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Guy Bouchitté and Ben Schweizer

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Fakultät für Mathematik
Technische Universität Dortmund
Vogelpothsweg 87
44227 Dortmund

tu-dortmund.de/MathPreprints

Plasmonic waves allow perfect transmission through sub-wavelength metallic gratings

Guy Bouchitté[1] and Ben Schweizer[2]

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Abstract: We perform a mathematical analysis of the transmission properties of a metallic layer with narrow slits. Our analysis is inspired by recent measurements and numerical calculations that report an unexpected high transmission coefficient of such a structure in a subwavelength regime. We analyze the time harmonic Maxwell's equations in the H-parallel case for a fixed incident wavelength. Denoting by η the typical size of the grated structure, we analyze the limit $\eta \to 0$ and derive effective equations that take into account the role of plasmonic waves. We obtain a formula for the effective transmission coefficient.

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key-words: plasmonic wave, Helmholtz equation, scattering, resonance, homogenization, effective tensor

1 Introduction

The interest to construct small scale optical devices for technical applications has initiated much research in the fields of micro- and nano-optics. In structures of sub-wavelength size, the behavior of electromagnetic waves is often counterintuitive and its mathematical understanding requires to develop new analytical tools. One example is the behavior of metamaterials with a negative index, see [17] for a first investigation and [7] for a mathematical analysis.

Another interesting and technically relevant example of the astonishing behavior of small scale structures is the high transmission of light through metallic layers with thin holes. As reported e.g. in [10], a metallic film with submicrometre cavities can display highly unusual transmittivity. Since the apertures are smaller than the wavelength of the incident photon, this high transmission is astonishing and contradicts classical aperture theory.

Many theoretical and numerical investigations of the effect are already available. The analysis given in [19] already establishes the connection of the effect to the excitation of

[1]Institut de Mathématiques IMATH, Université de Toulon et du Var, BP 20132, 83957 La Garde cedex, France. bouchitte@univ-tln.fr

[2]Technische Universität Dortmund, Fakultät für Mathematik, Vogelpothsweg 87, D-44227 Dortmund, Germany. ben.schweizer@tu-dortmund.de

surface plasmon polaritons. The photonic band structure of the surface plasmons is evaluated numerically, the contribution contains additionally two-dimensional calculations of typical electric fields in a neighborhood of the gratings. A semi-analytical calculation of transmission coefficients for lamellar grating is performed in [11], while the effect of surface plasmons on the upper and lower boundary of the layer is analyzed in [8]. Based on these investigations, the contribution [12] states that, in contrast to previously given explanations of the effect, the presence of surface plasmons has a negative effect on the transmission efficiency.

Further investigations focus on more specific topics. In [20], the effect of a finite conductivity is addressed. A relation between the high transmission effect and the negative index material obtained with a fishnet structure is made in [13]. An approach using homogenization theory is proposed in [18] where the authors emphasize the connection between the skin depth of evanescent modes in the metal and the period of the gratings.

The aim of the contribution at hand is to provide, through a mathematical analysis of the scattering problem, a new rigorous approach to transmission properties of heterogeneous media, enlightening the role of plasmonic resonances. We show that high transmission effects can survive in a metallic grating even in an extreme sub-wavelength regime.

We assume that the metallic obstacle is invariant in one direction and that the magnetic field is parallel to that direction *(transverse magnetic polarization)*. Accordingly, we work with a two dimensional model. The obstacle is described by a metallic slab of finite length and finite height in $\mathbb{R}^2$, the slits (vacuum) are repeated periodically with a small period denoted η. The period η will be infinitesimal with respect to the incident wavelength $\lambda = \frac{2\pi}{k}$, k denoting the wave number. In order to incorporate plasmonic effects in our model, we have to make two important assumptions on the permittivity coefficient in the metal: first, we need the permittivity $\varepsilon = \varepsilon_\eta$ to be scaled with factor $\frac{1}{\eta^2}$ as in [3, 4]. Second, in order to produce transverse evanescent modes in the metal, we require that the real part of ε_η is *negative.*

The change of sign of the (real part of the) permittivity implies that waves penetrate the metal only in a region that is determined by the skin-depth. At the same time, a surface plasmon can exist along the vertical boundaries of the slits. For certain wave-numbers k, the surface plasmon solution has a wave-length that is in resonance with the height of the metallic layer. As a result we obtain an effective scattering problem in which the metallic layer is replaced by an effective material with frequency dependent permittivity ε_{eff} and permeability μ_{eff}. The formulas for these effective parameters allow to evaluate the transmission coefficient $T = T(k)$ of the total structure in terms of the incident wave number k . We obtain a high transmission coefficient for certain resonant values of k and even $|T| = 1$ in the ideal case of a lossless metal with real negative ε_η.

This article proceeds as follows. The mathematical description of the metallic grating and of the scattering problem is given in Subsection 1.1, the effective system is described in Subsection 1.2. A rigorous analysis of the oscillating behavior of solutions as $\eta \to 0$ is performed in Section 2, where the proof for the main theorems is given. Section 3 contains the calculation of the transmission properties of the effective system.

The mathematical tools of this contribution are related to those of [2, 3, 5], and [7], where the Maxwell equations in other singular geometries have been investigated. Another application where the negative real part of the permittivity becomes relevant is cloaking by anomalous localized resonance, see [16] and the rigorous results in [6].

1.1 Mathematical description

Our interest is to study the Maxwell equations in a complex geometry and with high contrast permittivities. Since singular parameters will appear in the description of the geometry and of the permittivity, we introduce a number $\eta > 0$ as a parameter in the equations.

The H-parallel case in time-harmonic Maxwell equations. The time harmonic Maxwell equations read

$$\operatorname{curl} E_\eta = \quad i\omega\mu_0 H_\eta, \tag{1.1}$$
$$\operatorname{curl} H_\eta = -i\omega\varepsilon_\eta\varepsilon_0 E_\eta, \tag{1.2}$$

with fixed positive real constants ω, μ_0 and ε_0 that denote the frequency of the incoming light and the permeability and permittivity of vacuum. The inclusion of a material in a region Σ_η is described by a relative permittivity ε_η which is different from 1.

We study the situation of a wave with a polarized magnetic field $H_\eta = (0, 0, \bar{u}_\eta)$, such that the electric field has no third component, $E_\eta = (E_{x,\eta}, E_{y,\eta}, 0)$. The Maxwell equations then simplify to the two-dimensional system

$$\nabla^\perp \cdot (E_{x,\eta}, E_{y,\eta}) = \quad i\omega\mu_0 \bar{u}_\eta, \tag{1.3}$$
$$-\nabla^\perp \bar{u}_\eta = -i\omega\varepsilon_\eta\varepsilon_0 (E_{x,\eta}, E_{y,\eta}), \tag{1.4}$$

where we used the two-dimensional orthogonal gradient, $\nabla^\perp u = (-\partial_2 u, \partial_1 u)$, and the two-dimensional curl, $\nabla^\perp \cdot (E_x, E_y) = -\partial_2 E_x + \partial_1 E_y$. The system can be described equivalently by a scalar Helmholtz equation. We multiply the second equation with the space dependent coefficient $\varepsilon_\eta^{-1} = \varepsilon_\eta^{-1}(x)$ and apply the operator $\nabla^\perp\cdot$ to the result. Using the identity $\nabla^\perp \cdot (\varepsilon_\eta^{-1}\nabla^\perp \bar{u}_\eta) = \nabla \cdot (\varepsilon_\eta^{-1}\nabla \bar{u}_\eta)$ and setting $\bar{k}^2 = \omega^2\varepsilon_0\mu_0$ we obtain

$$\nabla \cdot \left(\frac{1}{\varepsilon_\eta}\nabla \bar{u}_\eta\right) = -\bar{k}^2 \bar{u}_\eta. \tag{1.5}$$

Nondimensionalization. Our mathematical analysis uses the aspect ratio $\eta := \bar{d}/\bar{h}$, where $\bar{d}$ is the distance between two gratings and $\bar{h}$ is the thickness of the layer. We derive asymptotic formulas for the transmission under the assumption that the dimensionless parameter $\eta > 0$ is small. We derive the effect of perfect transmission in the limiting case of small η, but we note that almost perfect transmission is also reported in studies where η is almost 1.

