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CHANGE POINT TESTING FOR THE DRIFT PARAMETERS OF A PERIODIC MEAN REVERSION PROCESS

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Abstract. In this paper we investigate the problem of detecting a change in the drift parameters of a generalized Ornstein-Uhlenbeck process which is defined as the solution of

$$dX_t = (L(t) - \alpha X_t)dt + \sigma dB_t$$

and which is observed in continuous time. We derive an explicit representation of the generalized likelihood ratio test statistic assuming that the mean reversion function $L(t)$ is a finite linear combination of known basis functions. In the case of a periodic mean reversion function, we determine the asymptotic distribution of the test statistic under the null hypothesis.

1. Introduction

The problem of testing for a change in the parameters of a stochastic process has been an important issue in statistical inference for a long time. Initially investigated for i.i.d. data, change point analysis has more recently been extended to time series of dependent data. For a general review of change-point analysis, see e.g. the book by Csörgő and Horvath [3].

In the present paper, we investigate the problem of detecting changes in the parameters of a diffusion process. Diffusion processes are a popular and widely studied class of models with applications in economics, finance, physics and engineering. Statistical inference for diffusion processes has been investigated by many authors, see e.g. the monographs by Liptser and Shiryaev [12] and by Kutoyants [11]. However, change-point analysis for diffusion processes has found little attention up to now.

In our paper, we focus on change-point analysis for a special class of diffusion processes, namely for so-called generalized Ornstein-Uhlenbeck processes. These processes are defined as solutions to the stochastic differential equation

$$dX_t = (L(t) - \alpha X_t)dt + \sigma dB_t, \quad t \geq 0, \tag{1}$$

where α and σ are positive constants and where the mean-reversion function $L(t)$ is non-random. $(B_t)_{t\geq 0}$ denotes standard Brownian motion and X_0 is a square-integrable real-valued random variable that is independent of $(B_t)_{t\geq 0}$. If $L(t) \equiv \mu$ is a constant, we obtain the classical Ornstein-Uhlenbeck process, introduced by Ornstein and Uhlenbeck [14]. Ornstein-Uhlenbeck processes are popular models for prices of commodities that exhibit a trend of reversion to a fixed mean level. Generalized Ornstein-Uhlenbeck processes can be used as models for the evolution of prices with a trend or seasonal component $L(t)$.

Dehling, Franke and Kott [5] have studied the problem of parameter estimation of a generalized Ornstein-Uhlenbeck process if the mean-reversion function $L(t)$ is a linear combination

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of known basis functions $\varphi_1(t), \ldots, \varphi_p(t)$, i.e. when

$$L(t) = \sum_{i=1}^{p} \mu_i \varphi_i(t) \tag{2}$$

In this model, the unknown parameter vector is $\theta = (\mu_1, \ldots, \mu_p, \alpha)^t$. We denote the corresponding parameter space by Θ, and observe that

$$\Theta = \mathbb{R}^p \times (0, \infty).$$

As is usual in the statistical inference for the drift of a time-continuously observed diffusion process, the diffusion parameter σ is supposed to be known. This can be justified by the fact that the volatility σ can be computed by the quadratic variation of the process.

We are interested in testing whether there is a change in the values of the parameters $\mu_1, \ldots, \mu_p$ and α, in the time interval $[0, T]$ during which the process is observed. In the first step, we will consider this problem assuming that the change-point $\tau \in (0, T)$ is known. For the asymptotic analysis, when $T \to \infty$, we write $\tau = s\,T$, where $s \in (0, 1)$ is known. The generalized Ornstein-Uhlenbeck process with change-point $\tau = s\,T$ is given by

$$dX_t = (S(\theta, t, X_t)\mathbf{1}_{\{t \le \tau\}} + S(\theta', t, X_t)\mathbf{1}_{\{t > \tau\}})dt + \sigma dB_t, \quad 0 \le t \le T, \tag{3}$$

where

$$S(\theta, t, X_t) = \sum_{i=1}^{p} \mu_i \varphi_i(t) - \alpha X_t, \tag{4}$$

and where $\mathbf{1}_A$ denotes the indicator function of the set A. The test problem of interest can be formulated as

$$H_0 : \ \theta = \theta' \text{ (no change)} \quad \text{vs.} \quad H_A : \ \theta \ne \theta' \text{ (change at time point } \tau) \ . \tag{5}$$

We want to study the generalized likelihood ratio test for this test problem. We denote by P_X the measure induced by the observable realizations $X^T = \{X_t, 0 \le t \le T\}$ on the measurable space $(C[0, T], \mathcal{B}[0, T])$, $C[0, T]$ being the space of continuous, real-valued functions on $[0, T]$ and $\mathcal{B}[0, T]$ the associated Borel σ-field. Moreover, let P_B be the measure generated by the Brownian motion on $(C[0, T], \mathcal{B}[0, T])$. Then the likelihood function $\mathcal{L}$ of observations X^T of the process with stochastic differential (3) is defined as the Radon-Nikodym derivative, i.e.

$$\mathcal{L}(\theta, \theta', X^T) := \frac{dP_X}{dP_B}(X^T).$$

The generalized likelihood ratio $\mathcal{R}(X^T)$ is given by

$$\mathcal{R}(X^T) = \frac{\sup_{\theta \in \Theta} \mathcal{L}(\theta, \theta, X^T)}{\sup_{\theta, \theta' \in \Theta} \mathcal{L}(\theta, \theta', X^T)}. \tag{6}$$

Note that this likelihood ratio depends on the suspected change point $\tau = sT$, where $s \in (0, 1)$. Eventually, we will study the log-transformed likelihood ratio

$$\Lambda_T(s) := -2 \log(\mathcal{R}(X^T)).$$

We will give an explicit expression of the process $(\Lambda_T(s))_{0 \le s \le 1}$ and study the asymptotic distribution of this process as $T \to \infty$.

The outline of the paper is as follows. In Section 2, we will first derive an explicit representation of the log-transformed generalized likelihood ratio test statistic $(\Lambda_T(s))_{0 \le s \le 1}$. We then formulate two theorems concerning the asymptotic distributions of sup-norm functionals of $(\Lambda_T(s))_{0\le s\le 1}$. It turns out that $\sup_{0\le s\le 1}\Lambda_T(s)$ does not have a non-degenerate limit distribution, as $T\to\infty$. In Theorem 1, we will prove convergence of $\sup_{s_1\le s\le s_2}\Lambda_T(s)$, when $0<s_1<s_2<1$ are fixed constants. In Theorem 2, we will show that there exist centering and norming sequences a_T and b_T such that $(\sup_{0\le s\le 1}\Lambda_T(s)-b_T)/a_T$ converges towards an extreme value distribution. The proofs of these theorems are given in Section 3 and Section 4, respectively.

