Hybrid dynamics in large-scale logistics networks

von Mykhaylo Kosmykov

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To my family

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Abstract

We study stability properties of interconnected hybrid systems with application to large-scale logistics networks.

Hybrid systems are dynamical systems that combine two types of dynamics: continuous and discrete. Such behaviour occurs in wide range of applications. Logistics networks are one of such applications, where the continuous dynamics occurs in the production and processing of material and the discrete one in the picking up and delivering of material. Stability of logistics networks characterizes their robustness to the changes occurring in the network. However, the hybrid dynamics and the large size of the network lead to complexity of the stability analysis.

In this thesis we show how the behaviour of a logistics networks can be described by interconnected hybrid systems. Then we recall the small gain conditions used in the stability analysis of continuous and discrete systems and extend them to establish input-to-state stability (ISS) of interconnected hybrid systems. We give the mixed small gain condition in a matrix form $\Gamma \circ \mathcal{D} \ngeq \mathrm{id}$, where the matrix Γ describes the interconnection structure of the system and the diagonal matrix $\mathcal{D}$ takes into account whether ISS condition for a subsystem is formulated in the maximization or the summation sense. The small gain condition is sufficient for ISS of an interconnected hybrid system and can be applied to an interconnection of an arbitrary finite number of ISS subsystems. We also show an application of this condition to particular subclasses of hybrid systems: impulsive systems, comparison systems and the systems with stability of only a part of the state.

Furthermore, we introduce an approach for structure-preserving model reduction for large-scale logistics networks. This approach supposes to aggregate typical interconnection patterns (motifs) of the network graph. Such reduction allows to decrease the number of computations needed to verify the small gain condition.

Contents

Introduction

Hybrid dynamical systems occur in many modern applications due to their ability to deal with a combination of both continuous and discrete types of dynamics in one model. The continuous dynamics is usually given by an ordinary differential equation with an input:

$$\dot{x} = f(x,u),\ (x,u) \in C,$$

and the discrete dynamics is given by an instantaneous change in the state (jump):

$$x^+ = g(x,u),\ (x,u) \in D.$$

This system behaves continuously in the points $(x,u) \in C$ and jumps in the points $(x,u) \in D$. Such behaviour occurs, for example, in control systems that combine digital and analog devices, e.g., robotics [5], network control systems [159], [110], reset systems [111] or engineering systems [62].
Logistics networks is another type of systems that possesses hybrid dynamics. These networks produce and move goods from suppliers to customers. In the literature, there are known many approaches for the modelling of logistics networks. These models differ in their ability to describe different network characteristics, to apply various analysis methods and to achieve desirable performance goals. Networks, where only continuous flows occur, are described by continuous dynamical systems, see [69], [41] and [65]. A network with discrete changes is given by a discrete-time model [7], [116], [113]. A network with random (stochastic) events can be modelled as a stochastic model, see [94] and [145]. In more general types of logistics networks there are usually continuous changes in production, processing or transportation of goods and discrete (discontinuous) changes in picking up and delivering of goods to other locations. In this case it is natural to describe the dynamics by a hybrid dynamical system, see [137] and [146].
The analysis of logistics networks can be performed also in different directions: optimization [69], [113], [137], where the main point is optimal performance of the network; stability, where stable behaviour under perturbation is desirable [146], [41], [116], [69]; control, where the tools for the control of the network are developed [113], [137].
However, the real-world logistics networks are large-scale and possess a complex structure. This implies large size and complex structure of their models. Analytical analysis of large-scale models is rather sophisticated and time-consuming. This motivates the question of reduction of the model size before its analysis. The best way of reduction is to approximate the model by a smaller one. It means that the reduced model has to possess similar characteristics as the original one. In mathematical systems theory there is a theory of model reduction that proposes different methods for reduction of large-scale systems [4], [123]. These methods are well-developed for linear systems. Their main advantages are small approximation error, preservation of dynamical properties (stability, observability, controllability) and numerical efficiency. However, the weak point of their application to logistics networks is that they, in general, do not preserve the structure of the network. This property is crucial for the analysis of logistics networks, because logistics networks consist of real physical objects

like production facilities, warehouses, retailers, transportation routes and thus information about them should not be lost. Furthermore, the dynamics of logistics networks is usually nonlinear. Compared to linear systems, the theory of model reduction of nonlinear systems is taking only the first steps in its development and is applied only to particular classes of nonlinear dynamics [10].
We start this thesis by surveying eleven known approaches for modelling of logistics networks in Chapter 1. These approaches cover four types of dynamics: discrete one in Section 1.2.1, continuous one in Section 1.2.2, hybrid one in Section 1.2.3 and stochastic one in Section 1.2.4. We present the main equations of each model that describe the network dynamics and recall the main results concerning their application to stability analysis and control. We support the survey by a comparison of main characteristics of the approaches in Table 1.1.
To study stability of a logistics network we consider one of the modelling approaches that proposes to model logistics networks as an interconnection of n hybrid subsystems. Then the dynamics of logistic location i is described by a hybrid system

$$\begin{cases} \dot{x}_i &= f_i(x_1, \ldots, x_n, u_i),\ (x_1, \ldots, x_n, u_1, \ldots, u_n) \in C_i, \\ x_i^+ &= g_i(x_1, \ldots, x_n, u_i),\ (x_1, \ldots, x_n, u_1, \ldots, u_n) \in D_i \end{cases}$$

with state x_i (e.g. queue of orders or stock level), external input u_i (e.g. customer orders or flow of raw material), continuous changes described by the function f_i, discontinuous changes described by the function g_i. The sets C_i, D_i define the type of the behaviour of the ith subsystem corresponding to the given states x_i and inputs u_i: continuous in case C_i or discontinuous in case D_i. Thus, this modelling approach includes two types of dynamics in one model and allows description of more general types of logistics networks.
We are concerned with stability of logistics networks, because this property guarantees persistence of the network to perturbations that occur, for example, in demand, cooperation between logistic partners or transportation. In particular, we are interested in input-to-state stability (ISS) introduced for continuous systems in [152] and extended to hybrid systems in [27] . This type of stability assures boundedness of the overall state $x = (x_1^T, \ldots, x_n^T)^T$ of the system under boundedness of the overall external input $u = (u_1^T, \ldots, u_n^T)^T$ for all times and state jumps:

$$|x(t,k)| \leq \max\{\beta(|x_0|, t, k), \gamma(\|u\|_{(t,k)})\}.$$

Here x_0 is the initial state, t is the time, k is the number of the interval between the jumps, $\|u\|_{(t,k)}$ is the norm of the hybrid input. The function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ increases in the first argument and tends to zero in the second and the third one. The function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is strictly increasing and $\gamma(0) = 0$.
In the stability analysis of the interconnection of hybrid subsystems we restrict us to the case where all subsystems has the ISS property, i.e.

$$|x_i(t,k)| \leq \max\{\beta_i(|x_i^0|, t, k), \max_{j, j\neq i} \gamma_{ij}(\|x_j\|_{(t,k)}), \gamma(\|u_i\|_{(t,k)})\}.$$

Functions $\gamma_{ij} : \mathbb{R}_+ \to \mathbb{R}_+$ are strictly increasing, unbounded and $\gamma(0) = 0$. Furthermore, γ_{ij} describes the influence of the jth subsystem on the ith subsystem and thus it is also called gain function.
Our aim is to use information about the interconnection structure of the network for checking whether the network is ISS. For continuous and discrete systems a well-established approach is to use the so-called small gain conditions [82], [54], [126], [50] and [86]. As hybrid systems combine both types of dynamics, this motivates us to adapt these small gain conditions to hybrid subsystems. The first attempts were done for an interconnection of two hybrid systems in [96], [110]. In Chapter 2 we

extend application of the small gain condition to an interconnection of more than two subsystems. Moreover, we extend this condition to the case where some subsystems has the ISS property with summations instead of maximizations in the definition of ISS. To this end, we consider the gain matrix $\Gamma := (\gamma_{ij})_{n\times n}$ that describes the interconnection structure of the network. To guarantee ISS of the network we impose a sufficient condition, the mixed small gain condition in a matrix form, see Theorem 2.4.5:

$$\Gamma \circ \mathcal{D} \not\geq \mathrm{id}$$

where $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ is a matrix operator corresponding to the gain matrix Γ, id is the identity operator and $\not\geq$ is the logical neglecting of $\geq$. The diagonal matrix operator $\mathcal{D} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ has id in the ith element of the diagonal in case the ith subsystem has the ISS property in terms of maximizations and $\mathrm{id} + \alpha$ in case the ith subsystem has the ISS property in terms of summations with some $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that α is strictly increasing, unbounded and $\alpha(0) = 0$. Thus, the mixed small gain condition impose a condition on the cooperation between logistics locations, given by the matrix Γ, that guarantees stability of a logistics network.

In Theorem 2.4.13 we apply this small gain condition to construct an ISS-Lyapunov function for the interconnected hybrid system. This function provides a useful tool for establishing ISS of the hybrid system. As a corollary, we present the small gain conditions for particular classes of hybrid systems: for systems where only parts of the states are stable in Corollary 2.4.20, for impulsive systems in Theorem 2.4.26 and for comparison systems in Theorem 2.4.30.

In the case of a large size of a logistics network, the verification of the small gain condition needs large amount of computations due to the large size of the corresponding gain matrix Γ. With aim to reduce the size of Γ, in Chapter 3 we introduce an approach for structure preserving model reduction of logistics networks. In this approach we consider the matrix Γ as a model of logistics networks that describes the interconnection structure of the network. To reduce the size of the gain matrix, we introduce three rules based on certain types of interconnections in the networks, so-called motifs [103], that allow to pass from the matrix Γ of dimension n to the matrix $\tilde{\Gamma}$ of dimension l with $l < n$. These rules suppose aggregation of the gains of the subsystems that belong to one of the following motifs: parallel connection, sequential connection and almost disconnected subgraph. In Theorems 3.2.1, 3.2.5 and 3.2.9 we derive that, if the small gain condition holds for $\tilde{\Gamma}$ obtained by an application of one of three aggregation rules, then the small gain condition holds also for Γ. Thus, we can establish ISS of logistics network of the size n by checking the small gain condition corresponding to the matrix $\tilde{\Gamma}$ of dimension $l < n$, see Corollaries 3.2.2, 3.2.6 and 3.2.10. As the matrix Γ consists in general of nonlinear gains this approach can be applied to networks with linear dynamics as well as with nonlinear one. Furthermore, these aggregation rules preserve the main structure of a logistics network.

Description of a model of a logistics network as an interconnected continuous system considered in Section 1.2.2 is published in [41], [43] and [44]. A survey on the known modelling approaches for logistics networks from Chapter 1 is partially published in [141], [146]. The result on the mixed small gain condition from Chapter 2 for interconnected continuous systems is published in [49], [50]. The small gain results for hybrid and impulsive systems are published in [45], [46], [47] and [48]. Application of the aggregation rules, considered in Chapter 3, to reduce the size of the model of a logistics network is published in [143], [144]. A result on an investigation of topological properties of logistics networks is published in [142].

Chapter 1

Mathematical models of logistics networks

Logistics network, called also production network or supply chain, is a system that moves products from suppliers to customers [36]. Modern logistics networks vary in their structural and dynamical properties. They may consist of locations geographically distributed all over the world as well of machines arranged inside one production facility [33]. The main performance indicators of such networks are stability, minimization of costs and ability to satisfy customer orders. Feature characteristics and dynamics of a logistics network can be modelled either by the simulation models [140] or by the mathematical models. We are interested in the mathematical models as they allow deep investigation of network dynamics. In the literature there is a wide choice of modelling approaches that vary in their properties. To summarize these approaches, there were performed several reviews of the known models in the literature. In [17], [104] authors consider simulation, game-theoretic, deterministic and stochastic models that are mostly static. Models describing the decision process in logistics networks were investigated in [100] and models that deal with information sharing were reviewed in [78]. However, the aforementioned papers do not consider the dynamical behaviour of the network. The dynamical properties of logistics networks were studied in [135]. The authors stress there on the review of the typical mathematical approaches for the analysis of dynamical effects in logistics networks without providing a detailed overview of the known modelling approaches. In this chapter we are going to fill this gap. To this end, we go through the main modelling approaches known in the literature and identify their modelling concepts, application areas and features. At the end of the chapter we provide a comparison table that highlights the main properties of each modelling approach.

1.1 Notation

First, we introduce the notation that will be used throughout the thesis.

1.1.1 Logistics network

The main activities of a logistics network include production, inventory control, storing and processing. Thus, the network consists of different objects: suppliers, production facilities, distributors, retailers, customers, machines at a production facility. We call such objects *locations*. We denote by n the number of locations and we number all the locations by $i = 1, \dots, n$. The decision, a location takes, on handling the orders relies on a certain *policy*. By x we understand the *state* of a location. Usually, it is the stock level (inventory level) of a location or a work content to be performed. The variable q denotes a *length of queue*, e.g., the queue of customer orders at a location or products to be processed by a machine. *The external input* denoted by u, describes usually the flow of customer orders or the flow of raw material from the external suppliers. The *output* is denoted by y. A typical output is consumption. *The customer demand* is described by the variable d. An example of a logistics network that illustrates our notation is shown in Figure 1.1.

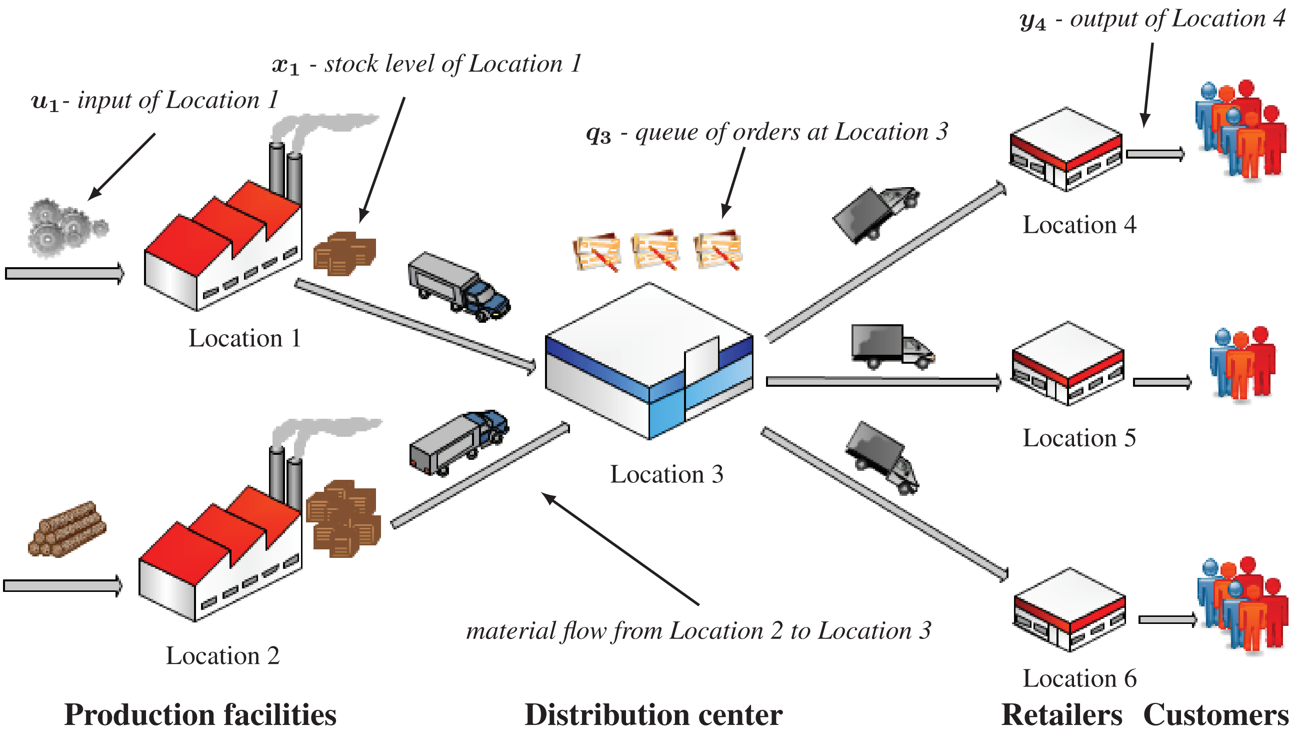


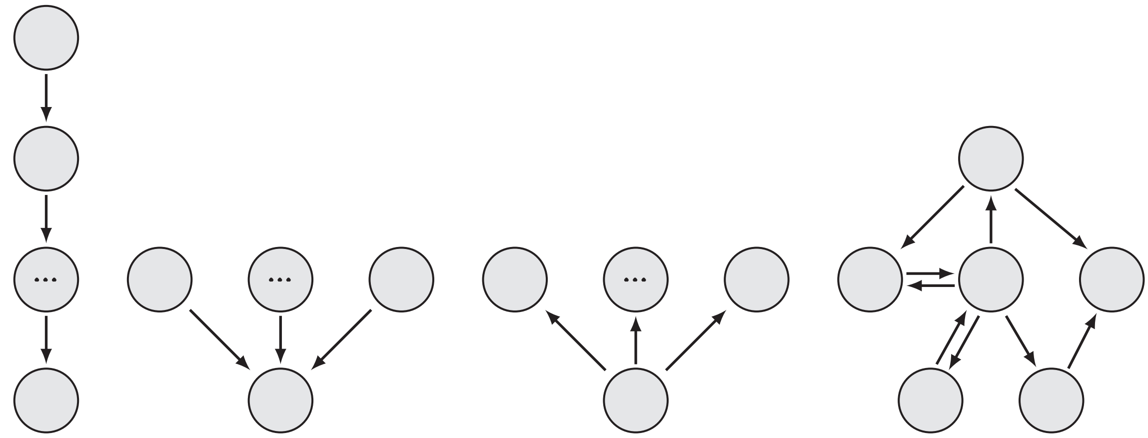
Figure 1.1: General description of a logistics network.

The production rate of a production facility is given by a production function f and the *maximal production rate* is denoted by α. The number of a *production step* is denoted by $k = 1, 2, \ldots$ and the *product type* is denoted by $p = 1, 2, \ldots$. We define by c_{ij} *the consumption rate* of products of location j by location i and we denote by d_{ij} *the delivery rate* of products of location i to location j. For usual *time* we write t. For the time needed to rearrange a location for production of another type of products we write τ and call this time *adaptation time*.

Note that this is only a general description of logistics networks and its parameters. Later in Section 1.2, where we present different approaches for mathematical modelling of dynamics of logistics networks, some of these parameters disappear or new ones appear depending on the features of a specific modelling approach.

Material, information and monetary flows connect locations of a logistics network and create the structure of the network. The structure of a flow is frequently characterized as linear, convergent, divergent, or nonlinear, see Figure 1.2. Here, linear denotes a simple chain of locations passed one after the other, convergent describes flows originating from a large number of locations and ending in a few end locations, divergent describes the opposite structure in which a few sources feed a larger number of end locations, while nonlinear in this context simply denotes a more intricate structure which does not fit into the other categories.

Lack of information between the locations, complexity of the network structure, nonlinearity of dynamics and large size of the network can bring the network to instability, e.g. [154], [37], [107] and [39]. Instability of logistics network means, roughly speaking, unboundedness of the overall state.



(a) Linear (b) Convergent (c) Divergent (d) Nonlinear

Figure 1.2: Structure of the network

Instability results in increasing of costs, in increasing of the number of unsatisfied orders and finally in the loss of profit. For example, a traffic jam causes the later pick up of the products from the warehouse. Thus, the amount of stored products increases, that increases the costs due to storing. The later pick up by-turn causes later delivery of the goods to customers. This decreases the customer satisfaction and thus implies the decreasing of the customer demand and of the profits of the logistics network. Thus, stability of the networks is the property of their survivability. This motivates the study of the network structure and behaviour.
There are many different types of stability. In this thesis our aim is to study input-to-state stability of logistics networks. In Section 1.1.4 we give a precise definition of input-to-state stability and in Chapter 2 we provide conditions that guarantee input-to-state stability of logistics networks.
In the following subsections we recall the notions from matrix, graph and control theory. These notions we are going to use to present different modelling approaches of logistics networks in Section 1.2 and to derive stability conditions for logistics networks in Chapter 2.

1.1.2 Vectors and matrices

In the following, we set $\mathbb{R}_+ := [0,\infty)$ and denote the positive orthant by $\mathbb{R}^n_+ := [0,\infty)^n$. A vector $v \in \mathbb{R}^n_+$ and a matrix $M \in \mathbb{R}^{n\times n}_+$ are called *nonnegative*. The transposition of a vector $v \in \mathbb{R}^n$ is denoted by v^T and the transposition of a matrix $M \in \mathbb{R}^{n\times n}$ is denoted by M^T. By $\langle \cdot,\cdot \rangle$ we denote the standard scalar product in $\mathbb{R}^n$.
On $\mathbb{R}^n$ we use the partial order [54] induced by the positive orthant given by

$$\begin{aligned} x \geq y &\iff x_i \geq y_i, \quad i = 1,\dots,n, \\ x > y &\iff x_i > y_i, \quad i = 1,\dots,n, \end{aligned}$$

for all $x, y \in \mathbb{R}^n$. Furthermore, for all $x, y \in \mathbb{R}^n$ we write $x \not\geq y \iff \exists\, i \in \{1,\dots,n\} : x_i < y_i$, i.e., the logical negation of $\geq$.
For a nonempty index set $I \subset \{1,\dots,n\}$ we denote by $|I|$ the number of elements of I. For a nonempty index set $J \subset \{1,\dots,n\}$ let P_J denote the projection of $\mathbb{R}^n_+$ onto $\mathbb{R}^{|J|}_+$. Let R_I be the anti-projection $\mathbb{R}^{|I|}_+ \to \mathbb{R}^n_+$, defined by

$$x \mapsto \sum_{k=1}^{|I|} x_{i_k} e_{i_k},$$

where $\{e_{i_k}\}_{k=1,\ldots,n}$ denotes the standard basis in $\mathbb{R}^n$ and $I = \{i_1, \ldots, i_{|I|}\}$.
$\mathbb{B}$ is the open unit ball in $\mathbb{R}^n$ and $\overline{\mathbb{B}}$ is its closure. The set $B \subset \mathbb{R}^n$ is *relatively closed in the set* $\chi \subset \mathbb{R}^n$, if $B = \overline{B} \cap \chi$.
We denote by $|\cdot|$ some vector norm in $\mathbb{R}^n$. In particular, $|v|_{\max} = \max_i |v_i|$ means the maximum norm and $|v|_1 = \sum_{i=1}^n |v_i|$ the 1-norm. The spectral radius of a matrix M is denoted by $\rho(M)$. A matrix $M \in \mathbb{R}^{n\times n}$ is called *reducible*, if there exists a permutation matrix $P \in \mathbb{R}^{n\times n}$ such that

$$P^T\, M\, P = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix},$$

where the matrices A and C are square. Otherwise, M is said to be *irreducible*. A matrix M is called *primitive*, if there exists a positive integer $k \in \mathbb{N}$ such that $M^k > 0$. Note that any primitive matrix is irreducible. The converse is false, in general. For the following connection between primitive and irreducible matrices we refer to [20, Theorem 2.1.7].

Lemma 1.1.1. *For $M \geq 0$ the following are equivalent.*

(a) M is irreducible and $\rho(M)$ is greater in magnitude than any other eigenvalue.

(b) M is primitive.

Another useful connection between irreducible and primitive matrices is the following, cf. [20, Corollary 2.2.28].

Lemma 1.1.2. *An irreducible matrix is primitive, if its trace is positive.*

1.1.3 Graphs

A useful tool to describe networks are graphs. Here, we introduce the notion of graphs from [14] and show how graphs can be described by matrices. A *directed graph with weights* consists of a finite *vertex set* V and an *edge set* E, where a *directed edge from vertex* i *to vertex* j is an ordered pair $(i, j) \in E \subset V \times V$. The weights can be represented by a $|V| \times |V|$ *weighted adjacency matrix* A, where $a_{ij} \geq 0$ denotes the weight of the directed edge from vertex i to vertex j. By convention $a_{ij} > 0$, if and only if $(i, j) \in E$. We will denote a directed graph with weights of this form by $G = (V, E, A)$. Additionally, we define for each vertex i the set of *successors* by

$$S(i) = \{j \,:\, (i, j) \in E\}$$

and the set of *predecessors* by

$$P(i) = \{j \,:\, (j, i) \in E\}.$$

A *path* from vertex i to j is a sequence of distinct vertices starting with i and ending with j such that there is a directed edge between consecutive vertices. A directed graph is said to be *strongly connected*, if for any ordered pair (i, j) of vertices, there is a path which leads from i to j. In terms of the weighted adjacency matrix this is equivalent to the fact that A is irreducible, [20].

1.1.4 Notions from control theory

To describe dynamics of a logistics network we need the notions from control theory [89].
We call a continuous function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ *positive definite*, if $\gamma(0) = 0$ and $\gamma(s) \neq 0$ for $s \neq 0$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{K}$, if it is continuous, strictly increasing and $\gamma(0) = 0$. Function $\gamma \in \mathcal{K}$ is of class $\mathcal{K}_\infty$, if, in addition, it is unbounded. Note that for any $\alpha \in \mathcal{K}_\infty$ its inverse function α^{-1} always exists and $\alpha^{-1} \in \mathcal{K}_\infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KL}$, if for each fixed t the function $\beta(\cdot, t)$ is of class $\mathcal{K}$ and, for each fixed s, the function $t \mapsto \beta(s, t)$ is continuous, non-increasing and tends to zero for $t \to \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ is said to be of class $\mathcal{KLL}$, if for each fixed $t \geq 0$ the function $\beta(\cdot, \cdot, t) \in \mathcal{KL}$ and for each fixed $r \geq 0$, $\beta(\cdot, r, \cdot) \in \mathcal{KL}$. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is *affine*, if there exist matrices $A_1, \dots, A_n \in \mathbb{R}^m \times \mathbb{R}^1$ and a vector $b \in \mathbb{R}^m$ such that for all $x \in \mathbb{R}^n$, $f(x) = A_1x_1 + \dots + A_nx_n + b$.
M^n denotes the n-fold composition of a map $M : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ denoted by $M \circ \ldots \circ M$. We denote by id the identity map. We define the restriction of a function $v : \mathbb{R}_+ \to \mathbb{R}^m$ to the interval $[s_1, s_2]$ by

$$v_{[s_1,s_2]}(t) = \begin{cases} v(t), & \text{if } t \in [s_1, s_2], \\ 0, & \text{otherwise.} \end{cases}$$

The essential supremum norm of a measurable function $\phi : \mathbb{R}_+ \to \mathbb{R}^m$ is denoted by $\|\phi\|_\infty := \operatorname{ess\,sup}\{|\phi(s)|, s \in \mathbb{R}_+\}$. L_∞ (or $L_\infty(\mathbb{R}_+, \mathbb{R}^m)$) is the set of measurable functions for which this norm is finite.
The *tangent cone to the set $C \subset \mathbb{R}^n$ at $x \in C$, $T_C(x)$*, is the set of all $v \in \mathbb{R}^n$ for which there exist real numbers $\alpha_i \searrow 0$ and vectors $v_i \to v$ such that for $i = 1, 2, \dots,\quad x + \alpha_i v_i \in C$, see [63], [125] and [13].
A function $f(x)$ is *locally Lipschitz continuous* on a domain (open and connected set) $D \subset \mathbb{R}^n$, if each point of D has a neighborhood D_0 such that f satisfies the Lipschitz condition

$$|f(x) - f(y)| \leq L_0|x - y|$$

for all points x, y in D_0 with some Lipschitz constant L_0 than can depend on the point in D_0. We denote the set of such functions by Lip_{loc}. Note that locally Lipschitz continuous functions are differentiable almost everywhere by the Rademacher's theorem. In the points, where such a function is not differentiable, we use the notion of Clarke's generalized gradient, see [34], [52]. The set

$$\partial f(x) = \text{conv}\{\zeta \in \mathbb{R}^n : \exists x_k \to x, \exists \nabla f(x_k) \text{ and } \nabla f(x_k) \to \zeta\} \tag{1.1}$$

is called *Clarke's generalized gradient of f at $x \in \mathbb{R}^n$*. Note that if f is differential at some point, then its Clarke generalized gradient coincides with the usual gradient at this point.

1.1.5 Dynamical systems and their stability

Here we briefly introduce four types of dynamical systems and the types of their stability.

Dynamical systems

We distinguish four main types of dynamical systems by the type of their behaviour:

- *Continuous dynamical system* given by an ordinary differential equation with an input [89]:

$$\dot{x}(t) = f(x(t), u(t)), \quad x(t_0) = x^0, \tag{1.2}$$

where t is the time, $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the measurable locally essentially bounded input, $f : \mathbb{R}^{n_x+n_u} \to \mathbb{R}^{n_x}$ describes the continuous dynamics and $x(t_0)$ is the initial condition at time t_0. We assume that f is continuous and for all $r \in \mathbb{R}_+$ it is locally Lipschitz continuous in x and uniformly in u for $|u| \leq r$.

- *Discrete dynamical system* given by a difference equation with an input [89]:

$$x(k+1) = g(x(k), u(k)), \quad x(0) = x^0, \quad k = 0, 1, 2, \ldots, \tag{1.3}$$

where k is the discrete time, $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the measurable locally essentially bounded input, $g : \mathbb{R}^{n_x+n_u} \to \mathbb{R}^{n_x}$ describes the discrete dynamics and $x(0)$ is the initial condition. We assume that g is continuous.

- *Hybrid dynamical system* given as a combination of continuous and discrete dynamical systems [60], [162]:

$$\begin{array}{rcll} \dot{x}(t) & = & f(x(t), u(t)), & \text{if } (x(t), u(t)) \in C, \\ x^+(t) & = & g(x(t), u(t)), & \text{if } (x(t), u(t)) \in D, \end{array} \quad x(t_0) = x^0, \tag{1.4}$$

where $x \in \mathbb{R}^{n_x}$ is the state, $x^+(t)$ is a state after its "immediate" change at time t ("jump"), $u \in \mathbb{R}^{n_u}$ is the measurable locally essentially bounded input, $f : \mathbb{R}^{n_x+n_u} \to \mathbb{R}^{n_x}$ describes the continuous dynamics at points $(x, u) \in C \subset \mathbb{R}^{n_x+n_u}$, $g : \mathbb{R}^{n_x+n_u} \to \mathbb{R}^{n_x}$ describes the discrete dynamics at points $(x, u) \in D \subset \mathbb{R}^{n_x+n_u}$ and x^0 is the initial condition. A point of the solution trajectory is denoted by $x(t, k)$ where t is the time and k is the number of the interval between the jumps, see Section 2 for a detailed description of hybrid systems. Functions f and g are assumed to be continuous, and set C and D closed.

- *Stochastic linear dynamical system* given by a stochastic difference equation [19]:

$$x(k+1) = Ax(k) + Bu(k) + Ed(k), \quad x(0) = x^0, \tag{1.5}$$

where k is the discrete time, $x \in \mathbb{R}^{n_x}$ is the state, $u \in \mathbb{R}^{n_u}$ is the measurable locally essentially bounded input, $d \in \mathbb{R}^{n_d}$ is the bounded random value, $A \in \mathbb{R}^{n_x \times n_x}$, $B \in \mathbb{R}^{n_x \times n_u}$ and $E \in \mathbb{R}^{n_x \times n_d}$, $x(0)$ is the initial condition.

We consider also subclasses of continuous (1.2) and discrete system (1.3) without inputs, so-called *unforced dynamical systems*:

$$\dot{x}(t) = f(x(t)), \tag{1.6}$$

and

$$x(k+1) = g(x(k)). \tag{1.7}$$

Consider system (1.6) with $x \in \mathbb{R}^n$. Let S be $n-1$ dimensional *surface of section*, i.e. hypersurface where all trajectories starting on S flow throught it, not parallel to it. The map $P : S \to S$ is called *Poincaré map* [157], if is obtained by following trajectories from one intersection with S to the next. If $x_k \in S$ denotes the kth intersection, then the Poincaré map is defined by

$$x_{k+1} = P(x_k). \tag{1.8}$$

Stability notions

We consider for continuous systems (1.2) or (1.6) the following types of stability [89], [152]:

Definition 1.1.3 (Stability of continuous dynamical systems)**.** *Consider a system of the form* (1.6)

- *A point x^* is* a fixed (or equilibrium) point *for f, if $f(x^*) = 0$;*
- *A fixed point x^** is stable*, if for each $\epsilon > 0$ there is a $\delta > 0$ such that whenever $|x^0 - x^*| < \delta$ the solution $x(t)$ exists and satisfies $|x(t) - x^*| < \epsilon$, for all $t \geq 0$;*
- *A fixed point x^** is attractive*, if there is a $\delta > 0$ such that whenever $|x^0 - x^*| < \delta$ the solution $x(t)$ of* (1.6) *exists and $\lim\limits_{t\to\infty} x(t) = x^*$;*
- *A fixed point x^** is globally attractive *if it is attractive for any $\delta > 0$;*
- *A fixed point x^** is asymptotically stable*, if it is stable and attractive;*
- *A fixed point x^** is unstable*, if it is not stable.*

Consider a system of the form (1.2)*:*

- *System* (1.2) *is called* input-to-state stable (ISS)*, if there exist functions $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all $x^0 \in \mathbb{R}^n$, $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$*

$$|x(t)| \leq \max\{\beta(|x^0|, t), \gamma(\|u\|_\infty)\}, \quad t \geq 0. \tag{1.9}$$

Remark 1.1.4. *By an abuse of notation the abbreviation "ISS" will mean throughout the thesis "input-to-state stable" or "input-to-state stability" depending on the context.*

We consider for a discrete system of the form (1.7) the following types of stability [89], [114].

Definition 1.1.5 (Stability of discrete dynamical systems)**.** *Consider a system of the form* (1.7)*:*

- *A point x^* is* a fixed (or equilibrium) point *for g, if $x^* = g(x^*)$;*
- *A fixed point x^** is stable*, if for each $\epsilon > 0$ there is a $\delta > 0$ such that whenever $|x^0 - x^*| < \delta$, the solution $\{x^k\}$ of* (1.7) *exists and satisfies $|x^k - x^*| < \epsilon$, for all $k \geq 1$;*
- *A point x^* is called a* period two point*, if $x^* = g^2(x^*)$;*
- *A fixed point x^** is attractive*, if there is a $\delta > 0$ such that whenever $|x^0 - x^*| < \delta$ the solution $\{x^k\}$ of* (1.7) *exists and $\lim\limits_{k\to\infty} x^k = x^*$;*
- *A fixed point x^** is asymptotically stable*, if it is stable and attractive;*
- *A fixed point x^** is unstable*, if it is not stable.*

Remark 1.1.6. *By stability of a dynamical system we will usually understand stability of its equilibrium points.*

In hybrid systems of the form (1.4) we are interested in input-to-state stability [96]:

Definition 1.1.7 (ISS of hybrid dynamical systems)**.** *System* (1.4) *is called* ISS*, if there exist functions* $\beta \in \mathcal{KLL}$ *and* $\gamma \in \mathcal{K}_\infty \cup \{0\}$ *such that for all initial conditions* $(x(0,0), u(0,0)) \in C \cup D$ *, bounded inputs* $u \in L_\infty(\mathbb{R}_+, \mathbb{R}^m)$ *and times* $t \geq 0$ *, number of intervals between the jumps* $k \geq 0$

$$|x(t,k)| \leq \max\{\beta(|x(0)|, t, k), \gamma(\|u\|_\infty)\}. \tag{1.10}$$

In the following section we will review the main known approaches for modelling the dynamics of logistics networks. Furthermore, we will show the main results concerning their stability and performance.

1.2 Review of the known modelling approaches

Here we review eleven modelling approaches for logistics networks found in the recent literature. For convenience, we group them by the type of dynamics they possess. In particular, we distinguish four groups of models corresponding to four types of dynamics: discrete, continuous, hybrid and stochastic one.
Investigations of stability properties of logistics networks conducted in Chapter 2 will be based on the model of logistics network as an interconnected hybrid system from Subsection 1.2.3.

1.2.1 Discrete deterministic systems

Discrete deterministic systems are used to model logistic networks where the considered time changes discretely.

Modelling of product lines as "bucket brigades"

A product line is a production system that consists of machines and workers. In [21] it was proposed to treat a product line as a so-called "bucket brigade", where each worker carries an item from machine to machine and then passes it to the next worker. This idea was further extended in [15], [16], [25], [8] and [7]. The main problem is to arrange the jobs between the workers, i.e. to assign the points of the passing of the items, to achieve certain performance aims like self-balancing or maximal throughput.
We introduce this modelling approach relying on [7] and show conditions that guarantee self-balancing.
We consider two workers denoted by A and B. Work content of a job to be performed on an item is denoted by w. When an item enters a product line $w = 0$ and then w changes continuously. At the end $w = 1$. Each worker works with his own speed. The speed for worker B is assumed to be 1 and the speed of worker A is assumed to be f_1 on the interval $[0, W]$ of the job and f_2 on the interval $[W, 1]$ such that $f_1 < 1 < f_2$.
The states at time t are positions of the workers along the product line: $x_A(t) \in [0,1]$ for worker A and $x_B(t) \in [0,1]$ for worker B. A new job is every time started by worker B. Worker A finishes the previous job at some point x^0 and then takes over the job from worker B. The portion of the job of worker B is

$$x_B(t) = t, \forall t \geq 0. \tag{1.11}$$

If worker A starts after the break point W, then the portion of a job of worker A is

$$x_A(t) = x^0 + f_2 t, \tag{1.12}$$

and if he starts before the break point

$$x_A(t) = \begin{cases} x^0 + f_1 t & , t < t_W \\ W + f_2(t - t_W) & , t \geq t_W, \end{cases} \quad (1.13)$$

where $t_W = \frac{W - x^0}{f_1}$ is the time it takes worker A to get to W.
Define $\bar{t}_n$ as the amount of time that worker A spends to process job n and assume that the time is reseted whenever a job is finished. Then the new starting point for worker A at a new job $n+1$ is given by

$$x^{n+1} = x_B(\bar{t}_n) = \begin{cases} \frac{1 - x^n}{f_2} & , \text{ for } x^n > W, \\ W\left(\frac{1}{f_1} - \frac{1}{f_2}\right) - \frac{x^n}{f_1} + \frac{1}{f_2} & , \text{ for } x^n \leq W. \end{cases} \quad (1.14)$$

A piecewise linear map given by (1.14) is a Poincaré map as in (1.8).
One of the aims is to obtain a self-balanced or a self-organized product line. Self-balancing and self-organizing are defined using the notion of a fixed point. In particular, existence of a fixed point x_s for a two worker bucket brigade means that worker B works always from 0 to x_s and worker A from x_s to 1. For the map given in (1.14) a fixed point always exists, but may be either stable, periodic or unstable [7]. The corresponding product line corresponding to a stable fixed point is called *self-balanced*, i.e. for every starting point of worker A they eventually end up working the same part of the product line. If there exists a period-two stable point, then the corresponding product line is called to be *self-organized* and handover of the product from the worker B to the worker A occurs alternatingly at two positions, i.e. at two fixed points.
The type of fixed point can be identified using a value L that is the time that worker A would need to get through the whole production line:

$$L := \frac{W}{f_1} + \frac{1 - W}{f_2}. \quad (1.15)$$

If $L > 1$, then worker B is *faster on average* than worker A.
In [7] conditions that establish the type of the fixed point were obtained. We formulate them in the following theorem.

Theorem 1.2.1. *Depending on W, x_s and L the fixed point is*

- *globally stable, if $W \leq x_s$ and worker A is faster on average than worker B ($L < 1$)*
- *stable period-two point, if $W \leq x_s$ and worker B is faster on average than worker A ($L \geq 1$)*
- *unstable period-two point, if $W > x_s$ and worker A is faster on average than worker B ($L < 1$)*
- *unstable, if $W > x_s$ and B is faster on average than worker A ($L \geq 1$)*

This modelling approach allows to describe a production line as a bucket brigade. We have considered two types of stable production lines: self-balanced and self-organized. Stability condition given in Theorem 1.2.1 depends on the workers speeds. Though the presented model is simplified by considering only two workers with constant speeds, it may be further modified and extended to achieve maximal throughput, by adding more workers, by implying another rules of job sharing like blocking or by introducing new properties for the workers like learning [8].

Decentralized supply chains

In decentralized (or autonomous) supply chains information is not shared between all locations and each supplier determines its order quantities based on the demand and inventory information of previous time periods, [116], [36], [37] and [118]. Due to this bounded information sharing the Bullwhip effect may occur. The main question is how to quantify this effect. Here we will show an approach to quantify this effect based on the model considered in [116].
Consider a supply chain that consists of $n+1$ suppliers connected sequentially, see Figure 1.2a. The suppliers are denoted by indices $i = 1, 2, \ldots, n+1$ starting from downstream and $i = 0$ corresponds to the final customer. The time is discrete and the time periods are denoted by $t = 0, 1, 2, \ldots$. At the beginning of every time period t, supplier i checks his inventory level during the period and orders the needed quantity u_i at the end of the period. The inventory level of the ith supplier at the period $t+1$, i.e., the difference between the placed and received orders, is described as follows

$$x_i(t+1) = x_i(t) + u_i(t) - u_{i-1}(t), i = 1, 2, \ldots. \tag{1.16}$$

Goods ordered by supplier i arrive after a constant lead time τ_i. The in-stock inventory level of supplier i during the period $t+1$, i.e., the difference between the received items and received orders, is given by

$$y_i(t+1) = y_i(t) + u_i(t-\tau_i) - u_{i-1}(t), i = 1, 2, \ldots. \tag{1.17}$$

The order quantity $u_i(t)$ of supplier i at the end of period t is calculated based on the information about its inventory levels x_i, y_i of all previous periods up to t and the order quantities u_{i-1} of all previous periods up to $t-1$. The next step is to focus on the ordering policy, which is based on the information above. Policies often used in practice are proper, linear and time-invariant (LTI). A policy is called *proper*, if the size of the orders received is constant over the time, the supplier inventory tends to a constant equilibrium value that is independent of the initial conditions and the orders placed tend to the value of orders received. Further, a policy is called *LTI*, if $u_i(t)$ is a time-dependent linear function of x_i, y_i and u_{i-1}. In order to give a simple description of a proper and LTI policy we introduce the unit shift operator R for the time series and let R_l denote its l-fold application, i.e.

$$R^l x_i(t) := x_i(t-l) \tag{1.18}$$

for all t and for all $l = 0, 1, \ldots$. Then the general expression is

$$u_i(t) = \gamma_i + A_i(R)x_i(t) + B_i(R)y_i(t) + C_i(R)u_{i-1}(t-1), i = 1, 2, \ldots. \tag{1.19}$$

Here γ_i is a real number and A_i, B_i and C_i are polynomials with real coefficients

$$A_i(R) = a_0^i + a_1^i R + a_2^i R^2 + \ldots, \tag{1.20}$$

$$B_i(R) = b_0^i + b_1^i R + b_2^i R^2 + \ldots, \tag{1.21}$$

$$C_i(R) = c_0^i + c_1^i R + c_2^i R^2 + \ldots. \tag{1.22}$$

The polynomials A_i and B_i indicate the influence of inventory history on the ordering decisions and C_i the influence of orders received. The exact choice of these polynomials depends on the application needs. For such an ordering policy it follows from the definition of the properness that the nominal equilibrium exists such that order sizes, inventory levels and in-stock inventories stay constant, say x_i^∞, y_i^∞ and u_i^∞ .

