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AN INVESTIGATION OF SAMPLED DATA INTERPOLATION ERROR

A THESIS

Presented to

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by

Joseph Alexander Knight

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AN INVESTIGATION OF SAMPLED

DATA INTERPOLATION ERROR

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SUMMARY

The basic concept that a function of time may be sampled, i.e., specified for particular values of its argument, and subsequently reconstructed, or interpolated, in some manner to form an approximation to the original function of time is fairly well-known. The idea is so intuitively appealing that any restrictions upon the nature of the sampled function or the interpolation technique are not at all apparent. Electrical engineers are perhaps most familiar with Shannon's Theorem, dealing with one aspect of sampling, to the effect that band-limited functions require only a finite number of samples per unit time for exact reconstruction.

When this idea is to be used practically, analysis quickly shows that the joint operations of sampling and interpolating can be viewed as a sort of frequency domain filter introducing distortion to the original spectrum as well as obliterating some frequency components. Under a suitable restriction, namely a sufficiently high sampling rate, it is possible to obtain an output spectrum which closely resembles that of the input and thus in some sense represents an approximation of the original signal. The advantages gained by having to deal only with samples rather than an entire function often counterbalance the loss of accuracy in the resultant approximation -- leading to an interpolation error versus sampling rate trade off. Any low pass device will interpolate a sampled input but some have more engineering interest than others. Such a device is the zero-order hold which maintains the constant value of the latest sample until the

next sample occurs.

The preceding remarks make clear that some error characterization is necessary to rate the performance of a given interpolator with a specific input in terms of the sampling interval. The commonly used error criterion of the expected value of the mean squared error, $\bar{\psi}(T, \lambda)$, is just such a figure of merit. For the zero-order hold, $\bar{\psi}(T, \lambda)$ is a functional of the variation, a basic second-order statistic of a random process defined by $V(\tau) = R(0) - R(\tau)$. The Fourier transform relationship between $R(\tau)$ and $S(\omega)$ is used to establish a frequency domain representation of $V(\tau)$ which is shown to be dominated by the behavior of $S(\omega)$ for large ω. It is then shown that $\bar{\psi}(T, \lambda)$ is well-behaved; however, a basic problem in evaluating error criteria, i.e., their relationship to actual error performance still remains. The quality of $\bar{\psi}(T, \lambda)$ is analyzed by the Bienayme Inequality for the general random process. Gaussian random processes possess sufficient tractability that several aspects of the relation of $\bar{\psi}(T, \lambda)$ to both the error in one sampling interval and the time average error along a sequence of such intervals are analyzed.

Simple bounds on $V(\tau)$, and hence on $R(\tau)$, are shown to exist for all band-limited processes and $V(\tau)$ is shown to be monotone and convex for sufficiently small τ. Upper bounds are shown to exist for general classes of non-band-limited processes. These relations are sufficient to establish bounds on zero-order hold interpolation error, and some general curves are presented which permit selection of a suitable sampling rate knowing only two basic parameters of the process, $R(0)$ and $|R''(0)|$. It is also shown that the effective delay introduced by the zero-order hold is not necessarily one-half sampling period unless a condition on $V'(\tau)$ is fulfilled.

The Bienayme Inequality is shown to yield a confidence level on the difference between the value of $\bar{\Psi}(T, \lambda)$ and the actual mean square error in a sampling interval for any process. Properties of Gaussian processes are used to show that the mean square error process has a variance expressed as a functional of the autocorrelation function. An approximation, valid for high sampling rates, is used to show that the behavior of Gaussian processes, during an observation interval consisting of several consecutive sampling intervals, is well-behaved and that the time average mean squared error should converge quickly to the value given by the quadratic interpolation error bound. The behavior of this time average error is examined for several common spectral densities.

CHAPTER I

INTRODUCTION

Definition of the Problem

Sampling with subsequent interpolation as a means of representing a function is based on the central idea that at least some functions are completely, i.e., uniquely, specified if a sufficient number of values per unit time are known. Most of the analyses arising from the problems posed by the sampling theorem are treatments of highly idealized, restrictive cases requiring such simplifications as band-limited functions or non-realizable interpolators.

One of the most common realizable interpolators is the zero-order sample-and-hold in which the output during an interval is a constant equal to the value of the sample representing that interval. Such an approach is intuitively acceptable if, in addition, it is recognized that the output is truly an approximation to the sampled input process due to the unavoidable presence of such phenomena as loss of frequency information resulting from spectral overlapping and distortion of frequency information resulting from the interpolator's filtering tendencies.

The approximate nature of any realizable interpolator output means that any discussion of the performance of an interpolator will necessarily require that some error criterion be defined and used as a figure of merit. Both the particular interpolator structure being analyzed as well as the statistical parameters of the input may influence the nature of this figure of merit.

The characterization of the inherent error between interpolator input and output is thus a key factor if some quantitative measure of the quality of the approximation is to be obtained, and it is this characterization which is to be investigated for a class of interpolators. In particular, two aspects of special significance will be dealt with. The first problem is to determine the general tendencies of zero-order sample-and-hold error, i.e., what is the nature of the mean of this error, and the second problem is to determine the relationship between the expected behavior and the actual behavior, i.e., what is the nature of the variance of this error.

Origin and History

The underlying concept of sampling theory seems to have been outlined first by Cauchy (1) in 1841 when he stated a relationship between frequency components and sampling rates roughly corresponding to the intuitive approach that if a function of time is band-limited, i.e., contains no frequency components outside some finite range, and is sampled at a rate at least twice as fast as the period of the highest frequency component, then it should be possible to at least construct a good approximation to the sampled function since it cannot change appreciably between sampling intervals of this order. Nyquist (2) pointed out the fundamental importance of a sampling period one-half the period of the highest frequency contained in a telegraph signal by using a Fourier series expansion as an approximation. Whittaker (3) showed that for a function f(t) with Fourier transform $F(j\omega)$, where $F(j\omega) = 0$ for $|\omega| > \pi$, knowledge of f(n) for n = 0, ±1, ±2 ... is sufficient to reconstruct the entire time function if a "cardinal" interpolation function is used and that such an f(t) may also be written as

$$f(t) = \sum_{-\infty}^{\infty} f(n) \frac{\sin[\pi(t-n)]}{\pi(t-n)}$$

where $\frac{\sin \pi t}{\pi t}$ may be considered to be the impulse response of the cardinal interpolator.

Electrical engineers are normally more familiar with Shannon's Theorem dealing with this property of band-limited functions, to wit:

> If a function f(t) contains no frequencies higher than W cps, it is completely specified by giving its ordinates at a series of points spaced 1/2W seconds apart.

which Shannon subsequently used to develop his formula for maximum error-free channel capacity (4). Balakrishnan (5) extended this concept to show that when sampling random processes with band-limited spectral densities, the Nyquist rate, in conjunction with a cardinal interpolation function, is sufficient to yield a reconstructed signal equal to the original in a mean square sense, i.e., that

$$\underset{N \to \infty}{\text{limit}}\ E\left\{\left[\left|x(t) - \sum_{-N}^{N} x\left(\frac{n}{2W}\right) \frac{\sin[\pi(2Wt - n)]}{\pi(2Wt - n)}\right|\right]^2\right\} = 0 .$$

The interpolation process is most easily understood when viewed as a frequency domain operation based upon the fact that any sampled signal defined as

$$f^{*}(t) = \sum_{-\infty}^{\infty} f(nT)\delta(t-nT),$$

where f(t) has the Fourier transform $F(j\omega)$, has the transform

$$F^{*}(j\omega) = \frac{1}{T} \sum_{-\infty}^{\infty} F[j(\omega - \frac{2\pi n}{T})] =$$

$$= \frac{1}{T} F(j\omega) + \frac{1}{T} \sum_{1}^{\infty} \left\{ F[j(\omega + \frac{2\pi n}{T})] + F[j(\omega - \frac{2\pi n}{T})] \right\} .$$

It is apparent that if $F(j\omega)$ is band-limited and $T \leq \pi/\omega_c$, then $F^{*}(j\omega)$ contains an undistorted version of the original spectrum as well as an infinite number of images centered about the $\frac{2\pi n}{T}$ points. Seen in this light, all that is needed for interpolation is a flat, low-pass filter with gain T which will remove the high frequency sidebands and pass only a resultant spectrum identical to the original $F(j\omega)$. In the frequency domain, such a filter may be shown to represent the ideal, or cardinal, interpolator discussed earlier. A similar interpretation may be made for the case of random process sampling. Effectively, any realizable low-pass device will filter from $F^{*}(j\omega)$ a frequency spectrum related to $F(j\omega)$ in a manner depending upon the filter rolloff, the sampling rate and $F(j\omega)$. In the frequency domain, the difference between the input and output can be attributed to some combination of three sources: first, distortion of the base-band frequencies by the low-pass characteristics of the interpolator filter; second, errors of omission, i.e., attenuation of the high frequency terms present in the original spectrum by the necessary cutoff tendencies of the interpolator; and, third, errors of commission, i.e., obliteration of low frequency terms in $F(j\omega)$ by the additional frequency terms in the base band which result from the overlapping of the high frequency image terms in $F^*(j\omega)$ with the low frequency $\frac{1}{T}F(j\omega)$ term. The latter two errors are normally present in the more general non-band-limited case but are also present in the degenerate band-limited case with insufficient sampling rate ($T > \frac{\pi}{\omega_c}$). In the time domain, although it is difficult to define specific error sources, a contributing factor is the causality restriction normally placed on the interpolator. In any case, these problems do not negate the utility of the sampling concept but they do demand that it not be used without an understanding of its limitations.

For various reasons, such as those above, the interpolator output is an approximation to the sampled input and some measure of the quality of the approximation is needed. The error comparator of Figure 1 yields a useful interpolator error parameter. A uniform sampling interval of T is assumed and a delay, d, is considered to be a variable in the error. The interpolator filter has impulse response $h_o(t)$ and, by inspection, the interpolator output, $\hat{x}(t)$, becomes

$$\hat{x}(t) = \sum_{-\infty}^{\infty} x(nT)h_o(t - nT)$$

leading to an instantaneous error defined as

$$e(t, d) = x(t - d) - \hat{x}(t) .$$

Further examination of this error, where the selection of $h_o(t)$

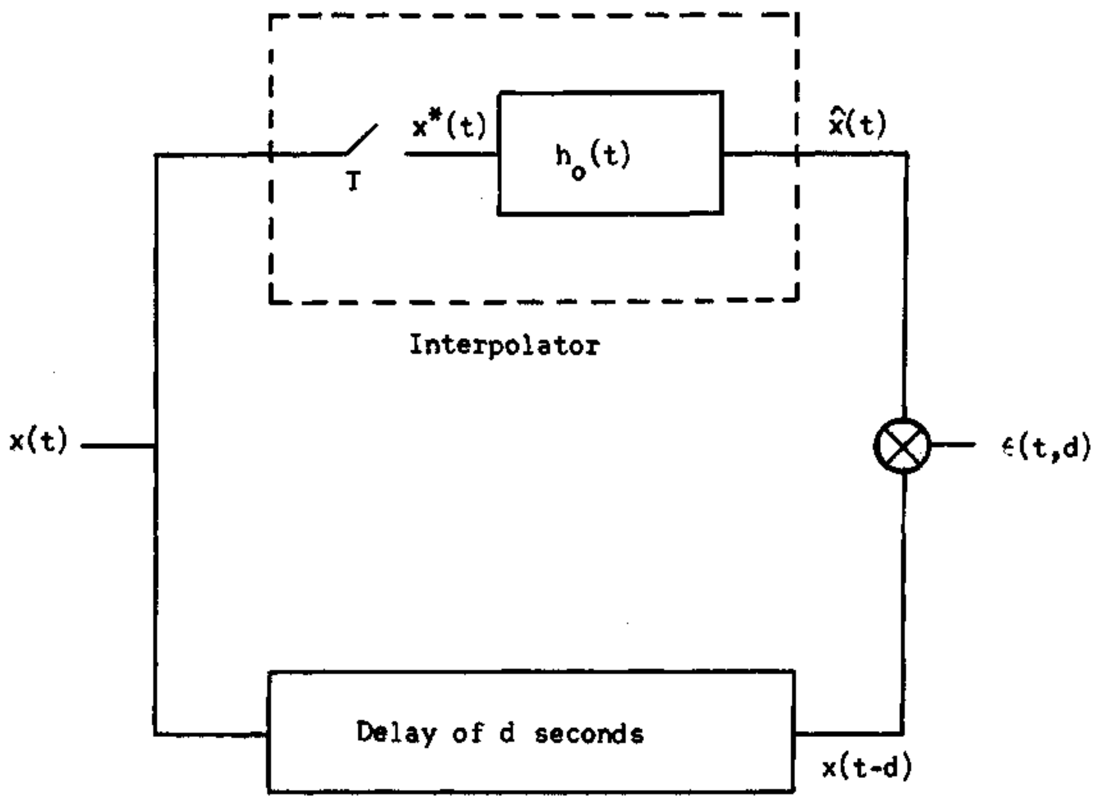

Figure 1. Block Diagram of Interpolator Error Comparator.

is to be made in such a manner that the error is minimized, leads to one of the two basic interpolator types, namely, the optimal interpolator. Such an attack is related to the familiar Weiner filter problem and seeks to minimize the expression

$$E\left\{\left[x(t) - \sum_{-\infty}^{\infty} x(nT)h_o(t - nT)\right]^2\right\}$$

by solving for a realizable interpolator response, $h_o(t)$, where the statistics of $x(t)$ are known. This problem has been analyzed in detail by several researchers. The first contributors to the area of optimal filtering of sampled data seem to have been Franklin (6) and Lloyd and McMillan (7) followed closely by Stewart (8), whose work along with that of Spilker (9), not only yields optimal filter criteria but also delineates some of the theoretical bounds and limiting behavior to be expected in these interpolators. Perhaps the best such analysis to date, as well as the most recent, is that of Leneman (10) who discusses a procedure for determining an optimal filter subject to several additional constraints which increase the generality of his solution.

Examination of this error, where $h_o(t)$ is chosen so as to be easily realizable, leads to the other basic interpolator type, the Taylor series interpolator. This general class of interpolators operates by using n sampled values to estimate n coefficients in an approximate Taylor series expansion about each sample point. The simplest, and perhaps most common, of these are the zero-order hold which retains only the constant term of the Taylor series, and its immediate offspring, the exponential hold, which is essentially a zero-order hold with an exponentially decaying output. The first-order hold uses two sample points to estimate the constant and first derivative which together approximate the function during a sampling period. The actual form and behavior of such interpolators is obviously dependent upon what sampled values of x(t) may be used to evaluate the coefficients in the expansion. If only past values are utilized to yield the interpolated output, then the output may be used as an approximation without delay, although such an interpolation procedure may introduce an effective decay -- a phenomenon further investigated in Chapter III. In some applications, past as well as future data may be used in selecting the series coefficients for the approximation and an actual delay is introduced into the interpolated output. For example, if an actual delay of one sampling period is permissible, the first-order hold may be used as a linear point connector which yields a linear approximation which is exact at both end points of the sampling interval. General n^{th} order hold circuits have been postulated, with and without delay, and should yield better and better approximations at the cost of increasing complexity.

A particular case of interest occurs when a wide sense stationary random process x(t) is sampled periodically every T seconds by an impulse sampler acting as the input to a zero-order hold whose impulse response has a duration of T seconds. The instantaneous error, for $nT \leq t < (n+1)T$, becomes

$$e(t, d) = x(t - d) - x(nT)$$

with an associated mean square error for the n^{th} sampling interval defined as

$$\psi(nT, d) = \frac{1}{T}\int_{nT}^{(n+1)T} e^2(t)\,dt .$$

For random processes restricted as above, it may be readily shown that

$$E\left\{e(t, d)\right\} = E[x(t-d)] - E[x(nT)] = 0$$

regardless of the probability distribution of x(t) and that

$$E\left\{\psi(nT, d)\right\} = \frac{1}{T} E\left\{\int_{nT}^{(n+1)T}[x(t-d) - x(nT)]^2 dt\right\} =$$

$$= \frac{2}{T}\int_{0}^{T}[R(0) - R(\tau - d)]\,d\tau .$$

This expected value is a functional of the variation of x(t), which is defined by

$$V(\tau) = R(0) - R(\tau) ,$$

and is a measure of one aspect of the interpolation error performance.

Investigation of the variation shows it to have a frequency domain integral form imposed by the Fourier transform relationship between $R(\tau)$ and $S(\omega)$, or

$$V(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)[1 - \cos \omega\tau]\,d\omega .$$

The error criteria defined above are two of the most basic, and the expected mean square error criterion is in general use as an interpolator figure of merit, but a further examination is needed to determine their characteristics as indicators of error performance.

Purpose of Research

The primary engineering problem which appears when sampling with subsequent interpolation is to be utilized is determination of the sampling rate. Such a determination must be made in light of the sampling rate versus interpolation error trade off which exists for any but the ideal unrealizable interpolator. This research is directed toward determination of interpolation error criteria, couched in terms of simple input process statistics, which will permit choice of a sampling rate sufficient to constrain this error to an acceptable level.

The zero-order sample-and-hold will be the basic interpolator to be investigated. Bounds on the behavior of the expected mean square error for both band-limited and non-band-limited sampled random processes will be shown to exist. This error criterion will also be analyzed to determine its dependability, i.e., a comparison of the error criterion to actual interpolator performance, for the general sampled process, and some additional observations will be made for the case where $x(t)$ is Gaussian.

The expected mean square error criterion formulated for the zero-order sample-and-hold in terms of the variation, and hence related to the spectral density of $x(t)$, will be shown to fall into one of several categories based on the behavior of $S(\omega)$ for large ω. Each category is based upon a set of non-restrictive conditions which insure the tractability of the variation, which in turn serves to bound the error criterion. The bounds so obtained may then be used to select a sampling rate sufficient to guarantee satisfaction of a constraint on expected mean square error.

A related problem in the determination of an expected value error criterion is to determine its relationship to the actual performance, since if the two differ greatly then the validity of the error criterion is suspect. For the interpolator discussed above, a mean square error, $\psi(nT, d)$, has been defined for each sampling period. It is apparent that $\psi(nT, d)$ is a random process derived from the sampled process $x(t)$ but, in addition, is dependent upon the time origin of the sampling process and the values of n and T.

The expected value of $\psi(nT, d)$ is an intuitive choice for a mean square error figure of merit since it is a valid criterion for any sampling period (due to the wide sense stationarity restriction imposed on $x(t)$) as well as for any ensemble member (due to the nature of the expected value operator). However, this expected value, by itself, has the serious inherent flaw that it yields neither information about the range of values that $\psi(nT, d)$ can assume, nor about the distribution of these values, nor about the behavior of $\psi(nT, d)$ along a specific ensemble member. Several approaches to this problem will be discussed and a specific expression for the variance of $\psi(nT, d)$ will be obtained for the Gaussian process in terms of a functional of the autocorrelation function of $x(t)$. Although the Gaussian process yields fourth-order moments in terms of second-order statistics and is obviously a natural area of investigation, the analysis will also include comments applicable to more general random processes.

The conventional mean square error criterion has dominated discussion of interpolators because of the ease of its formulation in terms of simple second-order statistics. The utility of additional error criteria is undeniable, especially in the context of interpolation where, for example, the instantaneous error between input and output is instinctively the most natural figure of merit assignable to an interpolator. Although the Gaussian process is often assumed as a model for many statistical problems, its tractability has yet to be utilized to analyze interpolation errors. For this case, knowledge of the second-order probability distribution will be shown to be sufficient to calculate the non-stationary, periodic probability distribution of e(t, d) in terms of a Gaussian distribution with a variance defined by V(t - nT) and thus provide some insight into the nature of the instantaneous error.

In brief, the research is aimed at examination of those characteristics of sample-and-hold interpolation error which will tend to define and clarify the relationship between the sample input, the interpolated output, and the sampling rate.

Review of the Literature

All previous analyses of sampling interpolation error found in the literature are limited in the sense that they have been constrained to studies of the band-limited case or to the non-band-limited case with exact input statistics or to limited examinations of the instantaneous error. However, there are several basic papers which should not be overlooked.

Papoulis has made two contributions related to the interpolation problem. The first is a discussion of errors in band-limited interpolation, although not for zero-order hold, where data, i.e., certain sample values, are altered by any one of several mechanisms -- among them sampling time jitter, round-off error in the samples, and a restricted case of high frequency spectrum overlapping (errors of commission) (11). A discussion of an approximation technique to realize ideal interpolator response is also included in this paper. His second contribution (12, 13) to this area concerns the nature of a band-limited random process, and presents some upper and lower bounds for the variation in terms of the statistics of x(t).

Liff (14), and Leneman and Lewis (15) have investigated the behavior of a number of the more common interpolator schemes for specific input statistics and the latter have presented curves relating their relative mean square error performance. McRae (16) has also investigated and compared the mean square error resulting from a number of conventional interpolation techniques under the assumption of an approximate spectral density.

Finn (17) has analyzed several aspects of the zero-order sample-and-hold interpolator, in particular, mean square error bounds for the general band-limited random process, analysis of expected interpolation error for some specific cases, and an instantaneous error analysis based upon use of the Tchebycheff Inequality.

CHAPTER II

ANALYSIS OF VARIATION

This chapter is devoted to discussion and development of bounds on the variation in terms of parameters of the spectral density for both the band-limited and non-band-limited cases. An investigation of the variation is worthwhile in itself since it serves as a measure of the mean square behavior of a random process; however, its primary importance here is its vital role in the interpolator error problem to be discussed in Chapter III, where it will be shown that the expected mean square error is a functional of the variation.

Definition of Variation

Consider $x(t)$ to be a wide sense stationary real valued random process with autocorrelation function $R(\tau)$ and spectral density $S(\omega)$ related by the Fourier Transform pair

$$R(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\epsilon^{j\omega\tau}\, d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\cos\omega\tau\, d\omega \;, \qquad \text{(2-1a)}$$

$$S(\omega) = \int_{-\infty}^{\infty} R(\tau)\epsilon^{-j\omega\tau}\, d\tau = \int_{-\infty}^{\infty} R(\tau)\cos\omega\tau\, d\tau \;, \qquad \text{(2-1b)}$$

where the cosine integrals result because $S(\omega)$ and $R(\tau)$ are even functions.

For such a random process, the variation has been defined to be

$$V(\tau) = R(0) - R(\tau) . \tag{2-2}$$

Intuitively, one can relate $V(\tau)$ to the mean square behavior of $x(t)$ since

$$V(\tau) = \frac{1}{2} E\left\{[x(t+\tau) - x(t)]^2\right\} . \tag{2-3}$$

$V(\tau)$ may also be written in terms of the spectral density as

$$V(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega - \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)\cos\omega\tau\, d\omega \tag{2-4}$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)[1 - \cos\omega\tau]d\omega .$$

The variation is obviously bounded above by $2R(0)$ and below by zero for all τ. These two bounds are rather crude, however, since they are based on the absolute maximum and minimum values of $[1 - \cos\omega\tau]$ and it seems that $V(\tau)$ should not jump from its zero value at $\tau = 0$ to its maximum value $2R(0)$ for arbitrarily small τ. Inherent restrictions in the integral formulation of (2-4) may be used to obtain more meaningful functional bounds.

Henceforth, $R(\tau)$, $V(\tau)$, and $S(\omega)$ will be considered to be related in the manner defined in this section. If $S(\omega)$ is termed band-limited to ω_c, then $S(\omega) = 0$ for all $|\omega| > \omega_c$. Only real valued $x(t)$ are to be considered.

The Variation of a Band-Limited Random Process

Upper Bounds

Differentiability of $R(\tau)$. Papoulis (18) and Finn (19) have analyzed the behavior of band-limited processes by making use of the fact that $R(\tau)$ so restricted must be infinitely differentiable, thus $|R''(0)|$ is finite -- a result easily seen in the frequency domain.

