# Advanced Integral Equation and Hybrid Methods for the Efficient Analysis of General Waveguide and Antenna Structures 

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# Advanced Integral Equation and Hybrid Methods for the Efficient Analysis of General Waveguide and Antenna Structures 

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## Chapter 1

## Introduction

The increasing complexity of the microwave components and antennas in wireless terrestrial and satellite based communications, as well as in radar and remote sensing applications, demands new levels of efficiency and accuracy of the software tools utilized for their design and validation. Tight specifications concerning performance, miniaturization, short production time and low costs represent continuous challenges in the development of the corresponding simulation/optimization algorithms. Moreover, a microwave device often contains geometrical features that vary in size from a tiny fraction of a wavelength to hundreds or even thousands of wavelengths, thus rendering the implementation of the required computer-aided design (CAD) tools even more challenging.

Approaches to improve the efficiency of the $3-D$ electromagnetic (EM) field solvers include the model order reduction (MOR) [1], [2], [3], [4] applied to the finite element method (FEM) or finite difference (FD) method, the multilevel fast mulipole algorithm (MLFMA) [5], [6] and the adaptive integral method (AIM) [7] applied for the fast solution of integral equations discretized by the method of moments (MoM), sub-grid and conformal techniques [8] used in the FD formulation of EM problems. Typical electromagnetic simulations using single method $3-D$ field solvers - usually based on FEM, FD or MoM - provide high flexibility but often prove impractical for the design/optimization of real-world microwave devices, even if the aforementioned acceleration techniques are considered. Fast algorithms like the mode-matching (MM) method [9], [10], [11], [12] and the boundary integral resonant mode expansion (BI-RME) [13], [14] represent efficient alternatives but ultimately at the expense of flexibility.

A simple procedure of combining different algorithms for the analysis of a complex device is the domain decomposition - the structure under investigation is divided into building blocks, and each building block is separately computed with the most efficient method. In a last step, the individual scattering matrices are combined to yield the generalized scattering matrix (GSM), therefore the simulation algorithms involved in the domain decomposition are required to be full-wave. Combining different techniques while retaining their advantages and largely removing their disadvantages is also referred as hybridization [15]. The procedure dramatically enhances the simulation efficiency even in the worst case scenario, i.e. a $3-D$ field solver is employed to calculate each block. To illustrate this, consider a computational domain discretized into $N$ unknowns and divided into $M$ regions of $N / M$ unknowns.

The overall complexity of $\mathrm{MoM}^{1}$ is now $\mathcal{O}\left(M \cdot(N / M)^{2}\right)=\mathcal{O}\left(N^{2} / M\right)$, or $M$ times smaller than the single domain computation. In the FEM and FD models the propagation of the waves takes place through a numerical grid. A small phase error in the field is usually committed in this mode of field propagation [16], [17]. This error is cumulative, thus the larger the computational domain the larger the calculation error. As a consequence, the mitigation of the phase error is automatically accomplished by the domain decomposition through the minimization of the problem size.

The numerical methods presented in this work are designed to be used in a comprehensive hybrid CAD tool [18] applying the domain decomposition approach. They are intended for the calculation of those building blocks for which the fast mode-matching $/ 2-D$ finite element technique cannot be applied. The algorithms introduced here are doubly higher order, that is higher order basis functions [19], [20] are considered for current/field modeling whereas geometry discretization is performed with triangular/tetrahedral elements of higher polynomial degree.

A numerical method based solely on integral equations is presented in Chapter 2. Here, the electric field integral equation (EFIE) is enforced at conducting surfaces, whereas the dielectric bodies are handled by the PMCHWT ${ }^{2}$ approach [16], [21], [22], [23], hitherto applied mainly to plane wave scattering problems. The technique of [24], [25], [26] is extended in the present work by including homogeneous dielectrics and the use of higher order methods. Moreover, the use of same (frequency independent) basis functions at the ports and at the conducting surfaces is facilitated by the introduction of a new formulation that exhibit none of the disadvantages noted in [24], [25], [26], thus yielding an algorithm independent on the modal excitation. Special attention is given to bodies of revolution (BoR) for which novel higher order basis functions are constructed.

The finite element - boundary integral (FE-BI) simulation of arbitrary passive microwave devices is presented in Chapter 3, and represents one step further concerning the flexibility of the structures that can be efficiently calculated. Finite elements are used to characterize the arbitrarily shaped, possibly anisotropic/inhomogeneous, domains. The algorithm [27], hitherto applied for free-space plane wave scattering, is extended here to truncate the computational domain of radiating structures. In contrast to classical FE-BI formulations [28], [29], the calculation of boundary integrals involving the surface divergence of $\hat{n} \times \mathbf{f}$ terms ${ }^{3}$ is avoided here, thus instabilities associated with artificial line charges are avoided. Model order reduction (MOR) techniques are applied for the efficient calculation of the wide-band frequency re-

[^0]sponse. Moreover, the passive reduced-order interconnect macro-modeling analysis (PRIMA) based MOR technique [3], [30] is extended for the treatment of structures with frequency dependent (inhomogeneous cross-section) waveguide ports, whereas the well-conditioned asymptotic waveform evaluation (WCAWE) algorithm [4], [31] is modified to allow multiple right-hand sides (modal excitations).

Rectangular cavities loaded with arbitrarily shaped conductors and and/or dielectric bodies (see Fig. 4.2) are useful key building blocks for the design of many common types of microwave components $[32,33,34,35,36,37,38,39,40,41,42$, $43,44,45]$, such as compact filters for terrestrial and space applications, cf. e.g. [32, 33, 34], or broad-band transitions [46]. Efficient approaches specialized to rectangular cavities include the boundary integral - resonant mode expansion (BI-RME) method [14] and the state-space integral equation method [43]. Although fast, the technique of [14] is limited to rectangular cavities loaded with radially symmetric insets whereas the approach presented in [43] can treat only metallic boxes loaded with cylindrical dielectrics. A novel method for the simulation of rectangular cavities loaded with conductors and/or dielectrics of arbitrary shape is thus presented in Chapter 4. Finite elements are employed to characterize the inhomogeneous and arbitrarily shaped material in the cavity, while integral equations deal with the necessary boundary conditions. The present algorithm extends the known finite element boundary integral formulation at radiators/scatterers [27] to shielded environments. All boundary integrals involving rectangular cavity Green's functions are efficiently evaluated utilizing the Ewald transform [47, 48, 49]. There are mainly two factors responsible for the efficiency of this approach. First, due to the separation of the Green's functions into static series (zero frequency limit), whose convergence is enhanced with the help of the Ewald transform, and an already convergent dynamic series (higher frequency correction), the most computationally intensive part of the algorithm is performed only once in a frequency sweep. Secondly, as a consequence of the use of the cavity Green's functions, only a small portion of the computational domain must be discretized, thus drastically reducing the number of unknowns.

The efficiency and validity of the present algorithms are demonstrated by numerous numerical examples. General conclusions are drawn in the last chapter.

# Integral equation analysis of general waveguide structures 

### 2.1 Introduction

This chapter presents the free-space integral equation (IE) analysis of general waveguide multiport structures (Fig. 2.1) including homogeneous dielectrics. The electric field integral equation (EFIE) is enforced at conducting surfaces, whereas the dielectric bodies are replaced by equivalent sources with the help of the equivalence principle formulated via the PMCHWT ${ }^{1}$ method. The integral equations are solved by the method of moments (MoM) to yield the surface current densities and other parameters of interest like the modal scattering matrix, radiated far fields, etc.

The PMCHWT method [21], [22], [16], [23], hitherto applied mainly to scattering problems, is extended here to yield a stable (resonance-free) algorithm when dealing with dielectric loaded multiports. The technique of [24], [25], [26] is further extended by the inclusion of homogeneous dielectrics and the use of higher order methods. Moreover, the use of same (frequency independent) basis functions at the ports and at the conducting surfaces is facilitated by the introduction of a new formulation that exhibit none of the disadvantages noted in [24], [25], [26], thus yielding an algorithm independent on the modal excitation. Special attention is given to bodies of revolution (BoR) for which novel higher order basis functions are constructed.

The described method yields the generalized admittance and scattering matrices of the structure under investigation. This allows the convenient combination with other powerful techniques, such as the hybrid mode-matching/finite-element techniques [15], for the effective further combination with common elements, e.g. irises, steps, transitions, etc., which yields the desirable high flexibility for the efficient analysis/optimization of structures like antennas, filters, diplexers, etc

As a further enhancement, the presented technique is fully higher order, i.e. the electric and magnetic sources are modeled by higher polynomial order basis functions while curved surfaces are discretized by second or third order curved triangles. Moreover, hierarchical divergence-conforming basis functions, are derived from hiererchal curl-conforming functions which are presented in the finite element method (FEM) literature (e.g. [20]).

[^1]Since many waveguide structures (especially antennas) exhibit rotational symmetry (e.g. dielectric loaded conical horn antennas, paraboloidal reflectors, etc.), special attention is given to bodies of revolution (BoR). Novel higher order BoR basis functions are constructed whereas the body's generatrix is discretized by higher degree spline curves.

### 2.2 MoM formulation

The problem depicted in Fig. 2.1 is formulated, in this section, with the help of mixed potential integral equations that are further discretized by MoM. A new formulation, finally stated in (2.66) in matrix form, is derived in this section. This new approach removes the drawbacks noted in [24], [25], [26], hence permitting the use of same (frequency independent) basis functions at the ports and at the conducting surfaces, thus yielding an algorithm independent on the number/type of waveguide modes.

Fig. 2.1 shows he geometry of a general multiport structure. An arbitrarily shaped cavity containing a number of prerfect electric conductors (PEC) and homogeneous dielectric bodies is fed by $n$ waveguides through $n$ apertures designated as $S_{p, 1}, S_{p, 2}, \ldots, S_{p, n}$. The cavity may be closed or open, that is, it may have an aperture through which it can radiate into the free space.


Figure 2.1: Multiport structure, original problem.

The equivalence principle $[50,51,52,53]$ will be first used at the port apertures. Accordingly, the structure under investigation is divided into two regions [26],[25]: Regions $I$ and $I I$, as shown in Fig. 2.2. Region $I$ is further divided in $n$ sub-


Figure 2.2: Multiport structure, equivalent problem.
regions corresponding to the $n$ waveguides connected to the multiport structure. Region $I$ and region $I I$ are separated by a (fictitious) infinitely thin perfectly electric conductor (PEC) of surface $S_{p}=S_{p, 1}+S_{p, 2}+\ldots+S_{p, n}$. In order to maintain the original problem, magnetic current densities $\mathbf{M}_{P}=\mathbf{E} \times \hat{n}$, of equal magnitude and opposite sign are introduced on both sides of the surface $S_{p}$, which restores the continuity of the tangential electric field on the surface $S_{p}$. Here, $\hat{n}$ is the ports unit normal vector directed from region $I$ towards region $I I$, and $\mathbf{M}_{P}=$ $\left\{\mathbf{M}_{P, i}, i=1,2, \ldots, n\right\}$.

The continuity of the tangential magnetic field on the surface $S_{p}$ reads

$$
\begin{equation*}
\mathbf{H}_{\mathrm{tan}}^{I}=\mathbf{H}_{\mathrm{tan}}^{I I} \tag{2.1}
\end{equation*}
$$

where $\mathbf{H}_{\mathrm{tan}}^{I}$ is the tangential magnetic field on the region $I$ side of $S_{p}$, and $\mathbf{H}_{\mathrm{tan}}^{I I}$ represents the tangential magnetic field on the region $I I$ side of $S_{p}$. Moreover

$$
\begin{equation*}
\mathbf{H}_{\mathrm{tan}}^{i n c}+\mathbf{H}_{\mathrm{tan}}^{I}\left(-\mathbf{M}_{p}\right)=\mathbf{H}_{\mathrm{tan}}^{I I}\left(\mathbf{M}_{P}, \mathbf{J}, \mathbf{M}\right), \tag{2.2}
\end{equation*}
$$

where $\mathbf{H}_{\mathrm{tan}}^{\text {inc }}, \mathbf{H}_{\mathrm{tan}}^{I}\left(-\mathbf{M}_{p}\right)$ and $\mathbf{H}_{\mathrm{tan}}^{I I}\left(\mathbf{M}_{P}, \mathbf{J}, \mathbf{M}\right)$ are the incident magnetic field tangential to $S_{p}$ in region $I$, the linear operator for the scattered tangential magnetic field in region $I$ (the field generated by the magnetic current density $-\mathbf{M}_{p}$ in region $I$ ) and the linear operator for the scattered tangential magnetic field in region $I I$
(the field generated by the magnetic current density $\mathbf{M}_{P}$, electric current density $\mathbf{J}$ and magnetic current density $\mathbf{M}$ in region $I I$ ), respectively. Furthermore, we have

$$
\begin{equation*}
\mathbf{H}_{\mathrm{tan}}^{i n c}=\mathbf{H}_{\mathrm{tan}}^{I}\left(\mathbf{M}_{P}\right)+\mathbf{H}_{\mathrm{tan}}^{I I}\left(\mathbf{M}_{P}, \mathbf{J}, \mathbf{M}\right) \tag{2.3}
\end{equation*}
$$

since $H^{I}$ and $H^{I I}$ are linear operators.
The magnetic current density $\mathbf{M}_{P}$ is approximated by set of $N_{P}$ linearly independent basis functions $\mathbf{M}_{P}=\left\{\mathbf{M}_{P, i}, i=0 \ldots N_{P}\right\}$

$$
\begin{equation*}
\mathbf{M}_{P}=\sum_{i=1}^{N_{P}} v_{i} \mathbf{M}_{P, i} \tag{2.4}
\end{equation*}
$$

where $v_{i}$ are unknown expansion coefficients for the magnetic current density on the surface $S_{p}=S_{p, 1}+S_{p, 2}+\ldots+S_{p, n}$.

Introducing a set of $N_{P}$ linearly independent test functions $\left\{\mathbf{T}_{i}, i=1 \ldots N_{P}\right\}$ and taking the inner product of equation (2.3), while considering the expansion (2.4), yields:

$$
\begin{equation*}
\left\langle\mathbf{T}_{i}, \mathbf{H}_{\mathrm{tan}}^{i n c}\right\rangle=\left\langle\mathbf{T}_{i}, \sum_{i=1}^{N_{P}} v_{j} \mathbf{H}_{\mathrm{tan}}^{I}\left(\mathbf{M}_{P, j}\right)\right\rangle+\left\langle\mathbf{T}_{i}, \sum_{i=1}^{N_{P}} v_{j} \mathbf{H}_{\mathrm{tan}}^{I I}\left(\mathbf{M}_{P, j}, \mathbf{J}, \mathbf{M}\right)\right\rangle \tag{2.5}
\end{equation*}
$$

or in matrix notation:

$$
\begin{equation*}
\left[I^{i n c}\right]=\left(\left[Y^{I}\right]+\left[Y^{I I}\right]\right)[V] \tag{2.6}
\end{equation*}
$$

where

$$
\begin{align*}
{\left[I^{i n c}\right]_{i} } & =\left\langle\mathbf{T}_{i}, \mathbf{H}^{i n c}\right\rangle  \tag{2.7a}\\
{\left[Y^{I}\right]_{i j} } & =\left\langle\mathbf{T}_{i}, \mathbf{H}_{\tan }^{I}\left(\mathbf{M}_{P, j}\right)\right\rangle  \tag{2.7b}\\
{\left[Y^{I I}\right]_{i j} } & =\left\langle\mathbf{T}_{i}, \mathbf{H}_{\tan }^{I I}\left(\mathbf{M}_{P, j}, \mathbf{J}, \mathbf{M}\right)\right\rangle  \tag{2.7c}\\
{[V]_{i} } & =v_{i} \tag{2.7~d}
\end{align*}
$$

The computation of the excitation vector $\left[I^{i n c}\right]$ and admittance matrices $\left[Y^{I}\right]$ and $\left[Y^{I I}\right]$ will be detailed in the next sections. Once $\left[I^{i n c}\right],\left[Y^{I}\right]$ and $\left[Y^{I I}\right]$ are known, the system (2.6) can be solved for $[V]$, and the magnetic current densities $\mathbf{M}_{P}$ at the ports are found by replacing the coefficients $v_{i}$ back into (2.4).

### 2.2.1 Computation of the admittance matrix of region $I$

The transversal electric and magnetic fields, in an $z$-directed infinitely long waveguide, are [54]

$$
\begin{align*}
\mathbf{E}_{\mathrm{tan}}(\mathbf{r}) & =\sum_{i} c_{i}^{+} e^{-i \beta z} \mathbf{e}_{i}(\mathbf{r})+\sum_{i} c_{i}^{-} e^{i \beta z} \mathbf{e}_{i}(\mathbf{r}), \\
\mathbf{H}_{\mathrm{tan}}(\mathbf{r}) & =\sum_{i} Y_{i} c_{i}^{+} e^{-i \beta z} \cdot \hat{z} \times \mathbf{e}_{i}(\mathbf{r})-\sum_{i} Y_{i} c_{i}^{-} e^{i \beta z} \cdot \hat{z} \times \mathbf{e}_{i}(\mathbf{r}), \tag{2.8}
\end{align*}
$$

where $\mathbf{r}, c_{i}^{+}, c_{i}^{-}, Y_{i}, \mathbf{e}_{i} \hat{z}, i, \beta$ are the position vector, the amplitude of the forward wave, the amplitude of the reflected wave, the characteristic modal admittance, waveguide eigenvectors, unit vector pointing in the positive direction of axis $z$, imaginary number and the waveguide propagation constant, respectively. The eigenvectors are presumed to satisfy the orthonormality relation

$$
\begin{equation*}
\int_{S_{p}} \mathbf{e}_{m}(\mathbf{r}) \mathbf{e}_{n}(\mathbf{r}) d S=\delta_{m n} \tag{2.9}
\end{equation*}
$$

with $\delta_{m n}$ being the Kronecker delta function.

Without loss of generality one can consider that the port surfaces $S_{p}$ are placed at the $z=0$ plane. Let us consider a forward (towards $S_{p}$ ) propagating modal field in the form

$$
\begin{align*}
\mathbf{E}_{\mathrm{tan}} & =\sum_{i} a_{i}^{i n c} e^{-i \beta z} \mathbf{e}_{i},  \tag{2.10}\\
\mathbf{H}_{\mathrm{tan}} & =\sum_{i} Y_{i} a_{i}^{i n c} e^{-i \beta z} \cdot \hat{z} \times \mathbf{e}_{i} . \tag{2.11}
\end{align*}
$$

As $S_{p}$ are perfect conducting surfaces, the fields are totally reflected, thus giving

$$
\begin{align*}
\mathbf{E}_{\tan }^{i n c}\left(\mathbf{r} \in S_{p}\right) & =0,  \tag{2.12}\\
\mathbf{H}_{\tan }^{i n c}\left(\mathbf{r} \in S_{p}\right) & =2 \sum_{i} Y_{i} a_{i}^{i n c} \cdot \hat{z} \times \mathbf{e}_{i} . \tag{2.13}
\end{align*}
$$

Expanding $\mathbf{M}_{P}$ in a modal series

$$
\begin{equation*}
\mathbf{M}_{P}=\sum_{i} a_{i}^{-} \mathbf{e}_{i} \tag{2.14}
\end{equation*}
$$

yields the total tangential fields at region $I$ side of the ports

$$
\begin{align*}
\mathbf{E}_{\tan }^{I}\left(\mathbf{r} \in S_{p}\right) & =\sum_{i} a_{i}^{-} \mathbf{e}_{i},  \tag{2.15}\\
\mathbf{H}_{\tan }^{I}\left(\mathbf{r} \in S_{p}\right) & =2 \sum_{i} Y_{i} a_{i}^{i n c} \cdot \hat{z} \times \mathbf{e}_{i}-\sum_{i} Y_{i} a_{i}^{-} \cdot \hat{z} \times \mathbf{e}_{i}, \tag{2.16}
\end{align*}
$$

where $a_{i}^{-}$are unknown amplitudes to be determined. From (2.15) we get

$$
\begin{equation*}
\mathbf{M}^{p}=\mathbf{E}^{I} \times \hat{n}=\sum_{i} a_{i}^{-} \mathbf{e}_{i} \times \hat{n} . \tag{2.17}
\end{equation*}
$$

The above relation is multiplied with $\mathbf{e}_{j} \times \hat{n}$ and integrated over $S_{p}$, yielding

$$
\begin{equation*}
\sum_{k} v_{k} \int_{S_{p}} \mathbf{M}_{P, k} \cdot \mathbf{e}_{i} \times \hat{n} \cdot d S=\sum_{i} a_{i}^{-} \int_{S_{p}} \mathbf{e}_{i} \times \hat{n} \cdot \mathbf{e}_{j} \times \hat{n} \cdot d S \tag{2.18}
\end{equation*}
$$

Due to the orthonormality of the eigenvectors (2.9), the amplitudes $a_{i}^{-}$can be extracted from the previous equation, giving

$$
\begin{equation*}
a_{i}^{-}=\sum_{j} v_{j} \underbrace{\int_{S_{p}} \mathbf{e}_{i} \times \hat{n} \cdot \mathbf{M}_{P, j} \cdot d S}_{A_{i j}}=\sum_{j} v_{j} A_{i j} . \tag{2.19}
\end{equation*}
$$

Replacing (2.19) in the expression of the transverse magnetic field from (2.16) yields

$$
\begin{equation*}
\mathbf{H}^{I}=\underbrace{2 \sum_{i} Y_{i} a_{i}^{i n c} \cdot \hat{n} \times \mathbf{e}_{i}}_{\mathbf{H}^{n n c}}+\underbrace{\sum_{i} Y_{i} \cdot \hat{n} \times \mathbf{e}_{i} \sum_{j} v_{j} \int_{S_{p}} \hat{n} \times \mathbf{e}_{i} \cdot \mathbf{M}_{P, j} \cdot d S}_{\mathbf{H}^{I}\left(\mathbf{M}_{P}\right)} . \tag{2.20}
\end{equation*}
$$

Testing the above equation with the functions $\left\{\mathbf{T}_{k}\right\}$ (see previous section), we get

$$
\begin{align*}
\left\langle\mathbf{T}_{k}, \mathbf{H}^{I}\right\rangle & =2 \sum_{i} Y_{i} a_{i}^{i n c}\left\langle\mathbf{T}_{k}, \hat{n} \times \mathbf{e}_{i}\right\rangle \\
& +\sum_{i} Y_{i}\left\langle\mathbf{T}_{k}, \hat{n} \times \mathbf{e}_{i}\right\rangle \sum_{j} v_{j} \int_{S_{p}} \hat{n} \times \mathbf{e}_{i} \cdot \mathbf{M}_{P, j} \cdot d S . \tag{2.21}
\end{align*}
$$

Finally, the expressions for the waveguide region admittance matrix and modal excitation matrix are, respectively

$$
\begin{align*}
{\left[Y^{I}\right] } & =[A]\left[Y^{W}\right][B],  \tag{2.22}\\
{\left[I^{\text {inc }}\right] } & =2 a_{i}^{\text {inc }}[A]\left[Y^{W}\right], \tag{2.23}
\end{align*}
$$

where

$$
\begin{align*}
{[A]_{k i} } & =\left\langle\mathbf{T}_{k}, \hat{n} \times \mathbf{e}_{i}\right\rangle,  \tag{2.24}\\
{[B]_{i n} } & =\left\langle\mathbf{M}_{P, n}, \hat{n} \times \mathbf{e}_{i}\right\rangle,  \tag{2.25}\\
{\left[Y^{W}\right] } & =\operatorname{diag}\left[Y_{i}\right] . \tag{2.26}
\end{align*}
$$

### 2.2.2 Computation of the admittance matrix of region $I I$

Until now, the equivalent problem at the ports has been formulated in terms of equivalent magnetic sources, which preserve the continuity of the tangential electric field at port apertures, while the tangential magnetic field continuity on $S_{p}$ has been enforced through the boundary equation (2.3).

The surface equivalence principle is further used to describe the homogeneous dielectrics in region $I I$. Fig. 2.3(a) depicts a homogeneous dielectric body. Region 1 represents the free space of permittivity $\varepsilon_{0}$ and permeability $\mu_{0}$, while region 2 denotes the interior of the body characterized by the relative permittivity $\varepsilon_{r}$ and relative permeability $\mu_{r}$. The fields in region 1 are $\mathbf{E}_{1}, \mathbf{H}_{1}$, whereas $\mathbf{E}_{2}, \mathbf{H}_{2}$ denote the fields throughout region 2 . Using the surface equivalence principle, the original problem depicted in Fig. 2.3(a) will be replaced by equivalent sources that replicate the original fields in both regions.


Figure 2.3: The equivalence principle.

The exterior problem, as shown in Fig. 2.3(b), is constructed by defining electric and magnetic sources

$$
\begin{align*}
\mathbf{J}_{1} & =\hat{n} \times \mathbf{H}_{1},  \tag{2.27}\\
\mathbf{M}_{1} & =\mathbf{E}_{1} \times \hat{n}, \tag{2.28}
\end{align*}
$$

and placing them, just outside, on the bounding surface of the body. Here $\hat{n}$ is the surface outward normal unit vector. These sources produce the fields $\mathbf{E}_{1}, \mathbf{H}_{1}$ throughout region 1 and null fields in region 2. Because null fields are produced in region 2 , medium 2 can be filled with the same material $\left(\varepsilon_{0}, \mu_{0}\right)$ as region 1 without changing $\mathbf{E}_{1}$ and $\mathbf{H}_{1}$. Thus, the sources $\mathbf{J}_{1}$ and $\mathbf{M}_{1}$, radiating in free space, replicate the fields throughout region 1, or, the exterior part of the original problem in Fig. 2.3(a) has been replaced by the equivalent problem depicted in 2.3(b).

A second equivalence is now needed to describe the interior problem. Again,
equivalent sources are introduced as

$$
\begin{align*}
\mathbf{J}_{2} & =(-\hat{n}) \times \mathbf{H}_{2},  \tag{2.29}\\
\mathbf{M}_{2} & =\mathbf{E}_{2} \times(-\hat{n}), \tag{2.30}
\end{align*}
$$

and placed, just inside, on the boundary of the original scatterer, as shown in Fig. 2.3(c). $\mathbf{J}_{2}$ and $\mathbf{M}_{2}$ replicate the fields in region 2 and produce zero fields in region 1 , hence, region 1 can be filled with the constitutive material of medium 2 , without perturbing $\mathbf{E}_{2}$ and $\mathbf{H}_{2}$. Thus, the newly introduced sources radiate in the infinite homogeneous space of parameters $\varepsilon_{r} \varepsilon_{0}, \mu_{r} \mu_{0}$.

The tangential continuity of the electric and magnetic fields at the interface between the dielectric and the free space reads

$$
\begin{equation*}
\mathbf{J}_{2}=-\mathbf{J}_{1} \tag{2.31}
\end{equation*}
$$

and

$$
\begin{equation*}
\mathbf{M}_{2}=-\mathbf{M}_{1} \tag{2.32}
\end{equation*}
$$

which brings us to the situation in Fig. 2.2, where electric and magnetic current densities of equal magnitude and opposite sign are used to replace the dielectrics in region $I I$.

In the case of perfectly conducting scatterers the boundary conditions read

$$
\begin{equation*}
\mathbf{E}_{\mathrm{tan}}=\hat{n} \times \mathbf{E}=0 \tag{2.33}
\end{equation*}
$$

on its surface, yielding

$$
\begin{align*}
\mathbf{J}_{1} & =\hat{n} \times \mathbf{H}_{1},  \tag{2.34}\\
\mathbf{M}_{1} & =0 \tag{2.35}
\end{align*}
$$

for the exterior problem, while no interior problem exists due to the impenetrability of a perfectly conducting body.

Returning to the case of Fig. 2.2, on each dielectric surface $S_{i}$ we have

$$
\begin{align*}
\hat{n} \times\left[\mathbf{E}^{I I}(\mathbf{J})+\mathbf{E}^{I I}(\mathbf{M})+\mathbf{E}^{I I}\left(\mathbf{M}_{P}\right)\right] & =-\mathbf{M},  \tag{2.36}\\
-\hat{n} \times\left[-\mathbf{E}^{i n}(\mathbf{J})-\mathbf{E}^{i n}(\mathbf{M})\right] & =\mathbf{M},  \tag{2.37}\\
\hat{n} \times\left[\mathbf{H}^{I I}(\mathbf{J})+\mathbf{H}^{I I}(\mathbf{M})+\mathbf{H}^{I I}\left(\mathbf{M}_{P}\right)\right] & =\mathbf{J},  \tag{2.38}\\
-\hat{n} \times\left[-\mathbf{H}^{i n}(\mathbf{J})-\mathbf{H}^{i n}(\mathbf{M})\right] & =-\mathbf{J}, \tag{2.39}
\end{align*}
$$

where $\mathbf{E}^{I I}, \mathbf{H}^{I I}$ and $\mathbf{E}^{i n}, \mathbf{H}^{i n}$ represent the field operators in region $I I$ and interior dielectric regions, respectively.