We use the two length scales $\bar{d}$ and $\bar{h}$ to non-dimensionalize the problem. Using the aspect ratio $\eta = \bar{d}/\bar{h}$ of the periodic structure as a non-dimensional variable, we can eliminate the grating width $\bar{a} < \bar{d}$ and the physical wave-length $\bar{\lambda}$ by setting

$$\eta = \frac{\bar{d}}{\bar{h}}, \qquad \alpha = \frac{\bar{a}}{\bar{d}}, \qquad \gamma = \frac{1-\alpha}{2}, \qquad \lambda = \frac{\bar{\lambda}}{\bar{h}} \qquad k = \frac{2\pi}{\lambda}.$$

The physical spatial parameter $\bar{x} \in \bar{\Omega}$ is replaced by $x = \bar{x}/\bar{h}$ in the dimension-less domain $\Omega := \bar{\Omega}/\bar{h} \subset \mathbb{R}^2$. In the non-dimensional variables, the layer has the height 1 and the periodicity length η, the relative aperture volume is α and the relative metal volume in the layer is 2γ, the dimension-less wave-length is λ. From now on, we work only with the dimension-less parameters.

The relative permittivity ε_η is dimension-less and remains unchanged. If the complex geometry is given by the set $\Sigma_\eta \subset \Omega$, such that the metal structure occupies the domain Σ_η, we assume that the relative permittivity is given by

$$\varepsilon_\eta = \begin{cases} -\dfrac{\sigma^2}{\eta^2} & \text{in } \Sigma_\eta, \\ 1 & \text{in } \Omega \setminus \Sigma_\eta, \end{cases} \tag{1.6}$$

where we normalize σ to have $\Re\sigma \geq 0$. A relative permittivity with positive imaginary part then corresponds to $\Im\sigma \leq 0$. We will study the Helmholtz equation (1.5) in dimension-less quantities, using the coefficient $a_\eta := \varepsilon_\eta^{-1} = -\sigma^{-2}\eta^2$. We emphasize that the coefficient can have a negative real part and that it vanishes in the limit $\eta \to 0$. Our sign convention in (1.6), using the expression σ^2/η^2 for the *negative* permittivity of the metal might be unusual, but it expresses the fact that the negative real part is a crucial fact in the plasmonic resonance. With our sign convention, a lossless metal with negative permittivity corresponds to a positive real number $\sigma > 0$.

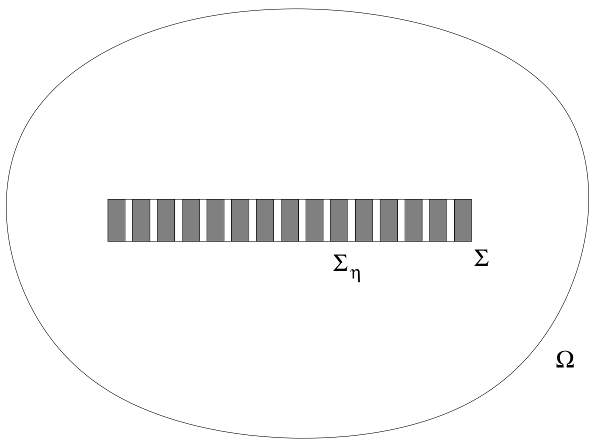

Figure 1: *Sketch of the non-dimensional geometry, showing the layer with the gratings. The physical variables, carrying dimensions, are the periodicity length $\bar{d}$, the layer thickness $\bar{h}$, the width of the grating $\bar{a}$, and the wave-length of the incoming light $\bar{\lambda}$. In the non-dimensional problem the layer has height* 1 *and periodicity length η.*

Typical physical parameters. To illustrate typical choices for the various parameters we refer to [8]. Figure 3 (b) of that work was obtained for periodicity length $\bar{d} = 3.5\mu m$, slit-width $\bar{a} = 0.5\mu m$, $\bar{h} = 3.0\mu m$, and wave-length $\bar{\lambda} = 7.5\mu m$. The corresponding quantities in the non-dimensional Helmholtz equation are

$$\eta = 7/6, \quad \alpha = 1/7, \quad \gamma = 3/7, \quad \lambda = 15/6, \quad k = 2\pi/\lambda \approx 2.51. \tag{1.7}$$

We use here the relative permittivity of silver as in [12], $\varepsilon_\eta = (0.12+3.7i)^2$. This amounts to setting $\sigma^2 = -\eta^2\varepsilon_\eta$ or $\sigma = \eta(3.7 - 0.12i)$.

1.2 Main results

Description of the complex geometry. We denote the macroscopic domain containing the scatterer and the free space around the scatterer by $\Omega \subset \mathbb{R}^2$. The scatterer is given by the metallic structure which is contained in the layer $\Sigma = (-l, l) \times (-h, 0) \subset \Omega$ for some $l > 0$ and $h = 1$. We denote the complement of Σ by $\Sigma^c := \Omega \setminus \Sigma$. We recall that $\gamma < 1/2$ stands for the half metal-width and define a collection of many small intervals by setting $\Gamma_\eta := \eta\mathbb{Z} + \eta(-\gamma, \gamma) \subset \mathbb{R}$. The metal occupies the complex domain $\Sigma_\eta \subset \Omega$,

$$\Sigma_\eta := (\Gamma_\eta \times (-h, 0)) \cap \Sigma. \tag{1.8}$$

This set Σ_η describes the geometry, the scattering problem is defined by the relative permittivity in (1.6). We will work with the inverse coefficient a_η,

$$a_\eta := (\varepsilon_\eta)^{-1} = \begin{cases} -\eta^2\sigma^{-2} & \text{in } \Sigma_\eta, \\ 1 & \text{in } \Omega \setminus \Sigma_\eta. \end{cases} \tag{1.9}$$

For notational convenience we always assume that all metal components have the same shape. We achieve this by demanding $l \in \eta\mathbb{N} + \gamma\eta$.

The shape function Ψ. We define $\Psi : \mathbb{R} \to \mathbb{C}$ as the continuous 1-periodic function that satisfies $\Psi(z) = 1$ for all $z \in [-1/2, -\gamma] \cup [\gamma, 1/2]$, and $\partial_z^2\Psi = k^2\sigma^2\Psi$ in $(-\gamma, \gamma)$. On the interval $[-1/2, 1/2]$ the function is given by an explicit formula, we will later also use average by β of this function,

$$\Psi(z) = \begin{cases} \frac{\cosh(k\sigma z)}{\cosh(k\sigma\gamma)} & \text{for } |z| \le \gamma, \\ 1 & \text{for } \gamma < |z| \le 1/2, \end{cases} \qquad \beta := \int_{-1/2}^{1/2} \Psi(z)\, dz. \tag{1.10}$$

Two further parameters $\rho \in \mathbb{C}$ and $\tau \in \mathbb{C}$ are defined as

$$\rho := \frac{2}{k\sigma\alpha} \frac{\sinh(k\sigma\gamma)}{\cosh(k\sigma\gamma)}, \quad \tau^2 := 1 + \rho\,. \tag{1.11}$$

The average β can be evaluated explicitly as $\beta = \frac{2}{k\sigma}\frac{\sinh(k\sigma\gamma)}{\cosh(k\sigma\gamma)} + \alpha$. We observe the following connection between β and ρ: the function $\tilde\Psi : (-1/2, 1/2) \to \mathbb{C}$, $\tilde\Psi(z) = \Psi(z)$ for $|z| < \gamma$ and $\tilde\Psi(z) = -\rho$ for $|z| \ge \gamma$, has a vanishing average.

The effective equation. With the help of the shape function Ψ we have defined two complex numbers $\beta, \tau \in \mathbb{C}$; they depend on the wave number k, the geometry parameter γ, and the permittivity parameter σ. The coefficient $\tau \in \mathbb{C}$ and the geometry parameter $\alpha = 1 - 2\gamma$ are now used to define the effective coefficients, the parameter β is useful in the interpretation of the effective system. We will formulate the limit problem with the x-dependent effective coefficients $a_{\text{eff}} = \varepsilon_{\text{eff}}^{-1} : \Omega \to \mathbb{R}^{2\times 2}$ and $\mu_{\text{eff}} : \Omega \to \mathbb{C}$,

$$\begin{aligned} a_{\text{eff}}(x) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mu_{\text{eff}}(x) = 1 \quad \text{for } x \in \Omega \setminus \Sigma, \\ a_{\text{eff}}(x) &= \begin{pmatrix} 0 & 0 \\ 0 & \alpha \end{pmatrix} \quad \text{and} \quad \mu_{\text{eff}}(x) = \alpha\tau^2 \quad \text{for } x \in \Sigma. \end{aligned} \tag{1.12}$$

We note that for a lossless metal of negative permittivity, the coefficient σ is real and can be chosen positive. In this case, the effective permeability $\mu_{\text{eff}} = \alpha\tau^2 = \alpha(1+\rho)$ is also real and positive. The derivative $\partial_\sigma \rho(\sigma)$ is negative and real for positive arguments σ; this implies that for slightly lossy material with small positive imaginary part of the permittivity and small negative imaginary part of σ, the imaginary part of the effective permeability satisfies $\Im \mu_{\text{eff}} > 0$.

