Measuring the Rectangularization of Life Tables Using the Gompertz Distribution

Peter Pflaumer[1]
[1]Technical University of Dortmund, 44221 Dortmund, Germany

Abstract
The rectangularization of life tables is defined as a trend toward a more rectangular shape of the survival curve, due to increased survival and concentration of deaths around the mean age at death. As the mortality of modern developed populations is largely the mortality of old age, the Gompertz model provides a good approximation of life tables in these populations, and can be used to estimate and forecast many parameters of the life table and the stationary population, such as, expectation of life, modal age, age-dependency ratios, and indices of the rectangularization. Formulas of known rectangularization indices are developed assuming the Gompertz distribution, whereas, some new indices are proposed, too. The mathematical relationships between the single indices are shown. It is demonstrated that some mentioned indices are a function of the coefficient of variation.

Key Words: Mortality, life table, Keyfitz entropy, Gini coefficient

1. Introduction

The rectangularization of the survival curve is defined as a trend toward a more rectangular shape of the survival curve, due to increased survival and concentration of deaths around the mean age at death. The variability in the age at death declines and deaths are being compressed into the upper years of life (Nussfelder et al, 1996). An overview of the different indices describing the rectangularization is given, for example, in Cheung et al. (2005), or Wilmoth and Horiuchi (1999).

Eakin and Witten (2005), introduced the so-called prolate rectangularity index kappa for describing rectangularization. The index was based on the angle located on the diagonal line, connecting the point of maximum acceleration of death in attrition to the point of maximum deceleration of death on the standardized survival curve. They obtained analytical expression for the prolate index using the Gompertz and Weibull distribution.

Other selected indices of rectangularization shall be calculated analytically using the Gompertz distribution for life tables with high life expectancy, and shall be compared. The Gompertz model was chosen because current life tables with low mortality could be explained quite well through this model (cf., for example, Pollard, 1998 or Pflaumer, 2007). Analytical formulas for the already well-known indices, such as, the Keyfitz entropy or the Gini coefficient, were developed. Moreover, some new indices were proposed: a modification of the index of Eakin and Witten (1995), an index based on the old-age dependency ratio, and an index that considered the potential years of life lost

(PYLL). The relationship between the different indices has been analyzed and shown. It appears that some indices are functions of the coefficient of variation.

2. The Gompertz Distribution and its Statistical Parameters

The British actuary, Benjamin Gompertz, proposed a simple formula in 1825 for describing the mortality rates of the elderly. This famous law states that death rates increase exponentially with age.

In terms of actuarial notation, this formula can be expressed as

$\mu(x)=A\cdot e^{kx}$,

where $\mu(x)$ is the force of mortality, A>0 and k>0. A represents the general mortality level, and k is the age-specific growth rate of the force of mortality. As

$\mu(x)=-\frac{d\ l(x)}{dx}\cdot\frac{1}{l(x)}=-\frac{d\ \ln l(x)}{dx}$,

one gets the survivor function of the Gompertz distribution through integration

$l(x)=\exp\left(\frac{A}{k}-\frac{A}{k}\cdot e^{k\cdot x}\right)\quad x\geq 0$, with the modal value $m=-\frac{\ln\left(\frac{A}{k}\right)}{k}$.

From

$A=k\cdot e^{-k\cdot m}=\mu(m)\cdot e^{-k\cdot m}$

it follows that

$\mu(x)=k\cdot e^{k\cdot(x-m)}=\mu(m)\cdot e^{k\cdot(x-m)}$.

In this representation the survivor function is given by

$l(x)=\exp\left(e^{-k\cdot m}-e^{k\cdot(x-m)}\right)=\exp\left(e^{-m/s}-e^{-(x-m)/s}\right)$ (cf., e.g., Fraser, 1911; Gumbel, 1937; Carriere, 1994).

One recognizes that the survivor function is characterized by the mode m and a spread parameter $s=\frac{1}{k}$.