2. Generalized likelihood ratio test

In this section, we will derive an explicit representation of $\Lambda_T(s)$. In order to do so, we need to calculate the maxima in the numerator and denominator in (6). Note that this is achieved by the corresponding maximum likelihood estimators. A corollary to Girsanov's theorem, see Theorem 7.6 in Lipster and Shiryayev [12], gives an explicit expression of the likelihood function of a diffusion process provided that

$$\mathbb{P}\left(\int_0^T S(\theta,t,X_t)^2dt<\infty\right)=1 \tag{7}$$

for all $0\le T<\infty$ and all θ.

Lemma 2.1. *Let $\mathcal{L}(\theta,X^T)$ denote the likelihood function of the observations $X^T=\{X_t,0\le t\le T\}$ of the generalized Ornstein-Uhlenbeck process $(X_t)_{t\ge 0}$, defined in (1), with mean reversion function (2). If the drift term (4) satisfies condition (7) then*

$$\arg\max_\theta \mathcal{L}(\theta,X^T)=\hat\theta_{ML}=Q_T^{-1}\tilde R_T.$$

Here $Q_T\in\mathbb{R}^{(p+1)\times(p+1)}$ and $\tilde R_T\in\mathbb{R}^{p+1}$ are defined as

$$\begin{aligned} Q_T &= \begin{pmatrix} G_T & -a_T\\ -a_T^t & b_T\end{pmatrix},\\ \tilde R_T &= \left(\int_0^T\varphi_1(t)dX_t,\ldots,\int_0^T\varphi_p(t)dX_t,-\int_0^T X_t\,dX_t\right)^t,\end{aligned}$$

where $G_T=(\int_0^T\varphi_j(t)\varphi_k(t)dt)_{1\le j,k\le p}\in\mathbb{R}^{p\times p}$, $a_T=(\int_0^T\varphi_1(t)X_t\,dt,\ldots,\int_0^T\varphi_p(t)X_t\,dt)^t$ and $b_T=\int_0^T X_t^2dt$.

Remark 1. Note that the integrals in $\tilde R_T$ can be rewritten as

$$\int_0^T\varphi_i(t)dX_t=\int_0^T\varphi_i(t)(L(t)-\alpha X_t)\,dt+\sigma\int_0^T\varphi_i(t)dB_t$$

where the latter is a well-defined Itô integral.

Proof. The likelihood function $\mathcal{L}$ of a general diffusion process

$$dX_t=S(\theta,t,X_t)dt+\sigma dB_t,\quad 0\le t\le T,$$

is given by

$$\mathcal{L}(\theta,X^T)=\frac{dP_X}{dP_B}(X^T)=\exp\left(\frac{1}{\sigma^2}\int_0^T S(\theta,t,X_t)dX_t-\frac{1}{2\sigma^2}\int_0^T S(\theta,t,X_t)^2dt\right) \tag{8}$$

if condition (7) is fulfilled; see Theorem 7.6 in Lipster and Shiryayev [12]. The maximum likelihood estimator is defined as the maximum of the functional $\theta \mapsto \mathcal{L}(\theta, X^T)$ and the partial derivatives of the logarithm of this functional are

$$\frac{\partial}{\partial \theta_i} \ln(\mathcal{L}(\theta, X^T)) = \frac{1}{\sigma^2} \int_0^T \frac{\partial}{\partial \theta_i} S(\theta, t, X^T) dX_t - \frac{1}{\sigma^2} \int_0^T S(\theta, t, X_t) \frac{\partial}{\partial \theta_i} S(\theta, t, X_t) dt. \tag{9}$$

The derivatives of the drift function specified in (4) can be computed to be

$$\frac{\partial}{\partial \theta_i} S(\theta, t, X_t) = \begin{cases} \varphi_i(t), & i = 1, \dots, p; \\ -X_t, & i = p+1. \end{cases}$$

Setting the partial derivatives of the log-likelihood function in (9) equal zero gives a system of linear equations which yields the assertion. □

Due to the linearity of the drift term, the log-likelihood function of the process (3) is given by

$$\begin{aligned} \ln\left(\mathcal{L}(\theta, \theta', X^T)\right) &= \frac{1}{\sigma^2}\left(\int_0^\tau S(\theta, t, X_t) dX_t + \int_\tau^T S(\theta', t, X_t) dX_t\right) \\ &\quad - \frac{1}{2\sigma^2}\left(\int_0^\tau S(\theta, t, X_t)^2 dt + \int_\tau^T S(\theta', t, X_t)^2 dt\right). \end{aligned}$$

Hence, defining $X^{\tau,T} = \{X_t, \tau \le t \le T\}$, we can write the generalized likelihood ratio (6) as

$$\mathcal{R}(X^T) = \frac{\sup_\theta \mathcal{L}(\theta, X^T)}{\sup_{\theta^*} \mathcal{L}(\theta^*, X^\tau) \sup_{\theta'} \mathcal{L}(\theta', X^{\tau,T})} \tag{10}$$

where $\mathcal{L}(\theta, X^T)$ is given in (8) with drift function specified in (4). The terms $\mathcal{L}(\theta^*, X^\tau)$ and $\mathcal{L}(\theta', X^{\tau,T})$ are defined analogously as integrals with integration regions 0 to τ and τ to T, respectively. It follows from Lemma 2.1 that

$$\mathcal{R}(X^T) = \frac{\mathcal{L}(\hat{\theta}^T_{ML}, X^T)}{\mathcal{L}(\hat{\theta}^\tau_{ML}, X^\tau) \mathcal{L}(\hat{\theta}^{\tau,T}_{ML}, X^{\tau,T})} \tag{11}$$

where the maximum likelihood estimates $\hat{\theta}^T_{ML}$, $\hat{\theta}^\tau_{ML}$ and $\hat{\theta}^{\tau,T}_{ML}$ are computed from the total, the pre- and post-change sample, respectively. This representation of the likelihood ratio is used to prove the following proposition.

