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Nonparametric comparison of quantile curves: a stochastic process approach

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Abstract

A new test for comparing conditional quantile curves is proposed which is able to detect Pitman alternatives converging to the null hypothesis at the optimal rate. The basic idea of the test is to measure differences between the curves by a process of integrated nonparametric estimates of the quantile curve. We prove weak convergence of this process to a Gaussian process and study the finite sample properties of a Kolmogorov-Smirnov test by means of a simulation study.

AMS Subject Classification:

Keywords and Phrases: nonparametric analysis of covariance, quantile regression, crossing quantile curves, monotone rearrangements

1 Introduction

An important problem in statistics is determining whether there exist differences between several populations. Typical situations in medicine or economics include such challenging problems as testing for gender differences with respect to therapy or race discrimination in earnings functions, respectively. Similarly, in other fields like engineering or agriculture it is often of interest to choose between two complementary courses of action. If the response depends on predictors these problems are related to comparing regression curves. In a parametric context the comparison of regression curves is well studied while in a nonparametric setup this problem has only been discussed more recently. Hall and Hart (1990) proposed a Cramer-von-Mises type statistic while Delgado (1993) studied a Kolmogorov-Smirnov type statistic. Alternatively, King et al. (1991) used the mean squared difference between nonparametric regression estimates [see also Dette and Neumeyer (2001)]. Young and Bowman (1995) compared several nonparametric regressions depending on a one-dimensional random variable with normal residuals and Kulasekera (1995) used empirical processes for this purpose [see also Cabus (1998) or Neumeyer and Dette (2003) for some alternative procedures based on empirical processes].

Usually the inference in the cited literature refers to a comparison between the conditional means $E[Y_1|X_1]$ and $E[Y_2|X_2]$ of the responses, say Y_1 and Y_2, given the predictors X_1 and X_2 in two (or more) populations. However, in many applications this quantity is not of primary importance, rather it is more of interest whether there exist differences between the conditional quantiles of the populations. This naturally leads to the comparison of quantile curves, which have been introduced by Koenker and Bassett (1978) as a supplement to least squares methods and yield a great extension of parametric and nonparametric regression methods [see Yu et al. (2003) or Koenker (2005)]. In contrast to mean regression the quantile curves are robust with respect to outliers and require weaker assumptions on the data generating process. Several nonparametric estimation methods for quantile curves have been proposed in the recent literature [see e.g. Yu and Jones (1997, 1998), De Gooijer and Zerom (2003) or Horowitz and Lee (2005) among others]. On the other hand, the problem of comparing (nonparametric) conditional quantile curves has only been investigated marginally in statistics. Batalgi et al. (1996) and Lavergne (2001) considered median regression. A test for comparing other conditional quantile curves has been investigated recently by Sun (2006) and Dette et al. (2011). These authors used L^2-type statistics based on nonparametric estimates and as a consequence the corresponding tests are not consistent with respect to local alternatives converging to the null hypothesis with a rate $1/\sqrt{n}$, where n denotes the total sample size.

The purpose of the present paper is the construction of a test that can detect such alternatives using an empirical process approach. We propose to consider stochastic processes of the integrated differences between the nonparametric estimates for the quantile curves from the different samples. In Section 2 we introduce the necessary notation and give a motivation of the stochastic process in the construction of our test statistic. Section 3 contains our main results and we prove weak convergence to a Gaussian process for two estimators of the quantile curves. Some simulation results are presented in Section 4, where we investigate finite sample properties of the Kolmogorov-Smirnov statistic and compare the test with the procedure proposed by Dette et al. (2011). Finally, some technical details regarding the proofs of the results in Section 3 are given in the appendix [see Section 5].

2 An empirical process build from quantile estimates

We consider J independent samples, say

$$\{(X_{i1}, Y_{i1})_{i=1}^{n_1}\}, \ldots, \{(X_{iJ}, Y_{iJ})_{i=1}^{n_J}\}, \tag{2.1}$$

where for each $j = 1, \ldots, J$ the random variables $(X_{1j}, Y_{1j}), \ldots, (X_{n_j j}, Y_{n_j j})$ are independent identically distributed. We assume that the explanatory variable X_{ij} has a continuous and positive density, say f_j, on the interval $[0, 1]$. The restriction to a one dimensional predictor is made for the sake of a transparent presentation, and the general case will be briefly mentioned in Remark 3.6. Throughout this paper let $F_j(y|x) = P(Y_{1j} \leq y|X_{1j} = x)$ denote the conditional distribution function of the random variable Y_{ij} given $X_{ij} = x$, and we assume that it has a density, say $f_{j,Y}(y|x)$, which is continuous in both arguments. For fixed $p \in (0, 1)$ let $F_j^{-1}(p|x)$ denote the corresponding conditional quantile function ($j = 1, \ldots, J$). We are interested in the hypothesis that the data can be pooled for inference regarding the conditional p-quantile, that is

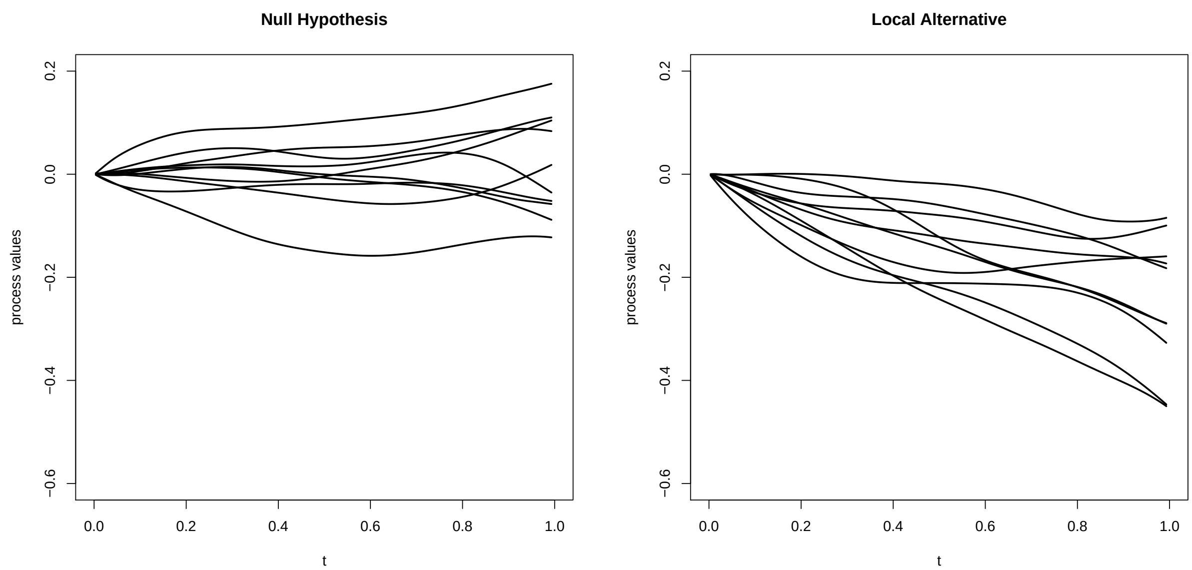
$$H_0 : F_1^{-1}(p|\cdot) = \cdots = F_J^{-1}(p|\cdot) \quad \text{versus} \quad H_1 : F_i^{-1}(p|\cdot) \neq F_j^{-1}(p|\cdot) \text{ for some } i \neq j. \tag{2.2}$$

In order to answer this question, we first define processes $(S_t^{i,j})_{t\in[0,1]}$ in the following way

$$S_t^{i,j} = \int_0^t (\hat{F}_i^{-1}(p|x) - \hat{F}_j^{-1}(p|x))\hat{w}_{ij}(x)dx, \tag{2.3}$$

where $\hat{F}_i^{-1}(p|x)$ is an estimate of the conditional quantile function and $\hat{w}_{ij}(x)$ are weight functions. Intuitively these random functions should approximately vanish for all t and for all pairs (i, j) if the null hypothesis of equal conditional quantile curves holds (we will specify this in the next section). A rigorous statement for this property will be given in Section 3 and typical realizations of the process $(S_t^{1,2})_{t\in[0,1]}$ are depicted in Figure 1, where we show 10 simulated processes of the integrated difference between the estimates (sample sizes 100 in both groups) of the 50%

Figure 1: 10 *simulated processes defined in* (2.3) *under the null hypothesis* (2.4) *and the alternative* (2.5)



conditional quantile curves under the null hypothesis

$$F_1^{-1}(\frac{1}{2}, x) = F_2^{-1}(\frac{1}{2}, x) = \cos(\pi t) \tag{2.4}$$

(left panel) and the alternative

$$F_2^{-1}(\frac{1}{2}, x) = F_1^{-1}(\frac{1}{2}, x) + 0.25 \tag{2.5}$$

(right panel).