Theorem 1 (Effective Helmholtz equation across Σ). *For $\Omega \subset \mathbb{R}^2$ open, let the geometry of the gratings be given by $\Sigma_\eta \subset \Omega$ of* (1.8) *and let the inverse permittivity $a_\eta := \varepsilon_\eta^{-1}$ be given by* (1.6)*. Let $u_\eta \in H^1_{\text{loc}}(\Omega)$ be a sequence of solutions to*

$$\nabla \cdot (a_\eta \nabla u_\eta) = -k^2 u_\eta \quad \text{in } \Omega. \tag{1.13}$$

Let $u \in L^2(\Omega)$ be a limit function, $u_\eta \rightharpoonup u$ in $L^2(\Omega)$ for $\eta \to 0$. Using $\beta \in \mathbb{C}$ from (1.10)*, we define $U : \Omega \to \mathbb{C}$ as $U(x) = u(x)$ for $x \in \Omega \setminus \Sigma$ and $U(x) = \beta^{-1} u(x)$ for $x \in \Sigma$. With this definition of $U \in L^2(\Omega)$, the function solves the effective Helmholtz equation*

$$\nabla \cdot (a_{\text{eff}} \nabla U) = -\mu_{\text{eff}} k^2 U \quad \text{in } \Omega, \tag{1.14}$$

where effective inverse permittivity and effective permeability are given by (1.12)*.*

We can also obtain the effective equation in a scattering problem. When considering the Helmholtz equation on $\mathbb{R}^2$, we will impose for the scattered field u^s the Sommerfeld condition

$$\partial_r u^s - iku^s = o\left(r^{-1/2}\right) \tag{1.15}$$

for $r = |x| \to \infty$, uniformly in the angle variable.

Theorem 2 (Effective scattering problem). *For $\Omega := \mathbb{R}^2$ we consider a bounded scatterer contained in $\Sigma := (-l, l) \times (-1, 0)$. Let the geometry of the gratings be given by Σ_η of* (1.8) *with coefficient $a_\eta := \varepsilon_\eta^{-1}$ of* (1.6)*. We assume that the permittivity in the metal satisfies $\Re\, \sigma^2 > 0$ and that the effective relative permeability coefficient ρ of* (1.11) *satisfies $\Im\, \rho > 0$. Let u^i be an incident wave, solving the free space equation $\Delta u^i = -k^2 u^i$ on $\mathbb{R}^2$. Let u_η be a sequence of solutions to* (1.13) *such that $u^s = (u_\eta - u^i)$ satisfies the outgoing wave condition* (1.15)*.*

Then there holds $u_\eta \mathbf{1}_{\Omega\setminus\Sigma} \rightharpoonup U \mathbf{1}_{\Omega\setminus\Sigma}$ in $L^2_{\text{loc}}(\Omega)$ as $\eta \to 0$. Here, the effective field $U : \Omega \to \mathbb{C}$ is given as the unique solution of the effective Helmholtz equation

$$\nabla \cdot (a_{\text{eff}} \nabla U) = -\mu_{\text{eff}} k^2 U \quad \text{in } \mathbb{R}^2 \tag{1.16}$$

with the boundary condition (1.15) *for the scattered field $(U - u^i)$. The effective parameters are given by* (1.12)*.*

2 Proof of the main results

2.1 Proof of Theorem 1

We will derive the effective equations with the tool of two-scale convergence as outlined in [1].

Lemma 2.1 (Two-scale limit of u_η)**.** *Let $u_\eta \rightharpoonup u$ in $L^2(\Omega)$ be a sequence of solutions to (1.13) as in Theorem 1. Then the sequence u_η converges strongly outside Σ,*

$$u_\eta \, \mathbf{1}_{\Omega'} \to u \, \mathbf{1}_{\Omega'} \text{ strongly in } L^2(\Omega') \text{ as } \eta \to 0 \tag{2.1}$$

for every compactly contained subset $\Omega' \subset \Omega \setminus \bar{\Sigma}$. In the entire set Ω, the sequence u_η converges weakly in two scales,

$$u_\eta(x) \stackrel{2}{\rightharpoonup} u_0(x,y) := \begin{cases} u(x) & \text{for } x \notin \Sigma, \\ \beta^{-1} u(x)\, \Psi(y_1) & \text{for } x \in \Sigma. \end{cases} \tag{2.2}$$

Proof. Strong convergence outside Σ. Let $\Omega' \subset \Omega$ satisfy $\bar{\Omega}' \subset \Omega \setminus \bar{\Sigma}$. Then the sequence u_η satisfies on Ω' the equation $\Delta u_\eta = f_\eta$ for $f_\eta := -k^2 u_\eta \rightharpoonup f := -k^2 u$ in $L^2(\Omega)$. The Cacciopoli inequality provides locally uniform bounds for ∇u_η and thus the strong convergence of the sequence u_η in $L^2(\Omega')$. The strong convergence implies, loosely speaking, that no oscillations occur outside Σ.

Two-scale convergence. By assumption, the sequence u_η is bounded in $L^2(\Omega)$. Compactness of bounded sequences in the sense of two-scale convergence implies that, for $Y = (-1/2, +1/2)^2$ and up to the choice of a subsequence, there exists a limit function $u_0 \in L^2(\Omega \times Y)$ with $u_0 = u_0(x,y) = u_0(x_1, x_2, y_1, y_2)$, such that the sequence u_η converges in two scales to u_0. It remains to show the characterization of u_0 in (2.2). For $x \notin \Sigma$, the property $u_0(x,y) = u(x)$ is a consequence of (2.1). Once that we have shown $u_0(x,y) = \frac{1}{\beta} u(x)\, \Psi(y_1)$ for $x \in \Sigma$, we have shown the characterization of u_0. In particular, we thus also obtain that the whole sequence converges and the proof of the lemma is complete.

Characterization of the two-scale limit for $x \in \Sigma$. We restrict the discussion to an arbitrary open subset Ω', compactly contained in Ω. For some constant C we have the energy estimate

$$\int_{\Omega'} a_\eta |\nabla u_\eta|^2 \le C \tag{2.3}$$

for all $\eta > 0$. This can be obtained easily as in the Cacciopoli inequality by testing equation (1.13) with $\Theta^2(x) u_\eta(x)$, where $\Theta \in C_c^\infty(\Omega)$ is a cut-off function. Estimate (2.3) implies, in particular, the two-scale convergence for the gradient, $\eta \nabla u_\eta \rightharpoonup \nabla_y u_0(x,y)$ in the sense of two scales in L^2. We note here that $\nabla_y u_0(x,.) \in L^2(Y)$ implies the regularity $u_0(x,.) \in H^1(Y)$.

On the set $\Omega' \setminus \Sigma_\eta$, where the coefficient is $a_\eta = 1$, we find the weak two-scale convergence in $L^2(\Omega')$

$$0 \leftharpoondown \eta\, \nabla u_\eta \, (1 - \mathbf{1}_{\Sigma_\eta}) \rightharpoonup \nabla_y u_0(x,y)\, \mathbf{1}_{\{|y_1| > \gamma\}} \, .$$

By periodicity of u_0, this implies that u_0 is constant on the strips $\{|y_1| > \gamma\}$. We next use an arbitrary function $\varphi_0 \in C^\infty(\Omega' \times Y)$ which is periodic in y and with compact support in $\{(x,y) : x \in \Sigma,\ |y_1| < \gamma\}$. Using $\varphi_\eta(x) = \varphi_0(x, x/\eta)$ as a test-function for equation (1.13) and inserting the coefficient $a_\eta = -\sigma^{-2}\eta^2$ we obtain

$$\begin{aligned} k^2 \int_\Sigma \int_Y u_0(x,y) \varphi_0(x,y)\, dy\, dx \leftarrow k^2 \int_\Sigma u_\eta \varphi_\eta = \int_\Sigma a_\eta \nabla u_\eta \nabla \varphi_\eta \\ \to -\sigma^{-2} \int_\Sigma \int_Y \nabla_y u_0(x,y) \cdot \nabla_y \varphi_0(x,y)\, dy\, dx. \end{aligned}$$

Since φ_0 can be chosen to be independent of y_2, we obtain

$$k^2 u_0(x,y) = \sigma^{-2}\partial_{y_1}^2 u_0(x,y) \quad \forall y_1 \in (-\gamma,\gamma).$$

From $u_0(x,.) \in H^1(Y)$ we conclude that the function u_0 is independent of y_2 and thus continuous in y_1. For fixed $x \in \Omega'$ and $y_2 \in (-1/2, 1/2)$, the function $y_1 \mapsto u_0(x, y_1, y_2)$ is periodic and satisfies the same equations as Ψ. We therefore obtain $u_0(x,y) = A(x)\,\Psi(y_1)$ for some factor $A(x)$. The factor is easily determined by evaluating the weak $L^2(\Omega')$-limit of the sequence u_η for $\eta \to 0$,

$$u(x) = \int_Y u_0(x,y)\,dy = A(x)\int_{-1/2}^{1/2}\Psi(y_1)\,dy_1 = \beta A(x).$$