Let $F(j\omega)$ be the Fourier transform of $f(t)$, then $(j\omega)^n F(j\omega)$ is the Fourier transform of $\frac{d^n f}{dt^n}$. Note that a derivative so determined might not always be finite, i.e., might contain impulses, therefore, $|\frac{d^n f}{dt^n}(0)|$ does not necessarily exist. The behavior of $R^{(n)}(\tau)$ may thus be investigated by examination of the inverse transform of $(j\omega)^n S(\omega)$, i.e.,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} (j\omega)^n S(\omega)\epsilon^{j\omega\tau}\, d\omega \ . \tag{2-5}$$

For band-limited spectral densities, (2-5) may be bounded as follows,

$$\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} |(j\omega)^n S(\omega)\epsilon^{j\omega\tau}|d\omega \le \frac{\omega_c^{\ n}}{2\pi}\int_{-\omega_c}^{\omega_c} S(\omega)d\omega = \omega_c^{\ n}R(0) \ . \tag{2-6}$$

Thus the inverse transform integral of (2-5) is bounded and $R^{(n)}(\tau)$ exists for any n. In particular, $|R''(0)| \le \omega_c^{\ 2}R(0)$. This property along with the trigonometric relations,

$$|\sin\varphi| \le \varphi \ , \tag{2-7a}$$

$$1 - \cos\varphi = 2\sin^2(\tfrac{\varphi}{2}) \ , \tag{2-7b}$$

may be used to obtain an upper bound on $V(\tau)$.

<u>Quadratic Bound</u>. Suppose that $S(\omega)$ is band-limited to ω_c, then

$$V(\tau) \leq |R''(0)| \frac{\tau^2}{2} \leq \frac{(\omega_c\tau)^2}{2} R(0) \ . \qquad (2\text{-}8)$$

Proof:

$$V(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S(\omega)[1 - \cos \omega\tau] \, d\omega \ .$$

This may be rewritten from (2-7b) as

$$V(\tau) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} S(\omega)[2 \sin^2(\tfrac{\omega\tau}{2})] d\omega \ . \qquad (2\text{-}9)$$

But $S(\omega) \geq 0$, and by using (2-7a) in (2-9),

$$V(\tau) \leq \frac{\tau^2}{2} \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \omega^2 S(\omega) d\omega = \frac{\tau^2}{2} |R''(0)| \ .$$

In addition, since $R(\tau)$ is differentiable, then $|R''(0)| \leq \omega_c^2 R(0)$, and

$$V(\tau) \leq \frac{\tau^2}{2} |R''(0)| \leq \frac{(\omega_c\tau)^2}{2} R(0) \ .$$

Sine Bound. Suppose that $S(\omega)$ is band-limited to ω_c, then, for $\tau \in [0, \pi/\omega_c]$,

$$V(\tau) \leq 2 \sin^2(\frac{\omega_c\tau}{2}) R(0) \ . \qquad (2\text{-}10)$$

Proof: Since $[1 - \cos x]$ is monotone increasing for $x \in [0, \pi]$, then

$$1 - \cos b \geq 1 - \cos x, \ 0 \leq x \leq b \leq \pi \ . \qquad (2\text{-}11)$$

From (2-11),

$$V(\tau) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} S(\omega)\,[1 - \cos\omega\tau\,]d\omega \leq$$

$$\leq \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} S(\omega)\,[1 - \cos\omega_c\tau\,]d\omega = 2\,\sin^2(\frac{\omega_c\tau}{2})\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega \,.$$

This bound was obtained independently by Papoulis (20). It is not necessarily the best bound for all variations since the quadratic bound of the preceding section may well be valid for $\tau > \frac{\pi}{\omega_c}$, but it does serve to define the extreme behavior possible for any variation.

A Lower Bound

Finn (21) obtained a lower bound on $V(\tau)$, and the following derivation yields his result and shows its relationship to a lower bound obtained by Papoulis.

Suppose that $S(\omega)$ is band-limited to ω_c, then

$$V(\tau) \geq 2\left[\frac{\sin(\frac{\omega_c\tau}{2})}{\omega_c}\right]^2 |R''(0)| \geq \frac{\tau^2}{2}\left[\frac{\sin\frac{\omega_c T}{2}}{\frac{\omega_c T}{2}}\right]^2 |R''(0)| \quad (2\text{-}12)$$

for $0 \leq \tau \leq T \leq \frac{2\pi}{\omega_c}$.

Proof: Since sin x is concave (has a negative second derivative) for $x \in [0, \pi]$, then sin x is greater than the secant line connecting sin a and sin b for $0 \leq a \leq x \leq b \leq \pi$. The equation of the secant line is $x\,\frac{\sin b}{b}$ and is a reasonable straight line approximation to sin x for $0 \leq a \leq b \leq \frac{\pi}{2}$. Thus,

$$\sin x \geq x \frac{\sin b}{b}, \quad 0 \leq x \leq b \leq \pi, \tag{2-13}$$

and

$$V(\tau) = \frac{1}{\pi}\int_{-\omega_c}^{\omega_c} S(\omega)\ \sin^2(\tfrac{\omega\tau}{2})d\omega \geq \frac{1}{\pi}\int_{-\omega_c}^{\omega_c} S(\omega)\left[(\tfrac{\omega\tau}{2})\ \frac{\sin(\frac{\omega_c\tau}{2})}{\frac{\omega_c\tau}{2}}\right]^2 d\omega =$$

$$= \left[\frac{\sin(\frac{\omega_c\tau}{2})}{\omega_c}\right]^2 \frac{1}{\pi}\int_{-\omega_c}^{\omega_c} \omega^2 S(\omega)d\omega = 2\left[\frac{\sin(\frac{\omega_c\tau}{2})}{\omega_c}\right]^2 |R''(0)| .$$

This may be further simplified since

$$V(\tau) \geq 2\left[\frac{\sin(\frac{\omega_c\tau}{2})}{\omega_c}\right]^2 |R''(0)| \geq 2\left[\frac{\frac{\omega_c\tau}{2}\ \frac{\sin\frac{\omega_c T}{2}}{\frac{\omega_c T}{2}}}{\omega_c}\right]^2 |R''(0)| =$$

$$= \frac{\tau^2}{2}\left[\frac{\sin(\frac{\omega_c T}{2})}{\frac{\omega_c T}{2}}\right]^2 |R''(0)| .$$

Papoulis (22), using the straight line approximation to $\sin x$ given by $\frac{2}{\pi} x$ for $x \in [0, \pi/2]$, also obtained a form of the latter bound for the case where $\tau \leq \frac{\pi}{2\omega_c}$. This quadratic bound is obviously not as tight a bound as the sine squared bound given in (2-12), at least for small τ, and they approach each other only as $\omega_c T \to 0$. Note that (2-12) implies that if $V(\tau_1) = 0$, then $\tau_1 \geq \frac{2\pi}{\omega_c}$.

<u>Derivative Behavior of $V(\tau)$</u>

<u>Monotonicity of $V(\tau)$</u>. Suppose that $S(\omega)$ is band-limited to ω_c,

then $V(\tau)$ is monotone increasing for $\tau \in [0, \pi/\omega_c]$.

Proof: It suffices to show that $V'(\tau)$ is non-negative in the interval $[0, \pi/\omega_c]$. Now

$$V'(\tau) = \frac{d}{d\tau}\left\{\frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} S(\omega)[1 - \cos\omega\tau]\,d\omega\right\} = \tag{2-14}$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \omega S(\omega)\ \sin\omega\tau\,d\omega\ .$$

Using (2-13) in this integral expression, a lower bound may be obtained as follows,

$$V'(\tau) \geq \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \omega S(\omega)[\omega\tau\ \frac{\sin\omega_c\tau}{\omega_c\tau}]\,d\omega = \tag{2-15}$$

$$= \frac{\sin\omega_c\tau}{\omega_c}\ |R''(0)|\ .$$

This lower bound on $V'(\tau)$ is non-negative for $\tau \in [0, \pi/\omega_c]$, thus $V(\tau)$ is monotone increasing.

<u>Convexity of $V(\tau)$</u>. Suppose that $S(\omega)$ is band-limited to ω_c, then $V(\tau)$ is convex for $\tau \in [0, \pi/2\omega_c]$.

Proof: The expression in (2-14) may be differentiated once more to yield

$$V''(\tau) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} \omega^2 S(\omega)\cos\omega\tau\,d\omega\ . \tag{2-16}$$

By inspection, the integrand is non-negative for $\tau \in [0, \pi/2\omega_c]$, thus

$V''(\tau)$ is non-negative and $V(\tau)$ is convex, i.e., has an increasing positive first derivative.

Derivative Bounds. Suppose that $S(\omega)$ is band-limited to ω_c, then, for $\tau \in [0, \pi/\omega_c]$,

$$\frac{\sin \omega_c \tau}{\omega_c} |R''(0)| \leq V'(\tau) \leq \omega_c \sin \omega_c \tau R(0) . \qquad (2\text{-}17)$$

Proof: The lower bound follows from (2-15). Since $\omega \sin \omega\tau$ is monotone increasing for $\tau \in [0, \pi/\omega_c]$ and $\omega \in [0, \omega_c]$, then

$$V'(\tau) = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} \omega S(\omega) \sin \omega\tau \, d\omega \leq \omega_c \sin \omega_c \tau \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} S(\omega) d\omega ,$$

and the upper bound follows. This technique could be used to bound higher order derivatives.

The Variation of a Non-Band-Limited Random Process

Immediate extension of the previous techniques to the non-band-limited case is not obvious. In particular, it seems impossible to formulate a lower bound without excessive restriction on the nature of $S(\omega)$. However, some meaningful results can be obtained.

Classification of Non-Band-Limited Processes

In the following sections, and throughout the remainder of the discussion, the only non-band-limited random processes considered are those with spectral densities which may be written as the ratio of two even polynomials in ω, i.e.,

$$S(\omega) = \frac{N(\omega^2)}{D(\omega^2)} = \frac{a_o + a_2\omega^2 + \ldots + a_{2m}\omega^{2m}}{b_o + b_2\omega^2 + \ldots + b_{2n}\omega^{2n}} \tag{2-18}$$

where $D(\omega^2)$ has no real roots and $m \leq n-1$. Such spectral densities will further be classified according to their relative high frequency behavior. The concept of the order, k, of $S(\omega)$, may be used where $k \equiv n-m$ and

$$S(\omega) \simeq \frac{a_{2m}}{b_{2n}} \omega^{-2k} \tag{2-19}$$

for large ω. The order also serves to define the rolloff rate of the spectral density. It can be shown that first-order spectral densities represent non-differentiable random processes while all higher order spectral densities represent differentiable processes.

Consider the inverse transform of $\omega^2 S(\omega)$

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \omega^2 S(\omega) \epsilon^{j\omega\tau} \, d\omega \ . \tag{2-20}$$

From Fourier Transform theory (23) if (2-20) exists then it must represent the second derivative of $-R(\tau)$. For first-order spectral densities, (2-20) does not exist in the normal sense since

$$\int_{-\infty}^{\infty} |\omega^2 S(\omega)| d\omega = \int_{-\infty}^{\infty} \frac{a_o\omega^2 + a_2\omega^4 + \ldots + a_{2m}\omega^{2n}}{b_o + b_2\omega^2 + \ldots + a_{2n}\omega^{2n}} \, d\omega$$

is undefined because the integrand approaches a non-zero constant for large ω. A more complete answer could be obtained from impulse theory;

however, for the purposes of this discussion, it is sufficient to note that $|R''(0)|$ is undefined for first-order spectral densities. The integral of (2-20) exists automatically for all higher order data since

$$\int_{-\infty}^{\infty} |\omega^2 S(\omega)|d\omega = \int_{-\infty}^{\infty} \frac{a_o\omega^2 + a_2\omega^4 + \ldots + a_{2m}\omega^{2m+2}}{b_o + b_2\omega^2 + \ldots + b_{2n}\omega^{2n}} d\omega < \infty$$

because $(2m+2) \leq 2n-2$ so that for large ω the integrand falls off at least as fast as some c/ω^2 and is thus integrable. Existence of this integral implies that $0 < |R''(0)| < \infty$ for all higher order $S(\omega)$. Now, granting an interchange of limits and expectation, the expected value of $x'(t)$ is

$$\lim_{\epsilon \to o} E\left\{\left[\frac{x(t+\epsilon) - x(t)}{\epsilon}\right]^2\right\} = \lim_{\epsilon \to o} \left\{\frac{2[R(0) - R(\epsilon)]}{\epsilon^2}\right\} = -R''(0)$$

which exists under the above conditions, hence $x(t)$ is differentiable in the mean square sense (24). It will be seen that the two broad classifications: first-order or non-differentiable, and higher order or differentiable are sufficient to determine bounds on the variations of the class of non-band-limited processes defined in (2-18).

Differentiable Random Processes

The following quadratic bound may be determined from the above discussion.

Suppose that $x(t)$ is a random process differentiable in the mean square sense, and has an autocorrelation function $R(\tau)$ with Fourier Transform $S(\omega)$, then

$$V(\tau) \leq \frac{|R''(0)|}{2}\tau^2 = \frac{(\omega_d\tau)^2}{2} R(0) \qquad (2\text{-}21)$$

where $\omega_d^{\,2} \equiv |R''(0)|/R(0)$.

Proof:

$$V(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)[1 - \cos\omega\tau]d\omega = \qquad (2\text{-}22)$$

$$= \frac{1}{\pi}\int_{-\infty}^{\infty} S(\omega)\ \sin^2(\frac{\omega\tau}{2})\ d\omega \leq$$

$$\leq \frac{1}{\pi}\int_{-\infty}^{\infty} (\frac{\omega\tau}{2})^2\ S(\omega)d\omega = \frac{\tau^2}{2}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2 S(\omega)d\omega\ ,$$

using (2-5) and the fact that both $S(\omega)$ and $\sin^2(\frac{\omega\tau}{2})$ are positive for all ω. Since $x(t)$ is differentiable, then the last integral in (2-22) must exist and equals $|R''(0)|$. Defining an artificial effective band-limited frequency

$$\omega_d \equiv [\frac{|R''(0)|}{R(0)}]^{1/2} \qquad (2\text{-}23)$$

and then substituting it in the above, the quadratic bound of (2-21) may be obtained. $V(\tau)$ might now be compared to the variation of a process band-limited to ω_d.

<u>Non-Differentiable Random Processes</u>

In general, random processes do not have to be differentiable, and the familiar exponential autocorrelation function, i.e., $R(\tau) = \epsilon^{-\alpha|\tau|}$, is just such a case. With additional restrictions, some results can be obtained for this situation.

Suppose that x(t) is a random process with autocorrelation function $R(\tau)$ and spectral density $S(\omega)$, where

$$\left[\frac{k}{\omega^2} - S(\omega)\right] \geq 0 \tag{2-24}$$

for all ω, then

$$V(\tau) \leq \frac{k|\tau|}{2} = \omega_n|\tau|\, R(0) \tag{2-25}$$

where $\omega_n \equiv \frac{k}{2R(0)}$.

Proof:

$$V(\tau) = \frac{1}{\pi}\int_{-\infty}^{\infty} S(\omega)\left[\sin^2\left(\frac{\omega\tau}{2}\right)\right]d\omega \leq$$

$$\leq \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{k}{\omega^2} \sin^2\left(\frac{\omega\tau}{2}\right) d\omega$$

since $\frac{k}{\omega^2} \geq S(\omega)$ for all ω. Let $\gamma = \frac{\omega\tau}{2}$ then

$$\frac{k}{\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2} \sin^2\left(\frac{\omega\tau}{2}\right)d\omega = \frac{k|\tau|}{2\pi}\int_{-\infty}^{\infty} \frac{\sin^2\gamma}{\gamma^2}\, d\gamma$$

but this last integral is a well-behaved one with a value equal to π. Substitution in the above leads to the bound of (2-25).

The form chosen to express the normalized bounds for both non-band-limited cases is related to the autocorrelation of the process, R(0), and defines an "effective" cutoff frequency, ω_n or ω_d. Since the variation itself is known to be bounded by 2R(0), neither of these bounds yields any information if their value is greater than 2R(0) -- a condition which occurs in both bounds if either $\omega_d\tau$ or $\omega_n\tau$ is greater than 2. Both bounds are valid for any τ, but obviously have an effective useful limit.

Lower Bounds

For the non-band-limited case, there do not seem to be any techniques leading to a lower bound similar to that obtained for the band-limited case even permitting excessive restrictions on the nature of $S(\omega)$.

Papoulis (25) lists a non-functional lower bound, i.e., $V(\tau) \geq \frac{1}{4^n} V(2^n\tau)$, but this yields no information as to the nature of $V(\tau)$. Basically, any integral bounding approach such as those used for the upper bounds breaks down when lower bounds for infinite integrals are sought.

Derivative Bounds

With the addition of a few more constraints on $S(\omega)$, a comment on the monotonicity of $R(\tau)$ can be made.

Upper Bound. Suppose that x(t) is mean square differentiable and has autocorrelation function $R(\tau)$ and spectral density $S(\omega)$, then

$$V'(\tau) \leq |R''(0)|\tau, \ \tau > 0 \ . \qquad (2\text{-}26)$$

Proof: The expression

$$V'(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega S(\omega)\sin\omega\tau \, d\omega$$

may be bounded since $\omega \sin\omega\tau \leq \omega^2\tau$, a consequence of (2-7a), and

$$V'(\tau) \le \frac{1}{2\pi}\int_{-\infty}^{\infty} \omega^2\tau S(\omega)d\omega = |R''(0)|\tau \ .$$

Since x(t) is differentiable, then existence of the integral of $\omega^2 S(\omega)$ is assured.

Monotonicity. Suppose x(t) has $R(\tau)$ with Fourier transform $S(\omega)$ and there exist k and ω_1 such that for $n \ge 2$

$$\frac{k}{\omega^{2n}} \ge S(\omega) \qquad (2\text{-}27)$$

and

$$\omega^{2n}S(\omega) > \omega_1^{\,2n}S(\omega) \quad \text{for } |\omega| > \omega_1 \qquad (2\text{-}28)$$

then $V'(\tau) > 0$ if $\tau < \frac{\pi}{\omega_1}$ and

$$\frac{S(\omega_1)}{k/(\omega_1)^{2n}} > \frac{\pi}{2(n-1)(1+\cos\omega_1\tau)} \ . \qquad (2\text{-}29)$$

Proof: The integral expression for $V'(\tau)$ may be written as a one-sided integral composed of two parts, i.e.,

$$V'(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\omega S(\omega)\sin\omega\tau\, d\omega = \frac{1}{\pi}\int_{0}^{\infty}\omega S(\omega)\sin\omega\tau\, d\omega =$$

$$= \frac{1}{\pi}\int_{0}^{\pi/\tau}\omega S(\omega)\sin\omega\tau\, d\omega + \frac{1}{\pi}\int_{\pi/\tau}^{\infty}\omega S(\omega)\sin\omega\tau\, d\omega \ .$$

Each of these two integrals may be bounded under the assumptions made above:

$$\frac{1}{\pi}\int_{o}^{\pi/\tau}\omega S(\omega)\sin\omega\tau\,d\omega \geq \frac{1}{\pi}\int_{\omega_1}^{\pi/\tau}\frac{\omega_1^{2n}S(\omega_1)}{\omega^{2n-1}}\sin\omega\tau\,d\omega \geq$$

$$\geq \frac{\omega_1^{2n}S(\omega_1)}{\pi}\int_{\omega_1}^{\pi/\tau}\frac{\sin\omega\tau\,d\omega}{(\pi/\tau)^{2n-1}} = \frac{\omega_1^{2n}S(\omega)[1+\cos\omega_1\tau]}{\pi^2(\frac{\pi}{\tau})^{2n-2}},$$

and,

$$\frac{1}{\pi}\int_{\pi/\tau}^{\infty}\omega S(\omega)\sin\omega\tau\,d\omega \geq -\frac{1}{\pi}\int_{\pi/\tau}^{\infty}\omega S(\omega)d\omega \geq$$

$$\geq -\frac{1}{\pi}\int_{\pi/\tau}^{\infty}\frac{k\,d\omega}{\omega^{2n-1}} = \frac{-k}{2\pi(n-1)(\frac{\pi}{\tau})^{2n-2}}\ .$$

Combining the bounds,

$$V'(\tau) \geq \frac{k}{\pi(\frac{\pi}{\tau})^{2n-2}}\left\{\frac{S(\omega_1)[1+\cos\omega_1\tau]}{\pi\left(\frac{k}{\omega_1^{2n}}\right)} - \frac{1}{2(n-1)}\right\},$$

but the term in brackets is positive for τ restricted as above, thus $V(\tau)$ is monotone increasing. No functional lower bound is obtainable for this general case.

Application to Characteristic Functions

Consider the characteristic function defined by

$$\Phi(u) = E\left\{\epsilon^{jux}\right\} = \int_{-\infty}^{\infty}\epsilon^{jux}\,p(x)dx\ .$$

$\Phi(u)$ and $p(x)$ have the same relationship as $R(\tau)$ and $S(\omega)$. Symmetry of

$p(x)$ means $\Phi(u)$ is real and corresponds to the case for $R(\tau)$ where $x(t)$ is real. Since the results obtained for $V(\tau)$ can be related to bounds on $R(\tau)$, and if $p(x)$ is symmetric and as tractable as the $S(\omega)$ considered previously, then bounds can be established for $\Phi(u)$.

Gnedenko and Kolmogoroff (26) established one property of such transforms (to be discussed for the variation in the next section), namely that if $|\Phi(u_k)| = 1$ for some $u_k \neq 0$, then $\Phi(u)$ is the characteristic function of an improper distribution requiring impulses, i.e., $x(t)$ may assume only discrete values.

<u>Mean Square Periodicity of $V(\tau)$</u>

The process $x(t)$ is said to be mean square periodic if there exists a T_o such that the following equivalent conditions may be satisfied for $T_o \neq 0$ and any value of τ:

$$E\left\{[x(t+\tau) - x(t)]^2\right\} = E\left\{[x(t+T_o+\tau) - x(t)]^2\right\},$$

$$V(\tau) = V(\tau + T_o),$$

$$R(\tau) = R(\tau + T_o).$$

Papoulis (27) showed that if $V(\tau_o) = 0$ for some $\tau_o \neq 0$, then $x(t)$ is mean square periodic since if

$$V(\tau_o) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)[1 - \cos\omega\tau_o]\,d\omega = 0, \qquad (2\text{-}30)$$

then either $S(\omega) \equiv 0$, a trivial case, or $S(\omega)$ is a collection of impulses occurring at the zeroes of $[1 - \cos\omega\tau]$, i.e.,

$$S(\omega) = \sum_{-\infty}^{\infty} A_n \delta(\omega - \frac{2n\pi}{\tau_o}) . \qquad (2\text{-}31)$$

Thus, $R(\tau)$ is periodic, since it has a Fourier Series representation, with a period at least as small as τ_o.

$V(\tau)$ may attain another extreme, $2R(0)$, and a related analysis yields some additional information about mean square periodic processes.

Suppose that $V(\tau_2) = 2R(0)$, then $x(t)$ is mean square periodic and for $\tau_o = 2n\tau_2$, $n=1,2,\ldots$, $V(\tau_o) = 0$.

Proof:

$$V(\tau_2) = \frac{1}{\pi}\int_{-\infty}^{\infty} S(\omega)\sin^2(\frac{\omega\tau_2}{2})d\omega = 2R(0),$$

but,

$$R(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)d\omega ,$$

thus $V(\tau_2)$ can equal $2R(0)$ if and only if

$$\frac{1}{\pi}\int_{-\infty}^{\infty} S(\omega)\sin^2(\frac{\omega\tau_2}{2})d\omega = \frac{1}{\pi}\int_{-\infty}^{\infty} S(\omega)d\omega .$$

Since $\sin^2(\frac{\omega\tau}{2}) \leq 1$, this equality can hold only if $S(\omega)$ consists of impulses occurring at the ones of $\sin^2(\frac{\omega\tau_2}{2})$, i.e.,

$$S(\omega) = \sum_{-\infty}^{+\infty} B_k \delta[\omega - \frac{(2k+1)\pi}{\tau_2}] , \qquad (2\text{-}32)$$

hence $R(\tau)$ is periodic.

Now

$$V(\tau_o) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\sum_{-\infty}^{\infty} B_k \delta[\omega - \frac{(2k+1)\pi}{\tau_2}]\right\}\sin^2(\frac{\omega\tau_o}{2})d\omega = V(2n\tau_2) .$$

Using the sifting property of the impulse,

$$V(\tau_o) = \sum_{-\infty}^{\infty}\frac{B_k}{\pi}\sin^2\left[\frac{(2k+1)\pi\tau_o}{2\tau_2}\right] = \sum_{-\infty}^{\infty}\frac{B_k}{\pi}\sin^2[(2k+1)n\pi] = 0 .$$

Similarly,

$$V[(2n+1)\tau_2] = 2R(0) \quad \text{for } n = 0,1,2,\dots .$$

The converse statement that $V(\tau_o) = 0$ implies existence of τ_2 such that $V(\tau_2) = 2R(0)$ is not true. Consider $S(\omega)$ as specified in (2-31), then

$$V(\tau_2) = \frac{1}{\pi}\int_{-\infty}^{\infty}\left\{\sum_{-\infty}^{\infty} A_n\delta[\omega - \frac{2n\pi}{\tau_o}]\right\}\sin^2(\frac{\omega\tau_2}{2})\,d\omega =$$

$$= \sum_{-\infty}^{\infty}\frac{A_n}{\pi}\sin^2[\frac{n\pi\tau_2}{\tau_o}] .$$

For $V(\tau_2) = 2R(0)$, $\frac{n\pi\tau_2}{\tau_o}$ must equal some odd multiple of $\frac{\pi}{2}$ for all n, a condition obviously not satisfied by the above series unless n takes on only odd values as would be the case if $A_n = 0$ for n even, i.e., $S(\omega)$ has the form given by (2-32).