As with human populations $e^{-k\cdot m}\approx 0$, the survivor function can be approximated for $-\infty<x<\infty$ by

$l(x)=\exp\left(-\frac{A}{k}\cdot e^{k\cdot x}\right)=\exp\left(-e^{k\cdot(x-m)}\right)$

(cf., e.g., Gumbel, 1937; Pollard, 1991; Pollard and Valcovics, 1992; Pollard, 1998), which is the survivor function of the Gumbel distribution (minimum) or the extreme value type I distribution for the minimum.

Carriere (1994) showed that with the cumulant generating function the cumulants of the Gumbel distribution are also approximately valid for the Gompertz distribution. One obtains,

Mean:

$\mu\approx-\frac{\ln\left(\frac{\mathrm{A}}{\mathrm{k}}\right)+\gamma}{\mathrm{k}}=\mathrm{m}-\frac{\gamma}{\mathrm{k}}$ with $\gamma=0{,}577221566...$ (Euler-Mascheroni-Constant)

Variance:

$$\sigma^2 \approx \frac{\pi^2/6}{k^2} = \frac{1,6449341}{k^2}$$

Skewness:

$$g_1 \approx -1,1395415$$

Kurtosis:

$$g_2 = \frac{\mu_4}{\sigma^4} - 3 \approx 2,4 .$$

From the distribution function it is easy to obtain the p-quantiles, which are,

$$x_p = \frac{\ln(1 - e^{k \cdot m} \cdot \ln(1-p))}{k} \approx m + \frac{\ln(-\ln(1-p))}{k} .$$

3. Rectangularization Measures

The methodical basis for the calculation of the measures shown in tables 1 and 2 is the exponential integral function

$$E_1(x) = \int_x^\infty \frac{e^{-t}}{t} \cdot dt$$

with the following series representation (cf. Abramowitz; Stegun, 1972)

$$E_1(x) = -\gamma - \ln x + x - \frac{1}{4} \cdot x^2 + \frac{1}{18} \cdot x^3 - \frac{1}{96} \cdot x^4 + \frac{1}{600} \cdot x^5 - \frac{1}{4320} \cdot x^6 ....$$

Obvious measures of rectangularization are the standard deviation, the coefficient of variation, and the interquartile range of the Gompertz distribution. All these measures tend to go to zero with increasing rectangularization. The coefficient of variation as a relative dispersion measure tends to go, even with constant variance, against zero, if the life expectation e_0 becomes larger.

The Keyfitz entropy H (Keyfitz, 1977) in table 1 is one of the best known mortality measures in demography. Assuming the Gompertz distribution one receives a very simple expression for the entropy, it depends only on the expectation of life e_0 and k, which represent the age specific growth rate of the force of mortality function. The entropy H can be interpreted clearly: If the death rates at all ages decrease by one percent, the expectation of life increases by $\frac{1}{k \cdot e_0}$ percent, which equals 0.118 percent, for example, if $k=0.1$ and $e_0=85$.

It can easily be shown that the Keyfitz entropy is a constant multiple of the coefficient of variation.

Substitution of $k=\frac{\gamma}{m-e_0}$ in above formula yields a very descriptive and simply explicable representation of the Keyfitz entropy $H=\frac{1}{\gamma}\left(\frac{m}{e_0}-1\right)$, where $\left(\frac{m}{e_0}-1\right)\cdot 100$ is the percentage extent to which the modal value is higher than the life expectancy.

Gumbel (1932) uses the ratio of the life expectancy and the average age of the stationary population as a measure of rectangularization

$$\mathrm{g}=\frac{\mathrm{e}_0}{2\cdot\mu_S} \text{ with } \mu_S=\frac{\int x\cdot l(x)dx}{\int l(x)dx}=\frac{\mu+\frac{\sigma^2}{\mu}}{2}=\frac{e_0+\frac{\sigma^2}{e_0}}{2}.$$

In a complete rectangular survival function the life expectancy at birth is twice as high as the mean age of a stationary population. In this limiting case the Gumbel ratio is one.

The Gini coefficient is a measure of statistical dispersion developed by the Italian statistician Corrado Gini in 1912. It is commonly used as a measure of inequality of income and wealth. However, it can be also used as an index of rectangularization. In this context it is represented by

$$R=1-\frac{\int_0^{\omega}(l(x))^2dx}{e_0}$$

(cf. Gumbel (1929), Dorfman (1979) and Hanada (1983)).