This determines the factor of the two-scale limit as $A(x) = \beta^{-1}u(x)$, and the lemma is shown. □

Lemma 2.2 (Special one-dimensional test-function Φ_η)**.** *Let $\varphi \in C_c^2(\mathbb{R},\mathbb{C})$ be a given function and $\eta \to 0$ a sequence. Then there exists a sequence of oscillatory functions $\Phi_\eta : \mathbb{R} \to \mathbb{C}$ with the following properties.*

1. *Φ_η is continuous and satisfies $\Phi_\eta(x) = \varphi(x)$ in all points $x \in \eta\mathbb{Z} \pm \eta\gamma = \partial\Gamma_\eta$.*
2. *Φ_η satisfies $\partial_x^2\Phi_\eta = \eta^{-2}k^2\sigma^2\Phi_\eta$ in all intervals $\Gamma_\eta = \eta\mathbb{Z} + \eta(-\gamma,\gamma)$.*
3. *With the coefficient $A_\eta(x) = -\eta^2\sigma^{-2}$ in Γ_η and $A_\eta(x) = 1$ in $\mathbb{R}\setminus\Gamma_\eta$, the flux expression $A_\eta\partial_x\Phi_\eta$ has a continuous representative. It satisfies $\|A_\eta\partial_x\Phi_\eta\mathbf{1}_{\Gamma_\eta}\|_{C^0} \le C\eta$ with C depending only on φ.*
4. *In all intervals $\eta\mathbb{Z}+\eta(\gamma, 1-\gamma)$, the function Φ_η is given by a third order polynome.*

The sequence Φ_η has the following strong convergence properties for $\eta \to 0$.

$$(\Phi_\eta - \varphi)\,\mathbf{1}_{\mathbb{R}\setminus\Gamma_\eta} \to 0 \text{ strongly in } L^2(\mathbb{R}), \tag{2.4}$$

$$(\partial_x^2\Phi_\eta - \rho k^2\varphi)\,\mathbf{1}_{\mathbb{R}\setminus\Gamma_\eta} \to 0 \text{ strongly in } L^2(\mathbb{R}). \tag{2.5}$$

Proof. The proof uses the explicit construction of the function Φ_η.

Step 1. Construction on intervals $m\eta + (-\gamma\eta, \gamma\eta)$ with $m \in \mathbb{Z}$. We fix a point $x_0 = m\eta \in \eta\mathbb{Z}$ in the center of a connected component of Γ_η. We will see that properties 1 and 2 determine the function Φ_η uniquely on the interval $(x_0 - \gamma\eta, x_0 + \gamma\eta)$. In fact, property 2 implies, for some coefficients $a_1, a_2 \in \mathbb{C}$, for all x in this interval the relation

$$\Phi_\eta(x) = a_1 e^{k\sigma(x-x_0)/\eta} + a_2 e^{-k\sigma(x-x_0)/\eta}.$$

Upon changing variables to $x = x_0 + \eta z$ and changing the variables to $b_1, b_2 \in \mathbb{C}$, we write the function as

$$\Phi_\eta(x_0 + \eta z) = b_1\cosh(k\sigma z) + b_2\sinh(k\sigma z). \tag{2.6}$$

The two boundary values $w_\pm^{(x_0)} = \varphi(x_0 \pm \gamma\eta)$ determine the coefficients $b_1 = b_1^{(x_0)}, b_2 = b_2^{(x_0)} \in \mathbb{C}$,

$$b_1^{(x_0)} = \frac{1}{2}(w_+^{(x_0)} + w_-^{(x_0)})\frac{1}{\cosh(k\sigma\gamma)}, \qquad b_2^{(x_0)} = \frac{1}{2}(w_+^{(x_0)} - w_-^{(x_0)})\frac{1}{\sinh(k\sigma\gamma)}.$$

These explicit formulas imply

$$\left| b_1^{(x_0)} - \frac{\varphi(x_0)}{\cosh(k\sigma\gamma)} \right| \le C\eta, \qquad \left| b_2^{(x_0)} \right| \le C\eta. \tag{2.7}$$

The constant C depends on k, γ, σ and on the Lipschitz constant of φ, but it is independent of η and x_0.

Step 2. The flux $A_\eta \partial_x \Phi_\eta$. We next evaluate the flux quantity $F_\eta = A_\eta \partial_x \Phi_\eta$ for $A_\eta = -\eta^2 \sigma^{-2}$ in the interval $(x_0 - \gamma\eta, x_0 + \gamma\eta)$. We find, for $|z| < \gamma$,

$$F_\eta(x_0 + \eta z) = A_\eta \partial_x \Phi_\eta(x_0 + \eta z) = -\frac{\eta^2}{\sigma^2} \left[b_1 \frac{k\sigma}{\eta} \sinh(k\sigma z) + b_2 \frac{k\sigma}{\eta} \cosh(k\sigma z) \right].$$

This implies the smallness of the fluxes on Γ_η. Furthermore, in the end-points $x_0 \pm \gamma\eta$ we find, in the sense of traces,

$$F_\eta(x_0 \pm \gamma\eta) = -\frac{\eta k}{\sigma} \left[\pm b_1 \sinh(k\sigma\gamma) + b_2 \cosh(k\sigma\gamma) \right].$$

Recalling the characterization (2.7) of b_1 and b_2, we therefore obtain

$$\left| F_\eta(x_0 \pm \gamma\eta) \pm \frac{\eta k}{\sigma} \frac{\sinh(k\sigma\gamma)}{\cosh(k\sigma\gamma)} \varphi(x_0) \right| \le C\eta^2.$$

Step 3. Construction on intervals $(m\eta + \gamma\eta, m\eta + \eta - \gamma\eta)$ with $m \in \mathbb{Z}$. It remains to extend the function Φ_η to the remaining intervals. By property 4, we want to choose a third order polynome in each of these intervals. Properties 1 and 3 determine the function Φ_η uniquely, since values and first derivatives of Φ_η are prescribed on both end-points of each interval $(x_0 + \gamma\eta, x_0 + \eta - \gamma\eta)$, for $x_0 = m\eta \in \eta\mathbb{Z}$. We write the function Φ_η in the form $P(z) := \Phi_\eta(x_0 + \eta/2 + z\eta) = c_0 + c_1 \eta z + c_2 \eta^2 z^2 + c_3 \eta^3 z^3$ for $z \in (-(1/2-\gamma), 1/2-\gamma)$. It remains to study the properties of the coefficients $c_j = c_j^{(x_0)} \in \mathbb{C}$, $j = 0, 1, 2, 3$.

The values of the coefficients are determined by the boundary conditions

$$\begin{aligned} P(z)\Big|_{z=-1/2+\gamma} &= \varphi(x_0 + \gamma\eta), & P(z)\Big|_{z=1/2-\gamma} &= \varphi(x_0 + \eta - \gamma\eta), \\ \frac{d}{dz} P(z)\Big|_{z=-1/2+\gamma} &= \eta F_\eta(x_0 + \gamma\eta), & \frac{d}{dz} P(z)\Big|_{z=1/2-\gamma} &= \eta F_\eta(x_0 + \eta - \gamma\eta). \end{aligned}$$

We observe that the coefficient c_2 can be evaluated directly with an integration of the second derivative,

$$\begin{aligned} 2c_2\eta^2(1-2\gamma) &= \int_{-1/2+\gamma}^{1/2-\gamma} \frac{d^2}{dz^2} P(z)\, dz = \eta F_\eta(x_0 + \eta - \gamma\eta) - \eta F_\eta(x_0 + \gamma\eta) \\ &= 2\frac{\eta^2 k}{\sigma} \frac{\sinh(k\sigma\gamma)}{\cosh(k\sigma\gamma)} \varphi(x_0) + O(\eta^3). \end{aligned}$$

Recalling the parameter ρ of equation (1.11), we find

$$\rho = \frac{2}{\alpha k\sigma} \frac{\sinh(k\sigma\gamma)}{\cosh(k\sigma\gamma)}, \qquad c_2 = \frac{\rho k^2}{2} \varphi(x_0) + O(\eta).$$

This expression for c_2 implies, in particular, the boundedness of c_2. The coefficient c_0 is then determined by the average of the boundary values and we obtain $|c_0 - \varphi(x_0)| \le C\eta$. The odd coefficients c_1 and c_3 are determined by the average slope of P, i.e. by $\varphi(x_0+\eta-\gamma\eta) - \varphi(x_0+\gamma\eta) = O(\eta)$, and by the average slope of P in the end-points, i.e. by $\eta F_\eta(x_0+\eta-\gamma\eta) + \eta F_\eta(x_0+\gamma\eta) = O(\eta^3)$. The interpolating third order polynome has the property that $c_1\eta$ and $c_3\eta^3$ are of order $O(\eta)$, hence $|c_1| \le C$ and $\eta^2|c_3| \le C$.