For the estimation of the conditional quantile curves in the example and throughout this paper we use the nonparametric estimator proposed in Dette and Volgushev (2008) which is given by

$$\widehat{F}_j^{-1}(p|x) = G^{-1}(\widehat{H}_j^{-1}(p|x)), \tag{2.6}$$

where $G : \mathbb{R} \to [0, 1]$ is a strictly increasing distribution function and

$$\widehat{H}_j^{-1}(p|x) = \frac{1}{N_j h_{j,d}} \sum_{k=1}^{N_j} \int_{-\infty}^{p} K_d \left(\frac{\widehat{F}_j \left(G^{-1}(\frac{k}{N_j})|x \right) - u}{h_{j,d}} \right) du. \tag{2.7}$$

Here $h_{j,d}$ is a bandwidth, K_d a nonnegative symmetric kernel, $n_j = O(N_j)$ and $\widehat{F}_j(y|x)$ is a nonparametric estimate of the conditional distribution function of Y_{ij}, or more precisely

$$\widehat{F}_j(y|x) = \sum_{i=1}^{n_j} \tilde{w}_{ij}(x) I\{Y_{ij} \leq y\}, \tag{2.8}$$

where $\tilde{w}_{ij}$ are either the Nadaraya-Watson weights, i.e.

$$\tilde{w}_{ij}(x) = \frac{K_r\left(\frac{X_{ij}-x}{h_{j,r}}\right)}{\sum_{l=1}^{n_j} K_r\left(\frac{X_{lj}-x}{h_{j,r}}\right)} \tag{2.9}$$

or the local linear weights, i.e.

$$\tilde{w}_{ij}(x) = \frac{K_r\left(\frac{X_{ij}-x}{h_{j,r}}\right)(S_{j,2}(x) - (x - X_{ij})S_{j,1}(x))}{S_{j,2}(x)S_{j,0}(x) - S_{j,1}^2(x)}, \tag{2.10}$$

$$S_{j,k}(x) = \sum_{l=1}^{n_j} K_r\Big(\frac{x - X_{lj}}{h_{j,r}}\Big)(x - X_{lj})^k \quad k = 0, 1, 2,$$

which were used in the example depicted in Figure 1. In (2.9) and (2.10) K_r denotes a nonnegative symmetric kernel and h_r is a bandwidth which converges to 0 with increasing sample size. In the following discussion we restrict ourselves mainly to the case of $J = 2$ samples, and we write S_t^{NW} for $S_t^{(1,2)}$ if we use Nadaray-Watson-weights in the initial estimate of the conditional distribution function and S_t^{LL} if we use local linear weights, respectively. The case of $J > 2$ samples is briefly discussed in Remark 3.5. The weight functions $\hat{w}_{ij}(x)$ in (2.3) are given by

$$\hat{w}_{ij}(x) = \frac{1}{n_i n_j h_{j,r}^2} S_{i,0}(x)S_{j,0}(x), \tag{2.11}$$

in the case of Nadaraya-Watson weights and by

$$\hat{w}_{ij}(x) = \frac{1}{n_i^2 n_j^2 h_{j,r}^8}(S_{i,2}S_{i,0} - S_{i,1}^2)(x)(S_{j,2}S_{j,0} - S_{j,1}^2)(x) \tag{2.12}$$

in the case of local linear weights, respectively.

Remark 2.1 Note that the estimation of the conditional curves depends on the choice of the distribution function G. It will be demonstrated in the following section in Theorem 3.2 and 3.3 that this choice does not play a role asymptotically. Some recommendations on how to choose G can be found in Dette and Volgushev (2008). In this reference it is also demonstrated that the impact of the choice of G is negligible in finite samples, provided that the interval

$[G^{-1}(0.05), G^{-1}(0.95)]$ contains most of the data $\{Y_j \mid X_j - x \mid < h_d\}$ [see Dette and Volgushev (2008) for more details]. A similar statement applies to the choice of the bandwidth h_d in (2.7). As long as this bandwidth is chosen sufficiently 'small', its effect is not visible in the resulting estimate [see Dette et al. (2006) or Dette and Volgushev (2008) for more details].

3 Weak convergence

For the investigations of the asymptotic properties of the statistics S_t^{NW} and S_t^{LL} we assume that all bandwidths $h_{j,r}$ and $h_{j,d}$ in the different estimators coincide, that is

$$h_{j,d} = h_d \quad \text{and} \quad h_{j,r} = h_r \qquad j = 1, 2. \tag{3.1}$$

This "simplification" is made for the sake of clear exposition and all results remain true in the general case with a substantial additional amount of notation. Besides the assumptions stated in Section 2, we require the following basic assumptions:

(A) The function $G : \mathbb{R} \to [0, 1]$ is strictly increasing, twice continuously differentiable with bounded second derivative, such that G^{-1} has also a bounded second derivative on every interval $[a, b]$ with $0 < a < b < 1$.

(B) The density f_j of X_{1j} is twice differentiable and f_j'' is bounded.

(C) The conditional distribution function $F_j(y|x)$ is three times differentiable with respect to both arguments. The kth partial derivatives with respect to y or x are denoted by ∂_1^k or ∂_2^k, respectively, and we assume that the derivatives $\partial_2(F_j(y|x))$, $\partial_2^2(F_j(y|x))$, $\partial_2^3(F_j(y|x))$ and $\partial_1^3(F_j(y|x))$ are uniformly bounded.

(D) The conditional quantile function $F_j^{-1}(y|x)$ is twice differentiable with respect to y in a neighborhood of p with bounded second derivative.

(E) The kernels K_r and K_d are symmetric, bounded, nonnegative and their support is given by the interval $[-1, 1]$. Additionally, K_d is twice continuously differentiable on the interval $(-1, 1)$ and K_d'' is Lipschitz continuous. For $i \in \mathbb{N}$ we use the notation

$$\mu_k(K) = \int K(u)u^k du$$

and assume $\mu_0(K_r) = \mu_0(K_d) = 1$.

(F) The bandwidths h_d and h_r satisfy

$$nh_r^4 = o(1), \quad h_d = o(h_r). \tag{3.2}$$

(G) For $j=1,\dots,J$ the sample size n_j relative to the total sample size $n=\sum_{j=1}^J n_j$ fulfils

$$\lim_{n\to\infty}\frac{n_j}{n}=a_j\in(0,1). \tag{3.3}$$

Our first result specifies the asymptotic bias of the process defined in (2.3).

