Life Table Forecasting with the Gompertz Distribution

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1. Introduction

There has been a dramatic increase in the life expectancy of humans in all industrial countries in the last century. In the first decades, the increase was due to decline in mortality rates of infants, and in later decades, was due to decline in mortality rates of adults. Because of the low mortality in young ages, the Gompertz model is suitable for forecasting a complete life table with its relevant parameters. In the present work, the Gompertz model is transformed into a time-dependent model with four parameters. Life table data of the USA from 1900 to 2000 serve as a basis for the estimation. The parametric model is then used for the forecast of the life table and its statistical and demographic parameters until 2100. The forecasts are compared with official forecasts. First, however, the Gompertz distribution is presented.

2. The Gompertz distribution and its statistical and demographic parameters

The British actuary, Benjamin Gompertz, proposed a simple formula in 1825 for describing the mortality rates of the elderly. This famous law states that death rates increase exponentially with age.

In terms of actuarial notation, this formula can be expressed as

$$\mu(x)=A\cdot e^{kx},$$

where $\mu(x)$ is the force of mortality, A>0 and k>0. A represents the general mortality level, and k is the age-specific growth rate of the force of mortality. Since

$$\mu(x)=-\frac{d\,l(x)}{dx}\cdot\frac{1}{l(x)}=-\frac{d\,\ln l(x)}{dx},$$

one gets through integration the survivor function of the Gompertz distribution

$$l(x)=\exp\left(\frac{A}{k}-\frac{A}{k}\cdot e^{k\cdot x}\right)\quad x\geq 0.$$

The mode can simply be determined by differentiation of the density function $-\frac{dl(x)}{dx}$ (cf., e.g., Fraser, 1911):

$$m=-\frac{\ln\left(\frac{A}{k}\right)}{k}.$$

From

$$A=k\cdot e^{-k\cdot m}=\mu(m)\cdot e^{-k\cdot m}$$

follows

$$\mu(x)=k\cdot e^{k\cdot(x-m)}=\mu(m)\cdot e^{k\cdot(x-m)}.$$

In this representation the survivor function is given by

$$l(x)=\exp\left(e^{-k\cdot m}-e^{k\cdot(x-m)}\right) \qquad = \exp\left(e^{-m/s}-e^{-(x-m)/s}\right)$$

(cf., e.g., Fraser, 1911; Gumbel, 1937; Carriere, 1994). One recognizes that the survivor function is characterized by the mode m and a spread parameter $s=\frac{1}{k}$.

Since with human populations $e^{-k\cdot m}\approx 0$, the survivor function can be approximated for $-\infty<x<\infty$ by

$$l(x)=\exp\left(-\frac{A}{k}\cdot e^{k\cdot x}\right)=\exp\left(-e^{k\cdot(x-m)}\right)$$

(cf., e.g., Gumbel, 1937; Pollard, 1991; Valcovics, 1992; Pollard, 1998), which is the survivor function of the Gumbel distribution (minimum) or the extreme value type I distribution for the minimum.

Carriere (1994) showed with the cumulant generating function that the cumulants of the Gumbel distribution are also approximately valid for the Gompertz distribution. One obtains,

Mean:

$$\mu\approx-\frac{\ln\left(\frac{\text{A}}{\text{k}}\right)+\gamma}{\text{k}}=\text{m}-\frac{\gamma}{\text{k}}$$ with $\gamma=0{,}577221566...$ (Euler-Mascheroni-Constant)

Variance:

$$\sigma^2\approx\frac{\pi^2/6}{\text{k}^2}=\frac{1{,}6449341}{\text{k}^2}$$

Skewness:

$$g_1\approx-1{,}1395415$$

Kurtosis:

$$g_2=\frac{\mu_4}{\sigma^4}-3\approx 2{,}4.$$

From the distribution function it is easy to obtain the p-quantiles, which are

$$x_p = \frac{\ln(1-e^{k\cdot m}\cdot\ln(1-p))}{k} \approx m + \frac{\ln(-\ln(1-p))}{k}.$$

Especially the median is

$$x_{0,5} = \frac{\ln\left(1+e^{k\cdot m}\cdot\ln 2\right)}{k} \approx m + \frac{\ln(\ln 2)}{k}.$$