Smits and Jonden (2009) have used the Gini coefficient as a measure of length of life inequality. They have shown in an empirical study that a strong negative correlation between the life expectancy and the Gini coefficient exists. Moreover, at each level of life expectancy they have observed considerable variations of the Gini coefficient. These results can also be explained by the Gompertz model through a simple relationship between the Gini coefficient R and the life expectancy e_0

$$\mathrm{R}=\frac{\ln 2}{\mathrm{e}_0\cdot\mathrm{k}}.$$

Gini also introduced a dispersion measure, which is called the mean difference
$\Delta=\iint|x-y|dF(x)dF(y)$.

The relative mean difference quantifies the mean difference in comparison to the size of the mean. The Gini coefficient is equal to half of the relative mean difference,

$R=\frac{\Delta}{2\cdot\mu}=\frac{\Delta}{2\cdot e_0}$ (cf. Nair, 1936). Using the Gompetz distribution, a simple analytical expression can be deduced. The Gini coefficient is a multiple of both the coefficient of variation and of the Keyfitz entropy. The mean difference is proportional to the standard deviation.

Table 1: Rectangularization Measures I

Measure	Gompertz Distribution	Remarks and Relationships
Standard Deviation $\sigma=\sqrt{-\int_0^\infty (x-e_0)^2 dl(x)}$	$\sigma=\frac{\sqrt{6}\cdot\pi}{6\cdot k}$	$e_0\approx m-\frac{\gamma}{k}$
Coefficient of Variation $V=\frac{\sigma}{e_0}$	$V=\frac{\sqrt{6}\cdot\pi}{6\cdot(k\cdot m-\gamma)}$	
Interquartile Range $Q=x_{0.75}-x_{0.25}$	$Q=\frac{\ln\left(\frac{\ln 0.25}{\ln 0.75}\right)}{k}\approx\frac{1.5725}{k}\approx 1{,}2261\cdot\sigma$	p-quantile $x_p=\frac{\ln(1-e^{k\cdot m}\cdot\ln(1-p))}{k}\approx m+\frac{\ln(-\ln(1-p))}{k}$
Keyfitz Entropy $H=-\frac{\int_0^\infty l(x)\ln l(x)dx}{\int_0^\infty l(x)dx}$	$H=\frac{1}{k\cdot m-\gamma}=\frac{1}{k\cdot e_0}=\frac{1}{\gamma}\left(\frac{m}{e_0}-1\right)$	$H=\frac{\sqrt{6}}{\pi}\cdot V$
Gumbel Ratio $g=\frac{e_0}{2\cdot\mu_S}=\frac{e_0^2}{e_0^2+\sigma^2}$	$g=\frac{6\cdot(\gamma-k\cdot m)^2}{6\cdot\gamma^2-12\gamma\cdot k\cdot m+6\cdot k^2\cdot m^2+\pi^2}$	$g=\frac{1}{1+V^2}\approx 1-V^2+V^4$ $g=\frac{1}{\frac{\pi^2}{6}\cdot H^2+1}\approx 1-\frac{\pi^2\cdot H^2}{6}$
Gini Coefficient $R=1-\frac{\int_0^\omega (l(x))^2 dx}{e_0}$	$R=\frac{\ln 2}{k\cdot m-\gamma}=\frac{\ln 2}{e_0\cdot k}$	$R=\frac{\sqrt{6}\cdot\ln 2}{\pi}\cdot V\approx 0{,}54\cdot V$ $R=\ln 2\cdot H$
Mean Difference $\Delta=\iint\lvert x-y\rvert dF(x)dF(y)$	$\Delta=2\cdot e_0\cdot R=\frac{2\cdot e_0\cdot\ln 2}{e_0\cdot k}=\frac{2\cdot\ln 2}{k}$	$\Delta=\frac{2\cdot\sqrt{6}\ln 2}{\pi}\cdot\sigma\approx 1{,}08088\cdot\sigma$ $R=\frac{\Delta}{2\cdot e_0}$
Potential Years of Lost Lifes $PYLL=\int_0^m l(x)\cdot\mu(x)\cdot(a-x)dx$	$PYLL=m\cdot\exp(-e^{-k\cdot m})+\frac{\gamma}{k}+\frac{E_1(1)}{k}-m$ $\approx\frac{\gamma}{k}+\frac{E_1(1)}{k}=\frac{0{,}7966}{k}$	$E_1(1)=\int_1^\infty\frac{e^{-t}}{t}dt\approx 0{,}2194$ $PYLLr=\frac{PYLL}{m}\approx\frac{0.7966}{k\cdot m}$
Percentage of Elderly People $POEP=\frac{\int_a^\omega l(x)dx}{\int_0^\omega l(x)dx}$, $m>a$	$POEP=\frac{k\cdot(m-a)-\gamma+\exp(-k\cdot(m-a))}{k\cdot m-\gamma}$ $\approx\frac{k\cdot(m-a)-\gamma}{k\cdot m-\gamma}=\frac{e_0-a}{e_0}$	$\lim_{k\to\infty} POEP=\frac{m-a}{m}$ $POEP_r=\frac{m\cdot(e_0-a)}{e_0\cdot(m-a)}$
Elasticity ε of POEP with respect to e_0	$\varepsilon=\frac{dPOEP}{de_0}\cdot\frac{e_0}{POEP}$; $\varepsilon\approx\frac{a}{e_0-a}$	$\lim_{e_0\to\infty}\varepsilon=0$