Lemma 3.1 *Let assumptions (A) - (E) be satisfied. Using the notation introduced in Section 2 the following statements are true.*

(a) Under the null hypothesis of equal quantile curves we have

$$E\left[S_t^{NW}\right]=h_r^2B_t^{NW}+o\left(\frac{1}{\sqrt{n}}\right)$$

with

$$B_t^{NW}=\int_0^t(f_1(x)C_2(x)-f_2(x)C_1(x))dx,$$

where the term $C_j(x)$ is given by

$$\begin{aligned}C_j(x)= \quad & \partial_1(F_j^{-1}(p|x))\mu_2(K_r)\Big(f_j^{'}(x)\partial_2\ (F_j(y|x))\Big|_{(y,x)=(F_j^{-1}(p|x),x)} \\ & +\frac{1}{2}f_j(x)\partial_2^2(F_j(y|x))\Big|_{(y,x)=(F_j^{-1}(p|x),x)}\Big)\end{aligned}$$

and

$$E\left[S_t^{LL}\right]=h_r^2B_t^{LL}+o\left(\frac{1}{\sqrt{n}}\right)$$

with

$$B_t^{LL}=\int_0^t f_1^2(x)f_2^2(x)(\tilde{C}_1(x)-\tilde{C}_2(x))dx,$$

where the term $\tilde{C}_j(x)$ is given by

$$\tilde{C}_j(x)=\frac{1}{2}\partial_1(F_j^{-1}(p|x))\mu_2^3(K_r)\partial_2^2(F_j(y|x))\Big|_{(y,x)=(F_j^{-1}(p|x),x)}.$$

(b) Under the alternative of unequal quantile curves we have

$$E\left[S_t^{NW}\right]=\int_0^t(F_1^{-1}(p|x)-F_2^{-1}(p|x))f_1(x)f_2(x)dx+O(h_r^2)$$

and

$$E\left[S_t^{LL}\right]=\mu_2^2(K_r)\int_0^t(F_1^{-1}(p|x)-F_2^{-1}(p|x))f_1^2(x)f_2^2(x)dx+O(h_r^2).$$

It follows from the previous lemma that the processes $(S_t^{NW})_{t\in[0,1]}$ and $(S_t^{LL})_{t\in[0,1]}$ are useful for testing the equality of quantile curves. For example, tests could be based on the Kolmogorov-Smirnov-type test statistics

$$K_n^{NW} = \sup_{t\in[0,1]} \mid S_t^{NW} \mid ; \tag{3.4}$$

$$K_n^{LL} = \sup_{t\in[0,1]} \mid S_t^{LL} \mid \; . \tag{3.5}$$

Our next results state the weak convergence of the process $(S_t^{NW})_{t\in[0,1]}$ and $(S_t^{LL})_{t\in[0,1]}$ in $C[0,1]$ and as a consequence of the continuous mapping theorem we obtain weak convergence of the statistics defined in (3.4) and (3.5).

Theorem 3.2 *Let assumptions (A) - (E) be satisfied. Then the process*

$$\sqrt{n}\left(S_t^{NW}\right)_{t\in[0,1]}$$

has continuous sample paths. Moreover, under the null hypothesis of equal conditional quantile curves it converges weakly to a centered Gaussian process with covariance function

$$H(s,t) = p(1-p)\int_0^{s\wedge t}\left(\left(\partial_1(F_2^{-1}(p|x))\right)^2 \frac{f_1^2(x)f_2(x)}{a_2} + \left(\partial_1(F_1^{-1}(p|x))\right)^2 \frac{f_1(x)f_2^2(x)}{a_1}\right) dx.$$

The proof of this theorem is complicated and therefore deferred to the Appendix. A similar result as in Theorem 3.2 holds for the process $(S_t^{LL})_{t\in[0,1]}$ and this is stated in the next Theorem.

Theorem 3.3 *Let assumptions (A) - (E) be satisfied. Then the process*

$$\sqrt{n}\left(S_t^{LL}\right)_{t\in[0,1]}$$

has continuous sample paths. Moreover, under the null hypothesis of equal conditional quantile curves it converges weakly to a centered Gaussian process with covariance function

$$\tilde{H}(s,t) = p(1-p)\mu_2^4(K_r)\int_0^{s\wedge t}\left(\left(\partial_1(F_2^{-1}(p|x))\right)^2 \frac{f_1^4(x)f_2^3(x)}{a_2} + \left(\partial_1(F_1^{-1}(p|x))\right)^2 \frac{f_1^3(x)f_2^4(x)}{a_1}\right) dx.$$

Remark 3.4 In the case of local alternatives of the form

$$F_2^{-1}(p|x) = F_1^{-1}(p|x) + \frac{g(x,p)}{\sqrt{n}},$$

where g denotes an integrable function, the processes $\sqrt{n}(S_t^{NW})_{t\in[0,1]}$ and $\sqrt{n}\left(S_t^{LL}\right)_{t\in[0,1]}$ also converge weakly in the space $C[0,1]$. The limiting processes are non-centered Gaussian processes with covariance functions given by Theorem 3.2 and Theorem 3.3, respectively, and expectation functions

$$E^{NW}(t) = \int_0^t g(x,p)f_1(x)f_2(x)dx$$

in the case of Nadaraya-Watson weights and

$$E^{LL}(t) = \mu_2^2(K_r)\int_0^t g(x,p)f_1^2(x)f_2^2(x)dx$$

in the case of local linear weights. These results follow by a careful inspection of the proof of Theorem 3.2 and Theorem 3.3, respectively. Thus, in contrast to the tests of Sun (2006) and Dette et al. (2011), the proposed test is consistent against local alternatives converging to the null hypothesis with rate $1/\sqrt{n}$.

Remark 3.5 The results above are easily generalized to the case of $J > 2$ groups. In this case the multidimensional process

$$\sqrt{n}(S_t^{1,2},\ldots,S_t^{J-1,J})_{t\in[0,1]}$$

with $S_t^{i,j} = \int_0^t(\hat{F}_i^{-1}(p|x) - \hat{F}_j^{-1}(p|x))\hat{w}_{ij}(x)dx$ for $i < j$ converges weakly to a multidimensional centered Gaussian process. The off-diagonal entries in the covariance function matrix corresponding to $S_t^{i,j}$ and $S_s^{i,k}$ are given by

$$\frac{p(1-p)}{a_i}\int_0^{s\wedge t}\left(\partial_1(F_i^{-1}(p|x))\right)^2 f_i(x)f_j(x)f_k(x)dx$$

if Nadaraya-Watson-weights are used in the initial estimate of the conditional distribution function and by

$$\mu_2^4(K_r)\frac{p(1-p)}{a_i}\int_0^{s\wedge t}\left(\partial_1(F_i^{-1}(p|x))\right)^2 f_i^3(x)f_j^2(x)f_k^2(x)dx$$

in the local linear case. The entries of the covariance function matrix corresponding to $S_t^{j,i}$ and $S_s^{k,i}$ are the same as above. Those corresponding to $S_t^{i,j}$ and $S_s^{k,i}$ and $S_t^{j,i}$ and $S_s^{i,k}$, respectively, are the same as above but with opposite sign. All other covariances are 0.

Remark 3.6 Note that the results of this section can easily be generalized to a multivariate predictor, by simply using a multivariate Nadaraya-Watson or local linear estimate of the conditional distribution function in the initial step as described e.g. in Härdle et al. (2004). The details are omitted for the sake of brevity. However, it should be mentioned here that some care is necessary if the test based on S_t is applied in the case of a multivariate predictor, because of the curse of dimensionality. If $d \geq 4$ it is usually difficult to estimate the conditional quantile curve with sufficient precision, such that a test for equality between conditional quantile curves with a high dimensional predictor will not be informative.