Step 4. Convergence properties of Φ_η. Regarding the values of $\Phi_\eta \mathbf{1}_{\Gamma_\eta}$, the contributions of $c_1\eta$, of $c_2\eta^2$, and of $c_3\eta^3$ vanish in the limit $\eta \to 0$. We therefore obtain

$$\Phi_\eta \rightharpoonup \Phi_0(x,z) = \varphi(x)\Psi(z)$$

in the sense of two scales for $\eta \to 0$. Furthermore, the estimates for the constants c_j imply the strong convergence (2.4). The derivative of the flux function satisfies because of the relation $2c_2 = \rho k^2 \varphi(x_0) + O(\eta)$ in each interval

$$\partial_x F_\eta(x) = \partial_x(A_\eta \partial_x \Phi_\eta)(x) = \begin{cases} -k^2\Phi_\eta(x) & \text{for } x \in \Gamma_\eta, \\ k^2\rho\varphi(x) + O(\eta) & \text{for } x \in \mathbb{R}\setminus\bar{\Gamma}_\eta\,. \end{cases}$$

This implies, in particular, the strong convergence (2.5). □

The oscillating test-function of Lemma 2.2 allows to derive the effective system. Relation (2.8) of the subsequent lemma provides the limit system (1.14) of Theorem 1 in its weak form. Therefore, with Lemma 2.3, Theorem 1 is shown.

Lemma 2.3 (Weak form of the limit problem)**.** *We consider coefficients* $a_\eta : \Omega \to \mathbb{C}$, $a_\eta = (-\eta^2\sigma^{-2})\mathbf{1}_{\Sigma_\eta} + \mathbf{1}_{\Omega\setminus\Sigma_\eta}$, *and solutions* $u_\eta : \Omega \to \mathbb{C}$ *to the Helmholtz equation* (1.13) *as in Theorem 1. We assume the weak convergence* $u_\eta \rightharpoonup u$ *in* $L^2(\Omega)$ *and set* $U := u\mathbf{1}_{\Sigma^c} + \beta^{-1}u\mathbf{1}_\Sigma \in L^2(\Omega)$. *Then* U *satisfies*

$$k^2\int_{\Sigma^c} U\,\varphi + \int_\Sigma \alpha k^2\tau^2 U\,\varphi = \int_{\Sigma^c} \nabla U\cdot\nabla\varphi + \int_\Sigma \alpha\partial_{x_2}U\,\partial_{x_2}\varphi \tag{2.8}$$

for every test-function $\varphi \in C^2_c(\Omega)$.

Proof. Let a smooth test-function $\varphi \in C^2_c(\Omega,\mathbb{R})$ be given. For the proof of the lemma we construct from the given function φ an oscillatory sequence $\varphi_\eta \in C^0_c(\Omega,\mathbb{R})$ of test-functions. The construction uses the one-dimensional profiles Φ_η.

For fixed height $x_2 \in \mathbb{R}$, we consider the one-dimensional function $\varphi(.,x_2) : \mathbb{R} \ni x_1 \mapsto \varphi(x_1,x_2) \in \mathbb{R}$ of class $C^2(\mathbb{R},\mathbb{R})$. Lemma 2.2 provides an oscillatory function

$$\Phi_\eta(.,x_2) : \mathbb{R}\to\mathbb{R} \text{ of Lemma 2.2 corresponding to } \varphi(.,x_2). \tag{2.9}$$

We have to modify this function, since $\Sigma = (-l,l)\times(-1,0)$ is bounded. We therefore modify $\Phi_\eta(.,x_2)$ outside the interval $[-l,l]$ and set $\Phi_\eta(x_1,x_2) := \varphi(x_1,x_2)$ for $|x_1| > l$. By construction of Φ_η and the choice $l \in \eta\mathbb{N} + \gamma\eta$, this provides a one-parameter family of functions $\Phi_\eta(.,x_2) \in C^0(\mathbb{R},\mathbb{R})$.

In order to construct a continuous test-function $\varphi_\eta : \Omega \to \mathbb{R}$, additional care is required at the upper and the lower boundary of the strip $\Sigma = (-l,l)\times(-h,0)$. As a convex combination of the two previously defined functions we obtain

$$\varphi_\eta(x_1,x_2) := \begin{cases} \varphi(x_1,x_2) & \text{if } x_2 \le -h \text{ or } x_2 \ge 0, \\ \Phi_\eta(x_1,x_2) & \text{if } -h+\eta \le x_2 \le -\eta, \\ \left(1+\frac{x_2}{\eta}\right)\varphi(x_1,x_2) - \frac{x_2}{\eta}\Phi_\eta(x_1,x_2) & \text{if } -\eta < x_2 < 0, \\ \left(1-\frac{h+x_2}{\eta}\right)\varphi(x_1,x_2) + \frac{h+x_2}{\eta}\Phi_\eta(x_1,x_2) & \text{if } -h < x_2 < -h+\eta. \end{cases}$$

Using φ_η as a test-function in equation (1.13) yields

$$k^2 \int_\Omega u_\eta \varphi_\eta = \int_\Omega a_\eta \nabla u_\eta \nabla \varphi_\eta. \tag{2.10}$$

We emphasize at this point that the function φ_η is continuous and piece-wise continuously differentiable by construction. It is therefore locally of class H^1 and hence admissible as a test-function. Performing the limit $\eta \to 0$ in (2.10) will provide the limit equation (2.8).

In a first step, we decompose the integrals distinguishing the layer Σ and the complement Σ^c. We perform one integration by parts for the partial derivative ∂_{x_1} in the layer Σ. In this integration by parts we exploit that the function $a_\eta(.,x_2)\partial_{x_1}\varphi_\eta(.,x_2)$ is continuous across internal interfaces. We obtain

$$\begin{aligned} k^2 \int_{\Sigma^c} u_\eta\varphi_\eta + k^2 \int_\Sigma u_\eta\varphi_\eta - \int_{\Sigma^c} a_\eta \nabla u_\eta \nabla\varphi_\eta - \int_\Sigma a_\eta\, \partial_{x_2} u_\eta\, \partial_{x_2}\varphi_\eta = \int_\Sigma a_\eta\, \partial_{x_1} u_\eta\, \partial_{x_1}\varphi_\eta \\ = -\int_\Sigma u_\eta\, \partial_{x_1}(a_\eta \partial_{x_1}\varphi_\eta) + \int_{\{l\}\times(-1,0)} u_\eta\, (a_\eta\partial_{x_1}\varphi_\eta) - \int_{\{-l\}\times(-1,0)} u_\eta\, (a_\eta\partial_{x_1}\varphi_\eta). \end{aligned}$$

Regarding the integrals over the lateral sides of Σ we observe that the sequence of traces u_η is bounded in $L^2(\{\pm l\}\times(-1,0))$ and that the fluxes $a_\eta\partial_{x_1}\varphi_\eta$ are uniformly bounded by $C\eta$, see Lemma 2.2, item 3. The last two integrals hence vanish in the limit $\eta \to 0$.

In the next step, we decompose the integrals over Σ into an integral over the metal structure $\Sigma_\eta \subset \Sigma$, and an integral over the slits $S_\eta := \Sigma \setminus \Sigma_\eta$. Inserting the coefficient a_η in the various domains we obtain

$$\begin{aligned} k^2 \int_{\Sigma^c} u_\eta\varphi_\eta + k^2 \int_{S_\eta} u_\eta\varphi_\eta + k^2 \int_{\Sigma_\eta} u_\eta\varphi_\eta - \int_{\Sigma^c} \nabla u_\eta \nabla\varphi_\eta + O(\eta) \\ = \int_{S_\eta} \partial_{x_2} u_\eta\, \partial_{x_2}\varphi_\eta + \int_{\Sigma_\eta} (-\eta^2\sigma^{-2}) \partial_{x_2} u_\eta\, \partial_{x_2}\varphi_\eta - \int_{S_\eta} u_\eta\, \partial^2_{x_1}\varphi_\eta + \int_{\Sigma_\eta} \eta^2\sigma^{-2}\, u_\eta\, \partial^2_{x_1}\varphi_\eta \end{aligned} \tag{2.11}$$

By construction, there holds $\eta^2\sigma^{-2}\,\partial^2_{x_1}\varphi_\eta(x_1,x_2) = k^2\varphi_\eta(x_1,x_2)$ for all $(x_1,x_2) \in \Gamma_\eta \times (-h+\eta,-\eta) \subset \Sigma_\eta$. This identity implies that two integrals in (2.11) are comparable, more precisely, that

$$\left| k^2 \int_{\Sigma_\eta} u_\eta\varphi_\eta - \int_{\Sigma_\eta} \eta^2\sigma^{-2}\, u_\eta\, \partial^2_{x_1}\varphi_\eta \right| \le C \int_{-\eta}^0 \int_{\Gamma_\eta} |u_\eta| + C \int_{-h}^{-h+\eta} \int_{\Gamma_\eta} |u_\eta| \le C\sqrt{\eta}$$

by the boundedness of u_η in $L^2(\Omega)$.