The extreme values attainable by any autocorrelation function, band-limited or non-band-limited, can occur only if the spectral densities are of the restricted impulse summation form discussed above and imply mean square periodicity of the random process.

CHAPTER III

SAMPLE AND HOLD INTERPOLATOR PERFORMANCE

The preceding chapter has investigated some of the properties of the second-order statistics of a random process, in particular, the behavior of the variation. This information will now be used to determine interpolation error bounds for some sample-and-hold interpolators.

A Definition of Interpolator Error

Sample-and-hold interpolation, in general, is based upon the premise that some finite number of samples of a time function, taken at equally spaced and sufficiently short intervals, can be used to create an approximate finite Taylor series expansion which will represent the original function adequately over one sampling interval. There is, of course, a complex relationship between the number of samples used in the expansion, the length of the sampling interval, the statistics of the random process, and the quality of the approximation. A block diagram of a structure yielding a useful interpolator error comparator was illustrated in Figure 1. An input $x(t)$ is sampled at a uniform rate, unless otherwise specified, so that one sample is taken every T seconds. The sampled input to the interpolator is given by

$$x^*(t) = \sum_{-\infty}^{\infty} x(t_o + nT)\delta(t - t_o - nT) \qquad (3\text{-}1)$$

where t_o represents any shift between the actual time origin and the sampling points. The interpolator response to a unit sample occurring at $t = t_o + nT$ is $h_o(t - t_o - nT)$. The output of the interpolator is

$$\hat{x}(t) = \int_{-\infty}^{\infty} \left\{ \sum_{-\infty}^{\infty} x(t_o + nT)\delta(\tau - t_o - nT) \right\} h_o(t - \tau)d\tau = \qquad (3\text{-}2)$$

$$= \sum_{-\infty}^{\infty} x(t_o + nT)h_o(t - t_o - nT) \; .$$

To obtain an error criterion, the difference between $\hat{x}(t)$ and an arbitrarily delayed version of $x(t)$ is used as a basis and leads to an instantaneous error

$$e(t, d) = x(t - d) - \hat{x}(t) \; . \qquad (3\text{-}3)$$

It will prove advantageous to define the delay in terms of a fractional delay, λ, where

$$d = \lambda T$$

so that the relationship of the delay to the sampling interval duration T remains clear.

Expected Mean Square Error Criterion

General Derivation

The following discussion will be restricted to consideration of those interpolators whose response to each input sample is non-zero only within that particular sampling interval. The expression of (3-2) simplifies considerably for such interpolators since, for a given t, only one term in the summation is non-zero; i.e., if $t \varepsilon [t_o + nT, t_o + (n+1)T]$ then

$$\hat{x}(t) = x(t_o + nT)h_o(t - t_o - nT) \tag{3-4}$$

and

$$e(t, \lambda T) = x(t - \lambda T) - x(t_o + nT)h_o(t - t_o - nT). \tag{3-5}$$

The n^{th} sampling interval has a mean squared error associated with it given by

$$\psi_h(nT, \lambda) = \frac{1}{T}\int_{t_o+nT}^{t_o+(n+1)T} e^2(t, \lambda T)\, dt = \tag{3-6}$$

$$= \frac{1}{T}\int_{t_o+nT}^{t_o+(n+1)T} [x(t-\lambda T)-x(t_o+nT)h_o(t-t_o-nT]^2 dt.$$

Making a change of variables, the expected value of $\psi_h(nT, \lambda)$ is

$$\bar{\psi}_h(T,\lambda) = \frac{1}{T} E\left\{\int_o^T [x^2(t' + t_o + nT - \lambda T) - 2x(t' + t_o + nT - \lambda T)x(t_o + nT) \cdot \right.$$

$$\left. \cdot\, h_o(t') + x^2(t_o + nT)h_o^2(t')]dt'\right\} =$$

$$= \frac{1}{T}\int_o^T \left\{R(0)[1 + h_o^2(t')] - 2R(t' - \lambda T)h_o(t')\right\} dt' , \tag{3-7}$$

where due to the stationarity of $x(t)$, $\bar{\psi}_h(T, \lambda)$ is the same for each sampling interval, and is a measure of the quality of the approximation. Since the expected mean square error criterion is independent of t_o, it is often assumed to be zero; however, the actual mean square error is related to t_o.

Zero-Order Hold

Consider the simplest Taylor series interpolator where only the most recent sample value is retained and used as an interpolated x(t). The response of such an interpolator to a unit input sample is unity within the appropriate sampling interval and zero elsewhere. From (3-4), it is readily seen that the interpolated output $\hat{x}(t)$ is equal to $x(t_o + nT)$ for $t \varepsilon [t_o + nT, t_o + (n+1)T)$.

The instantaneous error $e(t, \lambda T)$ for $t \varepsilon [t_o + nT, t_o + (n+1)T)$ becomes

$$e(t, \lambda T) = x(t - \lambda T) - x(t_o + nT) \tag{3-8}$$

and the expected mean square error of (3-7) becomes

$$\bar{\psi}(T,\lambda) = \frac{2}{T}\int_0^T [R(0) - R(t' - \lambda T)]dt' = \tag{3-9}$$

$$= \frac{2}{T}\int_{-\lambda T}^{(1-\lambda)T} [R(0) - R(\tau)]d\tau = \frac{2}{T}\int_{-\lambda T}^{(1-\lambda)T} V(\tau)d\tau \ .$$

Figure 2 illustrates some wave forms representative of the general behavior of a zero-order sample-and-hold.

Error Reduction by Delay

If in (3-9) above, T is considered fixed, then λ might be chosen such that $\bar{\psi}(T, \lambda)$ is a minimum. It is widely stated that $\lambda = 1/2$ yields this minimum; however, an additional condition needs to be satisfied.

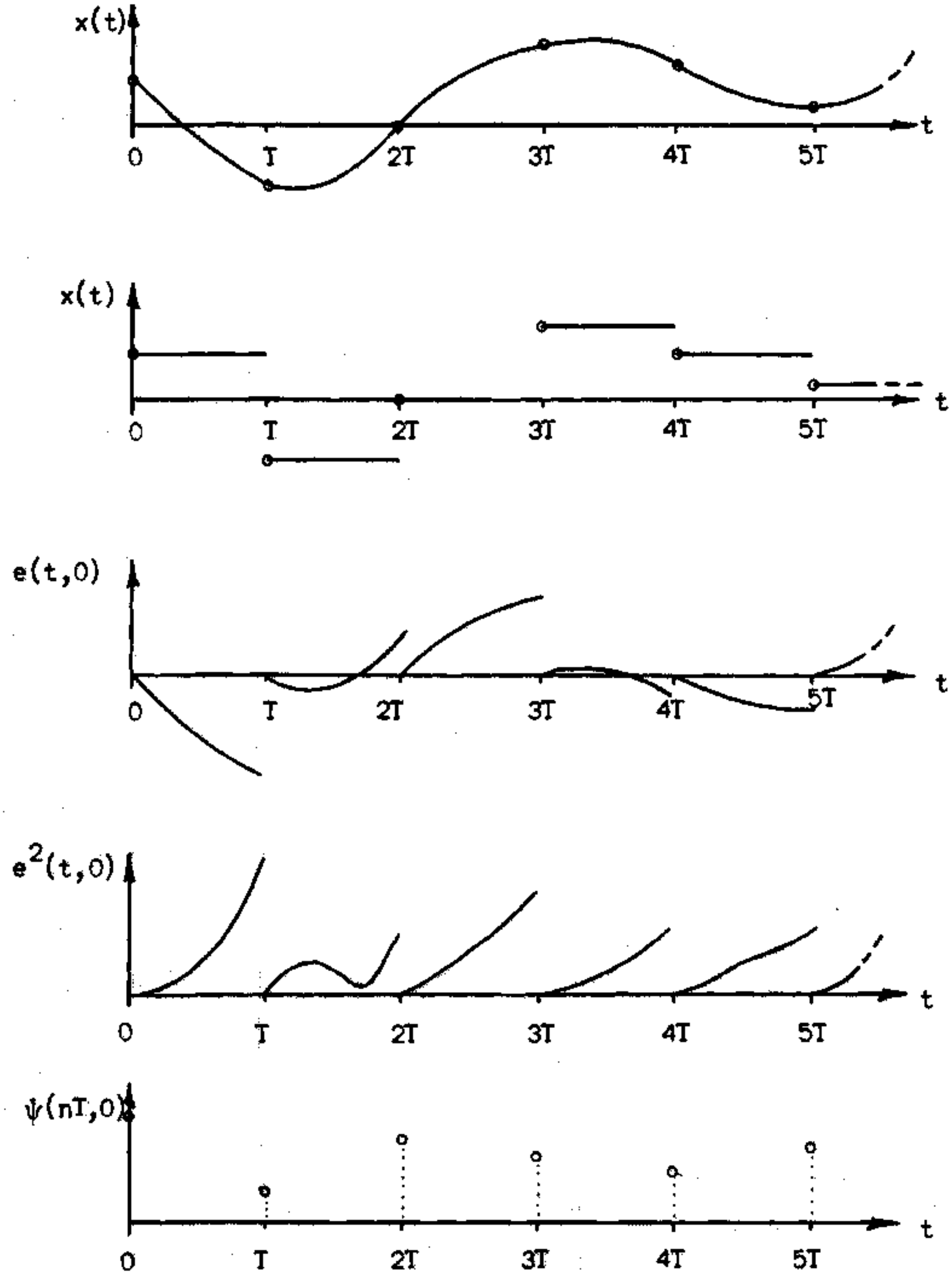

Figure 2. Typical Waveforms Present in a Zero-Order Sample-and-Hold Interpolator (λ = 0).

Maxima and minima of $\bar{\psi}(T, \lambda)$ occur at points where $\frac{d}{d\lambda}[\bar{\psi}(T, \lambda)]|_{\lambda=\lambda_m}$ either equals zero or ceases to exist. Assuming that an interchange of differentiation and integration is valid, the derivative may be evaluated as follows

$$\frac{d}{d\lambda}[\bar{\psi}(T,\lambda)]\,|_{\lambda=\lambda_m} = \frac{d}{d\lambda}\left[\frac{2}{T}\int_{-\lambda T}^{(1-\lambda)T} V(\tau)d\tau\right]|_{\lambda=\lambda_m} = \tag{3-10}$$

$$= \frac{2}{T}\left\{-TV[(1-\lambda_m)T] + TV[-\lambda_m T]\right\} =$$

$$= 2\left\{V[\lambda_m T] - V[(1-\lambda_m)T]\right\} = 0\ ,$$

where the last step follows from the evenness of $V(\tau)$. For continuous $V(\tau)$, a mild restriction, the first derivative of $\bar{\psi}(T, \lambda)$ must exist for all values of λ, thus any extrema must satisfy the condition of (3-10). Such points are minima of $\bar{\psi}(T, \lambda)$ if, in addition to (3-10), $\frac{d^2}{d\lambda^2}[\bar{\psi}(T, \lambda)]\,|_{\lambda=\lambda_m} > 0$, or,

$$\frac{d^2}{d\lambda^2}[\bar{\psi}(T, \lambda)]\,|_{\lambda=\lambda_m} = 2T\left\{V'(\lambda_m T) + V'[(1-\lambda_m)T]\right\} > 0. \tag{3-11}$$

Inspection of (3-10) shows $\lambda_m = \frac{1}{2}$ will always satisfy the first derivative condition. Evaluating (3-11) for this value of λ, it may be seen that the second derivative condition is satisfied only if $V'(\frac{T}{2}) > 0$ and is automatically satisfied for band-limited processes if $T < \frac{2\pi}{\omega_c}$ since, from (2-15), $V'(\tau)$ is monotone increasing for $\tau \in [0, \frac{\pi}{\omega_c}]$. It may also be satisfied by some non-band-limited processes as discussed in Chapter II.

The following conclusion may be drawn. In general, if $\hat{x}(t)$ represents the interpolation of $x(t)$ by a zero-order sample-and-hold, then, judged by the expected mean square error criterion, $\hat{x}(t)$ is a better approximation to $x(t - \frac{T}{2})$ than it is for any other value of delay, if $V'(\frac{T}{2}) > 0$. However, for this same criterion, if $V'(\frac{T}{2})$ is not positive, then the apparent delay in $\hat{x}(t)$ is not T/2 but whatever value of λ_m satisfies both (3-10) and (3-11).

Band-Limited Interpolation Error

The formula for $\bar{\psi}(T, \lambda)$ given in (3-9) is a functional of the variation, $V(\tau)$, discussed in Chapter II. Using the following well-known theorem from analysis (28): "If $f_1(x) \leq f_2(x)$ on $[a, b]$, then $\int_a^b f_1(x)dx \leq \int_a^b f_2(x)dx$.", the bounds on $V(\tau)$ can be extended to bounds on $\bar{\psi}(T, \lambda)$.

Lower Bounds

Suppose $S(\omega)$ is band-limited to ω_c, then for $T_\lambda \leq 2\pi/\omega_c$, where T_λ is the larger of λT and $(1 - \lambda)T$,

$$\bar{\psi}(T,\lambda) \geq \frac{2|R''(0)|}{\omega_c^2}\left\{\frac{\omega_c T - 2\sin(\frac{\omega_c T}{2})\cos[\frac{\omega_c T}{2}(1 - 2\lambda)]}{\omega_c T}\right\} \geq \qquad (3\text{-}12)$$

$$\geq \frac{|R''(0)|}{3}\left[\frac{\sin(\frac{\omega_c T_\lambda}{2})}{\frac{\omega_c T_\lambda}{2}}\right]^2 [1 - 3\lambda + 3\lambda^2]T^2 . \qquad (3\text{-}13)$$

Proof: Using the two bounds of (2-12), i.e.,

$$V(\tau) \geq 2\left[\frac{\sin(\frac{\omega_c\tau}{2})}{\omega_c}\right]^2 |R''(0)| \geq \left[\frac{\sin(\frac{\omega_c T_\lambda}{2})}{\omega_c}\right]^2 \frac{|R''(0)|}{2}\tau^2$$

in the expression of $\bar{\psi}(T, \lambda)$, (3-12) and (3-13) are obtained by integration over the interval $[-\lambda T, (1-\lambda)T]$ where the constraints on τ in the formulation of (2-12) are met since T_λ is the larger of λT and $(1-\lambda)T$. A bound of this form was obtained by Finn (29).

Upper Bounds

Quadratic Bound. Suppose $S(\omega)$ is band-limited to ω_c, then

$$\bar{\psi}(T, \lambda) \leq |R''(0)|(1 - 3\lambda + 3\lambda^2)\frac{T^2}{3}. \tag{3-14}$$

Proof: Using the bound of (2-8) on the expression for $\bar{\psi}(T, \lambda)$ and integrating, obtain (3-14). This bound was obtained by Finn (30). It is a useful bound only for τ such that the quadratic bound on the variation is less than $2R(0)$ or $\tau \leq \tau_1 = 2[\frac{R(0)}{|R''(0)|}]^{1/2}$.

Sine Bound. Suppose that $S(\omega)$ is band-limited to ω_c then, for $T_\lambda \leq \frac{\pi}{\omega_c}$ where T_λ is the larger of λT and $(1-\lambda)T$,

$$\bar{\psi}(T,\lambda) \leq 2R(0)\left\{\frac{\omega_c T - 2\sin(\frac{\omega_c T}{2})\cos[\frac{\omega_c T}{2}(1-2\lambda)]}{\omega_c T}\right\}. \tag{3-15}$$

Proof: Integrate the bound of (2-10) to obtain (3-15).

Inspection of the quadratic bound shows that for $\lambda = 1/2$, this bound is reduced by a factor of 1/4 over its value for $\lambda = 0$, i.e., no delay.

Figure 3 illustrates the nature of the interpolation error bounds for a general band-limited process. The shaded region between the tightest upper and lower bounds indicates where $\bar{\psi}(T, \lambda)$ is constrained to lie for smaller values of T.

Non-Band-Limited Interpolation Error

A similar extension of the variation bounds of non-band-limited random processes to interpolation error bounds is possible and a restriction on the limiting behavior of a zero-order sample-and-hold is noted.

First-Order Data

Suppose x(t) is a non-differentiable random process and there exists a k such that $\frac{k}{\omega^2} \geq S(\omega)$ for all ω, then

$$\bar{\psi}(T, \lambda) \leq \frac{kT}{2}\left[1 - 2\lambda + 2\lambda^2\right] = \omega_n T\left[1 - 2\lambda + 2\lambda^2\right]R(0) \qquad (3\text{-}16)$$

where $\omega_n \equiv \frac{k}{2R(0)}$.

Proof: The bound of (2-25), $V(\tau) \leq \frac{k|\tau|}{2}$, when used in the expression of (3-9) and integrated yields (3-16).

Higher-Order Data

Suppose x(t) is a mean square differentiable random process, then

$$\bar{\psi}(T, \lambda) \leq |R''(0)|\ (1 - 3\lambda + 3\lambda^2)\ \frac{T^2}{3} = \qquad (3\text{-}17)$$

$$= \frac{(\omega_d T)^2}{3}\ (1 - 3\lambda + 3\lambda^2)R(0)$$

where $\omega_d = \frac{|R''(0)|}{R(0)}$.

Proof: If x(t) is mean square differentiable, then $V(\tau) \leq \frac{|R''(0)|}{2}\tau^2$

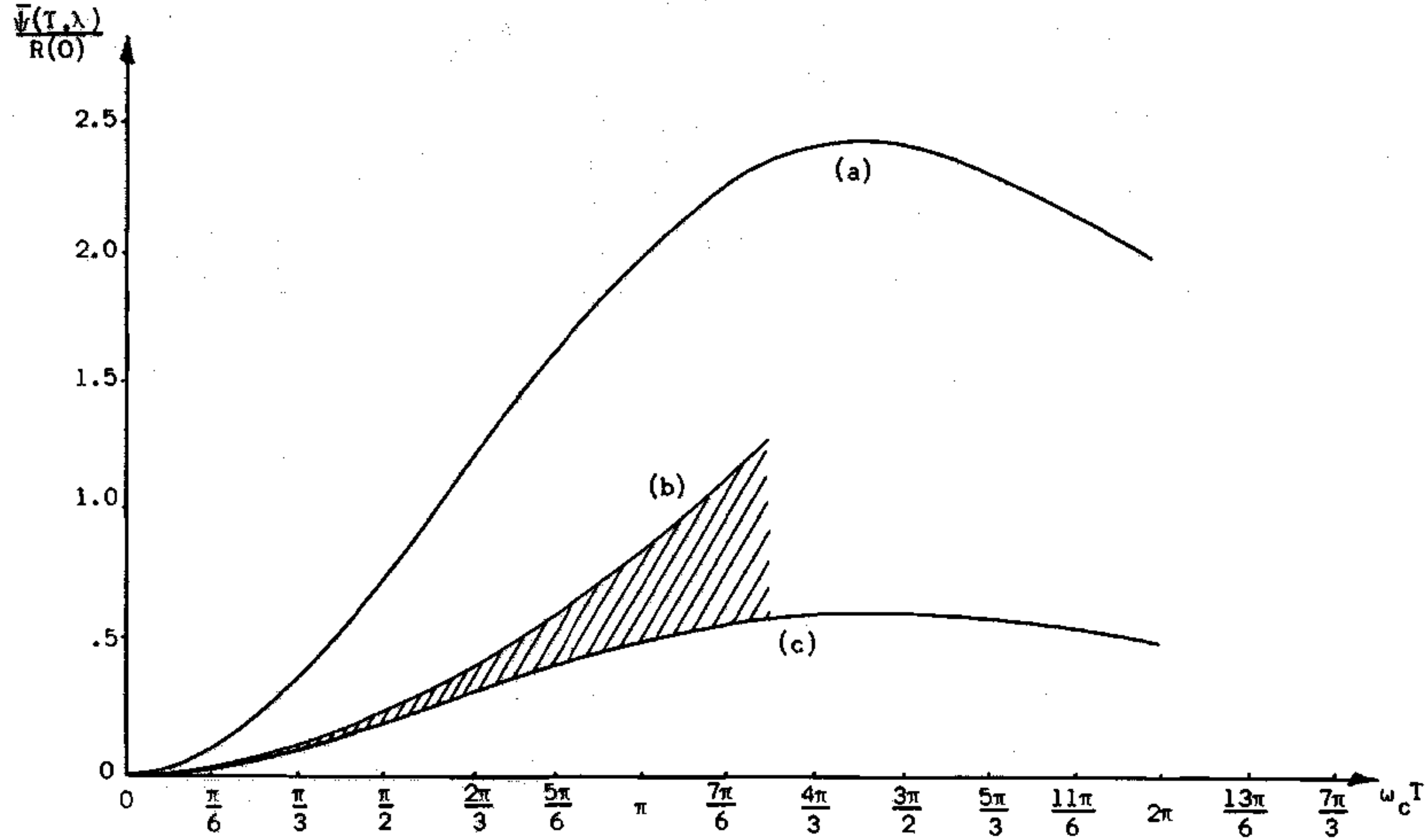

Figure 3. Normalized Interpolation Error Bounds for Band-Limited Random Processes ($\lambda = 0$ and $|R''(0)| = \omega_c^2 R(0)/4$).

a) $2[\omega_c T - 2 \sin(\omega_c T/2)\cos(\omega_c T/2)]/(\omega_c T)$ - (3-15),

b) $|R''(0)|T^2/3R(0) = \omega_c T^2/12$ -(3-14),

c) $[\omega_c T - 2 \sin(\omega_c T/2)\cos(\omega_c T/2)]/2\omega_c T$ - (3-12).

and integration over $[-\lambda T, (1-\lambda)T]$ yields (3-17).

Figure 4 illustrates the form of the normalized bounds for both non-band-limited cases. Neither bound is plotted past $\omega\tau = 2$ since the original variation bound equals $2R(0)$ at this point.