A negative effect occurring in such networks is an instability given by the Bullwhip effect: increasing of fluctuation of orders in direction from retailers to suppliers due to the changes in the customer demand [59], [95], [30]. To quantify the Bullwhip effect we consider the error between the current states and their corresponding equilibrium. That is, we denote

$$\begin{array}{l} \bar{u}_i(t) = u_i(t) - u^\infty = A_i(R)\bar{x}_i(t) + B_i(R)\bar{y}_i(t) + C_i(R)\bar{u}_{i-1}(t-1), i = 1, 2, \ldots \\ \bar{x}_i(t) = x_i(t) - x^\infty = \bar{x}_i(t) + \bar{u}_i(t) - \bar{u}_{i-1}(t), i = 0, 2, \ldots \\ \bar{y}_i(t) = y_i(t) - y^\infty = \bar{y}_i(t) + \bar{u}_i(t-\tau_i) - \bar{u}_{i-1}(t), 0 = 1, 2, \ldots \end{array} \tag{1.23}$$

and consider the ratio of the order sequences of the most upstream supplier and customer demand. This reflects the idea of the so-called *worst-case RMSE (root mean square errors) amplification factor* [116] that is given by

$$W_n = \sup_{\overline{u}_0(\cdot)\neq 0} \left[\frac{\left(\sum_{t=0}^{\infty} \overline{u}_n^2(t)\right)^{\frac{1}{2}}}{\left(\sum_{t=0}^{\infty} \overline{u}_0^2(t)\right)^{\frac{1}{2}}} \right]. \tag{1.24}$$

This factor allows to state whether supplier $n+1$ experiences the Bullwhip effect or not. To be precise, in a supply chain, that is described within the error framework, supplier $n+1$ is said *to experience no Bullwhip effect*, if $W_n \leq 1$, i.e. if the overall fluctuation of orders of the last supplier is less than or equal the overall fluctuation of the customer demand. In the case of proper LTI supply chains with $n+1$ locations the sufficient condition for the occurrence of the Bullwhip effect is stated in the following theorem, see [116, Theorem 3].

Theorem 1.2.2. *Supplier $n+1$ in an LTI supply chain described by* (1.23) *experiences the Bullwhip effect if*

$$W_n := \sum_{i=1}^{n} \frac{1 + B_i(1)\tau_i - C_i(1)}{A_i(1) + B_i(1)} > 0. \tag{1.25}$$

The following corollary establish the Bullwhip effect if the supply chain is *homogeneous*, i.e. all the locations are alike.

Corollary 1.2.3. *When the supply chain is homogeneous, the Bullwhip effect exists if*

$$\frac{1 + B(1)\tau - C(1)}{A(1) + B(1)} > 0. \tag{1.26}$$

Furthermore, there are similar analytical conditions for other policies (e.g. advanced demand information) to predict whether the Bullwhip effect will occur or not, see [116], [117], [115].
Though here only the case of sequentially connected suppliers was considered, it is possible to extend the modelling approach on general network structures, as well as to cover delays in delivery or uncertainties [118]. Explicitly there is no production processes involved. By production process one can understand only lead times τ_i.

Modelling of re-entrant lines as queueing systems

Products considered in this modelling approach pass through several production steps to be finished [113]. Such networks have nonlinear structure as in Figure 1.2d and are called *re-entrant lines*. The topology of the routes is determined by the set mappings. The state of the system is given by the length of the queues at the machines. The main aim is to arrange the machine processing time for different product types to achieve the maximal production rate.

We consider n machines denoted by U_i, $i = 1, \dots, n$, and p product flows denoted by P_k, $k = 1, \dots, p$. There are a finite fixed number of routes for each product. During the production cycle only one item of product type can be processed at a machine. A product flow P_k can be processed on n_k machines denoted by $1, \dots, n_k$ and has s_k sequential production steps. The mapping $s_k(l) : \{1, \dots, n_k\} \mapsto \{1, \dots, s_k\}$ defines the set of numbers of the production steps which can be performed by the machine l. The family of all mappings from $\{1, \dots, n_k\}$ into the set of all subsets $\{1, \dots, s_k\}$ is denoted by $M^{s_k}_{n_k}$.
For two mappings $a, b \in M^{s_k}_{n_k}$ by $h = a \cup b$ we denote such a mapping from $M^{s_k}_{n_k}$ that for any $l \in \{1, \dots, n_k\}$ it holds that $h(l) = a(l) \cup b(l)$. A mapping $s \in M^{s_k}_{n_k}$ is a *product-stream* P_k if for any $l \in \{1, \dots, n_k\}$ it holds that $s_k(l) \neq \emptyset$ and $\bigcup\limits_{l=1}^{n_k} s_k(l) = \{1, 2, \dots, s_k\}$.
By Q^{kj}_{il} we denote the queue of the items of the product P_k after being processed at the machine U_i and that are waiting to be processed by the machine U_l, where j is the production step of the product P_k. The length of the queue Q^{kj}_{il} at time t is $q^{kj}_{il}(t)$. The state of the machine U_l described by products P_k at stage j at time t is given by

$$q^{kj}_{l}(t) = \sum_{i \in in_k(l)} q^{kj}_{il}(t), \tag{1.27}$$

where $in_k(l)$ is the set of machines from which the products P_k arrive to machine U_l.
Let τ^{kj}_{il} be the time that machine U_i needs to perform a production step j of a product k produced for machine U_l, n^{kj}_{il} be the number of items of product P_k on production step j produced by the machine U_i for the machine U_l during the processing time τ^{kj}_{il}, m_{kj} be the batch size required for the production step $j+1$ of P_k, a^{kj}_{il} be the ratio of the time needed for machine U_i for working for machine U_l over product P_k on production step j. a^{kj}_{il}, n^{kj}_{il} satisfy the following natural conditions: $\sum\limits_{k,j,l} a^{kj}_{il} \leq 1$, $a^{kj}_{il} \geq 0$ for all $i, j > 0, i > 0, l$, $\sum\limits_{k,l} a^{k0}_{0l} \leq 1$ and $a^{k0}_{0l} \geq 0$ for all k, l. Denote also $N^{kj}_{i} = \sum\limits_{l \in out_k(i)} n^{kj}_{il}$ where $out_k(i)$ is the set of machines to which the products P_k arrive from machine U_i.
The dynamics of the machine U_l is described by the change of its queue length due to the arriving of the new items and shipping of the processed items to the next machines:

$$q^{kj}_{l}(t) = S\left\{ \left[\sum_{i \in in_k(l)} \frac{n^{kj}_{il}}{m_{kj}} \left[\frac{a^{kj}_{il}(t - t_0)}{\tau^{kj}_{il}} \right] \right] - \sum_{i \in out_k(l)} \left[\frac{a^{kj+1}_{li}(t - t_0)}{\tau^{kj+1}_{li}} \right] + q^{kj}_{l}(t_0) \right\} \tag{1.28}$$

where $S\{x\} = \frac{|x|+x}{2}$.
The Stinson-Smith Condition [155]

$$\sum_{i \in in_k(l)} \frac{n^{kj}_{il} a^{kj}_{il}}{\tau^{kj}_{il} m_{kj}} - \sum_{i \in out_k(l)} \frac{a^{kj+1}_{li}}{\tau^{kj+1}_{li}} = 0 \tag{1.29}$$

imposed on the dynamics of the machine U_l minimizes its idle time. Moreover, this condition guarantees boundedness of its queue q^{kj}_{l}.
Such a model is used to set an optimization (or quasi-optimization problem) [74] that finds n^{kj}_{il}, a^{kj}_{il} under which a production rate is maximal (or quasi-maximal). The quasi-optimization problem is formulated as a linear programming problem using the Stinson-Smith Condition (1.29). Let n be the total number of machines, ν be a number of product items considered in the optimization problem

in [113], $\hat{a}_{il}^{kj}$, $\hat{n}_{il}^{kj}$ be the solution of the optimization problem, $\widetilde{a}_{il}^{kj}$, $\widetilde{n}_{il}^{kj}$ be the solution of the quasi-optimization problem, $\hat{q}_{il}^{kj}(t)$, $\widetilde{q}_{il}^{kj}(T)$ be the states at time T for optimal and quasi-optimal solutions, and μ_k be the cost of one item per product p_k. Though the Stinson-Smith Condition guarantees the boundedness of the queues, it allows to obtain only the quasi-maximal production rate. The difference between production rate of optimal solution $\sum_{i=k}^{\nu} \mu_k \cdot \hat{q}_{n+1}^{ks_k}(T)$ and production rate of quasi-optimal solution $\sum_{k=1}^{\nu} \mu_k \cdot \widetilde{q}_{n+1}^{ks_k}(T)$ is estimated in the following theorem, see [113, Theorem 3.1].

Theorem 1.2.4. *For* $\sum_{k=1}^{\nu} \mu_k \cdot \hat{q}_{n+1}^{ks_k}(T)$ *and* $\sum_{k=1}^{\nu} \mu_k \cdot \widetilde{q}_{n+1}^{ks_k}(T)$ *the inequality*

$$0 \leq \sum_{k=1}^{\nu} \mu_k \cdot \hat{q}_{n+1}^{ks_k}(T) - \sum_{k=1}^{\nu} \mu_k \cdot \widetilde{q}_{n+1}^{ks_k}(T) \leq \sum_{k=1}^{\nu} \mu_k \sum_{k \in in_k(n+1)} (1 + \delta(\widetilde{n}_{kn+1}^{ks_k}, n_{ks_k}) \cdot \frac{n_{in+1}^{ks_k}}{n_{ks_k}})$$

holds, where

$$\delta\left(\widetilde{n}_{in+1}^{ks_k}, n_{ks_k}\right) = 1, \textit{ if } \widetilde{n}_{in+1}^{ks_k} \neq n_{ks_k},$$
$$\delta\left(\widetilde{n}_{in+1}^{ks_k}, n_{ks_k}\right) = 0, \textit{ if } \widetilde{n}_{in+1}^{ks_k} = n_{ks_k}.$$

This modelling approach describes re-entrant lines where the products pass through several production steps. Furthermore, there may be more than one route for a product. A machine is able to assign the processing time and the number of items of each product to be be processed in order to achieve certain performance goals. The Stinson-Smith Condition (1.29) guarantees boundedness of the queue at the machines. However, this condition restricts the choice of possible decisions of a machine to get the global optimal performance. Thus, only quasi-optimal solutions can be achieved under this condition.

1.2.2 Continuous deterministic systems

Here we consider modelling approaches that deal with continuous material flows in the network. Such a network can be described by ordinary or partial differential equations.

Ordinary differential equation

In this framework we model the whole network by modelling the dynamics of each single location by an ordinary differential equation with inputs as in (1.2) and by describing the dynamics of the whole network as a system of ordinary differential equations [42], [41], [43] and [44]. Such equations allow to study stability properties of the networks using known Lyapunov methods and recently established small gain conditions.
The dynamics of the ith locations, $i = 1, \ldots, n$, looks as follows

$$\dot{x}_i = \widetilde{f}_i(x_1, \ldots, x_n, u_i), \quad x_i(0) = x_i^0 \tag{1.30}$$

where $x_i(0)$ is an initial state and the functions $\widetilde{f}_i$ describe the changes at state x_i of location i and need not to be linear, [141], [42]. For example, the state x_i is the number of unsatisfied orders at location i, $f_i(x_i(t)) = \alpha_i \left(1 - e^{-x_i(t)}\right)$ is its actual production rate, where α_i is the maximal production rate [42]. If the state x_i is large, then the production rate f_i tends to α_i and if the state is small, then f_i tends to zero. This means, if there are many orders, the actual production rate is close to the maximum production rate and if there are no orders nothing will be produced. The state of a location i influences the states of the other locations $j \neq i$. The state of a location i is also subject to an external input u_i. This input might be caused by the new orders from the customers of the logistics network. This allows to model the dependence and interconnections between the locations more precise

$$\dot{x}_i = u_i + \sum_{j,j\neq i} c_{ji} f_j(x_j(t)) - f_i(x_i(t)), \tag{1.31}$$

where $0 \leq c_{ji} \leq 1$ is the share of orders of location j at location i, $\sum_{i=1}^{n} c_{ji} = 1$.
The state of the whole logistics network is obtained by combining the states of all locations in one vector, i.e. $x = (x_1^T, \dots, x_n^T)^T$. The dynamics of the logistics network is given by

$$\dot{x} = \widetilde{f}(x,u) = \begin{pmatrix} \widetilde{f}_1(x_1,\dots,x_n,u_1) \\ \dots \\ \widetilde{f}_n(x_1,\dots,x_n,u_n) \end{pmatrix}, \quad x(0) = x^0. \tag{1.32}$$

A well established notion to describe stability of interconnected nonlinear dynamical systems with inputs is the notion of ISS. In particular, ISS of (1.32) defined by (1.9) occurs when the state $x(t)$ is bounded by some function of the initial value $x(0)$ and of the overall input $u = (u_1^T, \dots, u_n^T)^T$ over the time.
For the individual location i described by (1.31) ISS is defined by

$$|x_i(t)| \leq \max\{\beta_i(|x_i^0|, t), \max_j\{\gamma_{ij}(\|x_{j[0,t]}\|_\infty)\}, \gamma_i(\|u_i\|_\infty)\}, \tag{1.33}$$

where $\gamma_{ij} \in \mathcal{K}_\infty$ are called *gains* from other locations j [54].
The notion of ISS is one possibility of defining stability. Here the external influences are addressed explicitly by γ_i. A further advantage of the ISS notion is that there are stability criteria for interconnected systems based on the gains γ_{ij} [54], [42]. To illustrate this criteria, all the gains are collected in a matrix $\Gamma := (\gamma_{ij})_{n\times n}$. This matrix describes interconnection structure of the network. The operator $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ is defined by

$$\Gamma(s) := \begin{pmatrix} \max\{\gamma_{1,2}(s_2), \dots, \gamma_{1,n}(s_n)\} \\ \vdots \\ \max\{\gamma_{n1}(s_1), \dots, \gamma_{n,n-1}(s_{n-1})\} \end{pmatrix}. \tag{1.34}$$

Inequality $\Gamma(s) \not\geq s$ means that for every s there exists at least one component of the vector $\Gamma(s)$ such that $(\Gamma(s))_i < s_i$. This inequality is called the small gain condition. A *r-cycle* in the matrix Γ is a sequence $\gamma_{i_0,i_1}, \gamma_{i_1,i_2}, \dots, \gamma_{i_{r-1}i_r}$ of size r, where $i_0 = i_r$. The cycle of size r in Γ is a *contraction* if

$$\gamma_{i_0,i_1} \circ \gamma_{i_1,i_2} \circ \dots, \circ \gamma_{i_{r-1},i_r} < \text{id}. \tag{1.35}$$

In [126, Corollary 3.3.5] the following *small gain theorem* was proved that guarantees ISS of the interconnected system (1.32).

Theorem 1.2.5. *Consider system* (1.30) *and suppose that each subsystem is ISS so that condition* (1.33) *holds. If* $\Gamma(s) \not\geq s$ *for all* , $s \neq 0$, *or equivalently, if every cycle in* Γ *is a contraction, then system* (1.32) *is ISS from* u *to* x.

Thus, this stability condition depends on the interconnection structure of the network given by matrix Γ. This modelling approach allows for a modularity principle. That is, the ISS concept allows to establish stability of the network from stability of single locations. Moreover, there are no restrictions on the interconnection structure.
From the practical point of view, this framework can cope with nonlinear dynamics in every location of the logistics network. Further, this approach provides a stability criteria to decide whether the interconnection of stable locations leads to a stable logistics network. The criterion takes the topology of the logistics network into account. This modelling approach can be further extended to capture delays in delivery [43]. Input-to-state stability and the small gain condition are discussed in more detail in Chapter 2.

Damped oscillator models

A modelling approach inspired by physics of interconnected oscillators [158] has been investigated in [69], [70], [105] and [151]. Here we model logistics networks that adapt their production speeds. Due to the fact that adaptation needs some time, instability effects like the Bullwhip effect occur.
The model uses an idea of a physical transport problem, where the flows of products are considered. The model is given by balance equations for the flows of products and by the adaptation of the production speeds. There are n logistics locations denoted by $i \in \{1, \dots, n\}$. Location i delivers d_{ji} products of kind i to other locations and consumes c_{ki} products of kind k per production cycle, i.e. for production of one unit. The production speed $f_i(t)$ of a location i is the number of production cycles per time unit (day, week,...). $x_i(t)$ denotes the number of products of kind i available in the logistics network (inventory). The function $y_i(t)$ represents an external flow like consumption, losses and import of resources

$$y_i(t) = c_{i,n+1} f_{n+1}(t) - d_{i0} f_0(t). \tag{1.36}$$

Here f_{n+1} reflects the customers demand while f_0 reflects the inflow of resources. It is assumed that c_{ij} and d_{ij} are normalized, such that $0 \le c_{ij}, d_{ij} \le 1$ and

$$d_{i0} = 1 - \sum_{j=1}^{n} d_{ij} \ge 0, \quad c_{i,n+1} = 1 - \sum_{j=1}^{n} c_{ij} \ge 0. \tag{1.37}$$

The inventory change of product i is given by the difference of supply and demand

$$\frac{dx_i}{dt} = f_i^{in}(t) - f_i^{out}(t) = \sum_{j=1}^{n} d_{ij} f_j(t) - \sum_{j=1}^{n} c_{ij} f_j(t) + y_i(t), \tag{1.38}$$

where the first term represents the supply and the second term denotes the demand. Variations of the consumption rate $y_i(t)$ enforce an adaptation of production speeds. This is based on information about the current inventory of all locations i, the change of inventory $x_i(t)$ and the current production speed $f_i(t)$. The adaptations are not instantaneous and require an adaptation time τ_i for adjustments. In the following we state an adaption for the case of sequential logistics networks. Let x_i^0 denote a desired inventory level, $f_j^0(x_i, \frac{dx_i}{dt})$ a desired rate, then the delivery rate is adapted to minimize changes in inventory $\frac{dx_i}{dt}$ according to

$$\frac{df_j}{dt} = \frac{1}{\tau_i} \left[\frac{x_i^0 - x_i(t)}{T_i} - \beta_i \frac{dx_i}{dt} + \epsilon_i [f_j^0 - f_i(t)] \right]. \tag{1.39}$$

The analysis of the Bullwhip effect is performed by linearizing the model described in (1.38), (1.39) around the equilibrium point $(x_i, \frac{dx_i}{dt}) = (x_i, 0)$ [89]. The size of the Bullwhip effect depends on the network topology and the adaptation of production rates. Instability condition for the case of a sequential supply chain, see Figure 1.2a, and feedback (1.39) was shown in [69]:

Theorem 1.2.6. *Variations in the consumption rate are magnified under the instability condition:*

$$\tau_i > \epsilon_i T_i(\beta_i + \epsilon_i/2). \quad (1.40)$$

This condition implies that the Bullwhip effect occurs, if the adaptation time τ_i is too large, if there is no adaptation to some desired production speed (corresponding to $\epsilon_i = 0$), or if the production management reacts too strong to deviations of the actual stock level x_i from the desired one x_i^0 (corresponding to small value of T_i).
In the damped oscillator model the dynamics of a logistics network is represented by the flow of products and by the adaptation of the production rates. The model can be extended to cover transport delays [151] and to a macroeconomic model with different economic sectors [69]. This approach contains qualitative models for the analysis of the Bullwhip effect, that may occur in a supply chain.

Multilevel network model

We consider a multilevel network consisted of logistics network, information network and financial network [108], [107] and [106]. Logistics locations that compose this supply chain compete with each other. The commodity is homogeneous. Manufactures produce goods and sell them to retailers. Retailers deliver then goods to consumers at demand markets. The main aim is to achieve stable behaviour of the material flows between locations and of the prices for the products.
We use the following notation for the logistics locations and delivery sizes: i is the number of a producer (manufacturer), $i = 1, ..., n_p$, j is the number of retailer, $j = 1, ..., n_r$, l is the number of a consumer, $l = 1, ..., n_c$, y_{ij}^1 is a nonnegative size of delivery between producer i and retailer j, y_{jl}^2 is a nonnegative size of delivery between retailer j and consumer l, $Y^1 = (y_{ij}^1)_{i=1,...,n_p; j=1,...,n_r} \in \mathbb{R}_+^{n_p \times n_r}$ is the overall delivery between producers and retailers, $Y^2 = (y_{jl}^2)_{j=1,...,n_r; l=1,...,n_c} \in \mathbb{R}_+^{n_r \times n_c}$ is the overall delivery between retailers and consumers, y_i is the overall amount of produced goods by manufacturer i, $y = (y_1, ..., y_i, ..., y_{n_p}) \in \mathbb{R}_+^{n_p}$ is the overall amount of produced goods.
The logistics network consists of 3 levels of nodes: manufactures, retailers and consumers. The nodes are connected by edges describing material flows between manufacturers and retailers (y_{ij}^1), and retailers and consumers (y_{jl}^2).
To describe the financial network we use the following notation: π_{1ij} is the product price of manufacturer i associated with retailer j, $\pi_{1i} \in \mathbb{R}_+^{n_p}$ is the price of manufacturer i for the product, $\pi_1 = (\pi_{11}, \pi_{12}, ..., \pi_{1n_p}) \in \mathbb{R}_+^{n_p}$ are the prices of all manufacturers; π_{2j} is the price of retailer j, $\pi_2 = (\pi_{21}, \pi_{22}, ..., \pi_{2n_r}) \in \mathbb{R}_+^{n_r}$ are the prices of all retailers; π_{3l} is the true price for the product as perceived by consumer l, $\pi_3 = (\pi_{31}, \pi_{32}, ..., \pi_{3n_c}) \in \mathbb{R}_+^{n_c}$.
Then the financial network consists of the nodes representing the same logistics locations but where the edges denote the prices for the products.
The information network consists also of the same logistics locations but is now bidirectional and shows delivery and price information over the time in order to adjust delivery size and prices for obtaining equilibrium, i.e. constant delivery sizes and prices.
Dynamics of the price π_{3k} is given as follows

$$\dot{\pi}_{3k} = \begin{cases} d_l(\pi_3) - \sum_{j=1}^{n_r} y_{jl}^2, & \text{if } \pi_{3l} > 0, \\ \max\{0, d_l(\pi_3) - \sum_{j=1}^{n_r} y_{jl}^2\}, & \text{if } \pi_{3l} = 0, \end{cases} \quad (1.41)$$

where d_l is a demand function that depends on the price π_{3l}. Thus, the change in the price is the difference between the demand and the available amount of goods at the consumers. The second equation in (1.41) guarantees that the price does not become negative.
Dynamics of the price π_{2l} is given by

$$\dot{\pi}_{2j} = \begin{cases} \sum_{l=1}^{n_c} y_{jl}^2 - \sum_{i=1}^{n_p} y_{ij}^1, & \text{if } \pi_{2j} > 0, \\ \max\{0, \sum_{l=1}^{n_c} y_{jl}^2 - \sum_{i=1}^{n_p} y_{ij}^1\}, & \text{if } \pi_{2j} = 0. \end{cases} \tag{1.42}$$

The change in price is now the difference between the available amount of goods at the manufacturers and at the retailers.
Dynamics of the delivery size y_{jl}^2 is described in the following way

$$\dot{y}_{jl}^1 = \begin{cases} \pi_{3l} - \theta_{jl}(Y^2) - \pi_{2j}, & \text{if } y_{jl}^2 > 0, \\ \max\{0, \pi_{3l} - \theta_{jl}(Y^2) - \pi_{2j}\}, & \text{if } y_{jl}^2 = 0, \end{cases} \tag{1.43}$$

where $\theta_{jl}(Y^2)$ are the transportation costs. The change in the delivery size is then the difference between the price the consumers are ready to pay and the transportation costs together with the price of a retailer. The second equation in (1.43) guarantees that the delivery size is not negative.
Consider production cost function $\eta_i = \eta_i(Y^1)$ and transportation cost function $\theta_{ij} = \theta_{ij}(y_{ij}^1)$. The overall cost of a manufacturer is the sum of production costs and transportation costs. The price for the product of the manufacturer is the marginal costs of production and transportation: $\frac{\partial \eta_i(Y^1)}{\partial y_{ij}^1} + \frac{\partial \theta_{ij}(y_{ij}^1)}{\partial y_{ij}^1}$. The costs of the retailer for handling are denoted by $\theta_j = \theta_j(\sum_{i=1}^{n_p} y_{ij}^1) = \theta_j(Y^1)$.
Dynamics of the material flows between the manufacturers and the retailers is given by the difference between the price of retailer and the marginal costs of the retailer and the manufacturer:

$$\dot{y}_{ij} = \begin{cases} \pi_{2j} - \frac{\partial \theta_j(Y^1)}{\partial y_{ij}^1} - \frac{\partial \eta_i(Y^1)}{\partial y_{ij}^1} - \frac{\partial \theta_{ij}(y_{ij}^1)}{\partial y_{ij}^1}, & \text{if } y_{jl}^1 > 0 \\ \max\{0, \pi_{2j} - \frac{\partial \theta_j(Y^1)}{\partial y_{ij}^1} - \frac{\partial \eta_i(Y^1)}{\partial y_{ij}^1} - \frac{\partial \theta_{ij}(y_{ij}^1)}{\partial y_{ij}^1}\}, & \text{if } y_{jl}^1 = 0. \end{cases} \tag{1.44}$$

To describe the overall dynamics of the network given in (1.41)-(1.44) we take $X = (Y^1, Y^2, \pi_2, \pi_3) \in K \equiv \mathbb{R}_+^{n_p n_r + n_r n_c + n_r + n_c}$, $F(X) \equiv (F_{ij}, F_{jl}, F_j, F_l)$, where $F_{ij} \equiv \frac{\partial \eta_i(Y^1)}{\partial y_{ij}} + \frac{\partial \theta_{ij}(y_{ij}^1)}{\partial y_{ij}^1} + \frac{\partial \theta_j(Y^1)}{\partial y_{ij}^1} - \pi_{2j}$; $F_{jl} \equiv \pi_{2j} + \theta_{jl}(Y^2) - \pi_{3l}$; $F_j \equiv \sum_{i=1}^{n_p} y_{ij}^1 - \sum_{l=1}^{n_c} y_{jl}^2$; and $F_l \equiv \sum_{j=1}^{n_r} y_{jl}^2 - d_l(\pi_3)$. Then we can represent the dynamics as the projected dynamical system (PDS) [109]

$$\dot{X} = \prod\nolimits_K (X, -F(X)), X(0) = X_0 \tag{1.45}$$

where $\prod_K$ is a projection operator of $-F(X)$ onto K at X and $X_0 = ({Y^1}^0, {Y^2}^0, {\pi_2}^0, {\pi_3}^0)$ is an initial condition. Note that this dynamical system has the discontinuous right-hand side as we guarantee in (1.41)-(1.44) that all the variables are nonnegative. Under assumption that functions η_i are additive and have bounded second-order derivative, functions θ_{ij}, θ_j have bounded second-order derivative and θ_{jl}, θ_l have bounded first-order derivative there exists a unique solution of (1.45) for any $X_0 \in K$.
Dynamical system (1.45) allows to study stability of the given logistics network. The following theorem establishes condition for stability of the system (1.45), see [108, Theorem 4].

Theorem 1.2.7. *Suppose that the production cost functions η_i are additive, internal production cost function η_i^1, transportation cost functions θ_{ij}, θ_j are convex, θ_{jk} are monotone increasing and demand functions d_l are monotone decreasing of the demand market prices, then the dynamical system* (1.45) *underlying the supply chain is stable.*

The given modelling approach allows to study three different levels of operation of a logistics network: logistical, financial and information. An approach was introduced for three types of locations but can be extended for an arbitrary number of location types and to cover uncertainties like uncertaint customer demand [55]. The behaviour of the logistics networks can be described as a projected dynamical system. The discontinuity of the right-hand side of (1.45) does not allow to apply classical tools for the stability analysis.

Modelling with partial differential equations

Here we investigate the dynamics of the material flows between locations using partial differential equations [58] and the conservation law of material flow [6], [64], [65], [71] and [112]. This modelling approach is usually used to pose optimization problems to minimize the queue length or to maximize the outflow of goods.
In the model introduced in [65] a logistics network is represented as a finite connected graph (V, E) [68]. Each $i \in V$ corresponds to a logistic location and f_i is a material flow on it. Locations are connected to each other at vertices $j \in E$. Each location consist of a processor that has fixed constant processing time τ_i, size(length) l_i and maximal capacity α_i. The location is modelled by a finite interval $[a_i, b_i]$, where $x \in [a_i, b_i]$ is the processing stage of a material at location i. Its queue q_i is located at $x = a_i$. The density of material $\rho_i(x, t)$ for location i at time t and stage $x \in [a_i, b_i]$ satisfies the advection equation (conservation law) with initial conditions $\rho_{i,0}(x)$, see [112]:

$$\begin{aligned} \partial_t \rho_i + \partial_x f_i(\rho_i) &= 0, \quad t \geq 0, && (1.46) \\ f_i(\rho_i) &:= \min\left\{\frac{l_i}{\tau_i}\rho_i, \alpha_i\right\}, && (1.47) \\ \rho_i(x, 0) &= \rho_{i,0}(x) \in [a_i, b_i]. && (1.48) \end{aligned}$$

The governing equation for the corresponding queue $q_i(t)$ depends on the connections of the vertex. In the simplest possible case with one incoming and one outgoing link the queue q_i buffers possible demands for the processor i

$$\begin{aligned} \partial_t q_i(t) &= f_{i-1}(\rho_{i-1}(b_{i-1}, t)) - f_i(\rho_i(a_i, t)), \quad t > 0, && (1.49) \\ q_i(0) &= q_{i,0}. && (1.50) \end{aligned}$$

The boundary condition for the outgoing location i at $x = a_i$:

$$f_i(\rho_i(a_i, t)) = \begin{cases} \min\{f_{i-1}(\rho_{i-1}(b_{i-1}, t)), \alpha_i\} &, \quad q_i(t) = 0, \\ \alpha_i &, \quad q_i(t) > 0. \end{cases} \tag{1.51}$$

To define the solution for (1.46)-(1.51), according to [64], we give first the notions of a solution for a single node without coupling, i.e. for (1.46)-(1.48), and of a solution for the dynamics on an edge (1.49)-(1.51).
Consider the Cauchy problem:

$$\partial_t \rho(x, t) + \partial_x f(\rho) = 0, \quad \rho(x, 0) = \rho(x). \tag{1.52}$$

Definition 1.2.8. *A locally bounded and measurable function* $\rho(x,t)$ *on* $\mathbb{R}\times\mathbb{R}_+$ *is called an* admissible weak solution to (1.52)*, if for any non-decreasing function* $h(\rho)$ *and any smooth non-negative function* ϕ *with compact support in* $\mathbb{R}\times\mathbb{R}_+$*,*

$$\int_0^\infty \int_{-\infty}^\infty (I(\rho)\phi_t + F(\rho)\phi_x) dx dt + \int_{-\infty}^\infty I(\rho_0)\phi(x,0)dx \geq 0,$$

where $I(\rho) = \int^\rho h(\xi)d\xi$ *and* $F(\rho) = \int^\rho h(\xi)df(\xi)$*.*

Compact support of a function is the set of points where the function is not zero.

Definition 1.2.9. *Let functions* $\rho_1(x,t), \rho_2(x,t)$ *in* $L^1([a,b]\times\mathbb{R}_+)$ *be given such that* $\rho_j(\cdot,t)$ *has bounded variation. Let* $q(t) := q_2(t) \geq 0$ *be a an absolutely continuous function on* $[0,T]$ *for* T *sufficiently large and let* $f_j(\rho_j) := \min\{\mu_j, \rho_j\}$*. Then we call* (ρ_1, ρ_2, q) *an* admissible solution at the vertex for all times $0 \leq t \leq T$*, if and only if*

$$\frac{d}{dt}q(t) = f_1(\rho_1(b-,t)) - f_2(\rho_2(a+,t)) \text{ for almost all } t$$

$$f_2(\rho_2(a+,t)) = \begin{cases} \mu_2, & q(t) > 0, \\ \min\{\mu_2, f_1(\rho_1(b-,t))\}, & q(t) = 0. \end{cases}$$

$L^1([a,b]\times\mathbb{R}_+)$ is the space of functions $\rho : [a,b]\times\mathbb{R}_+ \to \mathbb{R}$ with $\int\limits_{[a,b]\times\mathbb{R}_+} |\rho(s)| < \infty$, $a+$ is the right limit of a and $b-$ is the left limit of b. Function ρ_j has *bounded variation on* $[a,b]$ if the total variation over all partitions $P = \{x_0, \ldots, x_{n_P}\}$ of $[a,b]$ given by $\sup\limits_P \sum\limits_{k=1}^{n_P-1} |\rho_j(x_{k+1}) - \rho_j(x_k)|$ is bounded.

Solution of the whole network is then defined as follows.

Definition 1.2.10. *Let* $T > 0$*, values* $q_{i,0} \geq 0, i = 2,\ldots,n$ *and functions* $\rho_{i,0} : [a_i, b_i] \to \mathbb{R}$ *in* L^1 *and with bounded variation for all* $i = 1,\ldots,n$ *be given. The* supply chain problem *then reads with* $f_i = \min\{\mu_i, \rho_i\}$ *and* $\forall i = 1,\ldots,n \quad \forall (x,t) \in (a_j, b_j)\times(0,T)$*,* $\forall i = 2,\ldots,n$

$$\partial_t \rho_i + \partial_x f_i(\rho_i) = 0, \rho_i(x,0) = \rho_{i,0}(x),$$
$$\partial_t q_i(t) = f_{i-1}(\rho_{i-1}(b-,t)) - f_i(\rho_i(a+,t)).$$

We call a family $\rho_i : [a_i, b_i]\times[0,T]$ *of* L^1 *functions with bounded variation and functions* q_j*, absolutely continuous, an* admissible solution to the network problem*, if for each vertex,* $(\rho_j, q_j)_j$ *is an admissible solution at the node in the sense of Definition 1.2.9, and if for all* $i = 2,\ldots,n$*,* $q_i(0) = q_{i,0}$ *and if for all* i*,* ρ_i *is an admissible weak solution for the processor in the sense of Definition 1.2.8.*

Condition for the existence of solution for a network with linear structure, see Figure 1.2a, is given in the following theorem, see [64, Theorem 3.13].

Theorem 1.2.11. *Consider a network of* n *processors and assume:*
(A1) the processors are consecutively labelled, such that processor $i-1$ *is connected at* $x = b_{i-1}$ *to processor* i*;*
(A2) $l_i/\tau_i = 1$ *for all* i*;*
(A3) $\rho_{i,0}(x) \leq \alpha_i$ *for almost all* $x \in [a_i, b_i]$ *and all* i*.*
Consider the problem (1.47) *and* (1.52)*.*
Assume that the initial data $(\rho_{1,0})(x), \ldots, \rho_{n,0}(x))$ *are the step functions. Then the problem* (1.47)*,* (1.52) *has a weak admissible solution constructed by admissible network solutions in the sense of Definition 1.2.10.*

More general interconnections are considered in [65].
Optimization problems considered in this model include minimization of the queue length or maximization of outflow of the goods by controlling the production velocity or choosing the route for the flow [6]. In [40] the Bullwhip effect was studied.
This modelling approach considers dynamics of logistics networks as a continuous flow of goods between and inside locations. The model covers different properties of logistic locations like processing time, queue length and capacity. It is possible to consider different optimization problems [98]. Extension of the model on multiple products is considered in [9]. The use of numerical methods for solving optimization problems with PDEs causes long processing time, [65]. But if some restrictions on the model are introduced, then it is possible to find numerically quasi-optimal solutions.

1.2.3 Hybrid deterministic systems

Shipping of finished products, switching of production rate cause discontinuous changes at the stock level of locations or in the behaviour of production facilities. Here we will show how such discontinuous (hybrid) effects can be described by switched or more generally by hybrid dynamical systems [162].

Modelling by interconnected hybrid dynamical systems

The modelling and analysis framework using hybrid dynamical systems is similar to continuous dynamical systems in (1.32). Again the network is represented as an interconnection of several subsystems that describe logistics locations. However, the dynamics of a subsystem is more general now and is allowed to be additionally discontinuous (discrete) at some time instants. These discontinuities arise when there is an immediate change (jump) in the state of a location. This permits, for instance, a more detailed description of transportation processes. In particular, if the state represents the stock level, then modelling of discrete shipments of material is possible. Moreover, according to the state and the demand, a distinction of the kind of shipping can be drawn, e.g. shipping by a truck, a ship or an airplane.
To model such a hybrid behaviour we use the notion of hybrid dynamical systems from (1.4). We use the set $M_{C_i} \subset \mathbb{R}^{\sum_i N_i + \sum_i M_i}$ to define condition, when the state $x_i \in \chi_i \subset \mathbb{R}_+^{N_i}$, respectively the stock level of location i, changes continuously. Here χ_i describes the values that the state x_i can take. We denote by $u_i \in U_i \subset \mathbb{R}^{M_i}$ the input of the ith location, where U_i describes the values that the input can take. Then the dynamics of location i is given by

$$\dot{x}_i = \widetilde{f}_i(x_1, \ldots, x_n, u_i), \quad (x, u) \in M_{C_i}, \tag{1.53}$$

where $x := (x_1^T, \ldots, x_n^T)^T$, $u := (u_1^T, \ldots, u_n^T)^T$. Here again as in continuous systems in (1.31) and in [42], [43] and [44], we can choose, for example, function

$$\widetilde{f}_i(x_1, \ldots, x_n, u_i) := u_i + \sum_{j, j \neq i} c_{ji} f_j(x_j(t)) - f_i(x_i(t)),$$

where $f_i(x_i(t)) := \alpha_i \left(1 - e^{-x_i(t)}\right)$ and α_i describes the maximal production rate.
The discontinuous changes in the state occur, if $(x, u) \in M_{D_i} \subset \mathbb{R}^{\sum_i N_i + \sum_i M_i}$. And the jumps in the state follow the equation

$$x_i^+ = \widetilde{g}_i(x_1, \ldots, x_n, u_i), \quad (x, u) \in M_{D_i}. \tag{1.54}$$

Function $\widetilde{g}_i$ describes the discrete changes at location i. For example, think of a truck that arrives to location i, and delivers or takes material from it. In this case the time needed to load or upload the truck can be neglected with respect to the time that the location needs to produce one unit of material or the truck needs to transport material to another location. Furthermore, the change in the state depends usually on the state of the location and the truck's capacity. For example, if the capacity of the truck arriving to location i is denoted by b and if the truck picks up the ready-made material from location i, then $\widetilde{g}_i(x_1, \ldots, x_n, u_i) = x_i - \min\{b, x_i\}$ and

$$x_i^+ = x_i - \min\{b, x_i\}.$$

If the truck delivers material to location j, then $\widetilde{g}_j(x_1, \ldots, x_n, u_j) = x_j + \min\{b, x_i\}$ and

$$x_j^+ = x_j + \min\{b, x_i\}.$$

As we will see in Section 2.1, we can describe the overall behaviour of such logistics network as one large hybrid system (1.4)

$$\begin{array}{rcll} \dot{x} & = & \widetilde{f}(x,u), & (x,u) \in M_C, \\ x^+ & = & \widetilde{g}(x,u), & (x,u) \in M_D, \end{array} \qquad (1.55)$$

with $\chi := \chi_1 \times \ldots \times \chi_n$, $U := U_1 \times \ldots \times U_n$, $M_C := \cap M_{C_i}$, $M_D := \cup M_{D_i}$, $f{:=}(f_1^T, \ldots, f_n^T)^T$ and $g{:=}(\widetilde{g}_1^T, \ldots, \widetilde{g}_n^T)^T$, where

$$\widetilde{g}_i(x,u) := \begin{cases} g_i(x,u_i), & \text{if } (x,u) \in M_{D_i}, \\ x_i, & \text{otherwise .} \end{cases}$$

The properties of such a description of an interconnection of hybrid systems we will discuss in Chapter 2.
The perturbation in external inputs like fluctuation of customer demand may lead to instability of such network. Thus, the concept of input-to-state stability introduced in (1.10) needs to be applied to analyse stability of such networks. In this analysis we assume that each individual location i can be autonomously controlled to achieve input-to-state stability of its dynamics, i.e. according to (1.10):

$$|x_i(t,k)| \leq \max\{\beta_i(|x_i^0|, t, k), \max_{j, j\neq i} \gamma_{ij}(\|x_j\|_{(t,k)}), \gamma(\|u_i\|_{(t,k)})\}$$

holds for all times $t \geq 0$, number of intervals between the jumps $k \geq 0$, inputs from other subsystems x_j and external inputs u_i.
However, this property does not in general guarantee that the interconnection of these ISS subsystems will be ISS, see Example (2.4.16). This may occur due to non-effective cooperation between the logistic locations. To check, whether the network is nevertheless ISS, we can use similar approach as in interconnected continuous systems in Section 1.2.2. We consider the gain matrix $\Gamma := (\gamma_{ij})_{n\times n}$. This matrix describes the interconnection structure of the network. And we consider the corresponding matrix operator $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$. As corollary of Theorem 2.4.5 in Chapter 2, it can be shown that the following small gain theorem guarantees ISS of the interconnected hybrid system (1.55):

Theorem 1.2.12. *Consider interconnected system* (1.54). *Assume that $M_{D_i} = M_D$, $i = 1, \ldots, n$ and that the set $\{f(x,u) : u \in U \cap \epsilon\overline{\mathbb{B}}\}$ is convex for each $x \in \chi, \epsilon > 0$. If all subsystems in* (1.53) *are ISS and $\Gamma(s) \not\geq s$ for all s, $s \neq 0$, then the system* (1.55) *is ISS.*

Due to the combination of discrete and continuous dynamics, this modelling approach has more capabilities in precise description of logistics networks than those based only on continuous or discrete dynamical systems introduced in the previous sections. For example, the possibility that the discontinuous changes can be taken into account enables to model additional characteristics like transportation processes. However, the nature of hybrid systems implies more intricated analysis of their dynamics. In Chapter 2 we will discuss more precisely hybrid systems, their interconnections and show how stability of them can be established using the small gain condition. Furthermore, we will show how the Lyapunov technique, that is being effectively exploited in studying stability of continuous systems, can be applied to establish ISS of hybrid systems.

