb{Z}} \nu(\sigma(A)-k) = \sum_{i\in\underline{N}}\sum_{l\in\mathbb{Z}} \nu(\sigma(A) - Nl - i) = \sum_{l\in\mathbb{Z}} \nu\left(\bigcup_{i=0}^{N-1} \tau_i^{-1}(A-l)\right)$$

and

$$\nu\Big(\bigcup_{i\in\underline{N}} \tau_i^{-1}(A-l)\Big) = \sum_{i\in\underline{N}} \nu(\tau_i^{-1}(A-l)) = N \cdot \nu(A-l).$$

Thus,

$$\widetilde{\nu_{\mathbb{Z}}}(\sigma(A)) = \sum_{l\in\mathbb{Z}} N \cdot \nu(A-l) = N \cdot \widetilde{\nu_{\mathbb{Z}}}(A).$$

Since the measure is uniquely defined on $[0,1]$ by Hutchinson's theorem, compare Theorem A.9, or [**Hut81**], and it is translation invariant, so it follows that $\widetilde{\nu_{\mathbb{Z}}} = \nu_{\mathbb{Z}}$ if $\operatorname{supp}\left(\nu_{\mathbb{Z}}|_{[0,1]}\right) = [0,1]$. Otherwise we can still prove that $\widetilde{\nu_{\mathbb{Z}}}$ and $\nu_{\mathbb{Z}}$ take the same values on intervals in $[0,1]$ and hence they differ only by a set of measure zero. To see that both take the same measure on intervals consider the interval $[a,b] \subset [0,1]$. Then the measures of $[0,a)$ and $[0,b]$ can both be approximated by sets $\sigma^{-j}([k,k+1])$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$. We get $\nu_{\mathbb{Z}}([0,a)) = \sum_{i=1}^{\infty} r_i N^{-i}$ and $\nu_{\mathbb{Z}}([0,b]) = \sum_{i=1}^{\infty} s_i N^{-i}$, where the coefficients r_i are given as

$$\begin{aligned}
r_1 &= \max\{j \in \mathbb{N}_0 : \sigma^{-1}\left([j,j+1]\right) \le a\},\\
r_2 &= \max\{j \in \mathbb{N}_0 : \sigma^{-2}\left([r_1 + Nj, r_1 + N(j+1)]\right) \le a\},\\
r_n &= \max\left\{j \in \mathbb{N}_0 : \sigma^{-n}\left(\left[\sum_{i=1}^{n-1} r_i N^{i-1} + N^{n-1}j, \sum_{i=1}^{n-1} r_i N^{i-1} + N^{n-1}(j+1)\right]\right) \le a\right\},
\end{aligned}$$

and the coefficients s_i are defined in the analogous way for b, more precisely

$$s_1 = \max\{j \in \mathbb{N}_0 : \sigma^{-1}([j, j+1]) \leq b\},$$

$$s_n = \max\left\{j \in \mathbb{N}_0 : \sigma^{-n}\left(\left[\sum_{i=1}^{n-1} s_i N^{i-1} + N^{n-1}j, \sum_{i=1}^{n-1} s_i N^{i-1} + N^{n-1}(j+1)\right]\right) \leq b\right\}.$$

Since the measure $\widetilde{\nu_{\mathbb{Z}}}$ satisfies the same relations as $\nu_{\mathbb{Z}}$, we get exactly the same formulas for $\widetilde{\nu_{\mathbb{Z}}}([0,a))$ and $\widetilde{\nu_{\mathbb{Z}}}([0,b])$, and hence $\nu_{\mathbb{Z}} = \widetilde{\nu_{\mathbb{Z}}}$. $\square$

Proposition 10.33. *The measure $\nu_{\mathbb{Z}}$ is the Stieltjes measure for the homeomorphism $\phi : \mathbb{R} \to \mathbb{R}$ satisfying $\nu_{\mathbb{Z}} = \lambda \circ \phi^{-1}$, where λ is the Lebesgue measure on $\mathbb{R}$. Furthermore let $\widetilde{\sigma}(x) := Nx$, $x \in \mathbb{R}$, then $\phi \circ \sigma = \widetilde{\sigma} \circ \phi$.*

The connection of the last proposition can be illustrated in a commutative diagram as follows:

$$\begin{tikzcd} L^2(\nu_{\mathbb{Z}}) \arrow[r, "\sigma"] \arrow[d, "\phi"'] & L^2(\nu_{\mathbb{Z}}) \arrow[d, "\phi"] \\ L^2(\mathbb{R}, \lambda) \arrow[r, "\widetilde{\sigma}"] & L^2(\mathbb{R}, \lambda) \end{tikzcd}$$

Proof of Proposition 10.33. The homeomorphism ϕ is given in [**BK10**] as the fixed point of the operator $F : E \to E$ given by

$$(Ff)(x) = \sum_{i \in \underline{N}} \widetilde{\tau}_i \circ f \circ \tau_i^{-1}(x) \cdot \mathbb{1}_{[\tau_i(0), \tau_i(1))}(x) + \mathbb{1}_{\{1\}}(x), \ x \in [0,1],$$

where $E := \{f \in \mathcal{C}([0,1],[0,1]) : f(0) = 0, f(1) = 1, f : [0,1] \to [0,1]\}$ and $\widetilde{\tau}_i(x) = \frac{x+i}{N}$ for $i \in \underline{N}$, $x \in [0,1]$.

Now we show that for $k \in \mathbb{Z}$, $l \in \mathbb{N}$,

$$\nu_{\mathbb{Z}}\left([\sigma^{-l}(k), \sigma^{-l}(k+1)]\right) = \phi(\sigma^{-l}(k+1)) - \phi(\sigma^{-l}(k)).$$

This follows directly from the following two observations. First notice that $\nu_{\mathbb{Z}}(\sigma^{-l}([k,k+1])) = N^{-l}$ and the homeomorphism ϕ maps

$$\phi(\sigma^{-l}(k)) = \widetilde{\sigma}^{-l}(\phi(k)) = N^{-l}\phi(k) = N^{-l}k$$

by the conditions on ϕ.

For arbitrary intervals $[a,b]$, $a, b \in \mathbb{R}$, we consider the measure $\nu_{\mathbb{Z}}([a,b])$. The measures of $[0,a)$ and $[0,b]$ are calculated in the proof of Proposition 10.31. Thus we have

$$\nu_{\mathbb{Z}}([a,b]) = \sum_{i=1}^{\infty} s_i N^{-i} - \sum_{i=1}^{\infty} r_i N^{-i} = \phi(b) - \phi(a).$$

$\square$

Corollary 10.34. *The function $h(x) := \nu_{\mathbb{Z}}((-\infty, x])$, $x \in \mathbb{R}$, satisfies $h \circ \sigma = N \cdot h$.*

Proof. The property follows from $h(\sigma(x)) = \nu_{\mathbb{Z}}((-\infty, \sigma(x)]) = \nu_{\mathbb{Z}}(\sigma((-\infty, x])) = N \cdot h(x)$, $x \in \mathbb{R}$. $\square$

Now we start with the construction of the wavelet basis. This goes along the lines of Section 11.3 or [**BK10**] and consequently, we omit the proofs.

For this case the two unitary operators U and T are defined as $Tf(x) = f(x-1)$ and $Uf(x) = \sqrt{N}f(\sigma(x))$ for $f \in L^2(\nu_{\mathbb{Z}})$, $x \in \mathbb{R}$. The father wavelet is $\varphi = \mathbb{1}_{[0,1]}$ and so the corresponding filter function is $m_0(z) = \frac{1}{\sqrt{N}} \sum_{i \in \underline{N}} z^i$, $z \in \mathbb{T}$. These satisfy the properties in Proposition 10.18. In this case we have that $\varphi = \sum_{i \in \underline{N}} \varphi(\sigma(\cdot) - i) = Um_0(T)\varphi$.

Corollary 10.35. *Let the father wavelet be* $\varphi := \mathbb{1}_{[0,1]}$ *and for* $j \in \mathbb{Z}$ *let*

$$V_j := \operatorname{cl}\operatorname{span}\left\{U^j T^k \varphi :\ k \in \mathbb{Z}\right\},$$

then $(\nu_{\mathbb{Z}}, U, T)$ *allows a two-sided MRA with respect to* φ*,* V_j*,* $j \in \mathbb{Z}$*, as above. In particular,*

$$\operatorname{cl}\operatorname{span}\left\{U^n T^k \varphi :\ k, n \in \mathbb{Z}\right\} = L^2(\nu_{\mathbb{Z}}).$$

For the construction of the mother wavelets we have to get $N-1$ high-pass filters m_i, $i \in \underline{N}\backslash\{0\}$. These can be constructed as in [**BK10, DJ06**], compare Remark 10.21.

Corollary 10.36. *Define the mother wavelets for* $i \in \underline{N}\backslash\{0\}$ *as* $\psi_i := U m_i(T)\varphi$*. The set*

$$\{U^n T^k \psi_i :\ i \in \underline{N}\backslash\{0\},\ n,\ k \in \mathbb{Z}\}$$

is an ONB for $L^2(\nu_{\mathbb{Z}})$*.*

Remark 10.37. It is also possible to show that the cascade operator converges under the same conditions on the filter functions as in the case for the Lebesgue measure with a scaling by N. The proof given in [**BJ02**] can be given in an analogous way for this setting of the MRA. So we can take exactly the same filter functions to obtain a father wavelet. Hence we can obtain many different wavelet bases on the space $L^2(\nu_{\mathbb{Z}})$. More precisely, the wavelets that are carried over by ϕ from $L^2(\mathbb{R}, \lambda)$ with the scaling operator $Uf(x) = \sqrt{N} f(Nx)$, $f \in L^2(\mathbb{R}, \lambda)$, $x \in \mathbb{R}$, also give a basis in $L^2(\nu_{\mathbb{Z}})$.

Part 3

Wavelet bases on fractals in the plane

CHAPTER 11

Wavelet bases on enlarged fractals in two dimensions

In this chapter we construct wavelet bases on enlarged fractals in two dimensions. This construction is analogous to the one dimensional construction, but in two dimensions there are more restrictions on the underlying fractal. We start with the setup for the multiresolution analysis. First we define the enlarged fractal for an IFS and the measure on it. Then we continue with the construction of the wavelet basis. For this we define two unitary operators and prove the properties of the MRA. In the next step we define the mother wavelets and obtain an orthonormal basis.

11.1. Setting for the construction of wavelet bases on enlarged fractals

For the construction of the MRA we consider a compact set $D \subset \mathbb{R}^2$ that gives a tiling of $\mathbb{R}^2$. By this we mean that there are linearly independent vectors $\overrightarrow{v_1}$, $\overrightarrow{v_2} \in \mathbb{R}^2$ such that $\mathbb{R}^2 = D + \mathbb{Z}\overrightarrow{v_1} + \mathbb{Z}\overrightarrow{v_2}$, and such that distinct translates of D by elements of $\mathbb{Z}\overrightarrow{v_1} + \mathbb{Z}\overrightarrow{v_2}$ are essentially disjoint in the sense that their intersection has measure zero (the measure under consideration is given in Proposition 11.7).

We start with an IFS

$$\mathcal{S} = \left(\sigma_i : D \to D : i \in \{0, \ldots, p-1\} = \underline{p}\right),$$

$p > 2$, consisting of p contractions with a contraction constant $0 < c_{\mathcal{S}} < 1$ and the IFS satisfies the open set condition for $\mathring{D}$, the interior of D. This IFS gives a fractal $C \subset D$, satisfying $C = \bigcup_{i \in \underline{p}} \sigma_i(C)$, i.e. it is the limit set of $\mathcal{S}$, by Hutchinson's theorem (see Theorem A.8).

Furthermore, we assume that there exists an extension of the IFS $\mathcal{S}$ to an IFS

$$\mathbb{S} = \left(\tau_{(i,j)} : (i,j) \in K\right),$$

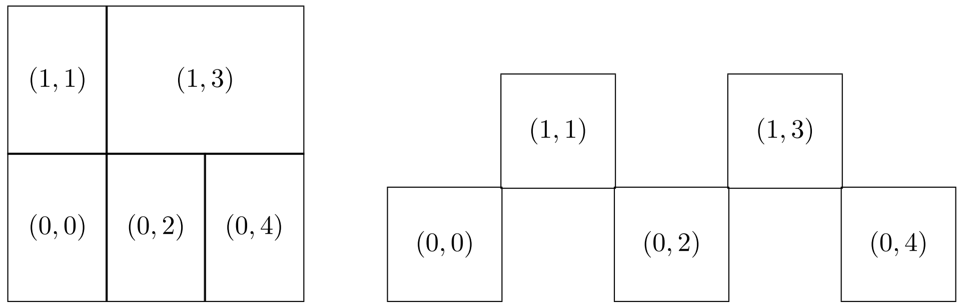
where $K \subset \mathbb{Z}^2$, $N := \operatorname{card} K \geq p$, consisting of contractions with contraction constant $0 < c_{\mathbb{S}} < 1$, such that $D = \bigcup_{(i,j) \in K} \tau_{(i,j)}(D)$, satisfying the OSC for $\mathring{D}$. In addition there exists a set $A \subset K$ such that

$$\left\{\sigma_i : i \in \underline{p}\right\} = \left\{\tau_{(i,j)} : (i,j) \in A\right\}$$

and there exist numbers $N_1, N_2 \in \mathbb{N}$ with $N = N_1 N_2$ and

$$\mathbb{R}^2 = \biguplus_{(k,l) \in \mathbb{Z}^2} \biguplus_{(i,j) \in K} D + (N_1 k + i)\overrightarrow{v_1} + (N_2 l + j)\overrightarrow{v_2}. \tag{11.1.1}$$

Example 11.1. An example for this setting is the division of the unit square as shown in Figure 11.1.1(A), where $K = \{(0,0), (0,2), (1,1), (0,4), (1,3)\}$ and the fractal is given by affine functions mapping to the parts $(1,1), (0,2), (1,3)$. The unit square $D = [0,1] \times [0,1]$ gives a tiling with the vectors $\overrightarrow{v_1} = (1,0)^t$ and $\overrightarrow{v_2} = (0,1)^t$. There are five subsets in the unit square (as illustrated in Figure 11.1.1(A)) and so $N = 5$, thus a choice for N_1, N_2 is $N_1 = 5$ and $N_2 = 1$. For $K = \{(0,0), (0,2), (1,1), (0,4), (1,3)\}$ we obtain $F = D \cup (D + 2\overrightarrow{v_2}) \cup (D + 4\overrightarrow{v_2}) \cup (D + \overrightarrow{v_1} + \overrightarrow{v_2}) \cup (D + \overrightarrow{v_1} + 3\overrightarrow{v_2})$ as shown in Figure 11.1.1(B) and this clearly satisfies the condition given in (11.1.1).



(A) Division of the unit square.

(B) Display of the set F.

FIGURE 11.1.1. Example of a division of the unit square and its set F.

In the following we focus on the special case

$$\mathbb{S} = \left(\tau_{(i,j)} \,:\, (i,j) \in \underline{N_1} \times \underline{N_2}\right).$$

The numbers N_1 and N_2 play a role in the definition of the scaling operator U for the MRA. If there exists an extended IFS, one possible choice for the pair N_1, N_2 is with $N_1 = 1$. If we want to stay close to the structure of fractal, that is the position of the parts belonging to the fractal, N_1 and N_2 should be chosen such that $\bigcup_{(i,j)\in\underline{N_1}\times\underline{N_2}} D + i\overrightarrow{v_1} + j\overrightarrow{v_2}$ is an augmented version of D. More precisely, $\bigcup_{(i,j)\in\underline{N_1}\times\underline{N_2}} D + i\overrightarrow{v_1} + j\overrightarrow{v_2} = \Psi(D)$, where Ψ is an affine transformation with $\Psi(\overrightarrow{v_1}) = a$, $\Psi(\overrightarrow{v_2}) = b$ for some $a, b \in \mathbb{R}^+$, $a, b > 1$. Another natural condition could be $\bigcup_{(i,j)\in\underline{N_1}\times\underline{N_2}} C + i\overrightarrow{v_1} + j\overrightarrow{v_2} = \Psi(C)$. This is satisfied e.g. in the example of [**DMP08**]. But this does not always exist since e.g. if the contractions $\tau_{(i,j)}$ are not affine, then it is not possible to obtain this relation $\bigcup_{(i,j)\in\underline{N_1}\times\underline{N_2}} C + i\overrightarrow{v_1} + j\overrightarrow{v_2} = \Psi(C)$. Furthermore, if the number N of functions in the extended IFS is prime, then the only choice is $N_1 = 1$, $N_2 = N$ (or $N_1 = N$, $N_2 = 1$).

Remark 11.2.
(1) Such an extension $\mathbb{S}$ need not to exist for a given $\mathcal{S}$, and when it does exist, it is not unique (compare Example 11.6).
(2) For the construction it is important that the indices (i,j) run through all of $\underline{N_1} \times \underline{N_2}$, not just through a subset of $\underline{N_1} \times \underline{N_2}$.
(3) For the extension $\mathbb{S}$ of the IFS we define contractions on the gaps of the original IFS $\mathcal{S}$.
(4) Notice that the possibility of "gap filling functions" and the tiling of the space are two separate conditions since e.g. the hexagon allows a tiling of the space but it is not possible to cover the hexagon with affine scaled copies of itself such that there is no overlap between the copies with respect to the Lebesgue measure.

On the fractal C we consider a measure given by Hutchinson's theorem, (see [**Hut81**] or Theorem A.9) with arbitrary weights on the subsets of the fractal. More precisely, we consider the measure μ on C for weights $p_{(i,j)} \in (0,1)$, $(i,j) \in A$ with $\sum_{(i,j)\in A} p_{(i,j)} = 1$ satisfying

$$\mu = \sum_{(i,j)\in A} p_{(i,j)} \cdot \mu \circ \tau_{(i,j)}^{-1}.$$

Remark 11.3. If μ has the same weights $p_{(i,j)} = \frac{1}{p}$ for $(i,j) \in A$, then μ is the measure of maximal entropy if the shift dynamical system corresponding to the IFS is considered.

11.2. Definitions of the enlarged fractal and a measure on it

The measure of the L^2-space on which the wavelet basis will be constructed has a dense set in $\mathbb{R}^2$ as support. This set is called the enlarged fractal. The enlarged fractal is first defined in D by filling the gaps with scaled copies of the fractal and then it is extended to $\mathbb{R}^2$ by translation. We start by clarifying some notation. For $\omega := ((i_0, j_0), \ldots, (i_{k-1}, j_{k-1})) \in \left(\underline{N_1} \times \underline{N_2}\right)^k$ let $\tau_\omega := \tau_{(i_0,j_0)} \circ \cdots \circ \tau_{(i_{k-1},j_{k-1})}$, let $\tau_\emptyset = \mathrm{id}$ be the identity on D and let $|\omega| = k$ denote the length of ω with $|\emptyset| := 0$. We let $\Sigma^{*,2}$ denote the set of all words of finite length with the alphabet $\underline{N_1} \times \underline{N_2}$, more precisely $\Sigma^{*,2} = \bigcup_{n=1}^{\infty} \left(\underline{N_1} \times \underline{N_2}\right)^n$. Furthermore, let $\mathcal{B}$ denote the Borel σ-algebra on $\mathbb{R}^2$.

Definition 11.4. Define the *enlarged fractal* in D in two dimensions by

$$S := \bigcup_{k \geq 0} \bigcup_{\omega \in \left(\underline{N_1} \times \underline{N_2}\right)^k} \tau_\omega(C) = \bigcup_{\omega \in \Sigma^{*,2}} \tau_\omega(C) \subset D,$$

and define the *enlarged fractal* in $\mathbb{R}^2$ by

$$R := \bigcup_{(k,l) \in \mathbb{Z}^2} S + k\overrightarrow{v_1} + l\overrightarrow{v_2}.$$

Fact 11.5. *Let*

$$\Sigma^{(2)} := \left\{ ((i_0, j_0), \ldots, (i_{k-1}, j_{k-1})) \in \left(\underline{N_1} \times \underline{N_2}\right)^k,\ k \in \mathbb{N},\ (i_{k-1}, j_{k-1}) \notin A \right\}.$$

Then S can be written as

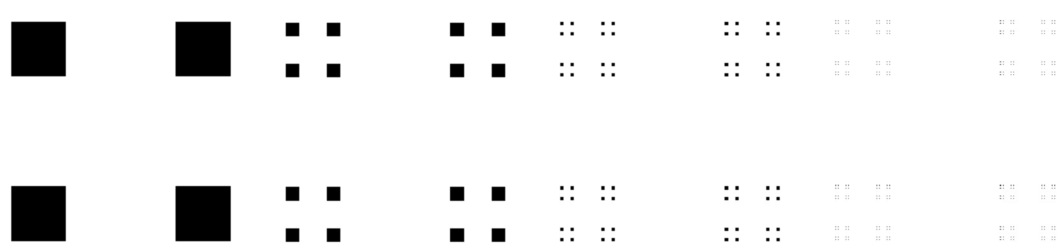
$$S = \biguplus_{\omega \in \Sigma^{(2)} \cup \{\emptyset\}} \tau_\omega(C).$$

Now we give two examples for non-trivial enlarged fractals R.

Example 11.6. We give two examples which both have as the underlying fractal the $1/4$-Cantor Dust. In this example we have $D = [0,1] \times [0,1]$, $\overrightarrow{v_1} = (1, 0)$ and $\overrightarrow{v_2} = (0, 1)$. The fractal is given by the IFS $\mathcal{S} = (\sigma_i : i \in \underline{4})$ with

$$\sigma_i(\overrightarrow{x}) = \begin{pmatrix} 1/4 & 0 \\ 0 & 1/4 \end{pmatrix} \cdot \overrightarrow{x} + a_i, \ \overrightarrow{x} \in [0,1] \times [0,1],$$

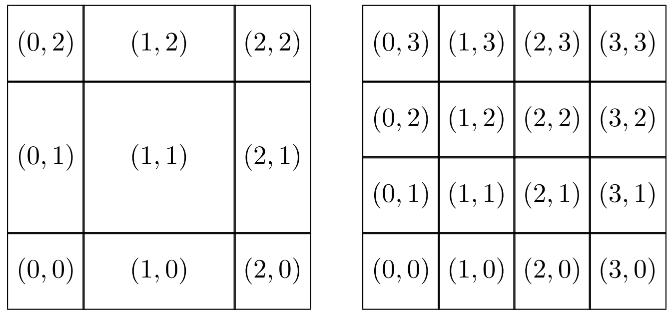
$a_i \in \left\{ \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 3/4 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 3/4 \end{pmatrix}, \begin{pmatrix} 3/4 \\ 3/4 \end{pmatrix} \right\}$. This fractal is obtained iteratively as shown in Figure 11.2.1.



(A) Prefractal of order 1. (B) Prefractal of order 2. (C) Prefractal of order 3. (D) Prefractal of order 4.

FIGURE 11.2.1. Prefractals of the Cantor Dust.

(a) In the first case we extend the IFS $\mathcal{S}$ to the IFS $\mathbb{S}_1 = \left(\tau_{(i,j)} : (i,j) \in \underline{3} \times \underline{3}\right)$ as indicated in Figure 11.2.2(A), with affine contractions on the gaps. Consequently, the set $A = \{(0,0), (2,0), (0,2), (2,2)\}$. In this example we have that $3 \cdot D = \bigcup_{(i,j) \in \underline{3} \times \underline{3}} D + i\overrightarrow{v_1} + j\overrightarrow{v_2}$. But on the other hand we have that $3 \cdot C \neq \bigcup_{(i,j) \in \underline{3} \times \underline{3}} C + i\overrightarrow{v_1} + j\overrightarrow{v_2}$.



(A) Division of square for $\mathbb{S}_1$. (B) Division of square for $\mathbb{S}_2$.

FIGURE 11.2.2. The division of unit square for the Cantor Dust.

(b) Another possible definition for the extended IFS $\mathbb{S}_2 = \left(\tau_{(i,j)} : (i,j) \in \underline{4} \times \underline{4}\right)$ is as indicated in Figure 11.2.2(B), with affine contractions on the gaps. In this case $A = \{(0,0),(3,0),(0,3),(3,3)\}$. In this example we have that $4 \cdot D = \bigcup_{(i,j)\in\underline{4}\times\underline{4}} D + i\overrightarrow{v_1} + j\overrightarrow{v_2}$. We also have that $4 \cdot C = \bigcup_{(i,j)\in\underline{4}\times\underline{4}} C + i\overrightarrow{v_1} + j\overrightarrow{v_2}$. So we consider this extension as the natural extension.

Now we have to define an appropriate measure $\nu_{\mathbb{Z}_v^2}$ on $\mathbb{R}^2$ with essential support R. The measure is defined in a way analogous to that of the enlarged fractal; first on D and then on $\mathbb{R}^2$. The subscript $\mathbb{Z}_v^2$ of the measure $\nu_{\mathbb{Z}_v^2}$ denotes the translation of the measure ν by $\mathbb{Z}^2$ and the vectors $\overrightarrow{v_1}, \overrightarrow{v_2}$. For the definition of the measure $\nu_{\mathbb{Z}_v^2}$ we fix weights $c_{(i,j)} \in \mathbb{R}^+$ for $(i,j) \in \underline{N_1} \times \underline{N_2}$.

Proposition 11.7. *The set function $\nu : \mathcal{B} \to \overline{\mathbb{R}}_0^+$ given by*

$$\nu := \sum_{\omega\in\{\emptyset\}\cup\Sigma^{(2)}} \left(\prod_{k=0}^{|\omega|-1} c_{(i_k,j_k)}\right) \cdot \mu \circ \tau_\omega^{-1},$$

where $c_{(i,j)} \in \mathbb{R}^+$, $(i,j) \in \underline{N_1} \times \underline{N_2}$, fixed, and $\omega = \left((i_0,j_0),\dots,(i_{|\omega|-1},j_{|\omega|-1})\right) \in \Sigma^{(2)}$, defines a measure on D. Also the sum of the translates

$$\nu_{\mathbb{Z}_v^2} : \mathcal{B} \to \overline{\mathbb{R}}_0^+,$$
$$B \mapsto \sum_{(k,l)\in\mathbb{Z}^2} \nu\left(B - k\overrightarrow{v_1} - l\overrightarrow{v_2}\right)$$

defines a measure. Its essential support is equal to R.

PROOF. This proof is analogous to the one-dimensional version (see Proposition 10.7) with the difference that we have a translation by $\mathbb{Z}\overrightarrow{v_1} + \mathbb{Z}\overrightarrow{v_2}$ instead of $\mathbb{Z}$ and use the IFS consisting of $\tau_{(i,j)}$, $(i,j) \in \underline{N_1} \times \underline{N_2}$ instead of τ_i, $i \in \underline{N}$. $\square$

Remark 11.8. Now we give some observations regarding these measures.

(1) Notice that for every $(i,j) \in A$ we also take a weight $c_{(i,j)} \in \mathbb{R}^+$on $\tau_{(i,j)}(D \backslash C)$ which may differ from $p_{(i,j)}$.

(2) Notice that for $B \in \mathcal{B}$, $B \subset D$, we have $\nu\left(\tau_{(i,j)}(B)\right) = c_{(i,j)} \cdot \nu(B)$, $(i,j) \in \left(\underline{N_1} \times \underline{N_2}\right) \backslash A$, and

$$\nu\left(\tau_{(i,j)}(B)\right) = c_{(i,j)} \cdot \nu(B) + \left(p_{(i,j)} - c_{(i,j)}\right) \cdot \mu(B)$$

for $(i,j) \in A$. In particular, $\nu\left(\tau_{(i,j)}(C)\right) = c_{(i,j)}$ for $(i,j) \in \left(\underline{N_1} \times \underline{N_2}\right) \backslash A$ and $\nu\left(\tau_{(i,j)}(C)\right) = p_{(i,j)}$ for $(i,j) \in A$.

(3) We also have that $\nu_{\mathbb{Z}_v^2}(C \cap \partial D) = 0$, where C is the fractal and ∂D is the boundary of the set D, since we assume that $\nu_{\mathbb{Z}_v^2}\left((D + k\overrightarrow{v_1} + l\overrightarrow{v_2}) \cap (D + n\overrightarrow{v_1} + m\overrightarrow{v_2})\right) = 0$ for $(k,l) \neq (n,m)$, $(k,l),(n,m) \in \mathbb{Z}^2$.

Proposition 11.9. *The measure $\nu_{\mathbb{Z}_v^2}$ is locally finite if and only if*

$$\sum_{(i,j)\in\underline{N_1}\times\underline{N_2}} c_{(i,j)} < 1.$$

In particular, $\nu_{\mathbb{Z}_v^2}$ is not locally finite if the weights $c_{(i,j)}$ are given by $c_{(i,j)} = p_{(i,j)}$ for $(i,j) \in A$.

Proof. This proof is analogous to the one-dimensional version (see the proof of Proposition 10.10) with the difference that in two dimensions a measure is locally finite if it is finite for sets $[a,b]\times[c,d]$, $a,b,c,d \in \mathbb{R}$. Since the set D is contained in some rectangle and it also contains some rectangle, we show that $\nu_{\mathbb{Z}_v^2}(D) < \infty$. To prove this we notice that $\mu\left(\tau_\omega^{-1}(D)\right) = 1$ for all $\omega \in \Sigma^{(2)} \cup \{\emptyset\}$. Then the argument goes as the one-dimensional one, where we use $c_{(i,j)}$, $(i,j) \in \underline{N_1}\times\underline{N_2}$, $p_{(i,j)}$, $(i,j)\in A$, and the sums run through $\Sigma^{(2)}\cup\{\emptyset\}$, $\underline{N_1}\times\underline{N_2}$ and $\left(\underline{N_1}\times\underline{N_2}\right)\setminus A$. $\square$

11.3. Definitions and properties for the MRA

For the MRA we have to define two unitary operators on $L^2(\nu_{\mathbb{Z}_v^2})$ and a father wavelet. We start by giving the unitary operators in terms of the extended IFS.

Definition 11.10. The *translation operator* T is defined for $(m,n) \in \mathbb{Z}^2$, $f \in L^2(\nu_{\mathbb{Z}_v^2})$, as

$$\left(T^{(m,n)}f\right)(\cdot) = f\left(\cdot - m\overrightarrow{v_1} - n\overrightarrow{v_2}\right).$$

The *scaling operator* U acting on $L^2(\nu_{\mathbb{Z}_v^2})$ is defined for $\overrightarrow{x}\in\mathbb{R}^2$, $f \in L^2(\nu_{\mathbb{Z}_v^2})$, as

$$\begin{aligned} Uf(\overrightarrow{x}) = \sum_{(k,l)\in\mathbb{Z}^2} \Bigg(& \sum_{(i,j)\in\left(\underline{N_1}\times\underline{N_2}\right)\setminus A} \sqrt{c_{(i,j)}}^{-1}\cdot \mathbb{1}_{\tau_{(i,j)}\left(\mathring{D}\right)}(\overrightarrow{x} - k\overrightarrow{v_1} - l\overrightarrow{v_2}) \\ & \cdot f\left(\tau_{(i,j)}^{-1}(\overrightarrow{x} - k\overrightarrow{v_1} - l\overrightarrow{v_2}) + (N_1k+i)\,\overrightarrow{v_1} + (N_2 l + j)\,\overrightarrow{v_2}\right) \\ & + \sum_{(i,j)\in A}\left(\sqrt{c_{(i,j)}}^{-1}\cdot\mathbb{1}_{\tau_{(i,j)}\left(\mathring{D}\setminus C\right)}(\overrightarrow{x} - k\overrightarrow{v_1} - l\overrightarrow{v_2}) + \sqrt{p_{(i,j)}}^{-1}\cdot\mathbb{1}_{\tau_{(i,j)}(C)}(\overrightarrow{x} - k\overrightarrow{v_1} - l\overrightarrow{v_2})\right) \\ & \cdot f\left(\tau_{(i,j)}^{-1}(\overrightarrow{x} - k\overrightarrow{v_1} - l\overrightarrow{v_2}) + (N_1k+i)\,\overrightarrow{v_1} + (N_2l+j)\,\overrightarrow{v_2}\right)\Bigg). \end{aligned}$$

Remark 11.11.

(1) We notice that Uf for $f \in L^2(\nu_{\mathbb{Z}_v^2})$ is not defined everywhere on $\operatorname{supp}(\nu_{\mathbb{Z}_v^2})$ since Uf is not supported on the boundary of D. Nevertheless, that does not have an impact since the boundary has measure zero.

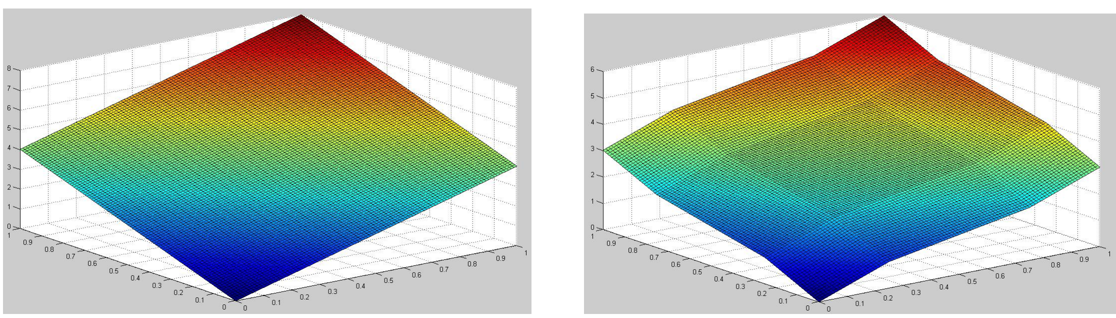
(2) In the definition of the operator U we observe that since $(i,j)\in\underline{N_1}\times\underline{N_2}$ the function Uf is supported on all of $\mathbb{R}$ (almost surely, depending on $\operatorname{supp}(f)$). If we had that $(i,j)\in B \subsetneq \underline{N_1}\times\underline{N_2}$, then the values of $f(\overrightarrow{x})$ for $\overrightarrow{x}\in\bigcup_{(i,j)\in\left(\underline{N_1}\times\underline{N_2}\right)\setminus B} D + i\overrightarrow{v_1} + j\overrightarrow{v_2}$ are omitted in Uf and so the operator U is not unitary.