Table 2a: Rectangularization Measures II

Measure	Gompertz Distribution
Prolate Rectangularity Angle of Eakin&Witten $\theta = \arctan\left(\frac{y_{c\max} - y_{c\min}}{l(y_{c\min}) - l(y_{c\max})}\right)$	$\theta = \arctan\left(\frac{-\frac{\ln\left(\frac{7}{2} - \frac{3\cdot\sqrt{5}}{2}\right)}{k\cdot m}}{\exp\left(\frac{\sqrt{5}}{2} - \frac{3}{2}\right) - \exp\left(-\frac{\sqrt{5}}{2} - \frac{3}{2}\right)}\right)$ $\approx \arctan\left(\frac{3.1577}{k\cdot m}\right)$
Prolate Rectangularity Index of Eakin&Witten $\kappa = \cos(\theta)$	$\kappa = \cos\left(\arctan\left(\frac{3.1577}{k\cdot m}\right)\right)$ $= \frac{k\cdot m}{\sqrt{9.9698 + k^2\cdot m^2}}$
Modified Prolate Rectangularity Angle $\theta^* = \arctan\left(\left(-\frac{dl(y^*)}{dy^*}\right)^{-1}\right)$	$\theta^* = \arctan\left(\frac{e}{k\cdot m}\right)$
Modified Prolate Rectangularity Index $\kappa^* = \cos(\theta^*)$	$\kappa^* = \cos\left(\arctan\left(\frac{e}{k\cdot m}\right)\right) = \frac{k\cdot m}{\sqrt{e^2 + k^2\cdot m^2}}$

Table 2b: Rectangularization Measures II (Remarks and Relationships)