Different MoM formulations are found by choosing different combinations of
(2.36) - (2.39) :

$$
\begin{array}{r}
\text { PMCHWT: }\left\{\begin{array}{l}
(2.36)+(2.37) \\
(2.38)+(2.39)
\end{array}\right. \\
\text { Mueller: }\left\{\begin{array}{l}
(2.36)-\varepsilon_{r}(2.37) \\
(2.38)-\mu_{r}(2.39)
\end{array}\right. \\
\text { CFIE: }\left\{\begin{array}{l}
\alpha(2.36)+(1-\alpha)(2.38) \\
\alpha(2.37)+(1-\alpha)(2.39)
\end{array}\right. \\
\text { EFIE: }\left\{\begin{array}{l}
(2.36) \\
(2.37)
\end{array}\right. \\
\text { MFIE: }\left\{\begin{array}{l}
(2.38) \\
(2.39)
\end{array}\right. \tag{2.44}
\end{array}
$$

The PMCHWT formulation [21], [22] is named after Poggio, Miller, Chew, Harrington, Wu and Tsai, who were among the first to investigate it. Both PMCHWT and Mueller [55] formulations combine interior and exterior equations. The combined field integral equation (CFIE) represents linear combinations of the electric field integral equation (EFIE) and magnetic field integral equation (MFIE). The combinations are performed separately for the exterior and interior regions and the parameter $\alpha \in[0,1]$ is chosen a priori. One easily remarks that the CFIE degenerates into EFIE or MFIE if $\alpha=1$ or $\alpha=0$, respectively. Using MoM to solve the EFIE or MFIE formulations is known to be prone to internal resonance problems [16], [28], [6], i.e. the algorithm breaks in the vicinity of the internal resonant frequencies of the conducting cavity formed by the boundary surface of the dielectric. All other formulations (PMCHWT, Mueller and CFIE) are resonance-free when discretized by MoM. When the CFIE is discretized with identical basis functions for both electric and magnetic sources, special testing procedures must be carried out [6], in order to catch the singular behavior of the integral operators, yielding an undesired feature: Line charges are introduced and line integrals must be evaluated to account for these charges. PMCHWT and Mueller formulations do no exhibit this drawback. PMCHWT is more attractive than Mueller in terms of practical implementation [23], when direct linear solvers are used, while a variation of the Mueller formulation, called N -Mueller [56], leads to a well-conditioned matrix equation suitable for iterative solving.

The PMCHWT formulation shall be further considered for dielectrics, since di-
rect equation solvers are used here. Thus, from (2.36) - (2.39) and (2.40), one gets

$$
\begin{align*}
{\left[\left(\mathbf{E}^{I I}+\mathbf{E}^{i n}\right)(\mathbf{J})+\left(\mathbf{E}^{I I}+\mathbf{E}^{i n}\right)(\mathbf{M})+\mathbf{E}^{I I}\left(\mathbf{M}_{P}\right)\right]_{\tan } } & =0  \tag{2.45a}\\
{\left[\left(\mathbf{H}^{I I}+\mathbf{H}^{i n}\right)(\mathbf{J})+\left(\mathbf{H}^{I I}+\mathbf{H}^{i n}\right)(\mathbf{M})+\mathbf{H}^{I I}\left(\mathbf{M}_{P}\right)\right]_{\tan } } & =0 \tag{2.45b}
\end{align*}
$$

Back to Fig. 2.2, in case of conducting bodies, the boundary conditions read

$$
\begin{array}{rll}
{\left[\mathbf{E}^{I I}(\mathbf{J})+\mathbf{E}^{I I}(\mathbf{M})+\mathbf{E}^{I I}\left(\mathbf{M}_{P}\right)\right]_{\mathrm{tan}}} & = \begin{cases}\hat{n} \times \mathbf{M}_{P} & \text { on port } \\
0 & \text { otherwise }\end{cases} & \text { EFIE } \\
{\left[\mathbf{H}^{I I}(\mathbf{J})+\mathbf{H}^{I I}(\mathbf{M})+\mathbf{H}^{I I}\left(\mathbf{M}_{P}\right)\right]_{\tan }} & =\mathbf{J} \times \hat{n} & \mathrm{MFIE} \tag{2.47}
\end{array}
$$

Only one of the equations $(2.46),(2.47)$ is sufficient to model the conducing parts of the structure, or a linear combination in form of $\alpha$ EFIE $+(1-\alpha)$ MFIE may be employed to yield the CFIE. When MFIE is enforced on open conducting surfaces, line charges accumulate on the boundary of the surface. Line integrals must be then calculated to account for these charges, rendering the MFIE an unsuitable choice. Since MFIE is part of CFIE, neither of these two formulations shall be used when dealing with open conducting bodies. EFIE does not exhibit this limitation, hence it can be safely considered for both open and closed conductors.

Summarizing the equations for region $I I$, we have

$$
\begin{equation*}
\left[\left(\mathbf{E}^{I I}+\mathbf{E}^{i n}\right)(\mathbf{J})+\left(\mathbf{E}^{I I}+\mathbf{E}^{i n}\right)(\mathbf{M})+\mathbf{E}^{I I}\left(\mathbf{M}_{P}\right)\right]_{\tan }=0 \quad \text { on dielectric, } \tag{2.48a}
\end{equation*}
$$

$$
\begin{equation*}
\left[\left(\mathbf{H}^{I I}+\mathbf{H}^{i n}\right)(\mathbf{J})+\left(\mathbf{H}^{I I}+\mathbf{H}^{i n}\right)(\mathbf{M})+\mathbf{H}^{I I}\left(\mathbf{M}_{P}\right)\right]_{\tan }=0 \quad \text { on dielectric, } \tag{2.48b}
\end{equation*}
$$

$$
\begin{equation*}
\left[\mathbf{E}^{I I}(\mathbf{J})+\mathbf{E}^{I I}(\mathbf{M})+\mathbf{E}^{I I}\left(\mathbf{M}_{P}\right)\right]_{\tan }=0 \quad \text { on PEC } \tag{2.48c}
\end{equation*}
$$

$$
\begin{equation*}
\left[\mathbf{E}^{I I}(\mathbf{J})+\mathbf{E}^{I I}(\mathbf{M})+\mathbf{E}^{I I}\left(\mathbf{M}_{P}\right)\right]_{\tan }=\hat{n} \times \mathbf{M}_{P} \quad \text { on port. } \tag{2.48d}
\end{equation*}
$$

The scattered fields $\mathbf{E}(\mathbf{J}, \mathbf{M})$ and $\mathbf{H}(\mathbf{J}, \mathbf{M})$ can be convenientely expressed as functions of two linear operators $\mathcal{L}$ and $\mathcal{K}[6]$

$$
\begin{align*}
\mathbf{E}(\mathbf{J}) & =-\eta_{\beta} \mathcal{L}^{\beta}(\mathbf{J})  \tag{2.49a}\\
\mathbf{E}(\mathbf{M}) & =-\mathcal{K}^{\beta}(\mathbf{M})  \tag{2.49b}\\
\mathbf{H}(\mathbf{J}) & =\mathcal{K}^{\beta}(\mathbf{J})  \tag{2.49c}\\
\mathbf{H}(\mathbf{M}) & =-\frac{1}{\eta_{\beta}} \mathcal{L}^{\beta}(\mathbf{M}), \tag{2.49~d}
\end{align*}
$$

with

$$
\begin{align*}
\mathcal{L}^{\beta}(S ; \mathbf{x}) & =i k_{\beta} \iint_{S} G_{\beta}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) \mathbf{x}\left(\mathbf{r}^{\prime}\right) d S^{\prime}-\frac{\nabla}{i k_{\beta}} \iint_{S} G_{\beta}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) \nabla^{\prime} \mathbf{x}\left(\mathbf{r}^{\prime}\right) d S^{\prime}  \tag{2.50a}\\
\mathcal{K}^{\beta}(S ; \mathbf{x}) & =\nabla \times \iint_{S} G_{\beta}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) \mathbf{x}\left(\mathbf{r}^{\prime}\right) d S^{\prime}=\iint_{S} \nabla G_{\beta}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) \times \mathbf{x}\left(\mathbf{r}^{\prime}\right) d S^{\prime} \tag{2.50b}
\end{align*}
$$

where

$$
\begin{equation*}
G_{\beta}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)=\frac{e^{-i k_{\beta}\left|\mathbf{r}-\mathbf{r}^{\prime}\right|}}{4 \pi\left|\mathbf{r}-\mathbf{r}^{\prime}\right|} \tag{2.51}
\end{equation*}
$$

is the free space Green function, $\beta$ denotes the homogeneous unbounded medium where the operators are evaluated, $\eta_{\beta}$ is the characteristic impedance of the respective medium and $\mathbf{r}, \mathbf{r}^{\prime}$ are the observation and source points, respectively. With these considerations, (2.48) becomes

$$
\begin{array}{r}
{\left[\left(\eta_{I I} \mathcal{L}^{I I}+\eta_{i n} \mathcal{L}^{i n}\right)(\mathbf{J})+\left(\mathcal{K}^{I I}+\mathcal{K}^{i n}\right)(\mathbf{M})+\mathcal{K}^{I I}\left(\mathbf{M}_{P}\right)\right]_{\tan }=0} \\
{\left[-\left(\mathcal{K}^{I I}+\mathcal{K}^{i n}\right)(\mathbf{J})+\left(\frac{1}{\eta_{I I}} \mathcal{L}^{I I}+\frac{1}{\eta_{i n}} \mathcal{L}^{i n}\right)(\mathbf{M})+\frac{1}{\eta_{I I}} \mathcal{L}^{I I}\left(\mathbf{M}_{P}\right)\right]_{\tan }=0} \\
{\left[\eta_{I I} \mathcal{L}^{I I}(\mathbf{J})+\mathcal{K}^{I I}(\mathbf{M})+\mathcal{K}^{I I}\left(\mathbf{M}_{P}\right)\right]_{\tan }=\left\{\begin{array}{cc}
-\hat{n} \times \mathbf{M}_{P} & \text { on port } \\
0 & \text { on PEC }
\end{array}\right.} \tag{2.54}
\end{array}
$$

Making the notation

$$
\mathcal{K}_{\tan }^{I I}\left(\mathbf{M}_{P}\right) \equiv\left\{\begin{array}{cc}
\mathcal{K}_{\tan }^{I I}\left(\mathbf{M}_{P}\right)+\hat{n} \times \mathbf{M}_{P} & \text { on port }  \tag{2.55}\\
\mathcal{K}_{\tan }^{I I}\left(\mathbf{M}_{P}\right) & \text { otherwise }
\end{array}\right.
$$

i.e. the port boundary condition in the in the EFIE (2.54) is incorporated in the $\mathcal{K}^{I I}\left(\mathbf{M}_{P}\right)$ operator. Equation $(2.54)$ can be considered a special case of $(2.52)$, with vanishing $\mathbf{M}, \mathcal{L}^{i n}$ and $\mathcal{K}^{i n}$. Therefore the EFIE will not be explicitely stated.

Similarly to the port expansion (2.4), the electric and magnetic current densities in region $I I$ are approximated by linearly independent basis functions $\left\{\mathbf{J}_{i}, i=\right.$ $\left.1 \ldots N_{J}\right\}$ and $\left\{\mathbf{M}_{i}, i=1 \ldots N_{M}\right\}$, respectively. Thus

$$
\begin{align*}
\mathbf{J} & =\sum_{i=1}^{N_{J}} u_{i} \mathbf{J}_{i}  \tag{2.56}\\
\mathbf{M} & =\sum_{i=1}^{N_{M}} w_{i} \mathbf{M}_{i} \tag{2.57}
\end{align*}
$$

The testing procedure of $(2.52),(2.53)$ will be carried out with the same basis functions used in (2.56) and (2.57). The use of identical test and expansion functions is usually referred as Galerkin testing.

Taking the inner products of (2.52) with $\left\{\mathbf{J}_{i}, i=1 \ldots N_{J}\right\}$, and of (2.53) with $\left\{\mathbf{M}_{i}, i=1 \ldots N_{M}\right\}$, yields

$$
\begin{align*}
& \left\langle\mathbf{J}_{m}, \sum_{n=1}^{N_{J}}\left(\eta_{I I} \mathcal{L}^{I I}+\eta_{i n} \mathcal{L}^{i n}\right)\left(\mathbf{J}_{n}\right)\right\rangle \\
+ & \left\langle\mathbf{J}_{m}, \sum_{n=1}^{N_{M}}\left(\mathcal{K}^{I I}+\mathcal{K}^{i n}\right)\left(\mathbf{M}_{n}\right)\right\rangle \\
+ & \left\langle\mathbf{J}_{m}, \sum_{n=1}^{N_{P}} \mathcal{K}^{I I}\left(\mathbf{M}_{P, n}\right)\right\rangle=0,  \tag{2.58a}\\
& -\left\langle\mathbf{M}_{m}, \sum_{n=1}^{N_{J}}\left(\mathcal{K}^{I I}+\mathcal{K}^{i n}\right)\left(\mathbf{J}_{n}\right)\right\rangle \\
& +\left\langle\mathbf{M}_{m}, \sum_{n=1}^{N_{M}}\left(\frac{1}{\eta_{I I}} \mathcal{L}^{I I}+\frac{1}{\eta_{i n}} \mathcal{L}^{i n}\right)\left(\mathbf{M}_{n}\right)\right\rangle \\
& +\left\langle\mathbf{M}_{m}, \sum_{n=1}^{N_{P}} \frac{1}{\eta_{I I}} \mathcal{L}^{I I}\left(\mathbf{M}_{P, n}\right)\right\rangle=0, \tag{2.58b}
\end{align*}
$$

In matrix notation:

$$
\begin{align*}
{\left[R^{J J}\right][U]+\left[R^{J M}\right][W]+\left[P^{J M}\right][V] } & =0,  \tag{2.59}\\
{\left[R^{M J}\right][U]+\left[R^{M M}\right][W]+\left[P^{M M}\right][V] } & =0 \tag{2.60}
\end{align*}
$$

or, in a more compact form

$$
\underbrace{\left[\begin{array}{ll}
{\left[R^{J J}\right]} & {\left[R^{J M}\right]}  \tag{2.61}\\
{\left[R^{M J}\right]} & {\left[R^{M M}\right]}
\end{array}\right]}_{[R]} \underbrace{\left[\begin{array}{c}
{[U]} \\
{[W]}
\end{array}\right]}_{[X]}+\underbrace{\left[\begin{array}{c}
{\left[P^{J M}\right]} \\
{\left[P^{M M}\right]}
\end{array}\right]}_{[P]}[V]=0 .
$$

The matrix equation

$$
\begin{equation*}
[R][X]+[P][V]=0 \tag{2.62}
\end{equation*}
$$

represents the coupling between the unknown electric and magnetic current densities in region $I I$ (matrix $[X]$ ) and the unknown port current density (matrix $[V]$ ). Moreover

$$
\begin{equation*}
[X]=-[R]^{-1}[P][V] \tag{2.63}
\end{equation*}
$$

The inner product matrices are given by

$$
\begin{align*}
{\left[R^{J J}\right]_{m n} } & =\left\langle\mathbf{J}_{m},\left(\eta_{I I} \mathcal{L}^{I I}+\eta_{i n} \mathcal{L}^{i n}\right)\left(\mathbf{J}_{n}\right)\right\rangle  \tag{2.64a}\\
{\left[R^{J M}\right]_{m n} } & =\left\langle\mathbf{J}_{m},\left(\mathcal{K}^{I I}+\mathcal{K}^{i n}\right)\left(\mathbf{M}_{n}\right)\right\rangle  \tag{2.64b}\\
{\left[R^{M J}\right]_{m n} } & =\left\langle\mathbf{M}_{m},\left(-\mathcal{K}^{I I}-\mathcal{K}^{i n}\right)\left(\mathbf{J}_{n}\right)\right\rangle  \tag{2.64c}\\
{\left[R^{M M}\right]_{m n} } & =\left\langle\mathbf{M}_{m},\left(\frac{1}{\eta_{I I}} \mathcal{L}^{I I}+\frac{1}{\eta_{i n}} \mathcal{L}^{i n}\right)\left(\mathbf{M}_{n}\right)\right\rangle  \tag{2.64~d}\\
{\left[P^{J M}\right]_{m n} } & =\left\langle\mathbf{J}_{m}, \mathcal{K}^{I I}\left(\mathbf{M}_{P, n}\right)\right\rangle,  \tag{2.64e}\\
{\left[P^{M M}\right]_{m n} } & =\left\langle\mathbf{M}_{m}, \frac{1}{\eta_{I I}} \mathcal{L}^{I I}\left(\mathbf{M}_{P, n}\right)\right\rangle  \tag{2.64f}\\
{[U]_{m} } & =u_{m}  \tag{2.64~g}\\
{[W]_{m} } & =w_{m} \tag{2.64h}
\end{align*}
$$

The total magnetic field in region $I I$ is

$$
H^{I I}\left(\mathbf{M}_{P}, \mathbf{J}, \mathbf{M}\right)=H^{I I}\left(\mathbf{M}_{P}\right)+H^{I I}(\mathbf{J})+H^{I I}(\mathbf{M})
$$

Testing the previous equation with the same set of basis $\left\{\mathbf{T}_{i}, i=1 \ldots N_{P}\right\}$ as (2.5), gives

$$
\begin{equation*}
\left\langle\mathbf{T}_{i}, \mathbf{H}^{I I}\left(\mathbf{M}_{P}, \mathbf{J}, \mathbf{M}\right)\right\rangle=\left\langle\mathbf{T}_{i}, \mathbf{H}^{I I}\left(\mathbf{M}_{P}\right)\right\rangle+\left\langle\mathbf{T}_{i}, \mathbf{H}^{I I}(\mathbf{J})\right\rangle+\left\langle\mathbf{T}_{i}, \mathbf{H}^{I I}(\mathbf{M})\right\rangle . \tag{2.65}
\end{equation*}
$$

Observing that $\left\langle\mathbf{T}_{i}, \mathbf{H}^{I I}\left(\mathbf{M}_{P}, \mathbf{J}, \mathbf{M}\right)\right\rangle=\left[Y^{I I}\right][V]$, the following matrix equation holds

$$
\left[Y^{I I}\right][V]=\left[F^{M M}\right][V]+\underbrace{\left[\left[G^{M J}\right]\left[G^{M M}\right]\right]}_{[G]} \underbrace{\left[\begin{array}{c}
{[U]} \\
{[W]}
\end{array}\right]}_{[X]} .
$$

Replacing $[X]$ from (2.63) and removing $[V]$, we get the final expression for $\left[Y^{I I}\right]$

$$
\begin{equation*}
\left[Y^{I I}\right]=\left[F^{M M}\right]-[G][R]^{-1}[P] \tag{2.66}
\end{equation*}
$$

where the involved matrices are

$$
\begin{align*}
{\left[F^{M M}\right]_{m n} } & =\left\langle\mathbf{T}_{m}, \mathbf{H}^{I I}\left(\mathbf{M}_{P, n}\right)\right\rangle=-\frac{1}{\eta_{I I}}\left\langle\mathbf{T}_{m}, \mathcal{L}^{I I}\left(\mathbf{M}_{P, n}\right)\right\rangle  \tag{2.67a}\\
{\left[G^{M J}\right]_{m n} } & =\left\langle\mathbf{T}_{m}, \mathbf{H}^{I I}\left(\mathbf{J}_{n}\right)\right\rangle=\left\langle\mathbf{T}_{m}, \mathcal{K}^{I I}\left(\mathbf{J}_{n}\right)\right\rangle  \tag{2.67b}\\
{\left[G^{M M}\right]_{m n} } & =\left\langle\mathbf{T}_{m}, \mathbf{H}^{I I}\left(\mathbf{M}_{n}\right)\right\rangle=-\frac{1}{\eta_{I I}}\left\langle\mathbf{T}_{m}, \mathcal{L}^{I I}\left(\mathbf{M}_{n}\right)\right\rangle \tag{2.67c}
\end{align*}
$$

and a notation similar to (2.55) is assumed, that is

$$
\mathcal{K}_{\tan }^{I I}(\mathbf{J}) \equiv\left\{\begin{array}{cc}
\mathcal{K}_{\tan }^{I I}(\mathbf{J})+\hat{n} \times \mathbf{J} & \text { on port }  \tag{2.68}\\
\mathcal{K}_{\tan }^{I I}(\mathbf{J}) & \text { otherwise }
\end{array} .\right.
$$

Another expression for the admittance matrix $\left[Y^{I I}\right]$ can be obtained by taking into consideration the linearity of the $\mathbf{H}^{I I}$ operator together with the magnetic field boundary condition at the ports [26], [25]

$$
\begin{equation*}
\hat{n} \times \mathbf{H}^{I I}(\mathbf{r})=\mathbf{J}(\mathbf{r}) \forall \mathbf{r} \in S_{p} \tag{2.69}
\end{equation*}
$$

Then

$$
\begin{equation*}
\sum_{n=1}^{N_{P}} v_{n}\left\langle\mathbf{T}_{m}, \mathbf{H}^{I I}\left(\mathbf{M}_{P, n}, \mathbf{J}, \mathbf{M}\right)\right\rangle=\sum_{k=1}^{N_{J}} u_{k}\left\langle\mathbf{T}_{m}, \mathbf{J}_{k} \times \hat{n}\right\rangle \tag{2.70}
\end{equation*}
$$

or, if matrix notation is used, we have

$$
\begin{equation*}
\left[Y^{I I}\right][V]=[Q][U] \tag{2.71}
\end{equation*}
$$

where

$$
\begin{equation*}
[Q]_{m n}=\left\langle\mathbf{T}_{m}, \mathbf{J}_{n} \times \hat{n}\right\rangle \tag{2.72}
\end{equation*}
$$

More convenient is to determine $\left[Y^{I I}\right]$ as a function of already available matrix $[X]$ from (2.63). This can be accomplished by zero-padding the matrix $[Q]$, yielding

$$
\left[Y^{I I}\right][V]=\left[\begin{array}{ll}
{[Q]} & {[0]}
\end{array}\right] \underbrace{\left[\begin{array}{c}
{[U]}  \tag{2.73}\\
{[W]}
\end{array}\right]}_{[X]}=-[[Q] \quad[0]][R]^{-1}[P][V]
$$

Finally

$$
\begin{equation*}
\left[Y^{I I}\right]=-[[Q] \quad[0]][R]^{-1}[P] \tag{2.74}
\end{equation*}
$$

The formulations (2.66) and (2.74) are theoretically equivalent, but numerically different. In (2.66) and (2.67) the magnetic field operator matrices $\left[F^{M M}\right],\left[G^{M J}\right]$ and $\left[G^{M M}\right]$ must be calculated, while formulation (2.74) needs only the computation of the inner products matrix $[Q]$. This renders formulation (2.66) slightly more computationally intensive than (2.74). However, a closer look to (2.74) reveals an undesired feature: If the testing functions $\left\{\mathbf{T}_{i}\right\}$ are identical to the electric current density basis functions $\left\{\mathbf{J}_{i}\right\}$, the testing procedure will fail for $[Q]$, because its self elements do always vanish, that is $\left\langle\mathbf{J}_{m}, \mathbf{J}_{m} \times \hat{n}\right\rangle=0$, as it represents the inner product of two spatially orthogonal vectors. This problem was also noted in [26]
and [25], as well as in [6] where the authors solve the CFIE for a homogeneous dielectric body and use Rao-Wilton-Glisson (RWG) basis functions [57] to model both electric and magnetic current densities. The new formulation (2.66) overcomes this limitation.

### 2.3 Basis functions

The rationales behind the choice of the basis functions are discussed in this section. Hierarchical divergence-conforming basis functions are derived, in a simple manner, from curl conforming higher order basis functions. For the case of bodies of revolution, novel bases of higher polynomial order are constructed along its generatrix.

### 2.3.1 Basis functions for triangular patches

The first step in the implementation of the boundary integral formulation is the discretization of the geometry of the structure under consideration, procedure called meshing, i.e. the division of the boundary of the structure into a number of surface elements on which the basis functions will be later defined.

The triangular mesh element is by far the most flexible element when complex, arbitrarily shaped, geometries need to be investigated. Moreover, curved geometries must be considered. Meshes composed of first order (flat) triangular patches cannot accurately model curved boundaries in a reasonable manner. Accurate modeling of curved surfaces demand an increases in the number of mesh elements resulting in the (sometimes unneeded) increase in the number of basis functions used for current density representation which, in turn, unnecessarily increases the computational effort. Moreover, higher order basis functions would make no sense in conjunction with very dense meshes as the number of unknowns can be extremely large.

Accurate meshes of minimal size require the use of higher order curved triangles. While curved patches introduce some additional computational complexity, their use is totally justified. Thus, an efficient implementation should use flat elements on the planar parts of the geometry, while curved triangles should model curved boundaries. Parametric triangles up to the third polynomial order are utilized in the implementation of the present MoM algorithm.

Another crucial step in the discretization of the integral equations is the selection of the basis and test functions. Divergence conforming basis functions, i.e. vector functions that preserve the continuity of the normal component of the current densities at the mesh edges, are the most suitable for the discretization of the inner product involving the EFIE operator $\mathcal{L}$. In case of non divergence-conforming functions, the continuity equations demand the presence of line charges at the mesh edges, which usually cause inconsistencies and anomalies in the solution.

The most popular triangular-patch divergence conforming functions are the ones introduced by Rao, Wilton and Glisson (RWG) [57]. Due to their constant diver-
gence, the RWG are zero order bases, i.e. the divergence is modeled by a zero order polynomial.

Higher order functions are often preferred due to their improved convergence properties. Much of the development of higher order functions has been done by the FEM comunity [58], [20], [59], as for the divergence-conforming basis we note those introduced by Graglia et al [19] (interpolatory) and Cai et al [60] (hierarchical) for triangular patches, and Kolundzija [23],[61] for quadrilateral elements (hierarchical).

The computational efficiency of the Kolundzija functions increases as their order increases, making them suitable for patches whose electrical dimensions are relatively large. Electrically large patches must be themselves of high geometric order(e.g. spline or NURB surfaces), in order to accurately model curved boundaries, thus augmenting the computational effort, while some geometries contain details not suitable for coarse meshing. These two drawbacks and the lack of robust higher order quadrilateral meshing tools render the Kolundzija basis somehow less attractive for this work. Numerical experiments have been performed with quadrilaterals basis. While very accurate results have been obtained, the computation time did not come close to expectations, mainly due to the bad quality of the quadrilateral mesh constructed from a triangular mesh, using triangle recombination.

Regarding the triangular basis functions of Graglia [19] and Cai [60], functions up to polynomial order of two have been used to analyse a large number of structures. The interpolatory basis of Graglia [19] were found to have better convergence.

The interpolatory basis have the advantage of excellent linear independence, but in the same time they share the drawback of being non-hierarchical, i.e. the $p$-th order basis set is totally different from the lower order sets, thus mixing different orders within the same mesh ( $p$-adaptation), while still preserving normal continuity, is impossible.

In the case of the hierarchical functions, the basis of order $p-1$ represent a subset of the basis of order $p$, hence $p$-adaptation can be used. Hierarchical curl conforming basis (basis that preserve tangential continuity) defined on triangles are widely used in FEM calculations [20], [62]. Divergence conforming functions can be simply constructed starting from the curl conforming ones by taking the cross product with the triangle's normal unit vector

$$
\begin{equation*}
\mathbf{F}^{d i v}=\hat{n} \times \mathbf{F}^{c u r l} \tag{2.75}
\end{equation*}
$$

where $\mathbf{F}^{\text {div }}$ and $\mathbf{F}^{\text {curl }}$ are the divergence and curl conforming basis, respectively.

### 2.3.2 Basis functions at the ports

The currents densities $\mathbf{J}$ and $\mathbf{M}$ in (2.56) and (2.57) are approximated by the interpolatory or hierarchical basis outlined earlier. The expansion of the port magnetic current density (2.4) can be performed using the same basis only if formulation (2.66) is used, as explained in Section 2.2.2. A choice that raises no difficulties with both
(2.66) and (2.74) formulations, is the use of the port eigenvectors as basis functions in (2.4). Since the magnetic current density must satisfy $\mathbf{M}=\mathbf{E} \times \hat{n}$, an obvious choice is

$$
\begin{equation*}
\mathbf{M}_{P}=\sum_{i=1}^{N_{P}} v_{i}\left(\mathbf{e}_{i} \times \hat{n}\right), \tag{2.76}
\end{equation*}
$$

where $\mathbf{e}_{i}$ are the electric eigenvectors of the port.
2-D FEM computed eigenvectors can be employed if ports are arbitrarily shaped. First order basis expansions of the electric potential throughout the FEM calculation yield divergenceless eigenvectors. Although there are no difficulties associated with the FEM calculation itself, the MoM algorithm needs the divergence of the port magnetic current density. In this case, we are left with the formulation (2.66) in conjuction with triangular patch basis for the modeling of the port magnetic current density, as formulation (2.74) yields a singular $\left[Y^{I I}\right]$ matrix if the same basis approximate both electric and magnetic current densities (see Section 2.2.2).

### 2.3.3 Basis functions for bodies of revolution (BoR)

The surface of a body of revolution (BoR) is generated by revolving a plane curve around an axis (e.g. $z$ axis). The surface and coordinate system of a BoR are shown in Fig. 2.4. Here $\varrho, \phi$ and $z$ are the usual cylindrical coordinates and $t$ is the length variable along the surface generatrix.

The discretization of a BoR resumes to the discretization of its generatrix. Thus, the generating curve of the body is divided into linear or curved segments along $t$. The problems outlined in the case of triangular surface meshes are valid here too. Briefly, accurate modeling of the generatrix demands the use of higher order curved segments, whereas accurate and efficient modeling of the sources require the corresponding basis functions are of higher order along $t$.