We observe that the other integral over Σ_η, containing x_2-derivatives of u_η, is of order η due to the uniform boundedness of $\partial_{x_2}\varphi_\eta$ and the $L^2(\Omega)$-boundedness of $\eta\nabla u_\eta$. We obtain

$$k^2\int_{\Sigma^c} u_\eta\varphi_\eta + k^2\int_{S_\eta} u_\eta\varphi_\eta = \int_{\Sigma^c}\nabla u_\eta\nabla\varphi_\eta + \int_{S_\eta}\partial_{x_2}u_\eta\partial_{x_2}\varphi_\eta - \int_{S_\eta} u_\eta\partial_{x_1}^2\varphi_\eta + O(\sqrt{\eta}).$$

In this equality, we will be able to evaluate the limit as $\eta\to 0$ for all terms.

Concerning integrals over Σ^c we observe that $\varphi_\eta = \varphi$ holds on that domain. Let us next analyze the error functions $e_\eta^0 := (\varphi_\eta - \varphi)\,\mathbf{1}_{S_\eta}$, $e_\eta^1 := (\partial_{x_2}\varphi_\eta - \partial_{x_2}\varphi)\,\mathbf{1}_{S_\eta}$, and $e_\eta^2 := (\partial_{x_1}^2\varphi_\eta - k^2\rho\varphi)\,\mathbf{1}_{S_\eta}$. By (2.4), (2.5), and the construction of φ_η, these error functions satisfy

$$e_\eta^0 \to 0 \text{ and } \ e_\eta^1 \to 0 \text{ and } \ e_\eta^2 \to 0, \text{ strongly in } L^2(\Omega).$$

This allows to replace, producing only small error terms, the functions φ_η by φ in the above integral identity. We obtain, provided that the single limits exist, the relation

$$\lim_{\eta\to 0} k^2\int_{\Sigma^c} u_\eta\varphi + \lim_{\eta\to 0} k^2\int_{S_\eta} u_\eta\varphi = \lim_{\eta\to 0}\int_{\Sigma^c}\nabla u_\eta\nabla\varphi + \lim_{\eta\to 0}\int_{S_\eta}\partial_{x_2}u_\eta\partial_{x_2}\varphi - \lim_{\eta\to 0}\int_{S_\eta} u_\eta k^2\rho\varphi\,.$$

The limits are obtained from the two-scale convergence property (2.2) of u_η. By definition of U we obtain

$$k^2\int_{\Sigma^c} U\varphi + \alpha k^2\int_{\Sigma} U\varphi = \int_{\Sigma^c}\nabla U\nabla\varphi + \alpha\int_{\Sigma}\partial_{x_2}U\partial_{x_2}\varphi - \alpha\int_{\Sigma} Uk^2\rho\varphi. \tag{2.12}$$

This provides the weak equation (2.8) by the definition of $\tau^2 = 1+\rho$. $\square$

2.2 Proof of Theorem 2

Uniqueness for the limit problem. For fixed incident field u^i we want to show that there exists at most one solution U of the limit problem described in Theorem 2. To this end we consider a solution $u:\mathbb{R}^2\to\mathbb{C}$ of the problem

$$\nabla\cdot(a_{\text{eff}}\nabla u) = -\mu_{\text{eff}}k^2 u \qquad \text{in } \mathbb{R}^2, \tag{2.13}$$

$$\partial_r u - iku = o\left(r^{-1/2}\right) \qquad \text{for } r\to\infty. \tag{2.14}$$

We have to verify the u vanishes identically. We will follow arguments that are outlined e.g. in [9]. Based on the formula $|\partial_r u - iku|^2 = |\partial_r u|^2 + k^2|u|^2 + 2k\Im\,(u\,\partial_r\bar{u})$, we observe that the radiation condition implies for spheres $S_r := \partial B_r(0)$

$$\int_{S_r}\{|\partial_r u|^2 + k^2|u|^2 + 2k\Im\,(u\partial_r\bar{u})\} = \int_{S_r}|\partial_r u - iku|^2 \to 0 \tag{2.15}$$

for $r\to\infty$. We furthermore notice that the divergence $\nabla\cdot(u\nabla\bar{u}) = -k^2|u|^2 + |\nabla u|^2$ is real outside Σ. This allows to write (2.15) with a surface integral over a fixed surface. We assume that the ball $B_s(0)\subset B_r(0)$ contains Σ and obtain

$$\int_{S_r}\{|\partial_r u|^2 + k^2|u|^2\} + 2k\int_{S_s}\Im\,(u\partial_r\bar{u}) = \int_{S_r}|\partial_r u - iku|^2 \to 0 \tag{2.16}$$

for $r \to \infty$. In particular, we obtain for the radius s

$$\int_{S_s} \Im\,(u\partial_r \bar{u}) \leq 0.$$

We now study the effective equation (2.13) on the ball $B_s(0)$ and use $\bar{u}$ as a test-function. Using in the first equation that a_{eff} is real, we can write

$$-\Im \int_{B_s(0)} k^2 \mu_{\text{eff}} |u|^2 = \Im \int_{B_s(0)} \{a_{\text{eff}} |\nabla u|^2 - k^2 \mu_{\text{eff}} |u|^2\} = \Im \int_{\partial B_s(0)} \partial_r u\, \bar{u} \geq 0. \tag{2.17}$$

We now use the fact that $\Im\,\mu_{\text{eff}}$ is positive on Σ, which was assumed in Theorem 2. This implies that the left hand side of (2.17) is non-positive and we conclude that all terms of (2.17) vanish. The strict inequality $\Im\,\mu_{\text{eff}} \neq 0$ on Σ implies $u = 0$ on Σ.

In order to conclude $u = 0$ on $\mathbb{R}^2$ we use the Green's formula. This formula expresses $u(x)$ in arbitrary points $x \in \mathbb{R}^2 \setminus \Sigma$ with the help of the fundamental solution of the Helmholtz equation in terms of $u|_{\partial\Sigma}$ and $\partial_n u|_{\partial\Sigma}$ as an integral over $\partial\Sigma$. Both, solution and normal derivative vanish on $\partial\Sigma$, hence the Green's formula implies $u = 0$ on $\mathbb{R}^2$.

Derivation of the limit problem assuming an L^2_{loc}-bound. We analyze a sequence u_η as given in Theorem 2 and a subsequence $\eta = \eta_j \to 0$. We choose a radius $r_0 > 0$ such that $\Sigma \subset B_{r_0}(0)$. In this step of the proof we assume that, for every radius $r > r_0$, there holds

$$t_\eta := \left(\int_{B_r(0)} |u_\eta|^2 \right)^{1/2} \leq C(r), \tag{2.18}$$

where $C(r)$ depends on r, but not on η.

In the situation with bounds as in (2.18), we can choose a sequence $r \to \infty$ and construct a subsequence $\eta \to 0$, such that for a limit function $U \in L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{C})$, there holds that $u_\eta|_{B_r(0)} \rightharpoonup U|_{B_r(0)}$ weakly in $L^2(B_r(0))$, for every radius r. Theorem 1 can be applied in each such ball $B_r(0)$ and we obtain that the limit function U is a solution of the effective system (1.14), hence also of system (1.16).

Regarding the outgoing wave condition (1.15) we note that each solution u_η can be represented, outside a large ball $B_r(0)$ with Green's formula through the values of u_η and $\partial_r u_\eta$ on the boundary $\partial B_r(0)$. Since values and derivatives converge strongly on compact subsets, also the limit function U is given by Green's formula through the values of U and $\partial_r U$ on the boundary $\partial B_r(0)$. In particular, U satisfies again the outgoing wave condition (1.15).

By uniqueness for the limit problem, we obtain the convergence of the entire sequence and Theorem 2 is shown.

Derivation of the L^2_{loc}-bound (2.18)**.** We use a contradiction argument. We assume that the solution sequence u_η posesses a radius $r > r_0$ such that the energies are not bounded in a ball $B_r(0)$. We fix the radius and consider on the ball $Q := B_r(0)$ the energies t_η which satisfy $t_\eta \to \infty$ for $\eta \to 0$. We rescale the solutions and consider

$$v_\eta := \frac{1}{t_\eta} u_\eta\,.$$

By linearity, the functions v_η satisfy again the original Helmholtz problem. Furthermore, the functions $v_\eta|_Q$ are bounded in $L^2(Q)$. Concerning the boundary condition we note that, outside the ball $Q = B_r(0)$, the solutions v_η are given by Green's formula with incident field $v^i = u^i/t_\eta \to 0$. In particular, for a subsequence $\eta \to 0$, the functions v_η are locally L^2-bounded and every weak L^2 limit function v on $\mathbb{R}^2$ satisfies again the outgoing wave condition. By the uniqueness result, the limit function vanishes, $v = 0$. On the other hand, on Q, the functions v_η solve the Helmholtz equation and have norm 1. Once that we show that the convergence $v_\eta \to v$ is a strong convergence in $L^2(Q)$, we have derived the desired contradiction.