4 Finite sample properties

In order to investigate the performance of the proposed test for finite samples, we have performed a small simulation study. Because the limiting processes derived in Section 3 depend on several unknown quantities of the data generating process which are difficult to estimate we propose to use a smoothed residual bootstrap to obtain critical values. To be precise, let

$$\widehat{U}_{ij} = Y_{ij} - \widehat{F}_j^{-1}(p|X_{ij}) \quad (i = 1, \dots, n_j;\ j = 1, \dots, J) \tag{4.1}$$

denote the estimated quantile-residuals, where $\widehat{F}_j^{-1}(p|\cdot)$ is the estimator of the $p-$th quantile-function, calculated from the $j-$th sample, with bandwidths $h_{j,r,B}$ and $h_{j,d,B}$ [for the definition of the estimator, see equations (2.7) - (2.10)]. We now randomly draw with replacement from the estimated residuals in each sample (name the resulting random variables U_{ij}^*) and add normally distributed random variables τ_{ij} independent conditionally on $\widehat{U}_j = (\widehat{U}_{1j}, \dots, \widehat{U}_{n_j j})$, with expectation $\mu_p(\delta, \widehat{U}_j)$ and variance δ^2, where $\mu_p(\delta, \widehat{U}_j)$ is chosen to guarantee that the distribution of $U_{ij}^B = U_{ij}^* + \tau_{ij}$ has $p-$quantile 0 conditional on the data [note that the value of $\mu_p(\delta, \widehat{U}_j)$ may vary depending on the values of the estimated residuals]. The bootstrap data (X_{ij}^B, Y_{ij}^B) are finally defined as

$$\begin{aligned} X_{ij}^B &= X_{ij}, \\ Y_{ij}^B &= \widehat{F}^{-1}(p|X_{ij}) + U_{ij}^B, \end{aligned} \tag{4.2}$$

where $\widehat{F}^{-1}(p|\cdot)$ is an estimate of the conditional quantile function calculated from the pooled data. The bandwidths used for the calculation of the estimate $\widehat{F}^{-1}(p|\cdot)$ are denoted by $h_{r,B}$ and $h_{d,B}$. From the bootstrap sample we calculate the bootstrap statistic T_n^*, and the $\alpha-$quantile of the test statistic T_n is estimated on the basis of R bootstrap replications. More precisely, if t^* denotes the $(1-\alpha)$-quantile of the bootstrap sample $T_n^{*(1)}, \dots, T_n^{*(R)}$, the null hypothesis is rejected if

$$T_n > t^*. \tag{4.3}$$

In the simulation study we compared $J = 2$ quantile curves with the Kolmogorov-Smirnov test. In all nonparametric estimates we used local-linear weights and the Epanechnikov kernel

$$K_r(x) = K_d(x) = \frac{3}{4}(1 - x^2) I_{[-1,1]}(x). \tag{4.4}$$

The estimates $\widehat{F}_j^{-1}(p|\cdot)$ in (4.1) were calculated using the bandwidths

$$h_{j,r,B} = \frac{3}{2}\left(\frac{p(1-p)}{\phi(\Phi^{-1}(p))^2}\right)^{.35}\left(\frac{\sigma_j^2}{n_j}\right)^{.35}, \quad h_{j,d,B} = h_{j,r,B}^{1.3} \tag{4.5}$$

where σ_j is the variance of the jth sample, and ϕ, Φ denote the density and distribution function of the standard normal distribution, respectively. The estimate $\widehat{F}^{-1}(p|\cdot)$ from the pooled sample in (4.2) was calculated with the bandwidth

$$h_{r,B} = \left(\frac{p(1-p)}{\phi(\Phi^{-1}(p))^2}\right)^{1/9}\left(\frac{(\sigma_1^2+\sigma_2^2)}{2(n_1+n_2)}\right)^{1/9}, \quad h_{d,B} = h_{r,B}^{1.3}. \tag{4.6}$$

Finally, the test statistic T_n, $T_n^{*(1)}, \ldots, T_n^{*(R)}$ were calculated with the bandwidth

$$h_{j,r} = \left(\frac{p(1-p)}{\phi(\Phi^{-1}(p))^2}\right)^{1/5}\left(\frac{\sigma_j^2}{n_j}\right)^{1/5}, \quad h_{j,d} = h_{j,r}^{1.3}$$

(see also Yu and Jones (1998), who used a similar bandwidth). Heuristically, the choice of these bandwidths can be motivated as follows. The bandwidth $h_{j,r,B}$ used for the calculation of the residuals should be relatively small because otherwise we would introduce additional bias. On the other hand the bandwidth used for the pooled data should be relatively large, because the derivatives of the corresponding quantile estimator should converge to the corresponding derivatives of the underlying quantile curve with a reasonable rate [see also Härdle and Marron (1991) and Sun (2006)]. Finally, for choosing $h_{j,d}, h_{d,B}, h_{j,d,B}$ we follow the recommendations of Dette and Volgushev (2008).

The choice of the distribution function G is not very critical [see Dette and Volgushev (2008)], and a normal distribution with mean μ_G and variance σ_G^2 was used, where μ_G and σ_G^2 were chosen as the sample mean and variance of $Y_{1j}, ..., Y_{n_j j}$ for the calculation of $\hat{F}_j^{-1}$, as the sample mean and variance of the pooled data for estimating $\hat{F}^{-1}$ and as the sample mean and variance of $Y_{1j}^B, ..., Y_{n_j j}^B$ for the estimators of the quantile curves in the bootstrap procedure. The data were generated by

$$Y_{ij} = g_j(X_{ij}) + \sigma_j U_{ij} \quad (i = 1, \ldots, n_j;\ j = 1, 2), \tag{4.7}$$

where the random variables X_{ij} were uniformly distributed on the interval $[0, 1]$ and U_{ij} were normally distributed with mean 0 and variance 1. For the smoothing of the bootstrap residuals we used different δ's for each group, i.e.

$$\delta_j = 0.35\frac{p(1-p)}{\phi(\Phi^{-1}(p))^2}\ \sigma_j^{1/2} n_j^{-1/8}, \qquad j = 1, 2. \tag{4.8}$$

All simulation results reported in the following discussion are based on 1000 simulation runs and 99 bootstrap replications. For the simulation of the nominal level of the test, we considered the five models

$$g_1(t) = g_2(t) = t^2, \tag{4.9}$$

$$g_1(t) = g_2(t) = \cos(\pi t), \tag{4.10}$$

$$g_1(t) = g_2(t) = 1, \tag{4.11}$$

$$g_1(t) = g_2(t) = \exp(t), \tag{4.12}$$

$$g_1(t) = g_2(t) = \sin(2\pi t). \tag{4.13}$$

The corresponding results are depicted in Table 1 and 2 for the 50% and 25% quantile curves. We observe a reasonable approximation of the nominal level in almost all cases under consideration. The level of the bootstrap test for comparing 25% curves is slightly larger than that of the test for the equality of the 50% curves. For an investigation of the power of the bootstrap test we also simulated local alternatives of the form

$$g_2(t) = g_1(t) + c/\sqrt{n}, \tag{4.14}$$

$$g_2(t) = g_1(t) + tc/\sqrt{n}, \tag{4.15}$$

$$g_2(t) = g_1(t) + (\exp(t) - 1)c/\sqrt{n} \tag{4.16}$$

with $g_1(t) = \cos(\pi t)$ in all three cases for various values of n and c. The corresponding results are depicted in Tables 3 - 6 for various regression functions, sample sizes and values for the parameter c. In these tables we also display (in brackets) the corresponding rejection probabilities of the L^2-type test proposed in Dette et al. (2011). We observe that for larger of c the test proposed in this paper usually yields larger power than the L^2-type test. The differences are substantial in the model (4.14) and less visible in model (4.15). On the other hand, in model (4.16) we find some cases where the L^2-type test yields slightly larger power. Investigating the results for the comparison of the 50% quantile curves (Table 3 and 4) and of the 25% quantile curves (Table 5 and 6) we observe a loss in power. This corresponds to intuition because the 25% quantile curves are estimated with less precision.