The surface current density on a BoR can be can be separated in $t$-directed and $\phi$-directed components, and due to the rotational invariance of Maxwell's equations, Fourier series expansion of the azimuthal variation of each component can be performed [63]. Although linear basis are usual for the modeling of $t$-directed current densities, higher order polynomial functions are introduced here for improving convergence. Therefore, on each segment $s_{n}$, the $t$ component of the current densities can be expanded in terms of:


Figure 2.4: BoR coordinate system.

$$
f_{n}^{k}(u \in[0,1])=\left\{\begin{array}{ccc}
u & \text { on } s_{n}^{+} & \text {and } k=1  \tag{2.77}\\
1-u & \text { on } s_{n}^{-} & \text {and } k=1 \\
u-u^{k} & \text { on } s_{n}^{ \pm} & \text {and } k>1 \\
0 & & \text { otherwise }
\end{array}\right.
$$

The junction functions $f_{n}^{1}(u)$ have a linear variation from zero, at the end point of segment $s_{n}^{+}$or $s_{n}^{-}$, to unity, at the opposite end point. They are used to ensure the current continuity at the segments interface, hence, the basis defined on two or more adjacent segments must be combined into doublets or multiplets. The segment or interior higher order functions $f_{n}^{k}(u), k>1$ vanish at both segment ends, hence they do not participate at the continuity of the junction currents, their role is to improve the current density representation within the segment.

### 2.4 Matrix element evaluation

Since the regular integrals involved in the present formulation are evaluated without any difficulty using Gaussian quadrature rules, the calculation of the corresponding singular integrals is briefly discussed here.

In the case of triangular meshes, a very popular method is the singularity extraction technique [64]. Unfortunately it cannot be applied to curved triangles, so we
turn out attention to more general singularity handling procedures, i.e. the singularity cancellation method [65], [66], [67], [68]. Probably the most popular numerical cancellation technique is the one known as the Duffy method [69]. It can handle singularities of order $R^{-1}$, which makes it suitable for the calculation of singular or near singular potential integrals, i.e. integral involving the $\mathcal{L}$ operator (equation (2.50a)). However it has two drawbacks. First, it produces an angular variation about the singular point in the resulting integrand, and second, it does not work well for the near-singular case [65]. The algorithms developed by Khayat and Wilton [65], [66] overcome the disadvantages of the Duffy method. The hypersingular integrals involving the $\mathcal{K}$ operator (see (2.50b)) are evaluated as Cahchy's principal value integrals, hence only the near-singular case is of interest. In a similar manner with the singular potential integrals case, numerical cancellation methods were developed in [67] and [68], to handle near-singularities of order $R^{-2}$. Details for the singularity cancellation method are presented at Appendix A.

For the case of axially symmetric bodies, the use of curved segments demands the development of a new singularity integration procedure. The thin wire kernel singularity handling algorithm [70] is extented at Appendix B to calculate the corresponding singular integrals involved in the BoR formulation.

### 2.5 Symmetry walls

Many structures to be analyzed have one or two symmetry planes, thus only $1 / 2$ or $1 / 4$ of the geometry can be discretized resulting in dramatic savings of both memory consumption and CPU time. Since the complexity of the MoM algorithm is of order $N^{2}$ ( $N$ is the number of unknowns), the introduction of one symmetry plane theoretically improves the overall performance by a factor of four. Considering two symmetry planes would yield a theoretical reduction of sixteen times of both computation time and required memory.


Figure 2.5: Electric and magnetic sources and symmetry walls.

In practice, the improvement almost matches the theoretical expectations because the images of the current sources must be integrated and added to the final result too. Depending on the excitation, a certain symmetry plane can translate into an electric or a magnetic wall, thus the sign of the images must be correctly taken into account as shown in Fig. 2.5.

### 2.6 Calculation of the scattering matrix

Once the magnetic current densities at the port(s) have been determined, the amplitude of the reflected wave at the port's plane can be computed with the help of equation (2.15), yielding

$$
c_{i}^{-}= \begin{cases}a_{i}^{-} & \text {for } i \neq j  \tag{2.78}\\ a_{i}^{-}-a_{i}^{\text {inc }} & \text { for } i=j\end{cases}
$$

Using the above relation and (2.19), gives the modal scattering matrix in the form

$$
\begin{equation*}
s_{i j}=-\sum_{m=1}^{M} A_{i m} v_{m j}-\delta_{i j} \tag{2.79}
\end{equation*}
$$

with $M$ being the total number of modes and $a_{i}^{i n c}$ has been assumed to be unity.
The eigenvector normalization in form of relation (2.9) has been considered so far. The scattering matrix elements, relative to the power normalization

$$
\begin{equation*}
\int_{S_{p}}\left[\mathbf{E}_{i}(\varrho) \times \mathbf{H}_{j}(\varrho)\right] \hat{n}_{p}=\delta_{i j} \tag{2.80}
\end{equation*}
$$

are given as a function of $s_{i j}$ and the modal admittances

$$
\begin{equation*}
S_{i j}=\frac{\sqrt{Y_{i}}}{\sqrt{Y_{j}}} s_{i j} \tag{2.81}
\end{equation*}
$$

Another way to calculate the global scattering matrix (GSM) is to first determine the generalized admittance matrix (GAM). From the network theory we know that the modal currents and the modal voltages are related by the relation

$$
\begin{equation*}
I_{m}=\int_{S_{p}} \mathbf{h}_{m} \mathbf{H}_{\mathrm{tan}} d S=\sum_{m=1}^{M} \sum_{n=1}^{M} Y_{m n} V_{n} \tag{2.82}
\end{equation*}
$$

where $\mathbf{h}_{m}, \mathbf{H}_{\text {tan }}, V_{n}$ and $I_{m}$ are the magnetic modal eigenvectors, total tangential port magnetic field, modal voltages and modal currents, respectively.

Now considering a single mode excitation with $V_{n}=1$ and all other modes
shorted, one can "invert" the previous equation, yielding

$$
\begin{equation*}
Y_{m n}=\frac{\sqrt{Y_{m}}}{\sqrt{Y_{n}}} \int_{S_{p}} \mathbf{h}_{m} \mathbf{H}_{n, \tan } d S=\frac{\sqrt{Y_{m}}}{\sqrt{Y_{n}}} \int_{S_{p}} \mathbf{h}_{m} \mathbf{J}_{n} d S \tag{2.83}
\end{equation*}
$$

where $\mathbf{H}_{n, \tan }$ and $\mathbf{J}_{n}$ are the total tangential port magnetic field due to $n$-th eigenmode and the port electric current due to $n$-th eigenmode, respectively. In the previous expression the power normalization (2.80) has been considered. Once the GAM is known, the GSM is calculated by

$$
\begin{equation*}
[S]=2([I]+[Y])^{-1}-[I] \tag{2.84}
\end{equation*}
$$

with $[I]$ being the identity matrix.

### 2.7 Calculation of the far field

Once the unknown coefficients in (2.4), (2.56) and (2.57) are determined using (2.6) and (2.63), the scattered electric field in the far region can be computed as the superposition of the radiated (far) fields due to electric and magnetic current densities:

$$
\begin{align*}
\mathbf{E}^{J}(\mathbf{r}) & =-i k_{0} \eta_{0} \iint_{S} G_{0}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) \mathbf{J}\left(\mathbf{r}^{\prime}\right) d S^{\prime} \\
& +\frac{\eta_{0}}{i k_{0}} \nabla \iint_{S} G_{0}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) \nabla^{\prime} \mathbf{J}\left(\mathbf{r}^{\prime}\right) d S^{\prime}  \tag{2.85}\\
\mathbf{E}^{M}(\mathbf{r}) & =-\nabla \times \iint_{S} G_{0}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) \mathbf{M}\left(\mathbf{r}^{\prime}\right) d S^{\prime} \tag{2.86}
\end{align*}
$$

Under the far field assumptions [24], the fields become

$$
\begin{align*}
\mathbf{E}^{f a r, J}(\mathbf{r}) & =i k_{0} \eta_{0} \frac{e^{-i k_{0} r}}{4 \pi r} \hat{r} \times \hat{r} \times \mathbf{\Pi}^{J}(\mathbf{r})  \tag{2.87a}\\
\mathbf{E}^{f a r, M}(\mathbf{r}) & =i k_{0} \frac{e^{-i k_{0} r}}{4 \pi r} \hat{r} \times \mathbf{\Pi}^{M}(\mathbf{r}) \tag{2.87~b}
\end{align*}
$$

where

$$
\begin{equation*}
\boldsymbol{\Pi}^{\Lambda}(\mathbf{r})=\int_{S} \boldsymbol{\Lambda}\left(\mathbf{r}^{\prime}\right) e^{i k_{0} \hat{r} \mathbf{r}^{\prime}} d S^{\prime} \tag{2.88}
\end{equation*}
$$

with $\boldsymbol{\Lambda}$ standing for $\mathbf{J}$ or $\mathbf{M}, \hat{r}$ and $\mathbf{r}^{\prime}$ are the unit vector field point in spherical coordinate system and source point, respectively.

In case of linearly polarized fields, Ludwig's third definition [71] extracts the
co-polar (reference) and cross-polar far field components, as follows

$$
\begin{align*}
& E_{c p}^{f a r}(r, \phi, \theta)=\hat{u}_{p 1} \mathbf{E}^{f a r}(r, \phi, \theta)=(\hat{\theta} \sin \phi+\hat{\phi} \cos \phi) \mathbf{E}^{f a r}(r, \phi, \theta)  \tag{2.89a}\\
& E_{x p}^{f a r}(r, \phi, \theta)=\hat{u}_{p 2} \mathbf{E}^{f a r}(r, \phi, \theta)=(\hat{\theta} \cos \phi-\hat{\phi} \sin \phi) \mathbf{E}^{f a r}(r, \phi, \theta) \tag{2.89b}
\end{align*}
$$

where $\{r, \phi, \theta\}$ represents the spherical coordinate system and $\{\hat{r}, \hat{\theta}, \hat{\phi}\}$ are the corresponding unit vectors.

In case of circularly polarized fields the right-hand and left-hand components are [72]

$$
\begin{align*}
E_{r h}^{f a r}(r, \phi, \theta) & =\frac{e^{-i \phi}}{\sqrt{2}}(\hat{\theta}-i \hat{\phi}) \mathbf{E}^{f a r}(r, \phi, \theta)  \tag{2.90a}\\
E_{l h}^{f a r}(r, \phi, \theta) & =\frac{e^{-i \phi}}{\sqrt{2}}(\hat{\theta}+i \hat{\phi}) \mathbf{E}^{f a r}(r, \phi, \theta) \tag{2.90b}
\end{align*}
$$

Now, the co-polar field is represented by the desired (right-hand or left-hand) component while the undesired component represents the cross-polar field.

### 2.8 Numerical examples

### 2.8.1 Dielectric loaded conical horn antennas

Dielectric loaded conical horns are known to have low cross-polarization over a wide band of frequencies. Being relatively easy to fabricate they are used as an alternative to corrugated horns, especially at small wavelengths. Fig. 2.6 shows the geometry of a conical horn loaded with a dielectric cone [73].

Two techniques have been used to analyze the structure:

- MoM-BoR: This method uses the specialized BoR basis functions defined in Section 2.3.3. The antenna is considered to be excited by the fundamental $T E_{11}$ mode, thus only $1 n, n>1$ eigenmodes can be excited anywhere in the structure, because of the axial symmetry of the antenna. Consequently, the Fourier number $M$ will take the values $M= \pm 1$, as all basis having $M \neq$ $\pm 1$ are orthogonal to any $1 n$ waveguide mode, thus having no influence in the computation. In other words, the fundamental $T E_{11}$ generates current distributions whose Fourier series representation contains only the $m= \pm 1$ terms.
- MoM-patch: The general MoM algorithm, with basis functions defined on triangular patches [19] are used to approximate the unknown current densities. The curvature of the horn requires a higher order triangular mesh (Fig. 2.7) for the accurate modeling of the geometry.


Figure 2.6: Conical dielectric loaded horn [73]: $d=26 \mathrm{~mm}, L=301.1 \mathrm{~mm}, D_{1}=154$ $\mathrm{mm}, D_{2}=129.5 \mathrm{~mm}, \epsilon_{r}=1.13$.


Figure 2.7: Conical dielectric loaded horn.

Fig. 2.8 shows the the normalized co- and cross-polar far field plots in the $45^{\circ}$ plane, obtained by the two MoM algorithms previously mentioned, and those calculated in [73]. Good agreement among the results is observed.


Figure 2.8: Normalized far field of the horn in Fig. 2.6, $\phi=45^{\circ}, f=10.5 \mathrm{GHz}$.
In order to reduce the computational burden, only $1 / 4$ of the antenna has been meshed (Fig. 2.7(b)), since two symmetry planes are available. Although the use of symmetry walls greatly improves the overall performance of the method, it is still clearly outperformed by the more specialized $\operatorname{BoR}$ implementation by roughly one order of magnitude.

Another example is a metalized dielectric loaded horn antenna [73], shown in Fig. 2.9. The inner dielectric core is surrounded by a dielectric layer with a metalized outer surface. A matching section is used at the throat of the horn in order to minimize the reflection. A $3 D$ snapshot and the mesh of a quater of the structure are shown in Fig. 2.10.

The computed co- and cross-polar far field plots for $\phi=45^{\circ}$ at $f=9 \mathrm{GHz}$ are compared, in Fig. 2.11, with the results obtained in [73].

### 2.8.2 Dielectric covered corrugated horn antenna

Fig. 2.12 shows a corrugated horn [74], with the aperture covered by a protecting dielectric material. The radius of the feeding waveguide and aperture are 5.6 mm and 34.4 mm , respectively. The dielectric covering is 4.65 mm thick and its relative


Figure 2.9: Conical metalized dielectric loaded horn [73]: $d=23 \mathrm{~mm}, D_{1}=90 \mathrm{~mm}$, $D_{2}=70 \mathrm{~mm}, L_{1}=111 \mathrm{~mm}, L_{2}=158 \mathrm{~mm}, L_{3}=180 \mathrm{~mm}, L_{4}=200 \mathrm{~mm}, \epsilon_{r 1}=1.8$, $\epsilon_{r 2}=1.4$.


Figure 2.10: Conical metalized dielectric loaded horn.


Figure 2.11: Normalized far field of the horn in Fig. 2.9, $\phi=45^{\circ}, f=9.5 \mathrm{GHz}$.
permittivity is $\epsilon_{r}=2.28$. The coordinates of the corrugations are presented as axial-radial pairs in [74].

Two approaches were employed to model the antenna:

- BoR: The whole antenna is modeled by MoM-BoR algorithm
- BoR-MM: The corrugated sections are modeled by mode matching (MM) and only the aperture and the outer geometry are computed by the MoM-BoR algorithm.

In Fig. $2.14(\mathrm{a})$, far field cuts obtained by the computation of the entire structure using MoM-BoR, are compared with the measurements available in [74]. Fig. 2.14 (b) shows the relative far fields calculated by the combined MoM and modematching methods, plotted against the same measurements. The calculated and measured reflection coefficients are shown in Fig. 2.13. In all cases, measurements and computations agree very well. We also note that the MoM-BoR combined with mode-matching performs about five times faster than the simulation using MoMBoR alone.

### 2.8.3 Dielectric loaded rectangular horn antenna

A pyramidal horn [75] is shown in Fig. 2.15. It has an input matching section formed by five rectangular-to-rectangular waveguide steps, while a dielectric loaded

(a) 3 D view

(b) BoR generatrix.

Figure 2.12: Dielectric covered corrugated horn antenna [74].


Figure 2.13: Return loss of the horn in Fig. 2.12.
$\left(\epsilon_{r}=2.2\right)$ rectangular waveguide is further connected to the pyramidal taper.
Two building blocks have been considered in the simulation model. One block is formed by the input steps and the rectangular-to-rectangular approximation of the horn's taper, computed by the mode matching method, whereas the remaining block is represented by the dielectric loaded waveguide, calculated by MoM.

Since the antenna has two symmetry planes only $1 / 4$ of it has been discretized, as shown in Fig. 2.16(b). Both interpolatory [19] and hierarchical basis functions [20] were employed, giving identical results.

Fig. 2.17 shows good agreement between MoM calculations and the measurements from [75]. We note the slight asymmetry of the measured cross-polar pattern; probably a consequence of small differences in the permittivity of the two dielectric slabs.

### 2.8.4 Dielectric resonator antennas

The geometry of a truncated tetrahedron dielectric resonator antenna [76], mounted on a ground plane, is shown in Fig. 2.18(a). The truncated tetrahedron dielectric has the relative permittivity $\epsilon_{r}$, equilateral base and top, with sidelengths $L_{L}$ and $L_{U}$, respectively, and height $h$. The excitation of the antenna is a $z$ directed wire probe coupled to a coaxial SMA connector. The wire probe is perpendicular to the ground plane and is located on the $x$ axis at $w_{x}$.


Figure 2.14: Far field of the horn in Fig. 2.12, at $f=22.5 \mathrm{GHz}$.


Figure 2.15: Dielectric loaded rectangular horn antenna [75].


Figure 2.16: Dielectric loaded rectangular horn antenna [75].


Figure 2.17: Far field of the horn antenna in Fig. 2.15 at $f=6 \mathrm{GHz}$.

Three different antenna configurations have been measured in [76] and analyzed here using MoM. In a first configuration, the truncated tetrahedral dielectric block is actually a prism (identical base and top), secondly, the base is larger than the top, while the in the last configuration the top is larger than the base.


Figure 2.18: Truncated tetrahedron dielectric resonator antenna [76].
Fig. 2.19 shows the measured and computed $S_{11}$ for the three geometries mentioned earlier. While very good agreement between measurement and calculation is observed in Fig.s 2.19(a) and 2.19(b), the computed return loss in Fig. 2.19(c) can be declared satisfactory.

The difference between experiment and theory in Fig. 2.19(c) can arise from the fact that the dimensions of the ground plate are not specified in [76], while a perfectly conducting, square screen $\left(\lambda_{\max } \times \lambda_{\max }\right)$ has been chosen in the MoM model. Here $\lambda_{\max }$ represents the maximum free space wavelength in the corresponding frequency sweep. Moreover, it was found that the geometry of the ground plane has considerable influence on the reflection coefficient for the case of Fig. 2.19(c), but has little influence on the remaining two cases.

The mesh used in the calculation contains first order triangular patches on the planar surfaces of the model and second order triangles to accurately model the wire probe and the coaxial port. Higher order interpolatory [19] and hierarchical basis functions [20] have been used, yielding indistinguishable results.

### 2.8.5 Dielectric resonator filter

A filter [32] composed of four aligned resonators, coupled by rectangular irises, is shown in Fig. 2.20(a). The inputs of the filter are represented by steps from a WR75 waveguide to a $6.91 \mathrm{~mm} \times 9 \mathrm{~mm}$ rectangular waveguide.

The analysis has been carried out by dividing the filter into building blocks. Thus, each rectangular cavity resonator have been analyzed by MoM while the

(a) $h=24 \mathrm{~mm}, L_{L}=66 \mathrm{~mm}, L_{U}=66$ $\mathrm{mm}, w_{x}=15 \mathrm{~mm}, w_{L}=15.5 \mathrm{~mm}, r_{P}=$ $0.381 \mathrm{~mm}, \epsilon_{r}=12$.

(b) $h=26.7 \mathrm{~mm}, L_{L}=63.2 \mathrm{~mm}, L_{U}=$ $28.8 \mathrm{~mm}, w_{x}=19.9 \mathrm{~mm}, w_{L}=14 \mathrm{~mm}$, $r_{P}=0.381 \mathrm{~mm}, \epsilon_{r}=12$.

(c) $h=26 \mathrm{~mm}, L_{L}=64 \mathrm{~mm}, L_{U}=25$ $\mathrm{mm}, w_{x}=5.5 \mathrm{~mm}, w_{L}=11.5 \mathrm{~mm}, r_{P}=$ $0.381 \mathrm{~mm}, \epsilon_{r}=12$.

Figure 2.19: Return loss of the dielectric resonator antenna in Fig. 2.18.
input waveguide steps and the irises were calculated by mode-matching. Finally, the individual scattering matrices are combined yielding the global scattering matrix of the filter.

A higher order mesh [Fig. 2.20(b)], composed of second order triangular patches on the cylindrical dielectrics and first order(flat) triangles on the cavity walls, has been employed. Higher order hierarchical basis function [20] were used for the current density modeling.

The calculated reflection and transmission coefficients are compared, in Fig. 2.21(a) and Fig. 2.21(b), with measurements available in [32]. There is some disagreement between theoretically obtained curves and the measured data, a fact that can be attributed to small errors in the construction of the filter, as pointed out in [32].


Figure 2.20: Rectangular cavity dielectric resonator filter [32].


Figure 2.21: S-parameters of the filter in Fig. 2.20(a).

# Hybrid FE-BI simulation of arbitrary microwave structures 

### 3.1 Introduction

A technique based solely on integral equations was presented in the previous chapter. This method, however, cannot be applied to anisotropic/inhomogeneous bodies. Moreover, the efficiency of the numerical algorithm drops with rising material permittivity/permeability because: First, it requires very fine meshing to account for rapidly varying fields at dielectric's surface, thus increasing the number of unknowns, and second, higher order quadrature rules must be considered to accurately integrate the rapidly varying kernels, thus increasing the evaluation time of the corresponding integrals.

Differential equation techniques, like the finite element method (FEM), have none of the aforementioned drawbacks. However, unlike integral equation methods, they do not satisfy the Sommerfeld's radiation conditions, therefore the truncation of the computational domain must be accompanied by the imposition of proper boundary conditions. In turn, these boundary conditions can be naturally enforced via integral equations (IE) formulated in the free space.

This chapter presents a hybrid finite element - boundary integral (FE-BI) method for the analysis of arbitrary microwave structures. Finite elements are used to characterize the arbitrarily shaped, possibly inhomogeneous, domains. The boundary conditions, at the ports, are imposed by the matching of the modal and interior fields, thus yielding a full-wave algorithm. For radiating structures, an advantageous algorithm [27], hitherto applied for free-space plane wave scattering, is employed to formulate the radiation boundary conditions. In contrast with classical FE-BI formulations, the calculation of boundary integrals involving the divergence of $\hat{n} \times \mathbf{f}$ terms ${ }^{1}$ is not required here, thus no instabilities associated with artificial line charges are present.

Model order reduction (MOR) techniques are applied here and a new procedure is given for the treatment of frequency-dependent (inhomogeneous cross-section) ports within the MOR framework. Due to the incompatibility, with the MOR formalism, of the exact formulation (via integral equations) of the required radiation boundary

[^2]

Figure 3.1: General waveguide structure.
conditions, absorbing boundary condtions ( ABC ) are used to truncate the computational domain, when the MOR procedure is applied to unbounded (radiating) structures.

However, both ABC and IE radiation conditions are introduced in the general formulation of the presented algorithm, thus allowing, for instance, the use of IE in the critical regions of a given structure whereas ABCs may be employed on the remaining radiating parts.

At the end of the chapter several examples are presented, in order to demonstrate the validity and the efficiency of the present approach.

### 3.2 Statement of the problem

Figure 3.1 shows the geometry of the investigated structure. An arbitrarily shaped cavity of volume $V$ is connected to $N$ waveguides through $N$ planar ports of cross section $S_{P, 1}, S_{P, 2}, \ldots, S_{P, N}$. The cavity may radiate through arbitrarily shaped, planar or non-planar, apertures(s) of surface $S_{R} \cup S_{A}$. Perfect electric conductors of
surface $S_{P E C}$, perfect magnetic conductors of surface $S_{P M C}$, imperfectly conducting surfaces $S_{Z}$ and inhomogeneous materials of permittivity $\overline{\bar{\epsilon}}_{r}(\mathbf{r})$ and permeability $\overline{\bar{\mu}}_{r}(\mathbf{r})$, may all be present within $V$.

The Maxwell's equations, for a source-free region, read

$$
\begin{align*}
\nabla \times \mathbf{E} & =-i \omega \overline{\bar{\mu}} \mathbf{H},  \tag{3.1a}\\
\nabla \times \mathbf{H} & =i \omega \overline{\bar{\epsilon}} \mathbf{E},  \tag{3.1b}\\
\nabla \cdot \bar{\epsilon} \mathbf{E} \mathbf{E} & =0,  \tag{3.1c}\\
\nabla \cdot \overline{\bar{\mu}} \mathbf{H} & =0 . \tag{3.1d}
\end{align*}
$$

Eliminating the magnetic field from (3.1a) and (3.1b) yields the homogeneous wave equation for the electric field, that is

$$
\begin{equation*}
\nabla \times \overline{\bar{\mu}}_{r}^{-1} \nabla \times \mathbf{E}-k_{0}^{2} \overline{\bar{\epsilon}}_{r} \mathbf{E}=0, \tag{3.2}
\end{equation*}
$$

The electric field $\mathbf{E}$ satisfies the following boundary conditions

$$
\begin{align*}
\hat{n} \times \mathbf{E} & =0 \text { on } S_{P E C},  \tag{3.3a}\\
\hat{n} \cdot \bar{\epsilon}_{r} \mathbf{E} & =0 \text { on } S_{P M C},  \tag{3.3b}\\
\hat{n} \times \mathbf{E} & =\eta_{Z} \hat{n} \times \hat{n} \times \mathbf{H} \text { on } S_{Z},  \tag{3.3c}\\
\hat{n} \times \mathbf{E} & =\eta_{0} \hat{n} \times \hat{n} \times \mathbf{H} \text { on } S_{A},  \tag{3.3d}\\
\hat{n} \times \mathbf{E}^{+} & =\hat{n} \times \mathbf{E}^{-} \text {on } S_{R} \text { and } S_{P} . \tag{3.3e}
\end{align*}
$$

Similarily, the magnetic field obeys

$$
\begin{equation*}
\nabla \times \bar{\epsilon}_{r}^{-1} \nabla \times \mathbf{H}-k_{0}^{2} \overline{\bar{\mu}}_{r} \mathbf{H}=0, \tag{3.4}
\end{equation*}
$$

and is subject to the boundary conditions

$$
\begin{align*}
\hat{n} \times \mathbf{H} & =0 \text { on } S_{P M C},  \tag{3.5a}\\
\hat{n} \cdot \overline{\bar{\mu}}_{r} \mathbf{H} & =0 \text { on } S_{P E C},  \tag{3.5b}\\
\hat{n} \times \mathbf{H} & =-\frac{1}{\eta_{Z}} \hat{n} \times \hat{n} \times \mathbf{E} \text { on } S_{Z},  \tag{3.5c}\\
\hat{n} \times \mathbf{H} & =-\frac{1}{\eta_{0}} \hat{n} \times \hat{n} \times \mathbf{E} \text { on } S_{A},  \tag{3.5d}\\
\hat{n} \times \mathbf{H}^{+} & =\hat{n} \times \mathbf{H}^{-} \text {on } S_{R} \text { and } S_{P} . \tag{3.5e}
\end{align*}
$$

Here $\eta_{Z}$ is the intrinsic impedance of the imperfect conductor, $\eta_{0}$ is the intrinsic impedance of the free space and $\mathbf{E}^{+}, \mathbf{H}^{+}$and $\mathbf{E}^{-}, \mathbf{H}^{-}$represent, respectively, the tangential fields at the boundary of $V$ and just inside $V$ and the tangential fields at the boundary of $V$ and just outside $V$. The radiating port has been divided into a surface $S_{R}$, on which the radition conditions are formulated via integral equations,
and a surface $S_{A}$ on which the first order absorbing boundary conditions (3.3d), (3.5d) are imposed.

Referring to Fig. 3.1, we define, similarly to [27], the solution spaces for the electric field $\mathbf{E}$, magnetic field $\mathbf{H}$, electric current density $\mathbf{J}$ and magnetic current density $\mathbf{M}$, respectively, as

$$
\begin{align*}
W^{E} & =\left\{\mathbf{a} \in H(\operatorname{curl}, V) \mid \hat{n} \times \mathbf{a}=0 \text { on } \Gamma_{P E C}\right\},  \tag{3.6a}\\
W^{H} & =\left\{\mathbf{a} \in H(\operatorname{curl}, V) \mid \hat{n} \times \mathbf{a}=0 \text { on } \Gamma_{P M C}\right\},  \tag{3.6b}\\
T^{H} & =\left\{\mathbf{a} \in \operatorname{Span}\left\{\hat{n} \times\left.\mathbf{b}\right|_{\partial V}\right\} \mid \mathbf{b} \in \mathbf{W}^{H}\right\},  \tag{3.6c}\\
T^{E} & =\left\{\mathbf{a} \in \operatorname{Span}\left\{\hat{n} \times\left.\mathbf{b}\right|_{\partial V}\right\} \mid \mathbf{b} \in \mathbf{W}^{E}\right\}, \tag{3.6~d}
\end{align*}
$$

where $\partial V$ represents the boundary of $V$.

### 3.3 Formulation

Different formulations are possible, depending on which of the equations (3.2) and/or (3.4) are used, and on how the required boundary conditions, at the surface $S_{R}$, are imposed. In the E-J formulation, the starting point is (3.2) and the unknowns, as the name suggests, are the electric field and the tangential magnetic field (electric current density). Employing (3.4) yields the H-M formulation, where the unknowns are the magnetic field and the tangential electric field (magnetic current density). The EJ formulation uses the magnetic field integral equation (MFIE) in the boundary integrals resulted from the Galerkin testing of (3.2) and the electric field integral equation (EFIE) as an additional equation. In the $\mathrm{H}-\mathrm{M}$ formulation, the EFIE is present in the boundary integrals and the MFIE is required as an independent equation. Combining both (3.2) and (3.4) yields the E-H formulation [27]. One can also discretize (3.2) and (3.4), and use the EFIE and MFIE, respectively, to account for the boundary conditions at $S_{R}$. A last possibility is the use of any of the wave equations plus the combined field integral equation (CFIE). We call these last three formulations E-EFIE, H-MFIE and E/H-CFIE, respectively.