Outside the structure Σ_η, the gradients of the sequence u_η are bounded by

$$\int_Q a_\eta |\nabla v_\eta|^2 = \int_Q k^2 |v_\eta|^2 + \int_{\partial Q} \partial_r v_\eta \, \bar{v}_\eta \leq C. \tag{2.19}$$

In the inequality we use the normalization of the $L^2(Q)$-norm of v_η and the fact that $\partial_r v_\eta$ and $\bar{v}_\eta$ on ∂Q can be expressed by the Green's formula with their values on a smaller ball. The compactness of the Rellich embedding implies the strong convergence $v_\eta \, \mathbf{1}_{Q \setminus \Sigma_\eta} \to 0$ in $L^2(Q)$. It remains to show

$$\int_Q |v_\eta \, \mathbf{1}_{\Sigma_\eta}|^2 = \sum_j \int_{\Sigma_\eta^j} |v_\eta|^2 \to 0. \tag{2.20}$$

We will show (2.20) with an analysis of the single connected components of Σ_η. We note that a similar approach was used to derive L^p-estimates for homogenization problems in [14], and to study the homogenization of another high contrast problem in [15]. As a preparation, we first consider a rescaled structure and formulate an extension lemma.

Lemma 2.4 (Extension)**.** *Let $M^\eta := (-1/2, 1/2) \times ((-1-h)/\eta, 1/\eta) \subset \mathbb{R}^2$ be a family of domains, depending on a coefficient $\eta \in (0,1)$. For $\gamma \in (0, 1/2)$ we consider subdomains $N^\eta := (-\gamma, \gamma) \times (-h/\eta, 0)$. Then every function $w : M^\eta \setminus N^\eta \to \mathbb{C}$ can be extended into N^η with comparable norm. More precisely, there exists a constant $C > 0$, depending on γ, but not on η, such that the following holds. To every $w \in H^1(M^\eta \setminus N^\eta, \mathbb{C})$, there exists $W \in H^1(M^\eta, \mathbb{C})$, such that*

$$W = w \text{ on } M^\eta \setminus N^\eta, \qquad \|W\|^2_{H^1(M^\eta)} \leq C \|w\|^2_{H^1(M^\eta \setminus N^\eta)} \,. \tag{2.21}$$

Proof. For the proof, one localizes the function w to domains of unit size, $M_k^\eta := (-1/2, 1/2) \times (k-1, k+2)$, $k \in \mathbb{N}$, and a subordinate decomposition of unity, Θ_k. On each subdomain M_k^η, one obtains an extension of $w\Theta_k$ with control of the H^1-norm by a standard construction. Adding the extensions over all k provides the function W. $\square$

We now give the proof for property (2.20). We set, for $j \in \mathbb{Z}$, $\Sigma_\eta^j := (\eta j - \eta\gamma, \eta j + \eta\gamma) \times (-h, 0)$, and use also the neighborhood $M_\eta^j := (\eta j - \eta/2, \eta j + \eta/2) \times (-h-1, 1)$. We denote by $J \subset \mathbb{Z}$ the indices j such that $\Sigma_\eta^j \subset \Sigma$ and may assume without loss of generality that all the sets M_η^j are contained in Q.

We now rescale the single connected component and define, for each $j \in J$,

$$w_j : M^\eta \to \mathbb{C}, \quad w_j(y) := v_\eta((\eta j, 0) + \eta y) \,, \tag{2.22}$$

with large domains M^η as in Lemma 2.4. By that lemma, we find extensions W_j of w_j such that

$$\int_{M^\eta} |W_j|^2 + |\nabla_y W_j|^2 \leq C \left\{ \int_{M^\eta \setminus N^\eta} |w_j|^2 + |\nabla_y w_j|^2 \right\} .$$

By rescaling the extensions W_j back to the small domains M_η^j, we obtain a function $V_\eta : Q \to \mathbb{C}$ that extends $v_\eta|_{Q\setminus\Sigma_\eta}$ to all of Q. This extension satisfies, for every $j \in J$,

$$\begin{aligned} \left\{ \int_{M_\eta^j} \frac{1}{\eta^2} |V_\eta|^2 + |\nabla V_\eta|^2 \right\} &= \int_{M^\eta} |W_j|^2 + |\nabla_y W_j|^2 \\ \leq C \left\{ \int_{M^\eta \setminus N^\eta} |w_j|^2 + |\nabla_y w_j|^2 \right\} &\leq C \left\{ \int_{M_\eta^j \setminus \Sigma_\eta^j} \frac{1}{\eta^2} |v_\eta|^2 + |\nabla v_\eta|^2 \right\} . \end{aligned} \tag{2.23}$$

In order to obtain (2.20), we use $v_\eta - V_\eta$ as a test-function in the Helmholtz equation for v_η. Since we can integrate by parts without boundary terms, we obtain

$$\int_Q a_\eta \nabla v_\eta \nabla (v_\eta - V_\eta) = k^2 \int_Q v_\eta (v_\eta - V_\eta).$$

We recall that $v_\eta - V_\eta$ vanishes outside Σ_η and that $a_\eta = -\eta^2/\sigma^2$ holds on Σ_η. Since the real part of σ^2 was assumed to be positive, we re-write the equation as

$$\int_{\Sigma_\eta} \frac{\eta^2}{\sigma^2} |\nabla v_\eta|^2 + k^2 \int_{\Sigma_\eta} |v_\eta|^2 = \int_{\Sigma_\eta} \frac{\eta^2}{\sigma^2} \nabla v_\eta \nabla V_\eta + k^2 \int_{\Sigma_\eta} v_\eta V_\eta \,.$$

With an application of the Cauchy-Schwarz inequality and the Young inequality, we obtain

$$\Re \int_{\Sigma_\eta} \frac{\eta^2}{\sigma^2} |\nabla v_\eta|^2 + k^2 \int_{\Sigma_\eta} |v_\eta|^2 \leq C \left\{ \int_{\Sigma_\eta} \eta^2 |\nabla V_\eta|^2 + \int_{\Sigma_\eta} |V_\eta|^2 \right\} .$$

We now exploit estimate (2.23) for the extensions V_η, which we sum over $j \in J$ and multiply by η^2, and obtain

$$\Re \int_{\Sigma_\eta} \frac{\eta^2}{\sigma^2} |\nabla v_\eta|^2 + k^2 \int_{\Sigma_\eta} |v_\eta|^2 \leq C \int_{Q\setminus\Sigma_\eta} |v_\eta|^2 + \eta^2 |\nabla v_\eta|^2 \,.$$

In our last step, we use the a priori estimate (2.19) with its consequence, that the L^2-norm outside Σ_η converges to 0. We obtain that the right hand side vanishes in the limit and thus, in particular, the convergence (2.20). This concludes the proof of Theorem 2.

3 Transmission properties of the effective layer

Our main Theorems 1 and 2 provide the effective Helmholtz equation describing the gratings. We have thus determined the optical properties of the layer $\Sigma \subset \Omega$. Our aim in this last section is to calculate the reflection and transmission properties of the effective structure.

For simplicity, we restrict to normal incidence and assume that the effective structure extends to infinity, i.e. $\Sigma = \mathbb{R} \times (-h, 0)$. Eventually, our aim is to construct a solution of the form $e^{-ikx_2} + Re^{ikx_2}$ above the layer $\Sigma \subset \Omega$, i.e. for $x_2 > 0$. This describes a normally incident wave with unit amplitude together with a reflected wave Re^{ikx_2}, where $R \in \mathbb{C}$ expresses amplitude and phase shift of the reflected wave. Below the layer Σ, i.e. for $x_2 < -h$, the solution should have the form of an outgoing transmitted wave $\tilde{T}e^{-ikx_2}$. Our aim is to determine $R, \tilde{T} \in \mathbb{C}$ and, in particular, the amplitude $|\tilde{T}|$ of the transmitted wave.