Proposition 2.2. *The log-transformed generalized likelihood ratio test statistic $\Lambda_T(s) = -2\ln(\mathcal{R}(X^T))$ of the test problem (5) can be represented under the null hypothesis as*

$$\Lambda_T(s) = -R_T^t Q_T^{-1} R_T + R_\tau^t Q_\tau^{-1} R_\tau + (R_T - R_\tau)^t (Q_T - Q_\tau)^{-1} (R_T - R_\tau)$$

where Q_T is given in Lemma 2.1 and

$$R_T = \left(\int_0^T \varphi_1(t) dB_t, \dots, \int_0^T \varphi_p(t) dB_t, -\int_0^T X_t dB_t\right)^t.$$

Proof. Our aim is to compute an explicit expression of the ratio given in (11). Note that the likelihood function in the numerator of (10) can be represented as

$$L(\theta, X^T) = \exp\left(\frac{1}{\sigma^2} \theta^t \tilde{R}_T - \frac{1}{2\sigma^2} \theta^t Q_T \theta\right)$$

where $\tilde{R}_T$ and Q_T are given in Lemma 2.1. Denoting by θ_0 the true value of θ, the representations

$$\hat{\theta}^T_{ML} = Q_T^{-1}\tilde{R}_T \quad \text{and} \quad \tilde{R}_T = Q_T\theta_0 + \sigma R_T,$$

where the latter can be obtained by plugging in the initial SDE (1) and (2), lead to

$$\begin{aligned} L(\hat{\theta}^T_{ML}, X^T) &= \exp\left(\frac{1}{2\sigma^2}\tilde{R}^t_T Q_T^{-1}\tilde{R}_T\right) \\ &= \exp\left(\frac{1}{2\sigma^2}\theta_0^t Q_T\theta_0 + \frac{1}{\sigma}R^t_T\theta_0 + \frac{1}{2}R^t_T Q_T^{-1}R_T\right). \end{aligned}$$

The same procedure yields an analog expression for $L(\hat{\theta}^\tau_{ML}, X^\tau)$. The additivity of the integrals provides

$$\hat{\theta}^{\tau,T}_{ML} = (Q_T - Q_\tau)^{-1}(\tilde{R}_T - \tilde{R}_\tau)$$

and

$$\tilde{R}_T - \tilde{R}_\tau = (Q_T - Q_\tau)\theta_0 + \sigma(R_T - R_\tau)$$

such that

$$\begin{aligned} L(\hat{\theta}^{\tau,T}_{ML}, X^{\tau,T}) &= \exp\left(\frac{1}{2\sigma^2}\theta_0^t(Q_T - Q_\tau)\theta_0 + \frac{1}{\sigma}(R_T - R_\tau)^t\theta_0\right. \\ &\quad \left. + \frac{1}{2}(R_T - R_\tau)^t(Q_T - Q_\tau)^{-1}(R_T - R_\tau)\right). \end{aligned}$$

Under the null hypothesis, cancelation of several terms in (11) proves the assertion. □

For the rest of our investigations, we study periodic functions

$$\varphi_j(t + \nu) = \varphi_j(t) \tag{12}$$

where ν is the period observed in the data. Under the null hypothesis of no change, this implies periodicity of the mean reversion function, i.e. $L(t + \nu) = L(t)$. We assume that we observe the process over some integer multiple of periods, i.e. $T = n\nu$, $n \in \mathbb{N}$. By Gram-Schmidt orthogonalization we may assume without loss of generality that the basis functions $\varphi_1(t), \ldots, \varphi_p(t)$ form an orthonormal system in $L^2([0, \nu], \frac{1}{\nu}d\lambda)$, i.e. that

$$\int_0^\nu \varphi_j(t)\varphi_k(t)dt = \begin{cases} \nu, & j = k \\ 0, & j \neq k. \end{cases} \tag{13}$$

Under these assumptions, the matrix Q_T appearing in the test statistic $\Lambda_T(s)$, see Proposition 2.2, simplifies to

$$Q_T = \begin{pmatrix} T\, I_{p\times p} & a_T \\ a_T^t & b_T \end{pmatrix}.$$

Theorem 1. *Let $X^T = \{X_t, 0 \le t \le T\}$ be observations of the mean reversion process (1) with mean reversion function of the form (2), satisfying (12) and (13). Denote by $\Lambda_T(s) = -2\ln(\mathcal{R}(X^T))$ the log-transformed generalized likelihood ratio test statistic for the test problem (5). Then, for any fixed $0 < s_1 < s_2 < 1$, under the null hypothesis,*

$$\sup_{s\in[s_1,s_2]} \Lambda_T(s) \xrightarrow{\mathcal{D}} \sup_{s\in[s_1,s_2]} \frac{\|W(s) - sW(1)\|^2}{s(1-s)}$$

as $T \to \infty$. Here $\|\cdot\|$ denotes the Euclidean norm and W is a $(p+1)$-dimensional standard Brownian motion.

The result stated in Theorem 1 is not satisfactory in application: First, it is not clear how to choose the interval $[s_1, s_2]$ potentially containing a change point if no information about the location of the change is available. Second, the distribution of the limit which is the squared length of a multi-dimensional Brownian bridge is not explicitly given such that further analysis or a simulation study are necessary in order to specify the limit distribution explicitly. In order to avoid this inconvenience it is possible to consider the *exact* test statistic $\sup_{0<s\leq 1}\Lambda_T(s)$. It turns out that, by means of some appropriate normalizing terms a_T and b_T, the distribution of the expression $(\sup_{0<s\leq 1}\Lambda_T(s) - b_T)/a_T$ converges to the Gumbel distribution.

Theorem 2. *Under the same assumptions as in Theorem 1 it holds under the null hypothesis that*

$$\big(\sup_{0<s\leq 1}\Lambda_T(s) - b_T\big)/a_T \stackrel{\mathcal{D}}{\longrightarrow} G,$$

as $T \to \infty$, where G denotes a real-valued random variable satisfying

$$\mathbb{P}(G \leq x) = \exp(-2e^{-x/2}).$$

Here $b_T = \left(2\ln\ln\frac{T}{\nu} + \frac{p+1}{2}\ln\ln\ln\frac{T}{\nu} - \ln\Gamma(\frac{p+1}{2})\right)^2/(2\ln\ln\frac{T}{\nu})$, $a_T = \sqrt{b_T/(2\ln\ln\frac{T}{\nu})}$ where Γ is the gamma function.

3. Proof of Theorem 1

Before we can complete the proof of Theorem 1, we have to establish some auxiliary results. First, we will study the asymptotic behavior of $\Lambda_T(s)$ in the case of a periodic mean reversion function, see (12) and (13). Note that by Proposition 2.2 we have the representation

$$\Lambda_T(s) = -R_T^tQ_T^{-1}R_T + R_{sT}^tQ_{sT}^{-1}R_{sT} + (R_T - R_{sT})^t(Q_T - Q_{sT})^{-1}(R_T - R_{sT}).$$

The first term,

$$R_T^tQ_T^{-1}R_T = \frac{1}{\sqrt{T}}R_T^t\Big(\frac{1}{T}Q_T\Big)^{-1}\frac{1}{\sqrt{T}}R_T,$$

has already been studied by Dehling, Franke and Kott [5]. The following proposition summarizes the results of Proposition 4.5, 5.1 and 5.2 in Dehling et al. [5].