Switched system

We consider manufacturing networks where the machines switch between processing different types of products. Such a switching is modelled using logical-differential equations [136], [137] or billiards [149], [120], [119]. The main problem is to schedule the switching to achieve stable behaviour.
We illustrate this modelling approach by means of logical-differential equations [136]. There are P part-types denoted by $1,\dots,P$ and n machines denoted by the set $M=\{1,\dots,n\}$. Parts of type p are processed at the machines $\mu_{p,1},\dots,\mu_{p,n_p}$ where $\mu_{p,i}\in M$. The parts may visit the machine more then once and then $\mu_{p,i}=\mu_{p,j}$. Raw parts of type p arrive to the system at the machine $\mu_{p,1}$ at a constant rate $u_p>0$. Parts of type p to be processed at the ith machine are waiting in the buffer $b_{p,i}$. From this buffer the products are processed at a given constant rate $\alpha_{p,i}>0$.
The level (state) of the buffer $b_{p,i}$ at time t is denoted by $x_{p,i}(t)$. Part of the type p needs a transportation time $\tau_{p,i}\geq 0$ to get from the machine i to the machine $i+1$. The buffer of the machine l is then given by $B_l:=\{b_{p,i}:\mu_{p,i}=l\}$. To switch from the processing of the part of one type to the part of another type the machine l needs a set-up time $\delta_l^0>0$.
The dynamics of the system we describe using logical-differential equations . The state of the machine l is described by a symbolic variable $q_l(t)$ that is given as follows

$$q_l(t):=\begin{cases}0, & \text{if the machine } l \text{ does not work at time } t,\\ b_{p,i}, & \text{if the machine } l \text{ works with the buffer } b_{p,i} \text{ at time } t.\end{cases}\tag{1.56}$$

The amount of parts of the type p fully processed at time t is denoted by $y_p(t)$.
The change of the state of the buffer $b_{p,1}$ is described as follows

$$\dot{x}_{p,1}(t):=\begin{cases}u_p-\alpha_{\mu_{p,1}}, & \text{if } q_{\mu_{p,1}}(t)=b_{p,1},\\ u_p, & \text{if } q_{\mu_{p,1}}(t)\neq b_{p,1}.\end{cases}\tag{1.57}$$

The change of the state of the buffer $b_{p,i}$, $i=2,\dots,n_p$ is given by

$$\dot{x}_{p,i}(t):=\begin{cases}\alpha_{\mu_{p,i-1}}-\alpha_{\mu_{p,i}}, & \text{if } q_{\mu_{p,i}}(t)=b_{p,1} \text{ and } q_{\mu_{p,i-1}}(t-\tau_{p,i-1})=b_{p,i-1},\\ -\alpha_{\mu_{p,i}}, & \text{if } q_{\mu_{p,i}}(t)=b_{p,1} \text{ and } q_{\mu_{p,i-1}}(t-\tau_{p,i-1})\neq b_{p,i-1},\\ \alpha_{\mu_{p,i-1}}, & \text{if } q_{\mu_{p,i}}(t)\neq b_{p,1} \text{ and } q_{\mu_{p,i-1}}(t-\tau_{p,i-1})=b_{p,i-1},\\ 0, & \text{if } q_{\mu_{p,i}}(t)\neq b_{p,1} \text{ and } q_{\mu_{p,i-1}}(t-\tau_{p,i-1})\neq b_{p,i-1}.\end{cases}\tag{1.58}$$

The change in the overall amount of parts of type p (cumulative output) processed by the machine $\mu_{p,i}$ is given as follows

$$\dot{y}_{p,i}(t):=\begin{cases}\alpha_{\mu_{p,i}}, & \text{if } q_{\mu_{p,1}}(t)=b_{p,1},\\ 0, & \text{if } q_{\mu_{p,1}}(t)\neq b_{p,1}.\end{cases}\tag{1.59}$$

The rule for machine l for switching from processing parts from buffer $b\in B_l$ to buffer $b'\in B_l$ is given by the set $\mathcal{T}_l(b\to b')$ and functions $F_l:([q_l(\cdot),x_b(\cdot)]|_0^t)\to[\delta_l^0,\infty)$. Then the feedback policy is given by

$$\text{if } (q_l(t) = b) \text{ and } [q_l(\cdot), x_b(\cdot)]|_0^t \in \mathcal{T}_l(b \to b') \text{ then} \tag{1.60}$$

$$\begin{pmatrix} \delta_l(t) := F_l([q_l(\cdot), x_b(\cdot)]|_0^t); \\ q_l(\hat{t}) := 0 \quad \forall \hat{t} \in (t, t + \delta_l(t)]; \\ q_l(t + \delta_l(t) + 0) := b' \end{pmatrix}. \tag{1.61}$$

The state of the manufacturing system at time t is then described by vectors $q(t) = [q_1(t), \dots, q_n(t)]$, $x(t) = \{x_{p,i}(t)\}$ and $y(t) = \{y_{p,i}(t)\}$.
This model is used for the calculating of stable or optimal scheduling policies by choosing appropriate arrival rates u_p and set-up times $\delta_l \geq \delta_l^0$.

Definition 1.2.13. *The closed-loop system* (1.56)-(1.60) *is said to be* stable *if for any solution* $[q(t), x(t)]$ *to the system with initial condition* $x(0) = x_0$, $y(0) = 0$, $q(t) = 0\ \forall t < 0$, *the vector function* $x(t)$ *is bounded on* $[0, \infty)$ *by a constant* $L(x_0) > 0$.

Definition 1.2.14. *The closed-loop system* (1.56)-(1.60) *is said to be* regular with the production levels $d_1, d_2, \dots, d_P$ and the scheduling period T *if it is stable and the following condition holds:*
For any solution $[q(t), x(t)]$ *to the system with initial conditions* $x(0) = x_0$, $y(0) = 0$, $q(t) = 0$ $\forall t < 0$, *the output* $y(t)$ *satisfies:*

$$\lim_{j \to \infty} (y_{p,i}((j+1)T) - y_{p,i}(jT)) = d_p, \forall p, \forall i.$$

Definition 1.2.15. *Assume that* $d_1, d_2, \dots, d_P$ *are given. The minimal time* T_0 *for which there exist constants* $u_1, \dots, u_P$ *and a feedback policy of the form* (1.60) *such that the closed-loop system is regular with the production levels* $d_1, d_2, \dots, d_P$ *and the scheduling period* T_0, *is called* the minimal scheduling period of the system with the production levels $d_1, d_2, \dots, d_P$.

The following theorem provides conditions for the regularity of system (1.56)-(1.60) and an estimate for the minimal scheduling period, see [137, Theorem 3.1].

Theorem 1.2.16. *Consider the flexible manufacturing system defined by its production paths* $\mu_{p,1}, \dots, \mu_{p,n_p}$ *of the part-types, machine rates* $\alpha_{p,i}$, *minimal machine set-up times* δ_l^0, *and transportation delays* $\tau_{p,i}$. *Let* $d_1, d_2, \dots, d_P$ *be given. Then, the following statements hold:*
(1) The minimal scheduling period T_0 *of this system with the production levels* $d_1, d_2, \dots, d_P$ *is defined by*

$$T_0 = \max_{l=1,\dots,n} \left[k_l \delta_l^0 + \sum_{b(p,i) \in B_l} \frac{d_p}{\alpha_{p,i}} \right],$$

where k_l *is the number of buffers in* B_l.
(2) For any $T \geq T_0$, *the closed-loop system with the part arrival rates* $u_p = \frac{d_p}{T}$ *is regular with the production levels* $d_1, d_2, \dots, d_P$ *and the scheduling period* T *if:*
the feedback policy is defined by:

$$\textit{if } q_l(t) = b(b, i) \textit{ and } \left(x_{p,i}(t) = 0 \textit{ or } t - \theta_l[q_l(\cdot)|_0^t] = \frac{d_p}{\alpha_{p,i}} \right),$$

$$\textit{then} \begin{pmatrix} \delta_l(t) := \delta_l^n + \frac{d_p}{\alpha_{p,i}} - t + \theta_l[q_l(\cdot)|_0^t] \\ q_l(\hat{t}) := 0 \quad \forall \hat{t} \in (t, t + \delta_l(t)] \\ q_l(t + \delta_l(t) + 0) := \mathbf{next}[b(p,i)] \end{pmatrix},$$

$$\textit{where } \theta_l[q_l(\cdot)|_0^t](t) := \inf\{t_0 \le t : q_l(s) = q_l(t) \quad \forall s \in (t_0, t]\};$$

$$\delta_l^n := \frac{T - \sum\limits_{b(p,i)\in B_l} \frac{d_p}{\alpha_{p,i}}}{k_l};$$

and cycling switching order is given by: $b_1 \mapsto b_2 \dots b_{k_l} \dots b_1$.

Here **next**[b] is the buffer that is the next to b in the cycling switching order.
This modelling approach allows to describe manufacturing networks that possess switching between processing of different types of products. Multiple processing of the product at the same machine is possible. The model allows to calculate processing policy of the machine to obtain stable or optimal behaviour. Queues and time delays are covered by the model. Queues are modelled by buffers and time delays by transportation times. The hybrid nature of the model does not allow to apply methods from continuous systems during their analysis. Moreover, chaotic effects ('strange' billiards) may occur due to the switching [119].

1.2.4 Stochastic models

Here we model logistics networks that experience random effects due to the fluctuation of customer demand or the changes in the manufacturing processes using the theory of stochastic systems [77], [12] and [26].

Modelling as a stochastic dynamical system

Stochastic dynamical systems describe behaviour of logistics networks driven by an unknown customer demand [94], [161], [56] and [23]. Usually, such models are used to consider optimization problems under certain assumptions on the unknown demand, e.g. maximum possible value. In [94] the network is modelled as a stochastic discrete-time controlled dynamical system [19]:

$$x(t+1) = \widetilde{f}(x(t), u(t), d(t)), \tag{1.62}$$

where $x(t) \in \mathbb{R}^{n_x}$ denotes the state of the system, $u(t) \in \mathbb{R}^{n_u}$ the control input and $d(t) \in \mathbb{R}^{n_d}$ the uncertain disturbances in customer demand at time t.
The linear map $\widetilde{f} : \mathbb{R}^{n_x+n_u+n_d} \longrightarrow \mathbb{R}^{n_x}$ depends on the structure of the network. The structure is described by a directed and connected graph $G = (V, E)$, where $V := \{v_1, \dots, v_{n_V}\}$ denotes the set of vertices and represents the logistics locations. $E := \{e_1, \dots, e_{n_s}\} \subseteq V \times V$ denotes the arcs and represents the material flows and additional arcs. $R := \{r_1, \dots, r_{n_r}\} \subseteq V \times V$ represents the informational flows of orders. If there exists an arc (material flow) from node v to node w in E, then there exists an arc (information flow) from w to v in R.
In order to obtain a first-order difference equation (1.62), the authors introduce additional artificial nodes between two nodes in such way that the transportation time between any nodes(real and artificial) is exactly one time unit, see [94]. Thus an arc $e_i \in E$ is replaced by a path(chain) of arcs $E_i := \{e_{i1}, \dots, e_{i\tau_{e_i}}\}$ and the set of additional nodes is defined by $V_i = \{\sigma_{i1}, \dots, \sigma_{i\tau_{e_i}}\}$, where $\tau_{e_i} \in \{1, 2, \dots, \}$ is the transportation time of arc e_i. In the same way arcs of the information flows are replaced by appropriate paths. The new vertex set is $\widetilde{V}$, the new arc set for material flow is $\widetilde{E}$ and the new set for the information flow is $\widetilde{R}$.
The state $x^{(v)}(t)$ of location v is its inventory level at time t. Its input is $u^{(v)}(t) = [u_E^{(v)}(t), u_R^{(v)}(t)]^T \in \mathbb{R}^{\check{\delta}^{(v)}+\check{\delta}'^{(v)}}$, where $\check{\delta}^{(v)}$ is the number of all incoming material flow arcs and $\check{\delta}'^{(v)}$ is the number of all

incoming information flow arcs. The input corresponds to the flow on the incoming arcs. Its output is $y^{(v)}(t) = [y_E^{(v)}(t), y_R^{(v)}(t)]^T \in \mathbb{R}^{\hat{\delta}^{(v)}+\hat{\delta}'^{(v)}}$ and corresponds to the flow on the outgoing arcs at time t. The dynamics is described then by the map

$$\begin{aligned} x^{(v)}(t+1) &= x^{(v)}(t) + \epsilon^{\check{\delta}^{(v)}} \cdot u_E^{(v)}(t) - \epsilon^{\hat{\delta}^{(v)}} \cdot z_E^{(v)}(t), &(1.63)\\ y_R^{(v)}(t) &= z_R^{(v)}(t), &(1.64)\\ y_E^{(v)}(t) &= z_E^{(v)}(t), &(1.65)\end{aligned}$$

where $\epsilon^k \in \mathbb{R}^k$ denotes a row vector of ones. The values $z_E^{(v)}(t)$ are the goods that node v decides to ship to its customers and $z_R^{(v)}(t)$ are the orders that node v decides to place to each of its suppliers. Thus, the dynamics of the network defined in (1.63) is determined by the change of inventory level due to arriving of material and due to shipping of goods. The values z_E and z_R are control variables for the system.
The nodes $v_C \in V$ with no outgoing material flow arc are called *consumers* and form set V_C. They generate the uncertain demand $d^{(v_C)}$ and thus are given by

$$y_R^{(v_C)}(t) = d^{(v_C)}(t). \tag{1.66}$$

The nodes $v_M \in V$ with no incoming material flows act as sources of infinite supply capacity and form set V_M. They transform incoming orders into outgoing goods:

$$y_S^{(v_M)} = u_R^{(v_M)}(t). \tag{1.67}$$

The dynamics of the whole system is derived by connecting the corresponding inputs $u^{(v)}$ and the outputs $y^{(v)}$. Thus, they are eliminated. Let the nodes $v_i \in V_z := \widetilde{V} \setminus (V_C \cup V_M), w_j \in V_x := V \setminus (V_C \cup V_M)$ and $w'_k \in V_C$ be indexed such that $1 \le i \le |V_z|, 1 \le j \le |V_x|$ and $1 \le k \le |V_C|$. Defining $x(t) := [x^{(w_1)}, \dots, x^{(w_{|V_x|})}]^T$ as the state vector, $u(t) := [z_1(t), \dots, z_{|V_z|}(t)]^T$ with $z_i(t) := [[z_S^{(v_i)}]^T [z_R^{(v_i)}(t)]]$ as the input vector, and $d(t) := [[d^{(w'_1)}(t)]^T \dots [d^{(w'_{|V_C|})}(t)]^T]^T$ as the disturbance vector gives a first-order difference equation of the form (1.62). As the equations (1.63) are linear, the dynamics of the entire system can be written as (1.5)

$$x(t+1) = Ax(t) + Bu(t) + Ed(t). \tag{1.68}$$

The model allows to find the optimal control $u(t)$ for the network that guarantees the minimum of costs and the maximum of demand satisfaction. We consider a discrete optimization problem with the horizon $S \in \mathbb{N}$ that looks for the optimal input sequence $(u^{(s)})_{s=0}^S$ and is described on the step $s \in \mathbb{N}$ as

$$J^{*(s)}(x^{(s)}) = \min_{u^{(s)}} J^{(s)}(x^{(s)}, u^{(s)}) \tag{1.69}$$

such that for all $d^{(s)} \in \mathcal{D}$ the following holds

$$Fx^{(s)} + Gu^{(s)} \le g, \tag{1.70}$$

$$Ax^{(s)} + Bu^{(s)} \in \mathcal{X}^{(s)}, \tag{1.71}$$

where the goal function J^* is given by

$$J^{(s)}(x^{(s)}, u^{(s)}) := \max_{d^{(s)} \in \mathcal{D}} \{|Wx^{(s)}|_1 + |Ru^{(s)}|_1 + J^{*(s)}(x^{(s+1)})(Ax^{(s)} + Bu^{(s)} + Ed^{(s)})\}. \tag{1.72}$$

This goal function minimizes the costs given at the step s by $Wx^{(s)}, W \in \mathbb{R}^{n_x \times n_x}$ for the state, $Ru^{(s)}, R \in \mathbb{R}^{n_u \times n_u}$ for the input and maximizes the satisfaction of the customer demand given by $Ed^{(s)}$. The disturbance of customer demand is bounded and given on $\mathcal{D} := \{d : \Lambda d \leq \gamma\} \subset \mathbb{R}^{n_d}$, where $\Lambda \in \mathbb{R}^{n_\gamma \times n_d}, \gamma \in \mathbb{R}^{n_\gamma}, n_\gamma \in \mathbb{N} \cup \{0\}$. Inequality (1.70) describes the constraint on the state and the input, where $F \in \mathbb{R}^{n_g \times n_x}, G \in \mathbb{R}^{n_g \times n_u}, n_g \in \mathbb{N} \cup \{0\}$. $\mathcal{X}$ is the set of the feasible states given by $\mathcal{X}^{(s)} := \{x \in \mathbb{R}^{n_x} : \forall d \in \mathcal{D}\, \exists u \in \mathbb{R}^{n_u}$ with $Fx + Gu \leq g$ and $Ax + Bu + Ev \in \mathcal{X}^{(s+1)}\}$. The initial state vector is given by $x^{(0)} = x(t_0) := x(0)$. The boundary condition is given by $J^{*S}(x^S) = 0$ and $\mathcal{X}^{(S)} = \{x \in \mathbb{R}^{n_x} : Fx \leq g\}$. The finite time S-step optimal control $u^* : \mathbb{R}^{n_x} \to \mathbb{R}^{n_d}$ is given by the first component $u^{(0)}$ for a fixed horizon S and the infinite-time optimal control law is its limit for $S \to \infty$.

Such discrete optimization problems are usually solved using the dynamic programming [18]. Furthermore, in [19, Theorem 3] it was proved that each dynamic programming step can be solved by a multi-parametric linear program.

Theorem 1.2.17. *The solution u^* with parametric uncertainties in the B matrix only, is a piecewise affine function of $x(0) \in \chi_0$, where χ_0 is the set of initial states for which a solution to* (1.69)-(1.71) *exists. It can be found by solving a multi-parametric linear program.*

This approach takes into account an uncertainty in a customer demand. The model deals with transportation times by adding "delay nodes". Such a transformation allows to consider only networks with linear dynamics.

Multiclass queueing networks and fluid approximations

Multiclass queueing networks are a well-established modelling approach to capture stochastic events that influence the discrete material flow of a supply chain [36], [145], [148]. The main problem is to arrange the production rates and policy to achieve the bounded queue length [147]. Here only a brief description of a multiclass queueing network is given, for details see [38].

The network consists of n locations that process P different types of products. The dynamics of the network can be described by the following stochastic processes. The arrival process $u_p(t)$ describes the number of external arrivals of type p products in the time period $[0, t]$. The production process $x_p(t)$ reflects the number of finished products of type p during the first t time units. For convenience we assume that each type of product is produced exclusively at one location. The mapping $s : \{1, \ldots, P\} \to \{1, \ldots, j\}$ determines which type is produced at which location and generates the constituency matrix C, where $c_{jp} = 1$ if $s(p) = j$ and 0 otherwise.

After being processed, products either change their type according to a given probability or leave the network. The routing process $d_p^l(n)$ denotes the number of type l products among the first n products that become products of type p. As each location can produce various product types, a policy is needed that determines in which order the products are processed. Typical examples of such service disciplines are first-in-first-out (FIFO), priority or processor sharing. The allocation process $\tau_p(t)$ denotes the total amount of time that location $s(p)$ has devoted on producing type p products. The initial amount of type p products is $q_p(0)$ and the number of type p products at time t is given by the flow-balance equation

$$q_p(t) = q_p(0) + u_p(t) + \sum_{l=1}^{P} d_p^l(x_l(\tau_l(t))) - x_p(\tau_p(t)). \tag{1.73}$$

To obtain a complete description of the network dynamics further conditions on q and τ that depend on the service discipline have to be taken into account, see e.g. [31] and [32]. The main question in such model is whether the network is stable. A queueing (service) discipline of a multiclass network is *stable*, if the underlying Markov process describing the network dynamics is positive Harris recurrent [38]. When there is no ambiguity in the underlying queueing discipline, we say that a queueing network is stable, if the queueing discipline under discussion is stable. Roughly speaking (without explaining of the Harris recurrence), a queueing network is said to be stable, if the total number of products in the network remains bounded over all time. This can also be interpreted that the long-run input rate of the network equals the long-run output rate. An approach to analyse the stability of multiclass queueing networks is to rescale the stochastic processes and to take limits [38]. The so-called fluid limit model is obtained by replacing the stochastic processes by their rates, i.e.

$$\begin{array}{l} \frac{1}{t}\alpha_p(t) \to \rho_p, \\ \frac{1}{t}x_p(t) \to \mu_p, \\ \frac{1}{t}u_p^l(t) \to \gamma_{lk}. \end{array} \tag{1.74}$$

and by imposing additional specific conditions on the network parameters and queueing discipline, see [38] and [39]. The flow-balance equation in the continuous deterministic fluid model takes the form

$$q_p(t) = q_p(0) + \rho_k(t) + \sum_{l=1}^{P} \gamma_{lp}\mu_l(\tau_l(t)) - \mu_p\tau_p(t). \tag{1.75}$$

Again there are additional conditions on q and τ that are specific to the service discipline, see e.g. [31], [32]. A fluid limit model is *stable*, if for all $p \in \{1, \dots, P\}$ there is a time $\tau > 0$ such that for any $q_p(\cdot)$ with $\sum_p q_p(0) = 1$ it holds that $q_p(\tau + \cdot) \equiv 0$ [38].

In [39, Theorem 1.1] it was shown the following relation between stability of fluid and corresponding multiclass queueing networks.

Theorem 1.2.18. *A queueing discipline is stable, if the corresponding fluid model is stable.*

In [39] several results on stability of fluid networks were proved. In particular, the following theorem establish stability of fluid networks with different disciplines, see [39, Theorems 4.3 and 4.4]

Theorem 1.2.19. *The fluid model corresponding to the First-Buffer-First- Served(FBFS) or Last-Buffer-First-Served(LBFS) discipline is stable.*

This modelling approach is suitable, if the supply chain has highly reentrant flows. Further, there is huge variety of different service disciplines, which can be explicitly modelled in this framework. So simulations of different scenarios allow the choice of a policy, that is suitable to the requirements. Moreover, the strength of this approach is that analysis of the influence of stochastic uncertainties (e.g. production times, transportations etc.) on the stability is possible by purely deterministic criteria. In [147] the robustness of such networks is investigated using the notion of the stability radii, i.e. the size of the smallest changes of the network parameters that destabilize the network.

1.3 Comparison of the modelling approaches

In the previous subsections we have reviewed eleven approaches for modelling of logistics networks. For short overview of the these applicabilities, we have collected the main properties of all the models in Table 1.1. These properties are classified in this comparison table according to ability to cover different logistics properties of the network, ability to describe production and transportation processes, type of behaviour the network possesses, type of equations that describe their dynamics, type of the analysis and abilities to conduct planning and control. According to Table 1.1, all the approaches vary in their characteristics, advantages and disadvantages. Thus, the choice of the model that describes the dynamics of certain logistics network depends on the type of behaviour it possesses, its properties and application needs.

In the following chapter we will study an approach that proposes to model logistics networks as interconnected hybrid system (1.55) as we are interested in logistics networks that combine both continuous and discontinuous dynamics of material flows. In particular, we will show how input-to-state stability of a hybrid system can be established using the small gain condition and Lyapunov methods.

Modelling approach	"Bucket brigade"	Decentralized supply chain	Re-entrant/queueing system	Ordinary differential equation	Damped oscillator model	Multilevel network	Partial differential equation	Hybrid system	Switched system	Stochastic system	Queueing/fluid network
Logistical properties											
Micro level	x		x	x	x			x	x	x	x
Macro level		x		x	x	x	x	x		x	x
Continuous material flow	x			x	x	x	x	x	x	x	x
Discrete material flow		x	x				x	x		x	
Production process	x		x	x	x	x	x	x	x	x	x
Transportation process		x	x	x		x	x	x	x	x	x
Warehouse		x	x	x	x	x	x	x	x	x	x
Re-entrant		x	x	x	x		x	x	x	x	x
Network		x	x	x	x	x	x	x	x	x	x
Production process											
Service disciplines	x	x	x	x			x	x	x		x
Different production rates	x		x	x	x	x	x		x		x
Different product types			x						x		x
Transportation process											
Routing disciplines			x	x			x	x	x	x	x
Restricted capacity				x			x	x			
Transportation time		x		x			x	x	x	x	
Behaviour											
Discrete		x	x					x		x	
Continuous				x	x		x	x			x
Hybrid	x					x		x	x		
Linear	x	x	x	x				x	x	x	x
Nonlinear				x	x	x	x	x			x
Deterministic	x	x	x	x	x	x	x	x	x		x
Stochastic		x								x	x
Type of equation											
Discrete	x	x	x					x		x	
Ordinary differential equation				x	x	x		x			
Partial differential equation						x	x				
Hybrid						x		x	x	x	
Queue			x								x
Fluid					x						x
Analysis											
Stability	x	x	x	x	x	x	x	x	x	x	x
Input-to-state stability				x				x			
Bullwhip effect		x			x		x				
Robustness	x	x		x	x		x	x	x	x	x
Planning and control											
Stabilizing control				x				x	x		
Optimal control			x	x			x	x	x		x
Planning	x	x	x	x	x	x	x	x	x	x	x

Table 1.1: Classification of the modelling approaches

Chapter 2

Stability of interconnected hybrid systems

Though a hybrid system of the form (1.55) appears to be a natural way to describe logistics networks, its analysis is rather sophisticated due to the complex interconnection topology and hybrid effects (Zeno solution, dwell time) occurring there.

The first notion on hybrid systems was probably given in [163] where a system with some continuous and some discontinuous states was investigated. The change (transition) in the discrete state occurs, when the continuous state reaches a predefined transition set. In [1], [156] authors considered hybrid systems, where again some states are continuous and some discontinuous. But there the dynamics of discontinuous state was described by the difference equation. In the last decades most of the authors consider hybrid systems where each state can have both continuous and discontinuous dynamics, see e.g. [62], [162], [131], [102], [160], [133], [67] and [35].

Starting from [99], many results were obtained in the stability analysis of hybrid systems, see for example [73], [101], [122]. ISS was first introduced for hybrid systems in [27]. However, ISS of interconnected systems was first studied for continuous and discrete systems, see [82], [81], [91], [3], [29], [79], [54], [52] and [126]. The first result of the small gain type was proved for continuous systems in [82] for a feedback interconnection of two ISS systems. The Lyapunov version of this result is given in [81]. The small gain condition in [81] states that the composition of the gains from the subsystems should be less than identity. The second small gain condition in [82] is similar, but it involves the composition of the gains and of further functions of the form $(\mathrm{id} + \alpha_i)$. This difference is due to the use of different definitions of ISS in both papers. Both definitions are equivalent but the gains enter as a maximum in the first definition, and a sum of the gains is taken in the second one. These results were generalized for an interconnection of $n \geq 2$ systems in [54], [52], [126], [85] and [86]. In [54] and [52] it was pointed out that the difference between the small gain conditions remains, i.e., if the gains of different inputs enter as a maximum of gains in the ISS definition or a sum of them is taken in the definition.

ISS in terms of trajectories for interconnection of two interconnected hybrid systems was firstly studied using the small gain condition in [96]. In [110] a stability condition of a small gain type was used for a construction of an ISS-Lyapunov function of a feedback connection of two hybrid systems. An interconnection of an arbitrary number of sampled-data systems that are a special class of hybrid systems was considered in [86]. The small gain condition was given there in terms of vector Lyapunov functions.

In this chapter we show how stability analysis of the system (1.55) can be conducted. In particular, we give a precise definition of the hybrid system, show how its interconnection can be described and introduce a more general definition of input-to-state stability for hybrid systems than in (1.10). Such formulation of ISS, that we will call mixed ISS, allows to consider more general types of interconnections, where a part of the subsystems is ISS in terms of maximizations and the rest is ISS in terms of summations. Then we extend the small gain conditions for continuous systems to be applied to interconnections of an arbitrary finite number of hybrid ISS systems in terms of mixed formulation of ISS. The obtained mixed small gain condition is based on information about the interconnection property of the network and as we will see guarantees ISS of an interconnected hybrid system. Furthermore, we show how a Lyapunov technique can be applied to establish ISS of hybrid systems. At the end of this chapter we apply this small gain condition to establish input-to-state stability of certain subclasses of hybrid systems.

2.1 Interconnected hybrid systems

As mentioned in the previous chapter, hybrid systems combine both continuous and discontinuous types of behaviour. The continuous dynamics of the system is usually described by ordinary differential equations and the discontinuous dynamics by an immediate change ("jump") in the state.
Consider a system that is an interconnection of n hybrid subsystems with states $x_i \in \chi_i \subset \mathbb{R}^{N_i}$ of the subsystems and external inputs $u_i \in U_i \subset \mathbb{R}^{M_i}$, $i = 1, \ldots, n$. The dynamics of the ith subsystem is given by

$$\begin{aligned} \dot{x}_i &= f_i(x_1, \ldots, x_n, u_1, \ldots, u_n), \quad (x_1, \ldots, x_n, u_1, \ldots, u_n) \in C_i, \\ x_i^+ &= g_i(x_1, \ldots, x_n, u_1, \ldots, u_n), \quad (x_1, \ldots, x_n, u_1, \ldots, u_n) \in D_i, \end{aligned} \tag{2.1}$$

where $f_i : C_i \to \mathbb{R}^{N_i}, g_i : D_i \to \chi_i$ and C_i, D_i are subsets of $\chi_1 \times \ldots \times \chi_n \times U_1 \times \ldots \times U_n$. We will consider further a particular case with $f_i(x_1, \ldots, x_n, u_1, \ldots, u_n) = f_i(x_1, \ldots, x_n, u_i)$, $g_i(x_1, \ldots, x_n, u_1, \ldots, u_n) = g_i(x_1, \ldots, x_n, u_i)$, i.e. the case where the ith input u_i influences only the ith the subsystem.
Each hybrid subsystem is described by $(f_i, g_i, C_i, D_i, \chi_i, U_i)$. The function f_i describes the continuous dynamics defined on the set C_i, the function g_i describes the instantaneous jumps defined on the set D_i. If $(x_1, \ldots, x_n, u_1, \ldots, u_n) \in C_i$, then system (2.1) "flows" continuously and the dynamics is given by the function f_i. If $(x_1, \ldots, x_n, u_1, \ldots, u_n) \in D_i$, then the system jumps instantaneously according to the function g_i. In points in $C_i \cap D_i$ the system may either jump or flow, the letter only if the flowing keeps $(x_1, \ldots, x_n, u_1, \ldots, u_n) \in C_i$. This yields the non-uniqueness of solutions. In Proposition 2.1.1 we recall the condition that guarantees the existence of solutions of a hybrid system and in Proposition 2.1.2 we recall the uniqueness condition. However, first we need to define the notion of the solution of the hybrid system.
Define $\chi := \chi_1 \times \ldots \times \chi_n$, $U := U_1 \times \ldots \times U_n$. Solutions of the hybrid systems are usually defined on hybrid time domains. Hybrid time domains are defined as follows, cf. [131], [60], [110]. A subset $\mathbb{R}_+ \times (\mathbb{N} \cup \{0\})$ is called *hybrid time domain* denoted by *dom*, if it is given as a union of finitely or infinitely many intervals $[t_k, t_{k+1}] \times \{k\}$, where the numbers $0 = t_0, t_1, \ldots$ form a finite or infinite, nondecreasing sequence of real numbers. The "last" interval is allowed to be of the form $[t_K, T) \times \{K\}$ with T finite or $T = +\infty$. Roughly speaking, the hybrid time domain contains two types of information: the whole time and the time of the state jumps.
One of the reasons to introduce the hybrid time domains was the ability to study the robustness of solutions of hybrid systems by comparing the neighbour trajectories. Without taking into account the jump times of solution trajectories, the measured pointwise distance between trajectories that are "graphically close" may be arbitrarily large at the points where one solution jumps and the neighbour one does not. This yields that these trajectories do not converge in terms of pointwise distance [125], [60]. Thus, the analysis of solution properties like the robustness becomes rather sophisticated. On

the other hand, the hybrid time domain concept allows to study the convergence of "graphically close" trajectories using the notion of graphical convergence, e.g., [125], [60] and [63].
The *hybrid signal* is a function defined on the hybrid time domain. For the ith subsystem the *hybrid input*

$$v_i := (x_1^T, \ldots, x_{i-1}^T, x_{i+1}^T, \ldots, x_n^T, u_i^T)^T, \tag{2.2}$$

consists of the hybrid signals u_i : dom $u_i \to U_i \subset \mathbb{R}^{M_i}$, x_j : dom $x_j \to \chi_j, j \neq i$ such that $u_i(\cdot, k), x_j(\cdot, k)$ are Lebesgue measurable and locally essentially bounded for each k. We call x_j the *internal input* for the subsystem i and u_i is called the *external input*. For a signal u_i : dom $u_i \to U_i \subset \mathbb{R}^{M_i}$ we define its restriction to the interval $[(t_1, j_1), (t_2, j_2)] \in$ dom u_i by

$$u_{i[(t_1,j_1),(t_2,j_2)]}(t,k) = \begin{cases} u_i(t,k), & \text{if } (t_1,j_1) \leq (t,k) \leq (t_2,j_2), \\ 0, & \text{otherwise,} \end{cases}$$

where for the elements of the hybrid time domain we define that $(s, l) \leq (t, k)$ means $s + l \leq t + k$. For convenience, we denote $u_{i(t,k)} := u_{i[(0,0),(t,k)]}$.

The *hybrid arc* of the subsystem i is such a hybrid signal x_i : dom $x_i \to \chi_i$ that $x_i(\cdot, k)$ is locally absolutely continuous for each k. Define $x := (x_1^T, \ldots, x_n^T)^T \in \chi \subset \mathbb{R}^N$, $u := (u_1^T, \ldots, u_n^T)^T \in U \subset \mathbb{R}^M$, $N := \sum N_i$, $M := \sum M_i$. A hybrid arc and a hybrid input is a solution pair (x_i, v_i) of the ith hybrid subsystem (2.1), if
(i) dom x_i = dom u_i = dom x_j, $j \neq i$ and $(x(0,0), u(0,0)) \in C_i \cup D_i$,
(ii) for all $k \in \mathbb{N} \cup \{0\}$ and almost all $(t,k) \in$ dom x_i, for $(x(t,k), u(t,k)) \in C_i$, it holds

$$\dot{x}_i(t,k) = f_i(x_1(t,k), ..., x_n(t,k), u_i(t,k)), \tag{2.3}$$

(iii) for all $(t,k) \in$ dom x_i such that $(t, k+1) \in$ dom x_i, for $(x(t,k), u(t,k)) \in D_i$ it holds

$$x_i(t,k+1) = g_i(x_1(t,k), ..., x_n(t,k), u_i(t,k)). \tag{2.4}$$

The variable t denotes the time and k is the number of the interval between the jumps.
In the hybrid time domain concept one needs to find a solution x_i first and then to determine the hybrid time domain for it [131]. Hybrid time domains allow also for a description of the so-called Zeno solutions that are solutions with infinitely many jumps in a finite amount of time, i.e. with $t < \infty$ and $k \to \infty$, see [164], [60] and [131].
A solution pair of a hybrid system is *maximal*, if it cannot be extended. It is *complete*, if its hybrid time domain is unbounded. In particular, a complete solution may have the bounded time and the unbounded number of jumps (in case of Zeno solutions), or the unbounded time and the bounded number of jumps, or both unbounded. Let $S_u(x^0)$ be the set of all maximal solution pairs (x, u) to (2.5) with $x(0,0) = x^0$.
For the existence of solutions we assume that the following basic regularity conditions [28], [63] hold throughout the thesis:

Assumption (Basic regularity condition)**.**
(i) χ_i is open, U_i is closed, and $C_i, D_i \subset \chi \times U$ are relatively closed in $\chi \times U$;
(ii) f_i, g_i are continuous.

Their sufficiency for the existence of solutions of a hybrid system of the form (2.1) without internal and external inputs was proved in [63, Proposition 2.4]:

Proposition 2.1.1 (Existence of solutions)**.** *Assume basic regularity conditions (i)-(ii) hold. If $x_i^0 \in D_i$ or the following condition holds:*
(VC) $x_i^0 \in C_i$ and for some neighborhood W_i of x_i^0, for all $x_i' \in W_i \cap C_i$, $T_{C_i}(x_i') \cap f_i(x_i') \neq \emptyset$, then there exists a solution x_i to (2.1) *with $x_i(0,0) = x_i^0$ and $\operatorname{dom} x_i \neq (0,0)$.*
If (VC) holds for all $x_i^0 \in C_i \setminus D_i$, then for any maximal solution x_i at least one of the following statements is true:
1) x_i is complete;
2) x_i eventually leaves every compact subset of χ_i: for any compact $K \subset \chi_i$, there exists $(T,J) \in \operatorname{dom} x_i$ such that for all $(t,j) \in \operatorname{dom} x_i$ with $(T,J) < (t,j)$, $x_i(t,j) \notin K$;
3) for some $(T,J) \in \operatorname{dom} x_i$, $(T,J) \neq (0,0)$, we have $x_i(T,J) \notin C_i \cup D_i$.
Case 3) above does not occur if additionally
(VD) for all $x_i^0 \in D_i$, $g_i(x_i^0) \in C_i \cup D_i$.

According to definition of tangent cone in Section 1.1.4, condition $T_{C_i}(x_i') \cap f_i(x_i') \neq \emptyset$ means that the vector $f_i(x_i')$ points towards the set C_i. This guarantees that we can construct a solution belonging to the set C_i.
The following proposition guarantees the uniqueness of a solution of a hybrid system of the form (2.1) without internal and external inputs and follows from [62, Proposition S5] as a particular case.

Proposition 2.1.2 (Uniqueness of solution)**.** *Uniqueness of solutions holds for a hybrid system (f_i, g_i, C_i, D_i) if and only if the following conditions hold:*
1) For each initial point $x_i^0 \in C_i \subset \chi_i$ there exists a unique maximal solution z_i to the differential equation $\dot{z}_i(t) = f_i(z_i(t))$ satisfying $z_i(0) = x_i^0$ and $z_i(t) \in C_i$;
2) For each initial point $x_i^0 \in C_i \cap D_i$, there are no nontrivial solutions to $\dot{z}_i(t) = f_i(z_i(t))$ satisfying $z_i(0) = x_i^0$ and $z_i(t) \in C_i$.

For further discussions on existence and uniqueness of solutions and their continuous dependence on initial conditions for hybrid systems we refer to [63], [62] and [97].
Let us now turn to interconnections of hybrid subsystems. We consider interconnections of the form (2.1) as one large hybrid system

$$\begin{array}{rcll} \dot{x} &=& f(x,u), & (x,u) \in C, \\ x^+ &=& g(x,u), & (x,u) \in D, \end{array} \tag{2.5}$$

with the state x and the input u defined above. It seems to be natural to define $C := \cap C_i$, $D := \cup D_i$, since a jump of any subsystem means a jump for the overall state x, and to define the function $f : C \to \mathbb{R}^N$ by $f := (f_1^T, \ldots, f_n^T)^T$ and function $g : D \to \chi$ as $g := (\widetilde{g}_1^T, \ldots, \widetilde{g}_n^T)^T$, where

$$\widetilde{g}_i(x,u) := \begin{cases} g_i(x,u_i), & \text{if } (x,u) \in D_i, \\ x_i, & \text{otherwise}. \end{cases} \tag{2.6}$$

Note that, in general, we lose some solutions of individual systems by interconnection. Furthermore, the solutions of (2.5) may have different hybrid time domains than the solutions of the individual systems (2.1), see [130]. The above choice of C and D was also used in [130] considering interconnections of two hybrid systems. However, this choice has certain drawbacks, see Remark 2.4.7, see also Remark 4.3 in [130]. The supremum norm of the hybrid signal u defined on $[(0,0),(t,j)] \in \operatorname{dom} u$ is defined by

$$\|u\|_{(t,k)} := \max \left\{ \operatorname*{ess\,sup}_{\substack{(s,l) \in \operatorname{dom} u \setminus \Phi(u), \\ (s,l) \leq (t,k)}} |u(s,l)|, \sup_{\substack{(s,l) \in \Phi(u), \\ (s,l) \leq (t,k)}} |u(s,l)| \right\},$$

where $\Phi(u)$ is the set of all $(s, l) \in \operatorname{dom} u$ such that $(s, l+1) \in \operatorname{dom} u$. If $t + k \to \infty$, then $\|u\|_{(t,k)}$ is denoted by $\|u\|_\infty$. This norm takes the maximum between the essential supremum of an input function between the jumps and the supremum of a function over the jumps.
One of the most typical examples of the hybrid system is the bouncing ball.