Lemma 11.12. *The operators T and U are unitary in $L^2(\nu_{\mathbb{Z}_v^2})$ and the inverse operator of U is given for $f \in L^2(\nu_{\mathbb{Z}_v^2})$ and $\overrightarrow{x} \in \mathbb{R}$ by*

$$\begin{aligned} U^{-1} f(\overrightarrow{x}) = \sum_{(k,l)\in\mathbb{Z}^2} \Bigg(& \sum_{(i,j)\in\left(\underline{N_1}\times\underline{N_2}\right)\setminus A} \Big(\sqrt{c_{(i,j)}} \cdot \mathbb{1}_{\overset{\circ}{D}} \left(\overrightarrow{x} - (i+N_1k)\,\overrightarrow{v_1} - (j+N_2l)\,\overrightarrow{v_2}\right) \\ & \qquad \cdot f\left(\tau_{(i,j)}\left(\overrightarrow{x} - (i+N_1k)\,\overrightarrow{v_1} - (j+N_2l)\,\overrightarrow{v_2}\right) + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right)\Big) \\ & + \sum_{(i,j)\in A} \Big(\sqrt{c_{(i,j)}} \cdot \mathbb{1}_{\overset{\circ}{D}\setminus C} \left(\overrightarrow{x} - (i+N_1k)\,\overrightarrow{v_1} - (j+N_2l)\,\overrightarrow{v_2}\right) \\ & \quad + \sqrt{p_{(i,j)}} \cdot \mathbb{1}_C \left(\overrightarrow{x} - (i+N_1k)\,\overrightarrow{v_1} - (j+N_2l)\,\overrightarrow{v_2}\right)\Big) \\ & \quad \cdot f\left(\tau_{(i,j)}\left(\overrightarrow{x} - (i+N_1k)\,\overrightarrow{v_1} - (j+N_2l)\,\overrightarrow{v_2}\right) + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right) \Bigg). \end{aligned}$$

We postpone the proof to behind Remark 11.14. First we explore the action of this operator U in the example of the Cantor Dust, compare Example 11.6 and give a different representation of the operators U and U^{-1}.

Example 11.13. For the example of the Cantor Dust the operator U for the two divisions of the unit square acts on the identity map on $[0,1]\times[0,1]$ as shown in Figure 11.3.1. For the measure we consider the weights $c_{(i,j)} = \frac{1}{4}$ for $(i,j) \in \underline{3}\times\underline{3}$, $(i,j)\in\underline{4}\times\underline{4}$, respectively. Notice that the identity map is not in $L^2(\nu_{\mathbb{Z}_v^2})$ since the measure $\nu_{\mathbb{Z}_v^2}$ is not locally finite.



(A) The operator U for the division (a) given in Example 11.6.

(B) The operator U for the division (b) given in Example 11.6.

Figure 11.3.1. The action of the operator U for the the Cantor Dust on id.

We now give a connection between the scaling operator U and representations of the Cuntz algebra.

Remark 11.14. We can write the operator U in terms of the representation of a Cuntz algebra. We consider the following representation of the Cuntz algebra $\mathcal{O}_p$: for $(i,j)\in A$, $f\in L^2(\mu)$, $\overrightarrow{x}\in\operatorname{supp}(\mu)$, define

$$S_{(i,j)} f(\overrightarrow{x}) = \sqrt{p_{(i,j)}}^{-1} \cdot \mathbb{1}_{\tau_{(i,j)}(C)}(\overrightarrow{x}) \cdot f\left(\tau_{(i,j)}^{-1}(\overrightarrow{x})\right).$$

Furthermore, define for $(i,j)\in\left(\underline{N_1}\times\underline{N_2}\right)\setminus A$, $f\in L^2(\nu)$, $\overrightarrow{x}\in\operatorname{supp}(\nu)$,

$$\widetilde{S}_{(i,j)} f(\overrightarrow{x}) = \sqrt{c_{(i,j)}}^{-1} \cdot \mathbb{1}_{\tau_{(i,j)}(\overset{\circ}{D})}(\overrightarrow{x}) \cdot f\left(\tau_{(i,j)}^{-1}(\overrightarrow{x})\right)$$

and for $(i,j)\in A$

$$\widetilde{S}_{(i,j)} f(\overrightarrow{x}) = \sqrt{c_{(i,j)}}^{-1} \cdot \mathbb{1}_{\tau_{(i,j)}(\overset{\circ}{D}\setminus C)}(\overrightarrow{x}) \cdot f\left(\tau_{(i,j)}^{-1}(\overrightarrow{x})\right).$$

Notice that if we instead defined for $(i,j) \in A$, $f \in L^2(\nu)$, $\overrightarrow{x} \in \operatorname{supp}(\nu)$,

$$\widetilde{S}_{(i,j)} f(\overrightarrow{x}) = \sqrt{c_{(i,j)}}^{-1} \cdot \mathbb{1}_{\tau_{(i,j)}(D)}(\overrightarrow{x}) \cdot f\left(\tau_{(i,j)}^{-1}(\overrightarrow{x})\right),$$

and $\sum_{(i,j)\in \underline{N_1}\times\underline{N_2}} c_{(i,j)} = 1$, then $\left(\widetilde{S}_{(i,j)}\right)_{(i,j)\in \underline{N_1}\times\underline{N_2}}$ would be a representation of the Cuntz algebra $\mathcal{O}_N$. This is the case if the extended IFS coincides with the original IFS.

The operators U and U^{-1} take the following form. Let $\overrightarrow{x} \in \mathbb{R}^2$, $f \in L^2\left(\nu_{\mathbb{Z}_v^2}\right)$, then

$$Uf(\overrightarrow{x}) = \sum_{(k,l)\in\mathbb{Z}^2} \left(\sum_{(i,j)\in\left(\underline{N_1}\times\underline{N_2}\right)\setminus A} T^{(k,l)} \widetilde{S}_{(i,j)} T^{(-(N_1k+i),-(N_2l+j))} f(\overrightarrow{x}) \right.$$
$$\left. + \sum_{(i,j)\in A} T^{(k,l)} \left(\widetilde{S}_{(i,j)} + S_{(i,j)}\right) T^{(-(N_1k+i),-(N_2l+j))} f(\overrightarrow{x}) \right).$$

and

$$U^{-1}f(\overrightarrow{x}) = \sum_{(k,l)\in\mathbb{Z}^2} \left(\sum_{(i,j)\in\left(\underline{N_1}\times\underline{N_2}\right)\setminus A} T^{(i+N_1k,j+N_2l)} \widetilde{S}^*_{(i,j)} T^{(-k,-l)} f(\overrightarrow{x}) \right.$$
$$\left. + \sum_{(i,j)\in A} T^{(i+N_1k,j+N_2l)} \left(\widetilde{S}^*_{(i,j)} + S^*_{(i,j)}\right) T^{(-k,-l)} f(\overrightarrow{x}) \right).$$

We can easily check that the two formulas for U and the two for U^{-1} define the same operator.

Proof of Lemma 11.12. T is a unitary operator since $\nu_{\mathbb{Z}_v^2}$ is translation invariant under $\mathbb{Z}\overrightarrow{v_1} + \mathbb{Z}\overrightarrow{v_2}$ by definition. Furthermore it can be easily verified that the two formulas for U^{-1} define the same operator. The proof of $U^{-1}Uf = f$ for all $f \in L^2(\nu_{\mathbb{Z}_v^2})$ goes analogous to the one-dimensional version. We consider the representation of U and U^{-1} in terms of $S_{(i,j)}$, $(i,j) \in A$, and $\widetilde{S}_{(i,j)}$, $(i,j) \in \underline{N_1} \times \underline{N_2}$, and notice that

$$\widetilde{S}^*_{(i,j)} \widetilde{S}_{(k,l)} = \begin{cases} \delta_{(i,j),(k,l)} \cdot \mathbb{1}_{\overset{\circ}{D}} \cdot I, & (i,j),(k,l) \in \left(\underline{N_1}\times\underline{N_2}\right)\setminus A, \\ \delta_{(i,j),(k,l)} \cdot \mathbb{1}_{\overset{\circ}{D}\setminus C} \cdot I, & (i,j),(k,l) \in A, \\ 0, & (i,j) \in \left(\underline{N_1}\times\underline{N_2}\right)\setminus A,\ (k,l) \in A, \\ 0, & (k,l) \in \left(\underline{N_1}\times\underline{N_2}\right)\setminus A,\ (i,j) \in A. \end{cases}$$

and $S^*_{(i,j)}\widetilde{S}_{(k,l)} = 0$ for $(i,j) \in A$, $(k,l) \in \underline{N_1} \times \underline{N_2}$, $\widetilde{S}^*_{(i,j)} S_{(k,l)} = 0$ for $(i,j) \in \underline{N_1} \times \underline{N_2}$, $(k,l) \in A$, and $S^*_{(i,j)} S_{(k,l)} = \delta_{(i,j),(k,l)} I$ for $(i,j),(k,l) \in A$. With these observations we obtain the result in the same way as the one-dimensional version.

We can show that the operator U is unitary in exactly the same way as for the one-dimensional version with the representation of U and U^{-1} in terms of $S_{(i,j)}$, $\widetilde{S}_{(i,j)}$. □

Now we turn to further properties of the operators U and T.

Proposition 11.15. *The operators T and U satisfy the following.*

(1) $U^{-1}T^{(k,l)}U = T^{(N_1k,N_2l)}$, $(k,l) \in \mathbb{Z}^2$,
(2) $U^{-1}\varphi = m_0(T)\varphi$, *where* $m_0(z,w) := \sum_{(i,j)\in A} \sqrt{p_{(i,j)}} \cdot z^i w^j$, $(z,w) \in \mathbb{T}^2 := \left\{\omega \in \mathbb{C}^2 : \mid \omega \mid = 1\right\}$, *and* $\varphi := \mathbb{1}_C$,
(3) $\langle T^{(k,l)}\varphi | T^{(n,m)}\varphi \rangle = \delta_{(k,l),(n,m)}$, $(k,l),(n,m) \in \mathbb{Z}^2$.

Remark 11.16. We define the function $m_0(z,w) := \sum_{(i,j)\in A} \sqrt{p_{(i,j)}} \cdot z^i w^j$, $(z,w) \in \mathbb{T}^2$, as the low-pass filter for the father wavelet φ. The filter functions are applied to the operator T as $m_0(T) = \sum_{(i,j)\in A} \sqrt{p_{(i,j)}} \cdot T^{(i,j)}$.

Proof. The properties given in (1) and (2) can be shown in an analogous way to the ones in Proposition 10.18 (1) and (2) by using the appropriate version U, T, φ and m_0. For (1) we consider $U^{-1}T^{(k,l)}U$ with $(k,l) \in \mathbb{Z}^2$ instead of 1 (in the one-dimensional version), which carries through the proof without further difficulties.

ad (3): This follows from $\nu_{\mathbb{Z}_v^2}\left((C + k\overrightarrow{v_1} + l\overrightarrow{v_2}) \cap (C + n\overrightarrow{v_1} + m\overrightarrow{v_2})\right) = \delta_{(k,l),(n,m)}$, $(k,l),(n,m) \in \mathbb{Z}^2$, which holds since $\nu_{\mathbb{Z}_v^2}(\partial D) = 0$. □

Now we turn to the statement of the MRA in two dimensions.

Theorem 11.17. *Set the father wavelet $\varphi = \mathbb{1}_C$ and for $j \in \mathbb{Z}$ let*

$$V_j = \operatorname{cl\,span}\left\{U^j T^{(k,l)}\varphi : \, (k,l) \in \mathbb{Z}^2\right\},$$

then $(\nu_{\mathbb{Z}_v^2}, U, T)$ allows a two-sided MRA with respect to φ and V_j, $j \in \mathbb{Z}$, as above. In particular,

$$\operatorname{cl\,span}\left\{U^n T^{(k,l)}\varphi : \, n \in \mathbb{Z}, \, (k,l) \in \mathbb{Z}^2\right\} = L^2\left(\nu_{\mathbb{Z}_v^2}\right).$$

Proof. For the proof we show the six properties given in Remark 1.5. The properties (1), (3), (4), (5) and (6) follow in the same way as the corresponding one-dimensional properties with the appropriate operators U and T and the closed subspaces V_j, $j \in \mathbb{Z}$, defined above.

ad (2): For this property we give more details, although it is analogous to the one-dimensional version and so it is similar to the one in [**BLP+10**]. First we consider the case where C is totally disconnected. We define the following function σ in terms of the IFS $\mathbb{S}$ for $\overrightarrow{x} \in \mathbb{R}^2$ by

$$\begin{aligned}\sigma(\overrightarrow{x}) := \sum_{(k,l)\in\mathbb{Z}^2} \Bigg(\sum_{(i,j)\in \underline{N_1}\times\underline{N_2}} & \mathbb{1}_{\tau_{(i,j)}(\mathring{D})}(\overrightarrow{x} - k\overrightarrow{v_1} - l\overrightarrow{v_2}) \\ & \left(\tau_{(i,j)}^{-1}(\overrightarrow{x} - k\overrightarrow{v_1} - l\overrightarrow{v_2}) + (i + N_1 k)\,\overrightarrow{v_1} + (j + N_2 l)\,\overrightarrow{v_2}\right)\Bigg).\end{aligned} \tag{11.3.1}$$

This function is bijective on $\mathbb{R}^2$ and its inverse is

$$\begin{aligned}\sigma^{-1}(\overrightarrow{x}) := \sum_{(k,l)\in\mathbb{Z}^2} \Bigg(\sum_{(i,j)\in \underline{N_1}\times\underline{N_2}} & \mathbb{1}_{\mathring{D}}(\overrightarrow{x} - (N_1 k + i)\,\overrightarrow{v_1} - (N_2 l + j)\,\overrightarrow{v_2}) \\ & \left(\tau_{(i,j)}\left(\overrightarrow{x} - (N_1 k + i)\overrightarrow{v_1} - (N_2 l + j)\overrightarrow{v_2}\right) + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right)\Bigg).\end{aligned}$$

With this function we can write

$$R = \bigcup_{(k,l)\in\mathbb{Z}^2} \left(\biguplus_{\omega\in\Sigma^{(2)}\cup\{\emptyset\}} \tau_\omega(C)\right) + k\overrightarrow{v_1} + l\overrightarrow{v_2} = \bigcup_{n\in\mathbb{N}} \sigma^{-n}\left(\biguplus_{(k,l)\in\mathbb{Z}^2} C + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right).$$

In terms of the function σ we have that R is an increasing union in n of disjoint unions. More precisely $\sigma^{-n}\left(\biguplus_{(k,l)\in\mathbb{Z}^2} C + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right)$ is an increasing set in $n \in \mathbb{N}$. We claim that it suffices to approximate functions f which have support in $\sigma^{-N}\left(C + K\overrightarrow{v_1} + L\overrightarrow{v_2}\right)$ for $N \geq 0$ and $(K,L) \in \mathbb{Z}^2$. To see why, consider an arbitrary non-negative function $f \in L^2(\nu_{\mathbb{Z}_v^2})$ and define for $n \in \mathbb{N}$

$$g_n = f \cdot \mathbb{1}_{\sigma^{-n}\left(\bigcup_{(r,s)\in\mathbb{Z}^2} C + r\overrightarrow{v_1} + s\overrightarrow{v_2}\right)},$$

and a second sequence for $k, n \in \mathbb{N}$ as

$$g_n^k = \sum_{(i,j)\in\{-k,\dots,0,\dots,k\}^2} f \cdot \mathbb{1}_{\sigma^{-n}\left(C + j\overrightarrow{v_1} + i\overrightarrow{v_2}\right)}.$$

We have that $g_n^k \nearrow g_n$ for $k \to \infty$ and $g_n \nearrow f$ for $n \to \infty$. We additionally have $|g_n^k| \leq f$ for all $n \in \mathbb{N}$, $k \in \mathbb{N}$. Consequently, by the dominated convergence theorem applied twice we obtain that the function f can be approximated by functions with support contained in sets $\sigma^{-n}\left(C + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right)$ for $n \geq 0$ and $(k,l) \in \mathbb{Z}^2$. For general $f \in L^2(\nu_{\mathbb{Z}^2_v})$, we can write $f = f^+ - f^-$ with $f^+, f^- \geq 0$, and apply the above argument to f^+ and f^-. Now we realize that $T^{(-K,-L)}U^N f$ has support in C, where $f \in L^2(\nu_{\mathbb{Z}^2_v})$ with $\operatorname{supp}(f) \subset \sigma^{-N}\left(C + K\overrightarrow{v_1} + L\overrightarrow{v_2}\right)$.

We consider the sets $\sigma^{-n}\left(C + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right)$, $n \in \mathbb{N}$, $(k,l) \in \mathbb{Z}^2$, which are contained in C. For each $n \geq 0$, there are exactly p^n sets of the form $\sigma^{-n}\left(C + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right)$, $(k,l) \in \mathbb{Z}^2$, which are contained in C. The sets $\sigma^{-n}\left(C + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right)$, $(k,l) \in \mathbb{Z}^2$, are disjoint and satisfy

$$\sigma^{-n}\left(C + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right) = \bigcup_{(i,j)\in A} \sigma^{-(n+1)}\left(C + (N_1 k + i)\overrightarrow{v_1} + (N_2 l + j)\overrightarrow{v_2}\right).$$

Thus, two such sets $\sigma^{-n}\left(C + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right) \subset C$, $n \in \mathbb{N}_0$, $(k,l) \in \mathbb{Z}$, are either disjoint or one is contained in the other, and

$$B := \operatorname{span}\left\{\mathbb{1}_{\sigma^{-n}\left(C+k\overrightarrow{v_1}+l\overrightarrow{v_2}\right)} : n \geq 0,\, (k,l) \in \mathbb{Z}^2,\, \sigma^{-n}\left(C + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right) \subset C\right\}$$

is a $*$-subalgebra of $\mathcal{C}(C,\mathbb{C})$ which can be easily checked. Furthermore we have that B separates points of C since if we consider two points $x, y \in C$ then we can always find two separate sets $\tau_\omega(C)$ and $\tau_{\widetilde{\omega}}(C)$, $\omega, \widetilde{\omega} \in \Sigma^{*,2}$, such that $\tau_\omega(C) \cap \tau_{\widetilde{\omega}}(C) = \emptyset$ and $x \in \tau_\omega(C)$, $y \in \tau_{\widetilde{\omega}}(C)$. Now we can apply the Stone-Weierstrass theorem and get that B is uniformly dense in $\mathcal{C}(C,\mathbb{C})$.

Furthermore we know that μ is a regular Borel measure on C and it now follows that $\mathcal{C}(C,\mathbb{C})$ is dense in $L^2(\mu)$. Thus we can find for any $\varepsilon > 0$ a function g in

$$\operatorname{span}\left\{\mathbb{1}_{\sigma^{-n}\left(C+k\overrightarrow{v_1}+l\overrightarrow{v_2}\right)} : n \in \mathbb{Z},\, (k,l) \in \mathbb{Z}^2\right\} = \operatorname{span}\left\{U^n T^{(k,l)}\varphi : n \in \mathbb{Z},\, (k,l) \in \mathbb{Z}^2\right\}$$

such that $\|T^{(-K,-L)}U^{-N}f - g\| < \varepsilon$. We know that $T^{(K,L)}$ and U^N are unitary and so

$$\|f - U^N T^{(K,L)} g\| = \|T^{(-K,-L)}U^{-N} f - g\| < \varepsilon.$$

Furthermore, we have that

$$U^N T^{(K,L)}\left(U^n T^{(k,l)}\varphi\right) = U^{N+n} T^{(N_1^n K + k, N_2^n L + l)}\varphi,$$

so $U^N T^{(K,L)} g$ has the required form.

If we do not have that C is totally disconnected, but $C \subsetneq D$, we can proof the result by defining $C' = C \backslash \left(\bigcup_{n\leq 0}\bigcup_{(k,l)\in\mathbb{Z}^2} \sigma^n\left((\partial D \cap C) + k\overrightarrow{v_1} + l\overrightarrow{v_2}\right)\right)$. We have that $\nu_{\mathbb{Z}^2_v}(C') = 1$ since $\nu_{\mathbb{Z}^2_v}(\partial D \cap C) = 0$ and for the set C' we can argue in the same way as above. In the last step we can come back to the original set C by adding a set of measure zero which does not have any influence.

If $C = D$, notice that every set $[a,b] \times [c,d] \subset D$, $a,b,c,d \in \mathbb{R}$, can be approximated by $\tau_\omega(C)$, $\omega \in \Sigma^{*,2}$. Hence it generates $\mathcal{B}$, thus every element $A \in \mathcal{B}$, $A \subset D$, can be approximated by elements of $\left\{\tau_\omega(C) : \omega \in \Sigma^{*,2}\right\}$. Consequently, every elementary function can be approximated by functions $T^{(k,l)}\mathbb{1}_{\tau_\omega(C)}$ and so all functions in $L^2(\nu_{\mathbb{Z}^2_v})$ can be approximated by elements of $\left\{T^{(k,l)}\mathbb{1}_{\tau_\omega(C)} : \omega \in \Sigma^{*,2}, (k,l) \in \mathbb{Z}^2\right\} = \left\{U^n T^{(k,l)}\varphi : n \in \mathbb{N}_0,\, (k,l) \in \mathbb{Z}^2\right\}$. □

11.3.1. Mother wavelets in two dimensions and the wavelet basis. Now we turn to the construction of the mother wavelets which are obtained via filter functions of the form $m_j : \mathbb{T}^2 \to \mathbb{C}$, $(z,w) \mapsto \sum_{(k,l)\in\mathbb{Z}^2} c^j_{(k,l)} \cdot z^k w^l$, $j \in \underline{N}\backslash\{0\}$, and $c^j_{(k,l)} \in \mathbb{C}$, such that specific conditions are satisfied for the coefficients $c^j_{(k,l)} \in \mathbb{C}$. The construction goes along the lines of [**BLP**$^+$**10, DMP08**]. In [**BLP**$^+$**10, BLM**$^+$**09**] it is done in a more general setting, more precisely in arbitrary Hilbert spaces. In [**DMP08**] it is done for the special case of a Hilbert space defined on the enlarged fractal of the Sierpinski Gasket.

Remark 11.18. First observe that there is a standard isometric isomorphism $J : V_0 \to L^2\left(\mathbb{T}^2, \lambda\right)$ given by

$$J\Big(\sum_{(m,n)\in\mathbb{Z}^2} c_{(m,n)} T^{(m,n)}(\mathbb{1}_C)\Big) = \sum_{(m,n)\in\mathbb{Z}^2} c_{(m,n)} e_{(m,n)},$$

where $c_{(m,n)} \in \mathbb{C}$ and $e_{(m,n)}(z,w) := z^m w^n$ for $(z,w) \in \mathbb{T}^2$. For further discussions about the isomorphism see [**DMP08**]. Thus, the filter functions, which are used for the definition of the mother wavelets, are in $L^2\left(\mathbb{T}^2, \lambda\right)$ and under application of J^{-1} and U we get the mother wavelets in $L^2(\nu_{\mathbb{Z}^2_v})$.

For the construction of the mother wavelets we first define $N-1$ high-pass filters. Recall that the low-pass filter is given by

$$m_0(z,w) = \sum_{(i,j)\in A} \sqrt{p_{(i,j)}}\, e_{(i,j)}(z,w),$$

where $(z,w) \in \mathbb{T}^2$ and $e_{(i,j)}(z,w) = z^i w^j$. With the coefficients of the low-pass filter m_0 we can define a vector v_0 of length N given by

$$(v_0)_{i\in\underline{N}} = a^0_{\left((i)_{N_1}, \lfloor \frac{i}{N_1} \rfloor\right)}$$

with $a^0_{(i,j)} = \sqrt{p_{(i,j)}}$ for $(i,j) \in A$, $a^0_{(i,j)} = 0$ for $(i,j) \in \left(\underline{N_1} \times \underline{N_2}\right) \setminus A$ and $(m)_N := m \bmod N$, $m \in \mathbb{N}$, and $\lfloor x \rfloor = \max_{k\in\mathbb{Z}, k\le x}(k)$. Thus, we can also write the low-pass filter as $m_0 = v_0 \cdot \mathcal{E}^t$, where

$$(\mathcal{E})_{i\in\underline{N}} = e_{\left((i)_{N_1}, \lfloor \frac{i}{N_1} \rfloor\right)}$$

and $e_{(n,m)}(z,w) = z^n w^m$, $(z,w) \in \mathbb{T}^2$, $(n,m) \in \mathbb{Z}^2$.

We can find $N-1$ vectors v_i, $i \in \underline{N}\backslash\{0\}$, of length N such that $(v_i)_{i\in\underline{N}}$ forms a orthonormal basis for $\mathbb{C}^N$, or equivalently, the matrix $(v_i)_{i\in\underline{N}}$ is unitary. (For example, consider the vector v_0 and any $N-1$ linear independent vectors, e.g. for $i \in \underline{N}\backslash\{0\}$ consider the vector $(\delta_{i,j})_{j\in\underline{N}}$, which is 1 at the ith coordinate and zero otherwise, then we apply the Gram-Schmidt process to these vectors.)

The high-pass filter functions are defined as $m_i := v_i \cdot \mathcal{E}^t$, $i \in \underline{N}\backslash\{0\}$. We denote the entries in the vectors v_i by

$$(v_i)_{j\in\underline{N}} = a^i_{\left((j)_{N_1}, \lfloor \frac{j}{N_1} \rfloor\right)}. \tag{11.3.2}$$

Proposition 11.19. *Define the mother wavelets by $\psi_i := U m_i(T)\varphi$, $i \in \underline{N}\backslash\{0\}$, where*

$$m_i(T) = v_i \cdot \left(T^{\left((j)_{N_1}, \lfloor \frac{j}{N_1} \rfloor\right)}\right)^t_{j\in\underline{N}}.$$

Then the set

$$\left\{U^n T^{(k,l)} \psi_i : n \in \mathbb{Z},\, (k,l) \in \mathbb{Z}^2,\, i \in \underline{N}\backslash\{0\}\right\}$$

is an ONB for $L^2(\nu_{\mathbb{Z}^2_v})$.

Remark 11.20. The mother wavelets have the following form for $i \in \underline{N}\backslash\{0\}$

$$\psi_i = \sum_{(k,l)\in\left(\underline{N_1}\times\underline{N_2}\right)\setminus A} a^i_{(k,l)} \sqrt{c_{(k,l)}}^{-1} \mathbb{1}_{\tau_{(k,l)}(C)} + \sum_{(k,l)\in A} a^i_{(k,l)} \sqrt{p_{(k,l)}}^{-1} \mathbb{1}_{\tau_{(k,l)}(C)}.$$

Proposition 11.19 does not follow directly from Theorem 8.1 as in the one-dimensional setting, since there we only consider an MRA in one dimension.

Proof of Proposition 11.19. First we prove that $T^{(k,l)}\psi_i$, $(k,l) \in \mathbb{Z}^2$, $i \in \underline{N}\backslash\{0\}$, are orthonormal. Iteratively we obtain the orthonormality of $U^n T^{(k,l)}\psi_i$ and $U^m T^{(r,s)}\psi_j$, $(k,l),(r,s) \in \mathbb{Z}^2$, $n,m \in \mathbb{Z}$, $i,j \in \underline{N}\backslash\{0\}$. Let $(u,v),(r,s) \in \mathbb{Z}^2$ and $i,j \in \underline{N}\backslash\{0\}$, then

$$\begin{aligned}
&\langle T^{(u,v)}\psi_i | T^{(r,s)}\psi_j\rangle \\
&= \Bigg\langle \sum_{(k,l)\in\left(\underline{N_1}\times\underline{N_2}\right)\setminus A} a^i_{(k,l)}\sqrt{c_{(k,l)}}^{-1} T^{(u,v)}\mathbb{1}_{\tau_{(k,l)}(C)} + \sum_{(k,l)\in A} a^i_{(k,l)}\sqrt{p_{(k,l)}}^{-1} T^{(u,v)}\mathbb{1}_{\tau_{(k,l)}(C)} \\
&\qquad \Bigg| \sum_{(k,l)\in\left(\underline{N_1}\times\underline{N_2}\right)\setminus A} a^j_{(k,l)}\sqrt{c_{(k,l)}}^{-1} T^{(r,s)}\mathbb{1}_{\tau_{(k,l)}(C)} + \sum_{(k,l)\in A} a^j_{(k,l)}\sqrt{p_{(k,l)}}^{-1} T^{(r,s)}\mathbb{1}_{\tau_{(k,l)}(C)} \Bigg\rangle \\
&= \delta_{(u,v),(r,s)} \cdot \Bigg(\sum_{(k,l)\in\left(\underline{N_1}\times\underline{N_2}\right)\setminus A} a^i_{(k,l)}\overline{a}^j_{(k,l)} \left(c_{(k,l)}\right)^{-1} \langle \mathbb{1}_{\tau_{(k,l)}(C)} | \mathbb{1}_{\tau_{(k,l)}(C)}\rangle \\
&\qquad + \sum_{(k,l)\in A} a^i_{(k,l)}\overline{a}^j_{(k,l)} \left(p_{(k,l)}\right)^{-1} \langle \mathbb{1}_{\tau_{(k,l)}(C)} | \mathbb{1}_{\tau_{(k,l)}(C)}\rangle \Bigg) \\
&= \delta_{(u,v),(r,s)} \cdot \Bigg(\sum_{(k,l)\in\left(\underline{N_1}\times\underline{N_2}\right)\setminus A} a^i_{(k,l)}\overline{a}^j_{(k,l)} + \sum_{(k,l)\in A} a^i_{(k,l)}\overline{a}^j_{(k,l)} \Bigg) = \delta_{(u,v),(r,s)} \cdot \delta_{i,j},
\end{aligned}$$

where we use in the second equality that the inner product is only non-zero if $(u,v) = (r,s)$ and in the last we use that the vectors $(v_i)_{i\in\underline{N}}$ are an orthonormal basis of $\mathbb{C}^N$.

Furthermore, for $i \in \underline{N}\setminus\{0\}$ we have

$$\langle \psi_i | \varphi\rangle = \sum_{(k,l)\in A} a^i_{(k,l)}\sqrt{p_{(k,l)}} = 0.$$

Now we turn to the orthonormality on different scales $n, m \in \mathbb{Z}$. This follows for the scales $n = -1$ and $m = 0$, since for $i, j \in \underline{N}\setminus\{0\}$ and $(k,l) \in \mathbb{Z}^2$ we have $\psi_i \in W_0 \subset V_1$, $UT^{(k,l)}\psi_j \in W_1$ and $W_1 = V_2 \ominus V_1$. Iteratively, it follows for all scales $n, m \in \mathbb{Z}$.

We show that

$$\left\{U^n T^{(k,l)}\psi_i : n \in \mathbb{Z}, (k,l) \in \mathbb{Z}^2, i \in \underline{N}\setminus\{0\}\right\}$$

spans $L^2(\nu_{\mathbb{Z}^2_v})$ by showing that

$$\left\{T^{(k,l)}\psi_i : (k,l) \in \mathbb{Z}^2, i \in \underline{N}\setminus\{0\}\right\}$$

is an orthonormal basis of W_0. We establish this by writing a basis element of V_1 of the form $UT^{(k,l)}\varphi$, $(k,l) \in \mathbb{Z}^2$, as a linear combination of elements of V_0 and W_0, and then considering the inner product. We use the property of Proposition 11.15 (1) and so let $(k,l) \in \underline{N_1} \times \underline{N_2}$ (instead of $\mathbb{Z}^2$), then

$$\begin{aligned}
&\Big\langle UT^{(k,l)}\varphi \Big| \sum_{j\in\underline{N}\setminus\{0\}} \overline{a}^j_{(k,l)}\psi_j + \mathbb{1}_A(k,l)\cdot\sqrt{p_{(k,l)}}\varphi \Big\rangle \\
&= \Big\langle T^{(k,l)}\varphi \Big| \sum_{j\in\underline{N}\setminus\{0\}} \overline{a}^j_{(k,l)} \Bigg(\sum_{(u,v)\in A} a^j_{(u,v)} T^{(u,v)}\varphi + \sum_{(u,v)\in\left(\underline{N_1}\times\underline{N_2}\right)\setminus A} a^j_{(u,v)} T^{(u,v)}\varphi \Bigg) \\
&\qquad + \mathbb{1}_A(k,l)\cdot\sqrt{p_{(k,l)}} \sum_{(u,v)\in A} \sqrt{p_{(u,v)}} T^{(u,v)}\varphi \Big\rangle \\
&= \sum_{j\in\underline{N}\setminus\{0\}} \overline{a}^j_{(k,l)} a^j_{(k,l)} + \mathbb{1}_A(k,l)\cdot p_{(k,l)} \\
&= 1.
\end{aligned}$$

So $\left\{T^{(k,l)}\psi_i : (k,l) \in \mathbb{Z}^2, i \in \underline{N}\setminus\{0\}\right\}$ is an orthonormal basis of W_0 and iteratively it follows that $\left\{U^j T^{(k,l)}\psi_i : (k,l) \in \mathbb{Z}^2, i \in \underline{N}\setminus\{0\}\right\}$, $j \in \mathbb{Z}$, is an orthonormal basis for W_j. Since we also have $\bigoplus_{j\in\mathbb{Z}} W_j = L^2(\nu_{\mathbb{Z}^2_v})$, the result follows. □

Remark 11.21. It is also possible to obtain an orthonormal basis for some two-dimensional L^2-spaces via the tensor product of two one-dimensional wavelet bases. These wavelet bases also fit in the construction of this section. The wavelet bases obtained via the tensor product are called separable. In Example 11.22 we give an example for the construction of a separable wavelet bases.

11.4. Construction of the wavelet bases for the measure of maximal entropy

In this section we consider a special case of the construction of a wavelet basis on an enlarged fractal, namely we consider the measure of maximal entropy on the fractal. So we consider the same setting as before with $p_{(i,j)} = 1/p$ for all $(i,j) \in A$ and we extend the measure with the same weights, i.e. $c_{(i,j)} = 1/p$ for all $(i,j) \in \underline{N_1} \times \underline{N_2}$. Consequently, we have a non-locally finite measure by Proposition 11.9. In this context we can define the unitary operator U for the MRA in terms of the scaling function σ given in (11.3.1). The operators U and T are then defined for $f \in L^2(\nu_{\mathbb{Z}_v^2})$ as

$$Uf(\cdot) = \sqrt{p} f(\sigma(\cdot))$$

and

$$T^{(k,l)} f(\cdot) = f\left(\cdot - k\overrightarrow{v_1} - l\overrightarrow{v_2}\right).$$

The operators T and U are unitary in $L^2(\nu_{\mathbb{Z}_v^2})$ and the inverse of U is given for $f \in L^2(\nu_{\mathbb{Z}_v^2})$ by

$$U^{-1} f(\cdot) = \frac{1}{\sqrt{p}} f(\sigma^{-1}(\cdot)).$$

One family of filter functions for the construction of the mother wavelets can be given explicitly as follows. Let $\mathcal{G} := \left\{(i,j) \in \left(\underline{N_1} \times \underline{N_2}\right) \setminus A\right\} = \left\{\left(d_i^1, d_i^2\right) : i \in \underline{N-p}\right\}$ and $\mathcal{A} = \left\{\left(\mathbf{a}_j^1, \mathbf{a}_j^2\right) \in A : j \in \underline{p}\right\}$. Then the first $N-p$ high-pass filters, $m_1, \ldots, m_{N-p}$ are defined on $\mathbb{T}^2$ by

$$m_{i+1} : (z,w) \mapsto z^{d_i^1} w^{d_i^2},\ i \in \underline{N-p}.$$

The remaining $p-1$ filter functions are defined by

$$m_{N-p+k} : (z,w) \mapsto \frac{1}{\sqrt{p}} \sum_{j \in \underline{p}} \eta^{kj} \cdot z^{\mathbf{a}_j^1} w^{\mathbf{a}_j^2},\ \text{for } k \in \underline{p}\setminus\{0\},\ \eta = e^{2\pi i/p}.$$

The vectors given by these filter functions form an orthonormal basis of $\mathbb{C}^N$.