Proposition 3.1. *We have that*

$$\frac{1}{\sqrt{T}}R_T \stackrel{\mathcal{D}}{\to} N(0,\Sigma)$$

and

$$\frac{1}{T}Q_T \to \Sigma, \textit{ almost surely,}$$

as $T \to \infty$. The matrix Σ is given by

$$\Sigma = \begin{pmatrix} \nu\, I_{p\times p} & \Lambda \\ \Lambda^t & \omega \end{pmatrix} \tag{14}$$

where $\Lambda_i = \int_0^\nu \varphi_i(t)\tilde{h}(t)dt$, $i = 1,\ldots,p$, $\omega = \int_0^\nu(\tilde{h}(t))^2dt + \frac{\nu\sigma^2}{2\alpha}$ and where $\tilde{h} : [0,\infty) \to \mathbb{R}$ is defined by

$$\tilde{h}(t) = e^{-\alpha t}\sum_{j=1}^p \mu_j \int_{-\infty}^t e^{\alpha s}\varphi_j(s)ds. \tag{15}$$

Here, $N(0,\Sigma)$ denotes a normally distributed random vector with zero-mean and covariance matrix Σ.

Now we want to investigate the second term of $\Lambda_T(s)$ which we rewrite as

$$R_{sT}^t Q_{sT}^{-1} R_{sT} = \frac{1}{\sqrt{T}} R_{sT}^t \left(\frac{1}{T} Q_{sT}\right)^{-1} \frac{1}{\sqrt{T}} R_{sT}.$$

We will show that the process $\left(\frac{1}{\sqrt{T}} R_{sT}\right)_{s\in[s_1,s_2]}$ converges in distribution to a Gaussian process on $[s_1,s_2]$, and that $\frac{1}{T}Q_{sT}$ converges in probability uniformly on $[s_1,s_2]$.

We need the following functional version of the asymptotic normality proved in [5]

Proposition 3.2. *As $T\to\infty$, the sequence of processes $(R_\tau^{(T)})_{0\le\tau\le 1}$, where*

$$R_\tau^{(T)} := \frac{1}{\sqrt{T}} R_{\tau T},$$

converges in distribution to a $(p+1)$-dimensional Wiener-process R^* with $R_s^* \sim N(0, s\Sigma)$ and where Σ is defined in (14). Thus the covariance function of R^* is of the form

$$\operatorname{Cov}\left(R_i^*(s), R_j^*(t)\right) = (s\wedge t)\Sigma_{ij}; \quad \text{for } i,j = 1,\dots,p+1.$$

Proof. Note that the vector-valued processes

$$R_t^{(n)} := \left(\frac{1}{\sqrt{n}}\int_0^{n\nu t} \varphi_1(s)dB_s, ..., \frac{1}{\sqrt{n}}\int_0^{n\nu t}\varphi_p(s)dB_s, \frac{1}{\sqrt{n}}\int_0^{n\nu t} X_s dB_s\right)$$

are martingales with respect to the filtrations $\mathcal{F}_t^{(n)} := \sigma(B_s; s\le n\nu t)$. The associated covariance processes are given by

$$\langle R^{(n)}, R^{(n)}\rangle_t = \begin{pmatrix} \frac{1}{n}\int_0^{n\nu t}\varphi_1\varphi_1 ds & \dots & \frac{1}{n}\int_0^{n\nu t}\varphi_1\varphi_p ds & \frac{1}{n}\int_0^{n\nu t}\varphi_1 X ds \\ \vdots & & \vdots & \vdots \\ \frac{1}{n}\int_0^{n\nu t}\varphi_p\varphi_1 ds & \dots & \frac{1}{n}\int_0^{n\nu t}\varphi_p\varphi_p ds & \frac{1}{n}\int_0^{n\nu t}\varphi_p X ds \\ \frac{1}{n}\int_0^{n\nu t}\varphi_1 X ds & \dots & \frac{1}{n}\int_0^{n\nu t}\varphi_p X ds & \frac{1}{n}\int_0^{n\nu t} X^2 ds \end{pmatrix}.$$

As was shown in [5] (see p.184), these matrices converge for $n\to\infty$ almost surely towards the matrix

$$\begin{pmatrix} t\int_0^\nu \varphi_1\varphi_1 ds & \dots & t\int_0^\nu \varphi_1\varphi_p ds & t\int_0^\nu \varphi_1\tilde{h} ds \\ \vdots & & \vdots & \vdots \\ t\int_0^\nu \varphi_p\varphi_1 ds & \dots & t\int_0^\nu \varphi_p\varphi_p ds & t\int_0^\nu \varphi_p\tilde{h} ds \\ t\int_0^\nu \varphi_1\tilde{h} ds & \dots & t\int_0^\nu \varphi_p\tilde{h} ds & t\left(\int_0^\nu \tilde{h}^2 ds + \frac{\nu\sigma^2}{2\alpha}\right) \end{pmatrix} = t\Sigma.$$

The functional central limit theorem for continuous martingales (p.339 in [6]) now implies that the sequence of continuous $\mathcal{F}_t^{(n)}$-martingales $R^{(n)}$ converges in distribution toward the unique continuous Gaussian martingale with covariance function $t\Sigma$. □

Proposition 3.3. *Let Q_t be defined as in Lemma 2.1. Then, as $T\to\infty$,*

$$\frac{1}{T} Q_{sT} \longrightarrow s\Sigma$$

almost surely uniformly on $[0,1]$, where Σ is given in (14).

Proof. By Proposition 3.1, we know that, almost surely, $\frac{1}{T}Q_T \to \Sigma$ as $T \to \infty$. Thus, given $\epsilon > 0$, there exists a T_0 such that for all $T \geq T_0$

$$\|\frac{1}{T}Q_T - \Sigma\| \leq \epsilon.$$

Let $B := \sup_{0\leq s\leq T_0} \|Q_s\|$. Then we get for any $T \geq T_0$ and $T_0/T \leq s \leq 1$

$$\|\frac{1}{T}Q_{Ts} - s\Sigma\| = s\|\frac{1}{Ts}Q_{Ts} - \Sigma\| \leq \epsilon.$$

For $s \leq T_0/T$ we obtain

$$\|\frac{1}{T}Q_{Ts} - s\Sigma\| \leq \frac{1}{T}B + \frac{T_0}{T}\|\Sigma\| \leq \epsilon,$$

for T large enough. The last two inequalities together show that for T large enough, we have $\|\frac{1}{T}Q_{Ts} - s\Sigma\| \leq \epsilon$, and this proves the statement of the proposition. □

We can finally finish the proof of Theorem 1.