Within the layer Σ, i.e. for $x_2 \in (-h, 0)$, every solution u of the effective system must satisfy the relation $\alpha\partial_{x_2}^2 u(x_1, x_2) = -\alpha\tau^2 k^2 u(x_1, x_2)$. Using the slightly more convenient modified transmission coefficient $T = \tilde{T}e^{ikh}$ with $|T| = |\tilde{T}|$, we hence seek a solution $u : \Omega \to \mathbb{C}$ of the form

$$u(x_1, x_2) = \begin{cases} e^{-ikx_2} + Re^{ikx_2} & \text{for } x_2 > 0, \\ A_1 \cos(\tau k x_2) + A_2 \sin(\tau k x_2) & \text{for } 0 > x_2 > -h, \\ Te^{-ik(x_2+h)} & \text{for } -h > x_2. \end{cases} \tag{3.1}$$

It remains to determine the complex constants R, T, A_1 and A_2 from the non-standard transmission conditions across the interfaces $x_2 = 0$ and $x_2 = -h$.

The transfer matrix. In the transfer matrix formalism one regards the slab Σ as an object that induces a map between the solution characteristics on the upper boundary $x_2 = 0$ and the corresponding values on the lower boundary $x_2 = -h$. The linearity of the equations allows to describe this map by a matrix $M \in \mathbb{C}^{2\times 2}$,

$$M : \begin{pmatrix} u(0) \\ \partial_{x_2} u(0) \end{pmatrix} \mapsto \begin{pmatrix} u(-h) \\ \partial_{x_2} u(-h) \end{pmatrix} . \tag{3.2}$$

We emphasize that the matrix M is independent of the parameter x_1. In equation (3.2), the solution derivatives $\partial_{x_2} u(0)$ and $\partial_{x_2} u(-h)$ indicate the corresponding traces from outside the structure Σ. The transfer matrix can be calculated explicitly.

First column of M. To calculate the first column of M, we investigate a solution $U : \Omega \to \mathbb{C}$ of the effective system with the properties that $U|_{x_2=0+} = 1$ and $\partial_{x_2} U|_{x_2=0+} = 0$. We have to determine parameters A_1 and A_2, such that, setting $U(x_1, x_2) = A_1 \cos(\tau k x_2) + A_2 \sin(\tau k x_2)$ for $x_2 \in (-h, 0)$, the function U satisfies the transmission conditions across $x_2 = 0$. The transmission conditions imply continuity of U and continuity of $e_2 \cdot (a_{\text{eff}} \nabla U)$, hence

$$\begin{aligned} 1 &= A_1 \cos(\tau k x_2) + A_2 \sin(\tau k x_2)|_{x_2=0} = A_1, \\ 0 &= \alpha\partial_{x_2}[A_1 \cos(\tau k x_2) + A_2 \sin(\tau k x_2)]|_{x_2=0} = \alpha\tau k A_2 \,. \end{aligned}$$

We find $A_1 = 1$ and $A_2 = 0$ and hence for the solution U at $-h - 0$ the value and the derivative

$$\begin{aligned} U|_{x_2=-h-} &= U|_{x_2=-h+} = A_1 \cos(-\tau k h) + A_2 \sin(-\tau k h) = \cos(\tau k h), \\ \partial_{x_2} U|_{x_2=-h-} &= \alpha\partial_{x_2} U|_{x_2=-h+} = \alpha\tau k \sin(\tau k h). \end{aligned}$$

These two values provide the first column of M.

Second column of M. The calculation of the second column follows the same lines, the solution for $x_2 \in (-h, 0)$ reads $U(x_2) = (\alpha\tau k)^{-1}\sin(\tau k x_2)$. We obtain

$$M = \begin{pmatrix} \cos(\tau kh) & -(\alpha\tau k)^{-1}\sin(\tau kh) \\ \alpha\tau k \sin(\tau kh) & \cos(\tau kh) \end{pmatrix}. \tag{3.3}$$

We recall that the parameter $\alpha = 1 - 2\gamma \in \mathbb{R}$ of the effective system stands for the relative slit width.

The transmission coefficient. To calculate the transmission coefficient T, we consider again the incoming wave together with the reflected wave, $e^{-ikx_2} + Re^{ikx_2}$ for $x_2 > 0$. In the spirit of the transfer matrix formalism, this means that we study at $x_2 = 0+$ the value-derivative-vector $(1 + R, ik(-1 + R))$. The matrix M maps these two data onto the corresponding values at $x_2 = -h-$, and, referring to (3.1), we want them to be $(T, -ikT)$. We therefore have to solve the linear system

$$M \cdot \begin{pmatrix} 1 + R \\ ik(-1 + R) \end{pmatrix} = T \begin{pmatrix} 1 \\ -ik \end{pmatrix}. \tag{3.4}$$

In order to eliminate R, we introduce two vectors $v \in \mathbb{C}^2$ and $w \in \mathbb{C}^2$ as

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} := M \cdot \begin{pmatrix} 1 \\ ik \end{pmatrix}, \quad w := v^\perp = \begin{pmatrix} -v_2 \\ v_1 \end{pmatrix} = \begin{pmatrix} -\alpha\tau k \sin(\tau kh) - ik\cos(\tau kh) \\ \cos(\tau kh) - i(\alpha\tau)^{-1}\sin(\tau kh) \end{pmatrix}.$$

The vectors are constructed such that the multiplication of (3.4) with the vector w eliminates R. We obtain

$$\begin{aligned} w \cdot M \cdot \begin{pmatrix} 1 \\ -ik \end{pmatrix} = T\, w \cdot \begin{pmatrix} 1 \\ -ik \end{pmatrix} &= T\left(-2ik\cos(\tau kh) - [\alpha\tau k + (\alpha\tau)^{-1}k]\sin(\tau kh)\right) \\ &= -ikT\, \left(2\cos(\tau kh) - i[\alpha\tau + (\alpha\tau)^{-1}]\sin(\tau kh)\right). \end{aligned}$$

We evaluate the left hand side,

$$\begin{aligned} w \cdot M \cdot \begin{pmatrix} 1 \\ -ik \end{pmatrix} &= \begin{pmatrix} -\alpha\tau k \sin(\tau kh) - ik\cos(\tau kh) \\ \cos(\tau kh) - i(\alpha\tau)^{-1}\sin(\tau kh) \end{pmatrix} \cdot \begin{pmatrix} \cos(\tau kh) + i(\alpha\tau)^{-1}\sin(\tau kh) \\ \alpha\tau k\sin(\tau kh) - ik\cos(\tau kh) \end{pmatrix} \\ &= -2ik\cos^2(\tau kh) - 2ik\sin^2(\tau kh) = -2ik. \end{aligned}$$

This provides the explicit relation for $T \in \mathbb{C}$,

$$T = \left(\cos(\tau kh) - i\frac{\alpha\tau + (\alpha\tau)^{-1}}{2}\sin(\tau kh)\right)^{-1}. \tag{3.5}$$

We have thus determined T in dependence of the wave number k, the layer height h, the relative slit size α, and the surface plasmon resonance factor τ. The latter has been introduced in (1.11). We emphasize that T depends on k also implicitly, since τ depends on k. The graph of $|T|^2$ against the wave number k can be evaluated from the explicit relations (1.11) and (3.5), see Figure 2. We observe the high transmission effect for wave numbers that are in resonance with the surface plasmon wave. The effect can already be deduced from the transfer matrix M of (3.3): for $\cos(\tau kh) \approx 1$, the transfer matrix M is approximately the identity, and a transfer matrix $M = \mathrm{id}$ implies perfect transmission.

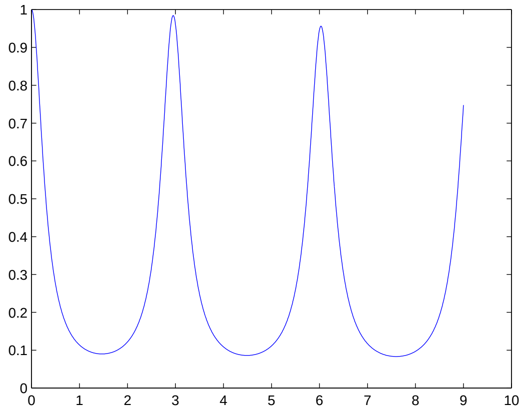

Figure 2: The transmission coefficient $|T|^2$ in dependence of the non-dimensional wave-number k. The non-dimensional geometrical quantities are $\eta = 7/6$, $\alpha = 1/7$, and $\gamma = (1-\alpha)/2 = 3/7$ as mentioned in (1.7), the frequency independent relative permittivity $\varepsilon_\eta = (0.12+3.7i)^2$ is obtained by setting $\sigma = 4.32 - 0.14i$. Pronounced peaks are clearly visible. The numerical experiments of [8] observed resonance at $k = 2.51$, our theory predicts a peak at $k = 2.95$. We recall at this point that our theory investigates the thin-slit limit $\eta \to 0$, such that even a qualitative agreement is remarkable for the above experimental parameters.

We recall that, for a lossless metal of negative permittivity, the numbers σ and τ are real and positive. In this case, perfect transmission necessarily occurs for appropriate wave-numbers k.

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