Example 2.1.3. *[Bouncing ball with air resistance] Consider a ball with unit mass that falls and bounces from the floor. We assume that the floor and the ball are elastic, i.e., there is no loss of energy of the ball during the bounce. However, there is an air resistance given by the coefficient $\nu \in (0, 1)$. The dynamics of the ball can be described then as a hybrid system* (2.5)*:*

$$\dot{x} = \begin{pmatrix} x_2 \\ -\gamma - \nu x_2 \end{pmatrix} =: f(x, u), \quad (x, u) \in C, \tag{2.7}$$

$$x^+ = \begin{pmatrix} x_1 \\ -x_2 \end{pmatrix} =: g(x, u), \quad (x, u) \in D, \tag{2.8}$$

where x_1 denotes the vertical position and x_2 the velocity of the ball, $C := \{(x, u) \in \mathbb{R}^2 \times U : x_1 \geq 0\}$ and $D := \{(x, u) \in \mathbb{R}^2 \times U : x_1 = 0, x_2 \leq 0\}$, γ is the gravitation force.
For a given initial condition $x_1(t_0) = x_1(0) = h > 0$ and $x_2(t_0) = x_2(0) = 0$ the solution can be written in terms of the hybrid time domain as follows. The hybrid arc until the first jump (continuous flow between t_0 and t_1) is given by

$$\begin{aligned} x_1(t, 0) &= -\tfrac{\gamma}{\nu} e^{-\nu t} - \tfrac{\gamma}{\nu} t + h + \tfrac{\gamma}{\nu}, \\ x_2(t, 0) &= -\tfrac{\gamma}{\nu}(e^{-\nu t} - 1), \end{aligned}$$

where t_1 the time of the first touch with the ground and is a solution of $x_1(t, 0) = 0$, i.e., $-\frac{\gamma}{\nu} e^{-\nu t_1} - \frac{\gamma}{\nu} t_1 + h + \frac{\gamma}{\nu} = 0$. The states after the jump are

$$\begin{aligned} x_1^+(t_1, 1) &= 0, \\ x_2^+(t_1, 1) &= \tfrac{\gamma}{\nu}(e^{-\nu t_1} - 1). \end{aligned}$$

The further states after the jump at t_j are $x_1^+(t_j, j) = 0$ and $x_2^+(t_j, j) = -x_2(t_j, j-1)$, $j = 1, 2, \ldots$. The arcs (between t_j and t_{j+1}) are given as solutions of the system of differential equations (2.7) with initial conditions $x_1(t_j, j) = 0$ and $x_2(t_j, j) = -x_2(t_j, j-1)$.
The trajectory of the ball released from the height $x_1^0 = 3$, with the initial velocity $x_2^0 = 0$, the coefficient of air resistance $\nu = 0.1$, gravity acceleration $\gamma = 9.81$ is shown in Figure 2.1. The time domain of a solution is illustrated in Figure 2.2.

Remark 2.1.4. *Hybrid systems are closely related to impulsive systems [67], switched [150] and sampled-data systems [86]. In impulsive systems discontinuous jumps in the state occur only at given time sequences. In switched systems a trajectory of the state is continuous but the right-hand side of a differential equation describing its dynamics changes discontinuously. In sampled-data systems the right-hand side of a differential equation changes at time instances calculated according to the systems evolution. Impulsive, switched and sampled-data systems can be represented in a form of a hybrid system. In particular, to describe impulsive systems we can introduce a variable that describes the time and the set D that describes the jump times, see Remark 2.4.21. To describe switched systems or sample-date systems as hybrid systems, a variable that determines the type of continuous dynamics can be introduced [61]. However, systems of all these types are studied usually separately to stress on their main features.*

For an explicit discussion on hybrid systems, their solutions, stability, control and applications we recommend tutorials [62], [162] and [35], and the references therein.

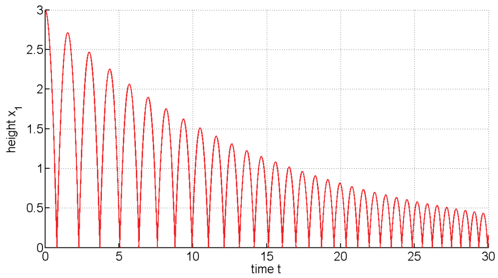


Figure 2.1: Trajectory of the bouncing ball.

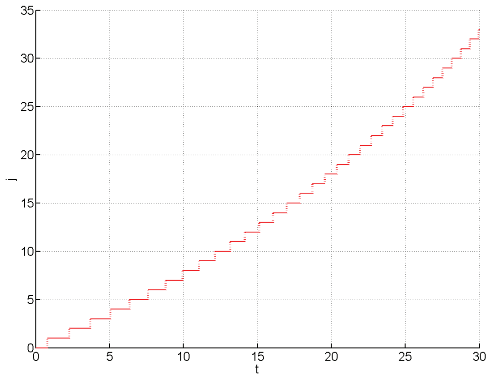


Figure 2.2: Time domain of the bouncing ball.

2.2 Stability notions

Here we introduce different stability notions for hybrid systems and show the relations between them. Furthermore, we give a more general formulation of input-to-state stability than those used in (1.10).

2.2.1 Input-to-state stability (ISS)

Consider a system of the form (2.5) with bounded inputs u. As these inputs are, in general, not equal to zero we are interested in a stability notion that takes into account the values of u. Input-to-state stability is a natural way to define such type of stability [27]:

Definition 2.2.1 (ISS)**.** *The system (2.5) is* input-to-state stable (ISS)*, if there exist* $\beta \in \mathcal{KLL}$ *,*$\gamma \in \mathcal{K}_\infty \cup \{0\}$ *such that for all initial values* x^0 *all solution pairs* $(x, u) \in S_u(x^0)$ *satisfy*

$$|x(t,k)| \le \max\{\beta(|x^0|, t, k), \gamma(\|u\|_{(t,k)})\}, \forall (t,k) \in dom\, x. \tag{2.9}$$

Function γ *is called ISS nonlinear gain.*

In particular, the properties of β and γ imply that an ISS system possesses a stable behaviour, where, in general, the state at the beginning is bounded by the function β depending on the initial condition x^0, and then by the function γ depending on the maximal value of the input u. The term "gain" here originally comes from electric networks, where an influence of a circuit on an increase of a signal is studied.

Remark 2.2.2. *Note that the ISS property can be equivalently defined replacing the maximizations in* (2.9) *by the sums:*

$$|x(t,k)| \le \bar{\beta}(|x^0|, t, k) + \bar{\gamma}(\|u\|_{(t,k)}), \tag{2.10}$$

where the comparison functions $\bar{\beta} \in \mathcal{KLL}$ *and* $\bar{\gamma} \in \mathcal{K}_\infty \cup \{0\}$ *may be different.*
The consideration of this equivalent formulation may yield more sharp estimations of stability that implies, for example, more sharp stability conditions for interconnections of hybrid systems, see Example 2.3.5. More general formulations of ISS, in terms of the so-called monotone aggregation functions (MAFs), were considered in [126] for continuous systems.

We also consider the following stability notions from [28]:

Definition 2.2.3 (0-input pre-stability)**.** *The system (2.5) is* 0-input pre-stable*, if for each $\epsilon > 0$ there exists a $\delta > 0$ such that each solution pair $(x, 0) \in S_u(x^0)$ with $|x^0| \leq \delta$ satisfies $|x(t,k)| \leq \epsilon$ for all $(t,k) \in dom\, x$.*

Definition 2.2.4 (pre-GS)**.** *The system (2.5) is* globally pre-stable (pre-GS)*, if there exist $\sigma, \hat{\gamma} \in \mathcal{K}_\infty \cup \{0\}$ such that for all initial values x^0 all solution pairs $(x,u) \in S_u(x^0)$ satisfy*

$$|x(t,k)| \leq \max\{\sigma(|x^0|), \hat{\gamma}(\|u\|_{(t,k)})\}, \forall (t,k) \in dom\, x. \tag{2.11}$$

The prefix "pre-" in the definitions above indicates that the maximal solutions of the system (2.5) are not necessarily complete [131].

Remark 2.2.5. *Note that pre-GS follows from ISS by taking $\sigma(|x^0|) := \beta(|x^0|, 0, 0)$ and 0-input pre-stability follows from pre-GS by considering $u \equiv 0$.*

Definition 2.2.6 (AG)**.** *The system (2.5) has the* asymptotic gain property *(AG), if there exists $\widetilde{\gamma} \in \mathcal{K}_\infty \cup \{0\}$ such that for all initial values x^0 all solution pairs $(x,u) \in S_u(x^0)$ are bounded and, if complete satisfy*

$$\limsup_{(t,k) \in dom\, x, t+k \to \infty} |x(t,k)| \leq \widetilde{\gamma}(\|u\|_\infty). \tag{2.12}$$

Note that in the definition of the AG property we take into account that the time t or the number of the intervals between the jumps k may be bounded, however the sum of them is not in the case of complete solutions.
In [28, Theorem 3.1] the following relation between ISS and AG with 0-input pre-stability was proved.

Theorem 2.2.7. *Assume that the set $\{f(x,u) : u \in U \cap \epsilon\overline{\mathbb{B}}\}$ is convex $\forall \varepsilon > 0$ and for any $x \in \chi$. Then system* (2.1) *is ISS if and only if it has the asymptotic gain property and it is* 0*-input pre-stable.*

We now intend to formulate ISS conditions for the subsystems of (2.1), where some conditions are in the sum formulation as in (2.10) while other are given in the maximum form as in (2.9). Consider the index set $I := \{1, \ldots, n\}$ partitioned into two subsets I_Σ, $I_{\max}$ such that $I_{\max} = I \setminus I_\Sigma$.
The ith subsystem (2.1) is ISS, if there exist β_i of class $\mathcal{KLL}$, γ_{ij}, $\gamma_i \in \mathcal{K}_\infty \cup \{0\}$ such that for all initial values x_i^0 each solution pair $(x_i, v_i) \in S_{v_i}(x_i^0)$ with v_i from (2.2) satisfies:

$$|x_i(t,k)| \leq \beta_i(|x_i^0|, t, k) + \sum_{j=1, j\neq i}^{n} \gamma_{ij}(\|x_j\|_{(t,k)}) + \gamma_i(\|u\|_{(t,k)}), \forall (t,k) \in \mathrm{dom}\, x_i, \tag{2.13}$$

$i \in I_\Sigma$, and

$$|x_i(t,k)| \leq \max\{\beta_i(|x_i^0|, t, k), \max_{j, j\neq i} \gamma_{ij}(\|x_j\|_{(t,k)}), \gamma_i(\|u\|_{(t,k)})\}, \forall (t,k) \in \mathrm{dom}\, x_i, \tag{2.14}$$

$i \in I_{\max}$.

Remark 2.2.8. *Note that, without loss of generality, we can assume that $I_\Sigma = \{1, \ldots, p\}$ and $I_{\max} = \{p+1, \ldots, n\}$ where $p := |I_\Sigma|$. This can be always achieved by a permutation of the subsystems in (2.1).*

Similarly, the ith system of the form (2.1) is pre-GS, if there exist $\sigma_i, \hat{\gamma}_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $\hat{\gamma}_i \in \mathcal{K}_\infty \cup \{0\}$ such that for all initial values x_i^0 each solution pair $(x_i, v_i) \in S_{v_i}(x_i^0)$ satisfies

$$|x_i(t,k)| \leq \sigma_i(|x_i^0|) + \sum_{j=1, j\neq i}^{n} \hat{\gamma}_{ij}(\|x_j\|_{(t,k)}) + \hat{\gamma}_i(\|u\|_{(t,k)}), \forall (t,k) \in \operatorname{dom} x_i, \tag{2.15}$$

$i \in I_\Sigma$, and

$$|x_i(t,k)| \leq \max\{\sigma_i(|x_i^0|), \max_{j, j\neq i} \hat{\gamma}_{ij}(\|x_j\|_{(t,k)}), \hat{\gamma}_i(\|u\|_{(t,k)})\}, \forall (t,k) \in \operatorname{dom} x_i, \tag{2.16}$$

$i \in I_{\max}$.
And the ith system of the form (2.1) has the AG property, if there exist $\widetilde{\gamma}_{ij} \in \mathcal{K}_\infty \cup \{0\}$, $\widetilde{\gamma}_i \in \mathcal{K}_\infty \cup \{0\}$ such that for all initial values x_i^0 each solution pair $(x_i, v_i) \in S_{v_i}(x_i^0)$ is bounded and, if complete satisfies

$$\limsup_{(t,k)\in \operatorname{dom} x_i, t+k\to\infty} |x_i(t,k)| \leq \sum_{j, j\neq i} \widetilde{\gamma}_{ij}(\|x_j\|_\infty) + \widetilde{\gamma}_i(\|u\|_\infty), \tag{2.17}$$

$i \in I_\Sigma$, and

$$\limsup_{(t,k)\in \operatorname{dom} x, t+k\to\infty} |x_i(t,k)| \leq \max\{\max_{j, j\neq i} \widetilde{\gamma}_{ij}(\|x_j\|_\infty), \widetilde{\gamma}_i(\|u\|_\infty)\}, \tag{2.18}$$

$i \in I_{\max}$.

Remark 2.2.9. *Note that using*

$$\max_{i=1,\ldots,n} \{x_i\} \leq \sum_{i=1}^{n} x_i \leq n \max_{i=1,\ldots,n} \{x_i\} \tag{2.19}$$

we can always pass to ISS-formulations with maximums or summations only, but with different gains in general.

2.2.2 ISS in terms of Lyapunov functions

Another notion useful for stability investigations of nonlinear systems is the notion of an ISS-Lyapunov function. In this section we give a definition of this function and recall the result showing that the existence of an ISS-Lyapunov function guarantees that the system is ISS. Thus we can use ISS-Lyapunov functions to check ISS of a hybrid system.
We consider locally Lipschitz continuous functions $V : \chi \to \mathbb{R}_+$ that are differentiable almost everywhere by the Rademacher's theorem. In points where such a function is not differentiable we use the notion of Clarke's generalized gradient from Section 1.1.4.

Definition 2.2.10 (ISS-Lyapunov function)**.** *A locally Lipschitz continuous function $V : \chi \to \mathbb{R}_+$ is an* ISS-Lyapunov function for the system (2.5)*, if*
1) there exist functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that:

$$\psi_1(|x|) \leq V(x) \leq \psi_2(|x|) \text{ for any } x \in \chi; \tag{2.20}$$

2) there exist a function $\gamma \in \mathcal{K}$, and continuous, positive definite functions α, λ with $\lambda(s) < s$ for all $s > 0$ such that:

$$V(x) \geq \gamma(|u|) \Rightarrow \forall \zeta \in \partial V(x) : \langle \zeta, f(x,u) \rangle \leq -\alpha(V(x)), (x,u) \in C, \tag{2.21}$$

$$V(x) \geq \gamma(|u|) \Rightarrow V(g(x,u)) \leq \lambda(V(x)), \quad (x,u) \in D. \tag{2.22}$$

Function γ is called ISS-Lyapunov gain corresponding to the input u.

If V is differentiable at x, then (2.21) can be written as

$$V(x) \geq \gamma(|u|) \Rightarrow \nabla V(x) \cdot f(x,u) \leq -\alpha(V(x)), (x,u) \in C.$$

Note that this definition is different from the definitions of an ISS-Lyapunov function used in [28], [46] and [110]. The equivalence between their existence for (2.5) is shown in Propositions 2.2.16 and 2.2.19. We will use such a formulation to show one of the main stability results in Section 2.4.

Remark 2.2.11. *Relations between the existence of a smooth ISS-Lyapunov function and the ISS property for hybrid systems were discussed in [28]: In particular, Proposition 2.7 in [28] shows that if a hybrid system has an ISS-Lyapunov function, then it is ISS. Example 3.4 in [28] shows that the converse is in general not true. In Theorem 3.1 [28] it was proved that if the system* (2.5) *is ISS with* f *such that the set* $\{f(x,u) : u \in U \cap \epsilon\overline{\mathbb{B}}\}$ *is convex* $\forall \varepsilon > 0$ *and for any* $x \in \chi$*, then it has an ISS-Lyapunov function.*

It turns out that the smoothness of an ISS-Lyapunov function is not necessary to guarantee the ISS property of the system (2.5) as the following proposition shows.

Proposition 2.2.12. *If the system* (2.5) *has a locally Lipschitz continuous ISS-Lyapunov function, then it is ISS.*

Proof. The proof of Proposition 2.7 in [28] stated with $\alpha \in \mathcal{K}_\infty$ works without change, if α is continuous and positive definite. As well, this proof can be extended to the nonsmooth V using the Clarke's generalized derivative. The assertion of the proposition follows then from this extension and Proposition 2.2.16. □

Remark 2.2.13. *If the set* C *has a nonempty interior, then it is enough to consider the classical derivative of* V *at the points of differentiability in* (2.21)*. But for the set* C *with an empty interior it is not enough, in general, to use the classical derivative of* V*, see the following example.*

Example 2.2.14. *Consider the hybrid system:*

$$\begin{aligned} \dot{x}_1 &= x_1, \\ \dot{x}_2 &= 0 \end{aligned}$$

for $x \in C = \mathbb{R} \times \{0\}$*, i.e., line* $x_2 = 0$*, and*

$$\begin{aligned} x_1^+ &= 0, \\ x_2^+ &= 0 \end{aligned}$$

for $x \in D = \mathbb{R}^2$.
The solution of this system corresponding to $x(0,0) = (x_1^0, 0)^T$ *is given by*

$$\begin{aligned} x_1(t,0) &= x_1^0 e^t, \\ x_2(t,0) &= 0. \end{aligned}$$

As $x_1(t,0)$ *is unbounded, the system is not ISS.*
Consider now locally Lipschitz continuous function $V(x) = |x_1| + |x_2|$*. Function* V *satisfies* (2.20) *and* (2.22)*. If we require for an ISS-Lyapunov function that its classical derivative has to satisfy* (2.21) *at the points of differentiability only, then function* V *will satisfy this condition as the set* C *is of zero measure. Thus we will obtain a contradiction to our previous conclusion that the system is not ISS.*

Remark 2.2.15. *Note that conditions* (2.21) *and* (2.22) *on an ISS-Lyapunov function require that the continuous and discontinuous dynamics of a hybrid system has to be always stable. This requirement implies some kind of conservativeness. Think of a hybrid system, where the continuous dynamics is stable and the discontinuous dynamics is unstable, i.e. the jumps increase the magnitude of the state of the system. Assume also that continuous dynamics stabilizes the system, i.e. the jumps occur so rarely, that during the time between the jumps the magnitude of the state decreases greater than it increases during the next jump. In this case, though the system is stable, we cannot apply Proposition 2.2.12 to establish this due to the violation of condition* (2.22)*. The same is in the reverse case, when the discontinuous dynamics is stable and the continuous dynamics is unstable, but the jumps occur so often that the system is nevertheless stable. To the best of our knowledge, there are no general approaches on ISS-Lyapunov functions that can be applied to show input-to-state stability in such case. However, we are sure that the development of methods that allow establishing of stability of such systems is of a great application importance. A possible starting point in this research may be an extension of the known stability results obtained for a certain class of hybrid systems and for a certain type of stability. In Section 2.4.4 we will discuss impulsive systems that are a subclass of a hybrid system. In particular, we will recall the dwell-time condition introduced for such a system. This condition restricts the time intervals between the jumps to guarantee input-to-state stability of an impulsive system where the flow or the jumps may be unstable. Other stability results, known in the literature, are obtained in the studying of asymptotic stability of hybrid systems. They are based on the LaSalle's invariance principle [132] and the nested Matrosov theorems [134].*

Equivalent definitions of an ISS-Lyapunov function

Here we present alternative definitions of an ISS-Lyapunov function used in [28], [46], [110]. We show that the existence of ISS-Lyapunov functions in terms of these definitions is equivalent to the existence of an ISS-Lyapunov function in terms of Definition 2.2.10. Consider a function $W : \chi \to \mathbb{R}_+$, $W \in Lip_{loc}$ that satisfies the following properties for the system of the form (2.5):
1) There exist functions $\bar{\psi}_1, \bar{\psi}_2 \in \mathcal{K}_\infty$ such that:

$$\bar{\psi}_1(|x|) \leq W(x) \leq \bar{\psi}_2(|x|) \text{ for any } x \in \chi. \tag{2.23}$$

2) There exist a function $\bar{\gamma} \in \mathcal{K}$, a continuous, positive definite function $\bar{\alpha}_1$ and a function $\bar{\alpha}_2 \in \mathcal{K}_\infty$ such that:

$$|x|\geq\bar{\gamma}(|u|)\Rightarrow\forall\zeta\in\partial W(x)\colon \langle\zeta, f(x,u)\rangle \leq -\bar{\alpha}_1(|x|), (x,u) \in C, \tag{2.24}$$

$$|x| \geq \bar{\gamma}(|u|) \Rightarrow W(g(x,u)) - W(x) \leq -\bar{\alpha}_2(|x|), (x,u) \in D. \tag{2.25}$$

In [28] the conditions (2.23)-(2.25) with $\bar{\alpha}_1 \in \mathcal{K}_\infty$ were used to define an ISS-Lyapunov function for a system of the form (2.5) and it was shown that existence of such a (smooth) function W implies that a system of the form (2.5) is ISS.

Proposition 2.2.16. *A system of the form* (2.5) *has an ISS-Lyapunov function V satisfying* (2.20)-(2.22) *if and only if there exists $W \in Lip_{loc}$ satisfying* (2.23)-(2.25)*.*

For the proof we need two auxiliary lemmas. The following lemma is an extension of [83, Lemma 2.8] on the case with external inputs.

Lemma 2.2.17. *Consider a system of the form* (2.5) *and assume that function $V \in Lip_{loc}$ satisfies* (2.23), (2.25) *with continuous, positive definite function $\bar{\alpha}_2$. Then there exists a smooth $\mathcal{K}_\infty$ function ρ such that with $W = \rho \circ V$ it holds that*

$$|x| \geq \bar{\gamma}(|u|) \Rightarrow W(g(x,u)) - W(x) \leq -\alpha(|x|), \quad \forall (x,u) \in \chi \times U, \tag{2.26}$$

for some $\alpha \in \mathcal{K}_\infty$.

Proof. Assume that $|x| \geq \bar{\gamma}(|u|)$. Then, applying [83, Lemma 2.8] we obtain a smooth function $\rho \in \mathcal{K}_\infty$ and a function $W = \rho \circ V$ that satisfies (2.26). □

Lemma 2.2.18. *For any $\mathcal{K}_\infty$-function α, there is a $\mathcal{K}_\infty$-function $\hat{\alpha}$ such that the following holds:*

- $\hat{\alpha}(s) \leq \alpha(s)$, $\forall s \geq 0$ *and*
- *id* $- \hat{\alpha} \in \mathcal{K}$.

See [84, Lemma B.1] for the proof.
Proof of Proposition 2.2.16
"⇒" Let V satisfy (2.20)-(2.22). Define

$$\bar{\gamma}(|u|) := \psi_1^{-1} \circ \gamma(|u|). \tag{2.27}$$

Let $|x| \geq \bar{\gamma}(|u|)$. Then from (2.20), (2.27) it follows $V(x) \geq \gamma(|u|)$. Applying (2.22) it follows that for all $(x,u) \in D$

$$V(g(x,u)) \leq \lambda(V(x)) = V(x) - \widetilde{\alpha}(x) \Rightarrow V(g(x,u)) - V(x) \leq -\hat{\alpha}(|x|), \tag{2.28}$$

with $\hat{\alpha}(r) := \min\limits_{|s|=r} \widetilde{\alpha}(s)$ that is a continuous, positive definite function, where $\widetilde{\alpha}(s) := V(s) - \lambda(V(s))$.
From (2.28) and Lemma 2.2.17 there exist $\rho, \bar{\alpha}_2 \in \mathcal{K}_\infty$, where ρ is smooth, such that $W := \rho \circ V$ satisfies (2.25) with $\bar{\gamma}(|u|)$ defined in (2.27).
As $V(x)$ satisfies (2.21), then W satisfies (2.24) with $\bar{\alpha}_1 := \widetilde{\rho} \cdot \alpha$ that is a continuous, positive definite function, where $\widetilde{\rho} \in \partial\rho(y)$, $y = V(x)$.
Thus, function W satisfies (2.23)-(2.25) with $\bar{\psi}_1 := \rho \circ \psi_1$, $\bar{\psi}_2 := \rho \circ \psi_2$, $\bar{\gamma}(|u|) := \psi_1^{-1} \circ \gamma(|u|)$ and $\bar{\alpha}_1 := \widetilde{\rho} \cdot \alpha$.
"⇐" Assume now that the function W satisfies (2.23)-(2.25) and define $V := W$, $\psi_1 := \bar{\psi}_1$ and $\psi_2 := \bar{\psi}_2$. Then condition (2.20) is satisfied. Let

$$\gamma(|u|) := \bar{\psi}_2 \circ \bar{\gamma}(|u|). \tag{2.29}$$

Consider $V(x) \geq \gamma(|u|)$. Then from (2.29) and (2.23) it follows $|x| \geq \bar{\gamma}(|u|)$. From (2.23) and (2.24) it holds for all $(x,u) \in C$

$$\forall \zeta \in \partial W(x) : \langle \zeta, f(x,u) \rangle \leq -\bar{\alpha}_1(|x|) \leq -\bar{\alpha}_1 \circ \bar{\psi}_2^{-1}(W(x)).$$

Thus V satisfies (2.21) with $\alpha := \bar{\alpha}_1 \circ \bar{\psi}_2^{-1}$.
From Lemma 2.2.18 for any $\bar{\alpha}_2 \circ \bar{\psi}_2^{-1} \in \mathcal{K}_\infty$ there exists $\widetilde{\alpha} \in \mathcal{K}_\infty$ such that $\widetilde{\alpha} \leq \bar{\alpha}_2 \circ \bar{\psi}_2^{-1}$ and $\mathrm{id} - \widetilde{\alpha} \in \mathcal{K}$. Applying (2.23) and (2.25) we obtain

$$\begin{aligned} W(g(x,u)) &\leq W(x) - \bar{\alpha}_2(|x|) \leq W(x) - \bar{\alpha}_2 \circ \bar{\psi}_2^{-1}(W(x)) \\ &\leq W(x) - \widetilde{\alpha}(W(x)) = (\mathrm{id} - \widetilde{\alpha})(W(x)) = \lambda(W(x)), \end{aligned}$$

where $\lambda := \mathrm{id} - \widetilde{\alpha}$. Hence, function V satisfies (2.20)-(2.22). □
The next proposition shows another way to introduce an ISS-Lyapunov function, used in [46], [110].

Proposition 2.2.19. *System of the form* (2.5) *has an ISS-Lyapunov function V satisfying* (2.20)-(2.22) *if and only if there exists $\widetilde{V} \in Lip_{loc}$ satisfying* (2.20)-(2.21) *and*

$$\widetilde{V}(g(x,u)) \leq \max\{\widetilde{\lambda}(\widetilde{V}(x)), \widetilde{\gamma}(|u|)\}, (x,u) \in D, \tag{2.30}$$

with $\widetilde{\gamma} \in \mathcal{K}$ and a continuous, positive definite $\widetilde{\lambda}$, $\widetilde{\lambda} < id$.

Proof. "$\Leftarrow$" We can always majorize a continuous, positive definite function $\widetilde{\lambda} <$ id from (2.30) by a function $\rho \in \mathcal{K}_\infty$ such that $\widetilde{\lambda}(r) < \rho(r) < r$, for example, taking $\rho(r) := \frac{1}{2}(\max_{[0,r]} \tilde{\lambda} + \text{id})(r)$, $r \geq 0$. Then for $(x,u) \in D$ from (2.30) we have

$$\widetilde{V}(g(x,u)) \leq \max\{\widetilde{\lambda}(\widetilde{V}(x)), \widetilde{\gamma}(|u|)\} \leq \max\{\rho(\widetilde{V}(x)), \widetilde{\gamma}(|u|)\}. \tag{2.31}$$

Define $\gamma := \rho^{-1} \circ \widetilde{\gamma} > \widetilde{\gamma}$ and $V := \widetilde{V}$. If $V(x) \geq \gamma(|u|)$, then $\rho(V(x)) \geq \widetilde{\gamma}(|u|)$ and using (2.31)

$$V(g(x,u)) = \widetilde{V}(g(x,u)) \leq \max\{\rho(\widetilde{V}(x)), \widetilde{\gamma}(|u|)\} = \rho(\widetilde{V}(x)) = \rho(V(x)).$$

Thus, function V satisfies the condition (2.22) with $\lambda := \rho <$ id.
"$\Rightarrow$" From (2.22) for $(x,u) \in D$, if $V(x) > \gamma(|u|)$, then $V(g(x,u)) \leq \lambda(V(x))$. Consider now $(x,u) \in D$ such that $V(x) \leq \gamma(|u|)$ and define the set

$$\mathcal{A}(|u|) := \{(x,u) \in D : V(x) \leq \gamma(|u|)\}.$$

Let us take now

$$\hat{\gamma}(|u|) := \max_{(x,u)\in\mathcal{A}(|u|)} V(g(x,u)).$$

Then $V(g(x,u)) \leq \hat{\gamma}(|u|)$ for $(x,u) \in \mathcal{A}(|u|)$. Note that $\hat{\gamma}(0) = 0$ as $V(x) \geq 0 = \gamma(0)$.
Furthermore, as the function V is nonnegative and $V \in Lip_{loc}$ and the function g is continuous, function $\hat{\gamma} \in Lip_{loc}$ is nonnegative. We can always majorize such function $\hat{\gamma}$ by a function $\check{\gamma} \in \mathcal{K}$ such that $\hat{\gamma} \leq \check{\gamma}$.
Thus, for $(x,u) \in D$ we have obtained that $V(g(x,u)) \leq \max\{\check{\gamma}(|u|), \lambda(V(x))\}$ and condition (2.30) is satisfied with $\widetilde{V} := V$ and $\widetilde{\gamma} := \max\{\check{\gamma}, \gamma\}$. □

Now, to establish ISS of the hybrid system (1.55) we can use Proposition 2.2.12, i.e. we need to find an ISS-Lyapunov function for the system (1.55). Note that the logistics network consists, in general of many locations. Therefore, the procedure of the looking for an ISS-Lyapunov function may be very sophisticated. In Section 2.4.2 we will show how this procedure can be facilitated by applying the small gain condition. To this end we need a description of an ISS-Lyapunov function for individual location, i.e. for subsystem (2.1).

ISS-Lyapunov functions for hybrid subsystems

Consider a system of the form (2.5) as an interconnection of n hybrid systems with several inputs. By an appropriate choice of $\gamma_{ij}, \gamma_i \in \mathcal{K}_\infty$, that depends on the used norms, $\gamma(|v_i|)$ can be written as a sum or a maximum over $\gamma_{ij}(|x_j|), j \neq i$ and $\gamma_i(|u_i|)$. If we assume that each subsystem i has a locally Lipschitz continuous ISS-Lyapunov function V_i, then $\gamma_{ij}(|x_j|)$ can be estimated from above and below by $\tilde{\gamma}_{ij}(V_j(x_j))$ and $\bar{\gamma}_{ij}(V_j(x_j))$ with an appropriate choice of $\tilde{\gamma}_{ij}, \bar{\gamma}_{ij} \in \mathcal{K}_\infty$. This follows from (2.20). Thus we obtain the following formulation of the ISS-Lyapunov function V_i for systems

of the form (2.1):
1) There exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ such that:

$$\psi_{i1}(|x_i|) \le V_i(x_i) \le \psi_{i2}(|x_i|) \text{ for any } x_i \in \chi_i. \tag{2.32}$$

2) There exist $\gamma_{ij}, \gamma_i \in \mathcal{K}_\infty$ and continuous, positive definite functions α_i, λ_i, with $\lambda_i(s) < s$ for all $s > 0$ such that for all $(x, u) \in C_i$

$$V_i(x_i) \ge \sum_{j, j\neq i} \gamma_{ij}(V_j(x_j)) + \gamma_i(|u|) \Rightarrow \forall \zeta_i \in \partial V_i(x_i) : \langle \zeta_i, f_i(x, u_i)\rangle \le -\alpha_i(V_i(x_i)), \tag{2.33}$$

$i \in I_\Sigma$,

$$V_i(x_i) \ge \max\{\max_{j, j\neq i}\{\gamma_{ij}(V_j(x_j))\}, \gamma_i(|u|)\} \Rightarrow \forall \zeta_i \in \partial V_i(x_i) : \langle \zeta_i, f_i(x, u_i)\rangle \le -\alpha_i(V_i(x_i)), \tag{2.34}$$

$i \in I_{\max}$, and for all $(x, u) \in D_i$

$$V_i(x_i) \ge \sum_{j, j\neq i} \gamma_{ij}(V_j(x_j)) + \gamma_i(|u|) \Rightarrow V_i(g_i(x, u_i)) \le \lambda_i(V_i(x_i)), \tag{2.35}$$

$i \in I_\Sigma$,

$$V_i(x_i) \ge \max\{\max_{j, j\neq i}\{\gamma_{ij}(V_j(x_j))\}, \gamma_i(|u|)\} \Rightarrow V_i(g_i(x, u_i)) \le \lambda_i(V_i(x_i)), \tag{2.36}$$

$i \in I_{\max}$.
Note that γ_{ij} in (2.33)-(2.36) are equal. This can be always achieved by taking maximums of separately obtained γ_{ij}'s for the continuous and discrete dynamics.

Remark 2.2.20. *Function V_i can be defined equivalently by replacing* (2.36) *with*

$$V_i(g_i(x, u)) \le \max\{\lambda_i(V_i(x_i)), \gamma_{ij}(V_j(x_j)), \gamma_i(|u|)\}, \tag{2.37}$$

for some continuous and positive definite $\lambda_i < id$ using Proposition 2.2.19. Certainly, these changes can lead to different Lyapunov gains.

An ISS-Lyapunov function provides a useful tool for checking ISS of a hybrid system as it does not require knowledge about the solution of the system. Note, however, that an interconnection of hybrid systems can be unstable, i.e., not ISS, even if each of its subsystems is ISS. In the following section we introduce conditions that guarantee stability for interconnections of ISS hybrid systems.

2.3 Gains

To establish ISS of an interconnected hybrid system of the form (2.5) we are going to extend an approach used for checking stability of interconnected continuous and discrete systems in [54], [52] and [86] to hybrid systems. This approach utilizes information about the interconnection structure of the network.
We assume that all subsystems that compose interconnection of the form (2.5) are ISS. We assume also that ISS estimates (2.13), (2.14) or at least ISS-Lyapunov estimates (2.33)- (2.36) are known for each subsystem. Function γ_{ij} occurring in these estimates describes the influence of jth subsystem on the ith one. We will call this function *gain*. We collect all the gains in a matrix $\Gamma = (\gamma_{ij})_{n\times n}$, with the convention $\gamma_{ii} \equiv 0,\ i = 1, \ldots, n$. Then this matrix describes the mutual influence between the subsystems of the interconnected hybrid system. We will call this matrix *gain matrix* and it will be the basis of our stability analysis. In particular, we will impose the so-called mixed small gain condition on the corresponding matrix operator that guarantees stability of an interconnected hybrid system.
In this section we will introduce this small gain condition and discuss its properties. Then we will show how this condition can be applied to check stability of an interconnection.

2.3.1 Gain operator

We define the *gain operator* $\Gamma : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ by

$$\Gamma(s) := (\Gamma_1(s), \dots, \Gamma_n(s))^T, \; s \in \mathbb{R}^n_+, \tag{2.38}$$

where the functions $\Gamma_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ are given by $\Gamma_i(s) := \gamma_{i1}(s_1) + \dots + \gamma_{in}(s_n)$ for $i \in I_\Sigma$ and $\Gamma_i(s) := \max\{\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n)\}$ for $i \in I_{\max}$. In particular, if $I_\Sigma = \{1, \dots, p\}$ and $I_{\max} = \{p+1, \dots, n\}$ we have

$$\Gamma(s) = \begin{pmatrix} \gamma_{12}(s_2) + \dots + \gamma_{1n}(s_n) \\ \vdots \\ \gamma_{k1}(s_1) + \dots + \gamma_{kn}(s_n) \\ \max\{\gamma_{p+1,1}(s_1), \dots, \gamma_{p+1,n}(s_n)\} \\ \vdots \\ \max\{\gamma_{n1}(s_1), \dots, \gamma_{n,n-1}(s_{n-1})\} \end{pmatrix}. \tag{2.39}$$

2.3.2 Mixed small gain condition

The small gain condition that we are going to use to establish stability of an interconnected hybrid system of the form (2.5) originates from the corresponding small gain conditions for interconnections of purely continuous systems, i.e. in case $D = \emptyset$, or interconnections of purely discrete systems, i.e. in case $C = \emptyset$. In [54] stability conditions of interconnected continuous systems are provided in terms of small gain conditions, where the cases $I_\Sigma = I = \{1, \dots, n\}$, respectively $I_{\max} = I$ are considered. In [126], [52] more general formulations of ISS are considered.
First, we recall the small gain conditions for the cases $I_\Sigma = I$, resp. $I_{\max} = I$, which imply ISS of an interconnected continuous system, [54]. If $I_\Sigma = I$, we need to assume that there exists an operator $\mathcal{D} := \operatorname{diag}_n(\mathrm{id} + \alpha)$, $\alpha \in \mathcal{K}_\infty$ such that

$$\Gamma \circ \mathcal{D}(s) \not\geq s, \; \forall s \in \mathbb{R}^n_+ \backslash \{0\}, \tag{2.40}$$

and if $I_{\max} = I$, then the small gain condition

$$\Gamma(s) \not\geq s, \; \forall s \in \mathbb{R}^n_+ \backslash \{0\} \tag{2.41}$$

is sufficient. The equivalence between the so-called cycle condition and the condition (2.41) was shown in [126, Lemma 2.3.14]:

Lemma 2.3.1. *Let the matrix* Γ *be given. Then the small gain condition* (2.41) *is equivalent to the cycle condition:*

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ \dots \circ \gamma_{k_{r-1}k_r} < id, \tag{2.42}$$

for all $(k_1, ..., k_r) \in \{1, ..., n\}^r$ *with* $k_1 = k_r$.

Since the matrix Γ describes the interconnection structure of a network, these conditions impose some restriction on interconnection properties. We will see that this restriction guarantees stability of the network. The intuitive meaning of (2.42) is that a signal going through the network is not amplified. See also [54] for further interpretations of the small gain conditions (2.40) and (2.41).
In the case that both I_Σ and $I_{\max}$ are not empty we can use (2.19) to pass to the situation with $I_\Sigma = \emptyset$ or $I_{\max} = \emptyset$. But this can lead to more conservative gains. To avoid this conservativeness we are

going to derive a new small gain condition for the case $I_\Sigma \neq I \neq I_{\max}$. As we will see, there are two essentially equivalent approaches to do this. We can use the weak triangle inequality

$$a + b \leq \max\{(\mathrm{id} + \eta) \circ a, (\mathrm{id} + \eta^{-1}) \circ b\}\,, \tag{2.43}$$

which is valid for all functions $a, b, \eta \in \mathcal{K}_\infty$ to pass to a pure maximum formulation of ISS. However, this method involves the right choice of a large number of weights in the weak triangular inequality which can be a nontrivial problem. Alternatively, tailor-made small gain conditions can be derived. The expressions in (2.40), (2.41) prompt us to consider the following small gain condition. For a given $\alpha \in \mathcal{K}_\infty$ let the diagonal operator $\mathcal{D} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be defined by

$$\mathcal{D}(s) := (\mathcal{D}_1(s_1), \ldots, \mathcal{D}_n(s_n))^T\,, \quad s \in \mathbb{R}^n_+\,, \tag{2.44}$$

where $\mathcal{D}_i(s_i) := (\mathrm{id} + \alpha)(s_i)$ for $i \in I_\Sigma$ and $\mathcal{D}_i(s_i) := s_i$ for $i \in I_{\max}$. The small gain condition on the operator Γ corresponding to a partition $I = I_\Sigma \cup I_{\max}$ is then

$$\exists\, \alpha \in \mathcal{K}_\infty \quad : \quad \Gamma \circ \mathcal{D}(s) \not\geq s,\ \forall s \in \mathbb{R}^n_+ \backslash \{0\}. \tag{2.45}$$

We will abbreviate this condition as $\Gamma \circ \mathcal{D} \not\geq \mathrm{id}$ and will call it *mixed small gain condition*.
The following lemmas that describe the properties of $\alpha \in \mathcal{K}_\infty$ were proved in [126].

Lemma 2.3.2. *Let $\alpha \in \mathcal{K}_\infty$. Then there exists a function $\widetilde{\alpha} \in \mathcal{K}_\infty$ such that $(id + \alpha)^{-1} = id - \widetilde{\alpha}$.*

Lemma 2.3.3. *For any $\alpha \in \mathcal{K}$, there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}$ such that*

$$id + \alpha = (id + \alpha_1) \circ (id + \alpha_2).$$

Moreover, if $\alpha \in \mathcal{K}_\infty$, then also $\alpha_i,\ i = 1, 2$ can be chosen to be of class $\mathcal{K}_\infty$.

The componentwise application of the last lemma implies the following property of the operator $\mathcal{D}$.

Lemma 2.3.4. *Let $\mathcal{D}$ be defined as in* (2.44) *for some $\alpha \in \mathcal{K}_\infty$. Then there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, such that for $\mathcal{D}_i$ defined as in* (2.44) *with corresponding α_i*

$$\mathcal{D} = \mathcal{D}_1 \circ \mathcal{D}_2.$$

In the following example we highlight the advantage of the new small gain condition (2.45). In order not to cloud the issue we keep the example as simple as possible.