Proof of Theorem 1. By Slutsky's theorem and Propositions 3.1, 3.2 and 3.3 we obtain

$$\begin{aligned}
\Lambda_T(s) &= -R_T^t Q_T^{-1} R_T + R_{sT}^t Q_{sT}^{-1} R_{sT} + (R_T - R_{sT})^t (Q_T - Q_{sT})^{-1} (R_T - R_{sT}) \\
&\xrightarrow{\mathcal{D}} -\|W(1)\|^2 + \frac{\|W(s)\|^2}{s} + \frac{\|W(1) - W(s)\|^2}{1-s} \\
&= \frac{\|W(s) - sW(1)\|^2}{s(1-s)}
\end{aligned}$$

in $C[s_1, s_2]$. Here we have used the fact that the process $W(t) := \Sigma^{-1/2} R_t^*$ is a Brownian motion with covariance matrix I_{p+1}, where I_{p+1} is the $(p+1) \times (p+1)$ identity matrix, and that $(R_t^*)^t \Sigma^{-1} R_t^* = \|W(t)\|^2$. The assertion about the supremum of $\Lambda_T(s)$ is justified by the continuous mapping theorem. □

4. Proof of Theorem 2

The proof of Theorem 2 is motivated by the proof of an analogous result for discrete time AR processes, given by Davis et al. [4]. We need the following result which is proved in in Davis et al. [4] and which relies on Lemma 2.2 in Horváth [7]).

Proposition 4.1 (Corollary A.2 in Davis et al. [4])**.** *Let $Y_1, Y_2, \ldots$ be an i.i.d. sequence of $(p+1)$-dimensional random vectors with $\mathbb{E}[Y_1] = 0$ and $\mathbb{E}[Y_1 Y_1^t] = I_{p+1}$. Define $S_k = \sum_{i=1}^k Y_i$. If $\max_{1\leq i\leq p+1} \mathbb{E}|Y_{i,1}|^{2+r} < \infty$ for some $r > 0$, then*

$$\big(\max_{1\leq k\leq n} \|S_k\|^2 - b_n\big)/a_n \xrightarrow{\mathcal{D}} G^*,$$

as $n \to \infty$, where G^* denotes a real-valued random variable satisfying

$$P(G^* \leq x) = \exp(-e^{-x/2}).$$

Thereby, it is $b_n = \left(2\ln\ln n + \frac{p+1}{2}\ln\ln\ln n - \ln\Gamma(\frac{p+1}{2})\right)^2/(2\ln\ln n)$, $a_n = \sqrt{b_n/(2\ln\ln n)}$.

Recall that

$$\Lambda_T(s) = -R_T^t Q_T^{-1} R_T + R_{sT}^t Q_{sT}^{-1} R_{sT} + (R_T - R_{sT})^t (Q_T - Q_{sT})^{-1} (R_T - R_{sT})$$

and $T = n\nu$, ν fixed. Let us assume for a moment that $\nu = 1$. We write $\Lambda_n(s)$, R_n and Q_n for $\Lambda_T(s)$, R_T and Q_T, respectively, since the asymptotic framework is $n \to \infty$.

Assume further that the solution of the stochastic differential equation (1) is stationnary. Note that this is not a constraint for our purposes since we may alternatively consider the stationary process

$$\tilde{X}_t := \tilde{h}(t) + \tilde{Z}_t$$

with

$$\tilde{h}(t) := e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s} h(s) ds$$

and

$$\tilde{Z}_t = \sigma e^{-\alpha t} \int_{-\infty}^{t} e^{\alpha s} d\tilde{B}_s$$

where

$$\tilde{B}_s := \check{B}_s \mathbf{1}_{\mathbb{R}_+}(s) + \hat{B}_{-s} \mathbf{1}_{\mathbb{R}_-}(s)$$

is bilateral Brownian motion defined through two independent standard Brownian motions $(\check{B}_t)_{t\geq 0}$ and $(\hat{B}_t)_{t\geq 0}$. The process $(\tilde{X}_t)_{t\geq 0}$ is stationary and does not depend on X_0. Furthermore, it was proved in [5] (Lemma 4.4) that one has $|\tilde{X}_t - X_t| \to 0$, almost surely, as $t \to \infty$.

The following proposition is essential for the proof of Theorem 2. First, define for two σ-algebras $\mathcal{A}$ and $\mathcal{B}$ the quantities

$$\alpha(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)|$$

and

$$\rho(\mathcal{A}, \mathcal{B}) = \sup_{F \in L^2(\mathcal{A}, \mathbb{P}), G \in L^2(\mathcal{B}, \mathbb{P})} \operatorname{Corr}(F, G).$$

It is known that

$$\alpha(\mathcal{A}, \mathcal{B}) \leq \frac{1}{4} \rho(\mathcal{A}, \mathcal{B}).$$

For a stationnary sequence of random variables $(\zeta_k)_{k\in\mathbb{N}}$ define the mixing coefficient α_ζ by

$$\alpha_\zeta(n) = \sup_{k\in\mathbb{N}} \alpha\big(\sigma(\zeta_i; i \leq k), \sigma(\zeta_i; i \geq k+n)\big).$$

The sequence $(\zeta_k)_{k\in\mathbb{N}}$ is called strongly mixing if $\alpha_\zeta(n) \to 0$ as $n \to \infty$.

Proposition 4.2. *The sequence of random vectors $(r_k)_{k\in\mathbb{N}}$ defined by*

$$\Delta R_k := (R_k - R_{k-1}) = \left(\int_{k-1}^{k} \varphi_1(t) dB_t, \ldots, \int_{k-1}^{k} \varphi_p(t) dB_t, -\int_{k-1}^{k} X_t dB_t \right)^t$$

is strongly mixing with mixing coefficient α of order

$$\alpha_{\Delta R}(n) = \mathcal{O}(e^{-\alpha(n-1)}).$$

Proof. Define the $C[0,1]$-valued stochastic process $(\xi^{(k)})_{k\in\mathbb{N}}$ by

$$\xi^{(k)} := \begin{pmatrix} X^{(k)} \\ B^{(k)} \end{pmatrix} := \begin{pmatrix} (X_{t+k})_{t\in[0,1]} \\ (B_{t+k})_{t\in[0,1]} \end{pmatrix}.$$