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Table 1: *Rejection probabilities of the bootstrap test (4.3) for the hypothesis of two equal* 50% *quantile curves for models (4.9) - (4.13) with normally distributed errors corresponding to the null hypothesis.*

$p = 0.5$						
$g_1(t) = g_2(t) = t^2$						
(n_1, n_2)	(25,25)	(25,50)	(25,100)	(50,50)	(50,100)	(100,100)
$\alpha = 5\%$	4.7 %	5.0 %	5.9 %	6.0 %	5.2 %	4.4 %
$\alpha = 10\%$	10.9 %	9.1 %	11.3 %	10.3 %	9.8 %	7.5 %
$\alpha = 20\%$	20.1 %	18.3 %	21.6 %	19.0 %	19.5 %	16.5 %
$g_1(t) = g_2(t) = \cos(\pi t)$						
$\alpha = 5\%$	4.1 %	3.9%	4.2 %	4.4 %	5.5 %	4.6 %
$\alpha = 10\%$	10.4 %	8.9 %	9.7 %	10.1 %	10.4 %	9.2 %
$\alpha = 20\%$	19.6 %	18.2 %	19.9 %	21.1 %	18.4 %	19.4 %
$g_1(t) = g_2(t) = 1$						
$\alpha = 5\%$	6.0 %	5.3 %	5.1 %	5.3 %	4.1 %	3.9 %
$\alpha = 10\%$	9.9 %	11.8 %	10.6 %	9.7 %	9.1 %	8.9 %
$\alpha = 20\%$	19.2 %	22.1 %	19.8 %	17.8 %	18.9 %	17.6 %
$g_1(t) = g_2(t) = \exp(t) + 1$						
$\alpha = 5\%$	5.6 %	6.1 %	7.0 %	4.8 %	5.0 %	4.3 %
$\alpha = 10\%$	12.0 %	10.2 %	11.1 %	10.0 %	9.6 %	8.8 %
$\alpha = 20\%$	22.3 %	20.0 %	20.9 %	21.0 %	19.1 %	17.0 %
$g_1(t) = g_2(t) = \sin(2\pi t) + 1$						
$\alpha = 5\%$	8.2 %	7.2 %	7.0 %	5.0 %	6.0 %	5.5 %
$\alpha = 10\%$	13.3 %	12.6 %	11.8 %	10.6 %	11.8 %	10.5 %
$\alpha = 20\%$	23.7 %	24.5 %	22.9 %	19.5 %	22.9 %	21.1 %

Table 2: *Rejection probabilities of the bootstrap test (4.3) for the hypothesis of two equal* 25% *quantile curves for models (4.9) - (4.13) with normally distributed errors corresponding to the null hypothesis.*

$p = 0.25$						
$g_1(t) = g_2(t) = t^2$						
(n_1, n_2)	(25,25)	(25,50)	(25,100)	(50,50)	(50,100)	(100,100)
$\alpha = 5\%$	4.5 %	6.0 %	5.4 %	6.0 %	4.5 %	5.3 %
$\alpha = 10\%$	8.2 %	11.8 %	9.3 %	11.8 %	8.4 %	9.2 %
$\alpha = 20\%$	16.6 %	21.2 %	20.3 %	21.2 %	17.9 %	17.7 %
$g_1(t) = g_2(t) = \cos(\pi t)$						
$\alpha = 5\%$	5.2 %	5.7 %	6.5 %	5.7 %	4.5 %	4.5 %
$\alpha = 10\%$	9.5 %	10.9 %	11.5 %	10.9 %	10.5 %	9.2 %
$\alpha = 20\%$	18.3 %	20.9 %	20.4 %	20.9 %	20.2 %	18.2 %
$g_1(t) = g_2(t) = 1$						
$\alpha = 5\%$	4.8 %	6.6 %	4.8 %	6.6 %	4.5 %	3.2 %
$\alpha = 10\%$	9.8 %	10.0 %	9.0 %	10.0 %	10.1 %	8.4 %
$\alpha = 20\%$	19.1 %	18.3 %	18.7 %	18.3 %	17.7 %	18.4 %
$g_1(t) = g_2(t) = \exp(t) + 1$						
$\alpha = 5\%$	4.5 %	4.6 %	5.3 %	4.6 %	4.4 %	6.1 %
$\alpha = 10\%$	9.4 %	9.5 %	10.5 %	9.5 %	8.2 %	11.6 %
$\alpha = 20\%$	17.9 %	17.3 %	28.8 %	17.3 %	17.5 %	19.6 %
$g_1(t) = g_2(t) = \sin(2\pi t) + 1$						
$\alpha = 5\%$	8.2 %	5.3 %	6.8 %	5.3 %	5.2 %	5.2 %
$\alpha = 10\%$	13.6 %	10.7 %	11.5 %	10.7 %	10.6 %	9.8 %
$\alpha = 20\%$	24.8 %	21.6 %	22.5 %	21.6 %	20.3 %	19.0 %

Table 3: *Rejection probabilities of the bootstrap test (4.3) for the hypothesis of equal* 50% *quantile curves under the local alternatives (4.14) - (4.16) with normally distributed errors. The numbers in brackets denote the corresponding rejection probabilities of the* L^2 *test of Dette et al. (2011) and the sample size is* $n_1 = n_2 = 50$.

$g_1(t) = \cos(\pi t),\ g_2(t) = g_1(t) + c/\sqrt{n}$					
c	.5	1	1.5	2	2.5
$\alpha = 5\%$	9.2(5.7)%	19.2(13.9)%	34(25)%	52.8(39.5)%	74.1(59.6)%
$\alpha = 10\%$	15.8(11.4)%	26.7(20.8)%	46.7(34.1)%	64(53)%	81.2(69.9)%
$\alpha = 20\%$	27.1(21.7)%	38.2(33.5)%	59.7(48.7)%	78(69.4)%	90.1(82.4)%
$g_1(t) = \cos(\pi t),\ g_2(t) = g_1(t) + tc/\sqrt{n}$					
c	.5	1	1.5	2	2.5
$\alpha = 5\%$	6.1(5.5)%	8.6(8.1)%	10.4(11.8)%	18.3(16.4)%	25(22.5)%
$\alpha = 10\%$	10.3(10.1)%	14.2(14.6)%	17.9(18.4)%	27.4(24.6)%	34.8(33)%
$\alpha = 20\%$	21.4(18)%	24.8(23.5)%	30.3(28.8)%	39.5(37.3)%	48.1(47.5)%
$g_1(t) = \cos(\pi t),\ g_2(t) = g_1(t) + (\exp(t) - 1)c/\sqrt{n}$					
c	.5	1	1.5	2	2.5
$\alpha = 5\%$	6(7.2)%	11.1(10.4)%	20.7(18.2)%	31.5(34.6)%	44.6(47.1)%
$\alpha = 10\%$	10.4(12.9)%	17.9(17.1)%	30.8(27.1)%	42.3(45)%	58.5(57.9)%
$\alpha = 20\%$	21.8(22.8)%	29.8(30.1)%	44.7(39.9)%	57.6(57.1)%	70.3(70.8)%

5 Appendix: Proofs

To keep the notation simply we concentrate on the case of $J = 2$ samples, $N_j = n_j$ and the Nadaraya-Watson estimate. The corresponding statements for the local linear estimate and more than 2 samples follow by exactly the same arguments with an additional amount of notation. For brevity we use the notation $S_t^{NW} = S_t$ throughout this section.

5.1 Proof of Lemma 3.1 and Theorem 3.2

We use the notation $H_j^{-1}(p|x) = G(F_j^{-1}(p|x))$, $\tilde{G}(x) = (G^{-1})'(H_1^{-1}(p|x))$ and obtain by a Taylor-expansion under the null hypothesis H_0 for all $t \in [0,1]$ (note that the distribution function G is strictly monotone)

$$S_t = \int_0^t (\widehat{F}_1^{-1}(p|x) - F_1^{-1}(p|x) + F_2^{-1}(p|x) - \widehat{F}_2^{-1}(p|x))\hat{w}_{12}(x)dx$$

(5.1)

$$= \int_0^t \tilde{G}(x) \left(\widehat{H}_1^{-1}(p|x) - H_1^{-1}(p|x) - (\widehat{H}_2^{-1}(p|x) - H_2^{-1}(p|x))\right) \hat{w}_{12}(t)dt$$

Table 4: *Rejection probabilities of the bootstrap test (4.3) for the hypothesis of equal* 50% *quantile curves under the local alternatives (4.14) - (4.16) with normally distributed errors. The numbers in brackets denote the corresponding rejection probabilities of the* L^2 *test of Dette et al. (2011) and the sample size is* $n_1 = n_2 = 100$.