Example 2.3.5. *We consider an interconnection of $n = 3$ continuous systems, i.e. $D = \emptyset$, given by*

$$\begin{aligned} \dot{x}_1 &= -\,x_1 + \gamma_{13}(|x_3|) + \gamma_1(|u|) \\ \dot{x}_2 &= -\,x_2 + \max\{\gamma_{21}(|x_1|), \gamma_{23}(|x_3|)\} \\ \dot{x}_3 &= -\,x_3 + \max\{\gamma_{32}(|x_2|), \gamma_3(|u|)\} \end{aligned} \tag{2.46}$$

where γ_{ij} and γ_i are the given $\mathcal{K}_\infty$ functions. Using the variation of constants method and the weak triangle inequality (2.43) *we see that the trajectories can be estimated by:*

$$\begin{aligned} |x_1(t,0)| &\leq \beta_1(|x_1(0)|, t, 0) + \gamma_{13}(||x_3||_{(t,0)}) + \gamma_1(\|u\|_\infty), \\ |x_2(t,0)| &\leq \max\{\beta_2(|x_2(0)|, t, 0), (id + \eta) \circ \gamma_{21}(||x_1||_{(t,0)}), (id + \eta) \circ \gamma_{23}(||x_3||_{(t,0)})\}, \\ |x_3(t,0)| &\leq \max\{\beta_3(|x_3(0)|, t, 0), (id + \eta) \circ \gamma_{32}(||x_2||_{(t,0)}), (id + \eta) \circ \gamma_3(\|u\|_\infty)\}\,, \end{aligned} \tag{2.47}$$

where β_i are appropriate $\mathcal{KLL}$ functions and $\eta \in \mathcal{K}_\infty$ is arbitrary. This estimation can be rather sharp by choosing an appropriate η in the weak triangle inequality (2.43).
This shows that each subsystem is ISS. In this case we have

$$\Gamma = \begin{pmatrix} 0 & 0 & \gamma_{13} \\ (id+\eta)\circ\gamma_{21} & 0 & (id+\eta)\circ\gamma_{23} \\ 0 & (id+\eta)\circ\gamma_{32} & 0 \end{pmatrix}.$$

Then the small gain condition (2.45) *requires that there exists an $\alpha \in \mathcal{K}_\infty$ such that*

$$\begin{pmatrix} \gamma_{13}(s_3) \\ \max\{(id+\eta)\circ\gamma_{21}\circ(id+\alpha)(s_1), (id+\eta)\circ\gamma_{23}(s_3)\} \\ (id+\eta)\circ\gamma_{32}(s_2) \end{pmatrix} \not\geq \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \tag{2.48}$$

for all $s \in \mathbb{R}^3_+\backslash\{0\}$.
If (2.48) *holds, then considering $s^T(r) := (\gamma_{13}\circ(id+\eta)\circ\gamma_{32}(r), r, (id+\eta)\circ\gamma_{32}(r))^T$, $r > 0$ we obtain that the following two inequalities are satisfied*

$$(id+\alpha)\circ\gamma_{13}\circ(id+\eta)\circ\gamma_{32}\circ(id+\eta)\circ\gamma_{21}(r) < r, \tag{2.49}$$

$$(id+\eta)\circ\gamma_{23}\circ(id+\eta)\circ\gamma_{32}(r) < r. \tag{2.50}$$

It can be shown by a contradiction that (2.49) *and* (2.50) *imply* (2.48).
Thus, (2.48) *is equivalent to* (2.49) *and* (2.50).
Assume that the gains are linear and are given by $\gamma_{13}(r) := \gamma_{21}(r) := \gamma_{23}(r) := \gamma_{32}(r) = 0.9\,r$, $r \geq 0$. Choosing $\alpha(r) = \eta(r) = 1/10\,r$ we see that the inequalities (2.49) and (2.50) *are satisfied. Thus, by Theorem 2.4.5 we can conclude that the system* (2.46) *is ISS. In this simple example we also see that a transformation to the pure maximum case would have been equally simple. A two times application of the weak triangle inequality for the first row with $\eta = \alpha$ would have led to the pure maximization case. In this case the small gain condition may be expressed as a cycle condition* (2.42)*, which just yields the conditions (2.49) and* (2.50).

We would like to note that the application of the small gain condition from [54] will not help us to prove stability for this example, as can be seen from the following example.

Example 2.3.6. *In order to apply the results from [45] we could (e.g., by using* (2.19)*) obtain estimates of the form*

$$\begin{aligned} |x_1(t,0)| &\leq \beta_1(|x_1(0)|,t,0) + \gamma_{13}(||x_3||_{(t,0)}) + \gamma_1(\|u\|_\infty), \\ |x_2(t,0)| &\leq \beta_2(|x_2(0)|,t,0) + \gamma_{21}(||x_1||_{(t,0)}) + \gamma_{23}(||x_3||_{(t,0)}), \\ |x_3(t,0)| &\leq \beta_3(|x_3(0)|,t,0) + \gamma_{32}(||x_2||_{(t,0)}) + \gamma_3(\|u\|_\infty). \end{aligned}$$

With the gains from the previous example the corresponding gain matrix is

$$\Gamma = \begin{pmatrix} 0 & 0 & 0.9 \\ 0.9 & 0 & 0.9 \\ 0 & 0.9 & 0 \end{pmatrix},$$

and in the summation case with linear gains the small gain condition is $\rho(\Gamma) < 1$, [54]. In our example $\rho(\Gamma) > 1.19$, so that using this criterion we cannot conclude ISS of the interconnection.

The previous examples motivate the use of the refined small gain condition for the case of different ISS characterizations of subsystems.
In particular, it will be shown that a mixed (or pure sum) ISS condition can always be reformulated as a maximum condition in such a way that the small gain property is preserved.
The following lemma recalls a fact, that was already noted in [126, Lemma 2.2.12].

Lemma 2.3.7. *For any $\alpha \in \mathcal{K}_\infty$ the small gain condition $\mathcal{D} \circ \Gamma \not\geq id$ is equivalent to $\Gamma \circ \mathcal{D} \not\geq id$.*

Proof. Note that $\mathcal{D}$ is a homeomorphism with inverse

$$v \mapsto \mathcal{D}_\alpha^{-1}(v) := \left(\mathcal{D}_1^{-1}(v_1), \dots, \mathcal{D}_n^{-1}(v_n)\right)^T.$$

By monotonicity of $\mathcal{D}$ and $\mathcal{D}^{-1}$ we have $\mathcal{D} \circ \Gamma(v) \not\geq v$ if and only if $\Gamma(v) \not\geq \mathcal{D}^{-1}(v)$. For any $w \in \mathbb{R}^n_+$ define $v = \mathcal{D}(w)$. Then $\Gamma \circ \mathcal{D}(w) \not\geq w$. This proves the equivalence. □

For convenience let us introduce $\mu : \mathbb{R}^n_+ \times \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by

$$\mu(w, v) := (\mu_1(w_1, v_1), \dots, \mu_n(w_n, v_n))^T, w \in \mathbb{R}^n_+,\ v \in \mathbb{R}^n_+, \tag{2.51}$$

where $\mu_i : \mathbb{R}^2_+ \to \mathbb{R}_+$ is such that $\mu_i(w_i, v_i) := w_i + v_i$ for $i \in I_\Sigma$ and $\mu_i(w_i, v_i) := \max\{w_i, v_i\}$ for $i \in I_{\max}$. The following counterpart of Lemma 13 in [54] provides the main technical step in the proof of main results.

Lemma 2.3.8. *Assume that there exists an $\alpha \in \mathcal{K}_\infty$ such that the operator Γ as defined in* (2.38) *satisfies $\Gamma \circ \mathcal{D} \not\geq id$ for a diagonal operator $\mathcal{D}$ as defined in* (2.44). *Then there exists a $\phi \in \mathcal{K}_\infty$ such that for all $w, v \in \mathbb{R}^n_+$,*

$$w \leq \mu(\Gamma(w), v) \tag{2.52}$$

implies $|w| \leq \phi(|v|)$.

Proof. Without loss of generality we assume $I_\Sigma = \{1, \dots, p\}$ and $I_{\max} = I \setminus I_\Sigma$, see Remark 2.2.8, and hence Γ is as in (2.39). Fix any $v \in \mathbb{R}^n_+$. Note that for $v = 0$ there is nothing to show, as then $w \neq 0$ yields an immediate contradiction to the small gain condition. So assume $v \neq 0$.
We first show, that for those $w \in \mathbb{R}^n_+$ satisfying (2.52) at least some components of w have to be bounded. To this end let $\widetilde{\mathcal{D}} : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ be defined by

$$\widetilde{\mathcal{D}}(s) := \left(s_1 + \alpha^{-1}(s_1), \dots, s_p + \alpha^{-1}(s_p), s_{p+1}, \dots, s_n\right)^T$$

for $s \in \mathbb{R}^n_+$ and let $s^* := \widetilde{\mathcal{D}}(v)$. Assume there exists $w = (w_1, \dots, w_n)^T$ satisfying (2.52) and such that $w_i > s^*_i$, $i = 1, \dots, n$. In particular, for $i \in I_\Sigma$ we have

$$s^*_i < w_i \leq \gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n) + v_i \tag{2.53}$$

and hence from the definition of s^* it follows that

$$s^*_i = v_i + \alpha^{-1}(v_i) < \gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n) + v_i.$$

And so $v_i < \alpha(\gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n))$. From (2.53) it follows

$$w_i \leq \gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n) + v_i < (id + \alpha) \circ (\gamma_{i1}(w_1) + \ldots + \gamma_{in}(w_n)). \tag{2.54}$$

Similarly, by the construction of w and the definition of s^* we have for $i \in I_{\max}$

$$v_i = s_i^* < w_i \le \max\{\gamma_{i1}(w_1), \dots, \gamma_{in}(w_n), v_i\}\,, \tag{2.55}$$

and hence

$$w_i \le \max\{\gamma_{i1}(w_1), \dots, \gamma_{in}(w_n)\}. \tag{2.56}$$

From (2.54), (2.56) we get $w \le \mathcal{D}\circ\Gamma(w)$. By Lemma 2.3.7 this contradicts the assumption $\Gamma\circ\mathcal{D} \not\ge \mathrm{id}$. Hence some components of w are bounded by the respective components of $s^1 := s^*$. Iteratively we will prove that all components of w are bounded.

Fix a w satisfying (2.52). Then $w \not> s^1$ and so there exists an index set $I_1 \subset I$, possibly depending on w, such that $w_i > s_i^1$, $i \in I_1$ and $w_i \le s_i^1$, for $i \in I_1^c = I \setminus I_1$. Note that by the first step I_1^c is nonempty. We now renumber the coordinates so that

$$w_i > s_i^1 \text{ and } \quad w_i \le \sum_{j=1}^{n} \gamma_{ij}(w_j) + v_i\,,\ i = 1, \dots, p_1, \tag{2.57}$$

$$w_i > s_i^1 \text{ and } \quad w_i \le \max\{\max_j \gamma_{ij}(w_j), v_i\}\,,\ i = p_1+1, \dots, n_1, \tag{2.58}$$

$$w_i \le s_i^1 \text{ and } \quad w_i \le \sum_{j=1}^{n} \gamma_{ij}(w_j) + v_i,\ ,i = n_1+1, \dots, n_1+p_2, \tag{2.59}$$

$$w_i \le s_i^1 \text{ and } \quad w_i \le \max\{\max_j \gamma_{ij}(w_j), v_i\}\,, i = n_1+p_2+1, \dots, n\,, \tag{2.60}$$

where $n_1 = |I_1|$, $p_1 + p_2 = p$. Using (2.59), (2.60) in (2.57), (2.58) we get

$$w_i \le \sum_{j=1}^{n_1} \gamma_{ij}(w_j) + \sum_{j=n_1+1}^{n} \gamma_{ij}(s_j^1) + v_i,\ i = 1, \dots, p_1, \tag{2.61}$$

$$w_i \le \max\{\max_{j=1,\dots,n_1} \gamma_{ij}(w_j), \max_{j=n_1+1,\dots,n} \gamma_{ij}(s_j^1), v_i\}, i = p_1+1, \dots, n_1\,. \tag{2.62}$$

Define $v^1 \in \mathbb{R}_+^{n_1}$ by

$$v_i^1 := \sum_{j=n_1+1}^{n} \gamma_{ij}(s_j^1) + v_i\,, \qquad i = 1, \dots, p_1\,,$$
$$v_i^1 := \max\{\max_{j=n_1+1,\dots,n} \gamma_{ij}(s_j^1), v_i\}\,, \quad i = p_1+1, \dots, n_1.$$

Now (2.61), (2.62) take the form:

$$w_i \le \sum_{j=1}^{n_1} \gamma_{ij}(w_j) + v_i^1\,, \qquad i = 1, \dots, p_1, \tag{2.63}$$

$$w_i \le \max\{\max_{j=1,\dots,n_1} \gamma_{ij}(w_j), v_i^1\}\,, \quad i = p_1+1, \dots, n_1. \tag{2.64}$$

Let us represent $\Gamma = \begin{pmatrix} \Gamma_{I_1I_1} & \Gamma_{I_1I_1^c} \\ \Gamma_{I_1^cI_1} & \Gamma_{I_1^cI_1^c} \end{pmatrix}$ and define the maps $\Gamma_{I_1I_1} : \mathbb{R}_+^{n_1} \to \mathbb{R}_+^{n_1}$, $\Gamma_{I_1I_1^c} : \mathbb{R}_+^{n-n_1} \to \mathbb{R}_+^{n_1}$, $\Gamma_{I_1^cI_1} : \mathbb{R}_+^{n_1} \to \mathbb{R}_+^{n-n_1}$ and $\Gamma_{I_1^cI_1^c} : \mathbb{R}_+^{n-n_1} \to \mathbb{R}_+^{n-n_1}$ analogous to Γ. Let

$$\mathcal{D}_{I_1}(s) := ((id+\alpha)(s_1), \dots, (id+\alpha)(s_{p_1}), s_{p_1+1}, \dots, s_{n_1})^T\ .$$

From $\Gamma\circ\mathcal{D}(s) \not\ge s$ for all $s \ne 0$, $s \in \mathbb{R}_+^n$ it follows by considering $s = (z^T, 0)^T$ that $\Gamma_{I_1I_1}\circ\mathcal{D}_{I_1}(z) \not\ge z$ for all $z \ne 0$, $z \in \mathbb{R}_+^{n_1}$. Using the same approach as for $w \in \mathbb{R}_+^n$ it can be proved that some components of $w^1 = (w_1, \dots, w_{n_1})^T$ are bounded by the respective components of $s^2 := \widetilde{\mathcal{D}}_{I_1}(v^1)$.

We proceed inductively, defining

$$I_{j+1} \subsetneqq I_j, \quad I_{j+1} := \{i \in I_j : w_i > s_i^{j+1}\}, \tag{2.65}$$

with $I_{j+1}^c := I \setminus I_{j+1}$ and

$$s^{j+1} := \widetilde{\mathcal{D}}_{I_j} \circ (\mu^j(\Gamma_{I_j I_j^c}(s^j_{I_j^c}), v_{I_j})), \tag{2.66}$$

where $\widetilde{\mathcal{D}}_{I_j}$ is defined analogously to $\widetilde{\mathcal{D}}$. The map $\Gamma_{I_j I_j^c} : \mathbb{R}_+^{n-n_j} \to \mathbb{R}_+^{n_j}$ acts analogously to Γ on vectors of the corresponding dimension, $s^j_{I_j^c} = (s_i^j)_{i \in I_j^c}$ is the restriction defined Section 1.1.4 and μ^j is appropriately defined similar to the definition of μ.
The nesting (2.65), (2.66) will end after at most $n-1$ steps: there exists a maximal $l \leq n$, such that

$$I \supsetneqq I_1 \supsetneqq \ldots \supsetneqq I_l \neq \emptyset$$

and all components of w_{I_l} are bounded by the corresponding

$$\begin{array}{rcl} s_\varsigma & := & \max\{s^*, R_{I_1}(s^2), \ldots, R_{I_l}(s^{l+1})\} \\ & := & \begin{pmatrix} \max\{(s^*)_1, (R_{I_1}(s^2))_1, \ldots, (R_{I_l}(s^{l+1}))_1\} \\ \vdots \\ \max\{(s^*)_n, (R_{I_1}(s^2))_n, \ldots, (R_{I_l}(s^{l+1}))_n\} \end{pmatrix} \end{array}$$

where R_{I_j} denotes the anti-projection $\mathbb{R}_+^{|I_j|} \to \mathbb{R}_+^n$ defined in Section 1.1.2.
By the definition of μ for all $v \in \mathbb{R}_+^n$ it holds

$$0 \leq v \leq \mu(\Gamma, \mathrm{id})(v) := \mu(\Gamma(v), v).$$

Applying $\widetilde{\mathcal{D}}$ we have

$$0 \leq v \leq \widetilde{\mathcal{D}}(v) \leq \widetilde{\mathcal{D}} \circ (\mu(\Gamma, \mathrm{id}))(v) \leq \cdots \leq [\widetilde{\mathcal{D}} \circ \mu(\Gamma, \mathrm{id})]^n(v). \tag{2.67}$$

From (2.66) and (2.67) for w satisfying (2.52) we have $w \leq s_\varsigma \leq [\widetilde{\mathcal{D}} \circ \mu(\Gamma, \mathrm{id})]^n(v)$. The term on the right-hand side does not depend on any particular choice of nesting of the index sets. Hence every w satisfying (2.52) also satisfies

$$w \leq [\widetilde{\mathcal{D}} \circ \mu(\Gamma, \mathrm{id})]^n (|v|_{\max}, \ldots, |v|_{\max})^T$$

and taking the maximum-norm on both sides yields $|w|_{\max} \leq \phi(|v|_{\max})$ for some function ϕ of class $\mathcal{K}_\infty$. For example, ϕ can be chosen as

$$\phi(r) := \max\{([\widetilde{\mathcal{D}} \circ \mu(\Gamma, \mathrm{id})]^n (r, \ldots, r))_1, \ldots, ([\widetilde{\mathcal{D}} \circ \mu(\Gamma, \mathrm{id})]^n (r, \ldots, r))_n\}.$$

This completes the proof of the lemma. □

Remark 2.3.9. *Note that if we use inequality $w \leq \Gamma(w) + v$ instead of $w \leq \mu(\Gamma(w), v)$ in Lemma 2.3.8, then the assertion of lemma does not holds in general, see [126, Example 2.5.3].*

Ω-path

We also introduce the important notion of Ω-paths [52]. This concept is useful for the construction of Lyapunov functions and will also be instrumental in obtaining a better understanding of the relation between small gain conditions (2.41) and (2.40).

Definition 2.3.10 (Ω-path)**.** *A continuous path $\sigma \in \mathcal{K}_\infty^n$ is called an* Ω-path with respect to Γ, *if*

(i) for each i, the function σ_i^{-1} is locally Lipschitz continuous on $(0, \infty)$;

(ii) for every compact set $K \subset (0, \infty)$ there are finite constants $0 < K_1 < K_2$ such that for all points of differentiability of σ_i^{-1} and $i = 1, \dots, n$ we have

$$0 < K_1 \leq (\sigma_i^{-1})'(r) \leq K_2, \quad \forall r \in K; \tag{2.68}$$

(iii) for all $r > 0$ it holds that

$$\Gamma(\sigma(r)) < \sigma(r). \tag{2.69}$$

By Theorem 2.3.11 the existence of an Ω-path σ follows from the small gain condition (2.41) provided that an irreducibility condition is satisfied. To define this notion we consider the directed graph $G(\mathcal{V}, \mathcal{E})$ corresponding to Γ with nodes $\mathcal{V} = \{1, \dots, n\}$. A pair $(i, j) \in \mathcal{V} \times \mathcal{V}$ is an edge in the graph if $\gamma_{ij} \neq 0$. Then, Γ is called irreducible, if the graph is strongly connected, see, e.g., the appendix in [54] for further discussions on this topic.
We note that if Γ is reducible, then it may be brought into upper block triangular form by a permutation of the indices

$$\Gamma = \begin{pmatrix} \Upsilon_{11} & \Upsilon_{12} & \dots & \Upsilon_{1d} \\ 0 & \Upsilon_{22} & \dots & \Upsilon_{2d} \\ \vdots & & \ddots & \vdots \\ 0 & \dots & 0 & \Upsilon_{dd} \end{pmatrix} \tag{2.70}$$

where each block $\Upsilon_{jj} \in (\mathcal{K}_\infty \cup \{0\})^{d_j \times d_j}$, $j = 1, \dots, d$, is either irreducible or 0.
Sufficient conditions for the existence of Ω-paths were proved in [52], [128] for a more general gain operator Γ. We specify these conditions for our case with mixed operator Γ in the following theorem:

Theorem 2.3.11. *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n \times n}$ be a gain matrix and $\Gamma : \mathbb{R}_+^n \to \mathbb{R}_+^n$ be the corresponding matrix operator defined as in* (2.45)*. Assume that one of the following assumptions is satisfied*
(i) Γ is linear and the spectral radius of Γ is less than one;
(ii) Γ is irreducible and $\Gamma \not\geq id$;
(iii) $I_{\max} = I$;
(iv) alternatively assume that Γ is bounded, i.e., $\Gamma \in ((\mathcal{K} \setminus \mathcal{K}_\infty) \cup \{0\})^{n \times n}$, and satisfies $\Gamma \not\geq 0$.
Then there exists an Ω-path σ with respect to Γ. This path can be chosen piecewise linear.

For the proof see [52, Theorem 5.2]. Note that the construction of an Ω-path is not explicit there.
Using the following lemma from [87] we can construct explicitly an Ω-path for the case of (iii) in Theorem 2.3.11 with a weaker property (2.69), i.e., such that an Ω-path satisfies:

$$\Gamma(\sigma(r)) \leq \sigma(r) \text{ for all } r > 0. \tag{2.71}$$

Lemma 2.3.12. *Let $I = I_{\max}$. If Γ satisfies* (2.41)*, then for all $s \in \mathbb{R}_+^n$ it holds that $\Gamma(s) \leq Q(s) := (Q_1(s), \dots, Q_n(s))^T$ where*

$$Q_i(s) := \max\{s_i, (\Gamma(s))_i, \dots, (\Gamma^{n-1}(s))_i\}. \tag{2.72}$$

For the proof see [87, Proposition 2.4]. For the construction we need the following lemma that shows how a $\mathcal{K}_\infty$-function can be approximated by a smooth function.

Lemma 2.3.13. *Consider a function $\theta \in \mathcal{K}_\infty$ and a bounded and continuous function $\delta : \mathbb{R}_+ \to \mathbb{R}_+$ with $\delta(0) = 0$ and $\delta(r) > 0$ for $r > 0$. Then there exists a function $\tilde{\theta} \in \mathcal{K}_\infty$ which is smooth on $\mathbb{R}_+$ and satisfies $|\theta(r) - \tilde{\theta}(r)| \leq \delta(\theta(r))$ for all $r \geq 0$ and $\frac{d}{dr}\tilde{\theta}(r) > 0$ for all $r > 0$.*

For the proof see [66, Lemma B.2.1].

Proposition 2.3.14. *Let $\Gamma \in (\mathcal{K}_\infty \cup \{0\})^{n\times n}$ be a gain matrix. Assume that $I_{\max} = I$ and Γ satisfies* (2.41). *Then there exists an Ω-path σ with respect to Γ satisfying (i), (ii) in Definition 2.3.10 and* (2.71). *This path can be chosen piecewise linear.*

Proof. Consider the map $Q : \mathbb{R}^n_+ \to \mathbb{R}^n_+$ defined by $Q(x) := (Q_1(x), \dots, Q_n(x))^T$ with Q_i from (2.72). From Lemma 2.3.12 the inequality $\Gamma(Q(x)) \leq Q(x)$ holds for all $x \geq 0$. Fix any positive vector $a > 0$ and consider $\sigma(t) := Q(at) \in \mathbb{R}^n_+$. Obviously, $\Gamma(\sigma(t)) \leq \sigma(t)$ and by the definition of Q it follows that $\sigma_i \in \mathcal{K}_\infty$ for all $i = 1, \dots, n$. Hence, σ satisfies the condition (2.71). As we can always estimate functions γ_{ij} by a smooth function applying Lemma 2.3.13 and as function σ is obtained through the composition of these functions, the condition (i) for an Ω-path in Definition 2.3.10 is satisfied. Furthermore, as σ is strictly increasing and locally Lipschitz continuous on $(0, \infty)$ function, its inverse σ^{-1} is also strictly increasing and locally Lipschitz continuous on $(0, \infty)$. Thus $(\sigma^{-1})'$ is positive and bounded on some set K and condition (ii) in Definition 2.3.10 is also satisfied. Piecewise linearity of σ follows from [127, Proposition 5.2]. □

Remark 2.3.15. *Note that it will be enough for a construction of an ISS-Lyapunov function for an interconnected hybrid system to have an Ω-path that satisfies* (2.71) *instead of* (2.69)*, see Theorem 2.4.11.*

The following is an immediate corollary to Theorem 8.11 in [52], where the result is only implicitly contained.

Corollary 2.3.16. *Assume that Γ defined as in* (2.38) *is irreducible. Then Γ satisfies the small gain condition if and only if there exists an Ω-path σ for $\mathcal{D} \circ \Gamma$.*

Proof. The implication that the small gain condition guarantees the existence of an Ω-path is shown in [52]. For the converse direction assume that an Ω-path exists for $\mathcal{D} \circ \Gamma$ and that for a certain $s \in \mathbb{R}^n_+, s \neq 0$ we have $\mathcal{D} \circ \Gamma(s) \geq s$. By continuity and unboundedness of σ we may choose a $\tau > 0$ such that $\sigma(\tau) \geq s$ and $\sigma(\tau) \not> s$. Then $s \leq \mathcal{D} \circ \Gamma(s) \leq \mathcal{D} \circ \Gamma(\sigma(\tau)) < \sigma(\tau)$. This contradiction proves the statement. □

2.3.3 From summation to maximization

Now, we use the previous considerations to show that an alternative approach is possible for the treatment of the mixed ISS formulation, which consists of the transforming of this formulation in a complete maximum formulation. Using the weak triangle inequality (2.43) iteratively, the conditions in (2.13) may be transformed into conditions of the form (2.14) with

$$|x_i(t,k)| \leq \beta_i(|x_i^0|, t, k) + \sum_{j,j\neq i} \gamma_{ij}(\|x_j\|_{(t,k)}) + \gamma_i(\|u_i\|_{(t,k)}) \tag{2.73}$$

$$\leq \max\{\tilde{\beta}_i(|x_i^0|, t, k), \max_{j,j\neq i}\{\tilde{\gamma}_{ij}(\|x_j\|_{(t,k)})\}, \tilde{\gamma}_i(\|u_i\|_{(t,k)})\} \tag{2.74}$$

for $i \in I_\Sigma$. To get a general formulation we let $j_1, \dots, j_{p_i}$ denote the indices j for which $\gamma_{ij} \neq 0$. Choose auxiliary functions $\eta_{i0}, \dots, \eta_{ip_i} \in \mathcal{K}_\infty$ and define $\chi_{i0} := (\mathrm{id} + \eta_{i0})$ and $\chi_{il} = (\mathrm{id} + \eta_{i0}^{-1}) \circ \dots \circ (\mathrm{id} + \eta_{i(l-1)}^{-1}) \circ (\mathrm{id} + \eta_{il})$, $l = 1, \dots, p_i$, and $\chi_{i(p_i+1)} = (\mathrm{id} + \eta_{i0}^{-1}) \circ \dots \circ (\mathrm{id} + \eta_{ip_i}^{-1})$. Choose a permutation $\pi_i : \{0, 1, \dots, p_i + 1\} \to \{0, 1, \dots, p_i + 1\}$ and define

$$\tilde{\beta}_i := \chi_{i\pi_i(0)} \circ \beta_i\,, \quad \tilde{\gamma}_{ij_l} := \chi_{i\pi_i(l)} \circ \gamma_{ij_l}\,,\ l = 1, \dots, p_i,\ \tilde{\gamma}_i := \chi_{i\pi_i(p_i+1)} \circ \gamma_i\,, \tag{2.75}$$

and of course $\tilde{\gamma}_{ij} \equiv 0,\ j \notin \{j_1, \dots, j_{p_1}\}$. In this manner the inequalities (2.74) are valid and a maximum ISS formulation is obtained. Performing this for every $i \in I_\Sigma$ we obtain an operator $\tilde{\Gamma} : \mathbb{R}_n^+ \to \mathbb{R}_n^+$ defined by

$$\left(\tilde{\Gamma}_1(s), \dots, \tilde{\Gamma}_n(s)\right)^T\,, \tag{2.76}$$

where the functions $\tilde{\Gamma}_i : \mathbb{R}^n_+ \to \mathbb{R}_+$ are given by $\tilde{\Gamma}_i(s) := \max\{\tilde{\gamma}_{i1}(s_1), \dots, \tilde{\gamma}_{in}(s_n)\}$ for $i \in I_\Sigma$ and $\tilde{\Gamma}_i(s) := \max\{\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n)\}$ for $i \in I_{\max}$. Here, the $\tilde{\gamma}_{ij}$'s are given by (2.75), whereas the γ_{ij}'s are the original gains.
As it turns out, the permutation is not really necessary and it is sufficient to peel off the summands one after the other. We will now show that given a gain operator Γ with a mixed or pure sum formulation, which satisfies the small gain condition $\mathcal{D} \circ \Gamma \not\geq \mathrm{id}$, it is always possible to switch to a maximum formulation, which also satisfies the corresponding small gain condition $\tilde{\Gamma} \not\geq \mathrm{id}$. In the following statement p_i is to be understood as defined just after (2.74).

Proposition 2.3.17. *Consider a gain operator Γ of the form* (2.38). *Then the following two statements are equivalent:*

(i) the small gain condition (2.45) *is satisfied,*

(ii) for each $i \in I_\Sigma$ there exist $\eta_{i,0}, \dots, \eta_{i,(p_i+1)} \in \mathcal{K}_\infty$, such that the corresponding small gain operator $\tilde{\Gamma}$ satisfies the small gain condition (2.41).

Remark 2.3.18. *We note that in the case that a system of the form* (2.1) *satisfies the mixed ISS condition* (2.45) *with operator Γ, then the construction in* (2.73) *shows that the ISS condition is also satisfied in the maximum sense with the operator $\tilde{\Gamma}$. On the other hand, the construction in the proof does not guarantee that, if the ISS condition is satisfied with gains from the operator $\tilde{\Gamma}$, i.e. in the maximum formulation, then it will also be satisfied for the original Γ the in mixed formulation. To the best of our knowledge, there is no sharp estimations, analogous to weak triangle inequality* (2.43)*, that allow to majorize maximizations by summations and to obtain equivalence between small gain conditions corresponding to mixed and summation cases.*

Proof. "⇒": We will show the statement under the condition that Γ is irreducible. In the reducible case we may assume that Γ is in upper block triangular form (2.70). In each of the diagonal blocks we can perform the transformation described below and the gains in the off-diagonal blocks are of no importance for the small gain condition.
In the irreducible case we may apply Corollary 2.3.16 to obtain a continuous map $\sigma : [0, \infty) \to \mathbb{R}^n_+$, where $\sigma_i \in \mathcal{K}_\infty$ for every component function of σ and so that

$$\mathcal{D} \circ \Gamma \circ \sigma(\tau) < \sigma(\tau)\,, \quad \text{for all} \quad \tau > 0. \tag{2.77}$$

Define the homeomorphism $T : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $T : s \mapsto (\sigma_1(s_1), \dots, \sigma_n(s_n))$. Then $T^{-1} \circ \mathcal{D} \circ \Gamma \circ T \not\geq \mathrm{id}$ and we have by (2.77) for $e = \sum_{i=1}^n e_i$, that

$$T(\tau e) = \sigma(\tau) > \mathcal{D} \circ \Gamma \circ \sigma(\tau) = \mathcal{D} \circ \Gamma \circ T(\tau e)\,,$$

so that for all $\tau > 0$

$$T^{-1} \circ \mathcal{D} \circ \Gamma \circ T(\tau e) < \tau e\,. \tag{2.78}$$

We will show that $T^{-1} \circ \tilde{\Gamma} \circ T(\tau e) < \tau e$ for an appropriate choice of the functions η_{ij}. By the converse direction of Corollary 2.3.16 this shows that $T^{-1} \circ \tilde{\Gamma} \circ T \not\geq \mathrm{id}$ and hence $\tilde{\Gamma} \not\geq \mathrm{id}$ as desired.
Consider now a row corresponding to $i \in I_\Sigma$ and let $j_1, \ldots, j_{p_i}$ be the indices for which $\gamma_{ij} \neq 0$. For this row (2.78) implies

$$\sigma_i^{-1} \circ (\mathrm{id} + \alpha) \circ \left(\sum_{j, j \neq i} \gamma_{ij}(\sigma_j(r)) \right) < r\,, \quad \forall r > 0\,, \tag{2.79}$$

or equivalently

$$(\mathrm{id} + \alpha) \circ \left(\sum_{j, j \neq i} \gamma_{ij} \circ \sigma_j \circ \sigma_i^{-1} \right) \circ \sigma_i(r) < \sigma_i(r)\,, \quad \forall r > 0\,. \tag{2.80}$$

This shows that

$$(\mathrm{id} + \alpha) \circ \left(\sum_{j, j \neq i} \gamma_{ij} \circ \sigma_j \circ \sigma_i^{-1} \right) < \mathrm{id}\,, \quad \text{on } (0, \infty)\,. \tag{2.81}$$

Note that this implies that $\left(\mathrm{id} - \sum_{j,j\neq i} \gamma_{ij} \circ \sigma_j \circ \sigma_i^{-1}\right) \in \mathcal{K}_\infty$ because $\alpha \in \mathcal{K}_\infty$. We may therefore choose $\hat{\gamma}_{ij} > \gamma_{ij} \circ \sigma_j \circ \sigma_i^{-1}, j = j_1, \ldots, j_{p_i}$ in such a manner that

$$\mathrm{id} - \sum_{l=1}^{p_i} \hat{\gamma}_{ij_l} \quad \in \mathcal{K}_\infty\,.$$

Now, define for $l = 1, \ldots, p_i$

$$\eta_{il} := \left(\mathrm{id} - \sum_{p, p \leq l} \hat{\gamma}_{ij_p} \right) \circ \hat{\gamma}_{ij_l}^{-1} \quad \in \mathcal{K}_\infty\,.$$

It is straightforward to check that

$$(\mathrm{id} + \eta_{il}) = \left(\mathrm{id} - \sum_{p, p < l} \hat{\gamma}_{ij_p} \right) \circ \hat{\gamma}_{ij_l}^{-1}\,,$$

$$(\mathrm{id} + \eta_{il}^{-1}) = \left(\mathrm{id} - \sum_{p, p < l} \hat{\gamma}_{ij_p} \right) \circ \left(\mathrm{id} - \sum_{p, p \leq l} \hat{\gamma}_{ij_p} \right)^{-1}\,.$$

With $\chi_{il} := (\mathrm{id} + \eta_{i1}^{-1}) \circ \ldots \circ (\mathrm{id} + \eta_{i(l-1)}^{-1}) \circ (\mathrm{id} + \eta_{il})$ it follows that

$$\begin{aligned} \chi_{il} \circ \gamma_{ij_l} \circ \sigma_{j_l} \circ \sigma_i^{-1} &= (\mathrm{id} + \eta_{i1}^{-1}) \circ \ldots \circ (\mathrm{id} + \eta_{i,l-1}^{-1}) \circ (\mathrm{id} + \eta_{il}) \circ \gamma_{ij_l} \circ \sigma_{j_l} \circ \sigma_i^{-1} \\ &= \hat{\gamma}_{ij_l}^{-1} \circ \gamma_{ij_l} \circ \sigma_{j_l} \circ \sigma_i^{-1} < \mathrm{id}\,. \end{aligned}$$

This shows that it is possible to choose $\eta_{ij}, i \in I_\Sigma$ such that all the entries in $T^{-1} \circ \tilde{\Gamma} \circ T$ are smaller than the identity. This shows the assertion.

"$\Leftarrow$": To show the converse direction let the small gain condition (2.41) be satisfied for the operator $\tilde{\Gamma}$. Consider $i \in I_\Sigma$.
We consider the following two cases for the permutation π used in (2.75). Define
$q := \min\{\pi(0), \pi(p_i+1)\}$. In the first case $\{\pi(0), \pi(p_i+1)\} = \{p_i, p_i+1\}$, i.e., $\pi(l) < q, \forall\, l \in \{1, \dots, p_i\}$. Alternatively, the second case is $\exists l \in \{1, \dots, p_i\} : \pi(l) > q$. We define $\alpha_i \in \mathcal{K}_\infty$ by

$$\alpha_i := \begin{cases} \eta_{iq}^{-1} \circ \sum\limits_{l,\pi(l)>q} \gamma_{ij_l} \circ \left(\sum\limits_j \gamma_{ij}\right)^{-1}, & \text{if } \exists j \in \{1,\dots,p_i\} : \pi(j) > q\,, \\ \eta_{i,q-1} \circ \gamma_{i,j_{\pi^{-1}(q-1)}} \circ \left(\sum\limits_j \gamma_{ij}\right)^{-1}, & \text{if } \forall j \in \{1,\dots,p_i\}, \quad \pi(j) < q\,. \end{cases} \tag{2.82}$$

Consider the ith row of $\mathcal{D} \circ \Gamma$ and the case $\exists j \in \{1, \dots, p_i\} : \pi(j) > q$. (Note that for no $l \in \{1, \dots, p_i\}$ we have $\pi(l) = q$).

$$\begin{aligned} (\text{id} + \alpha_i) \circ \sum_j \gamma_{ij} &= \sum_j \gamma_{ij} + \alpha_i \circ \sum_j \gamma_{ij} \\ &= \sum_j \gamma_{ij} + \eta_{iq}^{-1} \circ \sum_{l,\pi(l)>q} \gamma_{ij_l} \circ \left(\sum_j \gamma_{ij}\right)^{-1} \circ \sum_j \gamma_{ij} \\ &= \sum_j \gamma_{ij} + \eta_{iq}^{-1} \circ \sum_{l,\pi(l)>q} \gamma_{ij_l} \\ &= \sum_{l,\pi(l)<q} \gamma_{ij_l} + (\text{id} + \eta_{iq}^{-1}) \circ \sum_{l,\pi(l)>q} \gamma_{ij_l}\,. \end{aligned} \tag{2.83}$$

Applying the weak triangle inequality (2.43) first to the rightmost sum in the last line of (2.83) and then to the remaining sum, we obtain

$$\begin{aligned} \sum_{l,\pi(l)<q} \gamma_{ij_l} + (\text{id} + \eta_{iq}^{-1}) \circ \sum_{l,\pi(l)>q} \gamma_{ij_l} \le \sum_{l,\pi(l)<q-1} \gamma_{ij_l} + \max\{(\text{id} + \eta_{i,q-1}) \circ \gamma_{i,\pi^{-1}(q-1)}, \\ (\text{id} + \eta_{i,q-1}^{-1}) \circ (\text{id} + \eta_{iq}^{-1}) \circ \max_{l,\pi(l)>q} \{(\text{id} + \eta_{i,q+1}^{-1}) \circ \dots \\ \circ (\text{id} + \eta_{i,\pi(l)-1}^{-1}) \circ (\text{id} + \eta_{i\pi(l)}) \circ \gamma_{ij_l}\}\} \le \dots \\ \le \max_l \{\chi_{i\pi(l)} \circ \gamma_{ij_l}\}\,. \end{aligned} \tag{2.84}$$

The last expression is the defining equation for $\widetilde{\Gamma}_i(s_1, \dots, s_n) = \max\limits_{l=1,\dots,p_i} \{\chi_{i\pi(l)} \circ \gamma_{ij_l}(s_{j_l})\}$. Thus, from (2.83), (2.84) we obtain $\widetilde{\Gamma}_i \ge (\mathcal{D} \circ \Gamma)_i$.
Consider now the case $\forall l \in \{1, \dots, p_i\}$, $\pi(l) < q$. A similar approach shows that $\widetilde{\Gamma}_i \ge (\mathcal{D} \circ \Gamma)_i$. Following the same steps as in the first case we obtain

$$\begin{aligned} (\text{id} + \alpha_i) \circ \sum_j \gamma_{ij} &= \sum_j \gamma_{ij} + \eta_{i,q-1} \circ \gamma_{i,j_{\pi^{-1}(q-1)}} \\ &= \sum_{l,\pi(l)<q-1} \gamma_{ij_l} + (\text{id} + \eta_{i,q-1}) \circ \gamma_{i,j_{\pi^{-1}(q-1)}} \\ &\le \sum_{l,\pi(l)<q-2} \gamma_{ij_l} + \max\{(\text{id} + \eta_{i,q-2}) \circ \gamma_{ij_{\pi^{-1}(q-2)}}, \\ &\qquad (\text{id} + \eta_{i,q-2}^{-1}) \circ (\text{id} + \eta_{i,(q-1)}) \circ \gamma_{i,j_{\pi^{-1}(q-1)}}\} \\ &\le \dots \quad \le \max_l \{\chi_{i\pi(l)} \circ \gamma_{ij_l}\}\,. \end{aligned} \tag{2.85}$$

Again from (2.85), $\widetilde{\Gamma}_i \geq (\mathcal{D} \circ \Gamma)_i$.
Taking $\alpha = \min \alpha_i \in \mathcal{K}_\infty$ it holds that $\widetilde{\Gamma} \geq \mathcal{D} \circ \Gamma$. Thus if $\widetilde{\Gamma} \not\geq \mathrm{id}$, then $\mathcal{D} \circ \Gamma \not\geq \mathrm{id}$. □

2.4 Stability conditions

Now, we are going to use the small gain condition (2.45) to establish stability of an interconnection of hybrid systems. In particular, applying Lemma 2.3.8 we obtain that under the condition (2.45) ISS of a system of the form (2.5) follows. Furthermore, in Theorem 2.4.13 we illustrate a construction of an ISS-Lyapunov function of a system of the form (2.5) under the condition (2.45). Finally, we establish ISS of particular subclasses of hybrid systems under the mixed small gain condition.

2.4.1 Small gain theorems in terms of trajectories

To prove the stability results we follow similar steps as in the case of continuous systems [54], [126]. We first show that an interconnection of hybrid systems has the AG-property and is pre-GS using the result of Lemma 2.3.8. The following small gain theorems extend the results of [96], [126] and [54] to the case of an arbitrary number of interconnected hybrid subsystems.