The process $(\xi^{(k)})_{k\in\mathbb{N}}$ is both a Markov and a Gaussian process. Hence, by making use of the Markov property and by applying the correlation inequality for Gaussian processes from Lemma 4.3 we obtain

$$\begin{aligned}
\alpha_\xi(n) &\leq \sup_{t\in\mathbb{R}} \alpha\big(\sigma(\xi^{(k)}; k \leq m), \sigma(\xi^{(k)}; k \geq m+n)\big) \\
&= \sup_{m\in\mathbb{R}} \alpha\big(\sigma(\xi^{(m)}), \sigma(\xi^{(m+n)})\big) \\
&\leq \sup_{m\in\mathbb{N}} \sup_{a,b,c,d\in L^2[0,1]} \operatorname{Corr}(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \langle c, B^{(m+n)}\rangle + \langle d, X^{(m+n)}\rangle) \\
&\leq e^{-\alpha(n-1)} \sup_{a,b,c,d,\in L^2[0,1]} \operatorname{Corr}(\langle a, B^{(1)}\rangle + \langle b, X^{(1)}\rangle, \langle c, B^{(2)}\rangle + \langle d, X^{(2)}\rangle) \\
&= \mathcal{O}(e^{-\alpha(n-1)})
\end{aligned}$$

where the last equality is stated in Lemma 4.4. Note that each ΔR_k may be represented as

$$\Delta R_k = \mathfrak{f}_k(\xi^{(k)})$$

where $\mathfrak{f}_k : C[0,1] \times C[0,1] \to \mathbb{R}$ is a measurable function. Since the σ-algebra generated by $\mathfrak{f}_k(\xi^{(k)}) : \Omega \to \mathbb{R}$ is smaller or equal the σ-algebra generated by $\xi^{(k)} : \Omega \to C[0,1] \times C[0,1]$, the bound for $\alpha_\xi(n)$ established above is also valid for $\alpha_{\Delta R}(n)$. □

Lemma 4.3. *Let $(H, \langle .,. \rangle)$ be separable Hilbert-space and (X, Y) be a pair of random variables with Gaussian joint law. Then one has*

$$\rho(\sigma(X), \sigma(Y)) \leq \max_{a,b\in H} \operatorname{Corr}(\langle a, X\rangle, \langle b, Y\rangle).$$

Proof. Let $(e_i)_{i\in\mathbb{N}}$ be a system of orthonormal basis vectors for the Hilbert space H. If we set $V_i := \langle X, e_i\rangle$ and $W_j := \langle Y, e_j\rangle$ then we have the representations

$$X = \sum_{i=1}^{\infty} V_i e_i \quad \text{and} \quad Y = \sum_{j=1}^{\infty} W_j e_j.$$

Note that $\sigma(X) = \sigma(V_1, V_2, ...)$ and $\sigma(Y) = \sigma(W_1, W_2, ...)$. It follows from Prop. 3.18 and Thm. 9.2 in [2] that

$$\begin{aligned}
\rho(\sigma(X), \sigma(Y)) &= \lim_{n\to\infty} \rho(\sigma(V_1, .., V_n), \sigma(W_1, ..., W_n)) \\
&= \lim_{n\to\infty} \sup_{a_1,...,a_n,b_1,...,b_n\in\mathbb{R}} \operatorname{Corr}\Big(\sum_{i=1}^{n} a_i V_i, \sum_{j=1}^{n} b_j W_j\Big).
\end{aligned}$$

Since the correlation is homogeneous we can assume without loss of generality that $\sum a_i^2 = 1$ and $\sum b_j^2 = 1$ holds. From this then follows

$$\begin{aligned}
\rho(\sigma(X), \sigma(Y)) &\leq \sup_{(a_i)_{i\in\mathbb{N}},(b_j)_{j\in\mathbb{N}}: \sum a_i^2 = \sum b_j^2 = 1} \operatorname{Corr}\Big(\sum_{i=1}^{\infty} a_i V_i, \sum_{j=1}^{\infty} b_j W_j\Big) \\
&\leq \sup_{a,b\in H: \|a\|=\|b\|=1} \operatorname{Corr}\Big(\langle a, X\rangle, \langle b, Y\rangle\Big).
\end{aligned}$$

The second inequality follows since one has for $a \in H$ and $a_i := \langle a, e_i\rangle$ that

$$\langle a, X\rangle = \sum_{i=1}^{\infty} a_i V_i.$$

This finishes the proof of the lemma. □

Lemma 4.4. *For all $a, b, c, d \in L^2[0,1]$ we have*

$$\begin{aligned} & \mathrm{Corr}(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \langle c, B^{(m+n)}\rangle + \langle d, X^{(m+n)}\rangle) \\ = \; & e^{-\alpha(n-1)}\mathrm{Corr}(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \langle c, B^{(m+1)}\rangle + \langle d, X^{(m+1)}\rangle). \end{aligned}$$

Proof. Since $B^{(m+n)}$ is independent from $\sigma(B^{(m)}, X^{(m)})$ we have that

$$\begin{aligned} & \mathrm{Corr}(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \langle c, B^{(m+n)}\rangle + \langle d, X^{(m+n)}\rangle) \\ = \; & \mathrm{Corr}(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \langle d, X^{(m+n)}\rangle) \\ = \; & \frac{\mathrm{Cov}(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \langle d, X^{(m+n)}\rangle)}{\sqrt{\mathrm{Var}(\langle a, B^{(n)}\rangle + \langle b, X^{(n)}\rangle)}\sqrt{\mathrm{Var}(\langle d, X^{(m+n)}\rangle)}} \\ = \; & \frac{\mathrm{Cov}(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \langle d, X^{(m+n)}\rangle)}{\sqrt{\mathrm{Var}(\langle a, B^{(n)}\rangle + \langle b, X^{(n)}\rangle)}\sqrt{\mathrm{Var}(\langle d, X^{(n)}\rangle)}}. \end{aligned}$$

Note that we used the fact that the sequence $(X^{(m)})_{m\in\mathbb{N}}$ is stationary.

We will use the fact that $(X_t)_{t\geq m}$ is the unique solution of the SDE (1) with initial condition X_m to see that $X^{(m+n)}$ has the representation

$$X_{m+n+s} = e^{-\alpha(n+s)}X_m + h(n+s) + \sigma e^{-\alpha(n+s)}\int_0^{n+s} e^{\alpha r}dB_{m+r}.$$

This representation follows from Lemma 4.2 in Dehling, Franke, Kott (2010). We use this fact to compute the covariance in the above formula:

$$\begin{aligned} & \mathrm{Cov}(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \langle d, X^{(m+n)}\rangle) \\ = \; & e^{-\alpha n}\mathrm{Cov}\left(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \int_0^1 c(s)e^{-\alpha s}X_m ds\right) \\ & +\mathrm{Cov}\left(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \int_0^1 c(s)h(n+s)ds\right) \\ & +e^{-\alpha n}\mathrm{Cov}\left(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \int_0^1 c(s)\sigma e^{-\alpha s}\int_0^1 e^{\alpha r}dB_{m+r}ds\right) \\ & +e^{-\alpha n}\mathrm{Cov}\left(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \int_0^1 c(s)\sigma e^{-\alpha s}\int_1^{n+s} e^{\alpha r}dB_{m+r}ds\right) \end{aligned}$$