Effect of Delay

Quantitative statements concerning the effect of delay on $\bar{\psi}(T,\lambda)$ are seemingly impossible to make; however, the effect on the bounds of (3-16) and (3-17) may be seen by inspection. Let the bound for non-differentiable processes be denoted as

$$B_n(\lambda) = \frac{kT}{2}(1 - 2\lambda + 2\lambda^2) \qquad (3\text{-}18)$$

and that for differentiable processes as

$$B_d(\lambda) = |R''(0)|(1 - 3\lambda + 3\lambda^2)\frac{T^2}{3}. \qquad (3\text{-}19)$$

Both obviously have minimum value for $\lambda = 1/2$. However, the relative improvement (reduction of the bounds) for first-order data with delay is one half that for higher order data with delay since

$$\frac{B_d(\frac{1}{2})}{B_d(0)} \cdot \frac{B_n(0)}{B_n(\frac{1}{2})} = \frac{1}{2}.$$

Rate of Improvement with T

An Improvement Criterion

Any interpolator is expected to yield improved performance as the length of the sampling interval approaches zero. This is obviously true

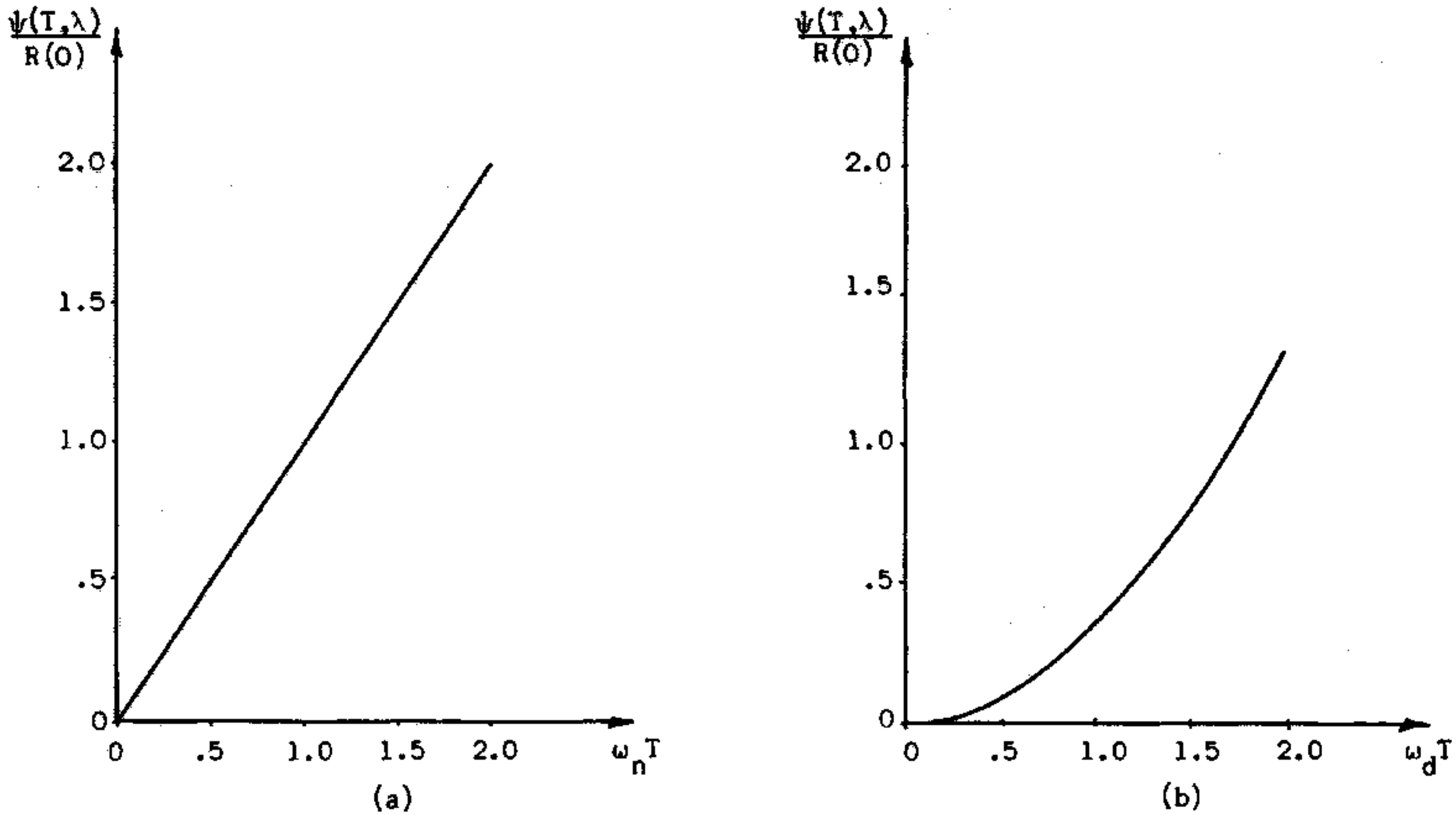

Figure 4. Normalized Interpolation Error Bounds for Non-Band-Limited Processes.
(a) Non-Differentiable Process ($\omega_n = k/2R(0)$,
(b) Differentiable Process ($\omega_d = [\frac{R''(0)}{R(0)}]^{1/2}$).

for the zero-order hold since the upper bounds on mean square interpolation error approach zero for small T. An additional factor, the rate of improvement, may be analyzed by L'Hospital's Rule. Consider an improvement criterion defined as

$$\eta = \frac{d\bar{\Psi}}{dT} \div \frac{\bar{\Psi}}{T} \tag{3-20}$$

then, for a given $V(\tau)$, $\lim_{T \to o} \eta = k$, where k has either the value one or two depending on the nature of $V(\tau)$. From the definition of $\bar{\Psi}(T, \lambda)$ and by using a theorem for differentiating through an integral,

$$\eta = \frac{\frac{2}{T}\left\{(1-\lambda)V[(1-\lambda)T] + \lambda V[\lambda T]\right\} - \frac{2}{T^2}\int_{-\lambda T}^{(1-\lambda)T} V(\tau)d\tau}{\frac{2}{T^2}\int_{-\lambda T}^{(1-\lambda)T} V(\tau)d\tau} .$$

In the limit as $T \to 0$, η has the form 0/0 and can be evaluated by repeatedly differentiating both numerator and denominator as required by L'Hospital's Rule until a value is found for the limit, i.e.,

$$\lim_{T \to 0} \eta \overset{H}{=} \lim_{T \to 0} \frac{T\left\{(1-\lambda)^2 V'[(1-\lambda)T] + \lambda^2 V'(\lambda T)\right\}}{(1-\lambda)V[(1-\lambda)T] + \lambda V[\lambda T]} \overset{H}{=}$$

$$\overset{H}{=} \lim_{T \to 0} \frac{(1-\lambda)^2 V'[(1-\lambda)T] + \lambda^2 V'[\lambda T] + T\left\{(1-\lambda)^3 V''[(1-\lambda)T] + \lambda^3 V''[\lambda T]\right\}}{(1-\lambda)^2 V'[(1-\lambda)T] + \lambda^2 V'[\lambda T]}$$

If $V'(0^+)$ is non-zero, as it would be for a non-differentiable process, then the above shows

$$\lim_{T \to 0} \eta_n = 1 .$$

If $V'(0^+)$ is zero, then one more application of L'Hospital's Rule yields

$$\lim_{T\to 0} \eta_d = \lim_{T\to 0} \frac{2\{(1-\lambda)^3 V''[(1-\lambda)T] + \lambda^3 V''[\lambda T]\} + T\{(1-\lambda)^4 V'''[(1-\lambda)T] + \lambda^4 V'''[\lambda T]\}}{(1-\lambda)^3 V''[(1-\lambda)T] + \lambda^3 V''[\lambda T]}$$

and

$$\lim_{T\to 0} \eta_d = 2 .$$

No further analysis is needed since if $V'(0^+) = 0$, then $V''(0) \neq 0$.

Limiting Behavior

An Interpretation of η. The interpolation error improvement for T approaching zero, as judged by η, has been shown to be a constant independent of data characteristics such as cut-off frequency or roll-off rate for the entire class of differentiable random processes, and a similar result holds for non-differentiable random processes although the value of η obtained in the limit is not the same for both classes.

The improvement criterion defined above may be viewed as the ratio of the percentage change in $\bar{\psi}(T, \lambda)$ to the percentage change in T and thus yields a quantitative measure of the utility of decreasing T to effect a decrease in the interpolator error. A change of variables yields another interesting result. Suppose that the interpolation error versus the sampling frequency, $f_s = \frac{1}{T}$, is plotted on a log-log scale. As $T \to 0$, $f_s \to \infty$, and the slope of the resulting curve may be determined from

$$\lim_{f_1 \to f_s} \left\{ \frac{\log[\bar{\psi}(\frac{1}{f_1}, \lambda)] - \log[\bar{\psi}(\frac{1}{f_s}, \lambda)]}{\log(f_1) - \log(f_s)} \right\} =$$

$$= \lim_{T_1 \to T} \left\{ \frac{\log[\bar{\psi}(T_1, \lambda)] - \log[\bar{\psi}(T, \lambda)]}{\log(\frac{1}{T_1}) - \log(\frac{1}{T})} \right\} =$$

$$\overset{H}{=} \frac{\bar{\psi}'(T)/\bar{\psi}(T)}{(-1/T)} = -\eta(T)$$

where L'Hospital's Rule has been used to determine the limit. Since $\eta(T)$ approaches a constant for small T, the interpolation error versus sampling frequency, on log-log coordinates, becomes a straight line for large f_s.

Comparison with Numerical Results. The value of $\eta(T)$ in the limit could have been used to predict some results obtained by McRae (31), who compared the performance of a number of interpolation schemes by calculating the error resulting when they sampled a set of approximate spectral densities of the form illustrated below in Figure 5.

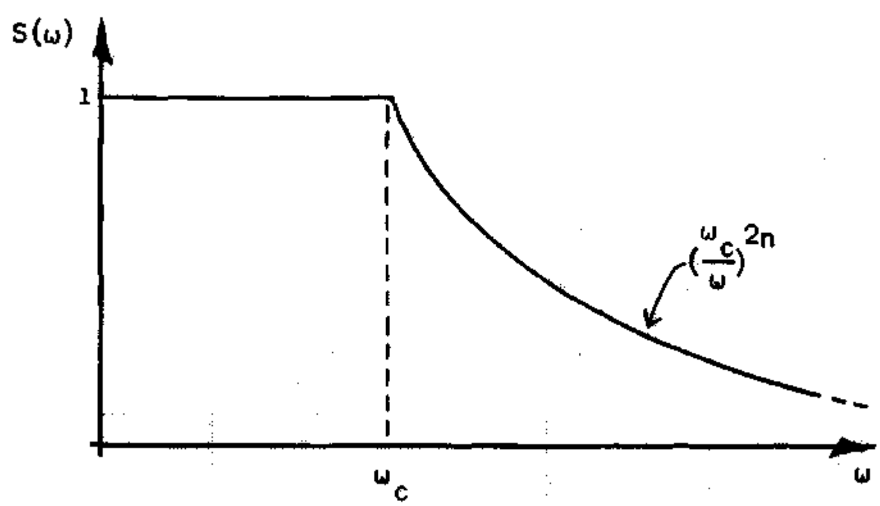

Figure 5. McRae's Approximate Spectral Density.

The spectrum is considered flat out to a break frequency of ω_c and rolls off as $(\frac{\omega_c}{\omega})^{2n}$ thereafter, thus approximating any type of data from first-order ($n = 1$) to band-limited ($n = \infty$). When the spectrum is known exactly, the interpolation error equation for the zero-order hold may be written as

$$\bar{\psi}(T, \lambda) = \frac{2}{T}\int_{-\lambda T}^{(1-\lambda)T} [R(0) - R(\tau)]d\tau = \tag{3-21}$$

$$= \frac{2}{T}\int_{-\lambda T}^{(1-\lambda)T} \left\{\frac{1}{2\pi}\int_{-\infty}^{\infty} S(\omega)[1 - \cos\omega\tau]\,d\omega\right\} d\tau$$

and evaluation of the error becomes a straightforward computational problem, which McRae solved. As indicated by the theoretical results, the slope of McRae's interpolation error curves versus sampling frequency plotted on a log-log scale turned out to be constant for sampling frequencies greater than about 10 f_c, i.e., $T \leq \frac{1}{10f_c}$. Furthermore, the slope for all data of order 2 or greater was twice the slope for first-order data. All these results bear out the intuitive feeling that first-order data, with its relatively high concentration of spectral power at large ω, should be more difficult to sample and interpolate than higher order data.

Exponential Hold

Error in the Exponential Hold

Some of the results of Chapter II may be used to determine a bound on the performance of the exponential hold. A typical interpolated output from such a device is shown in Figure 6. From the earlier

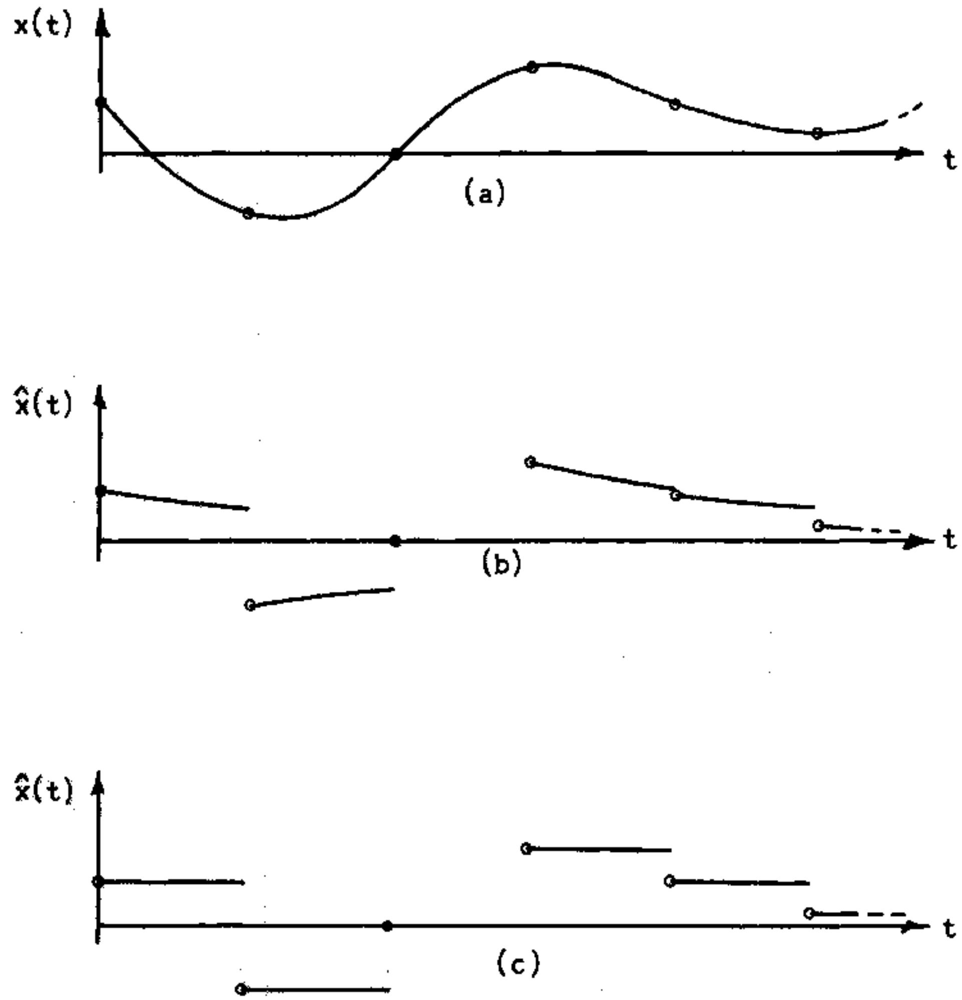

Figure 6. Typical Input and Output of Exponential Hold.

a) Input,
b) Exponential Hold Output,
c) Zero-Order Hold Output.

discussion of interpolators, the following equations hold:

$$h_e(t) = \epsilon^{-\alpha t}, \quad t \in [0, T) ;$$

$$\hat{x}(t) = x(nT)\epsilon^{-\alpha(t-nT)}, \quad t \in [nT, (n+1)T);$$

$$e(t,\lambda) = x(t-\lambda T)-x(nT)\epsilon^{-\alpha(t-nT)}, \quad t \in [nT,(n+1)T);$$

and, from (3-7)

$$\bar{\psi}_e(T,\lambda) = \frac{1}{T}\int_0^T \left\{R(0)[1+\epsilon^{-2\alpha\tau}] - 2R(\tau-\lambda T)\epsilon^{-\alpha\tau}\right\} d\tau . \qquad (3\text{-}22)$$

The zero-order hold is a special case where $\alpha = 0$.

<u>Upper Bound</u>

Suppose $x(t)$ is a mean square differentiable random process, then for $\alpha T \leq 1$

$$\bar{\psi}_e(T,\lambda) \leq \frac{R(0)}{3}\left\{(\omega_e T)^2[1-3\lambda+3\lambda^2 - \frac{\alpha T}{4}(3-8\lambda+6\lambda^2)]+2(\alpha T)^2\right\} = \psi \quad (3\text{-}23)$$

where $\omega_e^2 \geq |R''(0)|/R(0)$.

Proof: Consider the two exponential inequalities:

$$\epsilon^{-x} \geq 1 - x ;$$

and

$$\epsilon^{-2x} \leq 1 - 2x + 2x^2$$

valid for $x \geq 0$. Using these inequalities to bound (3-15),

$$\bar{\Psi}_e(T,\lambda) \leq \frac{1}{T}\int_0^T \left\{R(0)[2-2\alpha\tau+2(\alpha\tau)^2] - 2R(\tau-\lambda T)[1-\alpha\tau]\right\} d\tau = \tag{3-24}$$

$$= \frac{2}{T}\int_0^T \left\{[R(0)-R(\tau-\lambda T)][1-\alpha\tau] + (\alpha\tau)^2 R(0)\right\} d\tau .$$

However, Chapter II dealt with bounds of the variation, $R(0) - R(\tau)$, so that the following bound is known,

$$R(0) - R(\tau) \leq \frac{|R''(0)|}{2}\tau^2 \leq \frac{(\omega_e\tau)^2}{2} R(0)$$

where $\omega_e^2 \geq |R''(0)|/R(0)$. To preserve the sense of the inequality of (3-17), the sign of the variation must remain positive or $1-\alpha\tau \leq 0$. The result could be extended to larger $\alpha\tau$ for band-limited processes since both upper and lower bounds are known. Using this bound in (3-24) and carrying out the indicated integration, (3-23) follows.

<u>Least Upper Bound</u>

Consider $(\omega_e T)^2$, the sampling rate parameter, to be fixed at the value K. Inspection of (3-23) suggests that there might exist λ and αT such that this bound on the interpolation error is minimized or at least reduced. That such is the case is shown in the following analysis.

Let $\gamma = \alpha T$, then with $K = (\omega_e T)^2$

$$\psi = \frac{R(0)}{12}\left\{K[4-12\lambda+12\lambda^2 - \gamma(3-8\lambda+6\lambda^2)]\right\} + 8\gamma^2 . \tag{3-25}$$

For well behaved functions where the existence of all partial derivatives may be assumed, the necessary and sufficient conditions that $f(\gamma, \lambda)$ has a local minimum are that $f_\gamma = f_\lambda = 0$, $f_{\gamma\gamma}$ and $f_{\lambda\lambda}$ are positive and

$f_{\lambda\gamma}^2 - f_{\lambda\lambda} f_{\gamma\gamma} < 0$. Taking the necessary partials in (3-25)

$$\psi_\gamma = \frac{R(0)}{12} \left[-K(3 - 8\lambda + 6\lambda^2) + 16\gamma\right] , \qquad \text{(3-26a)}$$

$$\psi_\lambda = \frac{R(0)}{12} \left\{K[-12 + 24\lambda] - \gamma[-8 + 12\lambda]\right\} , \qquad \text{(3-26b)}$$

$$\psi_{\lambda\gamma} = \frac{KR(0)}{3} (2 - 3\lambda) , \qquad \text{(3-26c)}$$

$$\psi_{\gamma\gamma} = \frac{4}{3} R(0) , \qquad \text{(3-26d)}$$

$$\psi_{\lambda\lambda} = KR(0)[2 - \gamma] . \qquad \text{(3-26e)}$$

If $\psi_\gamma = \psi_\lambda = 0$, then (3-26a) and (3-26b) require that

$$\gamma = \frac{K}{16} (3 - 8\lambda + 6\lambda^2) \qquad \text{(3-27)}$$

$$\gamma = (3 - 6\lambda)/(2 - 3\lambda) .$$

For a minimum to have other than academic interest, K must be reasonably small and λ must be real. Equating the two expressions in (3-27) and rewriting, an equation in a form suitable for analysis by the inverse root locus technique may be obtained, i.e.,

$$\frac{3K}{16} \frac{(\lambda - 2/3)(\lambda^2 - 4/3\lambda + 1/2)}{(\lambda - 1/2)} - 1 = 0 .$$

This analysis shows that real λ exist between 0 and 1/2 for $K \leq 8$. Furthermore, for such λ, (3-27) requires $0 \leq \gamma \leq 3/2$, a reasonable range of decay rates. Since $\psi_{\gamma\gamma} > 0$, independent of λ and γ, and $\psi_{\lambda\lambda} > 0$ if $\gamma \leq 3/2$, then a minimum may be obtained if the following equation is also satisfied. Substituting for γ, then

$$\psi_{\lambda\gamma}^2 - \psi_{\lambda\lambda}\psi_{\gamma\gamma} = [\frac{KR(0)}{3}(2-3\lambda)]^2 - \frac{4}{3}R(0)\ KR(0)[2-\gamma] = \tag{3-28}$$

$$= [\frac{KR(0)}{6}]^2 [54\lambda^2 - 72\lambda + 25 - \frac{96}{K}].$$

Rewriting (3-28), and using the root locus technique again, it may be shown that for $\lambda \epsilon [0, \frac{1}{2}]$ and $K = (\omega_c T)^2 < 3.85$, then (3-28) is negative. Thus, for $\omega_e T < 1.96$ there exist real values for α, the decay rate of the exponential hold, and λ, the percentage delay, such that the bound on $\bar{\psi}_e(T, \lambda)$ has a minimum value.

Considering the restrictions imposed by the above and substituting for γ in (3-25) then

$$\psi = \frac{KR(0)}{12}\left\{(4 - 12\lambda + 12\lambda^2) - \frac{K}{32}(3 - 8\lambda + 6\lambda^2)^2\right\} = \tag{3-29}$$

$$= \frac{KR(0)}{3}[1 - 3\lambda + 3\lambda^2] - \frac{K^2R(0)}{384}(3 - 8\lambda + 6\lambda^2)^2 .$$

Recognizing the first term as that previously obtained for the quadratic bound on zero-order interpolation, then the second may be viewed as a measure of the improvement gained with exponential hold. For small K and $\lambda \simeq 1/2$, this improvement is relatively insignificant. However, since $(3 - 8\lambda + 6\lambda^2)$ has its maximum value at $\lambda = 0$ for $\lambda \epsilon [0, 1/2]$, then some appreciable improvement over the zero-order bound can be achieved with an exponential hold; in fact,

$$\psi|_{\lambda=0} = \frac{KR(0)}{3} - \frac{9K^2R(0)}{384} = \frac{KR(0)}{12}[4 - \frac{9K}{32}], \tag{3-30}$$

where $\gamma = \frac{3K}{16}$ and $K < 3.85$. This does not necessarily imply that the exponential hold is a better interpolator than the zero-order hold; however, its upper bound is smaller than the upper bound of the zero-order hold and in the limit as $T \to 0$ these bounds approach the actual performance of the respective interpolator errors. Figure 7 compares the zero-order hold bound ($\alpha T = 0$) to the least upper bound for the exponential hold.

Sampling Jitter

Elements of the preceding analysis may be used to examine a fairly general case of sampling jitter in zero-order interpolation.