Measure	Remarks and Relationships
Prolate Rectangularity Angle of Eakin&Witten $\theta=\arctan\left(\frac{y_{c\max}-y_{c\min}}{l(y_{c\min})-l(y_{c\max})}\right)$	$y=\frac{x}{m}$ (standardized age); $l(y)=\exp\left(-e^{k\cdot(m\cdot y-m)}\right)$ $(y_{c\max},l(y_{c\max}))$: point of maximum curvature $\frac{d^2l(y)}{dy^2}=\max$ $(y_{c\min},l(y_{c\min}))$: point of minimum curvature: $\frac{d^2l(y)}{dy^2}=\min$
Prolate Rectangularity Index of Eakin&Witten $\kappa=\cos(\theta)$	$\kappa=\frac{50\cdot\left(\sqrt{6}\cdot\gamma\cdot V+\pi\right)}{\sqrt{15000\cdot\gamma^2\cdot V^2+5000\cdot\sqrt{6}\cdot\pi\cdot\gamma\cdot V+149547\cdot V^2+2500\cdot\pi^2}}$ $\approx 1-\frac{149547}{5000\cdot\pi^2}\cdot V^2+\frac{149547\cdot\sqrt{6}\cdot\gamma}{2500\cdot\pi^3}\cdot V^3=1-3{,}030\cdot V^2+2{,}728\cdot V^3$ $=1-\frac{49849}{10000}\cdot H^2+\frac{49849}{5000}\cdot H^3\ \approx 1-4.9849\cdot H^2+5{,}7547\cdot H^3$
Modified Prolate Rectangularity Angle $\theta^*=\arctan\left(\left(-\frac{dl(y^*)}{dy^*}\right)^{-1}\right)$	$y=\frac{x}{m}$ (standardized age) $y^*=1$; $\frac{dl(y)}{dy}=-k\cdot m\cdot\exp\left(e^{-k\cdot m}-1\right)\approx -k\cdot m\cdot e^{-1}$
Modified Prolate Rectangularity Index $\kappa^*=\cos(\theta^*)$	$\kappa^*=\frac{\pi+\sqrt{6}\cdot\gamma\cdot V}{\sqrt{6\cdot\left(\gamma^2+e^2\right)\cdot V^2+2\cdot\sqrt{6}\cdot\gamma\cdot\pi\cdot V+\pi^2}}$ $\approx 1-\frac{3\cdot e^2}{\pi^2}\cdot V^2+\frac{6\cdot\sqrt{6}\cdot e^2\cdot\gamma}{\pi^3}\cdot V^3=1-\frac{e^2}{2}\cdot H^2+\gamma\cdot e^2\cdot H^3$

m: modal value
μ_S: mean of the stationary population
e_0: life expectancy at birth
k: force of mortality of modal age
V: coefficient of variation
$\gamma\approx 0.5772$:

The concept of the potential years of life lost (PYLL) involves estimating the average time a person would have lived had he or she not died prematurely (Gardner and Sanborn, 1990) . To calculate the years of potential life lost, one has to set an upper reference age. A simple measure for the Gompertz distribution results, if the upper limit is the modal age. A relative measure is obtained, if the PYLL is divided by that modal age.

The population of industrialized countries is aging. Over the last decade the percentage of the population aged 65 and over has increased in most countries. Thus, the percentage of elderly population (POEP) is a suitable indicator for rectangularization. The limit in a complete rectangular life table is

$$POEP=\frac{\omega-a}{\omega},$$

where a is the lower limit of elderly age (in general, a = 60 or a = 65) and ω is the maximum age, which also coincides in this limiting case with the modal age m and the life expectancy e_0. Dividing POEP by its limiting value leads to a relative measure $POEP_r$, with an upper limit of one.

The elasticity of POEP in respect to e_o is defined as

$$\varepsilon=\frac{dPOEP}{de_0}\cdot\frac{e_0}{POEP}.$$

It measures the percent change in the percentage of elderly people (POEP) for a one percent change in the life expectancy (e_o). The elasticity sinks with increasing life expectancy. For low mortality life tables, following simple approximations are possible:

$$POEP=\frac{e_0-a}{e_0} \text{ and } \varepsilon=\frac{a}{e_0-a}.$$

Table 3 shows the accuracy of the approximations.

Table 3: Calculated and Approximated Values of Percentage of Elderly People and their Elasticities with Respect to an Increase of the Life Expectation of one Percent (a=60; Elasticities ε calculated on the basis of one percent chance.)

e_0	80	90	100	110	120
POEP (k=0.1)	0.214	0.298	0.369	0.428	0.478
POEP (k=0.2)	0.229	0.316	0.386	0.443	0.490
$\frac{e_0-a}{e_0}$	0.25	0.333	0.4	0.455	0.5
ε (k=0.1)	3.31	2.32	1.74	1.37	1.13
ε (k=0.2)	3.37	2.19	1.61	1.27	1.05
$\frac{a}{e_0-a}$	3	2	1.5	1.2	1

Eakin and Witten (2005) introduced the so-called prolate rectangularity index kappa for describing the rectangularization, as already mentioned earlier. It is defined as the cosine of the prolate rectangularity index theta (complete rectangularizatin if theta=1). The representation of the index in this article uses other parameters and symbols, and therefore, the formula shown here is not identical with the original formula of Eakin and Witten (1995). However, the numerical results are the same.