The life expectancy at age x can be approximated by

$$e_x = -\frac{\dfrac{\gamma+\ln\left(\frac{A}{k}\right)+k\cdot x-\frac{A}{k}\cdot\exp(k\cdot x)}{k}}{\exp\left(\frac{A}{k}-\frac{A}{k}\cdot e^{k\cdot x}\right)}$$

$$= -\frac{\dfrac{\gamma+k\cdot(x-m)-\exp\left(k\cdot(x-m)\right)}{k}}{\exp\left(e^{-k\cdot m}-e^{k\cdot(x-m)}\right)}$$ (cf., e.g., Pollard, 1998).

Putting x=0, one gets the life expectancy at birth

$$\mathrm{e}_0 = \mu = -\frac{\ln\left(\frac{\mathrm{A}}{\mathrm{k}}\right)+\gamma-\frac{\mathrm{A}}{\mathrm{k}}}{\mathrm{k}} \approx -\frac{\ln\left(\frac{\mathrm{A}}{\mathrm{k}}\right)+\gamma}{\mathrm{k}} = \mathrm{m}-\frac{\gamma}{\mathrm{k}}.$$

Particularly simple is the life expectancy at the modal age, m,

$e_m = \frac{p}{k} = \frac{0{,}59634736..}{k}$ (cf. Pollard, 1998), where p is the Euler-Gompertz-Constant.

One can show that the mean, the variance, and the old-age dependency ratio of the stationary population are

$$\mu_S = \frac{\mu+\frac{\sigma^2}{2}}{2} = \frac{e_0+\frac{\sigma^2}{e_0}}{2} = \frac{-\frac{\ln\left(\frac{A}{k}\right)+\gamma}{k}+\frac{\pi^2}{12\cdot k^2}}{2}$$

$$= \frac{m-\frac{\gamma}{2}+\frac{\pi^2}{12\cdot k^2}}{2},$$

$$\sigma_S^2 = \frac{e_0^2}{12} - \left(\frac{\sigma^2}{2\cdot e_0}\right)^2 + \frac{\sigma^2}{2} + \frac{g_1\cdot\sigma^3}{3\cdot e_0},$$

and for big values of m,

$$ODR = \frac{\int_{60}^{\infty} l(x)dx}{\int_{20}^{60} l(x)dx} = \frac{\int_{60}^{\infty}\exp\left(\frac{A}{k}-\frac{A}{k}\cdot e^{k\cdot x}\right)dx}{\int_{20}^{60}\exp\left(\frac{A}{k}-\frac{A}{k}\cdot e^{k\cdot x}\right)dx} \approx \frac{m-\frac{\gamma}{k}-60}{40} = \frac{e_0-60}{40}.$$

The maximum life span can be estimated as the age of the last and single survivor of a population of size N according to a suggestion of Gumbel (1937) and Finch&Pike (1995). Assuming the Gompertz distribution one gets from

$$l(\omega) = \frac{1}{N},$$

the maximum life span as

$$\omega = \frac{\ln\left(1+\frac{k\cdot\ln N}{A}\right)}{k} = \frac{\ln\left(1+e^{k\cdot m}\cdot\ln N\right)}{k} = m + \frac{\ln\ln\left(N\cdot e\cdot l(m)\right)}{k}$$

From a life table with l(0)=100,000, the following approximation results

$$\omega \approx m + \frac{2.44347}{k} \approx e_0 + \frac{3}{k}.$$

As a measure for the rectangularization of a life table we use Keyfitz´s entropy (Keyfitz, 1977),

$$H = -\frac{\int_0^{\infty} l(x)\ln l(x)dx}{\int_0^{\infty} l(x)dx} \approx -\frac{1}{\ln\left(\frac{A}{k}\right)+\gamma} = \frac{1}{k\cdot m-\gamma}.$$

Rectangularization is defined as a trend towards a more rectangular shape of the survival curve due to increased survival and concentration of deaths around the mean age at death (cf., e.g., Eakin&Witten (1995); Wilmoth (1999) or Cheung, SLK et al. (2005)).