The wavelet basis for $L^2(\nu_{\mathbb{Z}_v^2})$ is then given as above for these filter functions and operators.

11.5. Examples

In this section we give two different examples of fractals on which we construct wavelet bases. The first example is the Cantor Dust and the second is based on Gosper Island.

Example 11.22. We construct wavelet bases for the Cantor Dust in two different ways. The first way is via the tensor product of two one-dimensional wavelet bases and the second way is the direct one described in Section 11.3. We consider the Cantor Dust with five gaps as in Example 11.6 (a) and we consider the measure of maximal entropy on the fractal which means we take the same weights on the gaps, $c_{(i,j)} = 1/4$ for all $(i,j) \in \underline{3} \times \underline{3}$.

For the tensor product approach we start by noticing that the Cantor Dust coincides with the tensor product of the one-dimensional 1/4-Cantor set C_4 given by the IFS $\left(\tau_0(x) = \frac{x}{4}, \tau_2(x) = \frac{x+3}{4}\right)$ with itself. Furthermore, the space $L^2(\nu_{\mathbb{Z}^2})$ based on the Cantor Dust coincides with $L^2(\nu_{\mathbb{Z}}) \otimes L^2(\nu_{\mathbb{Z}})$, where $\nu_{\mathbb{Z}}$ is the measure in one dimension obtained from the invariant measure μ on the Cantor set C_4 with the weights $(1/2, 1/2)$ and on the gap we also consider the weight $\frac{1}{2}$. For the definition of $\nu_{\mathbb{Z}}$ see Definition 10.6 or [**BK10**]. We have that $\nu_{\mathbb{Z}^2} = \nu_{\mathbb{Z}} \otimes \nu_{\mathbb{Z}}$. A wavelet basis is then constructed in $L^2(\nu_{\mathbb{Z}})$ with the operators T and U for $f \in L^2(\nu_{\mathbb{Z}})$ given by $(Tf)(x) := f(x-1)$ and $(Uf)(x) := \sqrt{2} f(\sigma(x))$, where the scaling function σ restricted to $[0,1]$ is given by

$$\sigma(x) := \sum_{k \in \mathbb{Z}} \mathbb{1}_{[0,\frac{1}{4})}(x-k) \cdot (4x-k) + \mathbb{1}_{[\frac{1}{4},\frac{3}{4})}(x-k) \cdot \left(2x + \frac{1}{2} + k\right) + \mathbb{1}_{[\frac{3}{4},1)}(x-k) \cdot (4x-1-k).$$

Let the father wavelet be $\varphi^{(1)} = \mathbb{1}_{C_4}$ and the mother wavelets be given for $x \in [0,1]$, by

$$\psi_1^{(1)}(x) = \sqrt{2}\mathbb{1}_{\frac{1}{2}C_4+\frac{1}{4}}(x),$$
$$\psi_2^{(1)}(x) = \mathbb{1}_{\frac{1}{4}C_4}(x) - \mathbb{1}_{\frac{1}{4}C_4+\frac{3}{4}}(x).$$

The corresponding orthonormal basis for $L^2(\nu_{\mathbb{Z}})$ is

$$\left\{U^n T^k \psi_i^{(1)} : i = 1,2,\, n,k \in \mathbb{Z}\right\}.$$

Consequently, we obtain a wavelet basis for the Cantor Dust with the father wavelet $\varphi((x,y)) = \varphi^{(1)}(x)\cdot\varphi^{(1)}(y)$ and the mother wavelets

$$\begin{aligned}
\psi_1((x,y)) &= \varphi^{(1)}(x)\cdot\psi_1^{(1)}(y), & \psi_2((x,y)) &= \varphi^{(1)}(x)\cdot\psi_2^{(1)}(y),\\
\psi_3((x,y)) &= \psi_1^{(1)}(x)\cdot\varphi^{(1)}(y), & \psi_4((x,y)) &= \psi_2^{(1)}(x)\cdot\varphi^{(1)}(y),\\
\psi_5((x,y)) &= \psi_1^{(1)}(x)\cdot\psi_1^{(1)}(y), & \psi_6((x,y)) &= \psi_1^{(1)}(x)\cdot\psi_2^{(1)}(y),\\
\psi_7((x,y)) &= \psi_2^{(1)}(x)\cdot\psi_1^{(1)}(y), & \psi_8((x,y)) &= \psi_2^{(1)}(x)\cdot\psi_2^{(1)}(y).
\end{aligned}$$

The values that these functions take on the subsets are shown in Figure 11.5.1. The subsets are viewed as scaled copies under application of $(\tau_i(C_4), \tau_j(C_4))$, $i,j \in \underline{3}$, of the Cantor Dust. So an orthonormal for $L^2(\nu_{\mathbb{Z}_v^2})$ is

$$\left\{\widetilde{U}^n T^{(k,l)}\psi_j : n \in \mathbb{Z}, (k,l) \in \mathbb{Z}^2, j \in \{1,\ldots,8\}\right\},$$

where $\widetilde{U}^n = (U^n, U^n)$ and $T^{(k,l)} = \left(T^k, T^l\right)$ such that e.g. $\widetilde{U}^n T^{(k,l)}\psi_1 = \left(U^n T^k \varphi^{(1)}\right)\cdot\left(U^n T^l \psi_1^{(1)}\right)$.

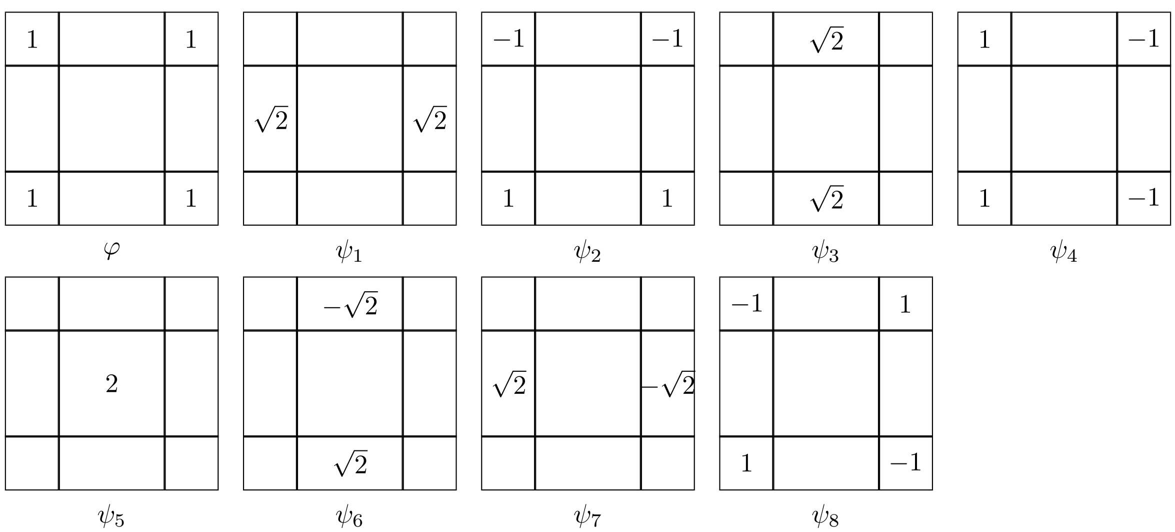


FIGURE 11.5.1. The mother wavelets for the Cantor Dust (construction via the tensor product).

Now we construct a wavelet basis directly in the way of Section 11.3 on the enlarged fractal of the Cantor Dust. In this way we have more freedom for the choice of the mother wavelets. A possible choice of coefficients for the filter functions is the following.

$$
\begin{array}{lccccccccc}
v_0: & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\
v_1: & \frac{1}{3} & \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{-1}{3} & \frac{1}{3} & \frac{1}{3} \\
v_2: & \frac{-1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & 0 & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{1}{2\sqrt{2}} & \frac{-1}{2\sqrt{2}} \\
v_3: & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 \\
v_4: & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 & 0 \\
v_5: & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} \\
v_6: & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & \frac{-1}{\sqrt{2}} & 0 & 0 \\
v_7: & \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{-1}{4} & \frac{-\sqrt{3}}{4} & 0 & \frac{-\sqrt{3}}{4} & \frac{-1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} \\
v_8: & \frac{-\sqrt{3}}{4} & \frac{1}{4} & \frac{\sqrt{3}}{4} & \frac{-1}{4} & 0 & \frac{-1}{4} & \frac{\sqrt{3}}{4} & \frac{1}{4} & \frac{-\sqrt{3}}{4}.
\end{array}
$$

The mother wavelets so obtained are visualized in Figure 11.5.2, where the values are only taken on images under the extended IFS of the Cantor Dust mapped to the subsets of the unit square. That is more precisely the values on $\tau_{(i,j)}(C)$ for $(i,j) \in \underline{3} \times \underline{3}$.

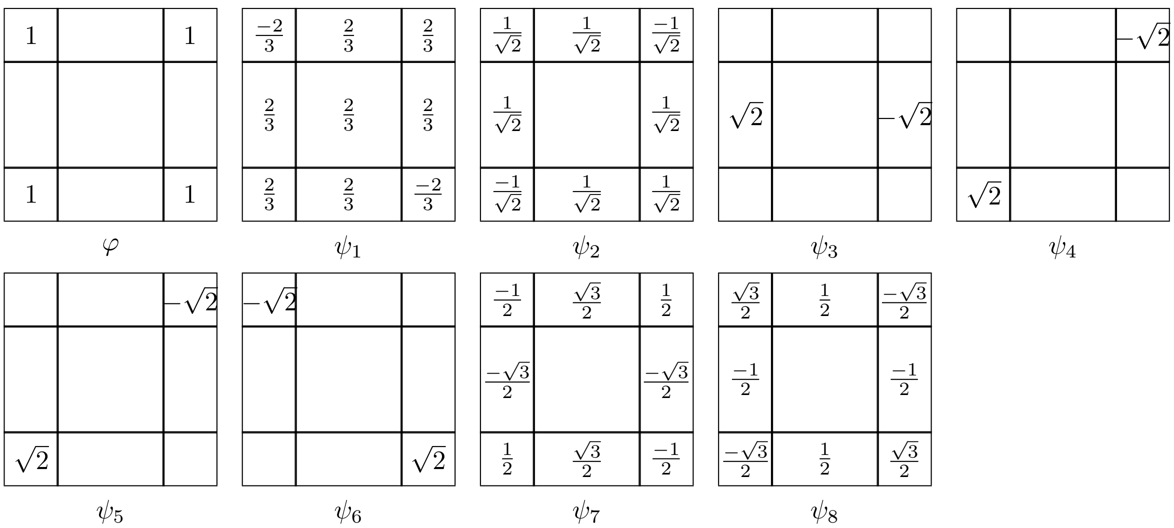
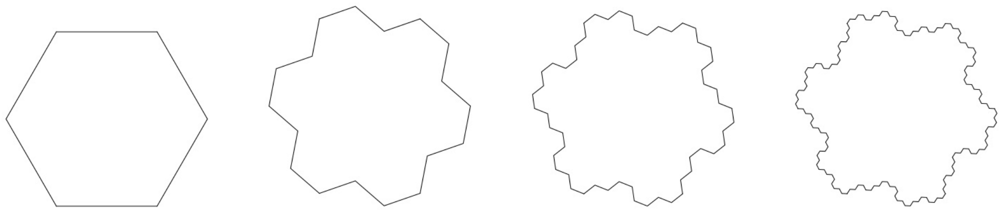


FIGURE 11.5.2. The mother wavelets on the Cantor Dust (direct construction).

In this example we can see that the wavelets obtained via the tensor product approach also satisfy the conditions for the direct approach and so we can also obtain these wavelets via the direct construction of two-dimensional wavelets. But the direct approach allows more freedom of choice in the coefficients of the high-pass filter functions, and hence we can obtain more different wavelet bases in the direct approach.

Example 11.23. We now consider an example based on the fractal called "Gosper Island". This fractal is constructed along the lines of Figure 11.5.3. For the construction of Gosper Island we start with a hexagon with the vertexes $(-2,0)$, $(-1,\sqrt{3})$, $(1,\sqrt{3})$, $(2,0)$, $(1,-\sqrt{3})$ and $(-1,-\sqrt{3})$. Then each edge is changed to four lines. Gosper Island is the limit set of continuing this process indefinitely.



(A) Gosper Island; order 0.
(B) Gosper Island; order 1.
(C) Gosper Island; order 2.
(D) Gosper Island; order 3.

FIGURE 11.5.3. The construction of Gosper Island.

Gosper Island is actually the boundary of the set and it has Hausdorff dimension $\frac{2\log 3}{\log 7} \approx 1.13$, the length of the boundary is infinite but the area bounded by Gosper Island is equal to the area of the starting hexagon. In the following we also call the set bounded by the curve Gosper Island. The underlying fractal for the construction of the wavelet basis is considered as lying in Gosper Island. The set D in our construction is taken to be Gosper Island and it allows a tiling of $\mathbb{R}^2$ with the two vectors $\overrightarrow{v_1} = \begin{pmatrix} 1 \\ \sqrt{3} \end{pmatrix}$ and $\overrightarrow{v_2} = \begin{pmatrix} 1 \\ -\sqrt{3} \end{pmatrix}$.

Seven copies of Gosper Island D fit inside itself, compare Figure 11.5.4, where it is shown for a prefractal of Gosper Island.

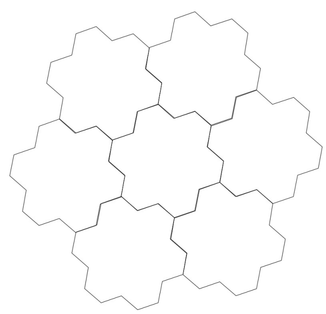


FIGURE 11.5.4. Scaled copies of Gosper Island in itself.

So the underlying fractal for the construction of the wavelet basis is given by the IFS $\mathcal{S} = (\sigma_0, \sigma_1, \sigma_2, \sigma_3)$ with

$$\sigma_0 : \overrightarrow{x} \mapsto \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \overrightarrow{x} + \begin{pmatrix} -2/3 \\ 0 \end{pmatrix},$$
$$\sigma_1 : \overrightarrow{x} \mapsto \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \overrightarrow{x} + \begin{pmatrix} 1/3 \\ 1/\sqrt{3} \end{pmatrix},$$
$$\sigma_2 : \overrightarrow{x} \mapsto \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \overrightarrow{x} + \begin{pmatrix} 1/3 \\ -1/\sqrt{3} \end{pmatrix},$$
$$\sigma_3 : \overrightarrow{x} \mapsto \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \overrightarrow{x} + \begin{pmatrix} 2/3 \\ 0 \end{pmatrix},$$

where the matrix and the vectors are given to the basis vectors $(1,0)^t$, $(0,1)^t$ of $\mathbb{R}^2$. The limit set for this IFS $\mathcal{S}$ is called C. The IFS satisfies the OSC for $\mathring{D}$ and it has Hausdorff dimension $\frac{\log 4}{\log 3}$. Consequently, the boundary of D has $\frac{\log 4}{\log 3}$-Hausdorff measure zero.

We define the extended IFS to be $\mathbb{S} = \left(\tau_{(i,j)} : (i,j) \in \underline{1} \times \underline{7}\right)$ given by

$$\tau_{(i,j)} : \overrightarrow{x} \mapsto \begin{pmatrix} 1/3 & 0 \\ 0 & 1/3 \end{pmatrix} \overrightarrow{x} + \begin{pmatrix} a_i \\ b_j \end{pmatrix},$$

with

$$\begin{pmatrix} a_i \\ b_j \end{pmatrix} \in \left\{ \begin{pmatrix} -2/3 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/3 \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 1/3 \\ 1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 2/3 \\ 0 \end{pmatrix}, \begin{pmatrix} 1/3 \\ -1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} -1/3 \\ -1/\sqrt{3} \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right\}.$$

Thus $A = \{(0,0), (0,2), (0,3), (0,4)\}$, $p = 4$ and $N = 7$ ($N_1 = 1$, $N_2 = 7$). So in this example we do not have that $\bigcup_{(i,j) \in \underline{1} \times \underline{7}} D + i\overrightarrow{v_1} + j\overrightarrow{v_2}$ is an affine scaled version of D, but seven copies of D are mapped in a line by translation with $j\overrightarrow{v_2}$, $j \in \underline{7}$.

The enlarged fractal is defined in two steps: first it is defined in D by

$$S = \bigcup_{\omega \in \Sigma^{(2)} \cup \{\emptyset\}} \tau_\omega(C),$$

where $\Sigma^{(2)} = \left\{ \omega = ((i_0, j_0), \dots, (i_{k-1}, j_{k-1})) \in (\underline{1} \times \underline{7})^k : k \in \mathbb{N},\ (i_{k-1}, j_{k-1}) \notin A \right\}$ and by translation it is defined in $\mathbb{R}^2$ by

$$R = \bigcup_{(m,n) \in \mathbb{Z}^2} S + m\,\overrightarrow{v_1} + n\,\overrightarrow{v_2}.$$

The measure on R is constructed in a way analogous to the construction of the enlarged fractal. The invariant measure μ is the $\frac{\log 4}{\log 3}$-Hausdorff measure restricted to the invariant set C for $\mathcal{S}$. This measure is the measure of maximal entropy. The measure ν on S is defined by

$$\nu = \sum_{\omega \in \Sigma^{(2)} \cup \{\emptyset\}} 3^{-|\omega|} \mu \circ \tau_\omega^{-1}.$$

The measure $\nu_{\mathbb{Z}_v^2}$ is obtained by translation of ν. This measure coincides with the $\frac{\log 4}{\log 3}$-Hausdorff measure restricted to the enlarged fractal R.

As the scaling function for the MRA we get for $\overrightarrow{x} \in \mathbb{R}^2$

$$\sigma(\overrightarrow{x}) = \sum_{(l,k) \in \mathbb{Z}^2} \mathbb{1}_{k\overrightarrow{v_1} + l\overrightarrow{v_2} + \overset{\circ}{D}}(\overrightarrow{x}) \sum_{(i,j) \in \underline{1} \times \underline{7}} \mathbb{1}_{\tau_{(i,j)}(\overset{\circ}{D})}(\overrightarrow{x} - (k\overrightarrow{v_1} + l\overrightarrow{v_2}))$$
$$\left(\tau_{(i,j)}^{-1} (\overrightarrow{x} - (k\overrightarrow{v_1} + l\overrightarrow{v_2})) + (i+k)\overrightarrow{v_1} + (j+7l)\overrightarrow{v_2} \right).$$

Then the operators are defined for $f \in L^2\left(\nu_{\mathbb{Z}_v^2}\right)$, $\overrightarrow{x} \in \mathbb{R}^2$, $(m,n) \in \mathbb{Z}^2$, by

$$\left(T^{(m,n)} f\right)(\overrightarrow{x}) = f\left(\overrightarrow{x} - m\overrightarrow{v_1} - n\overrightarrow{v_2}\right) \quad \text{and} \quad (Uf)(\overrightarrow{x}) = 2 \cdot f\left(\sigma(\overrightarrow{x})\right).$$

The father wavelet for the MRA is then as before the characteristic function on the fractal, i.e. $\varphi = \mathbb{1}_C$, and the low-pass filter is

$$m_0(z,w) = \frac{1}{2} \sum_{(i,j) \in A} e_{(i,j)}(z,w).$$

Now the 6 high-pass filters have to be constructed. The corresponding vector v_0 for the low-pass filter m_0 is

$$v_0 = \left(\frac{1}{2}, 0, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, 0, 0 \right)$$

and possible vectors for the high-pass filters are

$$\begin{array}{rlllllll} v_1 = & \big(0, & 1, & 0, & 0, & 0, & 0, & 0\big), \\ v_2 = & \big(0, & 0, & 0, & 0, & 0, & 1, & 0\big), \\ v_3 = & \big(0, & 0, & 0, & 0, & 0, & 0, & 1\big), \\ v_4 = & \big(0, & 0, & 0, & \frac{1}{\sqrt{2}}, & \frac{-1}{\sqrt{2}}, & 0, & 0\big), \\ v_5 = & \big(\frac{1}{\sqrt{2}}, & 0, & \frac{-1}{\sqrt{2}}, & 0, & 0, & 0, & 0\big), \\ v_6 = & \big(\frac{1}{2}, & 0, & \frac{1}{2}, & \frac{-1}{2}, & \frac{-1}{2}, & 0, & 0\big). \end{array}$$

The corresponding mother wavelets are given by

$$\begin{aligned} \psi_1 =& 2 \cdot \mathbb{1}_{\tau_{(0,1)}(C)}, & \psi_2 =& 2 \cdot \mathbb{1}_{\tau_{(0,5)}(C)}, \\ \psi_3 =& 2 \cdot \mathbb{1}_{\tau_{(0,6)}(C)}, & \psi_4 =& \sqrt{2} \cdot \mathbb{1}_{\tau_{(0,3)}(C)} - \sqrt{2} \cdot \mathbb{1}_{\tau_{(0,4)}(C)}, \\ \psi_5 =& -\sqrt{2} \cdot \mathbb{1}_{\tau_{(0,0)}(C)} + \sqrt{2} \cdot \mathbb{1}_{\tau_{(0,2)}(C)}, & \psi_6 =& \mathbb{1}_{\tau_{(0,0)}(C)} + \mathbb{1}_{\tau_{(0,2)}(C)} - \mathbb{1}_{\tau_{(0,3)}(C)} - \mathbb{1}_{\tau_{(0,4)}(C)}, \end{aligned}$$

and the wavelet basis is given by

$$\left\{ U^n T^{(k,l)} \psi_i : n \in \mathbb{Z}, \, (k,l) \in \mathbb{Z}^2, \, i \in \underline{7} \backslash \{0\} \right\}.$$

Gosper Island is not a good fractal to be used for image compression since it can only be applied to images of size (1×7^n), $n \in \mathbb{N}$. For further information about this see Remark 12.1 and Remark 12.4.

Remark 11.24.

(1) Notice that if we consider the extended IFS $\big(\tau_{(i,j)}\big)_{(i,j)\in K}$ with $K = \big\{(0,0), (1,0), (1,1), (0,1), (-1,0), (0,-1), (-1,-1)\big\}$, then

$$2 \cdot D = \bigcup_{(i,j)\in K} D + i\overrightarrow{v_1} + j\overrightarrow{v_2}.$$

So we obtain an augmented version of Gosper Island. Nevertheless, we do not have that $\bigcup_{(k,l)\in\mathbb{R}^2} \bigcup_{(i,j)\in K} D + (i + N_1 k)\overrightarrow{v_1} + (j + N_2 l)\overrightarrow{v_2}$ for any $N_1, N_2 \in \mathbb{N}$ is an essentially disjoint tiling of $\mathbb{R}^2$. For $N_1 = N_2 = 2$ we obtain a tiling but it is only essentially disjoint if we restrict the elements in K to the set $\{(0,0), (1,0), (0,1), (1,1)\}$. If we consider this restriction of K, the functions Uf are not supported on the subsets $\tau_{(i,j)}(D)$, $(i,j) \in K \backslash \{(0,0), (1,0), (0,1), (1,1)\}$ and so the operator U is not unitary.

(2) There are other interesting examples of fractals in two dimensions which satisfy the conditions given here for the construction of wavelet bases on fractals. An interesting class of fractals are Rosy fractals and the special class of Dragon curves. There the Dragon curve takes the place of the set D and the fractal lies inside D.

11.5.1. Multiresolution analysis for triangles and the Sierpinski Gasket. Now we turn to a slightly different construction than the one in Section 11.3. There will mainly be two differences in the construction. The first is that we include a rotation for the tiling instead of only a translation. The second is that we do not consider a fractal in the triangle but the triangle itself for the construction. To be more precise, we can also consider the triangle as the limit set of an IFS without any gaps. So the construction is analogous to the one for the two dimensional Haar wavelet, where the unit square can be regarded as the invariant set of an IFS with four functions. We consider the Lebesgue measure λ on $\mathbb{R}^2$, so we construct a wavelet basis in $L^2(\mathbb{R}^2, \lambda)$.

We omit the proofs here since they are similar to the ones in Section 11.3. The proofs for this section can be found in Appendix D.

We consider the triangle $\triangle$ with the vertices $(0,0), (1,0), (0.5,2)$. It is well known that it gives a tiling of $\mathbb{R}^2$ by translation and rotation. More precisely, we define the two vectors $\overrightarrow{v_1} = (1,0)^t$ and $\overrightarrow{v_2} = (0.5,2)^t$ and we consider the translation by $\overrightarrow{v_1}\mathbb{Z} + \overrightarrow{v_2}\mathbb{Z}$ and rotation by the matrix $\mathbf{R} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

(given with respect to the basis vectors $(1,0)^t$, $(0,1)^t$ of $\mathbb{R}^2$), which gives a tiling of $\mathbb{R}^2$ in the way shown in Figure 11.5.5.

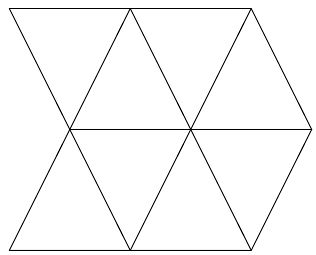


FIGURE 11.5.5. A tiling of $\mathbb{R}^2$ by triangles.

Hence we have to consider a different translation operator. More precisely, we consider two translation operators. The first translation operator is then given for $(k,l) \in \mathbb{Z}^2$, $f \in L^2(\mathbb{R}^2, \lambda)$, $\overrightarrow{x} \in \mathbb{R}^2$, by

$$T_1^{(k,l)} f(\overrightarrow{x}) := f\left(\overrightarrow{x} - k\overrightarrow{v_1} - l\overrightarrow{v_2}\right)$$

and the second by

$$T_2^{(k,l)} f(\overrightarrow{x}) := f\left(\mathbf{R}\overrightarrow{x} - k\overrightarrow{v_1} - l\overrightarrow{v_2}\right).$$

The dilation operator U is given by

$$U f(\overrightarrow{x}) := 2 f\left(\mathbf{A}\overrightarrow{x}\right),$$

where $\mathbf{A} := \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ (given with respect to the basis vectors $(1,0)^t$, $(0,1)^t$ of $\mathbb{R}^2$).

Hence it is the same dilation operator as for the MRA with the two-dimensional Haar wavelet, compare Example A.14.

Proposition 11.25. *The operators U, T_1 and T_2 are unitary.*

Notice that the Lebesgue measure of this triangle is 1. We consider the function $\varphi := \mathbb{1}_{\triangle}$, i.e. the characteristic function on the triangle, as the father wavelet, which satisfies for $\overrightarrow{x} \in \mathbb{R}^2$

$$\varphi(\overrightarrow{x}) = \varphi(\mathbf{A}\overrightarrow{x}) + \varphi(\mathbf{A}\overrightarrow{x} - \overrightarrow{v_1}) + \varphi(\mathbf{A}\overrightarrow{x} - \overrightarrow{v_2}) + \varphi(\mathbf{R}\mathbf{A}\overrightarrow{x} - \overrightarrow{v_1} + \overrightarrow{v_2})$$

or equivalently,

$$\varphi(\mathbf{A}^{-1}\overrightarrow{x}) = T_1^{(0,0)}\varphi(\overrightarrow{x}) + T_1^{(1,0)}\varphi(\overrightarrow{x}) + T_1^{(0,1)}\varphi(\overrightarrow{x}) + T_2^{(1,-1)}\varphi(\overrightarrow{x}).$$

This can be seen as the scaling relation for the father wavelet, where we have two low-pass filter functions, one applied to the operator T_1 and the second to the operator T_2. These filter functions are for $(z,w) \in \mathbb{T}^2$ given as

$$m_0^1((z,w)) = \frac{1}{2}\left(1 + e_{(1,0)}(z,w) + e_{(0,1)}(z,w)\right) \text{ and } m_0^2((z,w)) = \frac{1}{2} e_{(1,-1)}(z,w).$$

Proposition 11.26. *The operators U, T_1 and T_2 satisfy the following relations.*

(1) $\left\langle T_i^{(k,l)}\varphi \middle| T_j^{(n,m)}\varphi \right\rangle = \delta_{(i,k,l),(j,n,m)}$, *$(k,l), (n,m) \in \mathbb{Z}^2$, $i,j \in \{1,2\}$,*
(2) $U^{-1}\varphi = m_0^1(T_1)\varphi + m_0^2(T_2)\varphi$,
(3) $U^{-1}T_1^{(k,l)}U = T_1^{(2k,2l)}$, *$(k,l) \in \mathbb{Z}^2$, and* $U^{-1}T_2^{(k,l)}U = T_2^{(2k,2l)}$, *$(k,l) \in \mathbb{Z}^2$.*

Remark 11.27. If we combine the application of the two translation operators T_1 and T_2 we obtain the following relations: for $(k,l),(m,n) \in \mathbb{Z}^2$ we have

$$T_1^{(k,l)}T_1^{(n,m)} = T_1^{(k+n,l+m)}, \qquad T_2^{(k,l)}T_2^{(n,m)} = T_1^{(k+l+n,m-l)},$$
$$T_1^{(k,l)}T_2^{(n,m)} = T_2^{(k+l+n,m-l)}, \qquad T_2^{(k,l)}T_1^{(n,m)} = T_2^{(k+n,m+l)}.$$

Now we turn to the MRA which is as given in Remark 1.4 with the difference that we have two translation operators T_i, $i = 1, 2$, instead of one.

Theorem 11.28. *Let $\varphi := \mathbb{1}_\triangle$ be the father wavelet and for $j \in \mathbb{Z}$ let*

$$V_j := \operatorname{cl\,span}\left\{U^j T_i^{(k,l)}\varphi : (k,l) \in \mathbb{Z}^2,\ i \in \{1,2\}\right\}.$$

Then $(\lambda, U, (T_1, T_2))$ allows a two-sided MRA for φ and V_j, $j \in \mathbb{Z}$, as above. In particular, we have

$$\operatorname{cl\,span}\left\{U^n T_i^{(k,l)}\varphi : n \in \mathbb{Z},\ (k,l) \in \mathbb{Z}^2,\ i = 1,2\right\} = L^2(\mathbb{R}^2, \lambda).$$

Now we want to consider the mother wavelets. To define the mother wavelets we need three pairs of high-pass filter functions. These filter functions can be defined for $(z,w) \in \mathbb{T}^2$ as

$$m_1^1((z,w)) = -\frac{1}{2} + \frac{1}{2}e_{(1,0)}(z,w) - \frac{1}{2}e_{(0,1)}(z,w), \qquad m_1^2((z,w)) = \frac{1}{2}e_{(1,-1)}(z,w),$$
$$m_2^1((z,w)) = -\frac{1}{2} - \frac{1}{2}e_{(1,0)}(z,w) + \frac{1}{2}e_{(0,1)}(z,w), \qquad m_2^2((z,w)) = \frac{1}{2}e_{(1,-1)}(z,w),$$
$$m_3^1((z,w)) = -\frac{1}{2} + \frac{1}{2}e_{(1,0)}(z,w) + \frac{1}{2}e_{(0,1)}(z,w), \qquad m_3^2((z,w)) = -\frac{1}{2}e_{(1,-1)}(z,w).$$

The coefficients are chosen such that the (4×4)-matrix containing the four coefficients for each of the four pairs of filter functions is unitary.

Proposition 11.29. *Define the mother wavelets to be for $i = 1, 2, 3$*

$$\psi_i := U(m_i^1(T_1)\varphi + m_i^2(T_2)\varphi).$$

Then the set

$$\left\{U^n T_j^{(k,l)}\psi_i : i \in \{1,2,3\},\ n \in \mathbb{Z},\ (k,l) \in \mathbb{Z}^2,\ j \in \{1,2\}\right\}$$

is an ONB for $L^2\left(\mathbb{R}^2, \lambda\right)$.

Notice that these functions are weighted sums of the characteristic functions on the four sub-triangles of the original triangle taking the values as shown in Figure 11.5.6.

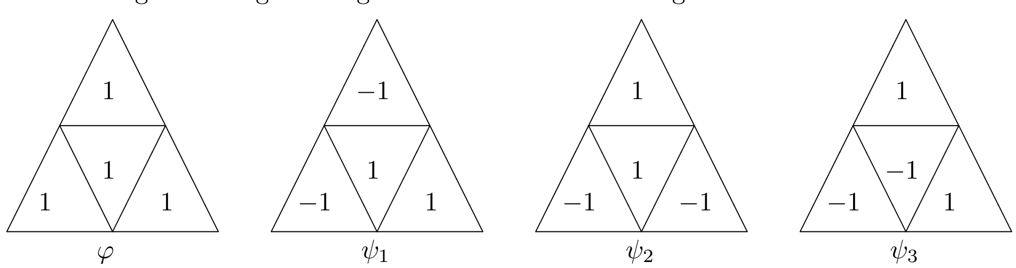


FIGURE 11.5.6. The mother wavelets on the triangle.

Remark 11.30. In this way we can construct an MRA for the Sierpinski Gasket. If we consider the Sierpinski Gasket in the triangle with the vertices $(0,0), (1,0), (0.5,2)$, then we combine the construction of the wavelet basis on triangles above with the construction of wavelet bases on fractals given in Section 11.3.

In **[DMP08]** D'Andrea, Merrill and Packer consider the construction of a wavelet basis on the Sierpinski Gasket with only one translation operator. They consider the triangle with vertices $(0,0)$,

$(1,0)$ and $(0,1)$ and obtain within the triangle the Sierpinski Gasket C given by the IFS $\mathcal{S} = (\sigma_0, \sigma_1, \sigma_2)$ with

$$\sigma_0(\overrightarrow{x}) = \mathbf{A}^{-1}\overrightarrow{x},$$
$$\sigma_1(\overrightarrow{x}) = \mathbf{A}^{-1}\overrightarrow{x} + \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$
$$\sigma_2(\overrightarrow{x}) = \mathbf{A}^{-1}\overrightarrow{x} + \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

where $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. The enlarged fractal that they consider is defined as

$$R = \bigcup_{n\in\mathbb{N}} \bigcup_{(k,l)\in\mathbb{Z}} \mathbf{A}^n \left(C + \begin{pmatrix} k \\ l \end{pmatrix} \right).$$

Consequently, the enlarged fractal is not dense in $\mathbb{R}^2$ since the gaps are not filled.

The wavelet basis is constructed in $L^2\left(H^{\log(3)/\log(2)}|_R\right)$, where $H^{\log(3)/\log(2)}$ is the $\log(2)/\log(3)$-Hausdorff measure on $\mathbb{R}^2$ and the unitary operators acting on $L^2\left(H^{\log(3)/\log(2)}|_R\right)$ are

$$T^{(k,l)} f(\overrightarrow{x}) = f\left(\overrightarrow{x} - k\overrightarrow{v_2} - l\overrightarrow{v_1}\right),$$

where $\overrightarrow{v_1} = (1,0)^t$, $\overrightarrow{v_2} = (0,1)^t$, $(k,l) \in \mathbb{Z}^2$, $\overrightarrow{x} \in \mathbb{R}^2$, and

$$Uf(\overrightarrow{x}) = \sqrt{3} f\left(\mathbf{A}\overrightarrow{x}\right).$$

If we consider only an approximation of the Sierpinski Gasket as shown in Figure 12.2.1, then the Sierpinski Gasket lies in the unit square $[0,1] \times [0,1]$ instead of the triangle.