Note that the second term on the right vanishes, since the right entry in the covariance is deterministic. Further, the fourth term vanishes, since the Brownian increments on the interval $[m+1, m+n+s]$ are independent with respect to $\sigma(B^{(m)}, X^{(m)})$. We thus have

$$\begin{aligned}
&\mathrm{Cov}(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \langle d, X^{(m+n)}\rangle) \\
=\ & e^{-\alpha n}\mathrm{Cov}\left(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \int_0^1 c(s)e^{-\alpha s}X_m ds\right) \\
&+e^{-\alpha n}\mathrm{Cov}\left(\langle a, B^{(m)}\rangle + \langle b, X^{(m)}\rangle, \int_0^1 c(s)\sigma e^{-\alpha s}\int_0^1 e^{\alpha r}dB_{m+r}ds\right).
\end{aligned}$$

The result follows since we can do the same reasoning for $n = 1$. □

Corollary 4.5. *There exists an iid-sequence of $\mathbb{R}^{p+1}$-valued Gaussian random variables $\zeta_i; i \in \mathbb{N}$ such that for $U_k := \sum_{i=1}^k \zeta_i$ one has*

$$R_k - U_k = O(k^{1/2-\lambda}) \quad \text{for some } \lambda > 0 \text{ as } k \to \infty.$$

Proof. This follows from Proposition 4.2 and the theorem from Kuelbs and Philipp on strong approximation of mixing random sequences (see [9]). □

Remark 2. In the following we will denote by Γ_{p+1} the covariance matrix of the Gaussian random variable ζ_1. It then follows that the sequence of random variables $\Gamma_{p+1}^{-1}R_{[nt]}/\sqrt{n}; t \in [0,1]$ converges in distribution toward a $p+1$-dimensional Brownian motion with covariance matric I_{p+1}. Here I_{p+1} denotes the identity matrix with $p+1$ rows. It follows from Proposition 3.2 that $\Gamma_{p+1} = \Sigma$.

Proposition 4.6. *For all $\delta > 0$ one has as $u \to 0$:*

$$\limsup_{T\to\infty} \mathbb{P}\left(\Big|\sup_{0<s\le u}\Lambda_T(s) - \sup_{0<s\le u} R_{sT}^t Q_{sT}^{-1} R_{sT}\Big| > a_T\delta\right) \to 0$$

and

$$\limsup_{T\to\infty} \mathbb{P}\left(\Big|\sup_{1-u<s\le 1}\Lambda_T(s) - \sup_{1-u<s\le 1}(R_T - R_{sT})^t(Q_T - Q_{sT})^{-1}(R_T - R_{sT})\Big| > a_T\delta\right) \to 0.$$

Proof. It holds that

$$\begin{aligned}
&a_T^{-1}\Big|\sup_{0<s\le u}\Lambda_T(s) - \sup_{0<s\le u} R_{sT}^t Q_{sT}^{-1} R_{sT}\Big| \\
\le\ & \sup_{0<s\le u} a_T^{-1}\Big|\Lambda_T(s) - R_{sT}^t Q_{sT}^{-1} R_{sT}\Big| \\
=\ & \sup_{0<s\le u} a_T^{-1}\Big|(R_T - R_{sT})^t(Q_T - Q_{sT})^{-1}(R_T - R_{sT}) - R_T^t Q_T^{-1} R_T\Big| \\
\stackrel{\mathcal{D}}{\to}\ & \sup_{0<s\le u}\left|\frac{\|W(1) - W(s)\|^2}{1-s} - \|W(1)\|^2\right| \quad (\text{as } T \to \infty) \qquad (16) \\
\to\ & 0, \quad \text{almost surely,}
\end{aligned}$$

as $u \to 0$, where the convergence in (16) is implicated by the proof of Theorem 1 and the fact that $a_T \to 1$. Analogously, one has

$$\begin{aligned} a_T^{-1} & \left| \sup_{1-u<s\leq 1} \Lambda_T(s) - \sup_{1-u<s\leq 1} (R_T - R_{sT})^t (Q_T - Q_{sT})^{-1} (R_T - R_{sT}) \right| \\ & \leq \sup_{1-u<s\leq 1} a_T^{-1} \left| R_{sT}^t Q_{sT}^{-1} R_{sT} - R_T^t Q_T^{-1} R_T \right| \\ & \xrightarrow{\mathcal{D}} \sup_{1-u<s\leq 1} \left| \frac{\|W(s)\|^2}{s} - \|W(1)\|^2 \right| \quad (\text{as } T \to \infty) \\ & \to 0, \quad \text{almost surely,} \end{aligned}$$

as $u \to 0$. □

Proposition 4.7. *Under the framework of Theorem 1 it holds under the null hypothesis that*

$$\frac{1}{a_T} \left(\sup_{0<s\leq u} R_{sT}^t Q_{sT}^{-1} R_{sT} - b_T \right) \xrightarrow{\mathcal{D}} G^*$$

and

$$\frac{1}{a_T} \left(\sup_{1-u<s\leq 1} (R_T - R_{sT})^t (Q_T - Q_{sT})^{-1} (R_T - R_{sT}) \right) \xrightarrow{\mathcal{D}} G^*,$$

as $n \to \infty$, where G^* denotes a real-valued random variable satisfying

$$\mathbb{P}(G^* \leq x) = \exp(-e^{-x/2})$$

and where a_T and b_T are given in Proposition 4.1.

Proof. The reasoning follows the lines of the proof of remark A3 presented in [4] (see page 297). We first note that

$$R_{sT}^t \Gamma_{p+1}^{-1} R_{sT} - U_{[sT]}^t \Gamma_{p+1}^{-1} U_{[sT]} = R_{sT}^t \Gamma_{p+1}^{-1} (R_{sT} - U_{[sT]}) + (R_{sT}^t - U_{[sT]}^t) \Gamma_{p+1}^{-1} U_{[sT]}.$$

The law of iterated logarithm implies $U_{[sT]}^t \Gamma_{p+1}^{-1} = O(([sT] \log[sT])^{1/2})$ and Corollary 4.5 then implies $R_{sT}^t \Gamma_{p+1}^{-1} = O(([sT] \log[sT])^{1/2})$. Using those facts and Corollary 4.5 again yields

$$R_{sT}^t \Gamma_{p+1}^{-1} R_{sT} - U_{[sT]}^t \Gamma_{p+1}^{-1} U_{[sT]} = O([sT]^{1-\lambda'}) \tag{17}$$

for some $\lambda' > 0$ as $T \to \infty$.
Since by Proposition 3.3 one has $Q_{sT}/sT \to \Gamma_{p+1}$ it follows that

$$R_{sT}^t Q_{sT}^{-1} R_{sT} - \frac{1}{sT} R_{sT} \Gamma_{p+1}^{-1} R_{sT} = \frac{R_{sT}^t}{(sT)^{1/2}} sT Q_{sT}^{-1} \left(\Gamma_{p+1} - \frac{Q_{sT}}{sT} \right) \Gamma_{p+1}^{-1} \frac{R_{sT}}{(sT)^{1/2}} \longrightarrow 0.$$