3. A simple, time-dependent Gompertz model

Life tables and their parameters depend on time. In the last decades, the life expectancy, the modal-age, and the survivor rates rose continuously. Therefore, it is obvious to take time t as variable explicitly into account in the Gompertz model.

Wetterstrand (1981) and Schoen et al. (2004) consider the time factor for a continuously sinking mortality, in which they modify the force of mortality function as follows

$\mu(x,t) = A\cdot e^{k\cdot x-c\cdot t} = e^{\ln A+k\cdot x-c\cdot t}$ with A, k >0, and $c \geq 0$.

This model certainly is suitable in order to explain a steadily increasing life expectancy. The life expectancy is in this case

$\mu(\mathrm{t}) = \mathrm{e}_0(\mathrm{t}) \approx -\frac{\ln\left(\frac{\mathrm{A}}{\mathrm{k}}\right)+\gamma-\mathrm{c}\cdot\mathrm{t}}{\mathrm{k}}$, with its first derivative

$$\frac{\mathrm{d}\mu(\mathrm{t})}{\mathrm{dt}} = \frac{\mathrm{de}_0(\mathrm{t})}{\mathrm{dt}} = \frac{\mathrm{c}}{\mathrm{k}}.$$

The life expectancy rises linearly with the factor c/k annually.

The cross-sectional average length of life, CAL, which indicates the population size at time t (Guillot, 2003), can be calculated in this time-dependent Gompertz model as

$$\mathrm{CAL}(t) \approx -\frac{\ln\left(\frac{A}{k-c}\right)+\gamma-c\cdot t}{k}.$$

The first derivative is $\frac{d\mathrm{CAL}(t)}{dt}=\frac{c}{k}$.

Life expectancy and CAL show the same change each year.

The previously observed model assumes that the growth rate k is independent of time. However, empirical analyses show that the mortality-decline in the last decades has happened with a sinking A, but an increasing k. This phenomenon is also called Strehler-Mildvan correlation (cf. Strehler&Mildvan, 1960). Therefore, the model shall be modified as follows

$$\mu(x,t)=A\cdot e^{(k+d\cdot t)\cdot x-c\cdot t}=A\cdot e^{k\cdot x+d\cdot t\cdot x-c\cdot t}=A\cdot e^{-c\cdot t}\cdot e^{k\cdot x+d\cdot t\cdot x}$$

with A, k > 0 and c, d ≥ 0.

Taking logarithms to base *e* yields

$$\ln\mu(x,t)=\ln A-c\cdot t+(k+d\cdot t)\cdot x.$$

The survivor function of the Gompertz distribution for this dynamic force of mortality is given by

$$l(x,t)=\exp\left(\frac{A\cdot e^{-c\cdot t}}{k+d\cdot t}-\frac{A\cdot e^{-c\cdot t}}{k+d\cdot t}\cdot e^{(k+d\cdot t)\cdot x}\right)$$

with the time-dependent mode, the time-dependent life expectancy, and the time-dependent variance

$$m(t)=-\frac{\ln\left(\frac{A}{k+d\cdot t}\right)-c\cdot t}{k+d\cdot t},$$

$$e_0(t)=-\frac{\gamma+\ln\left(\frac{A}{k+d\cdot t}\right)-c\cdot t}{k+d\cdot t},$$

$$\sigma^2(t)=\frac{\pi^2/6}{(k+d\cdot t)^2}.$$

Differentiation of the life expectancy at birth yields

$$\frac{de_0(t)}{dt}=\frac{d\cdot\ln\left(\frac{A}{k+d\cdot t}\right)+c\cdot k+d\cdot(\gamma+1)}{(k+d\cdot t)^2}.$$

The derivative $\frac{de_0(t)}{dt}$ is positive, if

$$c>-d\cdot\frac{\left(1+\gamma+\ln\left(\frac{A}{k+d\cdot t}\right)\right)}{k}.$$

Thus, the life expectancy at birth growths in contrast to the prior model with decreasing increases. The limiting value of the life expectancy is

$$e_0(\inf)=\lim_{t\to\infty}\left(-\frac{\gamma+\ln\left(\frac{A}{k+d\cdot t}\right)-c\cdot t}{k+d\cdot t}\right)=\frac{c}{d}.$$

In this theoretical case, a complete rectangularization of the life table is achieved. The life expectancy at birth is identical with the life span. All persons die in the same age. The variance $\sigma^2(t)$ is zero. The model, and later the empirical results suggest that increases in longevity reach some limit.