$g_1(t) = \cos(\pi t),\ g_2(t) = g_1(t) + c/\sqrt{n}$					
c	.5	1	1.5	2	2.5
$\alpha = 5\%$	4.9(5.5)%	13.8(7.0)%	22.0(15.4)%	41.5(27.6)%	58.8(47.3)%
$\alpha = 10\%$	9.4(10.1)%	23.4(13.4)%	32.7(23.1)%	55.2(38.0)%	71.1(60.0)%
$\alpha = 20\%$	21.7(19.4)%	33.7(26.3)%	48.3(39.0)%	70.3(53.5)%	82.2(74.4)%
$p = 0.5, \quad (n_1, n_2) = (100, 100)$					
$g_1(t) = \cos(\pi t),\ g_2(t) = g_1(t) + tc/\sqrt{n}$					
c	.5	1	1.5	2	2.5
$\alpha = 5\%$	6.4(5.7)%	7.4(5.9)%	12.4(8.7)%	18.4(14.3)%	25.2(20.5)%
$\alpha = 10\%$	10.9(9.2)%	13.5(10.7)%	20.6(15.6)%	25.5(24.6)%	34.5(29.7)%
$\alpha = 20\%$	20.5(19.9)%	24.7(20.6)%	31.4(28.2)%	39.5(37.4)%	50.4(44.2)%
$g_1(t) = \cos(\pi t),\ g_2(t) = g_1(t) + (\exp(t) - 1)c/\sqrt{n}$					
c	.5	1	1.5	2	2.5
$\alpha = 5\%$	6.4(6.8)%	12(9.3)%	17.8(18.1)%	32.1(29.7)%	42.8(43.6)%
$\alpha = 10\%$	11.3(12.6)%	18.5(15.7)%	26.5(27)%	42.6(40.1)%	57(55.1)%
$\alpha = 20\%$	19.9(21.7)%	28(28.2)%	40.5(39.3)%	55.5(55)%	69.1(68.3)%

Table 5: *Rejection probabilities of the bootstrap test (4.3) for the hypothesis of equal* 25% *quantile curves under the local alternatives (4.14) - (4.16) with normally distributed errors. The numbers in brackets denote the corresponding rejection probabilities of the* L^2 *test of Dette et al. (2011) and the sample size is* $n_1 = n_2 = 50$.

$g_1(t) = \cos(\pi t),\ g_2(t) = g_1(t) + c/\sqrt{n}$					
c	.5	1	1.5	2	2.5
$\alpha = 5\%$	4.9(5.5)%	13.8(7)%	22(15.4)%	41.5(27.6)%	58.8(47.3)%
$\alpha = 10\%$	9.4(10.1)%	23.4(13.4)%	32.7(23.1)%	55.2(38)%	71.1(60)%
$\alpha = 20\%$	21.7(19.4)%	33.7(26.3)%	48.3(39)%	70.3(53.5)%	82.2(74.4)%
$g_1(t) = \cos(\pi t),\ g_2(t) = g_1(t) + tc/\sqrt{n}$					
c	.5	1	1.5	2	2.5
$\alpha = 5\%$	3.7(4.4)%	6.1(6.1)%	9.3(8.6)%	15(11.2)%	21.5(19)%
$\alpha = 10\%$	7.5(8.7)%	11.5(11)%	14.6(14.7)%	22.8(19)%	32.2(28.6)%
$\alpha = 20\%$	15.9(16.9)%	23(20.4)%	28.1(24.9)%	36.3(30.8)%	44.1(42)%
$g_1(t) = \cos(\pi t),\ g_2(t) = g_1(t) + (\exp(t) - 1)c/\sqrt{n}$					
c	.5	1	1.5	2	2.5
$\alpha = 5\%$	4.1(5.5)%	8.8(8.6)%	16.9(16.6)%	26.6(24.6)%	43.8(41)%
$\alpha = 10\%$	9(11)%	15.4(15)%	24.1(25)%	38.6(34.6)%	55.5(52.2)%
$\alpha = 20\%$	17.8(19.4)%	27(27.8)%	39.1(40.9)%	52.5(49.8)%	68.5(64.9)%

$$+\frac{1}{2}\int_0^t \left(G^{-1}\right)''(\xi_1)\left(\widehat{H}_1^{-1}(p|x) - H_1^{-1}(p|x)\right)^2 - \left(G^{-1}\right)''(\xi_2)\left(\widehat{H}_2^{-1}(p|x) - H_2^{-1}(p|x)\right)^2 \hat{w}_{12}(x)dx,$$

where the random variables ξ_1 and ξ_2 satisfy $|\xi_j - H_j^{-1}(p|t)| \le |\widehat{H}_j^{-1}(p|t) - H_j^{-1}(p|t)|$. Under the assumptions of Theorem 3.2 it follows from Dette and Volgushev (2008) that

$$\widehat{H}_j^{-1}(p|x) - H_j^{-1}(p|x) = O_p(h_r^2) + O_p\left(\frac{1}{\sqrt{nh_r}}\right)$$

(this holds uniformly in x under our stronger assumptions) and as a consequence the last integral in (5.1) is of order $o_p(n^{-1/2})$. Therefore it remains to consider the first integral, which will be denoted by $S_t^{(1)}$ throughout this section. From the definition of $\widehat{H}_j^{-1}(p|x)$ in (2.7) we obtain by a further Taylor expansion

$$\widehat{H}_j^{-1}(p|x) - H_j^{-1}(p|x) = \Delta_j^{(1)}(p|x) + \Delta_j^{(2)}(p|x) + \Delta_j^{(3)}(p|x) + \Delta_j^{(4)}(p|x),$$

where

$$\Delta_j^{(1)}(p|x) = -\frac{1}{n_j h_d}\sum_{i=1}^{n_j} K_d\left(\frac{F_j(g_{ij}|x) - p}{h_d}\right)\left(\widehat{F}_j(g_{ij}|x) - F_j(g_{ij}|x)\right),$$

Table 6: *Rejection probabilities of the bootstrap test (4.3) for the hypothesis of equal* 25% *quantile curves under the local alternatives (4.9) - (4.11) with normally distributed errors. The numbers in brackets denote the corresponding rejection probabilities of the L^2 test of Dette et al. (2011) and the sample size is $n_1 = n_2 = 100$.*

$g_1(t) = \cos(\pi t),\ g_2(t) = g_1(t) + c/\sqrt{n}$					
c	.5	1	1.5	2	2.5
$\alpha = 5\%$	5.8(5.3)%	12.5(9.1)%	26.0(14.9)%	42.4(29.4)%	59.7(43.8)%
$\alpha = 10\%$	10.7(9.5)%	21.1(14.3)%	37.1(23.7)%	54.8(40.9)%	71.7(56.1)%
$\alpha = 20\%$	19.8(19.3)%	33.7(25.3)%	49.1(37.5)%	69.5(56.8)%	83.1(70.5)%
$g_1(t) = \cos(\pi t),\ g_2(t) = g_1(t) + tc/\sqrt{n}$					
c	.5	1	1.5	2	2.5
$\alpha = 5\%$	4.1(4.6)%	6.1(6.2)%	9.7(7.6)%	12.9(10.8)%	21.3(16.6)%
$\alpha = 10\%$	8.2(7.5)%	9.9(11.5)%	16(14.3)%	20.4(16.8)%	31.2(25.5)%
$\alpha = 20\%$	16.3(15)%	20.2(20)%	26.6(25.8)%	32.8(29.6)%	44.8(40.6)%
$g_1(t) = \cos(\pi t),\ g_2(t) = g_1(t) + (\exp(t) - 1)c/\sqrt{n}$					
c	.5	1	1.5	2	2.5
$\alpha = 5\%$	4.1(4.6)%	6.1(6.2)%	9.7(7.6)%	12.9(10.8)%	21.3(16.6)%
$\alpha = 10\%$	8.2(7.5)%	9.9(11.5)%	16(14.3)%	20.4(16.8)%	31.2(25.5)%
$\alpha = 20\%$	16.3(15)%	20.2(20)%	26.6(25.8)%	32.8(29.6)%	44.8(40.6)%

$$\Delta_j^{(2)}(p|x) = -\frac{1}{2n_j h_d^2} \sum_{i=1}^{n_j} K_d' \left(\frac{F_j(g_{ij}|x) - p}{h_d} \right) \left(\widehat{F}_j(g_{ij}|x) - F_j(g_{ij}|x) \right)^2,$$