CHAPTER 12

Application to image compression

In this chapter we apply the results for wavelet bases on enlarged fractals in two dimensions to image compression. This is done in a way very similar to [**DMP08**], where the authors do the same in the specific case of the Sierpinski Gasket fractal. Their treatment and ours are both closely analogous to compression using Haar wavelet. We apply different wavelet bases to image compression, with the "Lena" image[1] as an example. We start by giving a general explanation of how wavelet bases are used in image compression. Then we apply different wavelet bases on the Cantor Dust, a wavelet basis on the Sierpinski Gasket and the two-dimensional Haar wavelet basis to the image. We compare our results using the peak-signal-to-noise ratio (PSNR).

12.1. Theoretical background for the application to image compression

Here we explain how the application to images is done. This explanation of the theoretical background for the application is close to the one in [**DMP08**].

We consider greyscale images of size $(N_1^m \times N_2^m)$ pixels, $N_1, N_2, m \in \mathbb{N}$. The image takes values in $\{0, \dots, 255\}$. In correspondence with the size we consider a fractal C with an extended IFS $\left\{\tau_{(i,j)} : (i,j) \in \underline{N_1} \times \underline{N_2}\right\}$. Depending on the size of the image the vectors are $\overrightarrow{v_1} = (0,1)$ and $\overrightarrow{v_2} = (y,0)$, $y \in \mathbb{R}^+$, since the fractal is always seen as lying in a rectangle. In the image compression only the filter functions are used; not any other information about the underlying space, so the simplest underlying fractal to an IFS $\mathcal{S}$ (and extended IFS $\mathbb{S}$) consists of affine constractions and lies in a rectangle. There is the dependence of the image size on N_1, N_2 because we consider divisions of the image of size $N_1 \times N_2$. So if we have that $\bigcup_{(i,j)\in \underline{N_1}\times\underline{N_2}} D + i\overrightarrow{v_1} + j\overrightarrow{v_2} = \Psi(D)$ with Ψ an affine transformation, then the underlying structure of the fractal can have an impact on the image compression.

Remark 12.1. Here we see that for the example of the Cantor Dust we can apply the wavelet basis for the compression of images either to images of size $(3^m \times 3^m)$ or of size $(4^m \times 4^m)$, $m \in \mathbb{N}$, depending on whether we consider the square as divided in Figure 11.2.2(A) or in Figure 11.2.2(B). But then we consider different Cantor Dust fractals; for one the IFS consists of affine maps with a scaling by the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ and for the other it consists of affine maps with a scaling by the matrix $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$.

We can apply the wavelet basis for the Gosper Island only to images of size (1×7^m), $m \in \mathbb{N}$, which is not a common size of an image. The number of coefficients N in the filter functions have to be divided into $N_1, N_2 \in \mathbb{N}$ such that $N = N_1 \cdot N_2$. For a prime number like 7 the only possible choice is $1 \cdot 7$. Even the application of the wavelet basis to parts of the image of correct size does not give a good impact on image compression as can be seen in Figure 12.2.8.

Every pixel is considered as a scaled copy of the fractal C. In this way we regard the image as a function f in the closed subspace $V_m \subset L^2(\nu_{\mathbb{Z}^2})$, obtained by the multiresolution analysis. This function f can be viewed as having support in $R \cap ([0,1] \times [0,y])$, $y \in \mathbb{R}^+$, since the image is bounded. Furthermore, we have from the MRA that

[1] "Lena" (or "Lenna") image is the probably most widely used test image in image analysis. It is a part of a centerfold of the November 1972 issue of Playboy magazine. It is a photo of the Swedish model Lena Söderberg, taken by the photographer Dwight Hooker.

$$V_m = V_{m-1} \oplus W_{m-1} = V_0 \oplus \bigoplus_{j=1}^{m} W_{m-j}.$$

So we only consider a one-sided MRA. Consequently, for an image of size $(N_1^m \times N_2^m)$ we can consider at most m levels of decomposition. A level of decomposition is the projection on subspaces V_j and W_j. For $f \in V_m$ there are m pairs of subspaces (V_j, W_j), $j \in \{0, \ldots, m-1\}$, on which we can iteratively project the function f. We decompose f in terms of the MRA in the following way

$$\begin{aligned} f = & \sum_{(k,l)\in \underline{N_1^{m-1}} \times \underline{N_2^{m-1}}} \langle f | U^{m-1} T^{(k,l)} \varphi \rangle U^{m-1} T^{(k,l)} \varphi \\ & + \sum_{(k,l)\in \underline{N_1^{m-1}} \times \underline{N_2^{m-1}}} \sum_{j \in \underline{N}\backslash\{0\}} \langle f | U^{m-1} T^{(k,l)} \psi_j \rangle U^{m-1} T^{(k,l)} \psi_j \\ = & \langle f|\varphi\rangle \varphi + \sum_{n=1}^{m} \sum_{(k,l)\in \underline{N_1^{m-n}} \times \underline{N_2^{m-n}}} \sum_{j \in \underline{N}\backslash\{0\}} \langle f | U^{m-n} T^{(k,l)} \psi_j \rangle U^{m-n} T^{(k,l)} \psi_j . \end{aligned}$$

The inner products between f and the basis functions $U^{m-1}T^{(k,l)}\varphi$ and $U^{m-1}T^{(k,l)}\psi_j$ are taken as the discrete wavelet transform coefficients for the image compression in the first step. More precisely, for $(i,j) \in \underline{N_1^{m-1}} \times \underline{N_2^{m-1}}$ and $k \in \underline{N}\backslash\{0\}$ we take

$$a_{i,j}^1 = \langle f | U^{m-1} T^{(i,j)} \varphi \rangle \text{ and } d_{i,j}^{k,1} = \langle f | U^{m-1} T^{(i,j)} \psi_k \rangle.$$

We group these coefficients $a_{i,j}^1$, $d_{i,j}^{k,1}$, $(i,j) \in \underline{N_1^{m-1}} \times \underline{N_2^{m-1}}$, $k \in \underline{N}\backslash\{0\}$, in matrices $a_1 = \left(a_{i,j}^1\right)_{i\in \underline{N_1^{m-1}}, j \in \underline{N_2^{m-1}}}$ and $d_1^k = \left(d_{i,j}^{k,1}\right)_{i\in \underline{N_1^{m-1}}, j \in \underline{N_2^{m-1}}}$.

Iteratively, we obtain the coefficients for the next steps by application of UT to $V_{m-1} = V_{m-2} \oplus W_{m-2}$. Thus, we apply the operator U iteratively m times and obtain by using $T^{(k,l)}$ a complete decomposition of the image f in terms of coefficients $a_{i,j}^m$ and $d_{i,j}^{k,m}$, $k \in \underline{N}\backslash\{0\}$.

We call each step of the iterative decomposition of the image B into the matrices a_m and d_m^k one level of decomposition. So in Figure 12.1.2(B) five levels of decomposition are performed.

For the discrete image we can obtain the coefficients $a_{i,j}$, $d_{i,j}^k$ by multiplication with a matrix in the following way. We consider the image as a $(N_1^m \times N_2^m)$-matrix B. We decompose this matrix B in a coefficient matrix C by taking N^{m-1} sub-matrices of size $(N_1 \times N_2)$, we write these as column vectors and multiply these vectors with the matrix M consisting of the coefficients of the filter functions. More precisely, for $(i,j) \in \left\{(1+N_1k, 1+N_2l) :\ k \in \underline{N_1^{m-1}},\ l \in \underline{N_2^{m-1}}\right\}$ we rewrite the matrix

$$(b_{i+k,j+l})_{k\in \underline{N_1}, l \in \underline{N_2}} \backsimeq \left(b_{i+N_1-1-\lfloor \frac{k}{N_2} \rfloor, j+(k)_{N_2}}\right)_{k\in \underline{N}} =: \overrightarrow{b_{ij}}$$

as a vector. Now let the matrix M consist of the coefficients of the filter functions stored in the vectors $v_0, \ldots, v_{N-1}$ (compare (11.3.2)); more precisely, the vectors containing the coefficients of the filter give the rows of the matrix M:

$$M = \begin{pmatrix} v_0 \\ \vdots \\ v_{N-1} \end{pmatrix},$$

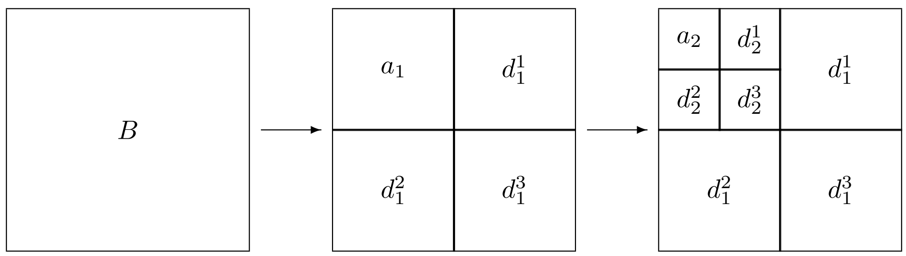
compare Section 11.3.1. The discrete decomposition of the image B (that is its projection on the subspaces (V_j, W_j), $j \in \underline{m}$) is done by multiplying $\overrightarrow{b_{ij}}$ with the matrix M, i.e.

$$\overrightarrow{c_{ij}} = M\overrightarrow{b_{ij}} = \begin{pmatrix} a^1_{\frac{i-1}{N_1}+1, \frac{j-1}{N_2}+1} \\ \left(d^{1,k}_{\frac{i-1}{N_1}+1, \frac{j-1}{N_2}+1}\right)^t_{k\in \underline{N}\backslash\{0\}} \end{pmatrix}.$$

In the next step we continue by transforming the sub-matrix a_1 given by

$$a_1 = \left(a^1_{\frac{i-1}{N_1}+1, \frac{j-1}{N_2}+1} \right)_{(i,j) \in \left\{ (1+N_1 k, 1+N_2 l) \colon k \in \underline{N_1^{m-1}},\, l \in \underline{N_2^{m-1}} \right\}},$$

which has the size $\left(N_1^{m-1} \times N_2^{m-1}\right)$. In the same way we consider N^{m-2} sub-matrices of a_1 of the size $(N_1 \times N_2)$ rewrite these as vectors and then we multiply these with the matrix M. The subdivision procedure is illustrated in Figure 12.1.1.



(A) The $(2^m \times 2^m)$ greyscale matrix B.

(B) The first decomposition in the coefficient matrix.

(C) The second iteration of the decomposition in the coefficient matrix.

FIGURE 12.1.1. Representation of (2×2)-sub-matrix decomposition for $N_1 = N_2 = 2$.

We illustrate the subdivision of an image in Figure 12.1.2(B), where we apply filter functions based on the Cantor Dust with nine equally sized sub-squares in the unit square to the "Lena" image, see Figure 12.1.2(A).

(A) "Lena" image

(B) Image of the decomposed "Lena" image with filters of the Cantor Dust.

FIGURE 12.1.2. "Lena" image and its decomposition.

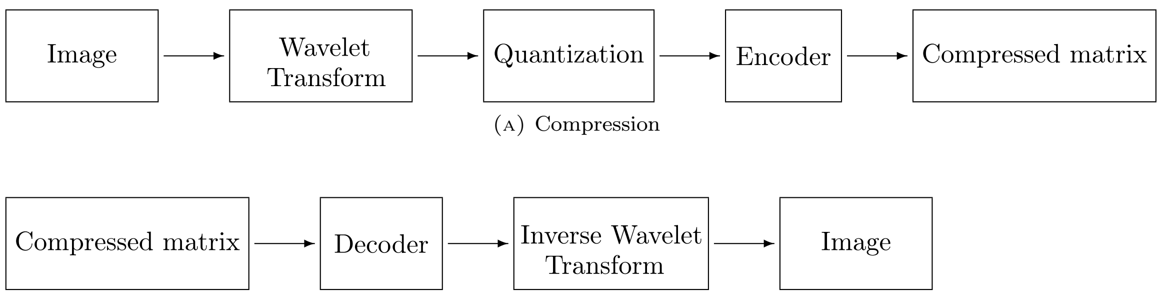
The reconstruction of the image after decomposition uses the inverse wavelet transform. More precisely, we obtain the reconstructed image in the following way if only one level of decomposition was done:

$$M^* \overrightarrow{c_{ij}} = M^* M \overrightarrow{b_{ij}} = \overrightarrow{b_{ij}}$$

for $(i,j) \in \left\{ (1+N_1 k, 1+N_2 l) \,:\, k \in \underline{N_1^{m-1}},\, l \in \underline{N_2^{m-1}} \right\}$ since the matrix M is unitary. Consequently, we have a perfect reconstruction if we keep all the transform coefficients on all levels.

The image compression takes place by keeping only a specific percentage P of the transform coefficients and setting the others to zero. For this we consider two different thresholds. In the *hard threshold,* we set the $P\%$ of the values with the smallest absolute value equal to zero and leave the others unchanged. In the *soft threshold,* we calculate the P-quantile of all absolute values after decomposition and set the values with the smallest absolute value to zero. The other values are then changed by the value of the P-quantile in the direction of zero (i.e. the P-quantile is either added or subtracted).

In this way we obtain a sparse matrix for which compression algorithms, e.g. entropy encoding, like Huffman coding or arithmetic coding, exist. (For further information about compression algorithms see e.g. [**Buc02**].) We illustrate the compression and reconstruction algorithms in a diagram schematically, see Figure 12.1.3.



(A) Compression

(B) Reconstruction

FIGURE 12.1.3. The steps of the compression and reconstruction of images.

Remark 12.2. Now we give some further remarks concerning the possible sizes of images, and a measure of similarity between images which gives us a way of quantifying the similarity between the original image and the reconstructed image.

(1) Many greyscale images have the size $(2^m \times 2^m)$, $m \in \mathbb{N}$. For these images the set of possible fractals is restricted in the following way. The extended IFSs for the fractals must have $N_1 = N_2 = 2^n$ and n must divide m. Then we can apply the matrix containing the coefficients of the filter functions to 4^{m-n} subsets of size $(2^n \times 2^n)$.
(2) We notice that the weights on the gaps do not have any influence on the possible filter functions and only the filter functions are used for the compression of images. The underlying space in which the wavelets lie is not considered explicitly.
(3) If we consider color images, we have three matrices of pixel values. One matrix has the values for the color red, one for green and one for blue. To each of these matrices we can apply the decomposition algorithm and the compression separately. But there are more efficient algorithms that consider the correlations between these three matrices.
(4) As a measure for the comparison between the compressed image and the original image B, we use the peak-signal-to-noise ratio (PSNR), see e.g. [**HTG08**]. This is defined as

$$\mathrm{PSNR} = 10 \cdot \log_{10}\left(\frac{(\max_B)^2}{\mathrm{MSE}}\right),$$

where $\max_B$ stands for the maximal possible pixel value of the image, i.e. for a greyscaled image it is 255, and MSE denotes the mean squared error between the original image B and the reconstructed image K defined as

$$\mathrm{MSE} = \frac{1}{N_1^m N_2^m} \sum_{i=0}^{N_1^m-1} \sum_{j=0}^{N_2^m-1} |B_{ij} - K_{ij}|^2 ,$$

where the images B and K are considered as matrices of size $(N_1^m \times N_2^m)$. The PSNR is given in decibel (dB) and the higher the PSNR the better the reconstruction. While studying the compressed images, we noticed that we did not see a great difference between the original image and the reconstructed image if the PSNR was bigger than 40dB. So if the PSNR is bigger than 40 dB we do not show the reconstructed images here.

12.2. Results of the application to images

Now we apply the wavelet bases in the L^2-spaces based on enlarged fractals to image compression. We start by considering which underlying fractals we can apply to image compression. We then apply different wavelets to the "Lena" image, see Figure 12.1.2(A), which is of the size (512×512) pixels. As a reference wavelet bases we consider the two-dimensional Haar wavelet (see Example A.14) and compare our results with the results obtained by this. We compare our results by using the PSNR as an index. The other wavelet bases we consider are based on the Sierpinski Gasket (as considered in [**DMP08**]) and the Cantor Dust.

We start by comparing the reconstruction results for fractals with different Hausdorff dimensions and analyze whether there is a correlation between the two.

In the example of the Sierpinski Gasket we compare the compression with the one of the Haar wavelet applied to the "Lena" image. In the example of the Cantor Dust we consider how different weights on the fractal influence the reconstruction results.

As our last example we apply the wavelet basis defined on Gosper Island to parts of the "Lena" image, see Example 11.23.

Results for underlying fractals. We cannot consider every fractal satisfying the conditions in Section 11.3 for the application to images. In the application to image compression only the coefficients of the filter functions are used and not any information about the underlying L^2-space and the operators on it. By the arrangement of the scaled coefficients the underlying fractal is considered to be in a rectangle and to consist of equally sized copies of it. Consequently, the only underlying fractals are those which lie in a rectangle and where the IFS consists of affine transformations.

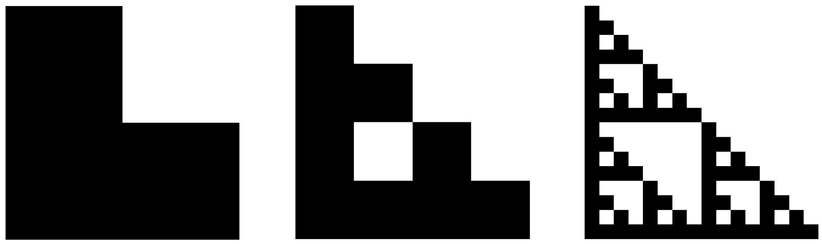
We can obtain a similar fractal to the Sierpinski Gasket in this way, as shown in Figure 12.2.1. For images there is no difference by using only an approximation of the fractal instead of the fractal itself because images are always only approximations since there is a limit for the different scales given by the size of the image, i.e. its number of pixels.

Results for different Hausdorff dimensions. One natural idea is that there is a correspondence between the Hausdorff dimension of the underlying fractal and its reconstruction quality. To study this connection we consider different fractals in the unit square with different Hausdorff dimensions, namely the ones of which the prefractals of order 1 are shown in Figure 12.2.2. On these fractals we consider the measure of maximal entropy, so every part of the fractal carries the same weight.

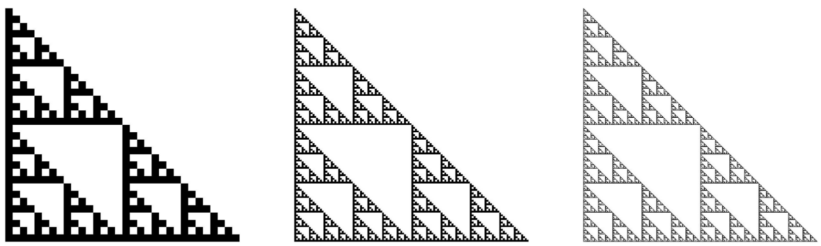
For each fractal we have a different low-pass filter for the construction of the wavelet basis. For each of these low-pass filters a vector of length 16 is given that contains the coefficients of the low-pass filter. We consider then the coefficients for the corresponding high-pass filters as stated in Appendix E.1. We apply these filter functions to a part of the "Lena" image of size (243×243) pixels. For the image compression we use 4 levels of decomposition of the image and set 80% of the pixels after decomposition to zero. We use the hard threshold. Then we obtain the reconstructed images shown in Figure 12.2.3.

These results indicate that there is a strong correlation between the Hausdorff dimension of the underlying fractal and the reconstruction performance of the wavelet. More precisely, this correlation holds if the image to which the compression is applied has smooth surfaces.

If we consider fractals with the same Hausdorff dimension than we notice that there is only a slight difference in the performance. We consider fractals with the Hausdorff dimension $\log(8)/\log(4)$ where parts of the fractal lie in different subsets of the unit square. We then obtain the reconstruction results

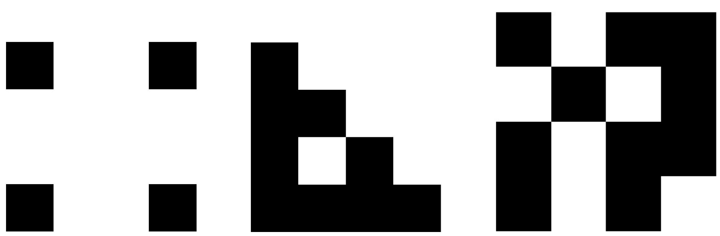


(A) Prefractal of order 1. (B) Prefractal of order 2. (C) Prefractal of order 4.

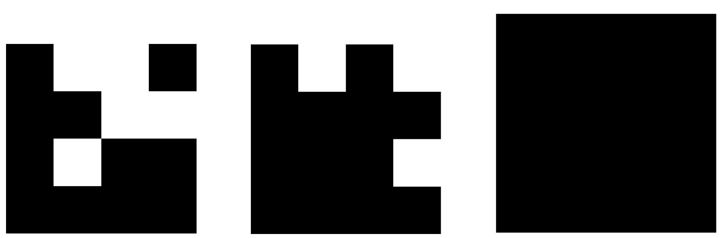


(D) Prefractal of order 5. (E) Prefractal of order 7. (F) Prefractal of order 9.

FIGURE 12.2.1. Approximation of the Sierpinski Gasket.



(A) Hausdorff dimension 1. (B) Hausdorff dimension $\log(3)/\log(2)$. (C) Hausdorff dimension $\log(10)/\log(4)$.

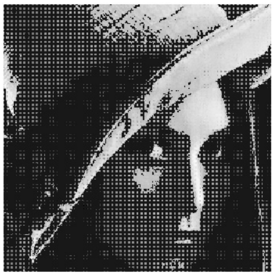


(D) Hausdorff dimension $\log(11)/\log(4)$. (E) Hausdorff dimension $\log(13)/\log(4)$. (F) Hausdorff dimension 2.

FIGURE 12.2.2. Prefractals of order 1 for different Hausdorff dimensions.

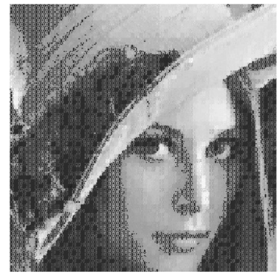
shown in Figure 12.2.4.

Results for the Sierpinski Gasket. Now we turn to the results that we obtain for the image compression with wavelets where the underlying fractal is the Sierpinski Gasket as viewed in Figure 12.2.1. We compare these results with those we obtain by using the Haar wavelet. The coefficients for



(A) Hausdorff dimension 1, PSNR: 13.47 dB.

(B) Hausdorff dimension $\log(3)/\log(2)$, PSNR: 18.35 dB.



(C) Hausdorff dimension $\log(10)/\log(4)$, PSNR: 20.02 dB.

(D) Hausdorff dimension $\log(11)/\log(4)$, PSNR: 22.23 dB.

(E) Hausdorff dimension $\log(13)/\log(4)$, PSNR : 29.09 dB.

(F) Hausdorff dimension 2, PSNR: 35.94 dB.

FIGURE 12.2.3. Reconstructed images for wavelets based on fractals with different Hausdorff dimensions.

the construction of the wavelet basis based on the Sierpinski Gasket are taken from [**DMP08**]. These coefficients are:

$$\begin{array}{rcccc}
v_0^{SG}: & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & 0 \\
v_1^{SG}: & 0 & 0 & 0 & 1 \\
v_2^{SG}: & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & 0 & 0 \\
v_3^{SG}: & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{2}{\sqrt{6}} & 0.
\end{array}$$

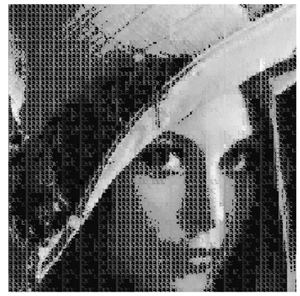
For the Haar wavelet we have the following coefficients for the filter functions:

$$\begin{array}{rcccc}
v_0^{Haar}: & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
v_1^{Haar}: & \frac{1}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} \\
v_2^{Haar}: & \frac{1}{2} & \frac{-1}{2} & \frac{1}{2} & \frac{-1}{2} \\
v_3^{Haar}: & \frac{1}{2} & \frac{-1}{2} & \frac{-1}{2} & \frac{1}{2}.
\end{array}$$

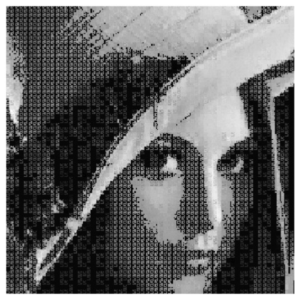
Here we have $N_1 = N_2 = 2$ and so for the "Lena" image of size (512×512) pixels we can consider at most 9 levels of decomposition. We notice that there is not a big difference in the reconstruction performance for different levels of decomposition (see Appendix E.2), so we only consider the maximal number of levels of decomposition, namely 9.

We keep 30%, 20%, 10% and 1% of the pixels, so we set 70% (80%, 90%, 99% respectively) of the pixel values after decomposition to zero. As the threshold option we will consider the hard threshold, since its results are often better than those of the soft threshold. For further information see Appendix E.3. For the image compression with the Haar wavelet we do not see any difference to the original image if we keep 30% or 20% of the coefficients since we then have an PSNR of 44.58 dB or 41.57 dB. So we omit the images here.

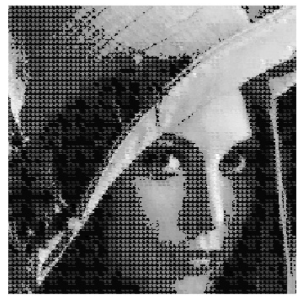
(A) PSNR: 16.92 dB.



(B) PSNR: 16.97 dB.



(C) PSNR: 16.99 dB.



(D) PSNR: 17 dB.

FIGURE 12.2.4. Reconstructed images for wavelets based on different fractals of Hausdorff dimension $\log(8)/\log(4)$.

Now we compare the compression for the Sierpinski Gasket wavelet and the Haar wavelet. The reconstructed images are viewed in Figure 12.2.5.

From the results in Figure 12.2.5 we notice, as expected, that the results are better if we use more coefficients. Furthermore we can clearly see how the underlying structures are induced to the images. Since the image consists of different smooth areas the results for the Haar wavelet are better than the ones for the Sierpinski Gasket wavelet.

Results for Cantor Dust. In the next step we apply the image compression with wavelets based on the Cantor Dust to a part of size (243×243) of the "Lena" image because for the Cantor Dust we divide the unit square in (3×3) equally sized sub-squares and so the image must have a size of $(3^n \times 3^n)$, $n \in \mathbb{N}$. We consider different probability measures on the Cantor Dust given via Hutchinson's theorem, see Theorem A.8. Consequently, we consider different low-pass filters. The coefficients of the low-pass filters are given in Figure 12.2.6.

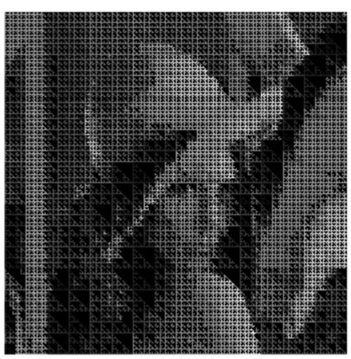
We consider these different coefficients to study how different weights on the subsets appear in the reconstructed images. The coefficients for the corresponding high-pass filters can be found in Appendix E.4. For this study we consider 4 levels of decomposition and set 70% of the pixel values after decomposition to zero. The reconstructed images can be seen in Figure 12.2.7.

(A) Sierpinski Gasket, 30%, PSNR: 30.57 dB.

(B) Sierpinski Gasket, 20%, PSNR: 22.04 dB.

(C) Sierpinski Gasket, 10%, PSNR: 12.32 dB.

(D) Haar wavelet, 10%, PSNR: 37.48 dB.



(E) Sierpinski Gasket, 1%, PSNR: 12.32 dB.

(F) Haar wavelet, 1%, PSNR: 28.42 dB.

FIGURE 12.2.5. Reconstructed images for the Haar wavelet and the Sierpinski Gasket wavelet.

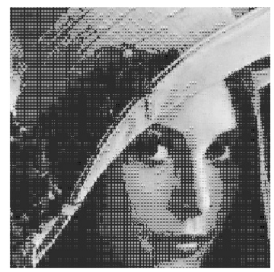
(A) CD1, PSNR: 25.77 dB.

(B) CD2, PSNR: 16.27 dB.

(C) CD3, PSNR: 15.73 dB.

(D) CD4, PSNR: 16.3 dB.

(E) CD5, PSNR: 16.33 dB.



(F) CD6, PSNR: 16.32 dB.

FIGURE 12.2.7. Reconstructed images for different wavelet bases based on the Cantor Dust.

α_3	0	α_4
0	0	0
α_1	0	α_2

CD1: $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = 1/2$

CD2: $\alpha_2 = \alpha_3 = \frac{1}{4}$, $\alpha_1 = \alpha_4 = \frac{\sqrt{7}}{4}$

CD3: $\alpha_2 = \alpha_3 = \frac{1}{100}$, $\alpha_1 = \alpha_4 = \frac{707}{1000}$

CD4: $\alpha_2 = \alpha_3 = \frac{\sqrt{7}}{4}$, $\alpha_1 = \alpha_4 = \frac{1}{4}$

CD5: $\alpha_1 = \alpha_2 = \frac{1}{4}$, $\alpha_3 = \alpha_4 = \frac{\sqrt{7}}{4}$

CD6: $\alpha_1 = \alpha_2 = \frac{\sqrt{7}}{4}$, $\alpha_3 = \alpha_4 = \frac{1}{4}$

FIGURE 12.2.6. The coefficients for different low-pass filters based on the Cantor Dust.

We observe again how the structure of the Cantor Dust as the underlying fractal is induced to the reconstructed image. Here we can also see differences in the reconstructed images for different wavelet bases based on the Cantor Dust. By comparing the results for the Cantor Dust 2 (CD2) and the Cantor Dust 4 (CD4), given in Figure 12.2.6, we notice how for Cantor Dust 2 a tendency from the lower left corner to the upper right one is induced on the image and for the Cantor Dust 4 form the upper left to the lower right corner, where there are the subsets with more weight. By comparing the results for Cantor Dust 5 and Cantor Dust 6 we notice the analogous structure for the upper and lower squares in one artifact.

Furthermore the reconstructed image for Cantor Dust 3 (CD3) shows more darker values than the other images. It concentrates the values more on parts of the image. On the other hand for Cantor Dust 1 we see that the values are more evenly distributed which coincides with the chosen filter functions.

Remark 12.3. In [**D'A08**] D'Andrea shows that the compression under the use of the Sierpinski Gasket wavelet is superior to the compression with the Haar wavelet when applied to an image of the Sierpinski Gasket itself. To obtain a perfect reconstruction with the Sierpinski Gasket wavelet we only need a few coefficients. We do notice the same result when the wavelet basis is applied to the Sierpinski Gasket. This reconstruction performance does not hold for all fractals. Only for fractals where there are not too big smooth surfaces in the image is the compression with a wavelet basis based on a fractal advantageous to the Haar wavelet basis.

Remark 12.4 (Result for Gosper Island)**.** Now we apply the wavelet basis on the fractal based on Gosper Island, compare Example 11.23, to an image. In this case we have $N_1 = 1$ and $N_2 = 7$, consequently the images on which we can apply these wavelets must be of the size $(7^n \times 1)$, $n \in \mathbb{N}$, pixels. So, we can only consider a part of the "Lena" image of size $\left(7^3 \times 1\right)$ pixels. If we want to consider an image of size $(7^n \times m)$, $m, n \in \mathbb{N}$, we can apply the wavelet basis to the subsets of size $(7^n \times 1)$ and for the reconstruction we merge these subsets together in the reconstruction. In this way the reconstructed image of size $\left(7^3 \times 512\right)$ with 30% of the coefficients and two levels of decompositions takes the form in Figure 12.2.8. The PSNR for this image is 16.84 dB. We clearly see that the other wavelet bases perform better. We observe that the artifacts are for each subset on similar pixels so that there are lines in the reconstructed image.

FIGURE 12.2.8. Reconstructed image for the wavelet basis based on Gosper Island.

APPENDIX A

Mathematical Introduction

Here we present some background material to the main mathematical areas used in the thesis, namely fractal geometry and wavelet analysis. In wavelet analysis we focus on multiresolution analysis, since this area is the dominating aspect in the second and third part. We also give some definitions and results concerning C^*-algebras. Since all of this material in this section is well known, we do not include proofs.

A.1. ... to Fractal geometry

Fractal geometry was introduced by Mandelbrot in 1975. It is the analysis of complex structures, like irregular and fragmented patterns, that occur in nature. These complex structures are assumed to be at all scales, so that the consideration of the set on a smaller scale does not simplify the problem. These sets cannot be studied by the techniques of classical geometry because they are too irregular. So different techniques are used for these sets. These sets appear for example as the shapes of clouds and coastlines. Their name "fractal" comes from the Latin word "fractus", meaning broken or scattered.

There were various attempts to give a precise definition of a fractal, but these were unsatisfactory. Consequently, we can only give some characteristics which fractals usually have. One of these properties is the irregularity at all scales. Another one is that they have non-integer Hausdorff dimension, which is defined in terms of the Hausdorff measure. The results stated here can be found in [**Fal97**].

The Hausdorff measure is defined by using δ-covers. If $F \subset \mathbb{R}^d$ a δ-cover of F is a countable collection $\{U_i : i \in \mathbb{N}\}$ of subsets of $\mathbb{R}^d$ such that $F \subset \bigcup_{i\in\mathbb{N}} U_i$ and $0 < |U_i| < \delta$, where $|U| = \sup\{d(x,y) : \ x, y \in U\}$ is the diameter of U, where d stands for the metric.

Definition A.1. For $s \geq 0$ and F a subset of $\mathbb{R}^d$ define

$$H^s_\delta(F) := \inf \left\{ \sum_{i\in\mathbb{N}} |U_i|^s : \bigcup_{i\in\mathbb{N}} U_i \supset F, \, |U_i| < \delta \right\}$$

as the δ-approximation of the Hausdorff measure. The *s-dimensional Hausdorff measure* for F is then defined as

$$H^s(F) := \lim_{\delta\to 0} H^s_\delta(F).$$

Remark A.2.

(1) This limit exists (may be infinite), because for $\delta' < \delta$ a δ'-cover of F is a δ-cover of F and so $H^s_{\delta'}(F) \leq H^s_\delta(F)$.
(2) H^s is a regular Borel measure for every $s \geq 0$.