In an analogy of the index of Eakin and Witten, a similar index will be proposed, which is easier to develop and understand. As a curve measure, the gradient of the inflection point of the standardized Gompertz survival function l(y) with mode m = 1 is determined. Standardization is achieved by dividing age x by the modal's age m. The first derivative at the mode is

$$\frac{dl(y)}{dy}=-k\cdot m\cdot\exp\left(e^{-k\cdot m}-1\right)\approx -k\cdot m\cdot e^{-1}<0\ .$$

Therefore, the tangent angle at the inflection point is

$$\theta^{*}=\arctan\left(\frac{1}{k\cdot m\cdot e^{-1}}\right).$$

The modified kappa is then defined as

$$\kappa^{*}=\cos\left(\theta^{*}\right)=\cos\arctan\left(\frac{e}{k\cdot m}\right)=\frac{k\cdot m}{\sqrt{e^{2}+k^{2}\cdot m^{2}}}\ .$$

The measure of Eakin and Witten and the modified index differ only through the constant under the root. The modified measure is somewhat larger than the measure of Eakin and Witten. However, with increasing rectangularization the difference becomes much more negligible. Both the measure of Eakin and Witten and the modified measure can be approximated by a quadratic or a cubic equation of the coefficient of variation.

4. Empirical Results

In order to forecast the measures for the United States, a time-dependent Gompertz model is used, which is described in Pflaumer (2007). The results of the important Gompertz model parameter forecasts are shown in table 4. According to the forecast, life expectancy and modal age will increase in future. Sinking dispersion measures indicate a further rectangularization of the life table.

Table 5 shows forecasts of the other rectangularization measures. They describe the degree of the rectangularization differently. Although the standard deviation sinks only by approximately 20% between 2000 and 2100, the coefficient of variation decreases around almost 30% because of the increasing life expectancy. Measures, such as the Keyfitz entropy or the Gini coefficient, which are functions of the coefficient of variation, therefore, sink more strongly than the standard deviation. Hill (1993) attributed the strong increase of the Keyfitz entropy of the last decades in industrialized countries mainly to the strong increase of life expectancy.

Table 4: Life Table Parameter Forecasts (Female) of the US

Year	k	m	eo	μ_S	σ	Q	V
2000	0.094	86.9	80.7	41.5	13.6	16.7	0.168
2010	0.097	88.1	82.1	42.1	13.3	16.3	0.162
2020	0.099	89.2	83.3	42.7	13.0	15.9	0.156
2030	0.101	90.3	84.6	43.2	12.7	15.5	0.150
2040	0.103	91.3	85.7	43.7	12.4	15.2	0.145
2050	0.106	92.3	86.8	44.2	12.1	14.9	0.140
2060	0.108	93.2	87.8	44.7	11.9	14.6	0.135
2070	0.110	94.1	88.8	45.2	11.6	14.3	0.131
2080	0.112	94.9	89.8	45.6	11.4	14.0	0.127
2090	0.115	95.7	90.7	46.0	11.2	13.7	0.123
2100	0.117	96.5	91.6	46.4	11.0	13.4	0.120

Table 5: Forecasts of Recangularization Measures (Female) of the US

Year	H	g	R	Δ	PYLL	POEP a=60	θ °	θ^* °	κ	κ^*
2000	0.131	0.972	0.091	14.7	8.4	0.212	21.1	18.3	0.933	0.949
2010	0.126	0.975	0.087	14.3	8.2	0.222	20.4	17.7	0.938	0.953
2020	0.121	0.976	0.084	14.0	8.1	0.231	19.7	17.1	0.941	0.956
2030	0.117	0.978	0.081	13.7	7.9	0.240	19.1	16.6	0.945	0.958
2040	0.113	0.979	0.078	13.4	7.7	0.249	18.5	16.1	0.948	0.961
2050	0.109	0.981	0.076	13.1	7.5	0.257	17.9	15.6	0.951	0.963
2060	0.105	0.982	0.073	12.8	7.4	0.265	17.4	15.1	0.954	0.965
2070	0.102	0.983	0.071	12.6	7.2	0.272	16.9	14.7	0.957	0.967
2080	0.099	0.984	0.069	12.3	7.1	0.280	16.5	14.3	0.959	0.969
2090	0.096	0.985	0.067	12.1	6.9	0.286	16.0	13.9	0.961	0.971
2100	0.093	0.986	0.065	11.9	6.8	0.293	15.6	13.5	0.963	0.972