E-EFIE and H-MFIE are known to bear the risk of internal resonance breakdown [29], [77], whereas E-J, H-M, E-H and E/H-CFIE formulations are free of internal resonance effects [27]. An undesired feature of the E/H-CFIE is that it requires the calculation of boundary surface integrals involving the divergence of curl-conforming basis functions [29]. Regarding the E-H formulation, one observes that it yields twice as many unknowns, compared to all other formulations. As a consequence, the E-J and $\mathrm{H}-\mathrm{M}$ formulations will be further considered.

The solution process is greatly simplified when dealing with a closed, non radiating structure, or if absorbing boundaries are exclusively used to account for the radiation conditions. In this case, the formulations outlined above would reduce to either an electric field (3.2) or magnetic field (3.4) formulation.

The E-J formulation is presented in this section. Since the derivation of the H-M formulation is similar to that of E-J, we only state the final equations of the $\mathrm{H}-\mathrm{M}$ formulation.

We begin by testing (3.2) with a set of linearly independent functions $\mathbf{W}^{E} \in W^{E}$, yielding

$$
\begin{equation*}
\underbrace{\int_{V}\left(\nabla \times \mathbf{W}^{E} \overline{\bar{\mu}}_{r}^{-1} \nabla \times \mathbf{E}-k_{0}^{2} \mathbf{W}^{E} \overline{\bar{\epsilon}}_{r} \mathbf{E}\right) d V}_{\mathcal{J}_{V}}-\underbrace{\int_{\partial V}\left(\hat{n} \times \mathbf{W}^{E} \times \overline{\bar{\mu}}_{r}^{-1} \nabla \times \mathbf{E}\right) d S}_{\mathcal{J}_{S}}=0 \tag{3.7}
\end{equation*}
$$

where $\hat{n}$ is the unit vector normal to $\partial V$. Considering that

$$
\begin{equation*}
\overline{\bar{\mu}}_{r}^{-1} \nabla \times \mathbf{E}=-i k_{0} \eta_{0} \mathbf{H} \tag{3.8}
\end{equation*}
$$

and the boundary conditions (3.3), the surface integral in (3.7) is developed to

$$
\begin{align*}
\mathcal{J}_{S}=-i k_{0} \eta_{0} & {[\underbrace{\int_{S_{Z}} \hat{n} \times \mathbf{W}^{E} \mathbf{H} d S}_{\mathcal{J}_{Z}}+\underbrace{\int_{S_{A}} \hat{n} \times \mathbf{W}^{E} \mathbf{H} d S}_{\mathcal{J}_{A}}}  \tag{3.9}\\
& +\underbrace{\int_{S_{P}} \hat{n} \times \mathbf{W}^{E} \mathbf{H} d S}_{\mathcal{J}_{P}}+\underbrace{\int_{S_{R}} \hat{n} \times \mathbf{W}^{E} \mathbf{H} d S}_{\mathcal{J}_{R}}]
\end{align*}
$$

Integrals $\mathcal{J}_{Z}$ and $\mathcal{J}_{A}$ account, respectively, for the incorporation of the impedance boundary conditions (3.3c) and absorbing boundary conditions (3.3d). They can be further developed to

$$
\begin{align*}
& \mathcal{J}_{Z}=\frac{1}{\eta_{Z}} \int_{S_{Z}} \hat{n} \times \mathbf{W}^{E} \cdot \hat{n} \times \mathbf{E} d S  \tag{3.10}\\
& \mathcal{J}_{A}=\frac{1}{\eta_{0}} \int_{S_{A}} \hat{n} \times \mathbf{W}^{E} \cdot \hat{n} \times \mathbf{E} d S \tag{3.11}
\end{align*}
$$

The third integral in (3.9) is taken over the port surface $S_{P}$. The magnetic field tangential to $S_{P}$ is given by the modal magnetic eigenvectors $\mathbf{h}_{i}$, in the form

$$
\begin{equation*}
\mathbf{H}\left(\mathbf{r} \in S_{P}\right)=\sum_{i}^{N_{P}} I_{P, i} \cdot \mathbf{h}_{i}, \tag{3.12}
\end{equation*}
$$

where the scalars $I_{P, i}$ denote the amplitude of the eigenvectors. Thus, $\mathcal{J}_{P}$ becomes

$$
\begin{equation*}
\mathcal{J}_{P}=\int_{S_{P}} \hat{n} \times \mathbf{W}^{E} \sum_{i}^{N_{P}} I_{P, i} \cdot \mathbf{h}_{i} d S=\sum_{i}^{N_{P}} I_{P, i} \int_{S_{P}} \hat{n} \times \mathbf{W}^{E} \cdot \mathbf{h}_{i} d S \tag{3.13}
\end{equation*}
$$

The evaluation of $\mathcal{J}_{R}$ is more complicated since it involves the exact (integral equation) formulation of the radiating boundary conditions. The magnetic field, produced by electric and magnetic sources, can be written as (see Chapter 2)

$$
\begin{align*}
\mathbf{H} & =\mathbf{H}(S ; \mathbf{J})+\mathbf{H}(S ; \mathbf{M}) \\
& =-\frac{1}{2} \hat{n} \times \mathbf{J}+\mathcal{K}(S ; \mathbf{J})-\frac{1}{\eta_{0}} \mathcal{L}(S ; \mathbf{M})  \tag{3.14}\\
& =-\frac{1}{2} \hat{n} \times \mathbf{J}+\mathcal{K}(S ; \mathbf{J})+\frac{1}{\eta_{0}} \mathcal{L}(S ; \hat{n} \times \mathbf{E}),
\end{align*}
$$

where the linear operators $\mathcal{K}$ and $\mathcal{L}$ have been already defined in (2.50). Introducing the expression of the magnetic field into (3.9), gives

$$
\begin{align*}
\mathcal{J}_{R}= & -\frac{1}{2} \int_{S_{R}} \hat{n} \times \mathbf{W}^{E} \hat{n} \times \mathbf{J} d S \\
& +\int_{S_{R}} \hat{n} \times \mathbf{W}^{E} \mathcal{K}\left(S_{R} ; \mathbf{J}\right) d S  \tag{3.15}\\
& -\frac{1}{\eta_{0}} \int_{S_{R}} \hat{n} \times \mathbf{W}^{E} \mathcal{L}\left(S_{R} ; \mathbf{M}\right) d S
\end{align*}
$$

Taking into account the expressions of the surface integrals $\mathcal{J}_{Z}, I_{A}, \mathcal{J}_{P}$ and $\mathcal{J}_{R}$, (3.7) becomes

$$
\begin{align*}
& \int_{V}\left(\nabla \times \mathbf{W}^{E} \overline{\bar{\mu}}_{r}^{-1} \nabla \times \mathbf{E}-k_{0}^{2} \mathbf{W}^{E} \overline{\bar{\epsilon}}_{r} \mathbf{E}\right) d V \\
& +i k_{0} \frac{\eta_{0}}{\eta_{Z}} \int_{S_{Z}} \hat{n} \times \mathbf{W}^{E} \hat{n} \times \mathbf{E} d S \\
& +i k_{0} \int_{S_{A}} \hat{n} \times \mathbf{W}^{E} \hat{n} \times \mathbf{E} d S \\
& -\frac{i k_{0} \eta_{0}}{2} \int_{S_{R}} \hat{n} \times \mathbf{W}^{E} \hat{n} \times \mathbf{J} d S  \tag{3.16}\\
& +i k_{0} \eta_{0} \int_{S_{R}} \hat{n} \times \mathbf{W}^{E} \mathcal{K}\left(S_{R} ; \mathbf{J}\right) d S \\
& -i k_{0} \int_{S_{R}}^{\hat{n}} \times \mathbf{W}^{E} \mathcal{L}\left(S_{R} ; \mathbf{M}\right) d S \\
& =-i k_{0} \eta_{0} \sum_{i}^{N_{P}} I_{P, i} \int_{S_{P}} \hat{n} \times \mathbf{W}^{E} \cdot \mathbf{h}_{i} d S .
\end{align*}
$$

A second equation is obtained by enforcing the EFIE on $S_{R}$, that is

$$
\begin{align*}
& -\hat{n} \times \hat{n} \times\left[\mathbf{E}\left(S_{R} ; \mathbf{J}\right)+\mathbf{E}\left(S_{R} ; \mathbf{M}\right)\right] \\
& =\hat{n} \times \hat{n} \times\left[\eta_{0} \mathcal{L}\left(S_{R} ; \mathbf{J}\right)+\mathcal{K}\left(S_{R} ; \mathbf{M}\right)-\frac{1}{2} \hat{n} \times \mathbf{M}\right]=\hat{n} \times \mathbf{M} \tag{3.17}
\end{align*}
$$

and testing it with a basis set $\mathbf{T}^{H} \in T^{H}$, yielding

$$
\begin{equation*}
\frac{1}{2} \int_{S_{R}} \mathbf{T}^{H} \hat{n} \times \mathbf{M} d S+\int_{S_{R}} \mathbf{T}^{H} \mathcal{K}\left(S_{R} ; \mathbf{M}\right) d S+\eta_{0} \int_{S_{R}} \mathbf{T}^{H} \mathcal{L}\left(S_{R} ; \mathbf{J}\right) d S=0 \tag{3.18}
\end{equation*}
$$

Now, we add (3.16) and (3.18), but before the addition is performed, (3.18) is multiplied with $-i k_{0} \eta_{0}$, in order to ensure the symmetry of the resulting equation. Thus, we get the the governing equation of the E-J formulation

$$
\begin{align*}
& \int_{V}\left(\nabla \times \mathbf{W}^{E} \overline{\bar{\mu}}_{r}^{-1} \nabla \times \mathbf{E}-k_{0}^{2} \mathbf{W}^{E} \overline{\bar{\epsilon}}_{r} \mathbf{E}\right) d V \\
& +i k_{0} \frac{\eta_{0}}{\eta_{Z}} \int_{S_{Z}} \hat{n} \times \mathbf{W}^{E} \cdot \hat{n} \times \mathbf{E} d S+i k_{0} \int_{S_{A}} \hat{n} \times \mathbf{W}^{E} \cdot \hat{n} \times \mathbf{E} d S \\
& -\frac{i k_{0} \eta_{0}}{2} \int_{S_{R}} \mathbf{T}^{H} \hat{n} \times \mathbf{M} d S-i k_{0} \eta_{0} \int_{S_{R}} \mathbf{T}^{H} \mathcal{K}\left(S_{R} ; \mathbf{M}\right) d S-i k_{0} \int_{S_{R}} \mathbf{T}^{E} \mathcal{L}\left(S_{R} ; \mathbf{M}\right) d S \\
& -\frac{i k_{0} \eta_{0}}{2} \int_{S_{R}} \mathbf{T}^{E} \hat{n} \times \mathbf{J} d S+i k_{0} \eta_{0} \int_{S_{R}} \mathbf{T}^{E} \mathcal{K}\left(S_{R} ; \mathbf{J}\right) d S-i k_{0} \eta_{0}^{2} \int_{S_{R}} \mathbf{T}^{H} \mathcal{L}\left(S_{R} ; \mathbf{J}\right) d S \\
& =-i k_{0} \eta_{0} \sum_{i}^{N_{P}} I_{P, i} \int_{S_{P}} \mathbf{T}^{E} \mathbf{h}_{i} d S . \tag{3.19}
\end{align*}
$$

In a similar manner, the H-M formulation is derived starting with (3.4), and enforcing the MFIE on $S_{R}$. The result reads

$$
\begin{align*}
& \int_{V}\left(\nabla \times \mathbf{W}^{H} \bar{\epsilon}_{r}^{-1} \nabla \times \mathbf{H}-k_{0}^{2} \mathbf{W}^{H} \overline{\bar{\mu}}_{r} \mathbf{H}\right) d V \\
& +\frac{i k_{0} \eta_{Z}}{\eta_{0}} \int_{S_{Z}} \hat{n} \times \mathbf{W}^{H} \cdot \hat{n} \times \mathbf{H} d S+i k_{0} \int_{S_{A}} \hat{n} \times \mathbf{W}^{H} \cdot \hat{n} \times \mathbf{H} d S \\
& -\frac{i k_{0}}{2 \eta_{0}} \int_{S_{R}} \mathbf{T}^{H} \hat{n} \times \mathbf{M} d S+\frac{i k_{0}}{2 \eta_{0}} \int_{S_{R}} \mathbf{T}^{H} \mathcal{K}\left(S_{R} ; \mathbf{M}\right) d S+i k_{0} \int_{S_{R}} \mathbf{T}^{H} \mathcal{L}\left(S_{R} ; \mathbf{J}\right) d S \\
& +\frac{i k_{0}}{2 \eta_{0}} \int_{S_{R}} \mathbf{T}^{E} \hat{n} \times \mathbf{J} d S+\frac{i k_{0}}{2 \eta_{0}} \int_{S_{R}} \mathbf{T}^{E} \mathcal{K}\left(S_{R} ; \mathbf{J}\right) d S-\frac{i k_{0}}{\eta_{0}^{2}} \int_{S_{R}} \mathbf{T}^{E} \mathcal{L}\left(S_{R} ; \mathbf{M}\right) d S \\
& =\frac{i k_{0}}{\eta_{0}} \sum_{i}^{N_{P}} I_{P, i} \int_{S_{P}} \mathbf{T}^{H} \mathbf{e}_{i} d S . \tag{3.20}
\end{align*}
$$

### 3.4 Numerical implementation

The reduction of (3.19) or (3.20) to a linear equation system is presented in this section. A drawback of the present algorithm is that it yields a fully populated block in the coefficient matrix. This fully populated sub-matrix is the direct result of the formulation of the radiation boundary conditions via integral equations. Moreover, there is no technique that can efficiently solve a linear equation system involving a coefficient matrix with both sparse and full blocks. To eliminate this drawback and take advantage of the sparsity of the FEM matrix, a simple domain splitting is perfomed.

In order to numerically solve (3.19) or (3.20), the electric and magnetic fields and the electric and magnetic current densities are approximated by, respectively

$$
\begin{align*}
& \mathbf{E}(\mathbf{r})=\sum_{i=1}^{N_{E}} E_{i} \mathbf{W}^{E}(\mathbf{r})  \tag{3.21a}\\
& \mathbf{H}(\mathbf{r})=\sum_{i=1}^{N_{H}} H_{i} \mathbf{W}^{H}(\mathbf{r})  \tag{3.21b}\\
& \mathbf{J}(\mathbf{r})=\sum_{i=1}^{N_{J}} J_{i} \hat{n} \times \mathbf{W}^{H}(\mathbf{r})=\sum_{i=1}^{N_{J}} J_{i} \mathbf{T}^{H}(\mathbf{r})  \tag{3.21c}\\
& \mathbf{M}(\mathbf{r})=\sum_{i=1}^{N_{M}} M_{i} \hat{n} \times \mathbf{W}^{E}(\mathbf{r})=\sum_{i=1}^{N_{M}} M_{i} \mathbf{T}^{E}(\mathbf{r}), \tag{3.21d}
\end{align*}
$$

where $\mathbf{W}^{E} \in W^{E}$ and $\mathbf{W}^{H} \in W^{H}$ are curl-conforming basis functions and $\mathbf{T}^{E} \in T^{E}$ and $\mathbf{T}^{H} \in T^{H}$ are divergence-conforming functions.

Replacing the expansions (3.21) into (3.19) or (3.20), yields the following matrix equation

$$
\left(\begin{array}{cc}
{\left[Y^{V V}\right]} & {\left[Y^{V S}\right]}  \tag{3.22}\\
{\left[Y^{S V}\right]} & {\left[Y^{S S}\right]}
\end{array}\right)\binom{\left[I^{V}\right]}{\left[I^{S}\right]}=\binom{\left[Y^{V P}\right]}{0}
$$

In the following, expressions for the matrix elements of (3.22) will be given for the E-J formulation only, as those of the H-M formulation can be derived in a similar manner.

Submatrix $[Y]^{V V}$ represents the interactions between the basis functions in $V$,
excluding the radiating port $S_{R}$, or

$$
\begin{align*}
{\left[Y^{V V}\right]_{m n}=} & \int_{V}\left(\nabla \times \mathbf{W}_{m}^{E} \overline{\bar{\mu}}_{r}^{-1} \nabla \times \mathbf{W}_{n}^{E}-k_{0}^{2} \mathbf{W}_{m}^{E} \overline{\bar{\epsilon}}_{r} \mathbf{W}_{n}^{E}\right) d V \\
& +i k_{0} \frac{\eta_{0}}{\eta_{Z}} \int_{S_{Z}} \hat{n} \times \mathbf{W}_{m}^{E} \cdot \hat{n} \times \mathbf{W}_{n}^{E} d S  \tag{3.23}\\
& +i k_{0} \int_{S_{A}} \hat{n} \times \mathbf{W}_{m}^{E} \cdot \hat{n} \times \mathbf{W}_{n}^{E} d S \\
& \text { for } \mathbf{W}_{m}^{E} \in V-S_{R} \text { and } \mathbf{W}_{n}^{E} \in V-S_{R}
\end{align*}
$$

Submatrix $\left[Y^{V S}\right]$ represents the interactions between the basis functions interior to $V$ and those nonvanishing on $S_{R}$ :

$$
\begin{align*}
{\left[Y^{V S}\right]_{m n}=} & \int_{V}\left(\nabla \times \mathbf{W}_{m}^{E} \overline{\bar{\mu}}_{r}^{-1} \nabla \times \mathbf{W}_{n}^{E}-k_{0}^{2} \mathbf{W}_{m}^{E} \overline{\bar{\epsilon}}_{r} \mathbf{W}_{n}^{E}\right) d V  \tag{3.24}\\
& \text { for } \mathbf{W}_{m}^{E} \in V-S_{R} \text { and } \mathbf{W}_{n}^{E} \in S_{R}
\end{align*}
$$

Submatrix $\left[Y^{S S}\right]$ incorporates, exclusively, the boundary interactions on $S_{R}$. It can be further decomposed into four submatrices as follows:

$$
\left[Y^{S S}\right]=\left(\begin{array}{cc}
{\left[Y^{M M}\right]} & {\left[Y^{M J}\right]}  \tag{3.25}\\
{\left[Y^{J M}\right]} & {\left[Y^{J J}\right]}
\end{array}\right)
$$

Submatrix $\left[Y^{M M}\right]$ involves the self interactions of the magnetic current densities M as well as the self interactions of the electric fields on $S_{R}$. Thus, it is given by

$$
\begin{align*}
{\left[Y^{M M}\right]_{m n}=} & \int_{V}\left(\nabla \times \mathbf{W}_{m}^{E} \overline{\bar{\mu}}_{r}^{-1} \nabla \times \mathbf{W}_{n}^{E}-k_{0}^{2} \mathbf{W}_{m}^{E} \overline{\bar{\epsilon}}_{r} \mathbf{W}_{n}^{E}\right) d V \\
& -i k_{0} \int_{S_{R}} \mathbf{T}_{m}^{E} \mathcal{L}\left(S_{R} ; \mathbf{T}_{n}^{E}\right) d S, \text { for } \mathbf{W}_{m}^{E} \in S_{R} \text { and } \mathbf{W}_{n}^{E} \in S_{R} \tag{3.26}
\end{align*}
$$

Submatrices $\left[Y^{M J}\right]$ and $\left[Y^{J M}\right]$ incorporate the interactions between the magnetic and electric current densities and vice-versa, or

$$
\begin{equation*}
\left[Y^{M J}\right]_{m n}=-\frac{i k_{0} \eta_{0}}{2} \int_{S_{R}} \mathbf{T}_{m}^{E} \hat{n} \times \mathbf{T}_{n}^{H} d S+i k_{0} \eta_{0} \int_{S_{R}} \mathbf{T}_{m}^{E} \mathcal{K}\left(S_{R} ; \mathbf{T}_{n}^{H}\right) d S \tag{3.27}
\end{equation*}
$$

From the symmetry of (3.19), one observes that

$$
\begin{equation*}
\left[Y^{J M}\right]=\left[Y^{M J}\right]^{T} \tag{3.28}
\end{equation*}
$$

Matrix $\left[Y^{J J}\right]$ represents the self interactions between the electric current densities.

It reads

$$
\begin{equation*}
\left[Y^{J J}\right]_{m n}=-i k_{0} \eta_{0}^{2} \int_{S_{R}} \mathbf{T}^{H} \mathcal{L}\left(S_{R} ; \mathbf{J}\right) d S \tag{3.29}
\end{equation*}
$$

Finally, matrix $\left[Y^{V P}\right]$ relates to the modal excitation. Its elements are

$$
\begin{equation*}
\left[Y^{V P}\right]_{m p}=-i k_{0} \eta_{0} \int_{S_{P}} \mathbf{T}_{m}^{E} \mathbf{h}_{p} d S \tag{3.30}
\end{equation*}
$$

The solution to (3.22) involves a partially full and partially sparse matrix. Namely, $\left[Y^{V V}\right],\left[Y^{V S}\right]$ and $\left[Y^{S V}\right]$ are sparse, whereas matrix $\left[Y^{S S}\right]$ is completely populated. The use of iterative methods to solve (3.22) might be an option. However, a direct solver for sparse linear equation systems can be used if (3.22) is simply rewritten as

$$
\begin{align*}
& {\left[Y^{V V}\right]\left[I^{V}\right]+\left[Y^{V S}\right]\left[I^{S}\right]=\left[Y^{V P}\right]} \\
& {\left[Y^{S V}\right]\left[I^{V}\right]+\left[Y^{S S}\right]\left[I^{S}\right]=0} \tag{3.31}
\end{align*}
$$

yielding

$$
\begin{align*}
& {\left[I^{S}\right]=\left(\left[Y^{S V}\right]\left[Y^{V V}\right]^{-1}\left[Y^{V S}\right]-\left[Y^{S S}\right]\right)^{-1}\left[Y^{S V}\right]\left[Y^{V V}\right]^{-1}\left[Y^{V P}\right]}  \tag{3.32}\\
& {\left[I^{V}\right]=\left[Y^{V V}\right]^{-1}\left(\left[Y^{V P}\right]-\left[Y^{V S}\right]\left[I^{S}\right]\right)}
\end{align*}
$$

Examinning (3.32), one observes that matrix products involving $\left[Y^{V V}\right]^{-1}$ are required. Therefore, the factorization of $\left[Y^{V V}\right]$ is performed once, and it is later used to calculate the required products, via a linear system solution of the form $\left[Y^{V V}\right][X]=[B]$.

The solution to (3.22) yields the unknown expansion coefficients, i.e. the matrices $\left[Y^{V}\right]$ and $\left[Y^{S}\right]$. Once these coefficients are found, they can be replaced back into (3.21) to give the electric or magnetic fields in $V$ and the electric and magnetic currents densities on $S_{R}$. Other quantities of practical interest, like scattering parameters and far-field radiation patterns, can be then determined.

### 3.5 Calculation of the scattering matrix

Since the E-J formulation yields the electric field in $V$, the generalized impedance matrix (GIM) of the multiport system can be convenientely computed as [78]

$$
\begin{align*}
{[Z]_{i j} } & =\int_{S_{P}} \mathbf{e}_{i} \mathbf{E}_{j} d S=\int_{S_{P}} \hat{n} \times \mathbf{h}_{i} \mathbf{E}_{j} d S \\
& =-\int_{S_{P}} \hat{n} \times \mathbf{E}_{j} \mathbf{h}_{i} d S=-\sum_{j} \int_{S_{P}} \mathbf{h}_{i} \mathbf{T}_{j}^{E} d S \tag{3.33}
\end{align*}
$$

where $\mathbf{e}_{i}$ and $\mathbf{h}_{i}$ represent the electric and magnetic port's eigenvectors, respectively. In matrix form, (3.33) becomes

$$
\begin{align*}
{[Z] } & =\left[Y^{P V}\right]\left[I^{V}\right] \\
{\left[Y^{P V}\right] } & =\frac{1}{i k_{0} \eta_{0}}\left[Y^{V P}\right]^{T} \tag{3.34}
\end{align*}
$$

with $\left[I^{V}\right]$ given in (3.32).
Similarly to the E-J formulation, the H-M formulation naturally yields the generalized admittance matrix (GAM) of the structure.

Finally, the $S$-matrix can be determined from the generalized impedance matrix or from the generalized admittance matrix

$$
\begin{align*}
& {[S]=([Z]+[I])^{-1}([Z]-[I]),} \\
& {[S]=([I]+[Y])^{-1}([I]-[Y]),} \tag{3.35}
\end{align*}
$$

respectively. $[I]$ represents the identity matrix.

### 3.6 Model order reduction (MOR)

Model order reduction is a process in which the number of unknowns, in a discrete mathematical representation of a problem, is drastically decreased while still maintaining accuracy. Thus, instead of solving a discrete problem having $N$ degrees of freedom, one formulates an equivalent problem of order $Q$, with $Q \ll N$.

Two MOR techniques are briefly presented in this section. The first method is based on the PRIMA (passive reduced-order interconnect macromodeling analysis) algorithm [3], [30]. However, PRIMA requires the ports to be frequency independent. A simple procedure that eliminates this drawback is given. The second method makes use of the well-conditioned asymptotic waveform evaluation (WCAWE) [4], [31] that is modified here to allow multiple right-hand sides (i.e. waveguide modes).

For a general structure containing frequency dependent materials, the impedance or admittance matrix can be cast in the form

$$
\begin{equation*}
[X]=\left[L\left(k_{0}\right)\right]^{H}\left(\left[G\left(k_{0}\right)\right]-k_{0}^{2}\left[C\left(k_{0}\right)\right]-k_{0}[D]\right)^{-1}\left[B\left(k_{0}\right)\right] . \tag{3.36}
\end{equation*}
$$

Here, $[G]$ and $[C]$ are the usual FEM matrices, given by

$$
\begin{align*}
& {\left[G\left(k_{0}\right)\right]_{m n}=\int_{V} \nabla \times \mathbf{W}_{m}^{E} \overline{\bar{\alpha}}\left(k_{0}\right) \nabla \times \mathbf{W}_{n}^{E}} \\
& {\left[C\left(k_{0}\right)\right]_{m n}=\int_{V} \mathbf{W}_{m}^{E} \overline{\bar{\beta}}\left(k_{0}\right) \mathbf{W}_{n}^{E} d V} \tag{3.37}
\end{align*}
$$

with $\{\overline{\bar{\alpha}}, \overline{\bar{\beta}}\}=\left\{\overline{\bar{\mu}}_{r}^{-1}, \overline{\bar{\epsilon}}_{r}\right\}$ for the electric field formulation, and $\{\overline{\bar{\alpha}}, \overline{\bar{\beta}}\}=\left\{\overline{\bar{\epsilon}}_{r}^{-1}, \overline{\bar{\mu}}_{r}\right\}$
in the case of the magnetic field formulation. Matrix $[D]$ relates to the impedance surfaces and/or surfaces with absorbing boundary conditions, $[X]$ stands for $[Y]$ or $[Z],[L]^{H}$ and $[B]$ are derived from $\left[Y^{P V}\right]$ and $\left[Y^{V P}\right]$, respectively. $[L]^{H}$ denotes the hermitian of $[L]$.

In order to find the frequency response of a given multiport, one has to solve the linear system (3.36) for every different value of $k_{0}$ in a frequency range. This can be a very time consuming procedure if the number of required frequency points is high. Model order reduction (MOR) can be applied to (3.36) to yield a dramatic decrease of the calculation time.

### 3.6.1 PRIMA based MOR

The PRIMA algorithm [3] requires a matrix equation that can be cast in the form

$$
\begin{align*}
{[X] } & =[L]^{H}([G]+\kappa[C])^{-1}[B] \\
& =[L]^{H}([I]-\delta[A])^{-1}[R], \tag{3.38}
\end{align*}
$$

where $\delta=\kappa-\kappa_{0}$ and $\kappa_{0}$ is the expansion frequency (chosen in the middle of the frequency range); $[L]$ and $[B]$ are now frequency independent, and

$$
\begin{align*}
& {[A]=\left([G]+\kappa_{0}[C]\right)^{-1}[C]} \\
& {[R]=\left([G]+\kappa_{0}[C]\right)^{-1}[B]} \tag{3.39}
\end{align*}
$$

One observes that (3.36) can be cast in the form of (3.38) only if

- frequency independent materials are considered,
- no impedance and/or ABC surfaces are present,
- the ports are frequency independent.

The first two requirements translate to

$$
\begin{equation*}
[X]=\left[L\left(k_{0}\right)\right]^{H}\left([G]-k_{0}[C]\right)^{-1}\left[B\left(k_{0}\right)\right] . \tag{3.40}
\end{equation*}
$$

Whereas the first aforementioned restriction cannot be removed, a procedure that eliminates the second drawback is given in [3]. Briefly, instead of working with one of the wave equations for the electric or magnetic field, the Maxwell's equations represent the starting point, and both field and flux are unknown quantities. The procedure yields an equation of the form (3.38), if the ports are freqeuncy independent. The number of unknowns is apparently doubled compared to the case when only the field represents the unknown quantity. However, by tanking into account the connections between the tangentially continuous finite element space (used to approximate the field) and the normally continuous finite element space (used to
approximate the flux), the number of unknowns is the same as for the formulation that uses the homogeneous wave equation. Moreover, the Maxwell's equations based PRIMA requires the assembly of the same matrices used in the wave equation formulation [3].