Proposition A.3. *If $F \subset \mathbb{R}^d$ and $c > 0$, then $H^s(cF) = c^s H^s(F)$, where $cF = \{cx : \ x \in F\}$, and H^s is translation invariant, i.e. $H^s(F + t) = H^s(F)$, $t \in \mathbb{R}^d$.*

The Hausdorff dimension of $F \subset \mathbb{R}^d$ is defined in terms of the Hausdorff measure.

Definition A.4. The *Hausdorff dimension* $\dim_H(F)$ of a set $F \subset \mathbb{R}^d$ is defined as

$$\dim_H(F) = \inf\{s : \ H^s(F) = 0\} = \sup\{s : \ H^s(F) = \infty\}.$$

Some fractals can be obtained as the invariant set of an iterated function system and for these fractals the Hausdorff dimension can be easily computed under some conditions. The considered fractals in this thesis are all given by iterated function systems. Iterated function systems are given by using contractions.

Definition A.5. A function $\tau\colon D \to D$, $D \subset \mathbb{R}^d$, is called a *contraction* if there is $0 < c < 1$ such that for $x, y \in D$ the following holds

$$\|\tau(x) - \tau(y)\| \leq c\,\|x - y\|.$$

A function τ is called a similarity if equality holds in the above formula.

A useful condition on IFS is the open set condition.

Definition A.6. An IFS $(\tau_i)_{i\in\underline{N}}$ is said to satisfy the *open set condition* (OSC) if there exists an open set V such that $\bigcup_{i\in\underline{N}} \tau_i(V) \subset V$ and $\tau_i(V) \cap \tau_j(V) = \emptyset$, $i \neq j$, $i, j \in \underline{N}$.

Definition A.7. A family of contractions $\tau_0, \ldots, \tau_{N-1} : D \to D$, where $D \subset \mathbb{R}^d$ is closed, is called an *iterated function system* (IFS). A compact set C is called the invariant set (limit set, fractal) of an IFS if

$$C = \bigcup_{i\in\underline{N}} \tau_i(C).$$

The following fundamental result is due to Hutchinson.

Theorem A.8 (**[Hut81]**)**.** *Let $(\tau_i)_{i\in\underline{N}}$ be a system of N contractive maps on a complete metric space $\mathcal{X}$. Then there is a unique compact subset $C \subset \mathcal{X}$ such that*

$$C = \bigcup_{i\in\underline{N}} \tau_i(C).$$

This set C is called the invariant set for the system of contractive maps.

This invariant set also has a probability measure uniquely determined by the IFS and an invariance condition.

Theorem A.9 (**[Hut81]**)**.** *Let $(\tau_i)_{i\in\underline{N}}$ be a contractive iterated function system on a complete metric space $\mathcal{X}$. Let $p_0, \ldots, p_{N-1} \in (0,1)$ be a list of probabilities such that $\sum_{i\in\underline{N}} p_i = 1$. Then there is a unique probability measure μ_p on $\mathcal{X}$ such that*

$$\mu_p(E) = \sum_{i\in\underline{N}} p_i \cdot \mu_p(\tau_i^{-1}(E)), \text{ for all Borel subsets } E.$$

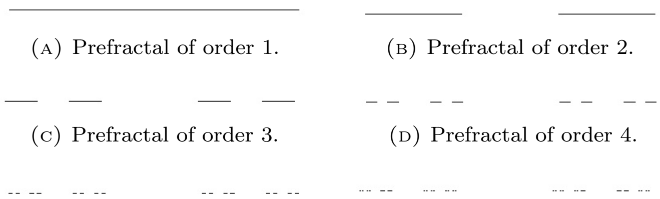
Moreover, the measure μ_p is supported on the invariant set of the iterated function system $(\tau_i)_{i\in\underline{N}}$.

Example A.10. One standard example is the middle-third Cantor set. This set is given by the aIFS $\left(\tau_0(x) = \frac{x}{3}, \tau_1(x) = \frac{x+2}{3}\right)$ and it has the Hausdorff dimension $\frac{\log 2}{\log 3}$.

By the construction of the fractal we call the single steps the prefractals of a specific order. So for the middle-third Cantor set we call the line segment from 0 to 1 the prefractal of order 1 and $[0, 1/3] \cup [2/3, 1]$ the prefractal of order 2. The prefractals up to order 6 are visualized in Figure A.1.1.

A.2. ... to classical wavelet analysis

Wavelet analysis is used in signal analysis and in some fields of physics. Via the wavelet transform a function can be reconstructed from a countable number of points. The wavelet transform is usually better than the classical Fourier transform at representing discontinuous functions, functions with peaks and non-periodic functions. In this analysis a square-integrable function is represented as a wavelet series with respect to an orthonormal basis or a frame.



(A) Prefractal of order 1. (B) Prefractal of order 2.

(C) Prefractal of order 3. (D) Prefractal of order 4.

(E) Prefractal of order 5. (F) Prefractal of order 6.

FIGURE A.1.1. Prefractals of the middle-third Cantor set.

So the aim of wavelet analysis is to find a countable basis for a Hilbert space. In the literature mainly the L^2-spaces with respect to the Lebesgue measure are considered. Often the countable bases are obtain via a multiresolution analysis. The classical multiresolution analysis is defined as follows. This definition differs from Definition 1.2 in that here we only consider unitary operators U and T acting on the $L^2(\mathbb{R}, \lambda)$, where λ is the Lebesgue measure on $\mathbb{R}$. For further information see [**Dau92**].

Definition A.11. A *multiresolution analysis (MRA)* consists of a family $\{V_j : j \in \mathbb{Z}\}$ of closed subspaces of $L^2(\mathbb{R}, \lambda)$, two unitary operators U and T on $L^2(\mathbb{R}, \lambda)$ satisfying $U^{-1}TU = T^N$ for some $N \in \mathbb{N}$, and a function $\varphi \in L^2(\mathbb{R}, \lambda)$ such that the following conditions are satisfied:
(1) $\cdots \subset V_{-2} \subset V_{-1} \subset V_0 \subset V_1 \subset V_2 \subset \cdots$,
(2) $\operatorname{cl} \bigcup_{j \in \mathbb{Z}} V_j = L^2(\mathbb{R}, \lambda)$,
(3) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$,
(4) $f \in V_j \Leftrightarrow Uf \in V_{j+1}$, $j \in \mathbb{Z}$,
(5) $\{T^n \varphi : n \in \mathbb{Z}\}$ is an orthonormal basis in V_0.

Remark A.12.
(1) The operators T and U are called the translation and scaling operator, respectively. T is usually defined as $Tf(\cdot) = f(\cdot - 1)$ acting on $L^2(\mathbb{R}, \lambda)$. If we consider $L^2(\mathbb{R}^2, \lambda)$ with λ being the Lebesgue measure on $\mathbb{R}^2$, the translation operator is usually defined as $T^{(k,l)} f(\cdot) = f\left(\cdot - \begin{pmatrix} k \\ l \end{pmatrix}\right)$, $(k, l) \in \mathbb{Z}^2$.
(2) In the standard case in $\mathbb{R}$ with respect to the Lebesgue measure, we have $Uf(\cdot) = \sqrt{N} f(N \cdot)$, $N \in \mathbb{N}$.
(3) The function φ is called the *father wavelet* and it satisfies a scaling relation of the form $U^{-1}\varphi = \sum_{k \in \mathbb{Z}} a_k T^k \varphi$ for some $a_k \in \mathbb{C}$. Define $m_0(z) = \sum_{k \in \mathbb{Z}} a_k z^k$, $z \in \mathbb{T}$, to be the low-pass filter to φ.

The mother wavelets, that give the basis, are obtained from the father wavelet φ in the following way. We have to obtain so-called high-pass filters m_j, $j \in \underline{N} \backslash \{0\}$, of the form $m_j : z \mapsto \sum_{k=0}^{N-1} a_k^j z^k$, $a_k^j \in \mathbb{C}$, from the low-pass filter m_0. The filter functions have to satisfy the condition that the matrix

$$M(z) := \frac{1}{\sqrt{N}} \left(m_j(\rho^l z) \right)_{j,l=0}^{N-1},$$

where $\rho = e^{2\pi i/N}$, is unitary for almost all $z \in \mathbb{T}$.

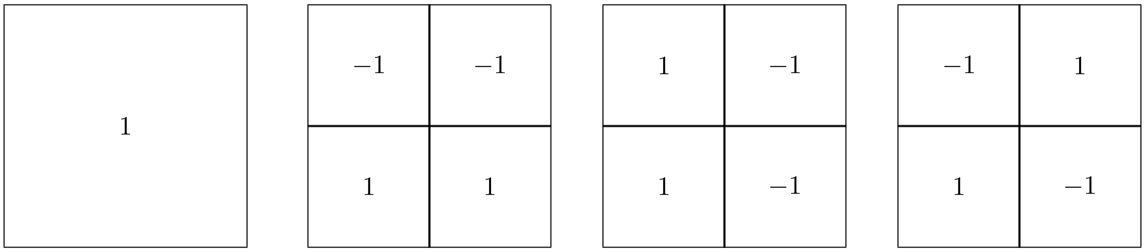
Then the mother wavelets are defined for $j \in \underline{N} \backslash \{0\}$ by

$$\psi_j := U m_j(T) \varphi.$$

For the mother wavelets we have that $\left\{T^k \psi_j : k \in \mathbb{Z}, j \in \underline{N} \backslash \{0\}\right\}$ is a basis for $W_0 := V_1 \ominus V_0$ and

$$\left\{U^n T^k \psi_i : n, k \in \mathbb{Z}, i \in \underline{N} \backslash \{0\}\right\}$$

is an ONB of $L^2(\mathbb{R}, \lambda)$.



(A) Father wavelet φ. (B) Mother wavelet ψ_1. (C) Mother wavelet ψ_2. (D) Mother wavelet ψ_3.

FIGURE A.2.1. Haar wavelet in two dimensions.

In the case of the Lebesgue measure, we have the advantage that we can apply the Fourier transform to the functions φ, ψ_j. Proofs that exploit the Fourier transform are often simpler than direct proofs.

Example A.13. The first constructed wavelet is the Haar wavelet. This gives a basis for $L^2(\mathbb{R}, \lambda)$ and it considers a dyadic MRA, i.e. $Uf(x) = \sqrt{2} f(2x)$, $x \in \mathbb{R}$. The father wavelet is the characteristic function on the unit interval, $\varphi = \mathbb{1}_{[0,1)}$. The corresponding low-pass filter is $m_0(z) = \frac{1}{\sqrt{2}}(1+z)$. Thus, we can take $m_1(z) = \frac{1}{\sqrt{2}}(1-z)$ as the high-pass filter and so the mother wavelet is $\psi = \sqrt{2}\left(\mathbb{1}_{[0,1/2)} - \mathbb{1}_{[1/2,1)}\right)$. The wavelet basis is then given by

$$\left\{x \mapsto 2^{j/2}\psi\left(2^j x - k\right) : j, k \in \mathbb{Z}\right\}.$$

Example A.14 (2 dimensional Haar wavelet)**.** The Haar wavelet in two dimensions is in analogy to the wavelet in one dimension and it gives a basis for $L^2(\mathbb{R}^2, \lambda)$. The scaling operator is $Uf(\vec{x}) = 2f(\mathbf{A}\vec{x})$, $\vec{x} \in \mathbb{R}^2$, with $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, and the translation operator is $T^{(k,l)}f(\vec{x}) = f\left(\vec{x} - \begin{pmatrix} k \\ l \end{pmatrix}\right)$, $(k,l) \in \mathbb{Z}^2$, $\vec{x} \in \mathbb{R}^2$. As the father wavelet φ we consider the characteristic function on the unit square which satisfies

$$U^{-1}\varphi = 2\left(\varphi + T^{(1,0)}\varphi + T^{(0,1)}\varphi + T^{(1,1)}\varphi\right).$$

The mother wavelets are chosen such that they take the values as shown in Figure A.2.1 on the unit square.

A.3. ... to other mathematical fields

In this section we give some further mathematical results from the literature that we use in this thesis.

Definition ([**BR87**])**.** Let $\mathcal{V}$ be a vector space over $\mathbb{C}$. The space $\mathcal{V}$ is called an *algebra* if it is equipped with a multiplication law which associates a product AB to each pair $A, B \in \mathcal{V}$. The product is assumed to be associative and distributive. Explicitly, one assumes for $A, B, C \in \mathcal{V}$ and $\alpha, \beta \in \mathbb{C}$ that

(1) $A(BC) = (AB)C$,
(2) $A(B + C) = AB + AC$,
(3) $\alpha\beta(AB) = (\alpha A)(\beta B)$.

A mapping $A \in \mathcal{V} \to A^* \in \mathcal{V}$ is called an *involution*, or adjoint operation, of the algebra $\mathcal{V}$ if it has the following properties:

(1) $(A^*)^* = A$,
(2) $(AB)^* = B^*A^*$,
(3) $(\alpha A + \beta B)^* = \overline{\alpha}A^* + \overline{\beta}B^*$.

An algebra with an involution is called a $*$-*algebra*.

The next result is from functional analysis. Let $\mathcal{C}(\mathcal{X}, \mathbb{C})$ be the set of continuous functions from $\mathcal{X}$ to $\mathbb{C}$.

Theorem (Stone-Weierstrass Theorem for complex functions)**.** *Let $\mathcal{X}$ be a locally compact Hausdorff space and S be a subset of $\mathcal{C}(\mathcal{X}, \mathbb{C})$ which separates points. Then the complex unital $*$-algebra generated by S is uniformly dense in $\mathcal{C}(\mathcal{X}, \mathbb{C})$.*

Definition. A Banach algebra is an algebra $\mathcal{V}$, equipped with a norm $\|\cdot\|$ making it into a Banach space, having the additional property that such that $\|AB\| \leq \|A\| \cdot \|B\|$ holds for all $A, B \in \mathcal{V}$.

Furthermore, it has a unit if there exists $I \in \mathcal{V}$ with $IA = AI = A$ for all $A \in \mathcal{V}$ and $\|I\| = 1$.

We also use specific C^*-algebras.

Definition. A C^*-algebra is a $*$-algebra with the extra conditions that $\mathcal{V}$ is a Banach algebra and for all A in $\mathcal{V}$ the condition $\|A^*A\| = \|A\|^2$ holds.

Now we turn to the specific C^*-algebras.

Definition (Cuntz algebra)**.** Let H be a separable infinite dimensional Hilbert space. For $n \in \mathbb{N}$ let $(S_i)_{i=0}^n$ be a family of isometries on H that satisfy

$$S_i^* S_j = \delta_{i,j} I,$$
$$\sum_{i=0}^{n} S_i S_i^* = I,$$

where I stands for the identity operator on H. Let $\mathcal{O}_{n+1}$ be the C^*-algebra generated by $(S_i)_{i=0}^n$. This C^*-algebra $\mathcal{O}_{n+1}$ is called the Cuntz algebra.

It is well known that the algebraic structure of the Cuntz algebra depends only on the algebraic relations above and not on the particular choice of isometries used to satisfy them. We will also need the Cuntz-Krieger algebras. These generalizations of Cuntz algebras are generated by *partial* isometries. Recall that an operator S is a partial isometry if and only if $S = SS^*S$.

Definition (Cuntz-Krieger algebra)**.** Let $(S_i)_{i=0}^n$ be a family of n non-zero partial isometries on a separable complex Hilbert space H satisfying the relations

$$S_i^* S_i = \sum_{j=0}^{n} A_{ij} S_j S_j^*$$
$$I = \sum_{i=0}^{n} S_i S_i^*,$$

where A is an $(n+1) \times (n+1)$, 0—1 matrix with no zero row or column and I is the identity. The C^*-algebra $\mathcal{O}_A$ generated by the family $(S_i)_{i=0}^n$ is called the Cuntz-Krieger algebra associated to A.

It is well-known that the algebraic structure of the algebra $\mathcal{O}_A$ depends only on the relations encoded in the matrix A in the definition above, and not on the particular choice of partial isometries $(S_i)_{i=0}^n$ used to satisfy them.

We end this section with some results of measure theory.

Proposition (Fatou's lemma)**.** *Let $0 \leq f_n : (\Omega, \mathcal{A}) \to \left(\overline{\mathbb{R}}, \overline{\mathcal{B}}\right)$, measurable, where $(\Omega, \mathcal{A})$ is a measurable space and $\overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$. Then*

$$\int \liminf_{n\to\infty} f_n d\mu \leq \liminf_{n\to\infty} \int f_n d\mu.$$

Theorem (Dominated convergence theorem)**.** *Let $g, f_n : (\Omega, \mathcal{A}) \to \left(\overline{\mathbb{R}}, \overline{\mathcal{B}}\right)$ with $g \in L^1(\mu)$, μ the measure on $(\Omega, \mathcal{A})$, $g \geq 0$ and $|f_n| \leq g$ a.e. and f_n a.e. convergent. Then there is a measurable real valued function $f = \lim_{n\to\infty} f_n$ a.e. For all such functions f, it holds that $f \in L^1(\mu)$ and*

$$\lim_{n\to\infty} \int |f - f_n| d\mu = 0, \text{ in particular } \int f d\mu = \lim_{n\to\infty} \int f_n d\mu.$$

APPENDIX B

A connection of the MRA to the Cuntz algebra

This section considers a different interpretation of the operators U in the one-dimensional and two-dimensional MRA on enlarged fractals of Chapter 10 and Chapter 11.

B.1. A representation of the one dimensional U in terms of the Cuntz algebra

In this section, we write the scaling operator from a one-dimensional MRA on enlarged fractals in terms of a tensor product of two representations of the Cuntz algebra $\mathcal{O}_N$. One of the representations is given in terms of the filter functions m_i, $i \in \underline{N}$, of the MRA and the second representation acts on $l^2(\mathbb{N}_0)$. It is given by $Z_i : |n\rangle \mapsto |Nn+i\rangle$, $i \in \underline{N}$, $n \in \mathbb{N}_0$, where we use Dirac's terminology for the natural basis $|n\rangle$ in $l^2(\mathbb{N}_0)$.

This approach was first considered for the special case of an aIFS with the same scaling, i.e. for a Cantor set and the Hausdorff measure, in **[Jor06]**. Here we will prove the analogous connection for the scaling operator U of Chapter 10 along the lines of **[Jor06]**.

To obtain this presentation we start by rewriting the wavelet basis of the Section 10.2, compare Proposition 10.26, as the sequence of functions given in Proposition B.1. Here we consider the basis as it can be obtained from a one-sided MRA.

Proposition B.1. *Define a sequence of functions for $i \in \underline{N}$, $n \in \mathbb{N}_0$, by*

$$\begin{aligned} \varphi_0 &= \mathbb{1}_C, \\ \varphi_{Nn+i} &= Um_i(T)\varphi_n. \end{aligned} \tag{B.1.1}$$

Then $\left\{T^k\varphi_n : n \in \mathbb{N}_0, k \in \mathbb{Z}\right\}$ is an ONB for $L^2(\nu_{\mathbb{Z}})$.

Remark B.2. In Chapter 10 the construction of the wavelet basis was done via a two-sided MRA. Thus, we have for $L^2(\nu_{\mathbb{Z}})$ the basis

$$\left\{U^n T^k \psi_i : n, k \in \mathbb{Z}, i \in \underline{N}\backslash\{0\}\right\},$$

where the functions ψ_i, $i \in \underline{N}\backslash\{0\}$ are defined in Corollary 10.26. If we use a one-sided MRA the corresponding basis is

$$\left\{U^n T^k \psi_i : n \in \mathbb{N}_0, k \in \mathbb{Z}, i \in \underline{N}\backslash\{0\}\right\} \cup \left\{T^k\varphi : k \in \mathbb{Z}\right\}.$$

Here we only take the closed spaces $V_0 \subset V_1 \subset \dots$ of $L^2(\nu_{\mathbb{Z}})$ and so we also need the father wavelet φ for the basis.

Proof of Proposition B.1. By the definition of the functions in (B.1.1) we have $\varphi = \varphi_0$. This definition is consistent with the iteration procedure since φ satisfies the scaling identity $\varphi = Um_0(T)\varphi$. Moreover, we have $\varphi_j = \psi_j$ for $j \in \underline{N}\backslash\{0\}$.

To show that $\left\{T^k\varphi_n : n \in \mathbb{N}_0, k \in \mathbb{Z}\right\}$ is an ONB, we first rewrite the formula for φ_n in terms of φ_0 and the filter functions m_i, $i \in \underline{N}$. For $n \in \mathbb{N}$ with $n = \sum_{i=0}^r n_i N^i$, $n_i \in \underline{N}$, $r \in \mathbb{N}$, we have

$$\begin{aligned} \varphi_n &= Um_{n_0}(T)\,\varphi_{\sum_{i=1}^r n_i N^{i-1}} \\ &= Um_{n_0}(T)\,Um_{n_1}(T)\,\varphi_{\sum_{i=2}^r n_i N^{i-2}} \\ &= U^{r+1}\prod_{i=0}^r m_{n_i}\left(T^{N^{r-i}}\right)\varphi_0 \end{aligned}$$

or equivalently

$$\varphi_n = U^r \prod_{i=0}^{r-1} m_{n_i}\left(T^{N^{r-i-1}}\right)\psi_{n_r}. \tag{B.1.2}$$

Now we turn to the orthonormality of $\left\{T^k\varphi_n : k\in\mathbb{Z}, n\in\mathbb{N}_0\right\}$. Notice that $\operatorname{supp}(\varphi_n)\subset[0,1]$, $n\in\mathbb{N}_0$. It therefore suffices to show that the set $\{\varphi_n:\ n\in\mathbb{N}_0\}$ is orthonormal. Let $n_1,n_2\in\mathbb{N}_0$ with $n_1=\sum_{i=0}^{r_1} k_iN^i$ and $n_2=\sum_{i=0}^{r_2} l_iN^i$ for some $k_i,l_i\in\underline{N}$, $r_1,r_2\in\mathbb{N}_0$. Since $\left\{U^mT^k\psi_j : m,k\in\mathbb{Z}, j\in\underline{N}\right\}$ is an orthonormal basis of $L^2(\nu_\mathbb{Z})$, it follows that for $r_1\neq r_2$ or $k_{r_1}\neq l_{r_2}$ we have $\langle\varphi_{n_1}|\varphi_{n_2}\rangle=0$ by its form, see (B.1.2). Thus now we consider the case $r_1=r_2=:r$ and $k_r=l_r=:p$. It follows that

$$\begin{aligned}\langle\varphi_{n_1}|\varphi_{n_2}\rangle &= \left\langle \prod_{i=0}^{r-1} m_{k_i}\left(T^{N^{r-i-1}}\right)\psi_p\middle|\prod_{i=0}^{r-1} m_{l_i}\left(T^{N^{r-i-1}}\right)\psi_p\right\rangle = \sum_{(j_0,\dots,j_{r-1})\in\underline{N}^r}\prod_{i=0}^{r-1} a_{j_i}^{k_i}\cdot\overline{a}_{j_i}^{l_i}\\ &= \prod_{i=0}^{r-1}\underbrace{\sum_{j_i\in\underline{N}} a_{j_i}^{k_i}\cdot\overline{a}_{j_i}^{l_i}}_{=\delta_{k_i,l_i}} = \delta_{n_1,n_2}.\end{aligned}$$

Now it remains to show that $\left\{T^k\varphi_n : k\in\mathbb{Z}, n\in\mathbb{N}_0\right\}$ spans $L^2(\nu_\mathbb{Z})$. To this end we show that every function $U^nT^k\psi_j$, $n\in\mathbb{N}_0$, $k\in\mathbb{Z}$, $j\in\underline{N}\backslash\{0\}$, can be written as a linear combination of functions $T^l\varphi_m$, $l\in\mathbb{Z}$, $m\in\mathbb{N}_0$. It then follows that $\left\{T^k\varphi_n : k\in\mathbb{Z}, n\in\mathbb{N}_0\right\}$ spans $L^2(\nu_\mathbb{Z})$ since

$$\left\{U^nT^k\psi_i:\ n\in\mathbb{N}_0, k\in\mathbb{Z}, i\in\underline{N}\backslash\{0\}\right\}\cup\left\{T^k\varphi:\ k\in\mathbb{Z}\right\}$$

does. First notice that we only have to consider $U^nT^k\psi_j$ with $k\in\underline{N^n}$ since for $l\in\mathbb{Z}\backslash\underline{N^n}$ we have $U^nT^l\psi_j=T^mU^nT^k\psi_j$ for some $m\in\mathbb{Z}$, $k\in\underline{N^n}$, with $l=k+N^nm$. For $U^nT^k\psi_j$, $n\in\mathbb{N}_0$, $k\in\underline{N^n}$, $j\in\underline{N}$, we only consider φ_m with $m=\sum_{i=0}^{n-1}k_iN^i+jN^n$ for $k_i\in\underline{N}$. Write $k=\sum_{i=0}^{n-1}l_iN^i$, $l_i\in\underline{N}$, then

$$\begin{aligned}U^nT^k\psi_j &= \sum_{(j_0,\dots,j_{n-1})\in\underline{N}^n}\prod_{i=0}^{n-1}\underbrace{\sum_{q_{n-1-i}\in\underline{N}}\overline{a}_{l_i}^{q_{n-1-i}}a_{j_i}^{q_{n-1-i}}}_{=\delta_{l_i,j_i}}U^nT^k\psi_j\\ &= \sum_{(q_0,\dots,q_{n-1})\in\underline{N}^n}\left(\prod_{i=0}^{n-1}\overline{a}_{l_i}^{q_{n-1-i}}\right)\sum_{(j_0,\dots,j_{n-1})\in\underline{N}^n}\left(\prod_{i=0}^{n-1}a_{j_i}^{q_{n-1-i}}\right)U^nT^{\sum_{i=0}^{n-1}j_iN^i}\psi_j\\ &= \sum_{(q_0,\dots,q_{n-1})\in\underline{N}^n}\left(\prod_{i=0}^{n-1}\overline{a}_{l_i}^{q_{n-1-i}}\right)U^n\prod_{i=0}^{n-1}m_{q_i}\left(T^{N^{n-i}}\right)\psi_j\\ &= \sum_{(q_0,\dots,q_{n-1})\in\underline{N}^n}\left(\prod_{i=0}^{n-1}\overline{a}_{l_i}^{q_{n-1-i}}\right)\varphi_{\sum_{i=0}^{n-1}q_iN^i+N^nj}.\end{aligned}$$

□

Now we turn to a correspondence between $L^2(\nu_\mathbb{Z})$ and $l^2(\mathbb{N}_0)\otimes l^2(\mathbb{Z})$.

Proposition B.3. *There exists a unitary isomorphism between $L^2(\nu_\mathbb{Z})$ and $l^2(\mathbb{N}_0)\otimes l^2(\mathbb{Z})$ which is the extension of $W:\ \varphi_n(\cdot-k)\mapsto|n\rangle\otimes|k\rangle$, $n\in\mathbb{N}_0$, $k\in\mathbb{Z}$, and furthermore*

$$L^2(\nu_\mathbb{Z})\backsimeq l^2(\mathbb{N}_0)\otimes l^2(\mathbb{Z})\backsimeq l^2(\mathbb{N}_0)\otimes L^2(\mathbb{T},\lambda).$$

Proof. From Proposition B.1 we have that $T^k\varphi_n=\varphi_n(\cdot-k)$, $n\in\mathbb{N}_0$, $k\in\mathbb{Z}$, is an ONB of $L^2(\nu_\mathbb{Z})$ and from **[Jor06]** we know that $|n\rangle\otimes|k\rangle$, $n\in\mathbb{N}_0$, $k\in\mathbb{Z}$, is an ONB of $l^2(\mathbb{N}_0)\otimes l^2(\mathbb{Z})$. Thus, we have that W maps the basis elements of $L^2(\nu_\mathbb{Z})$ to the ones of $l^2(\mathbb{N}_0)\otimes l^2(\mathbb{Z})$. Hence W gives a unitary isomorphism between these spaces. The last step of the chain comes from the Fourier isomorphism $l^2(\mathbb{Z}) \backsimeq L^2(\mathbb{T},\lambda)$, in which $|k\rangle$ corresponds to $e_k : x \mapsto e^{i2\pi kx}$, $k \in \mathbb{Z}$. □

Before we come to the association of the scaling operator U with the tensor product of two representations of the Cuntz algebra $\mathcal{O}_N$, we express the action of U on $T^k\varphi_n$, $n \in \mathbb{N}_0$, $k \in \mathbb{Z}$, in terms of the functions $T^j\varphi_{Nn+i}$ $n \in \mathbb{N}_0$, $j \in \mathbb{Z}$, $i \in \underline{N}$, and suitable coefficients depending on S_i, where $(S_i f)(z) := m_i(z) f\left(z^N\right)$, $i \in \underline{N}$, $z \in \mathbb{T}$, is a representation of $\mathcal{O}_N$ on $L^2(\mathbb{T},\lambda)$.

Lemma B.4. *The following holds for $n \in \mathbb{N}_0$, $k \in \mathbb{Z}$*

$$UT^k\varphi_n = \sum_{i\in\underline{N}}\sum_{j\in\mathbb{Z}} \langle e_k|S_i e_j\rangle T^j \varphi_{Nn+i}.$$

Proof. First we notice that $UT^k\varphi_n$, $n \in \mathbb{N}_0$, $k \in \mathbb{Z}$, is not an element of the orthonormal basis $\left\{T^k\varphi_n : k \in \mathbb{Z}, n \in \mathbb{N}_0\right\}$ of $L^2(\nu_{\mathbb{Z}})$ (compare Proposition B.1). Now we turn to the specific representation of $UT^k\varphi_n$, $k \in \mathbb{Z}$, $n \in \mathbb{N}_0$. Notice that for $j \in \mathbb{Z}$, $n \in \mathbb{N}_0$ and $i \in \underline{N}$

$$\varphi_{Nn+i}(\cdot - j) = T^j U m_i(T)\varphi_n$$

and recall that the filter functions are given as $m_k : z \mapsto \sum_{j\in\underline{N}} a_j^k z^j$, $z \in \mathbb{T}$, $k \in \underline{N}$, for suitable $a_j^k \in \mathbb{C}$. Now we show that $\langle UT^k\varphi_n|T^j\varphi_{Nn+i}\rangle = \langle e_k|S_i e_j\rangle$ for $j,k \in \mathbb{Z}$, $n \in \mathbb{N}_0$ and $i \in \underline{N}$. We have that

$$\begin{aligned}\langle UT^k\varphi_n|T^j\varphi_{Nn+i}\rangle &= \langle UT^k\varphi_n|T^jUm_i(T)\varphi_n\rangle = \langle\varphi_n|T^{Nj-k}m_i(T)\varphi_n\rangle\\ &= \begin{cases}\overline{a}_l^i, & \text{if there exists an } l \in \underline{N} \text{ such that } Nj-k+l=0,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

On the other hand we have

$$\begin{aligned}\langle e_k|S_i e_j\rangle &= \int \overline{m_i(z)} z^{-Nj}\cdot z^k dz = \sum_{l\in\underline{N}} \overline{a}_l^i \int z^{-Nj-l+k} dz\\ &= \begin{cases}\overline{a}_l^i, & \text{if there exists an } l \in \underline{N} \text{ such that } Nj-k+l=0,\\ 0, & \text{otherwise.}\end{cases}\end{aligned}$$

Consequently, $\langle UT^k\varphi_n|T^j\varphi_{Nn+i}\rangle = \langle e_k|S_i e_j\rangle$.

Now it follows for $k \in \mathbb{Z}$ with $k = k_0 + Nk_1$, $k_0 \in \underline{N}$, $k_1 \in \mathbb{Z}$, that

$$\begin{aligned}\sum_{i\in\underline{N}}\sum_{j\in\mathbb{Z}}\langle e_k|S_ie_j\rangle T^j\varphi_{Nn+i} &= \sum_{i\in\underline{N}}\sum_{j\in\mathbb{Z}} \overline{a}_{k_0}^i \delta_{j,k_1} UT^{Nj} m_i(T)\varphi_n = \sum_{i\in\underline{N}} \overline{a}_{k_0}^i \sum_{l\in\underline{N}} a_l^i UT^{Nk_1+l}\varphi_n\\ &= \sum_{l\in\underline{N}} \underbrace{\left(\sum_{i\in\underline{N}} \overline{a}_{k_0}^i a_l^i\right)}_{=\delta_{k_0,l}} UT^{Nk_1+l}\varphi_n = UT^{Nk_1+k_0}\varphi_n.\end{aligned}$$

□

The second representation of the Cuntz algebra $\mathcal{O}_N$ under consideration comes from the tuple of isometries $(Z_i)_{i\in\underline{N}}$ given by $Z_i|n\rangle = |Nn+i\rangle$, $i \in \underline{N}$, $n \in \mathbb{N}_0$, on $l^2(\mathbb{N}_0)$. With this we can give the interpretation of U.

Proposition B.5. *It holds that*

$$U = \sum_{i\in\underline{N}} Z_i \otimes S_i^*.$$

Notice that Z_i acts on $l^2(\mathbb{N}_0)$ and S_i on $L^2(\mathbb{T},\lambda)$. Consequently, here we use the association of $L^2(\nu_{\mathbb{Z}})$ as $l^2(\mathbb{N}_0) \otimes L^2(\mathbb{T},\lambda)$.

PROOF. This expression follows as in **[Jor06]** from

$$\begin{aligned}\sum_{i\in\underline{N}} Z_i\otimes S_i^*\left(|n\rangle\otimes|k\rangle\right) &= \sum_{i\in\underline{N}} |Z_i n\rangle\otimes|S_i^* e_k\rangle = \sum_{i\in\underline{N}}\sum_{j\in\mathbb{Z}}\langle S_i^* e_k|e_j\rangle|Z_i n\rangle\otimes|e_j\rangle \\ &= \sum_{i\in\underline{N}}\sum_{j\in\mathbb{Z}}\langle e_k|S_i e_j\rangle T^j\varphi_{Nn+i} = UT^k\varphi_n \\ &= U\left(|n\rangle\otimes|k\rangle\right).\end{aligned}$$

□

B.2. A representation of the two dimensional U in terms of the Cuntz algebra

In the same way we can interpret the operator U of the two dimensional MRA on enlarged fractals as the tensor product of two representations of Cuntz algebra $\mathcal{O}_N$. The construction is very similar to the one-dimensional case, so we only give the main steps in the proofs. Here we use the notation of Chapter 11. We start by rewriting the wavelet basis of $L^2(\nu_{\mathbb{Z}_v^2})$, compare Proposition 11.19, as a sequence of functions and their translates. Recall that $N = N_1 N_2$.

Proposition B.6. *Define a sequence of functions for $n\in\mathbb{N}_0$, $i\in\underline{N}$, by*

$$\begin{aligned}\varphi_0 &= \mathbb{1}_C,\\ \varphi_{Nn+i} &= Um_i(T)\varphi_n.\end{aligned}\tag{B.2.1}$$

Then $\left\{T^{(k,l)}\varphi_n : n\in\mathbb{N}_0, (k,l)\in\mathbb{Z}^2\right\}$ is an ONB for $L^2(\nu_{\mathbb{Z}_v^2})$.