5. Conclusions

Current life tables with low mortality to a high age can be approximated very well by the Gompertz distribution. Therefore, the knowledge of two parameters (e.g., life expectancy and modal age) of a life table is sufficient to determine measures of rectangularization in a simple manner. If k is unknown, it can be estimated through

$$k = \frac{\gamma}{m - e_0} \approx \frac{0.5772}{m - e_0} .$$

Under the assumption of the Gompertz distribution, functional relationships between single measures can be derived, which approximately apply to real life tables with high life expectancy. It is well known that the mortality rates at very old ages deviate from the Gompertz law. The mortality rates can be better approximated by a logistic distribution, for example. Nevertheless, the Gompertz model is suitable for the calculation of most rectangularization measures for real populations with low mortality, because the percentage of very old people in a life table is low, and therefore the error in the calculation is negligible.

Fundamentally, the measures can be classified into two types: The type of measures that can be represented as a function of the coefficient of variation, and the type of measures that only depend on an absolute dispersion measure, but not on life expectancy. For example, the Keyfitz entropy, the Gini coefficient or the index of Eakin and Witten belong to the first type, whereas, the interquartile range or the mean difference belong to the second type. In the first case, an increase in rectangularization can only be attributed to an increase in life expectancy without a change in dispersion.

References

Abramowitz, M; Stegun, I A (1972): Handbook of mathematical functions. Washington, D.C.
Carriere, JF (1994): An investigation of the Gompertz law of mortality, Actuarial Research Clearing House, 2, 161-177.
Cheung, SLK et al. (2005): Three dimensions of the survival curve: horizentalisation, verticalization, and longevity extension, Demography, 42.2, 243-258.
Dorfman, R (1979): A formula for the Gini coefficient, The Review of Economics and Statistics 61, 146-149.
Eakin, T; Witten, M (1995): How square is the survival curve of a given species? Experimental Gerontology, 30.1, 33-64.
Fraser, DC (1911): The curve of deaths, Journal of the Institute of Actuaries, 45, 476-483.
Gardner, J W.; Sanborn, J S (1990): Years of Potential Life Lost (YPLL)-What does it measure? Epidemiology 1:322-329.
Gumbel, EJ (1929): Das Konzentrationsmaß, Allg. Statistisches Archiv 18, 279-300.
Gumbel, EJ (1932): Das Zufallsgesetz des Sterbens, Leipzig und Berlin.
Gumbel EJ (1937): La durée extrême de la vie humaine, Paris.
Hanada, K (1983): A formula of Gini´s concentration ratio and its application to life table, J. Japan Statist. Soc. 13.2, 95-98.
Hill, G (1993): The entropy of the survival curve: An alternative measure, Canadian Studies of Population 20, 43-57.
Keyfitz, N (1977): Applied mathematical demography. New York.
Nair, U S (1936): Standard error of Gini's mean difference, Biometrika 28: 428–436.
Nussfelder, WJ; Mackenbach, JP (1996): Rectangularization of the survival curve in the Netherlands, 1959-1992, The Gerontologist, 36.6:773-782.
Pflaumer, P (2007): Life table forecasting with the Gompertz distribution, Proceedings of the Joint Statistical Meetings, Social Statistics Section, Salt Lake City, 3564-3571.
Pollard, JH (1991): Fun with Gompertz, Genus, XLVII-n.1-2, 1-20.
Pollard, JH (1998): An old tool – modern applications, Actuarial Studies and Demography, Research Paper No. 001/98, August.
Pollard JH; Valkovics EJ (1992): The Gompertz distribution and its applications, Genus, XLVIII-n3-4, 15-28.
Smits, J; Jonden C (2009): Length of life inequality around the globe, Social Science and Medicine 68(6): 1114-23.
Wilmoth, JR; Horiuchi, S (1999): Rectangularization revisted: Variability of age at death within human populations, Demography, 36.4, 475-495.