Nature of T_n

The following definitions will be used. The time of occurrence of the n^{th} sampling interval will be denoted as t_n. The duration of the n^{th} interval is defined by $T_n = t_{n+1} - t_n$. Normally, the mechanism of the sampling process would be set up to sample at some nominal rate with an interval duration denoted by $\bar{T}$. The actual sampling interval fluctuates, or jitters, about $\bar{T}$ from sample to sample. A typical interpolator output is compared to a uniform rate sample and hold interpolator in Figure 8. T_n may be considered to be a random variable, distributed on the interval $[\bar{T} - \Delta_1, \bar{T} + \Delta_2]$ according to some probability density $p_{T_n}(T_n)$. In order that $t_{n+1} - t_n$ be always positive, only those values of Δ_1 for which $\Delta_1 < \bar{T}$ will be used. The nominal interval duration $\bar{T}$ is not required to be the expected value of T_n although in most cases they will be identical due to the physical situation they model. T_n will be termed statistically independent if $\ldots = p_{T_{n-1}}(T) = p_{T_n}(T) = p_{T_{n+1}}(T) = \ldots$,

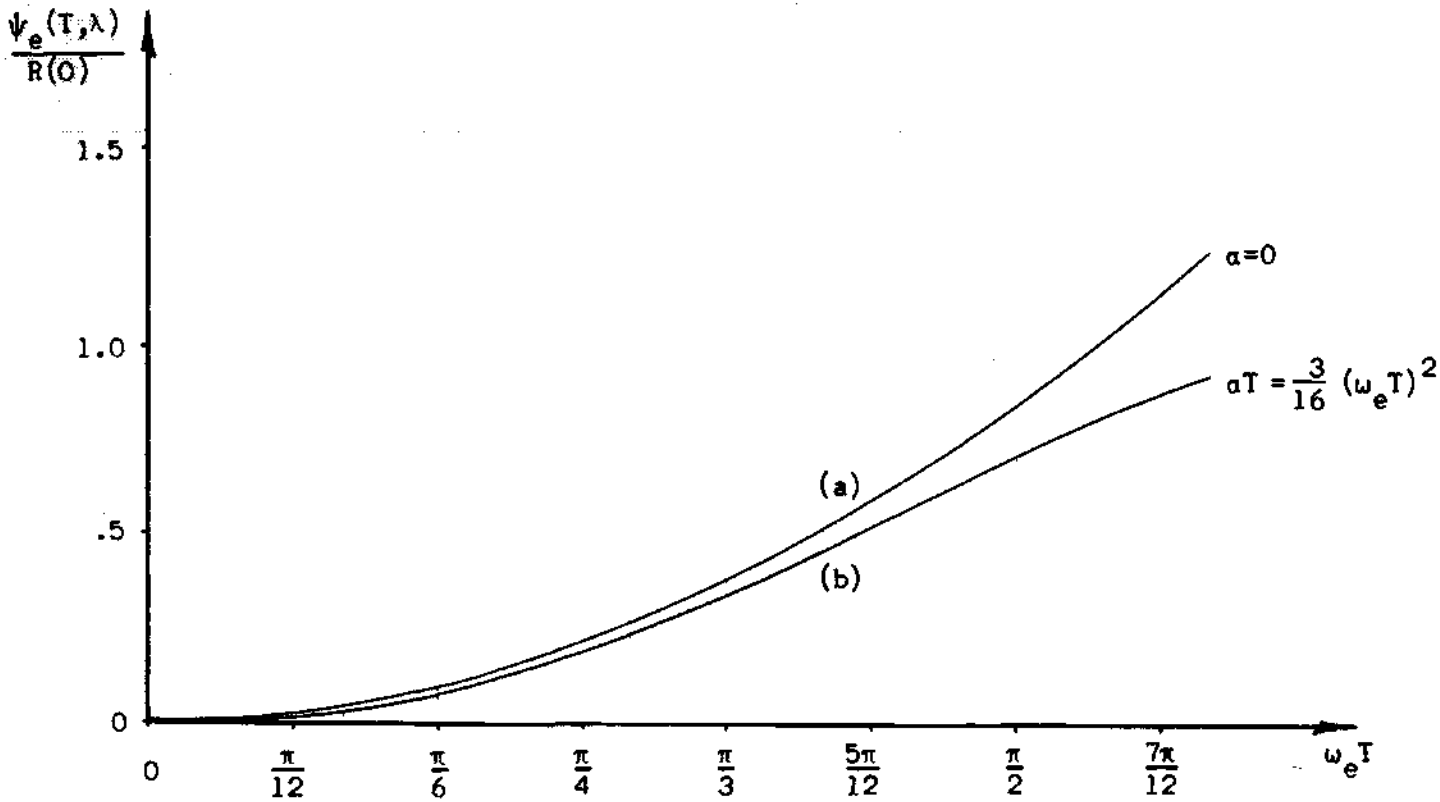

Figure 7. Comparison of Exponential and Zero-Order Hold Interpolation Error Bounds.
a. Zero-Order Hold Bound,
b. Exponential Hold Bound.

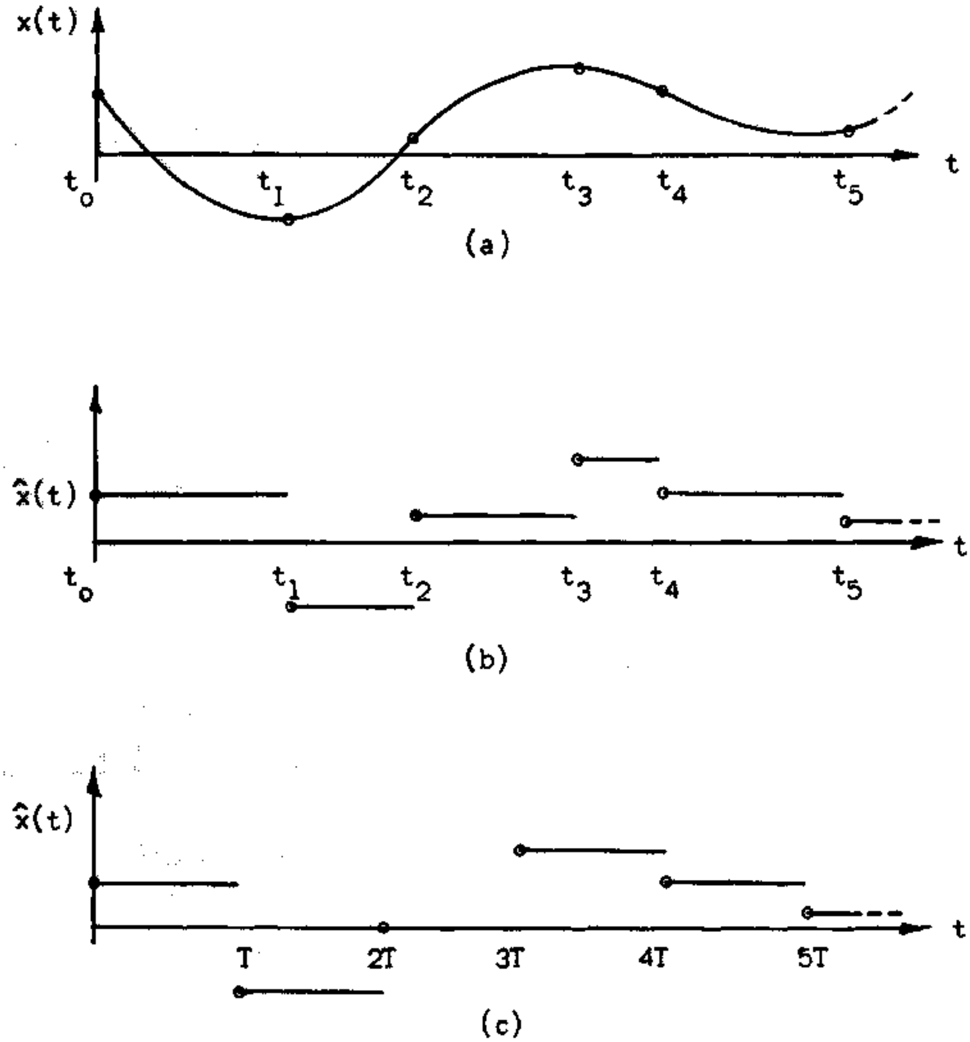

Figure 8. The Effect of Sampling Time Jitter on Zero-Order Hold Interpolation.
a) Sampling Instants,
b) Interpolator Output,
c) Interpolator Output of Uniform Rate, Zero-Order Sample-and-Hold.

i.e., $p_{T_n}(T) = p(T)$, and $p(T_j, T_{j+1}, T_{j+2}, \ldots, T_{j+k}) = p_{T_j}(T_j) p_{T_{j+1}}(T_{j+1}) \ldots p_{T_{j+k}}(T_{j+k})$ for any j and k.

A Bound on Interpolation Error

Suppose that the sampling interval T_n is distributed on the interval $[\bar{T} - \Delta_1, \bar{T} + \Delta_2]$ according to the probability density $p_{T_n}(T_n)$ where $\Delta_1 < \bar{T}$, and is statistically independent as defined above so that $p_{T_n}(T_n) = p(T_n)$, and x(t) is mean square differentiable, then

$$E_{x,T}\left\{\frac{1}{T_n}\int_{t_n}^{t_n+T_n}[x(t) - x(t_n)]^2 dt\right\} \leq \frac{|R''(0)|}{3} E_T\left\{T_n^2\right\}. \quad (3\text{-}31)$$

Proof:

$$E_{x,T}\left\{\frac{1}{T_n}\int_{t_n}^{t_n+T_n}[x(t) - x(t_n)]^2 dt\right\} = E_T\left\{E_x\left[\frac{1}{T_n}\int_{t_n}^{t_n+T_n}[x(t) - x(t_n)]^2 dt\right]\right\}$$

since T_n is independent of x(t). The expectation operator on x(t) is that of (3-5) and

$$E_{x,T}\left\{\frac{1}{T_n}\int_{t_n}^{t_n+T_n}[x(t) - x(t_n)]^2 dt\right\} = E_T\left\{\bar{\psi}(T_n, 0)\right\} = \int_{\bar{T}-\Delta_1}^{\bar{T}+\Delta_2}\left\{\bar{\psi}(T_n, 0)\right\} p(T_n) dT_n$$

which could be evaluated given $\bar{\psi}(T_n, 0)$ and $p(T_n)$. Both $\bar{\psi}(T_n, 0)$ and $p(T_n)$ are positive and $\bar{\psi}(T_n, 0)$ has an upper bound given by (3-14).

Using this bound in the above

$$E_{x,T}\left\{\frac{1}{T_n}\int_{t_n}^{t_n+T_n}[x(t) - x(t_n)]^2dt\right\} \leq \int_{\bar{T}-\Delta_1}^{\bar{T}+\Delta_2}\left\{\frac{|R''(0)|}{3}\ T_n^2\right\}p(T_n)dT_n =$$

$$= \frac{|R''(0)|}{3}\ E_T\ [T_n^2]$$

For T_n as restricted, $E_T[T_n^2]$ does not depend upon n.

Higher Order Systems

The above bounding techniques do not seem to be applicable to higher order hold interpolation since in the limit as $T \to 0$, these interpolators tend to approximate the derivative of the process and hence become independent of the second derivative of $R(\tau)$. For the zero-order interpolator, the $|R''(0)|$ bound obtained approaches the actual behavior as T becomes small.

CHAPTER IV

VARIANCE OF MEAN SQUARE ERROR

This chapter is devoted to a discussion and analysis of the dependability of the $\psi(nT, \lambda)$ defined earlier, i.e., how good an estimate of the actual behavior of interpolator error is given by $\bar{\psi}(T, \lambda)$. Several approaches to this problem will be made as well as an analysis of the interpolator error of a Gaussian random process.

The Interpolation Error Random Process

Interpolation Error Parameters

The interpolation error measure $\bar{\psi}(T, \lambda)$ discussed heretofore represents the mean value of a rather unorthodox random process (or random series since its arguments are discrete) because the actual value of $\psi(nT, \lambda)$ is dependent upon n, T, the value of the nT product, λ, and the phase relationship of the sampling process and x(t), as well as the nature of the particular ensemble member during the observation interval $[nT, (n+1)T)$. In particular, previous discussions of interpolator error have seemingly failed to investigate the relationship between that range of values which $\psi(nT, \lambda)$ may attain and the value of $\bar{\psi}(T, \lambda)$.

Range of $\bar{\psi}(T, \lambda)$

The difference in interpolator error from one sampling interval to the next is not necessarily negligible even for high sampling rates as may be seen from the following. Consider T to be sufficiently small that the error is a straight line during any sampling period or

$$e_1(t) = x'(nT + \lambda T)(t - nT - \lambda T),\ t \in [nT, (n+1)T), \tag{4-1}$$

where the one subscript will be used to distinguish those error parameters based upon this straight line approximation. Then

$$\psi_1(nT, \lambda) = \frac{1}{T}\int_{nT}^{(n+1)T} x'^2(nT + \lambda T)(t - \lambda T - nT)^2$$

$$= x'^2(nT + \lambda T)(1 - 3\lambda + 3\lambda^2)\ \frac{T^2}{3} \tag{4-2}$$

and the interpolation error for a given sampling interval is effectively determined by the samples of the derivative function and could vary substantially from sample to sample even for T sufficiently small that $x(t)$ does not change appreciably in an interval. Consequently, an examination of $\psi(nT, \lambda)$ to determine the range of its values about $\bar{\psi}(T, \lambda)$ is in order. Since $\psi(nT, \lambda)$ depends on a quadratic function of $x(t)$, then $Var[\psi]$ will depend upon the fourth-order moments of $x(t)$; however, a tractable form exists for Gaussian $x(t)$.

Properties of a Gaussian Process

In the following discussion, $x(t)$ is assumed to be a stationary, zero-mean, Gaussian process with normalized autocorrelation function

$$\rho(t_1, t_2) = \rho(t_2 - t_1) = \frac{R(t_2 - t_1)}{R(0)} \tag{4-3}$$

and thus has the following first and second order densities:

$$p[x(t)] = [2\pi R(0)]^{-1/2} \exp[-\frac{x^2}{2R(0)}] \tag{4-4}$$

and

$$p[x(t_1),x(t_2)] = \left\{(2\pi)^2R^2(0)[1-\rho^2(t_2-t_1)]\right\}^{-1/2} \exp\left\{-\frac{x_1^2 - 2\rho(t_2-t_1)x_1x_2+x_2^2}{2R(0)[1-\rho^2(t_2-t_1)]}\right\}. \tag{4-5}$$

For such a process, it can be shown that (32),

$$E\left\{x(t_1)x(t_2)x(t_3)x(t_4)\right\} = R(t_2-t_1)R(t_4-t_3)+R(t_3-t_1)R(t_4-t_2) +R(t_4-t_1)R(t_3-t_2) \ . \tag{4-6}$$

Instantaneous Error

Consider the difference between a random process shifted in time by an arbitrary amount, d, and its value at some fixed time, t_o. Motivated by the previous discussion, this will be termed instantaneous error and defined as

$$e(t,t_o,d) = x(t-d) - x(t_o) \ . \tag{4-7}$$

For any stationary process, it follows that $e(t,t_o,d)$ has zero mean. For a Gaussian process, the joint probability density of (4-5) is sufficient to yield $p[e(t,t_o,d)]$ by utilizing a transformation of variables (33). Since

$$P\left\{e(t,t_o,d) \leq e\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{e+x(t_o)} \rho[x(t-d),x(t_o)]dx(t-d)dx(t_o) \ , \tag{4-8}$$

then

$$p\left\{e(t,t_o,d)\right\} = \frac{d}{de}\left[P\left\{e(t,t_o,d) \leq e\right\}\right] =$$

$$= \int_{-\infty}^{\infty} p[e+x(t_o),\ x(t_o)]dx(t_o) =$$

$$= \int_{-\infty}^{\infty} p[x(t-d),\ x(t-d) - e]dx(t-d)\ .$$

Performing this substitution and integration on the second-order density of (4-5) yields

$$p\{e(t,t_o,d)\} = p\{e(t-d-t_o)\} = [4\pi V(t-d-t_o)]^{-1/2} \exp\left\{-\frac{e^2}{4V(t-d-t_o)}\right\}. \quad (4\text{-}9)$$

The instantaneous error, $e(t,t_o,d)$ is therefore a non-stationary, zero mean process and, in addition, is Gaussian since it is formed by a linear transformation of a Gaussian process and has the first-order density given in (4-9) in terms of the variation.

Squared Error

Consider the above difference squared and termed squared error and defined by

$$e^2(t,t_o,d) = [x(t-d) - x(t_o)]^2\ . \quad (4\text{-}10)$$

Again a transformation of variables may be used to determine $p\,[e^2(t,t_o,d)]$ and

$$p[e^2(t,t_o,d)] = [4\pi V(t-d-t_o)e^2]^{-1/2} \exp\left\{-\frac{e^2}{4V(t-d-t_o)}\right\}. \quad (4\text{-}11)$$

Thus, $e^2(t,t_o,d)$ has a first-order density in the form of the gamma density function, and

$$E\left\{e^2(t,t_o,d)\right\} = 2V(t-d-t_o) \tag{4-12}$$

$$\mathrm{Var}[e^2(t-d-t_o)] = 8V^2(t-d-t_o) . \tag{4-13}$$

Mean Squared Error

Consider the above squared error averaged on the interval $[t_1, t_2]$ and termed mean square error and defined by

$$\psi(t_o,t_1,t_2,d) = \frac{1}{t_2-t_1}\int_{t_1}^{t_2} [x(t-d)-x(t_o)]^2 dt$$

$$= \frac{1}{t_2-t_1}\int_{t_1}^{t_2} e^2(t,t_o,d)dt \tag{4-14}$$

For any stationary $x(t)$, $\psi(t_o,t_1,t_2,d)$ has the mean value

$$\bar{\psi}(t_o,t_1,t_2,d) = \frac{2}{t_2-t_1}\int_{t_1}^{t_2} [R(0) - R(t-d-t_o)]dt = \tag{4-15}$$

$$= \frac{2}{t_2-t_1}\int_{t_1-d-t_o}^{t_2-d-t_o} [R(0) - R(\tau)]d\tau .$$

For Gaussian $x(t)$, the variance of $\psi(t_o,t_1,t_2,d)$ may be stated in terms of products of the autocorrelation by substitution in the relationship,

$$\mathrm{Var}[y] = E[y^2] - E^2[y] . \tag{4-16}$$

Now

$$E\left\{\psi^2(t_o,t_1,t_2,d)\right\} = E\left\{\frac{1}{(t_2-t_1)^2}\int_{t_1}^{t_2}[x(\alpha-d)-x(t_o)]^2 d\alpha \int_{t_1}^{t_2}[x(\beta-d) -x(t_o)]^2 d\beta\right\} = \tag{4-17}$$

$$= E\left\{\frac{1}{(t_2-t_1)^2}\int\int_{t_1-d}^{t_2-d}[x^2(\alpha)x^2(\beta)-2x^2(\alpha)x(\beta)x(t_o) + x^2(\alpha)x^2(t_o)-2x(\alpha)x^2(\beta)x(t_o)+4x(\alpha)x(\beta)x^2(t_o) - 2x(\alpha)x^3(t_o)+x^2(\beta)x^2(t_o)-2x(\beta)x^3(t_o)+x^4(t_o)]d\alpha d\beta\right\}.$$

Interchanging the expectation operator and the double integral, and calling upon (4-6) to simplify the fourth-order moments, then (4-17) may be written as

$$E\left\{\psi^2(t_o,t_1,t_2,d)\right\} = \frac{1}{(t_2-t_1)^2}\int\int_{t_1-t_o-d}^{t_2-t_o-d}[6R^2(0)+4R(0)R(\alpha-\beta) - 16R(0)R(\alpha)+4R^2(\beta)+2R^2(\alpha-\beta)-8R(\beta)R(\alpha-\beta)+ + 8R(\alpha)R(\beta)]\, d\alpha d\beta . \tag{4-18}$$

Combining (4-15), (4-16), and (4-18), then

$$\mathrm{Var}[\psi(t_o,t_1,t_2,d)] = 2R^2(o) - \frac{1}{t_2-t_1}\int_{t_1-t_o-d}^{t_2-t_o-d}[8R(0)R(\alpha)-4R^2(\alpha)]d\alpha + = \frac{1}{(t_2-t_1)^2}\int\int_{t_1-t_o-d}^{t_2-t_o-d}[4R(0)R(\alpha-\beta)-8R(\beta)R(\alpha-\beta) + + 2R^2(\alpha-\beta)+4R(\alpha)R(\beta)]d\alpha d\beta .$$

By choosing t_o, t_1, t_2, and d appropriately, (4-18) may be interpreted as the variance of the $\psi(nT, \lambda)$ discussed in Chapter III. For this case, the interval of interest is $[nT,(n+1)T)$, or $t_o = t_1 = nT$, and $t_2 = t_o + T$, and the delay, d, is equal to λT. Making the appropriate substitutions,

$$\mathrm{Var}[\psi(T,\lambda)] = \frac{1}{T^2}\int_0^T\int_0^T [2R^2(0) - 8R(0)R(\alpha-\lambda T) + \tag{4-19}$$

$$+ 4R^2(\alpha-\lambda T) + 4R(0)R(\alpha-\beta) + 2R^2(\alpha-\beta) +$$

$$+ 8R(\alpha-\lambda T)R(\beta-\lambda T) - 8R(\beta-\lambda T)R(\alpha-\beta)]d\alpha d\beta \ .$$

Note that if T is considered small enough that only the first few terms in the Taylor series expansion are important, then

$$\mathrm{Var}[\psi(T,\lambda)] \simeq \frac{2}{9}\,|R''(0)|^2(1-3\lambda+3\lambda^2)^2 T^4 \tag{4-20}$$

where $|R''(0)|$ exists.

<u>The Derivative Approximation</u>

Suppose that T is considered small enough that the straight line error approximation discussed in (4-1) is valid where in order to deal with the derivative term, x(t) will be assumed to be mean square differentiable. Then for Gaussian x(t), under this assumption, where $t \in [nT, (n+1)T)$

$$e_1(t) = x'(nT + \lambda T)(t - nT - \lambda T), \tag{4-1}$$

$$\psi_1(nT,\lambda) = x'^2(nT + \lambda T)(1 - 3\lambda + 3\lambda^2)T^2/3 \tag{4-2}$$

$$\bar{\psi}_1(T,\lambda) = |R''(0)|(1 - 3\lambda + 3\lambda^2)T^2/3 \tag{4-21}$$

and

$$\mathrm{Var}[\psi_1(T,\lambda)] = \frac{2}{9}\,|R''(0)|^2(1-3\lambda+3\lambda^2)^2T^4 = 2\bar{\psi}_1(T,\lambda)\ , \tag{4-22}$$

where (4-6) has been used to simplify the fourth-order moment of $x'(nT+\lambda T)$. Note that (4-22) has the same form as (4-20). This simplified form will be used in the discussion immediately following.

<u>The Tchebycheff Inequality</u>

Suppose that $x(t)$ is Gaussian, and $\psi_1(nT,\lambda) = x'^2(nT - \lambda T)\ \frac{T^2}{3}$, then

$$P\left\{|\psi_1 - \bar{\psi}_1| \geq k\bar{\psi}_1\right\} \leq 2/k^2 \tag{4-23a}$$

or, equivalently,

$$P\left\{\psi_1 \geq k'\bar{\psi}_1\right\} \leq \frac{2}{(k'-1)^2}\ ,\ k' > 2. \tag{4-23b}$$

Proof: The Tchebycheff Inequality (34)

$$P\left\{|z - \bar{z}| \geq a\right\} \leq \frac{\mathrm{Var}[z]}{a^2} \tag{4-24}$$

becomes, upon substitution of $2\bar{\psi}_1^2$ for $\mathrm{Var}[\psi_1]$ from (4-22),

$$P\left\{|\psi_1 - \bar{\psi}_1| \geq a\right\} \leq 2\left(\frac{\bar{\psi}_1}{a}\right)^2$$

For $a = k\bar{\psi}_1$, (4-23a) follows. Further, since $\psi_1 \geq 0$ and $\bar{\psi}_1 \geq 0$, if $k > 1$ and $|\psi_1 - \bar{\psi}_1| \geq k\bar{\psi}_1$, then $\psi_1 > \bar{\psi}_1$, $|\psi_1 - \bar{\psi}_1| = \psi_1 - \bar{\psi}_1 \geq k\bar{\psi}_1$, and

$\psi_1 \geq (k+1)\bar{\psi}_1$. Substitution of this last inequality in (4-23a) yields

$$P\left\{|\psi_1 - \bar{\psi}_1| \geq k\bar{\psi}_1\right\} = P\left\{\psi_1 \geq (k+1)\bar{\psi}_1\right\} \leq \frac{2}{k^2} .$$

For $k+1 = k'$, (4-23b) follows.

If the bound is to have any meaning, then the bound must be less than 1, i.e., $k > \sqrt{2}$. For example, when $k = 2$, then

$$P\left\{|\psi_1 - \bar{\psi}_1| \geq 2\bar{\psi}_1\right\} = P\left\{\psi_1 \geq 3\bar{\psi}_1\right\} \leq \frac{1}{2}$$

Since ψ_1 is non-negative and $\bar{\psi}_1$ is small and $\mathrm{Var}[\psi_1]$ is couched in terms of $\bar{\psi}_1$, some further analysis is in order and leads to another bound.

<u>The Bienaymé Inequality</u>

Suppose that $\bar{\psi}(T, \lambda)$ is known, then

$$P\left\{\psi(nT,\lambda) \geq k\,\bar{\psi}(T,\lambda)\right\} \leq \frac{1}{k} . \tag{4-25}$$

Proof: Consider the non-negative random variable, x. Now

$$\bar{x} = \int_0^\infty xp(x)dx = \int_0^{k\bar{x}} xp(x)dx + \int_{k\bar{x}}^\infty xp(x)dx \geq$$

$$\geq \int_{k\bar{x}}^\infty xp(x)dx \geq k\bar{x}\int_{k\bar{x}}^\infty p(x)dx = k\bar{x}\,P\left\{x \geq k\bar{x}\right\}$$

thus

$$P\left\{x \geq k\bar{x}\right\} \leq 1/k ,$$

and (4-25) follows. This bound is superior to (4-23) in that it does not require either the Gaussian hypothesis or fourth-order moments, and, furthermore, for $k \leq 2 + \sqrt{3}$ and Gaussian x(t) it even yields a bound lower than that obtained by the Tchebycheff Inequality.

Both the Tchebycheff- and Bienaymé-derived bounds yield information about the probability that a specific $\psi(nT,\lambda)$ lies between 0 and $k\bar{\psi}(T, \lambda)$. The preceding chapter discussed techniques for bounding $\bar{\psi}(T, \lambda)$ and use of the upper bounds in (4-23a), (4-23b), or (4-25) serves only to make them more conservative.