PROOF. As in the one dimensional version we have that $\varphi=\varphi_0$ is consistent with the iteration procedure. Moreover, we have $\varphi_j=\psi_j$ for $j\in\underline{N}\backslash\{0\}$. In the next step we rewrite the formula for φ_n iteratively in terms of φ_0 and the filter functions, that is for $n=\sum_{i=0}^r k_i N^i$, $k_i\in\underline{N}$, $r\in\mathbb{N}$,

$$\varphi_n = U^{r+1}\prod_{i=0}^r m_{k_i}\left(T^{(N_1^{r-i},N_2^{r-i})}\right)\varphi_0,$$

or equivalently

$$\varphi_n = U^r\prod_{i=0}^{r-1} m_{k_i}\left(T^{(N_1^{r-i-1},N_2^{r-i-1})}\right)\psi_{k_r}.$$

Notice that $\operatorname{supp}(\varphi_n)\subset D$, $n\in\mathbb{N}$. Consequently, for the orthonormality we can consider without loss of generality $\langle\varphi_n|\varphi_m\rangle$, i.e. without the translation via $T^{(k,l)}$, $(k,l)\in\mathbb{Z}^2$. By considering $\langle\varphi_{n_1}|\varphi_{n_2}\rangle$, $n_1,n_2\in\mathbb{N}_0$, with $n_1=\sum_{i=0}^{r_1}k_iN^i$and $n_2=\sum_{i=0}^{r_2}l_iN^i$ for $k_i,l_i\in\underline{N}$, $r_1,r_2\in\mathbb{N}_0$, we have $\langle\varphi_{n_1}|\varphi_{n_2}\rangle=0$ if $r_1\neq r_2$ or $k_{r_1}\neq l_{r_2}$ since $\left\{U^mT^{(k,l)}\psi_j : m\in\mathbb{Z},(k,l)\in\mathbb{Z}^2,j\in\underline{N}\right\}$. So we assume that $r_1=r_2=:r$ and $k_r=l_r=:p$. Then

$$\begin{aligned}\langle\varphi_{n_1}|\varphi_{n_2}\rangle &= \langle\prod_{i=0}^{r-1}m_{k_i}\left(T^{(N_1^{r-i-1},N_2^{r-i-1})}\right)\psi_p|\prod_{i=0}^{r-1}m_{l_i}\left(T^{(N_1^{r-i-1},N_2^{r-i-1})}\right)\psi_p\rangle\\ &= \prod_{i=0}^{r-1}\underbrace{\sum_{j_i\in\underline{N}}a^{k_i}_{(j_i)_{N_1},\lfloor\frac{j_i}{N_1}\rfloor}\cdot\overline{a}^{l_i}_{(j_i)_{N_1},\lfloor\frac{j_i}{N_1}\rfloor}}_{=\delta_{k_i,l_i}} = \delta_{n_1,n_2}.\end{aligned}$$

Now it remains to be shown that $\left\{T^{(k,l)}\varphi_n : (k,l)\in\mathbb{Z}^2, n\in\mathbb{N}_0\right\}$ spans $L^2(\nu_{\mathbb{Z}_v^2})$. We show that we can write any element $U^nT^{(k,l)}\psi_j$ as a linear combination of $\left\{T^{(k,l)}\varphi_n : (k,l)\in\mathbb{Z}^2, n\in\mathbb{N}_0\right\}$. We notice that it is sufficient to consider $(k,l)\in\underline{N_1^n}\times\underline{N_2^n}$ by the condition $U^{-1}T^{(a,b)}U=T^{(N_1a,N_2b)}$. So let $n\in\mathbb{N}_0$ and $(k,l)\in\underline{N_1^n}\times\underline{N_2^n}$ with $(k,l)=\left(\sum_{i=0}^{n-1}k_iN_1^i,\sum_{i=0}^{n-1}l_iN_2^i\right)$, $k_i\in\underline{N_1}$, $l_i\in\underline{N_2}$, then

$$U^n T^{(k,l)}\psi_j = \sum_{(j_0,\ldots,j_{n-1})\in\underline{N}^n} \prod_{i=0}^{n-1} \underbrace{\sum_{q_{n-1-i}\in\underline{N}} \overline{a}^{q_{n-1-i}}_{k_i,l_i} a^{q_{n-1-i}}_{(j_i)_{N_1},\lfloor\frac{j_i}{N_1}\rfloor}}_{=\delta_{(k_i,l_i),\left((j_i)_{N_1},\lfloor\frac{j_i}{N_1}\rfloor\right)}} U^n T^{(k,l)}\psi_j$$
$$= \sum_{(q_0,\ldots,q_{n-1})\in\underline{N}^n} \left(\prod_{i=0}^{n-1} \overline{a}^{q_{n-1-i}}_{k_i,l_i}\right) \varphi_{\sum_{i=0}^{n-1} q_i N^i + N^n j}.$$

□

Now we turn to a correspondence between $L^2(\nu_{\mathbb{Z}^2_v})$ and $l^2(\mathbb{N}_0)\otimes l^2(\mathbb{Z}^2)$.

Proposition B.7. *There exists a unitary isomorphism between $L^2(\nu_{\mathbb{Z}^2_v})$ and $l^2(\mathbb{N}_0)\otimes l^2(\mathbb{Z}^2)$ which is the extension of $W : T^{(k,l)}\varphi_n \mapsto |n\rangle\otimes|k,l\rangle$, $n\in\mathbb{N}_0$, $(k,l)\in\mathbb{Z}^2$. Furthermore,*

$$L^2(\nu_{\mathbb{Z}^2_v}) \simeq l^2(\mathbb{N}_0)\otimes l^2(\mathbb{Z}^2) \simeq l^2(\mathbb{N}_0)\otimes L^2(\mathbb{T}^2,\lambda).$$

Proof. As in one dimension W maps an ONB of $L^2(\nu_{\mathbb{Z}^2_v})$ to one of $l^2(\mathbb{N}_0)\otimes l^2(\mathbb{Z}^2)$, so the argumentation goes as the one dimensional one.

□

In the following we want to express the scaling operator U of the MRA in terms of representations of the Cuntz algebra $\mathcal{O}_N$. The representations of $\mathcal{O}_N$ under consideration come from the tuple of isometries $(Z_i)_{i\in\underline{N}}$ given by $Z_i|n\rangle = |Nn+i\rangle$, $i\in\underline{N}$, $n\in\mathbb{N}_0$, on $l^2(\mathbb{N}_0)$ and from the tupel $(S_i)_{i\in\underline{N}}$ given by $(S_i f)(z,w) = m_i(z,w) f\left(z^{N_1},w^{N_2}\right)$, $i\in\underline{N}$, $(z,w)\in\mathbb{T}^2$, on $L^2(\mathbb{T}^2,\lambda)$, where m_i, $i\in\underline{N}$, are the filter functions defined in Proposition 11.19. First we will need the following connection between the basis in $L^2(\nu_{\mathbb{Z}^2_v})$ and the scalar product in $L^2(\mathbb{T}^2,\lambda)$.

Lemma B.8. *The following holds for $n\in\mathbb{N}_0$ and $(k,l)\in\mathbb{Z}^2$*

$$UT^{(k,l)}\varphi_n = \sum_{i\in\underline{N}} \sum_{(j_1,j_2)\in\mathbb{Z}^2} \langle e_{(k,l)}|S_i e_{(j_1,j_2)}\rangle T^{(j_1,j_2)}\varphi_{Nn+i}.$$

Proof. First we notice that $UT^{(k,l)}\varphi_n$ is not an element of the basis (compare Proposition B.6). Now we show that $\langle UT^{(k,l)}\varphi_n|T^{(j,m)}\varphi_{Nn+i}\rangle = \langle e_{(k,l)}|S_i e_{(j,m)}\rangle$. To do this first notice that for $(k,l)\in\mathbb{Z}^2$, $n\in\mathbb{N}_0$, $(j,m)\in\mathbb{Z}^2$,

$$\varphi_{Nn+i}(\cdot - j\overrightarrow{v_1} - k\overrightarrow{v_2}) = T^{(j,k)}Um_i(T)\varphi_n$$

and recall that the filter functions are given as $m_l : (z,w)\mapsto \sum_{(i,j)\in\underline{N_1}\times\underline{N_2}} a^l_{(i,j)}\cdot z^i w^j$, $(z,w)\in\mathbb{T}^2$, $l\in\underline{N}\backslash\{0\}$. Then it follows that

$$\langle UT^{(k,l)}\varphi_n|T^{(j,m)}\varphi_{Nn+i}\rangle = \begin{cases} \overline{a}^i_{(r,s)}, & \text{if there exists } (r,s)\in\underline{N_1}\times\underline{N_2} \\ & \text{ such that } N_1 j - k + r = 0 \text{ and } N_2 m - l + s = 0, \\ 0, & \text{otherwise.} \end{cases}$$

On the other hand

$$\langle e_{(k,l)}|S_i e_{(j,m)}\rangle = \begin{cases} \overline{a}^i_{(r,s)}, & \text{if there exists } (r,s)\in\underline{N_1}\times\underline{N_2} \\ & \text{ such that } N_1 j - k + r = 0 \text{ and } N_2 m - l + s = 0, \\ 0, & \text{otherwise.} \end{cases}$$

Consequently, $\langle UT^{(k,l)}\varphi_n|T^{(j,m)}\varphi_{Nn+i}\rangle = \langle e_{(k,l)}|S_i e_{(j,m)}\rangle$.

Now it follows for $(k,l) \in \mathbb{Z}^2$ with $(k,l) = (N_1k_1 + k_0, N_2l_1 + l_0)$, $k_1, l_1 \in \mathbb{Z}$, $k_0 \in \underline{N_1}$, $l_0 \in \underline{N_2}$, that

$$\begin{aligned}
&\sum_{i\in\underline{N}} \sum_{(j_1,j_2)\in\mathbb{Z}^2} \langle e_{(k,l)} | S_i e_{(j_1,j_2)} \rangle T^{(j_1,j_2)} \varphi_{Nn+i} \\
=&\sum_{i\in\underline{N}} \sum_{(j_1,j_2)\in\mathbb{Z}^2} \overline{a}^i_{(k_0,l_0)} \cdot \delta_{(j_1,j_2),(k_1,l_1)} T^{(j_1,j_2)} U m_i(T) \varphi_n \\
=&\sum_{i\in\underline{N}} \overline{a}^i_{(k_0,l_0)} U T^{(N_1k_1,N_2l_1)} m_i(T) \varphi_n \\
=&\sum_{i\in\underline{N}} \overline{a}^i_{(k_0,l_0)} \sum_{(p,q)\in\underline{N_1}\times\underline{N_2}} a^i_{(p,q)} U T^{(N_1k_1+p,N_2l_1+q)} \varphi_n \\
=&\sum_{(p,q)\in\underline{N_1}\times\underline{N_2}} \underbrace{\sum_{i\in\underline{N}} \overline{a}^i_{(k_0,l_0)} a^i_{(p,q)}}_{=\delta_{(k_0,l_0),(p,q)}} U T^{(N_1k_1+p,N_2l_1+q)} \varphi_n \\
=&\, U T^{(k,l)} \varphi_n.
\end{aligned}$$

□

With this we get the interpretation of the operator U.

Proposition B.9. *It holds that*

$$U = \sum_{i\in\underline{N}} Z_i \otimes S_i^*.$$

Proof. This proof follows in the same way as the one dimensional version.

□

Remark B.10. We have tried to find an analogous representation for the scaling operator U in the case of wavelet bases on Markov Interval Maps with a Markov measure, but there is not a natural expansion to this case. We suspect that such an expansion might be possible if one considers representations of the Cuntz-Krieger algebra instead of the Cuntz algebra.

B.3. Operator algebras for MIM

Here we want to give some remarks about the existence of analogous operators to the isometries $S_i f(z) = m_i(z) f(z^N)$, $z \in \mathbb{T}$, arising from the filter functions, for the construction of wavelet bases for MIMs. For the case of one father wavelet we obtain a so-called low-pass filter function and high-pass filter functions, in terms of which the mother wavelets are given. Via these filter functions we obtain a representation of the Cuntz algebra $\mathcal{O}_N$, where N is the number of filter functions. In the case of multiwavelets we can obtain weaker relations. Here we restrict to the case of an MIM with underlying Markov measure as treated in Section 9.3. These results are in correspondence to results in [**BFMP10**].

The relations for the father and the mother wavelets can be written in the following way:

For the following we introduce for $z \in \mathbb{T}$ the low-pass filter

$$H(z) = \left(\sqrt{\pi_{kl}} z^k\right)_{l,k\in\underline{N}}$$

and for each $k \in \underline{N}$ and $z \in \mathbb{T}$ the high-pass filter

$$G_k(z) = \left(A_{kl} c_l^{k,j} z^k\right)_{j\in\underline{q^k}\setminus\{0\},l\in\underline{N}}.$$

With these definitions we obtain the following immediate lemma.

Lemma B.11. *Let* $\Phi = (\varphi_j)^t_{j\in\underline{N}}$*, then* $\Phi = UH(T)\Phi$ *and let* $\Psi_k = \left(\psi^{k,j}\right)^t_{j\in\underline{q^k}\setminus\{0\}}$ *for* $k \in \underline{N}$*, then* $\Psi_k = UG_k(T)\Phi$*, where the operators* U *and* T *are applied to every entry in the vector.*

Remark B.12. It follows that for $z \in \mathbb{T}$

$$\overline{H(z)}H^t(z) = \left(\sum_{j \in \underline{N}} \sqrt{\pi_{kj}\pi_{lj}} z^{l-k} \right)_{k,l \in \underline{N}}$$

and for $k \in \underline{N}$, $z \in \mathbb{T}$,

$$\overline{G_k(z)} G_k^t(z) = I.$$

These filter functions lead us to the definitions of certain "isometries".

Definition B.13. For $z \in \mathbb{T}$ and $f = (f_0, \dots, f_{N-1})$, $f_j \in L^2(\mathbb{T}, \lambda)$, define

$$S_H f(z) = \sqrt{N} H^t(z) f\left(z^N\right)$$

and for $k \in \underline{N}$, $z \in \mathbb{T}$,

$$S_{G_k} f(z) = G_k^t(z) f\left(z^N\right).$$

For these "isometries" we have the following properties.

Proposition B.14. *The following relations hold:*

(1) $S_H^* S_H = I$,
(2) $S_{G_k}^* S_{G_k} = I$, $k \in \underline{N}$,
(3) $S_H^* S_{G_k} = 0$ *and* $S_{G_k}^* S_H = 0$, $k \in \underline{N}$,
(4) $S_{G_i}^* S_{G_j} = 0$, $i, j \in \underline{N}$, $i \neq j$.

Remark B.15. Realize that for $z \in \mathbb{T}$, $f = (f_0, \dots, f_{N-1})$, $f_j \in L^2(\mathbb{T}, \lambda)$,

$$S_H^* f(z) = \frac{1}{\sqrt{N}} \sum_{\omega^N = z} \overline{H(\omega)} f(\omega)$$

and for $k \in \underline{N}$, $z \in \mathbb{T}$, $f = (f_0, \dots, f_{N-1})$, $f_j \in L^2(\mathbb{T}, \lambda)$,

$$S_{G_k}^* f(z) = \frac{1}{N} \sum_{\omega^N = z} \overline{G_k(\omega)} f(\omega).$$

PROOF. ad (1): Let $z \in \mathbb{T}$, $f = (f_0, \dots, f_{N-1})$, $f_j \in L^2(\mathbb{T}, \lambda)$, then

$$S_H^* S_H f(z) = \sum_{\omega^N = z} \overline{H(\omega)} H^t(\omega) f(\omega^N) = \sum_{\omega^N = z} \overline{H(\omega)} H^t(\omega) f(z) = f(z).$$

ad (2): Let $k \in \underline{N}$, $z \in \mathbb{T}$, $f = (f_0, \dots, f_{N-1})$, $f_j \in L^2(\mathbb{T}, \lambda)$, then

$$S_{G_K}^* S_{G_k} f(z) = \frac{1}{N} \sum_{\omega^N = z} \overline{G_k(\omega)} G_k^t(\omega) f(z) = f(z).$$

ad (3): Let $k \in \underline{N}$, $z \in \mathbb{T}$, $f = (f_0, \dots, f_{N-1})$, $f_j \in L^2(\mathbb{T}, \lambda)$, then

$$S_H^* S_{G_k} f(z) = \frac{1}{N} \sum_{\omega^N = z} \overline{H(\omega)} G_k^t(\omega) f(z) = 0,$$

since $\sum_{\omega^N = z} \overline{H(\omega)} G_k^t(\omega) = 0$ by summing up the roots of unity.

For $S_{G_k}^* S_H$ we use that $\overline{G_k(\omega)} H^t(\omega) = 0$ by the choice of the coefficients $c_j^{k,l}$.

ad (4): Let $i, j \in \underline{N}$, $i \neq j$, $z \in \mathbb{T}$, $f = (f_0, \dots, f_{N-1})$, $f_j \in L^2(\mathbb{T}, \lambda)$, then

$$S_{G_i}^* S_{G_j} f(z) = \frac{1}{N} \sum_{\omega^N = z} \overline{G_i(\omega)} G_j^t(\omega) f(z) = 0,$$

by summing up the roots of unity. □

Here we have seen that in contrast to the filter functions for a usual MRA with one father wavelet and a unitary scaling operator U, we do not obtain a representation of a Cuntz algebra since we do not necessarily have that $S_H S_H^* + \sum_{k \in \underline{N} \setminus \{0\}} S_{G_k} S_{G_K}^* = I$. So we only obtain weaker relations between these filter functions

APPENDIX C

Direct limits

In this chapter we apply a direct limit approach as in [**BLP$^+$10**] to our construction of wavelet bases on enlarged fractals in one and two dimensions. The first connection between wavelets and direct limits was made by Larsen and Raeburn. In [**LR06**] Larsen and Raeburn used this approach to give an alternative proof of a theorem of Mallat which describes a construction of wavelets starting from a quadrature mirror filter. They mainly show how the father wavelet associated to the filter can be used to identify a direct limit of Hilbert spaces with $L^2(\mathbb{R}, \lambda)$ so that the wavelet basis can be directly identified. In [**Rae09**] Raeburn extends their results to show that wavelet-packet bases for $L^2(\mathbb{R}, \lambda)$ also fit naturally into the same direct limit framework.

Baggett et al. further extend this work to the construction of the wavelet basis via general MRA, compare [**BLP$^+$10, BLM$^+$09**]. In [**BLM$^+$09**] the authors show that for a general Hilbert space and an isometry S, the direct limit gives an increasing family of subspaces, whose union is dense in the direct limit space and whose intersection is $\{0\}$ if the isometry S is a pure isometry. These are exactly two of the properties for the MRA. But in the standard examples the direct limit space does not coincide with the space we are interested in, e.g. $L^2(\mathbb{R}, \lambda)$, because the isometries are e.g. defined on $L^2(\mathbb{T}, \lambda)$. Consequently, in [**BLP$^+$10**] Baggett et al. prove that with an isometry into another Hilbert space, e.g. from $L^2(\mathbb{T}, \lambda)$ into $L^2(\mathbb{R}, \lambda)$, that satisfies specific properties, the obtained properties can be carried over, so that one gets an MRA in the desired space. Wavelet bases constructed in the direct limit space can be mapped to wavelet bases in the space under consideration. The authors also apply this approach to wavelet bases on enlarged Cantor sets as constructed in [**DJ06**].

We apply the approach of [**BLP$^+$10**] to the wavelet bases constructed in Section 10.2 and Section 11.3. We start by clarifying the definitions and giving the main results of [**BLP$^+$10**].

C.1. Introduction to direct limits and their application to wavelet bases

We start with a short introduction to direct limits. So we give the precise definition of the direct limit.

Definition C.1. Suppose H_n, $n \in \mathbb{N}_0$, are Hilbert spaces and $T_n : H_n \to H_{n+1}$, $n \in \mathbb{N}_0$, are isometries. A direct limit $\big(H_\infty, (U_n)_{n\in\mathbb{N}_0}\big)$ is a Hilbert space H_∞ and a sequence of isometries $U_n : H_n \to H_\infty$, $n \in \mathbb{N}_0$, such that $U_{n+1} \circ T_n = U_n$, $n \in \mathbb{N}_0$, which satisfy the following universal property: for every family of isometries R_n of H_n, $n \in \mathbb{N}_0$, into another Hilbert space K such that $R_{n+1} \circ T_n = R_n$, $n \in \mathbb{N}_0$, there exists a unique isometry $R_\infty : H_\infty \to K$ such that $R_\infty \circ U_n = R_n$ for every $n \in \mathbb{N}_0$.

For the proof of the existence of this direct limit space see [**LR06, BLM$^+$09**]. By the uniqueness of R_∞ it also follows that $H_\infty = \operatorname{cl} \bigcup_{n\in\mathbb{N}_0} U_n H_n$.

The relations of Definition C.1 can be pictured as in Figure C.1.1. In the diagram all sub-diagrams commute. For further information see [**LR06, BLM$^+$09**].

In our application, we only need the direct limits in situations where there is a single Hilbert space H, and an isometry S on H, for which $H_n = H$ and $T_n = S$ for all $n \in \mathbb{N}_0$. In this case, applying the universal property of the direct limit to the sequence of isometries $R_n = U_n \circ S$ produces an isometry $S_\infty : H_\infty \to H_\infty$ satisfying $S_\infty \circ U_n = U_n \circ S$, and applying the same property to the sequence $U_{n+1} \circ S = U_n$ produces an isometry $Y_\infty : H_\infty \to H_\infty$ satisfying $Y_\infty \circ U_n = U_{n+1}$; and it is then immediate that for all $n \in \mathbb{N}_0$

$$S_\infty \circ Y_\infty \circ U_n = S_\infty \circ U_{n+1} = U_{n+1} \circ S = U_n$$

$$\begin{tikzcd}
H_0 \arrow[r, "T_0"] \arrow[rrrr, bend left, "U_0"] \arrow[drr, "R_0"'] & H_1 \arrow[r, "T_1"] \arrow[rrr, bend left, "U_1"] \arrow[dr, "R_1"] & H_2 \arrow[r, "T_2"] \arrow[rr, bend left, "U_2"] \arrow[d, "R_2"] & \cdots \arrow[r] & H_\infty, \arrow[dll, dotted, "R_\infty"'] \\
 & & K & &
\end{tikzcd}$$

FIGURE C.1.1. Direct limit for $((H_n)_{n\in\mathbb{N}_0}, (T_n)_{n\in\mathbb{N}_0})$.

and

$$Y_\infty \circ S_\infty \circ U_n = Y_\infty \circ U_n \circ S = U_{n+1} \circ S = U_n.$$

Since the identity operator I on H_∞ is also an isometry satisfying $I \circ U_n = U_n$ for all $n \in \mathbb{N}_0$, we conclude from the uniqueness property of the direct limit that $S_\infty \circ Y_\infty = I$ and hence that S_∞ is unitary. This situation is depicted in Figure C.1.2. If we choose to identify H with the subspaces $U_0 H$ of H_∞, the relation $S_\infty \circ U_0 = U_0 \circ S$, together with the facts that $Y_\infty = S_\infty^*$ maps $U_n H$ to $U_{n+1} H$ for all $n \in \mathbb{N}_0$ and $H_\infty = \operatorname{cl} \bigcup_{n\in\mathbb{N}_0} U_n H$, implies that S_∞ is the minimal unitary extension of S in the sense of [**Con00**]. Hence the direct limit process can be viewed as a way of turning S into a unitary.

$$\begin{tikzcd}
H \arrow[r, "S"] \arrow[d, "S"] \arrow[rrr, bend left, "U_n"] & H \arrow[r, "S"] \arrow[d, "S"] \arrow[rr, bend left, "U_{n+1}"] & \cdots & H_\infty \arrow[d, bend left, "S_\infty"] \\
H \arrow[r, "S"] \arrow[ur] \arrow[rrr, bend right, "U_n"'] & H \arrow[r, "S"] \arrow[ur] \arrow[rr, bend right, "U_{n+1}"'] & \cdots & H_\infty \arrow[u, bend left, "Y_\infty"]
\end{tikzcd}$$

FIGURE C.1.2. Direct limit for (H, S).

Now we state the main results from [**BLP$^+$10**] which we will apply to the one and two dimensional wavelets on enlarged fractals from Section 10.2 and Section 11.3. The main results of [**BLP$^+$10**] are conditions under which the direct limit space can be concretely identified via an isomorphism with another space (see Theorem C.4) and an application of this correspondence (Proposition C.5) to the construction of mother wavelets.

Let S be an isometry on a Hilbert space H and let $\left(H_\infty, (U_n)_{n\in\mathbb{N}_0}\right)$ be the Hilbert space direct limit of the direct system $((H_n, T_n) = (H, S), n \in \mathbb{N}_0)$. It was shown in [**BLP$^+$10**] that the corresponding unitary operator S_∞ on H_∞ has the following properties: $S_\infty U_n = U_n S = U_{n-1}$ for every $n \in \mathbb{N}$ and the subspaces

$$\mathcal{V}_n := \begin{cases} U_n(H), & n \geq 0, \\ S_\infty^{|n|}(V_0), & n < 0, \end{cases}$$

give an increasing sequence of subspaces in H_∞ such that $\operatorname{cl} \bigcup_{n\in\mathbb{Z}} \mathcal{V}_n = H_\infty$ and $S_\infty(\mathcal{V}_n) = \mathcal{V}_{n-1}$. Moreover, their intersection is just $\{0\}$ if and only if the isometry S is a pure isometry in the sense of the following definition.

Definition C.2. An isometry S on a Hilbert space H is a *pure isometry* if and only if $\bigcap_{n\in\mathbb{N}_0} S^n(H) = \{0\}$.

The following proposition is useful in showing that isometries of a form relevant to our setting are pure. First we start by fixing the notation. We let Γ denote a countable abelian group, regarded as a discrete topological space, and we let $\widehat{\Gamma}$ denote its dual, which is compact in the corresponding compact-open topology. The measure we use on $\widehat{\Gamma}$ is the normalized Haar measure. Furthermore, we fix an injective endomorphism α of Γ such that $\alpha(\Gamma)$ has finite index N in Γ and $\bigcap_{n\geq 1} \alpha^n(\Gamma) = \{0\}$.

Theorem C.3 ([**BLM**$^+$**09**])**.** *Suppose that B is a Borel subset of $\widehat{\Gamma}$ and $m : \widehat{\Gamma} \to \mathbb{C}$ is a Borel function such that*

$$\sum_{\alpha^*(\xi)=\omega} |m(\xi)|^2 = N\mathbb{1}_B(\omega), \text{ for almost all } \omega \in \widehat{\Gamma},$$

and define $S_m : L^2(B) \to L^2(B)$ by $(S_m f)(\omega) = m(\omega) f(\alpha^*(\omega))$. If either

(1) *$\widehat{\Gamma}\backslash B$ has positive measure, or*
(2) *$|m(\omega)| \neq 1$ on a set of positive measure,*

then S_m is a pure isometry.

Now we turn back to the main results of [**BLP**$^+$**10**] and the existence of an MRA.

Theorem C.4 ([**BLP**$^+$**10**])**.** *Let H and K be Hilbert spaces, Γ be a countable abelian group and let $\mathcal{W}(H)$, $\mathcal{W}(K)$ stand for the groups of unitary operators on H and K respectively. Suppose that $\varrho : \Gamma \to \mathcal{W}(H)$ is a unitary representation, and S is an isometry on H such that $S\varrho_\gamma = \varrho_{\alpha(\gamma)}S$ for $\gamma \in \Gamma$. Suppose that $\vartheta : \Gamma \to \mathcal{W}(K)$ is a unitary representation and D is a unitary operator on K such that $D\vartheta_\gamma D^* = \vartheta_{\alpha(\gamma)}$ for $\gamma \in \Gamma$. If there is an isometry $\mathcal{R} : H \to K$ such that*

(1) *$\mathcal{R}S = D\mathcal{R}$, and*
(2) *$\mathcal{R}\varrho_\gamma = \vartheta_\gamma \mathcal{R}$ for $\gamma \in \Gamma$,*

then there is an isomorphism $\mathcal{R}_\infty$ of H_∞ onto the subspace $\overline{\bigcup_{n=0}^{\infty} D^{-n}\mathcal{R}(H)}$ of K such that $\mathcal{R}_\infty S_\infty \mathcal{R}_\infty^* = D$ and $\mathcal{R}_\infty \varrho_\infty \mathcal{R}_\infty^* = \vartheta$. The subspaces $D^{-n}\mathcal{R}(H)$ form an MRA of $\mathcal{R}_\infty(H_\infty)$ relative to D and ϑ if and only if S is a pure isometry.

The next proposition gives a way of constructing an orthonormal basis for the limit Hilbert space H_∞.

Proposition C.5 ([**BLP**$^+$**10**])**.** *Suppose S is a pure isometry on H and suppose there are a Hilbert space L, a unitary representation $\rho : \Gamma \to \mathcal{W}(L)$, an orthonormal set B in L such that $\{\rho_\gamma l : l \in B, \gamma \in \Gamma\}$ is an orthonormal basis for L, and a unitary isomorphism S_1 of L onto $(SH)^\perp$ such that $S_1\rho_\gamma = \varrho_{\alpha(\gamma)}S_1$. Then*

$$\left\{S_\infty^{-j}\varrho_\infty(\gamma)\psi : j \in \mathbb{Z}, \gamma \in \Gamma, \psi \in U_1 S_1(B)\right\}$$

is an orthonormal basis for H_∞.

C.2. Application to wavelet bases on enlarged fractals in one dimension

Now we apply the results of [**BLP**$^+$**10**] to the construction of the wavelet bases on enlarged fractals, see Section 10.2. In our case, the countable abelian group Γ is $\mathbb{Z}$ and thus $\widehat{\Gamma} = \mathbb{T}$. The endomorphism α on $\mathbb{Z}$ is given by $\alpha(n) = Nn$. It is obvious that $\alpha(\mathbb{Z})$ has finite index N in $\mathbb{Z}$ and $\bigcap_{n\geq 1} \alpha^n(\mathbb{Z}) = \{0\}$. For the direct limit we consider the Hilbert space $L^2(\mathbb{T}, \lambda)$ with the isometry S acting on $L^2(\mathbb{T}, \lambda)$ defined by

$$S = S_{m_0} : f(z) \mapsto m_0(z) f\left(z^N\right), \ z \in \mathbb{T},$$

where m_0 is the low-pass filter from Section 10.2. Since m_0 is a filter function, it is well known that S is an isometry.

Proposition C.6. *The operator S is a pure isometry.*

Proof. We just apply Theorem C.3. We have that $m_0(z) = \sum_{i\in A}\sqrt{p_i}\cdot z^i$ and so

$$\sum_{\alpha^*(\xi)=\omega}|m_0(\xi)|^2 = \sum_{\xi^N=\omega}\sum_{i,j\in A}\sqrt{p_ip_j}\xi^{i-j} = \sum_{i,j\in A}\sqrt{p_ip_j}\underbrace{\sum_{\xi^N=\omega}\xi^{i-j}}_{=N\delta_{i,j}} = N\cdot\sum_{j\in A}p_j = N.$$

It is clear that $|m_0(\omega)| \neq 1$ on a set of positive measure. □

We further define $\varrho : \mathbb{Z} \to \mathcal{W}(L^2(\mathbb{T}))$ by $(\varrho_n f)(z) = z^n f(z)$. It is easily seen that ϱ_n is a unitary operator on $L^2(\mathbb{T},\lambda)$ for all $n \in \mathbb{Z}$. As the second Hilbert space we consider $L^2(\nu_{\mathbb{Z}})$, where the measure $\nu_{\mathbb{Z}}$ is defined in Definition 10.6 of Section 10.1. In this space we want to obtain a wavelet basis. Here we consider the operators used for the MRA in Section 10.1.1. These are the scaling and translation operator U and T (see Definition 10.13). We set $D = U^{-1}$ and $\vartheta = T$ in Theorem C.4. To apply Theorem C.4 to this setting we have to verify that the unitary operators ϱ_n, $n \in \mathbb{Z}$, satisfy the desired properties.

Lemma C.7. *The relation $S\varrho_n = \varrho_{Nn}S$, $n \in \mathbb{Z}$, is satisfied.*

Proof. Let $f \in L^2(\mathbb{T},\lambda)$ and $z \in \mathbb{T}$, $n \in \mathbb{Z}$, then $S\varrho_n f(z) = S(z^n f(z)) = m_0(z)z^{nN}f(z^N)$ and $\varrho_{Nn}Sf(z) = \varrho_{Nn}\left(m_0(z)f(z^N)\right) = z^{Nn}m_0(z)f(z^N)$. □

Before we can show that the direct limit $\left(\left(L^2(\mathbb{T},\lambda)\right)_\infty, S_\infty, \varrho_\infty\right)$ is isomorphic to $\left(L^2(\nu_{\mathbb{Z}}), U, T\right)$, we have to find an isometry of $L^2(\mathbb{T},\lambda)$ into $L^2(\nu_{\mathbb{Z}})$ such that the conditions of Theorem C.4 are satisfied.

Lemma C.8. *For $n \in \mathbb{Z}$, let $e_n : z \mapsto z^n$, $z \in \mathbb{T}$. Then there is an isometry $\mathcal{R}$ of $L^2(\mathbb{T},\lambda)$ into $L^2(\nu_{\mathbb{Z}})$ such that $\mathcal{R}e_n = T^n\mathbb{1}_C = \mathbb{1}_{C+n}$ for $n \in \mathbb{Z}$.*

Proof. This is analogous to the corresponding proof for the special setting of enlarged Cantor sets with the measure of maximal entropy (see [**DJ06**]) in [**BLP**$^+$**10**], so we will just sketch the arguments. First notice that $\{e_n : n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T},\lambda)$, so it suffices to check that $T^n\mathbb{1}_C = \mathbb{1}_{C+n}$ is an orthonormal set in $L^2(\nu_{\mathbb{Z}})$, and this follows from the results in Section 10.2, compare Proposition 10.18. □

We now show that the isometry $\mathcal{R}$ given in Corollary C.8 satisfies the conditions of Theorem C.4.

Lemma C.9. *The following relations hold:*
(1) $\mathcal{R}S = U^{-1}\mathcal{R}$,
(2) $\mathcal{R}\varrho_k = T^k\mathcal{R}$, $k \in \mathbb{Z}$.

Proof. ad (1): Let $n \in \mathbb{Z}$ then

$$\mathcal{R}Se_n = \mathcal{R}\left(m_0\cdot e_{Nn}\right) = \mathcal{R}\left(\sum_{j\in A}\sqrt{p_j}\cdot e_{j+Nn}\right) = \sum_{j\in A}\sqrt{p_j}\cdot T^{j+nN}\mathbb{1}_C$$

and on the other hand

$$\begin{aligned} U^{-1}\mathcal{R}e_n = U^{-1}T^n\mathbb{1}_C &= \sum_{k\in\mathbb{Z}}\sum_{i\in A}\sqrt{p_i}\cdot\mathbb{1}_C(\cdot - i - Nk)\cdot\mathbb{1}_C\left(\tau_i(\cdot - i - Nk) + k - n\right) \\ &= \sum_{i\in A}\sqrt{p_i}\cdot\mathbb{1}_C\left(\cdot - i - Nn\right) = \sum_{i\in A}\sqrt{p_i}\cdot T^{i+Nn}\mathbb{1}_C. \end{aligned}$$

ad (2): Let $k, n \in \mathbb{Z}$, then $\mathcal{R}\varrho_k e_n = \mathcal{R}e_{k+n} = T^{k+n}\mathbb{1}_C$, and $T^k\mathcal{R}e_n = T^kT^n\mathbb{1}_C = T^{k+n}\mathbb{1}_C$. □

Applying Theorem C.4, we then obtain the following.