Other time-dependent parameters of this version of the Gompertz distribution can be defined in a similar way. E.g., the Keyfitz entropy is

$$H(t)=-\frac{e^{-\left(\frac{A-c\cdot t}{k+d\cdot t}\right)}}{\ln\left(\frac{A-c\cdot t}{k+d\cdot t}\right)+\gamma}.$$

A closed form integration of the parameter CAL(t) is not possible with this expanded model. The integral can only be solved by numerical methods.

4. Empirical results

In order to estimate the parameters of the Gompertz distribution, different methods are available: the method of moments, the maximum-likelihood-method, and the method of least squares (cf., e.g., Pollard&Valkovics 1992; Carriere 1994; Kunimura 1997; Jong-Wuu et al. 2004).

In the present work, the parameters shall be estimated by a linear method of least squares using the function of the force of mortality. Taking logarithms yields,

$$\ln\mu(x,t)=\ln A-c\cdot t+(k+d\cdot t)\cdot x=\ln A-c\cdot t+k\cdot x+d\cdot(t\cdot x),$$

with following transformations

a) the force of mortality is estimated by $y=\mu(a,t)=-\ln(1-q(x,t))$, where q(x,t) is the age-specific death rate at age $a=x+0.5$ at time t.

b) $t=year-1900$, or, $t=0,10,20,....90,100$.

c) The variable $z=t\cdot a$ reflects the time influence of the parameter k.

The time-dependent regression model finally is

$$y=\ln A-c\cdot t+k\cdot a+d\cdot z+u$$
$$=(\ln A-c\cdot t)+k\cdot(a+d\cdot t)+u=\ln A(t)+k(t)+u$$

for, $a=35.5,36.5,.....and\, t=0,10,......,100$

with u as a stochastic influence. It is set up the hypothesis that c > 0 and d > 0; the general mortality-level $A(t)=A\cdot e^{-c\cdot t}$ will sink in the course of the time, whereas the age-specific mortality factor $k(t)=k+d\cdot t$ will rise.

Regression estimates were executed for both the female life table with n=808 observations and for the male life table with n=796 observations. The results are seen in table 1.
Data source are the decennial period life tables by calendar year and sex from 1900 to 2000 published by the Social Security Administration (SSA; c.f. Bell&Miller, 2005).

All estimators correspond to the theoretical assumptions, and are statistically significant. The coefficient of determination is very high. The model explains the development of the mortality between 1900 and 2000 very well. With the female population, the general mortality level A(t) decreases essentially more strongly when compared with the male one. On the other hand, the increase of k(t) is higher with the female population.

Table 2 shows the observed and estimated (forecast) values of some important life table parameters. Until approximately 1980, the model overestimates the life expectancy because of the still high infant mortality. From 1980, the differences between actual and estimated values clearly decrease because of the reduction of infant mortality. On the other hand, the Gompertz model reveals systematic overestimation at the oldest ages (cf., e.g., Thatcher et al., 1998). However, the overestimation at high ages has a small influence on many important life table parameters because of the relatively small proportion of persons in high age-classes. Despite these restrictions concerning infant and old age mortality, we think that the Gompertz model is a suitable tool for forecasting mortality of modern life tables (cf. also Pollard, 1998). The decreasing values of Keyfitz´s entropy H indicate the increasing rectangularization of the life tables.

The life expectancy grows with decreasing increases. Between 2000 and 2010 the yearly increase is 0.133 years with the female population and 0.105 years with the male population. However, between 2090 and 2100 the yearly increase will sink to 0.087 years (female population) and 0.076 years (male population). In the time-dependent model longevity or life expectancy finally reaches a limit ($t \longrightarrow \infty$), which is about 127 years with the female and about 123 years with the male population. The survivor function becomes more and more rectangular.