Error Behavior in a Run

The preceding analyses have been directed toward characteristics of the error in a single sampling interval. Another useful analysis is that of the multiple interval error behavior, or run error behavior, where a run is defined as N consecutive sampling intervals of duration T. In effect, an observation interval of length $T_o = NT$ is available for study. This is exactly the situation which arises in practice and leads to a comparison of the average error behavior of a finite run to the expected error criterion for the single interval.

Infinite Run

Suppose that x(t) is ergodic, then

$$\bar{\psi}(T,\lambda) = \lim_{T_o \to \infty} \frac{1}{2T_o} \int_{-T_o}^{T_o} e^2(t, \lambda T)dt \qquad (4\text{-}26)$$

with probability one.

Proof: Substituting the expression for $e^2(t, \lambda T)$ given in (3-8) and rewriting the infinite integral as an infinite summation of finite integrals, then

$$\lim_{T_o \to \infty} \frac{1}{2T_o}\int_{-T_o}^{T_o} e^2(t)dt = \lim_{N\to\infty}\left\{\frac{1}{(2N+1)T}\sum_{-N}^{N-1}\int_{nT}^{(n+1)T}[x(t-\lambda T) - x(nT)]^2 dt\right\} = \qquad (4\text{-}27)$$

$$= \frac{1}{T}\lim_{N\to\infty}\left\{\frac{1}{2N+1}\sum_{-N}^{N-1}\int_{o}^{T}[x^2(\tau-\lambda T+nT) - 2x(\tau-\lambda T+nT)x(nT)+x^2(nT)]\,d\tau\right\}.$$

Granting the validity of the interchange of the integration and limiting processes, then (4-27) may be written as

$$\frac{1}{T}\int_{o}^{T}\lim_{N\to\infty}\frac{1}{2N+1}\sum_{-N}^{N-1}\left\{x^2(\tau-\lambda T+nT) - 2x(\tau-\lambda T+nT)x(nT)+x^2(nT)\right\}d\tau. \quad (4\text{-}28)$$

But for ergodic processes (35)

$$\lim_{N\to\infty}\frac{1}{2N+1}\sum_{-N}^{N-1} x(nT+\theta)x(nT+\theta+\tau) = R(\tau),\ T > 0$$

with probability one, and, upon using this property in (4-28), the limit portion reduces to $2R(0) - 2R(\tau - \lambda T)$, thus

$$\lim_{T_o\to\infty}\frac{1}{2T_o}\int_{-T_o}^{T_o} e^2(t,\ \lambda T)dt = \frac{1}{T}\int_{o}^{T} 2[R(0) - R(\tau-\lambda T)]d\tau = \bar{\psi}(T,\lambda)$$

and (4-26) is proved.

The error parameter $\bar{\psi}(T,\lambda)$ is therefore valid as an estimate of the long term behavior of the interpolator. The remaining sections in this chapter will discuss the error existing in short runs where T_0 is finite.

Finite Runs

An Approximate Expression. Consider the derivative error approximation introduced in (4-1), (4-2), (4-21), and (4-22) with the resultant mean square error given in (4-2) as

$$\psi_1(nT,\lambda) = x'^2(nT+\lambda T)(1-3\lambda+3\lambda^2)\,\frac{T^2}{3} \tag{4-29}$$

which presupposes a fairly high sampling rate. This is a good approximation, especially in the sense of error analysis, since the expected value of $\psi_1(nT,\lambda)$ is identical to the quadratic bounds obtained for $\bar{\psi}(T,\lambda)$, the actual expected mean square error, in (3-14) and (3-17), for band-limited and differentiable processes respectively. Thus

$$\bar{\psi}(T,\lambda) \le |R''(0)|(1-3\lambda+3\lambda^2)\,\frac{T^2}{3} = \bar{\psi}_1(T,\lambda)\ .$$

Error in Approximation. For band-limited processes, it is possible to place a tighter bound upon the difference between the actual expected mean square error, $\bar{\psi}(T,\lambda)$, and the approximate expected mean square error, $\bar{\psi}_1(T,\lambda)$, since $\bar{\psi}_1(T,\lambda)$ is an upper bound on $\bar{\psi}(T,\lambda)$ and since $\bar{\psi}(T,\lambda)$ has a lower bound given in (3-12) then $\bar{\psi}(T,\lambda)$ must lie between the extremes given by these bounds. Rewriting (3-12), then

$$\bar{\psi}_1(T,\lambda) - \bar{\psi}(T,\lambda) \le |R''(0)|(1-3\lambda+3\lambda^2)\,\frac{T^2}{3} - \frac{2|R''(0)|}{\omega_c^2}\left\{\frac{\omega_c T - \sin(\lambda\omega_c T) - \sin[\omega_c T(1-\lambda)]}{\omega_c T}\right\}. \tag{4-30}$$

The sine functions may be replaced by a truncated series expansion which will preserve the inequality, i.e.,

$$\sin x \leq x - \frac{x^3}{3!} + \frac{x^5}{5!} ,$$

due to the property of alternating convergent series that the difference between the infinite and finite series is bounded by next term in the expansion. Making this substitution,

$$\bar{\Psi}_1(T,\lambda) - \bar{\Psi}(T,\lambda) \leq |R''(0)| \frac{T^2}{3} (\omega_c T)^2 [1-5\lambda+10\lambda^2-10\lambda^3+5\lambda^4] \leq \tag{4-31}$$

$$\leq \frac{R(0)}{3} (\omega_c T)^4 [1 - 5\lambda + 10\lambda^2 - 10\lambda^3 + 5\lambda^4]$$

where the latter inequality follows from (2-6). Therefore, the difference in the expected value of the true and derivative approximation mean square errors is bounded by an $(\omega_c T)^4$ term for band-limited random processes.

<u>The Sampled Mean</u>. Consider a finite run where N consecutive sampling intervals are observed. The data so obtained can be viewed as an estimate of the long run behavior of the interpolator. Defining φ in the following manner,

$$\varphi = \frac{1}{N} \sum_{n=o}^{N-1} \psi_1(nT,\lambda) = \frac{T^2}{3N} (1 - 3\lambda + 3\lambda^2) \sum_{n=o}^{N-1} x'^2(nT + \lambda T) , \tag{4-32}$$

then φ is an unbiased estimate of $\bar{\Psi}_1(T,\lambda)$, that is, $E\{\varphi\} = \bar{\Psi}_1(T,\lambda)$. The random portion of (4-29) may be isolated to form a sample mean which can be analyzed by standard techniques (36),

$$\mu = \frac{1}{N} \sum_{n=o}^{N-1} x'^2 (nT + \lambda T) \tag{4-33}$$

where $E\{\mu\} = |R''(0)|$. The variance of μ may be determined if

$$E\left\{[\mu - E\{\mu\}]^2\right\} = E\left\{\frac{1}{N^2} \sum_{n=o}^{N-1} \sum_{m=o}^{N-1} x'^2(nT + \lambda T) x'^2(mT + \lambda T)\right\} - E^2\{\mu\}$$

can be calculated. If $x'(t)$ is fourth-order stationary, then

$$\mathrm{Var}[\mu] = \frac{1}{N} E\left\{x'^4(0)\right\} + \frac{2}{N^2} \sum_{i=1}^{N-1} (N-i) E\left\{x'^2(0) x'^2(iT)\right\} - E^2\{\mu\} . \tag{4-34}$$

These preliminaries lead to the following results.

Variance of Mean Square Error. Suppose $x(t)$ is Gaussian and differentiable and T is sufficiently small that the straight line error approximation is valid, then the sample mean of a run of duration $T_o = NT$ seconds has a variance given by

$$\mathrm{Var}[\varphi] = \left[\frac{T^2(1-3\lambda+3\lambda^2)}{3}\right]^2 \left[\frac{2R''^2(0)}{N} + \frac{4}{N} \sum_{i=1}^{N-1} \left(1 - \frac{i}{N}\right) R''^2(iT)\right]. \tag{4-35}$$

Proof: Since $x(t)$ is Gaussian and differentiable, then

$$E\left\{x'^2(0) x'^2(iT)\right\} = R''^2(0) + 2R''^2(iT)$$

from (4-6). Using this equality, as well as the fact that $E^2\{\mu\} = R''^2(0)$, in (4-34), then

$$\mathrm{Var}[\mu] = \frac{3}{N} R''^2(0) + \frac{2}{N^2} \sum_{i=1}^{N-1} (N-i)[R''^2(0) + 2R''^2(iT)] - R''^2(0) .$$

Rearranging terms, this can be written as

$$\mathrm{Var}[\mu] = R''^2(0)\left[\frac{3}{N} - 1 + \frac{2}{N^2} \sum_{i=1}^{N-1} (N-i)\right] + \frac{4}{N^2} \sum_{i=1}^{N-1} (N-i) R''^2(iT) .$$

This can be simplified further since

$$\frac{2}{N^2} \sum_{i=1}^{N-1} (N-i) = \frac{2}{N^2} [(N-1) + (N-2) + \ldots + 1] =$$

$$= \frac{2}{N^2} \left[\frac{(N-1)^2 + (N-1)}{2}\right] = 1 - \frac{1}{N} ,$$

so that

$$\mathrm{Var}[\mu] = \frac{2}{N} R''^2(0) + \frac{4}{N^2} \sum_{i=1}^{N-1} (N-i) R''^2(iT).$$

Noting that

$$\varphi = \frac{T^2}{3} [1 - 3\lambda + 3\lambda^2] \mu ,$$

(4-35) immediately follows.

Run Variance in Error Analysis

Under the assumptions of high sampling rates, and differentiable random processes, the interpolation error in a sampling interval approaches a straight line and the time average run error on the interval $[0, nT)$ has been shown to be

$$\varphi = \frac{T^2}{3N}(1 - 3\lambda + 3\lambda^2) \sum_{n=o}^{N-1} x'^2(nT + \lambda T) . \qquad (4\text{-}32)$$

The expected value of φ is simply

$$\bar{\varphi} = \frac{T^2}{3}(1 - 3\lambda + 3\lambda^2)|R''(0)| . \qquad (4\text{-}36)$$

If the process is also Gaussian, it was shown that

$$\mathrm{Var}[\varphi] = \left[\frac{T^2(1 - 3\lambda + 3\lambda^2)}{3}\right]^2 \left[\frac{2R''(0)}{N} + \frac{4}{N}\sum_{i=1}^{N-1}(1 - i/N)R''^2(iT)\right]. \qquad (4\text{-}35)$$

Equation (4-35) may be rewritten as

$$\mathrm{Var}[\varphi] = \bar{\varphi}^2\left\{\frac{2}{N} + \frac{4}{N}\sum_{i=1}^{N-1}(1 - i/N)\left[\frac{R''(iT)}{R''(0)}\right]^2\right\}. \qquad (4\text{-}37)$$

This form indicates that the Tchebycheff Inequality might now be used to greater advantage in determining a confidence level on the difference in the run average and expected value of a run since although the variance is dependent on $\bar{\varphi}^2$ it is multiplied by a function which should decrease with increasing N. The Tchebycheff bound previously discussed, (4-23a) and (4-23b), could have been obtained from (4-37) for $N = 1$. For $N > 1$,

$$P\{|\varphi - \bar{\varphi}| \geq k\bar{\varphi}\} \leq \frac{\mathrm{Var}[\varphi]}{(k\bar{\varphi})^2} . \qquad (4\text{-}38)$$

Substitution of (4-37) in (4-38) leads to

$$P\left\{|\varphi - \bar{\varphi}| \geq k\bar{\varphi}\right\} \leq \frac{2}{k^2 N}\left\{1 + 2\sum_{i=1}^{N-1} (1 - i/N)\left[\frac{R''(iT)}{R''(0)}\right]^2\right\}. \qquad (4\text{-}39)$$

If a fixed observation interval $[0, T_o)$ is considered and N is allowed to increase without limit where $NT = T_o$, then (4-39) may be rewritten as an integral expression since

$$\lim_{N \to \infty}\left\{\frac{1}{N} + \frac{2}{N}\sum_{i=1}^{N-1} (1 - i/N)\left[\frac{R''(iT)}{R''(0)}\right]^2\right\} = \frac{2}{T_o}\int_0^{T_o} (1 - \tau/T_o)\left[\frac{R''(\tau)}{R''(0)}\right]^2 d\tau$$

thus

$$P\left\{|\varphi - \bar{\varphi}| \geq k\,\bar{\varphi}\right\} \leq \frac{4}{k^2 T_o}\int_0^{T_o} (1 - \tau/T_o)\left[\frac{R''(\tau)}{R''(0)}\right]^2 d\tau \qquad (4\text{-}40)$$

for sampling intervals approaching zero.

Neither the form of (4-39) nor that of (4-40) lends itself to general statements unless a specific $R''(\tau)$ is to be evaluated. However, it can be seen that for those $R''^2(\tau)$ which are monotone decreasing, an increase in N leads to a decrease in the bound given by (4-39) for $P\left\{|\varphi - \bar{\varphi}| \geq k\bar{\varphi}\right\}$ since the summation term could be rewritten as

$$\frac{1}{T_o}\sum_{i=1}^{N-1} \left(1 - \frac{iT}{T_o}\right)\left[\frac{R''(iT)}{R''(0)}\right]^2 T$$

which corresponds to the area of a monotonic staircase function and decreases with increasing N (37).

It is interesting to note that for either (4-39) or (4-40) if the observation interval, T_o, is small enough that $R''^2(\tau) \simeq R''^2(0)$ for $\tau \in [0, T_o)$, then both the summation and integral terms above have constant values and the bound on $P\{|\varphi - \bar{\varphi}| \geq k\bar{\varphi}\}$ becomes approximately $2/k^2$, and the confidence level on φ becomes the same as that obtained for $\psi_1(nT,\lambda)$ earlier (4-23).

Knowing $R(\tau)$, and thus $R''(\tau)$, the behavior of the run average error may be analyzed, generally leading to a confidence level about $\bar{\varphi}$ which improves as N, the number of samples considered in the run, is increased.

CHAPTER V

SAMPLE CALCULATIONS

Several of the techniques of error analysis developed in the preceding chapters will now be applied to the investigation of some common classes of spectral densities. The causes to be discussed are band-limited white noise, a non-band-limited but differentiable process, the non-differentiable exponential autocorrelation function, and the sampled sine wave. The latter case will be shown to have some interesting additional properties. The mean square error calculations may be simplified somewhat by considering only the two values of delay which are of prime interest, $\lambda = 0$ and $\lambda = 1/2$. Values for $\lambda = 1/2$ may be obtained if $\bar{\psi}(T, 0)$ is known by using a relation obtained by Liff (38),

$$\bar{\psi}\,(T,\ 1/2) = \bar{\psi}\,(T/2,\ 0)\ , \qquad (5\text{-}1)$$

which follows from the evenness of $V(\tau)$ and the integral formulation of $\bar{\psi}(T, \lambda)$.

The mean square error criterion has another property in the limit as T becomes large, at least for those processes with autocorrelation functions tending to zero for large τ. Obviously, for such processes $V(\tau)$ approaches $R(0)$ for large τ and since $\bar{\psi}(T, \lambda)$ is the average over T of $2V(\tau)$, then

$$\lim_{T \to \infty} \left\{ \bar{\psi}(T,\ \lambda) \right\} = 2R(0)\ .$$

Band-Limited White Noise

Suppose white noise is passed through an ideal flat lowpass filter of cutoff frequency, ω_c. The output has a spectral density given by

$$S(\omega) = \begin{cases} \pi/\omega_c \;\; N_o & \text{for } |\omega| \le \omega_c \\ 0 & \text{for } |\omega| > \omega_c \end{cases} \tag{5-2}$$

with the corresponding autocorrelation function

$$R(\tau) = N_o \frac{\sin \omega_c \tau}{\omega_c \tau} . \tag{5-3}$$

The integral formulation for $\bar{\psi}(T, \lambda)$ may be written since $R(\tau)$ is known and becomes

$$\bar{\psi}(T,\lambda) = \frac{2}{T} \int_{-\lambda T}^{(1-\lambda)T} N_o \left[1 - \frac{\sin \omega_c \tau}{\omega_c \tau}\right] d\tau \tag{5-4}$$

$$= \frac{2}{\omega_c T} \int_{-\lambda \omega_c T}^{(1-\lambda)\omega_c T} N_o \left[1 - \frac{\sin \alpha}{\alpha}\right] d\alpha .$$

For $\lambda = 0$, this reduces to

$$\bar{\psi}(T, \lambda) = \frac{2N_o}{\omega_c T} \left[\omega_c T - \mathrm{Si}(\omega_c T)\right]$$

where $\mathrm{Si}(x)$ represents the familiar $\frac{\sin x}{x}$ integral.

Differentiation of $R(\tau)$ yields

$$R'(\tau) = \omega_c N_o \frac{\omega_c \tau \cos(\omega_c \tau) - \sin(\omega_c \tau)}{(\omega_c \tau)^2} \tag{5-5}$$

and

$$R''(\tau) = \omega_c^2 N_o \frac{2\sin(\omega_c\tau) - 2(\omega_c\tau)\cos(\omega_c\tau) - (\omega_c\tau)^2\sin(\omega_c\tau)}{(\omega_c\tau)^3} . \qquad (5\text{-}6)$$

Application of L'Hospital's Rule readily shows that

$$R''(0) = -\frac{\omega_c^2}{3} N_o$$

and

$$\omega_d^2 = |R''(0)|/R(0) = \omega_c^2/3 . \qquad (5\text{-}7)$$

The interpolation error function and its bounds are plotted on a log-log scale in Figure 9. The units of the horizontal axis are in terms of the ratio of the sampling frequency to the cutoff frequency to more effectively illustrate the magnitude of the increase in the sampling rate necessary to reduce the interpolation error.

Either $\bar{\psi}(T, \lambda)$ or its upper bound might now be used in the Bienayme Inequality bound of (4-25) or, if x(t) is Gaussian, in the Tchebycheff Inequality bound of (4-23) to obtain results that are the same for all sampled processes in the sense that given a $\bar{\psi}(T, \lambda)$, these bounds are independent of other aspects of the process.

For Gaussian x(t), however, the run variance, which is a function of $R''(\tau)$, may be used to obtain the confidence level of (4-39). For sampled band-limited white noise, substitution of (5-6) into (4-39) yields

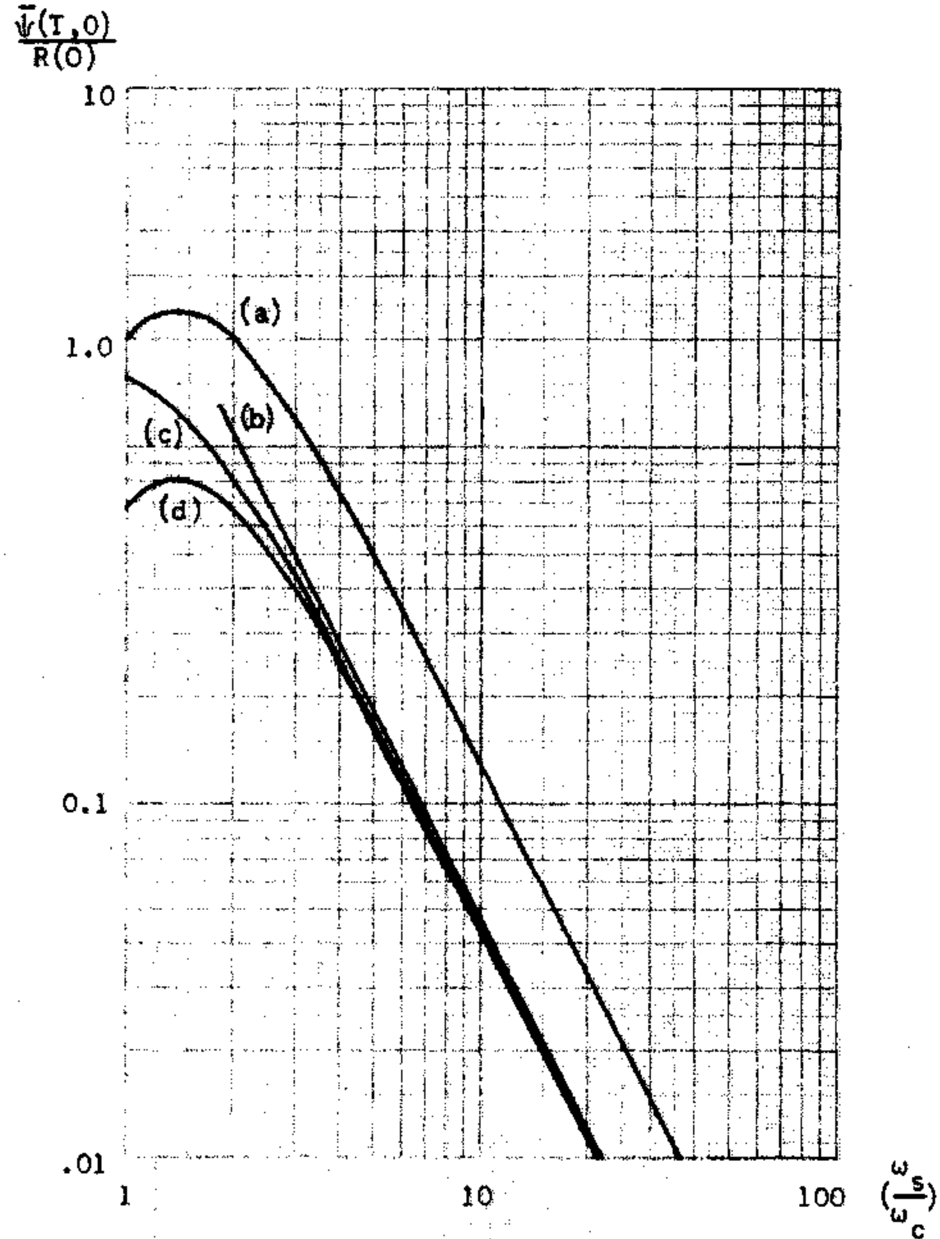

Figure 9. Normalized Interpolation Error Compared to its Theoretical Bounds for Band-Limited White Noise.

a) Upper Sine Bound,
b) Quadratic Upper Bound,
c) $\bar{\psi}(T,0)/R(0)$,
d) Lower Sine Bound.