The issue of frequency dependent (inhomogeneous cross section) ports is addressed here. In this respect we approximate the fields tangential to the port as follows

$$
\begin{equation*}
\mathbf{E}_{\mathrm{tan}}, \mathbf{H}_{\mathrm{tan}}=\sum_{i} v_{i} \mathbf{f}_{i}, \tag{3.41}
\end{equation*}
$$

where $\mathbf{f}_{i}$ represent frequency independent vector functions and $v_{i}$ are unknowns coefficients. By matching, at each frequency in a sweep, the modal fields with the expansion (3.41), we get

$$
\begin{equation*}
\sum_{i} v_{i} \mathbf{f}_{i}=\sum_{j} I_{P, j} \mathbf{x}_{j} . \tag{3.42}
\end{equation*}
$$

Here $I_{P, j}$ are known modal amplitudes and $\mathbf{x}_{j}$ represent the transverse modal electric or magnetic eigenvectors. Testing (3.42) with $\mathbf{f}_{i}$ yields the unknown coefficients $v_{i}$

$$
\begin{equation*}
[v]=[P]^{-1}[T]\left[I_{P}\right], \tag{3.43}
\end{equation*}
$$

with

$$
\begin{equation*}
[P]_{i j}=\int_{S_{P}} \mathbf{f}_{i} \cdot \mathbf{f}_{j} d S \tag{3.44}
\end{equation*}
$$

and

$$
\begin{equation*}
[T]_{i j}=\int_{S_{P}} \mathbf{f}_{i} \cdot \mathbf{x}_{j} d S \tag{3.45}
\end{equation*}
$$

With these considerations (3.40) becomes

$$
\begin{equation*}
[X]=\left[T\left(k_{0}\right)\right]^{H}[P]^{H}\left([G]-k_{0}[C]\right)^{-1}[P]\left[T\left(k_{0}\right)\right], \tag{3.46}
\end{equation*}
$$

and the PRIMA based model order reduction is now applied to

$$
\begin{equation*}
[X]=[P]^{H}\left([G]-k_{0}[C]\right)^{-1}[P] . \tag{3.47}
\end{equation*}
$$

### 3.6.2 WCAWE based MOR

While being very robust, the PRIMA method is restricted to the model order reduction of systems involving frequency independent materials. The asymptotic waveform evaluation has no such restrictions but, in its original form, the process is inherentely ill conditioned, i.e. the process stagnates, that is, increasing the order $q$ of the reduced model above some threshold does not necessarily increase accuracy as the moments slowly become linearly dependent. The Galerkin asymptotic waveform evaluation (GAWE) [79] and its multipoint version [80] increase the bandwidth of
approximation. The breakthrough is represented by the introduction of the wellconditioned asymptotic waveform evaluation (WCAWE), by Slone et al [4], [31]. In WCAWE, the moment generating process is modified in such a manner that the linear independence among moments is ensured.

However, the algorithms presented in [4], [31] consider a single exciation, i.e. a single waveguide mode. In order to allow multiple right-hand sides (modal excitations), the algorithm of [4], [31] is modified as follows. First, for a given order $Q$, the moments are independently generated for each waveguide mode $p$, yielding:

$$
\begin{equation*}
\left[V_{p}\right]=\operatorname{col}\left[v_{p, 1}, v_{p, 2}, \ldots, v_{p, Q}\right] \tag{3.48}
\end{equation*}
$$

The procedure is repeated for all modes $p=1 \ldots P$. In a second phase, a matrix $[V]_{N \times P \cdot Q}$ is formed as

$$
\begin{align*}
{[V] } & =\operatorname{col}\left[V_{1}, V_{2}, \ldots, V_{P}\right] \\
& =\operatorname{col}[\underbrace{v_{1,1}, v_{1,2}, \ldots, v_{1, Q}}_{V_{1}}, \underbrace{v_{2,1}, v_{2,2}, \ldots, v_{2, Q}}_{V_{2}}, \ldots \underbrace{v_{P, 1}, v_{P, 2}, \ldots, v_{P, Q}}_{V_{P}}] \tag{3.49}
\end{align*}
$$

and when a matrix $V_{p}$ is added to $V$, its columns are orthonormalized to all the columns of the previously added $V_{p}$ matrices, using the Gram-Schmidt process.

Once the moment matrix $[V]_{N \times P \cdot Q}, P \cdot Q \ll N$ is constructed, the frequency response (impedance/admittance matrix) can be calculated cf. [4].

### 3.7 Calculation of the far field

For radiating structures, that is structures having $S_{R} \neq 0$, the far-field is produced by the electric and magnetic current densities flowing on the surfaces $S_{R}$ and $S_{A}$, that is

$$
\begin{align*}
\mathbf{E}^{f a r, J}(\mathbf{r}) & =i k_{0} \eta_{0} \frac{e^{-i k_{0} r}}{4 \pi r} \hat{r} \times \hat{r} \times \boldsymbol{\Pi}^{J}(\mathbf{r}), \\
\mathbf{E}^{f a r, M}(\mathbf{r}) & =i k_{0} \frac{e^{-i k_{0} r}}{4 \pi r} \hat{r} \times \boldsymbol{\Pi}^{M}(\mathbf{r}) \tag{3.50}
\end{align*}
$$

where

$$
\begin{equation*}
\boldsymbol{\Pi}^{\Lambda}(\mathbf{r})=\int_{S_{R}+S_{A}} \boldsymbol{\Lambda}\left(\mathbf{r}^{\prime}\right) e^{i k_{0} \hat{r} \mathbf{r}^{\prime}} d S^{\prime} \tag{3.51}
\end{equation*}
$$

and $\boldsymbol{\Lambda}$ stands for $\mathbf{J}=\hat{n} \times \mathbf{H}$ or $\mathbf{M}=-\hat{n} \times \mathbf{E}, \hat{r}$ and $\mathbf{r}^{\prime}$ are the unit vector field point in spherical coordinate system and source point, respectively.

### 3.8 Numerical examples

Several examples are presented here. All computations have been performed on a Dual Core $2,2.5 \mathrm{GHz}$ PC with 4GB RAM.

### 3.8.1 Cylindrical cavity dielectric resonator filter

Fig. 3.2(a) shows a cylindrical cavity dielectric resonator filter. The radius and height of the cavity are 16.5 mm and 20.07 mm , respectively. The input/output ports of the filter are represented by standard SMA coaxial connectors, placed at a hight of 9.9 mm from the cavity's bottom. The inner conductors of the two coaxial connectors have a length of 5.2 mm and are further connected to $2.7 \mathrm{~mm} \times 0.5 \mathrm{~mm}$ $\times 8.9 \mathrm{~mm}$ conducting strips. The strips are bent at $18^{\circ}$ and $28^{\circ}$ angles. Dielectric pucks with $\epsilon_{r}=38,9.655 \mathrm{~mm}$ radius and 7.06 mm height, are placed on the same axis with the cavity housing, at 6.35 mm from the cavity's bottom. The two cylindrical cavities are connected by a $19.05 \mathrm{~mm} \times 20.07 \mathrm{~mm}$ rectangular aperture.

The filter has been discretized using tetrahedrons of polynomial order of two, as depicted in Fig. 3.3. Second order hierarchal basis functions were employed for field modeling, resulting in a discrete problem with 97836 unknowns. The PRIMA model order reduction technique has been used for a fast evaluation of the $S$-matrix. The total computation time was 20 second for 2000 frequency points. The calculated reflection and transmission coefficients agree very well with measurements, as shown in Fig. 3.4.

An notable feature of this filter is that one can easily implement transmission zeros (one below and one above the passband) by simply orienting the feeding strips in the same direction, as shown in Fig. 3.2(b), and therefore changing the sign of the coupling. Fig. 3.5 plots the computed $S$-parameters of the modified filter against the ones of the original design; no distortion in the passband is observed.

### 3.8.2 Microstrip filters

It is well-known that transmission zeros in the insertion loss of microwave filters can be accomplished by cross-coupling, i.e. electromagnetic coupling between nonadjacent resonators. A compact implementation of transmission zeros in filters realized in microstrip technology can be obtained if the resonators are printed on different substrates, thus cross-coupling between nonadjacent resonators is introduced in addition to the normal signal path of the filter [81]. A simple three layer structure that implements a transmission zero above the pass-band, has been designed in [81] and is shown in Fig. 3.6(a). The two substrates are separated by a foam of low dielectric permittivity ( $\epsilon_{r}=1.07$ ).

A delta gap source is employed to excite the filter, in the numerical method presented in [81]. However, a more realistic port model is often required. Therefore, we use, in the present algorithm, a coaxial port having the inner conductor tapped


Figure 3.2: Cylindrical cavity dielectric resonator filter.


Figure 3.3: Dielectric resonator filter, mesh view.


Figure 3.4: Measured vs computed S-parameters of the filter in Fig. 3.2(a).


Figure 3.5: S-parameters of the filters in Fig. 3.2(a) (dotted line) and Fig. 3.2(b) (continuous line).
to the input and output strips. The dimensions of the coaxial port are those of a coaxial K-connector: 0.15 mm inner radius, 0.99 mm outer radius and a relative permittivity of 5 . Finite conductivity (copper, $\sigma=58 \cdot 10^{6} \mathrm{~S} / \mathrm{m}$ ) has been considered for the metallic resonator strips and the box. The two dielectric substrates have a loss tangent of 0.005 .

The structure has been discretized using third order (second order complete) hierarchal basis functions yielding 122430 unknowns. The algorithm has been accelerated by a fast frequency sweep implemented via WCAWE. Fig. 3.6(b) compares the calculated reflection and transmission coefficients (obtained in less than half a minute) with measurements available in [81].

Besides of cross-coupling, introducing transmission zeros in the frequency response of a microwave filter can be accomplished by the use of dual-mode resonators. Very compact, dual-mode microstrip filters can be realized due to the fact that each dual-mode resonator can be used as a doubly tunned resonant circuit, thus the number of resonators for a given degree of the filter is reduced by half [82].

Two dual-mode, open-loop, microstrip filters, designed in [82], are presented in Fig.3.7(a) and 3.8(a). A loading element is tapped onto the open loop. The modal resonant characteristic of the filter can be changed by varying the geometry of the loading element. The input and output of the filters are implemented using two coaxial SMA connectors ( 0.635 mm inner radius, 2.05 outer radius and relative permittivity of 2.05) having the inner conductor soldered to the input and output strips.

Although the measurements in [82] were performed on open, possibly radiating structures, the FEM simulation model considers the filters are placed in conducting boxes. The frequency response of the two filters, shown in Fig. 3.7(b) and 3.8(b), was obtained, again, applying a fast frequency sweep via WCAWE. The number of unknowns was 98426 and 83542 for the filter in Fig. 3.7 and the one in Fig. 3.8, respectively. In both cases, the simulation time ( 500 frequency points) was less than 30 seconds.

A microstrip low-pass filter [2] is shown in Fig. 3.9(a). This time, the excitation is done in terms of microstrip modes, calculated using the transverse-longitudinalfield (TLF) [83], [17] $2 D$ finite element formulation. The computational domain is truncated with the help of first order absorbing boundary conditions (ABC). The ABC surfaces are placed at a distance of 100 mil from the top and sides of the bounding box of the filter. The modified (field-flux) PRIMA algorithm, presented in Section 3.6.1, was used to obtain a reduced order model of the filter. Second order hierarchical basis functions are used to approximate the fields, yielding 74123 unknowns. The computed $S$-parameters are plotted against measurements, in Fig. 3.9(c). The calculation time, for 500 frequency points, was less than 20 seconds.

The band-pass microstrip filter shown in Fig. 3.10, initially built and measured in [84], has been calculated in [85] by taking into consideration an anisotropic substrate having $\overline{\bar{\epsilon}}=$ diag $(3.45,5.12,3.45)$. The $S$-parameters of the filter, obtained using


Figure 3.6: Cross-coupled, boxed microstrip filter [81].


Figure 3.7: Dual-mode microstrip filter with a transmission zero above the pass band [82].

(a) 3D view.

(b) Measured vs. computed $S$-parameters.

Figure 3.8: Dual-mode microstrip filter with a transmission zero below the pass band [82].


Figure 3.9: Low-pass microstrip filter [2]. Dimensions [mils]: $l_{1}=65, l_{2}=45$, $l_{3}=w_{1}=25, w_{2}=60, w_{3}=15, w_{4}=125$. Substrate: $h=25 \mathrm{mil}, \epsilon_{r}=9.2$.
the present approach, are plotted versus measurements [84] and calculations [85] for the case of isotropic substrate [Fig. 3.11(a)] and anisotropic substrate [Fig. $3.11(\mathrm{~b})$ ], respectively. The MOR procedure, applied to the initial discrete system of 131720 unknowns, yielded a reduced order model with only ten degrees of freedom. Less than one minute and a half was needed to calculate the frequency response of the filter in 300 frequency points, most of the time being spent in the model order reduction. The fundamental mode of the microstrip port, calculated by the transverse-longitudinal-field (TLF) 2-D FEM approach [17], was employed to excite the structure.


Figure 3.10: Band-pass microstrip filter [84], [85]. Dimensions [mm]: $h=0.6$, $b=11 \cdot h, w=1.36, s=0.34, d=30.8$.

### 3.8.3 Dielectric loaded horn antenna

A last numerical example is presented in this section. The dielectric loaded rectangular horn antenna, described in Fig. 2.15, has already been computed at Chapter 2 using integral equations, and is now recalculated by the finite element - boundary integral (FE-BI) method.

Here too, the taper of the horn was computed by the mode matching method while only the dielectric loaded part (see Fig. 3.12) has been modeled by the FEBI algorithm. The $S$-matrices of the two parts were combined to find the global scattering matrix of the antenna and the modal excitation at the input of the FEBI section. The far fields were computed in a last step. The radiating surface $S_{R}$ is represented by the horn aperture and its outer geometry. Second order basis functions yielded 26508 interior (volume) unknowns, 3456 electric current density unknowns and 660 magnetic current density unknowns. The calculated far field patterns are in very good agreement with the measurements, as depicted in Fig. 3.13. The calculation time of about 40 seconds per frequency point compares well with MoM that needed roughly 30 seconds for one frequency point. However, the performance is expected to diminish with a rising number of boundary unknowns.


Figure 3.11: $S$-parameters of the filter in Fig. 3.10.


Figure 3.12: $3-D$ mesh view of the dielectric loaded rectangular horn.


Figure 3.13: Measured and computed radiation patterns at 6 GHz .

# Hybrid FE-BI simulation of boxed structures 

### 4.1 Introduction

Although very flexible, the methods presented in the previous two chapters share a common characteristic, that is the whole computational domain must be discretized. However, more efficient algorithms can be designed when dealing with structures for which the Green's functions are known (e.g. layered media, rectangular and spherical cavities etc.). In this case, the specialized Green's function is used to describe the electromagnetic behavior of the structure, whereas only those geometric parts that do not obey the Green's function formalism are meshed.

Rectangular cavities loaded with arbitrarily shaped conductors and and/or dielectric bodies (see Fig. 4.2) are useful key building blocks for the design of many common types of microwave components [32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, $43,44,45]$, such as compact filters for terrestrial and space applications, cf. e.g. [32, 33, 34], or broad-band transitions [46].

Hitherto reported approaches specialized to rectangular cavities include the boundary integral - resonant mode expansion method [14] and the state-space integral equation method [43] but ultimately exhibit limited flexibility as they merely allow radially symmetric insets or cylindrical dielectrics in rectangular boxes.

This chapter presents a hybrid finite-element boundary-integral (FE-BI) method formulated in a rectangular cavity environment. Finite elements are employed to characterize the inhomogeneous and arbitrarily shaped material in the cavity, while integral equations deal with the necessary boundary conditions. The present algorithm extends the known finite element - boundary integral formulation at radiators/scatterers [27] to shielded environments. All boundary integrals involving rectangular cavity Green's functions are efficiently evaluated utilizing the Ewald transform [47, 48, 49]. The described method yields the generalized admittance or scattering matrices, respectively, of the structure under investigation.

The major advantage of the present approach is the high efficiency, because: First, due to the separation of the Green's functions into static series (zero frequency limit), whose convergence is enhanced with the help of the Ewald transform, and fast convergent dynamic series (higher frequency correction), the most computational intensive part of the algorithm is performed only once in a frequency sweep, and
second, as a consequence of the use of the cavity Green's functions, only a small portion of the computational domain must be discretized, thus drastically reducing the number of unknowns.

Numerical examples, including rectangular and cylindrical cavity combline filters and dielectric resonator filters, are given to demonstrate the accuracy and the efficiency of the present approach.

### 4.2 Expressions for scattered fields

In a general cavity of volume $V_{C}$ and boundary surface $S_{C}$, the scattered electric and magnetic fields due to electric and magnetic sources, can be expressed in mixed potential form [77]

$$
\begin{align*}
& \mathbf{E}(S ; \mathbf{J} ; \mathbf{r})=-i k_{0} \eta_{0} \int_{S} \overline{\overline{\mathbf{G}}}_{A} \mathbf{J}\left(\mathbf{r}^{\prime}\right) d S^{\prime}-\frac{i \eta_{0}}{k_{0}} \nabla \int_{S} g_{v} \nabla^{\prime} \mathbf{J}\left(\mathbf{r}^{\prime}\right) d S^{\prime}  \tag{4.1a}\\
& \mathbf{E}(S ; \mathbf{M} ; \mathbf{r})=\frac{1}{2} \hat{n} \times \mathbf{M}(\mathbf{r})-V P \int_{S} \overline{\bar{G}}_{E M} \mathbf{M}\left(\mathbf{r}^{\prime}\right) d S^{\prime}  \tag{4.1b}\\
& \mathbf{H}(S ; \mathbf{J} ; \mathbf{r})=-\frac{1}{2} \hat{n} \times \mathbf{J}(\mathbf{r})+V P \int_{S} \overline{\bar{G}}_{H J} \mathbf{J}\left(\mathbf{r}^{\prime}\right) d S^{\prime}  \tag{4.2a}\\
& \mathbf{H}(S ; \mathbf{M} ; \mathbf{r})=-\frac{i k_{0}}{\eta_{0}} \int_{S} \overline{\overline{\mathbf{G}}}_{F} \mathbf{M}\left(\mathbf{r}^{\prime}\right) d S^{\prime}-\frac{i}{k_{0} \eta_{0}} \nabla \int_{S} g_{w} \nabla^{\prime} \mathbf{M}\left(\mathbf{r}^{\prime}\right) d S^{\prime} \tag{4.2b}
\end{align*}
$$

where $\overline{\overline{\mathbf{G}}}_{A}$ and $\overline{\overline{\mathbf{G}}}_{F}$ are the vector potential dyadic Green's functions, $g_{v}$ and $g_{w}$ are the scalar potential Green's functions and $\overline{\overline{\mathbf{G}}}_{E M}$ and $\overline{\overline{\mathbf{G}}}_{H J}$ represent the Green's function for the electric field due to magnetic current densities and the Green's function for the magnetic field due to electric current densities, respectively. The notations $i, k_{0}$ and $\eta_{0}$ denote the imaginary part, free space wavenumber and free space characteristic impedance, respectively. VP means Cauchy's principal value.

The potential Green's functions for the electric field, in the Lorentz gauge, obey

$$
\begin{align*}
\nabla^{2} \overline{\overline{\mathbf{G}}}_{A}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)+k_{0}^{2} \overline{\overline{\mathbf{G}}}_{A}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) & =-\overline{\overline{\mathbf{I}}} \delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right)  \tag{4.3a}\\
\hat{n} \times \overline{\overline{\mathbf{G}}}_{A}\left(\mathbf{r} \in S_{C}, \mathbf{r}^{\prime}\right) & =0  \tag{4.3b}\\
\nabla^{2} g_{v}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)+k_{0}^{2} g_{v}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) & =-\delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right)  \tag{4.3c}\\
g_{v}\left(\mathbf{r} \in S_{C}, \mathbf{r}^{\prime}\right) & =0 \tag{4.3~d}
\end{align*}
$$

while the following relations hold for the potential Green's functions for the magnetic
field, also in the Lorentz gauge

$$
\begin{align*}
\nabla^{2} \overline{\overline{\mathbf{G}}}_{F}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)+k_{0}^{2} \overline{\overline{\mathbf{G}}}_{F}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) & =-\overline{\overline{\mathbf{I}}} \delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right)  \tag{4.4a}\\
\hat{n} \times \nabla \times \overline{\overline{\mathbf{G}}}_{F}\left(\mathbf{r} \in S_{C}, \mathbf{r}^{\prime}\right) & =0  \tag{4.4b}\\
\nabla^{2} g_{w}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)+k_{0}^{2} g_{w}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) & =-\delta\left(\mathbf{r}-\mathbf{r}^{\prime}\right)  \tag{4.4c}\\
\hat{n} \cdot \nabla g_{w}\left(\mathbf{r} \in S_{C}, \mathbf{r}^{\prime}\right) & =0 \tag{4.4~d}
\end{align*}
$$

The field Green dyads are related to the potential dyads [86], [13] by

$$
\begin{align*}
\overline{\overline{\mathbf{G}}}_{E M}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) & =\nabla \times \overline{\overline{\mathbf{G}}}_{F}\left(\mathbf{r}, \mathbf{r}^{\prime}\right)  \tag{4.5a}\\
\overline{\overline{\mathbf{G}}}_{H J}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) & =\nabla \times \overline{\overline{\mathbf{G}}}_{A}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) \tag{4.5b}
\end{align*}
$$

and the reciprocity relations

$$
\begin{align*}
\overline{\overline{\mathbf{G}}}_{E M}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) & =\left[\overline{\overline{\mathbf{G}}}_{H J}\left(\mathbf{r}^{\prime}, \mathbf{r}\right)\right]^{T}  \tag{4.6a}\\
\overline{\overline{\mathbf{G}}}_{H J}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) & =\left[\overline{\overline{\mathbf{G}}}_{E M}\left(\mathbf{r}^{\prime}, \mathbf{r}\right)\right]^{T} \tag{4.6b}
\end{align*}
$$

hold too.

### 4.3 Rectangular cavity Green's functions

The solution to equations (4.3) and (4.4) is usually given in terms of infinite series of cavity eigenfunctions [86], [87]. Poisson summation formula can be applied to the eigenfunction expansion [88], yielding a solution in terms of image series.

It can be shown [86], [89], [90], [47] that, in the case of rectangular cavity filled with homogeneous and isotropic medium, the potential Green's function dyads are diagonal dyads, whereas the field Green's functions are antidiagonal dyads, i.e.

$$
\begin{gather*}
\overline{\overline{\mathbf{G}}}_{A, F}=\left(\begin{array}{lll}
G_{A, F}^{x x} & 0 & 0 \\
0 & G_{A, F}^{y y} & 0 \\
0 & 0 & G_{A, F}^{z z}
\end{array}\right)  \tag{4.7}\\
\overline{\overline{\mathbf{G}}}_{E M, H J}=\left(\begin{array}{lll}
0 & G_{E M, H J}^{x y} & G_{E M, H J}^{x z} \\
G_{E M, H J}^{y x} & 0 & G_{E M, H J}^{y z} \\
G_{E M, H J}^{z x} & G_{E M, H J}^{z y} & 0
\end{array}\right) \tag{4.8}
\end{gather*}
$$

Since the field Green's functions can be derived from the vector potential Green dyads via (4.5a) and (4.5b), we concentrate on the computation of $\overline{\overline{\mathbf{G}}}_{A}, \overline{\overline{\mathbf{G}}}_{F}$ and $g_{v}$, $g_{w}$.

Considering the axis system in Fig. 4.1, the eigenfunction expansion of the


Figure 4.1: Rectangular cavity coordinate system.
scalars $g_{v}, g_{w}$ and of any term of the dyadics $\overline{\overline{\mathbf{G}}}_{A}$ and $\overline{\overline{\mathbf{G}}}_{F}$, can be expressed in the form [89], [90]:

$$
\begin{align*}
G_{A, F}^{s s} & =\frac{1}{a b c} \sum_{m, n, p=0}^{\infty} \alpha_{m n p} f_{m}\left(k_{m x} x, k_{m x} x^{\prime}\right) g_{n}\left(k_{n y} y, k_{n y} y^{\prime}\right) h_{p}\left(k_{p z} z, k_{p z} z^{\prime}\right), \\
g_{v, w} & =\frac{1}{a b c} \sum_{m, n, p=0}^{\infty} \alpha_{m n p} f_{m}\left(k_{m x} x, k_{m x} x^{\prime}\right) g_{n}\left(k_{n y} y, k_{n y} y^{\prime}\right) h_{p}\left(k_{p z} z, k_{p z} z^{\prime}\right), \tag{4.9}
\end{align*}
$$

where $s$ stands for $x, y$ or $z$, and $f_{m} \cdot g_{n} \cdot h_{p}$ represents the $m n p$ cavity eigenmode, given in Table C. 1 (see Appendix C). The modal coefficients in (4.9) are given by

$$
\begin{equation*}
\alpha_{m n p}=\epsilon_{m} \epsilon_{n} \epsilon_{p} \frac{1}{k_{m n p}^{2}-k_{0}^{2}}, \tag{4.10}
\end{equation*}
$$

where

$$
\begin{align*}
\epsilon_{i} & =1+\delta_{0 i}, \\
k_{m n p} & =\sqrt{k_{m x}^{2}+k_{n y}^{2}+k_{p z}^{2}},  \tag{4.11}\\
k_{m x} & =\frac{m \pi}{a}, k_{n y}=\frac{n \pi}{b}, k_{p z}=\frac{p \pi}{c},
\end{align*}
$$

with $\delta$ standing for the Kronecker delta symbol.
Applying the Poisson summation formula [88] to the modal expansion (4.9),
yields the image series representation of the rectangular box Green functions in the form [90], [91]

$$
\begin{align*}
G_{A}^{s s} & =\frac{1}{4 \pi} \sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} A_{i}^{s s} \frac{e^{-i k_{0} R_{i, m n p}}}{R_{i, m n p}}  \tag{4.12a}\\
g_{v} & =\frac{1}{4 \pi} \sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} V_{i} \frac{e^{-i k_{0} R_{i, m n p}}}{R_{i, m n p}}  \tag{4.12b}\\
G_{F}^{s s} & =\frac{1}{4 \pi} \sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} F_{i}^{s s} \frac{e^{-i k_{0} R_{i, m n p}}}{R_{i, m n p}}  \tag{4.12c}\\
g_{w} & =\frac{1}{4 \pi} \sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} W_{i} \frac{e^{-i k_{0} R_{i, m n p}}}{R_{i, m n p}} \tag{4.12~d}
\end{align*}
$$

where $s$ stands for $x, y$ or $z, A_{i}, V_{i}, F_{i}$ and $W_{i}$ denote the sign of the $i$-th image (Tab. C. 3 and Tab. C.4). The distance to the $i$-th image is given by

$$
\begin{equation*}
R_{i, m n p}=\left[\left(X_{i}+2 m a\right)^{2}+\left(Y_{i}+2 n b\right)^{2}+\left(Z_{i}+2 p c\right)^{2}\right]^{\frac{1}{2}} \tag{4.13}
\end{equation*}
$$

There are, for each $m n p$ term, two sets of four images that sum up in the expansions (4.12), as shown in Fig. C. 1 and Fig. C. 3 for, respectively, $\overline{\bar{G}}_{A}$ and $g_{v}$ and Fig. C. 2 and Fig. C. 4 for $\overline{\overline{\mathbf{G}}}_{F}$ and $g_{w}$, in Appendix C. The position and sign of the eight images are summarized in Tab. C. 3 for electric sources and Tab. C. 4 for magnetic sources, also in Appendix C.

### 4.4 Green's functions evaluation via Ewald transform

The modal series (4.9) represent an infinite summation of cavity eigenmodes, thus, they automatically satisfy the boundary conditions at cavity walls. They exhibit fast convergence if the source-observer distance is far from the singular behavior of the Green's function, while thousands or tens of thousands of terms are needed to achieve convergence in the near-singular and singular regions. Contrary to the modal series, the image series (4.12) sums up rational terms that resemble the singular behavior of the Green's function, thus yielding good convergence in the case of small sourceobserver distances. The disadvantage of the image series is the slow convergence if the observation point is far from the source point.

As neither the modal or image series are suitable for a numerical implementation, the Ewald method [48], [49] is further employed to yield series representations that need a reasonable number of terms to achieve convergence.

We begin by splitting the Green function (generically named $\Psi$ below) into static
(zero frequency limit) and dynamic parts

$$
\begin{equation*}
\Psi=\underbrace{\Psi_{s t a}^{i m g}}_{\text {static }}+\underbrace{\left(\Psi^{m o d}-\Psi_{s t a}^{m o d}\right)}_{\text {dynamic }}, \tag{4.14}
\end{equation*}
$$

where $\Psi_{s t a}^{i m g}, \Psi^{\text {mod }}$ and $\Psi_{s t a}^{m o d}$ are the static image series, modal series and static modal series, respectively.