Corollary C.10. *The direct limit* $\left(\left(L^2(\mathbb{T},\lambda)\right)_\infty, S_\infty, \varrho_\infty\right)$ *is isomorphic to* $\left(L^2(\nu_{\mathbb{Z}}), U, T\right)$*. With the subspaces*

$$V_n = \operatorname{cl}\operatorname{span}\left\{U^n T^k \varphi : \, k \in \mathbb{Z}\right\}, \; n \in \mathbb{Z},$$

$\left(L^2(\nu_{\mathbb{Z}}), U, T\right)$ *allows a two-sided MRA and* $\left\{T^k\varphi : \, k \in \mathbb{Z}\right\}$ *is an ONB for* V_0*.*

Proof. In Section 10.2 we showed that $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\nu_{\mathbb{Z}})$. Since the range of $\mathcal{R}$ contains $T^k\varphi$, it follows that $\bigcup_{n\in\mathbb{N}_0} U^n \left(\mathcal{R}\left(L^2(\mathbb{T},\lambda)\right)\right)$ is dense in $L^2(\nu_{\mathbb{Z}})$. The result follows then directly from Lemma C.7, Lemma C.8 and Lemma C.9 under application of Theorem C.4. □

To obtain an orthonormal basis for $L^2(\nu_{\mathbb{Z}})$, we construct an orthonormal basis in $\left(L^2(\mathbb{T},\lambda)\right)_\infty$ via Proposition C.5 and map it to $L^2(\nu_{\mathbb{Z}})$ with $\mathcal{R}_\infty$. Recall that in Section 10.2 we have constructed the filter functions m_i, $i \in \underline{N}$, for the definition of the mother wavelets. With these filter functions we can define isometries S_{m_i} by $\left(S_{m_i} f\right)(z) = m_i(z) f\left(z^N\right)$, $i \in \underline{N}\backslash\{0\}$, $f \in L^2(\mathbb{T})$, $z \in \mathbb{T}$, such that $\left(S, (S_{m_i})_{i\in\underline{N}\backslash\{0\}}\right)$ generate the Cuntz algebra $\mathcal{O}_N$, where m_i, $i \in \underline{N}\backslash\{0\}$, are the high-pass filter functions of Section 10.2.

To apply Proposition C.5 to this setting we have to define the Hilbert space L, the unitary representation ρ, and we have to verify the conditions of Proposition C.5. We define L by

$$L := \underbrace{L^2(\mathbb{T},\lambda) \oplus \cdots \oplus L^2(\mathbb{T},\lambda)}_{N-1\,\text{times}}$$

and we give the isometry $F : \, L \to L^2(\mathbb{T},\lambda)$ in the following lemma.

Lemma C.11. *Let* $F : \, L \to L^2(\mathbb{T},\lambda)$ *be given by*

$$F\left(f_1, \ldots, f_{N-1}\right) := S_{m_1} f_1 + \cdots + S_{m_{N-1}} f_{N-1}.$$

This defines a unitary isomorphism of $L^2(\mathbb{T},\lambda) \oplus \cdots \oplus L^2(\mathbb{T},\lambda)$ *onto* $\left(S_{m_0}\left(L^2(\mathbb{T},\lambda)\right)\right)^\perp$*.*

Proof. First notice that F maps to $\left(S_{m_0}\left(L^2(\mathbb{T},\lambda)\right)\right)^\perp$ by the conditions on the filter functions. More precisely, it is well known that

$$L^2(\mathbb{T},\lambda) = S\left(L^2(\mathbb{T},\lambda)\right) \oplus \bigoplus_{i\in\underline{N}\backslash\{0\}} S_{m_i}\left(L^2(\mathbb{T},\lambda)\right)$$

since $SS^* + \sum_{i\in\underline{N}\backslash\{0\}} S_{m_i} S_{m_i}^* = I$. Furthermore the S_{m_i} are pure isometries satisfying $S_{m_i}^* S_{m_j} = \delta_{i,j}$ and $S_{m_i}^* S = 0$, and so it follows that F is a unitary isomorphism with inverse $F^{-1} = \sum_{i\in\underline{N}\backslash\{0\}} S_{m_i}^*$. □

Now we apply the Proposition C.5 with the set

$$B := \left\{\left(\delta_{i,j}\right)_{j\in\underline{N}} : i \in \underline{N}\right\},$$

and $\rho := \underbrace{\varrho \oplus \cdots \oplus \varrho}_{N-1}$, or more precisely $\rho_\gamma = \varrho_\gamma \oplus \cdots \oplus \varrho_\gamma$, $\gamma \in \mathbb{Z}$. It can be easily seen that ρ is a unitary representation on L since ϱ is a unitary representation on $L^2(\mathbb{T})$.

Lemma C.12. *The set* $\{\rho_\gamma l : \, l \in B, \gamma \in \mathbb{Z}\}$ *is an orthonormal basis for* L*.*

Proof. Let $l \in B$, then

$$\rho_\gamma l = (\varrho_\gamma \oplus \cdots \oplus \varrho_\gamma) l = \varrho_\gamma 0 \oplus \cdots \oplus \varrho_\gamma 1 \oplus \varrho_\gamma 0 \oplus \cdots \oplus \varrho_\gamma 0 = 0 \oplus \cdots \oplus z^\gamma \oplus \cdots \oplus 0$$

and this is obviously a basis since $e_\gamma(z) = z^\gamma$, $\gamma \in \mathbb{Z}$, $z \in \mathbb{T}$, is a basis for $L^2(\mathbb{T},\lambda)$. □

Lemma C.13. *It holds that* $F\rho_\gamma = \varrho_{\alpha(\gamma)} F$*,* $\gamma \in \mathbb{Z}$*.*

Proof. Let $z \in \mathbb{T}$ and $l \in B$, $l = (\delta_{i,j})_{j\in\underline{N}}$, $i \in \underline{N}$. First notice that l can be considered as a constant function. Then for $\gamma \in \mathbb{Z}$

$$F\left(\rho_\gamma l\right) = F\left((0, \ldots, 0, \varrho_\gamma, 0, \ldots, 0)\right) = S_{m_i}(z^\gamma) = m_i(z) \cdot z^{N\gamma}$$

and $\varrho_{\alpha(\gamma)}\left(Fl\right) = \varrho_{\alpha(\gamma)}\left(S_{m_i} 1(z)\right) = \varrho_{N\gamma}\left(m_i(z)\right) = m_i(z) \cdot z^{N\gamma}$. □

Corollary C.14. *The set*

$$\left\{S_\infty^{-j}\varrho_\infty(k)\psi:\ j,k\in\mathbb{Z},\ \psi\in U_1F(B)\right\}$$

is an orthonormal basis for $\left(L^2(\mathbb{T},\lambda)\right)_\infty$.

PROOF. In Lemma C.11, Lemma C.12 and Lemma C.13 the hypotheses of Proposition C.5 were checked. So the result follows directly from Proposition C.5. $\square$

As a last step we want to set the orthonormal basis of $\left(L^2(\mathbb{T},\lambda)\right)_\infty$ in correspondence to the wavelet basis for $L^2(\nu_{\mathbb{Z}})$ constructed in Section 10.2, Corollary 10.26. Therefore we apply the isomorphism $\mathcal{R}_\infty$ to the basis of $\left(L^2(\mathbb{T},\lambda)\right)_\infty$.

Proposition C.15. *The wavelet basis*

$$\left\{U^nT^k\psi_i : n,k\in\mathbb{Z}, i\in\underline{N}\backslash\{0\}\right\},$$

given in Chapter 10, Corollary 10.26, coincides with

$$\mathcal{R}_\infty\left(\left\{S_\infty^{-j}\varrho_\infty(k)\psi:\ j,k\in\mathbb{Z},\ \psi\in U_1F(B)\right\}\right).$$

PROOF. We start by noticing that $F(B)=\{m_i:\ i\in\underline{N}\backslash\{0\}\}$. The application of $\mathcal{R}_\infty$ gives for $k,j\in\mathbb{Z}$, $i\in\underline{N}\backslash\{0\}$,

$$\mathcal{R}_\infty\left(S_\infty^{-j}\varrho_\infty(k)\left(U_1m_i\right)\right)=U^j\mathcal{R}_\infty\varrho_\infty(k)\left(U_1m_i\right)=U^jT^k\mathcal{R}_\infty\left(U_1m_i\right),$$

since $\mathcal{R}_\infty S_\infty\mathcal{R}_\infty^*=U^{-1}$ and $\mathcal{R}_\infty\varrho_\infty\mathcal{R}_\infty^*=T$. Furthermore, $\mathcal{R}_\infty\left(U_1m_i\right)=U\mathcal{R}m_i=Um_i(T)\varphi=\psi_i$, $i\in\underline{N}\backslash\{0\}$. $\square$

C.3. Application to wavelet bases on enlarged fractals in two dimensions

In this section we give the application of the direct limit approach to the construction of wavelet bases on enlarged fractals in two dimensions, compare Section 11.3. We assume that $(0,0)\in D$. Since D allows a tiling of $\mathbb{R}^2$, the condition $(0,0)\in D$ is not a restriction.

In two dimensions we have the following ingredients: the countable group is $\Gamma=\overrightarrow{v_1}\mathbb{Z}+\overrightarrow{v_2}\mathbb{Z}$, where $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ give a basis of $\mathbb{R}^2$, and hence $\widehat{\Gamma}=\mathbb{T}^2$. Here the function α is defined on $\overrightarrow{v_1}\mathbb{Z}+\overrightarrow{v_2}\mathbb{Z}$ by $\alpha(k\overrightarrow{v_1}+l\overrightarrow{v_2})=N_1k\overrightarrow{v_1}+N_2l\overrightarrow{v_2}$, $(k,l)\in\mathbb{Z}^2$. For the direct limit approach we consider the Hilbert space $L^2(\mathbb{T}^2,\lambda)$ and a pure isometry on $L^2(\mathbb{T}^2,\lambda)$ is given by $S_{m_0}f(z,w)=m_0(z,w)f\left(z^{N_1},w^{N_2}\right)$, $(z,w)\in\mathbb{T}^2$. The unitary representation $\varrho:\overrightarrow{v_1}\mathbb{Z}+\overrightarrow{v_2}\mathbb{Z}\to\mathcal{W}(L^2(\mathbb{T}^2))$ is given for $(k,l)\in\mathbb{Z}^2$ by $\varrho_{l\overrightarrow{v_1}+k\overrightarrow{v_2}}f(z,w)=z^lw^kf(z,w)$, $(z,w)\in\mathbb{T}^2$.

On the second Hilbert space $L^2(\nu_{\mathbb{Z}_v^2})$ the corresponding operators are $\vartheta_{l,k}=T^{(l,k)}$, $(k,l)\in\mathbb{Z}^2$, and $D=U^{-1}$ defined in Definition 11.10.

We do not give any proofs in this section because they are analogous to the one dimensional results. We only have to consider the respective operators, spaces and unitary representations. First we start with the desired relations for the pure isometry S_{m_0} and the unitary representation ϱ.

Lemma C.16. *The relation* $S_{m_0}\varrho_{l\overrightarrow{v_1}+k\overrightarrow{v_2}}=\varrho_{N_1l\overrightarrow{v_1}+N_2k\overrightarrow{v_2}}S_{m_0}$, $(l,k)\in\mathbb{Z}^2$, *is satisfied.*

Before we can show that the direct limit $\left(\left(L^2(\mathbb{T}^2,\lambda)\right)_\infty,S_\infty,\varrho_\infty\right)$ is isomorphic to $\left(L^2(\nu_{\mathbb{Z}_v^2}),U,T\right)$, we have to give an isometry of $L^2(\mathbb{T}^2,\lambda)$ into $L^2(\mathbb{R}^2,\nu_{\mathbb{Z}_v^2})$ such that the conditions of Theorem C.4 are satisfied.

Corollary C.17. *For* $n,m\in\mathbb{Z}$, *let* $e_{n,m}:(z,w)\mapsto z^nw^m$, $(z,w)\in\mathbb{T}^2$. *Then there is an isometry* $\mathcal{R}$ *of* $L^2(\mathbb{T}^2,\lambda)$ *into* $L^2(\nu_{\mathbb{Z}_v^2})$ *such that* $\mathcal{R}e_{n,m}=T^{(n,m)}\mathbb{1}_C=\mathbb{1}_{C+n\overrightarrow{v_1}+m\overrightarrow{v_2}}$ *for* $n,m\in\mathbb{Z}$.

In the next step we consider the isometry $\mathcal{R}$ and check that it satisfies the conditions of Theorem C.4.

Lemma C.18. *The following relations hold:*

(1) $\mathcal{R}S_{m_0}=U^{-1}\mathcal{R}$,
(2) $\mathcal{R}\varrho_{n\overrightarrow{v_1}+m\overrightarrow{v_2}}=T^{(n,m)}\mathcal{R}$, $(n,m)\in\mathbb{Z}^2$.

Now we apply the Theorem C.4 to this setting.

Corollary C.19. *The direct limit* $\left(\left(L^2(\mathbb{T}^2,\lambda)\right)_\infty, S_\infty, \varrho_\infty\right)$ *is isomorphic to* $\left(L^2(\nu_{\mathbb{Z}_v^2}), U, T\right)$. *With the subspaces*

$$V_n = \operatorname{cl}\operatorname{span}\left\{U^n T^{(k,l)}\varphi : (k,l) \in \mathbb{Z}^2\right\}, n \in \mathbb{Z},$$

$\left(L^2(\nu_{\mathbb{Z}_v^2}), U, T\right)$ *allows a two-sided MRA and* $\{T^{(k,l)}\varphi : (k,l) \in \mathbb{Z}^2\}$ *is an ONB for* V_0.

We turn to the construction of the mother wavelets via the direct limit approach. For this we give a representation of the Cuntz algebra $\mathcal{O}_N$. We define the isometries S_{m_i} for $i \in \underline{N}$ by

$$(S_{m_i}f)(z,w) = m_i(z,w) f\left(z^{N_1}, w^{N_2}\right),$$

where the functions m_i are the high-pass filters (see Chapter 11). These generate the Cuntz algebra $\mathcal{O}_N$. Moreover, the isometries S_{m_i}, $i \in \underline{N}$, are pure.

We now apply Proposition C.5 to this setting. We start with the definition of an isometry $F : \underbrace{L^2(\mathbb{T}^2,\lambda) \oplus \cdots \oplus L^2(\mathbb{T}^2,\lambda)}_{N-1} \to L^2(\mathbb{T}^2,\lambda)$.

Lemma C.20. *Let* $F : L^2(\mathbb{T}^2,\lambda) \oplus \cdots \oplus L^2(\mathbb{T}^2,\lambda) \to L^2(\mathbb{T}^2,\lambda)$ *be given by*

$$F(f_1, \ldots, f_{N-1}) = S_{m_1} f_1 + \cdots + S_{m_N} f_{N-1}.$$

F *defines an unitary isomorphism from* $L^2(\mathbb{T}^2,\lambda) \oplus \cdots \oplus L^2(\mathbb{T}^2,\lambda)$ *onto* $\left(S_{m_0}(L^2(\mathbb{T}^2,\lambda))\right)^\perp$.

Now we apply the Proposition C.5 with the set

$$B = \left\{(\delta_{i,j})_{j\in\underline{N}} : i \in \underline{N}\right\},$$

and the unitary representation $\rho := \underbrace{\varrho \oplus \cdots \oplus \varrho}_{N-1}$, or more precisely the representation for $(\gamma,\delta) \in \mathbb{Z}^2$ given by $\rho_{\gamma\vec{v_1}+\delta\vec{v_2}} = \varrho_{\gamma\vec{v_1}+\delta\vec{v_2}} \oplus \cdots \oplus \varrho_{\gamma\vec{v_1}+\delta\vec{v_2}}$, to this setting.

Lemma C.21.

(1) *The set* $\left\{\rho_{\gamma\vec{v_1}+\delta\vec{v_2}} l : l \in B, (\gamma,\delta) \in \mathbb{Z}^2\right\}$ *is an orthonormal basis for* $L^2(\mathbb{T}^2,\lambda) \oplus \cdots \oplus L^2(\mathbb{T}^2,\lambda)$.
(2) *It holds that* $F\rho_{\gamma\vec{v_1}+\delta\vec{v_2}} = \varrho_{N_1\gamma\vec{v_1}+N_2\delta\vec{v_2}} F$, $(\gamma,\delta) \in \mathbb{Z}^2$.

Now we want to obtain the wavelet basis for $L^2(\nu_{\mathbb{Z}_v^2})$. In the discussion above we checked that all the conditions of Proposition C.5 are satisfied and thus for $\left(L^2(\mathbb{T}^2,\lambda)\right)_\infty$ we get the following orthonormal basis.

Corollary C.22. *The set*

$$\left\{S_\infty^{-j}\varrho_\infty(k,l)\psi : j \in \mathbb{Z}, (k,l) \in \mathbb{Z}^2, \psi \in U_1 F(B)\right\}$$

is an orthonormal basis for $\left(L^2(\mathbb{T}^2,\lambda)\right)_\infty$.

As a last step we want to set the orthonormal basis for $\left(L^2\left(\mathbb{T}^2,\lambda\right)\right)_\infty$ in correspondence to the wavelet basis for $L^2(\nu_{\mathbb{Z}_v^2})$ constructed in Section 11.3.1. Therefore we apply the isomorphism $\mathcal{R}_\infty$ to the basis of $\left(L^2\left(\mathbb{T}^2,\lambda\right)\right)_\infty$.

Proposition C.23. *The wavelet basis*

$$\left\{U^n T^{(k,l)}\psi_i : n \in \mathbb{Z}, (k,l) \in \mathbb{Z}^2, i \in \underline{N}\backslash\{0\}\right\},$$

given in Section 11.3.1, Proposition 11.19, coincides with

$$\mathcal{R}_\infty\left(\left\{S_\infty^{-j}\varrho_\infty(k,l)\psi : j \in \mathbb{Z}, (k,l) \in \mathbb{Z}^2, \psi \in U_1 F(B)\right\}\right).$$

C.4. Remarks on a general Fourier transform

In this section we explain how we can construct a more general Fourier transform for the wavelet construction on enlarged fractals via the direct limit approach. We will only give a general idea and for further details we refer to the literature. To obtain a general Fourier transform we give an alternative realization of the direct limit $\left(L^2(\mathbb{T},\lambda)\right)_\infty$ as a space of functions on the solenoid $\mathcal{S}_N$. This general Fourier transform was first introduced in [**Dut06**] in a direct way and in [**BLP$^+$10**] Baggett et al. gave a different construction via a realization of the direct limit $\left(L^2(\mathbb{T},\lambda)\right)_\infty$.

We explain this construction here because one difficulty in the proofs of the MRA on enlarged fractals is that we do not have a Fourier transform as in the case of an MRA for the L^2-space with respect to the Lebesgue measure. In the case of an MRA with respect to the Lebesgue measure many proofs are given in the frequency space and in the end the results are mapped back to $L^2(\mathbb{R},\lambda)$.

We start with the definition of the solenoid $\mathcal{S}_N$.

Definition C.24. The *solenoid* $\mathcal{S}_N$ for $N \in \mathbb{N}$ is defined as

$$\mathcal{S}_N := \left\{(z_0, z_1, \dots) \in \mathbb{T}^{\mathbb{N}} : z_{n+1}^N = z_n \text{ for } n \geq 0\right\}.$$

Remark C.25.

(1) The solenoid $\mathcal{S}_N$ is actually the dual of the group $\mathbb{Z}[1/N]$.
(2) The solenoid can also be regarded as the inverse limit $\mathcal{S}_N = \varprojlim(\mathbb{T},\, z \mapsto z^N)$.

Now we give a brief explanation how to obtain this general Fourier transform. Let pr_n be the projection from $\mathcal{S}_N$ onto the nth copy of $\mathbb{T}$. Then there exists a unique probability measure τ on $\mathcal{S}_N$ such that for all $f \in \mathcal{C}(\mathbb{T},\mathbb{C})$ we have

$$\int_{\mathcal{S}_N} (f \circ \mathrm{pr}_n) d\tau = \int_{\mathbb{T}} f(z) \left(\prod_{j=0}^{n-1} |m_0(\alpha^j(z))|^2\right) dz = \int_{\mathbb{T}} \frac{1}{N^n} \left(\sum_{w^{N^n}=z} f(w) \left(\prod_{j=0}^{n-1} |m_0(w^{N^j})|^2\right)\right) dz.$$

The existence of this measure follows by results of Kolmogorov, compare Lemma 6.1 and Proposition 6.2 of [**BLP$^+$10**].

Furthermore it is possible to obtain an isomorphism V_∞ from $L^2(\mathcal{S}_N,\tau)$ onto $\left(L^2(\mathbb{T},\lambda)\right)_\infty$ as it is described in [**BLP$^+$10**]. Thus, we obtain an isomorphism from $L^2(\nu_{\mathbb{Z}})$ onto $L^2(S_N,\tau)$ by concatenation. This isomorphism is the inverse of $\mathcal{R}_\infty \circ V_\infty : L^2(S_N,\tau) \to L^2(\nu_{\mathbb{Z}})$, where $\mathcal{R}_\infty : \left(L^2(\mathbb{T},\lambda)\right)_\infty \to L^2(\nu_{\mathbb{Z}})$ is given in Section C.2. Denote the isomorphism from $L^2(\nu_{\mathbb{Z}})$ onto $L^2(S_N,\tau)$ by $\mathcal{F} := (\mathcal{R}_\infty \circ V_\infty)^{-1}$.

This isomorphism satisfies the following relations: Let $\mathrm{pr}_0 : \mathcal{S}_N \to \mathbb{T}$ be the projection onto the first entry, and let θ denote the shift map on $\mathcal{S}_N$, i.e. $\mathrm{pr}_n(\theta(\xi)) = \mathrm{pr}_{n-1}(\xi)$. Then the isomorphism $\mathcal{F}$ satisfies the following relations for $f \in L^2(\mathcal{S}_N,\tau)$ and $\xi \in \mathcal{S}_N$:

(1) $(\mathcal{F}U\mathcal{F}^* f)(\xi) = m_0(\mathrm{pr}_0(\xi)) f(\theta(\xi))$,
(2) $(\mathcal{F}T^k\mathcal{F}^* f)(\xi) = \mathrm{pr}_0(\xi)^k f(\xi)$, $k \in \mathbb{Z}$,
(3) $\mathcal{F}(\mathbb{1}_C) = 1$.

Remark C.26.

(1) For the affine case, compare [**DJ06**], this general Fourier transform coincides with the "Fourier transform" Dutkay constructed in [**Dut06**].
(2) It can be easily checked that if we consider a case that is homeomorphic to the Cantor case, compare [**BK10, Boh09**], the Fourier transform obtained via the direct limit approach coincides with the one that is carried over from the Cantor case constructed by Dutkay in [**Dut06**].
(3) Further investigations of this Fourier transform are made by Baggett, Packer, Merrill and Furst in recent research.

APPENDIX D

Proofs of the MRA for triangles

Proof of Proposition 11.25. This is obvious for the operator U. For the operators T_1 it is obvious, too, and for T_2 realize that:

$$\begin{aligned}\int T_2^{(k,l)} f(\overrightarrow{x}) \cdot g(\overrightarrow{x}) d\nu_{\mathcal{L}_v}(\overrightarrow{x}) &= \int f(R\overrightarrow{x} - kv_1 - lv_2) \cdot g(\overrightarrow{x}) d\nu_{\mathcal{L}_v}(\overrightarrow{x}) \\ &= \int f(\overrightarrow{x}) \cdot g(R(\overrightarrow{x} + kv_1 + lv_2)) d\nu_{\mathcal{L}_v}(\overrightarrow{x}).\end{aligned}$$

□

Proof of Proposition 11.26. ad (1): Notice that these sets can just intersect in a line and thus this intersection has measure 0 with respect to the two-dimensional Lebesgue measure.

ad (2): This follows directly from the equation

$$\varphi(\mathbf{A}^{-1}\overrightarrow{x}) = T_1^{(0,0)}\varphi(\overrightarrow{x}) + T_1^{(1,0)}\varphi(\overrightarrow{x}) + T_1^{(0,1)}\varphi(\overrightarrow{x}) + T_2^{(1,-1)}\varphi(\overrightarrow{x}), \ \ \overrightarrow{x} \in \mathbb{R}^2.$$

ad (3): Let $(k,l) \in \mathbb{Z}^2$, $f \in L^2(\mathbb{R}^2, \lambda)$ and $\overrightarrow{x} \in \mathbb{R}^2$, then

$$U^{-1}T_1^{(k,l)}Uf(\overrightarrow{x}) = f(A(A^{-1}\overrightarrow{x} - k\overrightarrow{v_1} - l\overrightarrow{v_2})) = f(\overrightarrow{x} - 2k\overrightarrow{v_1} - 2l\overrightarrow{v_2}) = T_1^{(2k,2l)}f(\overrightarrow{x})$$

and

$$U^{-1}T_2^{(k,l)}Uf(\overrightarrow{x}) = f(A(RA^{-1}\overrightarrow{x} - k\overrightarrow{v_1} - l\overrightarrow{v_2})) = f(R\overrightarrow{x} - 2k\overrightarrow{v_1} - 2l\overrightarrow{v_2}) = T_2^{(2k,2l)}f(\overrightarrow{x}).$$

□

Proof of Theorem 11.28. We have to proof the six properties as they can be found in Remark 1.4. Let for $j \in \mathbb{Z}$

$$V_j := \operatorname{cl}\operatorname{span}\left\{U^j T_i^{(k,l)}\varphi : (k,l) \in \mathbb{Z}^2, i \in \{1,2\}\right\}.$$

ad (4): By the definition of the closed subspaces V_j it is satisfied that $\left\{T_i^{(k,l)}\varphi : (k,l) \in \mathbb{Z}^2, i \in \{1,2\}\right\}$ spans V_0. By Proposition 11.26 (1) it is also satisfied that it is an orthonormal basis.

ad (6): This was shown in Proposition 11.26 (3).

ad (1): Notice that $\varphi = U(m_0^1(T_1)\varphi + m_0^2(T_2)\varphi)$ and $U^{-1}T_1^{(k,l)}U = T_1^{(2k,2l)}$ and $U^{-1}T_2^{(k,l)}U = T_2^{(2k,2l)}$. Thus

$$U^j T_i^{(k,l)}\varphi = U^j T_i^{(k,l)} U(m_0^1(T_1)\varphi + m_0^2(T_2)\varphi) = U^{j+1}T_i^{(2k,2l)}(m_0^1(T_1)\varphi + m_0^2(T_2)\varphi.$$

Furthermore

$$\begin{aligned}T_1^{(k,l)}T_2^{(n,m)}f(x) &= f(R(x - kv_1 - lv_2) - nv_1 - mv_2) \\ &= f(Rx - (k+n-l)v_1 - (-l+m)v_2) \\ &= T_2^{(k+n-l,-l+m)}f(x)\end{aligned}$$

and

$$T_2^{(k,l)}T_1^{(n,m)}f(x) = f(Rx - kv_1 - lv_2 - nv_1 - mv_2) = T_2^{(k+n,l+m)}f(x)$$

since $Rv_1 = v_1$ and $Rv_2 = -v_2 + v_1$. Thus $V_j \subset V_{j+1}$.

ad (5): This follows from the definition of V_j.

ad (3): Clearly we have $0 \in \bigcap_{j\in\mathbb{Z}} V_j$. Let $f \in \bigcap_{j\in\mathbb{Z}} V_j$, $f \neq 0$, then $f \in V_j$ for all $j \in \mathbb{Z}$. So for all $j \in \mathbb{Z}$ there exist $(k,l) \in \mathbb{Z}^2$ such that $f|_{\mathbf{A}^j\left(\triangle + k\overrightarrow{v_1} + l \in \overrightarrow{v_2}\right)} = c\mathbb{1}_{\mathbf{A}^j\left(\triangle + k\overrightarrow{v_1} + l \in \overrightarrow{v_2}\right)}$, $c \neq 0$. But $\lambda\left(\mathbf{A}^j\left(\triangle + k\overrightarrow{v_1} + l \in \overrightarrow{v_2}\right)\right) \to \infty$, $j \to \infty$. Since $f \in L^2(\mathbb{R}^2, \lambda)$, it follows that $f = 0$.

ad (2): We have to show that for any $f \in L^2(\mathbb{R}^2,\lambda)$ $\lim_{j\to\infty} || \operatorname{pr}_{V_j} f - f|| = 0$. We will show this for characteristic functions on intervals. So realize that

$$\begin{aligned}
||\mathbb{1}_{[a,b]\times[c,d]}||^2 &= (b-a)(d-c) \\
&= || \operatorname{pr}_{V_j} \mathbb{1}_{[a,b]\times[c,d]} - \mathbb{1}_{[a,b]\times[c,d]} - \operatorname{pr}_{V_j} \mathbb{1}_{[a,b]\times[c,d]}||^2 \\
&= ||(\operatorname{pr}_{V_j} - I)\mathbb{1}_{[a,b]\times[c,d]} - \operatorname{pr}_{V_j} \mathbb{1}_{[a,b]\times[c,d]}||^2 \\
&= ||(\operatorname{pr}_{V_j} - I)\mathbb{1}_{[a,b]\times[c,d]}||^2 + || \operatorname{pr}_{V_j} \mathbb{1}_{[a,b]\times[c,d]}||^2
\end{aligned}$$

and so $||(\operatorname{pr}_{V_j} - I)\mathbb{1}_{[a,b]\times[c,d]}||^2 = (b-a)(d-c) - || \operatorname{pr}_{V_j} \mathbb{1}_{[a,b]\times[c,d]}||^2$. Thus in the following we will show that $\lim_{j\to\infty} || \operatorname{pr}_{V_j} \mathbb{1}_{[a,b]\times[c,d]}||^2 = (b-a)(d-c)$ for all $j \in \mathbb{Z}$. We clearly have that $\lim_{j\to\infty} || \operatorname{pr}_{V_j} \mathbb{1}_{[a,b]\times[c,d]}||^2 \le (b-a)(d-c)$ since $|| \operatorname{pr}_{V_j} \mathbb{1}_{[a,b]\times[c,d]}||^2 \le (b-a)(d-c)$. Furthermore,

$$\begin{aligned}
|| \operatorname{pr}_{V_j} \mathbb{1}_{[a,b]\times[c,d]}||^2 &= \sum_{(k,l)\in\mathbb{Z}^2, i\in\{1,2\}} |\langle \mathbb{1}_{[a,b]\times[c,d]} | U^j T_i^{(k,l)} \varphi\rangle|^2 \\
&\ge \left((b-a)2^j \cdot 2 \cdot (d-c)2^{j-1} - (b-a)2^j \cdot 4 - 2 \cdot (d-c)2^{j-1}\right) \cdot (4^j \cdot 16^{-j}) \\
&= (b-a)\cdot(d-c) - (b-a)2^{-j} \cdot 4 - (d-c)2^{-j} \\
&\to (b-a)\cdot(d-c),\ j \to \infty,
\end{aligned}$$

since there are minimal $\lceil (b-a)2^j \rceil \cdot 2 \cdot \lceil (d-c)2^{j-1} \rceil$ $(k,l) \in \mathbb{Z}^2$, such that $\langle \mathbb{1}_{[a,b]\times[c,d]} | U^j T_i^{(k,l)} \varphi\rangle \neq 0$ for $i \in \{1,2\}$ and the $(k,l) \in \mathbb{Z}^2$, $i \in \{1,2\}$, with $\operatorname{supp}\left(U^j T_i^{(k,l)}\varphi\right) \not\subseteq [a,b]\times[c,d]$ are omitted in the calculation above. There can be maximal $(b-a)2^j \cdot 4 + 2 \cdot (d-c)2^{j-1}$ which satisfy this. If $\operatorname{supp}\left(U^j T_i^{(k,l)}\varphi\right) \subset [a,b]\times[c,d]$, then $|\langle \mathbb{1}_{[a,b]\times[c,d]} | U^j T_i^{(k,l)} \varphi\rangle|^2 = \left(2^j \cdot 4^{-j}\right)^2$. $\square$

APPENDIX E

Appendix for image compression

Recall that the coefficients for the filter functions are given in a unitary $(N \times N)$-matrix $M = (M_{ij})_{i\in\underline{N},j\in\underline{N}}$, with $N = N_1N_2$.

E.1. Coefficients for the comparison of different Hausdorff dimensions

We include here the coefficients of the filter functions for the comparison of the reconstruction results for fractals with different Hausdorff dimensions.

The matrix M has the following coefficients of the filter functions for the underlying fractal of Hausdorff dimension 1: $M_{ij} = \frac{1}{2}$ for $(i,j) \in \{(0,0),(0,3),(0,12),(0,15),(15,0),(15,3)\}$, $M_{ij} = \frac{-1}{2}$ for $(i,j) \in \{(15,12),(15,15)\}$, $M_{ij} = \frac{1}{\sqrt{2}}$ for $(i,j) \in \{(13,0),(14,12)\}$, $M_{ij} = \frac{-1}{\sqrt{2}}$ for $(i,j) \in \{(13,3),(14,15)\}$, $M_{ij} = 1$ for $(i,j) \in \big\{(1,1),(2,2),(3,4),(4,5),(5,6),(6,7),(7,8),(8,9),(9,10),(10,11),(11,13),(12,14)\big\}$, $M_{ij} = 0$ otherwise.

The matrix M has the following coefficients of the filter functions for the underlying fractal of Hausdorff dimension $\log(2)/\log(3)$: $M_{ij} = \frac{1}{3}$ for $(i,j) \in \{0\} \times \{0,1,2,3,4,6,8,9,12\}$, $M_{ij} = \frac{1}{\sqrt{2}}$ for $(i,j) \in \{(7,0),(8,4),(10,9),(11,3)\}$, $M_{ij} = \frac{-1}{\sqrt{2}}$ for $(i,j) \in \{(7,2),(8,6),(10,12),(11,1)\}$, $M_{ij} = \frac{1}{2}$ for $(i,j) \in \{(12,9,),(12,12),(13,4),(13,6)\}$, $M_{ij} = \frac{-1}{2}$ for $(i,j) \in \{(12,1),(12,3),(13,0),(13,2\}$, $M_{ij} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{0,2,4,6\}$, $M_{ij} = \frac{-1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{1,3,9,12\}$, $M_{ij} = \frac{1}{\sqrt{72}}$ for $(i,j) \in \{15\} \times \{0,1,2,3,4,6,9,12\}$, $M_{ij} = \frac{-8}{\sqrt{72}}$ for $(i,j) = (15,8)$, $M_{ij} = 1$ for $(i,j) \in \big\{(1,7),(2,11),(3,5),(4,14),(5,10),(6,13),(9,15)\big\}$ and $M_{ij} = 0$ otherwise.