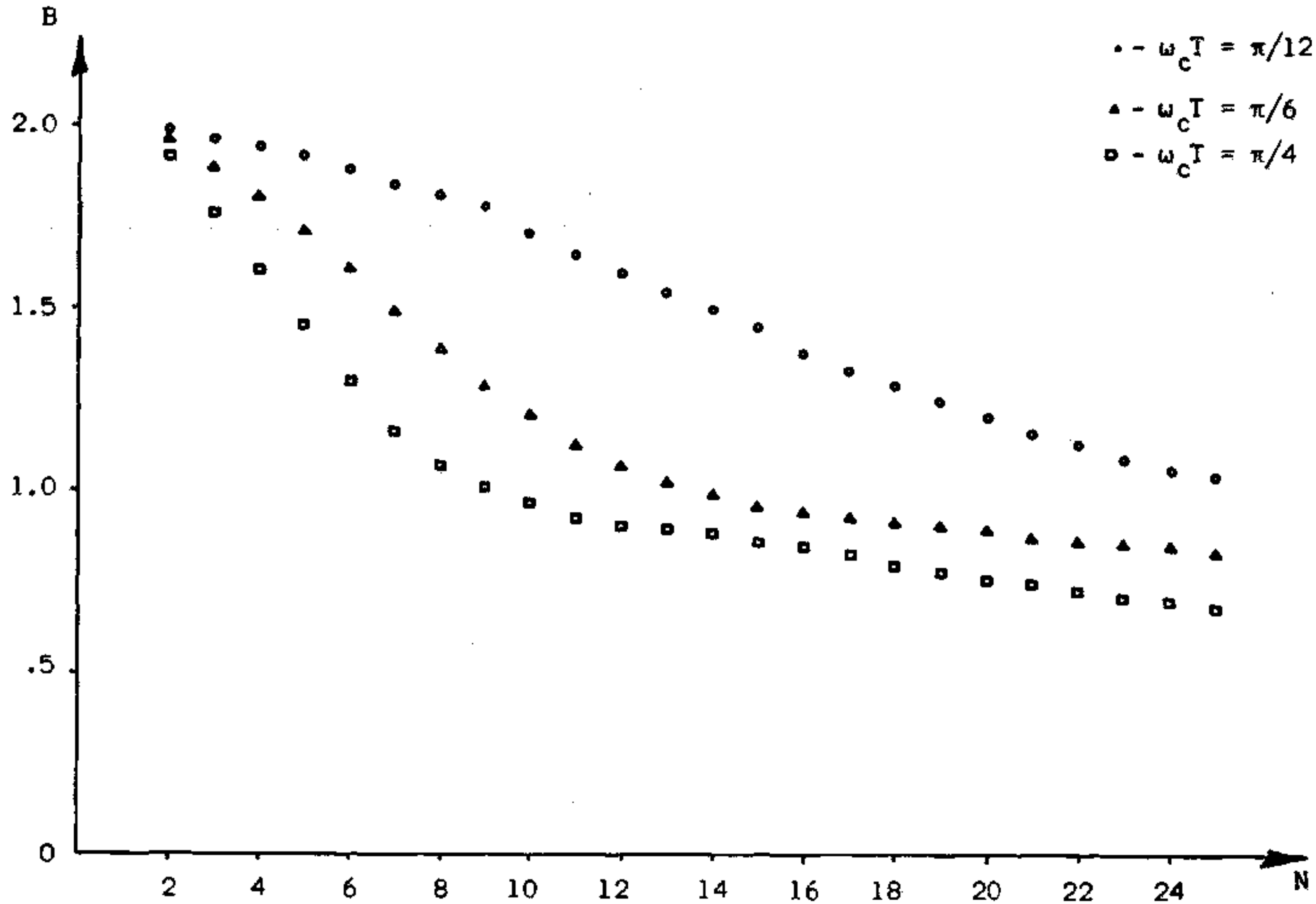

Figure 10. The Run Variance of Band-Limited White Noise Versus N.

$$B = \left\{\frac{2}{N} + \frac{4}{N^2}\sum_{i=1}^{N-1}(N-i)\left[\frac{R''(iT)}{R''(0)}\right]^2\right\} \text{ and } P\left\{|\varphi - \bar{\varphi}| \geq k\bar{\varphi}\right\} \leq B/k^2.$$

$$P\left\{|\varphi - \bar{\varphi}| \geq k\bar{\varphi}\right\} \leq \frac{1}{k^2}\left\{\frac{2}{N} + \frac{4}{N}\sum_{i=1}^{N-1}\frac{(1-i/N)}{(i\,\omega_c T)^6}[6\sin(i\,\omega_c T)\right. \qquad (5\text{-}8)$$

$$\left. - 6(i\,\omega_c T)\cos(i\omega_c T) - 3(i\omega_c T)^2\sin(i\omega_c T)]^2\right\}.$$

Given k, N, and T, (5-8) could be calculated and used to examine the run average behavior of sampled data.

Suppose it is desired to sample and interpolate an $x(t)$ with $R(\tau) = N_o \dfrac{\sin \omega_c\tau}{\omega_c\tau}$ where the expected interpolation error is to be less than 0.01 $R(0)$. From Figure 9, it may be seen that $\omega_s/\omega_c = 24$ is sufficient or $T = \pi/12\omega_c$. The Bienayme Inequality (4-25) may be used to show that $P\{\psi(nT, 0) \geq 2\bar{\psi}(T, 0)\} \leq 1/2$ and $P\{\psi(nT, 0) \geq 3\bar{\psi}(T, 0)\} \leq 1/3$. If the process is assumed Gaussian, then the Tchebycheff Inequality (4-23) may be used to show that $P\{\psi_1(nT, 0) \geq 3\bar{\psi}_1(T, 0)\} \leq \frac{1}{2}$ and no information is gained. However, from the curves presented for the run variance where the value of $\left\{\frac{2}{N} + \frac{4}{N^2}\sum_{i=1}^{N-1}(N-i)\left[\frac{R''(iT)}{R''(0)}\right]^2\right\}$ has been plotted versus N, it appears that $P\{\varphi \geq 3\bar{\varphi}\} \leq 0.301$ for runs of duration greater than ten sampling intervals. The average error behavior is thus rapidly converging to the expected error behavior.

A Non-Band-Limited Differentiable Process

Consider the non-band-limited but low pass spectrum

$$S(\omega) = \frac{4\alpha^3}{(\alpha^2 + \omega^2)^2} \qquad (5\text{-}9)$$

which could have resulted from the passage of white noise through an appropriate linear filter. A process with such a spectral density would also have the autocorrelation function

$$R(\tau) = \epsilon^{-\alpha|\tau|} + \alpha|\tau|\epsilon^{-\alpha|\tau|}. \qquad (5\text{-}10)$$

Substituting in $\bar{\psi}(T, \lambda)$, the expected interpolation error becomes

$$\bar{\psi}(T, \lambda) = \frac{2}{T}\int_{-\lambda T}^{(1-\lambda)T} [1 - \epsilon^{-\alpha|\tau|} - \alpha|\tau|\epsilon^{-\alpha|\tau|}]d\tau .$$

For $\lambda = 0$, this becomes

$$\bar{\psi}(T, \lambda) = \frac{2}{\alpha T}\int_{o}^{\alpha T} [1 - \epsilon^{-t} - t\epsilon^{-t}]dt \qquad (5\text{-}11)$$

$$= \frac{2}{\alpha T}[\alpha T + (2 + \alpha T)\epsilon^{-\alpha T} - 2] .$$

Differentiating $R(\tau)$,

$$R'(\tau) = -\alpha^2|\tau|\epsilon^{-\alpha|\tau|} \qquad (5\text{-}12)$$

$$R''(\tau) = -\alpha^2 \quad {}^{-\alpha|\tau|} + \alpha^3|\tau|\epsilon^{-\alpha|\tau|}$$

$$R''(0) = -\alpha^2$$

and

$$\omega_d^2 = \alpha^2 .$$

$\bar{\psi}(T, 0)$ and its bound are indicated in Figure 11.

If $x(t)$ is Gaussian, the run variance, can be calculated and leads to the following confidence level

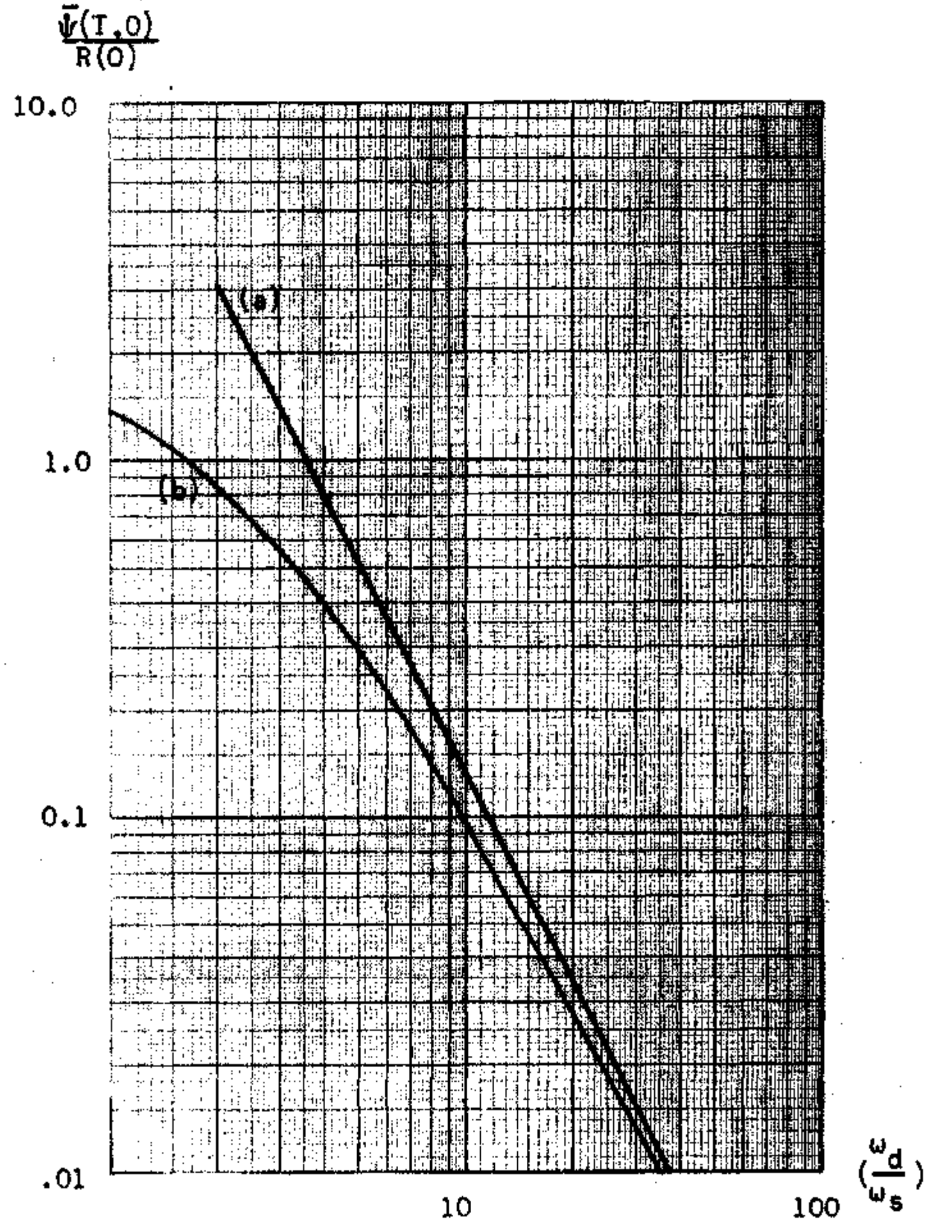

Figure 11. Normalized Interpolation Error Compared to its Theoretical Bound for a Second-Order Non-Band-Limited Spectral Density.

(a) Upper Quadratic Bound,

(b) $\bar{\Psi}(T,0)/R(0)$.

$$P\left\{|\varphi - \bar{\varphi}| \geq k\bar{\varphi}\right\} \leq \frac{2}{k^2}\left\{\frac{1}{N} + \frac{2}{N}\sum_{i=1}^{N-1} (1 - i/N)\left[\epsilon^{-2i\alpha T}(1 - 2i\alpha T + (i\alpha T)^2)\right]\right\}. \quad (5\text{-}13)$$

$\bar{\psi}(T,0)$ and its bound are indicated in Figure 12, plotted on a log-log scale. For an interpolation error less than 0.015 R(0), it may be seen that ω_s/ω_d = 31.4, or $T = \frac{2\pi}{31.4\alpha} = \frac{.2}{\alpha}$ is sufficient. Inspection of the run variance curve shows that for a run containing 4 or more samples where x(t) is Gaussian, $P\{\varphi \geq 3\bar{\varphi}\} \leq 0.23$ and, for runs of 9 or more samples, $P\{\varphi \geq 2\bar{\varphi}\} \leq 0.5$. The average error behavior is again seen to be rapidly converging to that predicted by the expected mean square error criterion.

The Exponential Autocorrelation Function

For the familiar exponential autocorrelation function

$$R(\tau) = \epsilon^{-\alpha(\tau)} \quad (5\text{-}14)$$

$$S(\omega) = 2\alpha/(\alpha^2 + \omega^2) \quad (5\text{-}15)$$

and substitution leads to

$$\bar{\psi}(T, \lambda) = \frac{2}{T}\int_{-\lambda T}^{(1-\lambda)T} [1 - \epsilon^{-\alpha|\tau|}]d\tau .$$

For $\lambda = 0$,

$$\bar{\psi}(T, 0) = \frac{2}{\alpha T}\int_{0}^{\alpha T} [1 - \epsilon^{-t}]dt =$$

$$= \frac{2}{\alpha T}\left[\alpha T - 1 + \epsilon^{-\alpha T}\right] . \quad (5\text{-}16)$$

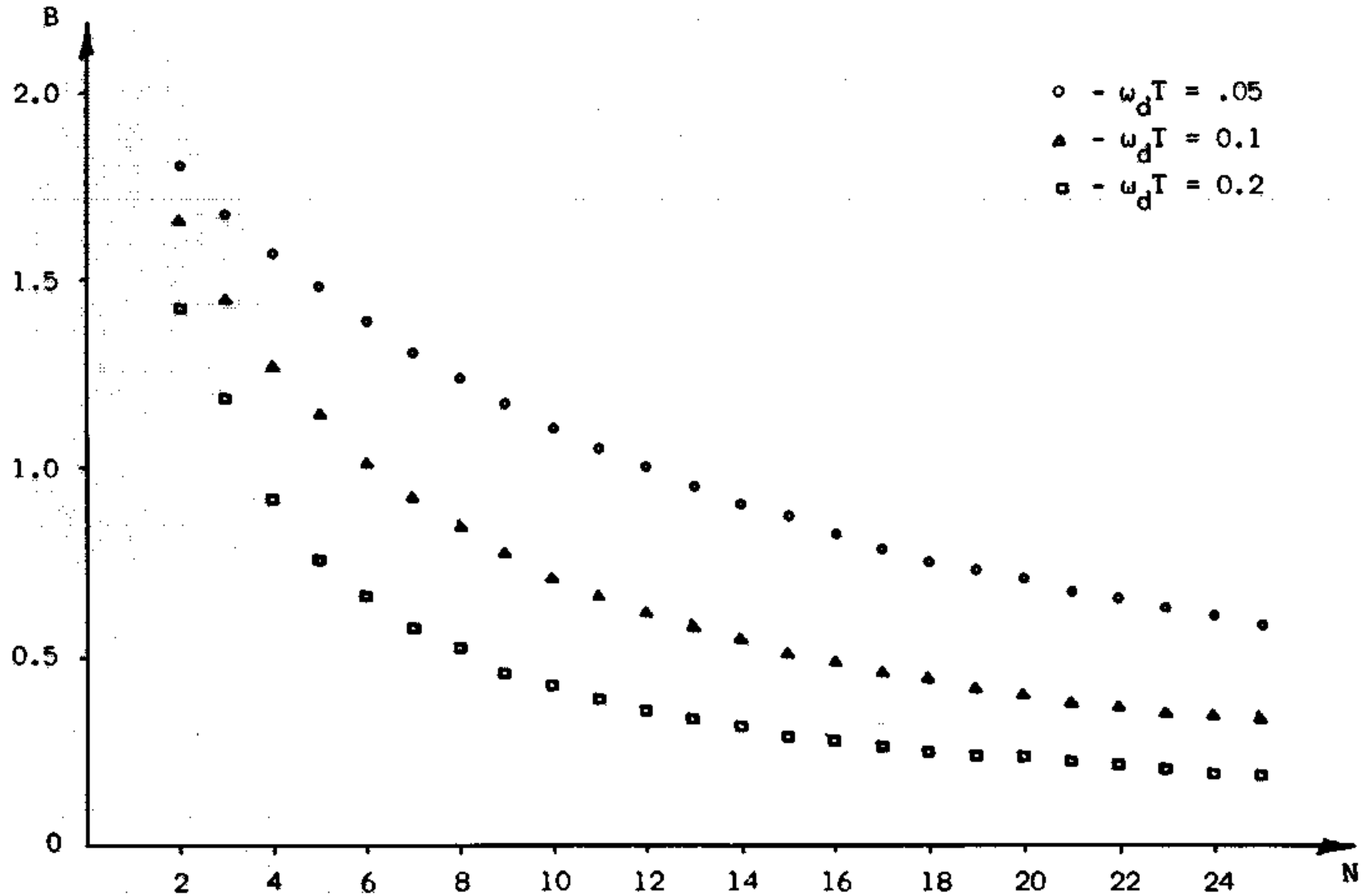

Figure 12. The Run Variance of a Second-Order Spectral Density Versus N.

$$B = \left\{ \frac{2}{N} + \frac{4}{N^2} \sum_{i=1}^{N-1} (N-i)\left[\frac{R''(iT)}{R''(0)}\right]^2 \right\} \text{ and } P\left\{ |\varphi - \bar{\varphi}| \geq k\bar{\varphi} \right\} \leq B/k^2.$$

Inspection shows that a process with such an autocorrelation function must be non-differentiable since $|R''(0)|$ does not exist. Examination of $S(\omega)$ further shows that

$$k/\omega^2 - S(\omega) \geq 0$$

for all ω if $k \geq 2\sigma$. For $k = 2\alpha$, then

$$\lim_{\omega \to \infty} \omega^2 S(\omega) = k = 2\alpha \ .$$

Thus, the bound of Chapter III can be applied yielding

$$\bar{\psi}(T, 0) \leq \alpha T = \omega_n T \qquad (5\text{-}17)$$

where $\omega_n = \frac{k}{2R(0)}$. $\bar{\psi}(T, 0)$ along with its bound is sketched in Figure 13. As predicted in Chapter III, for higher sampling rates, the rate of decrease in interpolation error for a given increase in sampling frequency is one half that obtained for the differentiable processes of Figures 9 and 11.

Since the discussion of the run variance depended upon the existence of a derivative approximation to the error, it cannot be used in this case.

Sampling of a Sine Wave with Random Phase

The investigation of the nature of the distribution of $\psi(nT, \lambda)$ was undertaken to evaluate $\bar{\psi}(T, \lambda)$'s value as an estimate of interpolator performance. As a single sample estimate, it is obvious that $E\{\psi(nT, \lambda)\}$ is the expected value of $\psi(nT, \lambda)$ for any ensemble member as well as for each sampling interval along any ensemble member. The

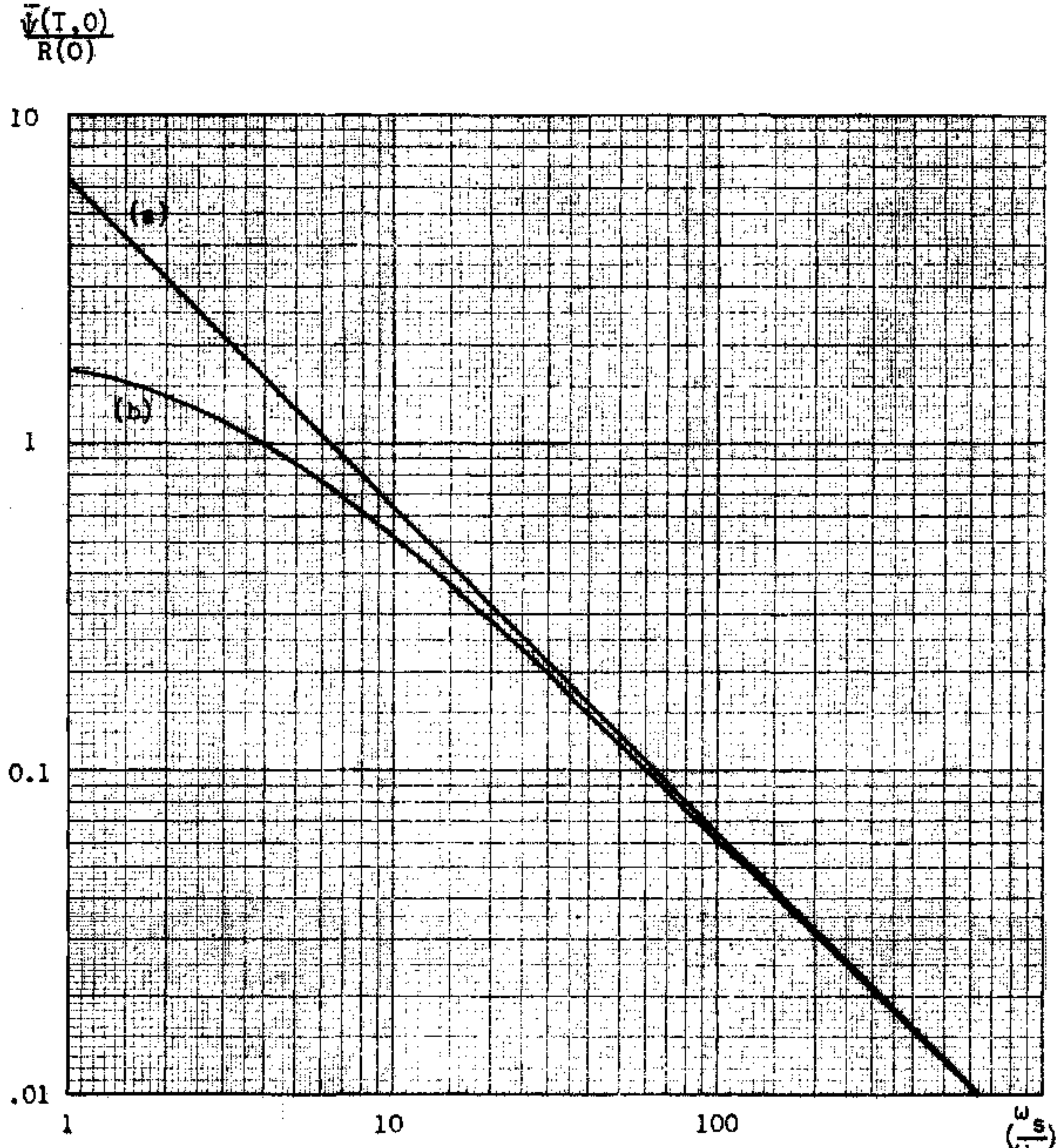

Figure 13. Normalized Interpolation Error Compared to its Theoretical Bound for the Exponential Autocorrelation Function.

a) Upper Linear Bound,

b) $\bar{\psi}(T,0)/R(0)$.

long run behavior of $\psi(nT, \lambda)$ for ergodic processes, i.e., the infinite time average of $\psi(nT, \lambda)$ along an ensemble member, has been shown to be the same as $\bar{\psi}(T, \lambda)$ for each ensemble member. The following pathological case involving sampled sine waves yields a result where a finite time average is sufficient to determine $\bar{\psi}(T, \lambda)$.

Consider the ensemble whose members may be represented by

$$x(t) = A \cos(\omega_o t + \theta) \tag{5-18}$$

where A and ω_o are known constants and θ is uniformly distributed on $[0, 2\pi)$. Since this ensemble is ergodic, then equality between $\bar{\psi}(T, \lambda)$ and the infinite time average along one ensemble number is expected. The following short run relation is also true.

Suppose x(t) is as above and is sampled at some rational multiple of the Nyquist rate, i.e., $T = \frac{\pi}{\omega_o} (\frac{m}{\ell})$, where $(\frac{m}{\ell}) < 1$, then the mean squared error is periodic with period $T_e = m \frac{\pi}{\omega_o}$, thus the average of the interpolation error over any consecutive ℓ intervals equals $\bar{\psi}(T, \lambda)$.