In order to assess the results, they are compared with life table forecasts of the Social Security Administration (SSA). The SSA mortality forecasts are mainly based on historical trends by age, sex, and cause of death (for details cf. Bell&Miller, 2005). Table 3 shows important results. Whereas similar forecasts for the male population can be observed, the forecasts differ for the female population with increasing forecast horizon. The SSA predicts lower differences between male and female mortality at the end of the 21st century.

Table 4 illustrates the development of the old-age dependency ratios. It is remarkable that for the male population, the Gompertz model and the SSA forecasts show the same trend with nearly identical values. Because of higher SSA mortality forecasts for the female population, the Gompertz model predicts substantially higher old-age dependency ratios for the female population at the end of the 21st century. Both the SSA and the Gompertz model come to the result that the youth dependency ratios (not shown here) decrease continuously to values which range between 0.5 and 0.51. Until 2050, the life span forecasts are almost same for both sexes; in later decades, the SSA forecasts of the life spans are slightly higher. These results are not implausible, because the Gompertz model overestimates mortality at very old ages. The SSA estimates the maximum life spans as 0.00001-quantiles.

5. Conclusion

Other possibilities, e.g., the models of Heligman and Pollard (1980), Lee and Carter (1992), or Bongaarts (2005) can also be used successfully to forecast mortality. The advantage of the generalized Gompertz model is the simple determination of the whole life table with all its statistical and demographic parameters, after the regression parameters have been estimated. In contrast to most other models, the forecast is not restricted only to death probabilities. Based on the fact that most integrals of the model can be expressed in closed form, one can deduce from the force of mortality function not only the survivor, the distribution, and the density function, but also formulas for relevant parameters of the life table, such as life expectancy, median, mode, variance, dependency ratio, and measures of the rectangularization (see Fig. 1). With the Gompertz model, one receives fast and consistent results. Implausible trend forecasts and unrealistic results of age-specific death rates are not possible, because intersections of trends cannot occur as in long term trend models. The disadvantages of the Gompertz model are well known: underestimation of mortality at young ages, and overestimation of mortality at old ages. But with the increasing rectangularization of the survival function, the Gompertz model becomes more and more suitable, because the goodness of fit of the model improves (see Fig. 2 and 3). Finally, with exception of the oldest ages, the life table can be summarized by two trend parameters, m(t) and k(t); the first providing an indication of the age, near which most deaths occur; and the second indicating the spread of the ages at death (cf. also Pollard, 1998).

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Table 1: Regression results of the female and male US life table between 1900 and 2000

	female			male		
Parameter	**Estimates**	**Std.-dev**	**t-Values**	**Estimates**	**Std.-dev**	**t-Values**
lnA	-7.67957	0.03162	-243	-7.47289	0.02879	-260
c	0.02877	0.0005137	-56	0.01837	0.0004731	-39
k	0.07174	0.0004299	167	0.06942	0.000393	177
d	0.0002261	0.00000683	33	0.0001495	0.00000636	24
R^2	0.9947			0.9947		
n	808			796		

Table 2: Observed (obs.) and forecast values

	female						male					
year	e_o	e_o (obs.)	μ_S	μ_S (obs.)	H	H (obs.)	e_o	e_o (obs.)	μ_S	μ_S (obs.)	H	H (obs.)
1900	62.28	48.96	33.70	33.70	0.2224	0.4769	60.91	46.41	33.26	33.00	0.2346	0.5130
1910	64.68	53.58	34.66	34.50	0.2080	0.4002	62.51	50.08	33.87	33.40	0.2241	0.4434
1920	66.93	56.27	35.58	34.60	0.1953	0.3562	64.05	54.51	34.47	34.30	0.2145	0.3769
1930	69.04	61.31	36.45	35.60	0.1840	0.2886	65.51	57.96	35.06	34.50	0.2056	0.3183
1940	71.02	65.74	37.28	36.70	0.1740	0.2350	66.92	61.43	35.62	35.10	0.1975	0.2678
1950	72.88	71.13	38.08	38.30	0.1650	0.1884	68.26	65.63	36.17	36.00	0.1900	0.2241
1960	74.64	73.24	38.83	39.00	0.1569	0.1723	69.55	66.66	36.70	36.30	0.1830	0.2131
1970	76.29	74.86	39.55	39.70	0.1496	0.1687	70.78	67.15	37.21	36.40	0.1765	0.2100
1980	77.86	77.52	40.24	40.60	0.1429	0.1535	71.97	69.94	37.71	37.40	0.1705	0.1904
1990	79.34	78.9	40.89	41.10	0.1368	0.1466	73.10	71.82	38.19	38.10	0.1648	0.1805
2000	80.75	79.39	41.52	41.20	0.1312	0.1391	74.20	74.03	38.66	38.80	0.1596	0.1656
2010	82.08		42.11		0.1261		75.25		39.11		0.1546	
2020	83.35		42.68		0.1213		76.26		39.54		0.1500	
2030	84.55		43.23		0.1169		77.24		39.97		0.1456	
2040	85.70		43.75		0.1128		78.18		40.38		0.1415	
2050	86.79		44.25		0.1090		79.08		40.78		0.1376	
2060	87.84		44.72		0.1055		79.96		41.16		0.1339	
2070	88.84		45.18		0.1022		80.80		41.53		0.1305	
2080	89.79		45.62		0.0991		81.62		41.90		0.1272	
2090	90.70		46.04		0.0961		82.41		42.25		0.1240	
2100	91.57		46.44		0.0934		83.17		42.59		0.1210	