Theorem 2.4.1. *Assume that all subsystems in* (2.1) *are pre-GS and a gain matrix is given by* $\Gamma = (\hat{\gamma}_{ij})_{n\times n}$ *with* $\hat{\gamma}_{ij}$ *as in* (2.15) *and* (2.16)*. If there exists a function* $\alpha \in \mathcal{K}_\infty$ *such that* Γ *satisfies* (2.45)*, then the system* (2.5) *is pre-GS.*

Proof. Let us take the supremum on both sides of (2.16) and (2.15) over $(\tau, l) \leq (t, k)$. For $i \in I_\sum$ we have

$$\|x_{i(t,k)}\|_{(\bar{\tau},\bar{l})} \leq \sigma_i(|x^0|) + \sum_{j,j\neq i} \hat{\gamma}_{ij}(\|x_{j(t,k)}\|_{(\bar{\tau},\bar{l})}) + \hat{\gamma}_i(\|u\|_{(\bar{\tau},\bar{l})}), \tag{2.86}$$

and for $i \in I_{\max}$ it follows

$$\|x_{i(t,k)}\|_{(\bar{\tau},\bar{l})} \leq \max\{\sigma_i(|x^0|), \max_{j,j\neq i} \hat{\gamma}_{ij}(\|x_{j(t,k)}\|_{(\bar{\tau},\bar{l})}), \hat{\gamma}_i(\|u\|_{(\bar{\tau},\bar{l})})\}, \tag{2.87}$$

where $(\bar{\tau}, \bar{l}) := \max_{(\tau,l)\in \mathrm{dom}\, x_i} (\tau, l)$. Let us denote $w := \left(\|x_{1(t,k)}\|_{(\bar{\tau},\bar{l})}, \ldots, \|x_{n(t,k)}\|_{(\bar{\tau},\bar{l})}\right)^T$,

$$v \;:=\; \begin{pmatrix} \mu_1(\sigma_1(|x^0|), \hat{\gamma}_1(\|u\|_{(\bar{\tau},\bar{l})})) \\ \vdots \\ \mu_n(\sigma_n(|x^0|), \hat{\gamma}_n(\|u\|_{(\bar{\tau},\bar{l})})) \end{pmatrix} = \mu(\sigma(|x^0|), \hat{\gamma}(\|u\|_{(\bar{\tau},\bar{l})}))\ ,$$

where we use notations μ and μ_i defined in (2.51), and denote $\sigma := (\sigma_1, \ldots, \sigma_n)^T$, $\hat{\gamma} := (\hat{\gamma}_1, \ldots, \hat{\gamma}_n)^T$. From (2.86), (2.87) we obtain $w \leq \mu(\Gamma(w), v)$. Then, by Lemma 2.3.8 there exists a function $\phi \in \mathcal{K}_\infty$ such that

$$\begin{aligned} |(\|x_{1(t,k)}\|_{(\bar{\tau},\bar{l})}, \ldots, \|x_{n(t,k)}\|_{(\bar{\tau},\bar{l})})^T| \;&\leq\; \phi(|\mu(\sigma(|x^0|), \hat{\gamma}(\|u\|_{(\bar{\tau},\bar{l})}))|) \\ &\leq\; \phi(2|\sigma(|x^0|)|) + \phi(2|\hat{\gamma}(\|u\|_{(\bar{\tau},\bar{l})})|) \end{aligned} \tag{2.88}$$

for all $(t, k) \in \mathrm{dom}\, x$. Hence, for every initial condition and essentially bounded input u the solution of the system (2.5) exists and is bounded, since the right-hand side of (2.88) does not depend on t, k. The estimate for pre-GS in terms of summations is then given by (2.88). □

As we check in Theorem 2.4.1 small gain condition (2.45) only, it seems that the theorem also holds if all σ_i in (2.16) and (2.15) are given as summands or arguments of maximization with respect to the rest of gains, i.e. in the case

$$|x_i(t,k)| \le \sigma_i(|x_i^0|) + \sum_{j=1, j\neq i}^{n} \hat{\gamma}_{ij}(\|x_j\|_{(t,k)}) + \hat{\gamma}_i(\|u\|_{(t,k)}), \forall (t,k) \in \text{dom}\, x_i,$$

$i \in I_\Sigma$, and

$$|x_i(t,k)| \le \sigma_i(|x_i^0|) + \max\{\max_{j, j\neq i} \hat{\gamma}_{ij}(\|x_j\|_{(t,k)}), \hat{\gamma}_i(\|u\|_{(t,k)})\}, \forall (t,k) \in \text{dom}\, x_i,$$

or

$$|x_i(t,k)| \le \max\{\sigma_i(|x_i^0|), \sum_{j=1, j\neq i}^{n} \hat{\gamma}_{ij}(\|x_j\|_{(t,k)}) + \hat{\gamma}_i(\|u\|_{(t,k)})\}, \forall (t,k) \in \text{dom}\, x_i,$$

$i \in I_\Sigma$, and

$$|x_i(t,k)| \le \max\{\sigma_i(|x_i^0|), \max_{j, j\neq i} \hat{\gamma}_{ij}(\|x_j\|_{(t,k)}), \hat{\gamma}_i(\|u\|_{(t,k)})\}, \forall (t,k) \in \text{dom}\, x_i,$$

However, in this case we cannot apply Lemma 2.3.8 to obtain GS-estimation (2.88), see Remark 2.3.9. Thus this question is still open.

Lemma 2.4.2. *Let $s : \mathbb{R}_+ \to \mathbb{R}_+^n$ defined on the time domain dom s be continuous between the jumps and bounded. Then*

$$\limsup_{(t,k)\in dom\, s,\, t+k\to\infty} s(t,k) = \limsup_{t+k\to\infty} \|s_{[(t/2,[k/2]), \lim\limits_{\tau+j\to\infty}(\tau,j)]}\|_\infty .$$

Proof. The proof goes along the lines of the proof of a similar result for continuous systems in Lemma 3.2 in [54], but instead of the time t we consider the points (t,k) of the time domain.
Let $\limsup\limits_{(t,k)\in \text{dom}\, s,\, t+k\to\infty} s(t,k) = a \in \mathbb{R}_+^n$ and $\limsup\limits_{t+k\to\infty} \|s_{[(t/2,[k/2]), \lim\limits_{\tau+j\to\infty}(\tau,j)]}\|_\infty = b \in \mathbb{R}_+^n$.
Note that we have

$$s(t,k) \le \|s_{[(t/2,[k/2]), \lim\limits_{\tau+j\to\infty}(\tau,j)]}\|$$

for all $t, k \ge 0$. Thus we obtain that $a \le b$ by taking $\limsup$ on both sides.
It remains to show that $a \ge b$. For all $\epsilon \in \mathbb{R}_+^n$, $\epsilon > 0$ there exist $T_a, T_b \ge 0$ such that

$$\forall\, t+k \ge T_a : \qquad \sup_{t+k\ge T_a} s(t,k) \le a + \epsilon, \tag{2.89}$$

$$\forall\, t+k \ge T_b : \qquad \sup_{t+k\ge T_b} \|s_{[(t/2,[k/2]), \lim\limits_{\tau+j\to\infty}(\tau,j)]}\| \le b + \epsilon. \tag{2.90}$$

If $s(\tilde{t},\tilde{j}) \le a+\epsilon$ for all $\tilde{t}+\tilde{j} \ge t+k$, then $\|s_{[(\tilde{t}/2,[\tilde{k}/2]), \lim\limits_{\tau+j\to\infty}(\tau,j)]}\| \le a+\epsilon$ for $\tilde{t}+\tilde{j} \ge 2(t+k+1)$.
Thus we obtain $a \ge b$. □

In the following small gain theorems we require additionally that all the subsystems have the same jump set $D_i = D$. According to (2.6), this additional condition implies that the subsystems can jump only simultaneously. This means, in particular, that the gains $\widetilde{\gamma}_{ii} = 0$ (resp. $\gamma_{ii} = 0$) for all $i \in \{1 \ldots, n\}$. In Remark 2.4.7 and Example 2.4.8 we explain the necessity of this requirement.

Theorem 2.4.3. *Assume that $D_i = D, i = 1, \dots, n$ and that each subsystem of* (2.1) *has the AG property and that solutions of the system* (2.5) *exist, are bounded and some of them are complete. Let the gain matrix Γ be given by $\Gamma = (\widetilde{\gamma}_{ij})_{n\times n}$ with $\widetilde{\gamma}_{ij}$ from* (2.17) *and* (2.18). *If there exists a function $\alpha \in \mathcal{K}_\infty$ such that Γ satisfies* (2.45), *then system* (2.5) *satisfies the AG property.*

If there is no complete solution of (2.5), then the system (2.5) is AG by definition.

Proof. Let (τ, l) be an arbitrary initial point of the time domain. From the definition of the AG property we have that any solution of (2.1) satisfies for $i \in I_\sum$

$$\limsup_{(t,k)\in \operatorname{dom} x_i, t+k\to\infty} |x_i(t,k)| \le \sum_{j,j\neq i} \widetilde{\gamma}_{ij}(\|x_{j[(\tau,l),\lim_{\bar\tau+\bar l\to\infty}(\bar\tau,\bar l)]}\|_\infty) + \widetilde{\gamma}_i(\|u\|_\infty) \tag{2.91}$$

and for $i \in I_{\max}$ it follows

$$\limsup_{(t,k)\in \operatorname{dom} x_i, t+k\to\infty} |x_i(t,k)| \le \max\{\max_{j,j\neq i} \widetilde{\gamma}_{ij}(\|x_{j[(\tau,l),\lim_{\bar\tau+\bar l\to\infty}(\bar\tau,\bar l)]}\|_\infty), \widetilde{\gamma}_i(\|u\|_\infty)\}. \tag{2.92}$$

Then, from Lemma 3.6 in [28] it follows for $i \in I_\sum$ that

$$\limsup_{(t,k)\in \operatorname{dom} x_i, t+k\to\infty} |x_i(t,k)| \le \sum_{j,j\neq i} \widetilde{\gamma}_{ij}(\limsup_{\tau+l\to\infty} \|x_{j[(\tau,l),\lim_{\bar\tau+\bar l\to\infty}(\bar\tau,\bar l)]}\|_\infty) + \widetilde{\gamma}_i(\|u\|_\infty) \tag{2.93}$$

and for $i \in I_{\max}$

$$\limsup_{(t,k)\in \operatorname{dom} x_i, t+k\to\infty} |x_i(t,k)| \le \max\{\max_{j,j\neq i} \widetilde{\gamma}_{ij}(\limsup_{\tau+l\to\infty} \|x_{j[(\tau,l),\lim_{\bar\tau+\bar l\to\infty}(\bar\tau,\bar l)]}\|_\infty), \widetilde{\gamma}_i(\|u\|_\infty)\}. \tag{2.94}$$

Since all solutions of (2.1) are bounded and continuous between the jumps, the following holds by Lemma 2.4.2:

$$\limsup_{\substack{(t,k)\in \operatorname{dom} x_i,\\ t+k\to\infty}} |x_i(t,k)| = \limsup_{\tau+l\to\infty}(\|x_{i[(\tau,l),\lim_{\bar\tau+\bar l\to\infty}(\bar\tau,\bar l)]}\|_\infty) =: l_i(x_i).$$

By this property from (2.93) for $i \in I_\sum$ it follows

$$l_i(x_i) \le \sum_{j,j\neq i} \widetilde{\gamma}_{ij}(l_j(x_j)) + \widetilde{\gamma}_i(\|u\|_\infty)$$

and for $i \in I_{\max}$ from (2.94) it follows

$$l_i(x_i) \le \max\{\max_{j,j\neq i} \widetilde{\gamma}_{ij}(l_j(x_j)), \widetilde{\gamma}_i(\|u\|_\infty)\}.$$

Using Lemma 2.3.8 for $\Gamma = (\widetilde{\gamma}_{ij})_{n\times n}$, $w_i := l_i(x_i)$ and $v_i := \widetilde{\gamma}_i(\|u\|_\infty)$ we conclude

$$\limsup_{(t,k)\in \operatorname{dom} x, t+k\to\infty} |x(t,k)| \le \phi(|(\widetilde{\gamma}_1(\|u\|_\infty), \dots, \widetilde{\gamma}_n(\|u\|_\infty))^T|) \tag{2.95}$$

for some ϕ of class $\mathcal{K}_\infty$, which is the desired AG property. □

Remark 2.4.4. *Boundedness of solutions of* (2.5) *is essential, otherwise the assertion is not true, see Example 14 in [54].*

The following theorem extends a result in [96]. In particular, in Theorem 1 in [96] it was shown that an interconnection of two hybrid systems that are ISS in terms of maximizations is ISS under the small gain condition (2.41). Here, we show that the same holds in terms of mixed formulation of ISS under the small gain condition (2.45) for an arbitrary finite number of hybrid systems.

Theorem 2.4.5. *Consider an interconnected system of the form* (2.5)*. Assume that* $D_i = D, i = 1, \dots, n$ *and that the set* $\{f(x,u) : u \in U \cap \epsilon\overline{\mathbb{B}}\}$ *is convex for each* $x \in \chi, \epsilon > 0$*. If all the subsystems* (2.1) *are ISS and there exists a function* $\alpha \in \mathcal{K}_\infty$ *such that a corresponding gain matrix* Γ *satisfies* $\Gamma \circ \mathcal{D}(s) \not\geq s$, $\forall s \in \mathbb{R}^n_+ \backslash \{0\}$*, then the system* (2.5) *is ISS.*

Proof. The proof follows the same steps as the proof of a similar theorem for continuous systems in [54, Theorem 4.4].
By Remark 2.2.5 and Theorem 2.2.7, since each subsystem is ISS, they are pre-GS with gains $\hat{\gamma}_{ij} \leq \gamma_{ij}$ and have the AG property with gains $\widetilde{\gamma}_{ij} \leq \gamma_{ij}$. By Theorem 2.4.1 the whole interconnection (2.5) is pre-GS and thus all solutions are bounded.
Then Theorem 2.4.3 implies that the system (2.5) has the AG property. From global pre-stability of (2.5) 0-input pre-stability follows, see Remark 2.2.5.
ISS of (2.5) follows then by Theorem 2.2.7. □

Remark 2.4.6. *In comparison to Theorem 1 in [96], we require additionally the convexity of* $f(x,u)$ *in Theorem 2.4.5. This is due to the fact that we use in our proof that ISS is equivalent to* 0*-input pre-stability and the AG property. To the best of our knowledge this property holds under the convexity assumption on* $f(x,u)$*, see [28, Thereom 3.1].*

Remark 2.4.7. *If we drop the requirements* $D_i = D$ *in Theorem 2.4.3 and Theorem 2.4.5 the assertion is not true, because otherwise it may happen for some* $(x,u) \in C_i$ *that* $(x,u) \notin C$*. According to our definition of the interconnection* (2.5) *this allows for a situation that in one of the solutions one subsystem undergoes infinitely many jumps and the subsystem* i *is "frozen". This implies that* β *never tends to zero though* k *tends to infinity, see the following example.*

Example 2.4.8. *Consider an interconnection of two hybrid systems* Σ_1 *and* Σ_2*:*

$$\Sigma_1 \quad : \begin{cases} \dot{x}_1 = -x_1 + \frac{1}{2}x_2 & =: f_1(x_1,x_2), \quad (x_1,x_2) \in C_1 = \{(s_1,s_2) \in \mathbb{R}^2 : s_1 \geq 1\}, \\ x_1^+ = \frac{1}{2}x_1 & =: g_1(x_1,x_2), \quad (x_1,x_2) \in D_1 = \{(s_1,s_2) \in \mathbb{R}^2 : s_1 \leq 1\}, \end{cases} \tag{2.96}$$

$$\Sigma_2 : \quad \begin{cases} \dot{x}_2 = -x_2 + \frac{1}{2}x_1 & =: f_2(x_1,x_2), \quad (x_1,x_2) \in C_2 = \{(s_1,s_2) \in \mathbb{R}^2 : s_2 \geq 1\}, \\ x_2^+ = \frac{1}{2}x_2 & =: g_2(x_1,x_2), \quad (x_1,x_2) \in D_2 = \{(s_1,s_2) \in \mathbb{R}^2 : s_2 \leq 1\}. \end{cases} \tag{2.97}$$

It can be easily shown that Σ_1 *and* Σ_2 *have the AG-property with* $\gamma_{12} = \gamma_{21} = \frac{1}{2}id$*. Let us now describe their interconnection* Σ *as in* (2.5)*:*

$$\Sigma : \quad \begin{cases} \dot{x} = (f_1, f_2)^T, \quad x \in C = C_1 \cap C_2 = \{(s_1,s_2) \in \mathbb{R}^2 : s_1, s_2 \geq 1\}, \\ x^+ = \begin{cases} \begin{pmatrix} x_1 \\ \frac{1}{2}x_2 \end{pmatrix}, & x_1 > 1, x_2 \leq 1, \\ \begin{pmatrix} \frac{1}{2}x_1 \\ x_2 \end{pmatrix}, & x_1 \leq 1, x_2 > 1, \\ \begin{pmatrix} \frac{1}{2}x_1 \\ \frac{1}{2}x_2 \end{pmatrix}, & x_1 \leq 1, x_2 \leq 1. \end{cases} \end{cases} \tag{2.98}$$

The cycle condition (2.42) *corresponding to the small gain condition* (2.45) *is satisfied:* $\gamma_{12} \circ \gamma_{21} = \frac{1}{4}id < id$*. Now, if we take initial conditions* $x_1^0 > x_2^0 > 1$*, then the trajectories of* x_1 *and* x_2 *are at the*

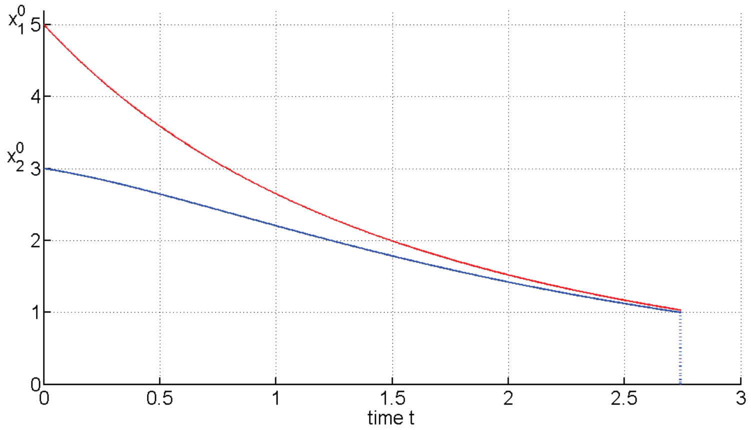


Figure 2.3: Trajectories of the subsystems Σ_1 and Σ_2.

beginning continuous, see Figure 2.3 for $x_1^0 = 5$, $x_2^0 = 3$. At some time instant t^*, the trajectory of x_2 reaches D_2, i.e. $x_2 \leq 1$ and subsystem Σ_2 begins to jump infinitely many times according to (2.98), *i.e., $x_1^+(t^*, l) = x_1^+(t^*, 0)$ and $x_2^+ = (t^*, l) = \frac{1}{2}x_2^+(t^*, l-1)$ for $l \geq 1$. Thus, with the number of jumps l tending to infinity, the trajectory of x_2 stays in D_2 and tends to zero, and trajectory of x_1 stays in C and is constant. Thus, the overall trajectory $x = (x_1, x_2)^T$ does not have the AG property. Note, however, that it is pre-GS.*

Now, according to Theorem 2.4.5, to establish input-to-state stability of the logistics network described as in (1.55) we need first to verify whether all logistics locations are ISS and to find their ISS estimates (2.13)-(2.14). Then, if their cooperation structure given by the matrix Γ satisfies the mixed small gain condition $\Gamma \circ \mathcal{D}(s) \not\geq s$, $\forall s \in \mathbb{R}^n_+ \backslash \{0\}$, then the logistics network is ISS.
From the practical point of view, it is easier to find first an ISS-Lyapunov function for the individual location. To establish ISS of the whole network in this case one needs a similar small gain theorem in terms of ISS-Lyapunov function. In the following section we prove such a theorem.

2.4.2 Construction of ISS-Lyapunov functions for interconnected hybrid systems

In this section we show how an ISS-Lyapunov function for an interconnected system of the form (2.5) can be constructed using the small gain condition (2.45). This allows to apply Proposition 2.2.12 to deduce ISS of (2.5).
First, we recall some known auxiliary results. The following lemma shows a property of the Clarke's gradient in the case of maximization.

Lemma 2.4.9. *Let q_i be Lipschitz in some neighbourhood of x, and $q(x) := \max\limits_{1\leq i\leq n} q_i(x)$. Then q is Lipschitz in this neighbourhood, with $\partial q(x) \subset$ conv $\left\{\bigcup_{i\in I(x)} \partial q_i(x)\right\}$, where $I(x) := \{i \in \{1, 2 \ldots, n\} : q_i(x) = q(x)\}$.*

For the proof see [34, p.83] and references therein.

Lemma 2.4.10. *[The chain rule] Let $Q_1 : X \to \mathbb{R}^n$ be Lipschitz near x, $Q_2 : \mathbb{R}^n \to \mathbb{R}$ be Lipschitz near $Q_1(x)$. Then the function $Q := Q_2 \circ Q_1$ is Lipschitz near x, and the following holds:*

$$\partial Q(x) \subset \overline{conv}\{\partial \langle \omega, Q_1(x)\rangle (x) : \omega \in \partial Q_2(Q_1(x))\}$$

where $\overline{conv}$ signifies the closed convex hull.

For the proof see [34, Theorem 2.5].
Now, consider the matrix $\overline{\Gamma}$ obtained from the matrix Γ by adding external gains γ_i as the last column and let the map $\overline{\Gamma} : \mathbb{R}_+^{n+1} \to \mathbb{R}_+^n$ be defined by:

$$\overline{\Gamma}(s,r) := \{\overline{\Gamma}_1(s,r), \dots, \overline{\Gamma}_n(s,r)\} \tag{2.99}$$

for $s \in \mathbb{R}_+^n$ and $r \in \mathbb{R}_+$, where $\overline{\Gamma}_i : \mathbb{R}_+^{n+1} \to \mathbb{R}_+$ is given by $\overline{\Gamma}_i(s,r) := \gamma_{i1}(s_1) + \cdots + \gamma_{in}(s_n) + \gamma_i(r)$ for $i \in I_\Sigma$ and by $\overline{\Gamma}_i(s,r) := \max\{\gamma_{i1}(s_1), \dots, \gamma_{in}(s_n), \gamma_i(r)\}$ for $i \in I_\Sigma$.
The following theorem establishes ISS of an interconnected hybrid system of the form (2.5) under the existence of an Ω-path with respect to Γ and shows a construction of an ISS-Lyapunov function.

Theorem 2.4.11. *Consider a system of the form* (2.5) *that is an interconnection of subsystems* (2.1). *Assume that $D_i = D, i = 1, \dots, n$ and that each subsystem i of* (2.1) *has an ISS-Lyapunov function V_i with corresponding ISS-Lyapunov gains $\gamma_{ij}, \gamma_i, i, j = 1, \dots, n$ as in* (2.33)-(2.36). *Let $\overline{\Gamma}$ be defined as in* (2.99). *Assume that there exists an Ω-path σ with respect to Γ and a function $\phi \in \mathcal{K}_\infty$ such that*

$$\overline{\Gamma}(\sigma(r), \phi(r)) \le \sigma(r), \quad \forall r > 0. \tag{2.100}$$

Then the system (2.5) *is ISS and an ISS-Lyapunov function is given by*

$$V(x) := \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i)). \tag{2.101}$$

Proof. We apply the properties of an Ω-path to show that the function V constructed in (2.101) satisfies (2.20)-(2.22).
Without loss of generality, the gains γ_{ij} can be assumed to be smooth on $\mathbb{R}_+$, see Lemma 2.3.13.
Define $\psi_1(|x|) := \min_{i=1,\dots,n} \sigma_i^{-1}(\psi_{i1}(L_1|x|))$ and $\psi_2(|x|) := \max_{i=1,\dots,n} \sigma_i^{-1}(\psi_{i2}(L_2|x|))$ for some suitable positive constants L_1, L_2 that depend on the the norm $|\cdot|$. For example if $|\cdot|$ denotes the maximum norm, then one can take $L_1 = L_2 = 1$. By this choice the condition (2.20) is satisfied.
Define the gain of the whole system by

$$\gamma(|u|) := \max_j \{\phi^{-1}(\gamma_j(|u|))\}. \tag{2.102}$$

Consider any $x \neq 0$, since the case $x = 0$ is obvious. Define by

$$\hat{I} := \{i \in \{1, \dots, n\} : V(x) = \sigma_i^{-1}(V_i(x_i)) \ge \max_{j, j\neq i} \sigma_j^{-1}(V_j(x_j))\} \tag{2.103}$$

the set of indices i for which the maximum in (2.101) is attained.
Note that $x_i \neq 0$ for $i \in \hat{I}$. Fix $i \in \hat{I}$. If $V(x) \ge \gamma(|u|)$, then by (2.102) it holds $\phi(V(x)) \ge \gamma_i(|u|)$ and from (2.100), (2.103) we have for $i \in I_{\max}$

$$\begin{aligned} V_i(x_i) = \sigma_i(V(x)) &\ge \max\{\max_{j,j\neq i} \gamma_{ij}(\sigma_j(V(x))), \phi(V(x))\} \\ &\ge \max\{\max_{j,j\neq i} \gamma_{ij}(V_j(x_j)), \gamma_i(|u|)\} \end{aligned}$$

and for $i \in I_{\sum}$

$$\begin{aligned} V_i(x_i) = \sigma_i(V(x)) &\ge \sum_{j,j\neq i} \gamma_{ij}(\sigma_j(V(x))) + \phi(V(x)) \\ &\ge \sum_{j,j\neq i} \gamma_{ij}(V_j(x_j)) + \gamma_i(|u|). \end{aligned}$$

To show (2.21) assume $(x, u) \in C$. As V is obtained through maximization (2.101), by Lemma 2.4.9 we have that

$$\partial V(x) \subset \text{conv}\left\{\bigcup_{i\in\hat{I}} \partial[\sigma_i^{-1} \circ V_i \circ P_i](x)\right\}, \tag{2.104}$$

where $P_i(x) = x_i$. Thus, we can use the properties of σ_i and V_i to estimate $\langle \zeta, f(x, u)\rangle$, $\zeta \in \partial V$. In particular, by the chain rule for Lipschitz continuous functions in Lemma 2.4.10, we have

$$\partial(\sigma_i^{-1} \circ V_i)(x_i) \subset \{c\zeta_i : c \in \partial\sigma_i^{-1}(y), y = V_i(x_i), \zeta_i \in \partial V_i(x_i)\}, \tag{2.105}$$

where c is bounded away from zero due to (2.68). Applying (2.34) we obtain for all $\zeta_i \in \partial V_i(x_i)$ that

$$\langle \zeta_i, f_i(x, u)\rangle \leq -\alpha_i(V_i(x_i)). \tag{2.106}$$

To get an estimate on the right-hand side of (2.106) independent on i, define for $\rho > 0$, $\widetilde{\alpha}_i(\rho) := c_{\rho,i}\alpha_i(\rho) > 0$, where the constant $c_{\rho,i} := K_1$ with K_1 corresponding to the set $K := \{x_i \in \chi_i : \rho/2 \leq |x_i| \leq 2\rho\}$ given by (2.68). And define $\hat{\alpha}(r) := \min\{\widetilde{\alpha}_i(V_i(x_i)) \,|\, |x| = r, V(x) = \sigma_i^{-1}(V_i(x_i))\} > 0$ for $r > 0$. Thus, using (2.105), (2.106) for all $\zeta \in \partial[\sigma_i^{-1} \circ V_i](x_i)$ we obtain

$$\langle \zeta, f_i(x, u)\rangle \leq -\hat{\alpha}(|x|). \tag{2.107}$$

The same argument can be applied for all $i \in \hat{I}$. Note that $x_i \neq 0$ for $i \in \hat{I}$. Let us now return to $\zeta \in \partial V(x)$. From (2.104) for any $\zeta \in \partial V(x)$ we have that $\zeta = \sum_{i\in\hat{I}} \delta_i c_i \zeta_i$ for suitable $\delta_i \geq 0, \sum_{i\in\hat{I}} \delta_i = 1$, and with $\zeta_i \in \partial(V_i \circ P_i)(x)$ and $c_i \in \partial\sigma_i^{-1}(V_i(x_i))$. Using (2.107) and that $c_i > 0$ due to (2.68), it follows that

$$\begin{aligned}\langle \zeta, f(x, u)\rangle &= \sum_{i\in\hat{I}} \delta_i \langle c_i\zeta_i, f(x, u)\rangle = \sum_{i\in\hat{I}} \delta_i \langle c_i P_i(\zeta_i), f_i(x, u)\rangle \\ &\leq -\sum_{i\in\hat{I}} \delta_i\hat{\alpha}(|x|) \leq -\hat{\alpha}(|x|) \leq -\hat{\alpha} \circ \psi_2^{-1} \circ V(x).\end{aligned}$$

Thus condition (2.21) is satisfied with $\alpha := \hat{\alpha} \circ \psi_2^{-1}$.
To show (2.22) assume that $(x, u) \in D$ now. Define

$$\lambda(t) = \max_i\{\sigma_i^{-1} \circ \lambda_i \circ \sigma_i(t)\} \tag{2.108}$$

for all $t > 0$. Note that $\sigma_i^{-1} \circ \lambda_i \circ \sigma_i(t) < \sigma_i^{-1} \circ \sigma_i(t) = t$ for all $t > 0$ as $\lambda_i(t) < t$. Thus $\lambda(t) < t$ for all $t > 0$. Let us show that such λ satisfies (2.22). The condition (2.36) for an ISS-Lyapunov function of subsystem i implies for $(x, u) \in D_i$

$$\begin{aligned}V(g(x, u)) &= \max_i \sigma_i^{-1} \circ V_i(g_i(x, u)) \leq \max_i \sigma_i^{-1} \circ \lambda_i(V_i(x_i)) \\ &= \max_i \sigma_i^{-1} \circ \lambda_i \circ \sigma_i \circ \sigma_i^{-1}(V_i(x_i)) = \lambda \circ \max_i \sigma_i^{-1}(V_i(x_i)) = \lambda(V(x)).\end{aligned} \tag{2.109}$$

Thus (2.22) is also satisfied and hence V is an ISS-Lyapunov function of the interconnected system (2.5). □

In the following theorem we show how for an irreducible Γ and a corresponding Ω-path a function $\phi \in \mathcal{K}_\infty$ can be constructed such that condition (2.100) is satisfied and thus the system is ISS.

Theorem 2.4.12. *Assume that $D_i = D, i = 1, \dots, n$ and that each subsystem of* (2.1) *has an ISS-Lyapunov function V_i and the corresponding gain matrix is given by* (2.99)*. If Γ is irreducible and if there exists $\alpha \in \mathcal{K}_\infty$ as in* (2.44) *such that $\Gamma \circ \mathcal{D}(s) \not\geq s$ for all $s \neq 0, s \geq 0$ is satisfied, then the whole system* (2.5) *is ISS and an ISS-Lyapunov function is given by $V(x) = \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i))$, where $\sigma \in \mathcal{K}_\infty^n$ is an arbitrary Ω-path with respect to $\mathcal{D} \circ \Gamma$.*

Proof. Recall that by Lemma 2.3.7 from $\Gamma \circ \mathcal{D} \not\geq \text{id}$ it follows $\mathcal{D} \circ \Gamma \not\geq \text{id}$. Furthermore, from the irreducibility of Γ it follows the irreducibility of $\mathcal{D} \circ \Gamma$. Then by (ii) of Theorem 2.3.11 there exists an Ω-path σ. From the structure of $\mathcal{D}$ it follows that

$$\begin{aligned} \sigma_i &> (\text{id} + \alpha) \circ \Gamma_i(\sigma), \qquad && i \in I_\Sigma, \\ \sigma_i &> \Gamma_i(\sigma), && i \in I_{\max}. \end{aligned}$$

The irreducibility of Γ ensures that $\Gamma(\sigma)$ is unbounded in all components. Let $\phi \in \mathcal{K}_\infty$ be such that for all $r \geq 0$ the inequality

$$\alpha(\Gamma_i(\sigma(r))) \geq \max_{i=1,\dots,n} \gamma_i(\phi(r)) \tag{2.110}$$

holds for $i \in I_\Sigma$ and

$$\Gamma_i(\sigma(r)) \geq \max_{i=1,\dots,n} \gamma_i(\phi(r)) \tag{2.111}$$

for $i \in I_{\max}$. Note that such ϕ always exists and can be chosen as follows. For any $\gamma_i \in \mathcal{K}$ we choose $\tilde{\gamma}_i \in \mathcal{K}_\infty$ such that $\tilde{\gamma}_i \geq \gamma_i$. Then ϕ can be taken as

$$\phi(r) := \frac{1}{2} \min\{ \min_{i \in I_\Sigma, j \in I} \tilde{\gamma}_j^{-1}(\alpha(\Gamma_i(\sigma(r)))), \min_{i \in I_{\max}, j \in I} \tilde{\gamma}_j^{-1}(\Gamma_i(\sigma(r)))\}.$$

Note that ϕ is a $\mathcal{K}_\infty$ function since the minimum over $\mathcal{K}_\infty$ functions is again of class $\mathcal{K}_\infty$. Then, using (2.110), we have for all $r > 0, i \in I_\Sigma$ that

$$\begin{aligned} \sigma_i(r) &> \mathcal{D}_i \circ \Gamma_i(\sigma(r)) = \Gamma_i(\sigma(r)) + \alpha(\Gamma_i(\sigma(r))) \\ &\geq \Gamma_i(\sigma(r)) + \gamma_i(\phi(r)) = \overline{\Gamma}_i(\sigma(r), \phi(r)) \end{aligned}$$

and, using (2.111), for all $r > 0, i \in I_{\max}$

$$\sigma_i(r) > \mathcal{D}_i \circ \Gamma_i(\sigma(r)) = \Gamma_i(\sigma(r)) \geq \max\{\Gamma_i(\sigma(r)), \gamma_i(\phi(r))\} = \overline{\Gamma}_i(\sigma(r), \phi(r)).$$

Thus $\sigma(r) > \overline{\Gamma}(\sigma(r), \phi(r))$ and the assertion follows from Theorem 2.4.11. □

The irreducibility assumption on Γ means, in particular, that the graph representing the interconnection structure of the whole system is strongly connected. To treat the reducible case we consider an approach using the irreducible components of Γ. Recall that if a matrix is reducible, it can be transformed to an upper block triangular form via a permutation of the indices.

Theorem 2.4.13. *Assume that $D_i = D, i = 1, \dots, n$ and that each subsystem of* (2.1) *has an ISS-Lyapunov function V_i and the corresponding gain matrix is given by* (2.99)*. If there exists a function $\alpha \in \mathcal{K}_\infty$ as in* (2.44) *such that $\Gamma \circ \mathcal{D}(s) \not\geq s$ for all $s \neq 0, s \geq 0$ is satisfied, then the whole system* (2.5) *is ISS. Moreover, there exist an Ω-path σ, a function $\phi \in \mathcal{K}_\infty$ satisfying $\overline{\Gamma}(\sigma(r), \phi(r)) < \sigma(r), \forall\, r > 0$ and an ISS-Lyapunov function for the whole system* (2.5) *is given by*

$$V(x) = \max_{i=1,\dots,n} \sigma_i^{-1}(V_i(x_i)).$$

Proof. Again, by Lemma 2.3.7 we have that $\mathcal{D} \circ \Gamma(s) \not\geq s$ for all $s \neq 0, s \geq 0$ holds. After a renumbering of the subsystems we can assume that $\overline{\Gamma}$ is of the form (2.70). Let D be the corresponding diagonal operator that contains id or id $+ \alpha$ on the diagonal depending on the new enumeration of the subsystems. Let the state x be partitioned into $z_i \in \mathbb{R}^{d_i}$ where d_i is the size of the ith diagonal block Υ_{ii}, $i = 1, \ldots, d$. Consider the subsystems Σ_j of the whole system (2.5) with the states

$$z_j := (x_{q_j+1}^T, x_{q_j+2}^T, \ldots, x_{q_{j+1}}^T)^T,$$

where $q_j = \sum_{l=1}^{j-1} d_l$, with the convention that $q_1 = 0$. So the subsystems Σ_j correspond exactly to the strongly connected components of the interconnection graph. Note that each $\Upsilon_{jj}, j = 1, \ldots, d$ satisfies the small gain condition of the form $\Upsilon_{jj} \circ \mathcal{D}_j \not\geq$ id, where $\mathcal{D}_j : \mathbb{R}^{d_j} \to \mathbb{R}^{d_j}$ is the corresponding part of $\mathcal{D}$.

For each Σ_j with the gain operator $\Upsilon_{jj}, j = 1, \ldots, d$ and external inputs $z_{j+1}, \ldots, z_d$, u, Theorem 2.4.12 implies that there is an ISS Lyapunov function

$$W_j(z_j) = \max_{i=q_j+1,\ldots,q_{j+1}} \widehat{\sigma}_i^{-1}(V_i(x_i)) \tag{2.112}$$

for Σ_j, where $(\widehat{\sigma}_{q_j+1}, \ldots, \widehat{\sigma}_{q_{j+1}})^T$ is an arbitrary Ω-path with respect to $\Upsilon_{jj} \circ \mathcal{D}_j$. We will show by induction over the number of blocks that an ISS-Lyapunov function for the whole system (2.5) of the form $V(x) = \max_{i=1,\ldots,n} \sigma_i^{-1}(V_i(x_i))$ exists, for an appropriate σ.

For one irreducible block there is nothing to show. Assume that for the system corresponding to the first $l-1$ blocks an ISS-Lyapunov function exists and is given by $\widetilde{V}_{l-1} = \max_{i=1,\ldots,q_l} \sigma_i^{-1}(V_i(x_i))$. Consider now the first l blocks with the state $(\widetilde{z}_{l-1}, z_l)$, where $\widetilde{z}_{l-1} := (z_1, \ldots, z_{l-1})^T$. Then we have the implication for $(x, u) \in C$

$$\begin{aligned} \widetilde{V}_{l-1}(\widetilde{z}_{l-1}) \;&\geq\; \widetilde{\gamma}_{l-1,l}(W_l(z_l)) + \widetilde{\gamma}_{l-1,u}(\|u\|) \quad \Rightarrow \\ \forall \widetilde{z}_{l-1} \in \partial \widetilde{V}_{l-1}(\widetilde{z}_{l-1}) &: \langle \zeta_{l-1}, f_{l-1}(\widetilde{z}_{l-1}, z_l, u)\rangle \leq -\widetilde{\alpha}_{l-1}(V_{l-1}(\widetilde{z}_{l-1})) \end{aligned}$$

and for $(x, u) \in D$

$$\begin{aligned} \widetilde{V}_{l-1}(\widetilde{z}_{l-1}) \;&\geq\; \widetilde{\gamma}_{l-1,l}(W_k(z_l)) + \widetilde{\gamma}_{l-1,u}(\|u\|) \quad \Rightarrow \\ &\widetilde{V}_{l-1}(g_{l-1}(\widetilde{z}_{l-1}, z_l, u)) \leq \widetilde{\lambda}_{l-1}(V_{l-1}(\widetilde{z}_{l-1})), \end{aligned}$$

where $\widetilde{\gamma}_{l-1,l}, \widetilde{\gamma}_{l-1,u}$ are the corresponding gains, $\widetilde{f}_{l-1}$, $\widetilde{g}_{l-1}$, $\widetilde{\alpha}_{l-1}$, $\widetilde{\lambda}_{l-1}$ are the right hand side and dissipation rate of the first $l-1$ blocks.

The gain matrix corresponding to the block l has the form

$$\overline{\Gamma}_l = \begin{pmatrix} 0 & \widetilde{\gamma}_{l-1,l} & \widetilde{\gamma}_{l-1,u} \\ 0 & 0 & \gamma_{l,u} \end{pmatrix}.$$

For $\overline{\Gamma}_l$ by [52, Lemma 6.1] there exists an Ω-path $\widetilde{\sigma}^l = (\widetilde{\sigma}_1^l, \widetilde{\sigma}_2^l)^T \in \mathcal{K}_\infty^2$ and $\phi \in \mathcal{K}_\infty$ such that $\overline{\Gamma}_l(\widetilde{\sigma}^l, \phi) < \widetilde{\sigma}^l$ holds. Applying Theorem 2.4.11, an ISS-Lyapunov function for the whole system exists and is given by

$$\widetilde{V}_l \;=\; \max\{(\widetilde{\sigma}_1^l)^{-1}(\widetilde{V}_{l-1}), (\widetilde{\sigma}_2^l)^{-1}(W_l)\}.$$

A simple inductive argument shows that the final Lyapunov function is of the form $V(x) = \max_{l=1,\ldots,d} (\sigma_l^{-1}(W_l(z_l))$ with W defined in (2.112), where for $l = 1, \ldots, d-1$ we have (setting $\sigma_2^0 = $ id)

$$\sigma_l^{-1} = \left(\tilde{\sigma}_1^{d-1}\right)^{-1} \circ \cdots \circ \left(\tilde{\sigma}_1^{l}\right)^{-1} \circ \left(\tilde{\sigma}_2^{l-1}\right)^{-1}$$

and $\sigma_d = \tilde{\sigma}_2^{d-1}$. This completes the proof. $\square$

Remark 2.4.14. *In the case $D = \emptyset$ we obtain the results from [50],[52], [51] and [53] as particular cases.*

These theorems provide a constructive method to derive Lyapunov functions for interconnected hybrid systems. In the following example we illustrate the construction of such Lyapunov functions.