The matrix M has the following coefficients of the filter functions for the underlying fractal of the Hausdorff dimension $\log(10)/\log(4)$: $M_{ij} = \frac{1}{\sqrt{10}}$ for $(i,j) \in \{0\} \times \{0,2,4,6,7,9,11,12,14,15\}$, $M_{ij} = \frac{1}{\sqrt{2}}$ for $(i,j) \in \{(7,0),(8,4),(9,7),(10,11),(11,14)\}$, $M_{ij} = \frac{-1}{\sqrt{2}}$ for $(i,j) \in \{(7,2),(8,6),(9,9),(10,12),(11,15)\}$, $M_{ij} = \frac{1}{2}$ for $(i,j) \in \{(12,11),(12,12),(13,4),(13,6)\}$, $M_{ij} = \frac{-1}{2}$ for $(i,j) \in \{(12,14),(12,15),(13,0),(13,2)\}$, $M_{ij} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{0,2,4,6\}$, $M_{ij} = \frac{-1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{11,12,14,15\}$, $M_{ij} = \frac{1}{\sqrt{40}}$ for $(i,j) \in \{15\} \times \{0,2,4,6,11,12,14,15\}$, $M_{ij} = \frac{-4}{\sqrt{40}}$ for $(i,j) \in \{15\} \times \{7,9\}$, $M_{ij} = 1$ for $(i,j) \in \{(1,1),(2,3),(3,5),(4,8),(5,10),(6,13)\}$ and $M_{ij} = 0$ otherwise.

The matrix M has the following coefficients of the filter functions for the underlying fractal of Hausdorff dimension $\log(11)/\log(4)$: $M_{ij} = \frac{1}{\sqrt{11}}$ for $(i,j) \in \{0\} \times \{0,1,2,3,4,6,7,8,9,12,15\}$, $M_{ij} = \frac{1}{\sqrt{110}}$ for $(i,j) \in \{1\} \times \{0,1,2,3,4,6,8,9,12,15\}$, $M_{ij} = \frac{-10}{\sqrt{110}}$ for $(i,j) = (1,7)$, $M_{ij} = \frac{1}{2}$ for $(i,j) \in \{(12,9),(12,12),(13,4),(13,6)\}$, $M_{ij} = \frac{-1}{2}$ for $(i,j) \in \{(12,1),(12,2),(13,0),(13,2)\}$, $M_{ij} = \frac{1}{\sqrt{2}}$ for $(i,j) \in \{(7,0),(8,4),(9,8),(10,9),(11,3)\}$, $M_{ij} = \frac{-1}{\sqrt{2}}$ for $(i,j) \in \{(7,2),(8,6),(9,15),(10,12),(11,1)\}$, $M_{i,j} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{0,2,4,6\}$, $M_{ij} = \frac{-1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{1,3,9,12\}$, $M_{ij} = \frac{1}{\sqrt{40}}$ for $(i,j) \in \{15\} \times \{0,1,2,3,4,6,9,12,15\}$, $M_{ij} = \frac{-4}{\sqrt{40}}$ for $(i,j) \in \{15\} \times \{8,15\}$, $M_{ij} = 1$ for $(i,j) \in \{(2,11),(3,5),(4,14),(5,10),(6,13)\}$ and $M_{ij} = 0$ otherwise.

The matrix M has the following coefficients of the filter functions for the underlying fractal of Hausdorff dimension $\log(13)/\log(4)$: $M_{ij} = \frac{1}{\sqrt{13}}$ for $(i,j) \in \{0\} \times \{0,1,2,3,4,5,6,8,9,10,11,12,14\}$, $M_{ij} = \frac{1}{\sqrt{2}}$ for $(i,j) \in \{(4,0),(5,2),(6,4),(7,6),(8,9)(9,11)\}$, $M_{ij} = \frac{-1}{\sqrt{2}}$ for $(i,j) \in \big\{(4,1),(5,3),(6,5),(7,8),(8,10),(9,13)\big\}$ $M_{ij} = \frac{1}{2}$ for $(i,j) \in \{(10,0),(10,1),(11,4),(11,5),(12,9),(12,10)\}$, $M_{ij} = \frac{-1}{2}$ for $(i,j) \in \{(10,2),(10,3),(11,6),(11,8),(12,11),(12,12)\}$, $M_{ij} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \{13\} \times \{0,1,2,3\}$, $M_{ij} = \frac{-1}{\sqrt{8}}$ for $(i,j) \in \{13\} \times \{4,5,6,8\}$, $M_{ij} = \frac{1}{\sqrt{20}}$ for $(i,j) \in \{14\} \times \{9,10,11,12\}$, $M_{ij} = \frac{-4}{\sqrt{20}}$ for $(i,j) = (14,14)$, $M_{ij} = \sqrt{\frac{5}{104}}$ for $(i,j) \in \{15\} \times \{0,1,2,3,4,5,6,8\}$, $M_{ij} = \frac{-8}{\sqrt{520}}$ for $(i,j) \in \{15\} \times \{9,10,11,12,14\}$, $M_{ij} = 1$ for $(i,j) \in \{(1,7),(2,13),(3,15)\}$ and $M_{ij} = 0$ otherwise.

The matrix M has the following coefficients of the filter functions for the underlying fractal of Hausdorff dimension 2: $M_{ij} = \frac{1}{4}$ for $(i,j) \in \{0\} \times \underline{16}$ and $M_{ij} = \frac{1}{4}$ for $(i,j) \in \{15\} \times \underline{8}$, $M_{ij} = \frac{-1}{4}$ for $(i,j) \in \{15\} \times \{8,9,10,11,12,13,14,15\}$, $M_{ij} = \frac{1}{\sqrt{2}}$ for $(i,j) \in \big\{(1,0),(2,2),(3,4),(4,6),(5,8),(6,10),(7,14),(8,12)\big\}$, $M_{ij} = \frac{-1}{\sqrt{2}}$ for $(i,j) \in \{(1,1),(2,3),(3,5),(4,7),(5,9),(6,11),(7,15),(8,13)\}$, $M_{ij} = \frac{1}{2}$ for $(i,j) \in \{(9,2),(9,3),(10,6),(10,7),(11,10),(11,11),(12,14),(12,15)\}$, $M_{ij} = \frac{-1}{2}$ for $(i,j) \in \big\{(9,0),(9,1),(10,4),(10,5),(11,8),(11,9),(12,12),(12,13)\big\}$, $M_{ij} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \big\{(13,0),(13,1),(13,2),(13,3),(14,8),(14,9),(14,10),(14,11)\big\}$, $M_{ij} = \frac{-1}{\sqrt{8}}$ for $(i,j) \in \big\{(13,4),(13,5),(13,6),(13,7),(14,12),(14,13),(14,14),(14,15)\big\}$ and $M_{ij} = 0$ otherwise.

Now we turn to the coefficients for different fractals of the same Hausdorff dimension. The fractals have the Hausdorff dimension $\log(8)/\log(4)$ and on the fractal the invariant measure is considered. So the different filter functions are just permuted in different ways.

The matrix M has the following coefficients of the filter functions for the first underlying fractal: $M_{ij} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \{0\} \times \{0,2,4,7,9,11,12,15\}$ and $M_{ij} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{0,2,4,7\}$, $M_{ij} = \frac{-1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{9,11,12,15\}$, $M_{ij} = \frac{1}{\sqrt{2}}$ for $(i,j) \in \{(8,0),(9,4),(10,11),(11,12)\}$, $M_{ij} = \frac{-1}{\sqrt{2}}$ for $(i,j) \in \{(8,2),(9,7),(10,9),(11,15)\}$, $M_{ij} = \frac{1}{2}$ for $(i,j) \in \{(12,9),(12,11),(13,4),(13,7)\}$, $M_{ij} = \frac{-1}{2}$ for $(i,j) \in \{(12,12),(12,15),(13,0),(13,2)\}$, $M_{ij} = 1$ for $(i,j) \in \big\{(1,1),(2,3),(3,5),(4,8),(5,10),(6,13),(7,14),(15,6)\big\}$ and $M_{ij} = 0$ otherwise.

The matrix M has the following coefficients of the filter functions for the second underlying fractal: $M_{ij} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \{0\} \times \{0,1,2,4,8,9,11,12,15\}$ and $M_{ij} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{0,2,4,8\}$, $M_{ij} = \frac{-1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{1,9,12,15\}$, $M_{ij} = \frac{1}{\sqrt{2}}$ for $(i,j) \in \{(8,0),(9,4),(10,9),(11,12)\}$, $M_{ij} = \frac{-1}{\sqrt{2}}$ for $(i,j) \in \{(8,2),(9,8),(10,15),(11,1)\}$, $M_{ij} = \frac{1}{2}$ for $(i,j) \in \{(12,9),(12,15),(13,4),(13,8)\}$, $M_{ij} = \frac{-1}{2}$ for $(i,j) \in \{(12,1),(12,12),(13,0),(13,2)\}$, $M_{ij} = 1$ for $(i,j) \in \big\{(1,7),(2,11),(3,5),(4,14),(5,10),(6,13),(7,3),(15,6)\big\}$ and $M_{ij} = 0$ otherwise.

The matrix M has the following coefficients of the filter functions for the third underlying fractal: $M_{ij} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \{0\} \times \{0,1,2,6,8,9,12,14\}$ and $M_{ij} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{0,1,6,8\}$, $M_{ij} = \frac{-1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{2,9,12,14\}$, $M_{ij} = \frac{1}{\sqrt{2}}$ for $(i,j) \in \{(8,0),(9,6),(10,12),(11,9)\}$, $M_{ij} = \frac{-1}{\sqrt{2}}$ for $(i,j) \in \{(8,1),(9,8),(10,14),(11,2)\}$, $M_{ij} = \frac{1}{2}$ for $(i,j) \in \{(12,12),(12,14),(13,6),(13,8)\}$, $M_{ij} = \frac{-1}{2}$ for $(i,j) \in \{(12,2),(12,9),(13,0),(13,1)\}$, $M_{ij} = 1$ for $(i,j) \in \big\{(1,7),(2,10),(3,4),(4,15),(5,11),(6,13),(7,3),(15,5)\big\}$ and $M_{ij} = 0$ otherwise.

The matrix M has the following coefficients of the filter functions for the fourth underlying fractal: $M_{ij} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \{0\} \times \{1,2,3,9,10,12,13,14\}$ and $M_{ij} = \frac{1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{2,3,10,14\}$, $M_{ij} = \frac{-1}{\sqrt{8}}$ for $(i,j) \in \{14\} \times \{1,9,12,13\}$, $M_{ij} = \frac{1}{\sqrt{2}}$ for $(i,j) \in \{(8,14),(9,3),(10,9),(11,12)\}$, $M_{ij} =$

$\frac{-1}{\sqrt{2}}$ for $(i,j) \in \{(8,2),(9,10),(10,13),(11,1)\}$, $M_{ij} = \frac{1}{2}$ for $(i,j) \in \{(12,9),(12,13),(13,3),(13,10)\}$, $M_{ij} = \frac{-1}{2}$ for $(i,j) \in \{(12,1),(12,12),(13,2),(13,14)\}$, $M_{ij} = 1$ for $(i,j) \in \big\{(1,7),(2,8),(3,5),(4,15),(5,11),(6,0),(7,4),(15,6)\big\}$ and $M_{ij} = 0$ otherwise.

E.2. Different levels of decomposition

We consider different levels of decomposition for the Sierpinski Gasket wavelet applied to the "Lena" image. We set 80% of the coefficients to zero. The reconstructed images for the decomposition levels 1 to 9 are in Figure E.2.1. We do not see any difference between the images E.2.1 (D) to (I), also they have slightly different PSNR.

E.3. Further results for the Sierpinski Gasket – Hard versus soft threshold

Here we consider the two different threshold procedures and compare their results. We apply the wavelet basis based on the Sierpinski Gasket and the Haar wavelet basis to the "Lena" image. We consider 9 levels of decomposition and we set 80% of the values after decomposition to zero. Then the resulting reconstructed images for the Sierpinski Gasket wavelet are as shown in Figure E.3.1. For the reconstruction with the Haar wavelet we obtain for the hard threshold a PSNR of 41.57 dB and for the soft threshold 37.42 dB.

E.4. Coefficients for different Cantor Dust wavelets

Here we give the coefficients of the filter functions for the different wavelet bases based on the Cantor Dust.

For Cantor Dust 1 we consider on the fractal the measure of maximal entropy and consider the following matrix M containing the coefficients of the filter functions: $M_{ij} = \frac{1}{2}$ for $(i,j) \in \{(0,0),(0,2),(0,6),(0,8),(2,0),(2,2),(4,0),(4,6),(8,0),(8,8)\}$, $M_{ij} = \frac{-1}{2}$ for $(i,j) \in \big\{(2,6),(2,8),(4,2),(4,8),(8,2),(8,6)\big\}$, $M_{ij} = \frac{1}{\sqrt{2}}$ for $(i,j) \in \{(1,3),(1,5),(3,1),(3,7),(6,1),(7,3)\}$, $M_{ij} = \frac{-1}{\sqrt{2}}$ for $(i,j) \in \{(6,7),(7,5)\}$, $M_{ij} = 1$ for $(i,j) = (5,4)$ and $M_{ij} = 0$ otherwise.

For Cantor Dust 2 we consider the measure given by Hutchinson's theorem with the weights $\left(\frac{7}{16}, \frac{1}{16}, \frac{1}{16}, \frac{7}{16}\right)$ and choose the following matrix M containing the coefficients of the filter functions: $M_{ij} = \frac{\sqrt{7}}{4}$ for $(i,j) \in \{(0,0),(0,8),(1,6),(2,0)\}$, $M_{ij} = \frac{-\sqrt{7}}{4}$ for $(i,j) \in \{(1,2),(2,8),(3,2),(3,6)\}$, $M_{ij} = \frac{1}{4}$ for $(i,j) \in \{(0,2),(0,6),(1,0),(2,2),(3,0),(3,8)\}$, $M_{ij} = \frac{-1}{4}$ for $(i,j) \in \{(1,8),(2,6)\}$, $M_{ij} = 1$ for $(i,j) \in \{(4,1),(5,3),(6,4),(7,5),(8,7)\}$ and $M_{ij} = 0$ otherwise.

For Cantor Dust 3 we consider the measure given by Hutchinson's theorem with the weights $\left(\left(\frac{707}{1000}\right)^2, \frac{1}{10000}, \frac{1}{10000}, \left(\frac{707}{1000}\right)^2\right)$ and choose the following matrix M containing the coefficients of the filter functions: $M_{ij} = \frac{707}{1000}$ for $(i,j) \in \{(0,0),(0,8),(1,6)\}$, $M_{ij} = \frac{-707}{1000}$ for $(i,j) \in \{(1,2),(3,2),(3,6)\}$, $M_{ij} = \frac{1}{100}$ for $(i,j) \in \{(0,2),(0,6),(1,0),(2,2),(3,0),(3,8)\}$, $M_{ij} = \frac{-1}{100}$ for $(i,j) \in \{(1,8),(2,6)\}$, $M_{ij} = 1$ for $(i,j) \in \{(4,1),(5,3),(6,4),(7,5),(8,7)\}$ and $M_{ij} = 0$ otherwise.

For Cantor Dust 4 we consider the measure given by Hutchinson's theorem with the weights $\left(\frac{1}{16}, \frac{7}{16}, \frac{7}{16}, \frac{1}{16}\right)$. These are the same weights as for Cantor Dust 2 only in a different order. We choose also the coefficients of the filter functions a permutation of the ones of Cantor Dust 2, namely: $M_{ij} = \frac{1}{4}$ for $(i,j) \in \{(0,0),(0,8),(1,6),(2,0)\}$, $M_{ij} = \frac{-1}{4}$ for $(i,j) \in \{(1,2),(2,8),(3,2),(3,6)\}$, $M_{ij} = \frac{\sqrt{7}}{4}$ for $(i,j) \in \{(0,2),(0,6),(1,0),(2,2),(3,0),(3,8)\}$, $M_{ij} = \frac{-\sqrt{7}}{4}$ for $(i,j) \in \{(1,8),(2,6)\}$, $M_{ij} = 1$ for $(i,j) \in \{(4,1),(5,3),(6,4),(7,5),(8,7)\}$ and $M_{ij} = 0$ otherwise.

For Cantor Dust 5 we consider again a permutation of the filter functions of Cantor Dust 2. Here the measure is given by Hutchinson's theorem with the weights $\left(\frac{7}{16}, \frac{7}{16}, \frac{1}{16}, \frac{1}{16}\right)$. These are

(A) Level 1, PSNR = 16.06, 2.3 seconds.

(B) Level 2, PSNR = 16.06, 2.86 seconds.

(C) Level 3, PSNR = 20.06, 3 seconds.

(D) Level 4, PSNR = 21.85, 3.08 seconds.

(E) Level 5, PSNR = 21.85, 3.09 seconds.

(F) Level 6, PSNR = 22.04, 3.05 seconds.

(G) Level 7, PSNR = 22.04, 3.11 seconds.

(H) Level 8, PSNR = 22.04, 3.18 seconds.

(I) Level 9, PSNR = 22.04, 3.09 seconds.

FIGURE E.2.1. Reconstructed image for wavelet based on the Sierpinski Gasket for different levels.

the same weights as for Cantor Dust 2 only in a different order. We choose also the coefficients of the filter functions a permutation of the ones of Cantor Dust 2, namely: $M_{ij} = \frac{\sqrt{7}}{4}$ for $(i,j) \in \{(0,0),(0,2),(1,6),(2,2)\}$, $M_{ij} = \frac{-\sqrt{7}}{4}$ for $(i,j) \in \{(1,8),(2,0),(3,6),(3,8)\}$, $M_{ij} = \frac{1}{4}$ for $(i,j) \in \{(0,6),(0,8),(1,0),(2,6),(3,0),(3,2)\}$, $M_{ij} = \frac{-1}{4}$ for $(i,j) \in \{(1,2),(2,8)\}$, $M_{ij} = 1$ for $(i,j) \in \{(4,1),(5,3),(6,4),(7,5),(8,7)\}$ and $M_{ij} = 0$ otherwise.

For Cantor Dust 6 we consider the last possible permutation of the weights for the measure given by Hutchinson's theorem of Cantor Dust 2. Namely we consider the weights $\left(\frac{1}{16}, \frac{1}{16}, \frac{7}{16}, \frac{7}{16}\right)$. We choose also the coefficients of the filter functions a permutation of the ones of Cantor Dust 2, namely: $M_{ij} = \frac{1}{4}$

(A) Hard threshold: PSNR: 22.04.

(B) Soft threshold: PSNR: 15.82.

FIGURE E.3.1. Results for the hard and the soft threshold; application to "Lena" image.

for $(i,j) \in \{(0,0),(0,2),(1,6),(2,0)\}$, $M_{ij} = \frac{-1}{4}$ for $(i,j) \in \{(1,8),(2,2),(3,6),(3,8)\}$, $M_{ij} = \frac{\sqrt{7}}{4}$ for $(i,j) \in \{(0,6),(0,8),(1,0),(2,8),(3,0),(3,2)\}$, $M_{ij} = \frac{-\sqrt{7}}{4}$ for $(i,j) \in \{(1,2),(2,6)\}$, $M_{ij} = 1$ for $(i,j) \in \{(4,1),(5,3),(6,4),(7,5),(8,7)\}$ and $M_{ij} = 0$ otherwise.

APPENDIX F

Code for the image compression

We give the code for the compression with the Sierpinski Gasket or the Haar wavelet. For the Cantor Dust the code is analogous, only the size of the image has to be changed and we have to consider the respective matrix containing the coefficients of the filter functions. We start with the code where we choose between the wavelet basis for the Haar wavelet and the Sierpinski Gasket wavelet. We also choose the percentage of coefficients, the threshold procedure and the levels of decomposition there. For the actual procedure that is applied to the image and does the compression, there is only a placeholder and the procedure is given in the next step.

```
% This program is to compress images by using different filter function
clear all
close all
% Input parameters for the user
m = input('Enter the decomposition level : ');
p = input('Enter the percentage of coefficients that should be set to zero : ');
q = input('Enter the wavelet that should be used, 0 for Haar, 1 for Sierpisnki Gasket :');
s = input('Soft or hard threshold, 0 for soft and 1 for hard :');
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

% Imread of the image that shall be compressed
A = imread('lena.bmp'); % The image must have the size N1^n x N2^n
                        % (i.e. the size and the filter functions must correspond)

A = im2double(A);
K = A; % copy the image for a comparison between the original
       % and the compressed image

n = 9; % Maximal number of decomposition levels
       % dependent on the size of the image and the chosen wavelets
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Definition of the different wavelets in terms of their filter functions. They are
% given as the vectors for the coefficients in the order
% (a(0,0),a(0,1),...,a(0,N1),a(1,0),...a(N1,N2)).
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Haar Wavelet
N1Haar = 2;
N2Haar = 2;

v0Haar = [1/2;1/2;1/2;1/2];
v1Haar = [1/2;1/2;-1/2;-1/2];
v2Haar = [1/2;-1/2;1/2;-1/2];
v3Haar = [1/2;-1/2;-1/2;1/2];

% Definition of the matrix M consisting of the vectors vi
MHaar = [transpose(v0Haar); transpose(v1Haar); transpose(v2Haar); transpose(v3Haar)];
```

```
clear v0Haar v1Haar v2Haar v3Haar
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Sierpinski Gasket
N1SG = 2;
N2SG = 2;

v0SG = [1/sqrt(3); 1/sqrt(3); 1/sqrt(3); 0];
v1SG = [0; 0; 0; 1];
v2SG = [1/sqrt(2); -1/sqrt(2); 0; 0];
v3SG = [-1/sqrt(6); -1/sqrt(6); 2/sqrt(6); 0];

% Definition of the matrix M consisting of the vectors vi
MSG = [transpose(v0SG); transpose(v1SG); transpose(v2SG); transpose(v3SG)];
clear v0SG v1SG v2SG v3SG

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
if q == 0 && m <= n;
  N1 = N1Haar;
  N2 = N2Haar;
  N = N1*N2;
  M = MHaar;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Do compression procedure %%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
elseif q == 1 && m <= n;
  N1 = N1SG;
  N2 = N2SG;
  N = N1*N2;
  M = MSG;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%% Do compression procedure %%
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
else
  break
end
```

Now the coefficients that we need are given and the compression algorithm can be applied.

```
% Decompostion of the matrix A
% Initialisation of matrices for the decomposition
B = zeros(N2,N1);
C = zeros(N,1);
D = zeros(N,1);
A1 = zeros(N2^n,N1^n);

for v = 0:m-1 % m decomposition levels
  for i = 0:(N1^(n-1-v)-1) % number of submatrizes
    for j = 0:(N2^(n-1-v)-1)
      for k = 1:N1
        for l = 1:N2
          B(N2-l+1,k) = A(l+N2*j,N1*i+k); % Taking the submatrices
        end
      end
```

```
      C = reshape(transpose(B),N,1); % writing the submatrix in a vector
      D = M*C; % Multiplying the vector with the matrix M

      for w = 0:N2-1
        for u = 1:N1
          A1(j+1+w*N2^(n-1-v),i+1+(u-1)*N1^(n-1-v)) = D(N1*w+u); % writing the decomposed
                                                                  % values in the matrix A1
        end
      end
    end
  end
   A = A1;
end

% Thresholding for less coefficients
t = prctile(reshape(abs(A),1,N^n),p);
% soft threshold
if s == 0;
  for i = 1:N1^(n)
    for j = 1:N2^(n)
      if abs(A(j,i)) < t
         A(j,i) = 0;
      elseif A(j,i) >= t
         A(j,i) = A(j,i)-t;
      else
         A(j,i) = A(j,i)+t;
      end
   end
  end

% hard threshold
elseif s == 1;
  for i = 1:N1^(n)
    for j = 1:N2^(n)
      if abs(A(j,i)) < t
         A(j,i) = 0;
      end
    end
  end
elso
  'no valid treshold option'
  break;
end

% Reconstruction of the image
C = zeros(N,1);
A1 = A;
for v = 1:m % different levels of decomposition
  for i = 1:N1^(n-m+v-1)
    for j = 1:N2^(n-m+v-1)
      for k = 0:N1-1
        for l = 0:N2-1
          C(N1*l+k+1) = A(j+l*N2^(n-m+v-1),i+k*N1^(n-m+v-1)); % taking submatrices and
                                                               % writing these as vectors
        end
      end
```

```
      b = transpose(M)*C; % multiplication of these vectors with M^(-1)
       for u = 0:N2-1
         for w = 1:N1
           A1(N2-u+(j-1)*N2,w+(i-1)*N1) = b(N1*u+w); % writing the reconstructed values
                                                      % in the matrix A1
         end
       end
     end
   end
 A = A1;
end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Construction of the compressed image after reconstruction
imagesc(A)
colormap(gray)
axis square
axis off

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
% Calculation of the PSNR
D = A-K;
D = D.^2;
MSE = 1/(length(A(:,1))*length(A(:,2)))*sum(sum(D));
PSNR = 10*log10(1/MSE)
```

If we use other wavelet bases, the algorithm stays the same; but the coefficients for the filter functions are different and if N_1 and N_2 are different, we have to consider an image of a different size.

Index

Nomenclature

aIFS	affine iterated function system
IFS	iterated function system
MIM	Markov Interval Map
GDMS	Graph directed Markov system
OSC	open set condition
ONB	orthonormal basis
MRA	multiresolution analysis
PSNR	peak-signal-to-noise ratio
a.e.	almost everywhere
dB	decibel
$\mathbb{N}$	natural numbers
$\mathbb{N}_0$	natural numbers including 0
$\mathbb{Z}$	integers
$\mathbb{R}$	real numbers
$\mathbb{R}^+$	$x \in \mathbb{R} : x \geq 0$
$\mathbb{C}$	complex numbers
$\mathbb{T}$	one-dimensional torus $\mathbb{T} = \{\omega \in \mathbb{C} :\mid \omega \mid = 1\}$
$\mathbb{T}^2$	two-dimensional torus, $\mathbb{T}^2 = \left\{\omega \in \mathbb{C}^2 :\mid \omega \mid = 1\right\}$
$\underline{N}$	$= \{0, \ldots, N-1\}$, $N \in \mathbb{N}$
card	cardinality of a set
cl	closure of a set
cos, sin	cosine, sine
exp	exponential function
ext	extreme points of a set
gcd	greatest common divisor
id	identity map
ideal	ideal generated by a measure
inf	infimum
lim	limit
ln	natural logarithm
log	logarithm
max	maximum
min	minimum
mod	modulus
span	linear span
sup	supremum
supp	support
$\mid \cdot \mid$	absolute value; also length of a word in the shift space
$(\cdot)_N$	$(m)_N = m \mod N$, $N \in \mathbb{N}$

$*$	involution; adjoint operator
$\circ$	catenation
$\delta_{i,j}$	Kronecker symbol
$\emptyset$	empty set
$\langle \cdot \mid \cdot \rangle$	inner product in $L^2(\mu)$, where μ is a meausre; $\langle f \mid g \rangle = \int f \cdot \overline{g} d\mu$
$\lfloor \cdot \rfloor$	$\lfloor x \rfloor = \max_{k\in\mathbb{Z}, k\leq x}(k)$, $x \in \mathbb{R}$
$\mathbb{1}$	characteristic function on a set
$\mid \cdot \rangle$	Dirac vector
$\oplus$	(internal) direct sum
$\otimes$	tensor product
$\overline{z}$, $z \in \mathbb{C}$	complex conjugate of z
$\overset{\circ}{D}$	interior of the set D
$\Vert \cdot \Vert$	norm, usually L^2-norm; $\Vert \cdot \Vert = \sqrt{\langle \cdot \mid \cdot \rangle}$
∂	boundary of a set
$\perp$	$f \perp g$ stands for $\langle f \mid g \rangle = 0$
$\simeq$	isormophic
$\star$	convolution of measures, $\mu \star \nu$
$\uplus$	disjoint union
v^t	transpose of the matrix or vector v
C^1	continuously differentiable functions
H^s	s-dimensional Hausdorff measure
I	identity operator, identity matrix
$L^1(\cdot)$	the set of integrable functions with respect to the mentioned measure
$L^2(\cdot)$	the set of square-integrable functions with image in $\mathbb{C}$ with respect to the mentioned measure
$\mathfrak{Re}$	real part
δ_x, $x \in \mathbb{R}$	Dirac measure
λ	Lebesgue measure or Haar measure
$\mathcal{B}$	Borel σ-algebra (it is used for different spaces like $\mathbb{R}$, $\mathbb{R}^2$, Σ_A)
$\mathcal{O}_N$	Cuntz algebra of order $N \in \mathbb{N}$
$\mathcal{O}_A$	Cuntz-Krieger algebra for the 0-1-matrix A
$l^2(\cdot)$	space of square-summable sequences
$\mathbb{B}$	Banach algebra of real signed Borel measures with finite variation
B_{abs}	$= \left\{\nu \in \mathcal{M} : \exists f \in L^1(\mathbb{R}, \lambda) \text{ such that } d\nu = f d\lambda\right\}$
$\widehat{\mu}$	Fourier transform of the measure μ, $\widehat{\mu}(t) = \int_{\mathbb{R}^d} e^{i2\pi t\cdot x} d\mu(x)$, $t \in \mathbb{R}^d$
e_x, $x \in \mathbb{R}$	$e_x : t \mapsto e^{i2\pi xt}$
$(e_\lambda)_{\lambda\in\Gamma}$	$= \{e_\lambda : \lambda \in \Gamma\}$
$\mathcal{M}$	set of all positive probability measures with compact support in $\mathbb{R}^d$
$\widetilde{\mathcal{M}}$	set of all probability measures on $\mathbb{R}$ (not necessarily compactly supported)
(μ, Γ)	spectral pair with measure μ and countable set $\Gamma \subset \mathbb{R}$
$\Gamma(L)$	$= \left\{\sum_{j=0}^k l_j R^j : l_j \in L, k \in \mathbb{N}_0\right\}$
(R, B, L)	Hadamard triple
$M_R(B, L)$	Hadmard matrix for the sets B and L with scaling R: $\frac{1}{\sqrt{N}}\left(e^{2\pi i R^{-1} bl}\right)_{b\in B, l\in L}$, $N = \operatorname{card} B$
$S(\mu, \Gamma)(t)$	$= \sum_{\gamma\in\Gamma} \lvert\widehat{\mu}(t-\gamma)\rvert^2$
$M^A(\Gamma)$	$= \left\{\mu \in \mathcal{M} : S(\mu, \Gamma)(t) \leq A \text{ for all } t \in \mathbb{R}^d\right\}$
$M^{OB}(\Gamma)$	$= \left\{\mu \in \mathcal{M} : S(\mu, \Gamma)(t) = 1 \text{ for all } t \in \mathbb{R}^d\right\}$
$M^{OB}(\mu)$	$= \left\{\Gamma \subset \mathbb{R} : S(\mu, \Gamma)(t) = 1 \text{ for all } t \in \mathbb{R}^d\right\}$
$M^\perp(\Gamma)$	$= M^1(\Gamma)$
$M^\perp(\mu)$	$= \left\{\Gamma \subset \mathbb{R} : S(\mu, \Gamma)(t) \leq 1 \text{ for all } t \in \mathbb{R}^d\right\}$

$\widetilde{M}^A(\Gamma)$	$= \left\{\mu \in \widetilde{\mathcal{M}} : \sum_{\gamma\in\Gamma} \mid \widehat{\mu}(t-\gamma) \mid^2 \leq A \text{ for all } t \in \mathbb{R}\right\}$
$\widetilde{M}^{\perp}$	$= \widetilde{M}^1$
$Z(f)$	zeros of the function f
$\mathcal{Z}(f)$	zeros of the function f in $[0,1]$, $Z(f) \cap [0,1]$
$\mathcal{S}$	iterated function system
$\mathbb{S}$	extended IFS
τ_i, σ_i	contraction of IFS or MIM
$\mu, \mu_B, \nu, \nu_{\mathbb{Z}}, \nu_{\mathbb{Z}_v^2}$	measures
T, $\mathcal{T}$	translation operator for MRA
U, $U^{(n)}$, $\mathcal{U}^{(n)}$	scaling operator for MRA
V_j	closed subspaces for MRA
W_j	$= V_{j+1} \ominus V_j$
φ, φ_j	father wavelet for MRA
ψ_j, $\psi_{n,j}$	mother wavelet
m_0	low-pass filter, $m_0 : \mathbb{T} \to \mathbb{C}$
m_j	high-pass filter, $m_j : \mathbb{T} \to \mathbb{C}$
π	coding map
Σ^n	$= \underline{N}^n$
Σ	$= \{(i_0, \dots, i_{k-1}) \in \underline{N}^k : k \in \mathbb{N}, i_{k-1} \notin A\}$
Σ^*	$= \bigcup_{n\in\mathbb{N}} \Sigma^n = \bigcup_{n\in\mathbb{N}} \underline{N}^n$
Σ^A	$= \left\{\omega \in \underline{N}^{\mathbb{N}} : \omega_i \in A \text{ for all } i\right\} = A^{\mathbb{N}}$
Σ_A	$= \{\omega = (\omega_0, \omega_1, \dots) \in \underline{N}^{\mathbb{N}} : A_{\omega_i\omega_{i+1}} = 1\, \forall i \geq 0\}$
Σ_A^n	$= \left\{\omega = (\omega_0, \dots, \omega_{n-1}) \in \underline{N}^n : A_{\omega_i\omega_{i+1}} = 1 \text{ for all } i \in \{0, \dots, n-1\}\right\}$
Σ_A^*	$= \bigcup_{n\geq 1} \Sigma_A^n$
$\Sigma^{*,2}$	$= \bigcup_{n=1}^{\infty} \left(\underline{N_1} \times \underline{N_2}\right)^n$
$\Sigma^{(2)}$	$= \left\{((i_0, j_0), \dots, (i_{k-1}, j_{k-1})) \in \left(\underline{N_1} \times \underline{N_2}\right)^k, k \in \mathbb{N}, (i_{k-1}, j_{k-1}) \notin A\right\}$
$[\omega_0 \dots \omega_k]$	$= \{(\omega_0', \omega_1', \dots) \in \Sigma_A : \omega_i = \omega_i', i \in \{0, \dots, k\}\}$
τ_ω	$= \tau_{\omega_0} \circ \tau_{\omega_1} \circ \dots \circ \tau_{\omega_{n-1}}$
θ	shift map
S_i, Z_i	isometry, representation of the Cuntz algebra
$\Pi = (\pi_{ij})_{i,j\in\underline{N}}$	stochastic matrix of Markov measure
$p = (p_0, \dots, p_n)$	probability vector

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