$$\Delta_j^{(3)}(p|x) = -\frac{1}{6n_j h_d^3} \sum_{i=1}^{n_j} K_d'' \left(\frac{\xi_{ij} - p}{h_d} \right) \left(\widehat{F}_j(g_{ij}|x) - F_j(g_{ij}|x) \right)^3,$$

$$\Delta_j^{(4)}(p|x) = \frac{1}{n_j h_d} \int_{-\infty}^{p} \sum_{i=1}^{n_j} K_d \left(\frac{F_j(g_{ij}|x) - u}{h_d} \right) du - H_j^{-1}(p|x),$$

we used the notation $g_{ij} = G^{-1}(i/n_j)$, and the random variables ξ_{ij} satisfy $|\xi_{ij} - F_j(g_{ij}|x)| \leq |\widehat{F}_j(g_{ij}|x) - F_j(g_{ij}|x)|$. For $k = 1, \ldots, 4$ we define

$$S_{t,k}^{(1)} = \int_0^t \tilde{G}(x) \left(\Delta_1^{(k)}(p|x) - \Delta_2^{(k)}(p|x) \right) \hat{w}_{12}(x).dx$$

We will show in the following that $\sqrt{n} S_{t,1}^{(1)}$ converges weakly to a Gaussian process while $S_{t,k}^{(1)} = o_p(n^{-1/2})$ for $k = 2, 3, 4$. Because of the definition $\hat{w}_{12}(x) = \hat{f}_1(x)\hat{f}_2(x)$, $\sup_x |\widehat{f}_j(x) - f_j(x)| =$

$O_P(\sqrt{\log n}/\sqrt{nh_r})$ and $\sup_x |\widehat{F}_j(y|x) - F_j(y|x)| = O_P(\sqrt{\log n}/\sqrt{nh_r})$ the result

$$\begin{aligned} S_{t,1}^{(1)} &= \tilde{S}_{t,1}^{(1)} + o_p\left(\frac{1}{\sqrt{n}}\right) \\ &= \int_0^t \tilde{G}(x)\left(f_2(x)\widehat{f}_1(x)\Delta_1^{(1)}(p|x) - f_1(x)\widehat{f}_2(x)\Delta_2^{(1)}(p|x)\right)dx + o_p\left(\frac{1}{\sqrt{n}}\right) \end{aligned}$$

holds, where we denote the first integral by $\tilde{S}_{t,1}^{(1)}$. Now we define the independent identically distributed random variables

$$Z_{kj}(x) = \frac{-1}{n_j^2 h_d h_r}\sum_{l=1}^{n_j} K_d\left(\frac{F_j(g_{lj}|x) - p}{h_d}\right)K_r\left(\frac{X_{kj} - x}{h_r}\right)(I\{Y_{kj} \le g_{lj}\} - F_j(g_{lj}|x))$$

and observe that by the definition of the Nadaraya-Watson-weights we get

$$\widehat{f}_j(x)\Delta_j^{(1)}(p|x) = \sum_{k=1}^{n_j} Z_{kj}(x).$$

First we calculate the expectation and covariance function of $\tilde{S}_{t,1}^{(1)}$. A straight forward but tedious calculation (use Taylor-expansions, Riemann-approximations and appropriate substitutions) yields

(5.2)

$$\begin{aligned} E\left[Z_{kj}(x)\right] = &\frac{-h_r^2}{n_j}\left(\partial_1(G(F_j^{-1}(y|x)))\right)_{y=p}\mu_2(K_r) \\ &\times\left(f_j'(x)\partial_2(F_j(y|x))\Big|_{(y,x)=(F_j^{-1}(p|x),x)} + \frac{1}{2}f_j(x)\partial_2^2(F_j(y|x))\Big|_{(y,x)=(F_j^{-1}(p|x),x)}\right) + O(h_r^3) \end{aligned}$$

uniformly with respect to $x \in [0,1]$ and therefore by the definition of $\tilde{G}$

(5.3) $$E\left[\tilde{S}_{t,1}^{(1)}\right] = h_r^2\int_0^t (f_1(x)C_2(x) - f_2(x)C_1(x))dx + O(h_r^3) = o\left(\frac{1}{\sqrt{n}}\right).$$

Further we define

(5.4) $$\tilde{Z}_k(x) = \begin{cases} f_2(x)\tilde{G}(x)(Z_{k1}(x) - E\left[Z_{k1}(x)\right]) & \text{for } 1 \le k \le n_1 \\ -f_1(x)\tilde{G}(x)(Z_{(k-n_1)2}(x) - E\left[Z_{(k-n_1)2}(x)\right]) & \text{for } n_1 + 1 \le k \le n \end{cases}$$

which are centered independent random variables. Using this notation we obtain from (5.3) the following representation for the statistic $S_{t,1}^{(1)}$

$$S_{t,1}^{(1)} = \sum_{k=1}^{n} \int_0^t \tilde{Z}_k(x)dx + o_p\left(\frac{1}{\sqrt{n}}\right). \tag{5.5}$$

Approximating sums by integrals and using Taylor-expansions one obtains by Fubini's theorem for $t \leq s$

$$\begin{aligned}
& E\left[\int_0^t \tilde{G}(x)Z_{kj}(x)dx \int_0^s \tilde{G}(x)Z_{kj}(x)dx\right] \\
&= \frac{1}{n_j^2 h_d h_r} \int_0^t \int_0^s \tilde{G}(x_1)\tilde{G}(x_2)\Big(\partial_y(G(F_j^{-1}(y|x_1)))\Big)_{y=p} f_j(x_1) \left(\int_{-1}^1 K_r(u)K_r\left(u + \frac{x_1 - x_2}{h_r}\right) du\right) \\
& \times \left(\int_0^1 K_d\left(\frac{F_j(G^{-1}(w)|x_2) - p}{h_d}\right) \big(p \wedge F_j(G^{-1}(w)|x_1) - pF_j(G^{-1}(w)|x_1)\big)\, dw\right) dx_2 dx_1 (1 + o(1)) \\
&= \frac{1}{n_j^2 h_d} \int_0^t \int_{\frac{x_1 - s}{h_r}}^{\frac{x_1}{h_r}} \tilde{G}(x_1)\tilde{G}(x_1 - h_r v)\Big(\partial_y(G(F_j^{-1}(y|x_1)))\Big)_{y=p} f_j(x_1) \left(\int_{-1}^1 K_r(u)K_r(u + v)\, du\right) \\
& \times \left(\int_0^1 K_d\left(\frac{F_j(G^{-1}(w)|x_1 - h_r v) - p}{h_d}\right) \big(p \wedge F_j(G^{-1}(w)|x_1) - pF_j(G^{-1}(w)|x_1)\big)\, dw\right) dv dx_1 \\
& \times (1 + o(1)).
\end{aligned}$$

Further Taylor-expansions of the terms having argument $x_1 - h_r v$ and a substitution in the integral over w now yields, using (5.2), (5.4) and the definition of $\tilde{G}$, that for large n the approximation

$$\begin{aligned}
& \operatorname{Cov}\left(\sum_{k=1}^n \int_0^t \tilde{Z}_k(x)dx, \sum_{k=1}^n \int_0^s \tilde{Z}_k(x)dx\right) \\
&= p(1-p) \int_0^{s \wedge t} \left(\big(\partial_1(F_2^{-1}(p|x))\big)^2 \frac{f_1^2(x)f_2(x)}{n_2} + \big(\partial_1(F_1^{-1}(p|x))\big)^2 \frac{f_1(x)f_2^2(x)}{n_1}\right) dx + o\left(\frac{1}{n}\right)
\end{aligned}$$

holds. From the multidimensional central limit theorem (note that $S_{t,1}^{(1)}$ is approximately a sum of two independent sums of independent identically distributed random variables) now the weak convergence of the finite dimensional distributions of $(S_{t,1}^{(1)})_{t \in [0,1]}$ follows, that is

$$\begin{pmatrix} \sqrt{n}(S_{t_1,1}^{(1)}) \\ \vdots \\ \sqrt{n}(S_{t_k,1}^{(1)}) \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_k(0, V)$$

for all $k \in \mathbb{N}$ and $t_1 \dots t_k \in [0,1]$ where the elements of the covariance matrix V are given by

$$V_{i,j} = p(1-p) \int_0^{t_i \wedge t_j} \left(\left(\partial_1 (F_2^{-1}(p|x))\right)^2 \frac{f_1^2(x) f_2(x)}{a_2} + \left(\partial_1 (F_1^{-1}(p|x))\right)^2 \frac{f_1(x) f_2^2(x)}{a_1} \right) dx.$$