Example 2.4.15. *We consider three interconnected hybrid systems:*

$$\begin{cases} \dot{x}_1 &= -x_1 + \max\left\{\frac{x_2^2+2x_2}{6x_2+8}, \frac{x_3^2+x_3}{8x_3+4}\right\} + u_1 &=: f_1(x_1,x_2,x_3,u_1), \\ \dot{x}_2 &= -x_2 + \max\left\{\frac{2x_1^2+x_1}{8x_1+5}, \frac{x_3^2+5x_3}{3x_3+11}\right\} + u_2 &=: f_2(x_1,x_2,x_3,u_2), \\ \dot{x}_3 &= -x_3 + \max\left\{\frac{x_1^2+x_1}{5x_1+3}, \frac{x_2^2+3x_2}{4x_2+7}\right\} + u_3 &=: f_3(x_1,x_2,x_3,u_3), \end{cases} \tag{2.113}$$

for $(x_1,x_2,x_3,u_1,u_2,u_3)^T \in C_i = C = \left\{(s_1,s_2,s_3,w_1,w_2,w_3)^T \in \mathbb{R}_+^3 \times U : s_1 \geq \frac{1}{2}\max\{s_2,s_3\}\right\}$ *and*

$$\begin{cases} x_1^+ &= \frac{1}{4}x_1 + \frac{1}{8}\max\{x_2,x_3\} &=: g_1(x_1,x_2,x_3), \\ x_2^+ &= \frac{1}{4}x_2 &=: g_2(x_1,x_2,x_3), \\ x_3^+ &= \frac{1}{4}x_3 &=: g_3(x_1,x_2,x_3), \end{cases} \tag{2.114}$$

for $(x_1,x_2,x_3,u_1,u_2,u_3)^T \in D_i = D = \left\{(s_1,s_2,s_3,w_1,w_2,w_3)^T \in \mathbb{R}_+^3 \times U : s_1 \leq \frac{1}{2}\max\{s_2,s_3\}\right\}$.
Consider functions $V_i(x_i) = |x_i|$, $i = 1,2,3$ as ISS-Lyapunov function candidates for the subsystems. With $\psi_{i1}(|x_i|) = \frac{1}{2}|x_i|$ and $\psi_{i2}(|x_i|) = 2|x_i|$ the condition (2.32) *is satisfied.*
Consider $(x_1,x_2,x_3,u_1,u_2,u_3)^T \in C$.
Taking $\gamma_{12}^c(r) := \dfrac{r(r+2)}{(3r+4)(1-\epsilon_1)}$, $\gamma_{13}^c(r) := \dfrac{r(r+1)}{(4r+2)(1-\epsilon_1)}$, $\gamma_{21}(r) := \dfrac{2r(2r+1)}{(8r+5)(1-\epsilon_2)}$, $\gamma_{23}(r) := \dfrac{2r(r+5)}{(3r+11)(1-\epsilon_2)}$, $\gamma_{31}(r) := \dfrac{2r(r+1)}{(5r+3)(1-\epsilon_3)}$, $\gamma_{32}(r) := \dfrac{2r(r+3)}{(4r+7)(1-\epsilon_3)}$, $\gamma_1(|u|) := \dfrac{2}{(1-\epsilon_1)}|u|$, $\gamma_2(|u|) := \dfrac{2}{(1-\epsilon_2)}|u|$, $\gamma_3(|u|) := \dfrac{2}{(1-\epsilon_3)}|u|$, $\epsilon_1 \in \left(0, \frac{1}{2}\right)$, $\epsilon_2 \in \left(0, \frac{1}{11}\right)$, $\epsilon_3 \in \left(0, \frac{1}{7}\right)$ *one can easily check that the condition* (2.34) *is satisfied with $\alpha_i(|x_i|) = \epsilon_i x_i$, $i = 1,\ldots,3$.*
Consider now $(x_1,x_2,x_3,u_1,u_2,u_3)^T \in D_i$.
With $\gamma_{12}^d(r) := \dfrac{1}{2(3-4\epsilon_1)}r$, $\gamma_{13}^d(r) := \dfrac{1}{2(3-4\epsilon_1)}r$, *the condition* (2.36) *is satisfied for $i = 1$ with $\lambda_1 = \frac{1}{2}$. And for any $(x,u) \in D$ it holds that*
$V_2(g_2(x_1,x_2,x_3)) = V_2\left(\frac{1}{4}x_2\right) = \frac{1}{4}|x_2| =: \lambda_2(V_2(x_2))$,
$V_3(g_3(x_1,x_2,x_3)) = V_3\left(\frac{1}{4}x_3\right) = \frac{1}{4}|x_3| =: \lambda_3(V_3(x_3))$.
Thus each subsystem is ISS and V_1, V_2, V_3 are ISS-Lyapunov functions for the corresponding subsystems with $I = I_{\max}$. To verify the stability of their interconnection we apply Theorem 2.4.13 with $\mathcal{D} = \mathrm{diag}(id,id,id)^T$ and Proposition 2.2.12. To this end define $x^T = (x_1,x_2,x_3)^T \in \mathbb{R}^3$, $u^T = (u_1,u_2,u_3)^T$, $f = (f_1,f_2,f_3)^T$, $g = (g_1,g_2,g_3)^T$. Then the whole system is of the form (2.5). Denote $\gamma_{12} := \max\{\gamma_{12}^c, \gamma_{12}^d\}$, $\gamma_{13} := \max\{\gamma_{13}^c, \gamma_{13}^d\}$. Then the gain matrix Γ is given by

$$\Gamma = \begin{pmatrix} 0 & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & 0 & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & 0 \end{pmatrix}.$$

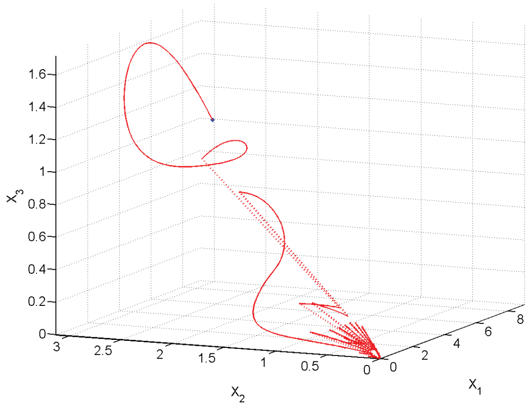


Figure 2.4: Trajectory of the whole system when the small gain condition is satisfied.

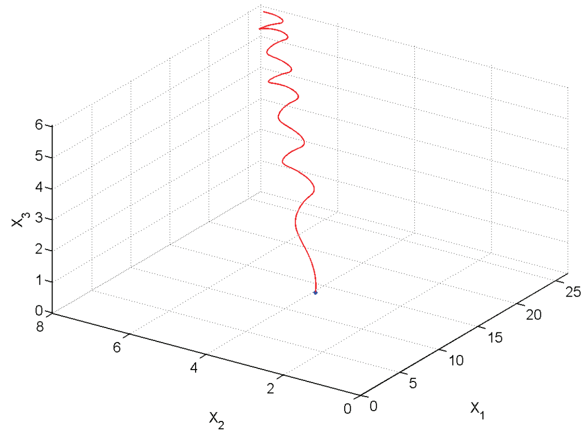


Figure 2.5: Trajectory of the whole system when the small gain condition is not satisfied

It can be easily checked for all the gains that $\gamma_{ij} < id$ holds, hence

$$\Gamma(s) < s \tag{2.115}$$

for all $s \neq 0$. It follows then that the small gain condition $\mathcal{D} \circ \Gamma(s) = \Gamma(s) \not\geq s$ is satisfied. Moreover it follows that σ can be taken as $\sigma := (id, id, id)^T$ for construction of an ISS-Lyapunov function of the interconnection (2.113), (2.114) *according to* (2.101)*:*

$$V(x) = \max\{V_1(x_1), V_2(x_2), V_3(x_3)\} = \max\{x_1, x_2, x_3\}.$$

Thus, by Theorem 2.4.13 and Proposition 2.2.12 the interconnection is ISS. The trajectory of the whole system corresponding to the initial condition $x(0) = (9; 3; 1)^T$, input $u(t) = (0.5(1+sin\, t); |2cos(t+2)|; cos^2\, t)^T$ is shown in Figure 2.4. We see that the trajectory approaches a bounded domain around the origin. However due the disturbances given by u it never reaches the origin and even do not approaches it arbitrary close.

The next example shows that if the small gain condition is not fulfilled, the system may possess an unstable behaviour.

Example 2.4.16. *Consider the interconnection of three hybrid systems with the same g, C and D as in the previous example, but with slight changes of the coefficients of the first subsystem describing continuous behaviour*

$$\begin{cases} \dot{x}_1 = -x_1 + \max\left\{\frac{7x_2^2+2x_2}{2x_2+1}, \frac{2x_3^2+3x_3}{5x_3+4}\right\} + u_1 \\ \dot{x}_2 = -x_2 + \max\left\{\frac{2x_1^2+x_1}{8x_1+5}, \frac{x_3^2+5x_3}{3x_3+11}\right\} + u_2 \\ \dot{x}_3 = -x_3 + \max\left\{\frac{x_1^2+x_1}{5x_1+3}, \frac{x_2^2+3x_2}{4x_2+7}\right\} + u_3 \end{cases} \tag{2.116}$$

$(x, u) \in C$.

Using the same approach as in the previous example it can be checked that all subsystems are still ISS in terms of maximizations, i.e. $I = I_{\max}$. The gains $\gamma_{12}^d(r)$, γ_{13}^d, γ_{23}, γ_{31}, γ_1, γ_2, γ_3 are the same as in the example before and the gains $\gamma_{12}^c(r) := \dfrac{2r(7r+2)}{(2r+1)(1-\epsilon_1)}$, $\gamma_{13}^c(r) := \dfrac{2r(r+3)}{(5r+4)(1-\epsilon_2)}$,

$\gamma_{12}(r) := \max\{\gamma_{12}^c(r), \gamma_{12}^d(r)\}$, $\gamma_{13}(r) := \max\{\gamma_{13}^c(r), \gamma_{13}^d(r)\}$, $\epsilon_1 \in (0,1)$, $\epsilon_2 \in \left(0, \frac{1}{11}\right)$, $\epsilon_3 \in \left(0, \frac{1}{7}\right)$. *Let us check the small gain condition $\Gamma(s) \not\geq s$, $s \neq 0$. Note that unlike the previous example not all γ_{ij}'s are less than identity, in particular for $r \geq 0$, $\gamma_{12}(r) \geq r$. Thus* (2.115) *does not hold for the first component of the inequality and we cannot use this property to prove the small gain condition. But we can use the fact that $\Gamma(s) \not\geq s$ is equivalent to the cycle condition* (2.42)*. However, for $r \geq 0$*

$$\begin{aligned}
\gamma_{12}^c \circ \gamma_{21}(r) &= \frac{4r(2r+1)(14r(2r+1)+2(8r+5)(1-\epsilon_2))}{(8r+5)(1-\epsilon_2)(4r(2r+1)+(8r+5)(1-\epsilon_2))(1-\epsilon_1)} \\
&= \frac{224r^4+(224+128(1-\epsilon_2))r^3+(56+144(1-\epsilon_2))r^2+40(1-\epsilon_2)r}{(64r^3+(72+64(1-\epsilon_2))r^2+(20+40(1-\epsilon_2))r+25(1-\epsilon_2))\,(1-\epsilon_1)(1-\epsilon_2)} \\
&\geq r.
\end{aligned}$$

Hence, the cycle condition is also violated and consequently the small gain condition is not satisfied. Figure 2.5 shows the trajectory of this interconnection corresponding to the initial condition $x(0) = (9;3;1)^T$, input $u(t) = (0.5(1+\sin t); |2\cos(t+2)|; \cos^2 t)^T$. We see that the trajectory grows unboundedly.

In the following sections we show an application of the small gain condition (2.45) for establishing stability of certain subclasses of hybrid systems.

2.4.3 Systems with stability of only a part of the state

In some applications one is interested in stability of only a part of the state. For example, one can use variables that describe time, counters or logical variables that never tend to zero and from a practical point of view there is no need in their stability, see [110], [131]. For such systems the definition of ISS and conditions (2.32) and (2.20) for ISS-Lyapunov functions have to be modified. Furthermore, we can adapt small gain theorem to study ISS of interconnection of such systems.
Assume that the system (2.1) can be represented as follows:

$$\begin{array}{ll} \begin{array}{l}\dot{x}_i^s = f_i^s(x^s, u_i)\\ \dot{x}_i^t = f_i^t(x^s, x^t, u_i)\end{array}, & (x,u) \in C_i \\ \begin{array}{l}x_i^{s+} = g_i^s(x^s, u_i)\\ x_i^{t+} = g_i^t(x^s, x^t, u_i)\end{array}, & (x,u) \in D_i \end{array} \tag{2.117}$$

with $x_i^s \in \chi_i^s \subset \mathbb{R}^{N_i^s}$, $x_i^t \in \chi_i^t \subset \mathbb{R}^{N_i^t}$, $x^s = ({x_1^s}^T, \ldots, {x_n^s}^T)^T \in \chi^s \subset \mathbb{R}^{N^s}$, $\chi^s = \chi_1^s \times \ldots \times \chi_n^s$, $N^s = \sum N_i^s$, $x^t = ({x_1^t}^T, \ldots, {x_n^t}^T)^T \in \chi^t \subset \mathbb{R}^{N^t}$, $\chi^t = \chi_1^t \times \ldots \times \chi_n^t$, $N^t = \sum N_i^t$, $x = ({x^s}^T, {x^t}^T)^T \in \chi \subset \mathbb{R}^N$, $\chi = \chi^s \times \chi^t$, $N = N^s + N^t$, $u_i \in U_i \subset \mathbb{R}^{M_i}$, $u = (u_1^T, \ldots, u_n^T)^T \in U \subset \mathbb{R}^M$, $U = U_1 \times \ldots \times U_n$, $M = \sum M_i$. Here x_i^s is the part of the state x_i, in stability of which we are interested. The interconnection of these subsystems can be represented as:

$$\begin{array}{ll} \begin{array}{l}\dot{x}^s = f^s(x^s, u)\\ \dot{x}^t = f^t(x^s, x^t, u)\end{array}, & (x,u) \in C \\ \begin{array}{l}x^{s+} = g^s(x^s, u)\\ x^{t+} = g^t(x^s, x^t, u)\end{array}, & (x,u) \in D \end{array} \tag{2.118}$$

where f^s, f^t, g^s, g^t, C and D are constructed analogous to (2.5).
We define ISS of such a system by a slight abuse of notation as in [28, 110]:

Definition 2.4.17. *The system* (2.118) *is ISS, if there exist β of class $\mathcal{KLL}$, $\gamma \in \mathcal{K}_\infty \cup \{0\}$ such that for all initial values x_0 each solution pair $(x, u) \in S_u(x_0)$ satisfies*

$$|x^s(t,k)| \le \max\{\beta(|x_0^s|, t, k), \gamma(\|u\|_{(t,k)})\}, \forall (t,k) \in dom\, x. \tag{2.119}$$

In the definition of an ISS-Lyapunov function we modify (2.20) only.

Definition 2.4.18. *A locally Lipschitz continuous function $V : \chi \to \mathbb{R}_+$ is an ISS-Lyapunov function for the system* (2.118) *if*
1) There exist functions $\psi_1, \psi_2 \in \mathcal{K}_\infty$ such that:

$$\psi_1(|x^s|) \le V(x) \le \psi_2(|x^s|) \quad \textit{for any } x \in \chi. \tag{2.120}$$

2) Conditions (2.21) *and* (2.22) *are satisfied.*

Thus, this definition is focused on the state x^s that has to be stable.
System of the form (2.118) allows to model and then to investigate logistics networks, where some of the parameters are not stable. For example, one can model logistics networks with switching production rates, where the switching is determined by a logical variable. In this case we are not interested in stability of this variable, but rather in stability of the variables describing the "physical" states, that we denote by x_s.
For such a definition of ISS there was shown a similar connection between an ISS-Lyapunov function and ISS in [28].

Remark 2.4.19. *From Proposition 2.7 in [28] it follows that if there exists an ISS-Lyapunov function satisfying* (2.120), (2.21) *and* (2.22), *then the system* (2.118) *is ISS.*

In case of (2.117) an ISS-Lyapunov function V_i satisfies:
1) There exist functions $\psi_{i1}, \psi_{i2} \in \mathcal{K}_\infty$ such that:

$$\psi_{i1}(|x_i^s|) \le V_i(x_i) \le \psi_{i2}(|x_i^s|). \tag{2.121}$$

2) Conditions (2.33)-(2.36) are satisfied.
Now, if we assume that the system of the form (2.118) is an interconnection of n hybrid ISS subsystems, then we can show the following small gain result.

Corollary 2.4.20. *Consider a system of the form* (2.118) *that is an interconnection of subsystems* (2.117). *Assume that $D_i = D, i = 1, \ldots, n$ and that each subsystem i of* (2.117) *has an ISS-Lyapunov function V_i that satisfy* (2.121) *and* (2.33)-(2.36) *with the corresponding ISS-Lyapunov gains γ_{ij}, γ_i. Let Γ be defined as in* (2.99). *If there exists $\alpha \in \mathcal{K}_\infty$ as in* (2.44) *such that $\mathcal{D} \circ \Gamma(s) \not\ge s$ for all $s \ne 0, s \ge 0$ is satisfied, then the hybrid system* (2.118) *has an ISS-Lyapunov function satisfying* (2.120), (2.21) *and* (2.22). *Furthermore, an ISS-Lyapunov function for the whole system of the form* (2.118) *can be constructed as in* (2.101).

Proof. The proof goes along the lines of the proof of Theorem 2.4.13 with
$\psi_1(|x^s|) := \min_{i=1,\ldots,n} \sigma_i^{-1}(\psi_{i1}(L_1|x^s|))$ and $\psi_2(|x^s|) := \max_{i=1,\ldots,n} \sigma_i^{-1}(\psi_{i2}(L_2|x^s|))$ in case of the maximum norm used. □

2.4.4 Impulsive dynamical systems

Impulsive systems are such hybrid systems that jump only at given time instances and between these time instances they change continuously. We consider an interconnection of n impulsive subsystems with inputs

$$\begin{aligned} \dot{x}_i(t) &= f_i(x_1(t), \ldots, x_n(t), u_i(t)),\ t \neq t_{i_{\tilde{k}}}, \\ x_i^+(t) &= g_i(x_1(t), \ldots, x_n(t), u_i(t)),\ t = t_{i_{\tilde{k}}}, \end{aligned} \tag{2.122}$$

$\tilde{k} \in \mathbb{N},\ i = 1, \ldots, n$, where $x_i(t) \in \mathbb{R}^{N_i}$ is the state of the ith subsystem; $u_i(t) \in \mathbb{R}^{M_i}$ is a locally bounded, Lebesgue-measurable input and $x_j(t) \in \mathbb{R}^{N_j},\ j \neq i$ can be interpreted as internal inputs of the ith subsystem. Given a sequence $\{t_{i_{\tilde{k}}}\}$ and a pair of times s, t satisfying $t_0 \leq s < t$, $\mathcal{N}_i(t, s)$ denotes the number of impulsive times $t_{i_{\tilde{k}}}$ in the semi-open interval $(s, t]$ of the ith subsystem.

We assume that functions f_i, g_i are from $\mathbb{R}^{N_1} \times \ldots \times \mathbb{R}^{N_n} \times \mathbb{R}^{M_i} \rightarrow \mathbb{R}^{N_i}$ and f_i are locally Lipschitz continuous. All signals (x_i and inputs u_i, $i = 1, \ldots, n$) are assumed to be right-continuous and to have left limits at all times.

We define $N := N_1 + \ldots + N_n,\ M := M_1 + \ldots + M_n$, $x := (x_1^T, \ldots, x_n^T)^T$, $u := (u_1^T, \ldots, u_n^T)^T$, $f := (f_1^T, \ldots,\ f_n^T)^T$ and the impulsive time sequence of the whole system $\{t_k\} := \left\{t | t = t_{i_{\tilde{k}}},\ \tilde{k} \in \mathbb{N}\right\}$, $k \in \mathbb{N}$.

It may happen that at an impulsive time $t_{i_{\tilde{k}}}$ there is a jump of the ith subsystem but not of the jth subsystem, $j \in \{1, ..., n\},\ j \neq i$, for example. This circumstance may lead to a conservative condition for stability of the whole system, obtained from the exponential Lyapunov functions of the subsystems. Therefore, we define $I_k := \{i | t_k = t_{i_{\tilde{k}}}\}$, which is the set of impulsive times of the ith subsystem and the whole system; $\overline{I}_k := \{i | t_k \neq t_{i_{\tilde{k}}}\}$, which is the set of impulsive times of the whole system, but not of the ith subsystem; and we denote $\mathcal{N}(t, s)$ as the number of impulsive times in the semi-open interval $(s, t]$ for the whole system.

Then we define $g := (\widetilde{g}_1^T, \ldots, \widetilde{g}_n^T)^T$, where

$$\widetilde{g}_i(x, u_i) := \begin{cases} g_i(x, u_i), & i \in I_k, \\ x_i, & i \in \overline{I}_k. \end{cases}$$

With these definitions the interconnected system (2.122) can be described as a system of the form

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)),\ t \neq t_k,\ k \in \mathbb{N}, \\ x^+(t) &= g(x(t), u(t)),\ t = t_k,\ k \in \mathbb{N}. \end{aligned} \tag{2.123}$$

Impulsive systems allow to consider time scheduling problems in logistics networks, i.e. the arrangement of the impulsive times to achieve certain performance aims. For example, one can look for the delivery times for certain locations that guarantee stable behaviour of the network.

Remark 2.4.21. *Note that we can describe a system of the form* (2.123) *as a hybrid system in the general form* (2.5) *by adding the time variable τ, the flow set $C = \{(x, \tau) : \tau \neq t_k, k \in \mathbb{N}\}$, the jump set $D = \{(x, \tau) : \tau = t_k, k \in \mathbb{N}\}$ and equations that describe the dynamics of τ:*

$$\dot{\tau} = 1, \quad (x, \tau) \in C,$$

$$\tau^+ = \tau, \quad (x, \tau) \in D.$$

Stability notions

We use the following notion of ISS adapted to impulsive systems in [72] as follows:

Definition 2.4.22. *Assume that a sequence $\{t_k\}$ is given. We call a system of the form* (2.123) *ISS, if there exist functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ such that for every initial condition $x(0)$ and every input u the corresponding solution to* (2.123) *exists globally and satisfies*

$$|x(t)| \leq \max\{\beta(|x(0)|, t), \gamma(\|u\|_{[0,t]})\}, \quad \forall t \geq 0. \tag{2.124}$$

The impulsive system (2.123) *is uniformly ISS over a given class $\mathcal{S}$ of admissible sequences of impulsive times, if* (2.124) *holds for every sequence in $\mathcal{S}$ with functions β and γ that are independent of the choice of the sequence.*

Here the supremum norm of an input u on the interval $[0, t]$ is defined by

$$\|u\|_{[0,t]} := \max\left\{\operatorname*{ess\,sup}_{s\in[0,t]} |u(s)|, \sup_{t_k\in[0,t]} |u(t_k)|\right\}.$$

For subsystems ISS can be formulated as follows:
Assume that a sequence $\{t_{i_{\tilde{k}}}\}$ is given. The ith subsystem of (2.123) is ISS, if there exist $\beta_i \in \mathcal{KL}$, $\gamma_{ij}, \gamma_i \in \mathcal{K}_\infty \cup \{0\}$ such that for every initial condition $x_i(0)$ and every input u_i the corresponding solution to (2.122) exists globally and satisfies for all $t \geq 0$

$$|x_i(t)| \leq \max\{\beta_i(|x_i(0)|, t), \max_{j,j\neq i} \gamma_{ij}(\|x_j\|_{[0,t]}), \gamma_i(\|u\|_{[0,t]})\} \tag{2.125}$$

for $i \in I_{\max}$, and

$$|x_i(t)| \leq \beta_i(|x_i(0)|, t) + \sum_{j,j\neq i} \gamma_{ij}(\|x_j\|_{[0,t]}) + \gamma_i(\|u\|_{[0,t]}) \tag{2.126}$$

for $i \in I_{\sum}$.
The impulsive system (2.122) is uniformly ISS over a given class $\mathcal{S}$ of admissible sequences of impulsive times, if (2.125), (2.125) hold for every sequence in $\mathcal{S}$ with functions β_i and γ_i, γ_{ij} that are independent of the choice of the sequence.
For the stability analysis of impulsive systems we use exponential Lyapunov functions, see [72]. Here, we assume that these functions are locally Lipschitz continuous.

Definition 2.4.23. *We say that a function $V : \mathbb{R}^N \to \mathbb{R}_+$ is an exponential ISS-Lyapunov function for* (2.123) *with rate coefficients $c, d \in \mathbb{R}$ if V is locally Lipschitz, positive definite, radially unbounded, and the following holds:*

$$V(x) \geq \gamma(|u|) \Rightarrow \nabla V(x) \cdot f(x, u) \leq -cV(x) \text{ for almost all } x, \text{ all } u, \tag{2.127}$$

$$V(x) \geq \gamma(|u|) \Rightarrow V(g(x, u)) \leq e^{-d}V(x) \text{ for all } x, u, \tag{2.128}$$

where γ is some function from $\mathcal{K}_\infty$.

Condition (2.127) states, that if c is positive then the function V decreases. On the other hand, if $c < 0$ then the function V can increase. Condition (2.128) states, that if d is positive, then the jump (impulse) decreases the magnitude of V. On the other hand, if $d < 0$, then the jump (impulse) can increase the magnitude of V.
Without loss of generality we use the same function γ in (2.127) and (2.128). Choosing $\gamma_c \in \mathcal{K}_\infty$ in (2.127) and $\gamma_d \in \mathcal{K}_\infty$ in (2.128) and taking the maximum of these two functions, we get γ.

Remark 2.4.24. *Note that in [72] the conditions* (2.127) *and* (2.128) *are in dissipative form. By Proposition 2.6 in [28] the conditions in dissipative form are equivalent to the conditions in implication form, used in Definition 2.4.23, but the coefficients* c, d *may be different.*

In [72] the following theorem was proved, which establishes stability of a single impulsive system.

Theorem 2.4.25. *Let* V *be an exponential ISS-Lyapunov function for (2.123) with rate coefficients* $c, d \in \mathbb{R}$ *with* $d \neq 0$. *For arbitrary constants* $\mu, \lambda > 0$, *let* $\mathcal{S}[\mu, \lambda]$ *denote the class of impulsive time sequences* $\{t_k\}$ *satisfying*

$$-d\mathcal{N}(t,s) - (c-\lambda)(t-s) \leq \mu, \ \forall t \geq s \geq 0. \tag{2.129}$$

Then the system (2.123) is uniformly ISS over $\mathcal{S}[\mu, \lambda]$.

The condition (2.129) is called *average dwell-time conditon*. If $d = 0$, then the jumps do not destabilize the system and the whole system will be ISS, if the corresponding continuous dynamics is ISS. This case was investigated in more detail in [72, Section 6], [153, Theorem 1].

Note that the condition (2.129) imposed on the intervals between the jumps guarantees stability of the impulsive system even if the continuous or discontinuous behaviour is unstable. For example, if the continuous behaviour is unstable, which means $c < 0$, then this condition assumes that the discontinuous behaviour has to stabilize the system $(d > 0)$ and the jumps have to occur often enough. Conversely, if the discontinuous behaviour is unstable $(d < 0)$ and the continuous behaviour is stable $(c > 0)$, then the jumps have to occur rarely, which stabilizes the system.

Similarly, we define Lyapunov functions for subsystems:

Assume that for each subsystem of the interconnected system (2.122) there is a given function $V_i : \mathbb{R}^{N_i} \to \mathbb{R}_+$, which is continuous, proper, positive definite and locally Lipschitz continuous on $\mathbb{R}^{N_i} \backslash \{0\}$. For $i = 1, \ldots, n$ the function V_i is called an exponential ISS-Lyapunov function for the ith subsystem of (2.122) with *rate coefficients* $c_i, d_i \in \mathbb{R}$, if

$$V_i(x_i) \geq \max\{\max_{j,j\neq i} \gamma_{ij}(V_j(x_j)), \gamma_i(|u_i|)\} \Rightarrow \nabla V_i(x_i) \cdot f_i(x, u_i) \leq -c_i V_i(x_i) \text{ for almost all } x, \text{ all } u_i \text{ and} \tag{2.130}$$

$$V_i(x_i) \geq \max\{\max_{j,j\neq i} \gamma_{ij}(V_j(x_j)), \gamma_i(|u_i|)\} \Rightarrow V_i(g_i(x,u_i)) \leq e^{-d_i} V_i(x_i) \text{ for all } x, u_i, \tag{2.131}$$

for $i \in I_{\max}$, and

$$V_i(x_i) \geq \sum_{j,j\neq i} \gamma_{ij}(V_j(x_j)) + \gamma_i(|u_i|) \Rightarrow \ \nabla V_i(x_i) \cdot f_i(x, u_i) \leq -c_i V_i(x_i) \text{ f.a.a. } x, \text{ all } u_i \text{ and} \tag{2.132}$$

$$V_i(x_i) \geq \sum_{j,j\neq i} \gamma_{ij}(V_j(x_j)) + \gamma_i(|u_i|) \Rightarrow \ V_i(g_i(x,u_i)) \leq e^{-d_i} V_i(x_i) \text{ for all } x, u_i, \tag{2.133}$$

for $i \in I_{\sum}$, where γ_{ij}, γ_i are some functions from $\mathcal{K}_\infty$.

Small gain theorems

In this section we show how an exponential ISS-Lyapunov function for an interconnected impulsive system can be constructed under the small gain condition (2.45) and the dwell-time condition (2.129). Note that not for all interconnections with nonlinear gains γ_{ij} one can construct the exponential Lyapunov function for the whole system, even if the small-gain condition is satisfied. Thus we will consider the case with linear gains γ_{ij}. By slight abuse of notation we denote $\gamma_{ij}(r) = \gamma_{ij} r$, where $\gamma_{ij} \geq 0$ and $r > 0$. Furthermore, as in Section 2.4.1 we assume that all subsystems jump simultaneously, i.e. $I_k = \{1 \ldots, n\}$, $\overline{I}_k = \emptyset$.

Theorem 2.4.26. *Consider system (2.123) with $I_k = \{1\dots,n\}$, $\overline{I}_k = \emptyset$. Assume that each subsystem of (2.122) has an exponential ISS-Lyapunov function V_i with corresponding linear ISS-Lyapunov gains γ_{ij} and rate coefficients c_i, d_i, $d_i \neq 0$. Define $c := \min_i c_i$ and $d := \min_i d_i$. For arbitrary constants $\mu, \lambda > 0$, let $\mathcal{S}[\mu,\lambda]$ denote the class of impulsive time sequences $\{t_k\}$ of the whole system. If the following holds*
i) $\mathcal{S}[\mu,\lambda]$ satisfies the condition (2.129),
ii) $\Gamma = (\gamma_{ij})_{n\times n}$ satisfies the small gain condition (2.45),
then the impulsive system (2.123) is uniformly ISS over $\mathcal{S}[\mu,\lambda]$ and the exponential ISS-Lyapunov function is given by

$$V(x) := \max_i\{\tfrac{1}{s_i}V_i(x_i)\}, \tag{2.134}$$

where $s = (s_1,\dots,s_n)^T$ is a linear Ω-path with $\Gamma(s) < s$.

Proof. From Theorem 2.3.11 there exists a linear Ω-path (vector) $s \in \mathbb{R}^n$, $s > 0$, with $\Gamma(s) < s$, see also proof of [52, Theorem 5.2]. Let us define $V(x) := \max_i\{\frac{1}{s_i}V_i(x_i)\}$ and show that this function is an exponential ISS-Lyapunov function for the system (2.123).
Define $\gamma(r) := \max_i \frac{1}{s_i}\gamma_i(r)$, $r > 0$.
Consider $x \neq 0$, as the case $x = 0$ is obvious. Define $\hat{I} := \{i \in \{1,\dots,n\} : \frac{1}{s_i}V_i(x_i) > \max_{j\neq i}\frac{1}{s_j}V_j(x_j)\}$.
Note that $x_i \neq 0$ for $i \in \hat{I}$. Fix $i \in \hat{I}$. Assume $V(x) \geq \gamma(|u|)$. Then

$$V_i(x_i) = s_iV(x) \geq \max\{\max_{j,j\neq i}\gamma_{ij}s_jV(x), s_i\gamma(|u|)\} \geq \max\{\max_{j,j\neq i}\gamma_{ij}V_j(x_j), \gamma_i(|u|)\},$$

and for $i \in I_{\sum}$

$$V_i(x_i) = s_iV(x) \geq \sum_{j=1}^n \gamma_{ij}s_jV(x) + s_i\gamma(|u|) \geq \sum_{j=1}^n \gamma_{ij}s_jV_j(x_j) + \gamma_i(|u|).$$

Then from (2.130), (2.132) we obtain for almost all x

$$\dot{V}(x) = \tfrac{1}{s_i}\nabla V_i(x_i)\cdot f_i(x,u_i) \leq -\tfrac{1}{s_i}c_iV_i(x_i) = -c_iV(x).$$

By the definition of $c := \min_i c_i$ the function V satisfies (2.127).
As $d := \min_i d_i$, it holds

$$V(g(x,u)) = \max_j\{\tfrac{1}{s_j}V_j(g_j(x_1,\dots,x_n,u_j))\} \leq \max_j\{\tfrac{1}{s_j}e^{-d_j}V_j(x_j)\} \leq e^{-d}\max_j\{\tfrac{1}{s_j}V_j(x_j)\} = e^{-d}V(x),$$

i.e., V satisfies condition (2.128).
All conditions of Definition 2.4.23 are satisfied and thus V is the exponential ISS-Lyapunov function of the system (2.123). By assumption i) there exist μ, $\lambda > 0$ such that $-d\mathcal{N}(t,s) - c(t-s) \leq \mu - \lambda(t-s)$, $\forall t \geq s \geq 0$. Thus, applying Theorem 2.4.25 the overall system is uniformly ISS over $S[\mu,\lambda]$. □

Remark 2.4.27. *Note that in the case of nonlinear gains γ_{ij}, if the small-gain condition holds, then we can construct non-exponential Lyapunov functions for the whole system as in Section 2.4.2. In this case the results from [72] cannot be applied and one has to develop more general conditions that guarantee ISS of the system.*

2.4.5 Comparison systems

Consider an interconnected system of the form (2.5). Assume that $D_i = D, i = 1, \ldots, n$ and that the set $\{f(x,u) : u \in U \cap \epsilon\overline{\mathbb{B}}\}$ is convex for each $x \in \chi, \epsilon > 0$. Assume also that all the subsystems (2.1) are ISS.
Consider the following interconnected discrete subsystems:

$$s_i(l+1) = \gamma_{i1}(s_1(l)) + \ldots + \gamma_{in}(s_n(l)) + v_i(l) \tag{2.135}$$

for $i \in I_{\sum}$ and

$$s_i(l+1) = \max\{\gamma_{i1}(s_1(l)), \ldots, \gamma_{in}(s_n(l)), v_i(l)\} \tag{2.136}$$

for $i \in I_{\max}$, where $s_i \in \mathbb{R}$, $v_i \in W_i \subset \mathbb{R}$, $\gamma_{ij} \in \mathcal{K}_\infty$ are from (2.13), (2.14), $l \in \mathbb{R}_+$.
Their interconnection can be written as:

$$s(l+1) = \mu(\Gamma(s(l)), u(l)), \tag{2.137}$$

where μ is defined in (2.51), Γ is taken from (2.38) and $v(l) \in \mathbb{R}^n_+$.
Using the definition of ISS for discrete systems in [84] we can formulate ISS of the system of the form (2.137) as follows:

Definition 2.4.28. *The system* (2.137) *is ISS from v to s if and only if $\exists\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}_\infty$ such that for every $s(0) \in \mathbb{R}^n_+$, $\{v(l)\}_{l=1}^\infty \subset \mathbb{R}^n_+$, and every $l \geq 0$*

$$|s(l+1)| \leq \mu\left((\beta(|s(l)|), l), \gamma(\sup_{l\geq 0} |v(l)|)\right). \tag{2.138}$$

For $I_{\max} = I$ and $I_{\sum} = I$ it was shown in [128] that (2.137) is ISS if and only if (2.41) resp. (2.40) hold. Thus, ISS of the hybrid system (2.5) can be established from ISS of the corresponding discrete system (2.137). This property can be extended to the case of $I_{\max}, I_{\sum} \neq \emptyset$.
First, we need to combine Theorem 5.10 from [128] and Proposition 2.3.17 in [126] to adapt them to the case of the mixed gains.

Theorem 2.4.29. *Let Γ be defined as in* (2.38) *and* (2.39). *Assume that there exists $\alpha \in \mathcal{K}_\infty$ such that for $\mathcal{D}$ defined as in* (2.44) *the operator $\mathcal{D} \circ \Gamma$ satisfies* (2.45). *Then there exists $\widetilde{\mathcal{D}}$ defined as in* (2.44) *with $\widetilde{\alpha} \in \mathcal{K}_\infty$, and $\sigma : \mathbb{R}_+ \to \mathbb{R}^n_+$, $\sigma : \mathcal{K}^n_\infty$, such that $\Gamma \circ \widetilde{\mathcal{D}}(\sigma(r)) < \sigma(r)$ for all $r > 0$. Moreover, σ can be chosen to be piecewise linear on* $(0, \infty)$.

Proof. The proof follows the same steps as the proofs of Theorem 5.10 in [128] for components $i \in I_{\sum}$ of σ and of Proposition 2.3.17 in [126] for $i \in I_{\max}$ using Lemma 2.3.4. □

Now, we can show ISS of the systems of the form (2.137) under the small gain condition (2.45).

Theorem 2.4.30. *System* (2.137) *is ISS from v to s if and only if* (2.45) *holds.*

Proof. The proof follows exactly the same steps as the proof of Theorem IV.1 in [128].
"$\Rightarrow$": If (2.137) is ISS, then the origin is globally attractive with respect to autonomous dynamics. From Proposition 4.1 in [127] it follows that $\Gamma(s) \not\geq s$ for all $s \neq 0$. By assumption there exists $\gamma \in \mathcal{K}_\infty$ such that for any $s(0) \in \mathbb{R}^n_+$ and any input signal v we have

$$\limsup_{l\to\infty} |s(l)|_1 \leq \gamma(|v|_1), \tag{2.139}$$

where the equivalence of norms on $\mathbb{R}^n$ was used. Define $\mathcal{D}$ as in (2.44) with $\alpha(r) := \frac{1}{2n}\gamma^{-1}(r) \in \mathcal{K}_\infty$. Fix $r > 0$ and consider the set $S_r = \{s \in \mathbb{R}^n_+ : |s|_1 = \sum_i s_i = r\}$. By construction, for $s \in S_r$ we have

$$\mathcal{D}(s) \le s + v_r, \tag{2.140}$$

where

$$v_{r_i} := \begin{cases} \frac{1}{2n}\gamma^{-1}(r), & i \in I_\Sigma, \\ 0, & i \in I_{\max}. \end{cases}$$

Observe that $|v_r|_1 \le \frac{1}{2}\gamma^{-1}(r)$, or equivalently, $r \ge \gamma(2|v_r|) > \gamma(|v_r|_1)$. Now (2.139) implies that

$$\Gamma(s + v_r) \not\geq s, \text{ for all } s \in S_r. \tag{2.141}$$

To prove this, assume the opposite. Then there exists $s^* \in S_r$ such that $\Gamma(s^* + v_r) \ge s^*$. Consider the trajectory ϕ of the dynamical system $w(l+1) = \Gamma(w(l) + v(l))$ with initial value $w(0) = s^*$ and input $v(l) \equiv v_r$. Assuming $w(0) \ge s^*$ we show inductively for $l \ge 0$ that $w(l+1) = \Gamma(w(l) + v(l)) \ge \Gamma(s^* + v_r) \ge s^*$. Since $|s^*|_1 = r > \gamma(|v_r|_1)$ we have a contradiction to (2.139). Thus (2.141) holds. Consider again an arbitrary $s \in S_r$. By (2.141) there exists an index $i \in \{1, \ldots, n\}$ such that $s_i > (\Gamma(s + v_r))_i \ge (\Gamma \circ \mathcal{D}(s))_i$, where the inequality (2.140) was used. This implies

$$\Gamma \circ \mathcal{D}(s) \not\geq s, \text{ for all } s \in S_r.$$

Since $r > 0$ was chosen arbitrary and $\cup_{r>0} S_r = \mathbb{R}^n_+ \setminus \{0\}$, the claim follows.

"$\Leftarrow$":
From Theorem 2.4.29 there exist $\sigma_i \in \mathcal{K}_\infty$ and $\widetilde{\alpha} \in \mathcal{K}_\infty$ such that with $\sigma(r) = (\sigma_1(r), \ldots, \sigma_n(r))^T$, $r \in [0, \infty)$, and $\widetilde{\mathcal{D}}$ as in (2.44) with corresponding $\widetilde{\alpha}$

$$\widetilde{\mathcal{D}} \circ \Gamma(\sigma(r)) < \sigma(r), \text{ for all } r > 0. \tag{2.142}$$

By Lemma 2.3.8 there exists $\mathcal{K}_\infty$ function ϕ such that

$$w \le \mu(\Gamma(w), v) \Rightarrow |w| \le \phi(|v|). \tag{2.143}$$

Let us show that bounded inputs yield bounded trajectories. To this end assume that $v(l) \le v \in \mathbb{R}^n_+$ for all $l \ge 0$. For any such v and arbitrary $s(0) \in \mathbb{R}^n_+$ by (2.142) there exists an $r > 0$ such that $\sigma(r) \ge s(0)$ and $\rho(\sigma(r)) \ge v$, where $\rho : \mathbb{R}^n_+ \to \mathbb{R}^n_+$, $\rho_i := \widetilde{\alpha}$ for $i \in I_\Sigma$, $\rho_i :=$ id for $i \in I_{\max}$. Now assume that

$$s(l) \le \mathcal{D} \circ \sigma(r), \forall l \ge 0.$$

This is obviously true for $l = 0$. For $l + 1$ we compute

$$\begin{aligned} s(l+1) &= \mu(\Gamma(s(l)), v(l)) \\ &\le \mu(\Gamma \circ \mathcal{D}(\sigma(r)), v) \\ &\le \mu(\sigma(r), \rho(\sigma(r))) \\ &= \mathcal{D} \circ \sigma(r), \end{aligned}$$

where (2.142) and Lemma 2.3.7 were used. Thus, by induction it follows that the trajectory $s(\cdot)$ is bounded.

Now, for a fixed initial condition $s(0)$ and an input bounded by $v(\cdot) \leq v$ let

$$\begin{aligned} s^* &:= s^*(s(0), v) \\ &:= \sup_{l \geq 0} s(l) \\ &\leq \sup_{l \geq 0} \{s(0), \mu(\Gamma(s(l)), v(l))\} \\ &\leq \max\{s(0), \mu(\Gamma(s^*), v)\} \\ &\leq \mu(s(0), \Gamma(s^*), v), \end{aligned}$$

where (2.135) and (2.136) were used. By Lemma 2.3.8 we have $|s^*| \leq \phi(|\mu(s(0), v)|) \leq \mu(\phi \circ \eta_1(|s(0)|), \phi \circ \eta_2(|v|))$, where the weak triangle inequality (2.43) was used and $\eta_{1,i} := (\mathrm{id} + \eta)$, $\eta_{2,i} := (\mathrm{id} + \eta^{-1})$ with arbitrary $\eta \in \mathcal{K}_\infty$, $\eta_{1,i} :=$ id for $i \in I_{\sum}$, $\eta_{2,i} :=$ id for $i \in I_{\max}$. Thus the GS property of (2.137) is obtained.
For the AG property we obtain

$$s^\sharp := \limsup_{l \to \infty} s(l) = \limsup_{l \to \infty} \mu(\Gamma(s(l)), v(l)) \leq \mu(\Gamma(s^\sharp), v).$$

By the weak triangle inequality (2.43) it follows that $s^\sharp \leq \phi(|v|)$. This is the AG property. Then the system (2.137) is ISS by Theorem 2 in [84], where AG is named $\mathcal{K}$-asymptotic gain and GS is named UBIBS. □

Chapter 3

Model reduction approach for large-scale networks

Verification of the small gain condition $\Gamma \circ \mathcal{D}(s) \not\geq s$ in (2.45) requires large amount of analytical computations in the case of the large size of logistics network. This procedure can be facilitated by reducing the size of the system (2.5), by applying a numerical method to verify (2.45) or applying a method that reduces the size of the gain matrix Γ used in condition (2.45).