This relation together with Equation (17) implies that as $T \to \infty$ one has

$$R_{sT}^t Q_{sT}^{-1} R_{sT} - U_{[sT]}^t \Gamma_{p+1}^{-1} U_{[sT]} / [sT] \longrightarrow 0. \tag{18}$$

Proposition 3.3 and the continuous mapping theorem yield

$$\sup_{s\in[u,1]} R_{sT}^t Q_{sT}^{-1} R_{sT} \xrightarrow{\mathcal{D}} \sup_{s\in[u,1]} \frac{\|W(s)\|^2}{s}.$$

It thus follows that

$$\sup_{s\in[u,1]} R_{sT}^t Q_{sT}^{-1} R_{sT} = O_P(1)$$

Moreover we have

$$\sup_{s\in(0,u]} R_{sT}^t Q_{sT}^{-1} R_{sT} \xrightarrow{P} \infty.$$

Thus with propability closer and closer to one the supremum is achieved in the interval $(0,u]$ and not in $[u,1]$. It then follows that

$$\mathbb{P}\Big(\sup_{s\in(0,1]} R_{sT}^t Q_{sT}^{-1} R_{sT} = \sup_{s\in(0,u]} R_{sT}^t Q_{sT}^{-1} R_{sT}\Big) \longrightarrow 1. \tag{19}$$

We also note that for a fixed $M>0$ one has

$$\mathbb{P}\Big(\sup_{s\in(0,1]} R_{sT}^t Q_{sT}^{-1} R_{sT} = \sup_{s\in(M/T,1]} R_{sT}^t Q_{sT}^{-1} R_{sT}\Big) \longrightarrow 1. \tag{20}$$

and

$$\mathbb{P}\Big(\sup_{s\in(0,1]} U_{[sT]}^t \Gamma_{p+1}^{-1} U_{[sT]} = \sup_{s\in(M/T,1]} U_{[sT]}^t \Gamma_{p+1}^{-1} U_{[sT]}\Big) \longrightarrow 1. \tag{21}$$

Let

$$R_T(M) := \sup_{s\in(M/T,1]} R_{sT}^t Q_{sT}^{-1} R_{sT}^t - \sup_{s\in(M/T,1]} U_{[sT]}^t \Gamma_{p+1}^{-1} U_{[sT]}^t \big/ [sT].$$

From Equation (18) we have

$$\begin{aligned} |R_T(M)| &= \Big|\sup_{s\in(M/n,1]} R_{sT}^t Q_{sT}^{-1} R_{sT}^t - \sup_{s\in(M/n,1]} U_{[sT]}^t \Gamma_{p+1}^{-1} U_{[sT]}^t \big/ [sT]\Big| \\ &\le \sup_{s\in(M/n,1]} \Big| R_{sT}^t Q_{sT}^{-1} R_{sT}^t - U_{[sT]}^t \Gamma_{p+1}^{-1} U_{[sT]}^t \big/ [sT]\Big| \end{aligned} \tag{22}$$

which goes to zero as $M \to \infty$ uniformly in $T \ge \nu$. It now follows from Eq.(19) and Proposition 4.1 that

$$\begin{aligned} & \lim_{T\to\infty} \mathbb{P}\Big(\sup_{s\in(0,u]} R_{sT}^t Q_{sT}^{-1} R_{sT} \le a_T x + b_T\Big) \\ =\; & \lim_{T\to\infty} \mathbb{P}\Big(\sup_{s\in(0,1]} R_{sT}^t Q_{sT}^{-1} R_{sT} \le a_T x + b_T\Big) \\ =\; & \lim_{M\to\infty} \lim_{T\to\infty} \mathbb{P}\Big(\sup_{s\in(M/T,1]} R_{sT}^t Q_{sT}^{-1} R_{sT} \le a_T x + b_T\Big) \\ =\; & \lim_{M\to\infty} \lim_{T\to\infty} \mathbb{P}\Big(\sup_{s\in(M/T,1]} U_{[sT]}^t \Gamma_{p+1}^{-1} U_{[sT]}^t \big/ [sT] \le a_T x + b_T + R_T(M)\Big) \\ =\; & \lim_{T\to\infty} \mathbb{P}\Big(\sup_{s\in(0,1]} U_{[sT]}^t \Gamma_{p+1}^{-1} U_{[sT]}^t \big/ [sT] \le a_T x + b_T\Big) \\ =\; & \lim_{T\to\infty} \mathbb{P}\Big(\sup_{s\in(0,1]} \|S_{[sT]}\|^2 / [sT] \le a_T x + b_T\Big) \longrightarrow \exp\big(-e^{-x/2}\big). \end{aligned}$$

This proves the first statement of the proposition. The second one is proved in an analogous way. □

Proof of Theorem 2. Since for fixed $x \in \mathbb{R}$ one has $a_T x + b_T \to \infty$ as $T \to \infty$ it follows from Theorem 1 for all $u \in (0, 1/2)$ that

$$\mathbb{P}\Big(\sup_{u<s<1-u} \Lambda_T(s) \le a_T x + b_T\Big) \longrightarrow 1.$$

Therefore, one has as $T \to \infty$ that

$$\mathbb{P}\Big(\sup_{0\le s\le 1} \Lambda_T(s) \le a_T x + b_T\Big) = \mathbb{P}\Big(\sup_{0\le s\le u} \Lambda_T(s) \le a_T x + b_T, \sup_{1-u\le s\le 1} \Lambda_T(s) \le a_T x + b_T\Big).$$

By Proposition 4.6 this has for $T \to \infty$ the same limit as

$$\mathbb{P}\left(\sup_{0<s<u} R_{sT}^t Q_{sT}^{-1} R_{sT} \le a_T x + b_T, \sup_{1-u<s<1} (R_T - R_{sT})^t (Q_T - Q_{sT})^{-1}(R_T - R_{sT}) \le a_T x + b_T\right).$$

Proposition 4.7 yields that this last expression converges toward $\exp(-2e^{-x/2})$ since the two sequences

$$\sup_{0<s<u} R_{sT}^t Q_{sT}^{-1} R_{sT}$$

and

$$\sup_{1-u<s<1} (R_T - R_{sT})^t (Q_T - Q_{sT})^{-1}(R_T - R_{sT})$$

are asymptotically independent by Proposition 4.2. □

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