In order to establish weak convergence of $(S_{t,1}^{(1)})_{t\in[0,1]}$ in the space $C[0,1]$ ($S_{t,1}^{(1)}$ has continuous paths) we finally show tightness of $T_n(t) = \sqrt{n} \sum_{k=1}^n \int_0^t \tilde{Z}_k(x) dx$. According to Theorem 15.6 in Billingsley (1968) it is sufficient to show that there exist constants $C > 0$, $\alpha > 0$ and $\beta > 0$ such that the inequality

$$E\left[|T_n(t) - T_n(s)|^\alpha\right] \le C|t-s|^{1+\beta} \quad \text{for all } n \in \mathbb{N} \text{ and all } s,\, t \in [0,1] \tag{5.6}$$

holds (continuity of the limiting process follows similar to that of Brownian motion). We set $\alpha = 4$ and use the independence of the random variables $\tilde{Z}_k(x)$ and Fubini's theorem to obtain for $t > s$

$$\begin{aligned} E\left[|T_n(t) - T_n(s)|^4\right] = 3n^4 &\left(\int_s^t \int_s^t E\left[\tilde{Z}_1(x_1)\tilde{Z}_1(x_2)\right] dx_1 dx_2 \right)^2 \\ &+ n^3 \int_s^t \int_s^t \int_s^t \int_s^t E\left[\tilde{Z}_1(x_1)\tilde{Z}_1(x_2)\tilde{Z}_1(x_3)\tilde{Z}_1(x_4)\right] dx_1 dx_2 dx_3 dx_4. \end{aligned}$$

Similar calculations to those used for the calculation of the covariance-structure yield

$$\int_s^t \int_s^t E\left[\tilde{Z}_1(x_1)\tilde{Z}_1(x_2)\right] dx_1 dx_2 \le \tilde{C}(t-s) O(n^{-2}),$$

and another straightforward (but tedious) calculation gives

$$\int_s^t \int_s^t \int_s^t \int_s^t E\left[\tilde{Z}_1(x_1)\tilde{Z}_1(x_2)\tilde{Z}_1(x_3)\tilde{Z}_1(x_4)\right] dx_1 dx_2 dx_3 dx_4 \le \frac{\tilde{C}(t-s)^4}{h_r^3} O(n^{-4})$$

with some constant $\tilde{C}$. Therefore (5.6) holds for some constant $C > 0$, $\alpha = 4$ and $\beta = 1$ and the weak convergence of $(S_{t,1}^{(1)})_{t\in[0,1]}$ follows.

Next we consider the term $S_{t,2}^{(1)}$. By a Riemann approximation and substitution we obtain

$$\begin{aligned} S_{t,2}^{(1)} = &\frac{1}{2h_d} \int_0^1 \int_0^t \frac{G'(F_2^{-1}(p + h_d v|x))}{G'(F_2^{-1}(p|x)) f_2(F_2^{-1}(p + h_d v|x))} K_d'(v) \\ &\times \left[\widehat{F}_2(G(F_2^{-1}(p + h_d v|x))|x) - F_2(G(F_2^{-1}(p + h_d v|x))|x)\right]^2 \widehat{w}_{12}(x) dx dv (1 + o_P(1)) \\ &- \frac{1}{2h_d} \int_0^1 \int_0^t \frac{G'(F_1^{-1}(p + h_d v|x))}{G'(F_1^{-1}(p|x)) f_1(F_1^{-1}(p + h_d v|x))} K_d'(v) \end{aligned}$$

$$\times \left[\widehat{F}_1(G(F_1^{-1}(p+h_d v|x))|x) - F_1(G(F_1^{-1}(p+h_d v|x))|x)\right]^2 \widehat{w}_{12}(x)dxdv(1+o_P(1)).$$

Similar calculations as in Dette et al. (2011) yield the weak convergence

$$\begin{pmatrix} n\sqrt{h_r}\left(S_{t_1,2}^{(1)} - (nh_r)^{-1}B_{t_1,2}^{(n)}\right) \\ \vdots \\ n\sqrt{h_r}\left(S_{t_k,2}^{(1)} - (nh_r)^{-1}B_{t_k,2}^{(n)}\right) \end{pmatrix} \xrightarrow{\mathcal{D}} \mathcal{N}_k(0, \overline{V})$$

for each $k \in \mathbb{N}$, where $\overline{V}$ is some positive definite matrix and the bias is given by

$$\begin{aligned} B_{t,2}^{(n)} = \frac{p(1-p)}{2h_d}\mu_2(K_r)\int_{-1}^{1} K_d'(v)\int_0^t \frac{1}{a_2}\frac{G'(F_2^{-1}(p+h_d v|x))}{G'(F_2^{-1}(p|x))}\frac{f_1(x)}{f_2(F_2^{-1}(p+h_d v|x))} \\ - \frac{1}{a_1}\frac{G'(F_1^{-1}(p+h_d v|x))}{G'(F_1^{-1}(p|x))}\frac{f_2(x)}{f_1(F_1^{-1}(p+h_d v|x))}dxdv. \end{aligned}$$

It can be shown by similar arguments as given above that the inequality

$$n^2 h_r E\left[\left|S_{t,2}^{(1)} - (nh_r)^{-1}B_{t,2}^{(n)} - \left(S_{s,2}^{(1)} - (nh_r)^{-1}B_{s,2}^{(n)}\right)\right|^2\right] \leq C|t-s|^2$$

holds for some constant C. Thus the process $n\sqrt{h_r}\left(S_{t,2}^{(1)} - (nh_r)^{-1}B_{t,2}^{(n)}\right)_{t\in[0,1]}$ is tight and therefore converges weakly to a centered Gaussian process. This yields

$$S_{t,2}^{(1)} = o_P\left(\frac{1}{\sqrt{n}}\right) + \frac{1}{nh_r}B_{t,2}^{(n)}$$

uniformly with respect to t. Using integration by parts in the integral with respect to v and a Taylor expansion we directly obtain

$$\frac{1}{nh_r}B_{t,2}^{(n)} = O\left(\frac{1}{nh_r}\right) = o\left(\frac{1}{\sqrt{n}}\right)$$

which shows that $S_{t,2}^{(1)}$ is asymptotically negligible.
Finally, straightforward calculations and similar arguments as given in Dette and Volgushev (2008) yield

$$\Delta_j^{(3)}(p|x) = o_P\left(\frac{1}{\sqrt{n}}\right)$$
$$\Delta_j^{(4)}(p|x) = O(h_d^2) = o\left(\frac{1}{\sqrt{n}}\right)$$

uniformly with respect to x thus showing $S_{t,k}^{(1)} = o_P(n^{-1/2})$ for $k = 3, 4$. This yields the assertion of Theorem 3.2. The last arguments together with (5.3) give part (a) of Lemma 3.1.
Part (b) of Lemma 3.1 simply follows because under H_1 we have

$$S_t = \int_0^t (\widehat{F}_1^{-1}(p|x) - F_1^{-1}(p|x) + F_2^{-1}(p|x) - \widehat{F}_2^{-1}(p|x))\hat{w}_{12}(x)dx + \int_0^t (F_1^{-1}(p|x) - F_2^{-1}(p|x))\hat{w}_{12}(x)dx.$$

Now we can apply similar methods as above to this statistic and the assertion of Lemma 3.1 follows.

□.

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