Table 3: SSA forecasts (until 2000 actual values) and Gompertz model estimates/forecasts

	female				male			
	Gompertz		SSA		Gompertz		SSA	
year	e_o	Median	e_o	Median	e_o	Median	e_o	Median
1900	62.28	65.21	48.96	58.17	60.91	63.94	46.41	55.15
1910	64.68	67.53	53.58	63.5	62.51	65.48	50.08	59.12
1920	66.93	69.69	56.27	65.27	64.05	66.96	54.51	63.75
1930	69.04	71.72	61.31	68.76	65.51	68.36	57.96	65.26
1940	71.02	73.63	65.74	72.08	66.92	69.71	61.43	67.53
1950	72.88	75.42	71.13	75.82	68.26	71.00	65.63	70.11
1960	74.64	77.11	73.24	77.79	69.55	72.24	66.66	70.7
1970	76.29	78.70	74.86	79.22	70.78	73.42	67.15	70.98
1980	77.86	80.21	77.52	81.29	71.97	74.56	69.94	73.47
1990	79.34	81.63	78.9	82.36	73.10	75.65	71.82	75.51
2000	80.75	82.98	79.39	82.69	74.20	76.70	74.03	77.58
2010	82.08	84.26	79.95	83.15	75.25	77.70	75.4	78.99
2020	83.35	85.48	80.8	83.96	76.26	78.68	76.5	80.04
2030	84.55	86.64	81.66	84.79	77.24	79.61	77.51	81
2040	85.70	87.74	82.47	85.55	78.18	80.51	78.46	81.91
2050	86.79	88.79	83.22	86.25	79.08	81.38	79.35	82.75

2060	87.84	89.79	83.93	86.88	79.96	82.22	80.18	83.53
2070	88.84	90.75	84.6	87.46	80.80	83.02	80.97	84.25
2080	89.79	91.66	85.23	88.01	81.62	83.81	81.71	84.91
2090	90.70	92.54	85.83	88.51	82.41	84.56	82.41	85.51
2100	91.57	93.37	86.4	88.99	83.17	85.29	83.07	86.07

Table 4: SSA forecasts (until 2000 actual values) and Gompertz model estimates/forecasts of the old dependency ratios ODR and maximum life spans ω