The zero frequency limit of $\Psi$ reads

$$
\begin{align*}
\Psi_{s t a}^{i m g} & =\lim _{k_{0} \rightarrow 0}\left[\frac{1}{4 \pi} \sum_{m, n, p=0}^{\infty} \sum_{i=0}^{7} S_{i} \frac{e^{-i k_{0} R_{i, m n p}}}{R_{i, m n p}}\right]  \tag{4.15}\\
& =\frac{1}{4 \pi} \sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} S_{i} \frac{1}{R_{i, m n p}} .
\end{align*}
$$

The dynamic term in (4.14) becomes

$$
\begin{align*}
\Psi_{d y n}^{m o d} & =\Psi^{m o d}-\lim _{k_{0} \rightarrow 0} \Psi^{m o d} \\
& =\frac{1}{a b c} \sum_{m, n, p=0}^{\infty} \beta_{m n p} f_{m}\left(k_{m x} x, k_{m x} x^{\prime}\right) g_{n}\left(k_{n y} y, k_{n y} y^{\prime}\right) h_{p}\left(k_{p z} z, k_{p z} z^{\prime}\right),  \tag{4.16}\\
\beta_{m n p} & =\epsilon_{m} \epsilon_{n} \epsilon_{p} \frac{k_{0}^{2}}{k_{m n p}^{2}\left(k_{m n p}^{2}-k_{0}^{2}\right)} .
\end{align*}
$$

The dynamic series in (4.16) has a rate of decay proportional to $k_{m n p}^{4}$, therefore it is clear that the static parts are responsible for the poor overall convergence.

The general form of the image series (4.12) is

$$
\begin{equation*}
\Psi^{i m g}=\frac{1}{4 \pi} \sum_{m, n, p=0}^{\infty} \sum_{i=0}^{7} S_{i} \frac{e^{-i k R_{i, m n p}}}{R_{i, m n p}}, \tag{4.17}
\end{equation*}
$$

and its static limit reads

$$
\begin{equation*}
\Psi_{s t a}^{i m g}=\frac{1}{4 \pi} \sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} S_{i} \frac{1}{R_{i, m n p}} \tag{4.18}
\end{equation*}
$$

Using the identity

$$
\begin{equation*}
\frac{1}{R}=\frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-R^{2} t^{2}} d t=\frac{2}{\sqrt{\pi}} \int_{0}^{E} e^{-R^{2} t^{2}} d t+\frac{2}{\sqrt{\pi}} \int_{E}^{\infty} e^{-R^{2} t^{2}} d t \tag{4.19}
\end{equation*}
$$

one can rewrite (4.18) as

$$
\begin{align*}
\Psi_{s t a}^{i m g} & =\underbrace{\frac{1}{2 \pi \sqrt{\pi}} \sum_{m, n, p=0}^{\infty} \sum_{i=0}^{7} S_{i} \int_{0}^{E} e^{-R^{2} t^{2} d t}}_{\Psi_{1}} \\
& +\underbrace{\frac{1}{2 \pi \sqrt{\pi}} \sum_{m, n, p=0}^{\infty} \sum_{i=0}^{7} S_{i} \int_{E}^{\infty} e^{-R^{2} t^{2}} d t}_{\Psi_{2}} \tag{4.20}
\end{align*}
$$

The functions $\Psi_{1}$ and $\Psi_{2}$ are evaluated to [47], [48]

$$
\begin{align*}
& \Psi_{1}=\frac{1}{a b c} \sum_{m, n, p=0}^{\infty} \gamma_{m n p} f_{m}\left(k_{m x} x, k_{m x} x^{\prime}\right) g_{n}\left(k_{n y} y, k_{n y} y^{\prime}\right) h_{p}\left(k_{p z} z, k_{p z} z^{\prime}\right)  \tag{4.21a}\\
& \Psi_{2}=\frac{1}{4 \pi} \sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} S_{i} \frac{\operatorname{erfc}\left(R_{i, m n p} E\right)}{R_{i, m n p}},  \tag{4.21b}\\
& \gamma_{m n p}=\epsilon_{m} \epsilon_{n} \epsilon_{p} \frac{e^{-\frac{k_{m n p}^{2}}{4 E^{2}}}}{k_{m n p}^{2}} \tag{4.22}
\end{align*}
$$

where erfc is the complementary error function.

The parameter $E \in[0, \infty)$ in (4.21) and (4.22) is called splitting parameter and its value must be established a priori. If $E=0, \Psi_{1}$ vanishes and $\Psi^{i m g}=\Psi_{2}$ becomes a pure image series. If $E \rightarrow \infty$, the situation is reversed: $\Psi_{2}$ vanishes and $\Psi^{i m g}=\Psi_{1}$ is a pure modal series. There is no straight-forward method to determine a value of $E$ that yields optimal overall convergence, although the formula

$$
\begin{equation*}
E=\left(\pi^{2} \frac{\frac{1}{a^{2}}+\frac{1}{b^{2}}+\frac{1}{c^{2}}}{a^{2}+b^{2}+c^{2}}\right)^{\frac{1}{4}} \tag{4.23}
\end{equation*}
$$

is suggested in [92] and will be adopted here as well.

With the potential Green functions brought to a convergent form, the field dyadics $\overline{\overline{\mathbf{G}}}_{E M}$ and $\overline{\overline{\mathbf{G}}}_{H J}$ can be obtained from (4.5a) and (4.5b)

$$
\overline{\overline{\mathbf{G}}}_{E M}=\nabla \times \overline{\overline{\mathbf{G}}}_{F}=\left(\begin{array}{lll}
0 & -\frac{\partial G_{F}^{y y}}{\partial z} & \frac{\partial G_{F}^{z z}}{\partial y}  \tag{4.24}\\
\frac{\partial G_{F}^{x x}}{\partial z} & 0 & -\frac{\partial G_{F}^{z z}}{\partial x} \\
-\frac{\partial G_{F}^{x x}}{\partial y} & \frac{\partial G_{F}^{y y}}{\partial x} & 0
\end{array}\right)
$$

$$
\overline{\overline{\mathbf{G}}}_{H J}=\nabla \times \overline{\overline{\mathbf{G}}}_{A}=\left(\begin{array}{lll}
0 & -\frac{\partial G_{A}^{y y}}{\partial z} & \frac{\partial G_{A}^{z z}}{\partial y}  \tag{4.25}\\
\frac{\partial G_{A}^{x x}}{\partial z} & 0 & -\frac{\partial G_{A}^{z z}}{\partial x} \\
-\frac{\partial G_{A}^{x x}}{\partial y} & \frac{\partial G_{A}^{y y}}{\partial x} & 0
\end{array}\right) .
$$

Like in the case of potential Green's functions, the field Green's functions are expressed as sums of dynamic and static series, i.e.

$$
\begin{align*}
\overline{\overline{\mathbf{G}}}_{E M} & =\underbrace{\left(\overline{\overline{\boldsymbol{\Upsilon}}}^{\text {mod }}-\overline{\bar{\Upsilon}}_{\text {sta }}^{m o d}\right)}_{\text {dynamic }}+\underbrace{\overline{\mathbf{\Upsilon}}_{\text {sta }}^{i m g}}_{\text {static }}  \tag{4.26}\\
& =\underbrace{\overline{\overline{\mathbf{\Upsilon}}}_{d y n}}_{\text {dynamic }}+\underbrace{\overline{\bar{\Upsilon}}_{1}}_{\text {static modal }}+\underbrace{\overline{\overline{\boldsymbol{\Upsilon}}}_{2}}_{\text {static image }}
\end{align*}
$$

Taking the derivative of $\Psi_{2}$ in (4.21), with respect to $u=x, y, z$, we get

$$
\begin{equation*}
\Upsilon_{2}^{u}=\frac{\partial \Psi_{2}}{\partial u}=-\sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} S_{i} u\left(\frac{\operatorname{erfc}\left(E R_{i, m n p}\right)}{4 \pi R_{i, m n p}^{3}}+\frac{e^{-E^{2} R_{i, m n p}^{2}}}{2 \pi \sqrt{\pi} R_{i, m n p}^{2}}\right) \tag{4.27}
\end{equation*}
$$

yielding the following expressions for the static image series of $\overline{\overline{\mathbf{G}}}_{E M}$ :

$$
\begin{align*}
& \Upsilon_{2}^{x y}=-\sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} F_{i}^{y y} \Upsilon_{2}^{Z_{i}+2 p c},  \tag{4.28a}\\
& \Upsilon_{2}^{y x}=\sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} F_{i}^{x x} \Upsilon_{2}^{Z_{i}+2 p c},  \tag{4.28b}\\
& \Upsilon_{2}^{x z}=\sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} F_{i}^{z z} \Upsilon_{2}^{Y_{i}+2 n b},  \tag{4.28c}\\
& \Upsilon_{2}^{z x}=-\sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} F_{i}^{x x} \Upsilon_{2}^{Y_{i}+2 n b}  \tag{4.28~d}\\
& \Upsilon_{2}^{y z}=-\sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} F_{i}^{z z} \Upsilon_{2}^{X_{i}+2 m a},  \tag{4.28e}\\
& \Upsilon_{2}^{z y}=\sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} F_{i}^{y y} \Upsilon_{2}^{X_{i}+2 m a} \tag{4.28f}
\end{align*}
$$

The dynamic and static modal series of $\overline{\overline{\mathbf{G}}}_{E M}$ have the same generic form like (4.16) and (4.21a), with $f_{m}, g_{n}$ and $h_{p}$ presented in Tab. C. 2 in Appendix C. The dyadic $\overline{\overline{\mathbf{G}}}_{H J}$ is determined with the help of the reciprocity relations (4.6).

### 4.5 Formulation

As the goal is the simulation of arbitrarily loaded rectangular cavities, the question that arises is: What method shall be used to model the fields inside penetrable bodies ? A first thought that comes to mind is the algorithm described in Chapter 2 . Let aside the fact that only homogeneous bodies can be handled, a closer look to the EFIE-PMCHWT method reveals a major drawback when formulated for boxed structures. There are two equivalent problems in the EFIE-PMCHWT formulation. The exterior problem considers the whole space (the whole cavity in our case) filled with a material having $k_{1}=k_{0} \sqrt{\epsilon_{r}^{e x t}} \mu_{r}^{\text {ext }}$. In turn, the formulation of the interior problem fills the whole space (also the whole cavity in our case) with a homogeneous material having now $k_{2}=k_{0} \sqrt{\epsilon_{r}^{i n t} \mu_{r}^{i n t}}$. Now, one clearly observes that the convergence rate of the dynamic modal series in (4.16) drastically diminishes with a rising $k_{0}$. But the interior equivalent problem fills the cavity with some $k_{2}>k_{0}$, thus thousands or even tens of thousands of cavity modes might be needed to reach convergence if, for instance, a high permittivity dielectric is present. In other words, the equivalent interior problem of the PMCHWT formulation might yield an overmoded rectangular cavity, thus drastically diminishing the convergence of the Green's functions. A numerical implementation of the EFIE-PMCHWT algorithm for boxed environments has been programmed. It was found that, indeed, its efficiency drops almost quadratically with rising material constants.

The next attempt was to formulate the fields inside penetrable bodies via volume integral equations (VIE), discretized by Schaubert-Wilton-Glisson (SWG) basis [93], or in conjunction with the solenoidal basis functions developed by Mendes et al [94], [95], [96], [97]. Although there is no theoretical difficulty in formulating the VIE inside a boxed environment, the discretization of the $3-D$ polarization currents associated with the VIE yields fully populated matrices. The numerical implementation of the VIE revealed the fact that the solenoidal basis of Mendes [94], [95], [96] yield faster convergence rates than the SWG basis. The algorithm works well if materials with small permittivity/permeability are analyzed, but a drastic augmentation of the number of unknowns is experienced if hight permittivity/permeability scatterers are present.

On the other side, differential equation based methods, like the Finite Element Method (FEM) [16], [29], [98], exhibit no difficulties if high permittivity/permeability materials are encountered. Moreover, sparse linear equation systems are involved in their numerical discretization. Therefore, the FEM is a good candidate to handle high permittivity/permeability materials. However, the mesh termination, in the case of FEM, must be accompanied by the fulfillment of the required boundary conditions [29]. These boundary conditions can be incorporated via integral equations, formulated in the region exterior to the material body. Hence, the combination of the finite element method with integral equation techniques appears to be the best choice.

Figure 4.2 shows the geometry of a general boxed structure. A rectangular cavity of dimensions $a \times b \times c$, loaded with arbitrarily shaped conductors and penetrable bodies, is connected to $n$ waveguides through $n$ planar ports of cross section $S_{p, 1}, S_{p, 2}, \ldots, S_{p, n}$. The total port surface is designated as $S_{P} \equiv \sum_{i=1}^{N} S_{p, i} ; V_{D}$ and $S_{D}$ represent, respectively, the volume and the boundary surface of the penetrable (possibly inhomogeneous) bodies and $S_{C}$ denotes all conducting surfaces outside $V_{D}$.

In conformity with the equivalence principle, electric current densities $\mathbf{J}=\hat{n} \times$ $\mathbf{H}$ are introduced on $S_{C}$, while electric current densities $\mathbf{J}$ and magnetic current densities $\mathbf{M}=\mathbf{E} \times \hat{n}$ are employed on the surface $S_{D}$. The port cross section $S_{P}$ is covered with a perfectly conducting sheet, and magnetic current densities $\mathbf{M}_{P}=\mathbf{E} \times \hat{n}_{p}$ are introduced in the usual way [26] [24], in order to preserve the continuity of the tangential electric field; with $\hat{n}_{p}$ being the unit normal vector to $S_{P}$ and pointing outwards.


Figure 4.2: Boxed structure with arbitrary loadings.

Inside the source-free region $V_{D}$ the electric field satisfies the homogeneous wave equation

$$
\begin{array}{r}
\nabla \times \overline{\bar{\mu}}_{r}^{-1} \nabla \times \mathbf{E}-k_{0}^{2} \overline{\bar{\epsilon}}_{\epsilon} \mathbf{E}=0,  \tag{4.29}\\
\hat{n} \times \mathbf{E}=0 \text { on } \Gamma_{P E C},
\end{array}
$$

where $\Gamma_{P E C}$ denotes any perfectly conducting surface in $V_{D}$ and $\overline{\bar{\epsilon}}_{r}$, $\overline{\bar{\mu}}_{r}$ denote, respectively, the relative permittivity and relative permeability dyads.

Similarly, inside the source-free region $V_{D}$, the magnetic field satisfies the homo-
geneous wave equation

$$
\begin{align*}
& \nabla \times \overline{\bar{\epsilon}}_{r}^{-1} \nabla \times \mathbf{H}-k_{0}^{2} \overline{\bar{\mu}}_{r} \mathbf{H}=0,  \tag{4.30}\\
& \hat{n} \times \bar{\epsilon}_{r}^{-1} \nabla \times \mathbf{H}=0 \text { on } \Gamma_{P E C} .
\end{align*}
$$

Using the terminology of [27], the solution spaces for the electric field $\mathbf{E}$, magnetic field $\mathbf{H}$, electric current density $\mathbf{J}$ and magnetic current density $\mathbf{M}$, are defined, respectively, as:

$$
\begin{align*}
W^{E} & =\left\{\mathbf{a} \in H\left(\operatorname{curl}, V_{D}\right) \mid \hat{n} \times \mathbf{a}=0 \text { on } \Gamma_{P E C}\right\},  \tag{4.31a}\\
W^{H} & =\left\{\mathbf{a} \in H\left(\operatorname{curl}, V_{D}\right)\right\},  \tag{4.31b}\\
T^{H} & =\left\{\mathbf{a} \in \operatorname{Span}\left\{\hat{n} \times\left.\mathbf{b}\right|_{S_{D}}\right\} \mid \mathbf{b} \in \mathbf{W}^{H}\right\},  \tag{4.31c}\\
T^{E} & =\left\{\mathbf{a} \in \operatorname{Span}\left\{\hat{n} \times\left.\mathbf{b}\right|_{S_{D}}\right\} \mid \mathbf{b} \in \mathbf{W}^{E}\right\} . \tag{4.31d}
\end{align*}
$$

Through a Galerkin testing procedure, the boundary value problems in (4.29) and (4.30) will be transformed into variational problems, using two different formulations:

- The E-J formulation: The starting point is (4.29). The unknowns are the electric field and the electric current density (tangential magnetic field).
- The H-M formulation: The starting point is (4.30). The unknowns are the magnetic field and the magnetic current density (tangential electric field).


### 4.5.1 The E-J formulation

Testing (4.29) with the solution space of the electric field and using the divergence theorem and Maxwell's equations, yields

$$
\begin{equation*}
\int_{V_{D}}\left(\nabla \times \mathbf{W}^{E} \overline{\bar{\mu}}_{r}^{-1} \nabla \times \mathbf{E}-k_{0}^{2} \mathbf{W}^{E} \overline{\bar{\epsilon}}_{r} \mathbf{E}\right) d S+i k_{0} \eta_{0} \int_{S_{D}} \hat{n} \times \mathbf{W}^{E} \mathbf{H}=0 . \tag{4.32}
\end{equation*}
$$

A second equation is obtained by enforcing the boundary conditions for the electric field, and testing it with the solution space of $\mathbf{J}$, i.e. $T^{H}$, yielding

$$
\begin{align*}
-\int_{S} \mathbf{T}^{H} \hat{n} \times \hat{n} \times\left[\mathbf{E}\left(S_{D}+S_{C} ; \mathbf{J}\right)\right. & \left.+\mathbf{E}\left(S_{D} ; \mathbf{M}\right)+\mathbf{E}\left(S_{P} ; \mathbf{M}_{P}\right)\right] d S \\
& =\left\{\begin{array}{ll}
0 & \text { on } S_{C} \\
\int_{S_{D}} \mathbf{T}^{H} \hat{n} \times \mathbf{M} d S & \text { on } S_{D}
\end{array},\right. \tag{4.33}
\end{align*}
$$

where the total electric field has been written as a superposition of individual electric fields produced by electric and magnetic sources in $V_{C}$.

Now, (4.32) and (4.33) are added together as each of them can be recovered by letting $\mathbf{W}^{E}=0$ or $\mathbf{T}^{H}=0$, but before the addition is performed (4.33) is multiplied
with $i k_{0} \eta_{0}$ in order to ensure the symmetry of the resulting equation. This procedure can be regarded as an extension to boxed environments excited by apertures of the E-J formulation presented in [27] for the free-space case.

### 4.5.2 The H-M formulation

Similarly to the E-J formulation, we start by testing (4.30) with the solution space $W^{H}$ of the magnetic field, yielding

$$
\begin{equation*}
\int_{V_{D}} \nabla \times \mathbf{W}^{H} \overline{\bar{\epsilon}}_{r}^{-1} \nabla \times \mathbf{H}-\frac{i k_{0}}{\eta_{0}} \int_{S_{D}} \hat{n} \times \mathbf{W}^{H} \mathbf{E}=0 \tag{4.34}
\end{equation*}
$$

This is an equation in $\mathbf{H}$ (that also includes the tangential magnetic field $\mathbf{J}=\hat{n} \times \mathbf{H}$ ), $\mathbf{M}$ and $\mathbf{M}_{P}$, tested with the solution space of $\mathbf{H}$. Two additional equations are obtained by enforcing the boundary conditions for the magnetic field on $S_{D}$, and for the electric field on $S_{C}$

$$
\begin{array}{r}
\int_{S} \mathbf{T}^{E} \hat{n} \times \hat{n} \times\left[\mathbf{H}\left(S_{D}+S_{C} ; \mathbf{J}\right)+\mathbf{H}\left(S_{D} ; \mathbf{M}\right)+\mathbf{H}\left(S_{P} ; \mathbf{M}_{P}\right)\right] d S \\
=\int_{S_{D}} \mathbf{T}^{E} \hat{n} \times \mathbf{J} d S \text { on } S_{D}(\mathrm{MFIE}) \\
\begin{aligned}
& \int_{S} \mathbf{T}^{H} \hat{n} \times \hat{n} \times\left[\mathbf{E}\left(S_{D}+S_{C} ; \mathbf{J}\right)+\mathbf{E}\left(S_{D} ; \mathbf{M}\right)+\mathbf{E}\left(S_{P} ; \mathbf{M}_{P}\right)\right] d S \\
&=0 \text { on } S_{C}(\mathrm{EFIE})
\end{aligned} \tag{4.36}
\end{array}
$$

This is one of the differences between the E-J and the H-M formulations: In the E-J formulation the MFIE is used in the boundary integral resulted from the testing of (4.29) and the EFIE represents an additional equation, while in the $\mathrm{H}-\mathrm{M}$ formulation the EFIE is replaced in the boundary integral resulted from the testing of (4.30) while both MFIE (on $S_{D}$ ) and EFIE (on $S_{C}$ ) are employed as additional equations. This choice, in the H-M formulation, is a consequence of the fact that $S_{C}$ might be an open surface, case which would render numerical difficulties in enforcing the MFIE on $S_{C}$ (see Chapter 2).

Again, (4.34), (4.35) and (4.36) can be added together, as each of them can be recovered back by simply letting $\mathbf{W}^{H}=0, \mathbf{T}^{E}=0$ or $\mathbf{T}^{H}\left(\mathbf{r} \in S_{C}\right)=0$. The MFIE in (4.35) is scaled by $\frac{i k 0}{\eta_{0}}$ and the EFIE in (4.36) is scaled by $-\frac{i k 0}{\eta_{0}}$ before the adition is performed. Again,the method outlined above can be viewed as an extension to shielded environments excited by apertures of the H-M formulation presented in [27] for the free-space case.

We must point out that there is no computational advantage of the E-J formulation over the H-M formulation or vice versa. However, both approaches have been
numerically implemented, the E-J formulation is used for calculation whereas the H-M algorithm is optionally employed for cross-checking.

### 4.6 Basis functions and matrix assembly

A numerical solution requires discrete approximations of the spaces $W^{E}, W^{H}, T^{E}$ and $T^{H}$ defined in (4.31). The electric field (space $W^{E}$ ) and the magnetic field (space $W^{H}$ ) require curl-conforming discretizations, whereas the electric and magnetic current densities demand div-conforming discretizations for $T^{H}$ and $T^{E}$. One observes that $T^{E}$ and $T^{H}$ can be obtained from $W^{E}$ and $W^{H}$. Consequently, defining appropriate discretizations for $W^{E}$ and $W^{H}$ will automatically ensure proper approximations of the spaces $T^{E}$ and $T^{H}$. Among the most popular basis functions are the ones defined in [19] (interpolatory) and [20] (hierarchical). In this chapter, the hierarchical basis from [20], up to mixed polynomial order of three are employed to discretize $W^{E}$ and $W^{H}$. Hierarchical basis functions were chosen here rather than the interpolatory ones because they allow $p$-adaptation, i.e. the mixing of basis functions of different orders within the same mesh. Tetrahedral and triangular elements of maximum polynomial order of two are used to represent the volume $V_{D}$ and the surfaces $S_{D}$ and $S_{C}$, respectively.

Thus, the electric and magnetic fields are approximated as series of curl-conforming basis functions

$$
\begin{align*}
\mathbf{E}(\mathbf{r}) & =\sum_{i=1}^{N_{E}} E_{i} \mathbf{W}^{E}(\mathbf{r}),  \tag{4.37}\\
\mathbf{H}(\mathbf{r}) & =\sum_{i=1}^{N_{H}} H_{i} \mathbf{W}^{H}(\mathbf{r}), \tag{4.38}
\end{align*}
$$

while the electric and magnetic current densities are expanded using div-conforming basis functions

$$
\begin{align*}
\mathbf{J}(\mathbf{r}) & =\sum_{i=1}^{N_{J}} J_{i} \hat{n} \times \mathbf{W}^{H}(\mathbf{r})=\sum_{i=1}^{N_{J}} J_{i} \mathbf{T}^{H}(\mathbf{r})  \tag{4.39}\\
\mathbf{M}(\mathbf{r}) & =\sum_{i=1}^{N_{M}} M_{i} \hat{n} \times \mathbf{W}^{E}(\mathbf{r})=\sum_{i=1}^{N_{M}} M_{i} \mathbf{T}^{E}(\mathbf{r}) . \tag{4.40}
\end{align*}
$$

Regarding the port current densitiy $\mathbf{M}_{P}$, a propper choice is [26], [24]

$$
\begin{equation*}
\mathbf{M}_{P}(\mathbf{r})=\sum_{i=1}^{N_{P}} V_{i} \mathbf{h}_{i}(\mathbf{r})=\sum_{i=1}^{N_{P}} V_{i} \mathbf{e}_{i}(\mathbf{r}) \times \hat{n} \tag{4.41}
\end{equation*}
$$

where $\mathbf{e}_{i}$ and $\mathbf{h}_{i}$ represent the 2-D port's electric and magnetic eigenvectors, respectively.


Figure 4.3: FE-BI matrix structure.

The approximations (4.37) - (4.41) yield a matrix equation of the form

$$
\begin{equation*}
\left[Y^{F E B I}\right]\left[U^{V}\right]=\left[U^{I}\right], \tag{4.42}
\end{equation*}
$$

where $\left[U^{V}\right]$ holds the unknown expansion coefficients in (4.37)-(4.41), and $\left[U^{I}\right]$ is the excitation vector.

The structure of matrix $\left[Y^{F E B I}\right]$ is shown in Fig. 4.3. The sparse matrix [ $Y^{V V}$ ] contains the interactions involving the interior basis only, while $\left[Y^{V S}\right]$ and $\left[Y^{S V}\right]=\left[Y^{V S}\right]^{T}$, also sparse, represent the interactions between the interior and the boundary basis. The full matrix $\left[Y^{S S}\right]$ incorporates exclusively the boundary interactions.

### 4.7 Determination of the admittance matrix

In order to take advantage of the sparse structure of matrices $\left[Y^{V V}\right],\left[Y^{V S}\right]$ and [ $\left.Y^{S V}\right]$, we rewrite the linear equation system (4.42) as follows

$$
\begin{align*}
{\left[Y^{V V}\right]\left[W^{V}\right]+\left[Y^{V S}\right]\left[V^{S}\right] } & =0, \\
{\left[Y^{S V}\right]\left[W^{V}\right]+\left[Y^{S S}\right]\left[V^{S}\right] } & =\left[Y^{S P}\right]\left[V^{P}\right], \tag{4.43}
\end{align*}
$$

yielding

$$
\begin{equation*}
\left[V^{S}\right]=\left(\left[Y^{S S}\right]-\left[Y^{S V}\right]\left[Y^{V V}\right]^{-1}\left[Y^{V S}\right]\right)^{-1}\left[Y^{S P}\right]\left[V^{P}\right] \tag{4.44}
\end{equation*}
$$

Above $[Y]^{S P}$ is the matrix that accounts for the interactions between the surface electric and magnetic current densities (on $S_{D}+S_{C}$ ) and port current densities (on
$\left.S_{P}\right)$.
The generalized admittance matrix (GAM) elements are given by [78]

$$
\begin{equation*}
[Y]_{i j}=\int_{S_{P}} \mathbf{h}_{i} \mathbf{H}_{j} d S \tag{4.45}
\end{equation*}
$$

where $\mathbf{h}_{i}$ is the port's $i$-th magnetic eigenvector, and $\mathbf{H}_{j}$ denote the total magnetic field produced by exciting the $j$-th magnetic eigenvector while all the remaining modes are short circuited.

The total magnetic field inside the cavity can be expressed as

$$
\begin{equation*}
\mathbf{H}=\mathbf{H}\left(S_{P}, \mathbf{M}_{P}\right)+\mathbf{H}\left(S_{D}+S_{C}, \mathbf{J}\right)+\mathbf{H}\left(S_{D}, \mathbf{M}\right) . \tag{4.46}
\end{equation*}
$$

To obtain the GAM, we consider a single mode excitation while all the other modes are short-circuited. Repeating this assumption for each mode and testing the total magnetic field in (4.46) with the ports magnetic eigenvectors, yields the GAM

$$
\begin{align*}
{[Y] } & =\left[Y^{A}\right]\left(\left[Y^{P P}\right]+\left[Y^{P S}\right]\left(\left[Y^{S S}\right]-\left[Y^{S V}\right]\left[Y^{V V}\right]^{-1}\left[Y^{V S}\right]\right)^{-1}\left[Y^{S P}\right]\right)\left[Y^{A}\right] \\
{\left[Y^{A}\right] } & =\operatorname{diag}\left(\frac{1}{\sqrt{Y_{i}}}\right) \tag{4.47}
\end{align*}
$$

where $Y_{i}$ is the admittance of the $i$-th port eigenmode, $\left[Y^{P S}\right]$ represents the interactions between the port current densities and the electric and magnetic currents densities on $S_{D}+S_{C}$ and

$$
\begin{equation*}
\left[Y^{P P}\right]_{m n}=\left\langle\mathbf{h}_{m}, \mathbf{H}\left(\mathbf{M}_{P, n}\right)\right\rangle \tag{4.48}
\end{equation*}
$$

Once the GAM is known, the global scattering matrix (GSM) is calculated using

$$
\begin{equation*}
[S]=2([I]-[Y])^{-1}-[I] \tag{4.49}
\end{equation*}
$$

where $[I]$ is the identity matrix.
We note that, as a consequence of the submatrix decomposition performed in (4.43), two linear systems need to be solved: first, corresponding to the product $\left[Y^{V V}\right]^{-1}\left[Y^{V S}\right]$ involving the sparse matrix $\left[Y^{V V}\right]$ and second, a fully populated matrix system whose size is given by the number of boundary unknowns. Iterative or direct algorithms, specialized for sparse matrices, may be used for the first case, while solving the full matrix system of the second case is not a major inconvenience, since the number of boundary unknowns is usually much smaller than the number of interior unknowns.