Proof: The expression for the average interpolation error may be written as

$$A_\ell\left\{e^2(t,d)\right\} = \frac{1}{\ell T} \sum_{n=N}^{N+\ell-1} \left\{\int_{nT}^{(n+1)T} [A \sin(\omega_o t-d+\theta) - A \sin(n\omega_o T+\theta)]^2 dt]\right\}. \tag{5-19}$$

This expression will be simplified by rearranging its terms using various trigonometric identities and then using the following two identities (39):

$$\sum_{i=o}^{\ell-1} \cos[i2\omega_o T]\Big|_{T=\frac{m\pi}{\ell\omega_o}} = \frac{1}{2}\left[1 + \frac{\sin[(\ell-1/2)2\omega_o T]}{\sin(\omega_o T)}\right]\Big|_{T=\frac{m\pi}{\ell\omega_o}} = \tag{5-20a}$$

$$= \frac{1}{2}\left[1 + \frac{\sin[2m\pi - \omega_o T]}{\sin \omega_o T}\right] = 0$$

and

$$\sum_{i=o}^{\ell-1} \sin[i2\omega_o T]\Big|_{T=\frac{m\pi}{\ell\omega_o}} = \frac{1}{2}\left[\cot(\omega_o T) - \frac{\cos[(\ell - 1/2)2\omega_o T]}{\sin(\omega_o T)}\right]\Big|_{T=\frac{m\pi}{\ell\omega_o}} = \tag{5-20b}$$

$$= \frac{1}{2}\left[\cot(\omega_o T) - \frac{\cos[2m\pi - \omega_o T]}{\sin(\omega_o T)}\right] = 0 \; .$$

Now,

$$A_\ell\{e^2(t, d)\} = \frac{A^2}{\ell T}\sum_{n=N}^{N+\ell-1}\Big\{\int_{nT}^{(n+1)T}[\sin^2(\omega_o t - d + \theta) + \sin^2(n\omega_o T + \theta)$$

$$- 2\sin(\omega_o t - d + \theta)\sin(n\omega_o T + \theta)]dt\Big\} \; .$$

The integrand may be rewritten as

$$[1 - \frac{1}{2}\cos(2n\omega_o T + 2\theta) - \frac{1}{2}\cos(2\omega_o t - 2\lambda T + 2\theta) - 2\sin(n\omega_o T + \theta)\cdot$$

$$\cdot \sin(\omega_o t - \lambda T + \theta)]$$

which becomes upon integration on $[nT, (n+1)T]$

$$[1 - \frac{1}{2}\cos(2n\omega_o T + 2\theta)]T - \frac{1}{4\omega_o T}\Big\{\sin[(2n+2)\omega_o T - 2d + 2\theta] -$$

$$- \sin[2n\omega_o T - 2d + 2\theta]\Big\} + \frac{2}{\omega_o}\sin(n\omega_o T + \theta)\Big\{\cos[(n+1)\omega_o T - d + \theta]$$

$$- \cos[n\omega_o T - d + \theta]\Big\} \; .$$

Now

$$2\sin(n\omega_o T + \theta)\cos[(n+1)\omega_o T - d + \theta] - 2\sin(n\omega_o T + \theta)\cos[n\omega_o T - d + \theta] =$$

$$= \sin[(2n+1)\omega_o T - d + 2\theta] - \sin[\omega_o T - d] - \sin[2n\omega_o T - d + 2\theta] - \sin(d) =$$

$$= \sin[(2n+1)\omega_o T]\cos(2\theta - d) + \cos[(2n+1)\omega_o T]\sin(2\theta - d)$$

$$- \sin[2n\omega_o T]\cos[2\theta - d] - \cos[2n\omega_o T]\sin(2\theta - d) - \sin[\omega_o T - d] - \sin(d).$$

The average may now be written as

$$A_\ell\left\{e^2(t, d)\right\} = \frac{A^2}{\ell T}\sum_{N}^{N+\ell-1}\left\{T - \frac{T}{2}\cos(2n\omega_o T + 2\theta) - \right. \tag{5-21}$$

$$- \frac{1}{4\omega_o}\left[\sin([2n+2]\omega_o T - 2d + 2\theta) - \sin(2n\omega_o T - 2d + 2\theta)\right] +$$

$$+ \frac{1}{\omega_o}\left[\cos(2\theta - d)\left\{\sin[(2n+1)\omega_o T] - \sin[2n\omega_o T]\right\} + \right.$$

$$+ \sin(2\theta - d)\left\{\cos[(2n+1)\omega_o T] - \cos[2n\omega_o T]\right\} -$$

$$\left.\left. - \sin(\omega_o T - d) - \sin d\right]\right\}.$$

For $T = \frac{m\pi}{\ell\omega_o}$, all the summations in (5-21) can be written in the form of either (5-19) or (5-20) and

$$A_\ell\left\{e^2(t, d)\right\} = A^2\left\{1 - \frac{1}{\omega_o T}\left[\sin(\omega_o T - d) + \sin(d)\right]\right\} \tag{5-22}$$

$$= A^2\left\{1 - \frac{1}{m\pi/\ell}\left[\sin\left(\frac{m\pi}{\ell} - d\right) + \sin d\right]\right\}.$$

For this ensemble, the autocorrelation function is

$$R(\tau) = \frac{A^2}{2} \cos(\omega_o \tau) \ ;$$

$$R'(\tau) = -\frac{A^2}{2\omega_o} \sin(\omega_o \tau) \ ;$$

$$R''(\tau) = -\frac{A^2}{2\omega_o^2} \cos(\omega_o \tau) \ ;$$

$$\omega_e^2 = \omega_o^2 ; \tag{5-23}$$

and

$$\bar{\psi}(T, \lambda) = \frac{2}{T} \int_{-\lambda T}^{(1-\lambda)T} \frac{1}{2}[A^2 - A^2 \cos \omega_o \tau] \, d\tau = \tag{5-24}$$

$$= A^2 \left\{ 1 - \frac{1}{\omega_o T} \left(\sin[(1-\lambda)\omega_o T] - \sin[\lambda \omega_o T] \right) \right\} .$$

For $T = \frac{m\pi}{\ell\omega_o}$ and $d = \lambda\omega_o T$, (5-22) and (5-24) are identical. Thus, any ℓ consecutive samples are sufficient to estimate $\bar{\psi}(T, \lambda)$ with zero error.

Note that for band-limited processes, the upper and lower sine bounds of (3-12) and (3-15) converge as $|R''(0)|$ approaches its maximum value of $\omega_c^2 R(0)$ and for $R(\tau) = A \cos(\omega\tau)$, since $|R''(0)| = \omega_c^2 R(0)$, they become identical and equal to (5-24).

Applications

General Procedure

Examination of the upper bounds on interpolator error for differentiable processes given in (3-14) and (3-17) indicates that the effect of the delay, λ, is separate and distinct from that of the sampling rate and furthermore that this quadratic bound is completely defined given

$R(0)$, $|R''(0)|$, λ, and T. Defining a percentage error bound as

$$\frac{\bar{\psi}(T,\ 0)}{R(0)} \leq \frac{|R''(0)|}{3R(0)}\, T^2 = \frac{(\omega_d T)^2}{3} = E^2 \qquad (5\text{-}25)$$

and using the effective band-limited frequency defined in (2-23), then

$$E^2 \approx \frac{(\omega_d T)^2}{3} = \frac{[\omega_d(\frac{1}{f_s})]^2}{3} = \frac{[2\pi(\frac{\omega_d}{\omega_s})]^2}{3}$$

which may be solved for ω_s, where $\omega_s = 2\pi f_s = 2\pi(\frac{1}{T})$, i.e.,

$$\omega_s = \frac{2\pi\omega_d}{\sqrt{3}\, E} = \frac{2\pi}{E}\left[\frac{|R''(0)|}{3R(0)}\right]^{1/2}, \qquad (5\text{-}26)$$

or, for T,

$$T = \left[\frac{3R(0)}{|R''(0)|}\right]^{1/2} E\ . \qquad (5\text{-}27)$$

In practice, the parameters $R(0)$ and $|R''(0)|$ could be determined by either of the following techniques: spectrum analysis or differentiation of $x(t)$. Suppose that a spectrum analyzer has been used to obtain $S(\omega)$ and that the order is at least two, then numerical analysis techniques may be used to evaluate the infinite integrals of $S(\omega)$ and $\omega^2 S(\omega)$. If $x(t)$ and $x'(t)$ are available, then an rms meter may be used to evaluate $\sqrt{R(0)}$ and $\sqrt{|R''(0)|}$. In either case, sufficient information is available to obtain ω_s, and thus a sampling rate.

Note that due to the normalization of both axes in Figure 12, the upper quadratic interpolation error bound sketched there is valid for any differentiable process. Knowing the percentage error level desired, a value of (ω_s/ω_d) can be read off and ω_s can be found in terms of ω_d.

The Butterworth Spectra

Consider the family of differentiable Butterworth spectra defined by

$$S(\omega) = \frac{1}{1 + (\omega/\omega_o)^{2n}} \tag{5-28}$$

where $n \geq 2$. The first-order spectrum $(n = 1)$ is the exponential autocorrelation function previously discussed. Both (5-26) and (5-27) require knowledge of $R(0)$ and $|R''(0)|$ which are readily obtainable from the integral

$$\int_{-\infty}^{\infty} \frac{x^{m-1}dx}{1 + x^{2n}} = \frac{\pi}{n \sin(\frac{m\pi}{2n})}$$

valid if $0 < m < 2n$ -- a condition satisfied in both the necessary integrals if $n \geq 2$. A change of variables yields

$$R(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{d\omega}{1 + (\frac{\omega}{\omega_o})^{2n}} = \frac{\omega_o}{2n \sin(\frac{\pi}{2n})}$$

and

$$|R''(0)| = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\omega^2 d\omega}{1 + (\frac{\omega}{\omega_o})^{2n}} = \frac{\omega_o^3}{2n \sin(\frac{3\pi}{2n})}$$

from which

$$\omega_d^{\ 2} = \frac{\sin(\pi/2n)}{\sin(3\pi/2n)}\,\omega_o^{\ 2}\ . \qquad (5\text{-}29)$$

Using these results in (5-26),

$$\omega_s = \frac{2\pi}{E}\left[\frac{\sin(\pi/2n)}{3\ \sin(3\pi/2n)}\right]^{1/2}\omega_o\ . \qquad (5\text{-}30)$$

Note that, for a given percentage error, ω_s has its maximum value, $\frac{2\pi}{\sqrt{3}\,E}\,\omega_o$ for n = 2, and rapidly converges to its limiting value of $\frac{2\pi}{3E}\,\omega_o$ ($n = \infty$). This means that, in order to achieve the same percentage interpolation error, the sampling frequency for a second-order Butterworth need only be $\sqrt{3}$ times greater than that required to sample and interpolate an ideal flat band-limited process with cutoff frequency ω_o, i.e., an infinite order Butterworth.

An Approximate Spectral Density

Analysis reveals similar results for the spectral form assumed by McRae and illustrated in Figure 6. For second-order or greater,

$$R(0) = \frac{1}{\pi}\int_0^{\omega_1} d\omega + \frac{1}{\pi}\int_{\omega_1}^{\infty}\left(\frac{\omega_1}{\omega}\right)^{2n} d\omega =$$

$$= \frac{\omega_1}{\pi}\left(\frac{2n}{2n-1}\right)$$

and

$$R''(0) = \frac{1}{\pi}\int_0^{\omega_1}\omega^2 d\omega + \frac{1}{\pi}\int_{\omega_1}^{\infty}\frac{\omega_1^{\ 2n}}{\omega^{2n-2}}\,d\omega =$$

$$= \frac{\omega_1^{\ 3}}{\pi}\left[\frac{2n}{3(2n-3)}\right]$$

from which

$$\omega_d^2 = \frac{2n-1}{3(2n-3)} \omega_1^2 .$$

Using these results in (5-26),

$$\omega_s = \frac{2\pi}{3E} \left[\frac{2n-1}{2n-3}\right]^{1/2} \omega_1 .$$

As expected, ω_s has its maximum value, $\frac{2\pi}{\sqrt{3}\,E}\,\omega_1$, for n = 2 and converges to its minimum value, $\frac{2\pi}{3E}\,\omega_1$, for $n = \infty$. Again it should be noted that the sampling frequencies vary by a factor of $\sqrt{3}$ and it appears that knowledge of an approximate rolloff point (ω_o, ω_1, etc.) is sufficient to estimate a suitable sampling rate.

<u>A Numerical Example</u>

As an example of the utility of the bounds obtained, consider the following problem. A signal having a Butterworth spectral density of the form of (5-28) is to be sampled and interpolated by a zero-order hold, and the sampling frequency is to be selected so that the percentage error meets an acceptable level. For purposes of illustration, several orders of data and error levels will be compared for the same break frequency, i.e., $\omega_o = 2\pi f_o = 2\pi \times 10^4$ rps. No attempt is made to normalize the signal power of different order spectra to the same level since the percentage error criterion takes this into account.

<u>First-Order Data</u>. The first-order Butterworth spectrum is recognizable as that of the exponential autocorrelation function with $R(0) = \omega_o/2$. The smallest applicable value of k which satisfies (2-24) is ω_o^2, thus $\omega_n = \frac{k}{2R(0)} = \omega_o$. Once ω_n is known, Figure 13 may be used to evaluate

(ω_s/ω_n) for a given error level. For this first-order signal, the sampling frequencies must be 6250 KC, 1250 KC, and 625 KC to bound percentage error to 1%, 5%, and 10% respectively, as defined by (5-25).

Second-Order Data. For higher order Butterworth spectra, ω_d is given by (5-29) or, for n = 2,

$$\omega_d = \left[\frac{\sin(\pi/4)}{\sin(3\pi/4)}\right]^{1/2} \omega_o = 2\pi \times 10^4 \ .$$

Once ω_d is known, the generalized bound in Figure 11 may be used to determine (ω_s/ω_d) for a given error level. Alternately, ω_s may be computed from (5-26). For the second-order signal, the sampling frequencies must be 362 KC, 162 KC, and 114 KC for percentage error levels of 1%, 5%, and 10% respectively.

Third-Order Data. For the third-order Butterworth spectrum, (5-29) may be used to evaluate ω_d,

$$\omega_d = \left[\frac{\sin(\pi/6)}{\sin(3\pi/6)}\right]^{1/2} \omega_o = \sqrt{2}\,\pi \times 10^4 \ .$$

Again either Figure 11 or (5-26) may be used to show that the sampling frequency must be 256 KC, 114.5 KC, or 80.5 KC for error levels of 1%, 5%, and 10% respectively.

Infinite-Order Butterworth Spectrum. As $n \to \infty$ in (5-28), the spectrum approaches the flat band-limited spectrum of (5-2) where $N_o = 1$ and

$$\omega_d = \lim_{n \to \infty} \left[\frac{\sin(\pi/2n)}{\sin(3\pi/2n)}\right]^{1/2} \omega_o = \frac{2\pi}{\sqrt{3}} \times 10^4 \ .$$

The necessary sampling rates may be determined by either method and are 209 KC, 94.6 KC, and 65.7 KC for error levels of 1%, 5%, and 10% respectively.

CHAPTER VI

CONCLUSIONS

The zero-order sample-and-hold interpolator is a widely used interpolation device whose performance is usually rated in terms of an expected mean square error criterion. The research was principally directed towards estimation of this error for the sampling of fairly general classes of random processes; however, several useful results pertaining to the general behavior of the second-order statistics of random processes were obtained as well as an analysis of the quality of the expected mean square error criterion as an estimate of the true error behavior.

Random processes which are either band-limited or have spectral densities expressible as ratios of even polynomials in ω have variations which are bounded by relatively simple functions of τ involving basic statistical parameters of the process. For a band-limited process, the variation is, of course, constrained to lie between 0 and 2R(0) but, in addition, has both a functional upper bound and a non-zero lower bound. The derivative behavior of band-limited variations is such that they are monotonic increasing for $\tau \in [0, \pi/\omega_c]$ and are convex (have non-negative second derivative) for $\tau \in [0, \pi/2\omega_c]$. Non-band-limited processes, while also constrained to lie between 0 and 2R(0), also possess either a quadratic or linear upper bound depending on the differentiability of the process. The variation of any process cannot equal either of its theoretical extremes, 0 or $2R(0)$, for $\tau \neq 0$, unless $x(t)$ is mean square periodic, i.e., $R(\tau) = R(\tau + T_o)$ in which case the spectral density is of the impulse summation form.

A meaningful expected value error criterion for a general interpolator may be defined as the expected value of the time averaged squared difference between the interpolated output and the delayed original input. For the zero-order sample-and-hold interpolator, this error criterion is a functional of the variation and may thus be bounded in terms of the relations obtained for the variation, thereby leading to the curves illustrated in Figures 11 and 13. The utility of these bounds has been enhanced by generalizing them in terms of percentage interpolation error and a normalized sampling rate. Once the effective bandwidth parameter, ω_n or ω_d, is determined from basic process statistics, it may be used in conjunction with these two curves to select a sampling rate which will satisfy a constraint on interpolation error.

It is well known that the operation of zero-order sample-and-hold interpolation introduces an effective delay, i.e., the interpolated output $\hat{x}(t)$ is a better approximation, in the mean square sense, to a delayed version of the input, $x(t-d)$, than it is to the original undelayed sampled process, $x(t)$. It is widely stated that the value of this delay is one-half the sampling period; however, for random processes there exists a condition on the first derivative of $V(\tau)$ which determines the value of this delay and is not necessarily satisfied for a delay of one-half sampling period. Although there do exist random processes for which the effective delay is not one-half sampling period, the derivative condition is automatically satisfied for any band-limited process sampled at a rate greater than one-half the Nyquist rate $(T \leq \frac{2\pi}{\omega_c})$.

For the classes of spectral densities considered, either band-limited or ratios of even polynomials, the limiting behavior of the zero-order hold judged in terms of the sampling rate versus interpolation error trade off is the same for all differentiable processes, band-limited or not, and is twice that for first-order data (non-band-limited spectral densities rolling off as $(\frac{1}{\omega})^2$).

The interpolation error in an exponential hold for differentiable random processes has a quadratic upper bound. Furthermore, there exists a decay rate which will minimize this bound for a given sampling rate yielding some improvement over the zero-order hold bound; however, for high sampling rates and low values of decay, the two bounds converge.

The expected mean square error behavior of a zero-order sample-and-hold interpolator with a randomly fluctuating sampling interval (sampling time jitter) is bounded by a function dependent upon the properties of the variation bound and the statistics of the jitter.

The expected mean square error criterion utilized to evaluate interpolator performance is a good estimate of the actual behavior of the error from interval to interval. Several approaches were used to point out the relationship between the expected value of the interpolator error, $\bar{\psi}(T, \lambda)$, and the range of values which $\psi(nT, \lambda)$ can assume. The Bienayme Inequality may be used to show that $P\{\psi(nT, \lambda) \geq k\bar{\psi}(T, \lambda)\} \leq \frac{1}{k}$.

For differentiable Gaussian processes, where the sampling rate is sufficiently high that the error in an interval is approximately a straight line, the Tchebycheff Inequality may be used to establish a similar relationship, $P\{\psi_1(nT, \lambda) \geq k\bar{\psi}_1(T, \lambda) \geq k\bar{\psi}_1(T, \lambda)\} \leq \frac{2}{(k-1)^2}$. In addition, for such processes, a confidence level exists which relates the time average interpolation error over a finite run (a number of consecutive sampling intervals) to the expected mean square error. This confidence level depends upon the duration of the run, the number of sampling intervals in the run, and the nature of the derivative of the sampled process ($R''(\tau)$).

The central conclusion drawn from the research is that the zero-order sample-and-hold interpolator possesses extremely well-behaved expected mean square error characteristics, which may be used in their simplified bounding forms to estimate a suitable sampling rate, yet depend only upon basic input process statistics. Furthermore, the overall performance of the interpolator is adequately represented by this criterion.

APPENDIX I

GENERAL BEHAVIOR OF THE AUTOCORRELATION FUNCTION

The results of Chapter II were obtained for the variation, not directly for the more familiar autocorrelation function, because the frequency domain integral for the variation is simpler to manipulate. The two functions are directly related, however, and differ only by a constant and a sign inversion. As the autocorrelation function is the standard second order statistical parameter, this appendix will present a synopsis of the results of Chapter II in terms of their effects upon $R(\tau)$. The following properties are the most important. Although two other quadratic bounds (an upper and a lower) were discussed in Chapter II, they cannot tighten the bounds given here and are therefore not included in this appendix.

Suppose $x(t)$ is a random process with autocorrelation function $R(\tau)$ with Fourier Transform $S(\omega)$ band-limited to ω_c, then $R(\tau)$ has the following properties:

$$R(\tau) \leq R(0) - 2\left[\frac{\sin(\frac{\omega_c\tau}{2})}{\omega_c}\right]^2 |R''(0)| \quad \text{for } \tau \in [0, \frac{2\pi}{\omega_c}], \qquad \text{(A1.1)}$$

$$R(\tau) \geq R(0) - |R''(0)| \frac{\tau^2}{2} \quad \text{for } \tau \in [0, 2\sqrt{R(0)/|R''(0)|}], \qquad \text{(A1.2)}$$

$$R(\tau) \geq R(0) - 2\sin^2(\frac{\omega_c\tau}{2})R(0) \quad \text{for } \tau \in [0, \frac{\pi}{\omega_c}], \qquad \text{(A1.3)}$$

$$R(\tau) \text{ is concave for } \tau \in [0, \pi/2\omega_c] , \quad \text{(A1.4)}$$

$$R(\tau) \text{ is monotonically decreasing for } \tau \in [0, \pi/\omega_c]. \quad \text{(A1.5)}$$

Proof: All of the above follow from the results of Chapter II and the fact that

$$R(\tau) = R(0) - V(\tau).$$

The implications of the above may best be appreciated by inspection of the sketch given in Figure 14. $R(\tau)$ is constrained to lie within the shaded region. Defining τ_1 as that value of τ for which the bound of (A1.2) intersects $-R(0)$, i.e.,

$$\tau_1 \equiv 2\left[\frac{R(0)}{|R''(0)|}\right]^{1/2}$$

and since $|R''(0)| \geq \omega_c^2 R(0)$, then $\tau_1 \geq \frac{2}{\omega_c}$. For those $R(\tau)$ with small values of τ_1, it is apparent that a combination of the bounds of (A1.2) and (A1.3) must be used to yield the best overall bound.

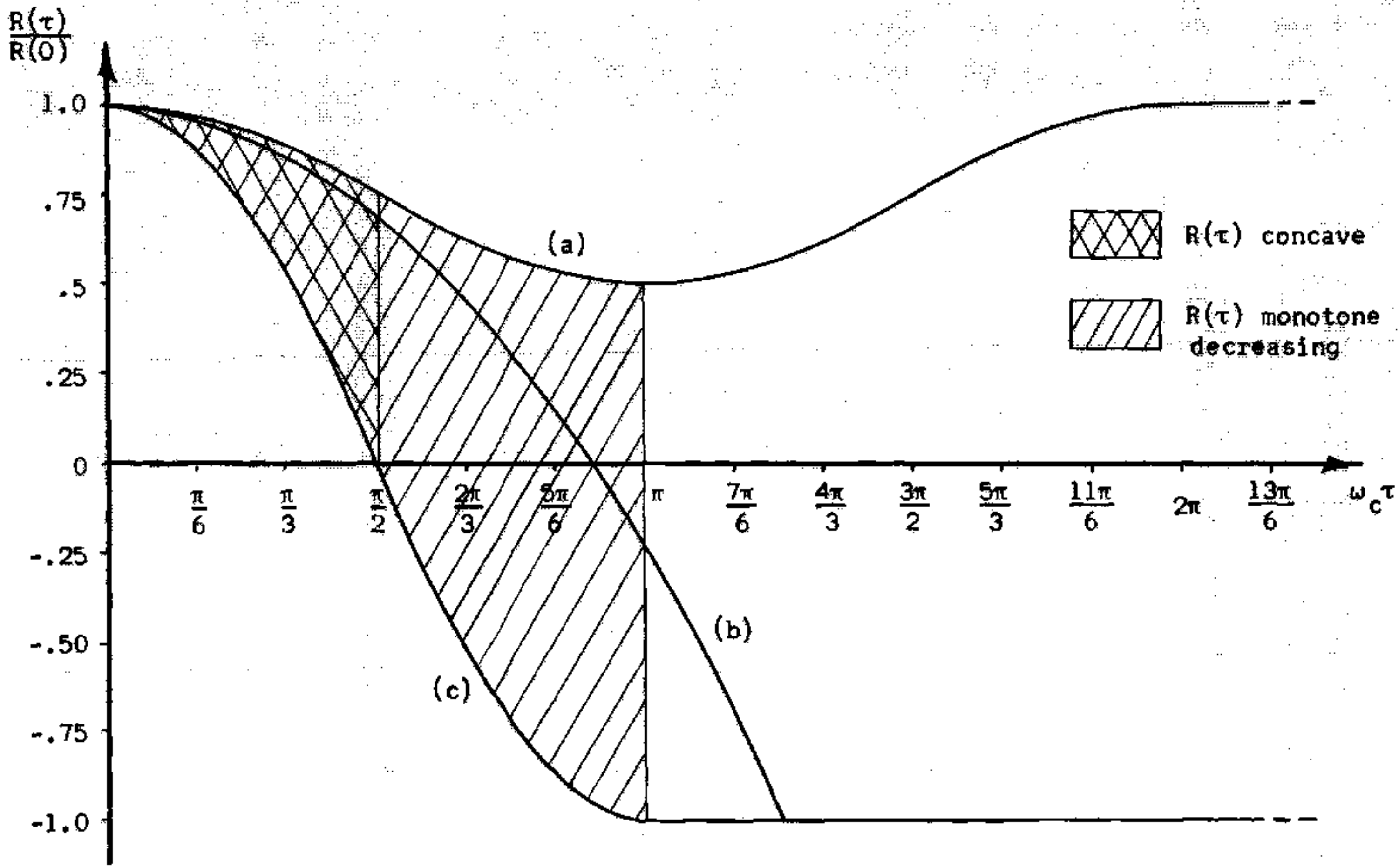

Figure 14. Bounds for a Band-Limited Autocorrelation Function with

$$|R''(0)| = \frac{\omega_c^2}{4} R(0).$$

a) Upper Sine Bound -- (A1.1),
b) Quadratic Lower Bound -- (A1.2),
c) Lower Sine Bound -- (A1.3).

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