Model reduction of linear large-scale systems is already a well-developed area. The most efficient approaches are balancing and moment matching (Krylov subspace methods), see [4]. In balancing methods state variables that are hard to control/observe are eliminated from the model. An approximation norm is usually given in terms of $\mathcal{H}_\infty$- or $\mathcal{H}_2$-norms. In moment matching methods a function that matches certain moment of the Laurent series expansion is being looked for. These methods are computationally efficient in comparison with balancing methods, however provide no approximation error bounds. Usually, one uses a combination of both methods where first the large size is reduced by the moment matching methods and then the balancing method is applied.

On the contrary, the methods for the reduction of nonlinear systems are still in the development. As of today, there exist many different approaches that provide first steps in the direction of the reduction of nonlinear systems. However, these approaches are applied only to certain subclasses of nonlinear systems. The most known methods are an extension of the balancing and moment matching methods to nonlinear systems, proper orthogonal decomposition, singular perturbations theory, trajectory piecewise linear approach, Volterra methods and the theory of global attractors. The balancing methods [138], [93] are applied to input-affine continuous-time nonlinear systems, and the moment matching to single-input single-output systems [10] and bilinear systems [24]. In the proper orthogonal decomposition (POD) [75], [11], [76] the original system is projected onto a subspace of a smaller dimension using the known set of data (snapshots). POD methods are usually applied to models describing physical systems. Singular perturbations theory [22], [90] is used for the systems, where parameters evolve in different time scales ("slow" and "fast" parameters). This approach assumes aggregation of the variables evolving in the fast time scale. The trajectory piecewise linear approach [124] is mostly applied to input-affine systems. The system is linearized several times along a trajectory and the final model is constructed as a weighted sum of all local linearized reduced systems. In Volterra methods [121] the reduction is performed by taking into account the first several terms of the Polynomial expansion of a nonlinear function. In the theory of global attractors [88] one searches for a slow-manifold, inertial manifold or center manifold, on which a restricted dynamical system represents the "interesting" behaviour of the dynamical system.

Note that, if these methods will be directly applied to a logistics network, then information about the

real physical objects of logistics network and of its structure will be, in general, lost. Therefore, a reduction method that preserves the main structure of the network is needed.
Structure preserving model reduction was studied in [139], [92], [129]. However, it is also applied only for particular classes of systems.
Another possibility to decrease the number of analytical computations in verifying the small gain condition (2.45) is an application of numerical methods. A first attempt to perform this was done in [126] by adapting the algorithm of Eaves [57]. There is considered a local version of ISS.
On the other hand, to the best of our knowledge, there exist no approaches on the reduction of the size of the gain matrix Γ in the small gain condition (2.45). In this chapter we make the first attempt in this direction. To this end, we consider the gain operator Γ used in the small gain condition as the gain model of the network that describes the interconnection between the subsystems of the network. This model consists of the subsystems and their relations, given by the gains γ_{ij} that are collected in the gain matrix Γ. By the model reduction we understand reduction of the gain model, i.e. transition from the gain matrix Γ of size n to the matrix $\widetilde{\Gamma}$ of size $l < n$.
To obtain the matrix $\widetilde{\Gamma}$ we propose to aggregate the subsystems and the gains γ_{ij} between the subsystems that belong to certain interconnection patterns. Aggregation of these patterns keeps the main structure of the mutual influences between the subsystems in the network, i.e. between locations of the logistics network. Thus the properties of the aggregated and the original models should be similar. This prompts us that ISS of the large-scale logistics network can be established by checking the aggregated small gain condition corresponding to the gain matrix $\widetilde{\Gamma}$.
In this chapter we introduce three aggregation rules for the reduction of the gain model. These rules are based on three interconnection patterns: sequentially connected nodes, nodes connected in parallel and almost disconnected subgraphs. We establish that fulfillment of the reduced small gain condition implies ISS of the large network. Furthermore, we show how an ISS-Lyapunov function for the large network can be constructed using Ω-path corresponding to the reduced small gain condition.

3.1 Gain model

Consider an interconnected hybrid system of the form (2.5). Without loss of generality, assume, for convenience, that all its subsystems are ISS in terms of maximizations with gains γ_{ij} collected in the gain matrix Γ, i.e. $I = I_{\max}$. Note that we can always pass from the summation formulation to the maximization one applying Proposition 2.3.17. Then, to establish ISS of the interconnection we can use Theorem 2.4.5, i.e. we need to verify the small gain condition (2.41):

$$\Gamma(s) \not\geq s, \forall s \in \mathbb{R}^n_+ \setminus \{0\},$$

with

$$\Gamma(s) := \begin{pmatrix} \max\{\gamma_{12}(s_2), \ldots, \gamma_{1n}(s_n)\} \\ \vdots \\ \max\{\gamma_{n1}(s_1), \ldots, \gamma_{n,n-1}(s_{n-1})\} \end{pmatrix}. \tag{3.1}$$

Recall also that by Lemma 2.3.1 the small gain condition (2.41) is equivalent to the cycle condition (2.42):

$$\gamma_{k_1k_2} \circ \gamma_{k_2k_3} \circ \cdots \circ \gamma_{k_{r-1}k_r} < \mathrm{id},$$

for all $(k_1, ..., k_r) \in \{1, ..., n\}^r$ with $k_1 = k_r$. The largest possible number of cycles to be checked in this condition can be calculated as $\sum_{k=2}^n \binom{n}{k} k!$, where $\binom{n}{k}$ is the binomial coefficient.

To reduce the size of the gain matrix in the small gain condition (2.41) we model the structure of the logistics network described in (2.5) as a directed graph with weights $G = (V, E, \Gamma)$. The vertex set $V = \{1, \dots, n\}$ corresponds to the subsystems of the network, the edge set E to the interconnection between subsystems, i.e.

$$e_{ij} = \begin{cases} 1, & \text{if } \gamma_{ij} \not\equiv 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

The weight of the edge e_{ij} from vertex i to j is given by γ_{ji} and describes the influence of subsystem i on subsystem j. All the weights are collected in the gain matrix Γ. Note that the matrix Γ is not static, i.e. the weights are in general nonlinear functions. Such model we call *gain model* of the interconnected system (2.5).

3.2 Aggregation rules

In our model reduction approach we propose to reduce the size of the gain matrix Γ in the small gain condition (2.41). In particular, we transform the graph $G = (V, E, \Gamma)$ by introducing aggregation rules for vertices for typical subgraphs occurring in the network. Such subgraphs we will call *motifs* [103]. By aggregation of the vertices we understand the construction of a smaller graph $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{\Gamma})$ in which the vertices may represent nonempty subsets of vertices in the original graph $G = (V, E, \Gamma)$. We single out the following motifs: parallel connections, sequential connections of vertices and almost disconnected subgraphs. These reduction rules are inspired by the properties of motifs in [2].

3.2.1 Aggregation of sequentially connected nodes

The vertices of the set $V_J = \{v_1, ..., v_l\}$ are called *sequentially connected*, see Figure 3.1, if there exist vertices $v, v' \in V \setminus V_J$ such that

$$P(v_i) = \begin{cases} v & i = 1, \\ v_{i-1} & i = 2, ..., l \end{cases}$$

and

$$S(v_i) = \begin{cases} v_{i+1} & i = 1, ..., l-1, \\ v' & i = l. \end{cases}$$

The predecessor set P and successor set S were defined in Section 1.1.3.

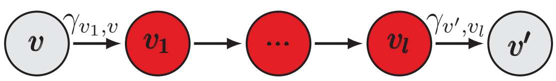


Figure 3.1: Sequential connection of vertices $v_1, \dots, v_l$.

The corresponding gain matrix is given by

$$\Gamma = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots \\ \dots & 0 & \gamma_{v_1,v} & 0 & \dots & 0 & 0 & 0 & \dots \\ \dots & 0 & 0 & \gamma_{v_2,v_1} & \dots & 0 & 0 & 0 & \dots \\ \dots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots \\ \dots & 0 & 0 & 0 & \dots & \gamma_{v_l,v_{l-1}} & 0 & 0 & \dots \\ \dots & \dots & \dots & 0 & \dots & 0 & \gamma_{v',v_l} & \dots & \dots \\ \dots & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{3.3}$$

The cycle condition (2.42) for the cycles that include nodes from $\{v_1, \dots, v_l\}$ looks as follows:

$$\dots \circ \gamma_{v',v_l} \circ \dots \circ \gamma_{v_2,v_1} \circ \gamma_{v_1,v} \dots < \mathrm{id}. \tag{3.4}$$

Aggregation of gains

To obtain a graph of a smaller size we aggregate the nodes $v_1, \dots, v_l$ with the node v. We denote the new vertex by J. A cut-out of the new reduced graph is shown in Figure 3.2. So, we consider the reduced graph $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{\Gamma})$, where the vertices are given by

$$\tilde{V} = (V \setminus (V_J \cup \{v\})) \cup J \tag{3.5}$$

and the edges are given by

$$\tilde{E} = E \setminus (\{(v,w),(w,v'),(w_1,w_2) : w, w_1, w_2 \in V_J\} \cup (v,v')) \\ \cup \{(J,v') \cup (u,J) : (u,v) \in E\}. \tag{3.6}$$

The corresponding weighted adjacency matrix $\tilde{\Gamma}$ of the dimension $n-l$ can be obtained from Γ, where the rows and columns corresponding to the vertices $v, v_1, \dots, v_l$ are replaced by a row and a column corresponding to the new vertex J. The weights are then given by

$$\widetilde{\gamma}_{v',J} := \max\{\gamma_{v',v_l} \circ \dots \circ \gamma_{v_2,v_1} \circ \gamma_{v_1,v}, \gamma_{v',v}\}, \tag{3.7}$$

$$\widetilde{\gamma}_{J,v'} := \gamma_{v,v'}, \quad \widetilde{\gamma}_{J,j} := \gamma_{v,j}, \quad \widetilde{\gamma}_{j,J} := \gamma_{j,J}, \quad j \in V \setminus (V_J \cup \{v, v'\}). \tag{3.8}$$

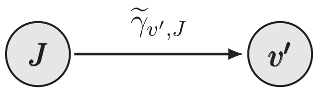


Figure 3.2: Vertices $v_1, \dots, v_l, v'$ are aggregated.

Other gains stay the same, i.e.

$$\widetilde{\gamma}_{ij} := \gamma_{ij}, \; i, j \neq J. \tag{3.9}$$

The small gain condition (2.41) corresponding to the reduced gain matrix $\tilde{\Gamma}$ has the following properties.

Theorem 3.2.1. *Consider a gain matrix* Γ *as in* (3.3)*. If condition* (2.41) *holds for the matrix* $\tilde{\Gamma}$ *with gains defined in* (3.7)-(3.9)*, then it holds also for the matrix* Γ*.*
Assume that there were p *cycles that include one of the nodes* v_i *from* V_J*. If* $\gamma_{v',v} \neq 0$*, then the number of cycles to be checked in the cycle condition* (2.42) *corresponding to the reduced matrix* $\tilde{\Gamma}$ *is decreased by* p *after the aggregation, otherwise it stays the same.*

Proof. Let condition (2.41) for the gains defined in (3.7)-(3.9) hold. Then the cycle condition (2.42) corresponding to these gains holds. In particular, for the cycles containing the gain $\widetilde{\gamma}_{v',J}$ the following inequality holds:

$$\ldots \circ \widetilde{\gamma}_{v',J} \circ \ldots < \mathrm{id}. \tag{3.10}$$

From the definition of the gain $\widetilde{\gamma}_{v',J}$ in (3.7) condition (3.4) holds. Condition (2.42) on other cycles corresponding to Γ is satisfied straightforwardly. Thus, the matrix Γ satisfies (2.41).
If $\gamma_{v',J} = 0$, then, as the cycle containing one of the nodes $\{v_1, \ldots, v_l\}$ contains necessarily all other nodes from $\{v_1, \ldots, v_l\}$, the number of cycles to be checked in the cycle condition is the same. Otherwise, these cycles will "coincide" with the cycles that include gain $\gamma_{v',v}$. Thus, the overall number of the cycles will decrease by p. □

Thus, to show that a system of the form (2.5) is ISS, it is enough to verify the small gain condition $\widetilde{\Gamma}(s) \not\geq s$ corresponding to the reduced gain matrix $\widetilde{\Gamma}$.

Corollary 3.2.2. *Consider interconnected system* (2.5) *and assume that the set* $\{f(x,u) : u \in U \cap \epsilon\mathbb{B}\}$ *is convex for each* $x \in \chi, \epsilon > 0$*. Assume also that* $D_i = D, i = 1, \ldots, n$ *and that all the subsystems in* (2.1) *are ISS with gains as in* (2.14)*. If condition* (2.41) *holds for the gains defined in* (3.7)-(3.9)*, then the system* (2.5) *is ISS.*

Proof. The assertion follows from Theorem 3.2.1 and Theorem 2.4.5. □

Construction of an Ω-path

To construct an ISS-Lyapunov function of the interconnected system (2.5), we can apply Theorem 2.4.11. However, for this purpose we need to have an Ω-path σ satisfying (2.71), i.e.

$$\Gamma(\sigma) \leq \sigma.$$

It appears, that if an Ω-path corresponding to the reduced gain matrix $\widetilde{\Gamma}$ is known, we can calculate an Ω-path for the large gain matrix Γ.

Proposition 3.2.3. *Consider a gain matrix* Γ *and the corresponding reduced gain matrix* $\tilde{\Gamma}$ *with gains defined in* (3.7)-(3.9)*. Let an* Ω*-path* $\widetilde{\sigma}$ *for* $\tilde{\Gamma}$ *satisfying* (2.71) *be given. Then an* Ω*-path* $\bar{\sigma}$ *for the matrix* Γ *can be constructed as*

$$\bar{\sigma}_w := \begin{cases} \gamma_{v_i,v_{i-1}} \circ \gamma_{v_{i-1},v_{i-2}} \circ \cdots \circ \gamma_{v_2,v_1} \circ \gamma_{v_1,v} \circ \widetilde{\sigma}_J, & \textit{if } w = v_i, i \in \{1, \ldots, l\}, \\ \widetilde{\sigma}_w, & \textit{otherwise}\ . \end{cases} \tag{3.11}$$

Proof. We assume that an Ω-path $\widetilde{\sigma}$ for the small gain matrix $\tilde{\Gamma}$ is known. In particular, by (2.71) $\widetilde{\Gamma}(\widetilde{\sigma}) \leq \widetilde{\sigma}$ holds. Let us check whether Ω-path $\bar{\sigma}$ defined in (3.11) is an Ω-path for the large gain matrix Γ. To this end we need to check (2.71) for $\bar{\sigma}$.

For the components $\Gamma(\bar{\sigma})_w$, $w \notin \{v_1, \ldots, v_l, v'\}$ the inequality (2.71) holds straightforwardly. Consider now $\Gamma(\bar{\sigma})_w$, $w = v_i, i \in \{1, \ldots, l\}$. Applying (3.7)-(3.9) and (3.11) we obtain:

$$\begin{aligned}
\Gamma(\bar{\sigma})_{v_i} &= \gamma_{v_i,v_{i-1}} \circ \bar{\sigma}_{v_{i-1}} = \gamma_{v_i,v_{i-1}} \circ \gamma_{v_{i-1},v_{i-2}} \circ \bar{\sigma}_{v_{i-2}} \\
&= \cdots = \gamma_{v_i,v_{i-1}} \circ \cdots \circ \gamma_{v_1,v} \circ \widetilde{\sigma}_J \\
&= \bar{\sigma}_{v_i}; \\
\Gamma(\bar{\sigma})_{v'} &= \max\{\gamma_{v',1}(\bar{\sigma}_1), \ldots, \gamma_{v',v_l}(\bar{\sigma}_{v_l}), \ldots, \gamma_{v',n}(\bar{\sigma}_n)\} \\
&= \max\{\widetilde{\gamma}_{v',1}(\widetilde{\sigma}_1), \ldots, \underbrace{\gamma_{v',v_l} \circ \cdots \circ \gamma_{v_1,v} \circ \widetilde{\sigma}_J}_{\widetilde{\gamma}_{v',J} \circ \widetilde{\sigma}_J}, \ldots, \widetilde{\gamma}_{v',n}(\widetilde{\sigma}_n)\} \\
&= \max\{\widetilde{\gamma}_{v',1}(\widetilde{\sigma}_1), \ldots, \widetilde{\gamma}_{v',J} \circ \widetilde{\sigma}_J, \ldots, \widetilde{\gamma}_{v',n}(\widetilde{\sigma}_n) \\
&\le \widetilde{\sigma}_{v'} = \bar{\sigma}_{v'}.
\end{aligned}$$

Thus $\Gamma(\bar{\sigma}) \le \bar{\sigma}$ and $\bar{\sigma}$ is an Ω-path corresponding to the large gain matrix Γ. □

The proposition above implies the following result concerning the construction of an ISS-Lyapunov function.

Corollary 3.2.4. *Consider a system of the form* (2.5) *that is interconnection of subsystems* (2.1). *Assume that $D_i = D, i = 1, \ldots, n$ and that each subsystem i of* (2.1) *has an ISS-Lyapunov function V_i with the corresponding ISS-Lyapunov gains $\gamma_{ij}, \gamma_i, i, j = 1, \ldots, n$ as in* (2.34), (2.36). *Let $\overline{\Gamma}$ be defined as in* (2.99) *with $I_{\max} = \{1, \ldots, n\}$. Assume that there exist an Ω-path $\tilde{\sigma}$ with respect to $\widetilde{\Gamma}$ defined by* (3.7)-(3.9) *and a function $\phi \in \mathcal{K}_\infty$ given by* (2.100). *Then the system* (2.5) *is ISS and an ISS-Lyapunov function is given by* (2.101) *with σ from* (3.11).

Proof. The assertion follows from Theorem 2.4.11 and Proposition 3.2.3. □

3.2.2 Aggregation of nodes connected in parallel

Parallel connections are characterized by the vertices having the same predecessor and successor sets consisting of a single vertex. Let the vertices $V_J := \{v_1, \ldots, v_l\} \subset V$ be *connected in parallel*, i.e. every vertex has only one ingoing and one outgoing edge and the ingoing edges originate from one vertex $v \in V$ and also the outgoing edges end in solely one vertex $v' \in V$, see Figure 3.3. To be precise, $V_J = \{i \in V \,:\, P(i) = v, S(i) = v'\}$.

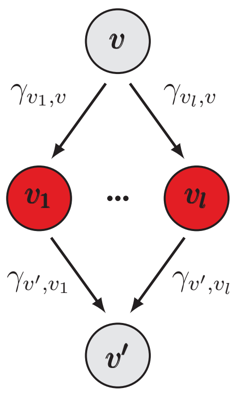


Figure 3.3: Parallel connection of vertices $v_1, \ldots, v_l$.

The corresponding gain matrix is given by

$$\Gamma = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & 0 & \dots & 0 & 0 & \dots & \dots \\ \dots & 0 & 0 & \dots & 0 & \gamma_{v_1,v} & 0 & \dots \\ \dots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \dots \\ \dots & 0 & 0 & \dots & 0 & \gamma_{v_l,v} & 0 & \dots \\ \dots & 0 & \gamma_{v',v_1} & \dots & \gamma_{v',v_l} & \dots & \dots & \dots \\ \dots & \dots & 0 & \dots & 0 & 0 & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{3.12}$$

The cycle condition (2.42) for the cycles that include nodes from $\{v_1, \dots, v_l\}$ looks as follows:

$$\dots \circ \gamma_{v',v_i} \circ \gamma_{v_i,v} \circ \dots < \mathrm{id}. \tag{3.13}$$

Aggregation of gains

Based on this structure a possibility to attain a graph of a smaller size is to aggregate the vertices connected in parallel to a single vertex and to leave the structure of the remaining graph as it is. We denote the new vertex by J. A cut-out of the new reduced graph is shown in Figure 3.4.

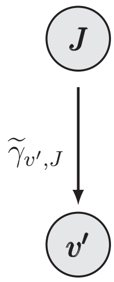


Figure 3.4: Aggregation of vertices $v_1, \dots, v_l, v$.

So, we consider the reduced graph $\widetilde{G} = (\widetilde{V}, \widetilde{E}, \widetilde{\Gamma})$, where the vertices are given by

$$\tilde{V} = (V \setminus (V_J \cup \{v\})) \cup J \tag{3.14}$$

and the edges are given by

$$\tilde{E} = E \setminus (\{(v,w),(w,v') : w \in V_J\} \cup (v,v')) \cup (J,v') \cup \{(u,J) : (u,v) \in E\}. \tag{3.15}$$

The corresponding weighted adjacency matrix $\tilde{\Gamma}$ of the dimension $n-l$ can be obtained from Γ, where the rows and columns corresponding to the vertices $v, v_1, \dots, v_l$ are replaced by a row and column corresponding to the new vertex J. The weights are then given by

$$\widetilde{\gamma}_{v',J} := \max\{\gamma_{v',v_1} \circ \gamma_{v_1,v}, \dots, \gamma_{v',v_l} \circ \gamma_{v_l,v}, \gamma_{v',v}\}, \tag{3.16}$$

$$\widetilde{\gamma}_{J,v'} := \gamma_{v,v'}, \quad \widetilde{\gamma}_{J,j} := \gamma_{v,j}, \quad \widetilde{\gamma}_{j,J} := \gamma_{j,J}, \quad j \in V \setminus (V_J \cup \{v,v'\}). \tag{3.17}$$

Other gains stay the same, i.e.

$$\widetilde{\gamma}_{ij} := \gamma_{ij}, i, j \neq J. \tag{3.18}$$

The small gain condition (2.41) corresponding to the reduced gain matrix $\tilde{\Gamma}$ has the following properties.

Theorem 3.2.5. *Consider a gain matrix* Γ *as in* (3.12)*. If condition* (2.41) *holds for the matrix* $\tilde{\Gamma}$ *with gains defined in* (3.16)-(3.18)*, then it holds also for the matrix* Γ*.*
Furthermore, if there were p *cycles that include node* v_i*, then the number of cycles to be checked in the cycle condition* (2.42) *corresponding to the reduced matrix* $\tilde{\Gamma}$ *is decreased by* $p(l-1-\delta_{v',v})$*, where* $\delta_{v',v} := 1$*, if* $\gamma_{v',v} \neq 0$ *and* $\delta_{v',v} := 0$ *otherwise.*

Proof. Let condition (2.41) for the gains defined in (3.16)-(3.18) hold. Then the cycle condition (2.42) for these gains holds. In particular, for the cycles containing the gain $\widetilde{\gamma}_{v',J}$ the following inequality holds:

$$\ldots \circ \widetilde{\gamma}_{v',J} \circ \ldots < \text{id}. \tag{3.19}$$

From the definition of the gain $\widetilde{\gamma}_{v',J}$ in (3.16), condition (3.13) holds. Condition (2.42) on the other cycles is satisfied straightforwardly. Thus Γ satisfies (2.41).
If there were p cycles that include node v_i in the large graph, then the number of the cycles that include a node from $\{v_1, \ldots, v_l\}$ is $p \cdot l$. If $\gamma_{v',v} \neq 0$, then the number of cycles with nodes $\{v_1, \ldots, v_l\}$ and gain $\gamma_{v',v}$ is $p \cdot (l+1)$. After the aggregation of the gains these cycles will "coincide", thus the number of the cycles to be checked in the small gain condition (2.41) is decreased by $p(l-1-\delta_{v',v})$. □

Again, to show that a system of the form (2.5) is ISS, it is enough to verify the small gain condition corresponding to the reduced gain matrix.

Corollary 3.2.6. *Consider interconnected system* (2.5) *and assume that the set* $\{f(x,u) : u \in U \cap \epsilon\overline{\mathbb{B}}\}$ *is convex for each* $x \in \chi, \epsilon > 0$*. Assume also that* $D_i = D, i = 1, \ldots, n$ *and that all subsystems in* (2.1) *are ISS with gains as in* (2.14)*. If condition* (2.41) *holds for the gains defined in* (3.16)-(3.18)*, then the system* (2.5) *is ISS.*

Proof. The assertion follows from Theorem 3.2.5 and Theorem 2.4.5. □

Construction of an Ω-path

Again we can calculate an Ω-path for a large gain matrix having an Ω-path corresponding for the reduced one.

Proposition 3.2.7. *Consider a gain matrix* Γ *and the corresponding reduced gain matrix* $\tilde{\Gamma}$ *with gains defined in* (3.16)-(3.18)*. Let an* Ω*-path* $\widetilde{\sigma}$ *for* $\tilde{\Gamma}$ *satisfying* (2.71) *be given. Then an* Ω*-path* $\bar{\sigma}$ *for the matrix* Γ *can be constructed as*

$$\bar{\sigma}_w := \begin{cases} \gamma_{w,v} \circ \widetilde{\sigma}_J, & \text{if } w \in \{v_1, \ldots, v_l\}, \\ \widetilde{\sigma}_w, & \text{otherwise .} \end{cases} \tag{3.20}$$

Proof. We assume that an Ω-path $\widetilde{\sigma}$ for the small gain matrix $\widetilde{\sigma}$ is known. In particular, by (2.71) $\widetilde{\Gamma}(\widetilde{\sigma}) \leq \widetilde{\sigma}$ holds. Let us check whether an Ω-path $\bar{\sigma}$ defined in (3.20) is an Ω-path for the large matrix Γ. To this end we need to check (2.71).
For the components $\Gamma(\bar{\sigma})_w$, $w \notin \{v_1, \ldots, v_l, v'\}$ the inequality (2.71) holds straightforwardly. Consider now $\Gamma(\bar{\sigma})_w$, $w \in \{v_1, \ldots, v_l\}$. Applying (3.16)-(3.18) and (3.20) we obtain:

$$\begin{aligned}
\Gamma(\bar{\sigma})_w &= \gamma_{w,v} \circ \bar{\sigma}_v = \widetilde{\sigma}_w; \\
\Gamma(\bar{\sigma})_{v'} &= \max\{\gamma_{v',1}(\bar{\sigma}_1), \ldots, \gamma_{v',v_1}(\bar{\sigma}_{v_1}), \ldots, \gamma_{v',v_1}(\bar{\sigma}_{v_l}), \ldots, \gamma_{v',n}(\bar{\sigma}_n)\} \\
&= \max\{\widetilde{\gamma}_{v',1}(\widetilde{\sigma}_1), \ldots, \underbrace{\gamma_{v',v_1} \circ \gamma_{v_1,v} \circ \widetilde{\sigma}_J, \ldots, \gamma_{v',v_l} \circ \gamma_{v_l,v} \circ \widetilde{\sigma}_J}_{\widetilde{\gamma}_{v',J} \circ \widetilde{\sigma}_J}, \ldots, \widetilde{\gamma}_{v',n}(\widetilde{\sigma}_n)\} \\
&= \max\{\widetilde{\gamma}_{v',1}(\widetilde{\sigma}_1), \ldots, \widetilde{\gamma}_{v',J} \circ \widetilde{\sigma}_J, \ldots, \widetilde{\gamma}_{v',n}(\widetilde{\sigma}_n) \\
&\leq \widetilde{\sigma}_{v'} = \bar{\sigma}_{v'}.
\end{aligned}$$

Thus $\Gamma(\bar{\sigma}) \leq \bar{\sigma}$ and $\bar{\sigma}$ is an Ω-path corresponding to the large gain matrix Γ. □

Corollary 3.2.8. *Consider a system of the form* (2.5) *that is an interconnection of the subsystems* (2.1)*. Assume that $D_i = D, i = 1, \ldots, n$ and that each subsystem i of* (2.1) *has an ISS Lyapunov function V_i with corresponding ISS-Lyapunov gains $\gamma_{ij}, \gamma_i, i, j = 1, \ldots, n$ as in* (2.34)*,* (2.36)*. Let $\overline{\Gamma}$ be defined as in* (2.99) *with $I_{\max} = \{1, \ldots, n\}$. Assume that there exists an Ω-path $\tilde{\sigma}$ with respect to $\widetilde{\Gamma}$ defined by* (3.16)*-*(3.18) *and a function $\phi \in \mathcal{K}_\infty$ given by* (2.100)*. Then the system* (2.5) *is ISS and an ISS-Lyapunov function is given by* (2.101) *with σ from* (3.20)*.*

Proof. The assertion follows from Theorem 2.4.11 and Proposition 3.2.7. □

3.2.3 Aggregation of almost disconnected subgraphs

A further structure in the network, that suggests itself to a reduction is given by subgraphs which are connected to the remainder of the network through just a single vertex. So, we consider a set of vertices $V_J = \{v_1, ..., v_l\}$ and a distinguished vertex $v^* \in V \setminus V_J$ such that any path from $v_i, i = 1, \ldots, l$ to the remainder of the vertices in $V \setminus V_J$, and any path from $V \setminus V_J$ to V_J necessarily passes through the vertex v^*. If we assume that the whole graph is strongly connected, this implies in particular, that the subgraph induced by $V_J \cup \{v^*\}$ is by itself strongly connected.
In Figure 3.5 an example graph is shown, where the vertices $V_J = \{v_1, \ldots, v_l\}$ are connected with the rest of the graph only through the vertex v^*.

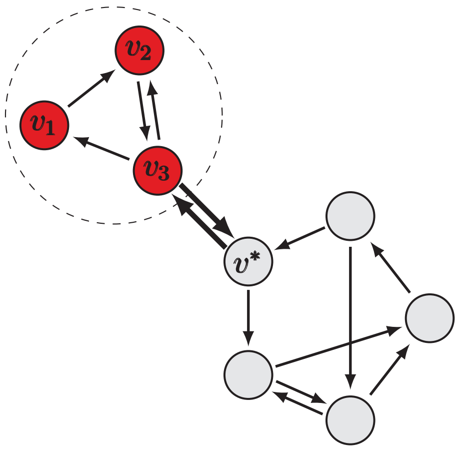


Figure 3.5: The subgraph consisting of the vertices $V_J = \{v_1, v_2, v_3\}$ is almost disconnected from the graph.

The cycles in (2.42) that include nodes only from $\{v_1, \ldots, v_l, v^*\}$ look as follows:

$$\gamma_{k_1,k_2} \circ \gamma_{k_2,k_3} \circ \cdots \circ \gamma_{k_{r-1},k_r} < \mathrm{id}, \tag{3.21}$$

for all $(k_1, ..., k_r) \in \{v_1, ..., v_l, v^*\}^r$ with $k_1 = k_r$.

Aggregation of gains

To reduce the network size we aggregate the vertices of the subgraph V_J with vertex v^* and do not change the remainder of the graph. We denote the new vertex by J. For the example in Figure 3.5 the reduced graph is shown in Figure 3.6. So we consider the reduced graph $\tilde{G} = (\tilde{V}, \tilde{E}, \tilde{A})$, where the vertices are given by

$$\tilde{V} = (V \setminus (V_J \cup \{v^*\})) \cup J \tag{3.22}$$

and the edges are given by

$$\begin{aligned}\tilde{E} = E \setminus \{(w_1, w_2), (v^*, w_1), (w_1, v^*) : w_1, w_2 \in V_J\} \\ \cup \{(J, u) : u \in \tilde{V}, (v^*, u) \in E\} \\ \cup \{(u, J) : u \in \tilde{V}, (u, v^*) \in E\}.\end{aligned} \tag{3.23}$$

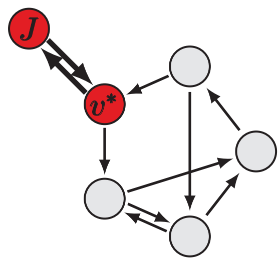


Figure 3.6: Subgraph V_J and node v^* are merged to vertex J.

The corresponding weighted adjacency matrix $\tilde{\Gamma}$ of the dimension $n - l + 1$ can be obtained from Γ, where the rows and columns corresponding to the vertices $v_1, \ldots, v_l$ are replaced by a row and column corresponding to new vertex J. The weights are then given by

$$\widetilde{\gamma}_{J,v^*} := \max_{(k_1,\ldots,k_r)\in\{v_1,\ldots,v_l,v^*\}^r, k_1=k_r} \{\gamma_{k_1,k_2} \circ \gamma_{k_2,k_3} \circ \cdots \circ \gamma_{k_{r-1},k_r}\}, \tag{3.24}$$

$$\widetilde{\gamma}_{v^*,J} = \mathrm{id}. \tag{3.25}$$

Other gains stay the same, i.e.

$$\widetilde{\gamma}_{ij} := \gamma_{ij},\ i, j \neq J. \tag{3.26}$$

Theorem 3.2.9. *Consider a gain matrix Γ. If condition* (2.41) *holds for the gain matrix $\tilde{\Gamma}$ with gains defined in* (3.24)-(3.26)*, then it holds also for the matrix Γ.*
If there were p cycles that include nodes only from $V_J \cup \{v^*\}$, then the number of cycles to be checked in the cycle condition (2.42) *corresponding to the reduced matrix $\tilde{\Gamma}$ is decreased by $p - 1$.*

Proof. Let condition (2.41) for the gains defined in (3.24)-(3.26) hold. Then the cycle condition (2.42) for these gains holds. In particular, for the cycles containing $\widetilde{\gamma}_{v^*,J}, \widetilde{\gamma}_{J,v^*}$ the following inequality holds:

$$\widetilde{\gamma}_{v^*,J} \circ \widetilde{\gamma}_{J,v^*} < \mathrm{id}. \tag{3.27}$$

From the definition of the gains $\widetilde{\gamma}_{J,v^*}$ and $\widetilde{\gamma}_{v^*,J}$ in (3.24) and (3.25), condition (2.41) for the large matrix Γ holds. Conditions on the other cycles in (2.42) are satisfied straightforwardly.
As instead of p cycles with nodes only from $V_J \cup \{v^*\}$ we consider only one cycle $\widetilde{\gamma}_{v^*,J} \circ \widetilde{\gamma}_{J,v^*}$, the number of cycles corresponding to the small gain matrix $\tilde{\Gamma}$ is decreased by $p - 1$. □

Corollary 3.2.10. *Consider interconnected system* (2.5) *and assume that the set $\{f(x, u) : u \in U \cap \epsilon\overline{\mathbb{B}}\}$ is convex for each $x \in \chi, \epsilon > 0$. Assume also that $D_i = D, i = 1, \ldots, n$ and that all subsystems in* (2.1) *are ISS with gains as in* (2.14)*. If condition* (2.41) *holds for the gains defined in* (3.24)-(3.26)*, then the system* (2.5) *is ISS.*

Proof. The assertion follows from Theorem 3.2.9 and Theorem 2.4.5. □

Construction of an Ω-path

Again, we can calculate an Ω-path for a large gain matrix having an Ω-path corresponding for a reduced one.

Proposition 3.2.11. *Consider a gain matrix* Γ *and the corresponding reduced gain matrix* $\tilde{\Gamma}$ *with gains defined in* (3.24)-(3.26). *Let an* Ω*-path* $\widetilde{\sigma}$ *for* $\tilde{\Gamma}$ *satisfying* (2.71) *be given. Then there exists an* Ω*-path* $\bar{\sigma}$ *for the matrix* Γ.

Proof. Using definitions of the gains $\widetilde{\gamma}_{v^*,J}$ and $\widetilde{\gamma}_{J,v^*}$ in (3.24)-(3.26) we obtain that all the cycles of the large network satisfy the cycle condition (2.42). Thus, we can construct an Ω-path for a large system using Proposition 2.3.14. □

Corollary 3.2.12. *Consider a system of the form* (2.5) *that is an interconnection of the subsystems* (2.1). *Assume that* $D_i = D, i = 1, \dots, n$ *and that each subsystem i of* (2.1) *has an ISS-Lyapunov function* V_i *with the corresponding ISS-Lyapunov gains* $\gamma_{ij}, \gamma_i, i, j = 1, \dots, n$ *as in* (2.34), (2.36). *Let* $\overline{\Gamma}$ *be defined as in* (2.99) *with* $I_{\max} = \{1, \dots, n\}$. *Assume that there exists* Ω*-path* $\tilde{\sigma}$ *with respect to* $\widetilde{\Gamma}$ *defined by* (3.24)-(3.26) *and a function* $\phi \in \mathcal{K}_\infty$ *given by* (2.100). *Then the system* (2.5) *is ISS and an ISS-Lyapunov function is given by* (2.101).

Proof. The assertion follows from Theorem 2.4.11 and Proposition 3.2.11. □

3.2.4 Notes on application of the aggregation rules

The aggregation rules described in the previous subsections preserve the main structure of a logistics network. Furthermore, in the case that there exist several motifs in one network, these rules can be applied step-by-step to reduce the size of the gain matrix Γ. The sequence of the application of these rules may be arbitrary or depend on some additional information about the network topology. For example, this sequence may depend on information about the most influential nodes of the network, see [144, Algorithm 1].
We restrict us on the maximum formulation of ISS. However, this reduction approach can be extended to more general formulations by applying, for example, Proposition 2.3.17.
The "quality" of the reduction is evaluated by the verifying, whether fulfillment of the small gain condition corresponding to the reduced gain matrix implies fulfillment of the small gain condition corresponding to the original gain matrix, and by comparing the number of cycles to be checked in the corresponding small gain conditions.

Chapter 4

Conclusion and outlook

Conclusion

The existing approaches to model logistics networks vary in their capabilities to describe network dynamics and in applicability to study network performance. In Chapter 1 we have reviewed eleven different approaches to model dynamics of logistics networks with application to stability analysis. Each approach possesses one of the four types of dynamics: discrete, continuous, stochastic or hybrid one. Comparison of the characteristics of these approaches is provided in Table 1.1.
Interconnected hybrid system of the form (1.55) provides a flexibility in the modelling of logistics networks due to its ability to model continuous and discrete processes occurring in the network. In the framework of this model, the state of the network is denoted by the variable x and the external inputs by the variable u. The continuous dynamics is described by the function $f(x, u)$ and the discrete one by the function $g(x, u)$. The set C determines when the state of the network changes continuously and the set D when the state of the network changes discretely. Complexity of the cooperation structure of logistics networks, perturbation of external inputs, nonlinearity of dynamics can lead to instability of the behaviour of logistics networks. The notion of input-to-state stability (ISS) allows to describe stability of dynamical systems with external inputs. Furthermore, there exists already a well-established method to study ISS of interconnected systems with only continuous or only discrete dynamics. This method is based on the so-called small gain condition that serves as a sufficient condition for ISS of an interconnection.
In Chapter 2 we have extended application of this condition to hybrid systems. Furthermore, we have extended this condition to the case when some of the subsystems are ISS in terms of the maximization formulation and other in terms of summations. Such mixed formulation can lead to more sharp stability conditions, see Examples 2.3.5 and 2.3.6. To guarantee ISS of interconnected hybrid system we require in Theorem 2.4.5 ISS of all its subsystems and that the condition $\Gamma \circ \mathcal{D} \not\geq \mathrm{id}$ holds, where the gain matrix Γ describes the interconnection structure of the system and the diagonal operator $\mathcal{D}$ has id on the ith component of the diagonal if the ith system is ISS in terms of maximizations and $\mathrm{id} + \alpha$ if the ith system is ISS in terms of summations. As well as in continuous or discrete systems, ISS-Lyapunov functions provide a useful tool to establish ISS of hybrid systems. By imposing the same small gain condition $\Gamma \circ \mathcal{D} \not\geq \mathrm{id}$ we have provided in Theorem 2.4.13 a construction of the ISS-Lyapunov function for the interconnection. This function is given by a scaling of ISS-Lyapunov functions for subsystems. Furthermore, we have presented application of this small gain condition to certain subclasses of hybrid systems: impulsive systems, comparison systems and systems with stability of only a part of the state. These subclasses of hybrid systems allow to model specific types of logistics networks.

In order to verify the small gain condition in the case of the large size of a logistics network, we have presented a method of the reduction of the size of the gain matrix Γ in Chapter 3. In this method we consider the interconnection structure of the network as a weighted graph where the nodes describe the subsystems and the weights describe the gains between the subsystems. Reduction of the gain matrix is performed by an aggregation of the gains of the gain matrix that correspond to certain interconnection motifs of the graph: sequentially connected nodes, nodes connected in parallel and almost disconnected subgraphs. We have established that ISS of a large-scale interconnected system can be concluded by the verifying the small gain condition corresponding to the reduced matrix $\tilde{\Gamma}$, see Corollaries 3.2.2, 3.2.6 and 3.2.10.

Outlook

Flexibility of interconnected hybrid systems in modelling of complex dynamical behaviour enables further extension and investigation of the dynamics and performance of logistics networks like considering of random effects, modelling of different production and service policies by adjusting the functions f, g and the sets C and D, and like imposing of control problems. Furthermore, the obtained stability results can be applied in other types of networks with hybrid dynamics like telecommunication, artificial neural or biological ones.

To stabilize a network with hybrid dynamics, methods for feedback control can be developed using the small gain condition and by extending the results from continuous systems, e.g. [80].

A restriction in the application of an ISS-Lyapunov function mentioned in Remark 2.2.15 suggests the development of more sharp Lyapunov-like functions and of stability conditions to be able to establish stability of hybrid systems with unstable continuous or discrete dynamics. We suppose to start by considering some subclasses of hybrid systems or particular types of stability.

The convexity assumption on the function f in Theorem 2.4.5 is needed to use the equivalence between the asymptotic gain property and 0-input pre-stability in the proof of Theorem 2.4.5 from [28, Theorem 3.1]. We suppose that in some cases we can omit this assumption. To get rid of this restriction one needs either to revise the proof of Theorem 2.4.5 or to prove the equivalence between the asymptotic gain property and 0-input pre-stability without the convexity assumption. Furthermore, we suggest that the requirement $D_i = D$ in the small gain theorems, see Theorem 2.4.5 for example, can be weakened for some particular cases of interconnections.

In Chapter 3 we have performed only initial steps in the development of a model reduction approach for large-scale networks with nonlinear dynamics. The next steps could be: extension of the aggregation rules to other types of motifs, introduction and estimation of the error measure that compares the reduced and the original models, and the development of a numerical algorithm that performs this reduction. Further improvement of the approach may be performed by the adapting of the ranking technique used in [142], [143] and [144] to identify the most influential logistic locations.

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