	female				male			
	Gompertz		SSA		Gompertz		SSA	
year	ODR	ω	ODR	ω	ODR	ω	ODR	ω
1900	0.265	104.39	0.279	104.91	0.259	104.4	0.256	104.41
1910	0.288	105.50	0.294	105.13	0.271	105.1	0.277	104.65
1920	0.314	106.54	0.302	105.71	0.285	105.8	0.286	105.36
1930	0.343	107.51	0.324	105.98	0.300	106.4	0.281	105.44
1940	0.372	108.41	0.354	106.43	0.317	107.0	0.290	105.63
1950	0.403	109.26	0.411	108.97	0.334	107.5	0.320	107.93
1960	0.433	110.05	0.442	109.54	0.352	108.1	0.327	108.23
1970	0.463	110.79	0.473	111.41	0.370	108.6	0.333	109.02
1980	0.493	111.49	0.513	112.62	0.389	109.1	0.372	110.38
1990	0.522	112.15	0.536	113.27	0.407	109.6	0.406	110.64
2000	0.551	112.76	0.540	111.99	0.426	110.0	0.439	109.7
2010	0.579	113.35	0.548	111.54	0.444	110.4	0.463	109.31
2020	0.606	113.90	0.565	112.26	0.463	110.8	0.480	110.11
2030	0.633	114.42	0.582	113.19	0.481	111.2	0.500	111.15
2040	0.658	114.92	0.599	114.11	0.499	111.6	0.518	112.19
2050	0.683	115.38	0.615	114.99	0.517	112.0	0.539	113.18
2060	0.707	115.83	0.630	115.91	0.534	112.3	0.555	114.13
2070	0.730	116.25	0.645	116.79	0.552	112.7	0.571	115.03
2080	0.752	116.65	0.658	117.63	0.569	113.0	0.586	115.94
2090	0.774	117.04	0.669	118.41	0.585	113.3	0.600	116.85
2100	0.795	117.40	0.686	119.11	0.602	113.6	0.614	117.71

$$e_0 = \int_0^\infty l(x)dx = \int_0^\infty x \cdot f(x)dx$$ **Parameters (e.g., life expectancy at birth)**

↑

$$f(x) = \frac{dF(x)}{dx} = -\frac{dl(x)}{dx}$$ **Density function**

$$F(x) = 1 - l(x)$$ **Distribution function**

$$l(x) = e^{-\int_0^x \mu(u)du}$$ **Survivor function**

↑

$$\mu(x)$$ **Force of mortality function**

Figure 1: Relationships between force of mortality function, survivor function, and parameters

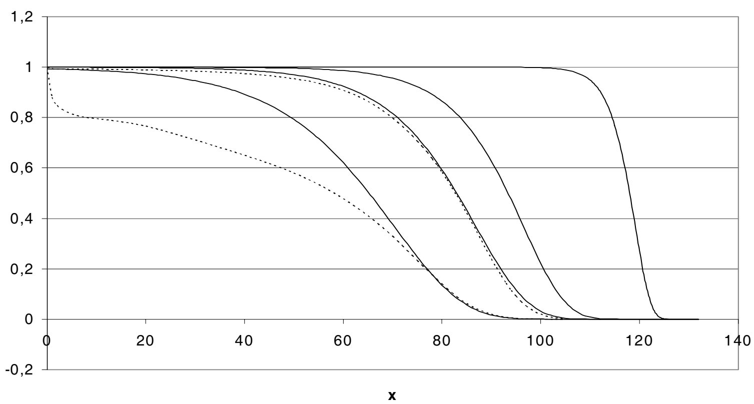

Figure 2: Gompertz life tables (survivor finctions) of the female population in 1900, 2000, 2100, 3000, and actual life tables in 1900 and 2000 (.......)

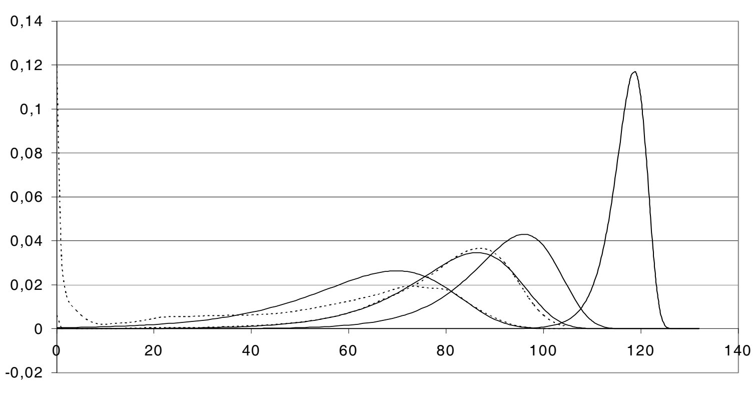

Figure 3: Gompertz density functions of the female population in 1900, 2000, 2100, 3000, and actual death density functions in 1900 and 2000 (.......)