### 4.8 Evaluation of the boundary integrals

The boundary integrals involved in the evaluation of $\left[Y^{S S}\right]$ can be written in a generic form

$$
\begin{equation*}
\mathcal{J}=\int_{S} \int_{S} f(\mathbf{r}) \cdot \mathcal{G} \cdot g\left(\mathbf{r}^{\prime}\right) d S^{\prime} d S \tag{4.50}
\end{equation*}
$$

where $f$ and $g$ are vector or scalar functions and $\mathcal{G}$ denotes a scalar potential Green's function or any term of the dyadic Green's function. Considering the splitting in (4.14) and the expressions of (4.21), $\mathcal{J}$ is expanded to

$$
\begin{align*}
\mathcal{J} & =\int_{S} \int_{S} f(\mathbf{r}) \cdot \mathcal{G}_{s t a} \cdot g\left(\mathbf{r}^{\prime}\right) d S^{\prime} d S \\
& +\underbrace{\int_{S} \int_{S} f(\mathbf{r}) \cdot \mathcal{G}_{d y n}^{m o d} \cdot g\left(\mathbf{r}^{\prime}\right) d S^{\prime} d S}_{\mathcal{J}_{1}} \\
& =\underbrace{\int_{S} \int_{S} f(\mathbf{r}) \cdot \mathcal{G}_{s t a}^{i m g} \cdot g\left(\mathbf{r}^{\prime}\right) d S^{\prime} d S}_{\mathfrak{J}_{2}}  \tag{4.51}\\
& +\underbrace{\int_{S} \int_{S} f(\mathbf{r}) \cdot \mathcal{G}_{s t a}^{m o d} \cdot g\left(\mathbf{r}^{\prime}\right) d S^{\prime} d S}_{\mathfrak{J}_{3}} \\
& =\mathcal{G}_{d y n}^{m o d} \cdot g\left(\mathbf{r}^{\prime}\right) d S^{\prime} d S
\end{align*}
$$

### 4.8.1 Modal series integration

Now, observe that the functions involved in the modal series (4.16) and (4.21a) can be completely separated relative to the source and observation points (see also Tables C. 1 and C.2), that is

$$
\begin{align*}
& f_{m}\left(k_{m x} x, k_{m x} x^{\prime}\right) g_{n}\left(k_{n y} y, k_{n y} y^{\prime}\right) h_{p}\left(k_{p z} z, k_{p z} z^{\prime}\right) \\
& =F_{m n p}\left(k_{m x} x, k_{n y} y, k_{p z} z\right) F_{m n p}^{\prime}\left(k_{m x} x^{\prime}, k_{n y} y^{\prime}, k_{p z} z^{\prime}\right), \tag{4.52}
\end{align*}
$$

yielding

$$
\begin{align*}
\binom{\mathcal{J}_{2}}{\mathcal{J}_{3}}=\frac{1}{a b c} \sum_{m n p=0}^{\infty}\left[\binom{\gamma_{m n p}}{\beta_{m n p}}\right. & \int_{S} f(\mathbf{r}) F_{m n p}\left(k_{m x} x, k_{n y} y, k_{p z} z\right) d S  \tag{4.53}\\
& \left.\cdot \int_{S} g\left(\mathbf{r}^{\prime}\right) F_{m n p}^{\prime}\left(k_{m x} x, k_{n y} y, k_{p z} z\right) d S^{\prime}\right]
\end{align*}
$$

Making the notations

$$
\begin{gather*}
\mathcal{V}_{m n p}=\int_{S} f(\mathbf{r}) F_{m n p}\left(k_{m x} x, k_{n y} y, k_{p z} z\right) d S  \tag{4.54}\\
\mathcal{V}_{m n p}^{\prime}=\int_{S} g\left(\mathbf{r}^{\prime}\right) F_{m n p}^{\prime}\left(k_{m x} x, k_{n y} y, k_{p z} z\right) d S^{\prime} \tag{4.55}
\end{gather*}
$$

we rewrite the sum (4.53) in the form

$$
\begin{equation*}
\binom{\mathcal{J}_{2}}{\mathcal{J}_{3}}=\frac{1}{a b c} \sum_{m n p=0}^{\infty}\left[\binom{\gamma_{m n p}}{\beta_{m n p}} \mathcal{V}_{m n p} \cdot \mathcal{V}_{m n p}^{\prime}\right] \tag{4.56}
\end{equation*}
$$

Here, $\mathcal{V}_{m n p}$ and $\mathcal{V}_{m n p}^{\prime}$ are frequency independent integrals, therefore they are computed only once in a frequency sweep and stored for later use.

### 4.8.2 Image series integration

Integral $\mathcal{J}_{1}$ in (4.51) involves the image series of the Green's function. Replacing (4.21b) and (4.27) in the expression of $\mathcal{J}_{1}$ in (4.51), yields

$$
\begin{align*}
\mathcal{J}_{1} & =\frac{1}{4 \pi} \int_{S} \int_{S} f(\mathbf{r})\left(\sum_{m n p=-\infty}^{\infty} S_{i} \frac{\operatorname{erfc}\left(R_{i, m n p} E\right)}{R_{i, m n p}}\right) \cdot g\left(\mathbf{r}^{\prime}\right) d S^{\prime} d S \\
& =\frac{1}{4 \pi} \int_{S} d S f(\mathbf{r}) \sum_{m n p=-\infty}^{\infty} S_{i} \int_{S} d S^{\prime} \frac{\operatorname{erfc}\left(R_{i, m n p} E\right)}{R_{i, m n p}} g\left(\mathbf{r}^{\prime}\right) \tag{4.57}
\end{align*}
$$

if potential Green's functions are involved, and

$$
\begin{align*}
\mathcal{J}_{1} & =\int_{S} \int_{S} f(\mathbf{r}) \sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} S_{i} u\left(\frac{\operatorname{erfc}\left(E R_{i, m n p}\right)}{4 \pi R_{i, m n p}^{3}}+\frac{e^{-E^{2} R_{i, m n p}^{2}}}{2 \pi \sqrt{\pi} R_{i, m n p}^{2}}\right) g\left(\mathbf{r}^{\prime}\right) d S^{\prime} d S \\
& =\int_{S} d S f(\mathbf{r}) \sum_{m, n, p=-\infty}^{\infty} \sum_{i=0}^{7} S_{i} u \int_{S} d S^{\prime}\left(\frac{\operatorname{erfc}\left(E R_{i, m n p}\right)}{4 \pi R_{i, m n p}^{3}}+\frac{e^{-E^{2} R_{i, m n p}^{2}}}{2 \pi \sqrt{\pi} R_{i, m n p}^{2}}\right) g\left(\mathbf{r}^{\prime}\right), \tag{4.58}
\end{align*}
$$

in the case of field Green's functions.
One observes, from (4.56) - (4.58), that singularities and near-singularities may occur exclusively in the evaluation of the source (inner) integrals involving the image series of the Green's functions in (4.57) and (4.58). The order of singularity is $\frac{1}{R}$ in (4.57), thus any of the transformations 1-4 in Table A. 1 may be employed to annihilate the singularity. The source integrals in (4.58) are hypersingular, i.e. the order of singularity is $\frac{1}{R^{2}}$. Hypersingular integrals, involving the field Green's
functions, are computed in principal value sense plus a residue term, i.e.

$$
\begin{align*}
\int_{S} \overline{\overline{\mathbf{G}}}_{E M, H J}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) \mathbf{f}\left(\mathbf{r}^{\prime}\right) d S^{\prime} & =\frac{1}{2} \hat{n}(\mathbf{r}) \times \mathbf{f}(\mathbf{r}) \\
& +P . V \cdot \int_{S} \overline{\overline{\mathbf{G}}}_{E M, H J}\left(\mathbf{r}, \mathbf{r}^{\prime}\right) \mathbf{f}\left(\mathbf{r}^{\prime}\right) d S^{\prime} \tag{4.59}
\end{align*}
$$

with P.V. standing for integration in principal value sense. For the accurate evaluation of (4.58), one must still pay attention to the near-singular cases. When near-singularities occur, the $R^{2} R A$ transformation (Table A.1) should be used to regularize the integrand.

The calculation of image series related integrals is much more time consuming than the ones involving the modal series, hence, one is interested in having as few as possible terms in the static image series of the Green's function.

The evaluation of the complementary error function, involved in the static Green's function series, is a computationally intensive operation, especially if observing that it is needed in double integration loops. Fortunately, erfc is a bounded function, thus lookup tables can be conveniently employed, yielding a dramatic reduction in the calculation time of the frequency independent integrals involving the image series.

The image series related boundary integrals are frequency independent, hence they need to be calculated only once in a frequency sweep. This is one of the major advantages of the present approach: The most time consuming part of the algorithm is performed only once and reused in subsequent calculations.

### 4.9 Numerical examples

Several numerical examples of practical interest are presented in this section, in order to demonstrate the accuracy and efficiency of the present algorithm. Further information, like the unknown count and timing, will be given. In this respect one needs to specify that all the calculations have carried out on a PC with 2.5 GHz Dual Core II, 6 MB L2 cache CPU, running a 64 bit OS. The calculations are fully parallelized using the OpenMP standard [99] implemented in the GNU C++ compiler.

### 4.9.1 Coaxial combline resonator

The first example is represented by a rectagular waveguide loaded with a conducting (re-entrant) coaxial resonator [100], as shown in Fig. 4.4. The coaxial resonator has been discretized into 44 second order triangular patches (Fig. 4.5), and hierarchical basis up to polynomial order of 2 were employed for current density modeling. This setup yielded 228 electric unknowns. The number of magnetic port unknowns is 20 , resulted from a prescribed maximum cutoff frequency of 10 GHz .

(a) Side view.

(b) Top View.

Figure 4.4: Re-entrant combline resonator [100]. Dimensions [mm]: $a=58.166$, $b=29.883, r_{1}=7.62, h_{1}=20.2692, r_{2}=6.3754, h_{2}=15.1384, r_{3}=3.8608$, $h_{3}=20.32$.

The calculation of the static boundary integrals needed 6 seconds and the total computation time, for 100 frequency points, was about 11 seconds. The present structure has been built and measured in [100]. Fig. 4.6 shows both the experimental and the calculated curves of the $S_{21}$ parameter. Excellent agreement between the numerical calculation and measured data is observed.

(a) Geometry.

(b) Mesh.

Figure 4.5: Resonator in Fig. 4.4.

### 4.9.2 Dielectric loaded waveguide

Fig. 4.8 shows a standard WR-90 waveguide ( $22.86 \times 10.16 \mathrm{~mm}$ ) loaded with two dielectric cylinders $\left(\epsilon_{r}=13.6\right)$ [40]. The cylinders are offset from the propagation axis by $d_{1}$ and $d_{2}$.

For testing purposes, two simulations models of the structure have been consid-


Figure 4.6: $S_{21}$ vs. frequency for the resonator in Fig. 4.4.


Figure 4.7: Cross section of the dielectric loaded waveguide [40]. Schematic front view. Dimensions [mm]: $a=22.86, b=10.16, h=8, \phi=2$.
ered. The mesh of the first model is presented in Fig. 4.8(a). Here, the two identical cylinders are meshed with second order curved tetrahedral elements. The mesh of the second model is shown in 4.8(b). Now, the dielectric cylinders have been enclosed in a larger dielectric cylinder having the relative dielectric permittivity equal to unity (phantom dielectric).

In the first model, the higher order mesh is formed by 118 tetrahedral elements and 278 triangular faces. Second order basis functions yielded 576 volume unknowns and 408 and 392 surface unknowns for the electric and magnetic current densities, respectively. When the phantom enclosure is considered, the mesh is composed of 674 tetrahedra and 1442 triangle on the boundary surface. The second order basis functions gave 3984 volume unknowns and 782 and 758 surface unknowns, for the electric and magnetic current densities, respectively.

The static parts of the algorithm needed about 25 seconds, for the first model, and less than one minute for the second model. The calculation of the scattering matrix consumed less than one second for the first simulation and about 4 second when the phantom dielectric is taken into account.

The $S$-parameters of the fundamental rectangular waveguide mode $H_{10}$ are shown in Fig. 4.9. Excellent agreement between computations and measurements [40] is observed.


Figure 4.8: Mesh models of dielectric loaded waveguide.

### 4.9.3 Combline filters

A more complex structure is the 4-resonator combline filter shown in Fig. 4.10. The filter operates in the evanescent $H_{10}$ rectangular waveguide mode. The input and output are coaxial SMA ports, having the inner conductor tapped to the circular posts.


Figure 4.9: Transmission coefficient of the dielectric loaded waveguide in Fig. 4.7, $d_{1}=3.5 \mathrm{~mm}$ and $d_{2}=8 \mathrm{~mm}$.

Two different models have been used for analysis. The filter is first divided into two building blocks. One building block is formed by the feeding rectangular cavity, while one of the mid-resonators represents the second block. Each structure is computed separately and their $S$-matrices are combined yielding a matrix $S_{1}$ of half of the filter. As the structure is symmetric with respect to the propagation direction, a second matrix $S_{2}$ is calculated by mirroring $S_{1}$. The global scattering matrix is then found by connecting $S_{1}$ and $S_{2}$. The mesh models, for this simulation, are shown in Fig. 4.11. The feeding resonator and the mid-resonator were discretized by 364 and 83 second order triangles, respectively. A denser mesh is needed for the first block, in order to accurately model the inner conductor of the coaxial port. The order of the hierarchical basis functions was set as follows: First order basis on the inner conductor of the coaxial port, second order basis on the rest of the feeding resonator and third order on the middle post resonator. This yielded a total of 1654 unknowns for the feed and 882 unknowns for the second block. The maximum cutoff was set to 20 GHz , resulting in 13 modes at the rectangular port and only the fundamental TEM mode at the the coaxial ports. The computation time, using this model, was 120 seconds for 200 frequency points. Very good agreement is noted between computations and experimental data, as shown in Fig. 4.12(a).

Another simulation was performed by considering a single block, formed by the the whole filter. The mesh density and cutoff frequency settings were kept the same as before, thus giving more than 4000 electric current density unknowns and only a single mode per port. The time consumption was about 18 minutes for 200

(a) Side view.

(b) 3D view.

Figure 4.10: Coaxial feed combline filter. Dimensions [mm]: $a=22.816, b=21.729$, $l_{1}=10.4005, l_{2}=30.78358, l_{3}=34.034, h=4.5, h_{1}=20.35, h_{2}=19.96$, SMA: $r_{\text {in }}=0.635, r_{\text {out }}=2.05, \epsilon_{r}=2.05$.


Figure 4.11: Coaxial feed combline filter mesh
frequency points. This is normally expected, since the algorithm complexity is of order $N^{2}$, with $N$ being the number of unknowns. The computed $S$ parameters are shown in $4.12(\mathrm{~b})$ versus the measured ones, and again, very good agreement between measurement and calculation is observed. The mesh of this simulation model, as well as the calculated normalized electric current density, are shown in Fig. 4.13. Two graphical representation of the current densities are presented: One plot at $f=$ 2 GHz (Fig. 4.13(b)), while the other plot shows the current density distribution in the resonance region of the filter. Referring to Fig. 4.13, the left side port is considered to be excited.

Another example is the combline filter of [42]. The filter is formed by 5 cascaded cylindrical resonators connected, via irises, to the input/output ports.

The analysis has been carried out, again, by first dividing the structure into building blocks. Here, the first and the last posts have identical dimensions, while the three middle cylinders are also identical (see [42]). As a consequence, only two loaded cavities need to be simulated. A cavity of dimensions $8 \times 10.16 \times 7.8$ [mm] has been chosen for both building blocks, and additional lengths are compensated by empty rectangular waveguides. The meshes are formed by 89 and 95 second order patches yielding 945 and 1008 third order hierarchical basis, respectively. The input waveguide steps were calculated by mode matching. Finally, the modal scattering matrix of the filter is obtained by cascading the scattering matrices of each building block.

The filter has been analyzed in [42], using the BI-RME (Boundary Integral Resonant Mode Expansion) algorithm. Fig. 4.14 presents the reflection and transmission coefficients of the fundamental $H_{10}$ rectangular waveguide mode. Good agreement between the present approach and the BI-RME method is observed.

To demonstrate the flexibility of the present method, an optimization example


Figure 4.12: $S$-parameters of the combline filter in Fig. 4.10.

(c) Current distribution at $f=2.3 \mathrm{GHz}$.

Figure 4.13: Mesh model and normalized current density distribution $\left(10 \log _{10} \frac{|\mathbf{J}|}{\left|\mathbf{J}_{\text {max }}\right|}\right)$ of the single block simulation of the combline filter in Fig. 4.10.


Figure 4.14: $S$-parameters of the combline filter of [42].
is further presented. A folded combline filter, composed of 6 cylindrical cavities, is shown in Fig. 4.15. The filter is excited through SMA coaxial ports having the inner conductor connected to a circular disc. Each cylindrical cavity is loaded with two stacked cylindrical posts with a circular hole on top. Tuning screws are also fitted on the top of the resonating cavities. The cavities are connected to each other via rectangular irises. A wire cross coupling, between the first and the last resonators, is added, in order to introduce transmission zeros.

Resonators 1 and 6 and resonators 2 and 5 have identical geometries. Cavities 3 and 4 are also identical with the exception of the port placement. Thus, in the simulation model, only cavities $1,2,3$ and 4 are used as building blocks. Each cavity is enclosed by its bounding box. The geometry of each building block is then meshed with second order triangular patches. The final mesh is formed by removing all triangles that are placed on the walls of the bounding box. The resulted mesh models are shown in Fig. 4.16. Each building block is simulated separately. Finally, the scattering matrix of the filter is found by cascading the individual $S$-matrices of each building block.

All the geometrical dimensions of the filter, except the cavities radii and heights, were set as optimization variables. $S_{11}<-20 \mathrm{~dB}$ in the pass band and $S_{21}<-35$ dB in the stop bands, were prescribed as optimization goals and the response in Fig. 4.17 was obtained after an overnight run. For comparison, the free-space


Figure 4.15: Cylindrical cavity combline filter.


Figure 4.16: Mesh models of the cylindrical cavity combline filter in Fig. 4.15.


Figure 4.17: $S$-parameters of the combline filter in Fig. 4.15.

MoM algorithm, presented in the previous chapter, needed about 10 hours only to analyse the structure in 100 frequency points. This example clearly demonstrates the efficiency of the present approach and the fact that the method is not restricted to rectangular geometries only.

### 4.9.4 Canonical ridged waveguide filter

The schematic layout of a ridged waveguide filter [101] is shown in Fig. 4.18. It is formed by the cascade of ridged waveguide cavities, arranged in two separate rows and coupled by rectangular waveguides operating in the evanescent $H_{10}$ mode. The two rows are coupled to each other by rectangular irises. The excitation is accomplished by standard coaxial SMA connectors with the inner conductor tapped to one of the ridges. If the filter needs to be integrated in a high-density packaging architecture, the structure can be built employing low-temperature cofired ceramics (LTCC) technology [101].

Because there are two symmetry planes, two building blocks were used to carry out the simulation. One block is contains the SMA connector, and has one coaxial port and two rectangular ports. The second block is formed by the remaining ridged cavity and has three rectangular ports. Each block is calculated separately yielding the scattering matrices $S_{1}$ and $S_{2} . S_{1}$ and $S_{2}$ are cascaded with respect to the common coupling port yielding a matrix $S_{3}$. A matrix $S_{4}$ is then obtained, by

(a) Schematic side view

(b) Schematic top view

Figure 4.18: Canonical ridged waveguide filter [101]. Dimensions [mm]: $a=45.72$, $b=20.32, w=24, h_{1}=4.14, t=2.54, w c_{1}=8, w s_{2}=10.86, d r_{1}=43.02$, $d r_{2}=24.34, d e_{1}=41.80, d e_{2}=11.77, d c_{1}=20.83, d s_{2}=20.08, l_{t}=9.45$, $y_{t}=4.12, \phi_{t}=1.06$. Input SMA connectors: $\phi_{\text {ext }}=4.11, \phi_{\text {int }}=1.27, \epsilon_{r}=1.98$.


Figure 4.19: $S$ parameters of the canonical ridged waveguide filter in Fig. 4.18.
mirroring $S_{3}$ relative to the port formed by the three irises. Finally, $S_{3}$ and $S_{4}$ are connected to yield the scattering matrix of the whole filter. The structure is mostly formed by planar faces, therefore there is no need for higher order patches, except for the modeling of the inner conductor of the coaxial ports. Hierarchical basis functions up to the polynomial order of three were used for electric current modeling. The number of unknowns was 1374 for the first block and 875 for the second block. The analysis needed about 160 seconds to find the response of the filter in 200 frequency points. The calculated reflection and transmission coefficients are compared with measured data in Fig. 4.19(a) and Fig. 4.19(b).

### 4.9.5 Coaxial to rectangular waveguide transition

The transition [46] is shown in Fig. 4.20. The inner conductor of the coaxial connector penetrates into the rectangular waveguide and is terminated with a circular disc. The dielectric material of the coaxial port is also extended to form a coating for the inner wire probe. Two tuning screws are placed on the bottom of the cavity for matching purposes.

In the simulation model, additional to the inner conductor coating, a phantom dielectric coating having $\epsilon_{r}=1$ is considered to enclose the circular disc (see Fig. 4.21). The structure has been meshed by tetrahedral/triangular elements of polynomial order of two. First order basis functions for the modeling of both volume field and surface current densities turned out to be sufficient for reaching convergence. The computed reflection coefficient is plotted, in Fig. 4.22, against the one measured in [46].

### 4.9.6 Dielectric resonator filters

A dielectric resonator filter [45] is presented in Fig. 4.23. It has two coaxial rods representing the ports; the two ring shaped dielectric cylinders (relative permittivity of 38 ) are coupled to each other through a rectangular waveguide below cutoff.

The structure was divided into two building blocks: The coaxial feed and the dielectric resonator. First order (linear) basis functions, defined on triangles of polynomial order of two, were used to model the current densities on the excitation rod. The dielectric cylinder has been meshed by second order tetrahedral/triangular elements. Second order functions were employed to model the interior (volume) fields, and linear functions have been used to model the tangential fields. The number of unknowns was for the dielectric resonator 2698 volume unknowns and $2 \times 366$ surface unknowns. The calculation time was 40 seconds for the frequency independent part and less than 1.5 seconds for the $S$-matrix calculation. The computed transmission coefficient is plotted in Fig. 4.25 against measured results of [45]. The peak at 6.6 GHz was reported in [45] as a measurement error.

Another dielectric resonator filter [43] is shown in Fig. 4.26. The filter is formed by the cascade of four resonating rectangular cavities connected to each other via

(a) Schematic view of the cross section. All dimesions are in millimeters. $\epsilon_{r}=1.9873$.

(b) 3-D wireframe view.

Figure 4.20: Coaxial to rectangular waveguide transition [46].


Figure 4.21: Coax to rectangular waveguide transition mesh.


Figure 4.22: Coax to rectangular waveguide transition return loss.

(a) Schematic top view.

(b) Schematic side view.

(c) 3D view.

Figure 4.23: Coaxial feed dielectric resonator filter [45]. Dimensions are expressed in millimeters.


Figure 4.24: Tetrahedral mesh model of the ring shaped dielectric resonator of the filter in Fig. 4.23.


Figure 4.25: Transmission loss of the filter in Fig. 4.23.
rectangular irises. The excitation is realized through standard WR-90 (22.86 $\times 10.16$ $\mathrm{mm})$ rectangular waveguides. Each cavity is loaded with a high permittivity $\left(\epsilon_{r}=\right.$ 50) dielectric cylinder which is supposedly mounted on a phantom dielectric support $\left(\epsilon_{r}=1\right)$ of height 3.25 mm . The geometrical dimensions of the structure are given in Table 4.1.


Figure 4.26: Dielectric resonator filter [43] - translucent view.

|  | width[mm] | height[mm] | length[mm] |
| :--- | :--- | :--- | :--- |
| Cavity 1 | 11. | 9 | 7.7 |
| Cavity 2 | 11 | 9 | 10.4 |
| Cavity 3 | 11 | 9 | 10.4 |
| Cavity 4 | 11. | 9 | 7.7 |
| Iris 1 | 6.08 | 4.43 | 0.5 |
| Iris 2 | 4.85 | 5.27 | 0.5 |
| Iris 3 | 6.08 | 4.43 | 0.5 |
|  | radius $[\mathrm{mm}]$ | height $[\mathrm{mm}]$ | $\epsilon_{r}$ |
| Resonators | 2.55 | 2.3 | 50 |

Table 4.1: Geometrical dimensions of the filter in Fig. 4.26.
Each resonating cavity has been computed by the present algorithm, whereas the input waveguide steps and the coupling irises have been modeled by the mode matching technique. The global scattering matrix of the filter is finally obtained by cascading the individual scattering matrices of its building blocks. The geometric modeling of the cylindrical pucks has been performed by second order triangular patches. Third order basis functions were employed to model the volume fields

(b) Wideband frequency response.

Figure 4.27: $S$ parameters of the filter in Fig. 4.26.
inside the dielectric resonators and the tangential fields on the resonator's boundary surface. About 45 seconds were spent in the calculation in the frequency independent part. About 1.5 seconds per frequency point were needed to evaluate the $S$ matrix. The computed $S$ - parameters of the fundamental rectangular waveguide mode are plotted, in Fig. 4.27(a) and Fig. 4.27(b) against the ones obtained by the BI-RME algorithm in [43].

The next example is a folded dielectric resonator filter shown in Fig. 4.28. The ports are represented by standard SMA connectors having the inner conductor connected to a bent metal strip. The four cavities are coupled by rectangular irises with rounded corners. Each loaded cavity represents a building block in the simulation model. Using the present algorithm, the filter has been optimized for a return loss $>15 \mathrm{~dB}$ in the $4.06-4.07 \mathrm{GHz}$ frequency range.

Second order functions were applied to model the interior (volume) fields and the tangential fields. The number of unknowns was 3698 volume unknowns together with 1553 electric and 735 magnetic surface unknowns, in the case of the input/output cavities, and 3698 volume unknowns and 735 electric and 735 magnetic surface unknowns in the case of the corner resonator. About 80 seconds were needed to evaluate the frequency independent parts and less than 5 seconds per frequency point in the calculation of the dynamic parts.

An own 3D finite element code has been applied for this structure in Fig. 4.28 for comparison purposes. Although a fast linear equation solver for sparse matrices has been utilized, more than 30 seconds were needed for each frequency point. The corresponding $S$-parameters are shown in Fig. 4.29.


Figure 4.28: Folded dielectric resonator filter.


Figure 4.29: $S$-parameters of the folded dielectric resonator filter.

## Conclusions

Three different algorithms, for the simulation of passive microwave components, are presented in this work. All three methods are doubly higher order, that is, higher order basis functions are used for current/field modeling whereas the geometry discretization is performed with curved triangular patches of higher polynomial degree. In order to allow the combination with other powerful techniques, such as the hybrid mode-matching/finite-element, the presented algorithms are full-wave, i.e. they yield the generalized admittance/scattering matrix of the structure under investigation. Moreover, the presented methods were implemented in a comprehensive software tool [18], for the CAD of passive microwave components and antennas.

The method of Chapter 2 is a pure integral equation technique. Here, the electric field integral equation (EFIE) is enforced at conducting surfaces, whereas the dielectric bodies are handled by the PMCHWT method [21], [22], [16], [23], hitherto applied mainly to plane wave scattering problems. The resulted integral equations are discretized by the method of moments (MoM). A new formulation, finally given by (2.66), is introduced in order to remove the drawback noted in [24], [25], [26]. This new procedure allows the use of identical types of (frequency-independent) basis functions for the modeling of both magnetic current densities at the waveguide ports and electric current densities at the conducting parts of a multiport. Hence, the algorithm complexity ${ }^{1}$ is independent of the number/type of port modes. As many passive microwave components exhibit rotational symmetry (e.g. conical horn antennas, paraboloidal dish reflectors, etc.), special attention is given to bodies of revolution (BoR), for which, novel higher order basis functions are constructed.

However, the MoM algorithm of Chapter 2 cannot be applied to inhomogeneous/anisotropic structures. The efficiency of the numerical algorithm also diminishes with rising material permittivity/permeability because: First, it requires very fine meshing to account for rapidly varying fields at dielectric's surface, thus increasing the number of unknowns, and secondly, higher order quadrature rules must be considered to accurately integrate the rapidly varying kernels associated to the interior problem, thus increasing the evaluation time of the corresponding integrals.

The finite-element boundary integral (FE-BI) algorithm introduced in Chapter 3 not only removes the aforementioned disadvantages of MoM but also increases flexibility by allowing anisotropic/inhomogeneous materials. Finite elements are used

[^3]here to characterize the arbitrarily shaped, possibly inhomogeneous, domains. The boundary conditions, at the waveguide ports, are imposed by the matching of the modal and interior fields. The known finite element - boundary integral formulation [27], is here extended for the analysis of multiport structures. In contrast to other FE-BI formulations [28], [29], the calculation of boundary integrals involving the surface divergence of $\hat{n} \times \mathbf{f}$ terms ${ }^{2}$ is avoided here, thus the instabilities associated with the introduction of artificial line charges are completely removed. Furthermore, model order reduction (MOR) techniques are applied for the expedient calculation of the wide-band frequency response of a multiport. Due to the incompatibility, with the MOR formalism, of the integral equation formulation of the radiation boundary conditions, absorbing boundary conditions (ABC) are alternatively employed to truncate the computational domain, when the MOR procedure is applied to unbounded (radiating) structures. Moreover, the PRIMA based MOR technique [3] has been extended for the treatment of structures with frequency dependent (inhomogeneous cross-section) waveguide ports, whereas the WCAWE algorithm of [4], [31] has been modified to allow multiple right-hand sides (port modes).

Both methods presented in Chapters 2 and 3 share a common characteristic, that is the whole structure under analysis must be discretized. However, many building blocks of microwave components of practical interest are composed of rectangular cavities with small metallic and/or dielectric loadings. Therefore, a new hybrid algorithm, that makes use of the rectangular box Green's function was designed in Chapter 4. Similarly to Chapter 3, the known finite element - boundary integral formulation [27], is now extended to shielded environments. All boundary integrals involving rectangular cavity Green's functions are efficiently evaluated utilizing the Ewald transform [47, 48, 49]. There are mainly two factors responsible for the efficiency of this approach. First, due to the separation of the Green's functions into static series (zero frequency limit), whose convergence is enhanced with the help of the Ewald transform, and an already convergent dynamic series (higher frequency correction), the most computational intensive part of the algorithm is performed only once in a frequency sweep. Second, as a consequence of the use of the cavity Green's functions, only a small portion of the computational domain must be discretized, thus drastically reducing the number of unknowns.

The algorithm of Chapter 2 solves for the sources and has the advantage of keeping the discretized domain at minimum, whereas its complexity is of the order of $N^{2}$, where $N$ is the number of unknowns. In contrast, differential equation techniques require a $3-D$ discretization of the electric or magnetic fields, but in turn they have a complexity of the order of $N$ and the advantage of very sparse coefficient matrices. Furthermore, in the case of integral equation methods, the Green's function, given in closed form or determined to the machine's precision,

[^4]accounts for the propagation of the waves from a point $A$ to a point $B$. In contrast, in the FEM model the propagation takes place through a numerical grid. A small error is usually introduced in this mode of field propagation [16], [17]. This error manifests as a phase error in the field and is cumulative. This phase error can be mitigated by increasing the grid density and/or by the use of higher order basis functions, thus the motivation of using higher order bases throughout this work. An additional problem is related to the truncation of the computational domain in differential equation techniques. In the last two chapters, exact radiation conditions, at the truncation boundary, are enforced via integral equations. In turn, the mesh truncation via boundary integrals represents the bottleneck of both FE-BI algorithms of chapters 3 and 4, mainly due to the time spent in the evaluation the boundary integrals and the presence of a fully populated block in the associated coefficient matrix. The impact of the boundary integral formulation is less severe for the method of Chapter 4 , as only a small portion of the computational domain is discretized. Absorbing boundary conditions (ABC) are introduced in Chapter 3 as an alternative to the boundary integral formulation. The ABCs are approximate boundary conditions, hence a small amount of the field is artificially reflected back into the computational domain, thus increasing the overall computation error. In turn, the ABCs have the advantage of preserving the sparsity of the coefficient matrix and they fulfill the MOR requirements.

It was found that the method of Chapter 2 is suitable for the analysis of radiating structures composed of conducting and/or homogeneous dielectrics while being outperformed by the FE-BI approach of Chapter 3 if arbitrarily shaped nonradiating devices are investigated. The analysis of arbitrarily loaded rectangular waveguides/cavities can be performed with any of the FEM-MOR or the specialized FE-BI algorithm of Chapter 4 as both methods are fast and equally accurate, although we note that the shielded environment formulation requires less memory. However, when arbitrarily shaped anisotropic/inhomogeneous structures must be simulated, we are left with the algorithm presented in Chapter 3. In this last case, ABCs can be used, as an alternative to boundary integrals, to truncate radiating structures. The ABCs sacrifice some of the accuracy but in turn they are compatible with the MOR formalism.

We end this chapter by stressing that a proper choice of numerical algorithms can have a tremendous impact on the efficiency and accuracy of the simulation/optimization of a passive microwave structures. Hybrid software tools [18] that employ the domain decomposition approach are usually orders of magnitude faster than their single method counterparts.

## Appendix A

## Singularity cancellation method

Consider the projection of an observation point $\mathbf{r}$ on the plane of a source triangle $T$, as in Fig A.1. The projected point divides $T$ into three subtriangles $T_{1}, T_{2}$ and $T_{3}$, and a local coordinate system is formed for each subtriangle (shown only for $T_{2}$ in Fig. A.1)


Figure A.1: Subdivision of the source triangle into three triangles and subtriangle local coordinate system.

We are interested in calculating integrals of the form

$$
\begin{equation*}
\iint_{T} f\left(\mathbf{r}^{\prime}\right) g(R) \frac{1}{R^{n}} d S^{\prime}=\sum_{k=1}^{3} \underbrace{\iint_{T_{k}} f\left(\mathbf{r}^{\prime}\right) g(R) \frac{1}{R^{n}} d S^{\prime}}_{I_{k}}, \tag{A.1}
\end{equation*}
$$

where $f$ and $g$ are arbitrary vector or scalar functions, $R$ represents source-observation distance and $n$ is a strictly positive integer. The above integrals are called singular
if $n=1$, and hypersingular if $n>1$. The subtriangle integral $I_{k}$ reads

$$
\begin{align*}
I_{k} & =\int_{0}^{h} \int_{x_{L}(y)}^{x_{U}(y)} f\left(\mathbf{r}^{\prime}\right) g(R) \frac{1}{R^{n}} d x d y  \tag{A.2}\\
& =\int_{0}^{h} \int_{y \cot \Phi_{L}}^{y \cot \Phi_{U}} f\left(\mathbf{r}^{\prime}\right) g(R) \frac{1}{R^{n}} d x d y .
\end{align*}
$$

A double variable transformation

$$
\begin{aligned}
& x^{\prime} \rightarrow u, \\
& y^{\prime} \rightarrow v,
\end{aligned}
$$

is introduced, yielding

$$
\begin{equation*}
I_{k}=\int_{v_{L}}^{v_{U}} \int_{u_{L}}^{u_{U}} f\left(\mathbf{r}^{\prime}\right) g(R) \frac{\mathcal{J}(u, v)}{R^{n}} d u d v \tag{A.3}
\end{equation*}
$$

which is regular and smooth, therefore can be efficiently evaluated using numerical quadratures. Several variable transformations [65], [66], [67], [68], are presented in table A.1. All except the last one are suitable for the calculation of the singular potential integrals, while the last one is suitable for the evaluation of the nearhypersingular field integrals.

When the source triangle is curved, the integrals are evaluated on the planar triangle tangential to the source triangle in the point $\left(u_{1}^{0}, u_{2}^{0}\right)$, which is chosen such as $\left|\mathbf{r}-\mathbf{r}^{\prime}\left(u_{1}^{0}, u_{2}^{0}\right)\right|=\mathrm{min}$. The minimization of $\left|\mathbf{r}-\mathbf{r}^{\prime}\left(u_{1}, u_{2}\right)\right|$ is nontrivial when higher order source triangles are considered. In the practical implementation of the algorithms from the present work the square distance function $\left[\mathbf{r}-\mathbf{r}^{\prime}\left(u_{1}, u_{2}\right)\right] \cdot\left[\mathbf{r}-\mathbf{r}^{\prime}\left(u_{1}, u_{2}\right)\right]$ is minimized with the help of the GSL [102]. If the projected point falls outside the source triangle then $\left(u_{1}^{0}, u_{2}^{0}\right)$ must be constrained to one of triangle's vertices or to one of triangle's edges, yielding only one or two subtriangles, respectively.

| no | $u$ | $v$ | $\mathcal{J}(u, v)$ | $\begin{gathered} u_{L, U} \\ v_{L, U} \end{gathered}$ |
| :---: | :---: | :---: | :---: | :---: |
| 1 <br> Asinh | $\sinh ^{-1} \frac{x}{\sqrt{y^{2}+z^{2}}}$ | $y$ | $R$ | $\begin{aligned} & \sinh ^{-1} \frac{y \cot \Phi_{L, U}}{\sqrt{y^{2}+z^{2}}} \\ & 0, h \end{aligned}$ |
| 2 extended Duffy | $\frac{x}{\sqrt{y^{2}+z^{2}}}$ | $y$ | $\sqrt{y^{2}+z^{2}}$ | $\begin{aligned} & \frac{y \cot \Phi_{L, U}}{\sqrt{y^{2}+z^{2}}} \\ & 0, h \end{aligned}$ |
| 3 <br> Radial <br> (extended polar) | $\Phi=\tan ^{-1} \frac{y}{x}$ | $R$ | $R$ | $\begin{aligned} & \Phi_{L, U} \\ & \|z\|, z^{2}+\sqrt{z^{2}+\left(\frac{h}{\sin u}\right)^{2}} \end{aligned}$ |
| $\begin{aligned} & 4 \\ & R A \\ & \text { (radial-angular) } \end{aligned}$ | $\ln \tan \frac{\Phi}{2}$ | $R$ | $\frac{R}{\cosh u}$ | $\begin{aligned} & \ln \tan \frac{\Phi_{L, U}}{2} \\ & \|z\|, z^{2}+\sqrt{z^{2}+(h \cosh u)^{2}} \end{aligned}$ |
| 5 <br> $R^{2} R A$ <br> (radial ${ }^{2}$-angular) | $\ln \tan \frac{\Phi}{2}$ | $\ln R$ | $\frac{R^{2}}{\cosh u}$ | $\begin{aligned} & \ln \tan \frac{\Phi_{L, U}}{2} \\ & \ln \|z\|, \frac{1}{2} \ln \left[z^{2}+(h \cosh u)^{2}\right] \end{aligned}$ |

Table A.1: Variable transformations in numerical cancellation method.

## Singular integrals for bodies of revolution

The double surface inner product integrals involving the $\mathcal{L}$ and $\mathcal{K}$ operators (see Section 2.2.2 ) can be greatly simplified (with the expense of greater analytical effort), when bodies of revolution are investigated. Considering the already defined basis functions from Section 2.3.3, the product integrals become

$$
\begin{align*}
& I_{m, p q, r s}^{\mathcal{L}, \alpha \beta}= C_{1} \int_{\Delta_{p}} \int_{0}^{2 \pi} \mathbf{T}_{m, p, q}^{\alpha}(t, \phi) \int_{\Delta_{r}} \int_{0}^{2 \pi} \mathbf{F}_{m, r, s}^{\beta}\left(t^{\prime}, \phi^{\prime}\right) G_{0}\left(t, \phi, t^{\prime}, \phi^{\prime}\right) d \phi^{\prime} d t^{\prime} d \phi d t, \\
&-C_{2} \int_{\Delta_{p}} \int_{0}^{2 \pi} \nabla \mathbf{T}_{m, p, q}^{\alpha}(t, \phi) \int_{\Delta_{r}} \int_{0}^{2 \pi} \nabla^{\prime} \mathbf{F}_{m, r, s}^{\beta}\left(t^{\prime}, \phi^{\prime}\right) G_{0}\left(t, \phi, t^{\prime}, \phi^{\prime}\right) d \phi^{\prime} d t^{\prime} d \phi d t,  \tag{B.1}\\
& I_{m, p q, r s}^{\mathcal{K}, \alpha \beta}=-\frac{1}{2} \int_{\Delta_{p=r}} \int_{0}^{2 \pi} \mathbf{T}_{m, p, q}^{\alpha}(t, \phi)\left[\hat{n} \times \mathbf{F}_{m, p, s}^{\beta}(t, \phi)\right] d \phi d t, \\
&+\int_{\Delta_{p}} \int_{0}^{2 \pi} \mathbf{T}_{m, p, q}^{\alpha}(t, \phi) \int_{\Delta_{r}} \int_{0}^{2 \pi} \mathbf{G}_{1}\left(t, \phi, t^{\prime}, \phi^{\prime}\right) \times \mathbf{F}_{m, r, s}^{\beta}\left(t^{\prime}, \phi^{\prime}\right) d \phi^{\prime} d t^{\prime} d \phi d t . \tag{B.2}
\end{align*}
$$

Here $\Delta$ is the domain of definition of the basis along the BoR generatrix, whereas $\alpha$ and $\beta$ stand for $t$ and $\phi$, respectively. After some analytic manipulations [63], the integral (B.1) becomes

$$
\begin{align*}
I_{m, p q, r s}^{\mathcal{L}, t t} & =\int_{\Delta_{p}} \int_{\Delta_{r}}\left[C_{1} f_{p}^{q} f_{r}^{\prime s} \sin \gamma_{p} \sin \gamma_{r}^{\prime} G_{2}^{m}+C_{1} f_{p}^{q} f_{r}^{\prime s} \cos \gamma_{p} \cos \gamma_{r}^{\prime} G_{1}^{m}\right. \\
& \left.-C_{2} \frac{\partial f_{p}^{q}}{\partial t} \frac{\partial f_{r}^{\prime s}}{\partial t^{\prime}} G_{1}^{m}\right] d t^{\prime} d t, \\
I_{m, p q, r s}^{\mathcal{L}, t \phi} & =\int_{\Delta_{p}} \int_{\Delta_{r}}\left[i C_{1} f_{p}^{q} f_{r}^{\prime s} \sin \gamma_{p} G_{3}^{m} d t^{\prime} d t+i m C_{2} \frac{\partial f_{p}^{q}}{\partial t} f_{r}^{\prime s} \frac{1}{\varrho^{\prime}} G_{1}^{m}\right] d t^{\prime} d t,  \tag{B.3}\\
I_{m, p q, r s}^{\mathcal{L}, \phi t} & =\int_{\Delta_{p}} \int_{\Delta_{r}}\left[-i C_{1} f_{p}^{q} f_{r}^{\prime s} \sin \gamma_{r}^{\prime} G_{3}^{m} d t^{\prime} d t-i m C_{2} f_{p}^{q} \frac{\partial f_{r}^{\prime s}}{\partial t^{\prime}} \frac{1}{\varrho} G_{1}^{m}\right] d t^{\prime} d t, \\
I_{m, p q, r s}^{\mathcal{L}, \phi \phi} & =\int_{\Delta_{p}} \int_{\Delta_{r}}\left[C_{1} f_{p}^{q} f_{r}^{\prime s} G_{2}^{m}-m^{2} C_{2} f_{p}^{q} f_{r}^{\prime s} \frac{1}{\varrho \varrho^{\prime}} G_{1}^{m}\right] d t^{\prime} d t,
\end{align*}
$$

and (B.2) is developed to

$$
\begin{align*}
I_{m, p q, r s}^{\mathcal{K}, t t} & =i \int_{\Delta_{p}} \int_{\Delta_{r}} f_{p}^{q} f_{r}^{\prime s}\left[\varrho \cos \gamma_{p} \sin \gamma_{r}^{\prime}-\varrho^{\prime} \sin \gamma_{p} \cos \gamma_{r}^{\prime}\right. \\
& \left.+\left(z-z^{\prime}\right) \sin \gamma_{p} \sin \gamma_{r}^{\prime}\right] G_{6}^{m} d t^{\prime} d t, \\
I_{m, p q, r s}^{\mathcal{K}, t \phi} & =\int_{\Delta_{p}} \int_{\Delta_{r}} f_{p}^{q} f_{r}^{\prime s}\left\{\left[\left(\varrho^{\prime}-\varrho\right) \cos \gamma_{p}-\left(z^{\prime}-z\right) \sin \gamma_{p}\right] G_{5}^{m}\right. \\
& \left.+2 \varrho^{\prime} \sin \gamma_{p} G_{4}^{m}\right\} d t^{\prime} d t,  \tag{B.4}\\
I_{m, p q, r s}^{\mathcal{K}, \phi t} & =\int_{\Delta_{p}} \int_{\Delta_{r}} f_{p}^{q} f_{r}^{\prime s}\left\{\left[\left(z^{\prime}-z\right) \sin \gamma_{r}^{\prime}-\left(\varrho^{\prime}-\varrho\right) \cos \gamma_{r}^{\prime}\right] G_{5}^{m}\right. \\
& \left.+2 \varrho \sin \gamma_{r}^{\prime} G_{4}^{m}\right\} d t^{\prime} d t, \\
I_{m, p q, r s}^{\mathcal{K}, \phi \phi} & =i \int_{\Delta_{p}} \int_{\Delta_{r}} f_{p}^{q} f_{r}^{\prime s}\left(z^{\prime}-z\right) G_{6}^{m} d t^{\prime} d t .
\end{align*}
$$

The prime superscripts are used to indicate source quantities, $\varrho, z, \gamma$ represent the radial coordinate, axial coordinate and the angle made by the tangent to the segment at the integration point with the axis of the body, respectively. The modal Green functions $G_{*}^{m}$ are given by

$$
\begin{align*}
& G_{1}^{m}=\frac{1}{2} \int_{0}^{\pi} \frac{e^{-i k_{0} R}}{R} \cos m \theta d \theta  \tag{B.5a}\\
& G_{2}^{m}=\frac{G_{m-1}+G_{m+1}}{2},  \tag{B.5b}\\
& G_{3}^{m}=\frac{G_{m-1}-G_{m+1}}{2},  \tag{B.5c}\\
& G_{4}^{m}=-\frac{1}{2} \int_{0}^{\pi} \frac{1+i k_{0} R}{R} e^{-i k_{0} R} \cos m \theta \sin ^{2}\left(\frac{\theta}{2}\right) d \theta,  \tag{B.5d}\\
& G_{5}^{m}=-\frac{1}{2} \int_{0}^{\pi} \frac{1+i k_{0} R}{R} e^{-i k_{0} R} \cos m \theta \cos \theta d \theta,  \tag{B.5e}\\
& G_{6}^{m}=-\frac{1}{2} \int_{0}^{\pi} \frac{1+i k_{0} R}{R} e^{-i k_{0} R} \sin m \theta \sin \theta d \theta . \tag{B.5f}
\end{align*}
$$

While $G_{4}, G_{5}$ and $G_{6}$ integrals are computed in Cauchy's principal value sense, extraction method is used to handle the singular and near-singular cases of $G_{1}$. Thus, we have

$$
\begin{equation*}
G_{1}^{m}=\underbrace{\frac{1}{2} \int_{0}^{\pi}\left[\frac{e^{-i k_{0} R}}{R} \cos m \theta-\frac{1}{R}\right] d \theta}_{G_{1}^{m, r e g}}+\underbrace{\frac{1}{2} \int_{0}^{\pi} \frac{1}{R} d \theta}_{G_{1}^{m, s i n g}} . \tag{B.6}
\end{equation*}
$$

First integral in (B.6) is regular, so it can be calculated using gaussian quadrature formulas. The second integral can be written as

$$
\begin{equation*}
G_{1}^{m, \text { sing }}=\frac{1}{2} \int_{0}^{\pi} \frac{d \theta}{\sqrt{\varrho^{2}+\varrho^{\prime 2}+\left(z-z^{\prime}\right)^{2}-2 \varrho \varrho^{\prime} \cos \theta}} . \tag{B.7}
\end{equation*}
$$

Making the notation $\alpha=\frac{\pi-\theta}{2}$, we have

$$
\begin{equation*}
G_{1}^{m, s i n g}=\frac{1}{R_{0}} \int_{0}^{\frac{\pi}{2}} \frac{d \alpha}{\sqrt{1-\beta^{2} \sin \alpha}}, \tag{B.8}
\end{equation*}
$$

with

$$
\begin{equation*}
R_{0}=\sqrt{\left(\varrho+\varrho^{\prime}\right)^{2}+\left(z-z^{\prime}\right)^{2}}, \tag{B.9}
\end{equation*}
$$

and

$$
\begin{equation*}
\beta=\frac{2 \sqrt{\varrho \varrho^{\prime}}}{R_{0}} . \tag{B.10}
\end{equation*}
$$

The integral in (B.8) is recognized as the complete elliptic integral of first kind, thus

$$
\begin{equation*}
G_{1}^{m, \text { sing }}=\frac{1}{R_{0}} K(\beta) . \tag{B.11}
\end{equation*}
$$

The source integrals in (B.3) still contain an integrable logarithmic singularity (when $\beta=1$ or $\varrho=\varrho^{\prime}$ and $z=z^{\prime}$ ) that can be handled using the Ma, Rokhlin and Wandzura scheme [103].

The BoR singularity handling procedure presented here completely differs from the one in [63] and is not restricted to linear segments. Again, with the source integrals regularized, the $\mathcal{L}$ and $\mathcal{K}$ related inner products in (B.3) and (B.4) are evaluated using gaussian quadratures.

# Modal functions and image sets for the Green's functions of a rectangular box 

| $G$ | $f_{m}\left(x, x^{\prime}\right)$ | $g_{n}\left(y, y^{\prime}\right)$ | $h_{p}\left(z, z^{\prime}\right)$ |
| :--- | :--- | :--- | :--- |
| $G_{A}^{x x}$ | $\cos x \cos x^{\prime}$ | $\sin y \sin y^{\prime}$ | $\sin z \sin z^{\prime}$ |
| $G_{A}^{y y}$ | $\sin x \sin x^{\prime}$ | $\cos y \cos y^{\prime}$ | $\sin z \sin z^{\prime}$ |
| $G_{A}^{z z}$ | $\sin x \sin x^{\prime}$ | $\sin y \sin y^{\prime}$ | $\cos z \cos z^{\prime}$ |
| $g_{v}$ | $\sin x \sin x^{\prime}$ | $\sin y \sin y^{\prime}$ | $\sin z \sin z^{\prime}$ |
| $G_{F}^{x x}$ | $\sin x \sin x^{\prime}$ | $\cos y \cos y^{\prime}$ | $\cos z \cos z^{\prime}$ |
| $G_{F}^{y y}$ | $\cos x \cos x^{\prime}$ | $\sin y \sin y^{\prime}$ | $\cos z \cos z^{\prime}$ |
| $G_{F}^{z z}$ | $\cos x \cos x^{\prime}$ | $\cos y \cos y^{\prime}$ | $\sin z \sin z^{\prime}$ |
| $g_{w}$ | $\cos x \cos x^{\prime}$ | $\cos y \cos y^{\prime}$ | $\cos z \cos z^{\prime}$ |

Table C.1: Modal functions for potential Green's functions.

| $G$ | $f_{m}\left(x, x^{\prime}\right)$ | $g_{n}\left(y, y^{\prime}\right)$ | $h_{p}\left(z, z^{\prime}\right)$ |
| :--- | :--- | :--- | :--- |
| $\Upsilon_{d y n}^{x y}, \Upsilon_{1}^{x y}$ | $\cos x \cos x^{\prime}$ | $\sin y \sin y^{\prime}$ | $-k_{z} \sin z \cos z^{\prime}$ |
| $\Upsilon_{d y n}^{y x} \Upsilon_{1}^{y x}$ | $\sin x \sin x^{\prime}$ | $\cos y \cos y^{\prime}$ | $k_{z} \sin z \cos z^{\prime}$ |
| $\Upsilon_{d y n}^{x z}, \Upsilon_{1}^{x z}$ | $\cos x \cos x^{\prime}$ | $k_{y} \sin y \cos y^{\prime}$ | $\sin z \sin z^{\prime}$ |
| $\Upsilon_{d y n}^{z x}, \Upsilon_{1}^{z x}$ | $\sin x \sin x^{\prime}$ | $-k_{y} \sin y \cos y^{\prime}$ | $\cos z \cos z^{\prime}$ |
| $\Upsilon_{d y n}^{y y n}, \Upsilon_{1 z}^{y z}$ | $-k_{x} \sin x \cos x^{\prime}$ | $\cos y \cos y^{\prime}$ | $\sin z \sin z^{\prime}$ |
| $\Upsilon_{d y n}^{z y}, \Upsilon_{1}^{z y}$ | $k_{x} \sin x \cos x^{\prime}$ | $\sin y \sin y^{\prime}$ | $\cos z \cos z^{\prime}$ |

Table C.2: Modal functions for $\overline{\overline{\mathbf{G}}}_{E M}$.

| $i$ | position | $X_{i}$ | $Y_{i}$ | $Z_{i}$ | $A_{i}^{x x}$ | $A_{i}^{y y}$ | $A_{i}^{z z}$ | $V_{i}$ |
| :---: | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | $\left(+x^{\prime},+y^{\prime},+z^{\prime}\right)$ | $\left(x-x^{\prime}\right)$ | $\left(y-y^{\prime}\right)$ | $\left(z-z^{\prime}\right)$ | +1 | +1 | +1 | +1 |
| 1 | $\left(+x^{\prime},+y^{\prime},-z^{\prime}\right)$ | $\left(x-x^{\prime}\right)$ | $\left(y-y^{\prime}\right)$ | $\left(z+z^{\prime}\right)$ | -1 | -1 | +1 | -1 |
| 2 | $\left(+x^{\prime},-y^{\prime},+z^{\prime}\right)$ | $\left(x-x^{\prime}\right)$ | $\left(y+y^{\prime}\right)$ | $\left(z-z^{\prime}\right)$ | -1 | +1 | -1 | -1 |
| 3 | $\left(+x^{\prime},-y^{\prime},-z^{\prime}\right)$ | $\left(x-x^{\prime}\right)$ | $\left(y+y^{\prime}\right)$ | $\left(z+z^{\prime}\right)$ | +1 | -1 | -1 | +1 |
| 4 | $\left(-x^{\prime},+y^{\prime},+z^{\prime}\right)$ | $\left(x+x^{\prime}\right)$ | $\left(y-y^{\prime}\right)$ | $\left(z-z^{\prime}\right)$ | +1 | -1 | -1 | -1 |
| 5 | $\left(-x^{\prime},+y^{\prime},-z^{\prime}\right)$ | $\left(x+x^{\prime}\right)$ | $\left(y-y^{\prime}\right)$ | $\left(z+z^{\prime}\right)$ | -1 | +1 | -1 | +1 |
| 6 | $\left(-x^{\prime},-y^{\prime},+z^{\prime}\right)$ | $\left(x+x^{\prime}\right)$ | $\left(y+y^{\prime}\right)$ | $\left(z-z^{\prime}\right)$ | -1 | -1 | +1 | +1 |
| 7 | $\left(-x^{\prime},-y^{\prime},-z^{\prime}\right)$ | $\left(x+x^{\prime}\right)$ | $\left(y+y^{\prime}\right)$ | $\left(z+z^{\prime}\right)$ | +1 | +1 | +1 | -1 |

Table C.3: Basic image set parameters of the potential Green functions for electric sources.

| $i$ | position | $X_{i}$ | $Y_{i}$ | $Z_{i}$ | $F_{i}^{x x}$ | $F_{i}^{y y}$ | $F_{i}^{z z}$ | $W_{i}$ |
| :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- | :--- |
| 0 | $\left(+x^{\prime},+y^{\prime},+z^{\prime}\right)$ | $\left(x-x^{\prime}\right)$ | $\left(y-y^{\prime}\right)$ | $\left(z-z^{\prime}\right)$ | +1 | +1 | +1 | +1 |
| 1 | $\left(+x^{\prime},+y^{\prime},-z^{\prime}\right)$ | $\left(x-x^{\prime}\right)$ | $\left(y-y^{\prime}\right)$ | $\left(z+z^{\prime}\right)$ | +1 | +1 | -1 | +1 |
| 2 | $\left(+x^{\prime},-y^{\prime},+z^{\prime}\right)$ | $\left(x-x^{\prime}\right)$ | $\left(y+y^{\prime}\right)$ | $\left(z-z^{\prime}\right)$ | +1 | -1 | +1 | +1 |
| 3 | $\left(+x^{\prime},-y^{\prime},-z^{\prime}\right)$ | $\left(x-x^{\prime}\right)$ | $\left(y+y^{\prime}\right)$ | $\left(z+z^{\prime}\right)$ | +1 | -1 | -1 | +1 |
| 4 | $\left(-x^{\prime},+y^{\prime},+z^{\prime}\right)$ | $\left(x+x^{\prime}\right)$ | $\left(y-y^{\prime}\right)$ | $\left(z-z^{\prime}\right)$ | -1 | +1 | +1 | +1 |
| 5 | $\left(-x^{\prime},+y^{\prime},-z^{\prime}\right)$ | $\left(x+x^{\prime}\right)$ | $\left(y-y^{\prime}\right)$ | $\left(z+z^{\prime}\right)$ | -1 | +1 | -1 | +1 |
| 6 | $\left(-x^{\prime},-y^{\prime},+z^{\prime}\right)$ | $\left(x+x^{\prime}\right)$ | $\left(y+y^{\prime}\right)$ | $\left(z-z^{\prime}\right)$ | -1 | -1 | +1 | +1 |
| 7 | $\left(-x^{\prime},-y^{\prime},-z^{\prime}\right)$ | $\left(x+x^{\prime}\right)$ | $\left(y+y^{\prime}\right)$ | $\left(z+z^{\prime}\right)$ | -1 | -1 | -1 | +1 |

Table C.4: Basic image set parameters of the potential Green functions for magnetic sources.

(a) $G_{A}^{x x}$


(b) $G_{A}^{y y}$


(c) $G_{A}^{z z}$

Figure C.1: Basic image set for $\overline{\overline{\mathbf{G}}}_{A}$.


Figure C.2: Basic image set for $\overline{\overline{\mathbf{G}}}_{F}$.


Figure C.3: Basic image set for $g_{v}$.


Figure C.4: Basic image set for $g_{w}$.

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[^0]:    ${ }^{1}$ The algorithm complexity is the number of required floating point operations. Algorithm complexity of MoM is of the order of $N^{2}$ [16], where $N$ is the number of unknowns.
    ${ }^{2}$ Named after Poggio, Miller, Chew, Harrington, Wu and Tsai, who were among the first to investigate it.
    ${ }^{3} \hat{n}$ is the surface normal unit vector and $\mathbf{f}$ represents a normally continuous (div-conforming) basis function.

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[^3]:    ${ }^{1}$ The number of required floating point operations.

[^4]:    ${ }^{2} \hat{n}$ is the surface normal unit vector and $\mathbf{f}$ represents a tangentially continuous basis function.

